Programming Research Group

CLASSICAL AND QUANTUM STRUCTURES

Bob Coecke, Éric O. Paquette and Dusko Pavlovic

PRG-RR-08-02

Oxford University Computing Laboratory
Wolfson Building, Parks Road, Oxford OX1 3QD

Abstract

Dagger-compact categories have been proposed as a categorical framework suitable for quantum reasoning [1, LiCS'04]. *Modelling classical operations in this framework seemed to require additional completeness assumptions, most notably biproducts. In the present paper, we show that classical operations naturally arise from the quantum structures, with no additional assumptions. Formally, the distinct capabilities of classical data – that they can be copied and deleted – are captured by means of special coalgebra structures of classical objects. Conceptually, this suggests that the connection of the classical and the quantum reasoning extends the connection of the classical and the resource sensitive logics. Technically, the connection of the classical and the quantum structures thus echoes the connection of the additive and the multiplicative connectives in Linear Logic. In this familiar conceptual framework, we propose a comprehensive formulation of axioms of quantum informatics, which essentially improves on previous work. The underlying graphic calculus allows a very succinct presentation of several quantum informatic protocols. The underlying structural analysis also provides the elements of an abstract stochastic calculus, and points towards possible refinements of resource sensitive logics, which arise from the quantitative content of quantum mechanics and the limited observability of quantum data.*

1 Introduction

One of the central logical concerns of Computer Science has been resource sensitive reasoning, in particular as captured through linear decomposition of logical operations [13]. From the outset, the central feature of linear logic was its distinction of the multiplicative and the additive connectives. The main technical and conceptual challenges of linear logic hinge on this distinction.

When Abramsky and Coecke [1] gave the long quest for "quantum logic" a categorical twist, and extended the framework of compact categories beyond its established use in semantics of programming languages, to encode quantum protocols, the interplay of the multiplicative structure of dagger-compact tensors with the additive structure of biproducts, played a crucial role. When Selinger [23] and Coecke [6] tackled the task of specifying a system of quantum types, an echo of the multiplicative/additive structure of Hilbert spaces still seemed appropriate. However, as pointed out in [6], the total additive operations, as given by the biproducts, do not yield to a natural physical interpretation. On the other hand, according to the No-Cloning and No-Deletion theorems [26, 20], the copying and deleting capabilities arise only locally, over classical data, and do not correspond to global operations over quantum types. A method to capture and analyze such local additive operations as a certain coalgebra structure of *classical objects* was proposed in [8].

In the present paper, we complete that conceptual development, and propose a comprehensive categorical axiomatics for the information flows relating the quantum and the classical data. While it echoes the structural distinction between the multiplicative and the additive operations of linear logic, there are also significant differences. The obvious one is that the multiplicative structure of the quantum operations, captured by compact categories, is a degenerate case of the multiplicative fragment of linear logic, where the two linear tensors boil down to one. But this simplification may be a feature of early work, and one can easily envision frameworks for quantum reasoning that would accomodate two different linear tensors, e.g. arising from non-commutative geometries [19]. A deeper, and more significant distinction of the additive operations in the quantum world is their *locality*. Just like in linear logic, the additive operations arise from internal comonoids over the multiplicative operations; in contrast with linear logic, we do not use the explicit additives, or the universal construction of cofree comonoids, viz the linear exponentials, but just arbitrary types with classical structure. On a technical level, this leads to significant practical simplifications, of the graphic calculus. As a consequence, even in the categories which do have biproducts and cofree comonoids, it turns out to be simpler to describe protocols in purely multiplicative terms.[1] On the conceptual side, the paradigm of *propositions-as-types* and *proofs-as-terms*, which provided a logical foundation for type theories, including linear, now extends in a new direction, towards *propositions-as-physical-systems* and *proofs-as-physical-operations*.

However, even with the categorical refinements, the logical paradigm of *morphisms-as-physical-operations* appears to be too coarse to capture a relevant fragment of quantum mechanics. Distinguishing the quantum operations from the classical operations, and even separating the quantum objects from the classical objects, does not suffice. In order to manipulate and combine the two kinds of data, we also need to distinguish and axiomatize *classical operations*, *control operations* and *measurements*.

The axiomatics offered in the present paper is a step in this direction. In summary, our contributions are as follows:

	pure	mixed
entanglement as $\otimes$	Abramsky & C [1]	Selinger [23]
single operations from $\otimes$	Abramsky & C [1]	Selinger [23]
classical data from $\otimes$	C & Pav [8]	this paper
classical operations from $\otimes$	this paper	this paper
measurements from $\otimes$	C & Pav [8]	C & Paq [7]
control operations from $\otimes$	this paper	this paper

Building upon the initial, in some case crude definitions from [8, 7], we now propose a unified, comprehensive axiomatics of quantum information flows. While measurements induce the information flows $Q \to Q \otimes C$, *from the quantum world to the classical world*, control operations induce information flows $Q \otimes C \to Q$, *from the classical world to the quantum world*. However, the apparent duality of these two types of operations stops at their type annotations, as their semantics turn out to be based on quite different mathematical structures, *viz* on †-Frobenius coalgebras in former case, and on relativized unitaries in the Kleisli category for a classical object in the latter case. The proposed axiomatic framework provides grounds for understanding their actual structural relationship.

In [1, 2] pictures were used to display the *quantum information flow*. Building upon the seminal work of Kelly and Laplaza [17], Selinger [23] provided the formal foundation, and the coherence theory for such graphical calculi. The importance and power of these graphic representations is presently illustrated by our protocol analyses, combining the classical and the quantum structures in a way which makes direct algebraic presentations practically unfeasible.

This paper also exposes that quantum structure and classical probability share a common high-level structure. This should not be a surprise. For example, when encoding a message as an encrypted message together with an encryption key, all data is transformed in pure correlations, leaving no trace of it with the individual components, exactly as it is the case for maximal quantum entanglement.

[1] In finitely dim. Hilbert spaces, every choice of a base gives a classical and hence a comonoid structure, but only an involved Grassman algebra construction [3] yields a cofree comonoid, and the linear exponentials.

Classical and quantum objects are defined in Sec. 2. Sec. 3 introduces fully comprehensive purely multiplicative categorical quantum axiomatics. Sec. 3.5 illustrates its use by providing a succinct description of several quantum protocols. We also extract axioms for an abstract theory of stochastic operators, as arising from the quantum world. *The relevant proofs are presented in the Appendix.*

2 Classical and quantum data

We will only consider *symmetric monoidal †-categories* i.e. symmetric monoidal categories with a strict monoidal involution [23]. Below $(\mathbf{C}, \otimes, \mathrm{I}, \dagger)$ is always such a category. Of key importance to this paper is the category of finite dimensional Hilbert spaces and linear maps **FdHilb**.

2.1 Quantum data

A *compact structure* on an object A of $\mathbf{C}$ is a quadruple

$$(A, A^*, \eta : \mathrm{I} \to A^* \otimes A, \varepsilon : A \otimes A^* \to \mathrm{I})$$

which is such that the diagrams

$$\begin{tikzcd} A^* \arrow[d, "\eta\otimes A^*"'] \arrow[dr, "id"] & \\ A^* \otimes A \otimes A^* \arrow[r, "A^*\otimes\varepsilon"'] & A^* \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "A\otimes\eta"] \arrow[dr, "id"'] & A \otimes A^* \otimes A \arrow[d, "\varepsilon\otimes A"] \\ & A \end{tikzcd}$$

commute [17]. Given such a compact structure, A^* is called the *dual* to A, and we call A *self-dual* whenever $A^* = A$.

Definition 2.1. A *quantum structure* in $\mathbf{C}$ is a pair

$$(A\,, \eta : \mathrm{I} \to A \otimes A)$$

for which $(A, A, \eta, \eta^\dagger)$ is a compact structure (and hence with A self-dual). We denote by $\mathbf{C}_q$ the category with quantum structures (A, η_A) in $\mathbf{C}$ as objects and with $\mathbf{C}_q((A, \eta_A), (B, \eta_B)) := \mathbf{C}(A, B)$.

Proposition 2.2. *Quantum structure induces two identity-on-object functors $(-)^* : \mathbf{C}_q^{op} \to \mathbf{C}_q$ and $(-)_* : \mathbf{C}_q \to \mathbf{C}_q$ such that $(f_*)^* = (f^*)_* = f^\dagger$, explicitly*

$$f^* := (1_A \otimes \eta_B^\dagger) \circ (1_A \otimes f \otimes 1_B) \circ (\eta_A \otimes 1_B)$$

$$f_* := (1_B \otimes \eta_A^\dagger) \circ (1_B \otimes f^\dagger \otimes 1_A) \circ (\eta_B \otimes 1_A)\,.$$

In $\mathbf{FHilb}$ the operation $(-)^*$ transposes matrices while $(-)_*$ conjugates the phases of all matrix entries. Hence invariance under $(-)_*$ stands for absence of phase data.

In [1] Abramsky & C axiomatised $\mathbf{C}_q$ in terms of the involutions $(-)^*$ and $(-)_*$ and defined *pure operations* or *unitaries* are morphisms which satisfy $U^\dagger = U^{-1}$. Selinger constructed (not normalised) *mixed operations*, showed that this construction preserves quantum structure, and that captured the passage from pure to mixed operations and states.

Definition 2.3. [23] Let $\mathrm{CP}_q(\mathbf{C})$ be the category with the same objects as $\mathbf{C}_q$ and with $\mathrm{CP}_q(\mathbf{C})(A, B)$ containing all morphisms $(1_B \otimes \eta_C^\dagger \otimes 1_B) \circ (f \otimes f_*) \in \mathbf{C}_q(A^{\otimes 2}, B^{\otimes 2})$ for $f \in \mathbf{C}_q(A, B \otimes C)$, with composition inherited from $\mathbf{C}_q$. Morphisms in $\mathbf{C}_q$ of the above form are *completely positive*.

Proposition 2.4. [23] *Objects in $\mathrm{CP}_q(\mathbf{C})$ inherit quantum structure from objects in $\mathbf{C}_q$ and the monoidal functor*

$$H_q : \mathbf{C}_q \to \mathrm{CP}_q(\mathbf{C}) :: f \mapsto f \otimes f_*$$

preserves this structure. Morphisms in $\mathrm{CP}_q(\mathbf{FdHilb})$ exactly are completely positive maps in the usual sense, and elements are (not normalised) density matrices.

2.2 Classical data

A *Frobenius algebra structure* in $\mathbf{C}$ is a quintuple

$$(X, m : X{\otimes}X \to X, e : \mathrm{I} \to X, \delta : X \to X{\otimes}X, \epsilon : X \to \mathrm{I})$$

where (X, m, e) is an internal commutative monoid and (X, δ, ϵ) is an internal commutative comonoid which satisfies the *Frobenius equation* i.e.

$$\begin{tikzcd} X \otimes X \arrow[r, "m"] \arrow[d, "\delta\otimes X"'] & X \arrow[d, "\delta"] \\ X \otimes X \otimes X \arrow[r, "X\otimes m"] & X \otimes X \end{tikzcd}$$

commutes [5, 18]. It is *special* whenever $m \circ \delta = 1_X$.

Definition 2.5. A *classical structure* in $\mathbf{C}$ is a triple

$$(X\,, \delta : X \to X \otimes X\,, \epsilon : X \to \mathrm{I})$$

for which $(X, \delta^\dagger, \epsilon^\dagger, \delta, \epsilon)$ is a special Frobenius algebra.

For Hilbert spaces, we set $\delta : |i\rangle \mapsto |ii\rangle$ and $\epsilon : |i\rangle \mapsto 1$, and specifying this classical structure means fixing a base, since the only vectors $|\psi\rangle$ which are not mapped on an entangled state by δ, i.e. $\delta(|\psi\rangle) = |\phi_1\rangle \otimes |\phi_2\rangle$ for some vectors $|\phi_1\rangle$ and $|\phi_2\rangle$, are exactly $\{|i\rangle\}_i$.

Proposition 2.6. *Each classical structure induces a quantum structure. Explicitly we have $\eta_X := \delta_X \circ \epsilon_X^\dagger$.*

Proposition 2.7. [8] *We have $\delta_* = \delta$ and $\epsilon_* = \epsilon$.*

The classical structures over X and Y induce a classical structure over $X \otimes Y$. The canonical classical structure of I consists of the isomorphism $\lambda_\mathrm{I} : \mathrm{I} \simeq \mathrm{I} {\otimes} \mathrm{I}$ and 1_I. We denote by $\mathbf{C}_c$ the category with classical structures $(X, \delta_X, \epsilon_X)$ in $\mathbf{C}$ as objects and with $\mathbf{C}_c((X, \delta_X, \epsilon_X), (Y, \delta_Y, \epsilon_Y)) := \mathbf{C}(X, Y)$. By proposition 2.6 the forgetful functor $\mathbf{C}_c \to \mathbf{C}$ thus factors through $\mathbf{C}_c \to \mathbf{C}_q$. Note that $\mathbf{C}_c$ differs from the cartesian category of classical objects and coalgebra homomorphisms, considered in [8].

Definition 2.8. The comonoid structure of each classical object X induces a *classical comonad* $X \otimes - : \mathbf{C}_q \to \mathbf{C}_q$. The counit and the comultiplication of the comonoid induce the counit and comultiplication of the comonad, and the categorical structure relativized over X in the Kleisli category $\mathbf{C}_X$ for this comonad [22]. If a morphism $f : A \to B$ of $\mathbf{C}_X$ has a property Φ, then we say that the underlying morphism $f : X \otimes A \to B$ of $\mathbf{C}$ has the property Φ *relative to* X, or that it is an X-Φ morphism.

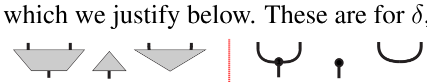
Proposition 2.9. *For every classical structure X, the induced Kleisli category $\mathbf{C}_X$ is a symmetric monoidal †-category. Its objects inherit the classical structure from $\mathbf{C}_c$.*

Definition 2.10. A *†-coalgebra* for a classical comonad is a (Eilenberg-Moore) coalgebra $f : A \to X \otimes A$ of which the adjoint $f^\dagger : X \otimes A \to A$ is self-adjoint in C_X.

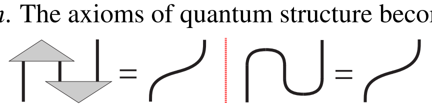
Similary to $\mathrm{CP}_q(\mathbf{C})$ we define $\mathrm{CP}_c(\mathbf{C})$ by only considering objects in $\mathbf{C}_c$. Objects in $\mathrm{CP}_c(\mathbf{C})$ now inherit classical structure from $\mathbf{C}_c$. We have $\mathrm{CP}_c(\mathbf{FdHilb}) \simeq \mathrm{CP}_q(\mathbf{FdHilb})$ as categories of quantum structures.

2.3 Graphical notation and normal form

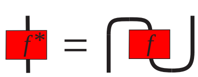
In what follows we will need to combine the classical and the quantum structures in a way which makes direct algebraic presentations unfeasible. Therefore we will rely on the well-established graphical representation of tensor calculi [17, 11, 16, 23, 18, 24]. In this notation morphisms in symmetric monoidal †-categories are represented by *boxes*, domain types by *input wires*, codomain types by *output wires*, and † by *reversal* of boxes [23]. As a convention we read pictures from bottom to top. We introduce two distinct but equivalent graphical notations for classical and quantum structure which we justify below. These are for δ, ϵ and η:



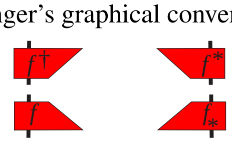
We refer to the first one as *box-form* and to the second as *wire-form*. The axioms of quantum structure become:



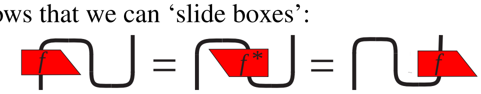
Observing that in wire-form we have:



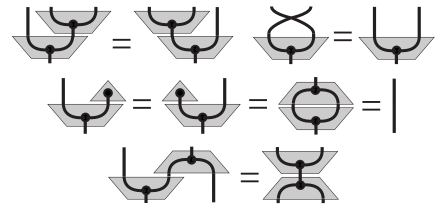
and adopting Selinger's graphical convention [23]:



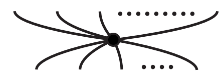
it follows that we can 'slide boxes':



and obtain a straightforward generalisation of the *compositionality lemmas* of [1]. Correctness of (postselected) logic gate teleportation and entanglement swapping are straightforward corollaries of this [1]. The axioms for classical structure depict in 'superposed' notation as:



Proposition 2.11. *If in graphic representation a morphism generated from classical structure and the symmetric monoidal †-structure is connected, then it can be reduced to the form of a "spider" with n input and m output wires,*



where we conventionally write

Setting $\delta_0 := \epsilon^\dagger$, $\delta_1 := 1_X$ and, for $n \geq 2$,

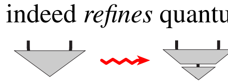
$$\delta_n := (\delta \otimes 1_{X^{\otimes n-2}}) \circ (\delta \otimes 1_{X^{\otimes n-3}}) \circ \ldots \circ (\delta \otimes 1_X) \circ \delta\,,$$

the "spider" is the graphical representation of $\delta_n \circ \delta_m^\dagger$.

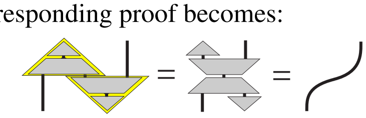
While the actual graphical calculus will be the one of wires and their connectors (dots) i.e. wire-form, the translation of the wire diagrams to algebraic expressions simplifies by annotating trapezoids, triangles etc. The triangles, i.e. morphisms of type $\mathrm{I} \to A$, also resemble Dirac-notation:

which makes the pictures more tangible for physicists. But in order to simplify the transformations of diagrams we omit these annotations in proofs: yanking a wire is more natural than substituting configurations of triangles into wires, and contracting the dots is more natural than transforming configurations of trapezoids into squares.

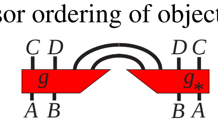
Our choice for the box-form of classical and quantum structure does immediately indicate, as in Prop. 2.6, that classical structure indeed *refines* quantum structure:



and the corresponding proof becomes:



Following [23], when $\mathrm{CP}_c(\mathbf{C})$ is represent in terms of graphical calculus it is very convenient to depict $(-)_*$ such that it reverses tensor ordering of objects e.g.:



resulting in a perfectly symmetric picture.

3 Interacting classical and quantum data

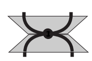
Below X, X_i and Y will always be classical structures and A, A_i and B will always be quantum structures. Without loss of generality, by relying on symmetry, we can represent combined classical-quantum data as $X \otimes A$, with $X = X_1 \otimes \ldots \otimes X_n$ and $A = A_1 \otimes \ldots \otimes A_m$. Given $\mathbf{C}$ we will now construct an new category $\mathbf{C}_{\Xi,q}$ that is sufficiently rich to capture full-blown classical-quantum interaction i.e. we can identify every physical concept as particular morphisms. Selinger's $\mathrm{CP}_q(\mathbf{C})$ will be the 'raw quantum fragment' of this. The remainder of the paper will mainly constitute the study of the other important fragments of $\mathbf{C}_{\Xi,q}$, namely the 'raw classical fragment'. We start by constructing this raw classical fragment, and then we will generalise this constructions to include all operations involving both classical and quantum data. From now on we denote the compound type $A \otimes B \otimes C \otimes \ldots$ by $ABC \ldots$.

3.1 Categorical semantics of raw classical data

Definition 3.1. The *diagonal structure* on X is

$$\Xi_X := \delta_X \circ \delta_X^\dagger : XX \to XX\,,$$

which depicts as:



and $f : XX \to YY$ is *diagonal* if $f = f \circ \Xi_X = \Xi_Y \circ f$.

Lemma 3.2. *Ξ_X is idempotent and completely positive.*

If $\rho : \mathcal{H} \to \mathcal{H}$ is a density matrix, then $(\rho \otimes 1_\mathcal{H}) \circ \eta_\mathcal{H}$ is diagonal exactly if there are no off-diagonal elements, and similarly, diagonal completely positive maps will send diagonal density matrices to diagonal density matrices.

Let $\mathrm{D}_\Xi(\mathbf{C})$ be the category with classical structures as objects and with $\mathrm{D}_\Xi(\mathbf{C})(X, Y)$ the set of all diagonal morphisms in $\mathbf{C}(XX, YY)$. Note that the identities in $\mathrm{D}_\Xi(\mathbf{C})$ are Ξ_X, not 1_X. Let G be identity-on-objects and

$$G : \mathrm{D}_\Xi(\mathbf{C})(X, Y) \to \mathbf{C}_c(X, Y) :: g \mapsto \delta_Y^\dagger \circ g \circ \delta_X\,.$$

Lemma 3.3. *G is an isomorphism with inverse*

$$F : \mathbf{C}_c(X, Y) \to \mathrm{D}_\Xi(\mathbf{C})(X, Y) :: f \mapsto \delta_Y \circ f \circ \delta_X^\dagger\,.$$

In wire-form Ff and Gg are

Let $\mathrm{CP}_\Xi(\mathbf{C})$ be the all-objects-including subcategory of $\mathrm{D}_\Xi(\mathbf{C})$ with morphisms restricted to completely positive ones and let $\mathbf{C}_\Xi$ be $\mathrm{CP}_\Xi(\mathbf{C})$'s image under G i.e.

$$\begin{tikzcd}
\mathbf{C}_\Xi \arrow[r, "F"'] \arrow[d, hook] & \mathrm{CP}_\Xi(\mathbf{C}) \arrow[l, bend right, "G"', "\simeq"] \arrow[d, hook] \\
\mathbf{C}_c \arrow[r, "F"] & \mathrm{D}_\Xi(\mathbf{C}) \arrow[l, bend left, "G", "\simeq"']
\end{tikzcd}$$

Proposition 3.4. *Morphisms in $\mathrm{CP}_\Xi(\mathbf{C})(X, Y)$ are exactly the $\mathbf{C}_c(XX, YY)$-morphisms of the form $\Xi_Y \circ g \circ \Xi_X$ for which $g : XX \to YY$ is completely positive.*

Theorem 3.5. *$\mathbf{C}_\Xi$ is a symmetric monoidal †-category and each classical structure in $\mathbf{C}_c$ induces a classical structure on the corresponding object in $\mathbf{C}_\Xi$.*

We call the morphisms of $\mathbf{C}_\Xi$ *classical maps*.

Proposition 3.6. *For $f : X \to Y$ setting*

$$\mathrm{Unf}(f) = (1_X \otimes \delta_Y^\dagger) \circ (1_X \otimes f \otimes 1_Y) \circ (\delta_X \otimes 1_Y)\,.$$

we have that $\mathrm{Unf}(f) : X \otimes X \to Y \otimes Y$ is positive if and only if $Ff : X \otimes X \to Y \otimes Y$ is completely positive.

Given an $n \times m$-matrix (f_{ij}), we have $\mathrm{Unf}(f_{ij}) : |ij\rangle \mapsto f_{ij}|ij\rangle$ i.e. we obtain a $(n \times m) \times (n \times m)$-matrix in which all entries of (f_{ij}) now appear on the diagonal. Requiring this matrix to be positive implies that all f_{ij} are positive. As a result we have $\mathbf{FdHilb}_\Xi \simeq \mathbf{Mat}_{\mathbb{R}^+}$ i.e. matrices with entries in the trivially involutive semiring $\mathbb{R}^+$.

Definition 3.7. A classical map is a *stochastic map* if it preserves ϵ i.e. $\epsilon_B \circ f = \epsilon_A$. A stochastic map is a *classical state* if it is of type $p : \mathrm{I} \to A$. A classical state p is a *pure classical state* if it preserves δ i.e. $\delta_A \circ p = (p \otimes p) \circ \lambda_\mathrm{I}$, and we call $\epsilon_A^\dagger : \mathrm{I} \to A$ the *maximally mixed classical state*.

The notions of stochastic map and classical state, purity and maximal mixedness coincide in this example with the usual notions. One should think of $\mathbf{FdHilb}_\Xi$ itself as the *positive convex cone* i.e. non-normalized probabilistic states and corresponding probabilistic processes. Hence in this key model classical maps are inherently qualitative.

3.2 Categorical semantics of all data

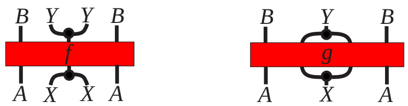
We call $f : AXXA \to BYYB$ *diagonal* if $f = f \circ \Xi_{(A,X)} = \Xi_{(B,Y)} \circ f$ where $\Xi_{(A,X)} := (1_A \otimes \Xi_X \otimes 1_A)$. Let $\mathrm{D}_{\Xi,q}(\mathbf{C})$ be the category with pairs (A, X) as objects, with $\mathrm{D}_{\Xi,q}(\mathbf{C})((A, X), (B, Y))$ being all diagonal morphisms in $\mathbf{C}(AXXA, BYYB)$, and identities $\Xi_{(A,X)}$. Let $\mathbf{C}_{c,q}$ be the category with pairs (A, X) as objects and with $\mathbf{C}_{c,q}((A, X), (B, Y)) := \mathbf{C}(AXA, BYB)$. Let $\delta_{(A,X)} := 1_A \otimes \delta_X \otimes 1_A$. Let G be identity-on-objects and

$$G : \mathrm{D}_{\Xi,q}(\mathbf{C}) \to \mathbf{C}_{c,q} :: \begin{cases} (A, X) \mapsto (A, X) \\ g \mapsto \delta_{(B,Y)}^\dagger \circ g \circ \delta_{(A,X)} \end{cases}$$

Lemma 3.8. *G is isomorphism with inverse*

$$F : \mathbf{C}_{c,q} \to \mathrm{D}_{\Xi,q}(\mathbf{C}) :: \left\{ \begin{array}{l} (A,X) \mapsto (A,X) \\ f \mapsto \delta_{(B,Y)} \circ f \circ \delta^\dagger_{(A,X)} \end{array} \right.$$

In wire-form Ff and Gg are



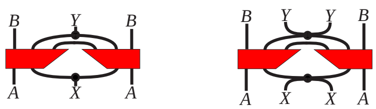
Let $\mathrm{CP}_{\Xi,q}(\mathbf{C})$ be the all-objects-including subcategory of $\mathrm{D}_{\Xi,q}(\mathbf{C})$ with morphisms restricted to completely positive ones and let $\mathbf{C}_{\Xi,q}$ be $\mathrm{CP}_{\Xi,q}(\mathbf{C})$'s image under G i.e.

$$\begin{tikzcd}
\mathbf{C}_{\Xi,q} \arrow[r, "F"'] \arrow[r, phantom, bend left, "\simeq"] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{CP}_{\Xi,q}(\mathbf{C}) \arrow[l, bend right, "G"'] \arrow[d, hook] \\
\mathbf{C}_{c,q} \arrow[r, "F"] \arrow[r, phantom, bend right, "\simeq"] & \mathrm{D}_{\Xi,q}(\mathbf{C}) \arrow[l, bend left, "G"]
\end{tikzcd}$$

Proposition 3.9. *The morphisms in $\mathbf{C}_{\Xi,q}$ and in $\mathrm{CP}_{\Xi,q}(\mathbf{C})$ are exactly those $\mathbf{C}_q$-morphisms of respective forms*

$$\delta^\dagger_{(B,Y)} \circ g \circ \delta_{(A,X)} \qquad \text{and} \qquad \Xi_{(B,Y)} \circ g \circ \Xi_{(A,X)}$$

for which $g : AXXA \to BYYB$ is completely positive. Hence, using Selinger's graphical representation for completely positive maps, in wire-form they respectively are:



Theorem 3.10. *$\mathbf{C}_{\Xi,q}$ is a symmetric monoidal $\dagger$-category and each pair consisting of a quantum structure A in $\mathbf{C}_q$ and a classical structure X in $\mathbf{C}_c$ induces a quantum structure on the object (A,X) in $\mathbf{C}_\Xi$.*

It also easily follows that for quantum structures A and classical structures X there are *faithful* canonical functors

$$C_A : \mathbf{C}_\Xi \to \mathbf{C}_{\Xi,q} :: Y \mapsto (A,Y)$$
$$Q_X : \mathrm{CP}_q(\mathbf{C}) \to \mathbf{C}_{\Xi,q} :: B \mapsto (B,X)\,.$$

Given $\mathbf{C}$ we constructed a category $\mathbf{C}_{\Xi,q}$ in which $\mathrm{CP}_q(\mathbf{C})$ can be embedded along a classical structure and in which the category $\mathbf{C}_\Xi$, which we also constructed from $\mathbf{C}$, can be embedded along a quantum structure.

3.3 Classical maps as quantum maps (twice)

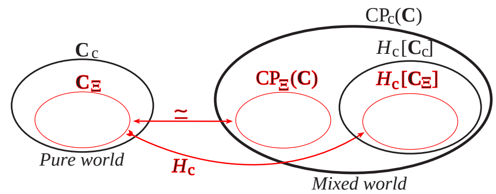
Since the category of classical maps $\mathbf{C}_\Xi$ is a subcategory of $\mathbf{C}_c$ it also embeds in $\mathrm{CP}_c(\mathbf{C})$ and hence in $\mathrm{CP}_q(\mathbf{C})$:

$$\begin{tikzcd}
\mathbf{C}_\Xi \arrow[r, hook] \arrow[rr, bend right, "\gamma_1"'] & \mathbf{C}_c \arrow[r, "H_c"] & \mathrm{CP}_q(\mathbf{C})
\end{tikzcd}$$

The functor γ_2 assigns to a classical map a *pure* quantum operation which 'extends this classical map by superposition'. However, another manifestation of classical maps and in particular, stochastic maps, within quantum theory is *mixedness*. Our categorical semantics also witnesses this:

$$\begin{tikzcd}
\mathbf{C}_\Xi \arrow[r, "F"] \arrow[rr, bend right, "\gamma_2"'] & \mathrm{CP}_\Xi(\mathbf{C}) \arrow[r, hook] & \mathrm{CP}_c(\mathbf{C})
\end{tikzcd}$$

The embedding $\mathrm{CP}_\Xi(\mathbf{C}) \hookrightarrow \mathrm{CP}_c(\mathbf{C})$ does not preserve identities and hence γ_2 is only a semifunctor.



We study how these two embeddings relate. We define Sqr and Diag by commutation of:

$$\begin{tikzcd}
\mathbf{C}_\Xi \arrow[rr, "\gamma_1"] \arrow[d, "\mathrm{Sqr}"'] & & \mathrm{CP}_c(\mathbf{C}) \arrow[d, "\mathrm{Diag}"] \arrow[dl, "\Xi\circ-\circ\Xi"'] \\
\mathbf{C}_\Xi \arrow[r, bend right, "F"'] \arrow[r, phantom, "\simeq"] & \mathrm{CP}_\Xi(\mathbf{C}) \arrow[l, bend right, "G"'] \arrow[r, hook] & \mathrm{CP}_c(\mathbf{C})
\end{tikzcd}$$

Note that $\Xi \circ - \circ \Xi$ does not preserve composition and that $\mathrm{CP}_\Xi(\mathbf{C}) \hookrightarrow \mathrm{CP}_c(\mathbf{C})$ does not preserve identities.

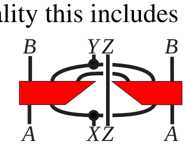
Proposition 3.11. *The following diagram commutes:*

$$\begin{tikzcd}
\mathbf{C}_\Xi \arrow[r, "\gamma_1"] \arrow[d, "\mathrm{Sqr}"'] & \mathrm{CP}_c(\mathbf{C}) \arrow[d, "\mathrm{Diag}"] \\
\mathbf{C}_\Xi \arrow[r, tail, "\gamma_2"] & \mathrm{CP}_c(\mathbf{C})
\end{tikzcd}$$

Example 3.12. Explicitly we have $\mathrm{Sqr}(f) = \delta^\dagger \circ (f \otimes f_*) \circ \delta$ so in **FdHilb** matrix (f_{ij}) becomes matrix $(|f_{ij}|^2)$. Since we restricted the domain of Sqr to $\mathbf{C}_\Xi$ the modulus is in fact superfluous, so Sqr squares the entries in the matrix of a linear map, hence merely does some *re-gauging*. This analysis may shed a new light on the l_1- vs. l_2-distinction.

3.4 Categorical quantum axiomatics

By Prop. 3.9 we can use wire-form within $\mathbf{C}_{\Xi,q}$ to define all quantum interaction concepts required for quantum protocols. By speciality this includes as a particular case:

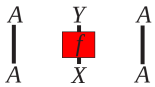


Definition 3.13. We call $\mathbf{C}_{\Xi,q} \simeq \mathrm{CP}_{\Xi,q}(\mathbf{C})$ the *quantum interaction theory* generated by $\mathbf{C}$. It includes as concepts:

1. *Raw classical operations* are morphisms

$$1_A \otimes f \otimes 1_A : AXA \to AYA$$

in $\mathbf{C}_{\Xi,q}$ where $f \in \mathbf{C}_\Xi(X,Y)$. In diagrammatic notation for symmetric monoidal †-categories that is:

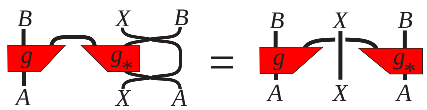


They constitute the range of the functors $\{C_A\}_A$. The operation $\delta_X : X \to X \otimes X$ stands for *copying* of classical data, $\epsilon_X : X \to \mathrm{I}$ stands for *deleting*, $\epsilon_X^\dagger$ stands for a *random variable*, and $\delta_X^\dagger$ stands for *comparing*.

2. *Raw quantum operations* are morphisms

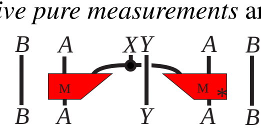
$$(1_B \otimes \eta_C^\dagger \otimes \sigma_{BX}) \circ (g \otimes g_* \otimes 1_X) \circ (1_A \otimes \sigma_{XA})$$

of type $AXA \to BXB$ in $\mathbf{C}_{\Xi,q}$ with $g \in \mathbf{C}_q(A, BC)$ and $\sigma_{AB} : AB \simeq BA$. Using wire-form for classical and quantum structure diagrammatically this becomes:

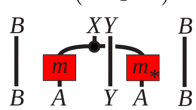


They constitute the range of the functors $\{Q_X\}_X$. Symbolic representation will become more and more involved below so we will restrict to the diagrammatic notation.

3. *Non-destructive pure measurements* are morphisms:

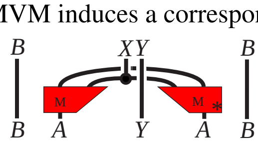


with $\mathcal{M}$ a †-coalgebra for $(X \otimes -)$. A destructive one is:

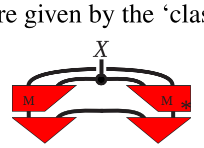


where now $m^\dagger : X \to A$ is an isometry i.e. $m \circ m^\dagger = 1_X$.

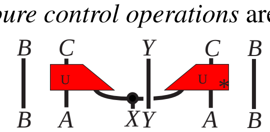
4. *Mixed measurements* or *PMVMs* depict the same as non-destructive pure measurements with now $\mathcal{M}^\dagger \circ \mathcal{M} = 1_A$ and $\mathcal{M}$ also X-polar-decomposable i.e. in Kleisli category $\mathbf{C}_X$ we have $\mathcal{M} = \mathcal{W} \circ_X \mathcal{N}$ where $\mathcal{W}$ is an isometry in $\mathbf{C}_X$ and $\mathcal{N}$ is positive in $\mathbf{C}_X$ i.e. $\mathcal{N} = \mathcal{L} \circ_X \mathcal{L}$ for some $\mathcal{L}$ in $\mathbf{C}_X$. Each PMVM induces a corresponding *POVM*:



5. *Outcome probabilities* for a measurement on a system in some mixed state are given by the 'classical state':

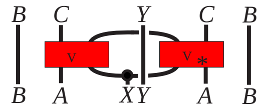


6. *Normalised pure control operations* are:



where $\mathcal{U}$ is X-unitary i.e. it is unitary in $\mathbf{C}_X$.

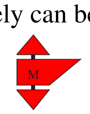
7. *Mixed control operations* are:



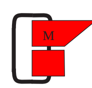
where $\mathcal{V}$ is now arbitrary.

8. *Controlled composition* is compostion in $\mathbf{C}_X$.

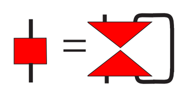
When in Def. 3.13 the category $\mathbf{C}$ is taken to be $\mathbf{FdHilb}$ then all concepts coincide with the ones in the usual quantum mechanical formalism. When passing from $\mathbf{C}_{\Xi,q}$ to $\mathrm{CP}_{\Xi,q}(\mathbf{C})$ via the isomorphism the proofs for measurements become those found in [8, 7]. In particular, the conditions for being a $(X \otimes -)$-coalgebra exactly boil down to having a family $\{\mathrm{P}_i : \mathcal{H} \to \mathcal{H}\}_i$ of mutulally orthogonal idempotents on a Hilbert space $\mathcal{H}$ i.e. $\mathrm{P}_i \circ \mathrm{P}_j = \delta_{ij}\mathrm{P}_i$, and X-self-adjointness enforces individual self-adjointness within this family i.e. $\mathrm{P}_i^\dagger = \mathrm{P}_i$, so we obtain the projector spectra of self-adjoint linear operators $H = \sum_i a_i \mathrm{P}_i$, the usual representatives for quantum measurements. The probabilities for pure measurements on pure and on mixed states respectively can be rewritten as:



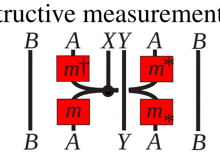
and



where



revealing the *Born* expressions $\langle\psi|\mathrm{P}_i|\psi\rangle$ and $\mathrm{Tr}(\rho \mathrm{P}_i)$. One verifies that for each destructive pure measurement a corresponding non-destructive measurement is definable:

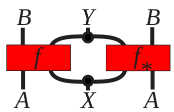


Operational significance. These quantum theoretical concepts are not merely defined by formal analogy but all have a clear operational significance. One condition for being an Eilenberg-Moore $(X \otimes -)$-coalgebra is idempotence in $\mathbf{C}_X^{op}$. This condition exactly captures *von Neumann's projections postulate* in a 'resource sensitive manner': performing a measurement twice is the same as performing a measurement once and copying the outcome. The other condition guarantees that they add up to one. Selinger's construction $\mathrm{CP}_q(\mathbf{C})$ captures *Stinespring's dilation theorem* [21] which states that completely positive maps can be obtained by performing pure operations on an extended system and then 'tracing out' the ancilla. Similary, by the theorem proved in [7], mixed measurements arise by performing pure measurements on an extended system, a result referred to in C^*-algebra as *Naimark's dilation theorem* [21].

Controlled measurements. The basic concepts of measurement (type $A \to Y \otimes A$) and control operation (type $X \otimes A \to A$) can be combined into *controlled measurements* $f : X \otimes A \to Y \otimes A$, that is, X-measurements. Here

X is a variable depending on which we perform a certain measurement. Controlled measurements play a central role in quantum key exchange (see below).

Purity. Both $\mathrm{CP}_q(\mathbf{C})$ and $\mathbf{C}_\Xi$ include a pure component. There is an obvious notion of *general pure operations*:



of which pure measurements are an example. If we delete the measurement outcome we obtain mixedness. However

$$\mathbf{C}_{\Xi,q} \xrightarrow{\ F\ } \mathrm{CP}_{\Xi,q}(\mathbf{C}) \xhookrightarrow{(A,B)\mapsto AB} \mathrm{CP}_q(\mathbf{C})$$

does not map pure operations within the range of H_q. Hence $\mathrm{CP}_q(\mathbf{C})$ plays a structural role in our construction which is not directly related to intoducing mixedness (cf. Prop. 3.6).

Scalars. When specifying protocols we will be needing dimensions, square-roots thereof, and inverses of both. Recall from [17] that each monoidal category admits a commutative monoid of scalars i.e. morphisms of type $\mathrm{I} \to \mathrm{I}$. *Scalar multiplication* is defined as [1]:

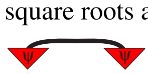
$$s \bullet f : A \simeq \mathrm{I} \otimes A \xrightarrow{s \otimes f} \mathrm{I} \otimes A \simeq B\,.$$

Since $\dim(A) := \eta_A^\dagger \circ \eta_A : \mathrm{I} \to \mathrm{I}$ is the *dimension* for quantum structure, for classical structure we set

$$\dim(X) := \epsilon_X \circ \epsilon_X^\dagger = \eta_X^\dagger \circ \eta_X : \mathrm{I} \to \mathrm{I}\,.$$

Proposition 3.14. *If s is a positive scalar in $\mathbf{C}_q$ i.e. $s = \psi^\dagger \circ \psi$ for some $\psi : \mathrm{I} \to A$, then $H_q(s)$ in $\mathrm{CP}_q(\mathbf{C})$ has a square-root i.e. there is t in $\mathrm{CP}_q(\mathbf{C})$ such that $t \circ t = s$.*

Interestingly, these square roots are in general mixed:



A scalar s is *zero* if for all scalars t holds $s \circ t = s$, and is a *divisor of zero* if there is a scalar t such that st is zero. Clearly, there is at most one zero scalar in a category, denoted o. We call a category with scalars *local* iff all of its positive scalars are either divisors of zero, or are invertible.

Proposition 3.15. *Each category of quantum structures $\mathbf{C}_q$ has a universal localisation $L\mathbf{C}_q$ equipped with a quantum structure preserving functor $\mathbf{C} \to L\mathbf{C}$, which is initial for all local categories of quantum structures with such a functor from $\mathbf{C}_q$. In particular, the objects of $L\mathbf{C}_q$ are those of $\mathbf{C}_q$, and a morphism in $L\mathbf{C}_q(A,B)$ is in the form $\frac{f}{s}$, where*

$$s \in \Sigma := \left\{ s \in \mathbf{C}(\mathrm{I},\mathrm{I}) \;\middle|\; \forall t \in \mathbf{C}(\mathrm{I},\mathrm{I}) : s \circ t \neq o \right\},$$

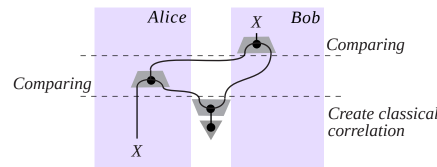
and these fractions are taken modulo the congruence

$$\frac{f}{s} = \frac{g}{t} \quad \Longleftrightarrow \quad \exists u, v \in \Sigma :\ u \circ s = v \circ t \ \&\ u \bullet f = v \bullet g\,.$$

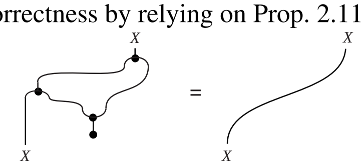
Hence we are entitled to assume that dimensions and square-roots thereof have inverses in our categories.

3.5 Protocol description and derivation

Classical teleportation. This protocol teleports classical bits by means of classical correlations, which we produce by copying a random variable (cf. Sec.4):

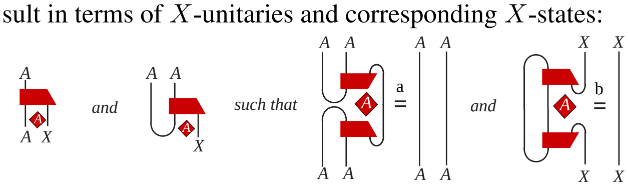


We prove correctness by relying on Prop. 2.11:

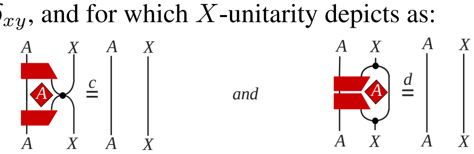


This protocol can be called classical teleportation since it exhibits exactly the same geometry as quantum teleportation [9], but in fact it is a *one-time pad* in disguise.

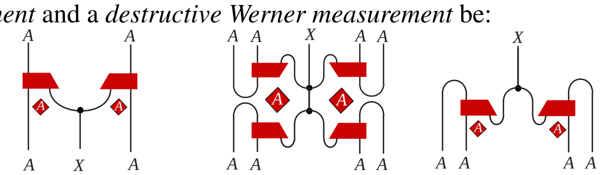
Mixed state teleportation. For $s_A := \dim(A)$ we represent $\frac{1}{s_A}$ and $\frac{1}{\sqrt{s_A}}$ respectively by a big and a small diamond (i.e. $small \circ small = big$). In [25] Werner establishes the one-to-one correspondence between quantum teleportation schemes, dense coding schemes, and certain orthonormal bases of maximally entangled vectors. We abstract his result in terms of X-unitaries and corresponding X-states:



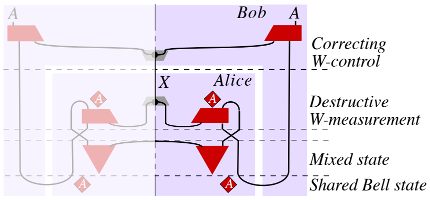
respective abstractions of $\dim(X) \geq (\dim(A))^2$ and $\mathrm{Tr}(U_x \circ U_y^\dagger) = \delta_{xy}$, and for which X-unitarity depicts as:



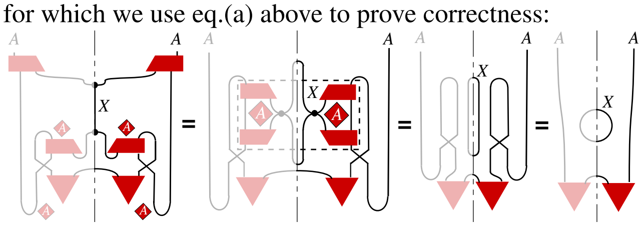
In **FdHilb** the *Pauli matrices* together with the identity are an example of such unitaries and the *Bell-basis* of such states. Let a *Werner control operation*, a *Werner measurement* and a *destructive Werner measurement* be:



One can consult [8] to see that Werner measurements indeed satisfy definition 3.13. Mixed state teleportation is:

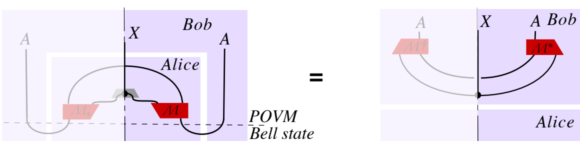


for which we use eq.(a) above to prove correctness:

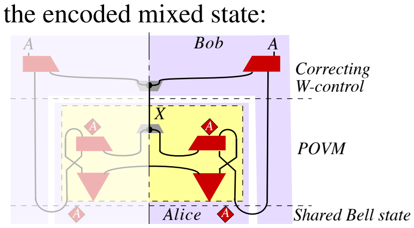


The resulting loop represents the number of possible scenarios that might have taken place.

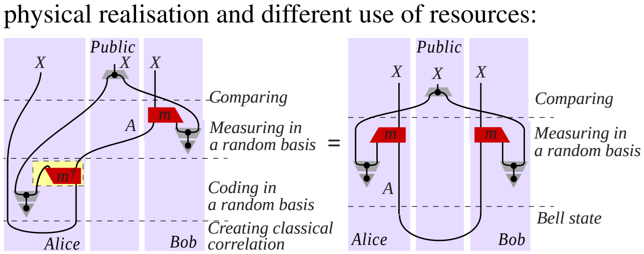
Mixed state generation and TelePOVM [15, 4]. If Alice and Bob share a Bell state then by performing a POVM she can create a mixed state in Bob's hand:



This state depends on the outcome of the POVM which in the above picture has been transmitted to Bob. With an appropriate choice of POVM, namely a W-measurement on an extended system, from correctness of the mixed state teleportation protocol (above) we know the appropriate correction to obtain the encoded mixed state:

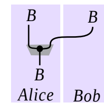


From BB84 to Ekert 91. We represented the above protocols in $\mathbf{C}_{\Xi,q}$. While this representation is 'more economical' in terms of 'wires' than representation in $\mathrm{CP}_{\Xi,q}(\mathbf{C})$, the latter allows for more spatio-temporal flexibility i.e. we can depict Bob *besides* Alice rather than *on top of* Alice. For clarity in the next picture we will somewhat abusively also not depict the 'shaded copy'. The quantum key distribution schemes BB84 to Ekert 91 are topologically equivalent and hence yield the same result despite their manifestly different physical realisation and different use of resources:

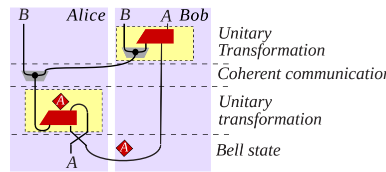


where the measurements m are controlled and destructive.

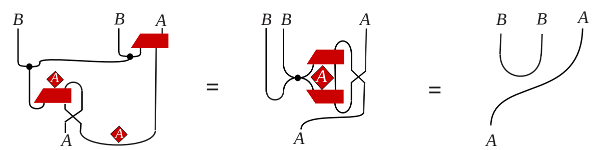
Communicating coherently. In [14] Harrow defined a 'between quantum and classical' mode of communication:



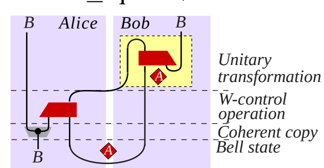
where B is a quantum structure $\eta_B = \delta_B \circ \epsilon_B^\dagger$ with some underlying classical structure. Hence we exploit the functor γ_1 of Sec. 3.3. If Alice and Bob share a Bell-state and have the ability to exchange such *cobits*:



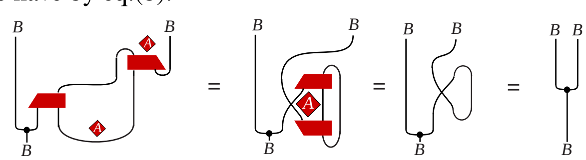
they can exchange a qubit and at the same time create *two* Bell-states (since $B \simeq A \otimes A$ by eq.(a,b)):



i.e. 2 cobits + Bell state $\geq$ qubit + 2 Bell states [14]. For:



we have by eq.(b):



so qubit + Bell state $\geq$ 2 cobits and hence we obtain *resource equality* qubit + Bell state $\stackrel{Bell}{=}$ 2 cobits. While we merely generalised a known fact to arbitrary dimension it should have become clear to the reader that Prop. 2.11 yields a very general statement on communication.

4 More classical species and orders

While in LL additives arise from multiplicatives by means of the cofree comonoid monad on a $*$-autonomous category, in our setting classical types arise from quantum types by means of a classical structure which induces a classical comonad on a category of quantum structures. Now consider the following result due to Fox.

Theorem 4.1. [10] *If* $\mathbf{C}$ *is symmetric monoidal then the category* $\mathbf{C}_\times$ *of its commutative comonoids and comonoid* *homomorphisms, with the forgetful functor $\mathbf{C}_\times \to \mathbf{C}$, is final among all cartesian categories with a monoidal functor to $\mathbf{C}$, mapping the cartesian product to the monoidal tensor.*

Categories of quantum structures are $*$-autonomous category in which $(-)^*$ preserves the tensor and to which an additional involution $(-)_*$ is adjoined [1]. Therefore in our case it is natural to require that morphisms also preserve $(-)_*$, i.e. physically, absence of phase data.

Proposition 4.2. *A comonoid homomorphism $f : X \to Y$ is also a classical map if and only if $f_* = f$.*

So in particular, classical maps do not carry any phase data. On the other hand, classical maps are in general not comonoid homomorphisms. For classical states this means that they cannot be copied nor deleted. Indeed, in the same manner as a linear operation which copies a base creates entanglement for all other states, i.e. from $|0\rangle \mapsto |00\rangle$ and $|1\rangle \mapsto |11\rangle$ follows $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \mapsto \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, an operation which copies pure classical data will create entanglement for probabilistic states, and for example maps the maximally mixed state $(1/2, 1/2)$ on 00 with weight $1/2$ and 11 with weight $1/2$, so we obtain non-deterministic but perfectly correlated data. Hence, as already had been established in Thm. 3.5 classical probability theory naturally embeds within categories of quantum structures.

Proposition 4.3. *A scalar in a monoidal category is a comonoid homomorphism of the comonoid structure $(\lambda_{\mathrm{I}}, 1_{\mathrm{I}})$ on I if and only if it is the identity.*

This indicates that there is a fundamental discrepancy between the notion of comonoid homomorphism and having non-trivial scalars. As an example, comonoid homomorphisms in $\mathbf{FdHilb}$ are functions — independent from the fact whether or not we require $f = f_*$.

Definition 4.4. We call $f : X \to Y$ a *partial map* if $f_* = f$ and it preserves δ i.e. $\delta_Y \circ f = (f \otimes f) \circ \delta_X$. A partial map f is *total* if it also preserves ϵ i.e. $\epsilon_Y \circ f = \epsilon_X$, and a total map f is a *permutation* iff also $f^\dagger$ is a total map.

One verifies that partial maps are classical maps. Our partial and total maps coincide in Carboni & Walters' *bicategory of relations* with theirs (Cor. 2.6 [5]) where $f = f_*$ always holds. Bicategories of relations were intended as an abstraction of **Rel** as a particular kind of locally posetal bicategory. We now show that $\mathbf{C}_c$ induces such a category.

Definition 4.5. We call $f : X \to Y$ a *relation* iff $f = \delta_Y^\dagger \circ (f \otimes f_*) \circ \delta_X$ and set $f \subseteq g \Leftrightarrow f = \delta_Y^\dagger \circ (f \otimes g) \circ \delta_X$.

Theorem 4.6. *If the number of relations in each hom-set of $\mathbf{C}_c$ is finite and if each object in $\mathbf{C}_c$ admits at most one classical structure for which both δ and ϵ are relations, then relations in $\mathbf{C}_c$ constitute a bicategory of relations $\mathbf{C}_r$ in Carboni & Walters' sense. In particular, relations are lax comonoid homomorphism w.r.t. local partial order $\subseteq$ i.e.*

$$\delta_Y \circ f \subseteq (f \otimes f) \circ \delta_X \quad \text{and} \quad \epsilon_Y \circ f \subseteq \epsilon_X$$

and δ (ϵ) is left-adjoint to $\delta^\dagger$ ($\epsilon^\dagger$) in bicategorical sense. Hence $\mathbf{C}_r$ also has local meets and tops (*cf.* [5] Thm. 1.6).

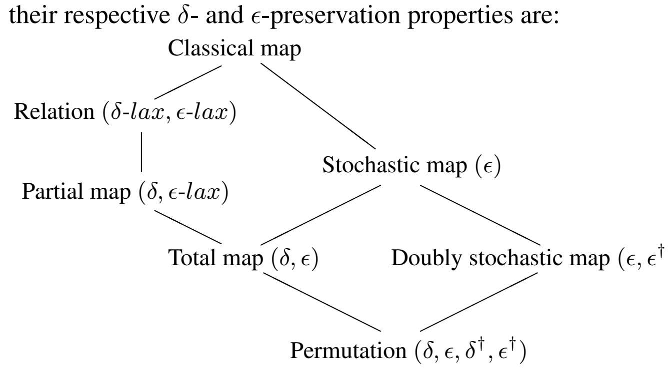
One easily verifies that each partial map is a relation and that each relation is a classical map. In **FdHilb** we obtain relations in the usual sense. Note in particular that the number of relations in each hom-set of **FdHilb** is finite. But neither the finiteness assumption nor the uniqueness assumption are needed for proving order-enrichment, local finite products, local top or lax-preservation of the comonoid structure. Finiteness only comes in when defining composition in $\mathbf{C}_r$ which is *not* inherited from $\mathbf{C}_c$ — cf. **FdHilb** and **Rel**. Uniqueness is only required as part of Carboni & Walters' definition of bicategory of relations. While composition in $\mathbf{C}_r$ is not inherited from $\mathbf{C}_c$, composition in $\mathbf{C}_r$ and $\mathbf{C}_c$ do coincide for partial maps.

Definition 4.7. We call $f : X \to Y$ *doubly stochastic* if both f and $f^\dagger$ are stochastic. Given classical structure X, a morphism $\mathcal{F} : X \otimes A \to B$ in $\mathbf{C}_q$, and a classical state $p : \mathrm{I} \to X$ with $\epsilon \circ p = 1_{\mathrm{I}}$, we call $c(\mathcal{F}, p) = \mathcal{F} \circ (p \otimes 1_A) \circ \lambda_A : A \to B$ with $\lambda_A : A \simeq \mathrm{I} \otimes A$ a *convex combination* of $\mathcal{F}$. A morphism $f : X_1 \to X_2$ is *majorized* by a morphism $g : Y_1 \to Y_2$ if there exist doubly stochastic maps $h_1 : X_1 \to Y_1$ and $h_2 : X_2 \to Y_2$ such that we have $g = h_2 \circ f \circ h_1^\dagger$.

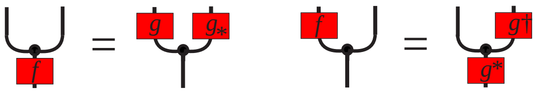
One verifies that if $h : X \to Y$ is doubly stochastic then $\dim(X) = \dim(Y)$, that permutations are doubly stochastic and that $G(U \otimes U_*)$ with $U : X \to Y$ is doubly stochastic. Let $\mathbf{C}_s$ be the subcategory of $\mathbf{C}_c$ of stochastic and of stochastic maps.

Theorem 4.8. **1.** *(doubly) stochastic maps are 'convex closed' i.e. if $\mathcal{F} : X \otimes A \to B$ is X-(doubly) stochastic then all convex combinations $c(\mathcal{F}, p)$ are (doubly) stochastic too.* **2.** *Majorization is a preordering on $\bigcup_{XY} \mathbf{C}_s(X, Y)$.*

The distinct *classical species* ordered by inclusion and their respective δ- and ϵ-preservation properties are:



While not a standard concept it also makes sense to introduce *weighted maps* as those $f : X \to Y$ for which there exists $g : X \to Y$ such that $\delta_Y \circ f = (g \otimes g_*) \circ \delta_X$. One verifies that each partial map is a weighted maps and that each weighted map is a classical map. Weighted map are in fact exactly those classical maps which *preserve purity*, and behave particularly well within graphical calculus:



References

[1] S. Abramsky and B. Coecke. A categorical semantics of quantum protocols. In *Proceedings of LiCS'04*, pages 415–425. IEEE Press, 2004. arXiv:quant-ph/0402130.

[2] S. Abramsky and B. Coecke. Abstract physical traces. *Theory and Applications of Categories*, 14:111–124, 2005.

[3] R. Blute, P. Panangaden, and R. A. G. Seely. Fock space: A model of linear exponential types. 1994.

[4] P. Brassard. G., Horodecki and T. Mor. Telepovm – a generalized quantum teleportation scheme. *IBM Journal of Research and Development*, 2004.

[5] A. Carboni and R. F. C. Walters. Cartesian bicategories I. *Journal of Pure and Applied Algebra*, 49:11–32, 1987.

[6] B. Coecke. De-linearizing linearity: projective quantum axiomatics from strong compact closure. *Electronic Notes in Theoretical Computer Science (To appear)*, 2006.

[7] B. Coecke and E. O. Paquette. Generalized measurements and Naimarks theorem without sums. In *Proceedings of Quantum Programming Languages IV*. Electronic Notes in Theoretical Computer Science, 2006. arXiv:quant-ph/0608072.v2.

[8] B. Coecke and D. Pavlovic. Quantum measurements without sums. In G. Chen, L. Kauffman, and S. Lamonaco, editors, *Mathematics of Quantum Computing and Technology*, pages 567–604. Taylor and Francis, 2006. arXiv:quant-ph/0608035.

[9] O. Cohen. Classical teleportation of classical states. arXiv: quant-ph/0310017, 2003.

[10] T. Fox. Coalgebras and cartesian categories. *Communications in Algebra*, 4:665–667, 1976.

[11] P. Freyd and D. Yetter. Braided compact closed categories with applications to low-dimensional topology. *Advances in Mathematics*, 77:156–182, 1989.

[12] P. Gabriel and M. Zisman. *Calculus of Fractions and Homotopy Theory*. Springer-Verlach, 1967.

[13] J.-Y. Girard. Linear logic. *Theoretical Computer Science*, 50:1–102, 1987.

[14] A. Harrow. Coherent communication of classical messages. *Physical Review Letters*, 92:097902, 2004.

[15] L. P. Hughston, R. Jozsa, and W. K. Wooters. A complete classification of quantum ensembles having a given density matrix. *Physics Letters A*, 183:14–18, 1993.

[16] A. Joyal and R. Street. The geometry of tensor calculus i. *Advances in Mathematics*, 88:55–112, 1991.

[17] G. M. Kelly and M. L. Laplaza. Coherence for compact closed categories. *Journal of Pure and Applied Algebra*, 19:193–213, 1980.

[18] J. Kock. *Frobenius Algebras and 2D Topological Quantum Field Theories*. Cambridge University Press, 2003.

[19] Y. I. Manin. *Topics in Noncommutative Geometry*. Princeton University Press, 1991.

[20] A. K. Pati and S. L. Braunstein. Impossibility of deleting an unknown quantum state. *Nature*, 404:164–165, 2000.

[21] V. Paulsen. *Completely Bounded Maps and Operator algebras*. Cambridge University Press, 2002.

[22] D. Pavlovic. Categorical logic of names and abstraction in action calculus. *Mathematical Structures in Computer Science*, 7:619–637, 1997.

[23] P. Selinger. Dagger compact closed categories and completely positive maps. *Electronic Notes in Theoretical Computer Science (To appear)*, 2006.

[24] V. Turaev. *Quantum Invariants of Knots and 3-Manifolds*. de Gruyter, 1994.

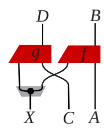
[25] R. F. Werner. All teleportation and dense coding schemes. *Journal of Physics A*, 34:7081–7094, 2001.

[26] W. Wootters and W. Zurek. A single quantum cannot be cloned. *Nature*, 299:802–803, 1982.

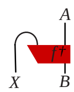
A Proofs

Proof of Prop. 2.6. The result follows from Frobenius identity together with the unit laws and commutativity of the (co)monoid structure. A diagrammatic derivation of this result is in Sec. 2.3.

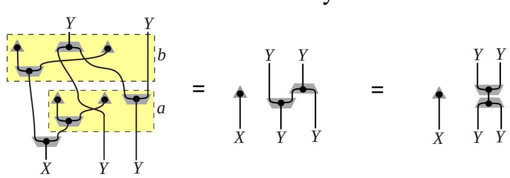
Proof of Prop. 2.9. The fact that $\mathbf{C}_X$ is a category is given. The tensor product on objects is as in $\mathbf{C}$ and on morphisms, $f : X \otimes A \to B$ and $g : X \otimes C \to D$ is:



For $f : X \otimes A \to B$, $f^\dagger$ is:

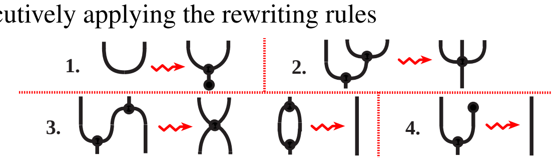


It's straightforward to see that the $\dagger$ endofunctor preserves the symmetric monoidal structure coherently hence, $\mathbf{C}_X$ is a symmetric monoidal $\dagger$-category. Classical structure in $\mathbf{C}_X$ is $\epsilon'_Y := \lambda_Y \circ (\epsilon_X \otimes \epsilon_Y)$ and $\delta'_Y := \lambda_Y \circ (\epsilon_X \otimes \delta_Y)$. With these, it's easy to check that the required identities hold and that the objects in $\mathbf{C}_X$ inherit their classical structure from $\mathbf{C}_c$. We show that Frobenius identity holds as an example:



where in the box a, we have $1_Y \otimes_X \delta'_Y$ and in the box b, we have $\epsilon'^{\dagger}_Y \otimes_X 1_Y$.

Proof (sketch) of Prop. 2.11. The result follows by consecutively applying the rewriting rules



as worked out in [7, §3]. Another proof can be extracted from [18, §1.4.16 & §1.4.37].

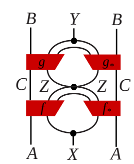
Proof of Lem. 3.2. Ξ_X is the normal form (cf. Prop. 2.11) both of the composite $\Xi_X \circ \Xi_X$ and of completely positive map $(1_X \otimes \eta_X^\dagger \otimes 1_X) \circ (\delta_X \otimes \delta_X)$.

Proof of Prop. 3.6. The result follows by Definition 4.11 and Remark 4.12(c) in [23].

Proof of Lem. 3.3 and Lem. 3.8. Well-definedness of F follows by speciality. Functoriality of F follow by speciality, and from speciality and diagonality of morphisms in $\mathrm{D}_{\Xi,q}(\mathbf{C})$ respectively follow $GF = 1_{\mathbf{C}_{c,q}}$ and $FG = 1_{\mathrm{D}_{\Xi,q}(\mathbf{C})}$. Lem. 3.3 is a special case of Lem. 3.8.

Proof of Prop. 3.4 and Prop. 3.9. We only proof Prop. 3.9 since Prop. 3.4 can be seen as a special case of this. First we proof consider $\mathrm{CP}_{\Xi,q}(\mathbf{C})$. If g is completely positive then so is $\Xi_{(B,Y)} \circ g \circ \Xi_{(A,X)}$ by complete positivity of Ξ_X, and it is diagonal by idempotence of Ξ_X. Conversely, each diagonal completely positive g can be rewritten in the form $\Xi_{(B,Y)} \circ g \circ \Xi_{(A,X)}$. By speciality we have $G(\Xi_{(B,Y)} \circ g \circ \Xi_{(A,X)}) = \delta^\dagger_{(B,Y)} \circ g \circ \delta_{(A,X)}$ so we obtain the desired result for $\mathbf{C}_{\Xi,q}$.

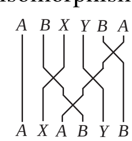
Proof of Thm. 3.5 and Thm. 3.10. **1.** We first check that $\mathbf{C}_{\Xi,q}$ is a symmetric monoidal $\dagger$-category. (i) *Identity:* The identity of an object $(A, X) \in |\mathrm{CP}_{\Xi,q}(\mathbf{C})|$ is $\Xi_{(A,X)}$. We get, $G(\Xi_{(A,X)}) = 1_{(A,X)}$ as the identity for $(A, X) \in |\mathbf{C}_{\Xi,q}|$. (ii) *Composition:* Let $f, g \in \mathbf{C}_{\Xi,q}$ and $f', g' \in \mathrm{CP}_{\Xi,q}(\mathbf{C})$ be such that $G(f') = f$ and $G(g') = g$ respectively. Then, $g \circ f$ is depicted as:



which is of the form $G(h)$ with $h = g' \circ f'$ via idempotence of $\Xi_{(Z,X)}$ and speciality. Since completely positive morphisms are closed under composition and h is diagonal, it follows that $G(h) = g \circ f \in \mathbf{C}_{\Xi,q}$. (iii) *Tensor product:* The tensor product of objects in $\mathbf{C}_{\Xi,q}$ is defined as $(A, X) \boxtimes (B, Y) := (AB, XY)$. For morphisms, let

$$s : AXABYB \xrightarrow{\sim} ABXYBA$$

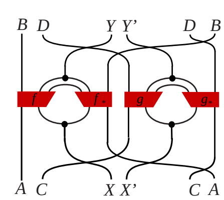
be the following natural isomorphism in $\mathbf{C}$:



If $f, g \in \mathbf{C}_{\Xi,q}$ then, their tensor product is given by

$$f \boxtimes g := s \circ (f \otimes g) \circ s^{-1}$$

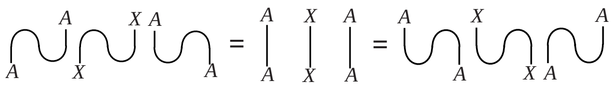
or, graphically



which is also in $\mathbf{C}_{\Xi,q}$. It is routine exercise to check that $\mathbf{C}_{\Xi,q}$ is symmetric monoidal. (iv) *Symmetric monoidal $\dagger$-structure:* Clearly, the $\dagger$-endofunctor preserves the symmetric monoidal structure coherently hence, $\mathbf{C}_{\Xi,q}$ is a symmetric monoidal $\dagger$-category.

2. We show that the objects of $\mathbf{C}_{\Xi,q}$ inherit their quantum structure from $\mathbf{C}_q$. Consider $f := \eta_A \otimes (\delta_{XX} \circ \eta_X) \otimes (\eta_A)_*$

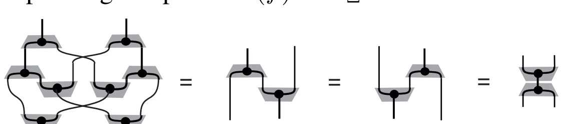
in $\mathrm{CP}_{\Xi,q}(\mathbf{C})$ then, the corresponding morphism in $\mathbf{C}_{\Xi,q}$ is $G(f) = \eta_A \otimes \eta_X \otimes (\eta_A)_*$. We also have $G(f^\dagger) = \eta_A^\dagger \otimes \eta_X^\dagger \otimes (\eta_A)^*$. Now, consider $G((f^\dagger \boxtimes \Xi_{(A,X)}) \circ (\Xi_{(A,X)} \boxtimes f))$; with easy graphical manipulation one sees it is equal to



which coincide with the quantum structure on AXA in $\mathbf{C}_q$ up to symmetry isomorphism.

3. We show that $\mathbf{C}_\Xi$ is a symmetric monoidal †- category. (i,ii) *Identity and composition:* The argument for $\mathbf{C}_\Xi$ is similar to the one for $\mathbf{C}_{\Xi,q}$. (iii) *Tensor product:* The tensor product on objects and morphisms in $\mathbf{C}_\Xi$ coincide with the one in $\mathbf{C}$ as well as the natural isomorphisms from the symmetric monoidal structure. Hence, the category is symmetric monoidal. (iv) *symmetric monoidal †-structure:* Follows directly from the definitions.

4. We show that the objects of $\mathbf{C}_\Xi$ inherit their classical structure from $\mathbf{C}_c$. First, $\Xi_X \circ (\epsilon_X \otimes (\epsilon_X)_*) \circ \Xi_I$ and $\Xi_{XX} \circ (\delta_X \otimes (\delta_X)_*) \circ \Xi_X$ are both in $\mathrm{CP}_\Xi(\mathbf{C})$; their respective images under G are ϵ_X and δ_X. We now need to check that the equation for the classical structure are holding in $\mathbf{C}_\Xi$. We show Frobenius identity: Let $f = \Xi_{XX} \circ (\Xi_X \otimes (\Xi_X)_*) \circ \Xi_{XX} \in \mathrm{CP}_\Xi(\mathbf{C})$. Then, it is a straightforward application of Prop. 2.11 to see that the corresponding morphism $G(f) \in \mathbf{C}_\Xi$ coincides with



thus yielding the Frobenius identity for $X \in |\mathbf{C}_\Xi|$. Speciality, symmetry, unit and counit law all holds by a similar argument and all coincide with the classical (and quantum) structure over X in $\mathbf{C}_c$ as required.

Proof of Prop. 3.14. See [8]§7 Prop. 7.3.

Proof of Prop. 3.15. It is easy to see that Σ is a multiplicative system allowing calculus of fractions in the sense of [12], and we took $L\mathbf{C}$ to be the 'category of fractions' $\mathbf{C}[\Sigma]$. The symmetric monoidal †-structure and quantum structure on the fractions is defined pointwise:

$$\frac{f}{s} \otimes \frac{g}{t} = \frac{f \otimes g}{s \circ t} \qquad \left(\frac{f}{s}\right)^\dagger = \frac{f^\dagger}{s^\dagger} \qquad \text{etc.}$$

The universal property is proven in [12].

Proof of Prop. 4.2. For $g : AA \to BB$ completely positive one verifies that $g_* = \sigma_{B,B} \circ Ff \circ \sigma_{A,A}$. Hence if Ff is completely positive then $(Ff)_* = \sigma_{Y,Y} \circ Ff \circ \sigma_{X,X} = Ff$ by commutativity of the comultiplication so $f = \delta^\dagger \circ Ff \circ \delta = \delta^\dagger \circ (Ff)_* \circ \delta = f_*$. If $f = f_*$ then $Ff = \delta \circ f \circ \delta^\dagger = (f \otimes f) \circ \Xi = (f \otimes f_*) \circ \Xi$ is completely positive by complete positivity of its components.

Proof of Prop. 4.3. From $\epsilon_\mathrm{I} \circ s = \epsilon_\mathrm{I}$ follows $s = 1_\mathrm{I}$.

Proof of Thm. 4.6. In [5] a bicategory of relations is defined to be a locally posetal bicategory, with symmetric monoidal structure, in which every object carries comonoid structure which satisfies Frobenius equation, in which comultiplication δ_X and comultiplicative unit ϵ_X have right (Galois) adjoints, and, in which $(X, \delta_X, \epsilon_X)$ is the only comonoid structure on X with structure morphisms having right (Galois) adjoints.

One verifies that $\subseteq$ defines a partial order on relations of the same type. Moreover, this partial order turns out to be a meet semilattice for $f \wedge g = \delta_Y^\dagger \circ (f \otimes g) \circ \delta_X$. For $f \in \mathbf{C}_\Xi$ and g a relation we set $f \trianglelefteq g \Leftrightarrow f = \delta_Y^\dagger \circ (f \otimes g) \circ \delta_X$. Set

$$f \circ_r g := \bigwedge \left\{ h \in \mathbf{C}_{rel} \;\middle|\; f \circ g \trianglelefteq h \right\}$$

which always exists due to the finiteness assumption. One verifies that $\circ_r$ and $\subseteq$ yield a posetal enriched category $\mathbf{C}_r$ of relations in a tedious but straightforward calculation.

The requirement $\delta \dashv \delta^\dagger$ can be rewritten as $1_X = \delta_X^\dagger \circ (1_X \otimes 1_X) \circ \delta_X$ and $\Xi_X = \delta_{XX}^\dagger \circ (1_{XX} \otimes \Xi_X) \circ \delta_{XX}$ which hold by Prop. 2.11, and the requirement $\epsilon \dashv \epsilon^\dagger$ reduces to $s = \delta_\mathrm{I}^\dagger \circ (s \otimes 1_\mathrm{I}) \circ \delta_\mathrm{I}$ where $s = \epsilon \circ \epsilon^\dagger$ which holds since $\delta_\mathrm{I} = \lambda_\mathrm{I}$ and $1_X = \delta_X^\dagger \circ (1_X \otimes \epsilon_X^\dagger \circ \epsilon_X) \circ \delta_X$ which holds by Prop. 2.11. It remains to be shown that uniqueness of classical structures forces $\delta^\dagger = \delta^\ddagger$ and $\epsilon^\dagger = \epsilon^\ddagger$ whenever $\delta \dashv \delta^\ddagger$ and $\epsilon \dashv \epsilon^\ddagger$. Given that

$$\begin{cases} 1_X = \delta_X^\dagger \circ (1_X \otimes (\delta_X^\ddagger \circ \delta_X)) \circ \delta_X \\ \delta_X \circ \delta_X^\ddagger = \delta_{XX}^\dagger \circ (1_{XX} \otimes (\delta_X \circ \delta_X^\ddagger)) \circ \delta_{XX} \end{cases}$$

pre-composing and post-composing with $\delta_X^\dagger$ respectively, and then applying Prop. 2.11, results in

$$\begin{cases} \delta_X^\dagger = \delta_X^\dagger \circ (1_X \otimes (\delta^\ddagger \circ \delta_X)) \circ \delta_X \\ \delta_X^\ddagger = \delta_X^\dagger \circ (1_X \otimes (\delta^\ddagger \circ \delta_X)) \circ \delta_X \end{cases}$$

which yields the desired result.

Proof of Thm. 4.8. **1.** For $\mathcal{F}$ to be X-doubly stochastic means $(\epsilon_X \otimes \epsilon_Z) \circ (1_X \otimes \mathcal{F}) \circ (\delta_X \circ 1_Y) = \epsilon_X \otimes \epsilon_Y$ which yields $\epsilon_Z \circ (1_X \otimes \mathcal{F}) = (\epsilon_X \otimes \epsilon_Y) \circ \lambda_\mathrm{I}^\dagger$. Hence $\epsilon_Z \circ c(\mathcal{F}, p) = (\epsilon_X \circ p) \bullet \epsilon_Y = \epsilon_Y$. Verification of $c(\mathcal{F}, p) \circ \epsilon_Y^\dagger = \epsilon_Z^\dagger$ proceeds similarly. **2.** Transitivity of majorization follows by the fact that doubly stochastic maps are closed under composition and reflexivity by the fact that identities are doubly stochastic.