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Duality notes

The goal of these notes is to explain the notion of a fully dualizable object in a symmetric monoidal (∞, n)-category. Along the way, we will review dualizability as it appears in various settings, including monoidal categories, adjoint functors, and bicategories. Our approach follows closely the approach of [4], but we also take ideas from [3].

Contents

1 Dualizability in monoidal categories

In this section, we review the notion of a dualizable object in a monoidal category, and we recall some basic facts concerning such objects.

1.1 Monoidal categories

A monoidal category is a category equipped with an additional structure called the monoidal product, which is a structure that generalizes the tensor product in the category of vector spaces. More precisely, we have the following definition.

Definition 1.1. A *monoidal category* $(\mathsf{C}, \otimes, e)$ consists of

- a category C,
- a bifunctor $\otimes : \mathsf{C} \times \mathsf{C} \to \mathsf{C}$, plus a family of isomorphisms

$$\alpha_{x,y,z} : (x \otimes y) \otimes z \xrightarrow{\sim} x \otimes (y \otimes z)$$

called the *associators*, which are natural in x, y, z, and

- a distinguished object e, plus two families of isomorphisms

$$\lambda_x : e \otimes x \xrightarrow{\sim} x \quad \text{and} \quad \rho_x : x \otimes e \xrightarrow{\sim} x$$

called the *left unitors* and *right unitors*, which are natural in x.

Moreover, these natural isomorphisms are required to satisfy certain coherence conditions, which can be stated in terms of commutative diagrams, namely, the *pentagon identity* and the *triangle identities*, which are pictured below.

$$\begin{array}{ccccc} & & (w \otimes x) \otimes (y \otimes z) & & \\ & \overset{\alpha_{w\otimes x,y,x}}{\nearrow} & & \overset{\alpha_{w,x,y\otimes z}}{\searrow} & \\ ((w \otimes x) \otimes y) \otimes z & & & & w \otimes (x \otimes (y \otimes z)) \\ \Big\downarrow {\scriptstyle \alpha_{w,x,y}\otimes\infty_{\ddagger}} & & & & \Big\downarrow {\scriptstyle 1_w \otimes \alpha_{x,y,z}} \\ (w \otimes (x \otimes y)) \otimes z & & \xrightarrow{\qquad \alpha_{w,x\otimes y,z} \qquad} & & w \otimes ((x \otimes y) \otimes z) \end{array}$$

$$\begin{array}{ccc} (x \otimes e) \otimes y & \xrightarrow{\alpha_{x,e,y}} & x \otimes (e \otimes y) \\ & {\scriptstyle \rho_x \otimes 1_y} \searrow & \Big\downarrow {\scriptstyle 1_x \otimes \lambda_y} \\ & & x \otimes y \end{array}$$

Remark 1.2. The existence of the associators ensures that the tensor product $\otimes$ is associative up to natural isomorphism. The existence of the unitors ensures that e serves as a unit for $\otimes$, up to natural isomorphism.

As examples, we have the following monoidal categories.

Example 1.3. The usual tensor product $\otimes$ endows the category Vect_k of k-vector spaces with the structure of a monoidal category, where the monoidal unit is taken to be the ground field k.

Example 1.4. For a commutative ring R, the tensor product $\otimes_R$ endows the category Mod_R of (left) R-modules with the structure of a monoidal category, where the monoidal unit is taken to be the ring R, viewed as a (left) R-module.

Example 1.5. More generally, for a commutative ring R, the category Chain_R of chain complexes of R-modules can be equipped with the structure of a monoidal category by taking the monoidal product to be the *graded* tensor product $\otimes$ and the monoidal unit to be the module R, viewed as a chain complex with non-trivial degree only in degree 0.

Example 1.6. The smash product $\wedge$ endows the category Sp of spectra with a monoidal structure, where the unit can be taken to be the sphere spectrum $\mathbb{S} = \Sigma^\infty S^0$.

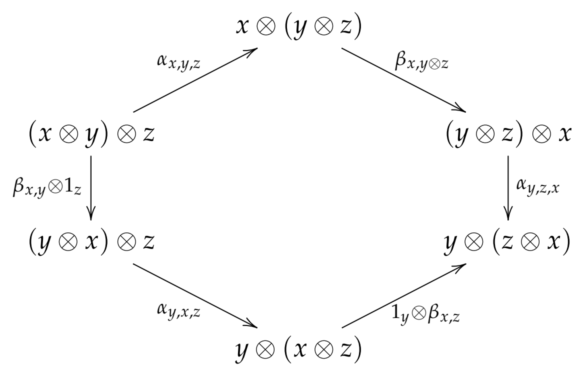
Example 1.7. The disjoint union $\sqcup$ endows the n-dimensional bordism category Bord_n with a monoidal product, and a unit for this product is $\varnothing$, viewed as an $(n-1)$-dimensional manifold.

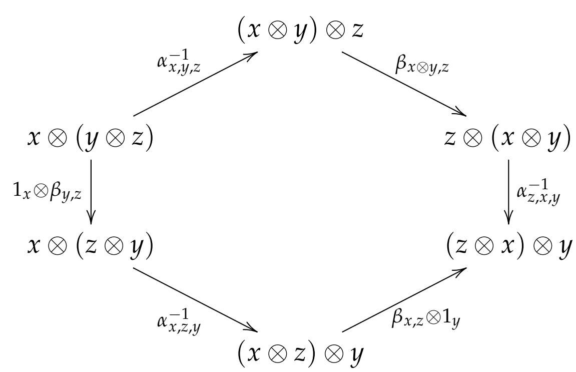
Example 1.8. For a category C, the endomorphism category $\mathrm{End}(\mathsf{C})$ of functors on C can be equipped with a monoidal product given by functor composition $\circ$. In this case, a monoidal unit is the identity functor 1_{C}.

Note that Examples 1.3 to 1.7 come equipped with a family of isomorphisms

$$\beta_{x,y} : x \otimes y \to y \otimes x,$$

which are natural in x and y. If a monoidal category is equipped with such a family of isomorphisms and these isomorphisms play well with the associators, in the sense that certain *hexagon diagrams commute* (pictured below), then such a monoidal category is called a *braided monoidal category*, and the $\beta_{x,y}$ are called the *braiding*.





We remark that the monoidal category $(\mathrm{End}(\mathsf{C}), \circ, 1_{\mathsf{C}})$ has no natural choice of braiding, since this would require natural isomorphisms $\beta_{F,G} : FG \Rightarrow GF$.

In addition, note that the braidings of the braided monoidal categories in Examples 1.3 through 1.7 are involutions in the sense that $\beta_{x,y} \circ \beta_{y,x} = 1_{y \otimes x}$. If the braidings in a monoidal category satisfy this property, then we call the braided monoidal category a *symmetric monoidal category*.

1.2 Dualizable objects in monoidal categories

Definition 1.9. Let $(\mathsf{C}, \otimes, e)$ be a monoidal category with associator α, and let x and y be objects in C. We say x *has a left dual* y (or y *has a right dual* x) if there is

- a morphism $\mathrm{ev} : y \otimes x \to e$ (called *evaluation*), and
- a morphism $\mathrm{coev} : e \to x \otimes y$ (called *coevaluation*)

such that the compositions

$$x \xrightarrow{\lambda_x^{-1}} e \otimes x \xrightarrow{\mathrm{coev} \otimes 1_x} (x \otimes y) \otimes x \xrightarrow{\alpha_{x,y,x}} x \otimes (y \otimes x) \xrightarrow{1_x \otimes \mathrm{ev}} x \otimes e \xrightarrow{\rho_x} x$$

and

$$y \xrightarrow{\rho_y^{-1}} y \otimes e \xrightarrow{1_y \otimes \mathrm{coev}} y \otimes (x \otimes y) \xrightarrow{\alpha_{y,x,y}^{-1}} (y \otimes x) \otimes y \xrightarrow{\mathrm{ev} \otimes 1_y} e \otimes y \xrightarrow{\lambda_y} y$$

are both the identity. If x has a left dual, we say that x is *left dualizable*.

Example 1.10. We can ask which objects in the monoidal category $(\mathsf{Vect}_k, \otimes, k)$ are left dualizable. The claim is that the set of left dualizable objects is the set of finite-dimensional vector spaces. Indeed, if V is a finite-dimensional vector space, then the space $W = \mathrm{Hom}(V, k)$ is a left dual for V, since we may define evaluation and coevaluation in the following manner. We let ev denote the linear map induced by the bilinear map

$$\begin{aligned} W \times V &\longrightarrow k \\ (f, v) &\longmapsto f(v) \end{aligned}$$

and, upon choosing a basis $\{v_i\}$ for V, we let $\mathrm{coev} : k \to V \otimes W$ denote the linear map determined by the assignment

$$1 \mapsto \sum_i v_i \otimes v_i^*$$

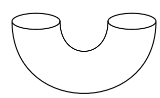
where $\{v_i^*\}$ denotes the basis dual to $\{v_i\}$. One can show that the necessary compositions are the identity with these definitions. On the other hand, it is not too difficult to show that if V has a left dual W, then V must be finite-dimensional, since the image of the coevaluation map will be spanned by a single vector, which is a finite sum of simple tensors, and hence factoring the identity on V by the composite will show that V is spanned by a finite set of vectors.

Example 1.11. More generally, one can show that in the monoidal category $(\mathsf{Mod}_R, \otimes, R)$, the dualizable objects are the finitely-generated projective modules.

Example 1.12. Even more generally, an object in Chain_R is dualizable if and only if it is a bounded chain complex of finitely-generated projective modules.

Example 1.13. In $(\mathsf{Bord}_n, \sqcup, \varnothing)$, every object M is dualizable. If $\overline{M}$ denotes M equipped with the opposite orientation, we let the evaluation map be the bordism $M \times [0,1]$ viewed as a bordism from $\overline{M} \sqcup M$ into $\varnothing$ as below.

$$\overline{M} \sqcup M \xrightarrow{\text{ev}} \varnothing$$



The coevaluation map is defined similarly. Checking that the necessary composites are the identity amounts to checking that a "snake-like tube" is diffeomorphic to the identity bordism on M. We encourage the reader to work out these compositions, if s/he has never done so before.

Example 1.14. In the monoidal category $(\mathrm{End}(\mathsf{C}), \circ, 1_{\mathsf{C}})$, an object (that is, a functor) has a left dual if and only if it has a left adjoint (see Section 2 for more information about adjoint functors). The evaluation and the co-evaluation maps are given by the co-unit and unit, respectively.

Example 1.15. In the homotopy category of spectra with the smash product, a spectrum is dualizable if and only if it is a finite spectrum, that is, a spectrum presented by sequences of finite CW complexes.

A left dualizable object gives rise to a certain adjunction, as we show now.

Lemma 1.16. *Suppose x is left dualizable with left dual y. Let $F(-) = - \otimes x$ and $G(-) = - \otimes y$ be the functors which tensor on the right by x and y respectively. Then F is a left adjoint for G, that is, there is a family of isomorphisms of the form*

$$\phi_{a,b} : \mathsf{C}(a \otimes x, b) \xrightarrow{\sim} \mathsf{C}(a, b \otimes y)$$

which is natural in a and b.

Proof. Let $\phi_{a,b}$ denote the map described by

$$(a \otimes x \xrightarrow{f} b) \mapsto (a \xrightarrow{\rho_a^{-1}} a \otimes e \xrightarrow{\text{coev}} a \otimes (x \otimes y) \xrightarrow{\alpha^{-1}} (a \otimes x) \otimes y \xrightarrow{f \otimes 1_y} b \otimes y).$$

Then we claim that it is easy to construct an inverse for $\phi_{a,b}$ and that $\phi_{a,b}$ can easily be shown to be natural in a and b. $\square$

As a consequence, the Yoneda Lemma implies that left duals are unique up to isomorphism. Let us first recall the Yoneda embedding.

The Yoneda embedding is an embedding of any category C into its category of presheaves $[\mathsf{C}^{\mathrm{op}}, \mathsf{Set}]$. More precisely, the Yoneda embedding is the functor $\mathsf{C} \to [\mathsf{C}^{\mathrm{op}}, \mathsf{Set}]$ described on objects by $y \mapsto \mathsf{C}(-, y)$. As a consequence of the Yoneda Lemma, this functor is fully faithful, meaning that for each pair of objects x, y, the map on morphisms $\mathsf{C}(y, y') \to \mathsf{Nat}(\mathsf{C}(-, y), \mathsf{C}(-, y'))$ is an isomorphism of sets. In particular, this implies that if there is a natural isomorphism from $\mathsf{C}(-, y)$ into $\mathsf{C}(-, y')$, then the objects y and y' are isomorphic.

Lemma 1.17. *Let y and y' be two left duals for x. Then y and y' are isomorphic objects.*

Proof. The previous lemma implies that we get natural isomorphisms of the form

$$\mathsf{C}(a \otimes x, b) \simeq \mathsf{C}(a, b \otimes y) \quad \text{and} \quad \mathsf{C}(a \otimes x, b) \simeq \mathsf{C}(a, b \otimes y').$$

Taking b to be the monoidal unit e, we get a natural isomorphism of the form

$$\mathsf{C}(a, y) \simeq \mathsf{C}(a, y'),$$

which, by the Yoneda Lemma, implies that y and y' are isomorphic, as desired. □

We finally remark that in the setting of symmetric monoidal categories, it is not too difficult to show that any left dual is also a right dual. Indeed, if y is a left dual for x, then we may define new (right) evaluation and (right) co-evaluation maps by

$$\mathrm{ev}^r = \mathrm{ev} \circ \beta_{x \otimes y} : x \otimes y \to e$$
$$\mathrm{coev}^r = \beta_{x \otimes y} \circ \mathrm{coev} : e \to y \otimes x.$$

Hence we are justified in speaking about a *dual* $x^\vee$ to an object x. By the previous lemma, this object is unique up to isomorphism, and so we can speak about *the* dual to an object.

2 Adjoint functors

In this section, we review the notion of a pair of adjoint functors. We refer the reader to [1] for a more complete discussion.

We begin with a standard definition in terms of hom-sets.

Definition 2.1. Let C and D be categories and let $F : \mathsf{C} \leftrightarrows \mathsf{D} : G$ be functors. We say that *G has a left adjoint F* (or *F has a right adjoint G*) if there is a family of isomorphisms of the form

$$\phi_{x,y} : \mathsf{D}(Fx, y) \xrightarrow{\sim} \mathsf{C}(x, Gy) \tag{1}$$

which is natural in x (in C) and y (in D).

Example 2.2. Let $\mathsf{C} = \mathsf{D} = \mathsf{Vect}_k$ the category of vectors spaces over a field k. Let F be the functor $- \otimes W$ for a fixed space W. Let G be the hom functor $\hom_k(W, -)$. Then we have the familiar adjunction

$$\hom_k(V \otimes W, U) \simeq \hom_k(V, \hom_k(W, U))$$

obtained by thinking of a bilinear map $V \otimes W \to U$ as a map $V \to \hom_k(W, U)$.

We now interpret this natural family of isomorphisms in a different manner. In particular, we will look at what this isomorphism gives for a fixed x and for a fixed y. We will use the Yoneda Lemma, which we recall now.

Lemma 2.3 (Yoneda)**.** *Let D be a category, d an object of D, and $K : \mathsf{D} \to \mathsf{Set}$ a functor. Then the following map is a bijection of sets*

$$\begin{aligned} \mathsf{Nat}(\mathsf{D}(d, -), K) &\to K(d) \\ \phi &\mapsto \phi_d(1_d). \end{aligned}$$

We use the Yoneda Lemma to interpret the family of isomorphisms in (1) in two different manners.

- For a fixed x, we get an isomorphism of functors

$$\phi_x : \mathsf{D}(Fx, -) \xrightarrow{\sim} \mathsf{C}(x, G-).$$

  Note that the left-hand functor is a co-representable functor with co-representing object Fx. By Yoneda's Lemma, the isomorphism ϕ_x corresponds to a unique element $\eta_x := \phi_x(1_{Fx})$ of $\mathsf{C}(x, GFx)$. One can show that the assignment $x \mapsto \eta_x$ defines a natural transformation from 1_{C} to GF.

- On the other hand, for a fixed y, one can similarly obtain a unique element ϵ_y of $\mathsf{D}(FGy, y)$. The assignment $y \mapsto \epsilon_y$ defines a natural transformation from FG into 1_{D}.

The natural transformation η is called the *unit* of the adjunction and ϵ is called the *co-unit*. It is not difficult to show that the definitions of η and ϵ imply that the compositions

$$F \xrightarrow{F\eta} FGF \xrightarrow{\epsilon F} F$$

and

$$G \xrightarrow{\eta G} GFG \xrightarrow{G\epsilon} G$$

are both the identity. These are called the *triangle identities*.

Conversely, given two natural transformations $\eta : 1_{\mathsf{C}} \Rightarrow GF$ and $\epsilon : FG \Rightarrow 1_{\mathsf{D}}$ satisfying the triangle identities, one might expect that we can construct a family of isomorphisms of the form

$$\phi_{x,y} : \mathsf{D}(Fx, y) \xrightarrow{\sim} \mathsf{C}(x, Gy)$$

which is natural in x (in C) and y (in D). Indeed, this is the case, as the following theorem asserts.

Theorem 2.4. *Let C and D be categories and let $F : \mathsf{C} \leftrightarrows \mathsf{D} : G$ be functors. If $\eta : 1_{\mathsf{C}} \Rightarrow GF$ and $\epsilon : FG \Rightarrow 1_{\mathsf{D}}$ are natural transformations satisfying the triangle identities, then the functor G has a left adjoint F.*

Proof. From η and ϵ we define natural maps

$$\begin{aligned} \phi : \mathsf{D}(Fx, y) &\to \mathsf{C}(x, Gy) \\ f &\mapsto Gf \circ \eta_x \end{aligned}$$

and

$$\begin{aligned} \psi : \mathsf{C}(x, Gy) &\to \mathsf{D}(Fx, y) \\ g &\mapsto \epsilon_y \circ Fg. \end{aligned}$$

Then ϕ and ψ are invertible. Indeed, we have

$$\phi\psi(g) = G(\psi g) \circ \eta_x = G\epsilon_y \circ GFg \circ \eta_x = G\epsilon_y \circ \eta_{Gy} \circ g = g$$

where the third equality follows from naturality of η and the last equality from the triangle identities. Similarly we can prove $\psi\phi(f) = f$. ☐

3 Dualizability in bicategories

We now discuss a setting of dualizability which is more general and which contains both of the examples above (duals in monoidal categories and adjoint functors) as particular examples. This setting is bicategories.

3.1 Bicategories

The notion of a bicategory extends the notion of a category in the sense that composition of morphisms in a bicategory is not required to be strictly associative, but rather associative up to higher isomorphism.

A prototypical example is the category Cat of small categories. Here the objects are categories, the morphisms are functors, and the 2-morphisms are natural transformations. Observe that for two categories C and D, the hom-set $\mathsf{Cat}(\mathsf{C}, \mathsf{D})$ is just the functor category $\mathsf{Fun}(\mathsf{C}, \mathsf{D})$. In particular $\mathsf{Cat}(\mathsf{C}, \mathsf{D})$ is an ordinary category.

Another structure which we would like to be a bicategory is the following.

Example 3.1. Consider $\Pi_{\leq 2}X$ for a topological space X. The objects are points of X, the 1-morphisms are paths in X, and the 2-morphisms are homotopy classes of homotopies between paths. The composition of paths is concatenation. But then in this structure, composition of paths is not *strictly associative*, it is only associative up to homotopy. Observe that

$$\Pi_{\leq 2}X(p,q) = \Pi_{\leq 1}P(X;p,q)$$

where $P(X;p,q)$ denotes the paths in X starting at p and ending at q.

Thus we should not ask that composition of 1-morphisms is strictly associative, but only associative up to a natural 2-isomoprhism. Somewhat more precisely, a *bicategory* is a category *weakly enriched* over the monoidal category Cat, meaning that the hom-objects of a bicategory form categories, but the associativity and identity axioms for an enriched category hold only up to coherent natural isomorphisms (called *associators* and *unitors* respectively). This means that a bicategory B has the following data

- A set of *objects* $x, y, z, \ldots$
- For each pair of objects x and y, a category $\mathsf{B}(x,y)$, whose objects are called 1-*morphisms* and whose morphisms are called 2-*morphisms*.
- For each triple of objects x, y, z in B, a *composition functor*

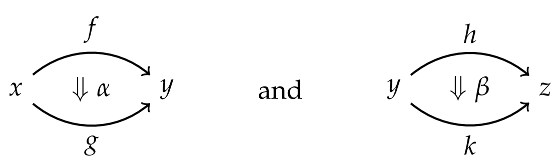
  $$c_{x,y,z} : \mathsf{B}(y,z) \times \mathsf{B}(x,y) \to \mathsf{B}(x,z)$$

  which is associative up to natural isomorphism (called the *associator* α).
- For each object x, an *identity functor* $1_x : \{*\} \to B(x,x)$ which is a left and right unit for composition $c_{x,y,z}$ up to natural isomorphisms (called the *left unitor* λ and *right unitor* ρ repsectively).

Moreover, all of this data is required to satisfy certain coherence conditions analogous to those of a monoidal category.

Remark 3.2. Note that in a bicategory, the 2-morphisms come equipped with two different types of compositions, namely, the usual composition inherited from the category $B(x,y)$ which we call *vertical composition* and the composition coming from the functor $c_{x,y,z}$, which we call *horizontal composition* and which we denote by $*$. The names of these different compositions are motivated by the following schematics.

Suppose that we have two 2-morphisms α and β pictured as



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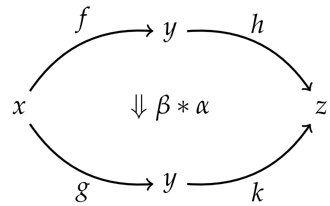
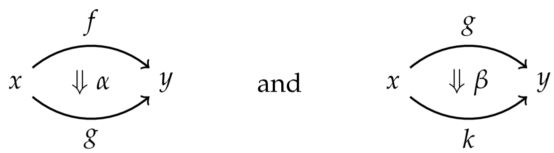
Then the horizontal composition allows us to obtain a new 2-morphism $\beta * \alpha$ from $h \circ f$ to $k \circ g$, pictured as

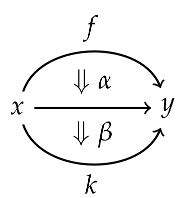


The pictures suggest that we are indeed composing horizontally, and hence the name horizontal composition.

On the other hand, suppose that we have 2-morphisms with compatible source and target



Then we should be able to compose these 2-morphisms vertically as pictured below



Example 3.3. Any monoidal category $(X, \otimes, e)$ gives rise to a bicategory $\mathsf{B}(X, \otimes, e)$, often referred to as the *delooping* of X, defined in the following way. The bicategory $\mathsf{B}(X, \otimes, e)$ has only a single object $*$. The 1-morphisms are defined to be the objects of X, while the 2-morphisms are the morphisms of X. The composition functor is defined by the monoidal structure $\otimes$. The resulting bicategory $\mathsf{B}(X, \otimes, e)$ is sometimes called the *delooping* of $(X, \otimes, e)$.

Example 3.4. We can form the bicategory of categories, functors, and natural transformations. The associators and unitors in this case can be taken to be the identity.

Example 3.5. (The bicategory of algebras) Fix a ground ring R. The objects of this bicategory are the R-algebras, the 1-morphisms are the bimodules, and the 2-morphisms are given by maps of bimodules. The vertical composition in this bicategory is the composition of maps of bimodules. The horizontal composition is given by the tensor product, i.e. given an (A, B)-bimodule M and a (B, C)-bimodule N, we have

$$M * N := M \otimes_B N$$

viewed as an (A, C)-bimodule. The identity bimodule of an algebra A is simply A viewed as an (A, A)-bimodule.

3.2 Dualizable 1-morphisms in bicategories

The example of delooping a monoidal category suggests that we should extend the notion of dualizable to 1-morphisms in a bicategory. We can do so in the following manner.

Let B be a bicategory with associator α, and let $f : x \to y$ and $g : y \to x$ be 1-morphisms. We say that f *has a left dual* g (or g *has a right dual* f) if there is

- a 2-morphism $\mathrm{ev} : g \circ f \to 1_x$ (called *evaluation*), and
- a 2-morphism $\mathrm{coev} : 1_y \to f \circ g$ (called *coevaluation*)

such that the compositions

$$f \xrightarrow{\lambda_f^{-1}} 1_y \circ f \xrightarrow{\mathrm{coev} 1_f} (f \circ g) \circ f \xrightarrow{\alpha_{f,g,f}} f \circ (g \circ f) \xrightarrow{1_f \mathrm{ev}} f \circ 1_x \xrightarrow{\rho_f} f$$

and

$$g \xrightarrow{\rho_g^{-1}} g \circ 1_y \xrightarrow{1_g \mathrm{coev}} g \circ (f \circ g) \xrightarrow{\alpha_{g,f,g}^{-1}} (g \circ f) \circ g \xrightarrow{\mathrm{ev} 1_g} 1_x \circ g \xrightarrow{\lambda_g} g$$

are both the identity. If f has a left dual, we say that f is *left-dualizable*. The notion of a *right dualizable* 1-morphism is formulated in an analogous manner.

Example 3.6. In the delooping $\mathsf{B}(X, \otimes, e)$ of a monoidal category $(X, \otimes, e)$, the left dualizable 1-morphisms coincide with the left dualizable objects of $(X, \otimes, e)$.

Example 3.7. We may view the strict 2-category Cat of categories, functors, and natural transformations as a bicategory. In this bicategory, a functor $F : \mathsf{C} \to \mathsf{D}$ has a left dual if and only if it has a left adjoint $G : \mathsf{D} \to \mathsf{C}$, and the 2-morphisms $\mathrm{ev} : GF \to 1_{\mathsf{C}}$ and $\mathrm{coev} : 1_{\mathsf{D}} \to FG$ are the counit and unit respectively.

4 Dualizability in (∞, n)-categories

We conclude by exploring dualizability in higher categories. We will discuss heuristically what we mean by an (∞, n)-category through sketching a definition and some examples. Following [4], we use the notion of dualizability in bicategories to introduce a notion of dualizability for k-morphisms in an (∞, n)-category. Finally, we will discuss what it means for an object in a symmetric monoidal (∞, n)-category to be fully dualizable.

4.1 (∞, n)-categories

Roughly, an ∞-category is an extension of a category where we have objects, 1-morphisms, 2-morphisms, ad infinitum. Moreover, just as we saw with bicategories that we have 2 types of composition for 2-morphisms, in an ∞-category, we will have k types of composition of k-morphisms.

Example 4.1. The ∞-groupoid of a topological space X, denoted $\Pi_{\leq\infty}X$ has

- Objects: points of X
- 1-morphisms: paths in X
- 2-morphisms: homotopies of paths
- 3-morphism: homotopies of homotopies

$\vdots$

This is called a groupoid because all morphisms are invertible.

It is a requirement of higher categories that the category of ∞-groupoids should come from topological spaces as $\Pi_{\leq\infty}X$, or at least this should be morally true. This is called the *homotopy hypothesis*.

By an (∞, n)-category, we mean an ∞-category in which all k-morphisms for $k > n$ are invertible. There are various more precise formulations of this notion (see [2,4] for formulations involving Segal spaces), but we will content ourselves with this "loose" formulation for now.

Just as in a bicategory C, for two objects $a, b \in \mathsf{C}$, we can form the hom-category $\mathsf{C}(a, b)$. However, in an ∞-category D, we should be able to form an ∞-category $\mathsf{D}(a, b)$ for any two objects a and b of D (and more generally for any two k-morphisms a and b). In particular, if we have an $(\infty, 1)$-category D and two objects a, b, then $\mathsf{D}(a, b)$ should be an $(\infty, 0)$-category. But an $(\infty, 0)$-category is one in which all morphisms are invertible, and so it is an ∞-groupoid. Therefore, $\mathsf{D}(a, b)$ is, as we noted earlier, the same thing as a topological space. Thus there is a close connection between higher categories and categories enriched in topological spaces.

Example 4.2. The ∞-groupoid $\Pi_{\leq\infty}X$ is an $(\infty, 0)$-category.

Example 4.3. The (∞, n)-category Bord_n has

- Objects: 0-manifolds with orientation (oriented points)
- 1-morphisms: oriented 1-bordisms (1-manifolds with boundary)
- 2-morphisms: oriented bordisms of 1-bordisms (2-manifolds with corners)

⋮

- n-morphisms: oriented bordisms of ... of bordisms (n-manifolds with corners)
- $(n+1)$-morphisms: diffeomorphisms of n-manifolds
- $(n+2)$-morphisms: isotopies of diffeommorphisms of n-manifolds

⋮

Example 4.4. Any category C can be regarded as an $(\infty, 1)$-category. Given objects $a, b \in \mathsf{C}$, we regard $\mathsf{C}(a, b)$ as a topological space with the discrete topology. Equivalently, we could add in higher morphisms by only adding in higher identity morphisms.

Finally, to any higher category C we can associate a regular category, called the *homotopy category* and denoted as $h\mathsf{C}$. The objects of $h\mathsf{C}$ are the objects of C, and the morphisms are the 1-morphisms of C up to isomorphism.

Example 4.5. The homotopy category of $\Pi_{\leq\infty}X$ is $\Pi_{\leq 1}X$.

4.2 Dualizability in (∞, n)-categories

To introduce a notion of dualizability for k-morphisms in an (∞, n)-category, we restrict ourselves to a certain bicategory at the k-th level, and ask whether the 1-morphisms in this bicategory are dualizable. More precisely, we do the following.

Definition 4.6. Let C be an (∞, n)-category. For $1 \leq k \leq n-1$, the *k-th level homotopy bicategory*, denoted $\mathsf{h}^k\mathsf{C}$, has

- Objects: $(k-1)$-morphisms of C
- 1-morphisms: k-morphisms of C
- 2-morphisms: isomorphism classes of $(k+1)$-morphisms of C.

We say that a k-morphism f *has a left dual* g if f has a left dual g when these are regarded as 1-morphisms in the k-th level homotopy bicategory $\mathsf{h}^k\mathsf{C}$.

What does this mean in concrete terms? A k-morphism $f : a \to b$ in C is left dualizable if there is a k-morphism $g : b \to a$ so that we have $(k+1)$-morphisms $\mathrm{ev} : g \circ f \to 1$ and $\mathrm{coev} : 1 \to f \circ g$ such that the usual composites are related to the identity $(k+1)$-morphisms by an invertible $(k+2)$-morphisms.

To proceed we need to give a description of what a symmetric monoidal (∞, n)-category is. We offer only one approach, and note that there are at least three different approaches in the literature [4].

Recall that a monoidal category is the same thing as a bicategory with a single object. Said in another way, a monoidal category may be delooped to get a bicategory, where the new "higher" composition that is introduced can be taken to be the monoidal product. This process demonstrates that we should think of a monoidal structure on a category C not merely as a type of product functor, but rather as a way of endowing the delooped category BC with a higher composition.

If a bicategory with a single object in addition has only one 1-morphism f, then the Eckmann-Hilton argument shows that the hom-set $\mathrm{Hom}(f, f)$ carries the structure of a commutative monoid. In other words, a commutative monoid may be delooped twice to form a bicategory, where the composition structure is taken to be the monoidal structure from the monoid.

Similarly, one can show that a 3-category C with a single object and single 1-morphism is the same as a braided monoidal category. To see this, note that if pt denotes the single object, then we have that $\mathsf{C}(\mathrm{pt}, \mathrm{pt})$ is a bicategory. Then composition yields a weak 2-functor

$$* : \mathsf{C}(\mathrm{pt}, \mathrm{pt}) \times \mathsf{C}(\mathrm{pt}, \mathrm{pt}) \to \mathsf{C}(\mathrm{pt}, \mathrm{pt})$$

So if f, g, h, k are 2-morphisms of C, i.e. morphisms in $\mathsf{C}(\mathrm{pt}, \mathrm{pt})$, then we have an isomorphism

$$(f \circ g) * (h \circ k) \simeq (f * h) \circ (g * k)$$

and then the Eckmann-Hilton argument yields for us a braiding (just replace $=$ with $\simeq$ in the proof of Eckmann-Hilton).

Similarly, it can be shown that a 4-category with a single object, 1-morphism, and 2-morphism is a symmetric monoidal category. So it seems that given some n-category, the more it can be delooped the more symmetric is the composition in the higher morphisms.

Therefore, in the ∞-categorical context, a symmetric monoidal ∞-category should be one which we can deloop any number of times and still get out an ∞-category. More precisely, we have the following definition.

Definition 4.7. An (∞, n)-category C is *symmetric monoidal* if for any $k \geq 0$, there is an $(\infty, n+k)$-category C_k with the first k layers trivial (that is, consisting of a single object, a single 1-morphism, ..., and a single $(k-1)$-morphism) so that C is equivalent to the hom-category $\mathrm{Hom}_{\mathsf{C}_k}(*, *)$, where $*$ denotes the unique $(k-1)$-morphism.

Example 4.8. We now argue that Bord_n carries a symmetric monoidal structure. First, suppose that we deloop Bord_n once, so that we have an $(\infty, n+1)$ category with a single object, and the layers of Bord_n have been shifted up and make up the rest of the morphisms in the category. We need a way to compose 1-morphisms (that is, objects of Bord_n), so we take this to be the disjoint union. Similarly, we need a way to compose k-morphisms (that is, $(k-1)$-morphisms of Bord_n), so we take this to be the disjoint union as well. In this way, we get an $(\infty, n+1)$-category which is the delooped version of Bord_n. In a similar fashion, we may deloop Bord_n k times, and endow this new set of data with the structure of an $(\infty, n+k)$-category by taking composition to be disjoint union. Therefore, Bord_n is a symmetric monoidal (∞, n)-category.

Definition 4.9. Let C be a symmetric monoidal (∞, n)-category. Because C is symmetric monoidal, the homotopy category is an ordinary symmetric monoidal category. We say that an object x in C *has a left dual* y if x has a left dual y when regarded as objects in the ordinary symmetric monoidal category hC.

Definition 4.10. A symmetric monoidal (∞, n)-category is *fully dualizable* if all k morphisms are left dualizable for $0 \leq k \leq n-1$ (in the case $k = 0$, we mean that the objects of C are left dualizable).

Example 4.11. We finish by arguing that Bord_n is a fully dualizable (∞, n)-category. Recall that each object in the ordinary category Bord_n is left dualizable, and essentially the same argument shows that every k-morphism in Bord_n is left dualizable: I take a bordism, multiply it by the unit interval I, and splice and bend this new bordism to get a snake!

References

[1] S. Mac Lane, *Categories for the Working Mathematician*, Second, Springer, New York, 1998.

[2] J. Lurie, $(\infty, 2)$-*Categories and the Goodwillie Calculus I* (2009), available at 0905.0462v2. preprint.

[3] ———, *On the classification of topological field theories*, Current Developments in Mathematics **2008** (2009), 129–280, available at 0905.0465.

[4] C. Schommer-Pries, *Dualizability in low-dimensional higher category theory*, Contemporary mathematics: Topology and field theories, 2014.