twining vielbein matrix $V^i_{\ \mu}$: every coordinate index can be transformed into an intrinsic one by $V^i_{\ \mu}$:

$$A^{i\cdots}_{\cdots\cdot} = V^i_{\ \mu}\, A^{\mu\cdots}_{\cdots\cdot} \tag{I.2.92}$$

and vice versa

$$A^{\mu\cdots}_{\cdots\cdot} = V^\mu_{\ i}\, A^{i\cdots}_{\cdots\cdot} \tag{I.2.93}$$

In particular

$$A^i A_i \equiv \eta_{ij} A^i A^j = \eta_{ij} V^i_{\ \mu} V^j_{\ \nu} A^\mu A^\nu = g_{\mu\nu} A^\mu A^\nu = A^\mu A_\mu \ . \tag{I.2.94}$$

where we have used Eq. (I.2.29).

Therefore coordinate scalars are also Lorentz scalars and vice versa.

Other useful relations are the following ones:

$$\mathcal{D} A^i = V^i_{\ \mu}\, \nabla A^\mu \tag{I.2.95a}$$

$$\nabla A^\mu = V^\mu_{\ i}\, \mathcal{D} A^i \tag{I.2.95b}$$

where $\mathcal{D}$ and ∇ are the covariant derivatives in the tangent or natural frames respectively.

Eqs. (I.2.95) can be proved by direct computation using Eqs. (I.2.90). Notice that the affine connection entering (I.2.95) is symmetric in its lower indices which implies that the torsion R^a is zero.

Therefore (I.2.95) is <u>not true</u> in presence of a <u>nonvanishing torsion</u>.

CHAPTER I.3

GROUP MANIFOLDS AND MAURER-CARTAN EQUATIONS

I.3.1 - Introduction

In this chapter we discuss Lie groups from a differential geometric point of view. As in previous chapters we just give those main definitions and properties which are relevant for the subsequent developments; previous knowledge of group theory is required.

The chapter is divided in two parts; in the first (Section 1 to 6) we concentrate on the study of those properties which are peculiar to group manifolds, like the existence of left- and right-invariant vector fields or 1-forms. This leads to the discussion of the Lie algebra associated to Lie groups and to the dual concept of Maurer-Cartan equations. Within the same framework we shortly discuss the adjoint and co-adjoint representations of groups and algebras and the Killing metric; finally a short account is given of the Riemannian geometry of semi-simple group manifolds.

The second part of this chapter is devoted to the study of manifolds which are locally diffeomorphic to group manifolds. They are obtained by softening the rigid metric structure of the group manifolds in the same way as the manifold of the translation group $\mathbb{R}^n$ can be locally softened to a general Riemannian manifold.

Manifolds obtained in this way are named soft-group manifolds. In particular we discuss the process of factorization of the curvatures which gives rise to the fiber bundle structure of the manifold.

The last section of the chapter is devoted to a detailed study of the important cases of (anti)-de Sitter and Poincaré soft group manifolds.

I.3.2 - Lie groups as manifolds: left and right invariant vector fields

A group G is a Lie group if it is a smooth manifold and if the map

$$G \times G \to G \tag{I.3.1}$$

defined by

$$(x,y) = xy \qquad x,y \in G \tag{I.3.2}$$

and the inverse mapping

$$G \to G \tag{I.3.3a}$$

defined by

$$x \to x^{-1} \tag{I.3.3b}$$

are both smooth.

In particular if a is a fixed element of G, then the left translation:

$$L_a : G \to G \tag{I.3.4a}$$

$$L_a(g) = ag \tag{I.3.4b}$$

and the right translation

$$R_a : G \to G \tag{I.3.5a}$$

$$R_a(g) = ga \tag{I.3.5b}$$

are diffeomorphisms.

From the associativity of the G multiplication

$$(ax)b = a(xb) \tag{I.3.6}$$

it follows that the left and right translations commute

$$[L_a, R_b] = 0 \quad . \tag{I.3.7}$$

Let us consider the tangent space $T_e(G)$ at the identity e. The diffeomorphism

$$e \to ge \equiv g \quad , \qquad g \in G \tag{I.3.8}$$

induces a map between $T_e(G)$ and $T_g(G)$ according to (I.1.123);

$$v_g = (L_g)_* \, v_e \quad . \tag{I.3.9}$$

The vector field obtained in this way is left-invariant. Indeed using (I.1.125) and (I.3.9):

$$(L_{g'})_* \, v_g = (L_{g'})_* \, (L_g)_* \, v_e = (L_{g'g})_* \, v_e = v_{g'g} \quad . \tag{I.3.10}$$

This shows that the functional form of $(L_{g'})_* \, v_g$ at $g'g$ is the same as that of v_g at g.

In an analogous way one can show that the vector field

$$v_g = (R_g)_* v_e \qquad \text{(I.3.11)}$$

is right invariant.

Since the left- (right-) translation is a diffeomorphism, by taking into account property (I.1.216), one sees that the subset of left- (right-) invariant vector fields is closed under the Lie bracket operation.

Hence the left- (right-) invariant vector fields form a Lie algebra.

Definition: The Lie algebra, $\mathfrak{G}$, of the left- (right-) invariant vector fields on G is called the Lie algebra of the group G. As a matter of convention in the following we shall mainly refer to left-invariant vector fields.

Since any left-invariant vector field is uniquely determined by its value at e, the identity element of G, $\mathfrak{G}$ can be identified with $T_e(G)$.

Let us introduce a basis, T_A, $A = 1, \ldots \dim(G)$ on $T_e(G)$: then

$$[T_A, T_B] = C_{AB}{}^C\, T_C \qquad \text{(I.3.12)}$$

where the $C_{AB}{}^C$ are constants. Indeed, since the left hand side of (I.3.12) is left-invariant, the same must be true for the right hand side; this implies that the $C_{AB}{}^C$ are left-invariant, that is constant. The $C_{AB}{}^C$ are called the structure constants of the Lie algebra of G. Actually the presence in Eq. (I.3.12) of structure constants instead of structure functions is what characterizes the Lie algebra of left-invariant vector fields on G.

From the Jacobi identity of vector fields:

$$[T_A,[T_B,T_C]] + [T_B,[T_C,T_A]] + [T_C,[T_A,T_B]] = 0 \qquad \text{(I.3.13a)}$$

one finds:

$$C^A{}_{B[C}\, C_{LM]}{}^B = 0 \qquad \text{(I.3.13b)}$$

which is the Jacobi identity for structure constants.

We now show that the left- (right-) invariant vector fields are the generators of right- (left-) translations and that each generator is in a one-to-one correspondence with the one-parameter subgroups of G. Let us consider a one-parameter subgroup H of G, that is the homomorphic map

$$\mathbb{R} \to H \subset G \quad . \qquad \text{(I.3.14)}$$

Let $\mathbb{R}$ be parametrized by t and g by the n coordinates x^i. Then the right translation

$$R_{H(t)}: g \to gH(t) = g' \qquad \text{(I.3.15)}$$

is a flow (I.1.203)

$$x'^i = H^t(t,x) \qquad \text{(I.3.16)}$$

where x' are the coordinates of g'.

According to the definition (I.1.207), there is an infinitesimal generator $t^{(R)}$ associated to the flow whose components are given by

$$t^{(R)i} = \left.\frac{d}{dt}\right|_{t=0} H^i(t,x) = \dot{H}^i(0,x) \quad . \qquad \text{(I.3.17)}$$

It is easy to see that this vector is left-invariant: indeed if $a \in G$, associativity of the group composition law implies:

$$L_a[g \cdot H(t)] = (ag)H(t) \quad . \qquad \text{(I.3.18)}$$

Hence, if we parametrize the left translation

$$g \to ag = g' \tag{I.3.19}$$

by

$$z = z(x,y) \tag{I.3.20}$$

where x,y and z are the coordinates of a,g and g' respectively, the value of the infinitesimal generator at g' is:

$$(L_a)_* t_g{}^{(R)} = \dot{H}^i(0,z(x,y)) \frac{\partial}{\partial z^i} \equiv t_{g'}{}^{(R)} \quad . \tag{I.3.21}$$

The last equality expresses the fact that the functional form of the components of $(L_a)_* t_g$ is the same as the one of t_g given in Eq. (I.3.17). In other words $t_g{}^{(R)}$ is left-invariant. If we had started with the left action of a one-parameter subgroup $H \subset G$.

$$g \to H(t)g = g' \tag{I.3.22}$$

the corresponding generator $t^{(L)}$ would have been right-invariant.

Hence we have shown that to each 1-parameter subgroup H of G there corresponds a generator of right- (left-) translations and the associated vector field is left- (right-) invariant.

It follows that the Lie derivative along the generator of right translations, $t^{(R)}$, of the right-invariant vector field $t^{(L)}$ is zero

$$\ell_{t^{(R)}} t^{(L)} = \left[t^{(R)}, t^{(L)}\right] = 0 \tag{I.3.23}$$

(Actually (I.3.23) is the infinitesimal form of (I.3.7)).

As an example let us consider the group

$$G = GL(n,\mathbb{R}) \tag{I.3.24}$$

The coordinates of a given element $g \in GL(n,\mathbb{R})$ can be taken to be the entries x^{ij} of the matrix g. The tangent space at the identity, $T_e(G)$, is spanned by the basis vectors

$$X_{ij} = \left.\frac{\partial}{\partial x^{ij}}\right|_e \tag{I.3.25}$$

so that a generic vector T at e can be written as

$$T = T^{ij} \left.\frac{\partial}{\partial x^{ij}}\right|_e \quad . \tag{I.3.26}$$

The Lie algebra of the tangent vectors at the identity (I.3.25) is thus isomorphic to the Lie algebra $M_n(R)$ of the $n \times n$ matrices T^{ij}.

In particular, the mapping:

$$t \to \exp tT \equiv H(t) \quad , \quad T \in M_n(R) \tag{I.3.27}$$

yields the one-parameter subgroup of $GL(n,\mathbb{R})$ whose infinitesimal generator is given by Eq. (I.3.26).

If $x^{ij}(t)$ denote the entries of the matrix e^{tT} then

$$x^{ij}(t) = \delta^{ij} + tT^{ij} + O(t^2) \quad . \tag{I.3.28}$$

Hence the components of the infinitesimal generator T^{ij} are given by

$$T^{ij} \doteq \left.\frac{d}{dt} x^{ij}\right|_{t=0} \quad . \tag{I.3.29}$$

Subalgebras of $M_n(R)$ give rise to subgroups of $GL(n,\mathbb{R})$. For instance if

$$T = - T^t \tag{I.3.31}$$

(i.e. T is antisymmetric) then the corresponding 1-parameter subgroup lies in SO(n).

In the following we shall often use the same notation T for both the tangent vector (I.3.26) and for the matrix of its components: $\{T^{ij}\} = T$, especially when we deal with a basis T_A, $(A = 1,\ldots,\dim G)$. When confusion can arise, the vector field T will be written with an arrow: $\vec{T}$.

I.3.3 - Maurer-Cartan equations

Now we introduce left-invariant 1-forms on a group-manifold G and we give a formulation of its Lie algebra in terms of 1-forms.

Let us consider again the diffeomorphic map L_g (I.3.8). On 1-forms ω we can define the pull-back map $L^*_{g^{-1}}$ (recall Eq. I.1.59)):

$$L^*_{g^{-1}}\omega_e = \omega_g \tag{I.3.32}$$

where

$$L_{g^{-1}}e = g^{-1}e = g^{-1} \quad . \tag{I.3.33}$$

Writing the mapping

$$h \to g^{-1} \cdot h \tag{I.3.34}$$

in coordinates

$$z = z(x,y) \tag{I.3.35}$$

where x are the fixed coordinates of g^{-1} and y are the coordinates of h, by setting

$$\omega_e = \omega_i(x,0)dy^i \tag{I.3.36}$$

we obtain

$$\omega_g = \omega_i \frac{\partial y^i}{\partial z^j} dz^j \quad . \tag{I.3.37}$$

The 1-form ω_g is left invariant. Indeed

$$\omega_{ag} = L^*_{(ag)^{-1}}\omega_e = L^*_{g^{-1}a^{-1}}\omega_e = L^*_{a^{-1}}L^*_{g^{-1}}\omega_e = L^*_{a^{-1}}\omega_g \tag{I.3.38}$$

where we have used property (I.1.169). Thus a field of left-invariant 1-forms is completely determined by its value at e.

Let us take a basis of left-invariant 1-forms at $T_e{}^*(G)$:

$$\{\sigma^A\}: \quad (A = 1,\ldots,\dim G) \quad . \tag{I.3.39}$$

Taking into account property (I.1.171) we have

$$L^*_a d\sigma^A = dL^*_a\sigma^A = (d\sigma^A)_{a^{-1}} \quad . \tag{I.3.40}$$

Since $d\sigma^A$ is also left-invariant we may expand it in the complete basis of 2-forms at e:

$$d\sigma^A = -\frac{1}{2} C^A{}_{BC}\sigma^B \wedge \sigma^C \tag{I.3.41}$$

where the $C^A{}_{BC}$ functions being left invariant are actually constants. Equations (I.3.41) are called the Maurer-Cartan equations for the left-invariant 1-forms σ^A. The content of the Maurer-Cartan equations (I.3.41) is completely equivalent to that of equations (I.3.12). Indeed, Eqs. (I.3.41) provide the dual formulation of the Lie algebra. To show this, let us introduce the basis of left-invariant vectors $T_A{}^{(R)}$ dual to the cotangent basis σ^A of the left-invariant 1-forms.

$$\sigma^A(T_B{}^{(R)}) = \delta^A{}_B \quad . \tag{I.3.42}$$

The label R is a reminder that $T_A^{(R)}$ generate right translations on G; for notational simplicity in the sequel we omit the label R. Evaluating both sides of (I.3.41) on the vectors T_M, T_N; we get:

$$d\sigma^A(T_M,T_N) = -\frac{1}{2} C^A{}_{BC}\sigma^B \wedge \sigma^C(T_M,T_N) \quad . \tag{I.3.43}$$

Using Eq. (I.1.244) one obtains

$$d\sigma^A(T_M,T_N) = \frac{1}{2}(T_M\sigma^A(T_N) - T_N\sigma^A(T_M) - \sigma^A([T_M,T_N])) =$$

$$= -\frac{1}{2} C^A{}_{BC}\sigma^B \wedge \sigma^C(T_M,T_N) \quad . \tag{I.3.44}$$

Since $T_M\sigma^A(T_N) = T_N\sigma^A(T_M) = 0$ because of Eq. (I.3.42), we have:

$$\sigma^A([T_M,T_N]) = C^A{}_{MN} \tag{I.3.45}$$

and therefore

$$[T_M,T_N] = C^A{}_{MN}T_A \quad . \tag{I.3.46}$$

In this way we also see that the constants entering the Maurer-Cartan equations are the structure constants defined by the Lie algebra.

In particular we notice that the Jacobi identity for the structure constants, Eq. (I.3.13), can be retrieved from the integrability condition $d^2 = 0$ of the Maurer-Cartan equations. Indeed, taking the exterior derivative of both sides of Eq. (I.3.41), one obtains

$$d(d\sigma^A) \equiv 0 = -\frac{1}{2}\, 2\, C^A{}_{BC}d\sigma^B \wedge \sigma^C =$$

$$= -C^A{}_{BC}C^B{}_{LM}\sigma^L \wedge \sigma^M \wedge \sigma^C \quad . \tag{I.3.47}$$

Taking into account the antisymmetry of $\sigma^L \wedge \sigma^M \wedge \sigma^C$ we get

$$C^A{}_{B[C}\, C^B{}_{LM]} = 0 \tag{I.3.48}$$

that is Eq. (I.3.41).

I.3.4 - Adjoint representation and Killing metric

From the commutativity of the left and right translations on G, (Eq. (I.3.7)) one easily deduces the commutativity of the corresponding mapping between tangent and cotangent vectors:*

$$[L_{a*},R_{a*}] = 0 \tag{I.3.49a}$$

$$[\overset{*}{L}_a,\overset{*}{R}_b] = 0 \quad . \tag{I.3.49b}$$

In particular if ω is a left invariant 1-form then

$$\overset{*}{L}_a(\overset{*}{R}_b\omega) = \overset{*}{R}_b\overset{*}{L}_a\omega = \overset{*}{R}_b\omega \tag{I.3.50}$$

that is, $\overset{*}{R}_b\omega$ is also left-invariant. The same is true for left invariant vector fields X

$$L_{a*}R_{b*}X = R_{b*}X \tag{I.3.51}$$

and in general for any left-invariant (k,ℓ) tensor on G.

Let us consider in particular the inner automorphism

$$I: G \to G \tag{I.3.52}$$

given by

---

* $(L_a)_*$ is now written as L_{a*}.

$$I: x \to b\,x\,b^{-1} \tag{I.3.53}$$

where b is some fixed element of G.

On the Lie algebra generators $T_A \in \mathbb{G}$, I induces the mapping

$$I_*(T_A) = L_{b*}\, R_{b^{-1}*} T_A = R_{b^{-1}*} T_A \quad . \tag{I.3.54}$$

This is a representation of G on the vector space $\mathbb{G}$ and is called the adjoint representation of G. It will be denoted by Adj (b).

In coordinates $R_{b^{-1}*}$ is the Jacobian of the map (I.3.53) evaluated at e.

As an example we take $G = GL(n,\mathbb{R})$. Using as coordinates the entries x^{ij} of $g \in GL(n,\mathbb{R})$, the map (I.3.53) becomes:

$$\mathrm{Adj}(b):\ x'^{ij} = b^{ik}\, x^{k\ell}(b^{-1})^{\ell j} \tag{I.3.55}$$

whose Jacobian is

$$J = \left.\frac{\partial x'^{ij}}{\partial x^{k\ell}}\right|_e = b^{ik}(b^{-1})^{\ell j} \quad . \tag{I.3.56}$$

Therefore if

$$T = T^{ij} \left.\frac{\partial}{\partial x^{ij}}\right|_e \tag{I.3.57}$$

is a generator of the Lie algebra, the components of Adj (b)T are given by

$$T'^{ij} = b^{ik}(b^{-1})^{\ell j} T^{k\ell} \tag{I.3.58}$$

that is, in matrix notation

$$T' = b\,T\,b^{-1} \quad . \tag{I.3.59}$$

If b is given by a one-parameter subgroup

$$b = e^{tB} \tag{I.3.60}$$

then the corresponding representation for the Lie algebra $\mathbb{G}$:

$$\mathrm{Adj}(B):\ \mathbb{G} \to \mathbb{G} \tag{I.3.61}$$

can be obtained using the analogue of Eq. (I.3.17) applied to the automorphism (I.3.59); one finds:

$$\mathrm{Adj}_B A = \frac{d}{dt}\,\mathrm{Adj}(e^{tB})A\Big|_{t=0} = \frac{d}{dt}\,(e^{tB}Ae^{-tB})\Big|_{t=0} = [B,A] \quad . \tag{I.3.62}$$

If A and B are two basis generators of $\mathbb{G}$:

$$A \equiv T_A \quad ; \quad B \equiv T_B \tag{I.3.63}$$

then

$$\mathrm{Adj}_B T_A = [T_B, T_A] = C^L{}_{BA} T_L \quad . \tag{I.3.64}$$

Hence

$$(T_B)^L{}_A = C^L{}_{BA} \quad . \tag{I.3.65}$$

In an analogous way one can represent the automorphism (I.3.53) on the left invariant 1-forms σ^B. In that case one deals with the coadjoint representation of G in $\mathbb{G}$. The coadjoint representation of the Lie algebra acts on forms rather than on vectors and can be defined by

$$[\mathrm{coadj}(T_A)]\sigma^B(T_C) \underset{\mathrm{def}}{=} \sigma^B(\mathrm{Adj}(T_A)T_C) = \sigma^B([T_A,T_C]) = $$
$$= C^L{}_{AC}\sigma^B(T_L) = C^B{}_{AC} \ . \tag{I.3.66}$$

Therefore

$$\mathrm{coadj}(T_A)\sigma^B = C^B{}_{AC}\sigma^C \quad . \tag{I.3.67}$$

We also notice that the adjoint and the coadjoint representations give the change of the left-invariant vector fields and 1-forms under the Lie derivative along the left-invariant generators $T_A{}^{(R)}$. Indeed, recalling that $T_A{}^{(R)}$ generate right translations and using definitions (I.1.223) and (I.1.227), we obtain

$$\ell_{T_A} T_B = [T_A, T_B] = C^L{}_{AB} T_L = \mathrm{Adj}(T_A) T_B \tag{I.3.68}$$

$$\ell_{T_A}\sigma^B = (T_A \rfloor d + d\, T_A \rfloor)\sigma^B = T_A \rfloor (-\frac{1}{2} C^B{}_{LM}\sigma^L \wedge \sigma^M) + d(\delta^B{}_A) =$$
$$= -\frac{1}{2} C^B{}_{LM}(\delta^L{}_A\sigma^M - \delta^M{}_A\sigma^L) = C^B{}_{LA}\sigma^L = -\,\mathrm{coadj}(T_A)\sigma^B \quad . \tag{I.3.69}$$

A set of independent 1-forms, i.e. a cotangent basis on G, can be obtained in terms of the group element g. Consider the 1-form:

$$\sigma = g^{-1} dg \quad . \tag{I.3.70}$$

This 1-form is left-invariant. Indeed, under translation through a fixed element $b \in G$ we have

$$L_b^*\sigma = (bg)^{-1} d(bg) = g^{-1}b^{-1}b dg = g^{-1}dg = \sigma \quad . \tag{I.3.71}$$

Differentiating both sides of Eq. (I.3.70) one obtains

$$d\sigma = dg^{-1} \wedge dg \tag{I.3.72}$$

and using

$$dg^{-1} \cdot g = -\,g^{-1}dg = -\,\sigma \tag{I.3.73}$$

one obtains

$$d\sigma + \sigma \wedge \sigma = 0 \quad . \tag{I.3.74}$$

We notice that

$$\sigma = g^{-1}dg \tag{I.3.75}$$

is a Lie algebra valued matrix of 1-forms and therefore can be expanded along the set of generators T_A (in their matrix realization):

$$\sigma = \sigma^A T_A \quad . \tag{I.3.76}$$

This can be proved by evaluating $g^{-1}dg$ at the origin e. The 1-forms σ^A span a cotangent basis.

Introducing the expansion (I.3.76) in (I.3.74) and using (I.3.12) one obtains again the Maurer-Cartan Eqs. (I.3.41). In a matrix representation of G, Eq. (I.3.74) is a matrix equation for a set of $\dim G$ linearly independent 1-forms, and can be used to explicitly compute the structure constants of G.

As an example let us derive the Maurer-Cartan equations for the Poincaré group in D dimensions (the group of rigid motions in D dimensions).

As in the four dimensional case it is defined to be the semi-direct product of the Lorentz group, $SO(1,D-1)$, with the D-dimensional translations and will be denoted by $ISO(1,D-1)$. It can be realized as the group of $(D+1) \times (D+1)$ matrices g of the form:

$$g = \begin{pmatrix} \Lambda & \xi \\ 0 & 1 \end{pmatrix} \qquad (I.3.77)$$

where Λ is a matrix of $SO(1,D-1)$ in the vector representation and $\xi \varepsilon \mathbb{R}^D$, the group of translations in D dimensions (with the vector addition as composition law).

The inverse of (I.3.77) is given by:

$$g^{-1} = \begin{pmatrix} \Lambda^{-1} & -\Lambda^{-1}\xi \\ 0 & 1 \end{pmatrix} \qquad (I.3.78)$$

so that the matrix $g^{-1}dg$ of left-invariant 1-forms is:

$$g^{-1}dg = \begin{pmatrix} \Lambda^{-1} & -\Lambda^{-1}\xi \\ 0 & 1 \end{pmatrix}\begin{pmatrix} d\Lambda & d\xi \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -\omega & V \\ 0 & 0 \end{pmatrix} \qquad (I.3.79)$$

where we have defined

$$\omega = - \Lambda^{-1}d\Lambda = d\Lambda^{-1}\Lambda \qquad (I.3.80)$$

$$V = \Lambda^{-1}d\xi \qquad (I.3.81)$$

By differentiation we immediately see that

$$d\omega = \omega \wedge \omega \qquad (I.3.82)$$

$$dV = - \Lambda^{-1}d\Lambda \wedge \Lambda^{-1}d\xi = \omega \wedge V \qquad (I.3.83)$$

that is we obtain the Maurer-Cartan equations of $ISO(1,D-1)$.

To arrive at the usual formulation of the Lie algebra of $ISO(1,D-1)$ in terms of commutation relations among generators, we introduce the dual tangent vectors T_A:

$$T_A: \{P_a;J_{ab}\} \qquad (I.3.84)$$

P_a and J_{ab} being respectively the generators of the translations and of the Lorentz transformations; by definition of dual algebra:

$$V^a(P_b) = \delta^a_b \quad ; \quad V^a(J_{cd}) = 0 \qquad (I.3.85)$$

$$\omega^{ab}(J_{cd}) = \delta^{ab}_{cd} \quad ; \quad \omega^{ab}(P_c) = 0 \quad . \qquad (I.3.86)$$

Evaluating Eqs. (I.3.83-4) on the couples of vectors $\{P_a,P_b\}$, $\{P_a,J_{bc}\}$ and $\{J_{ab},J_{cd}\}$ respectively and using Eq. (I.1.244) we obtain the commutation relations of the Poincaré group in D-dimensions:

$$[P_a,P_b] = 0 \qquad (I.3.87a)$$

$$[J_{ab},J_{cd}] = -\frac{1}{2}(J_{ac}\eta_{bd} + J_{bd}\eta_{ac} - J_{ad}\eta_{bc} - J_{bc}\eta_{ad}) \qquad (I.3.87b)$$

$$[J_{ab},P_c] = -\frac{1}{2}(\eta_{ac}P_b - \eta_{bc}P_a) \quad . \qquad (I.3.87c)$$

I.3.5 - Killing metric

Now we introduce a metric on G which is biinvariant, namely it is both left- and right-invariant.

Suppose we take a metric ϕ_e at $T_e(G)$:

$$\phi: X_e,Y_e \rightarrow \phi_e(X_e,Y_e) \varepsilon \mathbb{R} \quad . \qquad (I.3.88)$$

By left translation we determine a metric field on G

$$\phi_g = L^*_{g^{-1}} \phi_e \quad . \qquad (I.3.89)$$

As in the case of 1-forms one easily shows that ϕ_g is left-invariant:

$$L^*_a \phi_{ag} \equiv \phi_{a^{-1}ag} = \phi_g \quad . \tag{I.3.90}$$

Analogously $R^*_{g^{-1}}\phi_e$ is a right-invariant metric.

In order to get a biinvariant metric we need:

$$L^*_g R^*_{g^{-1}} \phi_e \equiv \mathrm{Adj}(g)\phi_e = \phi_e \quad . \tag{I.3.91}$$

To fulfill this requirement one defines the metric on $T_e(G)$ by the so-called Killing form:

$$g(X_e,Y_e) = \mathrm{Tr}(\mathrm{Adj}(X_e)\ \mathrm{Adj}(Y_e)) \quad . \tag{I.3.92}$$

This form is obviously bilinear and symmetric and therefore defines a metric on $T_e(G)$. Moreover it is biinvariant; indeed, using the cyclic property of the trace and recalling Eq. (I.3.89), one has

$$\mathrm{Adj}(a)g(X_e,Y_e) = \mathrm{Tr}(a\ \mathrm{Adj}(X_e)a^{-1}\, a\,\mathrm{Adj}(Y_e)a^{-1})$$
$$= g(X_e,Y_e) \quad . \tag{I.3.93}$$

If we take $X_e = T_A$, $Y_e = T_B$ then

$$\mathrm{Adj}(T_A)\cdot\mathrm{Adj}(T_B)(T_C) = \mathrm{Adj}(T_A)C^L{}_{BC}T_L =$$
$$= C^L{}_{BC}C^M{}_{AL}T_M \quad . \tag{I.3.94}$$

Hence

$$g_{AB} = g(T_A,T_B) = \mathrm{Tr}(\mathrm{Adj}(T_A)\ \mathrm{Adj}(T_B)) = C^L{}_{BM}C^M{}_{AL} \quad . \tag{I.3.95}$$

If the Killing form is non-degenerate the group is said to be semi-simple. For compact groups one can prove that g_{AB} is negative definite. To see what is the implication of biinvariance of g_{AB} let us rewrite Eq. (I.3.93) using $a \equiv e^{tA}$: one has

$$\mathrm{Tr}(e^{tA}\mathrm{Adj}(X_e)e^{-tA}e^{tA}\mathrm{Adj}(Y_e)e^{-tA}) = \mathrm{Tr}(\mathrm{Adj}(X_e)\mathrm{Adj}(Y_e)) \tag{I.3.96}$$

Differentiating with respect to t at $t=0$ one obtains:

$$\mathrm{Tr}([A,\ \mathrm{Adj}(X_e)\mathrm{Adj}(Y_e)]) = \mathrm{Tr}([A,\ \mathrm{Adj}(X_e)]\mathrm{Adj}\ Y_e + \mathrm{Adj}(X_e)\ \times$$
$$\times\ [A,\ \mathrm{Adj}(Y_e)]) = 0 \tag{I.3.97}$$

that is

$$g([A,X_e],Y_e) + g(X_e,[A,Y_e]) = 0 \quad . \tag{I.3.98}$$

Taking $A \equiv T_A$, $X_e = T_B$, $Y_e = T_C$, we obtain

$$C^L{}_{AB}g_{LC} + C^L{}_{AC}g_{BL} = 0 \tag{I.3.99}$$

Therefore defining

$$C_{ABC} = g_{AL}C^L{}_{BC} \tag{I.3.100}$$

one obtains

$$C_{ABC} + C_{ACB} = 0 \quad . \tag{I.3.101}$$

Taking into account the antisymmetry of $C^C{}_{AB}$ in A, B Eq. (I.3.101) implies complete antisymmetry of the lowered structure constants (I.3.100).

Finally we note that for semisimple groups the Killing metric can be used to lower or raise the indices of the Lie algebra; in particular the adjoint and coadjoint representations of the algebra are equivalent.

I.3.6 - Riemannian geometry of semisimple groups

On semisimple groups the Killing metric g_{AB} is non degenerate and, after some linear transformation on the generators, it can always be reduced to the diagonal form:

$$g_{AB} \equiv \eta_{AB} = \mathrm{diag}(\underbrace{+ \dots +}_{\text{p-times}} \quad \underbrace{- \dots -}_{\text{q-times}}) \qquad \text{(I.3.102)}$$

In this basis the Killing metric coincides with the tangent metric used in Chapter I.2 to define Riemannian geometry on an arbitrary manifold. For this reason in this section we confine ourselves to the study of the Riemannian geometry of semisimple Lie groups.

A Riemannian connection is introduced in the following way: let us consider again the Maurer-Cartan equations for the left-invariant 1-forms σ^A:

$$d\sigma^A + \frac{1}{2} C^A_{BC}\sigma^B \wedge \sigma^C = 0 \quad . \qquad \text{(I.3.103)}$$

Since the σ^A are a set of n independent 1-forms on G they can be used as a set of vielbeins on G. The associated dual moving frame is given by the Lie algebra vector fields T_A.

As connection we take the left-invariant 1-form $\omega^A{}_B$ defined by: by:

$$\omega^A{}_B = \frac{1}{2} C^A_{BC}\sigma^C \equiv \frac{1}{2} [\mathrm{Adj}(T_C)]^A{}_B \, \sigma^C \quad . \qquad \text{(I.3.104)}$$

Then the Maurer-Cartan equations merely express the fact the G is torsionless:

$$R^A = d\sigma^A - \omega^A{}_B \wedge \sigma^B \equiv 0 \quad . \qquad \text{(I.3.105)}$$

Moreover using the Killing metric:

$$\omega_{AB} = g_{AC}\omega^C{}_B = - \omega_{BA} \quad . \qquad \text{(I.3.106)}$$

We conclude that $\omega^A{}_B$ is a Riemannian connection (see Eqs. (I.2.37-38)).

Let us compute the corresponding curvature: using the definition (I.2.27b) we find:

$$R^A{}_B = d\omega^A{}_B - \omega^A{}_C \wedge \omega^C{}_B = - \frac{1}{4} (C^A_{BC}C^C_{DE} + C^A_{CD}C^C_{BE})\sigma^D \wedge \sigma^E =$$

$$= - \frac{1}{4} C^A_{CE}C^C_{BD}\sigma^D \wedge \sigma^E \quad . \qquad \text{(I.3.107)}$$

where in the last step we have used the Jacobi identity for the structure constants (I.3.13). Hence the intrinsic components of the curvature are constants.

$$R^A{}_{B|DE} = - \frac{1}{8} (C^A_{CE}C^C_{BD} - C^A_{CD}C^C_{BE}) \quad . \qquad \text{(I.3.108)}$$

Using the Killing metric g_{AB} to contract indices, one computes the Ricci tensor and the scalar curvature:

$$R_{BE} = R^A{}_{B|AE} = - \frac{1}{8} C^A_{CE}C^C_{BA} = \frac{1}{8} g_{BE} \qquad \text{(I.3.109)}$$

$$R = g^{BE} R_{BE} = \frac{1}{8} \dim G \quad . \qquad \text{(I.3.110)}$$

In particular from Eq. (I.3.109) we see that any semisimple group manifold is an Einstein space with respect to the Riemannian connection (I.3.104).

It is also possible to introduce a (non Riemannian) left-invariant connection on G such that $R^A{}_B$ is identically zero. It is sufficient to set

$$\hat{\omega}^A{}_B = C^A_{BC}\sigma^C \equiv (\mathrm{Adj}\ T_C)^A{}_B\sigma^C \quad . \tag{I.3.111}$$

Again we find:

$$\hat{\omega}^{AB} = -\hat{\omega}^{BA} \tag{I.3.112}$$

but now the corresponding torsion is different from zero:

$$R^A(\hat{\omega}) = d\sigma^A - \hat{\omega}^A{}_B \wedge \sigma^B = \frac{1}{2}C^A_{BC}\sigma^B \wedge \sigma^C \quad . \tag{I.3.113}$$

Hence $\omega^A{}_B$ is non Riemannian.

The intrinsic components of the torsion are given by the structure constants

$$R^A(\hat{\omega})_{BC} = \frac{1}{2}C^A{}_{BC} \quad . \tag{I.3.114}$$

The curvature tensor is:

$$R^A{}_B(\hat{\omega}) = C^A_{BC}d\sigma^C - C^A_{CD}\sigma^D \wedge C^C_{BF}\sigma^F = -\frac{1}{2}(C^A_{BC}C^C_{DF} + 2C^A_{CD}C^C_{BF})\sigma^D \wedge \sigma^F =$$

$$= (-\frac{1}{2}C^A_{BC}C^C_{DF} - \frac{1}{2}C^A_{CD}C^C_{BF} + \frac{1}{2}C^A_{CF}C^C_{BD})\sigma^D \wedge \sigma^F \equiv 0 \quad . \tag{I.3.115}$$

In deriving (I.3.115) we have used (I.3.103) and the Jacobi identities (I.3.13).

A manifold M is said to be parallelizable if one can find a connection 1-form $\omega^i{}_j$ such that the curvature $R^i{}_j(\omega)$ defined by (I.2.27b) is identically zero. $\overset{*}{\omega}{}^i{}_j$ is called the parallelizing connection. Thus Eq. (I.3.115) expresses the fact that every semisimple group manifold is parallelizable.

Any other left invariant connection not given by (I.3.104) or (I.3.111) gives rise to a non Riemannian manifold with non vanishing curvature.

I.3.7 - Soft group manifolds

Group manifolds G have a rigid structure: the left- or right-invariant vector fields and 1-forms have (in a given chart) a fixed coordinate dependence and, moreover, the Riemannian geometry of G is (locally) fixed in terms of its structure constants. As such they cannot be used as domains of definition of fields which should dynamically describe the space-time structure.

Nevertheless, as we show in a moment, the group manifolds G can be identified with the vacuum configurations of gravitational theories.

So we are led to consider manifolds $\tilde{G}$ in which the rigid topological and metric structure of G has been "softened" in order to describe non trivial physical configurations. $\tilde{G}$ manifolds are locally diffeomorphic to G and will be called soft group manifolds.

A well known example is space-time itself which, being diffeomorphic to $\mathbb{R}^4$, can be thought of as the soft group manifold of the four-dimensional translations. As a further example let us consider the soft Poincaré group manifold, naturally appearing in the vielbein formulation of gravity.

We first consider a flat Minkowskian space-time: its geometry is described by the vielbein V^a and a spin connection ω^{ab} fulfilling Eqs. (I.2.12). In a particular Lorentz gauge the solution is

$$V^a(x) = dx^a \tag{I.3.116a}$$

$$\omega^{ab}(x) \equiv 0 \tag{I.3.116b}$$

while in a general Lorentz gauge it reads:

$$V^a(x,\eta) = (\Lambda^{-1}(\eta)dx)^a \tag{I.3.117a}$$

$$\omega^{ab}(x,\eta) = (\Lambda^{-1}(\eta)d\Lambda(\eta))^{ab} \tag{I.3.117b}$$

(η^{ab} are the Lorentz parameters).

The solution (I.3.117) corresponds to the left-invariant 1-forms (I.3.81,82) of the Poincaré group (in four dimensions). Indeed we can identify the x^a and the η^{ab} with the parameters associated to the translations and the Lorentz rotations respectively. Therefore (I.3.117) satisfy the Maurer-Cartan equations (I.3.83-84). Moreover, since the Poincaré group, ISO(1,3), is locally isomorphic to $\mathbb{R}^4 \otimes SO(1,3)$, it can also be considered as a (trivial) principal bundle, $P(\mathbb{R}^4, SO(1,3))$, with base space given by

$$\mathbb{R}^4 \equiv ISO(1,3)/SO(1,3) \tag{I.3.118}$$

and SO(1,3) as fiber.

It follows that the rigid Poincaré group manifold describes the trivial configuration corresponding to flat Minkowski space.

Now suppose that the space-time M_4 is not flat: the fields V^a and ω^{ab}, subject to the gauge transformation laws (I.2.48) and (I.2.51) are now defined on a fiber bundle $P(M_4, SO(1,3))$. $P(M_4, SO(1,3))$ is <u>not isomorphic</u>, but just <u>locally diffeomorphic</u> to $G = ISO(1,3)$ due to the diffeomorphism $M_4 \sim \mathbb{R}^4$. In other words we have "softened" the rigid structure of the base space, $\mathbb{R}^4 \to M_4$, maintaining the structural group SO(1,3), which guarantees Lorentz covariance.

Notice that the curvatures R^a and R^{ab} associated to V^a and ω^{ab} are defined on the bundle through the gauge transformations (I.2.52) and these in turn imply "<u>horizontality</u>": the 2-forms R^a, R^{ab} do not contain the differential $d\eta^{ab}$; we express this by the equations:

$$\vec{J}_{ab} \rfloor R^{ab} = \vec{J}_{ab} \rfloor R^a = 0 \tag{I.3.119}$$

where J_{ab} is the left invariant vector field associated to the fiber SO(1,3).

If we now soften the Poincaré group manifold also in the direction of the fiber, we obtain the <u>soft Poincaré group manifold</u>, a manifold diffeomorphic to ISO(1,3) with no fiber bundle structure.

On this manifold the configurations of the V^a and ω^{ab} 1-forms are more general since their dependence on the parameters which were previously associated to Lorentz transformations is no longer factorized by a Lorentz transformation, and the curvatures R^a, R^{ab} are no longer horizontal.

In the particular case we are now discussing, this extension of the field configurations to the soft group manifold is not very significant from the physical point of view. Indeed a gravitational theory must be locally Lorentz invariant: hence we must end up with fields V^a, ω^{ab} having the correct gauge dependence on the Lorentz parameters.

The usual way to obtain this is to start directly with V^a and ω^{ab} defined on the principal bundle $P(M_4, SO(1,3))$. (In Chapter I.4 we will show that the fiber bundle structure can also be obtained from the variational principle starting with an action defined on the soft group manifold. This is certainly an interesting possibility from the point of view of the economy of concepts, but it is not of any fundamental importance).

In more general theories like supergravity theories, we will see that it is neither required nor desirable to factorize all the coordinates which are not associated to the translations. Indeed, starting from the super Poincaré group, only the Lorentz gauge transformations will be factorized: the gauge transformations of supersymmetry will not. The resulting theory will be described on a principal fiber bundle $P(M^{4/4}, SO(1,3))$ whose base space is the superspace $M^{4/4}$. This is discussed in detail in Chapters II.6 and III.3.

With this motivation in mind we now turn to the formal definitions and the important formulae concerning soft group manifolds.

Soft forms and curvatures

Let us start with the rigid group G and the set of left-invariant 1-forms σ^A satisfying the Maurer-Cartan equations (I.3.41):

$$d\sigma^A + \frac{1}{2} C^A_{BC}\sigma^B \wedge \sigma^C = 0 \quad . \qquad \text{(I.3.120)}$$

We soften G to the locally diffeomorphic soft group manifold $\tilde{G}$ by introducing new Lie algebra valued 1-forms

$$\mu = \mu^A T_A \qquad \text{(I.3.121)}$$

which are also soft, that is, non left-invariant. We note that the Lie algebra generators are considered in their matrix realization as in Eq. (I.3.76). The μ^A do not satisfy the Maurer-Cartan equations; the shift from zero of the l.h.s. of (I.3.120) defines the curvature of μ^A:

$$R^A \overset{\text{def}}{=} d\mu^A + \frac{1}{2} C^A_{BC}\mu^B \wedge \mu^C \qquad \text{(I.3.122)}$$

or, using $R = R^A T_A$

$$R = d\mu + \mu \wedge \mu \quad . \qquad \text{(I.3.123)}$$

We note that the definition of the soft 1-forms and of the associated curvature is the same as in Yang-Mills theory except that in our case μ^A is defined on a manifold $\tilde{G}$ which does not have an "a priori" fiber bundle structure.

When a fiber bundle structure is imposed on $\tilde{G}$ then μ^A becomes a Yang-Mills potential on $\tilde{G} = \tilde{G}(\tilde{G}/H,\ H)$ if H is the fiber.

The μ^A, $A = 1,\ldots,\dim G$, span a basis on the cotangent plane of $\tilde{G}$; therefore, they have to be interpreted as vielbein and not as Yang-Mills connections on $\tilde{G}$ (there is no structural group acting on $T_p(G)$).

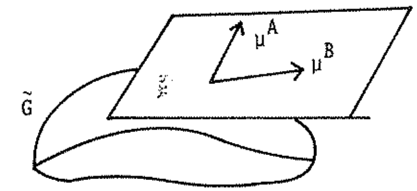

Fig. I.3.I

Let us take the exterior derivative of both sides of Eq. (I.3.122): from $d^2 = 0$ we get the Bianchi identity:

$$dR^A + C^A_{BC}\mu^B \wedge R^C = 0 \qquad \text{(I.3.124)}$$

which can also be rewritten as:

$$\nabla R^A = 0 \quad . \qquad \text{(I.3.125)}$$

In (I.3.125) we have introduced the covariant derivative operator ∇. On a tensor $A^{[\cdot]}$ with indices $[\cdot]$ in a representation D of $\mathbb{G}$, ∇ is defined as follows

$$\nabla A^{[\cdot]} = (d + \mu^B \wedge D(T_B))A^{[\cdot]} \quad . \qquad \text{(I.3.126)}$$

In the case of $A^{[\cdot]} \equiv R^A$:

$$[D(T_B)]^A{}_C = C^A_{BC} \qquad \text{(I.3.127)}$$

and (I.3.125) follows.

Applying the covariant exterior derivative to both sides of (I.3.126) one obtains

$$\begin{aligned}\nabla^2 A &= d(\nabla A) + \mu^B \wedge D(T_B)\nabla A = \\ &= d\mu^B \wedge D(T_B)A - \mu^B \wedge D(T_B)dA + \\ &\quad + \mu^B \wedge D(T_B)dA + \mu^B \wedge D(T_B)\mu^C \wedge D(T_C)A = \\ &= (d\mu^A + \frac{1}{2} C^A_{BC}\mu^B \wedge \mu^C)D(T_A)A \equiv \\ &\equiv R^A \wedge D(T_A)A \quad . \end{aligned} \tag{I.3.128}$$

We have introduced the soft forms starting from the dual formulation of the Lie algebra, namely the Maurer-Cartan equations. The same can be done in the language of vector fields. It suffices to consider a basis of soft vectors dual to the 1-forms μ^A:

$$\mu^A(\tilde{\vec{T}}_B) = \delta^A_B \quad . \tag{I.3.129}$$

The new vector fields* $\tilde{T}_B$, being the dual of the soft 1-forms μ^A are also soft (non left-invariant). Therefore they close a Lie algebra with structure functions rather than structure constants, according to the general formula (I.1.213).

Indeed the structure functions can be immediately computed evaluating Eq. (I.3.122) on two tangent vectors $\tilde{T}_L$, $\tilde{T}_M$. Expanding R^A along the intrinsic basis $\mu^A \wedge \mu^B$:

$$R^A = R^A_{BC}\mu^B \wedge \mu^C \tag{I.3.130}$$

* We omit the arrow on $\tilde{T}_A$ for notational clarity. However one must keep in mind the distinction between T_A, the Lie algebra generator in its matrix realization, and $\tilde{T}_A$ the vector field dual to μ^A. Notice that T_A, thought as a vector field, is dual to σ^A.

Eq. (I.3.122) can be rewritten as

$$d\mu^A + \frac{1}{2}(C^A_{BC} - 2R^A_{BC})\mu^B \wedge \mu^C = 0 \quad . \tag{I.3.131}$$

Therefore, in the same way as we derived Eq. (I.3.46) from (I.3.41), taking the value of both sides on $\tilde{T}_L$, $\tilde{T}_M$, we obtain:

$$[\tilde{T}_A,\tilde{T}_B] = (C^C_{AB} - 2R^C_{AB})\tilde{T}_C \quad . \tag{I.3.132}$$

We note that the structure functions are given in terms of the curvature intrinsic components.

<u>Lie derivative on soft group manifolds</u>

We now study the Lie derivative of the soft forms μ^A along the generic vector field $\tilde{T}_A$.

Let us consider a generic infinitesimal diffeomorphism on μ^A generated by

$$t = \varepsilon^A\tilde{T}_A \tag{I.3.133}$$

where $\varepsilon^A = \delta x^A$ is the infinitesimal parameter associated to the shift

$$x^A \to x^A + \delta x^A \quad (A = 1,2,\ldots,\dim G) \quad . \tag{I.3.134}$$

We want to compute the Lie derivative $\ell_t\, \mu^A$; using the definition (I.1.227), we obtain

$$\begin{aligned}\ell_t\mu^A &= (\underline{t}\rfloor d + d\,\underline{t}\rfloor)\mu^A = \underline{t}\rfloor d\mu^A + d(\varepsilon^B\, \underline{\tilde{T}_B}\rfloor \mu^A) = \\ &= \underline{t}\rfloor d\mu^A + d\varepsilon^A \quad . \end{aligned} \tag{I.3.135}$$

Adding and subtracting $1/2\; C^A_{BC}\mu^B \wedge \mu^C$ to $d\mu^A$ and using the definition (I.3.122) and (I.3.126), we reconstruct the curvatures R^A:

$$\ell_t \mu^A = \underline{t}\rfloor (d\mu^A + \frac{1}{2} C^A_{BC}\mu^B \wedge \mu^C) - \varepsilon^L C^A_{LC}\mu^C + d\varepsilon^A =$$

$$= (\nabla\varepsilon)^A + \underline{t}\rfloor R^A \qquad . \qquad (I.3.136)$$

The first term $(\nabla\varepsilon)^A$ corresponds to an infinitesimal gauge transformation of the group G.

Hence an infinitesimal diffeomorphism on the soft manifold $\tilde{G}$ is a G-gauge transformation plus curvature correction terms.

In particular if the curvature R^A has vanishing projection along a tangent vector t:

$$\underline{t}\rfloor R^A \equiv 2\varepsilon^B R^A_{BC}\mu^C = 0 \qquad (I.3.137)$$

then the action of the Lie derivative ℓ_t coincides with a gauge transformation.

Horizontality and factorization

Because of property (I.1.239), the set of vectors $\{t\}$ satisfying (I.3.137) must span a subalgebra H of the general algebra of diffeomorphisms (I.3.132). Let us denote by T_H, $(H=1,\dots,\dim H)$ a basis of vector fields in H. Then the condition

$$\underline{T_H}\rfloor R^A = 0 \Leftrightarrow R^A_{HB} = 0 \qquad (I.3.138)$$

reduces (I.3.132) to:

$$[T_H, T_{H'}] = C^{H''}_{HH'}\, T_{H''} \qquad . \qquad (I.3.139)$$

That is, H is the Lie algebra spanned by the left invariant vector fields of $H \subset G$. Condition (I.3.138) will be referred to as the H-horizontality condition for the curvatures R^A. We see that gauge transformations of μ^A can be considered as diffeomorphisms along the directions of the Lie algebra vector fields T_H.

This is strictly related to the fact that H-horizontality of the curvature is the condition under which the manifold $\tilde{G}$ assumes the structure of a principal fiber bündle with base space $M_D = \tilde{G}/H$ and fiber H:

$$\tilde{G} = \tilde{G}(\tilde{G}/H,\ H) \qquad . \qquad (I.3.140)$$

Indeed when (I.3.138) holds the gauge transformation generated by $t = \varepsilon^H T_H$

$$\delta^{(gauge)}_\varepsilon\ \mu^A = \ell_\varepsilon \mu^A = (\nabla\varepsilon)^A \qquad (I.3.141)$$

can be explicitly integrated.

To obtain the explicit expression of the finite transformation associated to (I.3.141) we split the coordinates x^A of the soft group manifold $\tilde{G}$ into coordinates x^K relative to the base space and coordinates η^H relative to the fiber, which is the rigid group-manifold of the subgroup H.

If the horizontality condition (I.3.138) holds then the dependence of $\mu^A(x,\eta)$ on η^H is factorized.

By factorization we mean that every $\mu^A(x,\eta)$ is determined by its boundary value on the base space:

$$\mu^A(x) \equiv \mu^A(x,\ \eta = 0) \qquad . \qquad (I.3.142)$$

Indeed taking any set of 1-forms

$$\mu^A(x) = \mu^A_\mu(x)dx^\mu \qquad (I.3.143)$$

on the base manifold we can lift them to forms defined on the whole $\tilde{G}$ via a finite gauge transformation of the subgroup H.

Let

$$h(\eta) = \exp(\eta^H\, T_H) \qquad (I.3.144)$$

be an element of H and let T_H be the H-generators in the coadjoint representation of G (see Eq. (I.3.66)): $h(\eta)$ is a $\dim G \times \dim G$ matrix with the block form:

$$h(\eta) = \mathrm{coadj}_G\, h(\eta) = \left(\begin{array}{c|c} \mathrm{coadj}_H\, h(\eta) & 0 \\ \hline 0 & M(h(\eta)) \end{array}\right) \begin{array}{l} \}\ \dim H \\ \\ \}\ \dim G/H \end{array} \qquad (I.3.145)$$

$\mathrm{coadj}_{(H)}\, h(\eta)$ is the coadjoint representation of H and $M(h(\eta))$ is the representation of H on the K subspace of the G-Lie algebra, K being the coset generators (see Chapter I.6).

Let us define the lifted 1-form

$$\mu'(x,\eta) = \mu^{A'}(x,\eta)T_A \qquad (I.3.146)$$

as the $h(\eta)$-gauge transform of $\mu(x) = \mu^A(x)T_A$:

$$\mu'(x,\eta) = h^{-1}(\eta)\mu(x)h(\eta) + h^{-1}(\eta)dh(\eta) \quad . \qquad (I.3.147)$$

Expanding along the generators we have:

$$\mu^H(x,\eta) = [\mathrm{coadj}_H(h(\eta))]^H{}_{H'}\, \mu^{H'}(x) + \sigma^H(\eta) \qquad (I.3.148a)$$

$$\mu^K(x,\eta) = M(h(\eta))^K{}_{K'}\, \mu^{K'}(x) \qquad (I.3.148b)$$

which gives the general expression of the 1-forms μ^A where the η dependence is factorized through a gauge transformation.

It is instructive to see the reason behind the integrability of the gauge transformation (I.3.137).

A complete set of initial data for the integration of the flow (I.3.137) is given by the boundary value of $\mu^A(x,0)$ and its "normal derivatives", that is

$$T_H \rfloor d\mu^A \qquad (I.3.149)$$

since the T_H span the orthogonal complement to $\tilde{G}/H$. Now since

$$d\mu^A(T_H,\tilde{T}_L) = (R^A + \frac{1}{2} C^A_{BC}\mu^B \wedge \mu^C)(T_H,\tilde{T}_L) \qquad (I.3.150)$$

the horizontality condition $R^A(T_H,\tilde{T}_L) = 0$ determines the "normal derivatives" making the flow integrable in terms of its boundary value

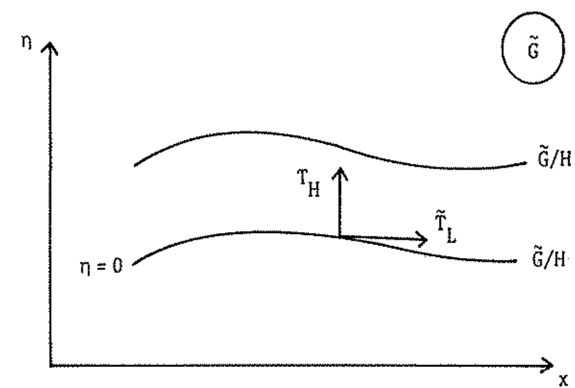

Fig. I.3.II

The conclusion is that we can interpret the original vielbein $\mu^A(x,\eta)$ on $\tilde{G}$ as a Yang-Mills G-Lie algebra valued connection $\mu^A(x)$ defined on a general base space $M_n = \tilde{G}/H$ and subject to the transformation law (I.3.147) or (I.3.148).

Let us observe that the definitions of curvatures and Bianchi identities on $\tilde{G}$ keep the same form when we restrict μ^A and R^A to the base space; this is a consequence of the fact that under the mapping

$$\phi: \tilde{G} \rightarrow \tilde{G}/H$$

the wedge-product and the exterior derivative commute with the pull back ϕ^* (see Eq. (I.1.170-171)).

Before concluding this section we give the explicit expression of the curvatures associated to the l.h.s. of (I.3.148) in the special case where the Lie algebra of G:

$$\mathbb{G} = \mathbb{H} + \mathbb{K}$$

is (weakly) reductive and symmetric, namely (see Chapter I.6) $C^{H}_{H'K} = C^{K}_{K',K''} = 0$ or equivalently:

$$[\mathbb{H},\mathbb{H}] \subseteq \mathbb{H} \quad (\mathbb{H} \text{ is a subalgebra}) \tag{I.3.150a}$$

$$[\mathbb{H},\mathbb{K}] \subseteq \mathbb{K} \quad (\mathbb{K} \text{ is weakly reductive}) \tag{I.3.150b}$$

$$[\mathbb{K},\mathbb{K}] \subseteq \mathbb{H} \quad (\mathbb{K} \text{ is symmetric}) \quad . \tag{I.3.150c}$$

In this case the curvatures of (I.3.148) read:

$$R^{H} = \mathcal{R}^{H} + \frac{1}{2} C^{H}_{KK'} \mu^{K} \wedge \mu^{K'} \tag{I.3.151a}$$

$$R^{K} = \mathcal{D}^{(H)} \mu^{K} \tag{I.3.151b}$$

where

$$\mathcal{R}^{H} = d\mu^{H} + \frac{1}{2} C^{H}_{H'H''} \mu^{H'} \wedge \mu^{H''} \tag{I.3.152}$$

is the curvature of the subgroup H and

$$\mathcal{D}^{(H)} = d(\mathbb{1})^{K}_{K'} + \mu^{H} \wedge [D(T_{H})]^{K}_{K'} \tag{I.3.153a}$$

$$[D(T_{H})]^{K}_{K'} = C^{K}_{HK'} \tag{I.3.153b}$$

defines the H-covariant derivative in the coadjoint representation.

The same decomposition on the Lie derivative (see Eq. (I.3.136)) gives:

$$\ell_{\varepsilon}\mu^{H} = d\varepsilon^{H} - \varepsilon^{H'} C^{H}_{H'H''}\mu^{H''} - \varepsilon^{K} C^{H}_{KK'}\mu^{K'} + \underline{\varepsilon|} R^{H} =$$

$$= \mathcal{D}^{(H)}\varepsilon^{H} - \varepsilon^{K} C^{H}_{KK'}\mu^{K'} + \underline{\varepsilon|} R^{H} \tag{I.3.154a}$$

$$\ell_{\varepsilon}\mu^{K} = d\varepsilon^{K} - \varepsilon^{K'} C^{K}_{K'H}\mu^{H} - \varepsilon^{H} C^{K}_{HK'}\mu^{K'} + \underline{\varepsilon|} R^{K} =$$

$$= \mathcal{D}^{(H)}\varepsilon^{K} - \varepsilon^{H} C^{K}_{HK'}\mu^{K'} + \varepsilon| R^{K} \tag{I.3.154b}$$

where

$$\varepsilon = \varepsilon^{H} T_{H} + \varepsilon^{K} \tilde{T}_{K} \quad . \tag{I.3.155}$$

If $\varepsilon = \varepsilon^{H} T_{H}$ and $\underline{\varepsilon|} R^{K} = \underline{\varepsilon|} R^{H} = 0$ then the Lie derivative coincides with the H-gauge transformation:

$$\ell_{\varepsilon}\mu^{H} = \delta^{(\text{gauge})}_{\varepsilon} \mu^{H} = \mathcal{D}^{(H)}\varepsilon^{H} \tag{I.3.156a}$$

$$\ell_{\varepsilon}\mu^{K} = \delta^{(\text{gauge})}_{\varepsilon} \mu^{K} = - \varepsilon^{H} C^{K}_{HK'}\mu^{K'} \quad . \tag{I.3.156b}$$

I.3.8 - The example of Poincaré and anti de Sitter soft group manifold

These groups are of particular relevance to the formulation of gravity and supergravity. We discuss first the de Sitter or anti-de Sitter group in D dimensions:

$$\text{de Sitter:} \quad G = SO(1,D) \tag{I.3.157a}$$

$$\text{Anti de-Sitter:} \quad G = SO(2,D-1) \tag{I.3.157b}$$

and their Lorentz subgroup $H = SO(1,D-1)$.

The Poincaré group

$$G = ISO(1,D-1) \tag{I.3.158}$$

will be discussed afterwards as an Inonü-Wigner contraction of G. We begin by computing the curvatures of the soft-manifold $\tilde{G}$.

De Sitter curvatures

Let A be an $SO(2,D-1)$ or an $SO(1,D)$ matrix, for convenience in the defining representation, and let $\sigma^{\hat{a}}{}_{\hat{b}}$ be the corresponding matrix of left-invariant 1-forms:

$$\sigma = \sigma^A T_A = A^{-1} dA \tag{I.3.159}$$

satisfying the Maurer-Cartan equations

$$d\sigma + \sigma \wedge \sigma = 0 \quad . \tag{I.3.160}$$

Being $SO(2,D-1)$ or $SO(1,D)$ Lie algebra valued σ is antisymmetric with respect to the metric $\eta_{\hat{a}\hat{b}}$:

$$\sigma^{\hat{a}\hat{b}} = - \sigma^{\hat{b}\hat{a}} \qquad ; \qquad (\sigma^{\hat{a}\hat{b}} = \eta^{\hat{b}\hat{c}} \sigma^{\hat{a}}{}_{\hat{c}}) \tag{I.3.160a}$$

$$\eta_{\hat{a}\hat{b}} = (1, \underbrace{-1,\ldots,-1}_{D-1 \text{ times}}, \pm 1) \quad ; \quad \hat{a}, \hat{b} \equiv 0,\ldots,D \tag{I.3.160b}$$

where the 2 signs in the D-D component of $\eta_{\hat{a}\hat{b}}$ distinguish between the anti-de Sitter and de Sitter case respectively.

The soft potential $\mu^A T_A \equiv \mu^{\hat{a}\hat{b}} J_{\hat{a}\hat{b}}$ has the same symmetry properties; the associated curvature is

$$R^{\hat{a}\hat{b}} = d\mu^{\hat{a}\hat{b}} - \mu^{\hat{a}\hat{c}} \wedge \mu_{\hat{c}}{}^{\hat{b}} \quad . \tag{I.3.161}$$

Decomposing the indices with respect to the Lorentz subgroup $SO(1,D-1)$:

$$\mu^{\hat{a}\hat{b}} \xrightarrow[SO(1,D-1)]{} \omega^{ab}, \omega^{aD} \tag{I.3.162a}$$

$$J_{\hat{a}\hat{b}} \xrightarrow[SO(1,D-1)]{} J_{ab}, J_{aD} \tag{I.3.162b}$$

$$R^{\hat{a}\hat{b}} \xrightarrow[SO(1,D-1)]{} R^{ab}, R^{aD} \tag{I.3.162c}$$

$$\omega^{aD} = - \omega^{Da} \quad ; \quad R^{aD} = - R^{Da} \quad ; \quad J_{aD} = - J_{Da} \equiv P_a \tag{I.3.162d}$$

$$\{\hat{a},\hat{b}\} \equiv \{0,1,\ldots,D\} \quad ; \quad \{a,b\} = \{0,\ldots,D-1\} \tag{I.3.162e}$$

(I.3.161) split as follows:

$$R^{ab} = d\omega^{ab} - \omega^{ac} \wedge \omega_c{}^b - \omega^{aD} \wedge \omega^{Db} \eta_{DD} \tag{I.3.163a}$$

$$R^{aD} = d\omega^{aD} - \omega^a{}_b \wedge \omega^{bD} \tag{I.3.163b}$$

We set

$$\omega^{aD} = 2 \bar{e} V^a \tag{I.3.164a}$$

$$R^{aD} = 2 \bar{e} R^a \tag{I.3.164b}$$

$$J_{aD} = \frac{1}{2\bar{e}} P_a \tag{I.3.164c}$$

where $\bar{e}$ is an arbitrary scaling factor (introduced in order to perform the contraction to the Poincaré group). Eqs. (I.3.163a-b) become

$$R^{ab} = d\omega^{ab} - \omega^{a}{}_{c} \wedge \omega^{cb} \pm 4\,\bar{e}^2\, V^a \wedge V^b \qquad \text{(I.3.165a)}$$

$$R^{a} = dV^{a} - \omega^{a}{}_{b} \wedge V^{b} = \mathscr{D} V^{a} \qquad \text{(I.3.165b)}$$

where $\mathscr{D}^a$ is the Lorentz covariant derivative

$$\mathscr{D} = d + \omega^{ab} D(J_{ab}) \qquad \text{(I.3.166)}$$

according to (I.3.126).

The plus and minus signs (in I.3.165a) refer to the SO(2,D-1) or SO(1,D) cases respectively. In the following we restrict our attention to the SO(2,D-1) (anti de Sitter) case only, bearing in mind that the SO(1,D) case can be obtained from SO(2,D-1) by the replacement

$$4\bar{e}^2 \to -4\bar{e}^2 \quad . \qquad \text{(I.3.167)}$$

Comparing (I.3.165) with (I.3.151-153) and using the identification

$$\mu^H \equiv \omega^{ab} \quad ; \quad R^H \equiv R^{ab} \quad ; \quad \mu^K \equiv V^a \quad ; \quad R^K \equiv R^a$$

we find that the Lie algebra valued curvatures of SO(2,D-1) are those of a weakly reductive and symmetric algebra.

With the same decomposition of indices the SO(2,D-1) Bianchi identities

$$\nabla^{(SO(2,D-1))} R^{\hat{a}\hat{b}} \equiv dR^{\hat{a}\hat{b}} - 2\omega_{\hat{c}}{}^{[\hat{a}}\, R^{\hat{b}]\hat{c}} = 0 \qquad \text{(I.3.168)}$$

split as follows:

$$2\,\mathscr{D}^{(SO(1,D-1))} R^{ab} + 8\bar{e}^2\, V^{[a} \wedge R^{b]} = 0 \qquad \text{(I.3.169a)}$$

$$\mathscr{D}^{(SO(1,D-1))} R^{a} + R^{ab} \wedge V_b = 0 \quad . \qquad \text{(I.3.169b)}$$

The explicit form of the curvature (I.3.165) completely specify the Lie algebra of SO(2,D-1); indeed one may extract the structure constants by comparing (I.3.165) with the general definition (I.3.122).

If we are interested in the commutation relations among the generators we set $R^{ab} = R^a = 0$ so that ω^{ab} and V^a become left-invariant; then setting

$$\sigma^{\hat{a}\hat{b}}(\vec{J}_{\hat{c}\hat{d}}) = \delta^{\hat{a}\hat{b}}_{\hat{c}\hat{d}} \qquad \text{(I.3.170)}$$

that is

$$\omega^{ab}(\vec{J}_{cd}) = \delta^{ab}_{cd} \quad ; \quad V^a(\vec{J}_{cd}) = 0 \qquad \text{(I.3.171a)}$$

$$\omega^{ab}(\vec{P}_c) = 0 \quad ; \quad V^a(\vec{P}_b) = \delta^a_b \qquad \text{(I.3.171b)}$$

one finds (see Eqs. (I.3.43-46) and (I.1.244)):

$$[J_{\hat{a}\hat{b}}, J_{\hat{c}\hat{d}}] = -\frac{1}{2}\left(J_{\hat{a}\hat{c}}\,\eta_{\hat{b}\hat{d}} + J_{\hat{b}\hat{d}}\,\eta_{\hat{a}\hat{c}} - J_{\hat{a}\hat{d}}\,\eta_{\hat{b}\hat{c}} - J_{\hat{b}\hat{c}}\,\eta_{\hat{a}\hat{d}}\right) \qquad \text{(I.3.172)}$$

$$[J_{ab}, J_{cd}] = -\frac{1}{2}\left(J_{ac}\,\eta_{bd} + J_{bd}\,\eta_{ac} - J_{ad}\,\eta_{bc} - J_{bc}\,\eta_{ad}\right) \qquad \text{(I.3.173a)}$$

$$[J_{ab}, P_c] = -\frac{1}{2}\left(\eta_{ac} P_b - \eta_{bc} P_a\right) \qquad \text{(I.3.173b)}$$

$$[P_a, P_b] = -2\bar{e}^2\, J_{ab} \qquad \text{(I.3.173c)}$$

Poincaré group curvatures

The Poincaré group in D dimensions, ISO(1,D-1) can be retrieved as the Inonü-Wigner contraction of SO(2,D-1). Indeed, performing the contraction limit $\bar{e}\to 0$ one obtains from (I.3.165) the Poincaré curvatures:

$$R^{ab} = \mathcal{R}^{ab} \equiv d\omega^{ab} - \omega^{a}{}_{c} \wedge \omega^{cb} \qquad (I.3.174)$$

$$R^{a} = \mathcal{D} V^{a} \equiv dV^{a} - \omega^{a}{}_{b} \wedge V^{b} \qquad (I.3.175)$$

and setting $R^{ab} = R^{a} = 0$ one recovers the Maurer-Cartan equations of the Poincaré group given in Eqs. (I.3.83-84) (for $D=4$).

The same limit applied to Eqs. (I.3.169) and to the Lie algebra (I.3.173) gives the Poincaré-Bianchi identities

$$\mathcal{D} R^{ab} = 0 \qquad (I.3.176a)$$

$$\mathcal{D} R^{a} + R^{ab} \wedge V_{b} = 0$$

and the Poincaré group algebra:

$$[J_{ab},J_{cd}] = -\frac{1}{2}(J_{ac}\eta_{bd} + J_{bd}\eta_{ac} - J_{ad}\eta_{bc} - J_{bc}\eta_{ad}) \qquad (I.3.177a)$$

$$[J_{ab},P_{c}] = -\frac{1}{2}(\eta_{ac}P_{b} - \eta_{bc}P_{a}) \qquad (I.3.177b)$$

$$[P_{a},P_{c}] = 0 \quad . \qquad (I.3.177c)$$

Fiber bundle structure

Let us now suppose that the H-horizontality conditions (I.3.138) are satisfied by $H = SO(1,D-1)$:

$$\underline{J_{\ell m}}| R^{A} = 0 \qquad A = \{ab;a\} \quad . \qquad (I.3.178)$$

Eq. (I.3.178) amounts to saying that the general expansion of $R^{A} \equiv (R^{ab},R^{a})$ on the cotangent basis:

$$R^{A} = R^{A}_{BC}\mu^{B} \wedge \mu^{C} \equiv R^{A}_{ab}V^{a} \wedge V^{b} + R^{A}_{\ell m,a}\omega^{\ell m} \wedge V^{a} + R^{A}_{\ell m,pq}\omega^{\ell m} \wedge \omega^{pq} \qquad (I.3.179)$$

reduces to

$$R^{A} = R^{A}_{ab}V^{a} \wedge V^{b} \quad . \qquad (I.3.180)$$

As we previously discussed, Eq. (I.3.178) implies that $\tilde{G}$ acquires a fiber bundle structure, where SO(1,D-1) is the gauge group and

$$M_{D} \equiv \widetilde{SO(2,D-1)}/SO(1,D-1) \qquad (I.3.181)$$

or

$$M_{D} \equiv \widetilde{ISO(1,D-1)}/SO(1,D-1) \qquad (I.3.182)$$

are the base manifolds, for the de Sitter or Poinçaré case respectively.

In this case Eqs. (I.3.174-175) and (I.3.176) become the curvatures and the Bianchi identities of the connections ω^{ab} and V^{a} gauging the Poincaré (or de Sitter) group.

Restricting now our attention to the Poincaré group we study the explicit form of the Lie derivative of ω^{ab}, V^{a} when condition (I.3.178) holds. Considering the effect of an infinitesimal diffeomorphism on $\tilde{G}$ generated by t we have that (I.3.136) split as follows:

$$\ell_{t}\omega^{ab} = \mathcal{D}\varepsilon^{ab} + \underline{t}| R^{ab} \qquad (I.3.183a)$$

$$\ell_{t}V^{a} = \mathcal{D}\varepsilon^{a} + \varepsilon^{ab}V_{b} + \underline{t}| R^{a} \qquad (I.3.183b)$$

where $t = \varepsilon^{ab}J_{ab} + \varepsilon^a\tilde{P}_a$.

Taking $\varepsilon^a = 0$, i.e.

$$t = \varepsilon^{ab}\tilde{J}_{ab} \qquad . \qquad (I.3.184)$$

we have that an infinitesimal coordinate transformation in the J_{ab} directions coincides with a Lorentz gauge transformation; indeed from (I.3.178)

$$\underline{t}|R^{ab} = R^{ab}(\tilde{J}_{cd},\tilde{T}_A) = 0 \qquad (I.3.185a)$$

$$\underline{t}|R^{a} = R^{a}(\tilde{J}_{bc},\tilde{T}_A) = 0 \qquad (I.3.185b)$$

and (I.3.183) become

$$\ell_t\omega^{ab} = \mathcal{D}\varepsilon^{ab} \qquad (I.3.186a)$$

$$\ell_t V^a = \varepsilon^{ab}V_b \qquad (I.3.186b)$$

that is an infinitesimal Lorentz gauge transformation of the fields. According to the general discussion given in Section I.3.6 the vector fields $\tilde{J}_{ab} \equiv J_{ab}$ are left-invariant and close the Lie algebra of SO(1,D-1).

Eqs. (I.3.186) can be integrated to a finite gauge transformation according to (I.3.148a); we obtain

$$\omega'(x,\eta) = \Lambda^{-1}(\eta)\omega(x,0)\Lambda(\eta) - \Lambda^{-1}(\eta)d\Lambda(\eta) \qquad (I.3.187a)$$

$$V'(x,\eta) = \Lambda^{-1}(\eta)V(x,0) \qquad . \qquad (I.3.187b)$$

Thus we recover the gauge transformation law of the vielbein and of the connection fields describing the Riemannian geometry of a D-dimensional space:

$$M_D = \tilde{G}/H \equiv \widetilde{ISO(1,D-1)}/SO(1,D-1) \qquad (I.3.188)$$

given in (I.2.48) and (I.2.51).

On the other hand the Lie derivatives on M_D along the tangent vectors

$$t = \varepsilon^a\,\tilde{P}_a \qquad (\varepsilon^{ab} = 0) \qquad (I.3.189)$$

generate infinitesimal coordinate transformations on M_D:

$$\ell_t\omega^{ab} = \underline{t}|R^{ab} = 2\varepsilon^c R^{ab}{}_{cm}V^m \qquad (I.3.190a)$$

$$\ell_t V^a = \mathcal{D}\varepsilon^a + \underline{t}|R^a = \mathcal{D}\varepsilon^a + 2\varepsilon^c R^a{}_{cm}V^m \qquad (I.3.190b)$$

Notice that the $\tilde{P}_a$ vector fields are not left-invariant since they are related to the generator of translations $P_\mu = \partial_\mu$ by

$$\partial_\mu = V^a_\mu\,\tilde{P}_a + \omega^{ab}_\mu\,J_{ab} \qquad (I.3.191a)$$

$$\Rightarrow \tilde{P}_a = V^\mu_a(\partial_\mu - \omega^{bc}_\mu\,J_{bc}) \qquad . \qquad (I.3.191b)$$

It is worth to see in more detail the relation between the coordinate transformation (I.3.190) and the Poincaré transformations (I.3.183).

Writing a generic tangent vector ε on $\tilde{G}$ in the intrinsic basis $\tilde{T}_A$ or in the coordinate basis $\tilde{T}_\Sigma$ one has

$$\varepsilon^A = \mu^A_\Sigma\,\varepsilon^\Sigma \qquad . \qquad (I.3.192)$$

μ^A_Σ being the components of the vielbein. Decomposing the indices A and Σ in (I.3.192) one finds:

$$\varepsilon^{ab} = \omega^{ab}_\mu \varepsilon^\mu + h^{ab} \qquad \text{(I.3.193a)}$$

$$\varepsilon^a = V^a_\mu \varepsilon^\mu \qquad \text{(I.3.193b)}$$

where we have set $\omega^{ab}_{\rho\sigma}\eta^{\rho\sigma} \equiv h^{ab}$, $\eta^{\rho\sigma}$ being the Lorentz parameters of a generic infinitesimal Lorentz transformation h^{ab} on the fiber, and $V^a_{(\rho\sigma)} \equiv 0$ by a coordinate choice.

Let us now substitute (I.3.193b) into the r.h.s. of (I.3.190b); recalling that $\underline{J_{ab}|} R^a = 0$ because of SO(1,3) factorization one finds:

$$\ell_\varepsilon V^a = \mathscr{D}(V^a_\mu \varepsilon^\mu) + \underline{\varepsilon^\mu \partial_\mu|} R^a_{\rho\sigma} dx^\rho \wedge dx^\sigma =$$

$$= \mathscr{D} V^a_\mu \varepsilon^\mu + V^a_\mu d\varepsilon^\mu + 2\varepsilon^\mu R^a_{\mu\sigma} dx^\sigma$$

$$(\ell_\varepsilon V^a)_\rho = (\mathscr{D}_\rho V^a_\mu - \mathscr{D}_\mu V^a_\rho)\varepsilon^\mu + \mathscr{D}_\mu V^a_\rho \varepsilon^\mu + V^a_\mu \partial_\rho \varepsilon^\mu +$$

$$+ 2\varepsilon^\mu R^a_{\mu\rho} = \mathscr{D}_\mu V^a_\rho \varepsilon^\mu + V^a_\mu \partial_\rho \varepsilon^\mu =$$

$$= (\partial_\mu V^a_\rho)\varepsilon^\mu + V^a_\mu \partial_\rho \varepsilon^\mu + \omega^{ab}_\mu V_{b|\rho} \varepsilon^\mu \quad . \qquad \text{(I.3.194)}$$

We see that the final result differs from the genuine general coordinate transformation (I.1.220) of the coordinate vector V^a_μ by the term $\varepsilon^\mu \omega^{ab}_\mu V_{b\rho}$, which can be interpreted as a field-dependent Lorentz transformation of parameter $\varepsilon^{ab} - h^{ab}$ since from Eq. (I.3.193a):

$$\varepsilon^\mu \omega^{ab}_\mu V_{b\rho} = (\varepsilon^{ab} - h^{ab}) V_{b\rho} \quad . \qquad \text{(I.3.195)}$$

In other words a diffeomorphism on the soft group manifold gives rise, on a Lorentz vector, to a diffeomorphism on the base space plus a field dependent Lorentz transformation.

CHAPTER I.4

POINCARE GRAVITY

I.4.1 - Poincaré Gravity

In this chapter we utilize the vielbein V^a and the spin connection ω^{ab} to describe the Einstein theory of gravitation.

On one hand this formalism reveals that gravity is a gauge theory, precisely the gauge theory of the Poincaré group ISO(1,3) (ISO(1,D-1) in a D-dimensional space-time); on the other hand, however, the action from which we deduce the gravitational field equations is essentially different from the Yang-Mills action utilized in ordinary gauge theories.

To understand this difference and to clarify the formal properties of "gravity" is essential for the formulation of its supersymmetric extension, namely "supergravity".

We begin by writing the Einstein-Cartan action:

$$A = \int_{M_4} R^{ab}(\omega) \wedge V^c \wedge V^d \varepsilon_{abcd} \quad . \qquad \text{(I.4.1)}$$