Bootstrapping M-theory

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Abstract

M-theory is a quantum theory of gravity in 11 dimensions that could possibly describe our world. Unfortunately, it is difficult to study and little is known about it beyond a few quantities that are protected by supersymmetry. The AdS/CFT conjecture provides a non-perturbative definition of M-theory, by relating a stack of N M2-branes in M-theory to a family of maximally supersymmetric conformal field theories (SCFTs) in three dimensions with $U(N) \times U(N)$ gauge group called ABJM theory. When N is large, ABJM theory is dual to classical supergravity in four dimensions with a negative cosmological constant, where sub-leading terms in N correspond to corrections from M-theory. ABJM theory itself is strongly coupled, and so is difficult to study using conventional methods. In this thesis, we use the recently discovered conformal bootstrap technique to compute quantities in ABJM theory, and thus M-theory via the AdS/CFT correspondence.

We begin by deriving a protected 1d topological sector in all 3d SCFTs with half maximal supersymmetry, which we use to compute certain protected observables in ABJM theory exactly for low N as well as in a large N expansion to all orders in $1/N$. For $N = 3$, we find a new duality between ABJM theory and another kind of maximally supersymmetric 3d SCFT that previously had no M-theory interpretation. We then use the conformal bootstrap to compute numerical bounds on observables in all maximally supersymmetric 3d SCFTs, and find that the previous analytic results for ABJM theory come close to saturating these bounds, which allows us to conjecturally read off the low-lying spectrum of this theory for all N. We then use the Mellin space formalism to compute the same quantities on the AdS side, and find that to leading order in N they match the predictions from the conformal bootstrap, providing a new check of the AdS/CFT conjecture. Finally, we outline a strategy to derive the M-theory S-Matrix from ABJM theory to all orders in N, and check that it works to sub-leading order in N using the previously derived analytic results.

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אתננה הראיתיך בעיניך ושמה לא תעבר

Contents

Chapter 1

Introduction

M-theory is a theory of quantum gravity in 11 spacetime dimensions. It is related by various dualities to the five varieties of String theory, which are all defined in 10 dimensions. Together, these interrelated theories comprise the only known consistent theory of quantum gravity. String theory has a well defined perturbative expansion in terms of the string coupling g_s. M-theory, in contrast, has no dimensionless coupling, and so is more difficult to study. The most complete definition of M-theory is given by the AdS/CFT conjecture, which relates M-theory on a spacetime background with negative curvature (Anti-de Sitter space, i.e. AdS) in $d+1$ dimensions to a quantum field theory with conformal symmetry (i.e. a CFT) in d dimensions. The CFT has no dynamical spacetime, i.e. no gravity, and so is a conventional well-defined quantum field theory, unlike a theory of quantum gravity like M-theory whose microscopic definition is still poorly understood.

In this thesis we will focus on a particular case of AdS/CFT that relates M-theory in four dimensional Anti-de Sitter space to a family of CFTs in three dimensions with the maximal amount of supersymmetry that was discovered by Aharony, Bergman, Jafferis, and Maldacena (ABJM) [2, 3]. ABJM with maximal supersymmetry has no small parameter, i.e. it is strongly coupled, and so cannot be studied using traditional methods like Feynman diagrams. Instead, we will study it using various non-perturbative methods such as the conformal bootstrap, which provides numerical bounds on physical observables in CFTs based on symmetry alone, without any reference to a Lagrangian. In CFTs with maximal supersymmetry, such as ABJM, these constraints can be especially powerful. Using the duality between ABJM theory and M-theory, the constraints we derive using the bootstrap for the former then apply to the latter, so we are bootstrapping M-theory.

In the first few chapters of this thesis, we will focus on deriving various physical observables in ABJM theory and other maximally supersymmetric CFTs in three dimensions. These results will include exact analytic computations, as well as numerical bounds. In the later chapters, we will use this information to provide evidence for the AdS/CFT conjecture, and to compute new quantities in M-theory from its dual ABJM theory. One of our primary results will be a method to compute the M-theory S-matrix, which describes the scattering of four gravitons in 11 dimensions, in terms of CFT data.

In this introductory chapter, we review some basic facts about superconformal field theories (SCFTs) in 3d, as well as the duality between ABJM theory and M-theory. In Chapter 2, we derive a protected 1d topological sector that exists for all 3d SCFTs with at least half maximal supersymmetry, and then use crossing symmetry and localization to compute observables analytically in maximally supersymmetric 3d SCFTs, including ABJM theory. In Chapter 3, we use this 1d sector to discover a duality between ABJM theory and another maximally supersymmetric 3d SCFT discovered by Bagger, Lamber, and Gustavson (BLG) [4, 5] that previously had no M-theory description in this case. In Chapter 4, we compute numerical bounds on observables in maximally supersymmetric 3d SCFTs by analyzing the crossing equations numerically, and find that these bounds are almost saturated by analytic results for ABJM theory computed in Chapter 2, which allows us to conjecturally read off all low-lying observables in ABJM theory for all N. In Chapter 5, we compute these same observable from the AdS side using the Mellin space formalism and find that they match the numerical bootstrap prediction for ABJM theory to leading order in $1/N$ even for unprotected observables. We also outline a strategy to compute the M-theory S-Matrix from ABJM theory, and check that it works to sub-leading order in $1/N$ using the observables computed analytically in Chapter 2. Finally, in Chapter 6, we summarize our results and discuss future directions.

1.1 Superconformal Field Theories in 3d

We begin by discussing some basic facts about SCFTs in 3d. We will start by discussing CFTs, including the all important crossing equations, and then the extra constraints due to supersymmetry. We will then focus on the four-point function of the stress tensor multiplet, which will be primary object of interest in this thesis. Finally, we discuss details of the known maximally supersymmetric SCFTs: ABJM and BLG.

1.1.1 Conformal Field Theories in 3d

The 3d conformal group consists of spacetime translations $x^\mu \to x^\mu + t^\mu$, Lorentz transformations $x^\mu = \Lambda^\mu{}_\nu x^\nu$, dilations $x^\mu \to \lambda x^\mu$, and special conformal transformations

$$x^\mu \to \frac{x^\mu + a^\mu x^2}{1 + 2x^\nu a_\nu + a^2 x^2}\,, \tag{1.1.1}$$

where spacetime indices $\mu, \nu = 0, 1, 2$ are raised and lowered with the standard $\mathbb{R}^{1,2}$ metric $\eta_{\mu\nu} = \text{diag}\{-1, 1, 1\}$. Let the anti-Hermitian generators of these transformations be P_μ, $M_{\mu\nu}$, and K_μ, respectively, normalized so that the commutation relations are

$$\begin{aligned}
[M_{\mu\nu}, P_\rho] &= -(\eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu)\,, \quad [M_{\mu\nu}, K_\rho] = -(\eta_{\mu\rho} K_\nu - \eta_{\nu\rho} K_\mu)\,, \\
[M_{\mu\nu}, M_{\rho\sigma}] &= -(\eta_{\mu\rho} M_{\nu\sigma} + \eta_{\nu\sigma} M_{\mu\rho} - \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\sigma})\,, \\
[D, P_\mu] &= -P_\mu\,, \quad [D, K_\mu] = K_\mu\,, \quad [K_\mu, P_\nu] = -2i M_{\mu\nu} + 2\eta_{\mu\nu} D\,.
\end{aligned} \tag{1.1.2}$$

Functions $f(x)$ on $\mathbb{R}^{1,2}$ transform under the differential representations, e.g. for infinitesimal dilations $\lambda = 1 + \epsilon$ we have $f(x) \to f(\lambda x) = f(x) + \epsilon x^\mu \partial_\mu f(x) + O(\epsilon^2)$, so that $x^\mu \partial_\mu$ is the generator of dilations in this representation. The other generators in the differential representation can be worked out similarly, and take the form

$$\begin{aligned}
P_\mu &= \partial_\mu\,, \qquad M_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu\,, \\
D &= x^\mu \partial_\mu\,, \qquad K_\mu = x^2 \partial_\mu - 2 x_\mu x^\nu \partial_\nu\,,
\end{aligned} \tag{1.1.3}$$

which satisfy the commutation relations (1.1.2).

The 3d conformal group is isomorphic to $SO(3,2)$, which can be seen at the level of generators by defining $\mathfrak{so}(3,2)$ generators $\tilde{M}_{IJ}$ with $I, J = 0, 1, 2, 3, 4$ and metric $\eta_{IJ} = \text{diag}\{-1, 1, 1, 1, -1\}$ as

$$\begin{aligned}
\tilde{M}_{\mu\nu} &= M_{\mu\nu}\,, \\
\tilde{M}_{\mu 3} &= \frac{K_\mu - P_\mu}{2}\,, \\
\tilde{M}_{\mu 4} &= \frac{K_\mu + P_\mu}{2}\,, \\
\tilde{M}_{43} &= D\,,
\end{aligned} \tag{1.1.4}$$

which from (1.1.2) have the usual $\mathfrak{so}(3,2)$ commutation relations

$$[\tilde{M}_{IJ}, \tilde{M}_{KL}] = \eta_{IL} \tilde{M}_{JK} + \eta_{JK} \tilde{M}_{IL} - \eta_{IK} \tilde{M}_{JL} - \eta_{JL} \tilde{M}_{IK}\,. \tag{1.1.5}$$

The maximally commuting set of generators of the conformal group is $\{M_{\mu,\nu}, D\}$, so we can choose a basis of operators in a conformal field theory to transform in a finite dimensional representation of the 3d Lorentz group, which is labeled by the spin j, and to be eigenfunctions of the dilation operator D with eigenvalue $-\Delta$, i.e.

$$[D, \mathcal{O}(x)] = (-\Delta + x^\mu \partial_\mu)\mathcal{O}(x)\,, \tag{1.1.6}$$

where Δ is called the scaling dimension of the operator.

The generators P_μ and K_μ act by raising and lowering, respectively, the scaling dimension of an operator by one. Primary operators $\mathcal{O}$ are defined as operators that are annihilated by K_μ, while all operators that can be obtained by acting with P_μ on a primary are called the descendants of that primary. In this thesis we will only discuss primary operators, which we will simply call operators for now on. The scaling dimension Δ_j of a spin j operator is constrained by unitarity to obey the unitarity bound

$$\Delta_j \geq j + 1 - \delta_{j,0}/2 - \delta_{j,1/2}/2\,. \tag{1.1.7}$$

Conformal symmetry constrains the two and three point functions of operators to take the following form in the standard basis

$$\begin{aligned} \langle \mathcal{O}_a(x_1)\mathcal{O}_b(x_2)\rangle &= \frac{C_{ab}\delta_{ab}}{|x_{12}|^{2\Delta_a}}\,, \\ \langle \mathcal{O}_a(x_1)\mathcal{O}_b(x_2)\mathcal{O}_c(x_3)\rangle &= \frac{\lambda_{abc}}{|x_{12}|^{\Delta_a+\Delta_b-\Delta_c}|x_{23}|^{\Delta_b+\Delta_c-\Delta_a}|x_{13}|^{\Delta_c+\Delta_a-\Delta_b}}\,, \end{aligned} \tag{1.1.8}$$

where $x_{ab} = x_a - x_b$. We can choose to normalize $C_{ab} = 1$, in which case it can be shown that a product of any two operators can be expanded as an operator product expansion (OPE) in terms of the rest of the operators in the theory as

$$\mathcal{O}_a(x)\mathcal{O}_b(0) = \sum_c \lambda_{abc} C_c(x, \partial_y)\mathcal{O}_c(y)\big|_{y=0}\,, \tag{1.1.9}$$

where $C_c(x, \partial_y)$ is a function completely fixed by conformal symmetry whose exact form we will not use, and λ_{abc} are the so called operator product expansion (OPE) coefficients, which are the same coefficients that appeared in the three point function (1.1.8). By taking successive OPEs, all higher point functions can be fixed by conformal symmetry in terms of the OPE coefficients and scaling dimensions of operators, which thus completely determine the local data of the theory. For instance, by taking the OPE of the first two and then the last two operators we can write a four point function as

$$\langle \mathcal{O}_a(x_1)\mathcal{O}_b(x_2)\mathcal{O}_c(x_3)\mathcal{O}_d(x_4)\rangle = \sum_{\mathcal{O}_e \in \mathcal{O}_a \times \mathcal{O}_b} \lambda_{abe}\lambda_{cde} T_{abcd} G_{\Delta_e, j_e}(U, V)\,, \tag{1.1.10}$$

where a, b, c, d are labels that we do not sum over, U, V are the conformal cross ratios

$$U \equiv \frac{\vec{x}_{12}^2 \vec{x}_{34}^2}{\vec{x}_{13}^2 \vec{x}_{24}^2} \,, \qquad V \equiv \frac{\vec{x}_{14}^2 \vec{x}_{23}^2}{\vec{x}_{13}^2 \vec{x}_{24}^2} \,, \tag{1.1.11}$$

T_{abcd} is a tensor structure that appears in case the operators transform under a global symmetry, and $G_{\Delta_e, j_e}(U, V)$ are called conformal blocks, which in 3d can be computed recursively [6]. If we work in Euclidean signature and $\mathcal{O}_a$ are bosonic operators, then the order of operators on the LHS of (1.1.10) doesn't matter, so we can exchange $\{x_1, a\} \leftrightarrow \{x_3, c\}$ on the RHS and equate it to the uncrossed version to get,

$$\sum_{\mathcal{O}_e \in \mathcal{O}_a \times \mathcal{O}_b} \lambda_{abe} \lambda_{cde} T_{abcd} G_{\Delta_e, j_e}(U, V) = \sum_{\mathcal{O}_e \in \mathcal{O}_a \times \mathcal{O}_b} \lambda_{abe} \lambda_{cde} T_{cbad} G_{\Delta_e, j_e}(V, U) \,, \tag{1.1.12}$$

which yields an an infinite set of constraints on the OPE coefficients and scaling dimensions of operators that appear in $\mathcal{O}_a \times \mathcal{O}_b$. These are called the crossing equations, and will be an important tool in this thesis.

1.1.2 Superconformal Symmetry in 3d with $\mathcal{N}$ supercharges

We can extend the conformal algebra $\mathfrak{so}(3, 2)$ by adding $2\mathcal{N}$ fermionic generators $Q_{\alpha r}$ and S_r^α for $\mathcal{N} \leq 8$, where α is a Lorentz spinor index and r is a fundamental index under the 3d R-symmetry algebra $\mathfrak{so}(\mathcal{N})_R$. This extended algebra is denoted as $\mathfrak{osp}(\mathcal{N}|4)$ and has a maximal bosonic subalgebra $\mathfrak{sp}(4) \oplus \mathfrak{so}(\mathcal{N})_R$, where $\mathfrak{sp}(4) \cong \mathfrak{so}(3, 2)$ is the conformal algebra.

To write the commutation relations it is convenient to first express the conformal generators in spinor notation:[1]

$$P_{\alpha\beta} = (\gamma^\mu)_{\alpha\beta} P_\mu \,, \quad K^{\alpha\beta} = (\bar{\gamma}^\mu)^{\alpha\beta} K_\mu \,, \quad M_\alpha{}^\beta = \frac{1}{2} (\gamma^\mu \bar{\gamma}^\nu)_\alpha{}^\beta M_{\mu\nu} \,, \tag{1.1.13}$$

where $(\gamma^a)_{\alpha\beta} \equiv (1, \sigma^1, \sigma^3)$ and $(\bar{\gamma}^a)^{\alpha\beta} \equiv (-1, \sigma^1, \sigma^3)$, so that

$$P_{\alpha\beta} = \begin{pmatrix} P_0 + P_2 & P_1 \\ P_1 & P_0 - P_2 \end{pmatrix} , \quad K^{\alpha\beta} = \begin{pmatrix} -K_0 + K_2 & K_1 \\ K_1 & -K_0 - K_2 \end{pmatrix} , \tag{1.1.14}$$

$$M_\alpha{}^\beta = \begin{pmatrix} M_{02} & M_{01} - M_{12} \\ M_{01} + M_{12} & -M_{02} \end{pmatrix} . \tag{1.1.15}$$

[1] The Clifford algebra is $\gamma^\mu \bar{\gamma}^\nu + \gamma^\nu \bar{\gamma}^\mu = \bar{\gamma}^\mu \gamma^\nu + \bar{\gamma}^\nu \gamma^\mu = 2\eta^{\mu\nu} \cdot 1$, and the completeness relation is $\gamma^\mu_{\alpha\beta} \bar{\gamma}_\mu^{\gamma\delta} = \delta_\alpha{}^\gamma \delta_\beta{}^\delta + \delta_\alpha{}^\delta \delta_\beta{}^\gamma$.

The Lorentz indices can be raised and lowered with the anti-symmetric symbol $\varepsilon^{12} = -\varepsilon^{21} = -\varepsilon_{12} = \varepsilon_{21} = 1$. Thus,

$$\underline{P}_{\alpha\beta} = P_{\alpha\beta}\,, \qquad \underline{K}_{\alpha\beta} = \varepsilon_{\alpha\gamma} K^{\gamma\delta} \varepsilon_{\delta\beta}\,, \qquad \underline{M}_{\alpha\beta} = M_\alpha{}^\beta \varepsilon_{\beta\gamma}\,. \tag{1.1.16}$$

We can then rewrite the conformal algebra commutation relations (1.1.2) as[2]

$$[M_\alpha{}^\beta, P_{\gamma\delta}] = \delta_\gamma{}^\beta P_{\alpha\delta} + \delta_\delta{}^\beta P_{\alpha\gamma} - \delta_\alpha{}^\beta P_{\gamma\delta}\,, \tag{1.1.17}$$

$$[M_\alpha{}^\beta, K^{\gamma\delta}] = -\delta_\alpha{}^\gamma K^{\beta\delta} - \delta_\alpha{}^\delta K^{\beta\gamma} + \delta_\alpha{}^\beta K^{\gamma\delta}\,, \tag{1.1.18}$$

$$[M_\alpha{}^\beta, M_\gamma{}^\delta] = -\delta_\alpha{}^\delta M_\gamma{}^\beta + \delta_\gamma{}^\beta M_\alpha{}^\delta\,, \quad [D, P_{\alpha\beta}] = P_{\alpha\beta}\,, \quad [D, K^{\alpha\beta}] = -K^{\alpha\beta}\,, \tag{1.1.19}$$

$$[K^{\alpha\beta}, P_{\gamma\delta}] = 4\delta_{(\gamma}{}^{(\alpha} M_{\delta)}{}^{\beta)} + 4\delta_{(\gamma}{}^{\alpha}\delta_{\delta)}{}^{\beta} D\,. \tag{1.1.20}$$

In this notation, the conjugation properties of the generators are

$$\begin{aligned} (P_{\alpha\beta})^\dagger &= K^{\alpha\beta}\,, \qquad (K_{\alpha\beta})^\dagger = P^{\alpha\beta}\,, \\ (M_\alpha{}^\beta)^\dagger &= M_\beta^\alpha\,, \qquad D^\dagger = D\,. \end{aligned} \tag{1.1.21}$$

The extension of the conformal algebra to the $\mathfrak{osp}(\mathcal{N}|4)$ superconformal algebra is then given by

$$\{Q_{\alpha r}, Q_{\beta s}\} = 2\delta_{rs} P_{\alpha\beta}\,, \qquad \{S^\alpha_r, S^\beta_s\} = -2\delta_{rs} K^{\alpha\beta}\,, \tag{1.1.22}$$

$$[K^{\alpha\beta}, Q_{\gamma r}] = -i\left(\delta_\gamma{}^\alpha S^\beta_r + \delta_\gamma{}^\beta S^\alpha_r\right)\,, \qquad [P_{\alpha\beta}, S^\gamma_r] = -i\left(\delta_\alpha{}^\gamma Q_{\beta r} + \delta_\beta{}^\gamma Q_{\alpha r}\right)\,, \tag{1.1.23}$$

$$[M_\alpha{}^\beta, Q_{\gamma r}] = \delta_\gamma{}^\beta Q_{\alpha r} - \frac{1}{2}\delta_\alpha{}^\beta Q_{\gamma r}\,, \qquad [M_\alpha{}^\beta, S^\gamma_r] = -\delta_\alpha{}^\gamma S^\beta_r + \frac{1}{2}\delta_\alpha{}^\beta S^\gamma_r\,, \tag{1.1.24}$$

$$[D, Q_{\alpha r}] = \frac{1}{2} Q_{\alpha r}\,, \qquad [D, S^\alpha_r] = -\frac{1}{2} S^\alpha_r\,, \tag{1.1.25}$$

$$[R_{rs}, Q_{\alpha t}] = i\left(\delta_{rt} Q_{\alpha s} - \delta_{st} Q_{\alpha r}\right)\,, \qquad [R_{rs}, S^\alpha_t] = i\left(\delta_{rt} S^\alpha_s - \delta_{st} S^\alpha_r\right)\,, \tag{1.1.26}$$

$$[R_{rs}, R_{tu}] = i\left(\delta_{rt} R_{su} + \cdots\right)\,, \qquad \{Q_{\alpha r}, S^\beta_s\} = 2i\left(\delta_{rs}\left(M_\alpha{}^\beta + \delta_\alpha{}^\beta D\right) - i\delta_\alpha{}^\beta R_{rs}\right)\,, \tag{1.1.27}$$

where R_{rs} are the anti-symmetric generators of the $\mathfrak{so}(\mathcal{N})$ R-symmetry. In addition to (1.1.21), we also have

$$\begin{aligned} (Q_{\alpha r})^\dagger &= -iS^\alpha_r\,, \qquad (S^\alpha_r)^\dagger = -iQ_{\alpha r}\,, \\ (R_{rs})^\dagger &= R_{rs}\,. \end{aligned} \tag{1.1.28}$$

From these commutation relations we see that the fermionic generators $Q_{\alpha r}$ and S^α_r raise and lower,

[2] Parentheses around indices means symmetrization by averaging over permutations.

respectively, operator scaling dimensions by half, analogously to P_μ and K_μ. We can thus define superconformal primaries as conformal primaries that are annihilated by S_r^α, which form a superconformal multiplet together with other conformal primaries that are related by $Q_{\alpha r}$.

1.1.3 Stress Tensor Four-Point Function for $\mathcal{N} = 8$ SCFTs

We will now specialize to the maximal amount of supersymmetry $\mathcal{N} = 8$, and discuss some general properties of the four-point function of the stress-tensor multiplet in an $\mathcal{N} = 8$ SCFT, and of the constraints imposed by the $\mathfrak{osp}(8|4)$ superconformal algebra.

Unitary irreps of $\mathfrak{osp}(8|4)$ are specified by the quantum numbers of their bottom component, namely by its scaling dimension Δ, Lorentz spin j, and $\mathfrak{so}(8)_R$ R-symmetry irrep with Dynkin labels $[a_1\, a_2\, a_3\, a_4]$, as well as by various shortening conditions. There are twelve different types of multiplets that we list in Table 1.1.[3] There are two types of shortening conditions denoted by the A and B families. The multiplet

Type	BPS	Δ	Spin	$\mathfrak{so}(8)_R$
$(A,0)$ (long)	0	$\geq \Delta_0 + j + 1$	j	$[a_1a_2a_3a_4]$
$(A,1)$	1/16	$\Delta_0 + j + 1$	j	$[a_1a_2a_3a_4]$
$(A,2)$	1/8	$\Delta_0 + j + 1$	j	$[0a_2a_3a_4]$
$(A,3)$	3/16	$\Delta_0 + j + 1$	j	$[00a_3a_4]$
$(A,+)$	1/4	$\Delta_0 + j + 1$	j	$[00a_30]$
$(A,-)$	1/4	$\Delta_0 + j + 1$	j	$[000a_4]$
$(B,1)$	1/8	Δ_0	0	$[a_1a_2a_3a_4]$
$(B,2)$	1/4	Δ_0	0	$[0a_2a_3a_4]$
$(B,3)$	3/8	Δ_0	0	$[00a_3a_4]$
$(B,+)$	1/2	Δ_0	0	$[00a_30]$
$(B,-)$	1/2	Δ_0	0	$[000a_4]$
conserved	5/16	$j + 1$	j	$[0000]$

Table 1.1: Multiplets of $\mathfrak{osp}(8|4)$ and the quantum numbers of their corresponding superconformal primary operator. The conformal dimension Δ is written in terms of $\Delta_0 \equiv a_1 + a_2 + (a_3 + a_4)/2$. The Lorentz spin can take the values $j = 0, 1/2, 1, 3/2, \ldots$. Representations of the $\mathfrak{so}(8)_R$ R-symmetry are given in terms of the four $\mathfrak{so}(8)_R$ Dynkin labels, which are non-negative integers.

denoted by $(A,0)$ is a long multiplet and does not obey any shortening conditions. The other multiplets of type A have the property that certain $\mathfrak{so}(2,1)$ irreps of spin $j - 1/2$ are absent from the product between the supercharges and the superconformal primary. The multiplets of type B have the property that certain $\mathfrak{so}(2,1)$ irreps of spin $j \pm 1/2$ are absent from this product, and consequently, the multiplets of type B are smaller.[4]

The stress-tensor multiplet is of $(B,+)$ type,[5], whose members are listed in Table 1.2. These include the

[3] The convention we use in defining these multiplets is that the supercharges transform in the $\mathbf{8}_v = [1000]$ irrep of $\mathfrak{so}(8)_R$.

[4] This description is correct only when $j > 0$. When $j = 0$, the definition of the multiplets also requires various conditions when acting on the primary with two supercharges.

[5] Whether it is $(B,+)$ or $(B,-)$ is a matter of convention. The two choices are related by the triality of $\mathfrak{so}(8)_R$.

dimension	spin	$\mathfrak{so}(8)_R$ irrep
1	0	$\mathbf{35}_c = [0020]$
3/2	1/2	$\mathbf{56}_v = [0011]$
2	0	$\mathbf{35}_s = [0002]$
2	1	$\mathbf{28} = [0100]$
5/2	3/2	$\mathbf{8}_v = [1000]$
3	2	$\mathbf{1} = [0000]$

Table 1.2: The operators comprising the $\mathcal{N} = 8$ stress-tensor multiplet along with their scaling dimension, spin, and R-symmetry representation.

spin-3/2 super-current, which in our convention transforms (like the supercharges) in the $\mathbf{8}_v$ of the $\mathfrak{so}(8)_R$ R-symmetry; and the spin-1 R-symmetry current, which transforms in the adjoint (i.e. the $\mathbf{28}$) of $\mathfrak{so}(8)_R$. In addition, the multiplet contains a spin-1/2 operator transforming in the $\mathbf{56}_v$, and two spin-0 operators with scaling dimension 1 and 2, which transform (in our conventions) in the $\mathbf{35}_c$ and $\mathbf{35}_s$, respectively. Let us denote the dimension 1 superconformal primary by $\mathcal{O}_{\text{Stress},IJ}(\vec{x})$. (The indices here are $\mathbf{8}_c$ indices, and $\mathcal{O}_{\text{Stress},IJ}(\vec{x})$ is a rank-two traceless symmetric tensor.) In order to not carry around the $\mathfrak{so}(8)_R$ indices, it is convenient to contract them with an auxiliary polarization vector Y^I that is constrained to be null $Y \cdot Y \equiv \sum_{I=1}^{8}(Y^I)^2 = 0$, thus defining

$$\mathcal{O}_{\text{Stress}}(\vec{x}, Y) \equiv \mathcal{O}_{\text{Stress},IJ}(\vec{x}) Y^I Y^J \,. \tag{1.1.29}$$

In the rest of this thesis we will only consider the four-point function of $\mathcal{O}_{\text{Stress}}(\vec{x}, Y)$. Conformal symmetry and $\mathfrak{so}(8)_R$ symmetry imply that the four point function of $\mathcal{O}_{\text{Stress}}(\vec{x}_3, Y_3)$ takes the form

$$\langle \mathcal{O}_{\text{Stress}}(\vec{x}_3, Y_3)\rangle = \frac{(Y_1 \cdot Y_2)^2 (Y_3 \cdot Y_4)^2}{|\vec{x}_{12}|^2\, |\vec{x}_{34}|^2} \mathcal{G}(U, V; \sigma, \tau) \,, \tag{1.1.30}$$

where σ and τ are $\mathfrak{so}(8)_R$ invariants formed out of the polarizations

$$\sigma \equiv \frac{(Y_1 \cdot Y_3)(Y_2 \cdot Y_4)}{(Y_1 \cdot Y_2)(Y_3 \cdot Y_4)} \,, \qquad \tau \equiv \frac{(Y_1 \cdot Y_4)(Y_2 \cdot Y_3)}{(Y_1 \cdot Y_2)(Y_3 \cdot Y_4)} \,. \tag{1.1.31}$$

In terms of $\mathcal{G}(U, V; \sigma, \tau)$, the crossing constraint corresponding to the exchange of (x_1, Y_1) with (x_3, Y_3) is

$$\mathcal{G}(U, V; \sigma, \tau) = \frac{U}{V} \tau^2 \mathcal{G}\left(V, U; \sigma/\tau, 1/\tau\right) \,. \tag{1.1.32}$$

Because (1.1.30) is a quadratic polynomial in each Y_i separately, the quantity $\mathcal{G}(U, V; \sigma, \tau)$ is a quadratic function of σ and τ, and so contains six distinct functions of U and V. It is helpful to exhibit explicitly these six functions by writing

$$\mathcal{G}(U,V;\sigma,\tau) = \sum_{a=0}^{2}\sum_{b=0}^{a} A_{ab}(U,V)Y_{ab}(\sigma,\tau)\,, \tag{1.1.33}$$

where the quadratic polynomials $Y_{ab}(\sigma,\tau)$ are defined as

$$\begin{aligned}
Y_{00}(\sigma,\tau) &= 1\,,\\
Y_{10}(\sigma,\tau) &= \sigma-\tau\,,\\
Y_{11}(\sigma,\tau) &= \sigma+\tau-\frac{1}{4}\,,\\
Y_{20}(\sigma,\tau) &= \sigma^2+\tau^2-2\sigma\tau-\frac{1}{3}(\sigma+\tau)+\frac{1}{21}\,,\\
Y_{21}(\sigma,\tau) &= \sigma^2-\tau^2-\frac{2}{5}(\sigma-\tau)\,,\\
Y_{22}(\sigma,\tau) &= \sigma^2+\tau^2+4\sigma\tau-\frac{2}{3}(\sigma+\tau)+\frac{1}{15}\,.
\end{aligned} \tag{1.1.34}$$

The definition (1.1.34) could be regarded simply as a convention. It has, however, a more profound meaning in terms of the $\mathfrak{so}(8)_R$ irreps that appear in the s-channel of the stress tensor four-point function. We have

$$\mathbf{35}_c\otimes\mathbf{35}_c = \mathbf{1}\oplus\mathbf{28}\oplus\mathbf{35}_c\oplus\mathbf{300}\oplus\mathbf{567}_c\oplus\mathbf{294}_c\,. \tag{1.1.35}$$

The six polynomials[6] in (1.1.34) correspond, in order, to the six terms on the right-hand side of (1.1.35). In terms of Dynkin labels, the indices (a,b) correspond to the irrep $[0\,(a-b)\,(2b)\,0]$.

The irreps $\mathbf{28}=[0100]=(1,0)$ and $\mathbf{567}_c=[0120]=(2,1)$ are in the anti-symmetric product of the two copies of $\mathbf{35}_c$, while the other irreps are in the symmetric product. Therefore only operators belonging to the $\mathcal{O}_{\mathbf{35}_c}(x_1,Y_1)\times\mathcal{O}_{\mathbf{35}_c}(x_2,Y_2)$ OPE with odd integer spin can contribute to the [0100] and [0120] channels. The other R-symmetry channels receive contributions only from operators with even integer spin.

By performing the OPE between the first two and last two operators in (1.1.30), one can decompose $\mathcal{G}(U,V;\sigma,\tau)$ into superconformal blocks $\mathfrak{G}_{\mathcal{M}}$,

$$\mathcal{G}(U,V;\sigma,\tau) = \sum_{\mathcal{M}\in\mathfrak{osp}(8|4)} \lambda^2_{\mathcal{M}}\mathfrak{G}_{\mathcal{M}}(U,V;\sigma,\tau)\,, \tag{1.1.36}$$

where $\mathcal{M}$ runs over all $\mathfrak{osp}(8|4)$ multiplets appearing in the $\mathcal{O}_{\text{Stress}}\times\mathcal{O}_{\text{Stress}}$ OPE, and the $\lambda^2_{\mathcal{M}}$ are the squared OPE coefficients for each such supermultiplet $\mathcal{M}$. In Table 1.3, we list the supermultiplets that may appear

[6] The polynomials in (1.1.34) are harmonic polynomials, which are eigenfunctions of the $\mathfrak{so}(8)_R$ Casimir. More details on these polynomials can be found in [7,8].

Type	(Δ, j)	$\mathfrak{so}(8)_R$ irrep	spin j	Name
$(B,+)$	$(2,0)$	$\mathbf{294}_c = [0040]$	0	$(B,+)$
$(B,2)$	$(2,0)$	$\mathbf{300} = [0200]$	0	$(B,2)$
$(B,+)$	$(1,0)$	$\mathbf{35}_c = [0020]$	0	Stress
$(A,+)$	$(j+2,j)$	$\mathbf{35}_c = [0020]$	even	$(A,+)_j$
$(A,2)$	$(j+2,j)$	$\mathbf{28} = [0100]$	odd	$(A,2)_j$
$(A,0)$	$\Delta \geq j+1$	$\mathbf{1} = [0000]$	even	$(A,0)_{j,n}$

Table 1.3: The possible superconformal multiplets in the $\mathcal{O}_{\text{Stress}} \times \mathcal{O}_{\text{Stress}}$ OPE. The $\mathfrak{so}(3,2) \oplus \mathfrak{so}(8)_R$ quantum numbers are those of the superconformal primary in each multiplet.

in this four-point function, following the constraints discussed in [9]. Since these are the only multiplets we will consider in this thesis, we denote the short multiplets other than the stress-tensor as $(B,+)$ and $(B,2)$, the semi short multiplets as $(A,2)_j$ and $(A,+)_j$ where j is the spin, and the long multiplet as $(A,0)_{j,n}$, where $n = 0,1,\ldots$ denotes the nth lowest multiplet with that spin—See the last column of Table 1.3. The superconformal blocks are fixed by the superconformal Ward identities [1]

$$\left(z\partial_z - \frac{1}{2}\alpha\partial_\alpha\right) \mathcal{G}(U,V;\sigma,\tau)\big|_{\alpha=z^{-1}} = \left(\bar{z}\partial_{\bar{z}} - \frac{1}{2}\alpha\partial_\alpha\right) \mathcal{G}(U,V;\sigma,\tau)\big|_{\alpha=\bar{z}^{-1}} = 0\,, \tag{1.1.37}$$

where we defined

$$U \equiv z\bar{z}\,, \qquad V \equiv (1-z)(1-\bar{z})\,, \qquad \sigma \equiv \alpha\bar{\alpha}\,, \qquad \tau \equiv (1-\alpha)(1-\bar{\alpha})\,. \tag{1.1.38}$$

In Chapter 3, we will derive explicit expressions for $\mathfrak{G}_{\mathcal{M}}$ using these Ward identities.

For the OPE coefficients $\lambda_{\mathcal{M}}$ in (1.1.36) to be meaninful, we must choose a normalization for $\mathcal{O}_{\text{Stress}}$ as well as for the conformal blocks. We can choose the normalization that corresponds to $\lambda_{\text{Id}} = 1$ and a unit superconformal block for the identity operator by requiring the two-point function of $\mathcal{O}_{\text{Stress}}$ to satisfy

$$\langle \mathcal{O}_{\text{Stress}}(x_1,Y_1)\mathcal{O}_{\text{Stress}}(x_2,Y_2)\rangle = \frac{(Y_1\cdot Y_2)^2}{|x_1-x_2|^2}\,. \tag{1.1.39}$$

We can now specify the normalization of the superconformal blocks. In our conventions, if the superconformal primary of $\mathcal{M}$ has conformal dimension Δ, spin j, and transforms as the $(c,d) = [0\,(c-d)\,(2d)\,0]$ of $\mathfrak{so}(8)_R$, then

$$A_{cd}(z,\bar{z}) \sim \frac{\Gamma(j+1/2)}{4^\Delta\sqrt{\pi}\,\Gamma(j+1)} z^{\frac{1}{2}(\Delta+j)}\bar{z}^{\frac{1}{2}(\Delta-j)}\,, \qquad \text{as } z,\bar{z}\to 0\,, \tag{1.1.40}$$

where $\bar{z}$ is taken to zero first.

With the normalization described above, we can relate the OPE coefficient of the stress-tensor multiplet to the central charge c_T, which is defined as the overall coefficient appearing in the two-point function of the canonically normalized stress tensor [10]:

$$\langle T_{\mu\nu}(x)T_{\rho\sigma}(0)\rangle = \frac{c_T}{64}\left(P_{\mu\rho}P_{\nu\sigma} + P_{\nu\rho}P_{\mu\sigma} - P_{\mu\nu}P_{\rho\sigma}\right)\frac{1}{16\pi^2 x^2}\,, \tag{1.1.41}$$

where $P_{\mu\nu} \equiv \eta_{\mu\nu}\nabla^2 - \partial_\mu\partial_\nu$. In (1.1.41), we normalized c_T such that it equals one for a real massless scalar or Majorana fermion. The relation between c_T and λ_{Stress} is[7]

$$\lambda^2_{\text{Stress}} = \frac{256}{c_T}\,. \tag{1.1.42}$$

It is worth pointing out two limits in which the four-point function (1.1.30) is known exactly and one can extract all OPE coefficients. The first limit is the free theory of eight real scalars X_I and eight Majorana fermions. The scalar $\mathcal{O}_{\text{Stress},IJ}$ in this case is given by

$$\mathcal{O}_{\text{Stress},IJ} = X_I X_J - \frac{\delta_{IJ}}{8}X_K X^K\,. \tag{1.1.43}$$

Performing Wick contractions with the propagator $\langle X_I(\vec{x})X_J(0)\rangle = \frac{\delta_{IJ}}{4\pi|\vec{x}|}$, we then find that (1.1.30) equals:

$$\frac{2}{(4\pi)^4}\frac{(Y_1\cdot Y_2)^2(Y_3\cdot Y_4)^2}{|\vec{x}_{12}|^2\,|\vec{x}_{34}|^2}\left[1 + u\sigma^2 + \frac{u}{v}\tau^2 + 4\sqrt{u}\sigma + 4\sqrt{\frac{u}{v}}\tau + 4\frac{u}{\sqrt{v}}\sigma\tau\right]. \tag{1.1.44}$$

By comparing this to the conformal block expansion, we can read off the OPE coefficients listed in Table 1.4, where the scaling dimensions of the long multiplet are given by

$$\Delta^{\text{free}}_{(A,0)_{j,n}} = j + \delta_{n,0} + 2n\,, \tag{1.1.45}$$

with $n = 0, 1, 2, \ldots$.

Another limit in which we can compute (1.1.30) explicitly is in the generalized free field theory (GFFT) where the dimension one operator $\mathcal{O}^{\text{GFFT}}_{\text{Stress},IJ}(\vec{x})$ is treated as a generalized free field with two-point function $\langle\mathcal{O}_{\text{Stress}}(\vec{x},Y_1)\mathcal{O}_{\text{Stress}}(0,Y_2)\rangle = \frac{(Y_1\cdot Y_2)^2}{|x|^2}$. The GFFT describes the $c_T\to\infty$, i.e. $\lambda^2_{\text{Stress}} = 0$, limit of $\mathcal{N} = 8$

[7] We stress that λ_{Stress} is not the OPE coefficient of the stress tensor in the $\mathcal{O}_{\text{Stress}}\times\mathcal{O}_{\text{Stress}}$ OPE, but instead the coefficient of the superconformal primary in the stress-tensor multiplet. The OPE coefficient of the stress tensor is $\lambda_{3,2} = \lambda_{\text{Stress}}/2$.

theories. Performing the Wick contractions, we then find

$$\frac{(Y_1 \cdot Y_2)^2 (Y_3 \cdot Y_4)^2}{|\vec{x}_{12}|^2 |\vec{x}_{34}|^2} \left[1 + u\sigma^2 + \frac{u}{v}\tau^2\right] . \tag{1.1.46}$$

By comparing this to (1.1.30), we can read off the OPE coefficients listed in Table 1.4, where the scaling dimensions of the long multiplet are given by

$$\Delta^{\text{GFFT}}_{(A,0)_{j,n}} = j + 2 + 2n \,, \tag{1.1.47}$$

with $n = 0, 1, 2, \ldots$.

Type $\mathcal{M}$	Free theory $\lambda^2_{\mathcal{M}}$	generalized free field theory $\lambda^2_{\mathcal{M}}$
$(B,2)$	0	$32/3 \approx 10.667$
$(B,+)$	16	$16/3 \approx 5.333$
$(A,2)_1$	$128/21 \approx 6.095$	$1024/105 \approx 9.752$
$(A,2)_3$	$2048/165 \approx 12.412$	$131072/8085 \approx 16.212$
$(A,2)_5$	$9273344/495495 \approx 18.715$	$33554432/1486485 \approx 22.573$
$(A,+)_0$	$32/3 \approx 10.667$	$64/9 \approx 7.111$
$(A,+)_2$	$20992/1225 \approx 17.136$	$16384/1225 \approx 13.375$
$(A,+)_4$	$139264/5929 \approx 23.489$	$1048576/53361 \approx 19.651$
$(A,0)_{0,0}$	4	$32/35 \approx 0.911$
$(A,0)_{2,0}$	4	$2048/693 \approx 2.955$
$(A,0)_{4,0}$	4	$1048576/225225 \approx 4.656$
$(A,0)_{6,0}$	4	$67108864/10669659 \approx 6.290$
$(A,0)_{8,0}$	4	$34359738368/4350310965 \approx 7.899$
$(A,0)_{10,0}$	4	$2199023255552/231618204675 \approx 9.494$
$(A,0)_{12,0}$	4	$2251799813685248/203176892887605 \approx 11.083$
$(A,0)_{0,1}$	$8/5 \approx 1.6$	$256/693 \approx 0.369$
$(A,0)_{2,1}$	$128/33 \approx 3.879$	$65536/45045 \approx 1.455$
$(A,0)_{4,1}$	$1263616/225225 \approx 5.611$	$8388608/3556553 \approx 2.359$
$(A,0)_{6,1}$	$4554752/627627 \approx 7.257$	$1073741824/334639305 \approx 3.209$
$(A,0)_{8,1}$	$38598606848/4350310965 \approx 8.873$	$274877906944/68123001375 \approx 4.035$
$(A,0)_{10,1}$	$2425742163968/231618204675 \approx 10.473$	$140737488355328/29025270412515 \approx 4.849$

Table 1.4: Values of OPE coefficients in the free and generalized free field theory limits for the $(B,2)$ and $(B,+)$ multiplets, the $(A,2)_j$ multiplet for spin $j = 1,3,5$, the $(A,+)_j$ multiplet for $j = 0,2,4$, and the $(A,0)_{j,n}$ multiplet for $j = 0,2,\ldots,12$ and $n = 0$, which is the lowest multiplet with that spin, as well as for $j = 0,2,\ldots,10$ and $n = 1$, which is the second lowest multiplet.

1.1.4 Known $\mathcal{N} = 8$ SCFTs

We will now discuss the known $\mathcal{N} = 8$ 3d SCFTs, which can all be described by a few infinite families of Chern-Simons (CS) theories with a product gauge group coupled to two pairs of matter chiral multiplets transforming in the bifundamental representation of the gauge group—see Figure 1.1. These families are:

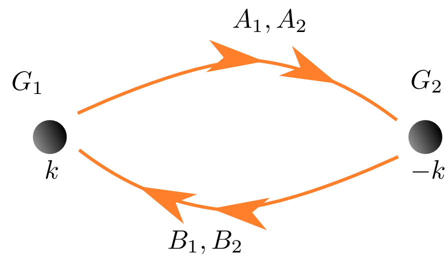

Figure 1.1: The field content of the two-gauge group description of $\mathcal{N} = 8$ SCFTs. The gauge group is $G_1 \times G_2$ with opposite Chern-Simons levels for the two factors. The matter content consists of two pairs of bifundamental chiral multiplets whose bottom components are denoted by A_1, A_2 and B_1, B_2. As explained in the main text, such theories have $\mathcal{N} = 8$ SUSY at the IR fixed point only for special values of k and/or for special gauge groups G_1 and G_2.

- BLG theories: These are $SU(2)_k \times SU(2)_{-k}$ (denoted BLG$'_k$) and $(SU(2)_k \times SU(2)_{-k})/\mathbb{Z}_2$ (denoted BLG$_k$) gauge theories, which preserve manifest $\mathcal{N} = 8$ supersymmetry for any integer Chern-Simons level k. This description of the BLG theories is a reformulation [11, 12] of the original work of Bagger, Lambert, [4, 13, 14] and Gustavsson [5] (BLG).

- ABJM or ABJ theories: These are $U(N)_k \times U(M)_{-k}$ gauge theories (denoted ABJM$_{N,k}$ if $N = M$ and ABJ$_{N,M,k}$ if $N \neq M$), which are believed to flow to IR fixed points with $\mathcal{N} = 8$ supersymmetry only if the Chern-Simons level is $k = 1$ or 2 and $|N - M| \leq k$. The theories with $M = N$ were first introduced by Aharony, Bergman, Jafferis, and Maldacena (ABJM) in [2], and those with $M \neq N$ by Aharony, Bergman, and Jafferis (ABJ) in [3]. Due to the dualities [2, 15]

$$\begin{aligned} \mathrm{ABJ}_{N+1,N,1} &\cong \mathrm{ABJM}_{N,1}\,, \\ \mathrm{ABJ}_{N+2,N,2} &\cong \mathrm{ABJM}_{N,2}\,, \end{aligned} \tag{1.1.48}$$

  the only independent theories in this family are the ABJM$_{N,1}$, ABJM$_{N,2}$, and ABJ$_{N+1,N,2}$ theories.

The case of the ABJM$_{1,1}$ theory is worth noting: this theory is equivalent to a free theory of 8 massless real scalars and 8 massless Majorana fermions. The case ABJM$_{N,1}$ for $N > 1$, flows to a product of two decoupled CFTs in the infrared (see for instance [16]). One of these CFTs is free (and equivalent to the ABJM$_{1,1}$ theory). The other CFT in the product is interacting and strongly coupled.

In addition to the dualities between ABJM / ABJ theories already mentioned, there are further dualities that relate the BLG and ABJM theories at certain small values of k. For instance [15, 17]:

$$\begin{aligned} \mathrm{BLG}_1 &\cong \mathrm{ABJM}_{2,1}\,, \\ \mathrm{BLG}'_2 &\cong \mathrm{ABJM}_{2,2}\,, \\ \mathrm{BLG}_4 &\cong \mathrm{ABJ}_{2,3,2}\,. \end{aligned} \tag{1.1.49}$$

Furthermore, it is possible to conjecture other dualities that come from the fact that the $k = 1, 2$ ABJM and the $k = 2$ ABJ theories can be thought of as the IR limits of the maximally supersymmetric Yang-Mills theory with gauge algebra $\mathfrak{u}(N)$, $\mathfrak{so}(2N)$, and $\mathfrak{so}(2N+1)$, respectively [18–21]. At small N, there are various coincidental isomorphisms between these Lie algebras, which themselves induce isomorphisms between the corresponding $\mathcal{N} = 8$ SCFTs. For instance, since $\mathfrak{u}(2) \cong \mathfrak{u}(1) \oplus \mathfrak{so}(3)$, one expects that the $\mathrm{ABJM}_{2,1}$ theory should be isomorphic to the product between the $\mathrm{ABJM}_{1,1}$ theory and the $\mathrm{ABJ}_{2,1,1}$ theory.

The matter content of BLG and ABJ(M), in $\mathcal{N} = 2$ notation, consists of chiral multiplets with scalar components A_1, A_2 and B_1, B_2 that transform under the product gauge group as $(\mathbf{N}, \overline{\mathbf{N}})$ and $(\overline{\mathbf{N}}, \mathbf{N})$, respectively. The theories have a quartic superpotential

$$W = \frac{2\pi}{k} \epsilon^{ab} \epsilon^{\dot{a}\dot{b}}\, \mathrm{Tr}(A_a B_{\dot{a}} A_b B_{\dot{b}})\,, \tag{1.1.50}$$

which preserves an $SU(2) \times SU(2)$ flavor symmetry under which A_a transforms as $(\mathbf{2}, \mathbf{1})$ and $B_{\dot{a}}$ transforms as $(\mathbf{1}, \mathbf{2})$. These theories also have a manifest $SU(2)_R$ symmetry (corresponding to $\mathcal{N} = 3$ SUSY) under which $(A_a, B^\dagger_{\dot{a}})$ form doublets, a $U(1)_R$ subgroup of which being the $\mathcal{N} = 2$ R-symmetry under which A_a and $B_{\dot{a}}$ have canonical R-charge $1/2$. The $SU(2) \times SU(2)$ flavor symmetry combines with the $SU(2)_R$ symmetry to form an $SU(4)_R$ R-symmetry. This is enhanced to the full $SO(8)_R$ R-symmetry in different ways depending on the gauge group. Theories with $U(N) \times U(N)$ gauge group have an additional topological $U(1)_T$ symmetry under which only monopole operators are charged. When $k = 1$ or 2, one can find additional R-symmetry generators, which together with $SU(4)_R$ and $U(1)_T$ combine into an $SO(8)_R$ symmetry. The theories with $(SU(N) \times SU(N))/\mathbb{Z}_N$ or $(SU(N) \times SU(N))$ gauge group in general do not have a similar R-symmetry enhancement. When $N = 2$, however, one can show that because A_a and $B_{\dot{a}}$ now transform in the same gauge representation, the superpotential (1.1.50) has an $SU(4)$ flavor symmetry (which contains the $SU(2) \times SU(2)$ flavor symmetry as well as a baryonic symmetry $U(1)_t$ under which the A_a have charge $+1$ and the $B_{\dot{a}}$ have charge -1), which combines with the $SU(2)_R$ symmetry mentioned above to form an $SO(8)_R$ R-symmetry. Such an enhancement occurs for any k.

We can construct operators in these theories using fields in the Lagrangian, as well as monopole operators.

The monopole operators $M^{n_1,\ldots,n_N}_{\tilde{n}_1,\ldots,\tilde{n}_N}$ create $\mathrm{diag}\{n_1,\ldots,n_N\}$ and $\mathrm{diag}\{\tilde{n}_1,\ldots,\tilde{n}_N\}$ units of magnetic flux through the two gauge groups, respectively. Here, we take both the n_r and $\tilde{n}_r$ to be in descending order. If the gauge groups are $U(N)$, then the equations of motion imply that

$$\sum_r n_r = \sum_r \tilde{n}_r = -2Q_T\,, \tag{1.1.51}$$

where Q_T is the charge under the $U(1)_T$ symmetry mentioned above, quantized in half-integer units.[8] If the gauge groups are $SU(N)$, then the n_r and $\tilde{n}_r$ must each sum to zero. We will only be considering BPS monopole operators, with zero R-charge. In general, the R-charge is

$$E = \sum_{r,s=1}^{N}\left[|n_r-\tilde{n}_s| - \frac{1}{2}|n_r-n_s| - \frac{1}{2}|\tilde{n}_r-\tilde{n}_s|\right]\,, \tag{1.1.52}$$

as was first proposed in [22] and derived in [16, 18, 23]. It is easy to see from (1.1.52) that $E=0$ only for $n_r=\tilde{n}_r$. In order to avoid clutter, we denote such operators simply by $M^{n_1,\cdots,n_N}$. For $k\neq 0$ these monopole operators transform nontrivially under the gauge group in a way that will be described in more detail in Chapter 3. To form gauge-invariant operators, the monopole operators $M^{n_1,\cdots,n_N}$ need to be dressed with the matter fields.

1.2 AdS/CFT for M-theory

We will now review how the AdS/CFT conjecture relates a stack of N M2 branes in M-theory on $(AdS_4\times S^7)/\mathbb{Z}_k$ to ABJ(M) theory with gauge group $U(N)_k\times U(N)_{-k}$ or $U(N)_k\times U(N+1)_{-k}$ for $k=1,2$. As in the previous section, we will use the coordinates x_μ with $\mu=0,1,2$ for the CFT$_3$, which can be thought as living on the boundary of AdS_4, which has the extra bulk coordinate r. We will begin by discussing AdS_4 and the general AdS$_4$/CFT$_3$ dictionary, and then discuss the specific holographic dual of ABJ(M). Finally, we discuss the small momentum expansion of the M-theory S-matrix, which is dual to the stress tensor four point function in ABJ(M) theory. Note that the BLG theories have no similar holographic interpretation, except for the cases where they are dual to certain ABJM theories.

[8] The conserved current associated to this charge is $J^\mu = -\frac{1}{16\pi}\epsilon^{\mu\nu\rho}\left(\mathrm{Tr}\,F_{\nu\rho}+\mathrm{Tr}\,\tilde{F}_{\nu\rho}\right)$. The other linear combination of gauge field strengths vanishes by the equations of motion.

1.2.1 Anti-de Sitter Space

AdS_4 is a hyperboloid in $\mathbb{R}^{3,2}$, with the metric η_{IJ} previously introduced in Section 1.1.1 for the $\mathfrak{so}(3,2)$ algebra isomorphic to the 3d conformal algebra. The embedding of AdS_4 in $\mathbb{R}^{3,2}$ is given by

$$\eta_{IJ}X^I X^j = -L^2 \,, \tag{1.2.1}$$

where X^I with $I = 0, 1, 2, 3, 4$ are the coordinates on $\mathbb{R}^{3,2}$ and L is the radius of AdS_4. Since both the metric η_{IJ} and the embedding equations (1.2.1) are invariant under $SO(3,2)$, the isometry group of AdS_4 is thus $SO(3,2)$, which is isomorphic to the 3d conformal group.

To describe the holographic dual we will use a different parameterization with coordinates $\{x_\mu, r\}$ described above, which are related to X^I as

$$\begin{aligned} &X^\mu = x^\mu e^r \,, \qquad X^3 = \frac{x^2 e^r}{2L} - L\sinh r \,, \\ &X^4 = \frac{x^2 e^r}{2L} + L\cosh r \,, \end{aligned} \tag{1.2.2}$$

where r and x^μ are unrestricted, and these coordinates parameterize only half the hyperboloid (1.2.1). The metric in these coordinates induced from η_{IJ} is then

$$ds^2 = e^{2r} dx_\mu dx^\mu + L^2 dr^2 \,, \tag{1.2.3}$$

and the Killing vectors

$$\begin{aligned} &P_\mu = \partial_\mu \,, \qquad M_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu \,, \\ &D = -\partial_r + x^\mu \partial_\mu \,, \qquad K_\mu = x^2 \partial_\mu - 2x_\mu x^\nu \partial_\nu + L^2 e^{-2r} \partial_\mu + 2x_\mu \partial_r \,, \end{aligned} \tag{1.2.4}$$

satisfy the commutation relations (1.1.2). Note that as $r \to \infty$ these Killing vector become the conformal Killing vectors (1.1.3) on $\mathbb{R}^{2,1}$, consistent with the idea that the CFT$_3$ lives on the boundary of the AdS$_4$ space.

1.2.2 AdS$_4$/CFT$_3$ Dictionary

We are now ready to make the precise statement of the AdS$_4$/CFT$_3$ duality, which is most easily expressed in Euclidean signature. The statement is that the generating functional for connected correlators of operators

$\mathcal{O}_\phi$ in the CFT$_3$ equals minus the on-shell gravitational action with the bulk field ϕ dual to $\mathcal{O}_\phi$:

$$W[J_\phi] = -S_{\text{on-shell}}[J_\phi]\,, \qquad W[J_\phi] = \log\langle \exp \int d^4x J_\phi(x)\mathcal{O}_\phi(x)\rangle\,, \tag{1.2.5}$$

where both are computed as functionals of the source J_ϕ, and the scaling dimension of $\mathcal{O}$ is related to the mass of ϕ as

$$\Delta(\Delta - 3) = m^2 L^2\,. \tag{1.2.6}$$

The gravity action is typically UV divergent, but can be made finite through a procedure called holographic renormalization. Connected correlation function of $\mathcal{O}_\phi$ can then be computed by taking functional derivatives of the renormalized on-shell action with respect to J_ϕ.

1.2.3 Holographic dual of ABJ(M) Theory

Let us begin by discussing the holographic dual of $U(N)_k \times U(N)_{-k}$ ABJM theory for small k, which as described in Section 1.1.4 has $\mathcal{N} = 6$ supersymmetry, which is promoted to $\mathcal{N} = 8$ for $k = 1, 2$. The dual of this theory is conjectured to be N coincident M2-branes probing a $\mathbb{C}^4/\mathbb{Z}_k$ orbifold singularity, which at large N should be described by classical supergravity on $AdS_4 \times S^7$.

We can define M2-branes as objects in the 11d theory that are electric sources for the field strength F_4 that appears in the 11d supergravity action:

$$S = \frac{1}{\kappa_{11}^2}\int d^{11}x\sqrt{-G}\left(R - \frac{1}{2}|F_4|^2\right) - \frac{1}{12\kappa_{11}^2}\int A_3 \wedge F_4 \wedge F_4\,, \tag{1.2.7}$$

where A_3 is a 3-form with $F_4 = dA_3$, R is the Einstein-Hilbert term for the 11d metric G, and κ_{11} is the 11d gravitational coupling constant, which is related to the Planck length ℓ_{11} by

$$2\kappa_{11}^2 = (2\pi)^5 \ell_{11}^9\,. \tag{1.2.8}$$

The solution that corresponds to N coincident M2-branes at the tip of the eight dimensional space $\mathbb{C}^4/\mathbb{Z}_k$ is

$$\begin{aligned} ds^2 =& H^{-2/3} dx_\mu dx^\mu + H^{1/3} ds^2_{\mathbb{C}^4/\mathbb{Z}_k}\,,\\ F_4 =& dH^{-1}\wedge dx^0 \wedge dx^1 \wedge dx^2\,, \end{aligned} \tag{1.2.9}$$

where $dx_\mu dx^\mu = (-dx^0)^2 + (dx^1)^2 + (dx^2)^2$, and H is a harmonic function on $\mathbb{C}^4/\mathbb{Z}_k$ away from the tip of the cone. The cone metric is

$$ds^2_{\mathbb{C}^4/\mathbb{Z}_k} = dr^2 + r^2 ds^2_{S^7/\mathbb{Z}_k}\,, \tag{1.2.10}$$

where r is the radial coordinate described in the previous sections, and the metric on $S^7/\mathbb{Z}_k$ is normalized so that $R_{mn} = 6g_{mn}$, where $m, n = 1, \ldots 6$. The most general harmonic function on $\mathbb{C}^4/\mathbb{Z}_k$ is

$$H = \alpha + \frac{(2L)^6}{r^6}\,, \tag{1.2.11}$$

for constants α and L. If we want the solution (1.2.9) to asymptote to $\mathbb{R}^{1,2} \times \mathbb{C}^4/\mathbb{Z}_k$ at large r, then we should take $\alpha = 1$. This solution corresponds to a stack of M2-branes extended along the 012 directions and located at the tip of the cone at $r = 0$.

If we look close to the branes, i.e. at small r, then we can neglect α in (1.2.11), and write the metric as

$$ds^2 = \frac{r^4}{(2L)^4} dx_\mu dx^\mu + \frac{(2L)^2}{r^2} dr^2 + (2L)^2 ds^2_{S^7/\mathbb{Z}_k}\,, \tag{1.2.12}$$

which in terms of the new radial coordinate $\rho = 2\log(r/2L)$ becomes

$$ds^2 = e^{2\rho} dx_\mu dx^\mu + L^2 d\rho^2 + (2L)^2 ds^2_{S^7/\mathbb{Z}_k}\,, \tag{1.2.13}$$

where the first two terms recover the metric (1.2.3) on AdS_4. We thus see that changing r from small to large interpolates between $\mathbb{R}^{1,2}$ and AdS_4, as expected from the $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence.

The $AdS_4 \times S^7/\mathbb{Z}_k$ classical background is a reliable approximation to the quantum mechanical M-theory dynamics when the smallest length scale in the geometry, which is the length $\approx L/k$ of the circle along which the $\mathbb{Z}_k$ isometry acts, is much larger than the Planck scale ℓ_{11}. To relate ℓ_{11} to N and k, we note that M2-branes are an electric source for F_4 with charge equal to their tension $T_{\mathrm{M2}} = 2\pi/(2\pi\ell_{11})^3$, so the integral of $\star_{11} F_4$ over any Gaussian surface enclosing the branes should equal N times $2\kappa_{11}^2 T_{\mathrm{M2}} = (2\pi)^3(\ell_{11})^6$, where $2\kappa_{11}^2$ comes from the normalization of the action (1.2.7). If we choose the Gaussian surface to be a section through the cone at fixed r then we get

$$\int_{S^7_{\mathbb{Z}_k}} \star_{11} F_4 = 128 L^6 \pi^4/k\,, \tag{1.2.14}$$

so that we can relate ℓ_{11} to N and k as

$$\left(\frac{L}{\ell_{11}}\right)^6 = \frac{Nk}{8} \,. \tag{1.2.15}$$

The condition that L/k be much bigger than ℓ_{11} is thus equivalent to $N \gg k^5$, i.e. large N and small k as we expected. We are interested in the special case $k = 1, 2$, in which case ABJM has maximal supersymmetry.

The construction for $U(N+l)_k \times U(N)_{-k}$ ABJ theory is the same as the one just described, with the extra ingredient of l fractional M2-branes, which are M5-branes wrapped on a vanishing 3-cycle at the orbifold point, where M5 branes are defined as the magnetic dual of M2 branes. This is a pure torsion cycle since $H_3(S^7/\mathbb{Z}_k, \mathbb{Z}) = \mathbb{Z}_k$, which by Poincare duality is equivalent to pure torsion flux of the F_4 in $H^4(S^7/\mathbb{Z}_k, \mathbb{Z}) = \mathbb{Z}_k$. We conclude that $U(N+l)_k \times U(N)_{-k}$ ABJ describes N M2-branes on $\mathbb{C}^4/\mathbb{Z}_k$ with l units of discrete torsion. In the large N limit, the effect of the torsion is invisible and the theory is described by the same classical supergravity as the $U(N)_k \times U(N)_{-k}$ ABJM theory. We are interested in the special case $k = 2$ and $l = 1$, in which case ABJ has maximal supersymmetry.

The above constructions can be motivated by comparing the moduli space and operators on each side, as well as by a complicated Type IIB construction [2,3] that is beyond the scope of this thesis.

1.2.4 M-theory S-matrix

Using the precise statement of the AdS/CFT duality in (1.2.5) applied to the M-theory dual of ABJ(M) just described, we see that the stress-tensor four point function is related to a four-graviton correlator in M-theory on $AdS_4 \times S^7$, which in the flat space limit becomes the 11d M-theory S-matrix. We will now review what is known about this S-matrix purely from the bulk perspective.

The M-theory S-matrix can be computed as an expansion in ℓ_{11}, i.e. a small momentum expansion. At low orders in the momentum expansion, beyond the tree level terms, the S-matrix elements have local terms such as the (supersymmetrized) R^4 vertex, and nonlocal terms that are determined by lower order terms through unitarity cuts. These nonlocal terms are what we loosely refer to as "loop amplitudes" in M-theory. Concretely, the S-matrix element involving 4 super-gravitons is constrained by supersymmetry Ward identities [24,25] to be of the form

$$\mathcal{A} = f(s,t)\mathcal{A}_{\text{SG,tree}} \,, \tag{1.2.16}$$

where $\mathcal{A}_{\text{SG,tree}}$ is the tree level scattering amplitude in 2-derivative supergravity, and f is a symmetric function of the Mandelstam invariants s, t, and $u = -s - t$. The tree-level supergravity scattering amplitude

$\mathcal{A}_{\text{SG,tree}}$ carries dependence on the polarization as well as the type of particles in the super-graviton multiplet. The function f admits a small momentum expansion, or equivalently, an expansion in the 11D Planck length ℓ_{11}, of the form

$$f(s,t) = 1 + \ell_{11}^6 f_{R^4}(s,t) + \ell_{11}^9 f_{\text{1-loop}}(s,t) + \ell_{11}^{12} f_{D^6R^4}(s,t) + \ell_{11}^{14} f_{D^8R^4}(s,t) + \cdots . \tag{1.2.17}$$

Here, $f_{D^{2n}R^4}$ refers to a local term which is a degree $n+3$ symmetric polynomial in s,t,u, whereas the loop terms are not analytic at zero momentum. In particular, f_{R^4}, $f_{1-\text{loop}}$, and $f_{D^6R^4}$ are known exactly [26–28], as they are protected by supersymmetry and can be determined by perturbative calculations either in type II string theory or in 11D supergravity [29, 30]. For instance,

$$f_{R^4}(s,t,u) = \frac{stu}{3\cdot 2^7}, \qquad f_{D^6R^4}(s,t,u) = \frac{(stu)^2}{15\cdot 2^{15}}. \tag{1.2.18}$$

We will now review the derivation of these coefficients.

We eschew the 11d action and instead work with type IIa amplitudes, which we uplift to 11d. The dilaton ϕ and the string parameter α' are related to the 11d Planck length as

$$e^{2\phi}(\alpha')^3 = \frac{\ell_{11}^6}{(2\pi)^2}\,, \tag{1.2.19}$$

where $e^\phi \equiv g_s$. We denote IIa amplitudes as A and 11d amplitudes as $\mathcal{A}$.

From the tree-level amplitude of type IIa [31],

$$A_{\text{tree}} = \widehat{K}\kappa_{10}^2 e^{-2\phi}\frac{2^6}{(\alpha')^3 stu}\exp\left[\sum_{k=1}^{\infty}\frac{2\zeta(2k+1)}{2k+1}(\alpha'/4)^{2k+1}(s^{2k+1}+t^{2k+1}+u^{2k+1})\right], \tag{1.2.20}$$

where $u=-s-t$ and $\widehat{K}$ is a universal term that encodes the polarizations of the particles. We have, up to the same universal term $\widehat{K}\kappa_{10}^2$,

$$\begin{aligned} A_{\text{tree}}\Big|_{R} &= \frac{64}{(\alpha')^3 stu}e^{-2\phi}\,, \\ A_{\text{tree}}\Big|_{R^4} &= 2\zeta(3)e^{-2\phi}\,, \\ A_{\text{tree}}\Big|_{D^6R^4} &= \frac{(\alpha')^3 stu}{32}\zeta(3)^2 e^{-2\phi}\,. \end{aligned} \tag{1.2.21}$$

First we address R^4. From the type IIa action [32],

$$S_{R^4} \propto 2\zeta(3)E_{3/2}(\phi) = 2\zeta(3)e^{-3\phi/2}(1 + \frac{\pi^2}{3\zeta(3)}e^{2\phi}) + (\text{non-perturbative}) \,, \tag{1.2.22}$$

which implies

$$\frac{A_{1-\text{loop}}|_{R^4}}{A_{\text{tree}}|_R} = g_s^2 \frac{\pi^2}{3\zeta(3)} \frac{A_{\text{tree}}|_{R^4}}{A_{\text{tree}}|_R} = g_s^2 (\alpha')^3 \frac{\pi^2}{96} stu \,. \tag{1.2.23}$$

Uplifting to 11d using (1.2.19),

$$\frac{\mathcal{A}|_{R^4}}{\mathcal{A}|_R} = \ell_{11}^6 \frac{stu}{3 \cdot 2^7} \,. \tag{1.2.24}$$

Next, we have [28]

$$S_{D^6R^4} \propto 4\zeta(3)^2 e^{-2\phi} + 8\zeta(2)\zeta(3) + \frac{48}{5}\zeta(2)^2 e^{2\phi} + \frac{8}{9}\zeta(6)e^{4\phi} + (\text{non-perturbative}) \,, \tag{1.2.25}$$

where $\zeta(2) = \pi^2/90$ and $\zeta(6) = \pi^6/945$. This implies

$$\frac{A_{2-\text{loop}}|_{D^6R^4}}{A_{\text{tree}}|_R} = g_s^4 \frac{12\zeta(2)^2}{5\zeta(3)^2} \frac{A_{\text{tree}}|_{D^6R^4}}{A_{\text{tree}}|_R} = g_s^4 (\alpha')^6 \, \frac{3\zeta(2)^2}{2560} (stu)^2 \,. \tag{1.2.26}$$

Uplifting to 11d using (1.2.19),

$$\frac{\mathcal{A}|_{D^6R^4}}{\mathcal{A}|_R} = \ell_{11}^{12} \frac{(stu)^2}{15 \cdot 2^{15}} \,. \tag{1.2.27}$$

Notice that this depends only on $(stu)^2$, not $(s^2 + t^2 + u^2)^3$.

Chapter 2

Protected 1d Topological Sector of $\mathcal{N} \geq 4$ 3d SCFTs

This chapter is an edited version of sections 2 and 3 of ref. [33], which was written in collaboration with Jaehoon Lee, Silviu S. Pufu, and Ran Yacoby, as well as section 3 of ref. [34], which was written in collaboration with Nathan B. Agmon and Silviu S. Pufu.

2.1 Introduction

In this chapter, we will derive a protected 1d topological sector in all 3d $\mathcal{N} \geq 4$ SCFTs, and then use it to compute OPE coefficients in 3d $\mathcal{N} = 8$ SCFTs.

Generically, any given four-point function of an (S)CFT can be expanded in (super)conformal blocks using the OPE, and this expansion depends on an infinite number of OPE coefficients. In $\mathcal{N} \geq 2$ SCFTs in 4d and $\mathcal{N} \geq 4$ SCFTs in 3d, the latter being the focus of our work, it was noticed in [35, 36] that it is possible to "twist" the external operators (after restricting them to lie on a plane in 4d or on a line in 3d) by contracting their R-symmetry indices with their position vectors.[1] The four-point functions of the twisted operators simplify drastically, as they involve expansions that depend only on a restricted set of OPE coefficients. When applied to these twisted four-point functions, crossing symmetry implies tractable relations within this restricted set of OPE coefficients.

The 3d construction starts with the observation that the superconformal algebra of an $\mathcal{N} = 4$ SCFT in three dimensions contains an $\mathfrak{su}(2|2)$ sub-algebra. This $\mathfrak{su}(2|2)$ is the superconformal algebra of a one-

[1] See also [37, 38] for similar constructions in 6d $(2, 0)$ theories and 4d class $\mathcal{S}$ theories.

dimensional SCFT with 8 real supercharges; its bosonic part consists of an $\mathfrak{sl}(2)$ representing dilatations, translations, and special conformal transformations along, say, the x^1-axis, as well as an $\mathfrak{su}(2)_R$ R-symmetry. From the odd generators of $\mathfrak{su}(2|2)$ one can construct a supercharge $\mathcal{Q}$ that squares to zero and that has the property that certain linear combinations of the generators of $\mathfrak{sl}(2)$ and $\mathfrak{su}(2)_R$ are $\mathcal{Q}$-exact. These linear combinations generate a "twisted" 1d conformal algebra $\widehat{\mathfrak{sl}(2)}$ whose embedding into $\mathfrak{su}(2|2)$ depends on $\mathcal{Q}$.[2]

If an operator $\mathcal{O}(0)$ located at the origin of $\mathbb{R}^3$ is $\mathcal{Q}$-invariant, then so is the operator $\widehat{\mathcal{O}}(x)$ obtained by translating $\mathcal{O}(0)$ to the point $(0,x,0)$ (that lies on the x^1-axis) using the twisted translation in $\widehat{\mathfrak{sl}(2)}$. A standard argument shows that the correlation functions

$$\langle \widehat{\mathcal{O}}_1(x_1)\widehat{\mathcal{O}}_2(x_2)\cdots\widehat{\mathcal{O}}_n(x_n)\rangle \tag{2.1.1}$$

of twisted operators $\widehat{\mathcal{O}}_i(x_i)$ may depend only on the ordering of the positions x_i where the operators are inserted.[3] Hence correlation functions like (2.1.1) can be interpreted as correlation functions of a 1d topological theory. If any of the $\widehat{\mathcal{O}}_i(x_i)$ happens to be $\mathcal{Q}$-exact, then the correlation function (2.1.1) vanishes. Indeed, we can obtain non-trivial correlation functions only if all $\widehat{\mathcal{O}}_i(x_i)$ are non-trivial in the cohomology of $\mathcal{Q}$.[4] We will prove that the cohomology of $\mathcal{Q}$ is in one-to-one correspondence with certain $\frac{1}{2}$-BPS superconformal primary operators[5] in the 3d $\mathcal{N}=4$ theory. Applying crossing symmetry on correlation functions like (2.1.1), one can then derive relations between the OPE coefficients of the $\frac{1}{2}$-BPS multiplets of an $\mathcal{N}=4$ SCFT.

We will apply this construction here to $\mathcal{N}=8$ SCFTs. From the $\mathcal{N}=8$ point of view, the local operators that represent non-trivial $\mathcal{Q}$-cohomology classes are Lorentz-scalar superconformal primaries that belong to certain $\frac{1}{4}$, $\frac{3}{8}$, or $\frac{1}{2}$-BPS multiplets of the $\mathcal{N}=8$ superconformal algebra—it is these $\mathcal{N}=8$ multiplets that contain $\frac{1}{2}$-BPS multiplets in the decomposition under the $\mathcal{N}=4$ superconformal algebra.[6]

An example of an operator non-trivial in $\mathcal{Q}$-cohomology that is present in any local $\mathcal{N}=8$ SCFT is the superconformal primary $\mathcal{O}_{\text{Stress}}$ of the $\mathcal{N}=8$ stress-tensor multiplet. This multiplet is $\frac{1}{2}$-BPS from the $\mathcal{N}=8$ point of view. The OPE of $\mathcal{O}_{\text{Stress}}$ with itself contains only three operators that are non-trivial in

[2] A similar construction was used in [39] in some particular 3d $\mathcal{N}=4$ theories. The difference between the supercharge $\mathcal{Q}$ and that used in [39] is that $\mathcal{Q}$ is a linear combination of Poincaré and superconformal supercharges of the $\mathcal{N}=4$ super-algebra, while the supercharge in [39] is built only out of Poincaré supercharges.

[3] The cohomology of $\mathcal{Q}$ is different from the one used in the construction of the chiral ring. In particular, correlation functions in the chiral ring vanish in SCFTs, while correlators of operators in the $\mathcal{Q}$-cohomology do not.

[4] In this chapter, we restrict our attention to $\mathcal{Q}$-cohomology classes that can be represented by a local operator in 3d.

[5] More precisely, the cohomology classes can be represented by operators that transform under the $\mathfrak{su}(2)_L$ and are invariant under the $\mathfrak{su}(2)_R$ sub-algebra of the $\mathfrak{so}(4)_R \cong \mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$ R-symmetry. There exists another cohomology where the roles of $\mathfrak{su}(2)_L$ and $\mathfrak{su}(2)_R$ are interchanged.

[6] These operators form a much smaller set of operators than the one appearing in the analogous construction in four dimensional $\mathcal{N}=4$ SCFTs, where the 1d topological theory is replaced by a 2d chiral algebra [36]. In that case, the stress-tensor OPE contains an infinite number of short representations that contribute to the 2d chiral algebra. In 3d $\mathcal{N}=8$ SCFTs, only finitely many short representations contribute to the 1d topological theory.

$\mathcal{Q}$-cohomology (in addition to the identity): $\mathcal{O}_{\text{Stress}}$ itself, the superconformal primary of a $\frac{1}{2}$-BPS multiplet we will refer to as "$(B,+)$", and the superconformal primary of a $\frac{1}{4}$-BPS multiplet we will refer to as "$(B,2)$". Using crossing symmetry of the four-point function of $\mathcal{O}_{\text{Stress}}$, one can derive the following relation between the corresponding OPE coefficients:

$$4\lambda^2_{\text{Stress}} - 5\lambda^2_{(B,+)} + \lambda^2_{(B,2)} + 16 = 0\,. \tag{2.1.2}$$

Eq. (2.1.2) is the simplest example of an exact relation between OPE coefficients in an $\mathcal{N}=8$ SCFT. In Section 2.3 we explain how to derive, at least in principle, many other exact relations that each relate finitely many OPE coefficients in $\mathcal{N}=8$ SCFTs. In doing so, we provide a simple prescription for computing any correlation functions in the 1d topological theory that arise from $\frac{1}{2}$-BPS operators in the 3d $\mathcal{N}=8$ theory.

We then use this 1d protected sector to derive the OPE coefficients of protected operators that appear in the stress tensor four point function. In particular, we argue that one can relate certain integrated correlators in the 1d theory to derivatives of the partition function of an $\mathcal{N}=4$-preserving mass deformation of the SCFT on S^3. For each theory, this mass-deformed S^3 partition function can be expressed as a matrix integral using the results of Kapustin, Willet, and Yaakov [40]. For BLG$_k$, the matrix integral can be computed exactly for all k. For ABJ(M) theory, the corresponding integrals can be computed either exactly at small N, or to all orders in the $1/N$ expansion using the Fermi gas methods in [41]. From the mass-deformed partition function, one can then determine the integrated four-point function in the 1d theory, and from it, as well as from crossing symmetry, one can extract the OPE coefficients of interest.

The rest of this chapter is organized as follows. In Section 2.2 we explain the construction of the 1d topological QFT from the 3d $\mathcal{N}\geq 4$ SCFT. In Section 2.3 we use this construction as well as crossing symmetry to derive exact relations between OPE coefficients in $\mathcal{N}=8$ SCFTs. In Section 2.4, we explain our method for computing the OPE coefficients, and we perform this computation for BLG and ABJ(M) theory. In Section 2.5 we conclude with some discussion on our results.

2.2 Topological Quantum Mechanics from 3d SCFTs

In this section we construct the cohomology announced in [36] in the case of three-dimensional SCFTs with $\mathcal{N}\geq 4$ supersymmetry. We start in Section 2.2.1 with a review of the strategy of [36]. In Section 2.2.2 we identify a sub-algebra of the 3d superconformal algebra in which we exhibit a nilpotent supercharge $\mathcal{Q}$ as well as $\mathcal{Q}$-exact generators. In Sections 2.2.3 and 2.2.4 we construct the cohomology of $\mathcal{Q}$ and characterize useful representatives of the non-trivial cohomology classes.

2.2.1 General Strategy

One way of phrasing our goal is that we want to find a sub-sector of the full operator algebra of our SCFT that is closed under the OPE, because in such a sub-sector correlation functions and the crossing symmetry constraints might be easier to analyze. In general, one way of obtaining such a sub-sector is to restrict our attention to operators that are invariant under a symmetry of the theory. In a supersymmetric theory, a particularly useful restriction is to operators invariant under a given supercharge or set of supercharges.

A well-known restriction of this sort is the chiral ring in $\mathcal{N}=1$ field theories in four dimensions. The chiral ring consists of operators that are annihilated by half of the Poincaré supercharges: $[Q_\alpha, \mathcal{O}(\vec{x})] = 0$, where $\alpha = 1, 2$ is a spinor index. These operators are closed under the OPE, and their correlation functions are independent of position. Indeed, the translation generators are Q_α-exact because they satisfy $\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = P_{\alpha\dot{\alpha}}$. Combined with the Jacobi identity, the Q_α-exactness of the translation generators implies that the derivative of a chiral operator, $[P_{\alpha\dot{\alpha}}, \mathcal{O}(\vec{x})] = \{Q_\alpha, \tilde{\mathcal{O}}_{\dot{\alpha}}(\vec{x})\}$ is also Q_α-exact. These facts imply that correlators of chiral operators are independent of position, because

$$
\begin{aligned}
\frac{\partial}{\partial x_1^{\alpha\dot{\alpha}}} \langle \mathcal{O}(\vec{x}_1) \cdots \mathcal{O}(\vec{x}_n) \rangle &= \langle [P_{\alpha\dot{\alpha}}, \mathcal{O}(\vec{x}_1)] \cdots \mathcal{O}(\vec{x}_n) \rangle = \langle \{Q_\alpha, \tilde{\mathcal{O}}_{\dot{\alpha}}(\vec{x}_1)\} \cdots \mathcal{O}(\vec{x}_n) \rangle \\
&= -\sum_k \langle \tilde{\mathcal{O}}_{\dot{\alpha}}(\vec{x}_1) \cdots [Q_\alpha, \mathcal{O}(\vec{x}_k)] \cdots \mathcal{O}(\vec{x}_n) \rangle = 0 \,,
\end{aligned} \tag{2.2.1}
$$

where in the third equality we used the supersymmetric Ward identity.

In fact, in unitary SCFTs correlation functions of chiral primaries are completely trivial. Indeed, in an SCFT, the conformal dimension Δ of chiral primaries is proportional to their $U(1)_R$ charge. Since all non-trivial operators have $\Delta > 0$ in unitary theories, all chiral primaries have non-vanishing $U(1)_R$ charges of equal signs, and, as a consequence, their correlation functions must vanish. Therefore, the truncation of the operator algebra provided by the chiral ring in a unitary SCFT is not very useful for our purposes.

One way to evade having zero correlation functions for operators in the cohomology of some fermionic symmetry $\mathcal{Q}$ satisfying $\mathcal{Q}^2 = 0$ (or of a set of several such symmetries) is to take $\mathcal{Q}$ to be a certain linear combination of Poincaré and conformal supercharges. Because $\mathcal{Q}$ contains conformal supercharges, at least some of the translation generators do not commute with $\mathcal{Q}$ now. Nevertheless, there might still exist a $\mathcal{Q}$-exact "R-twisted" translation $\widehat{P}_\mu \sim P_\mu + R^a$, where R^a is an R-symmetry generator. Let $\widehat{\mathcal{P}}$ be the set of $\mathcal{Q}$-exact R-twisted translations, and let $\mathcal{P} \subset \{P_\mu\}_{\mu=1}^d$ be the subset of translation generators which are $\mathcal{Q}$-closed but not $\mathcal{Q}$-exact, if any. It follows that if $\mathcal{O}(\vec{0})$ is $\mathcal{Q}$-closed, so that $\mathcal{O}(\vec{0})$ represents an equivalence class in $\mathcal{Q}$-cohomology, then

$$\widehat{\mathcal{O}}(\tilde{x};\hat{x}) \equiv e^{i\tilde{x}^a P_a + i\hat{x}^i \widehat{P}_i} \mathcal{O}(\vec{0}) e^{-i\tilde{x}^a P_a - i\hat{x}^i \widehat{P}_i} \tag{2.2.2}$$

represents the same cohomology class as $\mathcal{O}(\vec{0})$, given that $\widehat{P}_i \in \widehat{\mathcal{P}}$ and $P_a \in \mathcal{P}$. Here, the R-symmetry indices of $\mathcal{O}$ are suppressed for simplicity. In addition, a very similar argument to that leading to (2.2.1) implies that the correlators of $\widehat{\mathcal{O}}(\tilde{x};\hat{x})$ satisfy

$$\langle \widehat{\mathcal{O}}(\tilde{x}_1;\hat{x}_1) \cdots \hat{\mathcal{O}}(\tilde{x}_n;\hat{x}_n) \rangle = f(\tilde{x}_1, \ldots, \tilde{x}_n)\,, \tag{2.2.3}$$

for separated points $(\tilde{x}_i, \hat{x}_i)$. Now these correlators do not have to vanish since the R-symmetry orientation of each of the inserted operators is locked to the coordinates $\hat{x}_i$.[7]

The correlation functions (2.2.3) could be interpreted as correlation functions of a lower dimensional theory. In particular, in [36] it was shown that in 4d $\mathcal{N} = 2$ theories one can choose $\mathcal{Q}$ such that $\widehat{\mathcal{P}}$ and $\mathcal{P}$ consist of translations in a 2d plane $\mathbb{C} \subset \mathbb{R}^4$. More specifically, holomorphic translations by $z \in \mathbb{C}$ are contained in $\mathcal{P}$, while anti-holomorphic translations by $\bar{z} \in \mathbb{C}$ are contained in $\widehat{\mathcal{P}}$. The resulting correlation functions of operators in that cohomology are meromorphic in z and have the structure of a 2d chiral algebra. In the following section we will construct the cohomology of a supercharge $\mathcal{Q}$ in 3d $\mathcal{N} = 4$ SCFTs such that the set $\mathcal{P}$ is empty and $\widehat{\mathcal{P}}$ contains a single twisted translation. The correlation functions (2.2.3) evaluate to (generally non-zero) constants, and this underlying structure can therefore be identified with a topological quantum mechanics.

2.2.2 An $\mathfrak{su}(2|2)$ Subalgebra and $\mathcal{Q}$-exact Generators

We now proceed to an explicit construction in 3d $\mathcal{N} = 4$ SCFTs. We first identify an $\mathfrak{su}(2|2)$ sub-algebra of the $\mathfrak{osp}(4|4)$ superconformal algebra. This $\mathfrak{su}(2|2)$ sub-algebra represents the symmetry of a superconformal field theory in one dimension and will be the basis for the topological twisting prescription that we utilize in this work.

Let us start by describing the generators of $\mathfrak{osp}(4|4)$ in order to set up our conventions. The bosonic subalgebra of $\mathfrak{osp}(4|4)$ consists of the 3d conformal algebra, $\mathfrak{sp}(4) \simeq \mathfrak{so}(3,2)$, and of the $\mathfrak{so}(4) \simeq \mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$ R-symmetry algebra.[8] The 3d conformal algebra is generated by $M_{\mu\nu}$, P_μ, K_μ, and D, representing the

[7] We stress that (2.2.3) is valid only at separated $\hat{x}_i$ points. If this were not the case then we could set $\hat{x}_1 = \cdots = \hat{x}_n = 0$ in (2.2.3) and argue that $f(\tilde{x}_1, \ldots, \tilde{x}_n) = 0$, since due to (2.2.2) the R-symmetry weights of the $\widehat{\mathcal{O}}(\tilde{x}_i; 0)$ cannot combine to form a singlet. We will later see in examples that the limit of coincident $\hat{x}_i$ is singular. From the point of view of the proof around (2.2.1), these singularities are related to contact terms. Such contact terms are absent in the case of the chiral ring construction, but do appear in the case of our cohomology.

[8] In this thesis we will always take our algebras to be over the field of complex numbers.

generators of Lorentz transformations, translations, special conformal transformations, and dilatations, respectively. Here, $\mu,\nu=0,1,2$ are space-time indices. The generators of the $\mathfrak{su}(2)_L$ and $\mathfrak{su}(2)_R$ R-symmetries can be represented as traceless 2×2 matrices $R_a{}^b$ and $\bar{R}^{\dot a}{}_{\dot b}$ respectively, where $a,b=1,2$ are $\mathfrak{su}(2)_L$ spinor indices and $\dot a,\dot b=1,2$ are $\mathfrak{su}(2)_R$ spinor indices. In terms of the more conventional generators $\vec{J}^L$ and $\vec{J}^R$ satisfying $[J_i^L,J_j^L]=i\varepsilon_{ijk}J_k^L$ and $[J_i^R,J_j^R]=i\varepsilon_{ijk}J_k^R$, one can write

$$R_a{}^b=\begin{pmatrix} J_3^L & J_+^L \\ J_-^L & -J_3^L\end{pmatrix}\,,\qquad \bar{R}^{\dot a}{}_{\dot b}=\begin{pmatrix} J_3^R & J_+^R \\ J_-^R & -J_3^R\end{pmatrix}\,,\tag{2.2.4}$$

where $J_\pm^L=J_1^L\pm iJ_2^L$ and $J_\pm^R=J_1^R\pm iJ_2^R$. The odd generators of $\mathfrak{osp}(4|4)$ consist of the Poincaré supercharges $Q_{\alpha a\dot a}$ and conformal supercharges $S^\beta{}_{a\dot a}$, which transform in the $\mathbf{4}$ of $\mathfrak{so}(4)_R$, and as Majorana spinors of the 3d Lorentz algebra $\mathfrak{so}(1,2)\subset\mathfrak{sp}(4)$ with the spinor indices $\alpha,\beta=1,2$. The commutation relations of the generators of the superconformal algebra were described in Section 1.1.2 for general $\mathfrak{so}(\mathcal{N})_R$. We can set $\mathcal{N}=4$ in those relations and then project the $\mathfrak{so}(4)$ R-symmetry to $\mathfrak{su}(2)_L\oplus\mathfrak{su}(2)_R$ by dotting with quaternions represented by the matrices $\sigma^r_{a\dot a}\equiv(1,i\sigma^1,i\sigma^2,i\sigma^3)$ and $\bar\sigma^{r\dot a a}=\varepsilon^{\dot a\dot b}\varepsilon^{ab}\sigma^r_{b\dot b}=(1,-i\sigma^1,-i\sigma^2,-i\sigma^3)$, where $-\varepsilon_{12}=-\varepsilon^{21}=\varepsilon_{21}=\varepsilon^{12}=1$. We turn vectors into bi-spinors using $v_{a\dot a}\equiv\sigma^r_{a\dot a}v_r$. The $\mathfrak{so}(4)$ rotation generators R_{rs} can be decomposed into dual and anti-self-dual rotations using $R_a{}^b\equiv\frac{i}{4}(\sigma^r\bar\sigma^s)_a{}^bR_{rs}=\frac{i}{2}(\sigma^{rs})_a{}^bR_{rs}$ and $\bar R^{\dot a}{}_{\dot b}\equiv\frac{i}{4}(\bar\sigma^r\sigma^s)^{\dot a}{}_{\dot b}R_{rs}=\frac{i}{2}(\bar\sigma^{rs})^{\dot a}{}_{\dot b}R_{rs}$.

The $\mathcal{N}=4$ superconformal algebra in this notation is then given by[9]

$$\{Q_{\alpha a\dot a},Q_{\beta b\dot b}\}=4\varepsilon_{ab}\varepsilon_{\dot a\dot b}P_{\alpha\beta}\,,\qquad \{S^\alpha{}_{a\dot a},S^\beta{}_{b\dot b}\}=-4\varepsilon_{ab}\varepsilon_{\dot a\dot b}K^{\alpha\beta}\,,\tag{2.2.5}$$

$$[K^{\alpha\beta},Q_{\gamma a\dot a}]=-i\left(\delta_\gamma{}^\alpha S^\beta{}_{a\dot a}+\delta_\gamma{}^\beta S^\alpha{}_{a\dot a}\right)\,,\qquad [P_{\alpha\beta},S^\gamma{}_{a\dot a}]=-i\left(\delta_\alpha{}^\gamma Q_{\beta a\dot a}+\delta_\beta{}^\gamma Q_{\alpha a\dot a}\right)\,,\tag{2.2.6}$$

$$[M_\alpha{}^\beta,Q_{\gamma a\dot a}]=\delta_\gamma{}^\beta Q_{\alpha a\dot a}-\frac12\delta_\alpha{}^\beta Q_{\gamma a\dot a}\,,\qquad [M_\alpha{}^\beta,S^\gamma{}_{a\dot a}]=-\delta_\alpha{}^\gamma S^\beta{}_{a\dot a}+\frac12\delta_\alpha{}^\beta S^\gamma{}_{a\dot a}\,,\tag{2.2.7}$$

$$[D,Q_{\alpha a\dot a}]=\frac12 Q_{\alpha a\dot a}\,,\qquad [D,S^\alpha{}_{a\dot a}]=-\frac12 S^\alpha{}_{a\dot a}\,,\tag{2.2.8}$$

$$[R_a{}^b,Q_{\alpha c\dot c}]=\delta_c{}^bQ_{\alpha a\dot c}-\frac12\delta_a{}^bQ_{\alpha c\dot c}\,,\qquad [R_a{}^b,S^\alpha{}_{c\dot c}]=\delta_c{}^bS^\alpha{}_{a\dot c}-\frac12\delta_a{}^bS^\alpha{}_{c\dot c}\,,\tag{2.2.9}$$

$$[\bar R^{\dot a}{}_{\dot b},Q_{\alpha c\dot c}]=-\delta^{\dot a}{}_{\dot c}Q_{\alpha c\dot b}+\frac12\delta^{\dot a}{}_{\dot b}Q_{\alpha c\dot c}\,,\qquad [\bar R^{\dot a}{}_{\dot b},S^\alpha{}_{c\dot c}]=-\delta^{\dot a}{}_{\dot c}S^\alpha{}_{c\dot b}+\frac12\delta^{\dot a}{}_{\dot b}S^\alpha{}_{c\dot c}\,,\tag{2.2.10}$$

$$[R_a{}^b,R_c{}^d]=-\delta_a{}^dR_c{}^b+\delta_c{}^bR_a{}^d\,,\qquad [\bar R^{\dot a}{}_{\dot b},\bar R^{\dot c}{}_{\dot d}]=-\delta^{\dot a}{}_{\dot d}\bar R^{\dot c}{}_{\dot b}+\delta^{\dot c}{}_{\dot b}\bar R^{\dot a}{}_{\dot d}\,,\tag{2.2.11}$$

and also

$$\{Q_{\alpha a\dot a},S^\beta{}_{b\dot b}\}=4i\left[\varepsilon_{ab}\varepsilon_{\dot a\dot b}\left(M_\alpha{}^\beta+\delta_\alpha{}^\beta D\right)-\delta_\alpha{}^\beta\left((R\varepsilon)_{ab}\varepsilon_{\dot a\dot b}+(\varepsilon\bar R)_{\dot a\dot b}\varepsilon_{ab}\right)\right]\,.\tag{2.2.12}$$

[9]We only list the commutators which involve R-symmetry indices as the others remain as before.

In this notation, the conjugation properties (1.1.28) become

$$
\begin{aligned}
&(Q_{\alpha a\dot{a}})^\dagger = -i\varepsilon^{ab}\varepsilon^{\dot{a}\dot{b}}S^{\alpha}{}_{b\dot{b}}\,, \qquad S^{\alpha}{}_{a\dot{a}} = -i\varepsilon^{ab}\varepsilon^{\dot{a}\dot{b}}Q_{\alpha b\dot{b}}\,,\\
&(R_a{}^b)^\dagger = R_b{}^a\,, \qquad (\bar{R}^{\dot{a}}{}_{\dot{b}})^\dagger = \bar{R}^{\dot{b}}{}_{\dot{a}}\,.
\end{aligned}
\tag{2.2.13}
$$

In terms of more standard $\mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$ generators, $R_a{}^b$ and $\bar{R}^{\dot{a}}_{\dot{b}}$ can be written as in (2.2.4).

The embedding of $\mathfrak{su}(2|2)$ into $\mathfrak{osp}(4|4)$ can be described as follows. Since the bosonic sub-algebra of $\mathfrak{su}(2|2)$ consists of the 1d conformal algebra $\mathfrak{sl}(2)$ and an $\mathfrak{su}(2)$ R-symmetry, we can start by embedding the latter two algebras into $\mathfrak{osp}(4|4)$. The $\mathfrak{sl}(2)$ algebra is embedded into the 3d conformal algebra $\mathfrak{sp}(4)$, and without loss of generality we can require the $\mathfrak{sl}(2)$ generators to stabilize the line $x^0 = x^2 = 0$. This requirement identifies the $\mathfrak{sl}(2)$ generators with the translation $P \equiv P_1$, special conformal transformation $K \equiv K_1$, and the dilatation generator D. We choose to identify the $\mathfrak{su}(2)$ R-symmetry of $\mathfrak{su}(2|2)$ with the $\mathfrak{su}(2)_L$ R-symmetry of $\mathfrak{osp}(4|4)$. Using the commutation relations in Section 1.1.2 one can verify that, up to an $\mathfrak{su}(2)_R$ rotation, the fermionic generators of $\mathfrak{su}(2|2)$ can be taken to be $Q_{1a\dot{2}}$, $Q_{2a\dot{1}}$, $S^1{}_{a\dot{1}}$, and $S^2{}_{a\dot{2}}$. The result is an $\mathfrak{su}(2|2)$ algebra generated by

$$
\{P\,, K\,, D\,, R_a{}^b\,, Q_{1a\dot{2}}\,, Q_{2a\dot{1}}\,, S^1{}_{a\dot{1}}\,, S^2{}_{a\dot{2}}\}\,,
\tag{2.2.14}
$$

with a central extension given by

$$
\mathcal{Z} \equiv iM_{02} - R^{\dot{1}}{}_{\dot{1}}\,.
\tag{2.2.15}
$$

From the results of Section 1.1.2 and the commutation relations above, it is not hard to see that the inner product obtained from radial quantization imposes the following conjugation relations on these generators:

$$
\begin{aligned}
&P^\dagger = K\,, \qquad D^\dagger = D\,, \qquad \mathcal{Z}^\dagger = \mathcal{Z}\,, \qquad (R_a{}^b)^\dagger = R_b{}^a\,,\\
&(Q_{1a\dot{2}})^\dagger = -i\varepsilon^{ab}S^1{}_{b\dot{1}}\,, \qquad (Q_{2a\dot{1}})^\dagger = i\varepsilon^{ab}S^2{}_{b\dot{2}}\,,
\end{aligned}
\tag{2.2.16}
$$

where $\varepsilon^{12} = -\varepsilon^{21} = 1$.

Within the $\mathfrak{su}(2|2)$ algebra there are several nilpotent supercharges that can be used to define our cohomology. We will focus our attention on two of them, which we denote by $\mathcal{Q}_1$ and $\mathcal{Q}_2$, as well as their complex conjugates:

$$\begin{aligned} \mathcal{Q}_1 &= Q_{11\dot{2}} + S^2{}_{2\dot{2}}\,, \qquad \mathcal{Q}_1^\dagger = -i(Q_{21\dot{1}} - S^1{}_{2\dot{1}})\,, \\ \mathcal{Q}_2 &= Q_{21\dot{1}} + S^1{}_{2\dot{1}}\,, \qquad \mathcal{Q}_2^\dagger = i(Q_{11\dot{2}} - S^2{}_{2\dot{2}})\,. \end{aligned} \tag{2.2.17}$$

With respect to either of the two nilpotent supercharges $\mathcal{Q}_{1,2}$, the central element $\mathcal{Z}$ is exact, because

$$\mathcal{Z} = \frac{i}{8}\{\mathcal{Q}_1, \mathcal{Q}_2\}\,. \tag{2.2.18}$$

In addition, the following generators are also exact:

$$\widehat{L}_0 \equiv -D + R_1{}^1 = -\frac{1}{8}\{\mathcal{Q}_1, \mathcal{Q}_1^\dagger\} = -\frac{1}{8}\{\mathcal{Q}_2, \mathcal{Q}_2^\dagger\}\,, \tag{2.2.19}$$

$$\widehat{L}_- \equiv P + iR_2{}^1 = -\frac{1}{4}\{\mathcal{Q}_1, Q_{22\dot{1}}\} = \frac{1}{4}\{\mathcal{Q}_2, Q_{12\dot{2}}\}\,, \tag{2.2.20}$$

$$\widehat{L}_+ \equiv K + iR_1{}^2 = -\frac{1}{4}\{\mathcal{Q}_1, S^1{}_{1\dot{1}}\} = \frac{1}{4}\{\mathcal{Q}_2, S^2{}_{1\dot{2}}\}\,. \tag{2.2.21}$$

These generators form an $\mathfrak{sl}(2)$ triplet: $[\widehat{L}_0, \widehat{L}_\pm] = \pm\widehat{L}_\pm$, $[\widehat{L}_+, \widehat{L}_-] = -2\widehat{L}_0$. We will refer to the algebra generated by them as "twisted," and we will denote it by $\widehat{\mathfrak{sl}(2)}$. Note that $\widehat{L}_-$ is a twisted translation generator. Since it is $\mathcal{Q}$-exact (with $\mathcal{Q}$ being either $\mathcal{Q}_1$ or $\mathcal{Q}_2$), $\widehat{L}_-$ preserves the $\mathcal{Q}$-cohomology classes and can be used to translate operators in the cohomology along the line parameterized by x^1.

2.2.3 The Cohomology of the Nilpotent Supercharge

Let $\mathcal{Q}$ be either of the nilpotent supercharges $\mathcal{Q}_1$ or $\mathcal{Q}_2$ defined in (2.2.17), and let $\mathcal{Q}^\dagger$ be its conjugate. Let us now describe more explicitly the cohomology of $\mathcal{Q}$. The results of this section will be independent of whether we choose $\mathcal{Q} = \mathcal{Q}_1$ or $\mathcal{Q} = \mathcal{Q}_2$.

Since $-\widehat{L}_0 = D - R_1{}^1 \geq 0$ for all irreps of the $\mathfrak{osp}(4|4)$ superconformal algebra, and since $-\widehat{L}_0 = \frac{1}{8}\{\mathcal{Q}, \mathcal{Q}^\dagger\}$, one can show that each non-trivial cohomology class contains a unique representative $\mathcal{O}(0)$ annihilated by $\widehat{L}_0$. This representative is the analog of a harmonic form representing a non-trivial de Rham cohomology class in Hodge theory. Therefore, the non-trivial $\mathcal{Q}$-cohomology classes are in one-to-one correspondence with operators satisfying

$$\Delta = m_L\,, \tag{2.2.22}$$

where Δ is the scaling dimension (eigenvalue of the operator D appearing in (2.2.19)), and m_L is the $\mathfrak{su}(2)$

weight associated with the spin-j_L ($j_L \in \frac{1}{2}\mathbb{N}$) irrep of the $\mathfrak{su}(2)_L$ R-symmetry (eigenvalue of the operator ${R_1}^1$ appearing in (2.2.19)).

A superconformal primary operator of a unitary $\mathcal{N} = 4$ SCFT in three dimensions must satisfy $\Delta \geq j_L + j_R$. (See Table 2.1 for a list of multiplets of $\mathfrak{osp}(4|4)$.) It then follows from (2.2.22) and unitarity

Type	BPS	Δ	Spin	$\mathfrak{su}(2)_L$ spin	$\mathfrak{su}(2)_R$ spin
$(A, 0)$ (long)	0	$\geq j_L + j_R + j + 1$	j	j_L	j_R
$(A, 1)$	1/8	$j_L + j_R + j + 1$	j	j_L	j_R
$(A, +)$	1/4	$j_L + j_R + j + 1$	j	j_L	0
$(A, -)$	1/4	$j_L + j_R + j + 1$	j	0	j_R
$(B, 1)$	1/4	$j_L + j_R$	0	j_L	j_R
$(B, +)$	1/2	$j_L + j_R$	0	j_L	0
$(B, -)$	1/2	$j_L + j_R$	0	0	j_R
conserved	3/8	$j + 1$	j	0	0

Table 2.1: Multiplets of $\mathfrak{osp}(4|4)$ and the quantum numbers of their corresponding superconformal primary operator. The Lorentz spin can take the values $j = 0, 1/2, 1, 3/2, \ldots$. Representations of the $\mathfrak{so}(4) \cong \mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$ R-symmetry are given in terms of the $\mathfrak{su}(2)_L$ and $\mathfrak{su}(2)_R$ spins denoted j_L and j_R, which are non-negative half-integers.

that superconformal primaries that are non-trivial in the $\mathcal{Q}$-cohomology must have dimension $\Delta = j_L$ and they must be Lorentz scalars transforming in the spin $(j_L, 0)$ irrep of the $\mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$ R-symmetry. In addition, they must occupy their $\mathfrak{su}(2)_L$ highest weight state, $m_L = j_L$, when inserted at the origin. Such superconformal primaries correspond to the $\frac{1}{2}$-BPS multiplets of the $\mathfrak{osp}(4|4)$ superconformal algebra that are denoted by $(B, +)$ in Table 2.1.[10] These superconformal primaries are in fact all the operators of an $\mathcal{N} = 4$ SCFT satisfying (2.2.22), as we now show.

First, let us show that such operators cannot belong to A-type multiplets. A-type multiplets satisfy the unitarity bound

$$\Delta \geq j_L + j_R + s + 1 \,, \qquad \text{for SCPs of } A\text{-type multiplets} \,. \tag{2.2.23}$$

We can in fact show that

$$\Delta > j_L + j_R \,, \qquad \text{for all CPs of } A\text{-type multiplets} \,. \tag{2.2.24}$$

Indeed, let us consider the highest weight state of the various $\mathfrak{so}(4)$ irreps of all conformal primaries appearing in the supermultiplet. These states are related by the acting with the eight supercharges $Q_{\alpha i}$ for the long

[10]The $(B, -)$ type $\frac{1}{2}$-BPS multiplets are defined in the same way, but transform in the spin-$(0, j_R)$ representation of the $\mathfrak{so}(4)_R$ symmetry. We could obtain a cohomology based on $(B, -)$ multiplets by exchanging the roles of $\mathfrak{su}(2)_L$ and $\mathfrak{su}(2)_R$ in our construction, but we will not consider this possibility here.

multiplets, or a subset thereof for the semi-short multiplets. Here α is a Lorentz spinor index and $i = 1, \ldots, 4$ is an $\mathfrak{so}(4)$ fundamental index. The quantum numbers of these supercharges are $(\Delta, m_s, m_L, m_R) = \left(\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\right)$. The quantity $\Delta - m_s - m_L - m_R$ can thus take the following values: -1 (one supercharge), 0 (three supercharges), 1 (three supercharges), and 2 (one supercharge). By acting with the first supercharge, we can decrease the quantity $\Delta - j_L - j_R - s$ by one unit; the other supercharges don't decrease $\Delta - j_L - j_R - s$. Therefore, since the superconformal primary satisfies (2.2.23), we have

$$\Delta \geq j_L + j_R + s\,, \qquad \text{for all CPs of } A\text{-type multiplets}\,. \tag{2.2.25}$$

The inequality in (2.2.25) is saturated provided that the inequality in (2.2.23) is saturated and that we act with the first supercharge mentioned above. This supercharge has $m_s = +1/2$, therefore a state that saturates (2.2.25) must necessarily have $s > 0$. We conclude that (2.2.24) must hold. If (2.2.24) holds, then it is impossible to find a conformal primary in an A-type multiplet that has $\Delta = m_L$.

The superconformal primaries of B-type multiplets satisfy

$$\Delta = j_L + j_R\,, \qquad s = 0\,, \qquad \text{for SCPs of } B\text{-type multiplets}\,. \tag{2.2.26}$$

For these multiplets, the supercharge with $\Delta - m_s - m_L - m_R = -1$ and at least one supercharge with $\Delta - m_s - m_L - m_R = 0$ (namely the one with $m_L = m_R = +1/2$) annihilates the highest weight states of all CPs in these multiplets. Therefore, we have that all conformal primaries in these multiplets satisfy

$$\Delta \geq j_L + j_R + s\,, \qquad \text{for all CPs of } B\text{-type multiplets}\,. \tag{2.2.27}$$

The inequality is saturated either by the superconformal primary or by conformal primaries whose highest weights are obtained by acting with the supercharges that have $\Delta - m_s - m_L - m_R = 0$ on the highest weight state of the superconformal primary. These supercharges necessarily have $m_s = +1/2$, so these conformal primaries necessarily have $s > 0$. If we want to have $\Delta = m_L$, from (2.2.27) we therefore should have $j_R = s = 0$, and so the only option is a superconformal primary of a B-type multiplet with $j_R = 0$. This is a superconformal primary of a $(B, +)$ multiplet.

2.2.4 Operators in the 1d Topological Theory and Their OPE

We can now study the 1d operators defined by the twisting procedure in (2.2.2). Let us denote the $(B, +)$ superconformal primaries by $\mathcal{O}_{a_1 \cdots a_k}(\vec{x})$, where $k = 2j_L$. In our convention, setting $a_i = 1$ for all $i = 1, \ldots, k$

corresponds to the highest weight state of the spin-j_L representation of $\mathfrak{su}(2)_L$, and so the operator $\mathcal{O}_{11\cdots 1}(\vec{0})$ has $\Delta = j_L = m_L$ and therefore represents a non-trivial $\mathcal{Q}$-cohomology class. Since the twisted translation $\widehat{L}_-$ is $\mathcal{Q}$-exact, we can use it to translate $\mathcal{O}_{11\cdots 1}(\vec{0})$ along the x^1 direction. The translated operator is

$$\widehat{\mathcal{O}}_k(x) \equiv e^{-ix\widehat{L}_-}\mathcal{O}_{11\cdots 1}(\vec{0})\, e^{ix\widehat{L}_-} = u^{a_1}(x)\cdots u^{a_k}(x)\mathcal{O}_{a_1\cdots a_k}(\vec{x})\big|_{\vec{x}=(0,x,0)}\,, \tag{2.2.28}$$

where $u^a(x) \equiv (1, x)$. The translated operator $\widehat{\mathcal{O}}_k(x)$ represents the same cohomology class as $\mathcal{O}_{11\cdots 1}(\vec{0})$. The index k serves as a reminder that the operator $\widehat{\mathcal{O}}_k(x)$ comes from a superconformal primary in the 3d theory transforming in the spin-$j_L = k/2$ irrep of $\mathfrak{su}(2)_L$. From the 1d point of view, k is simply a label.

The arguments that led to (2.2.3) tell us correlation functions $\langle \widehat{\mathcal{O}}_{k_1}(x_1)\cdots\widehat{\mathcal{O}}_{k_n}(x_n)\rangle$ are independent of $x_i \in \mathbb{R}$ for separated points, but could depend on the ordering of these points on the real line. Therefore, they can be interpreted as the correlation functions of a topological theory in 1d.

Correlation Functions and 1d Bosons vs. Fermions

As a simple check, let us see explicitly that the two and three-point functions of $\widehat{\mathcal{O}}_{k_i}(x_i)$ depend only on the ordering of the x_i on the real line. Such a check is easy to perform because superconformal invariance fixes the two and three-point functions of $\mathcal{O}_{a_1\cdots a_k}(\vec{x})$ up to an overall factor. Indeed, let us denote

$$\mathcal{O}_k(x,y) \equiv \mathcal{O}_{a_1\cdots a_k}(\vec{x})\big|_{\vec{x}=(0,x,0)}\, y^{a_1}\cdots y^{a_k}\,, \tag{2.2.29}$$

where we introduced a set of auxiliary variables y^a in order to simplify the expressions below. The two-point function of $\mathcal{O}_k(x,y)$ is:

$$\langle \mathcal{O}_k(x_1,y_1)\mathcal{O}_k(x_2,y_2)\rangle \propto \left(\frac{y_1^a\varepsilon_{ab}y_2^b}{|x_{12}|}\right)^k\,, \tag{2.2.30}$$

where $x_{ij} \equiv x_i - x_j$ and $\varepsilon_{12} = -\varepsilon_{21} = -1$. In passing from $\mathcal{O}_k(x,y)$ to $\widehat{\mathcal{O}}(x)$, one should simply set $y^a = u^a(x) = (1,x)$, and then

$$\langle \widehat{\mathcal{O}}_k(x_1)\widehat{\mathcal{O}}_k(x_2)\rangle \propto (\operatorname{sgn} x_{12})^k\,. \tag{2.2.31}$$

Indeed, this two-point function only depends on the ordering of the two points x_1 and x_2. It changes sign under interchanging x_1 and x_2 if k is odd, and it stays invariant if k is even. Therefore, the one-dimensional operators $\widehat{\mathcal{O}}_k(x)$ behave as fermions if k is odd and as bosons if k is even.

To perform a similar check for the three-point function, we can start with the expression

$$\langle \mathcal{O}_{k_1}(x_1,y_1)\mathcal{O}_{k_2}(x_2,y_2)\mathcal{O}_{k_3}(x_3,y_3)\rangle \propto \left(\frac{y_1^a\varepsilon_{ab}y_2^b}{|x_{12}|}\right)^{\frac{k_1+k_2-k_3}{2}} \left(\frac{y_2^a\varepsilon_{ab}y_3^b}{|x_{23}|}\right)^{\frac{k_2+k_3-k_1}{2}} \left(\frac{y_1^a\varepsilon_{ab}y_3^b}{|x_{13}|}\right)^{\frac{k_3+k_1-k_2}{2}} \tag{2.2.32}$$

required by the superconformal invariance of the 3d $\mathcal{N}=4$ theory. This expression may be non-zero only if (2.2.32) is a polynomial in the y_i. This condition is equivalent to the requirement that k_1, k_2, and k_3 satisfy the triangle inequality and that they add up to an even integer. Setting $y_i^a = u_i^a = (1, x_i)$, we obtain

$$\langle \widehat{\mathcal{O}}_{k_1}(x_1)\widehat{\mathcal{O}}_{k_2}(x_2)\widehat{\mathcal{O}}_{k_3}(x_3)\rangle \propto (\operatorname{sgn} x_{12})^{\frac{k_1+k_2-k_3}{2}} (\operatorname{sgn} x_{23})^{\frac{k_2+k_3-k_1}{2}} (\operatorname{sgn} x_{13})^{\frac{k_3+k_1-k_2}{2}} \,. \tag{2.2.33}$$

Again, this expression depends only on the ordering of the points x_i on the real line. If we make a cyclic permutation of the three points, the three-point function changes sign if the permutation involves an exchange of an odd number of operators with odd k_i and remains invariant otherwise. Operators $\widehat{\mathcal{O}}_k(x)$ with odd k again behave as fermions and those with even k behave as bosons under cyclic permutations. Under non-cyclic permutations, the transformation properties of correlation functions may be more complicated.

The reason why cyclic permutations are special is the following. We can use conformal symmetry to map the line on which our 1d theory lives to a circle. After this mapping, the correlation functions of the untwisted operators $\mathcal{O}_{k_i}(x_i, y_i)$ depend only on the cyclic ordering of the x_i, because on the circle all such cyclic orderings are equivalent. In particular, an operator $\mathcal{O}_k(x, y)$ inserted at $x = +\infty$ is equivalent to the same operator inserted at $x = -\infty$. After the twisting by setting $y_i = (1, x_i)$, we have

$$\widehat{\mathcal{O}}_k(+\infty) = (-1)^k \widehat{\mathcal{O}}_k(-\infty)\,. \tag{2.2.34}$$

We can choose to interpret this expression as meaning that operators with even (odd) k behave as bosons (fermions) under cyclic permutations, as we did above. Equivalently, we can choose to interpret it as meaning that upon mapping from $\mathbb{R}$ to S^1 we must insert a twist operator at $x = \pm\infty$; the twist operator commutes (anti-commutes) with $\widehat{\mathcal{O}}_k$ if k is even (odd). The effect of (2.2.34) on correlation functions is that under cyclic permutations we have

$$\langle \widehat{\mathcal{O}}_{k_1}(x_1)\widehat{\mathcal{O}}_{k_2}(x_2)\ldots\widehat{\mathcal{O}}_{k_n}(x_n)\rangle = (-1)^{k_n}\langle \widehat{\mathcal{O}}_{k_n}(x_1)\widehat{\mathcal{O}}_{k_1}(x_2)\ldots\widehat{\mathcal{O}}_{k_{n-1}}(x_n)\rangle\,, \tag{2.2.35}$$

where we chose the ordering of the points to be $x_1 < x_2 < \ldots < x_n$. Eqs. (2.2.31) and (2.2.33) above obey this property.

The 1d OPE

To compute higher-point functions it is useful to write down the OPE of twisted operators in one dimension. From (2.2.31) and (2.2.33), we have, up to $\mathcal{Q}$-exact terms,

$$\widehat{\mathcal{O}}_{k_1}(x_1)\widehat{\mathcal{O}}_{k_2}(x_2) \sim \sum_{\widehat{\mathcal{O}}_{k_3}} \lambda_{\widehat{\mathcal{O}}_{k_1}\widehat{\mathcal{O}}_{k_2}\widehat{\mathcal{O}}_{k_3}} (\operatorname{sgn} x_{12})^{\frac{k_1+k_2-k_3}{2}}\, \widehat{\mathcal{O}}_{k_3}(x_2)\,, \qquad \text{as } x_1 \to x_2\,, \tag{2.2.36}$$

where the OPE coefficients $\lambda_{\widehat{\mathcal{O}}_{k_1}\widehat{\mathcal{O}}_{k_2}\widehat{\mathcal{O}}_{k_3}}$ do not depend on the ordering of the $\widehat{\mathcal{O}}_{k_1}(x_1)$ and $\widehat{\mathcal{O}}_{k_2}(x_2)$ insertions on the line. In this expression, the sum runs over all the operators $\widehat{\mathcal{O}}_{k_3}$ in the theory for which k_1, k_2, and k_3 obey the triangle inequality and add up to an even integer. Such an OPE makes sense provided that it is used inside a correlation function where there are no other operator insertions in the interval $[x_1, x_2]$. Note that (2.2.36) does not rely on any assumptions about the matrix of two-point functions. In particular, this matrix need not be diagonal, as will be the case in our $\mathcal{N} = 8$ examples below.

The OPE (2.2.36) is useful because, when combined with (2.2.35), there are several inequivalent ways to apply it between adjacent operators. Invariance under crossing symmetry means that these ways should yield the same answer. For instance, if we consider the four-point function

$$\langle \widehat{\mathcal{O}}_{k_1}(x_1)\widehat{\mathcal{O}}_{k_2}(x_2)\widehat{\mathcal{O}}_{k_3}(x_3)\widehat{\mathcal{O}}_{k_4}(x_4)\rangle\,, \tag{2.2.37}$$

with the ordering of points $x_1 < x_2 < x_3 < x_4$, one can use the OPE to expand the product $\widehat{\mathcal{O}}_{k_1}(x_1)\widehat{\mathcal{O}}_{k_2}(x_2)$ as well as $\widehat{\mathcal{O}}_{k_3}(x_3)\widehat{\mathcal{O}}_{k_4}(x_4)$. Using (2.2.35), one can also use the OPE to expand the products $\widehat{\mathcal{O}}_{k_4}(x_1)\widehat{\mathcal{O}}_{k_1}(x_2)$ and $\widehat{\mathcal{O}}_{k_2}(x_3)\widehat{\mathcal{O}}_{k_3}(x_4)$. Equating the two expressions as required by (2.2.35), one may then obtain non-trivial relations between the OPE coefficients.

2.3 Application to $\mathcal{N} = 8$ Superconformal Theories

The topological twisting procedure derived in the previous section for $\mathcal{N} = 4$ SCFTs can be applied to any SCFT with $\mathcal{N} \geq 4$ supersymmetry, and in this section we apply it to $\mathcal{N} = 8$ SCFTs. We start in Section 2.3.1 by determining how the operators chosen in the previous section as representatives of non-trivial $\mathcal{Q}$-cohomology classes sit within $\mathcal{N} = 8$ multiplets; we find that they are certain superconformal primaries of $\frac{1}{4}$, $\frac{3}{8}$, or $\frac{1}{2}$-BPS multiplets. We then focus on the twisted correlation functions of $\frac{1}{2}$-BPS multiplets, because these multiplets exist in all local $\mathcal{N} = 8$ SCFTs. For instance, the stress-tensor multiplet is of this type.

More specifically, in Section 2.3.2 we show explicitly how to project the $\mathcal{N}=8$ $\frac{1}{2}$-BPS operators onto the particular component that contributes to the cohomology of the supercharge $\mathcal{Q}$. The 1d OPEs of the twisted $\frac{1}{2}$-BPS operators are computed in Section 2.3.3 in a number of examples. We then compute some 4-point functions using these OPEs and show how to extract non-trivial relations between OPE coefficients from the resulting crossing symmetry constraints. Finally, in Section 2.3.4 we show how some of our results can be understood directly from the 3d superconformal Ward identity derived in [1].

2.3.1 The $\mathcal{Q}$-Cohomology in $\mathcal{N}=8$ Theories

In order to understand how the representatives of the $\mathcal{Q}$-cohomology classes sit within $\mathcal{N}=8$ supermultiplets, let us first discuss how to embed the $\mathcal{N}=4$ superconformal algebra, $\mathfrak{osp}(4|4)$, into the $\mathcal{N}=8$ one, $\mathfrak{osp}(8|4)$. Focusing on bosonic subgroups first, note that the $\mathfrak{so}(8)_R$ symmetry of $\mathcal{N}=8$ theories has a maximal sub-algebra

$$\mathfrak{so}(8)_R \supset \underbrace{\mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R}_{\mathfrak{so}(4)_R} \oplus \underbrace{\mathfrak{su}(2)_1 \oplus \mathfrak{su}(2)_2}_{\mathfrak{so}(4)_F} . \tag{2.3.1}$$

The $\mathfrak{so}(4)_R$ and $\mathfrak{so}(4)_F$ factors in (2.3.1) can be identified with an R-symmetry and a flavor symmetry, respectively, from the $\mathcal{N}=4$ point of view. In our conventions, the embedding of $\mathfrak{su}(2)^4$ into $\mathfrak{so}(8)_R$ is such that the following decompositions hold:

$$\begin{aligned}
[1000] &= \mathbf{8}_v \to (\mathbf{4},\mathbf{1}) \oplus (\mathbf{1},\mathbf{4}) = (\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2})\,,\\
[0010] &= \mathbf{8}_c \to (\mathbf{2},\mathbf{2}) \oplus (\overline{\mathbf{2}},\overline{\mathbf{2}}) = (\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2})\,,\\
[0001] &= \mathbf{8}_s \to (\mathbf{2},\overline{\mathbf{2}}) \oplus (\overline{\mathbf{2}},\mathbf{2}) = (\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{2}) \oplus (\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1})\,.
\end{aligned} \tag{2.3.2}$$

The first line in (2.3.2) is determined by the requirement that the supercharges of the $\mathcal{N}=8$ theory transform in the $\mathbf{8}_v$ of $\mathfrak{so}(8)_R$ and that four of them should transform in the fundamental representation of $\mathfrak{so}(4)_R$, as appropriate for an $\mathcal{N}=4$ sub-algebra. In general, for an $\mathfrak{so}(8)_R$ state with weights $[a_1a_2a_3a_4]$ (which is not necessarily a highest weight state as in (2.3.2)), one can work out the $\mathfrak{su}(2)^4$ weights (m_L, m_R, m_1, m_2):

$$[a_1a_2a_3a_4] \to \left(\frac{a_1+2a_2+a_3+a_4}{2}, \frac{a_1}{2}, \frac{a_3}{2}, \frac{a_4}{2}\right) . \tag{2.3.3}$$

It is now straightforward to describe which $\mathcal{N}=8$ multiplets can contribute to the $\mathcal{Q}$-cohomology of Section 2.2.3.[11] A list of all possible $\mathcal{N}=8$ multiplets was given in Table 1.1. Recall that from the $\mathcal{N}=4$

[11]Here $\mathcal{Q}$ can be chosen to be either $\mathcal{Q}_1$ or $\mathcal{Q}_2$, just as in Section 2.2.3.

perspective, each $\mathcal{Q}$-cohomology class is represented by a superconformal primary operator of a $(B,+)$ multiplet. Such a superconformal primary can only arise from a superconformal primary of a $(B,2)$, $(B,3)$, $(B,+)$, or $(B,-)$ multiplet in the $\mathcal{N}=8$ theory, as we now show.

We have already shown that such an operator must have

$$\Delta = m_L = j_L\,, \qquad j_R = s = 0\,. \tag{2.3.4}$$

Since such an operator is a superconformal primary of a B-type multiplet in $\mathcal{N}=4$, it must also be in a B-type multiplet in $\mathcal{N}=8$.

If (w_1, w_2, w_3, w_4) is an $\mathfrak{so}(8)_R$ weight, then we can take the $\mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$ quantum numbers to be

$$m_L = \frac{w_1 + w_2}{2}\,, \qquad m_R = \frac{w_1 - w_2}{2}\,. \tag{2.3.5}$$

An operator satisfying (2.3.4) must therefore have

$$w_1 = w_2 = \Delta\,, \qquad s = 0\,. \tag{2.3.6}$$

The states of the superconformal primary of any B-type multiplet satisfy

$$\Delta \geq w_1\,, \qquad \text{for any SCP of a } B\text{-type multiplet}\,, \tag{2.3.7}$$

or in other words $\Delta - w_1 \geq 0$. For the highest weight state we have $\Delta = w_1$. Now given the highest weight state of the superconformal primary, we can construct the highest weight states of the other conformal primaries by acting with the supercharges. In general there are 16 supercharges with $m_s = \pm 1/2$, and they have $\mathfrak{so}(8)_R$ weights $(\pm 1, 0, 0, 0)$, $(0, \pm 1, 0, 0)$, $(0, 0, \pm 1, 0)$, and $(0, 0, 0, \pm 1)$. The supercharges with weight vector $(1, 0, 0, 0)$ annihilate the highest weight states of all B-type multiplets, or generate conformal descendants that we're not interested in. The remaining supercharges all have $\Delta - w_1 > 0$. Therefore, all highest weight states of the conformal primaries other than the superconformal primary must have $\Delta - w_1 > 0$, and so

$$\Delta \geq w_1\,, \qquad \text{for all CPs of } B\text{-type multiplets}\,, \tag{2.3.8}$$

with the inequality being saturated only by superconformal primaries.

The condition (2.3.6) can therefore be obeyed only by superconformal primaries of $(B,2)$, $(B,3)$, $(B,+)$, or $(B,-)$ multiplets.

Since in $\mathcal{N}=4$ notation, the $\mathcal{N}=8$ theory has an $\mathfrak{so}(4)_F$ flavor symmetry, we should be more explicit about which $\mathfrak{so}(4)_F$ representation a $(B,+)$ multiplet of the $\mathcal{N}=4$ theory inherits from a corresponding $\mathcal{N}=8$ multiplet. From (2.3.3) it is easy to read off the (j_L, j_R, j_1, j_2) quantum numbers of the $\mathcal{N}=4$ $(B,+)$ superconformal primary:

$$\begin{aligned}
&\underline{\mathcal{N}=8} && && \underline{\mathcal{N}=4} \\
&(B,2):[0a_2a_3a_4] && \to && (B,+):\left(\frac{2a_2+a_3+a_4}{2},0,\frac{a_3}{2},\frac{a_4}{2}\right)\,, && (2.3.9)\\
&(B,3):[00a_3a_4] && \to && (B,+):\left(\frac{a_3+a_4}{2},0,\frac{a_3}{2},\frac{a_4}{2}\right)\,, && (2.3.10)\\
&(B,+):[00a_30] && \to && (B,+):\left(\frac{a_3}{2},0,\frac{a_3}{2},0\right)\,, && (2.3.11)\\
&(B,-):[000a_4] && \to && (B,+):\left(\frac{a_4}{2},0,0,\frac{a_4}{2}\right)\,. && (2.3.12)
\end{aligned}$$

Note that the $(B,+)$ multiplets in (2.3.9)–(2.3.12) have $j_R=0$, as they should, and that they transform in irreps of the flavor symmetry with $(j_1,j_2)=\left(\frac{a_3}{2},\frac{a_4}{2}\right)$, which in general are non-trivial. The operators in the topological quantum mechanics introduced in the previous section will therefore also carry these flavor quantum numbers. We will see below, however, that in the examples we study we will have only operators with $j_2=0$.

2.3.2 Twisted $(B,+)$ Multiplets

In this section we will construct explicitly the twisted version of $\mathcal{N}=8$ superconformal primaries of $(B,+)$ type. This construction will be used in the following sections to compute correlation functions of these operators in the 1d topological theory. Let us start by recalling some of the basic properties of these operators in the full three-dimensional theory. A $(B,+)$ superconformal primary transforming in the $[00k0]$ irrep will be denoted by $\mathcal{O}_{n_1\cdots n_k}(\vec{x})$, where the indices $n_i=1,\ldots,8$ label basis states in the $\mathbf{8}_c=[0010]$ irrep. This operator is symmetric and traceless in the n_i, and it is a Lorentz scalar of scaling dimension $\Delta=k/2$—see Table 1.1.

As is customary when dealing with symmetric traceless tensors, we introduce the polarizations Y^n that satisfy the null condition $Y\cdot Y=\sum_{n=1}^{8}Y^nY^n=0$. Thus we define

$$\mathcal{O}_k(\vec{x},Y)\equiv\mathcal{O}_{n_1\cdots n_k}(\vec{x})Y^{n_1}\cdots Y^{n_k} \tag{2.3.13}$$

and work directly with $\mathcal{O}_k(\vec{x}, Y)$ instead of $\mathcal{O}_{n_1\cdots n_k}(\vec{x})$. The introduction of polarizations allows for much more compact expressions for correlation functions of $\mathcal{O}_k(\vec{x}, Y)$. For example, the 2-point and 3-point functions, which are fixed by superconformal invariance up to an overall numerical coefficient, can be written as

$$\langle \mathcal{O}_k(\vec{x}_1, Y_1)\mathcal{O}_k(\vec{x}_2, Y_2)\rangle = \left(\frac{Y_1 \cdot Y_2}{|\vec{x}_{12}|}\right)^k , \tag{2.3.14}$$

$$\langle \mathcal{O}_{k_1}(\vec{x}_1, Y_1)\mathcal{O}_{k_2}(\vec{x}_2, Y_2)\mathcal{O}_{k_3}(\vec{x}_3, Y_3)\rangle = \lambda \left(\frac{Y_1 \cdot Y_2}{|\vec{x}_{12}|}\right)^{\frac{k_1+k_2-k_3}{2}} \left(\frac{Y_2 \cdot Y_3}{|\vec{x}_{23}|}\right)^{\frac{k_2+k_3-k_1}{2}} \left(\frac{Y_3 \cdot Y_1}{|\vec{x}_{31}|}\right)^{\frac{k_3+k_1-k_2}{2}} , \tag{2.3.15}$$

where the normalization convention for our operators is fixed by (2.3.14). Note that this normalization is different from that defined in (1.1.30). When we discuss an application of these methods to the stress tensor four point function, we will insert the correct numerical factors to match the normalization of (1.1.30). The coefficient λ in (2.3.15) may be non-zero only if k_1, k_2, and k_3 are such that the 3-point function is a polynomial in the Y_i.

The topologically twisted version of the $(B, +)$ operators $\mathcal{O}_k(\vec{x}, Y)$ can be constructed as follows. According to (2.3.11), the $\mathcal{N} = 4$ component of $\mathcal{O}_k(\vec{x}, Y)$ that is non-trivial in $\mathcal{Q}$-cohomology transforms in the $(\boldsymbol{k+1}, \mathbf{1}, \boldsymbol{k+1}, \mathbf{1})$ irrep of $\mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R \oplus \mathfrak{su}(2)_1 \oplus \mathfrak{su}(2)_2$. We can project $\mathcal{O}_k(\vec{x}, Y)$ onto this irrep by choosing the polarizations Y^n appropriately. In particular, Y^n transforms in the $\mathbf{8}_c$ of $\mathfrak{so}(8)_R$; as given in (2.3.2), this irrep decomposes into irreps of the four $\mathfrak{su}(2)$'s as

$$\mathbf{8}_c \to (\mathbf{2}, \mathbf{1}, \mathbf{2}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{2}) . \tag{2.3.16}$$

We can choose to organize the polarizations Y^n such that (Y^1, Y^2, Y^3, Y^4) transforms as a fundamental of $\mathfrak{so}(4)_{L,1} \cong \mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_1$ and is invariant under $\mathfrak{so}(4)_{R,2} \cong \mathfrak{su}(2)_R \oplus \mathfrak{su}(2)_2$, while (Y^5, Y^6, Y^7, Y^8) transforms as a fundamental of $\mathfrak{so}(4)_{R,2}$ and is invariant under $\mathfrak{so}(4)_{L,1}$. Since the k-th symmetric product of the $(\mathbf{2}, \mathbf{1}, \mathbf{2}, \mathbf{1})$ irrep in (2.3.16) is given precisely by the irrep $(\boldsymbol{k+1}, \mathbf{1}, \boldsymbol{k+1}, \mathbf{1})$ we want to obtain, setting $Y^5 = Y^6 = Y^7 = Y^8 = 0$ will project $\mathcal{O}_k(\vec{x}, Y)$ onto our desired $\mathfrak{su}(2)^4$ irrep.

Explicitly, we set

$$Y^i = \frac{1}{\sqrt{2}} y^a \bar{y}^{\dot{a}} \sigma^i_{a\dot{a}} , \quad Y^5 = Y^6 = Y^7 = Y^8 = 0 , \tag{2.3.17}$$

where $\sigma^i_{a\dot{a}}$ for $i = 1, \ldots, 4$, are defined in terms of the usual Pauli matrices as $\sigma^i_{a\dot{a}} \equiv (1, i\sigma^1, i\sigma^2, i\sigma^3)$, and we introduced the variables y^a and $\bar{y}^{\dot{a}}$ that play the role of $\mathfrak{su}(2)_L$ and $\mathfrak{su}(2)_1$ polarizations, respectively. It is easy to verify that the ansatz (2.3.17) respects the condition $Y \cdot Y = 0$ that the $\mathfrak{so}(8)_R$ polarizations Y^n must satisfy. We conclude that the $\mathcal{N} = 4$ superconformal primary that contributes to the cohomology is obtained from $\mathcal{O}_k(\vec{x}, Y)$ by plugging in the projection (2.3.17). It is given by

$$\mathcal{O}_k(\vec{x}, y, \bar{y}) \equiv \mathcal{O}_k(\vec{x}, Y)\big|_{(\mathbf{k+1},\mathbf{1},\mathbf{k+1},\mathbf{1})} = \frac{1}{2^{k/2}} \mathcal{O}_{i_1 \cdots i_k}(\vec{x})(y\sigma^{i_1}\bar{y}) \cdots (y\sigma^{i_k}\bar{y}) \,. \tag{2.3.18}$$

As we discussed in the previous section, the resulting operator $\mathcal{O}_k(\vec{x}, y, \bar{y})$ is a $(B,+)$-type operator in the $\mathcal{N} = 4$ sub-algebra of $\mathcal{N} = 8$. The twisted version of such $\mathcal{N} = 4$ operators was defined in (2.2.28) and is given by restricting $\vec{x}$ to the line $x^0 = x^2 = 0$ and twisting the $\mathfrak{su}(2)_L$ polarization y with the coordinate parameterizing this line. In summary, the twisted $\mathcal{N} = 8$ $(B,+)$ operators that participate in the 1d topological theory are given by

$$\widehat{\mathcal{O}}_k(x, \bar{y}) \equiv \mathcal{O}_k(\vec{x}, y, \bar{y})\big|_{\substack{\vec{x}=(0,x,0) \\ y=(1,x)}} \,. \tag{2.3.19}$$

Note that the twisted operator $\widehat{\mathcal{O}}_k(x, \bar{y})$ represents a collection of $k+1$ operators like the ones defined in Section 2.2.4, packaged together into a single expression with the help of the $\mathfrak{su}(2)_1$ polarization $\bar{y}$. Explicitly,

$$\widehat{\mathcal{O}}_k(x, \bar{y}) = \widehat{\mathcal{O}}_{k, a_1 \cdots a_{k+1}}(x)\bar{y}^{a_1} \cdots \bar{y}^{a_{k+1}} \,. \tag{2.3.20}$$

The components $\widehat{\mathcal{O}}_{k, a_1 \cdots a_{k+1}}(x)$ transform as a spin-$k/2$ irrep of $\mathfrak{su}(2)_1$.

By applying the projection (2.3.17) and (2.3.19) to the two-point and three-point functions in (2.3.14) and (2.3.15), we find that the corresponding correlators in the 1d theory are

$$\langle \widehat{\mathcal{O}}_k(x_1, \bar{y}_1)\widehat{\mathcal{O}}_k(x_2, \bar{y}_2)\rangle = \langle \bar{y}_1, \bar{y}_2\rangle^k \left(\operatorname{sgn} x_{12}\right)^k \,, \tag{2.3.21}$$

$$\begin{aligned}\langle \widehat{\mathcal{O}}_{k_1}(x_1, \bar{y}_1)\widehat{\mathcal{O}}_{k_2}(x_2, \bar{y}_2)\widehat{\mathcal{O}}_{k_3}(x_3, \bar{y}_3)\rangle &= \lambda \, \langle \bar{y}_1, \bar{y}_2\rangle^{\frac{k_1+k_2-k_3}{2}} \langle \bar{y}_2, \bar{y}_3\rangle^{\frac{k_2+k_3-k_1}{2}} \langle \bar{y}_3, \bar{y}_1\rangle^{\frac{k_3+k_1-k_2}{2}} \\ &\quad \times \left(\operatorname{sgn} x_{12}\right)^{\frac{k_1+k_2-k_3}{2}} \left(\operatorname{sgn} x_{23}\right)^{\frac{k_2+k_3-k_1}{2}} \left(\operatorname{sgn} x_{31}\right)^{\frac{k_3+k_1-k_2}{2}} \,,\end{aligned} \tag{2.3.22}$$

where the angle brackets are defined by

$$\langle \bar{y}_i, \bar{y}_j\rangle \equiv \bar{y}_i^a \varepsilon_{ab} \bar{y}_j^b \,. \tag{2.3.23}$$

The correlators (2.3.21) and (2.3.22) are equivalent to correlation functions of a 1d topological theory with an $\mathfrak{su}(2)$ global symmetry under which $\widehat{\mathcal{O}}_k$ transforms in the $\boldsymbol{k+1}$. The origin of this symmetry in the 3d $\mathcal{N} = 8$ theory is the $\mathfrak{su}(2)_1$ sub-algebra of $\mathfrak{so}(8)_R$.

2.3.3 Twisted Four Point Functions

As we discussed in Section 2.2.4 the 2-point and 3-point functions in (2.3.21) and (2.3.22) can be used to compute the OPE between two twisted operators up to $\mathcal{Q}$-exact terms. In this section we derive such OPEs in a number of examples and use them to compute 4-point functions in the 1d theory. In addition, we will see that applying crossing symmetry to these 4-point functions leads to a tractable set of constraints. These constraints allow us to derive simple relations between OPE coefficients that hold in any $\mathcal{N}=8$ theory.

The simplicity of the crossing constraints in the 1d theory is easy to understand from its 3d origin. In general the OPE between two $(B,+)$ operators in the 3d theory contains only a finite number of operators non-trivial in $\mathcal{Q}$-cohomology.[12] Indeed, there is a finite number of R-symmetry irreps in the tensor product $[00m0]\otimes[00n0]$, and multiplets of B-type are completely specified by their R-symmetry irrep.[13] A given correlator in the 1d theory therefore depends only on a finite number of OPE coefficients, and the resulting crossing constraints therefore also involve only a finite number of OPE coefficients of the 3d theory.

Let us discuss the representations in the OPE of two $(B,+)$ operators that transform as $[00n0]$ and $[00m0]$ of $\mathfrak{so}(8)_R$ in more detail.[14] The possible R-symmetry representations in this OPE are (assuming $m\geq n$)

$$\begin{aligned}
[00m0]\otimes[00n0] &= \bigoplus_{p=0}^{n}\bigoplus_{q=0}^{p}[0(q)(m+n-2q-2p)0] \\
&= \bigoplus_{p=0}^{n}\underbrace{[00(m+n-2p)0]}_{(B,+)}\oplus\bigoplus_{p=0}^{n}\bigoplus_{q=1}^{p}\underbrace{[0(q)(m+n-2q-2p)0]}_{(B,2)}\,,
\end{aligned} \tag{2.3.24}$$

where in the second line we have indicated the $\mathcal{N}=8$ multiplets that may be non-trivial in $\mathcal{Q}$-cohomology in each of the $\mathfrak{so}(8)_R$ irreps appearing in the product (see Table 1.1). There is an additional kinematical restriction on the OPE when $m=n$. In this case the tensor product decomposes into a symmetric and anti-symmetric piece corresponding to terms in (2.3.24) with even and odd q, respectively. Operators that appear in the anti-symmetric part of the OPE must have odd spin, and therefore cannot be of B-type (whose superconformal primary has zero spin). Passing to the cohomology, every term on the right-hand side of (2.3.24) represents a type of multiplet that is non-trivial in the $\mathcal{Q}$-cohomology and that contributes to the $\widehat{\mathcal{O}}_n\times\widehat{\mathcal{O}}_m$ OPE.

A few case studies are now in order.

[12] There may, however, be several 3d operators that contribute to the same cohomology class, but in general there is only a finite number of such degeneracies.

[13] This is not true, for instance, for semi-short multiplets of A-type, as those can have different Lorentz spins for a given R-symmetry irrep.

[14] The selection rules on the OPE of two $(B,+)$-type operators in $\mathcal{N}=8$ SCFTs were found in [9]. Our task is simpler here, since we are just interested in contributions that are non-trivial in cohomology.

The Free Multiplet

The simplest possible case to consider involves the OPE of $\widehat{\mathcal{O}}_1(x, \bar{y})$, which arises from twisting the superconformal primary $\mathcal{O}_1(\vec{x}, Y)$ of the free $\mathcal{N}=8$ multiplet consisting of 8 free real scalars and fermions. While it is trivial to write down the full correlation functions in this theory, it will serve as a good example for the general 1d twisting procedure.

According to (2.3.24) and the discussion following it, the relevant $\mathfrak{so}(8)_R$ irreps in the $\mathcal{O}_1 \times \mathcal{O}_1$ OPE appear in the symmetric tensor product:

$$[0010] \otimes_{\text{Sym}} [0010] = [0020] \oplus [0000] \,. \tag{2.3.25}$$

The contribution to the cohomology in the $\mathbf{35}_c = [0020]$ irrep comes from the superconformal primary of the stress-tensor $(B,+)$ multiplet that we will simply denote here by $\mathcal{O}_2$, and the only contribution from the [0000] multiplet is the identity operator $\widehat{1}$. After the twisting, the $\widehat{\mathcal{O}}_1 \times \widehat{\mathcal{O}}_1$ OPE can therefore be written as

$$\widehat{\mathcal{O}}_1(x_1, \bar{y}_1)\widehat{\mathcal{O}}_1(x_2, \bar{y}_2) = \operatorname{sgn} x_{12} \langle \bar{y}_1, \bar{y}_2 \rangle \widehat{1} + \frac{\lambda}{\sqrt{2}} \widehat{\mathcal{O}}_{\dot{a}_1 \dot{a}_2}(x_2) \bar{y}_1^{\dot{a}_1} \bar{y}_2^{\dot{a}_2} + (\mathcal{Q}\text{-exact terms}) \,, \tag{2.3.26}$$

where the factor $\sqrt{2}$ was chosen for later convenience. One can check that the twisted 2-point and 3-point functions in (2.3.21) and (2.3.22) are reproduced from this OPE.

Note that the OPE coefficient λ is fixed by the conformal Ward identity in terms of the coefficient c_T of the 2-point function of the canonically normalized stress-tensor. In particular, in the conventions of (1.1.30) $\lambda = 8/\sqrt{c_T}$ and a free real boson or fermion contributes one unit to c_T.[15] A free $\mathcal{N}=8$ multiplet therefore has $c_T = 16$, and as we will now see, this can be derived from the crossing symmetry constraints.

Using the invariance under the global $\mathfrak{su}(2)$ symmetry, and assuming $x_1 < x_2 < x_3 < x_4$, the 4-point function of $\widehat{\mathcal{O}}_1$ can be written as

$$\langle \widehat{\mathcal{O}}_1(x_1, \bar{y}_1)\widehat{\mathcal{O}}_1(x_2, \bar{y}_2)\widehat{\mathcal{O}}_1(x_3, \bar{y}_3)\widehat{\mathcal{O}}_1(x_4, \bar{y}_4) \rangle = \langle \bar{y}_1, \bar{y}_2 \rangle \langle \bar{y}_3, \bar{y}_4 \rangle \widehat{\mathcal{G}}_1(\bar{w}) \,. \tag{2.3.27}$$

The variable $\bar{w}$ should be thought of as the single $\mathfrak{su}(2)_1$-invariant cross-ratio, and is defined in terms of the polarizations as

$$\bar{w} \equiv \frac{\langle \bar{y}_1, \bar{y}_2 \rangle \langle \bar{y}_3, \bar{y}_4 \rangle}{\langle \bar{y}_1, \bar{y}_3 \rangle \langle \bar{y}_2, \bar{y}_4 \rangle} \,. \tag{2.3.28}$$

[15] In the next section we will make explicit the relation between the definition of the structure constants in (1.1.30) and the ones used in this section.

Applying the OPE (2.3.26) in the s-channel (i.e., (12)(34)) gives

$$\langle \widehat{\mathcal{O}}_1(x_1,\bar{y}_1)\cdots\widehat{\mathcal{O}}_1(x_4,\bar{y}_4)\rangle\big|_{\text{s-channel}} = \langle\bar{y}_1,\bar{y}_2\rangle\langle\bar{y}_3,\bar{y}_4\rangle\left[1+\frac{\lambda^2}{4}\frac{2-\bar{w}}{\bar{w}}\right]. \tag{2.3.29}$$

The only other OPE channel that does not change the cyclic ordering of the operators is the t-channel (i.e., (41)(23)). In computing it we should be careful to include an overall minus sign from exchanging the fermionic like $\widehat{\mathcal{O}}_1(x_4,\bar{y}_4)$ three times (see the discussion in section 2.2.4). The 4-point function in the t-channel is therefore obtained by exchanging $\bar{y}_1\leftrightarrow\bar{y}_3$ in (2.3.29) and multiplying the result by a factor of (-1), which gives

$$\langle \widehat{\mathcal{O}}_1(x_1,\bar{y}_1)\cdots\widehat{\mathcal{O}}_1(x_4,\bar{y}_4)\rangle\big|_{\text{t-channel}} = \langle\bar{y}_1,\bar{y}_4\rangle\langle\bar{y}_2,\bar{y}_3\rangle\left[1+\frac{\lambda^2}{4}\frac{1+\bar{w}}{1-\bar{w}}\right]. \tag{2.3.30}$$

In deriving (2.3.30) we used the identity

$$\langle\bar{y}_1,\bar{y}_2\rangle\langle\bar{y}_3,\bar{y}_4\rangle+\langle\bar{y}_1,\bar{y}_4\rangle\langle\bar{y}_2,\bar{y}_3\rangle=\langle\bar{y}_1,\bar{y}_3\rangle\langle\bar{y}_2,\bar{y}_4\rangle\,, \tag{2.3.31}$$

which implies that $\bar{w}\to 1-\bar{w}$ when exchanging $\bar{y}_1\leftrightarrow\bar{y}_3$.

Equating (2.3.29) to (2.3.30) we obtain (after a slight rearrangement) our first 1d crossing constraint:

$$\bar{w}\left[1+\frac{\lambda^2}{4}\frac{2-\bar{w}}{\bar{w}}\right]=(1-\bar{w})\left[1+\frac{\lambda^2}{4}\frac{1+\bar{w}}{1-\bar{w}}\right]. \tag{2.3.32}$$

This equation has the unique solution

$$\lambda^2 = 4\,. \tag{2.3.33}$$

Combined with $\lambda = 8/\sqrt{c_T}$ (in the conventions of (1.1.30), as mentioned above), (2.3.33) implies $c_T = 16$, as expected for a free theory with 8 real bosons and 8 real fermions. This is a nice check of our formalism.

The Stress-Tensor Multiplet

Moving forward to a non-trivial example we will now consider the OPE of the twisted version of the superconformal primary $\mathcal{O}_{\text{Stress}}(\vec{x},Y)=\mathcal{O}_2(\vec{x},Y)$ of the stress-tensor multiplet. The $\mathfrak{so}(8)_R$ irreps in the symmetric part of the $\mathcal{O}_{\text{Stress}}\times\mathcal{O}_{\text{Stress}}$ OPE are

$$[0020]\otimes_{\text{Sym}}[0020]=[0040]\oplus[0200]\oplus[0020]\oplus[0000]\,. \tag{2.3.34}$$

The possible contributions to this OPE that survive the topological twisting are a $(B,+)$-type operator transforming in the [0040], which we will simply denote by $\mathcal{O}_4$, the stress-tensor multiplet itself $\mathcal{O}_2$ in the [0020], and the identity operator $\widehat{1}$ in the trivial irrep [0000]. In addition, there may be a $(B,2)$-type multiplet transforming in the [0200] irrep. According to (2.3.9) the component of this $(B,2)$ operator that is non-trivial in cohomology transforms trivially under the global $\mathfrak{su}(2)_1 \oplus \mathfrak{su}(2)_2$ symmetry, and we will therefore denote it by $\widehat{\mathcal{O}}_0$.

Including all of the contributions mentioned above, the OPE of $\widehat{\mathcal{O}}_2$ can be written as

$$\begin{aligned}\widehat{\mathcal{O}}_2(x_1,\bar{y}_1)\widehat{\mathcal{O}}_2(x_2,\bar{y}_2) &= \langle \bar{y}_1,\bar{y}_2\rangle^2 \left(\widehat{1} + \frac{\lambda_{(B,2)}}{4}\widehat{\mathcal{O}}_0(x_2)\right) + \frac{\lambda_{\text{Stress}}}{\sqrt{2}} \operatorname{sgn} x_{12} \langle \bar{y}_1,\bar{y}_2\rangle \widehat{\mathcal{O}}_{\dot{a}_1\dot{a}_2}(x_2)\bar{y}_1^{\dot{a}_1}\bar{y}_2^{\dot{a}_2} \\ &+ \sqrt{\frac{3}{8}}\lambda_{(B,+)}\widehat{\mathcal{O}}_{\dot{a}_1\dot{a}_2\dot{a}_3\dot{a}_4}(x_2)\bar{y}_1^{\dot{a}_1}\bar{y}_1^{\dot{a}_2}\bar{y}_2^{\dot{a}_3}\bar{y}_2^{\dot{a}_4} + (\mathcal{Q}\text{-exact terms})\,, \end{aligned} \tag{2.3.35}$$

where the numerical factors were chosen such that the OPE coefficients match the conventions of (1.1.30). We emphasize again that up to these coefficients, the form of (2.3.35) is trivially fixed by demanding invariance under the global $\mathfrak{su}(2)_1$ symmetry.

Evaluating the $\widehat{\mathcal{O}}_2$ 4-point function in the s-channel gives ($x_1 < x_2 < x_3 < x_4$)

$$\begin{aligned}\langle \widehat{\mathcal{O}}_2(x_1,\bar{y}_1)\cdots\widehat{\mathcal{O}}_2(x_4,\bar{y}_4)\rangle = \langle \bar{y}_1,\bar{y}_2\rangle^2\langle \bar{y}_3,\bar{y}_4\rangle^2 \Bigg[1 &+ \frac{1}{16}\lambda^2_{(B,2)} + \frac{1}{4}\lambda^2_{\text{Stress}}\frac{2-\bar{w}}{\bar{w}} \\ &+ \frac{1}{16}\lambda^2_{(B,+)}\frac{6-6\bar{w}+\bar{w}^2}{\bar{w}^2}\Bigg]\,. \end{aligned} \tag{2.3.36}$$

The t-channel expression is obtained by taking $\bar{y}_1 \leftrightarrow \bar{y}_3$ under which $\bar{w} \to 1-\bar{w}$. Equating the two channels results in the crossing equation

$$\begin{aligned}&\bar{w}^2\left[1 + \frac{1}{16}\lambda^2_{(B,2)} + \frac{1}{4}\lambda^2_{\text{Stress}}\frac{2-\bar{w}}{\bar{w}} + \frac{1}{16}\lambda^2_{(B,+)}\frac{6-6\bar{w}+\bar{w}^2}{\bar{w}^2}\right] \\ &= (1-\bar{w})^2\left[1 + \frac{1}{16}\lambda^2_{(B,2)} + \frac{1}{4}\lambda^2_{\text{Stress}}\frac{1+\bar{w}}{1-\bar{w}} + \frac{1}{16}\lambda^2_{(B,+)}\frac{1+4\bar{w}+\bar{w}^2}{(1-\bar{w})^2}\right]\,. \end{aligned} \tag{2.3.37}$$

The solution of (2.3.37) is given by (2.1.2), which we reproduce here for the convenience of the reader:

$$4\lambda^2_{\text{Stress}} - 5\lambda^2_{(B,+)} + \lambda^2_{(B,2)} + 16 = 0\,. \tag{2.3.38}$$

Note that the OPE coefficients of the free theory and the GFFT listed in Table 1.4 satisfy this constraint. Moreover, the 5-point function of $\widehat{\mathcal{O}}_2$ depends only on the OPE coefficients appearing in (2.3.38), and it can be computed using (2.3.35) by taking the OPE in different ways. We have verified that the resulting crossing constraints for this 5-point function are solved only if (2.3.38) is satisfied.[16] We consider these facts to be non-trivial checks on our formalism.

The relation in (2.3.38) must hold in any $\mathcal{N}=8$ SCFT. In addition, (2.3.38) implies that in any unitary $\mathcal{N}=8$ theory $\lambda^2_{(B,+)}>0$; i.e. a $(B,+)$ multiplet transforming in the [0040] irrep must always exist and has a non-vanishing coefficient in the $\mathcal{O}_{\text{Stress}}\times\mathcal{O}_{\text{Stress}}$ OPE. In contrast, $\lambda_{(B,2)}$ can in principle vanish in which case $\lambda_{(B,+)}$ is determined in terms of λ_{Stress}. The free theory as well as the $U(2)_2\times U(1)_{-2}$ ABJ theory are examples for which $\lambda_{(B,2)}=0$. The latter case can be checked explicitly using the methods we will discuss Section 2.4.

2.3.4 4-point Correlation Functions and Superconformal Ward Identity

In this section we will show that in the particular case of 4-point functions of $(B,+)$ type operators $\mathcal{O}_k(\vec{x},Y)$ in $\mathcal{N}=8$ SCFTs, the results obtained by using the topological twisting procedure can be reproduced by using the superconformal Ward identity derived in [1]. This will provide a check on some of the computations of the previous sections that involve such 4-point functions. Note, however, that the topological twisting method applies more generally to any $\mathcal{N}\geq 4$ SCFT and to any n-point function of twisted operators.

Let us start by reviewing the constraints of superconformal invariance on 4-point functions of $\mathcal{O}_k(\vec{x},Y)$. These constraints were discussed in Section 1.1.3 for the specific case of the stress tensor multiplet, here we generalize these constraints to 4-point functions of general $(B,+)$ type operators. These 4-point functions are restricted by the $\mathfrak{sp}(4)$ conformal invariance and the $\mathfrak{so}(8)_R$ symmetry to take the form

$$\langle\mathcal{O}_k(\vec{x}_1,Y_1)\mathcal{O}_k(\vec{x}_2,Y_2)\mathcal{O}_k(\vec{x}_3,Y_3)\mathcal{O}_k(\vec{x}_4,Y_4)\rangle=\frac{(Y_1\cdot Y_2)^k(Y_3\cdot Y_4)^k}{|x_{12}|^k|x_{34}|^k}\mathcal{G}_k(z,\bar{z};\alpha,\bar{\alpha})\,, \tag{2.3.39}$$

where the variables $z,\bar{z}$ and $\alpha,\bar{\alpha}$ were defined in (1.1.38). The function $\mathcal{G}_k(z,\bar{z};\alpha,\bar{\alpha})$ in (2.3.39) is symmetric under $z\leftrightarrow\bar{z}$ and under $\alpha\leftrightarrow\bar{\alpha}$. Moreover, it is a general degree k polynomial in σ and τ, as follows from the fact that the 4-point function must be polynomial in all the Y_i variables. The full $\mathfrak{osp}(8|4)$ superconformal algebra imposes additional constraints on $\mathcal{G}_k(z,\bar{z};\alpha,\bar{\alpha})$, which are encapsulated in the superconformal Ward identity, which take the same form as the stress tensor 4-point function case (1.1.37). We will find it useful to rewrite these Ward identities in terms of the new variables

$$\tilde{U}=w\bar{w}=\frac{1}{\sigma}\,,\qquad\tilde{V}=(1-w)(1-\bar{w})=\frac{\tau}{\sigma} \tag{2.3.40}$$

[16] Correlators with 6 or more insertions of $\widehat{\mathcal{O}}_2$ depend on more OPE coefficients on top of the ones appearing in (2.3.38).

as

$$\left(z\partial_z + \frac{1}{2}w\partial_w\right)\mathcal{G}_k(z,\bar{z};w,\bar{w})\big|_{w\to z} = \left(\bar{z}\partial_{\bar{z}} + \frac{1}{2}\bar{w}\partial_{\bar{w}}\right)\mathcal{G}_k(z,\bar{z};w,\bar{w})\big|_{\bar{w}\to\bar{z}} = 0\,. \tag{2.3.41}$$

Let us now discuss how to obtain the 4-point function in the topologically twisted sector directly in terms of the variables $z, \bar{z}, w$, and $\bar{w}$. To do that we restrict the external operators in (1.1.30) to a line by taking $\vec{x}_i = (0, x_i, 0)$ with $0 = x_1 < x_2 < x_3 = 1$ and $x_4 = \infty$. In particular, this implies that $z\big|_{1d} = \bar{z}\big|_{1d} = x_2$. In addition, using the projection of the polarizations Y_i, which was given in (2.3.17) and (2.3.19), we find that

$$\tilde{U}\big|_{1d} = \frac{x_{12}x_{34}}{x_{13}x_{24}}\frac{\langle\bar{y}_1,\bar{y}_2\rangle\langle\bar{y}_3,\bar{y}_4\rangle}{\langle\bar{y}_1,\bar{y}_3\rangle\langle\bar{y}_2,\bar{y}_4\rangle} = z\frac{\langle\bar{y}_1,\bar{y}_2\rangle\langle\bar{y}_3,\bar{y}_4\rangle}{\langle\bar{y}_1,\bar{y}_3\rangle\langle\bar{y}_2,\bar{y}_4\rangle} = w\bar{w}\big|_{1d}\,, \tag{2.3.42}$$

$$\tilde{V}\big|_{1d} = \frac{x_{14}x_{23}}{x_{13}x_{24}}\frac{\langle\bar{y}_1,\bar{y}_4\rangle\langle\bar{y}_2,\bar{y}_3\rangle}{\langle\bar{y}_1,\bar{y}_3\rangle\langle\bar{y}_2,\bar{y}_4\rangle} = (1-z)\left(1-\frac{\langle\bar{y}_1,\bar{y}_2\rangle\langle\bar{y}_3,\bar{y}_4\rangle}{\langle\bar{y}_1,\bar{y}_3\rangle\langle\bar{y}_2,\bar{y}_4\rangle}\right) = (1-w)(1-\bar{w})\big|_{1d}\,. \tag{2.3.43}$$

Note that $z = \frac{x_{12}x_{34}}{x_{13}x_{24}}$ is the single $SL(2,\mathbb{R})$ cross-ratio, and for the ordering $x_1 < x_2 < x_3 < x_4$ we have that $0 < z < 1$. In addition, recall that $\frac{\langle\bar{y}_1,\bar{y}_2\rangle\langle\bar{y}_3,\bar{y}_4\rangle}{\langle\bar{y}_1,\bar{y}_3\rangle\langle\bar{y}_2,\bar{y}_4\rangle}$ is the single $SU(2)$ cross-ratio, which was denoted by $\bar{w}$ in (2.3.28) for reasons that now become obvious. We conclude that in terms of the variables z, $\bar{z}$, w and $\bar{w}$ the 1d topological twisting is equivalent to setting $z = \bar{z} = w$, and identifying $\bar{w}$ with the $SU(2)$ cross-ratio (2.3.28).

Since from our general arguments the full 4-point function in (2.3.39) must be constant after the 1d twisting, and the pre-factor of $\mathcal{G}_k(z,\bar{z};w,\bar{w})$ in (2.3.39) projects to a constant (up to ordering signs), we conclude that

$$\mathcal{G}_k(z,z;z,\bar{w}) \equiv \widehat{\mathcal{G}}_k(\bar{w}) = \sum_{j=0}^{k} a_j \bar{w}^{-j}\,, \tag{2.3.44}$$

where the a_j are some numbers and the same relation must hold for $\mathcal{G}_n(z,z;w,z)$ as follows from the $w \leftrightarrow \bar{w}$ symmetry of $\mathcal{G}_k$. In fact, one can prove (2.3.44) directly from the superconformal Ward identity (2.3.41) by a simple application of the chain rule.[17] Indeed,

$$\begin{aligned} z\partial_z\mathcal{G}_k(z,z;z,\bar{w}) &= (z\partial_z + \bar{z}\partial_{\bar{z}} + w\partial_w)\mathcal{G}_n(z,\bar{z};w,\bar{w})\big|_{\substack{\bar{z}\to z\\ w\to z}} \\ &\overset{(2.3.41)}{=} (\bar{z}\partial_{\bar{z}} + \frac{1}{2}w\partial_w)\mathcal{G}_k(z,\bar{z};w,\bar{w})\big|_{\substack{\bar{z}\to z\\ w\to z}} = (z\partial_z + \frac{1}{2}w\partial_w)\mathcal{G}_k(z,\bar{z};w,\bar{w})\big|_{\substack{\bar{z}\to z\\ w\to z}} \overset{(2.3.41)}{=} 0\,, \end{aligned} \tag{2.3.46}$$

[17]The analogous statement in the context of $\mathcal{N}=4$ theory in four dimensions is more familiar (see e.g. [8] and references therein). In that case the Ward identity for the 4-point functions of $\frac{1}{2}$-BPS operators transforming in the $[0k0] \in SU(4)_R$, is

$$(z\partial_z + w\partial_w)\mathcal{G}_k^{\mathcal{N}=4}(z,\bar{z};w,\bar{w})\big|_{w\to z} = 0 \Rightarrow \mathcal{G}_k^{\mathcal{N}=4}(z,\bar{z};z,\bar{w}) = f_k(z,\bar{w})\,. \tag{2.3.45}$$

The holomorphic functions $f_k(z,\bar{w})$ were interpreted in [36] as correlation function in 2d chiral CFT.

where in the next to last equality we used symmetry of $\mathcal{G}_k$ under $z \leftrightarrow \bar{z}$.

To make contact with the 1d OPE methods of Section 2.3.3 we must find the contribution of each superconformal multiplet to the function $\mathcal{G}_k(z,z;z,\bar{w}) = \widehat{\mathcal{G}}_k(\bar{w})$. For that purpose consider the s-channel expansion of the 4-point function (2.3.39):

$$\mathcal{G}_k(z,\bar{z};w,\bar{w}) \equiv \sum_{a=0}^{k}\sum_{b=0}^{a}\left[Y_{ab}\left(w,\bar{w}\right)\sum_{\mathcal{O}\in[0(a-b)(2b)0]}\lambda_{\mathcal{O}}^2\, G_{\Delta_{\mathcal{O}},j_{\mathcal{O}}}(z,\bar{z})\right]. \tag{2.3.47}$$

Each term in the triple sum of (2.3.47) corresponds to the contribution from a single conformal familiy in the $\mathcal{O}_k \times \mathcal{O}_k$ OPE, whose primary is an operator of dimension $\Delta_{\mathcal{O}}$, spin $j_{\mathcal{O}}$ and transforms in the $[0(a-b)(2b)0]$ irrep of $\mathfrak{so}(8)_R$. In particular, the outer double sum in (2.3.47) is over all irreps $[0(a-b)(2b)0]$ in the $[00k0] \otimes [00k0]$ tensor product, and the Y_{ab} are degree-a polynomials corresponding to the contribution arising from each of those irreps. Moreover, the functions $G_{\Delta,j}(z,\bar{z})$ are the conformal blocks, and $\lambda_{\mathcal{O}}$ are real OPE coefficients.

The Ward identity (2.3.41) imposes relations between OPE coefficients in (2.3.47) of primaries in the same superconformal multiplet. The full contribution to (2.3.47) from a single superconformal multiplet is called a superconformal block. It can be shown that the Ward identity holds independently for each superconformal block, and therefore those should evaluate to a constant after setting $z = \bar{z} = w$. We verified that this is true by using the explicit expressions for these blocks that will be given in Section 4.3.1 for the case $k = 2$. In particular, the $\mathcal{O}_2 \times \mathcal{O}_2$ OPE contains short multiplets of types $(B,+)$ and $(B,2)$, semi-short multiplets of type $(A,+)$ and $(A,2)$, and also long multiplets. One can check that the superconformal blocks corresponding to $(A,+)$, $(A,2)$, and long multiplets all vanish once we set $z = \bar{z} = w$, while contributions arising from the B-type multiplets are non-vanishing. This confirms the general cohomological arguments of section (2.3.1) that only those multiples survive the topological twisting.

Let us now compute the 1d projection of a given superconformal block. Superconformal primary operators of type B have zero spin and those that transform in the $[0(a-b)(2b)0]$ irrep have dimension $\Delta = a$. It follows that the full contribution to $\mathcal{G}_k(z,\bar{z};w,\bar{w})$ from such an operator is $\lambda^2 Y_{ab}(w,\bar{w})G_{a,0}(z,\bar{z})$ (see (2.3.47)). Our normalization convention for conformal blocks is defined to be

$$G_{\Delta,j}(z,z) = \left(\frac{z}{4}\right)^{\Delta}(1+O(z)) . \tag{2.3.48}$$

In addition, from the $SO(8)$ Casimir equation satisfied by the Y_{ab} (see e.g., [8]), one can show that

$$Y_{ab}(w,\bar{w}) = w^{-a} P_b\left(\frac{2-\bar{w}}{\bar{w}}\right) + O(w^{1-a})\,, \tag{2.3.49}$$

where $P_n(x)$ are the Legendre polynomials and the overall constant was fixed to match the conventions of (1.1.30).

We conclude that the contribution from any B-type multiplet to $\widehat{\mathcal{G}}_k$ is given by

$$\begin{aligned}\widehat{\mathcal{G}}_k(\bar{w})\big|_{\mathcal{O}\in[0(a-b)(2b)0]} &= \mathcal{G}_k(z,z;z,\bar{w})\big|_{\mathcal{O}\in[0(a-b)(2b)0]}\\ &= \lambda^2_{\mathcal{O}}\frac{1}{4^a}P_b\left(\frac{2-\bar{w}}{\bar{w}}\right) + O(z) = \lambda^2_{\mathcal{O}}\frac{1}{4^a}P_b\left(\frac{2-\bar{w}}{\bar{w}}\right)\,,\end{aligned} \tag{2.3.50}$$

where the higher order contributions in the expansion around $z = 0$ must all cancel, as the projected superconformal block is independent of z. One can verify that the contributions from each multiplet to the 4-point functions of $\widehat{\mathcal{O}}_1$ in (2.3.29) and those of $\widehat{\mathcal{O}}_2$ in (2.3.36), which were obtained by using the 1d OPE directly, match precisely those same contributions obtained using the prescription in (2.3.50).

2.4 Computation of OPE coefficients

We will now use the 1d sector derived in previous sections to compute the squared OPE coefficients $\lambda^2_{\text{Stress}}$, $\lambda^2_{(B,+)}$, and $\lambda^2_{(B,2)}$ in all Lagrangian $\mathcal{N} = 8$ SCFTs using supersymmetric localization. Conceptually, this computation can be split into several parts, each of which we discuss in separate subsections. In Section 2.4.1, we explain how, in $\mathcal{N} = 4$ SCFTs with flavor symmetries, one can relate the fourth derivative of the mass-deformed S^3 partition function with respect to the mass parameter to certain OPE coefficients. In Section 2.4.2, we apply this analysis to $\mathcal{N} = 8$ SCFTs. In Section 2.4.3, we use the existing results for the mass-deformed partition function of ABJM and BLG theories in order to extract $\lambda^2_{\text{Stress}}$, $\lambda^2_{(B,+)}$, and $\lambda^2_{(B,2)}$ from the results of the previous two sections.

2.4.1 Topological sector of $\mathcal{N} = 4$ SCFTs from mass-deformed S^3 partition function

While the discussion of the 1d sector in previous sections was for SCFTs defined on $\mathbb{R}^3$, one can perform a similar construction on any conformally flat space. In particular, using the stereographic projection, the 3d SCFT can also be placed on a round three-sphere of radius r such that the 1d line gets mapped to a great circle parameterized by $\varphi = 2\arctan\frac{x}{2r}$. The 1d operators are then defined as

$$\mathcal{O}(x) = \mathcal{O}_{a_1 \ldots a_{2J_L}}(0,0,x) u^{a_1}(x) \cdots u^{a_{2J_L}}(x) \,, \qquad u^1(x) = 1 \,, \qquad u^2(x) = \frac{x}{2r} \,. \tag{2.4.1}$$

Note that the previous definition of the 1d operators in (2.2.28) can be recovered by setting $r = 1/2$ here. On the circle, the 1d operators $\mathcal{O}(\varphi)$ are periodic on this circle if J_L is an integer and anti-periodic if J_L is a half odd-integer. Defining the 1d theory on a great circle of S^3 as opposed to a line in $\mathbb{R}^3$ has the benefit that when the 3d SCFT has a Lagrangian description, then it is possible to perform supersymmetric localization on S^3 in order to obtain an explicit Lagrangian description of the 1d sector itself. In the case where the Lagrangian of the 3d theory involves only hypermultiplets and vector multiplets, the 1d theory Lagrangian was derived in [42].

Regardless of whether the 1d theory has a Lagrangian description or not, let us describe a procedure for calculating certain integrated correlation functions in the 1d theory. We will be interested in the case where the 1d operator $J(\varphi)$ comes from a 3d operator $J_{ab}(\vec{x})$ with $\Delta = J_L = 1$. Such a 3d operator is the bottom component of a superconformal multiplet $(J_{ab}, K_{\dot{a}\dot{b}}, j_\mu, \chi_{a\dot{a}})$ that in addition to J_{ab} also contains the following conformal primaries: a pseudoscalar $K_{\dot{a}\dot{b}}$ of scaling dimension 2, a fermion $\chi_{a\dot{a}}$ of scaling dimension 3/2, and a conserved current j_μ. Thus, this is a conserved flavor current multiplet, and all its operators transform in the adjoint representation of the flavor symmetry. To exhibit the adjoint indices, we will write $(J^A_{ab}, K^A_{\dot{a}\dot{b}}, j^A_\mu, \chi^A_{a\dot{a}})$, where A runs from 1 to the dimension of the flavor symmetry Lie algebra.

Let us choose a basis for this Lie algebra where the two-point function of the current multiplets is diagonal in the adjoint indices:

$$\begin{aligned}
\langle j^A_\mu(\vec{x}) j^B_\nu(0)\rangle &= \delta^{AB} \frac{\tau}{64\pi^2}\left(\partial^2 \delta_{\mu\nu} - \partial_\mu \partial_\nu\right)\frac{1}{|\vec{x}|^2} \,, \\
\langle J^A_{ab}(\vec{x}) J^B_{cd}(0)\rangle &= -\delta^{AB} \frac{\tau}{256\pi^2} \frac{1}{|\vec{x}|^2}(\epsilon_{ac}\epsilon_{bd} + \epsilon_{ad}\epsilon_{bc}) \,, \\
\langle K^A_{\dot{a}\dot{b}}(\vec{x}) K^B_{\dot{c}\dot{d}}(0)\rangle &= -\delta^{AB} \frac{\tau}{128\pi^2} \frac{1}{|\vec{x}|^4}(\epsilon_{\dot{a}\dot{c}}\epsilon_{\dot{b}\dot{d}} + \epsilon_{\dot{a}\dot{d}}\epsilon_{\dot{b}\dot{c}}) \,.
\end{aligned} \tag{2.4.2}$$

Let us also normalize the current j^A_μ canonically, meaning that for an operator $\mathcal{O}$ transforming in a representation $\mathcal{R}$ of the flavor symmetry, we have $j^A_\mu(\vec{x})\mathcal{O}(0) \sim \frac{x_\mu}{4\pi|\vec{x}|^3} T^A \mathcal{O}(0)$, where T^A is the corresponding generator in representation $\mathcal{R}$. In particular, we then have $j^A_\mu(\vec{x}) j^B_\nu(0) \sim \frac{x_\mu}{4\pi|\vec{x}|^3} i f^{ABC} j^C_\nu(0)$, where the structure constants are defined by $[T^A, T^B] = i f^{ABC} T^C$.[18]

[18] For example, a free hypermultiplet has $\mathfrak{su}(2)$ flavor symmetry and a current multiplet as described. Indeed, we can write the hyper scalars as q_{ai} and the hyper fermions as $\psi_{\dot{a}i}$, with the two-point functions $\langle q_{ai}(\vec{x}) q_{bj}(0)\rangle = \frac{\epsilon_{ab}\epsilon_{ij}}{4\pi|x|}$ and $\langle \psi_{\alpha\dot{a}i}(\vec{x}) \psi_{\beta\dot{b}j}(0)\rangle = \frac{\epsilon_{ab}\epsilon_{ij} x^\mu \gamma_{\mu\alpha\beta}}{4\pi|\vec{x}|}$. Then $j^A_\mu = \frac{1}{2}\sigma^{Aij}\left[i\epsilon^{ab} q_{ai}\partial_\mu q_{bj} - \frac{1}{2}\epsilon^{\dot{a}\dot{b}}\psi_{\alpha\dot{a}i}\gamma^{\alpha\beta}_\mu \psi_{\beta\dot{b}j}\right]$, $J^A_{ab} = \frac{1}{8}\sigma^{Aij}(q_{ai}q_{bj} + q_{bi}q_{aj})$, and $K^A_{\dot{a}\dot{b}} = \frac{i}{8}\sigma^{Aij}(\psi_{\dot{a}i}\psi_{\dot{b}j} + \psi_{\dot{b}i}\psi_{\dot{a}j})$. We have $\tau = 2$ for the $\mathfrak{su}(2)$ flavor symmetry of a hyper.

At the linearized level, such a current multiplet couples to a background $\mathcal{N} = 4$ vector multiplet $(A^A_\mu, \Phi^A_{\dot{a}\dot{b}}, D^A_{ab}, \lambda^A_{a\dot{a}})$:

$$\int d^3x \left[A^{\mu A} j^A_\mu + iD^{abA} J^A_{ab} + \Phi^{\dot{a}\dot{b}A} K^A_{\dot{a}\dot{b}} + (\text{fermions})\right] . \tag{2.4.3}$$

(Quadratic terms in the background vector multiplet are also required in order to preserve gauge invariance and supersymmetry.)

Let us provide a prescription for computing correlation functions of the integrated operator $\int d\varphi\, J^A(\varphi)$. To obtain this prescription, first place the SCFT on a round S^3, then introduce an $\mathcal{N} = 4$-preserving (adjoint valued) real mass parameter $m = m^A T^A$. Introducing such a parameter requires the following background vector multiplet fields:

$$\Phi^A_{\dot{a}\dot{b}} = m^A \bar{h}_{\dot{a}\dot{b}} \,, \qquad D^A_{ab} = -\frac{m^A}{r} h_{ab} \,, \qquad A^A_\mu = \lambda^A_{a\dot{a}} = 0 \,. \tag{2.4.4}$$

Here, we follow the conventions of [42] for the hypermultiplet and vector multiplet fields and their SUSY variations. The quantities h and $\bar{h}$ are constant matrices, normalized such that $h_{ab}h^{ab} = \bar{h}_{\dot{a}\dot{b}}\bar{h}^{\dot{a}\dot{b}} = -2$. The mass-deformed theory is invariant under the superalgebra $\mathfrak{su}(2|1) \oplus \mathfrak{su}(2|1)$, and the mass parameter m appears as a central charge in this algebra. Up to linear order in m, the mass deformation amounts to adding

$$m^A \int d^3x \sqrt{g} \left[-i\frac{1}{r} h^{ab} J^A_{ab} + \bar{h}^{\dot{a}\dot{b}} K^A_{\dot{a}\dot{b}}\right] \tag{2.4.5}$$

to the conformal action on S^3. The S^3 partition function $Z(m)$ can be computed using supersymmetric localization [40] even for $\mathcal{N} = 4$ theories for which the localization to the 1d sector performed in [42] does not apply.

However, $Z(m)$ also computes the partition function of the 1d theory deformed by

$$-4\pi r^2 m^A \int_{-\pi}^{\pi} d\varphi J^A(\varphi) \,. \tag{2.4.6}$$

Such a statement can be proven explicitly[19] in the case where the results of [42] apply, but it should hold more generally. This statement should simply follow from the supersymmetric Ward identities, as was shown

[19]Let us consider N hypermultiplets $(q_{ai}, \psi_{\dot{a}i})$, where $i = 1, \ldots, 2N$, charged under a vector multiplet with gauge group G and generators t^α and flavor symmetry G_F with generators T^A, respectively. Both G and G_F are embedded into the flavor symmetry $USp(2N)$ of N ungauged free hypers. We then have $J^A_{ab} = \frac{1}{4} T^{Aij}(q_{ai}q_{bj} + q_{bi}q_{aj})$, where T^{Aij} is a symmetric matrix in the ij indices. Consequently, using (2.4.1), we have $J^A = J^A_{ab}u^a u^b = \frac{1}{2} T^{Aij} Q_i Q_j$, with $Q_i = q_{ai}u^a$. In [42], it was shown

in similar 4d examples in [43, 44]; it would be nice to investigate this more precisely in the future. In other words, we claim that the supersymmetric Ward identity must imply that the expressions (2.4.5) and (2.4.6) are equal up to Q-exact terms.

Consequently, we have that

$$\left\langle \int d\varphi J^{A_1}(\varphi) \cdots \int d\varphi J^{A_n}(\varphi) \right\rangle = \frac{1}{(4\pi r^2)^n} \frac{1}{Z} \frac{d^n Z}{dm^{A_1} dm^{A_2} \cdots dm^{A_n}} \bigg|_{m=0} . \tag{2.4.8}$$

This is the main result of this subsection.

As a particular case, we can consider $n = 2$. From (2.4.2), we see that on a line in $\mathbb{R}^3$, we have $\langle J^A(x) J^B(0) \rangle = -\frac{\tau}{512\pi^2 r^2} \delta^{AB}$, and so

$$\left\langle \int d\varphi J^A(\varphi) \int d\varphi J^B(\varphi) \right\rangle = -\frac{\tau}{128 r^2} \delta^{AB} . \tag{2.4.9}$$

Comparing to (2.4.8), we deduce

$$\tau = -\frac{8}{\pi^2} \frac{1}{Z} \frac{d^2 Z}{d(rm^A)^2} \bigg|_{m=0} = \frac{8}{\pi^2} \frac{d^2 F_{S^3}}{d(rm^A)^2} \bigg|_{m=0} \tag{2.4.10}$$

(with no summation over A), where we defined the S^3 free energy $F_{S^3} = -\log Z$. This formula agrees with the result of [45]. (For an $\mathcal{N} = 4$ mass-deformed SCFT on S^3, the free energy is real, so one does not have to take its real part as in [45].)

2.4.2 Application to $\mathcal{N} = 8$ SCFTs

In order to apply the above results to $\mathcal{N} = 8$ SCFTs, one would first have to go through the exercise of decomposing the various representations of the $\mathcal{N} = 8$ superconformal algebra into representations of the $\mathcal{N} = 4$ superconformal algebra in order to establish which of the $\mathcal{N} = 8$ irreps contain Higgs branch scalar operators with $\Delta = J_L$ and $J_R = 0$. In the notation introduced in Table 1.1, it can be checked that these irreps are of $(B, 2)$, $(B, 3)$, $(B, +)$, and $(B, -)$ type. So these are the $\mathcal{N} = 8$ multiplets that are captured by the 1d topological sector discussed above. In performing the decomposition from $\mathcal{N} = 8$ to $\mathcal{N} = 4$, one should also keep track of an $\mathfrak{su}(2)_1 \oplus \mathfrak{su}(2)_2$ flavor symmetry that commutes with $\mathfrak{so}(4)_R$ inside $\mathfrak{so}(8)_R$.

that the partition function of the 1d topological theory, defined on a circle, is described by the partition function

$$Z = \int_{\text{Cartan of } \mathfrak{g}} d\sigma \, \det{}_{\text{adj}}(2\sinh(\pi\sigma)) \int DQ \, e^{-2\pi r \int d\varphi \left(\Omega^{ij} Q_i \partial_\varphi Q_j - \sigma^\alpha t^{\alpha ij} Q_i Q_j - r m^A T^{Aij} Q_i Q_j \right)} . \tag{2.4.7}$$

So deforming the 3d theory by a mass parameter m is equivalent to deforming the 1d theory by (2.4.6). For a single free hyper, we have $N = 1$, $\Omega^{ij} = \epsilon^{ij}$ and $T^{Aij} = \frac{1}{2}\sigma^{Aij}$ for the $SU(2)$ flavor symmetry—see Footnote 18.

We are interested in analyzing the 4-point function of the $\mathbf{35}_c = [0020]$ scalar in the same $\mathcal{N}=8$ superconformal multiplet as the stress tensor, so we should only focus on the stress tensor multiplet ($(B,+)_{[0020]}$) as well as the multiplets $(B,+)_{[0040]}$ and $(B,2)_{[0200]}$ (for short referred to as $(B,+)$, and $(B,2)$ above) that appear in the OPE of two stress tensor multiplets. These multiplets contain the following Higgs branch operators (HBOs)

$$
\begin{aligned}
\text{Stress} &\supset \text{HBOs with } \Delta = J_L = 1 \text{ in } (\mathbf{3},\mathbf{1}) \text{ of flavor } \mathfrak{su}(2)_1 \oplus \mathfrak{su}(2)_2\,,\\
(B,+) &\supset \text{HBOs with } \Delta = J_L = 2 \text{ in } (\mathbf{5},\mathbf{1}) \text{ of flavor } \mathfrak{su}(2)_1 \oplus \mathfrak{su}(2)_2\,,\\
(B,2) &\supset \text{HBOs with } \Delta = J_L = 2 \text{ in } (\mathbf{1},\mathbf{1}) \text{ of flavor } \mathfrak{su}(2)_1 \oplus \mathfrak{su}(2)_2\,.
\end{aligned}
\tag{2.4.11}
$$

Thus, from the $\mathcal{N}=4$ perspective, each local $\mathcal{N}=8$ SCFT contains a conserved current multiplet $(J^A_{ab}, K^A_{\dot a\dot b}, j^A_\mu, \chi^A_{a\dot a})$, which transforms in the adjoint of an $\mathfrak{su}(2)_1$ flavor symmetry ($A=1,2,3$ in this case). This multiplet is embedded in the $\mathcal{N}=8$ stress tensor multiplet, with $\mathfrak{su}(2)_1$ embedded into $\mathfrak{so}(8)_R$. Consequently, the coefficient τ appearing in the two-point function (2.4.2) of the canonically normalized currents must be proportional to coefficient c_T appearing in the two-point function (1.1.41) of the canonically normalized stress tensor. In a free $\mathcal{N}=8$ theory we have[20] $c_T = 16$ and $\tau = 2$, so

$$
\tau = \frac{c_T}{8}\,. \tag{2.4.12}
$$

The precise projection to 1d was performed in the previous sections. Converting to the notation in this section, we have

$$
\begin{aligned}
\langle J^A(\varphi_1)J^B(\varphi_2)J^C(\varphi_3)J^D(\varphi_4)\rangle = \frac{\tau^2}{(512\pi^2r^2)^2}\Bigg[&\left(1+\frac{1}{16}\lambda^2_{(B,2)}\right)\delta^{AB}\delta^{CD}\\
&+\frac{1}{4}\,\mathrm{sgn}(\varphi_{12}\varphi_{13}\varphi_{24}\varphi_{34})\lambda^2_{\text{Stress}}(\delta^{AC}\delta^{BD}-\delta^{AD}\delta^{BC})\\
&+\frac{3}{16}\lambda^2_{(B,+)}\left(\delta^{AC}\delta^{BD}+\delta^{AD}\delta^{BC}-\frac{2}{3}\delta^{AB}\delta^{CD}\right)\Bigg]\,.
\end{aligned}
\tag{2.4.13}
$$

Let us now use the results of the previous section in order to extract the OPE coefficients $\lambda^2_{\text{Stress}}$, $\lambda^2_{(B,+)}$, and $\lambda^2_{(B,2)}$. Eq (2.4.12) gives a way to compute $\lambda^2_{\text{Stress}}$. From $\lambda^2_{\text{Stress}} = 256/c_T$ and (2.4.12) we obtain

[20]In $\mathcal{N}=4$ notation, an $\mathcal{N}=8$ free theory is a product between a theory of a free hypermultiplet and a free twisted hypermultiplet. The $\mathfrak{su}(2)_1$ acts on the hypermultiplet only, so τ has the same value as in the free hypermultiplet theory, namely $\tau=2$, as explained in Footnote 18.

$\lambda^2_{\text{Stress}} = 32/\tau$, and from (2.4.10) we further obtain

$$\lambda^2_{\text{Stress}} = -\frac{4\pi^2}{\frac{1}{Z}\frac{d^2Z}{d(rm^A)^2}\Big|_{m=0}}\,, \qquad c_T = -\frac{64}{\pi^2}\frac{1}{Z}\frac{d^2Z}{d(rm^A)^2}\Big|_{m=0}\,. \tag{2.4.14}$$

The other OPE coefficients can be calculated by specializing the four-point function (2.4.13) to $A = B = C = D$ and integrating over φ:

$$\left\langle \left(d\varphi \int J^A(\varphi) \right)^4 \right\rangle = \frac{9}{5}\frac{\tau^2}{(32r^2)^2}\left[1 + \frac{1}{16}\lambda^2_{(B,2)} + \frac{\lambda^2_{\text{Stress}}}{9} \right]. \tag{2.4.15}$$

Comparing with (2.4.8), we obtain

$$\lambda^2_{(B,2)} = 16\left[-1 + \frac{4\pi^2}{9}\frac{1}{\frac{1}{Z}\frac{d^2Z}{d(rm^A)^2}} + \frac{5}{9}\frac{\frac{1}{Z}\frac{d^4Z}{d(rm^A)^4}}{\left(\frac{1}{Z}\frac{d^2Z}{d(rm^A)^2}\right)^2} \right]\Bigg|_{m=0}\,. \tag{2.4.16}$$

From (2.1.2) we can then also obtain $\lambda^2_{(B,+)}$:

$$\lambda^2_{(B,+)} = \frac{16}{9}\left[-\frac{\pi^2}{\frac{1}{Z}\frac{d^2Z}{d(rm^A)^2}} + \frac{\frac{1}{Z}\frac{d^4Z}{d(rm^A)^4}}{\left(\frac{1}{Z}\frac{d^2Z}{d(rm^A)^2}\right)^2} \right]\Bigg|_{m=0}\,. \tag{2.4.17}$$

2.4.3 OPE coefficients in BLG and ABJ(M) theory

Let us now apply the formulas (2.4.14), (2.4.16), and (2.4.17) to the specific examples of ABJ(M) and BLG theories. For simplicity, let us turn on a mass parameter m through the Cartan of $\mathfrak{su}(2)_1$, thus dropping the superscript A from (2.4.14)–(2.4.17). We also set $r = 1$ for simplicity.

The mass-deformed partition function of the $U(N)_k \times U(M)_{-k}$ ABJ(M) theory takes the form

$$\begin{aligned} Z_{\text{ABJ(M)}}(m) = \frac{1}{N!M!}\int d^N\lambda d^M\mu e^{i\pi k[\sum_i \lambda_i^2 - \sum_j \mu_j^2]} \\ \times \frac{\prod_{i<j}\left(4\sinh^2[\pi(\lambda_i-\lambda_j)]\right)\prod_{i<j}\left(4\sinh^2[\pi(\mu_i-\mu_j)]\right)}{\prod_{i,j}\left(4\cosh[\pi(\lambda_i-\mu_j+m/2)]\cosh[\pi(\mu_i-\lambda_j)]\right)}\,, \end{aligned} \tag{2.4.18}$$

For BLG theory, we take $N = M = 2$ in the formula above and insert $\delta(\lambda_1+\lambda_2)\delta(\mu_1+\mu_2)$ in the integrand, thus obtaining

$$Z_{\text{BLG}}(m) = \frac{1}{32}\int d\lambda d\mu e^{2\pi ik[\lambda^2-\mu^2]}\frac{\sinh^2(2\pi\lambda)\sinh^2(2\pi\mu)}{\prod_{i,j}\cosh[\pi(\lambda_i-\mu_j+m/2)]\cosh[\pi(\mu_i-\lambda_j)]}\,, \tag{2.4.19}$$

where now $\lambda_i = (\lambda, -\lambda)$ and $\mu_i = (\mu, -\mu)$.

BLG theory

For BLG theory, one can use the identity

$$\frac{\sinh(2\pi\lambda)\sinh(2\pi\mu)}{4\prod_{i,j}\cosh[\pi(\lambda_i-\mu_j+m/2)]} = \det\begin{pmatrix} \frac{1}{2\cosh[\pi(\lambda-\mu+m/2)]} & \frac{1}{2\cosh[\pi(\lambda+\mu+m/2)]} \\ \frac{1}{2\cosh[\pi(-\lambda-\mu+m/2)]} & \frac{1}{2\cosh[\pi(-\lambda+\mu+m/2)]} \end{pmatrix} \tag{2.4.20}$$

and change variables to $x = (\lambda+\mu)/2$ and $y = (\lambda-\mu)/2$, to show that

$$Z_{\text{BLG}}(m) = \frac{k}{32}\int dx\, \frac{x}{\sinh(\pi k x)}\left[\operatorname{sech}^2\left(\frac{m\pi}{2}\right) - \operatorname{sech}\left(\frac{m\pi}{2}-x\right)\operatorname{sech}\left(\frac{m\pi}{2}+x\right)\right]. \tag{2.4.21}$$

One can then plug this expression into (2.4.14)–(2.4.17), which gives

$$\begin{aligned} \lambda^2_{\text{Stress}} &= \frac{8I_{2,k}}{2I_{2,k}-I_{4,k}}\,, \qquad c_T = 32\left(2-\frac{I_{4,k}}{I_{2,k}}\right)\,,\\ \lambda^2_{(B,2)} &= \frac{16\left(6I_{2,k}^2 - 3I_{4,k}^2 - 12 I_{2,k}I_{4,k} + 10 I_{2,k}I_{6,k}\right)}{3(I_{4,k}-2I_{2,k})^2}\,,\\ \lambda^2_{(B,+)} &= \frac{32 I_{2,k}\left(3I_{2,k}-3I_{4,k}+I_{6,k}\right)}{3(I_{4,k}-2I_{2,k})^2}\,, \end{aligned} \tag{2.4.22}$$

where we defined the integral

$$I_{n,k} \equiv \int_{-\infty}^{\infty} dx\, \frac{x}{\sinh(\pi k x)}\tanh^n(\pi x)\,. \tag{2.4.23}$$

This integral can be evaluated explicitly using contour integration. For $n = 2, 4, 6$ the expressions are

$$\begin{aligned} I_{2,k} &= \begin{cases} \frac{(-1)^{\frac{k-1}{2}}}{\pi} + \sum\limits_{s=1}^{k-1}(-1)^{s+1}\frac{k-2s}{2k^2}\tan\left[\frac{\pi s}{k}\right]^2 & \text{if } k \text{ is odd}\,,\\ -\frac{(-1)^{\frac{k}{2}}}{\pi^2 k} + \sum\limits_{\substack{s=1\\ s\neq k/2}}^{k-1}(-1)^{s+1}\frac{(k-2s)^2}{4k^3}\tan\left[\frac{\pi s}{k}\right]^2 & \text{if } k \text{ is even}\,, \end{cases}\\ I_{4,k} &= \begin{cases} \frac{(-1)^{\frac{k+1}{2}}(3k^2-8)}{6\pi} + \sum\limits_{s=1}^{k-1}(-1)^{s}\frac{k-2s}{2k^2}\tan\left[\frac{\pi s}{k}\right]^4 & \text{if } k \text{ is odd}\,,\\ \frac{(-1)^{\frac{k}{2}}(k^2-8)}{6\pi^2 k} + \sum\limits_{\substack{s=1\\ s\neq k/2}}^{k-1}(-1)^{s}\frac{(k-2s)^2}{4k^3}\tan\left[\frac{\pi s}{k}\right]^4 & \text{if } k \text{ is even}\,, \end{cases}\\ I_{6,k} &= \begin{cases} \frac{(-1)^{\frac{k-1}{2}}(184-120k^2+25k^4)}{120\pi} + \sum\limits_{s=1}^{k-1}(-1)^{s+1}\frac{k-2s}{2k^2}\tan\left[\frac{\pi s}{k}\right]^6 & \text{if } k \text{ is odd}\,,\\ -\frac{(-1)^{\frac{k}{2}}(552-120k^2+7k^4)}{360\pi^2 k} + \sum\limits_{\substack{s=1\\ s\neq k/2}}^{k-1}(-1)^{s+1}\frac{(k-2s)^2}{4k^3}\tan\left[\frac{\pi s}{k}\right]^6 & \text{if } k \text{ is even}\,. \end{cases} \end{aligned} \tag{2.4.24}$$

ABJ(M) theory

For ABJ(M) theory, one can use (2.4.18) and (2.4.14)–(2.4.17) to evaluate $\lambda^2_{\text{Stress}}$, $\lambda^2_{(B,+)}$, and $\lambda^2_{(B,2)}$. The number of integrals increases with N, however, and unlike the BLG case where analytical formulas were possible for the entire family of theories, in the ABJ(M) case we can perform these integrals analytically only for small values of N—See Table 2.2.

	$\frac{\lambda^2_{\text{Stress}}}{16} = \frac{16}{c_T}$	$\lambda^2_{(B,+)}$	$\lambda^2_{(B,2)}$
$\text{ABJM}_{1,k}$	1	16	0
$\text{ABJM}^{\text{int}}_{2,1} \cong \text{ABJ}_1$	$\frac{3}{4} \approx 0.75$	$\frac{64}{5} \approx 12.8$	0
$\text{BLG}_1 \cong \text{ABJM}_{2,1}$	$\frac{3}{7} \approx 0.429$	$\frac{384}{245} \approx 9.731$	$\frac{256}{49} \approx 5.224$
$\text{ABJM}_{2,2} \cong \text{BLG}_2 \cong \text{ABJ}_1^2$	$\frac{3}{8} \approx 0.375$	$\frac{136}{15} \approx 9.067$	$\frac{16}{3} \approx 5.333$
$\text{BLG}_3 \cong \text{ABJM}^{\text{int}}_{3,1}$	$\frac{\pi-3}{10\pi-31} \approx 0.340$	$\frac{16(\pi-3)(840\pi-2629)}{15(10\pi-31)^2} \approx 8.676$	$\frac{62208+16\pi(420\pi-2557)}{3(10\pi-31)^2} \approx 5.593$
$\text{BLG}_4 \cong \text{ABJ}_2$	$\frac{3\pi^2-24}{18\pi^2-160} \approx 0.318$	$\frac{32(\pi^2-8)(315\pi^2-2944)}{15(80-9\pi^2)^2} \approx 8.444$	$\frac{16(16384-3872\pi^2+225\pi^4)}{3(80-9\pi^2)^2} \approx 5.883$
BLG_5	0.302	8.300	6.156
$\vdots$	$\vdots$	$\vdots$	$\vdots$
BLG_∞	$\frac{1}{4} \approx 0.25$	8	8
ABJ(M)_∞	0	$\frac{16}{3} \approx 5.333$	$\frac{32}{3} \approx 10.666$

Table 2.2: OPE coefficients of $\frac{1}{2}$ and $\frac{1}{4}$ BPS operators that appear in $\mathcal{O}_{\text{Stress}} \times \mathcal{O}_{\text{Stress}}$ for $\mathcal{N} = 8$ theories. "$\cong$" denotes that theories have the same stress tensor four-point function.

One can also perform a $1/N$ expansion, where $M = N$ or $M = N + 1$. There are several approaches for developing a $1/N$ expansion: one can either work more generally in the 't Hooft limit where N is taken to be large while N/k is held fixed [46], and then take N/k large; or one can work at fixed k while taking N large [47, 48]. We will follow the approach originating in [48], where for $m = 0$ it was noticed in [48] that the S^3 partition function for ABJM theory can be rewritten as a partition function of a non-interacting Fermi gas of N particles with kinetic energy $T(p) = \log\cosh(\pi p)$ and potential energy $U(q) = \log\cosh(\pi q)$. Phase space quantization and statistical physics techniques allow one to calculate the S^3 partition function to all orders in $1/N$, and this expansion resums into an Airy function. The S^3 partition function in the presence of a mass deformation was computed using the same method in [41].

Up to non-perturbative corrections in $1/N$ and an overall m-independent prefactor, the result of [41] gives[21,22]

$$
\begin{aligned}
&Z(m) \approx e^{A} C^{-\frac{1}{3}} \operatorname{Ai}\left[C^{-\frac{1}{3}}(N-B)\right] , \\
&C=\frac{2}{\pi^2 k(1+m^2)} , \qquad B=\frac{\pi^2 C}{3}-\frac{2+m^2}{6k(1+m^2)}-\frac{k}{12}+\frac{k}{2}\left(\frac{1}{2}-\frac{M-N}{k}\right)^2 , \\
&A=\frac{1}{4}\left(\mathcal{A}[k(1+im)]+\mathcal{A}[k(1-im)]+2\mathcal{A}[k]\right) ,
\end{aligned}
\tag{2.4.25}
$$

where $k>0$, $M \geq N$, and the function $\mathcal{A}$ is given by

$$
\mathcal{A}(k)=\frac{2\zeta(3)}{\pi^2 k}\left(1-\frac{k^3}{16}\right)+\frac{k^2}{\pi^2}\int_0^\infty dx \frac{x}{e^{kx}-1}\log\left(1-e^{-2x}\right) . \tag{2.4.26}
$$

In order to plug this expression into (2.4.14)–(2.4.17), one needs the following derivatives of $\mathcal{A}$:

$$
\begin{aligned}
\mathcal{A}''(1)&=\frac{1}{6}+\frac{\pi^2}{32} , &\qquad \mathcal{A}''(2)&=\frac{1}{24} , \\
\mathcal{A}''''(1)&=1+\frac{4\pi^2}{5}-\frac{\pi^4}{32} , &\qquad \mathcal{A}''''(2)&=\frac{1}{16}+\frac{\pi^2}{80} .
\end{aligned}
\tag{2.4.27}
$$

Using (2.4.14), we then find

$$
\begin{aligned}
c_T^{\mathrm{ABJM}_{N,1}}&=1-\frac{112}{3\pi^2}-\frac{8(9+8N)\operatorname{Ai}'\left[(N-3/8)\left(\pi^2/2\right)^{1/3}\right]}{3(\pi^2/2)^{2/3}\operatorname{Ai}\left[(N-3/8)\left(\pi^2/2\right)^{1/3}\right]} , \\
c_T^{\mathrm{ABJM}_{N,2}}&=-\frac{112}{3\pi^2}-\frac{64(1+2N)\pi^{2/3}\operatorname{Ai}'\left[(N-1/4)\,\pi^{2/3}\right]}{3\pi^2\operatorname{Ai}\left[(N-1/4)\,\pi^{2/3}\right]} , \\
c_T^{\mathrm{ABJ}_N}&=-\frac{112}{3\pi^2}-\frac{32(3+4N)\pi^{2/3}\operatorname{Ai}'\left[N\pi^{2/3}\right]}{3\pi^2\operatorname{Ai}\left[N\pi^{2/3}\right]} ,
\end{aligned}
\tag{2.4.28}
$$

[21] The result of [41] is only for $N=M$, but here we generalize it to $N \neq M$ using the results of [49].

[22] In the notation of [41], we can take $\zeta_1 = im$ and $\zeta_2=0$, or equivalently $\xi=-im$ and $\eta=0$.

which are expressions valid to all orders in $1/N$. The analogous expressions for $\lambda^2_{(B,2)}$ are

$$\begin{aligned}
\left(\lambda_{(B,2)}^{\mathrm{ABJM}_{N,1}}\right)^2 &= \left(\frac{32}{3}\right)\left((112+45\pi^2)\mathrm{Ai}[(N-3/8)(\pi^2/2)^{1/3}]+8(9+8\pi)(2\pi)^{2/3}\mathrm{Ai}'[(N-3/8)(\pi^2/2)^{1/3}]\right)^{-2} \\
&\times\Big[(-94976+8(3373+1080N+4800N^2+2560N^3)\pi^2+3465\pi^4)\mathrm{Ai}\left[(N-3/8)(\pi^2/2)^{1/3}\right]^2 \\
&\quad +16(2\pi)^{2/3}(-5712-1664N+(981+872N)\pi^2)\mathrm{Ai}\left[(N-3/8)(\pi^2/2)^{1/3}\right]\mathrm{Ai}'\left[(N-3/8)(\pi^2/2)^{1/3}\right] \\
&\quad -192(9+8N)^2(2\pi^2)^{1/3}\mathrm{Ai}'\left[(N-3/8)(\pi^2/2)^{1/3}\right]^2\Big]\,, \\
\left(\lambda_{(B,2)}^{\mathrm{ABJM}_{N,2}}\right)^2 &= \left(\frac{32}{3}\right)\left(7\mathrm{Ai}[(N-1/4)\pi^{2/3}]+4(1+2N)\pi^{2/3}\mathrm{Ai}'[(N-1/4)\pi^{2/3}]\right)^{-2} \\
&\times\Big[(-371+(58+120N^2+160N^3)\pi^2)\mathrm{Ai}\left[(N-1/4)\pi^{2/3}\right]^2 \\
&\quad +8\pi^{2/3}(-43-26N+(4+8N)\pi^2)\mathrm{Ai}\left[(N-1/4)\pi^{2/3}\right]\mathrm{Ai}'\left[(N-1/4)\pi^{2/3}\right] \\
&\quad -24(1+2N)^2\pi^{4/3}\mathrm{Ai}'\left[(N-1/4)\pi^{2/3}\right]^2\Big]\,, \\
\left(\lambda_{(B,2)}^{\mathrm{ABJ}_N}\right)^2 &= \left(\frac{32}{3}\right)\left(7\mathrm{Ai}[N\pi^{2/3}]+2(3+4N)\pi^{2/3}\mathrm{Ai}'[N\pi^{2/3}]\right)^{-2} \\
&\times\Big[(-371+(68+90N+240N^2+160N^3)\pi^2)\mathrm{Ai}\left[N\pi^{2/3}\right]^2 \\
&\quad +4\pi^{2/3}(-99-52N+4(3+4N)\pi^2)\mathrm{Ai}\left[N\pi^{2/3}\right]\mathrm{Ai}'\left[N\pi^{2/3}\right]-6(3+4N)^2\pi^{4/3}\mathrm{Ai}'\left[N\pi^{2/3}\right]^2\Big]\,.
\end{aligned} \tag{2.4.29}$$

The formulas for $\lambda^2_{(B,+)}$ can then be determined from (2.1.2).

A comment is in order for the $\mathrm{ABJM}_{N,1}$ theory. This theory is a direct product between a free sector, identified with $\mathrm{ABJM}_{1,1}$, and an interacting sector. Since the free sector has $c_T = 16$, we have that the interacting sector of the $\mathrm{ABJM}_{N,1}$ theory has

$$c_{T,\mathrm{int}}^{\mathrm{ABJM}_{N,1}} = c_T^{\mathrm{ABJM}_{N,1}} - 16\,. \tag{2.4.30}$$

Extracting the value of $\lambda^2_{(B,2)}$ of just the interacting sector knowing $\lambda^2_{(B,2)}$ for the full theory requires more thought. In the free theory one has $\lambda^2_{(B,2)} = 0$, as such a multiplet does not exist as can be checked explicitly by decomposing the 4-point function of $\mathcal{O}_{\mathrm{Stress}}$ in superconformal blocks. Using this fact, and the general formulas for how squared OPE coefficients combine when taking product CFTs, which we will discuss later in Section 4.4.4, one has

$$\left(\lambda_{(B,2)}^{\mathrm{ABJM}_{N,1}}\right)^2=\frac{2c_{T,\mathrm{free}}^{\mathrm{ABJM}_{N,1}}c_{T,\mathrm{int}}^{\mathrm{ABJM}_{N,1}}\lambda^2_{(B,2),\mathrm{GFFT}}+\left(c_{T,\mathrm{int}}^{\mathrm{ABJM}_{N,1}}\right)^2\left(\lambda_{(B,2),\mathrm{int}}^{\mathrm{ABJM}_{N,1}}\right)^2}{\left(c_T^{\mathrm{ABJM}_{N,1}}\right)^2}\,. \tag{2.4.31}$$

where $\lambda^2_{(B,2),\mathrm{GFFT}}=32/3$ is the generalized free theory value of the $\lambda^2_{(B,2)}$ OPE coefficient as given in (1.4).

In Table 2.3 we compare $\lambda^2_{(B,2)}$, $\lambda^2_{(B,+)}$ and c_T to the exact values for $N=2$ and find excellent agreement.

	$\frac{\lambda^2_{\mathrm{Stress}}}{16}=\frac{16}{c_T}$	$\lambda^2_{(B,+)}$	$\lambda^2_{(B,2)}$
large N $\mathrm{ABJM}^{\mathrm{int}}_{2,1}$	0.7500	12.7982	−0.0100
exact $\mathrm{ABJM}^{\mathrm{int}}_{2,1}$	$\frac{3}{4}\approx 0.75$	$\frac{64}{5}\approx 12.800$	0
large N $\mathrm{ABJM}_{2,2}$	0.3759	9.0513	5.1995
exact $\mathrm{ABJM}_{2,2}$	$\frac{3}{8}\approx 0.375$	$\frac{136}{15}\approx 9.0667$	$\frac{16}{3}\approx 5.3333$
large N ABJ_2	0.3173	8.4533	5.9618
exact ABJ_2	$\frac{3\pi^2-24}{18\pi^2-160}\approx 0.3177$	$\frac{32(\pi^2-8)(315\pi^2-2944)}{15(80-9\pi^2)^2}\approx 8.4436$	$\frac{16\left(16384-3872\pi^2+225\pi^4\right)}{3(80-9\pi^2)^2}\approx 5.8831$

Table 2.3: Comparison of the large N formulae to the exact values for $N=2$ for OPE coefficients of $\frac{1}{2}$ and $\frac{1}{4}$ BPS operators that appear in $\mathcal{O}_{\mathrm{Stress}}\times\mathcal{O}_{\mathrm{Stress}}$ for the interacting sector of ABJ(M). The excellent agreement shows that the asymptotic formulas are reliable for all $N\geq 2$.

We can also expand $\lambda^2_{(B,2)}$ and $\lambda^2_{(B,+)}$ directly in terms of c_T, by comparing the large N expansions. When expanding the Airy functions in (2.4.28) and (2.4.29) for large N, one should be careful to expand in terms of the entire argument of the Airy function, and not just N, which is how these functions were originally defined in [48]. We find that the large c_T expansion is the same for ABJ_N and $\mathrm{ABJM}_{N,2}$, while the expansions for $\mathrm{ABJM}_{N,k}$ are

$$\begin{aligned}\left(\lambda_{(B,2)}^{\mathrm{ABJM}_{N,k}}\right)^2&=\frac{32}{3}-\frac{1024(4\pi^2-15)}{9\pi^2}\frac{1}{c_T^{\mathrm{ABJM}_{N,k}}}+\frac{40960}{\pi^{\frac{8}{3}}}\left(\frac{2}{9k^2}\right)^{\frac{1}{3}}\frac{1}{\left(c_T^{\mathrm{ABJM}_{N,k}}\right)^{\frac{5}{3}}}+O\left(\left(c_T^{\mathrm{ABJM}_{N,k}}\right)^{-2}\right)\,,\\ \left(\lambda_{(B,+)}^{\mathrm{ABJM}_{N,k}}\right)^2&=\frac{16}{3}+\frac{1024(\pi^2+3)}{9\pi^2}\frac{1}{c_T^{\mathrm{ABJM}_{N,k}}}+\frac{8192}{\pi^{\frac{8}{3}}}\left(\frac{2}{9k^2}\right)^{\frac{1}{3}}\frac{1}{\left(c_T^{\mathrm{ABJM}_{N,k}}\right)^{\frac{5}{3}}}+O\left(\left(c_T^{\mathrm{ABJM}_{N,k}}\right)^{-2}\right)\,,\end{aligned} \tag{2.4.32}$$

where note that the leading order correction is independent of k.

2.5 Discussion

In this Chaper, we have studied a certain truncation [36] of the operator algebra of three-dimensional $\mathcal{N}=4$ SCFTs obtained by restricting the spectrum of operators to those that are nontrivial in the cohomology of a certain supercharge $\mathcal{Q}$. The local operators that represent non-trivial cohomology classes are certain $\frac{1}{2}$-BPS operators that are restricted to lie on a line, and whose correlation functions define a topological quantum mechanics. More specifically, these $\frac{1}{2}$-BPS operators are superconformal primaries that are charged only under one of the $\mathfrak{su}(2)$ factors in the $\mathfrak{so}(4)_R \cong \mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$ R-symmetry. These are precisely the operators that contribute to the Higgs (or Coulomb) limits of the superconformal index [50]. What is special about the truncation we study is that the correlation functions in the 1d theory are very easy to compute and are in general non-vanishing. In particular, the crossing symmetry constraints imposed on these correlation functions can be solved analytically and may lead to non-trivial constraints on the full 3d $\mathcal{N} = 4$ theory.

We worked out explicitly some of these constraints in the particular case of $\mathcal{N} = 8$ SCFTs. These $\mathcal{N} = 8$ SCFTs can be viewed as $\mathcal{N} = 4$ SCFTs with $\mathfrak{so}(4)$ flavor symmetry. One of our main results is the relation (2.1.2) between the three OPE coefficients λ_{Stress}, $\lambda_{(B,+)}$, and $\lambda_{(B,2)}$ that appear in the OPE of the $\mathcal{N} = 8$ stress-tensor multiplet with itself. Since every local $\mathcal{N} = 8$ SCFT has a stress-tensor multiplet, the relation (2.1.2) is universally applicable to all local $\mathcal{N} = 8$ SCFTs! As explained in Section 2.3.2, this relation is only a particular case of more general relations that also apply to all local $\mathcal{N} = 8$ SCFTs and that can be easily derived using the same technique.

The main application of the 1d theory was to compute the OPE coefficients of protected operators that appear in the stress tensor four point function. For the BLG theories, we obtained exact expressions, while for the ABJ(M) theories we obtained all orders in $1/N$ results. It would be useful to include the non-perturbative corrections in $1/N$ to the results presented in this Chapter. These non-perturbative corrections have already been calculated for the S^3 free energy in many cases [41, 46, 48, 51–60], but to extract the OPE coefficients from these results we would need an expression for these corrections as a smooth function of m. With these non-perturbative corrections included, we would have exact values of these OPE coefficients also for $\mathcal{N} = 6$ ABJ(M) theories with gauge group $U(N)_k \times U(M)_{-k}$, so that we could see how these quantities interpolate between the $N, M \to \infty$ and fixed k M-theory limit, the $N, M, k \to \infty$ and fixed N/k Type IIA string theory limit, and the $N, k \to \infty$ and fixed M higher spin theory [61, 62] limit.

Lastly, it would also be interesting to calculate more BPS OPE coefficients in ABJ(M) theory in a large N expansion using the Fermi gas approach [48]. For half and quarter-BPS operators that appear in n-point functions of the stress tensor, this could be done by taking more derivatives of the free energy as a function of the mass parameter m. For instance, three new OPE coefficients, $\lambda^{(B,2)}_{(B,2),(B,2)}$, $\lambda^{(B,+)}_{(B,+),(B,+)}$, and $\lambda^{(B,2)}_{(B,+),(B,+)}$,[23] appear in the 6-point function, and crossing of the projection of this 6-point function to the 1d theory supplies two new constraints.[24] Thus, by taking 6 derivatives of the mass deformed S^3 partition

[23] $\lambda^{\text{Stress}}_{(B,+),(B,+)}$ also appears, but this OPE coefficient is related to c_T.

[24] We thank Ran Yacoby for pointing this out to us.

function we can compute the integrated 6-point function in the 1d theory, and thereby determine all these OPE coefficients. For BPS operators that do not appear in any n-point functions of the stress tensor, such as operators in the $[00a0]$ irrep for odd a, we could still express their OPE coefficients as matrix integrals using the 1d methods of [42, 63, 64]. These matrix integrals could then be computed as expectation values of n-body operators in the Fermi gas, along the lines of [65].

Chapter 3

A New Duality Between ABJM and BLG

This chapter is an edited version of ref. [63], which was written in collaboration with Nathan B. Agmon and Silviu S. Pufu.

3.1 Introduction and Summary

In Section 1.1.4 of the Introduction to this thesis, we described the known families of $\mathcal{N} = 8$ theories: $\text{ABJM}_{N,1}$, $\text{ABJM}_{N,2}$, ABJ_N and BLG_k. We described how the ABJ(M) type theories have holographic interpretations in terms of N M2-branes in M-theory, while the BLG_k theories have no known M-theory interpretation except for the cases $k = 1, 2, 4$, in which case they are dual to $\text{ABJM}_{2,1}$, $\text{ABJM}_{2,2}$, and ABJ_2. In this Chapter, we present yet another duality between the ABJ(M) and BLG theories that is not included in the list above. It is:

$$\boxed{\text{BLG}_3 \otimes \text{ABJM}_{1,1} \quad \cong \quad \text{ABJM}_{3,1}\ .} \tag{3.1.1}$$

Recalling that the $\text{ABJM}_{3,1}$ theory has a decoupled free sector isomorphic to $\text{ABJM}_{1,1}$ theory as well as an interacting one, this duality can be rephrased as

$$\boxed{\text{BLG}_3 \quad \cong \quad \text{interacting sector of } \text{ABJM}_{3,1}\ .} \tag{3.1.2}$$

Thus, our new duality (3.1.1)–(3.1.2) provides an interpretation for the BLG theory at level $k = 3$: it is the interacting sector of the theory on three coincident M2-branes. Quite curiously, this duality casts the $k = 3$ BLG theory as a theory on three coincident M2-branes, unlike the original intuition that BLG theories should be related to theories on two M2-branes.

It is worth mentioning that the $\mathcal{N} = 8$ SCFTs mentioned above may have other descriptions that are not two-node gauge quivers. An important example is that the $\text{ABJM}_{N,1}$ theory has the same IR fixed point as an $\mathcal{N} = 4$ $U(N)$ gauge theory with a fundamental hypermultiplet and an adjoint hypermultiplet [15,16]. In fact, it is this latter description that we will use in some of our computations in the $\text{ABJM}_{3,1}$ theory that we perform in order to check (3.1.1)–(3.1.2).

As was checked in previous dualities, for our proposed duality we match the moduli spaces and superconformal indices on each side of the duality. We also match the values of the S^3 partition functions of the two theories. In addition, we provide a new check using the recently proposed supersymmetric localization of 3d $\mathcal{N} = 4$ theories to a topological 1d sector [42]. Using this method, we calculate the two- and three-point functions of low-lying half and quarter BPS operators, which we use to extract their OPE coefficients, listed in (3.5.32), (3.5.33), and (3.5.36).

The Chapter is organized as follows. In Section 3.2 we demonstrate the explicit operator matching for low-lying BPS operators, including matching the superconformal index. In Section 3.3 we match the moduli spaces. In Section 3.4 we compute and match the values of the S^3 partition functions. In Section 3.5 we study certain 1d topological sectors of each theory, and extract the OPE coefficients of low-lying BPS operators.

3.2 Operator Spectrum and the Superconformal Index

In Section 1.1.4, we reviewed the ABJ(M) and BLG theories, and sketched how operators in these theories can be built from fields in the Lagrangian as well as monopole operators. In this section will study the cases of $\text{ABJM}_{3,1}$ and BLG_3 in more detail, and how to precisely match the low-lying spectra of operators for the dual theories. We will then show how the superconformal index of each theory is the same.

3.2.1 Low-lying BPS Operator Spectrum

ABJM$_{3,1}$

For the ABJM$_{3,1}$ theory, the monopole operators M^{n_1,n_2,n_3} transform under the $U(3)\times U(3)$ gauge group as

$$U(3)\times U(3)\text{ irrep:}\quad \left(\Upsilon_\nu, \overline{\Upsilon}_{-\nu}\right)\,, \quad \Upsilon = \begin{array}{l} \overbrace{\square\square\square}^{n_1-n_3} \\ \underbrace{\square\square}_{n_2-n_3} \end{array}\,, \quad \nu = \sum_r n_r\,, \tag{3.2.1}$$

where we have denoted a $U(3)$ irrep by Υ_ν, where Υ is an $SU(3)$ Young diagram and ν is the charge under the diagonal $U(1)$, normalized such that the fundamental of $U(3)$ is $\square_1$. In (3.2.1), $\overline{\Upsilon}$ denotes the conjugate tableau to Υ. In particular, we can write M^{n_1,n_2,n_3} more explicitly as a symmetric traceless tensor with n_1-n_2 fundamental and n_2-n_3 anti-fundamental indices under the first gauge group, and n_1-n_2 anti-fundamental and n_2-n_3 fundamental indices under the second gauge group. Using a notation in which $U(3)$ fundamental indices are upper and anti-fundamental indices are lower, this is $(M^{n_1,n_2,n_3})^{\alpha_1\ldots\alpha_{n_1-n_2}\dot\beta_1\ldots\dot\beta_{n_2-n_3}}_{\beta_1\ldots\beta_{n_2-n_3}\dot\alpha_1\ldots\dot\alpha_{n_1-n_2}}$.

We can construct gauge invariant BPS states with nonzero Q_T by dressing M^{n_1,n_2,n_3} with appropriate products of $C^I = (A_1, A_2, B_1^\dagger, B_2^\dagger)$ and $C_I^\dagger$, where upper/lower $I = 1,2,3,4$ is a fundamental/anti-fundamental index for $SU(4)_R$. In the notation above, the C^I transform in the gauge irrep $(\Upsilon_\nu, \overline{\Upsilon}_{-\nu})$, with $\Upsilon = \square$ and $\nu = 1$. Including explicit gauge indices, we would write $(C^I)^\alpha_{\dot\alpha}$ and $(C_I^\dagger)^{\dot\alpha}_\alpha$.

Using a single matter field, we find that $C_I^\dagger M^{1,0,0}$ and $C^I M^{0,0,-1}$ (with the gauge indices contracted in the only possible way, namely $(C_I^\dagger)^{\dot\alpha}_\alpha (M^{1,0,0})^\alpha_{\dot\alpha}$ and $(C^I)^\alpha_{\dot\alpha}(M^{0,0,-1})^{\dot\alpha}_\alpha$) are the only gauge-invariant combinations. They transform under $SU(4)_R\times U(1)_T$ as $\overline{\mathbf{4}}_{-\frac{1}{2}}$ and $\mathbf{4}_{-\frac{1}{2}}$, respectively. These operators have scaling dimension $\Delta = \frac{1}{2}$, and are thus free. They are part of the free sector of ABJM theory, which also contains all operators that appear in the OPE of $C_I^\dagger M^{1,0,0}$ and $C^I M^{0,0,-1}$. The lowest few scalar operators in this free sector are given schematically in Table 3.1.[1] The hallmark of the free sector is the $OSp(8|4)$ irrep $(B,+)^{[0010]}$ whose scalar operators were mentioned above. Another feature is the presence of a stress tensor multiplet $(B,+)^{[0020]}$.

The interacting sector, whose lowest few scalar operators are given schematically in Table 3.2, consists of all operators that decouple from the free sector. For instance, the first operator in Table 3.2, $C^\dagger_{(I}C^\dagger_{J)}M^{1,1,0}$,

[1]The relegation of operators to the free, mixed, and interacting sectors is schematic, as there may be mixing between operators in the same representations.

$\mathcal{O}$	Δ	$SU(4)_R \times U(1)_T$	$OSp(8\|4)$ irrep
$C_I^\dagger M^{1,0,0}$	$\frac{1}{2}$	$\overline{\mathbf{4}}_{-\frac{1}{2}}$	$(B,+)^{[0010]}$
$C^I M^{0,0,-1}$	$\frac{1}{2}$	$\mathbf{4}_{\frac{1}{2}}$	
$C_{(I}^\dagger C_{J)}^\dagger M^{2,0,0}$	1	$\overline{\mathbf{10}}_{-1}$	$(B,+)^{[0020]}$
$C^{(I}C^{J)} M^{0,0,-2}$	1	$\mathbf{10}_1$	
$C_I^\dagger C^J M^{1,0,-1}$	1	$\mathbf{15}_0$	
$C_{(I}^\dagger C_J^\dagger C_{K)}^\dagger M^{3,0,0}$	$\frac{3}{2}$	$\mathbf{20}''_{-\frac{3}{2}}$	$(B,+)^{[0030]}$
$C^{(I}C^J C^{K)} M^{0,0,-3}$	$\frac{3}{2}$	$\overline{\mathbf{20}}''_{\frac{3}{2}}$	
$C^I C_{(J}^\dagger C_{K)}^\dagger M^{2,0,-1}$	$\frac{3}{2}$	$\overline{\mathbf{36}}_{-\frac{1}{2}}$	
$C_I^\dagger C^{(J}C^{K)} M^{1,0,-2}$	$\frac{3}{2}$	$\mathbf{36}_{\frac{1}{2}}$	

Table 3.1: BPS operators with $\Delta \leq \frac{3}{2}$ in the free sector of the $\mathrm{ABJM}_{3,1}$ theory.

can be written more explicitly as:

$$\epsilon^{\alpha\beta\gamma}\epsilon_{\dot\alpha\dot\beta\dot\gamma} C^{\dagger\;[\dot\alpha}_{(I\,[\alpha} C^{\dagger\;\dot\beta]}_{J)\,\beta]} M^{\dot\gamma}_{\gamma} \,. \tag{3.2.2}$$

Note that the flavor indices are symmetrized, because the gauge indices for both gauge groups are simultaneously anti-symmetrized, and thus this operator transforms in the $\overline{\mathbf{10}}$ of $SU(4)_R$ and has $U(1)_T$ charge -1. Also note the presence of another stress tensor multiplet $(B,+)^{[0020]}$, which is different from the one appearing in the free sector. Thus, this ABJM theory has two $\mathcal{N}=8$ stress tensor multiplets, each corresponding to a decoupled sector.

$\mathcal{O}$	Δ	$SU(4)_R \times U(1)_T$	$OSp(8\|4)$ irrep
$C_{(I}^\dagger C_{J)}^\dagger M^{1,1,0}$	1	$\overline{\mathbf{10}}_{-1}$	$(B,+)^{[0020]}$
$C^{(I}C^{J)} M^{0,-1,-1}$	1	$\mathbf{10}_1$	
$C_I^\dagger C^J$	1	$\mathbf{15}_0$	
$C_{(I}^\dagger C_J^\dagger C_{K)}^\dagger M^{1,1,1}$	$\frac{3}{2}$	$\mathbf{20}''_{-\frac{3}{2}}$	$(B,+)^{[0030]}$
$C^{(I}C^J C^{K)} M^{-1,-1,-1}$	$\frac{3}{2}$	$\overline{\mathbf{20}}''_{\frac{3}{2}}$	
$C^I C_{(J}^\dagger C_{K)}^\dagger M^{1,0,0}$	$\frac{3}{2}$	$\overline{\mathbf{36}}_{-\frac{1}{2}}$	
$C_I^\dagger C^{(J}C^{K)} M^{0,0,-1}$	$\frac{3}{2}$	$\mathbf{36}_{\frac{1}{2}}$	

Table 3.2: BPS operators with $\Delta \leq \frac{3}{2}$ in the interacting sector of the $\mathrm{ABJM}_{3,1}$ theory.

Lastly, there is a mixed sector whose lowest few scalar operators are given in Table 3.3,[2] which consists of all operators built using both free and interacting sector operators. Note that there are no free or stress tensor multiplets in the mixed sector, as expected, but there are now both $(B,+)$ and $(B,2)$ operators with dimension $\frac{3}{2}$.

[2]The appearance of $C^I C^\dagger_{(J} C^\dagger_{K)} M^{1,0,0}$ in both the mixed and interacting sector is because there are two singlets in the product $\mathbf{3}\otimes\mathbf{3}\otimes\bar{\mathbf{3}}\otimes\bar{\mathbf{3}}$ of gauge irreps, and thus two inequivalent ways of contracting the gauge indices.

$\mathcal{O}$	Δ	$SU(4)_R \times U(1)_T$	$OSp(8\|4)$ irrep
$C^\dagger_{(I} C^\dagger_J C^\dagger_{K)} M^{2,1,0}$	$\frac{3}{2}$	$\mathbf{20}''_{-\frac{3}{2}}$	$(B,+)^{[0030]}$
$C^{(I} C^J C^{K)} M^{0,-1,-2}$	$\frac{3}{2}$	$\overline{\mathbf{20}}''_{\frac{3}{2}}$	
$C^I C^\dagger_{(J} C^\dagger_{K)} M^{1,1,-1}$	$\frac{3}{2}$	$\overline{\mathbf{36}}_{-\frac{1}{2}}$	
$C^\dagger_I C^{(J} C^{K)} M^{1,-1,-1}$	$\frac{3}{2}$	$\mathbf{36}_{\frac{1}{2}}$	
$C^\dagger_{(I} C^\dagger_{[J)} C^\dagger_{K]} M^{2,1,0}$	$\frac{3}{2}$	$\mathbf{20}_{-\frac{3}{2}}$	$(B,2)^{[0110]}$
$C^{(I} C^{[J)} C^{K]} M^{0,-1,-2}$	$\frac{3}{2}$	$\overline{\mathbf{20}}_{\frac{3}{2}}$	
$C^I C^\dagger_{(J} C^\dagger_{K)} M^{1,0,0}$	$\frac{3}{2}$	$\overline{\mathbf{36}}_{-\frac{1}{2}}$	
$C^\dagger_I C^{(J} C^{K)} M^{0,0,-1}$	$\frac{3}{2}$	$\mathbf{36}_{\frac{1}{2}}$	
$C^I C^\dagger_{[J} C^\dagger_{K]} M^{1,0,0}$	$\frac{3}{2}$	$\overline{\mathbf{20}}_{-\frac{1}{2}}$	
$C^\dagger_I C^{[J} C^{K]} M^{0,0,-1}$	$\frac{3}{2}$	$\mathbf{20}_{\frac{1}{2}}$	
$C^I C^\dagger_{[I} C^\dagger_{J]} M^{1,0,0}$	$\frac{3}{2}$	$\overline{\mathbf{4}}_{-\frac{1}{2}}$	
$C^\dagger_I C^{[I} C^{J]} M^{0,0,-1}$	$\frac{3}{2}$	$\mathbf{4}_{\frac{1}{2}}$	

Table 3.3: BPS operators with $\Delta \le \frac{3}{2}$ in the mixed sector of $\mathrm{ABJM}_{3,1}$ theory.

BLG$_3$

A similar construction holds for the BLG$_3$ theory. One difference between this theory and the $\mathrm{ABJM}_{3,1}$ example we studied above is that the BLG$_3$ theory has a different set of monopole operators with $E = 0$, labeled by only a single positive half-integer GNO charge n. They transform in the $SU(2) \times SU(2)$ gauge irrep

$$SU(2) \times SU(2) \text{ irrep:} \qquad (\mathbf{6n+1}, \mathbf{6n+1}) \; . \tag{3.2.3}$$

(For the BLG$_k$ theory with arbitrary k, the gauge irrep is $(\mathbf{2kn+1}, \mathbf{2kn+1})$.) These monopole operators must be combined with the matter fields C^I and $C^\dagger_I$, each of which transform as $(\mathbf{2}, \mathbf{2})$ under the gauge group.

The lowest dimension gauge invariant operators are quadratic in C^I and $C^\dagger_I$ and do not require monopole operators. The next lowest are cubic in the C^I and $C^\dagger_I$ and require the monopole operator with $n = 1/2$. See Table 3.4. These operators are in one-to-one correspondence with operators from the interacting sector of the $\mathrm{ABJM}_{3,1}$ theory given in Table 3.2. We take this match to be the first piece of evidence for the duality (3.1.1)–(3.1.2) between the two theories.

$\mathcal{O}$	Δ	$SU(4)_R \times U(1)_t$	$OSp(8\|4)$ irrep
$C^\dagger_{(I}C^\dagger_{J)}$	1	$\overline{\mathbf{10}}_{-1}$	$(B,+)^{[0020]}$
$C^{(I}C^{J)}$	1	$\mathbf{10}_1$	
$C^\dagger_I C^J$	1	$\mathbf{15}_0$	
$C^\dagger_{(I}C^\dagger_J C^\dagger_{K)}M^{1/2}$	$\frac{3}{2}$	$\mathbf{20}''_{-\frac{3}{2}}$	$(B,+)^{[0030]}$
$C^{(I}C^J C^{K)}M^{1/2}$	$\frac{3}{2}$	$\overline{\mathbf{20}}''_{\frac{3}{2}}$	
$C^I C^\dagger_{(J}C^\dagger_{K)}M^{1/2}$	$\frac{3}{2}$	$\overline{\mathbf{36}}_{-\frac{1}{2}}$	
$C^\dagger_I C^{(J}C^{K)}M^{1/2}$	$\frac{3}{2}$	$\mathbf{36}_{\frac{1}{2}}$	

Table 3.4: BPS operators with $\Delta \leq \frac{3}{2}$ in the BLG$_3$ theory.

3.2.2 Superconformal Index

As an alternative to the explicit construction given in the previous section, one can use the superconformal index. The superconformal index, to be defined more precisely shortly, captures information about protected representations of the superconformal algebra. Its advantage over the explicit construction of the previous section is that it can be rigorously computed using supersymmetric localization. Its disadvantage is that the information it encodes does not unambiguously identify all the $\mathfrak{osp}(8|4)$ representations.

In order to define the superconformal index, it is convenient to view an $\mathcal{N} = 8$ SCFT as an $\mathcal{N} = 2$ SCFT with $SU(4)$ flavor symmetry. One can then consider a supercharge Q within the $\mathcal{N} = 2$ superconformal algebra such that $\{Q, Q^\dagger\} = \Delta - R - j_3$, where Δ is the scaling dimension, j_3 is the third component of the angular momentum, and R is the $U(1)_R$ charge. (There is a unique such supercharge, and it has $\Delta = 1/2$, $R = 1$, and $j_3 = -1/2$.) The superconformal index with respect to Q is defined as the trace over the $S^2 \times \mathbb{R}$ Hilbert space

$$I(x, z_f) = \mathrm{Tr}\left[(-1)^F x^{\Delta + j_3} \prod_{f=1}^{3} z_f^{F_f}\right], \tag{3.2.4}$$

where $F = (-1)^{2j_3}$ is the fermion number and F_f are the charges under the Cartan of the $SU(4)$ flavor symmetry. Standard arguments imply that the only states contributing to the trace in (3.2.4) obey $\Delta = R + j_3$; all others cancel pairwise.

The indices for the theories we are interested in have been computed using supersymmetric localization in [66], following the general computation in [18]. It can be shown that $I_{\mathrm{ABJM}_{3,1}} = I_{\mathrm{BLG}_3} I_{\mathrm{free}}$, where $I_{\mathrm{ABJM}_{3,1}}$ is the index of the ABJM$_{3,1}$ theory, I_{BLG_3} is that of the BLG$_3$ theory, and I_{free} is that of the ABJM$_{1,1}$ theory, which is free. For instance, keeping only one fugacity z corresponding to the Cartan element of $SU(4)$ given by either $U(1)_T$ or $U(1)_t$, we have[3]

$$
\begin{aligned}
I_{\mathrm{ABJM}_{3,1}} &= 1 + 8x + 71x^2 + 320x^3 + 2z\left(x^{1/2} + 12x^{3/2} + 78x^{5/2}\right) \\
&\quad + z^2\left(6x + 56x^2 + 295x^3\right) + z^3(14x^{3/2} + 114x^{5/2}) + O(z^4, x^{7/2}) + \left(z \leftrightarrow z^{-1}\right) , \\
I_{\mathrm{free}} &= 1 + 4x + x^2 + 4x^3 + 2z\left(x^{1/2} + 2x^{3/2}\right) + z^2\left(3x + 4x^2\right) \\
&\quad + 4z^3\left(x^{3/2} + x^{5/2}\right) + O(z^4, x^{7/2}) + \left(z \leftrightarrow z^{-1}\right) , \\
I_{\mathrm{BLG}_3} &= 1 + 4x + 12x^2 + 24x^3 + 2z\left(3x^{3/2} + 11x^{5/2}\right) \\
&\quad + z^2\left(3x + 8x^2 + 27x^3\right) + 2z^3(2x^{3/2} + 10x^{5/2}) + O(z^4, x^{7/2}) + \left(z \leftrightarrow z^{-1}\right) .
\end{aligned}
\tag{3.2.5}
$$

One can indeed check that these expressions obey $I_{\mathrm{ABJM}_{3,1}} = I_{\mathrm{BLG}_3} I_{\mathrm{free}}$ up to the order given. We regard this match of the indices as the second piece of evidence supporting our conjectured duality (3.1.1)–(3.1.2).

3.3 Moduli Space

We now show how to relate the (classical) moduli space of vacua of the $\mathrm{ABJM}_{3,1}$ theory to that of the BLG_3 theory. The moduli space can be found by modding out the zero locus of the scalar potential by the gauge transformations. For both theories, one can check that the scalar potential vanishes provided that [2, 11]

$$
\langle A^{\dot\alpha}_{a\beta} \rangle = a_{\beta a} \delta^{\dot\alpha}_{\beta} , \qquad \langle B^{\beta}_{\dot a\dot\alpha} \rangle = b^{\beta}_{\dot a} \delta^{\dot\alpha}_{\beta} , \tag{3.3.1}
$$

where $a_{\beta a}, b^{\beta}_{\dot a}$ are complex numbers, and where we used part of the gauge symmetry to put $A^{\dot\alpha}_{a\beta}$ and $B^{\beta}_{\dot a\dot\alpha}$ in diagonal form. For a gauge group of rank N, the moduli space is thus parameterized by $4N$ complex numbers $z_r = \{a_{r1}, a_{r2}, b^r_1, b^r_2\}$ for $r = 1, \ldots, N$, modulo residual gauge transformations.

The residual gauge symmetry gives further relations on z_r. For the $\mathrm{ABJM}_{3,1}$ theory, these relations are [2]

$$
z_r \sim z_{\sigma(r)} , \qquad \sigma \in S_3 , \tag{3.3.2}
$$

where $r = 1, 2, 3$ and S_3 is the symmetric group of order six. The moduli space is thus $(\mathbb{C}^4)^3/S_3$. From the M-theory perspective, this is the moduli space of three M2-branes in flat space, where the S_3 corresponds to permuting the indistinguishable branes.

For the BLG_3 theory, for which we denote the corresponding coordinates by z'_r instead of z_r, the relations

[3] We fix a typo in [66] for the coefficient of z^2x^3 in the expression for $I_{\mathrm{ABJM}_{3,1}}$.

are [17, 67, 68]

$$z'_1 \sim z'_2\,, \qquad z'_1 \sim e^{2\pi i/3} z'_1\,, \qquad z'_2 \sim e^{-2\pi i/3} z'_2\,. \tag{3.3.3}$$

The first relation comes from permuting the identical gauge groups, while the last two come from identifications that depend on the Chern-Simons coupling. These relations define the moduli space $(\mathbb{C}^4)^2/\mathbf{D}_3$, where $\mathbf{D}_3$ is the dihedral group of order six. We wish to identify this with the interacting sector of $\mathrm{ABJM}_{3,1}$. To distinguish between the free and interacting sector of the latter, consider the reparameterization

$$w_1 = e^{-2\pi i/3} z_1 + e^{2\pi i/3} z_2 + z_3\,, \qquad w_2 = e^{2\pi i/3} z_1 + e^{-2\pi i/3} z_2 + z_3\,, \qquad w_3 = z_1 + z_2 + z_3\,. \tag{3.3.4}$$

The parameter w_3 is invariant under S_3 and thus parameterizes the moduli space of the free theory. The interacting sector is parameterized by w_1, w_2, which transform under the permutations $(12), (123) \in S_3$ as

$$\begin{aligned} (12):& \qquad w_1 \sim w_2\,, \\ (123):& \qquad w_1 \sim e^{2\pi i/3} w_1\,, \qquad w_2 \sim e^{-2\pi i/3} w_2\,, \end{aligned} \tag{3.3.5}$$

where (12) permutes $z_1 \leftrightarrow z_2$ and (123) permutes $z_1 \to z_2\,, z_2 \to z_3\,, z_3 \to z_1$. These relations are the same as (3.3.3), which establishes the isomorphism

$$(\mathbb{C}^4)^3/S_3 \cong (\mathbb{C}^4)^2/\mathbf{D}_3 \times \mathbb{C}^4\,, \tag{3.3.6}$$

where $\mathbb{C}^4$ corresponds to the free sector of the $\mathrm{ABJM}_{3,1}$ theory, and $(\mathbb{C}^4)^2/\mathbf{D}_3$ corresponds to the interacting sector as well as to the BLG_3 theory. We regard the match between the moduli spaces (3.3.6) as the third piece of evidence supporting our conjectured duality (3.1.1)–(3.1.2).

3.4 The S^3 Partition Function

We will now compare the S^3 partition functions of the two theories. The partition function for the $\mathrm{ABJM}_{N,k}$ theory can be written as the following finite dimensional integral [40]:

$$Z_{\mathrm{ABJM}_{N,k}} = \frac{1}{(N!)^2} \int d^N\sigma d^N\widetilde{\sigma}\, e^{\pi i k \sum_{\alpha=1}^N (\sigma_\alpha^2 - \widetilde{\sigma}_\alpha^2)} \left(\frac{\prod_{\alpha<\beta} 2\sinh(\pi(\sigma_\alpha - \sigma_\beta)) 2\sinh(\pi(\widetilde{\sigma}_\alpha - \widetilde{\sigma}_\beta))}{\prod_{\alpha,\beta} 2\cosh(\pi(\sigma_\alpha - \widetilde{\sigma}_\beta))} \right)^2 , \tag{3.4.1}$$

where $\sigma_\alpha, \widetilde{\sigma}_\alpha$ are integration variables that can be interpreted as the eigenvalues of the scalars in the vector multiplets associated with the two $U(N)$ gauge groups. For $k=1$ and $N=1,3$ we find

$$Z_{\text{ABJM}_{3,1}} = \frac{\pi-3}{64\pi}\,, \qquad Z_{\text{ABJM}_{1,1}} = Z_{\text{free}} = \frac{1}{4}\,, \tag{3.4.2}$$

where recall that the $\text{ABJM}_{1,1}$ theory is free.

The partition function of the BLG_k theory can be derived from the $\text{ABJM}_{N,k}$ partition function (3.4.1) by setting $N=2$, imposing the constraints $\sigma_1+\sigma_2=\widetilde{\sigma}_1+\widetilde{\sigma}_2=0$, and multiplying by 2 to take into account the $\mathbb{Z}_2$ quotient in the $(SU(2)\times SU(2))/\mathbb{Z}_2$ gauge group. The result is

$$Z_{\text{BLG}_k} = \frac{1}{64\pi^2}\int d^2\sigma_\pm\, e^{\frac{2ki\sigma_+\sigma_-}{\pi}} \left(\frac{\sinh(\sigma_++\sigma_-)\sinh(\sigma_+-\sigma_-)}{\cosh^2(\sigma_+)\cosh^2(\sigma_-)}\right)^2\,, \tag{3.4.3}$$

where we have made the change of variables $\sigma_\pm = \pi(\sigma_1 \pm \widetilde{\sigma}_1)$. For $k=3$, we find that

$$Z_{\text{BLG}_3} = \frac{\pi-3}{16\pi} = \frac{Z_{\text{ABJM}_{3,1}}}{Z_{\text{free}}}\,, \tag{3.4.4}$$

as we expect from our duality. We regard (3.4.4) as our fourth piece of evidence supporting the conjectured duality (3.1.1)–(3.1.2).

3.5 One-Dimensional Topological Sector

Lastly, let us attempt to make a more detailed check of the duality (3.1.1)–(3.1.2) at the level of correlation functions of BPS operators. As explained in Chapter 2, abstract arguments based on the superconformal algebra show that all three-dimensional $\mathcal{N}\geq 4$ SCFTs have two one-dimensional topological sectors (defined either on a line in flat space or on a great circle within S^3), one associated with the Higgs branch and the other with the Coulomb branch.[4] More precisely, these topological sectors arise as follows. All $\mathcal{N}=4$ SCFTs have an $SU(2)_H\times SU(2)_C$ R-symmetry. In general, there can be two types of 1/2-BPS scalar operators in these theories: "Higgs branch operators" that are invariant under $SU(2)_C$ and have scaling dimension Δ equal to the $SU(2)_H$ spin j_H, and "Coulomb branch operators" that are invariant under $SU(2)_H$ and have scaling dimension Δ equal to the $SU(2)_C$ spin j_C. The operators belonging to the Higgs (Coulomb) branch topological sector are linear combinations of the first (second) class of 1/2-BPS operators above with specific position-dependent coefficients. These operators, when inserted on a line in flat space or on a great circle on S^3, have topological correlation functions because they represent non-trivial cohomology classes of

[4] In Chapter 2 these were called $SU(2)_L$ and $SU(2)_R$.

a nilpotent supercharge with respect to which translations along the line / circle are exact. Concretely, in the case where the 1d Higgs branch theory is defined on a great circle parameterized by $\varphi \in [-\pi/2, \pi/2)$ that sits within a round S^3 of radius r, the 1d operators are

$$\mathcal{O}(\varphi) = \mathcal{O}_{i_1 \ldots i_{2j_H}}(\varphi) u^{i_1}(\varphi) \ldots u^{i_{2j_H}}(\varphi) , \qquad u^i(\varphi) \equiv \begin{pmatrix} \cos(\varphi/2) \\ \sin(\varphi/2) \end{pmatrix} , \tag{3.5.1}$$

where $\mathcal{O}_{i_1 \ldots i_{2j_H}}(\varphi)$ is a 3d operator with $\Delta = j_H$ and $j_C = 0$, written as a symmetric, rank-$2j_H$ tensor of $SU(2)_H$.

For the particular case of $\mathcal{N} = 8$ SCFTs, the Higgs and Coulomb topological sectors are isomorphic, so without loss of generality we will study the Higgs one. In [42], it was shown that for $\mathcal{N} = 4$ SCFTs described by a Lagrangian with a vector multiplet with gauge algebra $\mathfrak{g}$ and a hypermultiplet in representation $\mathcal{R}$ of $\mathfrak{g}$, it is possible to use supersymmetric localization to obtain an explicit description of the 1d sector associated with the Higgs branch. When the 1d topological sector is defined on a great circle within S^3 parameterized by φ, as above, its explicit description takes the form of a Gaussian 1d theory coupled to a matrix model:

$$Z = \int_{\text{Cartan of } \mathfrak{g}} d\sigma \, \det\nolimits'_{\text{adj}}(2 \sinh(\pi\sigma)) \int DQ \, D\widetilde{Q} \, e^{4\pi r \int d\varphi \, \left(\widetilde{Q}\partial_\varphi Q + \widetilde{Q}\sigma Q\right)} . \tag{3.5.2}$$

Here, σ is the matrix degree of freedom that has its origin in the 3d vector multiplet and was diagonalized to lie within the Cartan of the gauge algebra. The 1d fields $Q(\varphi)$ and $\widetilde{Q}(\varphi)$ have their origin in the 3d hypermultiplet and transform in the representations $\mathcal{R}$ and $\overline{\mathcal{R}}$, respectively. Their definition in terms of the hypermultiplet scalars is as in (3.5.1), with $\mathcal{O}$ replaced by the hypermultiplet scalars transforming in the fundamental of $SU(2)_H$. Upon integrating out Q and $\widetilde{Q}$ one obtains the Kapustin-Willett-Yaakov matrix model [40] for the S^3 partition function of the $\mathcal{N} = 4$ SCFT:

$$Z = \int_{\text{Cartan of } \mathfrak{g}} d\sigma \, \frac{\det'_{\text{adj}}(2 \sinh(\pi\sigma))}{\det_{\mathcal{R}}(2 \cosh(\pi\sigma))} . \tag{3.5.3}$$

The description (3.5.2) can be used to calculate arbitrary n-point functions of operators belonging to the 1d sector, so this result opens up the possibility of performing more detailed tests of our proposed duality (3.1.1)–(3.1.2) involving correlation functions captured by the 1d sector. Unfortunately, the ABJM and BLG theories we are interested in do not have Lagrangian descriptions in terms of just vector multiplets and hypermultiplets (one cannot accommodate non-zero Chern-Simons levels with just vector multiplets and hypermultiplets), so the result (3.5.2) quoted above does not directly apply to these theories.

Fortunately, there is a way around this difficulty. The right-hand side of (3.1.1)–(3.1.2), or more generally the ABJM$_{N,1}$ theory, has a dual description as an $\mathcal{N}=4$ $U(N)$ gauge theory coupled to an adjoint hypermultiplet and a fundamental hypermultiplet [2, 15, 69]. So if we worked with this dual description we could use (3.5.2) to compute correlation functions in the Higgs branch topological sector, and we will do so in the case of interest $N=3$. For the BLG theories no such dual description is available, but we will conjecture that a modification of (3.5.2) will allow us to compute some of the correlation functions in the Higgs branch sector. Our conjecture is that to the integrand of (3.5.2) we should insert

$$e^{i\pi k \operatorname{tr} \sigma^2} \tag{3.5.4}$$

for every gauge group factor that has a Chern-Simons level k, where the trace is taken in the fundamental representation of that gauge group factor and in the trivial representation of the rest. This conjecture is motivated by the fact that this is the correct prescription in the matrix model (3.5.3). Importantly, it allows us to compute correlation functions of gauge-invariant operators built from Q and $\widetilde{Q}$. However, unlike when $k=0$, these operators are not the most general operators in the 1d theory; some of the operators in the 1d theory descend from 3d monopole operators, and these are not captured by (3.5.2) supplemented by (3.5.4). Nevertheless, we will still be able to compute correlation functions of non-monopole operators in the BLG$_3$ theory and compare them with the analogous correlators in the ABJM$_{3,1}$ theory. As we will see, the results of these computations are consistent with our proposed duality in (3.1.1)–(3.1.2).

From the $\mathcal{N}=8$ perspective, the operators in the Higgs branch topological theory are specific linear combinations of at least 1/4-BPS short representations. To be concrete, let us consider an $SU(2)_H \times SU(2)_C \times SU(2)_F \times SU(2)_{F'}$[5] subgroup of the $\mathcal{N}=8$ $SO(8)_R$ R-symmetry, where, from the $\mathcal{N}=4$ point of view, $SU(2)_H \times SU(2)_C$ is interpreted as the R-symmetry and $SU(2)_F \times SU(2)_{F'}$ as a flavor symmetry. One can consider this embedding such that the fundamental representations of $SO(8)$ have the following decompositions:

$$\begin{aligned}
[1000] = \mathbf{8}_v &\to (\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2})\,,\\
[0010] = \mathbf{8}_c &\to (\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2})\,,\\
[0001] = \mathbf{8}_s &\to (\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{2}) \oplus (\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1})\,.
\end{aligned} \tag{3.5.5}$$

As shown in Chapter 2 the only operators in the 1d theory come from the superconformal primaries of $\mathcal{N}=8$ multiplets that are at least 1/4-BPS—in our case, these will be the $(B,+)^{[00m0]}$ and $(B,2)^{[0nm0]}$ representations. The superconformal primaries of these multiplets are scalars with scaling dimension $\Delta =$

[5]In Chapter 2 $SU(2)_F \times SU(2)_{F'}$ was denoted as $SU(2)_1 \times SU(2)_2$.

$n+m/2$ and $SO(8)_R$ irrep with Dynkin labels $[0nm0]$. They give 1d operators that are singlets of $SU(2)_{F'}$ and that transform in the spin-$m/2$ representation of $SU(2)_F$:

$$\begin{aligned} (B,+)^{[00m0]}: &\qquad \mathcal{O}^{(\Delta,j_F)}_{a_1\ldots a_{2j_F}}(\varphi) \qquad \Delta = j_F = \frac{m}{2}\,, \\ (B,2)^{[0nm0]}: &\qquad \mathcal{O}^{(\Delta,j_F)}_{a_1\ldots a_{2j_F}}(\varphi) \qquad \Delta = j_F + n = \frac{m}{2} + n\,, \end{aligned} \tag{3.5.6}$$

where we have denoted the 1d operators as $\mathcal{O}^{(\Delta,j_F)}_{a_1\ldots a_{2j_F}}(\varphi)$, writing them explicitly as rank-$2j_F$ symmetric tensors of the $SU(2)_F$. This $SU(2)_F$ is thus a global symmetry of the 1d topological theory.

As in Chapter 2, in order keep track of the $SU(2)_F$ indices more efficiently, we introduce polarization variables y^a, $a=1,2$, and denote the operators in the 1d theory by

$$\mathcal{O}^{(\Delta,j)}(\varphi,y) = \mathcal{O}^{(\Delta,j)}_{a_1\ldots a_{2j}}(\varphi,y)y^{a_1}\cdots y^{a_{2j}}\,, \tag{3.5.7}$$

where in order to avoid clutter we simply denote $j_F = j$. We consider a basis of 1d operators with diagonal two-point functions, normalized such that

$$\begin{aligned} \langle \mathcal{O}^{(\Delta,j)}(\varphi_1,y_1)\mathcal{O}^{(\Delta,j)}(\varphi_2,y_2)\rangle &= \langle y_1,y_2\rangle^{2j}\,(\operatorname{sgn}\varphi_{21})^{2\Delta}\,, \\ \langle \mathcal{O}^{(\Delta_1,j_1)}(\varphi_1,y_1)\mathcal{O}^{(\Delta_2,j_2)}(\varphi_2,y_2)\rangle &= 0 \qquad \text{if } \mathcal{O}^{(\Delta_1,j_1)} \neq \mathcal{O}^{(\Delta_2,j_2)}\,, \end{aligned} \tag{3.5.8}$$

where $\varphi_{21} \equiv \varphi_2 - \varphi_1$, and the product between $SU(2)_F$ polarizations is defined as

$$\langle y_1,y_2\rangle \equiv \epsilon_{ab}y_1^a y_2^b\,, \qquad (\epsilon^{12} \equiv -\epsilon_{12} \equiv 1)\,. \tag{3.5.9}$$

The form of the three point functions is fixed by the $SU(2)_F$ symmetry up to an overall coefficient that we denote by $\lambda_{(\Delta_1,j_1),(\Delta_2,j_2),(\Delta_3,j_3)}$:

$$\begin{aligned} \langle \mathcal{O}^{(\Delta_1,j_1)}(\varphi_1,y_1)\mathcal{O}^{(\Delta_2,j_2)}(\varphi_2,y_2)\mathcal{O}^{(\Delta_3,j_3)}(\varphi_3,y_3)\rangle &= \lambda_{(\Delta_1,j_1),(\Delta_2,j_2),(\Delta_3,j_3)} \\ \times \langle y_1,y_2\rangle^{j_{123}}\langle y_2,y_3\rangle^{j_{231}}\langle y_3,y_1\rangle^{j_{312}}\,(\operatorname{sgn}\varphi_{21})^{\Delta_{123}}&(\operatorname{sgn}\varphi_{32})^{\Delta_{231}}(\operatorname{sgn}\varphi_{13})^{\Delta_{312}}\,, \end{aligned} \tag{3.5.10}$$

where $j_{k_1k_2k_3} \equiv j_{k_1} + j_{k_2} - j_{k_3}$. Eq. (3.5.10) is correct as long as j_1, j_2, and j_3 obey the triangle inequality. If this requirement is not fulfilled, the RHS of (3.5.10) vanishes. Note that the definition of the OPE coefficients here differs from that in Section 1.1.3. In that notation, when $\varphi_1 < \varphi_2 < \varphi_3 < \varphi_4$ and $j_1 \geq j_2$, the four point function of $(B,+)^{[00(2j)0]}$ operators $\mathcal{O}^{(j,j)}(\varphi,y)$ can be decomposed in the s-channel as

$$\begin{aligned}\langle \mathcal{O}^{(j_1,j_1)}(\varphi_1,y_1)\mathcal{O}^{(j_2,j_2)}(\varphi_2,y_2)\mathcal{O}^{(j_3,j_3)}(\varphi_3,y_3)\mathcal{O}^{(j_4,j_4)}(\varphi_4,y_4)\rangle &= \langle y_1,y_2\rangle^{j_1+j_2}\langle y_3,y_4\rangle^{j_3+j_4} \\ \times\left[\frac{\langle y_1,y_4\rangle}{\langle y_2,y_4\rangle}\right]^{j_{12}}\left[\frac{\langle y_1,y_3\rangle}{\langle y_1,y_4\rangle}\right]^{j_{34}} &\sum_{\Delta=j_1-j_2}^{j_1+j_2}\sum_{j=j_1-j_2}^{\Delta}\frac{t_j(w)}{4^\Delta}\bar{\lambda}_{2j_1,2j_2,\Delta,j}\bar{\lambda}_{2j_3,2j_4,\Delta,j}\,,\end{aligned} \tag{3.5.11}$$

where w is the $SU(2)_F$ cross-ratio

$$w=\frac{\langle y_1,y_2\rangle\langle y_3,y_4\rangle}{\langle y_1,y_3\rangle\langle y_2,y_4\rangle}\,, \tag{3.5.12}$$

and here we denote the OPE coefficients in the conventions of Section 1.1.3 by $\bar{\lambda}$. For $j_1=j_2$ we have the extra constraint $\Delta+j\in$ Even, because scalar Higgs branch operators can only appear in the symmetric product of identical operators. The function $t_j(w)$ obeys the eigenvalue equation:

$$(1-w)w^2\frac{d^2t_j}{dw^2}+(j_{34}-j_{12}-1)w^2\frac{dt_j}{dw}+j_{12}j_{34}wt_j=j(j+1)t_j\,. \tag{3.5.13}$$

Up to normalization, the regular solution can be written in terms of the Jacobi polynomials $P_n^{(a,b)}(x)$ as

$$t_j(w)=w^{j_{34}}P_{j+j_{34}}^{(j_{12}-j_{34},-j_{12}-j_{34})}\left(\frac{2}{w}-1\right)\,. \tag{3.5.14}$$

Note that when this expression is plugged into (3.5.11), the total expression is a polynomial in the y's. The OPE coefficients $\bar{\lambda}$ in (3.5.11) are related to λ in (3.5.10) by

$$\lambda_{(j_1,j_1),(j_2,j_2),(\Delta,j)}\lambda_{(j_3,j_3),(j_4,j_4),(\Delta,j)}=\lim_{w\to 0}\frac{w^jt_j(w)}{4^\Delta}\bar{\lambda}_{2j_1,2j_2,\Delta,j}\bar{\lambda}_{2j_3,2j_4,\Delta,j}\,, \tag{3.5.15}$$

where here we do not sum over repeated indices.

3.5.1 ABJM$_{3,1}$

Let us now apply the formalism introduced above to the $U(3)_1\times U(3)_{-1}$ ABJM theory in its dual description as a $U(3)$ gauge theory with both an adjoint and fundamental $\mathcal{N}=4$ hypermultiplet. The result (3.5.2) reads in this case

$$Z_{\mathrm{ABJM}_{3,1}}=\frac{1}{3!}\int d^3\sigma\prod_{\alpha<\beta}4\sinh^2(\pi\sigma_{\alpha\beta})\int\mathcal{D}Q^\alpha\mathcal{D}\widetilde{Q}_\alpha\int\mathcal{D}X^\alpha{}_\beta\mathcal{D}\widetilde{X}_\alpha{}^\beta e^{-S_{\mathrm{ABJM}_{3,1}}} \tag{3.5.16}$$

with

$$
\begin{aligned}
S_{\text{ABJM}_{3,1}} = -4\pi r \int_{-\pi}^{\pi} d\varphi \Big[&\widetilde{Q}_\alpha \dot{Q}^\alpha + \widetilde{X}_\alpha{}^\beta \dot{X}^\alpha{}_\beta + \sigma_\alpha \widetilde{Q}_\alpha Q^\alpha + \sigma_{12}(\widetilde{X}_1{}^2 X^1{}_2 - \widetilde{X}_2{}^1 X^2{}_1) \\
&+ \sigma_{23}(\widetilde{X}_2{}^3 X^2{}_3 - \widetilde{X}_3{}^2 X^3{}_2) + \sigma_{31}(\widetilde{X}_3{}^1 X^3{}_1 - \widetilde{X}_1{}^3 X^1{}_3) \Big] \,,
\end{aligned} \tag{3.5.17}
$$

where $\alpha, \beta = 1, 2, 3$. The 1d fields $X^\alpha{}_\beta$ and $\widetilde{X}_\alpha{}^\beta$ correspond to the adjoint hypermultiplet, Q^α and $\widetilde{Q}_\alpha$ correspond to the fundamental hypermultiplet, and σ_α are the matrix degrees of freedom in the Cartan of the $U(3)$.

The D-term relations of the 3d theory allow us to rewrite the Q's in terms of the X's, so we will only use the latter to construct operators. Correlation functions of such operators can be computed by performing Wick contractions at fixed σ with the propagator

$$
\langle X^\alpha{}_\beta(\varphi_1, y_1) \widetilde{X}_\gamma{}^\delta(\varphi_2, y_2) \rangle_\sigma = -\delta^\alpha{}_\gamma \delta^\delta{}_\beta \frac{\operatorname{sgn}\varphi_{12} + \tanh(\pi\sigma_{\alpha\beta})}{8\pi r} e^{-\sigma_{\alpha\beta}\varphi_{12}} \,. \tag{3.5.18}
$$

and then integrating over σ:

$$
\begin{aligned}
\langle \mathcal{O}_1(\varphi_1, y_1) \cdots \mathcal{O}_n(\varphi_n, y_n) \rangle &= \frac{1}{Z_{\text{ABJM}_{3,1}}} \int d^3\sigma \, Z^\sigma_{\text{ABJM}_{3,1}} \langle \mathcal{O}_1(\varphi_1, y_1) \cdots \mathcal{O}_n(\varphi_n, y_n) \rangle_\sigma \,, \\
Z^\sigma_{\text{ABJM}_{3,1}} &= \frac{1}{2^6 \cdot 3!} \frac{\tanh^2(\pi\sigma_{12}) \tanh^2(\pi\sigma_{13}) \tanh^2(\pi\sigma_{23})}{\cosh(\pi\sigma_1) \cosh(\pi\sigma_2) \cosh(\pi\sigma_3)} \,,
\end{aligned} \tag{3.5.19}
$$

where $\langle \cdots \rangle_\sigma$ is the correlation function for the Gaussian theory in (3.5.17) at fixed σ computed using (3.5.19).

Being a 1d sector of an $\mathcal{N} = 8$ SCFT, the theory (3.5.16) must have a flavor $SU(2)_F$ symmetry. Indeed, it is not hard to see that the fields $(\widetilde{X}, X^T)$ transform as a doublet under $SU(2)_F$. It is thus convenient to define

$$
\mathcal{X}(\varphi, y) = y^1 \widetilde{X}(\varphi, y) + y^2 X^T(\varphi, y) \,, \tag{3.5.20}
$$

where the y^a are the same polarization variables introduced earlier in (3.5.7).

Free Sector

As explained above, the ABJM$_{3,1}$ theory has a decoupled free sector. Consequently, the 1d theory (3.5.16) also has a decoupled free sector. It is generated by the gauge invariant operator

$$
\mathcal{O}^{(\frac{1}{2},\frac{1}{2})}_{\text{free}}(\varphi, y) = \operatorname{tr} \mathcal{X}(\varphi, y) \,, \tag{3.5.21}
$$

which has its origin in the free multiplet $(B,+)^{[0010]}$, whose superconformal primaries are scalars of scaling dimension $\Delta = 1/2$.

Since $\operatorname{tr} X$ and $\operatorname{tr} \widetilde{X}$ only appear in the kinetic term of (3.5.17), we can simply read off the propagator

$$\langle \mathcal{O}_{\text{free}}^{(\frac{1}{2},\frac{1}{2})}(\varphi_1, y_1) \mathcal{O}_{\text{free}}^{(\frac{1}{2},\frac{1}{2})}(\varphi_2, y_2) \rangle = \frac{3}{8\pi r} \langle y_1, y_2 \rangle \operatorname{sgn} \varphi_{21} \,. \tag{3.5.22}$$

All other 1d operators belonging to the free sector are powers of $\mathcal{O}_{\text{free}}^{(\frac{1}{2},\frac{1}{2})}(\varphi, y)$:

$$\mathcal{O}_{\text{free}}^{(j,j)}(\varphi, y) = [\mathcal{O}_{\text{free}}^{(\frac{1}{2},\frac{1}{2})}(\varphi, y)]^{2j} \,. \tag{3.5.23}$$

It follows that all free theory correlations functions can be computed using Wick contractions with the propagator (3.5.22). For the two and three point functions, we find

$$\langle \mathcal{O}_{\text{free}}^{(j_1,j_1)}(\varphi_1, y_1) \mathcal{O}_{\text{free}}^{(j_2,j_2)}(\varphi_2, y_2) \rangle = \delta_{j_1,j_2} (2j_1)! \left(\frac{3}{8\pi r} \langle y_1, y_2 \rangle \operatorname{sgn}(\varphi_{21}) \right)^{2j_1} \tag{3.5.24}$$

and, when j_1, j_2, j_3 obey the triangle inequality,

$$\begin{aligned} &\langle \mathcal{O}_{\text{free}}^{(j_1,j_1)}(\varphi_1, y_1) \mathcal{O}_{\text{free}}^{(j_2,j_2)}(\varphi_2, y_2) \mathcal{O}_{\text{free}}^{(j_3,j_3)}(\varphi_3, y_3) \rangle = j_{123}! j_{231}! j_{321}! \binom{2j_1}{j_{123}} \binom{2j_2}{j_{231}} \binom{2j_3}{j_{312}} \\ &\times \left(\frac{3}{8\pi r} \operatorname{sgn}\ \varphi_{32} \langle y_1, y_2 \rangle \right)^{j_{123}} \left(\frac{3}{8\pi r} \operatorname{sgn}\ \varphi_{32} \langle y_2, y_3 \rangle \right)^{j_{321}} \left(\frac{3}{8\pi r} \operatorname{sgn}\ \varphi_{13} \langle y_3, y_1 \rangle \right)^{j_{312}} . \end{aligned} \tag{3.5.25}$$

Rescaling the $\mathcal{O}_{\text{free}}^{(j,j)}$ by a positive factor in order to match (3.5.8) and comparing with (3.5.10), we extract the OPE coefficients

$$\lambda^{\text{free}}_{(j_1,j_1),(j_2,j_2),(j_3,j_3)} = \frac{j_{123}! j_{231}! j_{321}!}{\sqrt{(2j_1)!(2j_2)!(2j_3)!}} \binom{2j_1}{j_{123}} \binom{2j_2}{j_{231}} \binom{2j_3}{j_{312}} . \tag{3.5.26}$$

Interacting Sector

Let us now discuss operators in the interacting sector in increasing order of the number of $\mathcal{X}$'s they are built from. The interacting sector cannot have any operators linear in $\mathcal{X}$, because such operators would have originated from $\Delta = 1/2$ operators in 3d, which are free. So, the first non-trivial operator in the interacting sector must involve two $\mathcal{X}$'s. It must also be orthogonal to the free theory operator that is quadratic in $\mathcal{X}$, namely $\mathcal{O}_{\text{free}}^{(1,1)}$ defined in (3.5.23). From this, one can show that such an operator is proportional to

$$\mathcal{O}_{\text{int}}^{(1,1)}(\varphi, y) = (\operatorname{tr} \mathcal{X}^2)(\varphi, y) - \frac{1}{3} (\operatorname{tr} \mathcal{X})^2 (\varphi, y) \,. \tag{3.5.27}$$

Next, we consider operators with three $\mathcal{X}$'s. It can be shown that the interacting sector contains only one such operator, which by assumption must be orthogonal to the operator $\mathcal{O}_{\text{free}}^{(\frac{3}{2},\frac{3}{2})}$ of the free sector as well as the operator $\mathcal{O}_{\text{free}}^{(\frac{1}{2},\frac{1}{2})}\mathcal{O}_{\text{int}}^{(1,1)}$ of the mixed sector. It follows that this operator in the interacting sector is proportional to

$$\mathcal{O}_{\text{int}}^{(\frac{3}{2},\frac{3}{2})}(\varphi, y) = (\operatorname{tr}\mathcal{X}^3)(\varphi, y) - \left(\operatorname{tr}\mathcal{X}^2 \operatorname{tr}\mathcal{X}\right)(\varphi, y) + \frac{2}{9}(\operatorname{tr}\mathcal{X})^3(\varphi, y)\,. \tag{3.5.28}$$

Next, we can construct operators with four $\mathcal{X}$'s. It can be shown that the interacting sector contains two such operators. One of them has $j = 2$ and is $\mathcal{O}_{\text{int}}^{2,2} = (\mathcal{O}_{\text{int}}^{(1,1)})^2$. The other has $j = 0$, and is given by:

$$\mathcal{O}_{\text{int}}^{(2,0)}(\varphi) = \epsilon^{ac}\epsilon^{bd}\mathcal{O}_{\text{int},ab}^{(1,1)}(\varphi)\mathcal{O}_{\text{int},cd}^{(1,1)}(\varphi) - \frac{3(2\pi-7)}{2(\pi-3)(4\pi r)^2}\,, \tag{3.5.29}$$

where here we have used explicit $SU(2)_F$ indices. The second term in the above expression ensures that this operator is orthogonal to the unit operator. It is straightforward to continue and construct operators with more than four $\mathcal{X}$'s.

We can now use the propagator (3.5.18) and the matrix model partition function (3.5.19) to compute two and three point functions. For instance, for $\mathcal{O}_{\text{int}}^{(1,1)}(\varphi, y)$ we compute the two point function

$$\begin{aligned}\langle\mathcal{O}_{\text{int}}^{(1,1)}(\varphi_1, y_1)\mathcal{O}_{\text{int}}^{(1,1)}(\varphi_2, y_2)\rangle &= \frac{\langle y_1, y_2\rangle^2}{Z_{\text{ABJM}_{3,1}}(4\pi r)^2}\int d^3\sigma\, Z_{\text{ABJM}_{3,1}}^{\sigma}\left(1 + \sum_{\alpha<\beta}\operatorname{sech}^2(\pi\sigma_{\alpha\beta})\right)\\ &= \frac{10\pi - 31}{2(\pi-3)(4\pi r)^2}\langle y_1, y_2\rangle^2\,.\end{aligned} \tag{3.5.30}$$

A similar calculation gives the three point function

$$\begin{aligned}\langle\mathcal{O}_{\text{int}}^{(1,1)}(\varphi_1, y_1)\mathcal{O}_{\text{int}}^{(1,1)}(\varphi_2, y_2)\mathcal{O}_{\text{int}}^{(1,1)}(\varphi_3, y_3)\rangle &= \frac{10\pi-31}{(\pi-3)(4\pi r)^3}\\ &\times \operatorname{sgn}\varphi_{21}\operatorname{sgn}\varphi_{32}\operatorname{sgn}\varphi_{13}\langle y_1, y_2\rangle\langle y_2, y_3\rangle\langle y_3, y_1\rangle\,.\end{aligned} \tag{3.5.31}$$

Rescaling $\mathcal{O}_{\text{int}}^{(1,1)}$ by a positive factor in order to match (3.5.8) and comparing with (3.5.10), we extract the OPE coefficient

$$\lambda_{(1,1),(1,1),(1,1)} = \sqrt{\frac{8(\pi-3)}{10\pi-31}}\,. \tag{3.5.32}$$

Two other Higgs branch operators appear in the $\mathcal{O}_{\text{int}}^{(1,1)} \times \mathcal{O}_{\text{int}}^{(1,1)}$ OPE: $\mathcal{O}_{\text{int}}^{(2,0)}$ and $\mathcal{O}_{\text{int}}^{(2,2)}$. Performing the analogous calculation for these other operators yields the OPE coefficients

$$\begin{aligned}\lambda_{(1,1),(1,1),(2,2)} &= \sqrt{\frac{2(\pi-3)(840\pi-2629)}{5(10\pi-31)^2}}\,,\\ \lambda_{(1,1),(1,1),(2,0)} &= \sqrt{\frac{3888+\pi(420\pi-2557)}{3(10\pi-31)^2}}\,.\end{aligned} \tag{3.5.33}$$

As a consistency check, these OPE coefficients satisfy the relations

$$3\lambda^2_{(1,1),(1,1),(1,1)} - 5\lambda^2_{(1,1),(1,1),(2,2)} + 6\lambda^2_{(1,1),(1,1),(2,0)} + 6 = 0\,, \tag{3.5.34}$$

which is just the relation (2.1.2) derived from crossing in the 1d theory, now written in the different notation of this Chapter. We can convert these OPE coefficients to the conventions of Section 1.1.3 as described above to find

$$\begin{aligned}\bar{\lambda}^2_{2,2,1,1} =& \frac{16(\pi-3)}{10\pi-31}\,,\\ \bar{\lambda}^2_{2,2,2,2} =& \frac{16(\pi-3)(840\pi-2629)}{15(10\pi-31)^2}\,,\\ \bar{\lambda}^2_{2,2,2,0} =& 16\frac{3888+\pi(420\pi-2557)}{3(10\pi-31)^2}\,.\end{aligned} \tag{3.5.35}$$

We also computed the OPE coefficients for Higgs branch operators in the $\mathcal{O}^{(1,1)}_{\text{int}} \times \mathcal{O}^{(2,2)}_{\text{int}}$ and $\mathcal{O}^{(2,2)}_{\text{int}} \times \mathcal{O}^{(2,2)}_{\text{int}}$ OPEs, and found

$$
\begin{aligned}
\lambda_{(2,2),(2,2),(1,1)} &= \sqrt{\frac{32(\pi-3)}{10\pi-31}}\,, \\
\lambda_{(2,2),(2,2),(2,2)} &= \sqrt{\frac{40(521767-166320\pi)^2(\pi-3)}{49(840\pi-2629)^3}}\,, \\
\lambda_{(2,2),(2,2),(3,3)} &= \sqrt{\frac{14400(\pi-3)(9520\pi-29877)}{7(2629-840\pi)^2}}\,, \\
\lambda_{(2,2),(2,2),(4,4)} &= \sqrt{\frac{30(\pi-3)(4583040\pi-14394049)}{7(2629-840\pi)^2}}\,, \\
\lambda_{(2,2),(2,2),(4,2)} &= \sqrt{\frac{960(\pi-3)(4447712646+35\pi(12972960\pi-81205777))}{49(840\pi-2629)^3}}\,, \\
\lambda_{(2,2),(2,2),(4,0)} &= \sqrt{\frac{4(\pi-3)(\pi(8530357644+35\pi(8648640\pi-79544233))-8707129344)}{(2629-840\pi)^2(3888+\pi(420\pi-2557))}}\,, \\
\lambda_{(2,2),(2,2),(3,1)} &= \sqrt{\frac{64(\pi-3)(2675592+5\pi(55440\pi-344503))}{(2629-840\pi)^2(10\pi-31)}}\,, \\
\lambda_{(2,2),(2,2),(2,0)} &= \sqrt{\frac{(847584+\pi(90720\pi-554797))^2}{3(2629-840\pi)^2(3888+\pi(420\pi-2557))}}\,, \\
\lambda_{(1,1),(2,2),(1,1)} &= \sqrt{\frac{2(\pi-3)(840\pi-2629)}{5(31-10\pi)^2}}\,, \\
\lambda_{(1,1),(2,2),(2,2)} &= \sqrt{\frac{32(\pi-3)}{10\pi-31}}\,, \\
\lambda_{(1,1),(2,2),(3,3)} &= \sqrt{\frac{45(\pi-3)(9520\pi-29877)}{7(10\pi-31)(840\pi-2629)}}\,, \\
\lambda_{(1,1),(2,2),(3,1)} &= \sqrt{\frac{4(\pi-3)(2675592+5\pi(55440\pi-344503))}{5(31-10\pi)^2(840\pi-2629)}}\,.
\end{aligned}
\tag{3.5.36}
$$

3.5.2 BLG$_3$

As explained above, the 1d theory corresponding to the BLG theory requires a generalization of [42]. If we are not interested in correlation functions of operators arising from monopole operators in 3d, we conjecture that we can simply insert (3.5.4) into (3.5.2) and compute correlation functions of gauge-invariant operators built from Q and $\widetilde{Q}$. For the BLG$_3$ theory, this conjecture produces the 1d theory

$$
Z_{\mathrm{BLG}_3} = \frac{1}{16\pi^2}\int d^2\sigma_\pm e^{\frac{6i\sigma_+\sigma_-}{\pi}}\left(\frac{\sinh(\sigma_++\sigma_-)\sinh(\sigma_+-\sigma_-)}{\cosh(\sigma_+)\cosh(\sigma_-)}\right)^2\int D\widetilde{Q}_\alpha{}^{\dot\beta}DQ_{\dot\beta}{}^{\alpha}e^{-S_{\mathrm{BLG}_3}}\,, \tag{3.5.37}
$$

with

$$S_{\mathrm{BLG}_3} = -4r\int_{-\pi}^{\pi} d\varphi\ \left[\pi \widetilde{Q}_\alpha{}^{\dot\beta}\partial_\varphi Q_{\dot\beta}{}^\alpha + \sigma_- \widetilde{Q}_1{}^{\dot 1}Q_{\dot 1}{}^1 - \sigma_+\widetilde{Q}_1{}^{\dot 2}Q_{\dot 2}{}^1 + \sigma_+\widetilde{Q}_2{}^{\dot 1}Q_{\dot 1}{}^2 - \sigma_-\widetilde{Q}_2{}^{\dot 2}Q_{\dot 2}{}^2\right]\,, \tag{3.5.38}$$

where α, β and $\dot\alpha, \dot\beta$ are fundamental indices for each gauge group, $Q_{\dot\beta}{}^\alpha$ and $\widetilde{Q}_\alpha{}^{\dot\beta}$ correspond to the bifundamental hypermultiplets, and $\sigma_\pm$ are the same integration variables as in (3.4.3). (Eq. (3.4.3) is obtained after integrating out Q and $\widetilde{Q}$ in (3.5.37).)

We can rewrite the action in terms of the mass matrix-like quantity

$$M_\alpha{}^{\dot\beta} = \begin{pmatrix} \sigma_- & -\sigma_+ \\ \sigma_+ & -\sigma_- \end{pmatrix}\,, \tag{3.5.39}$$

to read off the propagator

$$\langle Q_{\dot\beta}{}^\alpha(\varphi_1, y_1)\widetilde{Q}_\gamma{}^{\dot\delta}(\varphi_2, y_2)\rangle_\sigma = -\delta_{\dot\beta}{}^{\dot\delta}\delta_\gamma{}^\alpha\ \frac{\operatorname{sgn}\varphi_{12} + \tanh(\pi M_\alpha{}^{\dot\beta})}{8\pi r}e^{-M_\alpha{}^{\dot\beta}\varphi_{12}}\,, \tag{3.5.40}$$

where there is no sum over the gauge indices. We then compute correlation functions as

$$\begin{aligned}\langle \mathcal{O}_1(\varphi_1, y_1)\cdots\mathcal{O}_n(\varphi_n, y_n)\rangle =& \frac{1}{Z_{\mathrm{BLG}_3}}\int d^2\sigma_\pm\ Z^\sigma_{\mathrm{BLG}_3}\,\langle \mathcal{O}_1(\varphi_1, y_1)\cdots\mathcal{O}_n(\varphi_n, y_n)\rangle_\sigma\,,\\ Z^\sigma_{\mathrm{BLG}_3} =& \frac{e^{\frac{6i\sigma_+\sigma_-}{\pi}}}{64\pi^2}\left(\frac{\sinh(\sigma_+ + \sigma_-)\sinh(\sigma_+ - \sigma_-)}{\cosh^2(\sigma_+)\cosh^2(\sigma_-)}\right)^2\,,\end{aligned} \tag{3.5.41}$$

where $\langle \mathcal{O}_1(\varphi_1, y_1)\cdots\mathcal{O}_n(\varphi_n, y_n)\rangle_\sigma$ is the correlation function for the Gaussian theory (3.5.38) at fixed σ, given in (3.5.40).

Since the 1d theory (3.5.38) arises from an $\mathcal{N} = 8$ SCFT, it must have a flavor $SU(2)_F$ symmetry. Indeed, it can be checked that such a symmetry is present and that $(Q_{\dot\beta}{}^\alpha, \epsilon^{\alpha\gamma}\epsilon_{\dot\beta\dot\delta}\widetilde{Q}_\gamma{}^{\dot\delta})$ form a doublet. It is thus convenient to combine the 2×2 matrices Q and $\widetilde{Q}$ into the matrix

$$\mathbf{Q}(\varphi, y) = \begin{pmatrix} Q_{\dot 1}{}^1y^1 - \widetilde{Q}_2{}^{\dot 2}y^2 & Q_{\dot 1}{}^2y^1 + \widetilde{Q}_2{}^{\dot 1}y^2 \\ Q_{\dot 2}{}^1y^1 + \widetilde{Q}_1{}^{\dot 2}y^2 & Q_{\dot 2}{}^2y^1 - \widetilde{Q}_1{}^{\dot 1}y^2 \end{pmatrix}\,, \tag{3.5.42}$$

where y^a are our usual $SU(2)_F$ polarization variables.

Let us see what gauge-invariant operators we can construct using an increasing number of $\mathbf{Q}$'s. There are no gauge-invariant operators built from only one $\mathbf{Q}$. With two $\mathbf{Q}$'s we can construct operators, which taken together have $SU(2)_F$ spin $j = 1$ and can be written compactly as

$$\mathcal{O}^{(1,1)}_{\mathrm{BLG}_3}(\varphi, y) = \det \mathbf{Q}(\varphi, y)\,. \tag{3.5.43}$$

With three $\mathbf{Q}$'s we again cannot construct any gauge-invariant operators. With four $\mathbf{Q}$'s we can construct two operators: one with $j = 2$, namely $\mathcal{O}^{(2,2)}_{\mathrm{BLG}_3} = (\mathcal{O}^{(1,1)}_{\mathrm{BLG}_3})^2$, and one with $j = 0$, namely

$$\mathcal{O}^{(2,0)}_{\mathrm{BLG}_3}(\varphi) = \epsilon^{ac}\epsilon^{bd}\mathcal{O}^{(1,1)}_{\mathrm{BLG}_3,ab}(\varphi)\mathcal{O}^{(1,1)}_{\mathrm{BLG}_3,cd}(\varphi) - \frac{3(2\pi - 7)}{2(\pi - 3)(4\pi r)^2}\,, \tag{3.5.44}$$

where here we have again used explicit $SU(2)_F$ indices and have included a second term to ensure that it is orthogonal to the unit operator. It is straightforward to proceed further using five $\mathbf{Q}$'s and higher.

The operators constructed so far, namely $\mathcal{O}^{(1,1)}_{\mathrm{BLG}_3}$, $\mathcal{O}^{(2,2)}_{\mathrm{BLG}_3}$, and $\mathcal{O}^{(2,0)}_{\mathrm{BLG}_3}$, match a subset of the operators we constructed in Section 3.5.1 for the interacting sector of the $\mathrm{ABJM}_{3,1}$ theory, namely $\mathcal{O}^{(1,1)}_{\mathrm{int}}$, $\mathcal{O}^{(\frac{3}{2},\frac{3}{2})}_{\mathrm{int}}$, $\mathcal{O}^{(2,2)}_{\mathrm{int}}$, $\mathcal{O}^{(2,0)}_{\mathrm{int}}$. We were not able to construct the BLG_3 analog of $\mathcal{O}^{(\frac{3}{2},\frac{3}{2})}_{\mathrm{int}}$ using only the $\mathbf{Q}$'s because this operator requires monopole operators.

Nevertheless, we can use the propagator (3.5.40) and the matrix model partition function (3.5.41) to compute two and three point functions of the operators we were able to construct in the 1d theory (3.5.38), and compare them to the analogous expressions from the interacting sector of the $\mathrm{ABJM}_{3,1}$ theory. For instance, for $\mathcal{O}^{(1,1)}_{\mathrm{BLG}_3}(\varphi, y)$ we compute the two point function

$$\begin{aligned}\langle \mathcal{O}^{(1,1)}_{\mathrm{BLG}_3}(\varphi_1, y_1)\mathcal{O}^{(1,1)}_{\mathrm{BLG}_3}(\varphi_2, y_2)\rangle &= \frac{\langle y_1, y_2\rangle^2}{4Z_{\mathrm{BLG}_3}(4\pi r)^2}\int d^2\sigma_\pm\, Z^\sigma_{\mathrm{BLG}_3}(\operatorname{sech}^2(\sigma_-) + \operatorname{sech}^2(\sigma_+))\\ &= \frac{10\pi - 31}{8(\pi - 3)(4\pi r)^2}\,\langle y_1, y_2\rangle^2\,.\end{aligned} \tag{3.5.45}$$

A similar calculation gives the three point function

$$\begin{aligned}\langle \mathcal{O}^{(1,1)}_{\mathrm{BLG}_3}(\varphi_1, y_1)\mathcal{O}^{(1,1)}_{\mathrm{BLG}_3}(\varphi_2, y_2)\mathcal{O}^{(1,1)}_{\mathrm{BLG}_3}(\varphi_3, y_3)\rangle &= \frac{10\pi - 31}{4(\pi - 3)(4\pi r)^3}\\ &\times \operatorname{sgn}\varphi_{21}\operatorname{sgn}\varphi_{32}\operatorname{sgn}\varphi_{13}\,\langle y_1, y_2\rangle\langle y_2, y_3\rangle\langle y_3, y_1\rangle\,.\end{aligned} \tag{3.5.46}$$

By comparing to (3.5.8) and (3.5.10) (we rescale $\mathcal{O}^{(1,1)}_{\mathrm{BLG}_3}$ by a positive factor in order to match (3.5.8)), we extract the OPE coefficient

$$\lambda_{(1,1),(1,1),(1,1)} = \sqrt{\frac{8(\pi - 3)}{10\pi - 31}}\,, \tag{3.5.47}$$

which agrees with (3.5.32) for the interacting sector of the $\mathrm{ABJM}_{3,1}$ theory. We can similarly check that the OPE coefficients of all the other Higgs branch operators that appear in the $\mathcal{O}^{(1,1)}_{\mathrm{BLG}_3} \times \mathcal{O}^{(2,2)}_{\mathrm{BLG}_3}$, $\mathcal{O}^{(1,1)}_{\mathrm{BLG}_3} \times \mathcal{O}^{(2,2)}_{\mathrm{BLG}_3}$, and $\mathcal{O}^{(1,1)}_{\mathrm{BLG}_3} \times \mathcal{O}^{(2,2)}_{\mathrm{BLG}_3}$ OPEs match those of $\mathrm{ABJM}_{3,1}$ theory, given in (3.5.33) and (3.5.36).

Chapter 4

Numerical Bootstrap for 3d $\mathcal{N} = 8$ Theories

This chapter is an edited version of ref. [70] and section 4 of ref. [33], which were written in collaboration with Jaehoon Lee, Silviu S. Pufu, and Ran Yacoby, as well as sections 4 and 5 of ref. [34], which was written in collaboration with Nathan B. Agmon and Silviu S. Pufu.

4.1 Introduction

The conformal bootstrap [71–73] is an old idea that uses the associativity of the operator algebra to provide an infinite set of constraints on the operator dimensions and the operator product expansion (OPE) coefficients of abstract conformal field theories (CFTs). For two-dimensional CFTs, this idea was used to compute the correlation functions of the minimal models [74] and of Liouville theory [75]. In more than two dimensions, conformal symmetry is much less restrictive, and as a consequence it is difficult to extract such detailed information from the bootstrap. Recently, it has been shown by the authors of [76] that the constraints arising from the conformal bootstrap can be reformulated as a numerical problem.[1] This provides a new method to exclude CFTs with a large enough gap in the operator spectrum and to obtain non-perturbative bounds on certain OPE coefficients [6, 35, 76, 79–96].

The goal of this chapter is to set up and develop the conformal bootstrap program in three-dimensional SCFTs with $\mathcal{N} = 8$ supersymmetry, which is the largest amount of supersymmetry in three dimensions. At some level, our work parallels that of [35], who developed the numerical bootstrap program in four-

[1] See also [77, 78] for a different recent method.

dimensional theories with $\mathcal{N} = 4$ superconformal symmetry. The authors of [35] studied the implications of unitarity and crossing symmetry on the four-point function of the superconformal primary operator $\mathcal{O}_{\mathbf{20'}}$ of the $\mathcal{N} = 4$ stress-tensor multiplet.[2] This superconformal primary is a Lorentz scalar that transforms in the $\mathbf{20'}$ irrep under the $\mathfrak{so}(6)$ R-symmetry. In the present work, we study the analogous question in three-dimensional $\mathcal{N} = 8$ SCFTs. In particular, we analyze the four-point function of the superconformal primary $\mathcal{O}_{\text{Stress}}$ of the $\mathcal{N} = 8$ stress-tensor multiplet.

Upon using the OPE, the four-point function of $\mathcal{O}_{\text{Stress}}$ can be written as a sum of contributions, called superconformal blocks, coming from all superconformal multiplets that appear in the OPE of $\mathcal{O}_{\text{Stress}}$ with itself. In addition, this four-point function can be decomposed into the six R-symmetry channels corresponding to the $\mathfrak{so}(8)_R$ irreps that appear in the product $\mathbf{35}_c \otimes \mathbf{35}_c$. Generically, each superconformal multiplet contributes to all six R-symmetry channels. These superconformal blocks can be determined by analyzing the superconformal Ward identity written down in [1]. Crossing symmetry then implies six possibly independent equations that mix the R-symmetry channels amongst themselves. The situation described here is analogous to the case of 4-d $\mathcal{N} = 4$ theories where one also has six R-symmetry channels and, consequently, six possibly independent crossing equations.

A major difference between the 4d and 3d cases, is that in the case of 4-d $\mathcal{N} = 4$ theories, the solution to the superconformal Ward identity involves algebraic relations between the six R-symmetry channels. As a consequence, after solving for the BPS sector, it turns out that the six crossing equations reduce algebraically to a single equation. In 3-d, the solution to the superconformal Ward identity can be written formally in terms of non-local operators acting on a single function [1]. As we will show, despite the appearance of these non-local operators, the various R-symmetry channels can be related to one another with the help of local second order differential operators. These relations show that the six crossing equations are mostly redundant, but still no single equation implies the others, as was the case in 4-d.

Our numerical results will be compared to the exact results of the previous chapter in several ways. Firstly, we will show that the relations between protected OPE coefficients given in (2.1.2) are exactly satisfied by our numerics. Secondly, we will show that a kink that appears in our numerics corresponds to the ABJ_1 theory, which is the theory with highest c_T at which a certain OPE coefficient becomes zero.

We will also use the analytic results to improve our numerical study. For a generic point within the allowed region[3] formed by the numerical bootstrap bounds, there are generally many different solutions to the crossing equations obeying unitarity constraints. At the boundary of the allowed region, however, there is believed to be a unique such solution, which can then be used to read off the CFT data (scaling dimensions

[2]The OPE of the stress-tensor multiplet in $\mathcal{N} = 4$ SYM was first analyzed in [97–100].

[3]Here, we mean the limit of the allowed region as we remove the cutoff that controls the truncation of the crossing equation to a finite number.

and OPE coefficients) that enters the conformal block decomposition of the given four-point function(s). This solution can be found, for instance, using the extremal functional method of [84, 90, 101]. If we have reasons to believe that a known CFT lives on this boundary, we can therefore potentially determine at least part of its CFT data.

A notable application of this method is to the 3d Ising model. In [90], it was argued that the critical Ising model has the minimal value of the stress tensor coefficient c_T (to be defined more precisely shortly) in the space of possible 3d CFTs with $\mathbb{Z}_2$ symmetry, and thus it is believed to sit at the boundary of the region of allowed values of c_T. Reconstructing the corresponding unique solution of the crossing equations using the extremal functional method, one can then read off all low-lying CFT data in the critical Ising model. See [102–105] for other cases where this method was applied.

In this chapter, we will apply the extremal functional method to maximally supersymmetric ($\mathcal{N} = 8$) superconformal field theories (SCFTs) in 3d. To do so, we show that the protected OPE coefficients in the 1d sector discussed in the previous chapter for $\mathcal{N} = 8$ SCFTs with holographic duals come close to saturating their numerical bootstrap bounds. We conjecture that these OPE coefficients for some of these theories precisely saturate the bootstrap bounds in the limit of very precise numerics, which allows us to read off the spectrum for these theories.

The remainder of this Chapter is organized as follows. In Section 4.2 we describe the differential relations that the crossing equations satisfy. Section 4.3 is devoted to the derivation of the superconformal blocks building on the results of [1]. In Section 4.4 we study the crossing equations using the semi-definite programing method introduced in [87] and present our findings. In Section 4.5, we present our evidence for the conjecture that holographic theories saturate the bootstrap bounds on the OPE coefficients in the protected 1d sector. In Section 4.5.2, we use the extremal functional method to read off all the low-lying CFT data for theories that saturate the bootstrap bounds. Finally, in Section 4.6, we end with a discussion of our results and of future directions.

4.2 Relation Between the Crossing Equations

The crossing equations in terms of the stress tensor 4-point function $\mathcal{G}$, which can be written as either a function of the variables U, V, σ, τ or $z, \bar{z}, \alpha, \bar{\alpha}$ as defined in (1.1.12), (1.1.38) and (1.1.31), was written down in (1.1.32). By expanding (1.1.32) in U and V one obtains six crossing equations, mixing the different R-symmetry channels (1.1.33). However, these crossing equations cannot be used in the numerical bootstrap program as they stand, for the following reason. The different R-symmetry channels are related by supersymmetry, so these equations are not independent. Using these dependent equations in a semidefinite program solver like `sdpb` [106] (as we will discuss in detail in Section 4.4) results in a numerical instability.

4.2.1 Formal Solution to the Superconformal Ward Identity

To understand the dependencies between these crossing equations we have to study the solution of the Ward identity (1.1.3), which can be written in terms of a single arbitrary function $a(z, \bar{z})$ as[4]

$$\mathcal{G}(z, \bar{z}; \alpha, \bar{\alpha}) = (z\bar{z}) \left(\mathcal{D}_{\frac{1}{2}}\right)^{-\frac{1}{2}} \left[(z\alpha - 1)(\bar{z}\alpha - 1)(z\bar{\alpha} - 1)(\bar{z}\bar{\alpha} - 1)a(z, \bar{z})\right] , \tag{4.2.1}$$

where

$$\mathcal{D}_\varepsilon \equiv \frac{\partial^2}{\partial z \partial \bar{z}} - \frac{\varepsilon}{z - \bar{z}} \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right) . \tag{4.2.2}$$

The appearance of the non-local operator $\left(\mathcal{D}_{\frac{1}{2}}\right)^{-\frac{1}{2}}$ makes using (4.2.1) rather subtle. However, we can demystify the operator $\mathcal{D}_\varepsilon$ and its non-integer powers by interpreting $\mathcal{D}_\varepsilon$ as the Laplacian in $d = 2(\varepsilon + 1)$ dimensions.[5]

Using conformal transformations we can fix three of the coordinates of the four-point function on a line, such that: $x_1 = 0$, $x_3 = (0, \ldots, 0, 1) \equiv \hat{z}$ and $x_4 = \infty$. (We denote the unit vector $(0, \ldots, 0, 1) \in \mathbb{R}^d$ by $\hat{z}$ because we will eventually be interested in working in three dimensions where we denote the third coordinate by z.) We write the remaining unfixed point $x_2 \equiv \vec{r} \in \mathbb{R}^d$ in spherical coordinates $\vec{r} = (r, \theta, \Omega_{d-2})$, where θ is the angle between $\vec{r}$ and $\hat{z}$, and Ω_{d-2} parameterizes S^{d-2}. The four-point function does not depend on Ω_{d-2} because of the additional rotation symmetry which fixes the line determined by x_1, x_3, and x_4. The cross-ratios in these coordinates are given by

$$U = r^2 , \qquad V = |\hat{z} - \vec{r}|^2 = 1 + r^2 - 2r\cos\theta , \tag{4.2.3}$$

$$z = re^{i\theta} , \qquad \bar{z} = re^{-i\theta} . \tag{4.2.4}$$

In other words, U can be interpreted as the square of the distance to the origin of $\mathbb{R}^d$, while V is the square of the distance to the special point $(0, \ldots, 0, 1)$.

The operator $\mathcal{D}_\varepsilon$ can then be written as

$$\mathcal{D}_\varepsilon = \frac{1}{4} \left[\frac{1}{r^{2\varepsilon+1}} \partial_r \left(r^{2\varepsilon+1} \partial_r\right) + \frac{1}{r^2 \sin^{2\varepsilon}\theta} \partial_\theta \left(\sin^{2\varepsilon}\theta \partial_\theta\right)\right] . \tag{4.2.5}$$

[4] The function $a(z, \bar{z})$ that appears in this equation equals $(x\bar{x})^{-\frac{1}{2}} a(x, \bar{x})$ in the notation of [1] with $x \equiv z, \bar{x} \equiv \bar{z}$.

[5] That $\mathcal{D}_\varepsilon$ is the Laplacian in $d = 2(\varepsilon + 1)$ dimensions was first observed by Dolan and Osborn in [107].

Up to an overall factor of 1/4, $\mathcal{D}_\varepsilon$ is nothing but the d-dimensional Laplacian $\boldsymbol{\Delta}$ acting on functions that are independent of the azimuthal directions $\Omega_{d-2} \in S^{d-2}$.

In $d = 3$, the solution (4.2.1) to the Ward identity can then be written formally as

$$\mathcal{G}(\vec{r}; \alpha, \bar{\alpha}) = r^2 \frac{2}{\sqrt{\boldsymbol{\Delta}}} \left[|\alpha \vec{r} - \hat{z}|^2 \, |\bar{\alpha} \vec{r} - \hat{z}|^2 \, a(\vec{r}) \right], \tag{4.2.6}$$

for some undetermined function $a(\vec{r})$. Here, both $\mathcal{G}(\vec{r}; \alpha, \bar{\alpha})$ and $a(\vec{r})$ should be taken to be invariant under rotations about the z-axis. This expression will become quite useful when we analyze the crossing symmetry in the next section.

In terms of (4.2.1) the crossing equation (1.1.32) takes the form[6]

$$\frac{1}{\sqrt{\boldsymbol{\Delta}}} \left[|\alpha \vec{r} - \hat{z}|^2 \, |\bar{\alpha} \vec{r} - \hat{z}|^2 \left(a(U, V) - a(V, U) \right) \right] = 0 \,. \tag{4.2.7}$$

This expression seems to suggest that there is only one independent crossing equation given by $a(U, V) - a(V, U) = 0$. However, it is not easy to calculate $a(U, V) - a(V, U)$ by acting with the non-local operator $\sqrt{\boldsymbol{\Delta}}$ on (4.2.7), because currently there is too little global information available about the four point function of $\mathcal{O}_{\text{Stress}}$ and its (super)conformal block expansion. It would be interesting to explore this avenue in future work.

Despite the appearance of a non-local operator in the solution of the superconformal Ward identity, we can in fact show that the six R-symmetry channels and, consequently the six crossing equations, satisfy certain differential equations that relate them to one another. These relations will be crucial for the implementation of the numerical bootstrap program in Section 4.4.

4.2.2 Relations Between R-Symmetry Channels

The inverse square root of the Laplacian appearing in (4.2.6) can be defined by its Fourier transform

$$\frac{1}{\sqrt{\boldsymbol{\Delta}}} = \left(-p^2 \right)^{-1/2} \,. \tag{4.2.8}$$

[6]In deriving (4.2.7) we use the fact that under crossing $\vec{r} \to \hat{z} - \vec{r}$ and $\boldsymbol{\Delta}$ is invariant.

In expressions of the form $\boldsymbol{\Delta}^{-\frac{1}{2}} f(r,\theta)\boldsymbol{\Delta}^{\frac{1}{2}}$, we can then use the canonical commutation relation of quantum mechanics, $[x,p]=i$, to commute $\boldsymbol{\Delta}^{\frac{1}{2}}$ through $f(r,\theta)$. For example, it is straightforward to show that

$$\boldsymbol{\Delta}^{-\frac{1}{2}} r^2 \boldsymbol{\Delta}^{\frac{1}{2}} = r^2 - \boldsymbol{\Delta}^{-1}\left(4 + 2r\partial_r\right)\,, \tag{4.2.9}$$

$$\boldsymbol{\Delta}^{-\frac{1}{2}} \tilde{z} \boldsymbol{\Delta}^{\frac{1}{2}} = \tilde{z} - \boldsymbol{\Delta}^{-1}\partial_{\tilde{z}}\,, \tag{4.2.10}$$

where we defined $\tilde{z} \equiv r\cos\theta$.

To proceed, it is convenient to decompose the solution of the Ward identity (4.2.1) in the basis

$$\begin{aligned}
e_1 &\equiv \frac{1}{\sqrt{\boldsymbol{\Delta}}} a(U,V) \quad &,\quad e_2 &\equiv \frac{1}{\sqrt{\boldsymbol{\Delta}}}\big[(U-V)\,a(U,V)\big]\,,\\
e_3 &\equiv \frac{1}{\sqrt{\boldsymbol{\Delta}}}\big[(U+V)\,a(U,V)\big] \quad &,\quad e_4 &\equiv \frac{1}{\sqrt{\boldsymbol{\Delta}}}\big[(U^2-V^2)\,a(U,V)\big]\,,\\
e_5 &\equiv \frac{1}{\sqrt{\boldsymbol{\Delta}}}\big[(U-V)^2\,a(U,V)\big]\,, &\quad e_6 &\equiv \frac{1}{\sqrt{\boldsymbol{\Delta}}}\big[(U+V)^2\,a(U,V)\big]\,.
\end{aligned} \tag{4.2.11}$$

These e_i are simply related to the different R-symmetry channels A_{ab} by

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{pmatrix} = \frac{1}{U} \begin{pmatrix} 1 & -1 & \frac{3}{4} & \frac{5}{7} & -\frac{3}{5} & \frac{2}{5} \\ -1 & 0 & \frac{1}{4} & \frac{20}{21} & -1 & \frac{14}{15} \\ 1 & 0 & -\frac{1}{4} & \frac{22}{21} & -1 & \frac{16}{15} \\ -1 & -1 & -\frac{3}{4} & -\frac{5}{7} & -\frac{3}{5} & \frac{28}{5} \\ 1 & 1 & \frac{3}{4} & -\frac{9}{7} & -\frac{7}{5} & \frac{22}{5} \\ 1 & 1 & \frac{3}{4} & \frac{19}{7} & \frac{13}{5} & \frac{42}{5} \end{pmatrix} \begin{pmatrix} A_{00} \\ A_{10} \\ A_{11} \\ A_{20} \\ A_{21} \\ A_{22} \end{pmatrix}. \tag{4.2.12}$$

Defining the operators

$$\boldsymbol{\mathcal{D}}_\pm \equiv \frac{1}{4}\sqrt{\boldsymbol{\Delta}}(U \pm V)\sqrt{\boldsymbol{\Delta}}\,, \tag{4.2.13}$$

it can be seen from (4.2.11) that the following relations hold:

$$\boldsymbol{\mathcal{D}}_+ e_1 = \boldsymbol{\Delta} e_3\,, \qquad \boldsymbol{\mathcal{D}}_- e_1 = \boldsymbol{\Delta} e_2\,, \tag{4.2.14}$$

$$\boldsymbol{\mathcal{D}}_+ e_2 = \boldsymbol{\Delta} e_4\,, \qquad \boldsymbol{\mathcal{D}}_- e_2 = \boldsymbol{\Delta} e_5\,, \tag{4.2.15}$$

$$\boldsymbol{\mathcal{D}}_+ e_3 = \boldsymbol{\Delta} e_6\,, \qquad \boldsymbol{\mathcal{D}}_- e_3 = \boldsymbol{\Delta} e_4\,, \tag{4.2.16}$$

$$\boldsymbol{\mathcal{D}}_+ e_4 = \boldsymbol{\mathcal{D}}_- e_6\,, \qquad \boldsymbol{\mathcal{D}}_- e_4 = \boldsymbol{\mathcal{D}}_+ e_5\,. \tag{4.2.17}$$

It is easy to convince oneself that these are the most general relations between the e_i that can be obtained by acting with $\boldsymbol{\mathcal{D}}_\pm$. Moreover, instead of thinking of the solution to the Ward identity as given in terms of a single unconstrained function $a(U,V)$, we can think of it as given in terms of the six constrained functions e_i, with the constraints given by (4.2.14)–(4.2.17).

The advantage of this formulation of the solution is that the constraints (4.2.14)–(4.2.17) only involve local differential operators. Indeed, using (4.2.9), (4.2.10), and the coordinate transformation (4.2.4), we find

$$\begin{aligned}
\boldsymbol{\mathcal{D}}_- &= \frac{2\tilde{z}-1}{4}\boldsymbol{\Delta} + \frac{1}{2}\partial_{\tilde{z}}\,, \\
\boldsymbol{\mathcal{D}}_+ &= \frac{1+2r^2-2\tilde{z}}{4}\boldsymbol{\Delta} + r\partial_r - \frac{1}{2}\partial_{\tilde{z}} + 1\,.
\end{aligned} \tag{4.2.18}$$

In terms of the z, $\bar{z}$ coordinates, we have

$$\begin{aligned}
\boldsymbol{\mathcal{D}}_- &= (z+\bar{z}-1)\mathcal{D}_{\frac{1}{2}} + \frac{1}{2}\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}\right)\,, \\
\boldsymbol{\mathcal{D}}_+ &= (1+2z\bar{z}-z-\bar{z})\,\mathcal{D}_{\frac{1}{2}} + \left(z-\frac{1}{2}\right)\frac{\partial}{\partial z} + \left(\bar{z}-\frac{1}{2}\right)\frac{\partial}{\partial \bar{z}} + 1\,,
\end{aligned} \tag{4.2.19}$$

where $\mathcal{D}_{\frac{1}{2}}$ was defined in (4.2.2).

4.2.3 Independent Set of Crossing Equations

Define $\tilde{e}_i$ to be the same as the e_i in (4.2.11), but with the factors of $a(U,V)$ replaced by $a(U,V)-a(V,U)$. It is clear that the $\tilde{e}_i$ also satisfy the differential equations (4.2.14)–(4.2.17). The crossing symmetry constraints are simply given by $\tilde{e}_i = 0$.

One can solve the differential equations (4.2.14)–(4.2.17) by using series expansions around the crossing symmetric point. In particular, define $\tilde{e}^i_{n,m}$ through the expansions

$$\tilde{e}_i(z,\bar{z}) = \sum_{n,m=0}^{\infty} \frac{1}{n!m!}\left(z-\frac{1}{2}\right)^n \left(\bar{z}-\frac{1}{2}\right)^m \tilde{e}^i_{n,m}\,, \tag{4.2.20}$$

$$\tilde{e}^i_{n,m} \equiv \partial^n \bar{\partial}^m e_i(z,\bar{z})\big|_{z=\bar{z}=\frac{1}{2}}\,. \tag{4.2.21}$$

From $z \leftrightarrow \bar{z}$ symmetry and (anti-)symmetry under $U \leftrightarrow V$ we have

$$\tilde{e}^i_{n,m} = \tilde{e}^i_{m,n}\,, \tag{4.2.22}$$

$$\tilde{e}^i_{n,m} = 0 \quad \text{if} \begin{cases} m+n = \text{even}\,, & i = 1,3,5,6\,, \\ m+n = \text{odd}\,, & i = 2,4\,. \end{cases} \tag{4.2.23}$$

We can now plug the expansions (4.2.20) into the differential equations (4.2.14)–(4.2.17) and solve for the coefficients $\tilde{e}^i_{n,m}$ order by order. The results can be stated as follows. If we assume only the crossing equation $\tilde{e}_2 = 0$, then equations (4.2.14)–(4.2.17) imply

$$\tilde{e}_1 = 0\,, \tag{4.2.24}$$

$$\tilde{e}_3 = 0\,, \tag{4.2.25}$$

$$\tilde{e}^4_{n,m} = a_{nm}\tilde{e}^4_{m+n,0}\,, \tag{4.2.26}$$

$$\tilde{e}^5_{n,m} = b_{nm}\tilde{e}^4_{m+n+1,0}\,, \tag{4.2.27}$$

$$\tilde{e}^6_{n,m} = c_{nm}\tilde{e}^4_{m+n-1,0}\,, \tag{4.2.28}$$

for some constants a_{nm}, b_{nm}, and c_{nm} that can be determined order by order in the expansion. We conclude that the maximal set of independent crossing equations can be taken to be $\tilde{e}^2_{n,m} = 0$ and $\tilde{e}^4_{n,0} = 0$ for all integers $n, m \geq 0$.

4.3 Superconformal Blocks

In this section we will derive the $\mathcal{N} = 8$ superconformal blocks $\mathfrak{G}_{\mathcal{M}}$ of the stress tensor four-point function as defined in (1.1.36). Any given superconformal block represents the total contribution to the four-point function coming from all operators appearing in the $\mathcal{O}_{\text{Stress}} \times \mathcal{O}_{\text{Stress}}$ OPE that belong to a given superconformal multiplet. Since superconformal multiplets are made of conformal multiplets, the superconformal blocks are just linear combinations of the usual conformal blocks. Our task is to determine which conformal blocks appear in a given superconformal block and with which coefficients.

A common approach to deriving superconformal blocks involves analyzing the detailed structure of the three-point function between two $\mathcal{O}_{\text{Stress}}$ and a third generic superconformal multiplet. In this approach one has to construct the most general superconformal invariants out of the superspace variables appearing in this three-point function (see e.g., [84]). However, it is difficult to implement this method in theories with extended supersymmetry due to complications in using superspace techniques in such theories.

In practice, we will compute the superconformal blocks in our case of interest using two different methods. One method involves expanding the solution of the Ward identity given in (4.2.6) in conformal blocks.[7] Even though this method is hard to implement due to the appearance of the non-local operator $1/\sqrt{\mathbf{\Delta}}$ in (4.2.1), significant progress was made in [1] and we will build on it in Section 4.3.2. In the next subsection we will introduce a new strategy for computing the superconformal blocks. This second method relies on the fact that the superconformal Ward identity (1.1.37) holds separately for each superconformal block. As we will see momentarily in Section 4.3.1, this approach is simpler and more systematic than working directly with the full solution to the Ward identity.

In the following, we will sometimes denote the superconformal multiplets by $(\Delta, j)_X^{[a_1\, a_2\, a_3\, a_4]}$, with (Δ, j) and $[a_1\, a_2\, a_3\, a_4]$ representing the $\mathfrak{so}(3,2)$ and $\mathfrak{so}(8)_R$ quantum numbers of the superconformal primary, and the subscript X denoting the type of shortening condition (for instance, $X = (A, 2)$ or $X = (B, +)$).

4.3.1 Superconformal Blocks from Ward Identity

Our strategy to compute the superconformal blocks is very simple. Let $\mathcal{G}_{\Delta,j}^{(a,b)}$ denote the contribution to the four-point function of a multiplet whose primary has dimension and spin (Δ, j) and transforms in the $(a,b) \equiv [0\,(a-b)\,(2b)\,0]$ irrep of $\mathfrak{so}(8)_R$. This contribution can be written as some linear combination of a finite number of conformal blocks:

$$\mathcal{G}_{\Delta,j}^{(a,b)}(z,\bar{z},\alpha,\bar{\alpha}) = \sum_{c=0}^{2}\sum_{d=0}^{c}\left[Y_{cd}(\alpha,\bar{\alpha})\sum_{\mathcal{O}\in(\Delta,j)_{a,b}}\lambda_{\mathcal{O}}^2\, G_{\Delta_{\mathcal{O}},j_{\mathcal{O}}}(z,\bar{z})\right]\,, \tag{4.3.1}$$

where $G_{\Delta,j}(z,\bar{z})$ is the conformal block corresponding to the exchange of an operator with scaling dimension Δ and Lorentz spin j. (We will determine precisely which conformal blocks appear in this sum shortly.) The innermost sum runs over all conformal primaries in the superconformal multiplet $(\Delta, j)_{a,b}$ transforming in the R-symmetry channel (c, d) (specified by the outer sums).

By using the OPE one can show that the superconformal Ward identity (1.1.37) is satisfied on each $\mathcal{G}_{\Delta,j}^{(a,b)}$ contribution independently. We can expand (4.3.1) in a Taylor series around $z = \bar{z} = 0$ using the known expansions of the conformal blocks (see, for example, [109]). Plugging in this expansion in the superconformal Ward identity (1.1.37), we can generate infinitely many equations for the undetermined coefficients $\lambda_{\mathcal{O}}^2$. These equations must be consistent if in (4.3.1) we sum over all the operators $\mathcal{O}$ belonging to a given superconformal multiplet.

[7] The superconformal blocks of $\mathcal{N} = 2$ and $\mathcal{N} = 4$ theories in $d = 4$ were first derived in this way [108].

Character Decomposition

We begin by recalling the supermultiplets that appear in the OPE, which were listed in Table 1.3. We have to identify the conformal primaries belonging to the superconformal multiplets listed in Table 1.3. For each such superconformal multiplet, we can decompose its corresponding $\mathfrak{osp}(8|4)$ character [110] into characters of the maximal bosonic sub-algebra $\mathfrak{so}(3,2) \oplus \mathfrak{so}(8)_R$.

The $\mathfrak{osp}(8|4)$ characters are defined by

$$\chi_{(\Delta;j;\mathrm{r})}(s,x,y) \equiv \mathrm{Tr}_{\mathcal{R}_{(\Delta;j;\mathrm{r})}}\left(s^{2D}x^{2J_3}y_1^{H_1}\cdots y_4^{H_4}\right)\,, \tag{4.3.2}$$

where Δ, j, and $\mathrm{r} = (r_1\,,\dots\,,r_4) \in \frac{1}{2}\mathbb{Z}^4$ are, respectively, the conformal dimension, spin and $\mathfrak{so}(8)_R$ highest weights defining the $\mathfrak{osp}(8|4)$ representation. Moreover, H_i and J_3 are the Cartan generators of $\mathfrak{so}(8)_R$ and the $\mathfrak{su}(2)$ Lorentz algebra, respectively, and D is the dilatation operator. The Dynkin labels are related to $(r_1,\dots,r_4)$ by

$$[a_1\,a_2\,a_3\,a_4] = [r_1 - r_2\,, r_2 - r_3\,, r_3 + r_4\,, r_3 - r_4]\,. \tag{4.3.3}$$

The characters are most easily computed by first computing the Verma module characters. Verma modules are infinite (reducible) representations obtained from highest weights by acting unrestrictedly with lowering ladder operators. For instance, the $\mathfrak{su}(2)$ and $\mathfrak{so}(8)_R$ Verma module characters are given by

$$C_j(x) = \frac{x^{j+1}}{x - x^{-1}}\,, \tag{4.3.4}$$

$$C_\mathrm{r}(y) = \frac{\prod_{j=1}^4 y_j^{r_j+4-j}}{\Delta(y+y^{-1})}\,, \tag{4.3.5}$$

$$\Delta(y) \equiv \prod_{1\le i<j\le 4}(y_i - y_j)\,. \tag{4.3.6}$$

The characters of irreducible representations are obtained from the Verma module characters by Weyl symmetrization, which projects out all the null states in the Verma module. For $\mathfrak{su}(2)$ and $\mathfrak{so}(8)_R$, these symmetrizations are given, respectively, by

$$\mathfrak{W}^{\mathcal{S}_2} f(x) = f(x) + f(x^{-1})\,, \tag{4.3.7}$$

$$\mathfrak{W}^{\mathcal{S}_4\ltimes(\mathcal{S}_2)^3} f(y) = \sum_{\substack{\epsilon_1\,,\dots\,,\epsilon_3=\pm1\\ \prod\epsilon_i=1}}\ \sum_{\sigma\in\mathcal{S}_4} f(y_{\sigma(1)}^{\epsilon_1}\,,\dots\,,y_{\sigma(4)}^{\epsilon_4})\,. \tag{4.3.8}$$

Indeed, acting with $\mathfrak{W}^{\mathcal{S}_2}$ and $\mathfrak{W}^{\mathcal{S}_4\ltimes(\mathcal{S}_2)^3}$ on (4.3.4) and (4.3.5), one obtains the standard expressions for the $\mathfrak{su}(2)$ and $\mathfrak{so}(8)_R$ characters,

$$\chi_j(x) = \mathfrak{W}^{\mathcal{S}_2} C_j(x) = \frac{x^{j+1} - x^{-j-1}}{x - x^{-1}}\,, \tag{4.3.9}$$

$$\begin{aligned}\chi_{\rm r}(y) &= \mathfrak{W}^{\mathcal{S}_4 \ltimes (\mathcal{S}_2)^3} C_r(y) \\ &= \left(\det\left[y_i^{r_j+4-j} + y_i^{-r_j-4+j}\right] + \det\left[y_i^{r_j+4-j} - y_i^{-r_j-4+j}\right]\right)/2\Delta(y+y^{-1})\,.\end{aligned} \tag{4.3.10}$$

Defining $\mathfrak{W} = \mathfrak{W}^{\mathcal{S}_2}\mathfrak{W}^{\mathcal{S}_4 \ltimes (\mathcal{S}_2)^3}$, the $\mathfrak{osp}(8|4)$ characters are given by

$$\chi^{(i,n)}_{(\Delta;j;r_1,\ldots,r_1,r_{n+1},\ldots,r_4)}(s,x,y) = s^{2\Delta} P(s,x)\mathfrak{W}\left(C_{2j}(x) C_{\rm r}(y) \mathcal{R}^{(i,n)}(s,x,y) \prod_{\epsilon=\pm1} \bar{\mathcal{Q}}_4(s^{-1}y, x^{\epsilon})\right)\,, \tag{4.3.11}$$

$$\chi^{(i,+)}_{(\Delta;j;r,r,r,r)}(s,x,y) = s^{2\Delta} P(s,x)\mathfrak{W}\left(C_{2j}(x) C_{\rm r}(y) \mathcal{R}^{(i,+)}(s,x,y) \prod_{\epsilon=\pm1} \bar{\mathcal{Q}}_3(s^{-1}y, x^{\epsilon})\right)\,, \tag{4.3.12}$$

where

$$\mathcal{R}^{(i,n)} = \begin{cases} \mathcal{Q}_0(sy,x)\mathcal{Q}_n(sy,x^{-1}) & i = A\,, \\ \mathcal{Q}_n(sy,x)\mathcal{Q}_n(sy,x^{-1}) & i = B\,, \end{cases} \tag{4.3.13}$$

$$\mathcal{R}^{(i,+)} = \begin{cases} \mathcal{Q}_0(sy,x)(1+sy_4^{-1}x)(1+sy_4^{-1}x^{-1}) & i = A\,, \\ (1+sy_4^{-1}x)(1+sy_4^{-1}x^{-1}) & i = B\,, \end{cases} \tag{4.3.14}$$

$$\mathcal{Q}_n(y,x) = \prod_{j=n+1}^{4}(1+y_j x)\,, \qquad \bar{\mathcal{Q}}_n(y,x) = \prod_{j=1}^{n}(1+y_j^{-1}x)\,, \tag{4.3.15}$$

$$P(s,x) = \frac{1}{1-s^4}\sum_{n=0}^{\infty} s^{2n}\chi_{2n}(x)\,. \tag{4.3.16}$$

The function $P(s,x)$ in (4.3.11) and (4.3.12) is related to the $\mathfrak{so}(3,2)$ characters $\mathcal{A}_{\Delta,j}$, computed in [111,112]:

$$\mathcal{A}_{\Delta,j} = \mathrm{Tr}_{(\Delta,j)}\left(s^{2D}x^{2J_3}\right) = s^{2\Delta}\chi_{2j}(x)P(s,x)\,. \tag{4.3.17}$$

Note that since conformal representations decompose at unitarity as

$$(\Delta,j) \xrightarrow{\Delta\to j+1} (j+1,j)_{\rm short} + (j+2,j-1)\,, \tag{4.3.18}$$

the $\mathfrak{so}(3,2)$ character of a spin-j conserved current is actually $\mathcal{A}_{j+1,j} - \mathcal{A}_{j+2,j-1}$.

In order to expand the $\mathfrak{osp}(8|4)$ characters as a sum of products of conformal characters (4.3.17) times

R-symmetry characters (4.3.10), we need to disentangle the s, x and y dependence in (4.3.11), and (4.3.12). Explicitly, it is straightforward to show that[8]

$$\chi^{(A,+)}_{(\Delta;j;r,r,r,r)}(s,x,y) = s^{2\Delta} P(s,x) \sum_{a_1,\dots,a_4=0}^{2} \sum_{\bar{a}_1,\dots,\bar{a}_4=0}^{1} s^{a_1+\cdots+a_4+\bar{a}_1+\cdots+\bar{a}_4} \chi_{2j+\bar{a}_1+\cdots+\bar{a}_4}(x) \times \left(\prod_{i=1}^{4} \chi_{j_{a_i}}(x)\right) \chi_{(r+\bar{a}_1-a_1,\dots,r+\bar{a}_4-a_4)}(y)\,, \tag{4.3.19}$$

$$\chi^{(B,+)}_{(\Delta;0;r,r,r,r)}(s,x,y) = s^{2\Delta} P(s,x) \sum_{a_1,\dots,a_4=0}^{2} s^{a_1+\cdots+a_4} \left(\prod_{i=1}^{4} \chi_{j_{a_i}}(x)\right) \chi_{(r-a_1,\dots,r-a_4)}(y)\,, \tag{4.3.20}$$

$$\chi^{(A,n)}_{(\Delta;j;r_1,\dots,r_1,r_{n+1},\dots,r_4)}(s,x,y) = s^{2\Delta} P(s,x) \sum_{a_1,\dots,a_4=0}^{2} \sum_{\bar{a}_{n+1},\dots,\bar{a}_4=0}^{2} \sum_{\bar{a}_1,\dots,\bar{a}_n=0}^{1} s^{a_1+\cdots+a_4+\bar{a}_1+\cdots+\bar{a}_4} \chi_{2j+\bar{a}_1+\cdots+\bar{a}_n}(x) \times \left(\prod_{i=n+1}^{4} \chi_{j_{\bar{a}_i}}(x)\right) \left(\prod_{i=1}^{4} \chi_{j_{a_i}}(x)\right) \chi_{(r_1+\bar{a}_1-a_1,\dots,r_4+\bar{a}_4-a_4)}(y)\,, \tag{4.3.21}$$

$$\chi^{(B,n)}_{(\Delta;0;r_1,\dots,r_1,r_{n+1},\dots,r_4)}(s,x,y) = s^{2\Delta} P(s,x) \sum_{a_1,\dots,a_4,\bar{a}_{n+1},\dots,\bar{a}_4=0}^{2} s^{a_1+\cdots+a_4+\bar{a}_{n+1}+\cdots+\bar{a}_4} \left(\prod_{i=n+1}^{4} \chi_{j_{\bar{a}_i}}(x)\right) \times \left(\prod_{i=1}^{4} \chi_{j_{a_i}}(x)\right) \chi_{(r_1-a_1,\dots,r_1-a_n,r_{n+1}+\bar{a}_{n+1}-a_{n+1},\dots,r_4+\bar{a}_4-a_4)}(y)\,, \tag{4.3.22}$$

where $j_a \equiv a \pmod 2$.

The products of the $\mathfrak{su}(2)$ characters in (4.3.19)–(4.3.22) are easily transformed into sums of such characters by decomposing $\mathfrak{su}(2)$ tensor products. After doing so, we see that (4.3.19)–(4.3.22) become sums over $\mathfrak{so}(3,2) \oplus \mathfrak{so}(8)$ characters, as desired.[9]

Using these character formulae, we find the following decompositions of supermultiplets. The conformal primaries of the stress-tensor multiplet $(1,0)^{[0020]}_{(B,+)}$ were already given in Table 1.2. The conformal primaries of all the other multiplets appearing in Table 1.3 are given in Tables 4.1–4.5. The first column in these tables contains the conformal dimensions and the other columns contain the possible values of the spins in the various R-symmetry channels. In each table, we only list the operators which could possibly contribute to our OPE, namely only operators with R-symmetry representations in the tensor product (1.1.35), and only even (odd) integer spins for the representations (a,b) with even (odd) $a+b$.

[8]Note that for the B series $\Delta = r_1$, while for the A series $\Delta = r_1 + j + 1$ except for the long multiplet $(A,0)$ for which $\Delta \geq r_1 + j + 1$.

[9]Sometimes the $\mathfrak{so}(8)_R$ characters in (4.3.19)–(4.3.22) appear with negative Dynkin labels. One can then try to use the identity

$$\chi_{\mathrm{r}^\omega}(y) = (-)^{\ell(\omega)} \chi_{\mathrm{r}}(y)\,,$$

to obtain a character with non-negative Dynkin labels. In this identity $\omega \in \mathcal{S}_4 \ltimes (\mathcal{S}_2)^3$ is a Weyl transformation, $\mathrm{r}^\omega = \omega(\mathrm{r}+\rho)-\rho$ is a Weyl reflection, $\rho = (3,2,1,0)$ is the Weyl vector, and $(-)^{\ell(\omega)}$ is the signature of the Weyl transformation. If there is no Weyl transformation such that r^ω correspond to non-negative integer Dynkin labels, then $\chi_{\mathrm{r}} = 0$.

$(2,0)^{[0040]}_{(B,+)}$	spins in various $\mathfrak{so}(8)_R$ irreps					
dimension	**1** [0000]	**28** [0100]	$\mathbf{35}_c$ [0020]	**300** [0200]	$\mathbf{567}_c$ [0120]	$\mathbf{294}_c$ [0040]
2	–	–	–	–	–	0
3	–	–	–	–	1	–
4	–	–	2	0	–	–
5	–	1	–	–	–	–
6	0	–	–	–	–	–

Table 4.1: All possible conformal primaries in $\mathcal{O}_{\text{Stress}} \times \mathcal{O}_{\text{Stress}}$ corresponding to the $(2,0)^{[0040]}_{(B,+)}$ superconformal multiplet.

$(2,0)^{[0200]}_{(B,2)}$	spins in various $\mathfrak{so}(8)_R$ irreps					
dimension	**1** [0000]	**28** [0100]	$\mathbf{35}_c$ [0020]	**300** [0200]	$\mathbf{567}_c$ [0120]	$\mathbf{294}_c$ [0040]
2	–	–	–	0	–	–
3	–	1	–	0	1	–
4	0	1	0, 2	0, 2	1	0
5	0	1, 3	2	0	1	–
6	0, 2	1	2	0	–	–
7	0	1	–	–	–	–
8	0	–	–	–	–	–

Table 4.2: All possible conformal primaries in $\mathcal{O}_{\mathbf{35}_c} \times \mathcal{O}_{\mathbf{35}_c}$ corresponding to the $(2,0)^{[0200]}_{(B,2)}$ superconformal multiplet.

$(j+2,j)^{[0020]}_{(A,+)}$	spins in various $\mathfrak{so}(8)_R$ irreps					
dimension	**1** [0000]	**28** [0100]	$\mathbf{35}_c$ [0020]	**300** [0200]	$\mathbf{567}_c$ [0120]	$\mathbf{294}_c$ [0040]
$j+2$	–	–	j	–	–	–
$j+3$	–	$j\pm1$	j	–	$j+1$	–
$j+4$	$j\pm2,\ j$	$j\pm1$	$j+2,\ j$	$j+2,\ j$	$j+1$	$j+2$
$j+5$	$j+2$	$j+3,\ j\pm1$	$j+2,\ j$	$j+2$	$j+3,\ j+1$	–
$j+6$	$j+2$	$j+3,\ j+1$	$j+4,\ j+2,\ j$	$j+2$	–	–
$j+7$	$j+2$	$j+3,\ j+1$	–	–	–	–
$j+8$	$j+2$	–	–	–	–	–

Table 4.3: All possible conformal primaries in $\mathcal{O}_{\mathbf{35}_c} \times \mathcal{O}_{\mathbf{35}_c}$ corresponding to the $(j+2,j)^{[0020]}_{(A,+)}$ superconformal multiplet, with $j \geq 2$ even. For $j=0$ one should omit the representations with negative spins as well as $(4,0)^{[0000]}$.

$(j+2,j)_{(A,2)}^{[0100]}$	spins in various $\mathfrak{so}(8)_R$ irreps					
dimension	**1** [0000]	**28** [0100]	$\mathbf{35}_c$ [0020]	**300** [0200]	$\mathbf{567}_c$ [0120]	$\mathbf{294}_c$ [0040]
$j+2$	–	j	–	–	–	–
$j+3$	$j\pm1$	j	$j\pm1$	$j+1$	–	–
$j+4$	$j+1$	$j\pm2,j$	$j\pm1$	$j+1$	$j+2,j$	–
$j+5$	$j\pm3,j\pm1$	$j\pm2,j$	$j+3,j\pm1$	$j+3,j\pm1$	$j+2,j$	$j+1$
$j+6$	$j+3,j+1$	$j+4,j\pm2,j$	$j+3,j\pm1$	$j+1$	$j+2,j$	–
$j+7$	$j+3,j+1$	$j+2,j$	$j+3,j\pm1$	$j+1$	–	–
$j+8$	$j+1$	$j+2,j$	–	–	–	–
$j+9$	$j+1$	–	–	–	–	–

Table 4.4: All possible conformal primaries in $\mathcal{O}_{\text{Stress}}\times\mathcal{O}_{\text{Stress}}$ corresponding to the $(j+2,j)_{(A,2)}^{[0100]}$ superconformal multiplet, with j odd. For $j=1$ one should omit $(6,0)^{[0000]}$ and representations with negative spin.

$(\Delta,j)_{(A,0)}^{[0000]}$	spins in various $\mathfrak{so}(8)_R$ irreps					
dimension	**1** [0000]	**28** [0100]	$\mathbf{35}_c$ [0020]	**300** [0200]	$\mathbf{567}_c$ [0120]	$\mathbf{294}_c$ [0040]
Δ	j	–	–	–	–	–
$\Delta+1$	j	$j\pm1$	–	–	–	–
$\Delta+2$	j	$j\pm1$	$j\pm2,j$	j	–	–
$\Delta+3$	j	$j\pm3,j\pm1$	$j\pm2,j$	j	$j\pm1$	–
$\Delta+4$	$j\pm4,j\pm2,j$	$j\pm3,j\pm1$	$j\pm2,j$	$j\pm2,j$	$j\pm1$	j
$\Delta+5$	j	$j\pm3,j\pm1$	$j\pm2,j$	j	$j\pm1$	–
$\Delta+6$	j	$j\pm1$	$j\pm2,j$	j	–	–
$\Delta+7$	j	$j\pm1$	–	–	–	–
$\Delta+8$	j	–	–	–	–	–

Table 4.5: All possible conformal primaries in $\mathcal{O}_{\mathbf{35}_c}\times\mathcal{O}_{\mathbf{35}_c}$ corresponding to the $(\Delta,j)_{(A,0)}^{[0000]}$ (long) superconformal multiplet, with j even, $\Delta\geq j+1$. The decomposition of this multiplet at unitarity contains a conserved current multiplet, which, in turn, contains higher-spin conserved currents.

Explicit Formulae for Superconformal Blocks

Using these decompositions and the Ward identity we can now determine the superconformal blocks. In practice, we expand (4.3.1) to a high enough order so that we get an overdetermined system of linear equations in the $\lambda_{\mathcal{O}}^2$. We can then solve for the OPE coefficients in terms of one overall coefficient. The fact that we can successfully solve an overdetermined system of equations is a strong consistency check on our computation.

As an interesting feature of the superconformal blocks, we find that the OPE coefficients of all the operators which are marked in red in Tables 4.1–4.5 vanish. These operators are precisely the super-descendants obtained by acting on the superconformal primary with $\varepsilon^{\alpha\beta} Q_{a\alpha} Q_{b\beta}$ an odd number of times. This combination of supercharges is odd under parity, while $\mathcal{O}_{\text{Stress}}$ is even. There is no a priori reason, however, why an $\mathcal{N} = 8$ SCFT should be invariant under parity, even though all known examples do have this property. Our findings show that even if parity is not a symmetry of the full theory, it is a symmetry of the $\mathcal{O}_{\text{Stress}} \times \mathcal{O}_{\text{Stress}}$ OPE.[10]

Let us write our results for the superconformal blocks in order of increasing complexity. In all the supermultiplets, we normalize the coefficient of the superconformal primary to one. The results are presented in terms of the R-symmetry channels $A_{ab}(U, V)$, which were defined in (1.1.33).

For $(1,0)^{[0020]}_{(B,+)}$, corresponding to the stress-tensor multiplet, we have

$$A_{11}(U,V) = G_{1,0}(U,V)\,, \tag{4.3.23}$$

$$A_{10}(U,V) = -G_{2,1}(U,V)\,, \tag{4.3.24}$$

$$A_{00}(U,V) = \frac{1}{4} G_{3,2}(U,V)\,. \tag{4.3.25}$$

The superconformal blocks corresponding to $(2,0)^{[0040]}_{(B,+)}$ are

$$A_{22}(U,V) = G_{2,0}(U,V)\,, \tag{4.3.26}$$

$$A_{21}(U,V) = -\frac{4}{3} G_{3,1}(U,V)\,, \tag{4.3.27}$$

$$A_{20}(U,V) = \frac{16}{45} G_{4,0}(U,V)\,, \tag{4.3.28}$$

$$A_{11}(U,V) = \frac{256}{675} G_{4,2}(U,V)\,, \tag{4.3.29}$$

$$A_{10}(U,V) = -\frac{128}{875} G_{5,1}(U,V)\,, \tag{4.3.30}$$

$$A_{00}(U,V) = \frac{256}{18375} G_{6,0}(U,V)\,. \tag{4.3.31}$$

[10] A similar phenomenon occurs in four dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mills theory [108]. There, the operators that decouple are the ones which are not invariant under the "bonus symmetry" discussed in [113, 114].

For $(2,0)^{[0200]}_{(B,2)}$, the superconformal blocks are

$$A_{22}(U,V) = \frac{8}{9} G_{4,0}(U,V)\,, \tag{4.3.32}$$

$$A_{21}(U,V) = -\frac{8}{3} G_{3,1}(U,V) - \frac{192}{175} G_{5,1}(U,V)\,, \tag{4.3.33}$$

$$A_{20}(U,V) = G_{2,0}(U,V) + \frac{16}{63} G_{4,0}(U,V) + \frac{64}{45} G_{4,2}(U,V) + \frac{256}{1225} G_{6,0}(U,V)\,, \tag{4.3.34}$$

$$A_{11}(U,V) = \frac{32}{135} G_{4,0}(U,V) + \frac{512}{945} G_{4,2}(U,V) + \frac{8192}{25725} G_{6,2}(U,V)\,, \tag{4.3.35}$$

$$A_{10}(U,V) = -\frac{12}{35} G_{3,1}(U,V) - \frac{128}{525} G_{5,1}(U,V) - \frac{2304}{6125} G_{5,3}(U,V) - \frac{1024}{11319} G_{7,1}(U,V)\,, \tag{4.3.36}$$

$$A_{00}(U,V) = \frac{16}{735} G_{4,0}(U,V) + \frac{512}{56595} G_{6,0}(U,V) + \frac{1024}{25725} G_{6,2}(U,V) + \frac{5120}{539539} G_{8,0}(U,V)\,. \tag{4.3.37}$$

For $(j+2,j)^{[0200]}_{(A,+)}$, we find

$$A_{22}(U,V) = \frac{16}{3} G_{j+4,j+2}(U,V)\,, \tag{4.3.38}$$

$$\begin{aligned} A_{21}(U,V) &= -4G_{j+3,j+1}(U,V) - \frac{32(j+2)(j+3)^2}{(2j+5)^2(2j+7)} G_{j+5,j+1}(U,V) \\ &\quad - \frac{64(j+3)^4}{(4j^2+24j+35)^2} G_{j+5,j+3}(U,V)\,, \end{aligned} \tag{4.3.39}$$

$$\begin{aligned} A_{20}(U,V) &= \frac{4(j+1)}{2j+3} G_{j+4,j}(U,V) + \frac{32(j+2)(j+3)}{3(2j+3)(2j+7)} G_{j+4,j+2}(U,V) \\ &\quad + \frac{64(j+3)^3(j+4)^2}{(2j+5)(2j+7)^3(2j+9)} G_{j+6,j+2}(U,V)\,, \end{aligned} \tag{4.3.40}$$

$$\begin{aligned} A_{11}(U,V) &= G_{j+2,j}(U,V) + \frac{16(j+1)(j+2)(j+3)}{3(2j+3)^2(2j+7)} G_{j+4,j}(U,V) \\ &\quad + \frac{64(j+2)^2(j+3)^2}{9(2j+3)^2(2j+7)^2} G_{j+4,j+2}(U,V) + \frac{48(j+1)(j+2)(j+3)^2(j+4)^2}{(2j+3)(2j+5)^2(2j+7)^2(2j+9)} G_{j+6,j}(U,V) \\ &\quad + \frac{256(j+2)(j+3)^4(j+4)^2}{3(2j+3)(2j+5)(2j+7)^4(2j+9)} G_{j+6,j+2}(U,V) + \frac{256(j+3)^4(j+4)^4}{(2j+5)^2(2j+7)^4(2j+9)^2} G_{j+6,j+4}(U,V)\,, \end{aligned} \tag{4.3.41}$$

$$\begin{aligned} A_{10}(U,V) &= -\frac{j}{2j+1} G_{j+3,j-1}(U,V) - \frac{12(j+1)(j+3)}{5(2j+1)(2j+7)} G_{j+3,j+1}(U,V) \\ &\quad - \frac{6j(j+1)(j+3)^2}{(2j+1)(2j+3)(2j+5)(2j+7)} G_{j+5,j-1}(U,V) \\ &\quad - \frac{48(j+2)(2j(j+5)(4j(j+5)+35)+137)(j+3)^2}{5(2j+1)(2j+3)(2j+5)^2(2j+7)^2(2j+9)} G_{j+5,j+1}(U,V) \\ &\quad - \frac{192(j+2)(j+3)^4(j+4)}{5(2j+3)(2j+5)^2(2j+7)^2(2j+9)} G_{j+5,j+3}(U,V) \\ &\quad - \frac{96(j+2)(j+3)^3(j+4)^2(j+5)^2}{(2j+5)^2(2j+7)^3(2j+9)^2(2j+11)} G_{j+7,j+1}(U,V) \\ &\quad - \frac{256(j+3)^4(j+4)^3(j+5)^2}{(2j+5)^2(2j+7)^3(2j+9)^3(2j+11)} G_{j+7,j+3}(U,V)\,, \end{aligned} \tag{4.3.42}$$

$$\begin{aligned} A_{00}(U,V) &= \frac{3(j-1)j}{32j^2-8} G_{j+4,j-2}(U,V) + \frac{4j(j+1)(j+3)}{7(2j-1)(2j+3)(2j+7)} G_{j+4,j}(U,V) \\ &\quad + \frac{72(j+1)(j+2)(j+3)(j+4)}{35(2j+1)(2j+3)(2j+7)(2j+9)} G_{j+4,j+2}(U,V) \\ &\quad + \frac{64(j+2)(j+3)^3(j+4)^2(j+5)}{7(2j+3)(2j+5)(2j+7)^3(2j+9)(2j+11)} G_{j+6,j+2}(U,V) \\ &\quad + \frac{96(j+3)^3(j+4)^3(j+5)^2(j+6)^2}{(2j+5)(2j+7)^3(2j+9)^3(2j+11)^2(2j+13)} G_{j+8,j+2}(U,V)\,. \end{aligned} \tag{4.3.43}$$

The blocks for $(j+2,j)^{[0100]}_{(A,2)}$ are given by

$$A_{22}(U,V) = \frac{32(j+2)}{6j+15} G_{j+5,j+1}(U,V)\,, \tag{4.3.44}$$

$$\begin{aligned} A_{21}(U,V) = &-\frac{8(j+1)}{2j+3} G_{j+4,j}(U,V) - \frac{32(j+2)^2}{(2j+3)(2j+5)} G_{j+4,j+2}(U,V) \\ &- \frac{48(j+1)(j+2)(j+4)^2}{(2j+3)(2j+5)(2j+7)(2j+9)} G_{j+6,j}(U,V) - \frac{128(j+2)^2(j+3)(j+4)^2}{(2j+3)(2j+5)(2j+7)^2(2j+9)} G_{j+6,j+2}(U,V)\,, \end{aligned} \tag{4.3.45}$$

$$\begin{aligned} A_{20}(U,V) = &\, 4G_{j+3,j+1}(U,V) + \frac{6j(j+1)}{4j(j+2)+3} G_{j+5,j-1}(U,V) + \frac{64(j+2)\left(j^2+5j+3\right)}{3(2j+5)\left(4j^2+20j+9\right)} G_{j+5,j+1}(U,V) \\ &+ \frac{64(j+2)^2(j+3)^2}{(2j+3)(2j+5)^2(2j+7)} G_{j+5,j+3}(U,V) + \frac{96(j+2)(j+3)(j+4)^2(j+5)^2}{(2j+5)(2j+7)^2(2j+9)^2(2j+11)} G_{j+7,j+1}(U,V)\,, \end{aligned} \tag{4.3.46}$$

$$\begin{aligned} A_{11}(U,V) = &\, \frac{2j}{2j+1} G_{j+3,j-1}(U,V) + \frac{16(j+1)(j+2)}{3(2j+1)(2j+5)} G_{j+3,j+1}(U,V) \\ &+ \frac{8j(j+1)(j+3)(j+4)}{(2j+1)(2j+3)(2j+5)(2j+9)} G_{j+5,j-1}(U,V) \\ &+ \frac{32(j+2)^2(j+3)(j(j+5)(52j(j+5)+445)+822)}{9(2j+1)(2j+3)(2j+5)^3(2j+7)(2j+9)} G_{j+5,j+1}(U,V) \\ &+ \frac{256(j+2)^2(j+3)^3(j+4)}{3(2j+3)(2j+5)^3(2j+7)(2j+9)} G_{j+5,j+3}(U,V) \\ &+ \frac{80j(j+1)(j+2)(j+4)^2(j+5)^2}{(2j+1)(2j+3)(2j+5)(2j+7)(2j+9)^2(2j+11)} G_{j+7,j-1}(U,V) \\ &+ \frac{128(j+1)(j+2)^2(j+3)(j+4)^2(j+5)^2}{(2j+1)(2j+5)^2(2j+7)^2(2j+9)^2(2j+11)} G_{j+7,j+1}(U,V) \\ &+ \frac{512(j+2)^2(j+3)^2(j+4)^3(j+5)^2}{(2j+3)(2j+5)^2(2j+7)^2(2j+9)^3(2j+11)} G_{j+7,j+3}(U,V)\,, \end{aligned} \tag{4.3.47}$$

$$
\begin{aligned}
A_{10}(U,V) = &-G_{j+2,j}(U,V) - \frac{3(j-1)j}{8j^2-2}G_{j+4,j-2}(U,V) - \frac{4(j+1)(j+2)^2(44j(j+4)-75)}{5(2j-1)(2j+3)^2(2j+5)(2j+9)}G_{j+4,j}(U,V) \\
&- \frac{48(j+2)^2(2j(j+5)(4j(j+5)+35)+137)}{5(2j+1)(2j+3)^2(2j+5)(2j+7)(2j+9)}G_{j+4,j+2}(U,V) \\
&- \frac{10j\left(j^2-1\right)(j+4)^2}{(2j-1)(2j+1)(2j+3)(2j+7)(2j+9)}G_{j+6,j-2}(U,V) \\
&- \frac{72(j+1)(j+2)(2j(j+5)-3)(j+4)^2}{5(2j-1)(2j+3)(2j+5)(2j+7)(2j+9)(2j+11)}G_{j+6,j}(U,V) \\
&- \frac{64(j+2)^2(j+3)^3(44j(j+6)+145)(j+4)^2}{5(2j+1)(2j+3)(2j+5)^2(2j+7)^3(2j+9)(2j+11)}G_{j+6,j+2}(U,V) \\
&- \frac{256(j+2)^2(j+3)^2(j+4)^4}{(2j+3)(2j+5)^2(2j+7)^3(2j+9)^2}G_{j+6,j+4}(U,V) \\
&- \frac{160(j+1)(j+2)(j+3)(j+5)^2(j+6)^2(j+4)^2}{(2j+3)(2j+5)(2j+7)^2(2j+9)^2(2j+11)^2(2j+13)}G_{j+8,j}(U,V) \\
&- \frac{384(j+2)^2(j+3)(j+4)^3(j+5)^2(j+6)^2}{(2j+3)(2j+5)(2j+7)^2(2j+9)^3(2j+11)^2(2j+13)}G_{j+8,j+2}(U,V)\,,
\end{aligned} \tag{4.3.48}
$$

$$
\begin{aligned}
A_{00}(U,V) = &\frac{j}{8j+4}G_{j+3,j-1}(U,V) + \frac{4(j+1)(j+4)}{7(2j+1)(2j+9)}G_{j+3,j+1}(U,V) \\
&+ \frac{5(j-2)(j-1)j}{8(2j-3)(2j-1)(2j+1)}G_{j+5,j-3}(U,V) + \frac{6j(j+2)\left(j^2-1\right)}{7(2j+5)\left(8j^3+4j^2-18j-9\right)}G_{j+5,j-1}(U,V) \\
&+ \frac{144(j+2)^2(j+3)(j(j+5)(4j(j+5)+5)-14)}{35(2j-1)(2j+1)(2j+3)(2j+5)(2j+7)(2j+9)(2j+11)}G_{j+5,j+1}(U,V) \\
&+ \frac{64(j+1)(j+2)^2(j+3)^2(j+4)}{7(2j+1)(2j+3)(2j+5)^2(2j+7)(2j+9)}G_{j+5,j+3}(U,V) \\
&+ \frac{96(j+2)(j+3)^2(j+5)^2(j+6)(j+4)^2}{7(2j+5)^2(2j+7)^2(2j+9)^2(2j+11)(2j+13)}G_{j+7,j+1}(U,V) \\
&+ \frac{64(j+2)^2(j+3)^2(j+4)^3(j+5)^2}{(2j+3)(2j+5)^2(2j+7)^2(2j+9)^3(2j+11)}G_{j+7,j+3}(U,V) \\
&+ \frac{160(j+2)(j+3)(j+5)^2(j+6)^2(j+7)^2(j+4)^3}{(2j+5)(2j+7)^2(2j+9)^3(2j+11)^2(2j+13)^2(2j+15)}G_{j+9,j+1}(U,V)\,.
\end{aligned} \tag{4.3.49}
$$

Finally, for the long multiplet $(\Delta, j)^{[0000]}_{(A,0)}$ we find

$$
A_{22}(U,V) = \frac{128(\Delta-j+1)(\Delta-j-1)(\Delta+j)(\Delta+j+2)}{3(\Delta-j+2)(\Delta-j)(\Delta+j+1)(\Delta+j+3)}G_{\Delta+4,j}(U,V)\,, \tag{4.3.50}
$$

$$
\begin{aligned}
A_{21}(U,V) = &-\frac{64j(\Delta-j+1)(\Delta-j-1)(\Delta+j)}{(2j+1)(\Delta-j+2)(\Delta-j)(\Delta+j+1)}G_{\Delta+3,j-1}(U,V)\\
&-\frac{64(j+1)(\Delta-j-1)(\Delta+j)(\Delta+j+2)}{(2j+1)(\Delta-j)(\Delta+j+1)(\Delta+j+3)}G_{\Delta+3,j+1}(U,V)\\
&-\frac{256(\Delta+3)^2 j}{(2\Delta+5)(2\Delta+7)(2j+1)}\\
&\times\frac{(\Delta-j+3)(\Delta-j+1)(\Delta-j-1)(\Delta+j)(\Delta+j+2)}{(\Delta-j+4)(\Delta-j+2)(\Delta-j)(\Delta+j+1)(\Delta+j+3)}G_{\Delta+5,j-1}(U,V)\\
&-\frac{256(\Delta+3)^2(j+1)}{(2\Delta+5)(2\Delta+7)(2j+1)}\\
&\times\frac{(\Delta-j+1)(\Delta-j-1)(\Delta+j)(\Delta+j+2)(\Delta+j+4)}{(\Delta-j+2)(\Delta-j)(\Delta+j+1)(\Delta+j+3)(\Delta+j+5)}G_{\Delta+5,j+1}(U,V)\,,
\end{aligned}
\tag{4.3.51}
$$

$$
\begin{aligned}
A_{20}(U,V) = &\frac{16(\Delta-j-1)(\Delta+j)}{(\Delta-j)(\Delta+j+1)}G_{\Delta+2,j}(U,V)\\
&+\frac{64(j-1)j(\Delta-j+3)(\Delta-j+1)(\Delta-j-1)(\Delta+j)}{(4j^2-1)\,(\Delta-j+4)(\Delta-j+2)(\Delta-j)(\Delta+j+1)}G_{\Delta+4,j-2}(U,V)\\
&+\frac{8(\Delta-j+1)(\Delta-j-1)(\Delta+j)(\Delta+j+2)\left(\frac{3}{2\Delta+3}-\frac{3}{2\Delta+7}+\frac{4(8j(j+1)-3)}{4j(j+1)-3}\right)}{3(\Delta-j+2)(\Delta-j)(\Delta+j+1)(\Delta+j+3)}G_{\Delta+4,j}(U,V)\\
&+\frac{64(j+1)(j+2)(\Delta-j-1)(\Delta+j)(\Delta+j+2)(\Delta+j+4)}{(2j+1)(2j+3)(\Delta-j)(\Delta+j+1)(\Delta+j+3)(\Delta+j+5)}G_{\Delta+4,j+2}(U,V)\\
&+\frac{256(\Delta+3)^2(\Delta+4)^2}{(2\Delta+5)(2\Delta+7)^2(2\Delta+9)}\\
&\times\frac{(\Delta-j+3)(\Delta-j+1)(\Delta-j-1)(\Delta+j)(\Delta+j+2)(\Delta+j+4)}{(\Delta-j+4)(\Delta-j+2)(\Delta-j)(\Delta+j+1)(\Delta+j+3)(\Delta+j+5)}G_{\Delta+6,j}(U,V)\,,
\end{aligned}
\tag{4.3.52}
$$

$$
\begin{aligned}
A_{11}(U,V) = &\frac{32(j-1)j(\Delta-j+1)(\Delta-j-1)}{(2j-1)(2j+1)(\Delta-j+2)(\Delta-j)}G_{\Delta+2,j-2}(U,V)\\
&+\frac{64j(j+1)(\Delta-j-1)(\Delta+j)}{3(2j-1)(2j+3)(\Delta-j)(\Delta+j+1)}G_{\Delta+2,j}(U,V)\\
&+\frac{32(j+1)(j+2)(\Delta+j)(\Delta+j+2)}{(4j^2+8j+3)\,(\Delta+j+1)(\Delta+j+3)}G_{\Delta+2,j+2}(U,V)\\
&+\frac{512(\Delta+2)(\Delta+3)j(j+1)(\Delta-j+1)(\Delta-j-1)(\Delta+j)(\Delta+j+2)}{9(2\Delta+3)(2\Delta+7)(2j-1)(2j+3)(\Delta-j+2)(\Delta-j)(\Delta+j+1)(\Delta+j+3)}G_{\Delta+4,j}(U,V)\\
&+\frac{256(\Delta+2)(\Delta+3)(j+1)(j+2)}{3(2\Delta+3)(2\Delta+7)(2j+1)(2j+3)}\\
&\times\frac{(\Delta-j-1)(\Delta+j)(\Delta+j+2)(\Delta+j+4)}{(\Delta-j)(\Delta+j+1)(\Delta+j+3)(\Delta+j+5)}G_{\Delta+4,j+2}(U,V)\\
&+\frac{256(\Delta+2)(\Delta+3)(j-1)j(\Delta-j+3)(\Delta-j+1)(\Delta-j-1)(\Delta+j)}{3(2\Delta+3)(2\Delta+7)\,(4j^2-1)\,(\Delta-j+4)(\Delta-j+2)(\Delta-j)(\Delta+j+1)}G_{\Delta+4,j-2}(U,V)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{512(\Delta+3)^2(\Delta+4)^2(j-1)j}{(2\Delta+5)(2\Delta+7)^2(2\Delta+9)(2j-1)(2j+1)} \\
&\times\frac{(\Delta-j+5)(\Delta-j+3)(\Delta-j+1)(\Delta-j-1)(\Delta+j)(\Delta+j+2)}{(\Delta-j+6)(\Delta-j+4)(\Delta-j+2)(\Delta-j)(\Delta+j+1)(\Delta+j+3)}G_{\Delta+6,j-2}(U,V)+ \\
&+\frac{1024(\Delta+3)^2(\Delta+4)^2j(j+1)}{3(2\Delta+5)(2\Delta+7)^2(2\Delta+9)(2j-1)(2j+3)} \\
&\times\frac{(\Delta-j+3)(\Delta-j+1)(\Delta-j-1)(\Delta+j)(\Delta+j+2)(\Delta+j+4)}{(\Delta-j+4)(\Delta-j+2)(\Delta-j)(\Delta+j+1)(\Delta+j+3)(\Delta+j+5)}G_{\Delta+6,j}(U,V) \\
&+\frac{512(\Delta+3)^2(\Delta+4)^2(j+1)(j+2)}{(2\Delta+5)(2\Delta+7)^2(2\Delta+9)(2j+1)(2j+3)} \\
&\times\frac{(\Delta-j+1)(\Delta-j-1)(\Delta+j)(\Delta+j+2)(\Delta+j+4)(\Delta+j+6)}{(\Delta-j+2)(\Delta-j)(\Delta+j+1)(\Delta+j+3)(\Delta+j+5)(\Delta+j+7)}G_{\Delta+6,j+2}(U,V)\,, \qquad (4.3.53)
\end{aligned}
$$

$$
\begin{aligned}
A_{10}(U,V) = &-\frac{8j(\Delta-j-1)}{(2j+1)(\Delta-j)}G_{\Delta+1,j-1}(U,V)-\frac{8(j+1)(\Delta+j)}{(2j+1)(\Delta+j+1)}G_{\Delta+1,j+1}(U,V) \\
&-\frac{32(j-2)(j-1)j(\Delta-j+3)(\Delta-j+1)(\Delta-j-1)}{(2j-3)(2j-1)(2j+1)(\Delta-j+4)(\Delta-j+2)(\Delta-j)}G_{\Delta+3,j-3}(U,V)- \\
&-\frac{96j}{5(2j-3)(2j+1)(2j+3)(2\Delta+1)(2\Delta+7)} \\
&\times\frac{(\Delta-j+1)(\Delta-j-1)(\Delta+j)\left((8\Delta(\Delta+4)+19)j^2-13\Delta(\Delta+4)-34\right)G_{\Delta+3,j-1}(U,V)}{(\Delta-j+2)(\Delta-j)(\Delta+j+1)} \\
&-\frac{96(j+1)\left((8\Delta(\Delta+4)+19)j^2+2(8\Delta(\Delta+4)+19)j-5(\Delta+1)(\Delta+3)\right)}{5(2j-1)(2j+1)(2j+5)(2\Delta+1)(2\Delta+7)} \\
&\times\frac{(\Delta-j-1)(\Delta+j)(\Delta+j+2)}{(\Delta-j)(\Delta+j+1)(\Delta+j+3)}G_{\Delta+3,j+1}(U,V) \\
&-\frac{32(j+1)(j+2)(j+3)(\Delta+j)(\Delta+j+2)(\Delta+j+4)}{(2j+1)(2j+3)(2j+5)(\Delta+j+1)(\Delta+j+3)(\Delta+j+5)}G_{\Delta+3,j+3}(U,V) \\
&-\frac{128(j-2)(j-1)j(\Delta+3)^2}{(2j-3)(2j-1)(2j+1)(2\Delta+5)(2\Delta+7)} \\
&\times\frac{(\Delta-j+5)(\Delta-j+3)(\Delta-j+1)(\Delta-j-1)(\Delta+j)}{(\Delta-j+6)(\Delta-j+4)(\Delta-j+2)(\Delta-j)(\Delta+j+1)}G_{\Delta+5,j-3}(U,V) \\
&-\frac{384j\left((8\Delta(\Delta+6)+59)j^2-13\Delta(\Delta+6)-99\right)(\Delta+3)^2}{5(2j-3)(2j+1)(2j+3)(2\Delta+3)(2\Delta+5)(2\Delta+7)(2\Delta+9)} \\
&\times\frac{(\Delta-j+3)(\Delta-j+1)(\Delta-j-1)(\Delta+j)(\Delta+j+2)}{(\Delta-j+4)(\Delta-j+2)(\Delta-j)(\Delta+j+1)(\Delta+j+3)}G_{\Delta+5,j-1}(U,V) \\
&-\frac{384(j+1)\left((8\Delta(\Delta+6)+59)j^2+2(8\Delta(\Delta+6)+59)j-5(\Delta+2)(\Delta+4)\right)}{5(2j-1)(2j+1)(2j+5)(2\Delta+3)(2\Delta+5)(2\Delta+7)(2\Delta+9)} \\
&\times\frac{(\Delta-j+1)(\Delta-j-1)(\Delta+3)^2(\Delta+j)(\Delta+j+2)(\Delta+j+4)}{(\Delta-j+2)(\Delta-j)(\Delta+j+1)(\Delta+j+3)(\Delta+j+5)}G_{\Delta+5,j+1}(U,V)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{128(j+1)(j+2)(j+3)(\Delta+3)^2}{(2j+1)(2j+3)(2j+5)(2\Delta+5)(2\Delta+7)} \\
&\times \frac{(\Delta-j-1)(\Delta+j)(\Delta+j+2)(\Delta+j+4)(\Delta+j+6)}{(\Delta-j)(\Delta+j+1)(\Delta+j+3)(\Delta+j+5)(\Delta+j+7)} G_{\Delta+5,j+3}(U,V) \\
&- \frac{512j(\Delta+4)^2(\Delta+5)^2(\Delta+j)(\Delta+j+2)(\Delta+j+4)(\Delta+3)^2}{(2j+1)(2\Delta+5)(2\Delta+7)^2(2\Delta+9)^2(2\Delta+11)} \\
&\times \frac{(\Delta-j+5)(\Delta-j+3)(\Delta-j+1)(\Delta-j-1)}{(\Delta-j+6)(\Delta-j+4)(\Delta-j+2)(\Delta-j)(\Delta+j+1)(\Delta+j+3)(\Delta+j+5)} G_{\Delta+7,j-1}(U,V) \\
&- \frac{(\Delta-j+3)(\Delta-j+1)(\Delta-j-1)(\Delta+j)(\Delta+j+2)(\Delta+j+4)(\Delta+j+6)}{(\Delta-j+4)(\Delta-j+2)(\Delta-j)(\Delta+j+1)(\Delta+j+3)(\Delta+j+5)(\Delta+j+7)} \\
&\times \frac{512(j+1)(\Delta+4)^2(\Delta+5)^2(\Delta+3)^2}{(2j+1)(2\Delta+5)(2\Delta+7)^2(2\Delta+9)^2(2\Delta+11)} G_{\Delta+7,j+1}(U,V)\,,
\end{aligned}
\tag{4.3.54}
$$

$$
\begin{aligned}
A_{00}(U,V) = {}& G_{\Delta,j}(U,V) + \frac{16(\Delta-j-1)\Delta(\Delta+3)(\Delta+j)G_{\Delta+2,j}(U,V)}{7(\Delta-j)(\Delta+j+1)(2\Delta-1)(2\Delta+7)} \\
&+ \frac{16(j-3)(j-2)(j-1)j(\Delta-j-5)(\Delta-j+3)(\Delta-j+1)(\Delta-j-1)G_{\Delta+4,j-4}(U,V)}{(2j-5)(2j-3)(2j-1)(2j+1)(\Delta-j-6)(\Delta-j+4)(\Delta-j+2)(\Delta-j)} \\
&+ \frac{64(j-2)(j-1)j(j+1)(\Delta-j+3)(\Delta-j+1)(\Delta-j-1)(\Delta+j)G_{\Delta+4,j-2}(U,V)}{7(2j-5)(2j-1)(2j+1)(2j+3)(\Delta-j+4)(\Delta-j+2)(\Delta-j)(\Delta+j+1)} \\
&+ \frac{288}{35(2\Delta+1)(2\Delta+3)(2\Delta+7)(2\Delta+9)(2j-3)(2j-1)(2j+3)(2j+5)} \Big[\\
&8\Delta^2(\Delta+5)^2 j(j+1)(4j(j+1)-13) + 40\Delta(\Delta+5)j(j+1)(7j(j+1)-24) \\
&+ 3(15(\Delta+1)(\Delta+2)(\Delta+3)(\Delta+4) + j(j+1)(191j(j+1)-702))\Big] \\
&\times \frac{(\Delta-j+1)(\Delta-j-1)(\Delta+j)(\Delta+j+2)}{(\Delta-j+2)(\Delta-j)(\Delta+j+1)(\Delta+j+3)} G_{\Delta+4,j}(U,V) \\
&+ \frac{64j(j+1)(j+2)(j+3)(\Delta-j-1)(\Delta+j)(\Delta+j+2)(\Delta+j+4)}{7(2j-1)(2j+1)(2j+3)(2j+7)(\Delta-j)(\Delta+j+1)(\Delta+j+3)(\Delta+j+5)} G_{\Delta+4,j+2}(U,V) \\
&+ \frac{16(j+1)(j+2)(j+3)(j+4)}{(2j+1)(2j+3)(2j+5)(2j+7)} \\
&\times \frac{(\Delta+j)(\Delta+j+2)(\Delta+j+4)(\Delta+j+6)}{(\Delta+j+1)(\Delta+j+3)(\Delta+j+5)(\Delta+j+7)} G_{\Delta+4,j+4}(U,V) \\
&+ \frac{256(\Delta+2)(\Delta+3)^2(\Delta+5)(\Delta+4)^2}{7(2\Delta+3)(2\Delta+5)(2\Delta+7)^2(2\Delta+9)(2\Delta+11)} \\
&\times \frac{(\Delta-j+3)(\Delta-j+1)(\Delta-j-1)(\Delta+j)(\Delta+j+2)(\Delta+j+4)}{(\Delta-j+4)(\Delta-j+2)(\Delta-j)(\Delta+j+1)(\Delta+j+3)(\Delta+j+5)} G_{\Delta+6,j}(U,V) \\
&+ \frac{(\Delta-j+1)(\Delta-j-1)(\Delta-j+3)(\Delta-j+5)(\Delta+j)(\Delta+j+2)(\Delta+j+4)(\Delta+j+6)}{(\Delta-j-6)(\Delta-j+4)(\Delta-j+2)(\Delta-j)(\Delta+j+1)(\Delta+j+3)(\Delta+j+5)(\Delta+j+7))} \\
&\times \frac{256(\Delta+3)^2(\Delta+5)^2(\Delta+6)^2(\Delta+4)^2}{(2\Delta+5)(2\Delta+7)^2(2\Delta+9)^2(2\Delta+11)^2(2\Delta+13)} G_{\Delta+8,j}(U,V)\,.
\end{aligned}
\tag{4.3.55}
$$

4.3.2 Derivation of Superconformal Blocks Using the Results of [1]

The superconformal blocks can also be computed using the solution (4.2.6) of the Ward identity.[11] One first observes that for all multiplets listed in Tables 4.1–4.5, the [0040] channel receives contributions from a single

[11] The superconformal blocks of $\mathcal{N}=2,4$ SCFTs in $d=4$ were derived in this way in [108].

operator. The projection of the four-point function onto this channel is then given by a single conformal block. The other channels are related to the [0040] channel by (4.2.6), and their conformal block expansion can be determined by using certain recurrence relations obeyed by the conformal blocks.

Let us first write (4.2.6) in terms of the decomposition into $\mathfrak{so}(8)_R$ representations in (1.1.33),

$$
\begin{aligned}
A_{22} &= \frac{U}{3}\frac{1}{\sqrt{\boldsymbol{\Delta}}}U^2 a\,,\\
A_{21} &= U\frac{1}{\sqrt{\boldsymbol{\Delta}}}U(V-1)a\,,\\
A_{20} &= \frac{U}{3}\frac{1}{\sqrt{\boldsymbol{\Delta}}}U\left(3(V+1)-U\right)a\,,\\
A_{11} &= U\frac{1}{\sqrt{\boldsymbol{\Delta}}}\left((V-1)^2-\frac{2}{3}U(V+1)+\frac{1}{9}\right)a\,,\\
A_{10} &= U\frac{1}{\sqrt{\boldsymbol{\Delta}}}(V-1)\left((V+1)-\frac{3}{5}U\right)a\,,\\
A_{00} &= \frac{U}{2}\frac{1}{\sqrt{\boldsymbol{\Delta}}}\left((V+1)^2-\frac{1}{2}(V-1)^2-\frac{3}{7}U(V+1)+\frac{3}{70}U^2\right)a\,.
\end{aligned}
\tag{4.3.56}
$$

For the long multiplet A_{22} is determined (up to an overall coefficient) to be

$$
A_{22}^{(\mathrm{long})}(U,V) = \frac{1}{6}G_{\Delta+4,j}(U,V)\,. \tag{4.3.57}
$$

Then, for example, the A_{21} channel is given by

$$
A_{21}^{(\mathrm{long})}(U,V) = \frac{1}{2}U\frac{1}{\sqrt{\boldsymbol{\Delta}}}\frac{V-1}{U}\sqrt{\boldsymbol{\Delta}}\,\frac{G_{\Delta+4,j}(U,V)}{U}\,, \tag{4.3.58}
$$

and the other channels are given by similar expressions. This expression can be expanded in conformal blocks by using recurrence relations derived in [1]. The final result matches precisely the long multiplet superconformal block that we found using the method of the previous section.

It turns out that the superconformal blocks of the short multiplets can be derived by taking limits of the long superconformal block. These limits consist of taking Δ and j in the long block to certain values below unitarity, i.e. $\Delta < j+1$. For instance, we can try to obtain the superconformal block of the $(2,0)^{[0040]}_{(B,+)}$ multiplet (see Table 4.1) by taking $\Delta \to -2$ and $j \to 0$ in the long superconformal block. In this limit

$$
A_{22}^{(\mathrm{long})} \propto G_{\Delta+4,j} \to G_{2,0} \sim A_{22}^{(B,+)}\,, \qquad \text{as } \Delta \to -2 \text{ and } j \to 0\ . \tag{4.3.59}
$$

Note that such limits have to be taken with great care for two reasons. The first reason is that some of the conformal blocks $G_{\Delta,j}$ are divergent in this limit, but the coefficients multiplying them vanish, so the limit is finite. The divergence arises because the conformal blocks $G_{\Delta,j}$, viewed as functions of Δ, have poles below unitarity. The location and residues of these poles were computed in [6]. For example, there is a "twist-0" pole at $\Delta = j$ given by

$$G_{\Delta,j} \sim -2\,\frac{j(j-1)}{4j^2-1}\,\frac{G_{j+2,j-2}}{\Delta - j}\,, \qquad \text{as } \Delta \to j\ . \tag{4.3.60}$$

The second reason why the limits have to be taken with care is that the limits $\Delta \to 2$ and $j \to 0$ do not commute, so the result is ambiguous. We parameterize this ambiguity by taking first $\Delta = -2 + cj$ and later sending $j \to 0$. The constant c is kept arbitrary at this stage.

Taking the above considerations into account, for the $(2,0)^{[0040]}_{(B,+)}$ multiplet we find[12]

$$-\frac{1}{128}\lim_{j\to 0}\lim_{\Delta\to -2+cj} A^{(\text{long})}_{22} = \frac{c+1}{c-1}G_{2,0}\,, \tag{4.3.61}$$

$$-\frac{1}{128}\lim_{j\to 0}\lim_{\Delta\to -2+cj} A^{(\text{long})}_{21} = -\frac{4(c+1)}{3(c-1)}G_{3,1} - \frac{3}{2(c-1)}G_{1,0}\,, \tag{4.3.62}$$

$$-\frac{1}{128}\lim_{j\to 0}\lim_{\Delta\to -2+cj} A^{(\text{long})}_{20} = \frac{8(2c-1)(c+1)}{45c(c-1)}G_{4,0} + \frac{3(2c-1)}{4c(c-1)}G_{2,1}\,, \tag{4.3.63}$$

$$-\frac{1}{128}\lim_{j\to 0}\lim_{\Delta\to -2+cj} A^{(\text{long})}_{11} = \frac{256(c+1)}{675(c-1)}G_{4,2} + \frac{3}{8(c-1)}G_{0,1}\,, \tag{4.3.64}$$

$$\begin{aligned}-\frac{1}{128}\lim_{j\to 0}\lim_{\Delta\to -2+cj} A^{(\text{long})}_{10} = &-\frac{64(2c-1)(c+1)}{875c(c-1)}G_{5,1} - \frac{2c-1}{4c(c-1)}G_{3,2} - \frac{1}{10(c-1)}G_{1,0}\\ &- \frac{2c-1}{8c(c-1)}G_{1,2}\,,\end{aligned} \tag{4.3.65}$$

$$-\frac{1}{128}\lim_{j\to 0}\lim_{\Delta\to -2+cj} A^{(\text{long})}_{00} = \frac{128(2c-1)(c+1)}{18375c(c-1)}G_{6,0} + \frac{2c-1}{70c(c-1)}G_{2,1} + \frac{9(2c-1)}{320c(c-1)}G_{2,3}\,. \tag{4.3.66}$$

This result is, in general, inconsistent with unitarity because of the appearance of conformal blocks with negative twists such as $G_{2,3}$. These unphysical blocks can be removed in the limit $c \to \infty$. In this limit, the result matches precisely the $(2,0)^{[0040]}_{(B,+)}$ superconformal block in (4.3.26)–(4.3.31), and we conclude that

$$\mathcal{G}^{(2,2)}_{2,0} = -\frac{1}{128}\lim_{c\to\infty}\lim_{j\to 0}\lim_{\Delta\to -2+cj}\mathcal{G}^{(0,0)}_{\Delta,j}\,. \tag{4.3.67}$$

All other short superconformal blocks can be obtained from the long block in a similar fashion. Hence all the superconformal blocks can be derived from the solution (4.2.6) of the Ward identity, because we derived the long superconformal block by using this solution and all the short blocks are limits of the long block. This derivation provides a strong consistency check on the expressions for the superconformal blocks given above and on the solution (4.2.6) of the Ward identity.

[12] We use the identity $G_{\Delta,-j-1} = G_{\Delta,j}$, which can be derived from the conformal Casimir equation.

4.4 Numerics

All ingredients are now in place for our numerical study of the crossing equations (1.1.32). Explicitly, in terms of the functions $A_{ab}(u,v)$ defined in (1.1.33) and expanded in superconformal blocks in Section 4.3, these equations are:

$$\begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{pmatrix} \equiv \begin{pmatrix} \mathcal{F}^+_{10} + \mathcal{F}^+_{11} + \frac{5}{3}\mathcal{F}^+_{20} - \frac{2}{5}\mathcal{F}^+_{21} - \frac{14}{3}\mathcal{F}^+_{22} \\ \mathcal{F}^+_{00} - \frac{1}{4}\mathcal{F}^+_{11} - \frac{20}{21}\mathcal{F}^+_{20} + \mathcal{F}^+_{21} - \frac{14}{15}\mathcal{F}^+_{22} \\ \mathcal{F}^-_{20} + \mathcal{F}^-_{21} + \mathcal{F}^-_{22} \\ \mathcal{F}^-_{11} + \frac{4}{3}\mathcal{F}^-_{21} + \frac{8}{3}\mathcal{F}^-_{22} \\ \mathcal{F}^-_{10} + \frac{3}{5}\mathcal{F}^-_{21} + 3\mathcal{F}^-_{22} \\ \mathcal{F}^-_{00} - \frac{12}{7}\mathcal{F}^-_{21} + \frac{24}{35}\mathcal{F}^-_{22} \end{pmatrix} = 0\,, \tag{4.4.1}$$

where we defined

$$\mathcal{F}^{\pm}_{ab}(U,V) \equiv \frac{1}{U}A_{ab}(U,V) \pm \frac{1}{V}A_{ab}(V,U)\,. \tag{4.4.2}$$

Recall that the contribution to A_{ab} coming from each superconformal block takes the form of a linear combination of conformal blocks. Note that the basis of equations $d_i = 0$ used here is different from the basis $\tilde{e}_i = 0$ of Section 4.2.3. The two bases are related by the linear transformation

$$\begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & -1 & 0 & \frac{1}{6} & \frac{1}{3} \\ -\frac{1}{2} & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{8} & 0 & \frac{3}{4} & 0 & \frac{17}{56} & -\frac{5}{28} \end{pmatrix} \begin{pmatrix} \tilde{e}_1 \\ \tilde{e}_2 \\ \tilde{e}_3 \\ \tilde{e}_4 \\ \tilde{e}_5 \\ \tilde{e}_6 \end{pmatrix}. \tag{4.4.3}$$

Crossing equations such as (4.4.1) have been used many times recently to rule out the existence of (S)CFTs whose spectrum of operators satisfies certain additional assumptions. We will perform several such studies with or without additional assumptions besides locality (i.e. existence of a stress tensor), unitarity, and invariance under the $\mathcal{N} = 8$ superconformal algebra $\mathfrak{osp}(8|4)$. The main observation is that, when expanded in superconformal blocks, the crossing equations (4.4.1) take the form

$$d_i = \sum_{\mathcal{M} \in \mathfrak{osp}(8|4) \text{ multiplets}} \lambda_{\mathcal{M}}^2 \, d_{i,\mathcal{M}} = 0 \,, \tag{4.4.4}$$

where $\mathcal{M}$ ranges over all the superconformal multiplets that appear in the OPE of $\mathcal{O}_{\text{Stress}}$ with itself—see Table 1.3. In (4.4.4), $d_{i,\mathcal{M}}$ should be identified with the middle expression in (4.4.1) in which one uses only the contributions to the $\mathcal{F}_{ab}^{\pm}$ coming from the superconformal block of the multiplet $\mathcal{M}$.

In many of our numerical studies we will compare a bound on a CFT datum to a given $\mathcal{N} = 8$ SCFT, which we specify by its value of c_T. These values can be computed exactly for some low-lying ABJ(M) or BLG theories using (2.4.22)[13], and we list these values in Table 4.6.

SCFT	c_T
$U(1)_k \times U(1)_{-k}$ ABJM	16.0000
$U(2)_2 \times U(1)_{-2}$ ABJ	21.3333
$U(2)_1 \times U(2)_{-1}$ ABJM	37.3333
$U(2)_2 \times U(2)_{-2}$ ABJM	42.6667
$SU(2)_3 \times SU(2)_{-3}$ BLG	46.9998
$SU(2)_4 \times SU(2)_{-4}$ BLG	50.3575
$SU(2)_5 \times SU(2)_{-5}$ BLG	52.9354
$\vdots$	$\vdots$

Table 4.6: A few values of c_T for known SCFTs.

The approach for excluding (S)CFTs first introduced in [76] starts with constructing linear functionals of the expressions d_i that are required to vanish by crossing symmetry. One can construct such linear functionals by considering linear combinations of the d_i and of their derivatives at the crossing-symmetric point $z = \bar{z} = 1/2$. Denoting such a functional by α, we have

$$\alpha(\vec{d}) = \sum_i \sum_{m \geq n} \alpha_{i,mn} \left(\partial^m \bar{\partial}^n d_i\right)\Big|_{z=\bar{z}=\frac{1}{2}} \,, \tag{4.4.5}$$

where $\alpha_{i,mn}$ are numerical coefficients. In (4.4.5), we restricted the second sum to run only over $m \geq n$ because $\partial^m \bar{\partial}^n d_i = \partial^n \bar{\partial}^m d_i$, as follows from the fact that all conformal blocks are chosen to be invariant under $x \leftrightarrow \bar{x}$. Without this restriction, we would be double counting all derivatives with $m \neq n$.

Note that still not all the terms in the sum (4.4.5) are linearly independent. There are two additional sources of linear dependencies between the various terms in (4.4.5). The first such source can be seen from the definitions (4.4.1)–(4.4.2) whereby d_1 and d_2 are even under $z \to 1 - z$ and $\bar{z} \to 1 - \bar{z}$, while the other

[13]This formulae is only for the BLG theories, but we can obtain all the ABJ(M) values listed in Table 4.6 from BLG values using the dualities discussed in Section 1.1.4.

d_i are odd. Therefore, at the crossing-symmetric point $z = \bar{z} = 1/2$, we have $\partial^m \bar{\partial}^n d_i = 0$ for $i = 1, 2$ and $m + n$ odd or $i = 3, 4, 5, 6$ and $m + n$ even. We should not include these terms that vanish in (4.4.5).

The second source of dependencies is more subtle and follows from the discussion in Section 4.2.3. Indeed, in Section 4.2.3 we have shown that the derivatives of the $\tilde{e}_i$ were not all independent. The linear relation (4.4.3) then shows that the derivatives of the d_i are also not all independent. It is straightforward to check based on the results of Section 4.2.3 that a possibly independent set of derivatives of the d_i consists of the derivatives of d_2 as well as the holomorphic derivatives of d_1. There are many other such choices, but we make this one for convenience.

We can now attempt to find linear functionals (4.4.5) that satisfy certain positivity properties in order to obtain bounds on operator dimensions and OPE coefficients.

4.4.1 Obtaining a Lower Bound on c_T

In the Introduction we discussed that the $U(1) \times U(1)$ ABJM theory is free and has $c_T = 16$. This value can be obtained by adding up the equal unit contributions from the eight real scalars and eight Majorana fermions. One may then wonder if there exist other $\mathcal{N} = 8$ SCFTs with $c_T < 16$, or, given (1.1.42), with $\lambda^2_{\text{Stress}} > 16$. Let us therefore use the bootstrap to find an upper bound on $\lambda^2_{\text{Stress}}$.

The first step is to separate out the contributions from the identity multiplet and from the stress-tensor multiplet in (4.4.5). Since crossing requires $\vec{d} = 0$, we must have

$$0 = \alpha(\vec{d}) = \alpha(\vec{d}_{\text{Id}}) + \lambda^2_{\text{Stress}} \alpha(\vec{d}_{\text{Stress}}) + \sum_{\mathcal{M} \neq \text{Id,Stress}} \lambda^2_{\mathcal{M}} \alpha(\vec{d}_{\mathcal{M}}) \,. \tag{4.4.6}$$

An upper bound on $\lambda^2_{\text{Stress}}$ can be obtained by considering the space of functionals α that satisfy

$$\alpha(\vec{d}_{\text{Stress}}) = 1 \,, \qquad \text{and} \qquad \alpha(\vec{d}_{\mathcal{M}}) \geq 0 \,, \qquad \text{for all } \mathcal{M} \neq \text{Id, Stress} \,. \tag{4.4.7}$$

The conditions (4.4.7) and the equation (4.4.6) imply the bound

$$\lambda^2_{\text{Stress}} \leq -\alpha(\vec{d}_{\text{Id}}) \,. \tag{4.4.8}$$

To obtain the most stringent bound we should minimize $-\alpha(\vec{d}_{\text{Id}})$ under the constraints (4.4.7).

The minimization problem described above needs to be truncated for a numerical implementation. There are two truncations that should be performed: one in the number of derivatives used to construct α and one in the range of multiplets $\mathcal{M}$ that we consider. Instead of (4.4.5), we can consider the truncated version

$$\alpha_\Lambda(\vec{d}) = \sum_i \sum_{m+n\leq\Lambda} \alpha_{i,mn} \left(\partial^m \bar{\partial}^n d_i\right)\Big|_{z=\bar{z}=\frac{1}{2}} , \tag{4.4.9}$$

where the sum over m and n should only contain independent terms. In practice, the cutoff Λ that determines the size of our search space will be taken to be $\Lambda = 15$, 17, or 19. We can then minimize $-\alpha_\Lambda(\vec{d}_{\text{Id}})$ under the constraints

$$\begin{aligned} &\alpha_\Lambda(\vec{d}_{\text{Stress}}) = 1 \,, \\ &\alpha_\Lambda(\vec{d}_{\mathcal{M}}) \geq 0 \,, \qquad \text{for all other } \mathcal{M} \text{ with } j \leq j_{\max} \text{ and } \Delta \geq j+1 \end{aligned} \tag{4.4.10}$$

Here, Δ and j refer to the conformal dimension and spin of the superconformal primary, and $\Delta \geq j+1$ is just the unitarity condition. The second equation refers to all multiplets $\mathcal{M}$ other than the identity and the stress-tensor multiplet. In practice, we found that taking $j_{\max} = 20$ provides fairly accurate results.

For the long multiplet $(\Delta, j)^{[0000]}_{(A,0)}$ (henceforth referred to as "long") the quantity $\alpha_\Lambda(\vec{d}_{\text{long}})$ can further be approximated, for each spin, by a positive function times a polynomial in Δ. Such expansion is obtained by expanding the conformal blocks that comprise the long superconformal block in a Taylor series around $z = \bar{z} = 0$ using the recursion formula given in [6], and then approximating some of the poles as a function of Δ that appear in this expansion in terms of a smaller set of poles, as explained in the Appendix of [6].

The minimization of $-\alpha_\Lambda(\vec{d}_{\text{Id}})$ under the constraints (4.4.10) can then be rephrased as a semidefinite programing problem using the method developed in [87]. This problem can be solved efficiently by freely available software such as `sdpb` [106]. Implementing it as a dual problem, we obtain $\lambda^2_{\text{Stress}} \leq 17.02, 16.95, 16.67$, or equivalently, $c_T \geq 15.04, 15.11, 15.35$, for $\Lambda = 15, 17, 19$, respectively. Clearly, it would be desirable to increase Λ further, but we take these numerical results as good evidence that $c_T \geq 16$ in all local unitary SCFTs with $\mathcal{N} = 8$ supersymmetry. In the rest of this Chapter we only study such SCFTs with $c_T \geq 16$.

4.4.2 Bounds on Scaling Dimensions of Long Multiplets

A small variation on the method presented in the Section 4.4.1 yields upper bounds on the lowest scaling dimension Δ^*_j of spin-j superconformal primaries in a long multiplet. Such superconformal primaries must all be singlets under the $\mathfrak{so}(8)_R$ R-symmetry—see Table 1.3, where the long multiplet is in the last line. It is worth emphasizing that, as was the case in Section 4.4.1, these bounds do not depend on any assumptions about our $\mathcal{N} = 8$ SCFTs other than locality and unitarity.

The variation on the method presented in Section 4.4.1 is as follows. Let us fix c_T and look for functionals

α satisfying the following conditions:

$$\begin{aligned}
&\alpha(\vec{d}_{\rm Id}) + \frac{256}{c_T}\alpha(\vec{d}_{\rm Stress}) = 1\,, \\
&\alpha(\vec{d}_{\mathcal{M}}) \geq 0\,, \qquad \text{for all short and semi-short } \mathcal{M} \notin \{\text{Id}, \text{Stress}\}\,, \\
&\alpha(\vec{d}_{\mathcal{M}}) \geq 0\,, \qquad \text{for all long } \mathcal{M} \text{ with } \Delta \geq \Delta_j^*\,.
\end{aligned} \tag{4.4.11}$$

The existence of any such functional α would prove inconsistent all SCFTs with the property that superconformal primaries of spin-j long multiplets all have conformal dimension $\Delta \geq \Delta_j^*$, because if this were the case, then equation (4.4.6) could not possibly hold. If we cannot find a functional α satisfying (4.4.11), then we would not be able to conclude anything about the existence of an SCFT for which superconformal primaries of spin-j long multiplets all have conformal dimension $\Delta \geq \Delta_j^*$—such SCFTs may or may not be excluded by other consistency conditions we have not examined. An instance in which a functional α with the properties (4.4.11) should not exist is if c_T is chosen to be that of an ABJ(M) or a BLG theory and if we only impose restrictions coming from unitarity, namely if we take $\Delta_j^* = j+1$ for all j. Indeed, we should not be able to exclude the ABJ(M) and/or BLG theories, assuming that these theories are consistent as is believed to be the case.

As in the previous section, in order to make the problem (4.4.11) amenable to a numerical study, we should truncate the number of spins used in the second and third lines to $j \leq j_{\rm max}$ (where in practice we take $j_{\rm max} = 20$) and replace α by α_Λ such that our search space becomes finite-dimensional. We can then use `sdpb` to look for functionals α_Λ satisfying (4.4.11) for various choices of Δ_j^*. In practice, we will take $\Lambda = 15$, 17, and 19.

We present three numerical studies:

1. We first find an upper bound on the lowest dimension Δ_0^* of a spin-0 long multiplet assuming that all long multiplets with spin $j > 0$ are only restricted by the unitarity bound. In other words, we set $\Delta_j^* = j+1$ for all $j > 0$. This upper bound is plotted as a function of c_T in Figure 4.1 for $\Lambda = 15$ (in light brown), $\Lambda = 17$ (in black), and $\Lambda = 19$ (in orange). As can be seen from Figure 4.1, there is very good agreement between the latter two values of Λ, especially at large c_T.

   The upper bound on Δ_0^* interpolates monotonically between $\Delta_0^* \lesssim 1.02$ at $c_T = 16$ and $\Delta_0^* \lesssim 2.03$ as $c_T \to \infty$ when $\Lambda = 19$. As we will now explain, these bounds are very close to being saturated by the $U(1)_k \times U(1)_{-k}$ ABJM theory at $c_T = 16$ and by the large N $U(N)_k \times U(N)_{-k}$ ABJM theory (or its supergravity dual) at $c_T = \infty$.

   Let us denote the real and imaginary parts of the bifundamental scalar matter fields in $U(N) \times U(N)$

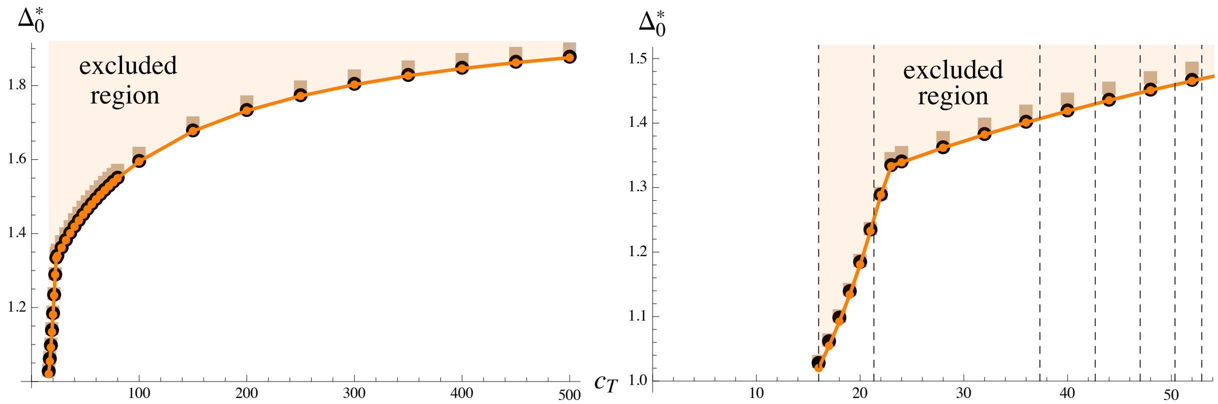

Figure 4.1: Upper bounds on Δ_0^*, which is the smallest conformal dimension of a long multiplet of spin-0 appearing in the $\mathcal{O}_{\text{Stress}} \times \mathcal{O}_{\text{Stress}}$ OPE. The long multiplets of spin $j > 0$ are only restricted by unitarity. These bounds are computed with $j_{\max} = 20$ and $\Lambda = 19$ (orange), $\Lambda = 17$ (black), and $\Lambda = 15$ (light brown). The plot on the right is a zoomed-in version of the plot on the left. The dashed vertical lines correspond to the values of c_T in Table 4.6.

ABJM theory with Chern-Simons levels ± 1 or ± 2 by X_i, with $i = 1, \ldots, 8$. In our convention, the X_i transform as the $\mathbf{8}_c$ of the emergent $\mathfrak{so}(8)_R$. The operator $\mathcal{O}_{ij}$ whose four-point function we have been analyzing transforms in the $\mathbf{35}_c$ of $\mathfrak{so}(8)_R$. It can be written schematically as[14]

$$\mathcal{O}_{ij} = \operatorname{tr}\left[X_i X_j - \frac{1}{8}\delta_{ij} X_k X^k\right], \tag{4.4.12}$$

up to an overall normalization. There are two $\mathfrak{so}(8)_R$ singlets appearing in the $\mathcal{O}_{ij} \times \mathcal{O}_{kl}$ OPE as the bottom components of long multiplets that are worth emphasizing: the single trace operator $\mathcal{O}_K = \operatorname{tr} X_k X^k$, which is the analog of the Konishi operator in 4-d $\mathcal{N} = 4$ SYM, and the double trace operator $\mathcal{O}_{ij}\mathcal{O}^{ij}$. When $N = 1$, the theory is free, and $\mathcal{O}_K$ has scaling dimension 1, while $\mathcal{O}_{ij}\mathcal{O}^{ij}$ has dimension 2. In this case $\Delta_0^* = 1$, and therefore this theory almost saturates our numerical bound. When $N = \infty$, $\mathcal{O}_K$ is expected to acquire a large anomalous dimension,[15] while $\mathcal{O}_{ij}\mathcal{O}^{ij}$ still has dimension 2 by large N factorization. Therefore, in this case $\Delta_0^* = 2$, and so the large N ABJM theory also almost saturates our numerical bound.

There is another feature of the bounds in Figure 4.1 that is worth noting: as a function of c_T, the bound on Δ_0^* has a kink. The location of the kink is approximately at $c_T \approx 22.8$ and $\Delta_0^* \approx 1.33$. This is close to the value of the ABJ$_1$ theory with $c_T \approx 21.333$, and we conjecture that with infinite precision the kink would like precisely at that value. We will discuss more evidence for this conjecture

[14] For Chern-Simons levels $k = 1, 2$, the products $X_i X_j$ must be combined with monopole operators into gauge invariant combinations.

[15] Single trace long multiplets are not part of the supergravity spectrum. The only single-trace operators that are dual to supergravity fluctuations around $AdS_4 \times S^7$ are part of the half-BPS multiplets $(n/2, 0)^{[00n0]}_{(B,+)}$ with $n \geq 2$ [115].

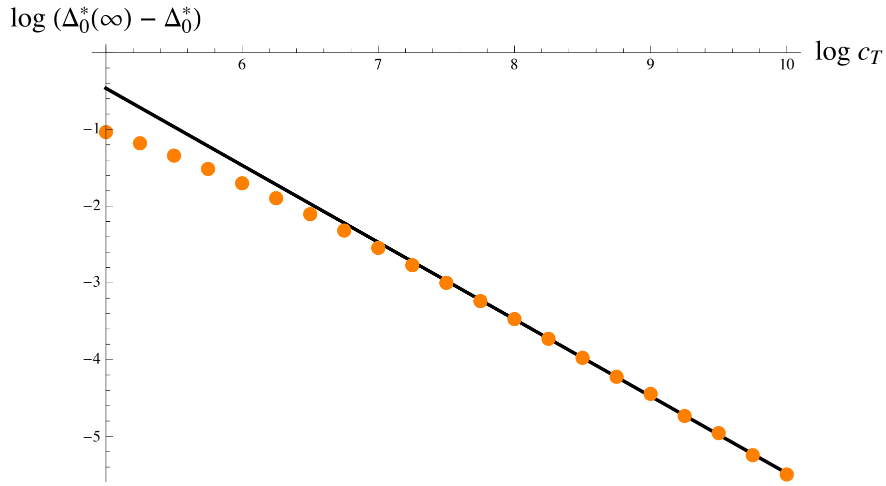

Figure 4.2: Upper bounds on Δ_0^* (the smallest conformal dimension of a spin-0 long multiplet appearing in the $\mathcal{O}_{\text{Stress}} \times \mathcal{O}_{\text{Stress}}$ OPE) for large values of c_T. The bounds are computed with $j_{\max} = 20$ and $\Lambda = 19$. The long multiplets of spin $j > 0$ are only restricted by unitarity. The best fit for the last ten points (shown in black) is $\log(\Delta_0^*(\infty) - \Delta_0^*) = 4.55 - 1.00 \log c_T$.

in the next section.

From a fit at large values of c_T we obtain $\Delta_0^* \gtrsim 2.03 - 94.6/c_T + \ldots$. See Figure 4.2. In particular, the first subleading term at large c_T scales as $1/c_T$. Such a behavior is also what would be expected from supergravity. Indeed, in radial quantization, the anomalous dimension of the double trace operator $\mathcal{O}_{ij}\mathcal{O}^{ij}$ takes the form of a binding energy, and, within supergravity, one expects such binding energies to be of the order of the effective 4-d Newton constant $G_4 \propto 1/c_T$.[16]

2. Our second numerical study is similar to the first. Instead of obtaining an upper bound on Δ_0^*, we now obtain an upper bound on Δ_2^*, which is the lowest scaling dimension of a spin-2 long multiplet. We obtain the bound on Δ_2^* under the assumption that long multiplets of spin $j \neq 2$ are only restricted by the unitarity condition. In other words, we set $\Delta_j^* = j + 1$ for all $j \neq 2$. In Figure 4.3, we plot the upper bound on Δ_2^* as a function of c_T for $\Lambda = 15$ (in light brown), $\Lambda = 17$ (in black), and $\Lambda = 19$ (in orange). The convergence as a function of Λ is poorer than in the Δ_0^* case, but it is still reasonably good throughout, especially at large c_T.

   A main feature of the plot in Figure 4.3 is that it interpolates monotonically between $\Delta_2^* \lesssim 3.11$ at $c_T = 16$ and $\Delta_2^* \lesssim 4.006$ at $c_T = \infty$. It is likely that as one increases Λ, the bound at $c_T = 16$ will become stronger still, since at this value of c_T the bound obtained when $\Lambda = 19$ is still noticeably different from that obtained when $\Lambda = 17$ and convergence has not yet been achieved.

   As was the case for the bounds on Δ_0^*, the bounds on Δ_2^* are also almost saturated by ABJM theory

[16] We thank I. Klebanov for a discussion on this issue.

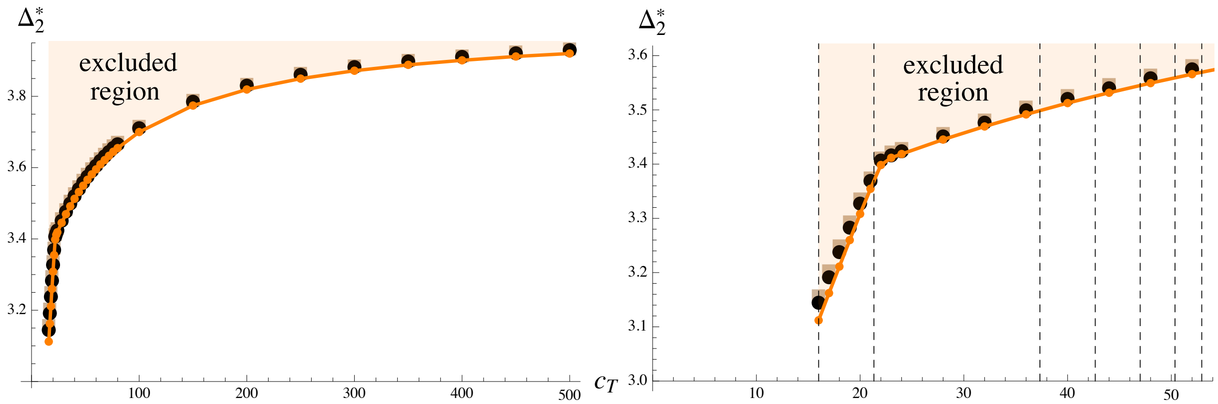

Figure 4.3: Upper bounds on Δ_2^*, which is the smallest conformal dimension of a long multiplet of spin-2 appearing in the $\mathcal{O}_{\text{Stress}} \times \mathcal{O}_{\text{Stress}}$ OPE. The long multiplets of spin $j \neq 2$ are only restricted by unitarity. These bounds are computed with $j_{\max} = 20$ and $\Lambda = 19$ (orange), $\Lambda = 17$ (black), and $\Lambda = 15$ (light brown). The plot on the right is a zoomed-in version of the plot on the left. The dashed vertical lines correspond to the values of c_T in Table 4.6.

at $c_T = 16$ and $c_T = \infty$. Indeed, two of the spin-2 $\mathfrak{so}(8)_R$ singlets that appear in the $\mathcal{O}_{ij} \times \mathcal{O}_{kl}$ OPE as bottom components of long multiplets are the single trace operator $\operatorname{tr} X_k \partial_\mu \partial_\nu X^k$ and the double trace operator $\mathcal{O}_{ij} \partial_\mu \partial_\nu \mathcal{O}^{ij}$. For $U(1) \times U(1)$ ABJM theory, they have scaling dimensions 3 and 4, respectively; in ABJM theory at infinite N, the first has a large anomalous dimension, while the second has scaling dimension 4 because of large N factorization. Therefore, the $N = 1$ theory has $\Delta_2^* = 3$, while the large N theory has $\Delta_2^* = 4$, in agreement with our numerical bounds.

Note that just as in the Δ_0^* case, our upper bound on Δ_2^* in Figure 4.3 also exhibits a kink for $c_T \approx 22.8$. Within our numerical precision, this kink is in the same location as that in Figure 4.1.

3. Our last numerical study yields combined upper bounds on Δ_0^* and Δ_2^* under the assumption that all long multiplets with spin $j > 2$ are restricted only by the unitarity bound, i.e. $\Delta_j^* = j + 1$ for all $j > 2$. In Figure 4.4 we provide such combined upper bounds only for a few values of c_T corresponding to the ABJ(M) / BLG theories whose values of c_T we listed in Table 4.6.

   As can be seen from Figure 4.4, the combined bounds take the form of a rectangle in the Δ_0^*-Δ_2^* plane, suggesting that these bounds are set by a single $\mathcal{N} = 8$ SCFT, if such an SCFT exists. A similar feature is present for the $\mathcal{N} = 4$ superconformal bootstrap in 4-d [35].

   Note that for $c_T = \infty$, the combined Δ_0^*-Δ_2^* bound comes very close to the values $(\Delta_0^*, \Delta_2^*) = (2, 4)$ of the large N ABJM theory.

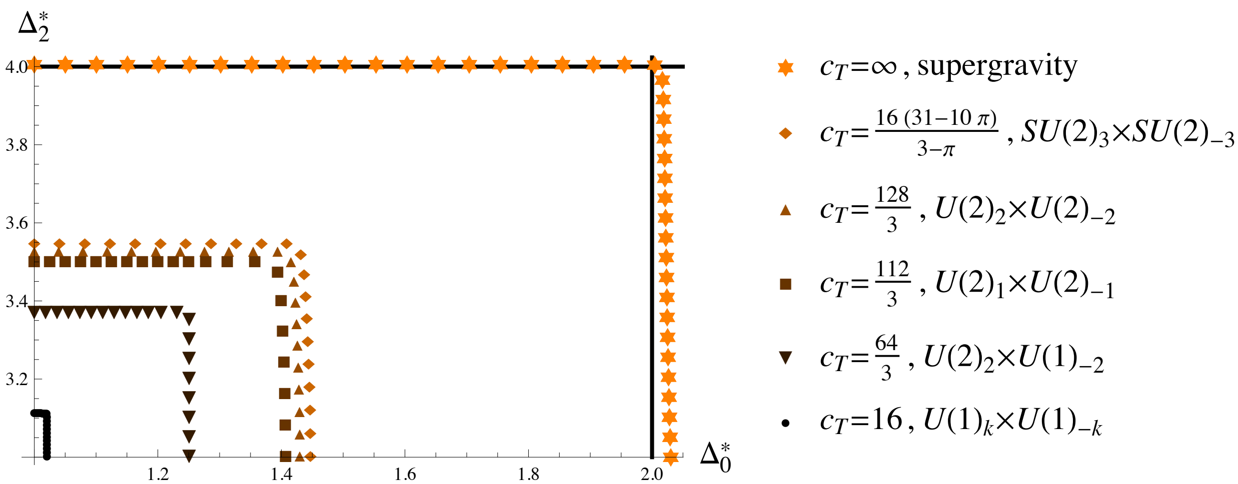

Figure 4.4: Combined upper bounds on Δ_0^* and Δ_2^*, which are the smallest scaling dimensions of spin-0 and spin-2 long multiplets appearing in the $\mathcal{O}_{\text{Stress}} \times \mathcal{O}_{\text{Stress}}$ OPE. The long multiplets of spin $j > 2$ are only restricted by unitarity. The bounds are computed with $j_{\max} = 20$ and $\Lambda = 19$. The solid lines correspond to the expected scaling dimensions in ABJM theory at large N.

4.4.3 Bounds on OPE Coefficients

We can also find upper and lower bounds on the OPE coefficients of both short and semi-short multiplets. To find upper/lower bounds on a given OPE coefficient of a multiplet $\mathcal{M}^*$ that appears in the $\mathcal{O}_{\text{Stress}} \times \mathcal{O}_{\text{Stress}}$ OPE, let us consider linear functionals α satisfying

$$\begin{aligned}
&\alpha(\vec{d}_{\mathcal{M}^*}) = s\,, && s = 1 \text{ for upper bounds, } s = -1 \text{ for lower bounds}\\
&\alpha(\vec{d}_{\mathcal{M}}) \geq 0\,, && \text{for all short and semi-short } \mathcal{M} \notin \{\text{Id}, \text{Stress}, \mathcal{M}^*\}\,,\\
&\alpha(\vec{d}_{\mathcal{M}}) \geq 0\,, && \text{for all long } \mathcal{M} \text{ with } \Delta \geq \Delta_j^*\,.
\end{aligned} \tag{4.4.13}$$

If such a functional α exists, then this α applied to (4.4.4) along with the positivity of all $\lambda^2_{\mathcal{M}}$ except, possibly, for that of $\lambda^2_{\mathcal{M}^*}$ implies that

$$\begin{aligned}
&\text{if } s = 1, \text{ then} && \lambda^2_{\mathcal{M}^*} \leq -\alpha(\vec{d}_{\text{Id}}) - \lambda^2_{\text{Stress}}\alpha(\vec{d}_{\text{Stress}})\\
&\text{if } s = -1, \text{ then} && \lambda^2_{\mathcal{M}^*} \geq \alpha(\vec{d}_{\text{Id}}) + \lambda^2_{\text{Stress}}\alpha(\vec{d}_{\text{Stress}})
\end{aligned} \tag{4.4.14}$$

provided that the scaling dimensions of each long multiplet satisfies $\Delta \geq \Delta_j^*$. Here we choose the spectrum to only satisfy unitarity bounds $\Delta_j^* = j + 1$, which provides no restrictions on the set of $\mathcal{N} = 8$ SCFTs. To obtain the most stringent upper/lower bound on $\lambda^2_{\mathcal{M}^*}$, one should then minimize/maximize the RHS of (4.4.14) under the constraints (4.4.13). Note that a lower bound can only be found this way for OPE coefficients of protected multiplets, as shown in [87]. For long multiplets, the condition $\alpha(\vec{d}_{\mathcal{M}^*}) = -1$ is

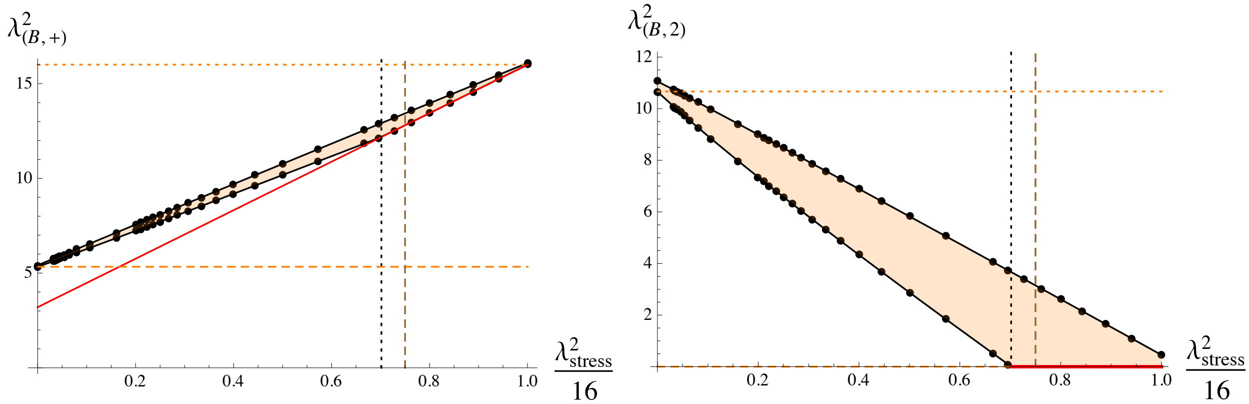

Figure 4.5: Upper and lower bounds on $\lambda^2_{(B,+)}$ and $\lambda^2_{(B,2)}$ OPE coefficients, where the orange shaded regions are allowed. These bounds are computed with $j_{\max} = 20$ and $\Lambda = 19$. The red solid line denotes the exact lower-bound (4.4.15) obtained from the exact relation (2.1.2). The black dotted vertical lines correspond to the kink at $\lambda^2_{\text{Stress}}/16 \approx 0.701$ ($c_T \approx 22.8$). The brown dashed vertical lines correspond to the $U(2)_2 \times U(1)_{-2}$ ABJ theory at $\lambda^2_{\text{Stress}}/16 = .75$ ($c_T = 21.333$). The orange horizontal lines correspond to known free (dotted) and mean-field (dashed) theory values listed in Table 1.4. The $\lambda^2_{(B,+)}$ bounds can be mapped into the $\lambda^2_{(B,2)}$ bounds using (2.1.2).

inconsistent with the requirement $\alpha(\vec{d}_{\mathcal{M}}) \geq 0$, because it is possible to have a continuum of long multiplets arbitrarily close to $\mathcal{M}^*$.

The numerical implementation of the minimization/maximization problem described above requires two truncations: one in the number of derivatives used to construct α and one in the range of multiplets $\mathcal{M}$ that we consider. We have found that considering multiplets $\mathcal{M}$ with spins $j \leq 20$ and derivatives parameter $\Lambda = 19$ leads to numerically convergent results. The truncated minimization/maximization problem can now be rephrased as a semidefinite programing problem using the method developed in [87]. This problem can be solved efficiently by freely available software such as `sdpb` [106].

In Figure 4.5 we show upper and lower bounds for $\lambda^2_{(B,+)}$ and $\lambda^2_{(B,2)}$ in $\mathcal{N} = 8$ SCFTs, and in Figure 4.6 we show upper and lower bounds on OPE coefficients in the semi-short $(A, 2)$ and $(A, +)$ multiplet series for the three lowest spins $1, 3, 5$ and $0, 2, 4$, respectively. We plot these bounds in terms of $\lambda^2_{\text{Stress}}/16$ instead of c_T (as was done for the long multiplet scaling dimensions above), because the allowed region becomes bounded by straight lines. Recall that for an SCFT with only one stress-tensor multiplet, $\lambda^2_{\text{Stress}}/16$ can be identified with $16/c_T$; this quantity ranges from 0, which corresponds to the GFFT obtained from large N limit of ABJ(M) theories with $c_T \to \infty$, to 1, which corresponds to the free $U(1)_k \times U(1)_{-k}$ ABJM theory with $c_T = 16$ that we showed to be the minimal possible c_T for any consistent 3d SCFT—see Table 4.6. For SCFTs with more than one stress tensor, one can also identify $\lambda^2_{\text{Stress}}/16$ with $16/c_T$, where c_T is the coefficient appearing in the two-point function of the canonically-normalized diagonal stress tensor, but, as we will see in the next subsection, more options are allowed.

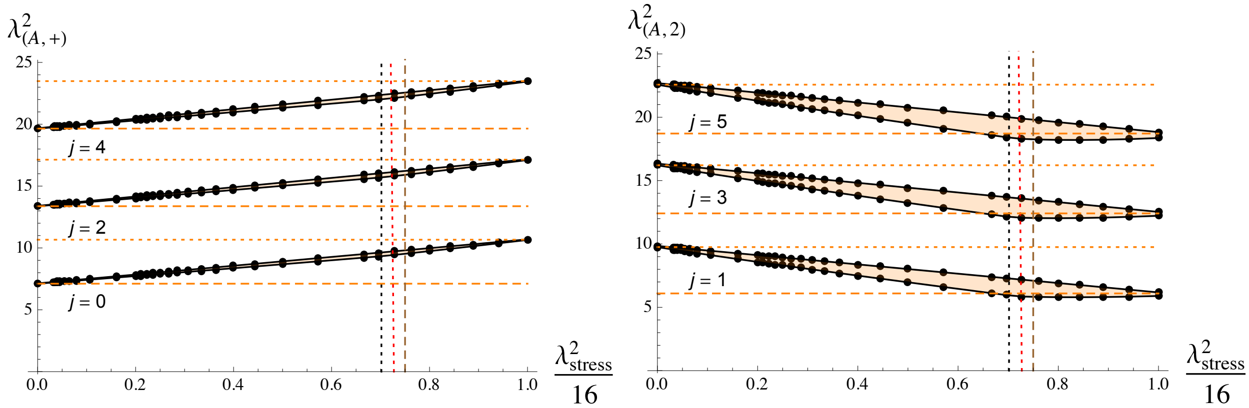

Figure 4.6: Upper and lower bounds on $(A,+)$ and $(A,2)$ OPE coefficients for the three lowest spins, where the orange shaded regions are allowed. These bounds are computed with $j_{\max} = 20$ and $\Lambda = 19$. The red dotted vertical lines correspond to the kink observed at $\lambda^2_{\text{Stress}}/16 \approx 0.727$ ($c_T \approx 22.0$) for bounds on OPE coefficients for the $(A,+)$ and $(A,2)$ multiplets. The black dotted vertical lines that correspond to the kink observed at $\lambda^2_{\text{Stress}}/16 \approx 0.701$ ($c_T \approx 22.8$) for the $(B,+)$ and $(B,2)$ multiplet OPE coefficient bounds and the long multiplet scaling dimension bounds. The brown dashed vertical lines correspond to the $U(2)_2 \times U(1)_{-2}$ ABJ theory at $\lambda^2_{\text{Stress}}/16 = 0.75$ ($c_T = 21.333$). The orange horizontal lines correspond to known free (dotted) and mean-field (dashed) theory values listed in Table 1.4.

There are a few features of these plots that are worth emphasizing:

- The bounds are consistent with and nearly saturated by the free and GFFT limits. In these limits, the OPE coefficients of the $(B,+)$ and $(B,2)$ multiplets are given in Table 1.4.

- The numerical bounds for $\lambda^2_{(B,+)}$ and $\lambda^2_{(B,2)}$ can be mapped onto each other under the exact relation (2.1.2) that is implied by crossing symmetry in $\mathcal{Q}$-cohomology. This mapping suggests that the relation (2.1.2) is already encoded in the numerical bootstrap constraints, and indeed, we checked that the numerical bounds do not improve by imposing it explicitly before running the numerics. The apparent visual discrepancy in the size of the allowed region between the two plots in Figure 4.5 comes from the factor of 5 difference between $\lambda^2_{(B,+)}$ and $\lambda^2_{(B,2)}$ in (2.1.2).

- The lower bounds for $\lambda^2_{(B,+)}$ as well as for the OPE coefficients of the A-series are *strictly* positive for all $\mathcal{N} = 8$ SCFT. Therefore, at least one multiplet of each such kind must exist in any $\mathcal{N} = 8$ SCFT—the absence, for instance, of $(A,2)$ multiplets of spin $j = 3$ would make the theory inconsistent.

- The lower bounds in Figures 4.5 and 4.6 are saturated (within numerical uncertainties) in the mean field theory limit $c_T \to \infty$, while the upper bounds are less tight. In the free theory limit $c_T = 16$, it is the upper bounds that are saturated (within numerical uncertainties), while the lower bounds are less tight for the A-series OPE coefficients. In the case of the $(B,+)$ and $(B,2)$ multiplets, the lower bounds are also saturated in the free theory limit $c_T = 16$, simply because there the relation

(2.1.2) combined with $\lambda^2_{(B,2)} \geq 0$ forces the lower bounds to coincide with the precise values of the OPE coefficients.

- The lower bound for $\lambda^2_{(B,2)}$ vanishes everywhere above $\lambda^2_{\text{Stress}}/16 \approx 0.701$ (or, equivalently, below $c_T \approx 22.8$). Consequently, the lower bound for $\lambda^2_{(B,2)}$ shows a kink at $c_T \approx 22.8$, and upon using (2.1.2) this kink also produces a kink in the lower bound for $\lambda^2_{(B,+)}$. Indeed, below $c_T \approx 22.8$ (above $\lambda^2_{\text{Stress}}/16 \approx 0.701$), the lower bound for $\lambda^2_{(B,+)}$ that we obtained from the numerics coincides with the analytical expression

$$\lambda^2_{(B,+)} \geq \frac{4}{5}\left(\lambda^2_{\text{Stress}} + 4\right) \tag{4.4.15}$$

obtained from (2.1.2) and the condition $\lambda^2_{(B,2)} \geq 0$.

The feature of the kink mentioned above is also present in the other bounds obtained using the numerical bootstrap. For instance, the upper bounds on dimensions of long multiplets also show kinks at the same value of c_T as in Figure 4.5. The lower bounds on OPE coefficients of A-type multiplets in Figure 4.6 exhibit kinks that are shifted slightly towards lower values of c_T relative to the location of the kink in the other plots.

The previous analysis suggests that the kink is caused by the disappearance of $(B, 2)$ multiplets, and therefore $\lambda^2_{(B,2)} = 0$. The only known $\mathcal{N} = 8$ SCFT aside from the free theory that has vanishing $\lambda_{(B,2)}$ is the ABJ_1 theory, which motivates our conjecture that the kink corresponds to that theory.

4.4.4 Product SCFTs

As all known constructions of $\mathcal{N} = 8$ SCFTs provide discrete series of theories, one may expect that only discrete points in Figures 4.5 and 4.6 correspond to consistent theories. Even if one assumes that there are no unknown constructions of $\mathcal{N} = 8$ SCFTs, this expectation is not correct—given two SCFTs there exists a whole curve that is realized in the product SCFT, which must lie within the region allowed by the bounds. It follows that any three $\mathcal{N} = 8$ SCFTs generate a two-dimensional allowed region in plots like those in Figures 4.5 and 4.6. Let us now derive the shape of these allowed regions and compare them with the numerical bounds shown in these figures.

Suppose we start with two $\mathcal{N} = 8$ SCFTs denoted SCFT_1 and SCFT_2 that each have a unique stress-tensor multiplet whose bottom component is a scalar in the $\mathbf{35}_c$ irrep of $\mathfrak{so}(8)_R$. Let us denote these scalars by $\mathcal{O}_1(\vec{x}, Y)$ and $\mathcal{O}_2(\vec{x}, Y)$ for the two SCFTs, respectively, where $\vec{x}$ is a space-time coordinate and Y is an

$\mathfrak{so}(8)_R$ polarization. Moreover, let us normalize these operators such that

$$\langle \mathcal{O}_1(\vec{x}_1, Y_1) \mathcal{O}_1(\vec{x}_2, Y_2) \rangle = \frac{(Y_1 \cdot Y_2)^2}{x_{12}^2} \,, \qquad \langle \mathcal{O}_2(\vec{x}_1, Y_1) \mathcal{O}_2(\vec{x}_2, Y_2) \rangle = \frac{(Y_1 \cdot Y_2)^2}{x_{12}^2} \,. \tag{4.4.16}$$

In the product SCFT we can consider the operator

$$\mathcal{O}(\vec{x}, Y) = \sqrt{1-t}\, \mathcal{O}_1(\vec{x}, Y) + \sqrt{t}\, \mathcal{O}_2(\vec{x}, Y) \,, \tag{4.4.17}$$

for some real number $t \in [0,1]$. The linear combination of $\mathcal{O}_1$ and $\mathcal{O}_2$ in (4.4.17) is such that $\mathcal{O}$ satisfies the same normalization condition as $\mathcal{O}_1$ and $\mathcal{O}_2$, namely

$$\langle \mathcal{O}(\vec{x}_1, Y_1) \mathcal{O}(\vec{x}_2, Y_2) \rangle = \frac{(Y_1 \cdot Y_2)^2}{x_{12}^2} \,. \tag{4.4.18}$$

Apart from this normalization condition, the linear combination in (4.4.17) is arbitrary.

We can easily calculate the four-point function of this operator given (4.4.16) and the four-point functions of $\mathcal{O}_1$ and $\mathcal{O}_2$:

$$\begin{aligned} \langle \mathcal{O}(x_1, Y_1) \mathcal{O}(x_2, Y_2) \mathcal{O}(x_3, Y_3) \mathcal{O}(x_4, Y_4) \rangle &= (1-t)^2 \langle \mathcal{O}_1(x_1, Y_1) \mathcal{O}_1(x_2, Y_2) \mathcal{O}_1(x_3, Y_3) \mathcal{O}_1(x_4, Y_4) \rangle \\ &+ t^2 \langle \mathcal{O}_2(x_1, Y_1) \mathcal{O}_2(x_2, Y_2) \mathcal{O}_2(x_3, Y_3) \mathcal{O}_2(x_4, Y_4) \rangle + 2t(1-t) \left[1 + u \frac{1}{U^2} + \frac{u}{v} \frac{V^2}{U^2} \right] . \end{aligned} \tag{4.4.19}$$

The term in the parenthesis is the four point function of a $\mathbf{35}_c$ operator in mean field theory.

In the $\mathcal{O} \times \mathcal{O}$ OPE we have both the operators appearing in the $\mathcal{O}_1 \times \mathcal{O}_1$ OPE and those in the $\mathcal{O}_2 \times \mathcal{O}_2$ OPE. Because $\mathcal{N} = 8$ supersymmetry fixes the dimensions of many operators, some of the operators in the $\mathcal{O}_1 \times \mathcal{O}_1$ OPE are identical to those in the $\mathcal{O}_2 \times \mathcal{O}_2$ OPE, and so in the four-point function (4.4.19) they contribute to the same superconformal block. The bootstrap equations are only sensitive to the total coefficient multiplying that superconformal block.

Let us denote by λ_1^2, λ_2^2, and λ^2 the coefficients multiplying a given superconformal block in the four-point function of $\mathcal{O}_1$, $\mathcal{O}_2$, and $\mathcal{O}$, respectively. Similarly, let λ_{MFT}^2 be the coefficient appearing in such a four-point function in mean field theory. Eq. (4.4.19) implies

$$\lambda^2(t) = (1-t)^2\, \lambda_1^2 + t^2\, \lambda_2^2 + 2\, t\, (1-t)\, \lambda_{\mathrm{MFT}}^2 \,. \tag{4.4.20}$$

In particular, if we are looking at the coefficient of the stress tensor block itself, we have

$$\lambda^2_{\text{Stress}}(t) = (1-t)^2 \lambda^2_{\text{Stress},1} + t^2 \lambda^2_{\text{Stress},2}\,, \tag{4.4.21}$$

because $\lambda^2_{\text{Stress, MFT}} = 0$.

It follows that if we have two $\mathcal{N} = 8$ SCFTs with $\left(\frac{\lambda^2_{\text{Stress},1}}{16}, \lambda_1^2\right)$ and $\left(\frac{\lambda^2_{\text{Stress},2}}{16}, \lambda_2^2\right)$, where $\lambda^2_{1,2}$ is the squared OPE coefficient of a given multiplet such as $(B,2)$ or $(B,+)$, then it is not just the points $\left(\frac{\lambda^2_{\text{Stress},1}}{16}, \lambda_1^2\right)$ and $\left(\frac{\lambda^2_{\text{Stress},2}}{16}, \lambda_2^2\right)$ that must lie within the region allowed by our bounds. Instead, the curve

$$\left(\frac{\lambda^2_{\text{Stress}}(t)}{16}, \lambda^2(t)\right)\,, \qquad t \in [0,1] \tag{4.4.22}$$

must lie within the allowed region. This curve is an arc of a parabola.

4.5 Bootstrap bound saturation

We will now show how the analytic values for the OPE coefficients of protected operators in the stress tensor multiplet come close to saturating the bootstrap bounds for ABJ(M) theories, which allows us to conjecturally extract the spectra of these theories using the extremal functional method.

In the following we will find upper/lower bounds on both the OPE coefficients of short and semi-short multiplets, as was done in the previous section, as well as the OPE coefficients of long multpliets. For a long multiplet $\mathcal{M}'$ labelled as $(A,0)_{j',n'}$, we consider linear functionals α satisfying

$$\begin{aligned}
&\alpha(\vec{d}_{\mathcal{M}'}) = s\,, && s = 1 \text{ for upper bounds, } s=-1 \text{ for lower bounds}\,,\\
&\alpha(\vec{d}_{\mathcal{M}}) \geq 0\,, && \text{for all short and semi-short } \mathcal{M} \notin \{\text{Id}, \text{Stress}\}\,,\\
&\alpha(\vec{d}_{(A,0)_{j,0}}) \geq 0\,, && \text{for all } j \neq j' \text{ with } \Delta_{(A,0)_{j,0}} \geq j+1\,,\\
&\alpha(\vec{d}_{(A,0)_{j',n}}) \geq 0\,, && \text{for all } n < n', \text{ and fixed } \Delta_{(A,0)_{j',n}}\,,\\
&\alpha(\vec{d}_{(A,0)_{j',n'+1}}) \geq 0\,, && \text{with } \Delta_{(A,0)_{j',n'+1}} > \Delta_{(A,0)_{j',n'}}\,.
\end{aligned} \tag{4.5.1}$$

If such a functional α exists, then this α applied to (4.4.4) along with the positivity of all $\lambda^2_{\mathcal{M}}$ except, possibly, for that of $\lambda^2_{\mathcal{M}'}$ implies that

$$\begin{aligned}
&\text{if } s=1, \text{ then} && \lambda^2_{\mathcal{M}'} \leq -\alpha(\vec{d}_{\text{Id}}) - \lambda^2_{\text{Stress}}\alpha(\vec{d}_{\text{Stress}})\,,\\
&\text{if } s=-1, \text{ then} && \lambda^2_{\mathcal{M}'} \geq \alpha(\vec{d}_{\text{Id}}) + \lambda^2_{\text{Stress}}\alpha(\vec{d}_{\text{Stress}})\,.
\end{aligned} \tag{4.5.2}$$

Note that the final condition $\Delta_{(A,0)_{j',n'+1}} > \Delta_{(A,0)_{j',n'}}$ when $\mathcal{M}'$ is a long multiplet $(A,0)_{j',n'}$ is so that $\mathcal{M}'$ is isolated from the continuum of possible long multiplets. To obtain the most stringent upper/lower bound on $\lambda^2_{\mathcal{M}'}$, one should then minimize/maximize the RHS of (4.5.2) under the constraints (4.4.13).

The numerical implementation of the minimization/maximization problem described above requires two truncations: one in the number of derivatives used to construct α and one in the range of multiplets $\mathcal{M}$ that we consider. We used the same parameters as in [102], namely spins in $\{0,\ldots,64\} \cup \{67,68,71,72,75,76,79,80,83,84,87,88\}$ and derivatives parameter $\Lambda = 43$. The truncated minimization/maximization problem can now be rephrased as a semidefinite programing problem using the method developed in [87]. This problem can be solved efficiently using `SDPB` [106].

4.5.1 Bounds on OPE coefficients

Let us now compare the analytical values of the OPE coefficients $\lambda^2_{(B,2)}$, $\lambda^2_{(B,+)}$, and $\lambda^2_{\text{Stress}} = 256/c_T$ found in Section 2.4 to the numerical bootstrap bounds obtained using the method outlined in the previous subsection. As noted in the previous section, the numerical bounds on these OPE coefficients exactly satisfy the constraint (2.1.2), so it suffices to discuss the bounds on just two of them, which for simplicity we choose to be $\lambda^2_{(B,2)}$ and $\lambda^2_{\text{Stress}}$. The main lesson from this comparison will be that $\lambda^2_{(B,2)}$ saturates the lower bounds for all $\mathcal{N}=8$ theories with holographic duals at large c_T, so we can use the extremal functional method to read off the spectrum of all operators in the OPE $\mathcal{O}_{\text{Stress}} \times \mathcal{O}_{\text{Stress}}$ in this regime. At smaller values of c_T, we expect that one of these holographic theories saturates the bounds, so the results for the extremal functional hold for that theory.

In Figure 4.7, we show upper and lower bounds on $\lambda^2_{(B,2)}$ as a function of $\lambda^2_{\text{Stress}}/16 = 16/c_T$. (The quantity $16/c_T$ ranges from 0 (GFFT limit) to 1 (free theory limit).) We show our most accurate bounds with $\Lambda = 43$ (solid line) as well as less accurate bounds with $\Lambda = 19$ (dashed line), to show how converged the bounds are. The upper bounds seem to be converging at the same rate for all c_T, whereas the lower bounds seem more converged for larger c_T. The vertical dotted line shows the numerical point where $\lambda^2_{(B,2)} = 0$. The red, gray, blue, and green dots denote some exact values listed in Table 2.2 for the interacting sector of $\text{ABJM}_{N,1}$, $\text{ABJM}_{N,2}$, BLG_k, and ABJ_N, respectively, where in all cases the dots go right to left for increasing k, N. We also list the free theory $\text{ABJM}_{1,1}$ as a magenta dot[17]. The red, gray, and green dotted lines show the large N values for these theories for $N \geq 2$ as given in (2.4.28) and (2.4.29).

There are several features of the BLG_k plot that we would like to emphasize. For $k = 1, 2$, which are the values where BLG_k theory is dual to a product theory (see footnote 6), the OPE coefficients lie in bulk of the allowed region. This is expected, because as described in Section 4.4.4 all product theories generically

[17] $\text{ABJM}_{1,2}$ is not a free theory, but has the same stress tensor four-point function as a free theory.

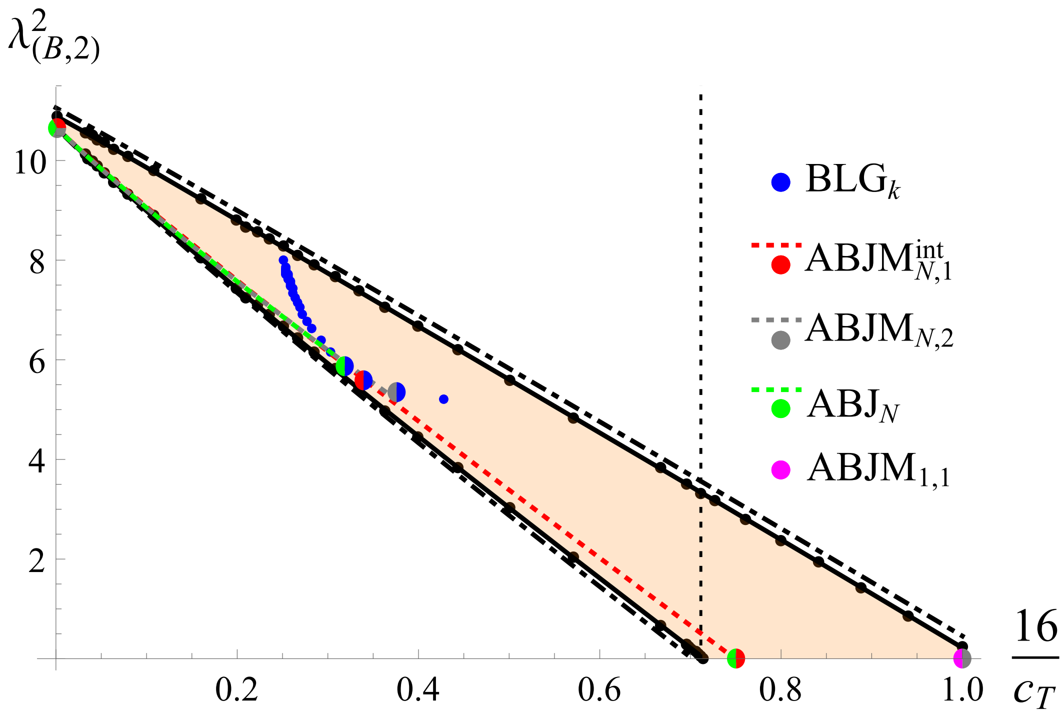

Figure 4.7: Upper and lower bounds on the $\lambda^2_{(B,2)}$ OPE coefficient in terms of the stress-tensor coefficient c_T, where the orange shaded region are allowed, and the plot ranges from the generalized free field theory limit $c_T \to \infty$ to the free theory $c_T = 16$. The blue dots denote the exact values in Table 2.2 in BLG_k for $k \geq 1$. The magenta dot denotes the free $\mathrm{ABJM}_{1,1}$ theory, the gray and green dots denote the exact values in Table 2.2 for $\mathrm{ABJM}_{N,2}$ and ABJ_N, respectively, for $N = 1, 2, \infty$, and the red dots denote $\mathrm{ABJM}^{\mathrm{int}}_{N,1}$ for $N = 2, 3, \infty$. The red, gray, and green dotted lines show the large N formulae (2.4.28) and (2.4.29) for these theories for all $N \geq 2$. The black dotted line denotes the numerical point $\frac{16}{c_T} \approx .71$ above which $\lambda^2_{(B,2)} = 0$. The solid lines were computed with $\Lambda = 43$. To show the level of convergence, the dashed lines are upper and lower bounds that were computed with $\Lambda = 19$.

lie in the bulk region. On the other hand, for $k = 3, 4$, which are the values where BLG_k theory is dual to the interacting sector of $\mathrm{ABJM}_{3,1}$ and ABJ_2, respectively, the OPE coefficients are close to saturating the lower bound. Lastly, for $k > 4$, where it is not known whether the BLG_k theories have an M-theory interpretation, the OPE coefficients of the BLG_k theories interpolate between the lower and upper bounds. The $k \to \infty$ value is a little off from the upper bound, which is likely explained by the fact that the upper bound numerics are not fully converged.

The ABJ(M) plot also has two interesting features. We first note that the $\mathrm{ABJ}_1 \cong \mathrm{ABJM}^{\mathrm{int}}_{2,1}$ theory is close to the numerical point where $\lambda^2_{(B,2)} = 0$. In fact, this is the only known interacting theory with $\lambda^2_{(B,2)} = 0$, so we suspect that with infinite accuracy the numerics would converge to this theory. We next note that all the ABJ(M) values seem to saturate the lower bound up to numerical error, with the exception of $\mathrm{ABJM}_{2,2}$, which as explained before has the same stress tensor four-point function as a product theory and so must lie in the bulk.

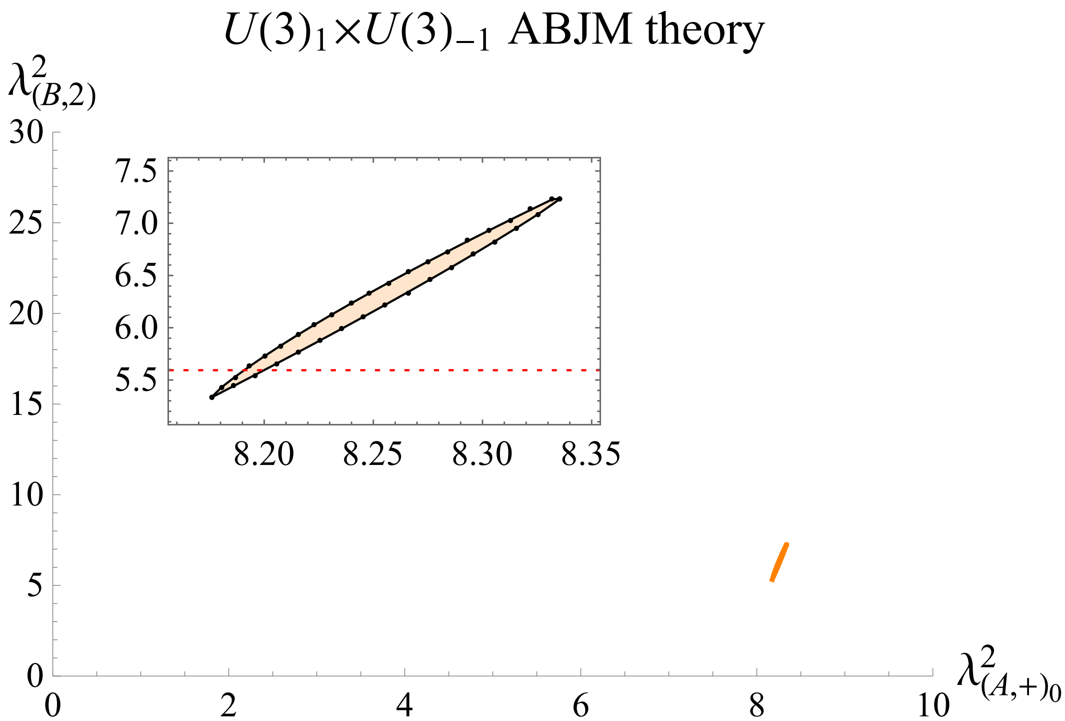

Figure 4.8: Bounds on $\lambda^2_{(B,2)}$ in terms of the $\lambda^2_{(A,+)_0}$ OPE coefficients at the ABJM$_{3,1}$ point with $\frac{16}{c_T} \approx 0.340$. The orange shaded region is the allowed island, while the red dotted line shows the exactly known value given in Table 2.2 for $\lambda^2_{(B,2)}$ in this theory. These bounds were computed with $\Lambda = 43$.

The fact that $\lambda^2_{(B,2)}$ for all unique ABJ(M) theories is close to saturating its lower bound may at first suggest that inputing any value of this OPE coefficient (within the bounds in Figure 4.7) into the numerical bootstrap code could uniquely specify that theory. To test this idea, in Figure 4.8 we plot upper/lower bounds of $\lambda^2_{(B,2)}$ as a function of $\lambda^2_{(A,+)_0}$ at the ABJM$_{3,1}$ point with $\frac{16}{c_T} \approx 0.340$ as given in Table 2.2. While the allowed region is a small island, it does not shrink to a point. On the other hand, as the zoomed in plot shows, when $\lambda^2_{(B,2)}$ is at its extremal values then $\lambda^2_{(A,+)_0}$ is uniquely fixed. This matches the general numerical bootstrap expectation that all CFT data in the relevant four-point function is fixed at the boundary of an allowed region. Since the extremal value is very close to the exactly known value, as shown by the red dotted line, if we assume that it would exactly saturate the bound at infinite precision, then we can read off the spectrum of ABJM$_{3,1}$ by looking at the functional α that extremizes $\lambda^2_{(B,2)}$. Similar plots can be made for all the other unique ABJ(M) theories, so that $\lambda^2_{(B,2)}$ minimization gives the spectra of all theories with holographic duals that saturate the lower bound.

4.5.2 Operator spectrum from numerical bootstrap

We now report our numerical results for the scaling dimensions and OPE coefficients of low-lying operators that appear in the OPE of $\mathcal{O}_{\text{Stress}}$ with itself. We are interested in theories with holographic duals, and the lowest such known theories are $\text{ABJM}_{2,1}^{\text{int}}$ and ABJ_1 with $\frac{16}{c_T} = .75$ and $\lambda^2_{(B,2)} = 0$. As we see from Figure 4.7, our numerics are not completely converged in that region, so we find that $\lambda^2_{(B,2)} = 0$ at the numerical point $\frac{16}{c_T} \approx .71$. As such, in the following plots we will show results for $\frac{16}{c_T} > .71$.

Let us describe the $(A,0)$ unprotected operators that we expect to see in the spectrum. At the $c_T \to \infty$ generalized free field value we have the dimension $j+2+2n$ double trace operators $[\mathcal{O}_{\text{Stress}}\mathcal{O}_{\text{Stress}}]_{n,j}$ of the schematic form

$$[\mathcal{O}_{\text{Stress}}\mathcal{O}_{\text{Stress}}]_{n,j} = \mathcal{O}_{\text{Stress}}\Box^n\partial_{\mu_1}\ldots\partial_{\mu_j}\mathcal{O}_{\text{Stress}} + \ldots\,, \tag{4.5.3}$$

where $n = 0,1,2,\ldots$ and μ_i are space-time indices. The OPE coefficients of these operators are given in Table 1.4. At infinite N, these are the only operators with nonzero OPE coefficients. At large but finite N, there are also m-trace operators $[\mathcal{O}_{\text{Stress}}]^m_{n,j}$, with $m > 1$, whose scaling dimension $\Delta^m_{n,j}$ and OPE coefficients $\lambda^m_{n,j}$ scale as [116]

$$\Delta^m_{n,j} = j + m + 2n + O(1/c_T)\,, \qquad (\lambda^m_{n,j})^2 = O(1/c_T^m)\,, \tag{4.5.4}$$

as well as single trace operators whose scaling dimension scales with N. For all ABJ(M) theories, $c_T \sim N^{3/2}$ [46, 117] to leading order in large N, so the OPE coefficient squared of m-trace operators is suppressed as $N^{-3m/2}$. Even for the lowest trace operator after $[\mathcal{O}_{\text{Stress}}]^2_{n,j}$, i.e. the triple trace operator $[\mathcal{O}_{\text{Stress}}]^3_{n,j}$, this suppression is extremely strong for even $N \sim 10$. As a result, we do not expect the numerical bootstrap bounds to be sensitive to these higher trace operators at the currently feasible levels of precision. The situation is similar to high spin operators, which also have OPE coefficients that are highly suppressed [109, 118, 119], and so one can restrict to a finite number of operators with spin below some cutoff without affecting the numerics. It is the ability to ignore higher spin operator which in fact makes the numerical bootstrap possible at all.

For small N, we would expect the OPE coefficients of these higher trace operators to become large enough that they start to affect the numerics. However, in this regime there is no clear distinction between higher trace and single trace operators because of trace relations. Moreover, since the unprotected single trace operators are expected to have large scaling dimensions at large N, it is really not clear whether at small N there should be an operator of small dimension that is continuously connected to the, say, triple trace

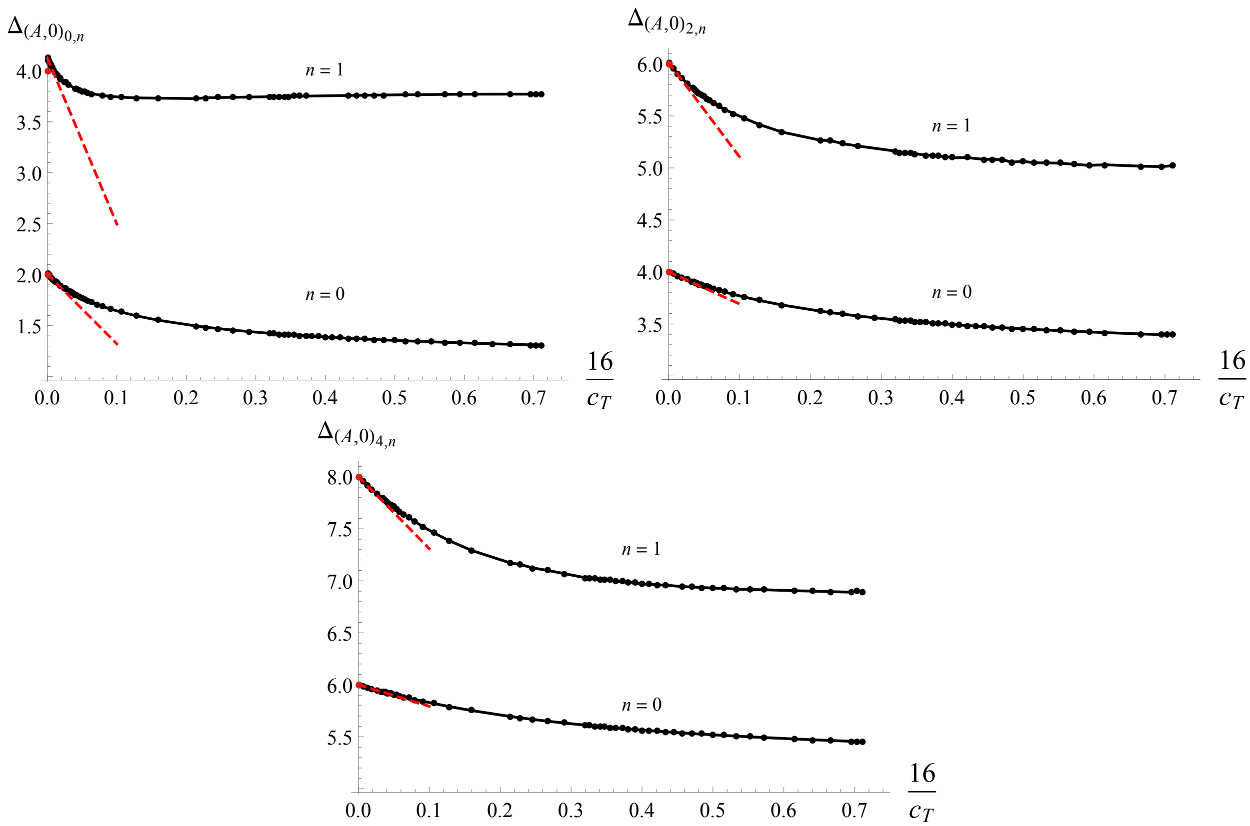

Figure 4.9: The scaling dimensions $\Delta_{(A,0)_{j,n}}$ for the two lowest $n = 0, 1$ long operators with spins $j = 0, 2, 4$ in terms of the stress-tensor coefficient c_T, where the plot ranges from the generalized free field theory limit $c_T \to \infty$ to the numerical point $\frac{16}{c_T} \approx .71$ where $\lambda^2_{(B,2)} = 0$. The red dots denote the known values $\Delta_j^{(n),\text{GFFT}} = j + 2 + 2n$ for the generalized free field theory, while the red dotted lines show the linear fit for large c_T given in (4.5.5). These bounds were computed with $\Lambda = 43$.

operator at large N.

$(A, 0)$ scaling dimensions

We can read off the scaling dimensions by looking at the zeros of the functional $\alpha(\Delta_{(A,0)_{j,n}})$ that minimizes $\lambda^2_{(B,2)}$. We trust those scaling dimensions that remain stable as we increase the number of derivatives Λ in the bootstrap numerics. We observed that $\Delta_{(A,0)_{j,n}}$ for $j = 0, 2, 4$ and $n = 0, 1$ are stable, and have values that in fact coincide with the upper bounds that we can independently compute for these quantities.

In Figure 4.9 we show our numerical results for $\Delta_{(A,0)_{j,n}}$ for $n = 0, 1$ and $j = 0, 2, 4$. All three of these plots show the same qualitative features. As described above, we only observe double trace operators, whose OPE coefficients are not suppressed at large N, i.e. small c_T. We can gauge how accurate these plots are by comparing to the $c_T \to \infty$ generalized free field values given in (1.1.45). The plots seem to match the

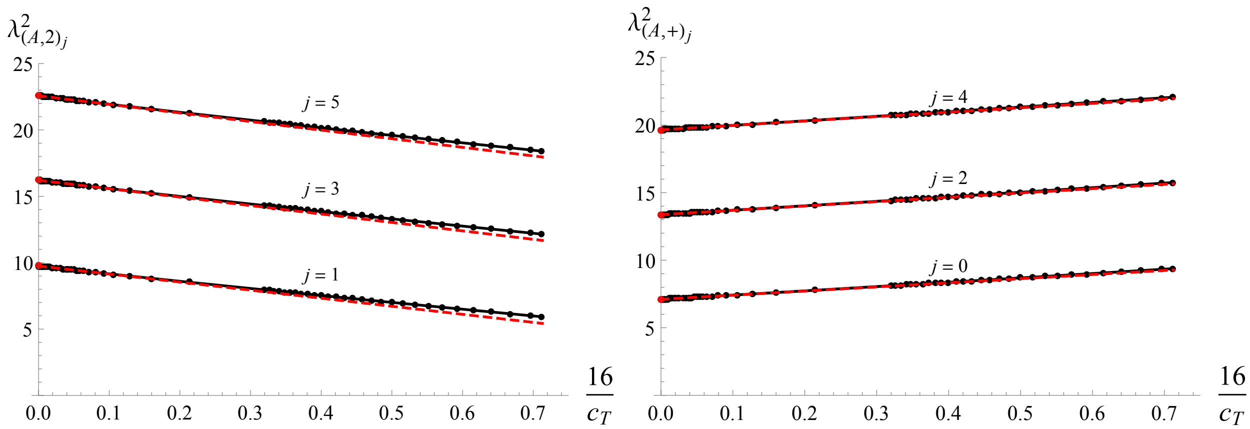

Figure 4.10: The $\lambda^2_{(A,2)_j}$ and $\lambda^2_{(A,+)_j}$ OPE coefficients with spins $j = 1, 3, 5$ and $j = 0, 2, 4$, respectively, in terms of the stress-tensor coefficient c_T, where the plot ranges from the generalized free field theory limit $c_T \to \infty$ to the numerical point $\frac{16}{c_T} \approx .71$ where $\lambda^2_{(B,2)} = 0$. The red dots denotes denote the known values at the generalized free field theory points given in Table 1.4, while the red dotted lines show the linear fit for large c_T given in (4.5.6). These bounds were computed with $\Lambda = 43$.

generalized free field theory values quite accurately. For large c_T, we find the following best fits

$$\begin{aligned}
&\Delta_{(A,0)_{0,0}} \approx 2.01 - \frac{109}{c_T}\,, \qquad \Delta_{(A,0)_{2,0}} \approx 4.13 - \frac{49}{c_T}\,, \qquad \Delta_{(A,0)_{4,0}} \approx 6.00 - \frac{33}{c_T}\,,\\
&\Delta_{(A,0)_{0,1}} \approx 4.03 - \frac{261}{c_T}\,, \qquad \Delta_{(A,0)_{2,1}} \approx 6.02 - \frac{145}{c_T}\,, \qquad \Delta_{(A,0)_{4,1}} \approx 8.00 - \frac{111}{c_T}\,.
\end{aligned} \tag{4.5.5}$$

As we see from Figure 4.9, these linear fits are only accurate for large c_T.

$(A,0)$, $(A,2)$, and $(A,+)$ OPE coefficients

Now that we have read off the low-lying scaling dimensions $\Delta_{(A,0)_{j,n}}$ from the extremal functional α, we can compute low-lying OPE coefficients in the $(A,0)$, $(A,2)$, and $(A,+)$ multiplets by inputing $\Delta_{(A,0)_{j,n}}$ back into the bootstrap and computing upper and lower bounds on a given OPE coefficient. Since in the previous sections we only computed long multiplets with $n = 0, 1$, we will input the exact values for $n = 0$ and then bound the continuum above the $n = 1$ value, so that we can only extract long multiplet OPE coefficients with $n = 0$. We find that the upper and lower bounds coincide, which matches our expectation that the extremal functional fixes these values. Note that in principle we could have extracted the OPE coefficients directly from α following the algorithm of [101, 102], but we found that this algorithm was very numerically unstable in our case.

In Figure 4.10 we show our numerical results for $\lambda^2_{(A,2)_j}$ and $\lambda^2_{(A,+)_j}$ with $j = 1, 3, 5$ and $j = 0, 2, 4$, respectively. Just as with the $\Delta_{(A,0)_{j,n}}$ plots, these plots accurately match the generalized free field theory

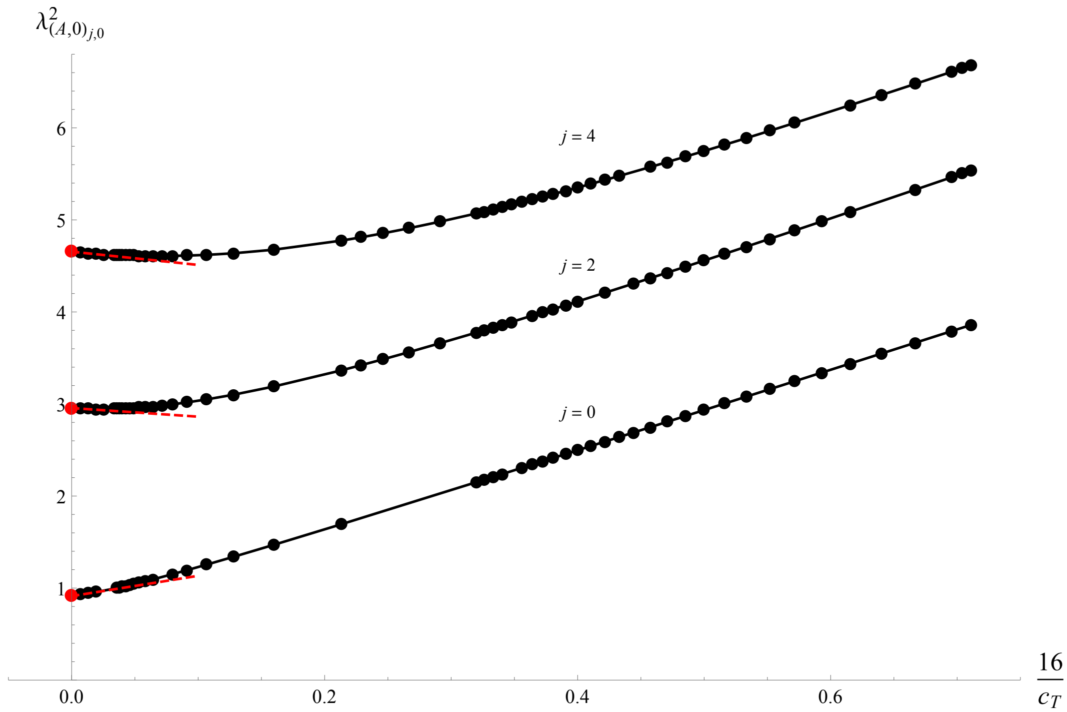

Figure 4.11: The $\lambda^2_{(A,0)}$ OPE coefficients for the three lowest spins in terms of the stress-tensor coefficient c_T, where the plot ranges from the generalized free field theory limit $c_T \to \infty$ to the numerical point $\frac{16}{c_T} \approx .71$ where $\lambda^2_{(B,2)} = 0$. The red dots denotes denote the known values at the generalized free field theory points given in Table 1.4, while the red dotted lines show the linear fit for large c_T given in (4.5.7). These bounds were computed with $\Lambda = 43$.

values listed in Table 1.4. For large c_T, we find the following best fits

$$\begin{aligned} \lambda^2_{(A,+)_0} &\approx 7.11 + \frac{49}{c_T}\,, \quad \lambda^2_{(A,+)_2} \approx 13.37 + \frac{51}{c_T}\,, \quad \lambda^2_{(A,+)_4} \approx 19.65 + \frac{52}{c_T}\,, \\ \lambda^2_{(A,2)_1} &\approx 9.75 - \frac{97}{c_T}\,, \quad \lambda^2_{(A,2)_3} \approx 16.21 - \frac{102}{c_T}\,, \quad \lambda^2_{(A,2)_5} \approx 22.57 - \frac{104}{c_T}\,. \end{aligned} \tag{4.5.6}$$

As we see from Figure 4.10, these linear fits seem to be accurate for all values of c_T.

In Figure 4.11 we show our numerical results for $\lambda^2_{(A,0)_{j,n}}$ with $j = 0, 2, 4$ and $n = 0$. Just as with the $\Delta_{(A,0)_{j,n}}$ plots, these plots accurately match the generalized free field theory values listed in Table 1.4. For large c_T, we find the following best fits

$$\lambda^2_{(A,0)_{0,0}} \approx 0.91 + \frac{35}{c_T}\,, \qquad \lambda^2_{(A,0)_{2,0}} \approx 2.96 - \frac{15}{c_T}\,, \qquad \lambda^2_{(A,0)_{4,0}} \approx 4.65 - \frac{23}{c_T}\,. \tag{4.5.7}$$

As we see from Figure 4.11, these linear fits are only accurate for very large c_T.

4.6 Discussion

Our conformal bootstrap analysis provides us with true non-perturbative information about $\mathcal{N} = 8$ SCFTs. Generically these theories are strongly coupled, and the conformal bootstrap is possibly the only available method to study them. Indeed, except for the $U(1) \times U(1)$ ABJM theory (which is trivial) and BLG theory at large k (which has no known gravity description), all known $\mathcal{N} = 8$ SCFTs are strongly interacting. In addition, while the large N limit of the ABJM theory can be studied through its weakly coupled supergravity dual, it is hard to obtain detailed information directly from the field theory side.

We described numerical results computed both without any assumptions, and assuming that ABJ(M) theories saturate the numerical bounds. The former bounds match analytic results in several interesting ways. Firstly, the bounds in the free theory and $c_T \to \infty$ GFFT limits match the expected values at those points. Secondly, the bounds show a kink at a value $c_T \approx 22.8$ very close to the value $c_T = \frac{64}{3} \approx 21.33$ for the ABJ_1, which seems to be explained by the fact that this is the only theory except the free theory for which a certain OPE coefficient vanishes. Thirdly, we found the analytic values of the OPE coefficients of the protected operators derived in Section 2.4 for the ABJ(M) theories came close to saturating the corresponding numerical bounds.

By assuming that one of these ABJ(M) theories in fact saturates the numerical bounds in the limit of infinite numerical precision, we were able to read off the low-lying spectrum of these theories. For large c_T, we were then able to extract numerical predictions for the leading $1/c_T$ correction to unprotected CFT data.

Looking ahead, it would be useful to impose additional assumptions that would automatically exclude the theories that do not saturate the lower bounds. For instance, in order to exclude the BLG_k theories with $k > 4$, one can apply the bootstrap to a mixed correlator between $\mathcal{O}_{\text{Stress}}$ and the half BPS multiplet in $\mathfrak{so}(8)_R$ irrep [0030]. As one can check from the superconformal index, this latter operator does not exist for BLG_k with $k > 4$, while it does for generic ABJ(M) theories. Another feature of this mixed correlator is that the free multiplet appears in it, so by setting its OPE coefficient to zero one could also exclude the free theory.

Chapter 5

M-theory from ABJ(M) Theory

This chapter is an edited version of ref. [120], which was written in collaboration with Silviu S. Pufu and Xi Yin, as well as ref. [121].

5.1 Introduction

M-theory can be understood as a quantum theory of interacting super-gravitons in 11 dimensions with no dimensionless coupling constant [122]. While some of its dynamics can be understood through a combination of its relation to superstring theories via compactification and the fact that certain observables are protected by supersymmetry [26–28, 123], there has not been a systematic way to produce, for instance, the small momentum expansion of the graviton S-matrix in 11D Minkowskian spacetime. Neither has there been much understanding of the particle spectrum of M-theory, or lack thereof, beyond super-gravitons.

It has been long anticipated that the AdS/CFT correspondence allows for extracting the full S-matrix of gravitons in the flat spacetime limit from correlation functions of the CFT [124–129]. In practice, this approach has been hardly tractable. Recently the 4-graviton S-matrix of tree level supergravity in $AdS_4 \times S^7$ has been computed in Mellin space [130] (see also [131–133] for similar computations in AdS_5 and AdS_7) and the anomalous dimension of the lowest double trace operator has been matched to the leading result of the large c_T expansion[1] of the stress-energy tensor 4-point function in ABJM theory [33, 34, 70]. Naturally one may wish to match the rest of the observables to this order to CFT results, and even extend this agreement to higher orders in the large c_T expansion, which amounts to going beyond supergravity in the bulk.

[1] c_T is the coefficient of the two-point function of the canonically-normalized stress-energy tensor, as defined in Section 5.3. It scales like $N^{\frac{3}{2}}$ in the large N, fixed k limit of ABJM theory. We prefer to think about the expansion in $1/c_T$ rather than $1/N$, because the former is what is more closely related to the expansion in Newton's constant in the flat space limit. Note that the correlator in question is not analytic in $1/c_T$, as fractional powers and logarithmic dependence will appear in the expansion.

In this Chapter, we will extend the leading order match to the rest of the CFT data in the stress tensor 4-point function, and outline a strategy for uncovering the small momentum expansion of the 4-graviton S-matrix in M-theory from the CFT data. This expansion was reviewed in Section 1.2.4. Our goal here is to reproduce the expansion (1.2.17) by taking the flat space limit of the CFT correlators. We will carry out this strategy to the first nontrivial order beyond two-derivative supergravity, and produce the R^4 effective coupling of M-theory from the large c_T expansion of a known BPS OPE coefficient in ABJM theory, in the following steps:

(1) We focus on the 4-point function of dimension $\Delta = 1$ scalar primaries $\mathcal{O}_{\text{Stress}}$ in the stress tensor supermultiplet that transform in the $\mathbf{35}_c$ representation of $\mathfrak{so}(8)_R$ R-symmetry, in ABJM theory with $k = 1$ or 2. Its Mellin transform, to be defined in Section 5.2, admits a large c_T expansion of the form

$$M(s,t;\sigma,\tau) = c_T^{-1} M_{\text{tree}} + c_T^{-\frac{5}{3}} M_{R^4} + c_T^{-2} M_{1-\text{loop}} + \cdots . \tag{5.1.1}$$

Here, s, t are Mellin space kinetic variables (not to be confused with the Mandelstam invariants), and σ, τ are the usual $\mathfrak{so}(8)_R$ invariants. M_{tree} represents the tree-level supergravity contribution, recently computed in [130]. M_{R^4} is a polynomial expression in s, t, whose large s, t limit will be related to the 4-graviton vertex that corresponds to the R^4 effective coupling in flat spacetime. $M_{1-\text{loop}}$ is the 1-loop supergravity contribution in $AdS_4 \times S^7$, which is free of logarithmic divergences. The higher order terms in the expansion may involve logarithmic dependence on c_T, as we will discuss later.

(2) At each order in the $1/c_T$ expansion, the Mellin amplitude is subject to the $\mathcal{N} = 8$ superconformal Ward identity. If the amplitude is a polynomial in s, t of known maximal degree, e.g. M_{R^4} is a degree 4 polynomial expression, then the Ward identity allows for finitely many solutions, thereby constraining the Mellin amplitude at this order in terms of finitely many unknown coefficients. Some linear combinations of these coefficients will be related to flat space amplitudes through the large s, t limit. The "loop amplitudes" will be determined by lower order terms in the $1/c_T$ expansion up to residual polynomial terms. Note that the loop Mellin amplitudes involve sums over poles that correspond to multi-trace operators in the OPE, and in the flat space limit the poles turn into branch cuts.

(3) Transforming the Mellin amplitude back to the correlation function, one would recover from (5.1.1) the $1/c_T$ expansion of the OPE coefficients as well as the scaling dimensions of various unprotected superconformal primaries. Some of these OPE coefficients, namely those of certain 1/2-BPS and 1/4-BPS multiplets, are known exactly as a function of c_T from supersymmetric localization computations

[33, 34, 70]. Other OPE coefficients, as well as the scaling dimension of long multiplets, are not known exactly but can be constrained by conformal bootstrap bounds.

(4) We will see that the OPE coefficient of the 1/4-BPS $(B, 2)$ multiplet, expanded to order $c_T^{-\frac{5}{3}}$, determines the coefficient of M_{R^4} in (5.1.1). Taking its large s, t limit then determines the R^4 effective coupling of M-theory in flat spacetime.[2] Our result is in perfect agreement with the known R^4 coefficient in (1.2.18), previously derived by combining toroidal compactification of M-theory, comparison to perturbative type II string amplitudes, and protection by supersymmetry.

It is worth noting that previously, in the AdS/CFT context, the R^4 coupling of M-theory has been probed through the study of conformal anomaly of the 6D $(2, 0)$ theory [134]. In this approach, one makes use of the bulk Lagrangian, including R^4 coupling as well as other terms related by supersymmetry. However, it is difficult to justify whether one has accounted for all the relevant terms in the effective Lagrangian, which is further subject to the ambiguity of field redefinitions. In contrast, our strategy produces from CFT data terms in the flat space S-matrix element, it is not subject to complications of the bulk effective Lagrangian, and all supersymmetries are manifest [135, 136].

A related comment concerns the structure of the derivative expansion of M-theory in 11D flat spacetime. Absent a dimensionless coupling constant, one could either speak of a Wilsonian effective Lagrangian, which is subject to the ambiguity of a floating cutoff scheme, or the 1PI/quantum effective Lagrangian, which amounts to a generating functional for the graviton S-matrix and is nonlocal. It is accidental, thanks to supersymmetry, that low order terms in the derivative expansion of the 1PI effective Lagrangian of M-theory can be separated into local terms, such as $t_8 t_8 R^4$, and nonlocal terms that correspond to loop amplitudes. This distinction ceases to exist starting at 20-th derivative order, where the supergravity 2-loop amplitude has a logarithmic divergence that is cut off at the Planck scale and mixes with a local term of the schematic form $D^{12}R^4$ [137, 138]. As mentioned above, it is clearer to phrase all of this directly in the language of the graviton S-matrix, and its expansion at small momenta as given in Eq. (1.2.17).

Finally, we should note that the idea that a large N CFT has a finite number of solutions to the conformal Ward identities at each order in N was first stated in [139]. In subsequent work [140, 141], this idea was generalized to maximally supersymmetric SCFTs in 4D and 6D, respectively, where the superconformal Ward identities further constrain the number of solutions. In 4D, [142] related the flat space limit of the Mellin amplitude to the S-matrix of type IIB string theory in 10D, but a precise reconstruction of the 10D

[2] One may contemplate, in principle, a more powerful approach for determining the couplings in the M-theory effective action, as follows. In principle, 11d SUSY determines the supersymmetric completion of the $D^{2k}R^4$ terms (perhaps up to a few coefficients). One can then reduce the 11d action on S^7 to obtain an effective action in AdS_4, which can then be used to calculate the CFT data via Witten diagrams. In practice, none of these steps are currently achievable without a tremendous effort. We thank Ofer Aharony for this comment.

S-matrix was not possible because of a lack of known CFT data that can fix the undetermined parameter in the CFT 4-point function. In the present work, we provide the first application of these ideas to 3D, and, as mentioned above, we can further recover the R^4 term in 11D from the CFT correlators by making use of nontrivial CFT data that can be computed using supersymmetric localization.[3]

The rest of this Chapter is organized as follows. In Section 5.2 we use the superconformal Ward identity as well as the asymptotic growth conditions on the Mellin amplitude in order to determine, up to a few constants, the Mellin amplitude order by order in $1/c_T$ in the case of the M2-brane theory. In Section 5.3 we explain how to extract various scaling dimensions and OPE coefficients from the Mellin amplitude constructed in Section 5.2, and show how to match all the supergravity order observables to previously computed CFT data, as well as reproduce the known correction to the supergravity scattering amplitude of four super-gravitons in 11D. Lastly, we end in Section 5.4 with a brief summary as well as a discussion of future directions.

5.2 The holographic four-point function

Let us now discuss the 4-point correlator of the operators $\mathcal{O}_{\text{Stress}}$ in the particular case of ABJM theory at CS level $k = 1$ or 2. In this section, we will use the AdS/CFT duality to study this correlator from the bulk side of the duality, without making any reference to the ABJM Lagrangian. We will use, however, that this theory is the low-energy theory on N coincident M2-branes placed at a $\mathbb{C}^4/\mathbb{Z}_k$ singularity, and that perturbatively at large N the back-reacted geometry is $AdS_4 \times S^7/\mathbb{Z}_k$. As reviewed in Section 1.2.3, the radius L of AdS_4 is given by

$$\frac{L^6}{\ell_{11}^6} = \frac{Nk}{8} + O(N^0) = \left(\frac{3\pi c_T k}{2^{11}}\right)^{\frac{2}{3}} + O(c_T^0)\,, \tag{5.2.1}$$

where ℓ_{11} is the 11D Planck length [2].[4] At leading order in $1/N$, the radius of $S^7/\mathbb{Z}_k$ is equal to $2L$.

Note that the subleading corrections in (5.2.1) depend on the precise definition of L beyond the supergravity solution. This ambiguity will not be important for us, as the precise large radius expansion will be performed in $1/c_T$ rather than in ℓ_{11}/L.

[3] In 4D and 6D there exists a protected part of the 4-point function of the 1/2-BPS scalar in the stress tensor multiplet that can be computed exactly [36, 37]. This sector, however, is completely fixed at order $1/c_T$, i.e. supergravity, for the stress tensor four point function.

[4] In the ABJM paper [2], the radius of AdS is L is denoted by $R/2$. Eq. (4.2) in that paper then implies $L^6/\ell_p^6 = \pi^2 Nk/2$. The scattering amplitudes in the main text were written in the convention $2\kappa_{11}^2 = (2\pi)^5 \ell_{11}^9$ whereas the ABJM paper uses the Polchinski [143] convention $2\kappa_{11}^2 = (2\pi)^8 \ell_p^9$. Thus, $\ell_p = \ell_{11}(2\pi)^{-1/3}$, so $L^6/\ell_{11}^6 = Nk/8$.

5.2.1 Holographic correlator in tree level supergravity

We will use the Mellin space representation to evaluate this correlator. Any 4-point function of scalar operators can be equivalently expressed in Mellin space. We will find it useful to separate out the disconnected piece of the correlator, which in a convenient normalization for $\mathcal{O}_{\text{Stress}}$ takes the form

$$\mathcal{G}_{\text{disc}}(U,V;\sigma,\tau) = 1 + U\sigma^2 + \frac{U}{V}\tau^2\,, \tag{5.2.2}$$

and then define the Mellin transform just for the connected part $\mathcal{G}_{\text{conn}} \equiv \mathcal{G} - \mathcal{G}_{\text{disc}}$:

$$\mathcal{G}_{\text{conn}}(U,V;\sigma,\tau) = \int_{-i\infty}^{i\infty} \frac{ds\,dt}{(4\pi i)^2} U^{\frac{s}{2}} V^{\frac{t}{2}-\Delta} M(s,t;\sigma,\tau)\Gamma^2\left[\Delta - \frac{s}{2}\right]\Gamma^2\left[\Delta - \frac{t}{2}\right]\Gamma^2\left[\Delta - \frac{u}{2}\right]. \tag{5.2.3}$$

Here, the Mellin space variables s, t, and u satisfy the constraint $s + t + u = 4\Delta$, and recall that for our 4-point function $\Delta = 1$. The two integration contours run parallel to the imaginary axis, such that all poles of the Gamma functions are on one side or the other of the contour.

The main advantage of the Mellin space representation is that in a theory with a holographic dual one can easily write down the *tree level* expression for the connected part of the four-point function. Indeed, the simplicity comes about as follows. At tree level, the relevant Witten diagrams are contact diagrams and exchange diagrams, so

$$M_{\text{tree}} = M_{s\text{-exchange}} + M_{t\text{-exchange}} + M_{u\text{-exchange}} + M_{\text{contact}}\,, \tag{5.2.4}$$

while the t- and u-channel exchange diagrams are related to the s-channel one as

$$\begin{aligned} M_{t\text{-exchange}}(s,t;\sigma,\tau) &= \tau^2 M_{s\text{-exchange}}(t,s;\sigma/\tau,1/\tau)\,,\\ M_{u\text{-exchange}}(s,t;\sigma,\tau) &= \sigma^2 M_{s\text{-exchange}}(u,t;1/\sigma,\tau/\sigma)\,. \end{aligned} \tag{5.2.5}$$

In Mellin space, the contact diagrams corresponding to vertices with n derivatives are order n polynomials in s, t, u. The exchange diagrams are slightly more complicated. An exchange diagram for a bulk field ϕ dual to a boundary conformal primary operator $\mathcal{O}$ of dimension $\Delta_{\mathcal{O}}$ and spin $\ell_{\mathcal{O}}$ has a meromorphic piece whose form is fixed up to an overall constant by the requirement that the residue at each pole agrees with the residue of the conformal block corresponding to the exchange of the operator $\mathcal{O}$, as well as a polynomial piece in s, t, u. The degree of the polynomial is given by $p_1 + p_2 - 1$, where p_1 and p_2 are half the numbers of derivatives in the two vertices connecting the ϕ internal line to the external lines. The meromorphic piece is independent of the vertices, and it has poles at $s = 2m + \tau_{\mathcal{O}}$, where $\tau_{\mathcal{O}} = \Delta_{\mathcal{O}} - \ell_{\mathcal{O}}$ is the twist of the conformal primary $\mathcal{O}$, and $m = 0, 1, 2, \ldots$. For example, if we denote

$$M^{\phi}_{s\text{-exchange}} = \widehat{M}^{\phi}_{s\text{-exchange}} + (\text{analytic})\,, \tag{5.2.6}$$

then the meromorphic pieces for various bulk fields that will be of interest to us can be taken to be:[5]

$$\begin{aligned}
\widehat{M}^{\text{graviton}}_{s\text{-exchange}} &= \frac{t^2+u^2-6tu+6(t+u)-8}{s(s+2)}\left(\frac{-(s+4)}{8}+\widehat{M}^{\Delta\,=\,1\text{ scalar}}_{s\text{-exchange}}\right)-\frac{(3s-4)}{8}\,,\\
\widehat{M}^{\text{gauge field}}_{s\text{-exchange}} &= \frac{t-u}{s}\left(-\frac{1}{2}+\widehat{M}^{\Delta\,=\,1\text{ scalar}}_{s\text{-exchange}}\right)\,,\\
\widehat{M}^{\Delta\,=\,1\text{ scalar}}_{s\text{-exchange}} &= \frac{\Gamma\left(\frac{1-s}{2}\right)}{2\sqrt{\pi}\,\Gamma\left(1-\frac{s}{2}\right)}\,.
\end{aligned} \tag{5.2.7}$$

In addition, we note that the contribution from any bulk field ϕ dual to an even-twist conformal primary must vanish:

$$\widehat{M}^{\text{even twist }\phi}_{s\text{-exchange}}(s,t) = 0\,, \tag{5.2.8}$$

because a non-zero meromorphic piece for such an exchange would have poles at even values of s, and that would produce third order poles when inserted in (5.2.3).

Going back to the situation of interest to us, i.e. the four-point function of the $\mathcal{O}_{\text{Stress}}$ operators in the $k = 1$ ABJM theory,[6] we should think about which exchange and contact diagrams we should write down. The scalar operators $\mathcal{O}_{\text{Stress}}$ are dual to certain components of the 11D graviton and 3-form in the S^7 directions. As is well known, the spectrum of fluctuations around $AdS_4 \times S^7$ organizes into representations of the supersymmetry algebra $\mathfrak{osp}(8|4)$ [115] (which is the same as the 3D $\mathcal{N} = 8$ superconformal algebra). As shown in Table 1.3, the $\mathcal{O}_{\text{Stress}} \times \mathcal{O}_{\text{Stress}}$ OPE contains two half-BPS operators: the stress tensor multiplet whose bottom component is $\mathcal{O}_{\text{Stress}}$ itself, and the $(B, +)$ multiplet whose component operators all have even twist. From the discussion above, it follows that the only bulk fields that contribute a meromorphic piece in

[5]These expressions are just rescaled versions of (3.31) of [130]. In particular, we have

$$\begin{aligned}
\widehat{M}^{\text{graviton}}_{s\text{-exchange}} &= -\sum_{n=0}^{\infty}\frac{\cos(n\pi)\Gamma(-\frac{3}{2}-n)}{4\sqrt{\pi}n!\Gamma(1/2-n)^2}\frac{4n^2-8ns+8n+4s^2+8st-20s+8t^2-32t+35}{s-(2n+1)} = -\frac{M^{\text{Zhou}}_{\text{graviton}}}{3\pi}\,,\\
\widehat{M}^{\text{gauge field}}_{s\text{-exchange}} &= -\sum_{n=0}^{\infty}\frac{\cos(n\pi)}{\sqrt{\pi}(1+2n)\Gamma(\frac{1}{2}-n)\Gamma(1+n)}\frac{2t+s-4}{s-(2n+1)} = -\frac{M^{\text{Zhou}}_{\text{vector}}}{\pi}\,,\\
\widehat{M}^{\Delta\,=\,1\text{ scalar}}_{s\text{-exchange}} &= -\sum_{n=0}^{\infty}\frac{\cos(n\pi)}{\sqrt{\pi}n!\Gamma(\frac{1}{2}-n)}\frac{1}{s-(2n+1)} = -\frac{M^{\text{Zhou}}_{\text{scalar}}}{\pi}\,.
\end{aligned}$$

[6]The computation for the $k = 2$ ABJ(M) theory is identical at leading order in the $1/c_T$ expansion.

the exchange diagrams are those in the stress tensor multiplet: the scalar fields dual to $\mathcal{O}_{\text{Stress}}$, the $\mathfrak{so}(8)_R$ gauge fields, and the graviton.[7] Consequently, $M_{s\text{-exchange}}$ is (up to an overall normalization that we will introduce later) a linear combination[8]

$$M_{s\text{-exchange}} = Y_{\mathbf{35}_c}(\sigma,\tau) M^{\Delta=1 \text{ scalar}}_{s\text{-exchange}} + bY_{\mathbf{28}}(\sigma,\tau) M^{\text{gauge}}_{s\text{-exchange}} + cY_{\mathbf{1}}(\sigma,\tau) M^{\text{graviton}}_{s\text{-exchange}}\,, \tag{5.2.9}$$

for some constants b and c. To determine the relative coefficients one can use the superconformal Ward identity, which was written down in position space in (1.1.37). To implement the Ward identities in Mellin space, we take the Mellin transform (5.2.3) of $\mathcal{G}(U,V;\sigma,\tau)$ expanded in terms of R-symmetry channels (1.1.33) to get

$$M(s,t;\sigma,\tau) = \sum_{a=0}^{2}\sum_{b=0}^{a} Y_{ab}(\sigma,\tau) M_{ab}(s,t)\,. \tag{5.2.10}$$

If we add up the two equations in (1.1.37), and expand in powers of $\bar{\alpha}$, then z and $\bar{z}$ always appear in the combination $z^m + \bar{z}^m$ for some integer m, which can then be turned into rational functions of U, V. The resulting equation involves a set of differential operators in U, V acting on $A_{ab}(U,V)$ in (1.1.33), organized in powers of $\bar{\alpha}$. Finally, we convert the Ward identity to Mellin space by setting

$$A_{ab}(U,V) \to M_{ab}(s,t), \quad U\partial_U \to \widehat{U\partial_U}, \quad V\partial_V \to \widehat{V\partial_V}, \quad U^m V^n \to \widehat{U^m V^n}, \tag{5.2.11}$$

where the hatted operators act on $M_{ab}(s,t)$ as

$$\begin{aligned}
\widehat{U\partial_U} M_{ab}(s,t) &= \frac{s}{2} M_{ab}(s,t)\,,\\
\widehat{V\partial_V} M_{ab}(s,t) &= \left[\frac{t}{2} - 1\right] M_{ab}(s,t)\,,\\
\widehat{U^m V^n} M_{ab}(s,t) &= M_{ab}(s-2m, t-2n) \left(1-\frac{s}{2}\right)^2_m \left(1-\frac{t}{2}\right)^2_n \left(1-\frac{u}{2}\right)^2_{-m-n}\,,
\end{aligned} \tag{5.2.12}$$

where $u = 4 - s - t$ and we will have independent constraints on each coefficient in the expansion in powers of $\bar{\alpha}$.

In [130], these Mellin space Ward identities were used to show that $b = -4$ and $c = 4$ in (5.2.9), so

$$M_{s\text{-exchange}} = Y_{\mathbf{35}_c}(\sigma,\tau) M^{\Delta=1 \text{ scalar}}_{s\text{-exchange}} - 4Y_{\mathbf{28}}(\sigma,\tau) M^{\text{gauge}}_{s\text{-exchange}} + 4Y_{\mathbf{1}}(\sigma,\tau) M^{\text{graviton}}_{s\text{-exchange}}\,. \tag{5.2.13}$$

[7] There is no bulk coupling between three scalars in the gravity multiplet, but there exists a boundary term that couples them (see for instance [144]). Therefore in the scalar exchange diagram the two intermediate points are located on the boundary.

[8] In the notation of [130], we have $\lambda_s = -1/\pi$, $\lambda_v = -b/\pi$, and $\lambda_g = -c/(3\pi)$.

Consequently, we can write a general tree-level Mellin amplitude as

$$M_{\text{tree}} = C \left[\widehat{M}_{\text{exchange}} + M_{\text{residual}} \right] , \tag{5.2.14}$$

where $\widehat{M}_{\text{exchange}} = \widehat{M}_{s\text{-exchange}} + \widehat{M}_{t\text{-exchange}} + \widehat{M}_{u\text{-exchange}}$, $\widehat{M}_{s\text{-exchange}}$ is given by (5.2.13) with all M's replaced by $\widehat{M}$'s, and C is an overall normalization factor.

The superconformal Ward identity also partly determines M_{residual} under the assumption that M_{residual} has a certain polynomial growth. For instance, if we require that M_{residual} has at most linear growth, as would be the case in a bulk theory of supergravity, then the analytic term is completely fixed in terms of (5.2.9) to be [130]

$$M_{\text{residual}}^{\text{SUGRA}} = \frac{1}{2} \left(s + u\sigma^2 + t\tau^2 - 4(t+u)\sigma\tau - 4(s+u)\sigma - 4(s+t)\tau \right) . \tag{5.2.15}$$

Thus, the supergravity tree level amplitude takes the form

$$M_{\text{tree}}^{\text{SUGRA}} = C \left[\widehat{M}_{\text{exchange}} + M_{\text{residual}}^{\text{SUGRA}} \right] . \tag{5.2.16}$$

For future reference, the linear growth at large s, t, u is given by[9]

$$M_{\text{tree}}^{\text{SUGRA}} \approx C \left[\frac{(tu + st\sigma + su\tau)^2}{stu} \right] . \tag{5.2.17}$$

The value of the overall coefficient C depends on the normalization of the operators $\mathcal{O}_{\text{Stress}}$ whose 4-point function we are considering. It is customary to normalize these operators such that their 2-point function is $\mathcal{O}(c_T^0)$ at large c_T, and then the connected 4-point function scales as c_T^{-1}. In particular, if the normalization of $\mathcal{O}$ is such that the disconnected piece of the 4-point function is given precisely by (5.2.2), then the overall coefficient C is fixed to be [130][10]

$$C = \frac{32}{\pi^2 c_T} = \frac{3}{2\sqrt{2k}\pi N^{3/2}} + O(N^{-5/2}) . \tag{5.2.18}$$

[9] At large s, t, u, we have

$$\widehat{M}_{\text{exchange}} \approx -\frac{1}{2} \left[\frac{t^2+u^2}{s} + \frac{s^2+t^2}{u}\sigma^2 + \frac{s^2+u^2}{t}\tau^2 \right] .$$

[10] In the notation of [130], we have $C = -\lambda_s/\pi$.

5.2.2 Contribution from higher derivative local terms

Now suppose the 11D supergravity Lagrangian is deformed by a local term of higher than 2-derivative order. The supersymmetric completion of higher derivative couplings are difficult to write off-shell, but are easily classified through local terms in the flat S-matrix elements of higher momentum powers. In $AdS_4 \times S^7$, they give rise to a contribution to the Mellin amplitude that is a polynomial expression in s, t, of the form

$$(tu + st\sigma + su\tau)^2 \, F(s^2 + t^2 + u^2, stu) + \dots \,, \tag{5.2.19}$$

where F is a homogeneous polynomial in s, t, u, determined by the corresponding flat space vertex, and $\cdots$ represents lower degree terms in s, t. One can check that the expression (5.2.19) solves the Mellin space superconformal Ward identity discussed above, after the latter is expanded to leading non-trivial order in large s and t. The number of polynomial solutions to the superconformal Ward identities of degree $p \geq 0$ is thus equal to the number of monomials in P and Q,

$$P \equiv s^2 + t^2 + u^2 \,, \qquad Q \equiv stu \tag{5.2.20}$$

of degree $d_P \geq 0$ in P and degree $d_Q \geq 0$ in Q such that $p \geq 2d_P + 3d_Q + 4$. This number is

$$n(p) = \left\lfloor \frac{6 + (p-1)^2}{12} \right\rfloor . \tag{5.2.21}$$

See the first two lines of Table 5.1, where for each degree $p \leq 10$ in s, t, u we listed the number of local solutions of the Ward identity with that growth at large s, t, u.

degree $\leq p$	3	4	5	6	7	8	9	10	$\cdots$
# of solutions	0	1	1	2	3	4	5	7	$\cdots$
11D vertex		R^4		D^4R^4	D^6R^4	D^8R^4	$D^{10}R^4$	$D^{12}R^4$ (2 types)	$\cdots$
scaling in M-theory		$c_T^{-\frac{5}{3}}$		$(0\times)c_T^{-\frac{19}{9}}$	$c_T^{-\frac{7}{3}}$	$c_T^{-\frac{23}{9}}$	$c_T^{-\frac{25}{9}}$	c_T^{-3}, $c_T^{-3}\log c_T$	$\cdots$
spin truncation		0		2	3	4	5	6	$\cdots$

Table 5.1: Number of solutions to the Ward identity of degree p polynomial growth at large s, t, u. At each order we can always have the solutions from previous orders. The solution corresponding to $p = 1$ is non-analytic; all other new solutions are purely polynomial in s, t, u and their number is given by $n(p)$ in (5.2.21). Spin truncation refers to the maximum spin of operators that receive contributions at this order. In the second to last row, we indicate the order of appearance of the *maximal degree* solution in the large c_T expansion of the Mellin amplitude of M-theory on $AdS_4 \times S^7$. Note that D^4R^4 is expected to be absent in M-theory, while one specific linear combination of the two possible $D^{12}R^4$ terms mixes with the 2-loop logarithmic divergence which is cut off at Planck scale.

Thus, the most general local term in the Mellin amplitude that solves the Ward identity is of the form

$$M_{\text{local}} = C \sum_{p \geq 4} \sum_{k=1}^{n(p)-n(p-1)} B_{p,k} M_{\text{local}}^{(p,k)} \,, \tag{5.2.22}$$

where $M_{\text{local}}^{(p,k)}$ is a polynomial solution to the Ward identity of degree p, labeled by the index k. We left out the overall constant C by convention. A well defined flat space limit would require the coefficients $B_{p,k}$ to scale with the AdS radius L like

$$B_{p,k} \sim L^{-2(p-1)}, \qquad \text{as } L \to \infty \,. \tag{5.2.23}$$

Beyond the leading large s, t, u asymptotics, the polynomial solutions are quite complicated. To simplify their form a bit, let us first note that any function $M(s,t;\sigma,\tau)$ that is crossing invariant can be written as

$$\begin{aligned} M &= (1+\sigma^2+\tau^2)f_1 + (s+u\sigma^2+t\tau^2)f_2 + (s^2+u^2\sigma^2+t^2\tau^2)f_3 \\ &\quad + (\sigma+\tau+\sigma\tau)f_4 + (t\sigma+u\tau+s\sigma\tau)f_5 + (t^2\sigma+u^2\tau+s^2\sigma\tau)f_6 \,, \end{aligned} \tag{5.2.24}$$

where the f_i are symmetric functions of s,t,u, or equivalently functions of P and Q as defined in (5.2.20). The first purely polynomial solution to the Ward identity, which is the unique solution of degree 4 we denoted by $M_{\text{local}}^{(4,1)}$ in (5.2.22), can then be written as

$$\begin{aligned} f_1^{(4,1)} &= \frac{P^2}{4} + \frac{6}{7}Q - \frac{22}{5}P + \frac{96}{5} \,, \\ f_2^{(4,1)} &= Q + 2P - \frac{736}{35} \,, \\ f_3^{(4,1)} &= -\frac{P}{2} + \frac{228}{35} \,, \\ f_4^{(4,1)} &= -\frac{104}{7}Q - \frac{40}{7}P + \frac{4672}{35} \,, \\ f_5^{(4,1)} &= 2Q - \frac{18}{7}P - \frac{496}{7} \,, \\ f_6^{(4,1)} &= \frac{832}{35} \,. \end{aligned} \tag{5.2.25}$$

In this normalization, the solution $M_{\text{local}}^{(4,1)}$ has the asymptotic form (5.2.19) with $F(P,Q)=1$. For $p=6,7,8,9$ we find one new solution for each p, while for $p=10$ we find two new solutions. We will write these polynomials in the notation of (5.2.20) and (5.2.24), so that in the large s,t limit they take the form

$$\begin{aligned} &(tu + st\sigma + su\tau)^2 \, F_{p,d}(P,Q) \,, \\ &P \equiv s^2+t^2+u^2 \,, \qquad Q \equiv stu \,, \end{aligned} \tag{5.2.26}$$

where they are normalized so that

$$F_{6,1} = P\,, \quad F_{7,1} = Q\,, \quad F_{8,1} = P^2\,, \quad F_{9,1} = QP\,, \quad F_{10,1} = P^3\,, \quad F_{10,2} = Q^2\,. \tag{5.2.27}$$

The full polynomials are then

$$F_{6,1} = P\,, \quad F_{7,1} = Q\,, \quad F_{8,1} = P^2\,, \quad F_{9,1} = QP\,, \quad F_{10,1} = P^3\,, \quad F_{10,2} = Q^2\,. \tag{5.2.28}$$

The full polynomials are then

$$\begin{aligned}
&M^{(6,1)}_{\text{local}} :\\
&\quad f_1^{(6,1)} = \frac{P^3}{4} - \frac{102P^2}{11} - \frac{10PQ}{11} + \frac{1152P}{11} + \frac{608Q}{77} - \frac{4096}{11}\,,\\
&\quad f_2^{(6,1)} = 2P^2 + PQ - \frac{544P}{11} - 16Q + \frac{21760}{77}\,,\\
&\quad f_3^{(6,1)} = -\frac{P^2}{2} + \frac{180P}{11} + \frac{40Q}{11} - \frac{8192}{77}\,,\\
&\quad f_4^{(6,1)} = -\frac{64P^2}{11} - \frac{208PQ}{11} + \frac{14912P}{77} - \frac{8000\,Q}{77} - \frac{1536}{7}\,,\\
&\quad f_5^{(6,1)} = -\frac{50P^2}{11} + 2PQ - \frac{11904P}{77} + \frac{592\,Q}{11} + \frac{44416}{77}\,,\\
&\quad f_6^{(6,1)} = 56P - \frac{40Q}{11} - \frac{16320}{77}\,.
\end{aligned} \tag{5.2.29}$$

$$\begin{aligned}
&M^{(7,1)}_{\text{local}} :\\
&\quad f_1^{(7,1)} = \frac{13P^3}{4} + \frac{P^2Q}{4} - \frac{1326P^2}{11} - \frac{2074P\,Q}{143} + \frac{14976P}{11} + \frac{36Q^2}{13} + \frac{15344\,Q}{143} - \frac{53248}{11}\,,\\
&\quad f_2^{(7,1)} = 26P^2 + 15PQ - \frac{7136P}{11} + Q^2 - \frac{2592Q}{11} + \frac{40960}{11}\,,\\
&\quad f_3^{(7,1)} = -\frac{13P^2}{2} - \frac{PQ}{2} + \frac{31172P}{143} + \frac{8500\,Q}{143} - \frac{204032}{143}\,,\\
&\quad f_4^{(7,1)} = -\frac{11568P^2}{143} - \frac{37288PQ}{143} + \frac{394816\,P}{143} - \frac{296Q^2}{13} - \frac{97664Q}{143} - \frac{820736}{143}\,,\\
&\quad f_5^{(7,1)} = -\frac{8460P^2}{143} + \frac{278PQ}{13} - \frac{22256P}{11} + 2\,Q^2 + \frac{58368Q}{143} + \frac{1271936}{143}\,,\\
&\quad f_6^{(7,1)} = \frac{105056P}{143} + \frac{2720Q}{143} - \frac{40704}{13}\,.
\end{aligned} \tag{5.2.30}$$

$M_{\text{local}}^{(8,1)}$:

$$
\begin{aligned}
f_1^{(8,1)} &= \frac{P^4}{4} - \frac{118906P^3}{8775} - \frac{303226P^2\ Q}{114075} + \frac{779296P^2}{2925} + \frac{123799376P\ Q}{1482975} - \frac{6478336P}{2925} \\
&\quad + \frac{893008\ Q^2}{54925} - \frac{776857216Q}{1482975} + \frac{57749504}{8775}\,, \\
f_2^{(8,1)} &= 2P^3 + P^2Q - \frac{635168P^2}{8775} - \frac{214816P\ Q}{7605} + \frac{96849664P}{114075} + \frac{64736\ Q^2}{114075} + \frac{7101056Q}{38025} \\
&\quad - \frac{74051584}{22815}\,, \\
f_3^{(8,1)} &= -\frac{P^3}{2} + \frac{219092P^2}{8775} + \frac{819392P\ Q}{114075} - \frac{482840128P}{1482975} - \frac{16172032\ Q}{296595} + \frac{1918862848}{1482975}\,, \\
f_4^{(8,1)} &= -\frac{88P^3}{15} - \frac{344P^2Q}{15} + \frac{92261504\ P^2}{494325} - \frac{807268288PQ}{1482975} + \frac{669270016\ P}{1482975} + \frac{20749184Q^2}{1482975} \\
&\quad + \frac{246585856Q}{1482975} + \frac{3609677824}{1482975}\,, \\
f_5^{(8,1)} &= -\frac{98P^3}{15} + 2P^2Q - \frac{11447968P^2}{32955} + \frac{257584528\ PQ}{1482975} + \frac{22751488P}{8775} + \frac{129472\ Q^2}{114075} \\
&\quad + \frac{638850176Q}{494325} - \frac{15066962944}{1482975}\,, \\
f_6^{(8,1)} &= \frac{6768P^2}{65} - \frac{112PQ}{15} - \frac{883842304\ P}{1482975} - \frac{116637824Q}{296595} + \frac{305705984}{164775}\,.
\end{aligned}
\tag{5.2.31}
$$

$M_{\text{local}}^{(9,1)}$:

$$
\begin{aligned}
f_1^{(9,1)} &= -\frac{4P^4}{182699} + \frac{P^3 Q}{4} + \frac{256\ P^3}{182699} - \frac{1470246 P^2 Q}{215917} - \frac{6144\ P^2}{182699} + \frac{16 P Q^2}{17} + \frac{2169915952 P\ Q}{30876131} \\
&\quad + \frac{65536 P}{182699} + \frac{13084704\ Q^2}{30876131} - \frac{7754383488 Q}{30876131} - \frac{245760}{182699}\,, \\
f_2^{(9,1)} &= -\frac{32 P^3}{182699} + \frac{365382 P^2 Q}{182699} + \frac{814496\ P^2}{182699} + P Q^2 - \frac{77230848 P Q}{2375087} - \frac{250089728\ P}{2375087} \\
&\quad - \frac{14173656 Q^2}{2375087} + \frac{246246272 Q}{2375087} + \frac{1292496896}{2375087}\,, \\
f_3^{(9,1)} &= \frac{8P^3}{182699} - \frac{P^2 Q}{2} + \frac{991112\ P^2}{182699} + \frac{49147600 P Q}{2375087} - \frac{2231011968\ P}{30876131} + \frac{64 Q^2}{17} \\
&\quad - \frac{3759586080\ Q}{30876131} + \frac{7327562240}{30876131}\,, \\
f_4^{(9,1)} &= -\frac{90136 P^3}{16609} - \frac{2773792 P^2\ Q}{182699} + \frac{7526374144 P^2}{30876131} - \frac{456 P\ Q^2}{17} + \frac{21774069376 P Q}{30876131} \\
&\quad - \frac{53524939264\ P}{30876131} - \frac{14438168832 Q^2}{30876131} + \frac{996369408\ Q}{165113} - \frac{561678131200}{30876131}\,, \\
f_5^{(9,1)} &= -\frac{9926 P^3}{182699} - \frac{1203696 P^2 Q}{182699} - \frac{175354688\ P^2}{30876131} + 2 P Q^2 - \frac{16185059776 P\ Q}{30876131} \\
&\quad + \frac{248339584 P}{182699} + \frac{237662432\ Q^2}{2375087} - \frac{77421998592\ Q}{30876131} + \frac{24500873216}{2806921}\,, \\
f_6^{(9,1)} &= \frac{1265488 P^2}{182699} + \frac{20966800 P\ Q}{182699} - \frac{13022172672 P}{30876131} - \frac{64\ Q^2}{17} + \frac{1028671104 Q}{1816243} \\
&\quad - \frac{57606243328}{30876131}\,.
\end{aligned}
\tag{5.2.32}
$$

$$
\begin{aligned}
&M_{\text{local}}^{(10,1)}: \\
&f_1^{(10,1)} = \frac{P^5}{4} - \frac{15560102282570\ P^4}{788812391521} - \frac{1352721664990P^3\ Q}{46400728913} + \frac{488239952498560\ P^3}{788812391521} \\
&\quad + \frac{761509603801472P^2\ Q}{788812391521} - \frac{7568372894878720\ P^2}{788812391521} - \frac{36285564992176P\ Q^2}{788812391521} \\
&\quad - \frac{8944052927693568P\ Q}{788812391521} + \frac{56120322339061760\ P}{788812391521} - \frac{93811894726016\ Q^2}{788812391521} \\
&\quad + \frac{34293760356859904\ Q}{788812391521} - \frac{157293341281452032}{788812391521}, \\
&f_2^{(10,1)} = 2P^4 + P^3Q - \frac{86930417851808\ P^3}{788812391521} - \frac{193714236162928P^2\ Q}{788812391521} + \frac{1555772227023104\ P^2}{788812391521} \\
&\quad - \frac{4531714954240P\ Q^2}{46400728913} + \frac{3584920148835072P\ Q}{788812391521} - \frac{737911375243264\ P}{46400728913} \\
&\quad + \frac{660575198396864\ Q^2}{788812391521} - \frac{17697616105197568\ Q}{788812391521} + \frac{2575160352276480}{46400728913}, \\
&f_3^{(10,1)} = -\frac{P^4}{2} + \frac{29459546898780\ P^3}{788812391521} + \frac{2793360500552P^2\ Q}{46400728913} - \frac{100360584961920\ P^2}{71710217411} \\
&\quad - \frac{1858335910889184P\ Q}{788812391521} + \frac{13493643403236352\ P}{788812391521} - \frac{15338759568448\ Q^2}{41516441659} \\
&\quad + \frac{12456326461020928\ Q}{788812391521} - \frac{53235484975194112}{788812391521}, \\
&f_4^{(10,1)} = -\frac{112P^4}{19} - \frac{512P^3Q}{19} + \frac{560077576316736\ P^3}{788812391521} + \frac{206913777028736P^2\ Q}{788812391521} \\
&\quad - \frac{1708367914422272\ P^2}{71710217411} + \frac{2132575731403584P\ Q^2}{788812391521} - \frac{53535047508136448P\ Q}{788812391521} \\
&\quad + \frac{15759894134800384\ P}{71710217411} + \frac{3255946003526144\ Q^2}{71710217411} - \frac{54552662836060160\ Q}{112687484503} \\
&\quad + \frac{937350835787825152}{788812391521}, \\
&f_5^{(10,1)} = -\frac{162P^4}{19} + 2P^3Q - \frac{73978710764208\ P^3}{112687484503} + \frac{781824468030880P^2\ Q}{788812391521} \\
&\quad + \frac{7762246710240640\ P^2}{788812391521} - \frac{9063429908480P\ Q^2}{46400728913} + \frac{45210666149624576P\ Q}{788812391521} \\
&\quad - \frac{157261767221821440\ P}{788812391521} - \frac{7746117391574656\ Q^2}{788812391521} + \frac{132057277767493632\ Q}{788812391521} \\
&\quad - \frac{30032651544567808}{71710217411}, \\
&f_6^{(10,1)} = \frac{54328P^3}{323} - \frac{216P^2Q}{19} - \frac{1729256379496320\ P^2}{788812391521} - \frac{538093588640640P\ Q}{41516441659} \\
&\quad + \frac{39075972258892288\ P}{788812391521} + \frac{15338759568448\ Q^2}{41516441659} - \frac{31559299392227840\ Q}{788812391521} \\
&\quad + \frac{75001766595411968}{788812391521}.
\end{aligned}
\tag{5.2.33}
$$

$M_{\text{local}}^{(10,2)}$:

$$
\begin{aligned}
f_1^{(10,2)} &= \frac{299520P^4}{112687484503} - \frac{195812761P^3\ Q}{26514702236} - \frac{23003136P^3}{112687484503} + \frac{P^2\ Q^2}{4} + \frac{28123710360P^2Q}{112687484503} \\
&+ \frac{705429504\ P^2}{112687484503} - \frac{111982834408P\ Q^2}{112687484503} + \frac{741648981488P\ Q}{112687484503} - \frac{11777605632P}{112687484503} \\
&+ \frac{90\ Q^3}{19} + \frac{284961517968Q^2}{112687484503} - \frac{7688398979328\ Q}{112687484503} + \frac{59624128512}{112687484503}\,, \\
f_2^{(10,2)} &= \frac{2396160P^3}{112687484503} - \frac{350338726P^2\ Q}{5930920237} + \frac{521837758752\ P^2}{112687484503} + \frac{13061538357P\ Q^2}{6628675559} \\
&- \frac{78313654944P\ Q}{112687484503} - \frac{12715684780800\ P}{112687484503} + Q^3 - \frac{3295800890632\ Q^2}{112687484503} \\
&- \frac{3955386769280\ Q}{112687484503} + \frac{69834972106752}{112687484503}\,, \\
f_3^{(10,2)} &= -\frac{599040P^3}{112687484503} + \frac{195812761P^2Q}{13257351118} + \frac{47804861328P^2}{10244316773} - \frac{P\ Q^2}{2} + \frac{2089851982888PQ}{112687484503} \\
&- \frac{6420168418176\ P}{112687484503} + \frac{2447753248668\ Q^2}{112687484503} - \frac{14510922184416\ Q}{112687484503} + \frac{16952863051776}{112687484503}\,, \\
f_4^{(10,2)} &= -\frac{525817052976P^3}{112687484503} - \frac{2126292211816P^2\ Q}{112687484503} + \frac{3829309249536\ P^2}{10244316773} - \frac{3084100585440P\ Q^2}{112687484503} \\
&+ \frac{158246717908608P\ Q}{112687484503} - \frac{101933809566720P}{10244316773} - \frac{584\ Q^3}{19} + \frac{21726894668416\ Q^2}{10244316773} \\
&- \frac{2789434673572352\ Q}{112687484503} + \frac{461843423772672}{5930920237}\,, \\
f_5^{(10,2)} &= -\frac{3406879532P^3}{112687484503} + \frac{7280563630P^2\ Q}{112687484503} - \frac{234903259776\ P^2}{112687484503} - \frac{44350210808P\ Q^2}{6628675559} \\
&- \frac{1631675597600P\ Q}{112687484503} + \frac{278557348028672P}{112687484503} + 2\ Q^3 - \frac{80859508899296Q^2}{112687484503} \\
&+ \frac{1258912624870784\ Q}{112687484503} - \frac{403649143517184}{10244316773}\,, \\
f_6^{(10,2)} &= \frac{624467978240P^2}{112687484503} + \frac{2192783584032P\ Q}{112687484503} - \frac{78555860196864\ P}{112687484503} + \frac{14514678629152\ Q^2}{112687484503} \\
&- \frac{294378521704192\ Q}{112687484503} + \frac{1144844596080640}{112687484503}\,.
\end{aligned}
\tag{5.2.34}
$$

5.2.3 Loop contributions

While the large c_T expansion of the M-theory Mellin amplitude in $AdS_4 \times S^7$ contains local terms that correspond to higher derivative vertices in the flat space limit, there must also be "loop terms" that are required by unitarity. The loop terms are determined, up to local terms, in terms of lower order terms in the large c_T expansion [116, 129, 145, 146].

Unlike the loop terms in the flat spacetime S-matrix, a loop term in the Mellin amplitude involves an infinite series of poles rather than a branch cut in the s, t, u variables. For instance, the supergravity 1-loop Mellin amplitude can be expressed as a sum over poles in s at $s = 2\Delta + 2n$, $n = 0, 1, 2, \cdots$, whose residues are polynomials in t, together with cross terms related by permutation on s, t, u. In the flat space limit, the sum of poles turns into an integral, which is nothing but a representation of the supergravity 1-loop S-matrix in the form of a dispersion relation.

The flat space loop amplitudes can typically be expressed as loop integrals that are UV divergent; the UV divergence can be renormalized by local counter terms up to logarithmic divergences. Similarly, the Mellin loop amplitudes typically involve a divergent sum over poles, that can be regularized by subtracting off polynomials in s, t term by term in the sum, up to logarithmic divergences. The log divergence is physical and is cut off at Planck scale in M-theory, resulting in a $\log c_T$ dependence in the Mellin amplitude. In this Chapter, we will not compute the M-theory loop Mellin amplitudes explicitly, but illustrate the general structure in a few examples, as follows.

The 1-loop 4-super-graviton amplitude in 11D supergravity has only power divergences that can be renormalized away, resulting in a contribution to the S-matrix element that scales with energy like $\ell_{11}^{18}(\sqrt{s})^{11}$. The 1-loop supergravity contribution to the Mellin amplitude likewise can be written as a convergent sum over double trace poles. It comes with an overall coefficient that scales like $(\ell_{11}/L)^{18} \sim c_T^{-2}$.

In the flat space S-matrix of M-theory, there is a higher momentum order 1-loop amplitude that scales like $\ell_{11}^{24}(\sqrt{s})^{\frac{17}{2}}$, whose unitarity cut factorizes into a tree level supergravity amplitude and an R^4 vertex. It gives rise to another 1-loop Mellin amplitude that sums up double trace poles, with an overall coefficient that scales like $(\ell_{11}/L)^{24} \sim c_T^{-\frac{8}{3}}$.

The 2-loop 4-super-graviton amplitude of 11D supergravity has a log divergence of the form [137] $(\log \Lambda) \frac{7}{5 \cdot 2^8 \cdot 13!} \ell_{11}^{18} stu \left[438(s^6 + t^6 + u^6) - 53s^2t^2u^2\right] \mathcal{A}_{\text{SG,tree}}$. The cutoff Λ is taken to be at Planck scale in M-theory. This gives rise to a local term in the Mellin amplitude of degree 10 in s, t, u, whose coefficient scales like $(\ell_{11}/L)^{18} \log(L/\ell_{11}) \sim c_T^{-3} \log c_T$, as indicated in Table 5.1.

5.2.4 The large radius expansion of the Mellin amplitude of M-theory on $AdS_4 \times S^7$

As shown in [127, 129], the relation between the large s, t limit of the Mellin amplitude $M(s,t)$ and the flat spacetime scattering amplitude $\mathcal{A}(s,t)$ takes the form

$$\lim_{L\to\infty} (2L)^7 V_7 \, M(L^2\tilde{s}, L^2\tilde{t}) = \frac{1}{\sqrt{\pi}} \int_0^\infty d\beta \, \beta^{-1/2} e^{-\beta} \mathcal{A}\left(2\beta\tilde{s}, 2\beta\tilde{t}\right) \,, \tag{5.2.35}$$

where $V_7 = \pi^4/3$ is the volume of the unit S^7. The amplitude $\mathcal{A}$ appearing on the RHS is the 11D flat spacetime amplitude of four supergravitons, with momenta restricted to a 4D sub-spacetime, integrated against four supergraviton Kaluza-Klein mode wave functions on the S^7, normalized by multiplying with an S^7 volume factor so that the $L \to \infty$ limit is finite. Indeed, the scaling in (5.2.23) is such that only the most divergent term in each $M^{(p,k)}_{\text{local}}$ contributes to the limit on the LHS of (5.2.35).

More precisely, if we label by i, j, k, ℓ the four supergraviton KK modes, then the amplitude $\mathcal{A}_{ijk\ell}(s,t)$ appearing on the RHS of (5.2.35) is related to the 11D scattering amplitude $\mathcal{A}^{11D}_{\alpha\beta\gamma\delta}(s,t)$ by

$$\mathcal{A}_{ijk\ell}(s,t) = \sum_{\alpha,\beta,\gamma,\delta} \mathcal{A}^{11D}_{\alpha\beta\gamma\delta}(s,t)\, V_7 \int_{S^7} d^7x \sqrt{g}\, \Psi_i^\alpha(x) \Psi_j^\beta(x) \Psi_k^\gamma(x) \Psi_\ell^\delta(x). \tag{5.2.36}$$

Here $\mathcal{A}^{11D}_{\alpha\beta\gamma\delta}(s,t)$ is an invariant tensor in the supergraviton polarizations $\alpha, \beta, \gamma, \delta$. $\Psi_i^\alpha(x)$ is the normalized KK mode wave function for the particle i on a unit S^7.

Since on the 3D SCFT side we are studying scalar operators transforming as the $\mathbf{35}_c$ of $\mathfrak{so}(8)_R$, the flat space limit of the 4-point function of these operators corresponds to the scattering amplitude $\mathcal{A}(s,t)$ of the 11D gravitons in their lowest KK modes, with momenta concentrated in a 4-dimensional sub-spacetime and polarization in the transverse directions. After contraction with $\mathfrak{so}(8)_R$ polarization vectors and rewriting in terms of the $\mathfrak{so}(8)_R$ invariants σ, τ (after stripping out a factor of $(Y_1 \cdot Y_2)^2 (Y_3 \cdot Y_4)^2$), the scattering amplitude will be denoted by $\mathcal{A}(s,t;\sigma,\tau)$. Rather than evaluating the integral in (5.2.36) directly, we can obtain the answer by reducing the tree level amplitude of the lowest KK modes on $AdS_4 \times S^7$ to that of the $\mathcal{N} = 8$ gauged supergravity in AdS_4 [147] (see also [115], as well as [148] for a review), whose flat spacetime limit gives the tree amplitude in 4D ungauged $\mathcal{N} = 8$ supergravity [149–151].

The 4D $\mathcal{N} = 8$ gravity multiplet consists of 128 bosonic and 128 fermionic massless states that can be conveniently represented as anti-symmetric tensors of the $SU(8)$ R-symmetry as follows: the helicity $h = +2$ and $h = -2$ states of the graviton can be represented as $SU(8)$ singlets h^+ and $h^- = h^{ABCDEFGH}$; the helicity $h = +3/2$ and $h = -3/2$ states of the gravitino can be represented as ψ^A and $\psi^{ABCDEFG}$; the helicity $h = +1$ and $h = -1$ states of the gravi-photon can be represented as v^{AB} and v^{ABCDEF}; the helicity $h = +1/2$ and $h = -1/2$ states of the gravi-photino can be represented as χ^{ABC} and χ^{ABCDE}; and the scalars, of helicity $h = 0$, can be represented as S^{ABCD}. Here $A = 1, \ldots 8$ are $SU(8)$ fundamental indices.

The 4-point scattering amplitude of any four particles from the gravity multiplet can be succinctly described by first introducing auxiliary Grassmann variables η_A and grouping all the particles of the gravity multiplet into an $\mathcal{N} = 8$ superfield (see for example [25])

$$\Phi = h^+ + \eta_A \psi^A - \frac{1}{2}\eta_A \eta_B v^{AB} - \frac{1}{6}\eta_A \eta_B \eta_C \psi^{ABC} + \frac{1}{4!}\eta_A \eta_B \eta_C \eta_D S^{ABCD} + \cdots. \tag{5.2.37}$$

The expression for Φ is designed such that one can extract a state of a given helicity by taking derivatives with respect to the auxiliary Grassmann variables η_A. For the 70 scalars, we have

$$S^{ABCD} = \frac{\partial}{\partial\eta^A}\frac{\partial}{\partial\eta^B}\frac{\partial}{\partial\eta^C}\frac{\partial}{\partial\eta^D}\Phi\,. \tag{5.2.38}$$

The tree-level 4-point scattering amplitude in supergravity can then be written as (see [25])[11]

$$\mathcal{A}_{\text{tree, SG}}(s,t;\eta_i) = \frac{1}{256}\prod_{A=1}^{8}\left(\sum_{i,j=1}^{4}\langle ij\rangle\eta_{iA}\eta_{jA}\right)\frac{[34]^4}{\langle 12\rangle^4}\frac{1}{stu}\,, \tag{5.2.39}$$

Here, η_{iA}, $i=1,\dots 4$, are the auxiliary polarization variables associated with the ith particle. The scattering amplitude of 4 scalars can be extracted by acting with derivatives on (5.2.39):

$$\mathcal{A}_{\text{tree, SG}}(\mathcal{O}_{\text{Stress}}\mathcal{O}_{\text{Stress}}\mathcal{O}_{\text{Stress}}\mathcal{O}_{\text{Stress}})^{A_1\cdots D_4}(s,t) = \partial_1^{A_1B_1C_1D_1}\partial_2^{A_2B_2C_2D_2}\partial_3^{A_3B_3C_3D_3}\partial_4^{A_4B_4C_4D_4}\mathcal{A}_{\text{tree, SG}}(s,t;\eta_i)\,, \tag{5.2.40}$$

where $\partial_i^{ABCD} \equiv \frac{\partial}{\partial\eta_{iA}}\frac{\partial}{\partial\eta_{iB}}\frac{\partial}{\partial\eta_{iC}}\frac{\partial}{\partial\eta_{iD}}$.

To obtain the flat space limit of the scattering amplitude of the $\mathbf{35}_c$ scalars in *gauged* supergravity, we should identify which of the 70 scalars S^{ABCD} of ungauged supergravity correspond to the $\mathbf{35}_c$ ones. To do so, note that the $SO(8)$ R-symmetry in AdS_4 is embedded into the $SU(8)$ flat space R-symmetry in such a way that the supercharges, transforming in the $\mathbf{8}$ of $SU(8)$, should also transform as the $\mathbf{8}_v$ of $SO(8)$ according to the convention we use in this Chapter. The 70 S^{ABCD} scalars transform then as an irreducible representation of $SU(8)$, namely the $\mathbf{70}$, which decomposes as $\mathbf{35}_s \oplus \mathbf{35}_c$ under $SO(8)$—the $\mathbf{35}_s$ and $\mathbf{35}_c$ can be identified with self-dual and anti-self-dual rank-4 anti-symmetric tensors, respectively.

To connect this discussion to our notation, we should convert between the representation of the $\mathbf{35}_c$ as a rank-4 anti-self-dual tensor of the $\mathbf{8}_v$ and its representation as a rank-2 symmetric traceless tensor of the $\mathbf{8}_c$ that we have been using. The conversion is realized through a tensor $E^{IJ}{}_{ABCD}$, which is symmetric traceless in the $\mathbf{8}_c$ indices I,J and anti-symmetric in the $\mathbf{8}_v$ indices obeying the anti-self-duality condition

$$E^{IJ}{}_{ABCD} = -\frac{1}{24}\epsilon_{ABCD}{}^{A'B'C'D'}E^{IJ}{}_{A'B'C'D'}\,. \tag{5.2.41}$$

Here, ϵ is the totally anti-symmetric tensor defined such that $\epsilon^{12345678}=1$, and all indices are raised and lowered with the Kronecker symbol.

To obtain $E^{IJ}{}_{ABCD}$, one can start with the Clebsch-Gordan coefficients $E^{I}{}_{aA}$ for obtaining an $SO(8)$ singlet out of the product $\mathbf{8}_v\otimes\mathbf{8}_c\otimes\mathbf{8}_s$: the coefficients $E^{I}{}_{aA}$ have the property that for any three quantities

[11] For the scattering amplitudes corresponding to higher derivative interactions in 4D, see [135, 136, 152].

u_I, v^a, and w^A transforming as $\mathbf{8}_v$, $\mathbf{8}_s$, and $\mathbf{8}_c$, respectively, the product $u_I v^a w^A E^I{}_{aA}$ is an $SO(8)$ singlet. As is well-known, the $E^I{}_{aA}$ can be identified with the coefficients in the multiplication table of the generators e_α ($\alpha = 1, \dots 8$) of the octonion algebra: $e_\alpha \cdot e_\beta = E^\gamma{}_{\beta\alpha} e^\gamma$, where $e_1 = 1$ and $e_\alpha \cdot e_\alpha = 1$ for any given α. Explicit formulas for the $E^I{}_{aA}$ are given in (A.12) of [153]. From the $E^I{}_{aA}$, we can construct

$$E^{IJ}{}_{AB} = E^{[I}{}_{a[A} E^{J]a}{}_{B]} \,, \tag{5.2.42}$$

which is a tensor that converts between the adjoint representation of $SO(8)$ written as either an anti-symmetric tensor of the $\mathbf{8}_v$ or as an anti-symmetric tensor of the $\mathbf{8}_c$. Then, using $E^{IJ}{}_{AB}$, we can further construct our desired tensor

$$E^{IJ}{}_{ABCD} = E^{IK}{}_{AB} E^{JK}{}_{CD} + E^{JK}{}_{AB} E^{IK}{}_{CD} - \frac{1}{4} \delta^{IJ} E^{KL}{}_{AB} E^{KL}{}_{CD} \,, \tag{5.2.43}$$

which has all the properties we required.

From any anti-self-dual anti-symmetric tensor T^{ABCD} we can obtain a symmetric traceless tensor $E^{IJ}{}_{ABCD} T^{ABCD}$, which can be further contracted with the null polarizations Y^I to obtain a quadratic function of Y:

$$T(Y) = Y_I Y_J E^{IJ}{}_{ABCD} T^{ABCD} \,. \tag{5.2.44}$$

Using this procedure for the amplitude (5.2.40), we can extract

$$\begin{aligned} \mathcal{A}_{\text{tree, SG}}(\mathcal{O}_{\text{Stress}}\mathcal{O}_{\text{Stress}}\mathcal{O}_{\text{Stress}}\mathcal{O}_{\text{Stress}})(s,t;Y_i) = &\left(\prod_{i=1}^{4} Y_{iI} Y_{iJ} E^{IJ}{}_{A_i B_i C_i D_i} \right) \\ &\mathcal{A}_{\text{tree, SG}}(\mathcal{O}_{\text{Stress}}\mathcal{O}_{\text{Stress}}\mathcal{O}_{\text{Stress}}\mathcal{O}_{\text{Stress}})^{A_1 \cdots D_4}(s,t) \,. \end{aligned} \tag{5.2.45}$$

Due to the $SO(8)$ R-symmetry, this expression can be written as $(Y_1 \cdot Y_2)^2 (Y_3 \cdot Y_4)^2$ times a function of the $SO(8)$ invariants σ and τ introduced in (1.1.11). To uncover this form, it is easier to set Y_i to some particular values, for instance

$$Y_i = \begin{pmatrix} \frac{1-\vec{y}_i^2}{2} & \vec{y} & i\frac{1+\vec{y}_i^2}{2} \end{pmatrix} \,, \tag{5.2.46}$$

for some 6-vectors $\vec{y}_i$ that we can further take to be

$$\begin{aligned}
\vec{y}_1 &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\,,\\
\vec{y}_2 &= \begin{pmatrix} \infty & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\,,\\
\vec{y}_3 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\,,\\
\vec{y}_4 &= \begin{pmatrix} x & y & 0 & 0 & 0 & 0 \end{pmatrix}\,,
\end{aligned} \tag{5.2.47}$$

for some parameters x and y. Plugging these expressions in (5.2.45) one finds that

$$\begin{aligned}
\frac{\mathcal{A}_{\text{tree, SG}}(\mathcal{O}_{\text{Stress}}\mathcal{O}_{\text{Stress}}\mathcal{O}_{\text{Stress}}\mathcal{O}_{\text{Stress}})(s,t;Y_i)}{(Y_1\cdot Y_2)^2(Y_3\cdot Y_4)^2} = \frac{1120}{stu}\langle 34\rangle^4[34]^4\Big[&1-4xA+4(1-x)B+2(3x^2+y^2)A^2\\
&+\left(4x(x-1)+12y^2\right)AB+\left(7(x-1)^2+2y^2\right)B^2-4x(x^2+y^2)A^3\\
&+4\left(x^2(x-1)+(x-3)y^2\right)A^2B+4\left(x(x-1)^2+(2+x)y^2\right)AB^2\\
&+4(1-x)\left((x-1)^2+y^2)\right)B^3-4(x^2+y^2)\left(x(x-1)+y^2\right)A^3B\\
&+(x^2+y^2)^2A^4+2\left(y^2+3\left(y^2+x(x-1)\right)^2\right)A^2B^2\\
&-4\left((x-1)^2+y^2\right)\left(x(x-1)+y^2\right)AB^3+\left((x-1)^2+y^2\right)^2B^4\Big]\,,
\end{aligned} \tag{5.2.48}$$

where

$$A\equiv\frac{\langle 13\rangle\langle 24\rangle}{\langle 12\rangle\langle 34\rangle}\,,\qquad B\equiv\frac{\langle 14\rangle\langle 23\rangle}{\langle 12\rangle\langle 34\rangle}\,. \tag{5.2.49}$$

Making use of the $SO(8)$ symmetry, the x and y dependence can be rewritten in terms of σ and τ through

$$x=\frac{1+\sigma-\tau}{2}\,,\qquad y^2=\frac{2\sigma(1+\tau)-\sigma^2-(1-\tau)^2}{4}\,. \tag{5.2.50}$$

Using that

$$\begin{aligned}
&s=(p_3+p_4)^2=-\langle 34\rangle[34]\,,\qquad t=(p_2+p_3)^2=-\langle 23\rangle[23]\,,\\
&u=(p_2+p_4)^2=-\langle 24\rangle[24]\,,
\end{aligned} \tag{5.2.51}$$

as well as the relations

$$\langle 12\rangle[24] = -\langle 13\rangle[34]\,, \qquad \langle 12\rangle[23] = \langle 14\rangle[34] \tag{5.2.52}$$

that follow from momentum conservation, it can be shown that

$$A = -\frac{u}{s}\,, \qquad B = \frac{t}{s}\,. \tag{5.2.53}$$

Plugging (5.2.50) and (5.2.53) into (5.2.48) and using that $s+t+u=0$, it can be shown that (5.2.48) can be rewritten as

$$\frac{\mathcal{A}_{\text{tree, SG}}(\mathcal{O}_{\text{Stress}}\mathcal{O}_{\text{Stress}}\mathcal{O}_{\text{Stress}}\mathcal{O}_{\text{Stress}})(s,t;Y_i)}{(Y_1\cdot Y_2)^2(Y_3\cdot Y_4)^2} = 1120\frac{(tu+st\sigma+su\tau)^2}{stu}\,. \tag{5.2.54}$$

Using this result for the overall factor of $\frac{(tu+st\sigma+su\tau)^2}{stu}$, we can write the 11d scattering amplitude as

$$\begin{aligned}\mathcal{A}(s,t;\sigma,\tau) = \ell_{11}^9\frac{(tu+st\sigma+su\tau)^2}{stu}\Big[&1+\ell_{11}^6 f_{R^4}(s,t)+\ell_{11}^9 f_{1-\text{loop}}(s,t)+\ell_{11}^{12}f_{D^6R^4}(s,t)\\ &+\ell_{11}^{14}f_{D^8R^4}(s,t)+\ell_{11}^{15}f_{1-\text{loop},R^4}(s,t)+\ell_{11}^{16}f_{D^{10}R^4}(s,t)+\ell_{11}^{18}f_{2-\text{loop}}(s,t)+\ell_{11}^{18}f_{D^{12}R^4}(s,t)+\cdots\Big]\,,\end{aligned} \tag{5.2.55}$$

with $f_{R^4} = \frac{stu}{3\cdot 2^7}$ and $f_{D^6R^4}(s,t,u) = \frac{(stu)^2}{15\cdot 2^{15}}$ as given in (1.2.18). $f_{1-\text{loop}}$ and $f_{2-\text{loop}}$ are known 11D supergravity loop amplitudes. The latter comes with a log divergence, whose counter term can be absorbed into $f_{D^{12}R^4}(s,t)$. $f_{1-\text{loop},R^4}(s,t)$ is the 1-loop amplitude, whose unitarity cut involves an R^4 vertex and a tree amplitude, as already mentioned; it is given by a known loop integral with only power divergences that can be regularized in the standard way. The coefficients of the local terms $f_{D^8R^4}$, $f_{D^{10}R^4}$, $f_{D^{12}R^4}$ are not protected by supersymmetry and are unknown.

At each order in c_T^{-1}, the large s,t limit of the Mellin amplitude (at this specific order) is determined by the flat space limit, i.e. by a corresponding term in the small momentum expansion of $\mathcal{A}(s,t)$. As such, the large c_T expansion of the Mellin amplitude is expected to be of the form

$$\begin{aligned}M(s,t;\sigma,\tau) = {}& c_T^{-1}M_{\text{tree}}^{\text{SUGRA}} + c_T^{-\frac{5}{3}}M_{R^4} + c_T^{-2}M_{1-\text{loop}} + c_T^{-\frac{7}{3}}M_{D^6R^4}(s,t)\\ &+ c_T^{-\frac{23}{9}}M_{D^8R^4} + c_T^{-\frac{8}{3}}M_{1-\text{loop},R^4} + c_T^{-\frac{25}{9}}M_{D^{10}R^4} + c_T^{-3}M_{2-\text{loop}} + c_T^{-3}M_{D^{12}R^4} + \cdots\,.\end{aligned} \tag{5.2.56}$$

While M_{R^4}, for instance, is proportional to the unique solution to the superconformal Ward identity of degree 4 in s,t, the term $M_{D^6R^4}$ is a linear combination of three independent solutions to the Ward identity, of degree 7, 6, and 4 respectively. We must be careful about the interpretation of the loop terms on the RHS. $M_{1-\text{loop}}$ is determined by the tree level supergravity Mellin amplitudes[12] up to the ambiguity of a term proportional to M_{R^4}. $M_{1-\text{loop},R^4}$ and $M_{2-\text{loop}}$ are subject to similar ambiguities. Note that $c_T^{-3}M_{2-\text{loop}}$ contains a log divergence that is cut off at Planck scale, resulting in a local term proportional to $c_T^{-3}\log(c_T)$ that is of the same degree as $M_{D^{12}R^4}$.

Based on superconformal Ward identities and the flat space limit, a priori one may expect that other terms suppressed by further powers of $(\ell_{11}/L)^2$, such as terms of the form $c_T^{-\frac{17}{9}}M_{R^4}$ or $c_T^{-\frac{19}{9}}M_{R^4}$, would be allowed on the RHS of (5.2.56). As we will see later, such terms are ruled out by comparison with the known CFT data, namely the $1/c_T$ expansion of the OPE coefficient $\lambda^2_{(B,+)}$. At low derivative orders, this can be understood from the supersymmetry protected terms in the bulk effective action as follows. A term suppressed by extra powers of $(\ell_{11}/L)^2$ in comparison to those that survive the flat space limit should come from the reduction of higher-than-4-point effective coupling of the super-graviton on $AdS_4\times S^7$, e.g. terms in the effective action of the schematic form R^5, R^6, etc. As explained in [154], the R^5 type coupling is not compatible with supersymmetry, whereas an R^6 coupling should be tied to D^4R^4 by supersymmetry Ward identities, but the latter is absent in the M-theory effective action. This leaves R^7, which is tied to D^6R^4, and its reduction on $AdS_4\times S^7$ may lead to a contribution to the 4-super-graviton Mellin amplitude that is down by $(\ell_{11}/L)^6\sim c_T^{-\frac{2}{3}}$ in comparison to the R^4 contribution. This is indeed consistent with the powers of c_T^{-1} appearing in the expansion of $\lambda^2_{(B,+)}$ on the CFT side.[13]

Comparing (5.2.35), (5.2.55), and (5.2.56), we can determine, up to an overall normalization constant,

$$M(s,t;\sigma,\tau)=C\left[\widehat{M}_{\text{exchange}}+M^{\text{SUGRA}}_{\text{residual}}+B_{4,1}M^{(4,1)}_{\text{local}}+(1-\text{loop})+B_{6,1}M^{(6,1)}_{\text{local}}+B_{7,1}M^{(7,1)}_{\text{local}}+\cdots\right] \tag{5.2.57}$$

where with the normalizations $F(P,Q)=1$ and $F(P,Q)=Q$ for $M^{(4,1)}_{\text{local}}$ and $M^{(7,1)}_{\text{local}}$, respectively, we have

$$B_{4,1}\approx\frac{35}{2^7}\frac{\ell_{11}^6}{L^6}\,,\qquad B_{6,1}=o\left(\frac{\ell_{11}^{10}}{L^{10}}\right)\,,\qquad B_{7,1}\approx\frac{9009}{2^{15}}\frac{\ell_{11}^{12}}{L^{12}} \tag{5.2.58}$$

in the large radius limit. Using the relation (5.2.1) given by the AdS/CFT dictionary, we can write (5.2.58) as

$$B_{4,1}\approx\frac{70}{(6\pi c_T k)^{\frac{2}{3}}}\,,\qquad B_{6,1}=o(c_T^{-10/9})\,,\qquad B_{7,1}\approx\frac{1001}{2^7}\left(\frac{3}{2\pi c_T k}\right)^{\frac{4}{3}}\,. \tag{5.2.59}$$

[12] To determine the polar part of $M_{1-\text{loop}}$, we need not only the 4-super-graviton amplitude in AdS_4, but also the amplitudes involving 2 gravitons and 2 KK modes in AdS_4.

[13] Beyond order $c_T^{-\frac{7}{3}}$, however, it is not clear from the bulk why the contributions from, say $c_T^{-\frac{23}{9}}M_{D^8R^4}$, to $\lambda^2_{(B,+)}$ should vanish. We will return to this point in Section 5.4.

In the normalization of $\mathcal{O}_{\text{Stress}}$ in which the disconnected piece of the 4-point function is given precisely by (5.2.2), the overall coefficient C is given by (5.2.18), which is exact in $1/c_T$. This is essentially because the exchange of the stress tensor multiplet only appears in $M_{\text{tree}}^{\text{SUGRA}}$, and hence the coefficient of the latter in the Mellin amplitude is exactly proportional to c_T^{-1}. All other terms on the RHS of (5.2.57) involve exchange of multi-trace operators.

So far, using the known part of the M-theory effective action, we have determined the following terms in the large c_T expansion of the super-graviton Mellin amplitude in $AdS_4 \times S^7$: order c_T^{-1} (tree level supergravity), order $c_T^{-\frac{5}{3}}$ (degree 4 in s, t, related to R^4 coupling), and the coefficient of the maximal degree 7 polynomial in s, t at order $c_T^{-\frac{7}{3}}$ (however, we cannot fix the three other coefficients, of degree $6, 5, 4$ polynomials in $c_T^{-\frac{7}{3}} M_{D^6 R^4}$). In principle, one can fix the non-analytic part of $M_{1-\text{loop}}$ and $M_{1-\text{loop},R^4}$ in terms of the lower order Mellin amplitude (that involves super-gravitons as well as KK modes in AdS_4). We also know the order $c_T^{-3} \log c_T$ term that is fixed by the logarithmic divergence of 2-loop amplitude in 11D supergravity. Other coefficients, such as those appearing in $M_{D^8 R^4}$, are entirely unknown due to our ignorance of the higher order terms in the small momentum expansion of the M-theory S-matrix.

In the next section, we show how to relate these coefficients to CFT data, namely the OPE coefficients and scaling dimensions. Thus, if one has an independent way of computing those CFT data, one can reconstruct the corresponding part of the Mellin amplitude.

5.3 Comparison with CFT data

We will now extract CFT data from the tree-level Mellin amplitudes computed above. We will focus on the OPE coefficients squared $a_{\mathcal{M}}$ of the protected multiplets $\mathcal{M}$ in Table 1.3, as well as the scaling dimension $\Delta_{A_{(n,j)}}$ of the nth lowest twist long multiplet with spin j. Note that if $n > 0$, there can be several operators with the same twist n, in which case $\Delta_{A_{(n,j)}}$ refers to the average of these operators. The supergravity contribution to these quantities is order c_T^{-1} by definition, while the higher derivative Mellin amplitudes $M_{\text{local}}^{(p,d)}$ discussed above will contribute starting at order $c_T^{-\frac{7+2p}{9}}$, and then will generically include all subleading powers of $c_T^{-2/9}$ corresponding to powers of ℓ_{11}^2 in the flat space limit.

As discussed in [139, 140, 155], a flat space vertex with $2p$ derivatives for $p > 1$, which corresponds to an AdS_4 Mellin amplitude of maximal degree p, contributes to operators with spin $j \leq p - 4$. From the list of conformal primaries for $(A, +)_j$ and $(A, 2)_j$ in Section 4.3.1 we see that these supermultiplets contain a superconformal descendent with spin $j + 2$ that is the only operator with these quantum numbers, so these multiplets receives contribution only for $p \geq j + 6$. We will now show how to fix the $n(p) - n(p-1)$ coefficients $B_{p,d}$, indexed by d, of each degree p tree level term $M_{\text{local}}^{(p,d)}$ in (5.2.57) by extracting at least $n(p) - n(p-1)$

different pieces of CFT data from these amplitudes.

We begin by writing the position space $\mathcal{G}^{(p,d)}$ corresponding to a given $M^{(p,d)}_{\text{local}}$ as

$$\mathcal{G}^{(p,d)}(U,V;\sigma,\tau) = \sum_{\mathcal{M}_{\Delta,j}} \left[a^{(p,d)}_{\mathcal{M}} \mathfrak{G}_{\mathcal{M}}(U,V;\sigma,\tau) + a^{(0)}_{\mathcal{M}} \Delta^{(p,d)}_{\mathcal{M}} \partial_\Delta \mathfrak{G}_{\mathcal{M}}(U,V;\sigma,\tau) \right]_{\Delta^{(0)}_{\mathcal{M}}} , \tag{5.3.1}$$

where the subscript $\Delta^{(0)}_{\mathcal{M}}$ denotes that the blocks for the unprotected operators should be evaluated with the leading order scaling dimension. Note that this expression only holds for tree level amplitudes that scale as some fraction of c_T^{-1}; for loop terms there would be additional terms. The superblocks $\mathfrak{G}_{\mathcal{M}}(U,V;\sigma,\tau)$ can be further expanded into $\mathfrak{so}(8)_R$ structures $Y_{ab}(\sigma,\tau)$ and conformal blocks $G_{\Delta',j'}(U,V)$ as in (1.1.33). To compare to the Mellin space amplitude, we will furthermore take the lightcone expansion $U \ll 1$ for fixed V, so that the conformal blocks can be written as

$$G_{\Delta,j}(U,V) = \sum_{k=0}^{\infty} U^{\frac{\Delta-j}{2}+k} g^{[k]}_{\Delta,j}(V) , \tag{5.3.2}$$

where the lightcone blocks $g^{[k]}_{\Delta,j}(V)$ are labeled by the $k+1$-th lowest twist, and are only functions of V. For instance, for $k=0,1$ the lightcone blocks are

$$\begin{aligned}
g^{[0]}_{\Delta,j}(V) &= \frac{\Gamma(j+1/2)}{4^\Delta \sqrt{\pi} j!} (1-V)^j \, {}_2F_1\left(\frac{\Delta+j}{2}, \frac{\Delta+j}{2}, \Delta+j, 1-V\right) , \\
g^{[1]}_{\Delta,j}(V) &= \frac{\Gamma(j+1/2)(1-V)^{j-2}}{2(2j-1)(2\Delta-1)4^\Delta \sqrt{\pi} j!} \Bigg[2(j+\Delta)(j+\Delta-2j\Delta) \, {}_2F_1\left(\frac{\Delta+j-2}{2}, \frac{\Delta+j}{2}, \Delta+j, 1-V\right) \\
&\quad -(1+V)(\Delta^2+j^2(2\Delta-1)-2j(\Delta^2+\Delta-1)) \, {}_2F_1\left(\frac{\Delta+j}{2}, \frac{\Delta+j}{2}, \Delta+j, 1-V\right) \Bigg] .
\end{aligned} \tag{5.3.3}$$

Note that $g^{[k]}_{\Delta,j}(V)$ goes like $(1-V)^{j-2k}$ in the $V \to 1$ limit.

Putting these ingredients together, we can now expand $\mathcal{G}^{(p,d)}$ to get the final expression

$$\begin{aligned}
\mathcal{G}^{(p,d)}(U,V;\sigma,\tau) &= \sum_{a=0}^{2}\sum_{b=0}^{a} Y_{ab}(\sigma,\tau) \sum_{\mathcal{M}_{\Delta,j}} \sum_{(\Delta',j')\in\mathcal{M}} \sum_{k=0}^{\infty} U^{\frac{\Delta'-j'}{2}+k} \\
&\left[a^{(p,d)}_{\mathcal{M}} A^{\mathcal{M}}_{ab\Delta'j'}(\Delta,j) g^{[k]}_{\Delta',j'}(V) + a^{(0)}_{\mathcal{M}} \Delta^{(p,d)}_{\mathcal{M}} \left[\partial_\Delta + \frac{\log U}{2}\right] \left[A^{\mathcal{M}}_{ab\Delta'j'}(\Delta,j) g^{[k]}_{\Delta',j'}(V) \right] \right]_{\Delta^{(0)}_{\mathcal{M}}} .
\end{aligned} \tag{5.3.4}$$

The utility of the lightcone expansion is that the U-dependence corresponds to the twist $\Delta - j$ of a conformal primary, and the $\log U$ term distinguishes between the scaling dimension and the OPE coefficient of that primary. In the Mellin transform (5.2.3), one can isolate the $U^{\frac{\Delta'-j'}{2}+k}$ factor by taking the residue of the pole $s = \Delta' - j' + 2k$. The t-integral can then be performed by summing all the poles, which yields a function of V. We can then extract the coefficients of a set of lightcone block using the orthogonality relations for hypergeometric functions [139]

$$
\begin{aligned}
\delta_{r,r'} &= -\oint_{V=1} \frac{dV}{2\pi i}(1-V)^{r-r'-1}F_r(1-V)F_{1-r'}(1-V)\,,\\
F_r(x) &\equiv {}_2F_1(r,r,2r,x)\,,
\end{aligned}
\tag{5.3.5}
$$

where the integration contour is chosen to encircle only the pole $V=1$. For instance, by multiplying $\mathcal{G}^{(p,d)}(U,V;\sigma,\tau)$ with $-(1-V)^{-1-\widetilde{j}}F_{1-\frac{\Delta'+\widetilde{j}}{2}}(1-V)$ and then evaluating the residue at $V=1$, we will collect contributions from all terms in $\mathcal{G}^{(p)}(U,V;\sigma,\tau)$ that involve the lightcone blocks $g^{[k]}_{\Delta',j'}(V)$ with $j'=\widetilde{j},\widetilde{j}+2,\ldots,\widetilde{j}+2k$, as well as those involving $\partial_{\Delta'}g^{[k]}_{\Delta',j'}(V)$ with $j'<\widetilde{j}+2k-1$. Combined with our ability to select the twist $\Delta'-j'$ and R-symmetry structure $Y_{ab}(\sigma,\tau)$, as well as our knowledge of how each conformal primary contributes to the superconformal multiplet, this is enough to recursively solve for all $\Delta^{(p)}_{\mathcal{M}}$ and $a^{(p)}_{\mathcal{M}}$ for each superconformal multiplet $\mathcal{M}_{\Delta,j}$. To extract the anomalous dimension $\Delta^{(p,d)}_{(A,0)_{n,j}}$, we will also need the leading order OPE coefficient squared $a^{(0)}_{(A,0)_{(n,j)}}$, which were listed in Table 1.4.

5.3.1 Supergravity

Let's start by extracting the $p=1$ tree level CFT data, which corresponds to the supergravity term in AdS.

$a^{(1,1)}_{(B,2)}$ and $a^{(1,1)}_{(B,+)}$

We begin with the short multiplets $(B,+)$ and $(B,2)$. For $(B,+)$, we choose the conformal primary $(2,0)_{[0040]}$, which happens to be the superconformal primary. This is a convenient choice, because it is the only conformal primary in any $\mathcal{M}$ with these quantum numbers, unlike e.g. $(3,1)_{[0120]}$ which appears in $(B,2)$ and $(A,+)_0$. We now take the residue of the pole $s=2$ in (5.2.3), and find that the coefficient of UY_{22} in $\mathcal{G}(U,V;\sigma,\tau)$ is

$$
\begin{aligned}
\mathcal{G}\big|_{UY_{22}}[V] = \int \frac{dt}{2\pi i}\frac{8}{c_T}\csc\left[\frac{\pi t}{2}\right]^2 V^{\frac{t}{2}-1}\Bigg[&(1-2\gamma-2\psi(t/2)) - \sum_{m=0}^{\infty}\Bigg[\frac{32\pi^{-\frac{1}{2}}(-1)^m}{3\Gamma[-m-3/2]m!}\\
&(3+4m(2+m))^{-2}\left(3+\frac{4}{1+2m-t}+\frac{4}{-1+2m+t}-4(1+m)(\gamma+\psi(t/2))\right)\Bigg]\Bigg]\,,
\end{aligned}
\tag{5.3.6}
$$

where γ is the Euler-Mascheroni constant and ψ is the Digamma function. This expression has $t>0$ poles for $t\in 2\mathbb{Z}^+$, and $t=2m+1$ in the sum. We sum the residues from these poles, and then multiply by

$\frac{F_0(1-V)}{V-1} = \frac{1}{V-1}$ and take the residue at $V = 1$ to get

$$\begin{aligned}\oint_{V=1} \frac{dV}{2\pi i} \frac{\mathcal{G}\big|_{UY_{22}}[V]}{V-1} &= \frac{1}{c_T}\left[-\frac{48}{\pi^2} + \sum_{m=0}^{\infty} \frac{(-1)^m 512 \left(7+4m+4\psi^{(1,1)}\left(\frac{1}{2}+m\right)\right)}{3\pi^{\frac{5}{2}}(3+4m(2+m))^2\Gamma\left[-\frac{3}{2}-m\right] m!}\right] \\ &= \frac{64}{c_T}\left(\frac{1}{9}+\frac{1}{3\pi^2}\right).\end{aligned} \tag{5.3.7}$$

From the block expansion for $\mathcal{G}(U,V;\sigma,\tau)$ in (5.3.4), we see that integrating against $\frac{1}{V-1}$ and taking the coefficient of UY_{22} isolates the term $\frac{\lambda^2_{(B,+)}}{16}$, where $A^{(B,+)}_{2220}(2,0) = 1$ because we chose the superconformal primary. We thus find

$$a^{(1,1)}_{(B,+)} = \frac{1024}{c_T}\left(\frac{1}{9}+\frac{1}{3\pi^2}\right). \tag{5.3.8}$$

Performing the analogous calculation for $(B,2)$, by choosing the superconformal primary $(2,0)_{[0200]}$, which is also the only the only conformal primary in any $\mathcal{M}$ with these quantum numbers, yields

$$a^{(1,1)}_{(B,2)} = \frac{1024}{c_T}\left(-\frac{4}{9}+\frac{5}{3\pi^2}\right). \tag{5.3.9}$$

$a^{(1,1)}_{(A,+)_j}$ for $j = 0,2,4,6$ and $a^{(1,1)}_{(A,2)_j}$ for $j = 1,3,5$

For the semi-short operator $(A,+)_j$, we choose the conformal primary $(j+4,j+2)_{[0040]}$. Note that this is not the superconformal primary, but it has the advantage of being the only conformal primary in $\mathcal{M}$ with these quantum numbers for any j. If we had chosen the superconformal primary $(j+2,j)_{[0020]}$, then for $j = 2$ this primary would have appeared in both $(A,+)_0$ and $(A,+)_2$. Another advantage of $(j+4,j+2)_{[0040]}$ is that it has the same twist and irrep as the conformal primary $(2,0)_{[0040]}$ that we chose for $(B,+)$, so we can use the same expression $\mathcal{G}\big|_{UY_{22}}[V]$ that was computed in (5.3.6). We now extract $g^{[0]}_{j+4,j+2}(V)$ by integrating with $\frac{F_{-j-2}(1-V)}{(V-1)^{j+3}}$, and perform the sum in m to find

$$\oint_{V=1} \frac{dV}{2\pi i} \frac{F_{-j-2}(1-V)}{(V-1)^{j+3}} \mathcal{G}\big|_{UY_{22}}[V] = \begin{cases} -\frac{160}{27}+\frac{560}{9\pi^2} & j=0 \\ -\frac{608}{315}+\frac{2596}{135\pi^2} & j=2 \\ -\frac{656}{2079}+\frac{44278}{14175\pi^2} & j=4 \\ -\frac{2272}{57915}+\frac{82517779}{212837625\pi^2} & j=6 \end{cases}. \tag{5.3.10}$$

From the block expansion (5.3.4) we find

$$\oint_{V=1} \frac{dV}{2\pi i} \frac{F_{-j-2}(1-V)}{(V-1)^{j+3}} \mathcal{G}\big|_{UY_{22}}[V] = a^{(1,1)}_{(A,+)_j} \frac{\Gamma(j+5/2)}{4^{j+2} 3\sqrt{\pi}(j+2)!}\,, \tag{5.3.11}$$

where we used $A^{(A,+)_j}_{22\,j+4\,j+2}(j+2,j) = \frac{16}{3}$. Comparing to (5.3.10) we get

$$\begin{aligned}
a^{(1,1)}_{(A,+)_0} &= -\frac{20480}{27} + \frac{71680}{9\pi^2}\,,\\
a^{(1,1)}_{(A,+)_2} &= -\frac{19922944}{3675} + \frac{85065728}{1575\pi^2}\,,\\
a^{(1,1)}_{(A,+)_4} &= -\frac{2751463424}{160083} + \frac{185715392512}{1091475\pi^2}\,,\\
a^{(1,1)}_{(A,+)_6} &= -\frac{4879082848256}{124227675} + \frac{177205581071777792}{456536705625\pi^2}\,.
\end{aligned} \tag{5.3.12}$$

The calculation for $(A,2)_j$ is more subtle, because there is no longer a twist 2 conformal primary that only appears in $(A,2)_j$. We choose the conformal primary $(j+4,j+2)_{[0120]}$, because it overlaps with fewer multiplets than other choices. Performing the usual first few steps, we find

$$\oint_{V=1} \frac{dV}{2\pi i} \frac{F_{-j-2}(1-V)}{(V-1)^{j+3}} \mathcal{G}\big|_{UY_{21}}[V] = \begin{cases} \frac{208}{15} - \frac{6104}{45\pi^2} & j=1 \\ \frac{496}{189} - \frac{366278}{14175\pi^2} & j=3 \\ \frac{152}{429} - \frac{5507939}{1576575\pi^2} & j=5 \end{cases}\,. \tag{5.3.13}$$

From the block expansion (5.3.4) and the tables in Section 4.3.1 we find

$$\begin{aligned}
&\oint_{V=1} \frac{dV}{2\pi i} \frac{F_{-j-2}(1-V)}{(V-1)^{j+3}} \mathcal{G}\big|_{UY_{21}}[V] = \\
&-\frac{\Gamma(j+5/2)}{4^{j+4}\sqrt{\pi}(j+2)!}\left[a^{(1,1)}_{(A,2)_j}\frac{32(2+j)^2}{(3+2j)(5+2j)} - a^{(1,1)}_{(A,+)_{j-1}}\frac{64(3+j)^4}{(35+24j+4j^2)^2} - 4a^{(1,1)}_{(A,+)_{j+1}}\right]\,,
\end{aligned} \tag{5.3.14}$$

where now we must already know $a^{(1,1)}_{(A,+)_{j\pm1}}$ to determine $a^{(1,1)}_{(A,2)_j}$. Using the formulae for the former in (5.3.12) and comparing to (5.3.13), we find

$$\begin{aligned}
a^{(1,1)}_{(A,2)_1} &= -\frac{262144}{105} + \frac{212992}{9\pi^2}\,,\\
a^{(1,1)}_{(A,2)_3} &= -\frac{16777216}{1617} + \frac{1117782016}{11025\pi^2}\,,\\
a^{(1,1)}_{(A,2)_5} &= -\frac{17179869184}{637065} + \frac{47746882994176}{180093375\pi^2}\,.
\end{aligned} \tag{5.3.15}$$

$\Delta^{(1,1)}_{(A,0)_{j,0}}$ **for** $j = 0, 2, \ldots, 12$ **and** $\Delta^{(1,1)}_{(A,0)_{j,1}}$ **for** $j = 0, 2, \ldots, 10$

We will now demonstrate how to compute the sub-leading scaling dimension for the unprotected operator $(A,0)_{j,n}$ with twist $2n+2$ and spin j. At order $1/c_T$, there are $n+1$ distinct operators of this form, which can be written as double traces of $\frac{1}{2}$-BPS operators:

$$[\mathcal{O}_p\mathcal{O}_p]_{j,m} = \mathcal{O}_p \Box^q \partial_{\mu_1}\ldots\partial_{\mu_j}\mathcal{O}_p + \ldots\,, \qquad \text{for } q = p/2-1\,, p/2+1\ldots\,, n\,, \tag{5.3.16}$$

where $\mathcal{O}_p$ for $p = 2, 4, \ldots$ are $\frac{1}{2}$-BPS $(B,+)$ operators in $\mathfrak{so}(8)_R$ irrep $[00p0]$ with $\Delta = p/2$, as shown in Table 1.1.[14] For instance, $\mathcal{O}_2 \equiv \mathcal{O}_{\text{Stress}}$ and $\mathcal{O}_4 \equiv \mathcal{O}_{(B,+)}$ in our shorthand notation. In the strict $c_T \to \infty$ limit, all such operators with the same n were indistinguishable and so we could refer to them all by the $p = 2$ operator, with scaling dimension (1.1.47). At order $1/c_T$, however, we expect each operators with different q to have different scaling dimensions and OPE coefficients, just like in the maximally supersymmetric $\text{AdS}_5/\text{CFT}_4$ case [141, 156]. For $n > 0$, our results are thus weighted averages of all $n+1$ operators of this form.

Let us begin with the lowest operator $n = 0$ for a given spin j, which has $\Delta^{(0)}_{(A,0)_{j,0}} = j+2$, i.e twist 2. Since only $(A,0)_{j,n}$ operators have anomalous dimensions, when choosing a conformal primary we need only check how many times it appears in $(A,0)_{j,n}$. From Section 4.3.1, we see that for $\Delta = j+2$ the only unique conformal primary is $(j+4, j+2)_{[0020]}$. We now perform the usual steps of projecting to Y_{11}, taking the $s = 2$ pole, performing the sum over poles in t, extracting $g^{[0]}_{j+4,j+2}(V)$, and then performing the sum over m, except we now choose the $U\log U$ coefficient because that is what multiples $\Delta^{(1,1)}_{(A,0)_{j,0}}$ in (5.3.4). We find

$$\oint_{V=1}\frac{dV}{2\pi i}\frac{F_{-j-2}(1-V)}{(V-1)^{j+3}}\mathcal{G}\big|_{U\log U\, Y_{11}}[V] = \begin{cases} -\frac{64}{15\pi^2} & j=0 \\ -\frac{16}{105\pi^2} & j=2 \\ -\frac{32}{3003\pi^2} & j=4 \\ -\frac{76}{109395\pi^2} & j=6 \\ -\frac{32}{734825\pi^2} & j=8 \\ -\frac{8}{2982525\pi^2} & j=10 \\ -\frac{1168}{7125711075\pi^2} & j=12 \end{cases}\,. \tag{5.3.17}$$

[14] We can also construct double traces of $\mathcal{O}_p$ for odd p, but these do not show up in the stress tensor four-point function.

From the block expansion (5.3.4) and the tables in Section 4.3.1 we find

$$\begin{aligned}&\oint_{V=1}\frac{dV}{2\pi i}\frac{F_{-j-2}(1-V)}{(V-1)^{j+3}}\mathcal{G}\big|_{U\log U\,Y_{11}}[V]=\\&\Delta^{(1,1)}_{j,0,0}\left(\frac{\Gamma(j+5/2)}{4^{j+4}\sqrt{\pi}(j+2)!}\right)\left(\frac{a^{(0)}_{(A,0)_{j,0,0}}}{2}\right)\left(\frac{128(1+j)^2(2+j)^2}{(1+2j)(3+2j)^2(5+2j)}\right),\end{aligned}\tag{5.3.18}$$

where $a^{(0)}_{(A,0)_{j,0}}$ are listed for $j=0,2,\ldots,12$ in Table 1.4. Comparing this to (5.3.17) we get

$$\begin{aligned}&\Delta^{(1,1)}_{(A,0)_{0,0}}=-\frac{1120}{\pi^2}\,,\quad\Delta^{(1,1)}_{(A,0)_{2,0}}=-\frac{2464}{5\pi^2}\,,\quad\Delta^{(1,1)}_{(A,0)_{4,0}}=-\frac{2288}{7\pi^2}\,,\quad\Delta^{(1,1)}_{(A,0)_{6,0}}=-\frac{5168}{21\pi^2}\,,\\&\Delta^{(1,1)}_{(A,0)_{8,0}}=-\frac{97888}{495\pi^2}\,,\quad\Delta^{(1,1)}_{(A,0)_{10,0}}=-\frac{165600}{1001\pi^2}\,,\quad\Delta^{(1,1)}_{(A,0)_{12,0}}=-\frac{64728}{455\pi^2}\,,\end{aligned}\tag{5.3.19}$$

where $\Delta^{(1,1)}_{0,0}$ was already obtained by [130] using the superconformal primary $(2,0)_{[0000]}$.

We now move on to the second lowest twist operators $(A,0)_{j,1}$, which has $\Delta^{(0)}_{(A,0,m)_{j,1}}=j+4$, i.e. twist 4. While there is no twist 4 conformal primary that only appears in $(A,0)_{j,1}$, we choose $(j+6,j+2)_{[0120]}$, because it overlaps with fewer multiplets than other choices. Performing the same first few steps as with $(A,0)_{j,0}$, except now choosing the $U^2\log U$ coefficient and integrating against $\frac{F_{-j-3}(1-V)}{(V-1)^{j+3}}$, we find

$$\oint_{V=1}\frac{dV}{2\pi i}\frac{F_{-j-3}(1-V)}{(V-1)^{j+3}}\mathcal{G}\big|_{U^2\log U\,Y_{11}}[V]=\begin{cases}-\frac{128}{75\pi^2} & j=0\\ -\frac{256}{3003\pi^2} & j=2\\ -\frac{3904}{853281\pi^2} & j=4\\ -\frac{20992}{82447365\pi^2} & j=6\\ -\frac{15424}{1064761425\pi^2} & j=8\\ -\frac{63872}{76346904375\pi^2} & j=10\end{cases}\,.\tag{5.3.20}$$

In the block expansion (5.3.4) we expect to receive contributions from other twist 4 blocks $g^{[0]}_{j+4,j}(V)$, as well as the $k=1$ correction to twist 2 blocks $g^{[1]}_{j'+2,j'}(V)$ for $j'=j,j+2$. Using the explicit formula for these blocks in (5.3.3), as well as the tables in Section 4.3.1, we get

$$\begin{aligned}&\oint_{V=1}\frac{dV}{2\pi i}\frac{F_{-j-3}(1-V)}{(V-1)^{j+3}}\mathcal{G}\big|_{U^2\log U\,Y_{11}}[V]=\Delta^{(1,1)}_{(A,0)_{j,1,\bar{q}}}a^{(0)}_{(A,0)_{j,1}}\frac{(2+j)(3+j)\Gamma\left(j+\frac{1}{2}\right)}{4^{j+4}(2j+5)(2j+7)\sqrt{\pi}j!}\\&+\Delta^{(1,1)}_{(A,0)_{j,0,0}}a^{(0)}_{(A,0)_{j,0}}\frac{2(j+4)!(3j^2+25j+46)\Gamma\left(j+\frac{5}{2}\right)}{4^{j+2}(6j+3)(2j+3)^2(2j+5)^2(2j+7)(2j+11)\sqrt{\pi}j!^2}\\&-\Delta^{(1,1)}_{(A,0)_{j+2,0,0}}a^{(0)}_{(A,0)_{j+2,0}}\frac{2(j+3)^2(3j^2+17j+18)\Gamma\left(j+\frac{9}{2}\right)}{4^{j+3}(6j+9)(2j+5)(2j+7)^2(2j+9)\sqrt{\pi}(j+2)!}\,,\end{aligned}\tag{5.3.21}$$

where we must already know $\Delta^{(1,1)}_{(A,0)_{j,0}}$ and $\Delta^{(1,1)}_{(A,0)_{j+2,0}}$ to determine $\Delta^{(1,1)}_{(A,0)_{j,1}}$. Using the formulae for the former in (5.3.19) and comparing to (5.3.20), we find

$$\begin{aligned}
&\Delta^{(1,1)}_{(A,0)_{0,0}} = -\frac{3584}{\pi^2}\,, \qquad \Delta^{(1,1)}_{(A,0)_{2,0}} = -\frac{59488}{35\pi^2}\,, \qquad \Delta^{(1,1)}_{(A,0)_{4,0}} = -\frac{367744}{315\pi^2}\,,\\
&\Delta^{(1,1)}_{(A,0)_{6,0}} = -\frac{444448}{495\pi^2}\,, \qquad \Delta^{(1,1)}_{(A,0)_{8,0}} = -\frac{942080}{1287\pi^2}\,, \qquad \Delta^{(1,1)}_{(A,0)_{10,0}} = -\frac{619440}{1001\pi^2}\,.
\end{aligned} \tag{5.3.22}$$

5.3.2 Comparison to exact results and numerical bootstrap

We now compare these tree level AdS_4 supergravity results to CFT_3 results. Recall the $1/c_T$ expansion of the short operator OPE coefficients $\lambda^2_{(B,2)}$ and $\lambda^2_{(B,+)}$ given in (2.4.32), which was computed from the 1d sector using the explicit Lagrangian of the ABJM theory. Note that the $1/c_T$ term exactly matches the the supergravity results (5.3.8) and (5.3.9).

There are no exact results for the other operators in $\mathcal{O}_{\text{Stress}} \times \mathcal{O}_{\text{Stress}}$, but the conformal bootstrap was used to estimate their correction at large c_T in Section 4.5.2. In Table 5.2, we compare the numerical CFT_3 predictions to the analytic AdS_4 results computed here. For the semi-short operators $(A,2)_j$ and $(A,+)_j$ and the lowest unprotected operator $(A,0)_{j,0,0}$, we find precise agreement for every value of j. In Figure 5.1 we compare the numerical plots of the semi-short OPE coefficients $\lambda^2_{(A,2)_j}$ and $\lambda^2_{(A,+)_j}$ from Section 4.5.2 to the exact $1/c_T$ correction (5.3.15) and (5.3.12). The $\lambda^2_{(A,+)_j}$ plots appears to be linear in $1/c_T$, while the $\lambda^2_{(A,2)_j}$ plots depart from linearity for large $1/c_T$. The plots for the other CFT data in Section 4.5.2 are not nearly linear, so we do not reproduce them here.

For the second to lowest $(A,0)_{j,1}$, we have only been able to compute the average quantity $\Delta^{(1,1)}_{(A,0)_{j,1}}$ of the two such operator given in (5.3.16). The numerical bootstrap was used to compute the anomalous dimension of the lower of these two operators, and so a direct comparison is not possible with this information. Nevertheless, by analogy to the explicit answer for all distinct operators with n,j in the $\text{AdS}_5/\text{CFT}_4$ case [141, 156], we expect that degeneracy is suppressed at large j. This expectation is confirmed in Table 5.2, where we find that $\Delta^{(1,1)}_{(A,0)_{j,1}}$ and the bootstrap result are very different for small j, but become quite similar for larger j, e.g. $j = 10$.

5.3.3 Matching the R^4 term

We now extract the CFT data that receives corrections from the degree 4 polynomial Mellin amplitude $M^{(4,1)}_{\text{local}}$ that corresponds to the R^4 term. From the discussion above, the multiplets that receive corrections at this order are $(B,+)$, $(B,2)$, and $(A,0)_{n,0}$. Since $\lambda^2_{(B,+)}$ and $\lambda^2_{(B,2)}$ are related by (2.1.2), we will only discuss the former.

CFT data	ABJ(M) numerical bootstrap	AdS$_4$ Supergravity
$a^{(1,1)}_{(A,2)_1}$	-97	-98.765
$a^{(1,1)}_{(A,2)_3}$	-102	-102.045
$a^{(1,1)}_{(A,2)_5}$	-104	-103.470
$a^{(1,1)}_{(A,+)_0}$	49	48.448
$a^{(1,1)}_{(A,+)_2}$	51	51.147
$a^{(1,1)}_{(A,+)_4}$	52	52.155
$\Delta^{(1,1)}_{(A,0)_{0,0}}$	-109	-113.480
$\Delta^{(1,1)}_{(A,0)_{2,0}}$	-49	-49.931
$\Delta^{(1,1)}_{(A,0)_{4,0}}$	-33	-33.118
$\Delta^{(1,1)}_{(A,0)_{6,0}}$	-25	-24.935
$\Delta^{(1,1)}_{(A,0)_{8,0}}$	-20	-20.037
$\Delta^{(1,1)}_{(A,0)_{10,0}}$	-17	-16.762
$\Delta^{(1,1)}_{(A,0)_{0,1}}$	-261	$\overline{-363.135}$
$\Delta^{(1,1)}_{(A,0)_{2,1}}$	-145	$\overline{-172.211}$
$\Delta^{(1,1)}_{(A,0)_{4,1}}$	-111	$\overline{-118.287}$
$\Delta^{(1,1)}_{(A,0)_{6,1}}$	-88	$\overline{-90.974}$
$\Delta^{(1,1)}_{(A,0)_{8,1}}$	-70	$\overline{-74.167}$
$\Delta^{(1,1)}_{(A,0)_{10,1}}$	-60	$\overline{-62.700}$

Table 5.2: The $1/c_T$ correction to the scaling dimensions $\Delta^{(1,1)}_{(A,0)_{j,n}}$ for unprotected operators with spin j and twist $2n+2$, as well as the OPE coefficients squared $a^{(1,1)}_{(A,+)_j}$ and $a^{(1,1)}_{(A,2)_j}$ for the semi-short operators of spin j, computed from the numerical conformal bootstrap for ABJ(M) in Section 4.5.2 and the supergravity calculation in this Chapter. Exact formulae for supergravity are given in (5.3.12), (5.3.15), (5.3.19), and (5.3.22). For $n > 0$, the exact results refer to averages over $n+1$ distinct operators with the same quantum numbers, as denoted by the overline, while the bootstrap results refers to the lowest of these mixed operators.

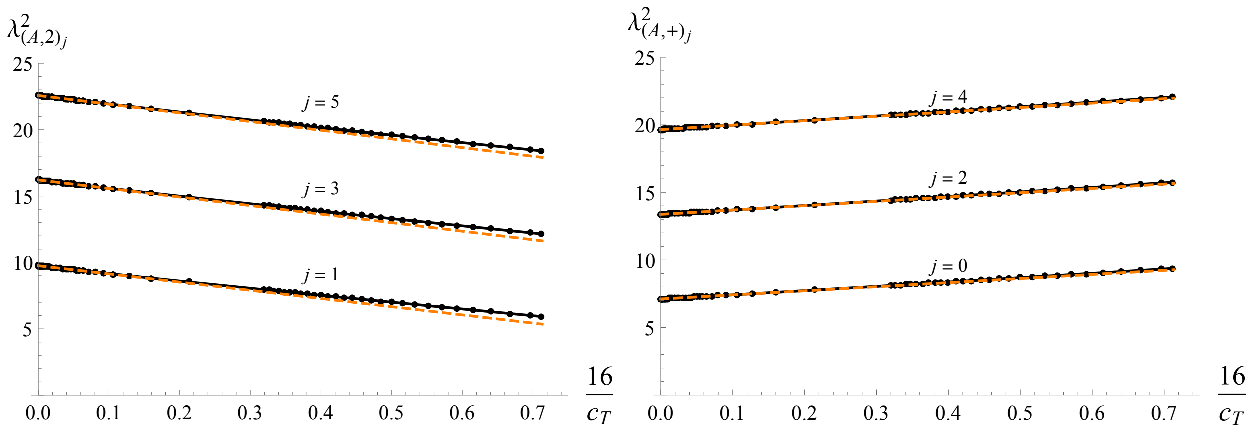

Figure 5.1: The $\lambda^2_{(A,2)_j}$ and $\lambda^2_{(A,+)_j}$ OPE coefficients with spins $j = 1, 3, 5$ and $j = 0, 2, 4$, respectively, in terms of the stress-tensor coefficient c_T, where the plot ranges from the generalized free field theory limit $c_T \to \infty$ to the numerical point $\frac{16}{c_T} \approx .71$ where $\lambda^2_{(B,2)} = 0$, which is near the lowest interacting theory ABJ$_1$ with $c_T = .75$. The orange dotted lines show the analytic $1/c_T$ corrections (5.3.15) and (5.3.12).

For $a^{(4,1)}_{(B,+)}$, we take the $s = 2$ pole in the Mellin transform (5.2.3) of $M^{(4,1)}_{\text{local}}$ given in (5.2.25) and find that the UY_{22} coefficient is

$$\begin{aligned} \mathcal{G}^{(4)}\big|_{UY_{22}}[V] &= -\frac{8\pi^2}{35}\int \frac{dt}{2\pi i} V^{t/2-1}\csc(\pi t/2)^2 \\ &= -\frac{16}{35}\frac{\log V}{1-V}\,, \end{aligned} \tag{5.3.23}$$

where we closed the contour to include all positive poles in t. From the expansion (5.3.4), we then extract the coefficient of $g^{[0]}_{2,0}(V)$ by integrating against $\frac{16F_0(1-V)}{V-1} = \frac{16}{V-1}$ to find

$$a^{(4,1)}_{(B,+)} = CB_{4,1}\oint_{V=1}\frac{dV}{2\pi i}\frac{16\mathcal{G}^{(5/3)}\big|_{UY_{22}}}{V-1} = \frac{256}{35}CB_{4,1}\,, \tag{5.3.24}$$

where we used $A^{(B,+)}_{2220}(2,0) = 1$ for the superconformal primary. We now compare to the localization result (2.4.32), and using the SUGRA normalization (5.2.18) we find that the leading $c_T^{-5/3}$ term in $B_{4,1}$ precisely agrees with the result (5.2.59) obtained from the R^4 effective coupling in 11D.

We can similarly extract the anomalous dimension at order $c_T^{-\frac{5}{3}}$ for the lowest $j = 0$ $(A,0)$ long multiplet by taking the $s = 2$ pole in the $SO(8)_R$ singlet channel and using the leading order OPE coefficient $a^{(0)}_{(A,0)_{0,0}} = \frac{32}{35}$ from Table 1.4. We obtain

$$\Delta^{(4,1)}_{(A,0)_{0,0}} = -192CB_{4,1} = -71680\left(\frac{6}{\pi^8 k^2}\right)^{1/3} c_T^{-5/3} + O(c_T^{-17/9})\,, \tag{5.3.25}$$

where we inputted the value of $CB_{4,1}$ determined above.

5.3.4 Higher derivative corrections

We now show how to extract CFT data from higher degree Mellin amplitudes $M^{(p,d)}_{\text{local}}$ in terms of their coefficents $B_{p,d}$ for $p = 6, 7, 8, 9, 10$, where $d = 1$ except for $p = 10$ where $d = 1, 2$. For $p < 10$ the leading order in $1/c_T$ contributions can be unambiguously extracted from these terms, as they do not mix with loop contributions. For $p = 10$, the c_T^{-3} contribution is affected by the as yet unknown 2-loop term, but there is a $c_T^{-3} \log c_T$ that one could unambiguously extract. For all higher terms, the tree level contribution is indistinguishable from the 2-loop and higher contributions.

Since $\lambda^2_{(B,+)}$ has already been used to fix $B_{4,1}$ in (5.3.24), and $\lambda^2_{(B,2)}$ is related to $\lambda^2_{(B,+)}$ by crossing symmetry, we will use the semi-short $\lambda^2_{(A,2)_j}$ and $\lambda^2_{(A,+)_j}$ as well as the lowest twist unprotected $\Delta_{(A,0)_{0,j}}$ for the allowed spin. These calculations will closely follow the SUGRA calculations in the previous section, except that we use $M^{(p,d)}_{\text{local}}$ in Section 5.2.2. As such we will only briefly sketch the calculations.

For $(A,+)_j$, we extract its OPE coefficient using the superconformal descendent $(j+4, j+2)_{[0040]}$, which has the advantage of being the only conformal primary in $\mathcal{M}$ with these quantum numbers for any j. If we had chosen the superconformal primary $(j+2, j)_{[0020]}$, then for $j = 2$ this primary would have appeared in both $(A,+)_0$ and $(A,+)_2$. Using the explicit coefficients in Section 4.3.1 and the formula for $M^{(p,d)}_{\text{local}}$ in Section 5.2.2, we can compute $a^{(p,d)}_{(A,+)_j}$ in terms of $CB_{p,d}$, which we list in Table 5.3.

The calculation for $(A,2)_j$ is more subtle, because there is no longer a twist 2 conformal primary that only appears in $(A,2)_j$. We choose the conformal primary $(j+4, j+2)_{[0120]}$, which overlaps with superconformal descendents of $(A,+)_{j\pm1}$. Since we have already computed $a^{(p,d)}_{(A,+)_j}$, we can remove them to find the answers for $a^{(p,d)}_{(A,2)_j}$ as given in Table 5.3.

For $(A,0)_{0,j}$, since we are considering its anomalous dimension, we only need to worry about mixing with other superconformal descendents of $(A,0)_{0,j'}$ for some other j'. If we choose the superconformal primary $(j+2, j)_{[0000]}$, then from Section 4.3.1 we see that a superconformal descendent of $(A,0)_{0,j}$ mixes with $(A,0)_{0,j+4}$. We can take into account this mixing by computing each j starting from $j = 0$, which yields the answers in Table 5.3.

Note that all the OPE coefficients and anomalous dimensions in Table 5.3 receive contributions from non-local terms in the Mellin amplitude, such as the tree level amplitude at order c_T^{-1}, the 1-loop amplitude at order c_T^{-2}, etc.

CFT data:	$M^{(4,1)}_{\text{local}}$	$M^{(6,1)}_{\text{local}}$	$M^{(7,1)}_{\text{local}}$	$M^{(8,1)}_{\text{local}}$	$M^{(9,1)}_{\text{local}}$	$M^{(10,1)}_{\text{local}}$	$M^{(10,2)}_{\text{local}}$
$a_{(A,+)_0}$	0	$\frac{16384}{1485}$	$\frac{950272}{6435}$	$-\frac{131396796416}{467137125}$	$-\frac{422304284672}{5835588759}$	$\frac{1363121203815907328}{82825301109705}$	$\frac{577700480155648}{35496557618445}$
$a_{(A,+)_2}$	0	0	0	$\frac{67108864}{557375}$	$\frac{499222839296}{7833219625}$	$-\frac{4522949604055470571 52}{33820331286462875}$	$\frac{10719544062509056}{4831475898066125}$
$a_{(A,+)_4}$	0	0	0	0	0	$\frac{687194767360}{273854581}$	0
$a_{(A,2)_1}$	0	$-\frac{533430272}{28875}$	$-\frac{7141523456}{28875}$	$\frac{584011250925568}{2096128125}$	$\frac{2720900729798656}{130926670875}$	$-\frac{27580985666129789924 1472}{43483283082595125}$	$-\frac{1917936427628851363 84}{6211897583227875}$
$a_{(A,2)_3}$	0	0	0	$-\frac{2813168548052992}{695269575}$	$-\frac{41882310723386736 64}{1954231632045}$	$\frac{402366911473941378853985320 96}{109687451241507854715}$	$-\frac{18018624516515449274368}{241071321409907373}$
$a_{(A,2)_5}$	0	0	0	0	0	$-\frac{10329687302395721154 56}{654551570673}$	0
$\Delta_{(A,0)_{0,0}}$	-192	$\frac{15360}{11}$	$\frac{192000}{11}$	$-\frac{18059264}{1521}$	$\frac{12428820480}{2375087}$	$-\frac{11167368386150400}{46400728913}$	$-\frac{31945339699200}{5930920237}$
$\Delta_{(A,0)_{0,2}}$	0	-1536	-18432	$\frac{509591552}{12675}$	$-\frac{78840397824}{2375087}$	$-\frac{21440053347492298752}{10254561089773}$	$\frac{3851013479989248}{112687484503}$
$\Delta_{(A,0)_{0,4}}$	0	0	0	-32768	$\frac{5987205120}{182699}$	$-\frac{628968771433267200}{788812391521}$	$-\frac{3801661424271360}{112687484503}$
$\Delta_{(A,0)_{0,6}}$	0	0	0	0	0	$-\frac{176947200}{143}$	0

Table 5.3: Contributions to the OPE coefficients squared a and anomalous dimensions Δ of various multiplets appearing in $\mathcal{O}_{\text{Stress}} \times \mathcal{O}_{\text{Stress}}$ from *local* terms in the Mellin amplitude. Each polynomial $M^{p,k}_{\text{local}}$ in the Mellin amplitude contributes to a given quantity in the left column an amount equal to the number in the corresponding entry of the table.

5.4 Discussion

In this Chapter, we compared the large c_T expansion of the stress tensor 4-point function in ABJM theory, to the dual M-theory amplitude on AdS_4 in Mellin space. Our first result was a successful match between all the CFT data at order $1/c_T$, i.e. supergravity, which on the CFT side was computed from the numerical bootstrap, and on the AdS side was computed using the Mellin space formalism and the assumption that the Lagrangian had a two derivative Einstein-Hilbert kinetic term (which is why we call it a gravity calculation). This match is the first example of AdS/CFT for unprotected operators.

Our second result was a strategy to recover the M-theory effective action, i.e. the small momentum expansion of the flat spacetime S-matrix, from the CFT data of ABJM theory using the large c_T expansion of the flat space limit of the AdS_4 Mellin amplitude. We determined certain low order terms in the latter expansion using the OPE coefficient of the $(B, +)$ multiplet, previously computed exactly as a function of c_T via the supersymmetric localization method. The known CFT data are enough for us to recover the correct R^4 effective coupling of M-theory, but not enough for a nontrivial check against the next two known coefficients of the M-theory effective action allowed by supersymmetry, namely D^4R^4 (whose coefficient is zero) and D^6R^4. It is plausible that there may be other protected OPE coefficients, say of semi-short multiplets, in the $\mathcal{O}_{\text{Stress}} \times \mathcal{O}_{\text{Stress}}$ OPE that could be determined using CFT methods, and tested against the absence of the D^4R^4 term and the coefficient of the D^6R^4 term in M-theory.

More importantly, our hope is that bootstrap bounds on unprotected OPE coefficients or anomalous dimensions at large c_T could be used to bound the coefficients of higher order terms in the M-theory effective action, such as D^8R^4, $D^{10}R^4$, etc. It has been suggested [27], based on naive power counting arguments, that the independent local terms in the M-theory effective action only arise at momentum order $D^{6k}R^4$ for non-negative integer k. It is not clear to us why this should be the case beyond D^6R^4, where supersymmetry no longer constrains the moduli dependence of the higher derivative couplings upon toroidal compactifications of M-theory [154]. Nonetheless, we saw that a certain cancelation in the contribution from local terms in the Mellin amplitude of the form $c_T^{-\frac{23}{9}} M_{D^8R^4}$ and $c_T^{-\frac{25}{9}} M_{D^{10}R^4}$ to the $(B, +)$ OPE coefficient is required, and we do not have an explanation of this from the bulk perspective. This does not imply the absence of D^8R^4 or $D^{10}R^4$ terms in M-theory, however, since the local Mellin amplitudes $M_{D^8R^4}$ and $M_{D^{10}R^4}$ are not entirely fixed by their flat space limits. An intriguing possibility is that perhaps such terms are absent in the Mellin amplitude altogether (which would imply the absence of D^8R^4 and $D^{10}R^4$ in the flat space limit). It would be extremely interesting to understand if this is the case.

In Section 4.5, it was noticed that the $(B, 2)$ (or $(B, +)$) OPE coefficients of ABJ(M) theory, as computed using supersymmetric localization, come close to saturating the numerical bootstrap bounds on these quantities obtained for general $\mathcal{N} = 8$ SCFTs. Such a bound saturation would imply that one may extract numerically all the CFT data encoded in the $\langle \mathcal{O}_{\text{Stress}}\mathcal{O}_{\text{Stress}}\mathcal{O}_{\text{Stress}}\mathcal{O}_{\text{Stress}} \rangle$ 4-point function,[15] thus allowing us in principle to recover the entire M-theory super-graviton S-matrix using the procedure outlined in this Chapter. However, as was pointed out in Section 4.5, the values of the $(B, 2)$ OPE coefficients as a function of $1/c_T$ start to depend on k at order $1/c_T^{5/3}$, with the value for the $k = 2$ ABJ and ABJM theories being closer to the numerical bound.[16] So it is possible that one of these $k = 2$ theories could in fact saturate the bootstrap bound for all values of c_T, and the strategy of determining the CFT data numerically and feeding it into the procedure described in this Chapter could work. As far as the $k = 1$ ABJM theory is concerned, while (at least at large c_T) this theory certainly does not saturate the bootstrap bound discussed in Section 4.5.2, it is possible that an improved bootstrap analysis could generate different stronger bounds that apply only to the $k = 1$ theory. For instance, a mixed correlator study of the lowest dimension scalars in the $(B, +)$ [0020] and [0030] multiplets would single out the $k = 1$ theory because the $(B, +)$ [0030] multiplet does not exist in the $k = 2$ theories. It would be very interesting to investigate these issues in the future.

So far we have focused entirely on 4-particle S-matrix elements. Our strategy based on the flat space limit of ABJM correlators allows us, in principle, to recover the M-theory S-matrix elements of n supergravitons, provided that their momenta are aligned within a 4D sub-spacetime of the 11D Minkowskian spacetime. This determines all n-point S-matrix elements for $n \leq 5$, but not for $n \geq 6$. To recover the $(n \geq 6)$-point S-matrix elements for general 11D momenta from the Mellin amplitudes of ABJM theory would be much more difficult, as it would require taking a flat space limit of the Mellin amplitudes for operators of large $\mathfrak{so}(8)_R$ quantum numbers.

It would be interesting to generalize the construction in this Chapter to theories with lower amounts of supersymmetry. In particular, it should be possible to extend the arguments of this Chapter to the full family of ABJM theories, which have only $\mathcal{N} = 6$ supersymmetry for CS level $k > 2$. The supersymmetric localization calculations extend to this case too, and one can perform both an expansion in large N at fixed k, as we did in this Chapter, or at large N and fixed $\lambda = N/k$ [46]. The latter expansion would allow us to probe scattering amplitudes in type IIA string theory directly.

[15] If the numerical bounds are only close to being saturated, then we cannot reconstruct the $\langle \mathcal{O}_{\text{Stress}}\mathcal{O}_{\text{Stress}}\mathcal{O}_{\text{Stress}}\mathcal{O}_{\text{Stress}} \rangle$ 4-point function, but we can still obtain stringent bounds on the CFT data.

[16] As already mentioned, the OPE coefficients of $k = 2$ ABJM and $k = 2$ ABJ theories have identical perturbative expansions in $1/c_T$.

Chapter 6

Conclusion

This thesis used the conformal bootstrap and other tools to study ABJM theory, which could then be used to study M-theory via the AdS/CFT correspondence. The results can be divided into numerical results derived from the numerical conformal bootstrap algorithm and analytic results derived using the superconformal algebra and supersymmetric localization.

On the analytical side, the main result was a derivation of the 1d protected topological subsector that exists for all 3d $\mathcal{N} \geq 4$ SCFTs. This 1d sector along with the mass deformed S^3 partition function computed from localization was then used to compute the OPE coefficients of half and quarter BPS operators that appear in the stress tensor four-point function for the ABJ(M)$_N$ and BLG$_k$ theories. For the BLG$_k$ theories, we were able to compute these quantities exactly for all k, while for ABJ(M)$_N$ we computed them exactly for small N, and to all orders in a large N expansion. We used a match between some of these small N and k results to motivate a conjectured duality between BLG$_3$ and the interacting sector of ABJM$_{3,1}$, which we also motivated by matching the moduli spaces, S^3 partition function, and superconformal indices. We then used the large N results for these protected OPE coefficients for ABJ(M)$_N$ as well as the superconformal Ward identities to derive the stress tensor four-point function to the first order beyond tree level in this theory in the large N expansion, whose flat space limit could be matched to the R^4 correction to the M-theory S-matrix. This precise match confirmed a general strategy we laid out to recover the entire M-theory S-matrix from CFT data in a large N expansion.

On the numerical side, we computed the superconformal blocks for the stress tensor four point function for 3d $\mathcal{N} = 8$ SCFTs, and used them to perform a numerical bootstrap study of these SCFTs. With no additional assumptions, we derived bounds on all low-lying scaling dimensions and OPE coefficients as a function of $c_T \sim N^{3/2}$, and found that they correctly interpolated between the free theory at $c_T = 16$ and the supergravity limit at $c_T \to \infty$. For operators in the protected 1d sector, we found that the previously derived analytic results for ABJ(M)$_N$ came close to saturating the numerical bounds for all c_T, which allowed us to conjecturally read off all the low-lying CFT data for whichever ABJ(M)$_N$ theory saturates the bounds in the infinite numerical precision limit.

Looking ahead, it would be nice to improve the numerics performed in this thesis, so that one could see which ABJ(M)$_N$ theory, if any, truly saturates the bounds, and then be able to precisely read off the low-lying CFT data in this theory, and extract the large N expansion coefficients of this data. This could then be used to derive higher terms in the M-theory S-Matrix using the Mellin space methods we discuss. One strategy to improve the numerics is to look at mixed correlators between the stress tensor and the next lowest half-BPS multiplet, so that one could automatically eliminate all known 3d $\mathcal{N} = 8$ SCFTs except for ABJM$_{N,1}$. This could even collapse the allowed region into a line, which would allow us to read off CFT data without needing to assume any saturation of bounds.

All of the results of this thesis could also be extended to $\mathcal{N} = 6$ ABJM theory, which can be used to study both M-theory and Type IIa string theory in various regimes N and k. It would also be nice to extend the strategy of extracting S-matrices from CFT four point functions in other contexts with less supersymmetry or in other dimensions, such as 4d $\mathcal{N} = 4$ SYM dual to Type IIb string theory or certain 3d $\mathcal{N} = 4$ SCFTs that are dual to Type IIa string theory. To complete the Mellin space program outlined in this thesis, we will also need to compute loop Mellin amplitudes, which is an open problem in any dimension with any amount of supersymmetry currently.

More ambitiously, the numerical results for 3d $\mathcal{N} = 8$ SCFTs in principle contain non-perturbative information about ABJM theory, and thus non-perturbative information about M-theory. For instance, operators with high twist should be related to black holes on the gravity side. It would be interesting to make this statement precise, and to extend the numerics to the high twist regime so as to probe this new physics.

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