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The Homotopy Hypothesis

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Abstract

The goal of this thesis is to give an introduction of the main concepts about model categories and their homotopy theory. There will also be a brief discussion about left Bousfield localisations. Furthermore we will establish a Quillen equivalence between the model category on 2-truncated simplicial sets and the model category on bi-groupoids with strict 2-functors. Later we discuss the similarities between the homotopy theory of the 2-truncations and the homotopy theory of bi-groupoids with weak 2-functors. Before giving this result we will establish two additional Quillen equivalences for the case of 0 and 1-truncated simplicial sets respectively and also a more restrictive one for the case of 2-truncations.

'Not all those who wander are lost.'

-J.R.R. Tolkien, The Fellowship of the Ring

'En memoria da miu tat Theodosi.'

Introduction

First we should deal with the question

‘*What is abstract homotopy theory?*’

The concept of homotopy is most likely known from topology or algebraic topology, for those familiar with these areas. The idea of a homotopy, sometimes also referred to as a deformation, seems to appear in a lot of different contexts. It aims to give a setting, abstract enough, allowing comparisons between certain structures. It is often very useful to generalise something in a more abstract concept in order to be able to give a proof of certain desired properties, it may well happen that the proof becomes easier in such generality. This theory seems to arise in a lot of mathematics, an example would for instance be chain homotopy, so the main motivation was to generalise such concepts.

Back to the question about abstract homotopy theory. The main idea is to start with some setting where one has a notion of equivalence, usually it is some kind of weak equivalence. That is, one usually starts with a category $\mathscr{C}$ where one chooses some class of morphisms $\mathcal{W} \subseteq \mathrm{mor}(\mathscr{C})$ ($\mathcal{W}$ for weak equivalence), the next step is then to localise this category $\mathscr{C}$ with respect to this class of morphisms. The result will be a new category, which we will refer to as the homotooy category of $\mathscr{C}$, denoted by $\mathrm{Ho}(\mathscr{C})$.

There are several different approaches to describe this kind of process - or better theory - we mentioned above. One of the more powerful ones, and also the one we will use throughout this thesis, will be the concept of model categories. A model category is asked to have all limits and colimits and comes equipped with three classes of morphisms called weak equivalences, fibrations and cofibrations respectively. These classes of morphisms have to satisfy some rules or axioms. This kind of categories are particularly nice for the idea of creating a homotopy theory in the sense described above i.e. the homotopy category of a model category. All objects in this homotopy category will be fibrant and cofibrant and furthermore a special version of Whitehead's theorem will hold, namely that the weak equivalences become invertible by becoming homotopy equivalences. In some sense we turn weak equivalences into isomorphisms and since we are working with model categories it can also be shown that every isomorphism in the homotopy category actually comes from such a weak equivalence.

A classical example would be the ‘classical homotopy theory for topological spaces’. In this particular case one deals with the category of topological spaces where objects are topological spaces and morphisms are continuous maps between them. The homotopies between continuous maps then correspond to the usual concept of homotopy which should be known from topology. This classic theory is also closely related to algebraic topology. Another very important example is the one of simplicial sets, both will be discussed in this thesis.

Next it is time to address another important question.

‘*What is the homotopy hypothesis?*’

First of all, there should be a notion making it possible to compare different homotopy theories. Indeed such a tool exists, we say that two of these theories are equivalent if the respective homotopy categories are equivalent as categories and the underlying categories form a Quillen pair. Actually we can also set this directly in the context of model categories, though we may not require that these model categories are equivalent as categories, this notion would simply be too strong. Instead we introduce the concept of Quillen equivalence, this notion will be weak enough for the setting of model categories and equivalent to the above idea that the homotopy categories should be an equivalence of categories.

Time to answer the above question. The homotopy hypothesis is the assertion, that ∞-groupoids are equivalent to topological spaces up to weak homotopy equivalences (these are the class of weak equivalences in **Top**) i.e. a Quillen equivalence. There is also a stronger statement, which sometimes also falls into this notion of 'the homotopy hypothesis', namely that n-groupoids are equivalent to homotopy n-types. The homotopy theory of simplicial sets is modeling ∞-groupoids (i.e. they are Kan complexes). The homotopy hypothesis tells us that this is actually Quillen equivalent to topological spaces.

The present work will consist of three main parts. In the first part, we will introduce a lot of basic notions and machinery needed for the above discussed theory. Starting with some set theoretical preliminaries, we introduce the notion of a model category and discuss several types of such structures. Next we introduce the concept of homotopy theory for model categories to be able to define the so called homotopy category of a model category. This leads to the definition and discussion of Quillen functors leading us towards the definition of a Quillen equivalence between model categories. For the final stage of this first part, we introduce the concept of Bousfield localisations, where a Bousfield localisation is a new model structure on the underlying category in which we add more weak equivalences than we had before but keep the same cofibrations.

In the second part we apply most of the theory defined in the first part. Especially we discuss two particular cases, namely the categories **Top** of topological spaces and **sSet** of simplicial sets. We will show that they have some very nice model category structures and therefore may be Bousfield localised with respect to any class of morphisms from the respective category. These first two parts are a collection of known results, provided here in order to justify the statements and theories given in the last part.

The last part is really the core of this thesis. At the beginning we state the 'classical' homotopy hypothesis establishing a Quillen equivalence between $\mathbf{sSet}_Q$ and $\mathbf{Top}_Q$, which are the above categories equipped with the nice model category structure developed in the second part. We then proceed to Bousfield localise these categories with respect to a special morphism, giving us the definition of a category of n-truncations for $\mathbf{Top}_Q$ and $\mathbf{sSet}_Q$ respectively.

With this new definitions we proof some different version of the homotopy hypothesis, namely that these truncations also form Quillen equivalences. It will allow us to connect 0-groupoids, 1-groupoids and 2-groupoids, as given by the homotopy hypothesis, to the more classical notions.

In the final section of this thesis, we introduce the concept of a weak **n-Grpd**, and show that at least for the cases $n = 0, 1$, there is a Quillen equivalence between n-truncated simplicial sets and n-groupoids.

For $n = 2$ we need to consider bi-groupoids with strict 2-functors rather than weak 2-functors for this to work, the reason being that it would not be a model category otherwise. In the end though, we will still relate the homotopy theories of 2-truncations with the one of bi-groupoids with weak functors, just not in the form of a Quillen equivalence.

Conventions and Notations

We generally assume the ZFC axiomatic from set theory.

When we talk about the natural numbers $\mathbb{N}$, we include 0.

If not explicitly stated otherwise, a category will usually denote a locally small category. A category $\mathscr{C}$ is said to be locally small, if for any objects A, B in $\mathscr{C}$, $\mathscr{C}(A, B)$ is a proper set rather than a class.

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Part I
Theory

As the title suggests, this first part of the present thesis, contains a lot of theory and fundamental concepts about model categories and homotopy theory.

The first section deals with the foundations. It will introduce helpful concepts like transfinite composition, this may be thought of as replacing induction arguments with arguments using Zorn's lemma. After that we will introduce a concept called the small object argument, this argument is particularly useful when one deals with cofibrantly generated model categories. In practice a lot of important model categories are indeed cofibrantly generated. We also deal with a concept of compactness, which is introduced in [PHir03] and is needed in order to deal with cellular model categories.

The second section introduces the definition of a model category and discusses some properties about them. Further, this section introduces some ot the more important notions of model category structures. We are especially interested in cofibrantly generated ones, cellular ones and left proper model categories. The reason for this is that if a category carries all of these structures, it will behave very well under left Busfield localisations.

The third section discusses homotopy in model categories and inspects the relation of homotopy. It will turn out that the concepts of right and left homotopy coincide if certain conditions are met.

The fourth section is one of the more important ones, especially for the homotopy hypothesis. In a first step, we will define what the homotopy category of a model category is and explain why this concept is well behaved. We will use the theory of localisations to define the homotopy category. Towards the end of the section we introduce the concept of Quillen functors, Quillen pairs and Quillen equivalences. The last being the main tool for the present work, giving a weaker notion of equivalence between categories in the sense that it relates the homotopy categories of the respective categories. That is, two categories are Quillen equivalent iff their homotopy categories are equivalent as categories. As the name may suggest these concepts were first discussed by Quillen in [GQui67].

The fifth section deals with Bousfield localisations, though we will restrict to the case of left Bousfield localisations, as these turn out to be the more natural choice in practice. The section is more of a brief introduction of the concepts as the whole theory is very involved. It was first introduced by Bousfield but was really developed by Hirschhorn in [PHir03]. We will provide two theorems providing two large classes of model categories for which the left Bousfield localisations exist with respect to any class of morphisms of the underlying category. They are also the two largest known classes with such a property. The first class (the one we will mostly use) considers cellular left proper model categories and the second one considers simplicial combinatorial left proper model categories.

Finally, in the sixth section we give a general construction of a nerve realisation adjunction. The main theorem which we will state there will provide us with lots of adjunctions needed throughout the whole thesis.

1 Foundations

We start with some set theoretic notions needed for the next section. It will be especially helpful for the small object argument, this argument provides (under good assumptions) a more or less convenient way to construct functorial factorisations if one faces the challenge to show that a certain category has to be a model category. Furthermore we introduce the notion of compactness, used to define cellular model categories in the next section. We follow [PHir03] and [MHov91] very closely, that is all results provided here and some proofs are taken from there. We also use some inspiration from [nLab].

1.1 Ordinals, Cardinals and Transfinite Composition

We will be interested in the concept of transfinite composition in order to be able to define small objects. To do so we first have to understand the idea of ordinals and cardinals, of course this discussion will be very brief.

1.1.1 Ordinals

Definition 1.1.
1. *A* ***preordered set*** *is a set with a relation that is reflexive and transitive.*
2. *A* ***partially ordered set*** *is a preordered set in which the relation is also antisymmetric.*
3. *A* ***totally ordered set*** *is a partially ordered set in which every pair of elements is comparable.*
4. *A* ***well ordered set*** *is a totally ordered set in which every nonempty subset has a first element.*

Definition 1.2 (Lesser Ordinal)**.** *If η is a well ordered set, then a* ***lesser ordinal*** *of η is a well ordered set which is an element of η.*

Definition 1.3 (Ordinal)**.** *An* ***ordinal*** *is a well ordered set in such a way that it is the well ordered set of all lesser ordinals and every well ordered set is isomorphic to a unique ordinal.*

Remark 1.4.
1. *Often an ordinal is just defined to be a well ordered set.*
2. *The union of a set of ordinals is an ordinal and it is the least upper bound of the set.*
3. *We use the usual convention that the ordering is the member relation "$\in$".*

♦

Example 1.5. *Examples of ordinals include $1, 2, \ldots,$ the natural numbers $\mathbb{N}$, the real numbers $\mathbb{R}$. Some more fancy examples involve $\mathbb{N}^{\mathbb{N}}$, where $\mathbb{N}^{\mathbb{N}} = \mathbb{R}$, or $\mathbb{R}^{\mathbb{R}}$ and so on. And there is of course also Ω which is the set of all ordinals.* ◀

Definition 1.6 (Successor)**.** *If η is an ordinal then the* ***successor*** *of η is the first ordinal greater than η, denoted by $\mathrm{Succ}(\eta)$. In the classical von Neumann notation one would denote the successor of an ordinal η by $\eta \cup \{\eta\}$.*

Definition 1.7 (Limit Ordinal). *A* ***limit ordinal*** *is any ordinal which is not a successor of any other ordinal.*

Example 1.8. *The empty set $\emptyset$ is a limit ordinal. Another example are the natural numbers $\mathbb{N}$.* ◀

For us it is useful to consider a preordered set as a category.

Remark 1.9. *We consider a preordered set S as a small category, where the objects are just the elements of S and a single morphism from s to t for $s, t \in S$ if $s \leq t$.* ◆

Definition 1.10. *If S is a totally ordered set and T a subset of S, then T will be called* ***right-cofinal*** *in S if $\forall\ s \in S\ \ \exists\ t \in T\ :\ s \leq t$.*

Theorem 1.11 ([PHir03]). *If $\mathscr{C}$ is a cocomplete category and S a totally ordered set, T a right cofinal subset of S and $X : S \to \mathscr{C}$ is a functor, then the natural map*

$$\operatorname{colim}_T X \to \operatorname{colim}_S X$$

is an isomorphism.

Proof. We construct a map which is inverse to the map $\operatorname{colim}_T X \to \operatorname{colim}_S X$. First we choose for any $s \in S$ an element $t \in T$ such that $s \leq t$ and define a map $X_s \to \operatorname{colim}_T X$ as the composition $X_s \to X_t \to \operatorname{colim}_T X$. If we happen to choose a different $t' \in T$ such that $s \leq t'$, then we must either have $t \leq t'$ or $t' \leq t$ but then the map we just defined is independent of the choice of $t \in T$.

Now, with a similar argument for a different $s' \in S$ such that $s \leq s'$ then for $t \in T$ such that $s' \leq t$ the composition $X_s \to X_{s'} \to X_t \to \operatorname{colim}_T X$ equals the composition $X_s \to X_t \to \operatorname{colim}_T X$. Now the combination of these maps define a map $\operatorname{colim}_S X \to \operatorname{colim}_T X$. This is now our candidate for the inverse map.

Let $s \in S$ but then the composition

$$X_s \to \operatorname{colim}_S X \to \operatorname{colim}_T X \to \operatorname{colim}_S X$$

equals the map $X_s \to \operatorname{colim}_S X$ which then means that the composition $\operatorname{colim}_S X \to \operatorname{colim}_T X \to \operatorname{colim}_S X$ is the identity. Nearly the same argument yields that the composition $\operatorname{colim}_T X \to \operatorname{colim}_S X \to \operatorname{colim}_T X$ is the identity, giving the desired result. ■

Proposition 1.12 ([PHir03]). *If S is a totally ordered set, then there is a right cofinal subset T of S that is well ordered.*

Proof. The idea is to consider the set of well ordered subsets of S. In a next step one may show, that this set has a maximal element, furthermore one argues that a maximal element has to be right cofinal. A more detailed proof may be found as Proposition 10.1.6 in [PHir03]. ■

1.1.2 Cardinals

Definition 1.13 (Cardinality)**.** *Given a set S we define the **cardinality** of S, denoted by* $\mathrm{card}(S)$*, to be the smallest ordinal for which there is a bijection* $\mathrm{card}(S) \to S$*.*

Definition 1.14 (Cardinal)**.** *A **cardinal** is an ordinal that is of greater cardinality than any lesser ordinal.*

Definition 1.15. *If S is a set then the **cardinal of** S is the unique cardinal whose underlying set has a bijection with S.*

Remark 1.16. *As for ordinals there is of course also a successor for a cardinal i.e. if γ is a cardinal the successor of γ, denoted* $\mathrm{Succ}(\gamma)$ *is the first cardinal greater than γ. Again with von Neumann notation one has* $\mathrm{Succ}(\gamma) = \gamma \cup \{\gamma\}$*.* ⧫

Definition 1.17 (Regular, γ-filtered)**.** *A cardinal γ is **regular** if whenever A is a set whose cardinal is less than γ and $\forall\ a \in A\ \ \exists\ S_a$, where S_a is a set for every $a \in A$, whose cardinal is less than γ, the cardinality of the set $\bigcup_{a \in A} S_a$ is less than γ. Equivalently we will also use the following formulation. Let γ be a cardinal. An ordinal α is γ**-filtered** if it is a limit ordinal, and if $A \subseteq \alpha$ and* $\mathrm{card}(A) \leq \gamma$*, then* $\sup(A) < \alpha$*.*

Remark 1.18. *If γ is finite, a γ-filtered ordinal is just a limit ordinal.* ⧫

Example 1.19. *Since the finite union of finite sets is finite, an example for a regular cardinal is given by the countable cardinal $\aleph_0$.* ◀

Some useful properties about cardinals can be found in [PHir03] p. 187 - 188.

1.1.3 Transfinite Composition

Definition 1.20 (λ-sequence, Composition)**.** *Let $\mathscr{C}$ be a category that is closed under colimits.*

1. *If λ is an ordinal, then a λ**-sequence** in $\mathscr{C}$ is a functor $X : \lambda \to \mathscr{C}$ i.e. it is a diagram*

$$X_0 \to X_1 \to \cdots \to X_\beta \to \dots \qquad (\beta < \lambda)$$

   in $\mathscr{C}$, such that for every ordinal $\gamma < \lambda$ the induced map

$$\mathrm{colim}_{\beta<\lambda} X_\beta \to X_\gamma$$

   is an isomorphism.
2. *The **composition** of the λ-sequence is the map* $X_0 \to \mathrm{colim}_{\beta<\lambda} X_\beta$*.*

Definition 1.21 (Transfinite Composition)**.** *Let $\mathscr{C}$ be a cocomplete category.*

1. *If $\mathcal{D}$ is a class of maps in $\mathscr{C}$ and λ is an ordinal, then a λ**-sequence of maps in** $\mathcal{D}$ is a λ-sequence*

$$X_0 \to X_1 \to \cdots \to X_\beta \to \dots \qquad (\beta < \lambda)$$

   in $\mathscr{C}$ such that the map $X_\beta \to X_{\beta+1}$ is in $\mathcal{D}$ for $\beta + 1 < \lambda$.

2. *If $\mathcal{D}$ is a class of maps in $\mathscr{C}$, then a* ***transfinite composition*** *of maps in $\mathcal{D}$ is a map in $\mathscr{C}$ that is the composition of a λ-sequence in $\mathcal{D}$ for some possibly finite ordinal λ.*
3. *If $\mathcal{D}$ is a subcategory of $\mathscr{C}$, then a* ***transfinite composition*** *of maps in $\mathcal{D}$ is a transfinite composition of maps in the class of maps of $\mathcal{D}$.*

Lemma 1.22 ([PHir03])**.** *Let $\mathscr{C}$ be a category, λ a limit ordinal and let $X : \lambda \to \mathscr{C}$ be a functor. If the functor $Y : \lambda \to \mathscr{C}$ is defined by: $Y_0 = X_0$, $Y_{\beta+1} = X_\beta$ if $\beta + 1 < \lambda$, $Y_\beta = \operatorname{colim}_{\gamma<\beta} X_\gamma$ if $\beta < \gamma$ and β is a limit ordinal. Then Y is a λ-sequence in $\mathscr{C}$ and*

$$\operatorname{colim}_{\beta<\lambda} X_\beta = \operatorname{colim}_{\beta<\lambda} Y_\beta.$$

Proof. This follows from the universal property of colimits. ■

Definition 1.23 (Reindexing)**.** *If $\mathscr{C}$ is a category and λ a limit ordinal and $X : \lambda \to \mathscr{C}$ is a functor, then the λ-sequence Y, obtained from the functor X as in the above lemma, will be called the* ***reindexing*** *of X.*

The next proposition may be found as Proposition 10.2.7 in [PHir03], the proof will not be given here.

Proposition 1.24 ([PHir03])**.** *If $\mathscr{C}$ is a category, S a set and $g_s : C_s \to D_s$ a map in $\mathscr{C}$ for every $s \in S$, then the coproduct*

$$\coprod g_s : \coprod C_s \to \coprod D_s$$

is a transfinite composition of pushouts of the g_s. If S is infinite, then it is a transfinite composition indexed by an ordinal whose cardinal equals that of S.

Some further properties about transfinite composition may be found in [PHir03] p. 190 - 191.

1.2 Small Objects

Definition 1.25 (Small)**.** *Let $\mathscr{C}$ be a cocomplete category and let $\mathcal{D}$ be a subcategory of $\mathscr{C}$.*

1. *If κ is a cardinal, then an object W in $\mathscr{C}$ is* ***κ-small relative to*** *$\mathcal{D}$ if, for every regular cardinal $\lambda > \kappa$ and every λ-sequence*

   $$X_0 \to X_1 \to \cdots \to X_\beta \to \ldots \qquad (\beta < \lambda)$$

   in $\mathscr{C}$ such that the map $X_\beta \to X_{\beta+1}$ is in $\mathcal{D}$ for every ordinal β such that $\beta + 1 < \lambda$ the map of sets

   $$\operatorname{colim}_{\beta<\lambda} \mathscr{C}(W, X_\beta) \to \mathscr{C}(W, \operatorname{colim}_{\beta<\lambda} X_\beta)$$

   is an isomorphism.
2. *An object is* ***small relative to*** *$\mathcal{D}$ if it is κ-small relative to $\mathcal{D}$ for some cardinal κ.*
3. *An object is* ***small*** *if it is small relative to $\mathscr{C}$.*

Remark 1.26. *If $\kappa < \kappa'$ and A is κ-small relative to $\mathcal{D}$ then A is also κ'-small relative to $\mathcal{D}$, since κ and κ' are regular.* ♦

Definition 1.27 (Finite)**.** *Let $\mathscr{C}$ be a cocomplete category, $\mathcal{D}$ a collection of morphisms of $\mathscr{C}$ and A an object of $\mathscr{C}$.*

1. *We say that A is **finite relative to** $\mathcal{D}$ if A is κ-small relative to $\mathcal{D}$ for a finite cardinal κ.*
2. *We say that A is **finite** if it is finite relative to $\mathscr{C}$ itself.*

The following example will be needed later.

Example 1.28 ([MHov91])**.** *Every object in* **Set** *is small.*

*Indeed, if $A \in$ **Set** we claim that A is $\mathrm{card}(A)$-small. To see this suppose that λ is a $\mathrm{card}(A)$-filtered ordinal and X is a λ-sequence of sets. Given a map $f : A \to \mathrm{colim}_{\beta<\lambda} X_\beta$ we find for each $a \in A$ an index $\beta(a)$ such that $f(a)$ is in the image of $X_{\beta(a)}$ (this comes from the construction of the filtered colimit of sets). Then we let γ be the supremum of the $\beta(a)$. Because λ is $\mathrm{card}(A)$-filtered $\gamma < \lambda$ and the map f will factor through a map $g : A \to X_\gamma$ as required. A similar argument shows that if two maps $A \to X_\beta$ and $A \to X_\gamma$ are equal in the colimit they must be equal at some stage of the colimit.* ◀

Definition 1.29 (Lifting)**.** *Let $\mathscr{C}$ be a category and consider the commutative diagram of solid arrows*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & B \arrow[d, "p"] \\ C \arrow[r, "g"'] \arrow[ur, dotted, "h"] & D. \end{tikzcd}$$

*A **lift** or **lifting** in the diagram is a (dotted) morphism $h : C \to B$ in $\mathscr{C}$ such that $h \circ i = f$ and $p \circ h = g$.*

Definition 1.30 (Lifting Properties)**.** *We consider a category $\mathscr{C}$. A morphism $i : A \to C$ in $\mathscr{C}$ is said to have the **left lifting property (LLP)** with respect to another morphism $p : B \to D$ in $\mathscr{C}$ and p is said to have the **right lifting property (RLP)** with respect to i if there is a lift $h : C \to B$ in $\mathscr{C}$ such that for some $f : A \to B$ and $g : C \to D$ in $\mathscr{C}$ the diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & B \arrow[d, "p"] \\ C \arrow[r, "g"'] \arrow[ur, "h"] & D \end{tikzcd}$$

commutes i.e. $h \circ i = f$ and $p \circ h = g$.

Notation 1.31. *Let $\mathscr{C}$ be a category and $\mathcal{S}$ a class of morphisms in $\mathscr{C}$. We sometimes write $RLP(\mathcal{S})$ for the collection of morphisms with the RLP with respect to $\mathcal{S}$. Similarly we write $LLP(\mathcal{S})$ for the collection of morphisms with the LLP with respect to $\mathcal{S}$.*

Definition 1.32. *Let I be a class of maps in a category $\mathscr{C}$.*

1. *A map is **I-projective** if it has the LLP with respect to every map in I. This class will be denoted I-proj i.e. I-proj $= LLP(I)$.*
2. *A map is **I-injective** if it has the RLP with respect to every map in I. This class of maps will be denoted I-inj i.e. I-inj $= RLP(I)$.*
3. *A map is an **I-cofibration** if it has the LLP with respect to every I-injective map. This class will be denoted I-cof i.e. I-cof $= LLP(RLP(I))$.*
4. *A map is an **I-fibration** if it has the RLP with respect to every I-projective map. This class will be denoted I-fib i.e. I-fib $= RLP(LLP(I))$.*

Remark 1.33.

1. *Let $\mathscr{C}$ be a model category and I the class of cofibrations then I-inj is the class of acyclic fibrations and I-cof $= I$. Dually if I is the class of fibrations then I-proj is the class of acyclic cofibrations and I-fib $= I$. The definition of a model category will be defined in the next section, but it is best to state this remark here.*
2. *It is also worth mentioning that $I \subseteq I$-cof and $I \subseteq I$-fib. Furthermore $(I$-cof$)$-inj $= I$-inj and $(I$-fib$)$-proj $= I$-proj. If $I \subseteq J$ then J-inj $\subseteq I$-inj and J-proj $\subseteq I$-proj. Thus I-cof $\subseteq J$-cof and I-fib $\subseteq J$-fib.*

♦

Lemma 1.34 ([MHov91])**.** *Suppose that $F : \mathscr{C} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathscr{D} : U$ is an adjunction, I a class of maps in $\mathscr{C}$ and J a class of maps in $\mathscr{D}$. Then*

1. *$U(FI$-inj$) \subseteq I$-inj.*
2. *$F(I$-cof$) \subseteq FI$-cof.*
3. *$F(UJ$-proj$) \subseteq J$-proj.*
4. *$U(J$-fib$) \subseteq UJ$-fib.*

Proof.

1. Let $g \in FI$-inj and $f \in I$. Then g has the RLP with respect to Ff and so by adjointness Ug has the RLP with respect to f. Thus $Ug \in I$-inj as required.
2. Let $f \in I$-cof and $g \in FI$-inj. Then by the previous part $Ug \in I$-inj and so f has the LLP with respect to Ug. Adjointness implies that Ff has the LLP with respect to g and so $Ff \in (FI$-inj$)$-proj $= FI$-cof.

The other two properties hold by duality. ■

Definition 1.35 (Cell Complex)**.** *Let $\mathscr{C}$ be a cocomplete category and I a class of maps in $\mathscr{C}$.*

1. *The subcategory of **relative I-cell complexes** is the subcategory of maps that can be constructed as a transfinite composition of pushouts of elements of I.*
2. *An object X in $\mathscr{C}$ is an **I-cell complex** if the map $\emptyset \to X$ is a relative I-cell complex.*
3. *A map is an **inclusion of I-cell complexes** if it is a relative I-cell complex whose domain is an I-cell complex.*

Remark 1.36. *The collection of relative I-cell complexes is denoted by **I-cell**.*

The first condition of the above definition means, that if $f : A \to B$ is a relative I-cell complex, then there is an ordinal λ and a λ-sequence $X : \lambda \to \mathscr{C}$ such that f is the composition of X and such that, for each β for which $\beta + 1 < \lambda$, there is a pushout square

$$\begin{tikzcd} C_\beta \arrow[r] \arrow[d, "g_\beta"'] & X_\beta \arrow[d] \\ D_\beta \arrow[r] & X_{\beta+1} \end{tikzcd}$$

where $g_\beta \in I$.

The identity map at A is the transfinite composition of the trivial 1-sequence A, so identity maps are relative I-cell complexes. If $f : A \to B$ is an isomorphism then f is also the composition of the 1-sequence A so f is a relative I-cell complex. ♦

We give some useful properties about I-cell.

Lemma 1.37. *Suppose I is a class of maps in a category $\mathscr{C}$ with all small colimits. Then I-cell $\subseteq$ I-cof.*

Proof. Remember that I-cof is defined by a lifting property, so it is closed under transfinite composition and pushouts which concludes the proof. ■

Lemma 1.38 ([MHov91])**.** *Suppose λ is an ordinal and $X : \lambda \to \mathscr{C}$ is a λ-sequence such that every map $X_\beta \to X_{\beta+1}$ is either a pushout of a map of I or an isomorphism. Then the transfinite composition of X is a relative I-cell complex.*

Proof. We define an equivalence relation $\sim$ on λ. If $\alpha \leq \beta$ define $\alpha \sim \beta$ if for all γ such that $\alpha \leq \gamma \leq \beta$ the map $X_\gamma \to X_{\gamma+1}$ is an isomorphism. Then each equivalence class $[\alpha]$ under $\sim$ is a closed interval $[\alpha', \alpha'']$ of λ and if $\alpha \leq \beta$ and $\alpha \sim \beta$ then the map $X_\alpha \to X_\beta$ is an isomorphism. The set of equivalence classes is a well-ordered set and so is isomorphic to a unique ordinal μ. The functor X descends to a functor $Y : \mu \to \mathscr{C}$ where $Y_{[\alpha]} = X_{\alpha'}$. Each map $Y_\beta \to Y_{\beta+1}$ is a pushout of a map of I. Y is a μ-sequence. Indeed, if $[\beta]$ is a limit ordinal of μ then β' must be a limit ordinal of λ. Since the transfinite composition of Y is isomorphic to the transfinite composition of X we are done. ■

Lemma 1.39 ([MHov91])**.** *Suppose $\mathscr{C}$ is a cocomplete category, and I is a set of maps of $\mathscr{C}$. Then I-cell is closed under transfinite compositions.*

Proof. Let $X : \lambda \to \mathscr{C}$ be a λ-sequence of relative I-cell complexes so that each map $X_\beta \to X_{\beta+1}$ is a relative I-cell complex. Then $X_\beta \to X_{\beta+1}$ is the composition of a λ-sequence $Y : \gamma_\beta \to \mathscr{C}$ of pushouts of maps of I. Consider the set S of all pairs of ordinals (β, γ) such that $\beta < \gamma$ and $\gamma < \gamma'$. Put a total order on S by defining $(\beta, \gamma) < (\beta', \gamma')$ if $\beta < \beta'$ or $\gamma < \gamma'$. Then S becomes a well-ordered set so is isomorphic to a unique ordinal μ. We therefore get a functor $Z : \mu \to \mathscr{C}$ which one can readily verify is a μ-sequence. Each map $Z_\alpha \to Z_{\alpha+1}$ is either one of the maps $Y_\gamma \to Y_{\gamma+1}$ or else is an isomorphism. Since a composition of X is also a composition of Z the previous lemma implies that a composition of X is a relative I-cell complex. ■

Lemma 1.40 ([MHov91])**.** *Suppose $\mathscr{C}$ is a cocomplete category, and I is a set of maps of $\mathscr{C}$. Then any pushout of coproducts of maps of I is in I-cell.*

Proof. Let λ be an ordinal, since every set K is isomorphic to an ordinal and $g_k : C_k \to D_k$ a map of I for all k in K. Assume f is the pushout of the diagram

$$\begin{tikzcd} \coprod C_k \arrow[r] \arrow[d, "\coprod g_k"'] & X \arrow[d, "f"] \\ \coprod D_k \arrow[r] & Y. \end{tikzcd}$$

We want that f is a relative I-cell complex. We then form a λ-sequence by letting $X_0 = X$ and $X_{\beta+1}$ be the pushout $X_\beta \coprod_{C_\beta} D_\beta$ over g_β and by letting $X_\beta = \operatorname{colim}_{\alpha<\beta} X_\alpha$ for limit ordinals β. The transfinite composition $X \to X_\lambda$ of this λ-sequence is now isomorphic to the map f and hence f is a relative I-cell complex. ■

Definition 1.41 (Presentation)**.** *Let $\mathscr{C}$ be a cocomplete category and let I be a set of maps in $\mathscr{C}$. If $f : X \to Y$ is a relative I-cell complex, then a **presentation** of f is a pair consisting of a λ-sequence*

$$X = X_0 \to X_1 \to \cdots \to X_\beta \to \dots \quad (\beta < \lambda)$$

(for some ordinal λ) and a sequence of ordered triples

$$\{(T^\beta, e^\beta, h^\beta)\}_{\beta<\lambda}$$

such that

1. *the composition of the λ-sequence is isomorphic to f*
2. *for every $\beta < \lambda$*
   (a) *T^β is a set*
   (b) *e^β is a function $e^\beta : T^\beta \to I$*
   (c) *if $i \in T^\beta$ and e^β_i is the element $C_i \to D_i$ of I, then h^β_i is a map $h^\beta_i : C_i \to X_\beta$ such that there is a pushout diagram*

$$\begin{tikzcd} \coprod_{T^\beta} C_i \arrow[r] \arrow[d, "\coprod h^\beta_i"'] & \coprod_{T^\beta} D_i \arrow[d] \\ X_\beta \arrow[r] & X_{\beta+1}. \end{tikzcd}$$

*If the map $f : \emptyset \to Y$ ($\emptyset$ is the initial object) is a relative I-cell complex, then a presentation of f is called **presentation of** Y.*

Definition 1.42 (Presented Relative I-cell Complex)**.** *Let $\mathscr{C}$ be a category and I a set of maps in $\mathscr{C}$, then a **presented relative I-cell complex** is a relative I-cell complex $f : X \to Y$ together with a particular presentation*

$$(X = X_0 \to X_1 \to \cdots \to X_\beta \to \dots \quad (\beta < \lambda), \{T^\beta, e^\beta, h^\beta\}_{\beta<\gamma})$$

*of it. A presented relative I-cell complex in which $X = \emptyset$ (the initial object) will be called **presented I-cell complex**.*

Definition 1.43. *Let $\mathscr{C}$ be a cocomplete category and I a set of maps in $\mathscr{C}$. If*

$$(f : X \to Y, X = X_0 \to X_1 \to \cdots \to X_\beta \to \dots \quad (\beta < \lambda), \{T^\beta, e^\beta, h^\beta\}_{\beta<\gamma})$$

is a presented relative I-cell complex then

1. *the **presentation ordinal** of f is λ*
2. *the **set of cells** of f is $\coprod_{\beta<\lambda} T^\beta$*
3. *the **size** of f is the cardinal of the set of cells of f*
4. *if e is a cell of f the **presentation ordinal** of e is the ordinal β such that $e \in T^\beta$*
5. *if $\beta < \lambda$, then the β**-skeleton** of f is X_β.*

1.3 Compactness

This idea of compactness is due to [PHir03], needed for his notion of cellular model categories. Compactness is actually a somewhat weaker notion of smallness.

Definition 1.44. *Let $\mathscr{C}$ be a cocomplete category and let I be a set of maps in $\mathscr{C}$.*

1. *If γ is a cardinal then an object W in $\mathscr{C}$ is γ**-compact relative to** I if for every presented relative I-cell complex $f : X \to Y$ every map from W to Y factors through a subcomplex of f of size at most γ.*
2. *An object W in $\mathscr{C}$ is **compact relative to** I if it is γ-compact relative to I for some cardinal γ.*

Definition 1.45. *Let $\mathscr{C}$ be a cofibrantly generated model category with generating cofibrations I.*

1. *If γ is a cardinal, then an object $W \in \mathscr{C}$ is γ**-compact** if it is γ-compact relative to I.*
2. *An object $W \in \mathscr{C}$ is **compact** if there is a cardinal γ for which it is γ-compact.*

The definition of a cofibrantly generated model category will follow in section 2.

1.4 The Category of Finite Ordinals

We introduce this category to be able to define the category of cosimplicial and simplicial objects. A tool also needed to dedfine the category of simplicial sets later on. We will follow [GoJa09] and use inspiration from [nLab].

Definition 1.46 (Category of Finite Ordinals)**.** *The **category of finite ordinals** is the category $\mathbf{\Delta}$ with objects*

$$[n] = \{0 \to 1 \to 2 \to \cdots \to n\} \quad \textit{for} \quad n \geq 0$$

and morphisms

$$\mathbf{\Delta}([n], [k])$$

the set of weakly order-preserving maps.

Usually, this category is called the simplex category in common literature. I prefer the term category of finite ordinals due to its nature.

Remark 1.47. *There are two subcategories of $\mathbf{\Delta}$, namely $\mathbf{\Delta}_+$ the category of injective order-preserving maps and $\mathbf{\Delta}_-$ the category of surjective order preserving maps.*

Any morphism in $\mathbf{\Delta}$ can be factored uniquely into a morphism of $\mathbf{\Delta}_-$ followed by a morphism of $\mathbf{\Delta}_+$. ⧫

Definition 1.48 (Cofaces, Codegeneracies and Cosimplicial Identities)**.** *We consider two special morphisms in $\mathbf{\Delta}$*

1. $d^i : [n-1] \to [n] \in \mathbf{\Delta}_+$ *for* $n \geq 1$ *and* $0 \leq i \leq n$, *where the image of* d^i *does not include* i *(cofaces).*
2. $s^i : [n] \to [n-1] \in \mathbf{\Delta}_-$ *for* $n \geq 1$ *and* $0 \leq i \leq n-1$, *where* s^i *identifies* i *and* $i+1$ *(codegeneracies).*

*All relations between these two maps are implied by the **cosimplicial identities**:*

$$\begin{aligned} d^j d^i &= d^i d^{j-1} && i < j \\ s^j d^i &= d^i s^{j-1} && i < j \\ &= id && i = j, j+1 \\ &= d^{i-1} s^j && i > j+1 \\ s^j s^i &= s^{i-1} s^j && i > j \end{aligned}$$

1.5 The Category of Cosimplicial and Simplicial Objects

The following categories are especially nice to work with, as they preserve a lot of properties initially given by the category which we first considered, before applying the definition.

Definition 1.49 (Category of Cosimplicial and Simplicial Objects)**.** *Let $\mathscr{C}$ be any category.*

1. *The **category of cosimplicial objects** in $\mathscr{C}$ is the functor category $\mathbf{Fun}(\mathbf{\Delta}, \mathscr{C})$.*
2. *The **category of simplicial objects** in $\mathscr{C}$ is the functor category $\mathbf{Fun}(\mathbf{\Delta}^{op}, \mathscr{C})$.*

Remark 1.50. *Since $\mathbf{Fun}(\mathbf{\Delta}, \mathscr{C})$ and $\mathbf{Fun}(\mathbf{\Delta}^{op}, \mathscr{C})$ are functor categories they have all colimits and limits that exist in $\mathscr{C}$ taken objectwise.* ⧫

2 Model Categories

We will define model categories with the help of weak factorisation systems. In the definition of a model category we will impose the use of functorial weak facorisation systems, this is justified with the fact that the most important categories we will be using in the thesis satisfy the small object argument (as we will see later), which on the other hand guarantees us this kind of functorial factorisations.

It will turn out that defining model categories in this way (with the use of weak factorisation systems) is indeed the same as the "classic" way of defining them, which is for example used in [MHov91] or [DwSp95].

Later in the section we give a lot of different types of model category structures. At some point we will deal with Bousfield localisations. At the moment there are two large classes of model categories, where it is known, that the Bousfield localisations exist with respect to any class of morphisms in the respective category. These classes are either proper cellular model categories or proper combinatorial simplicial model categories, depending on left and right Bousfield localisations. We only need left and right proper respectively in the before mentioned cases.

After the definition of the respective model categories we usually state a recognition theorem in order to show that a certain category is indeed of the desired type.

This section contains material from [MHov91], [PHir03], [JLur09], [ClBa], [DaDu], [DwSp95], [ERie09] and quite some inspiration from [nLab].

2.1 Weak Factorisation Systems

A weak factorization system on a category is a pair $(\mathbf{L}, \mathbf{R})$ of classes of morphisms such that every morphism of the category factors as the composite of a morphism in $\mathbf{L}$ followed by a morphism in $\mathbf{R}$, and $\mathbf{L}$ and $\mathbf{R}$ are closed under having the lifting property against each other.

Definition 2.1 (Retract)**.** *Let $\mathscr{C}$ be a category and $X \in \mathscr{C}$ an object. X is said to be a* ***retract*** *of an object Y if there exist morphisms $i : X \to Y$ and $r : Y \to X$ such that $r \circ i = id_X$.*

If $f, g \in \mathscr{C}$ are morphisms, we say that f is a ***retract*** *of g if the object of $\operatorname{mor}(\mathscr{C})$ represented by f is a retract of the object of $\operatorname{mor}(\mathscr{C})$ represented by g. Here $\operatorname{mor}(\mathscr{C})$ is the category of morphisms of $\mathscr{C}$. In other words we have the following commutative diagram*

$$\begin{array}{ccccc} A & \xrightarrow{i} & B & \xrightarrow{r} & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ A' & \xrightarrow[i']{} & B' & \xrightarrow[r']{} & A' \end{array}$$

such that $r \circ i = id_A$, $r' \circ i' = id_{A'}$.

Definition 2.2 (Weak Factorisation System - WFS)**.** *A **weak factorisation system (WFS)** on a category $\mathscr{C}$ is a tuple $(\mathbf{L}, \mathbf{R})$ of classes of morphisms of $\mathscr{C}$ such that*

1. *Any morphism $f : X \to Y$ of $\mathscr{C}$ may be factored as $f : X \xrightarrow{\in \mathbf{L}} Z \xrightarrow{\in \mathbf{R}} Y$.*
2. *Furthermore we want that $\mathbf{L}$ has the LLP against every morphism in $\mathbf{R}$ and $\mathbf{R}$ has the RLP against every morphism in $\mathbf{L}$.*

Remark 2.3. *If $(\mathbf{L}, \mathbf{R})$ is a WFS for a category $\mathscr{C}$, then $(\mathbf{L}^{op}, \mathbf{R}^{op})$ is a WFS on $\mathscr{C}^{op}$.* ♦

The following is a nice example, which can be found on Joyale's CatLab. Later one we will see more examples in the discussions of Part II.

Example 2.4. *Let R a ring. A morphism of left R-modules is projective if it has the LLP with respect to epimorphisms. An R-module M is projective iff $0 \to M$ is projective i.e. a map of R-modules $u : M \to N$ is projective iff it is monic and its cokernel is a projective R-module. Then the category of R-modules admits a weak factorisation system $(\mathbf{L}, \mathbf{R})$ in which $\mathbf{L}$ is the class of projective morphisms and $\mathbf{R}$ is the class of epimorphisms.* ◀

2.1.1 Properties of Weak Factorisation Systems

We introduce the closure properties from [nLab] and provide a more detailed proof.

Theorem 2.5 (Closure Properties, [nLab])**.** *Let $(\mathbf{L}, \mathbf{R})$ be a weak factorisation system on some category $\mathscr{C}$. Then*

1. *All isomorphisms and identities of $\mathscr{C}$ are in $\mathbf{L}$ and $\mathbf{R}$.*
2. *$\mathbf{L}$ and $\mathbf{R}$ are closed under composition in $\mathscr{C}$. Furthermore $\mathbf{L}$ is closed under transfinite composition.*
3. *$\mathbf{L}$ and $\mathbf{R}$ are closed under retracts in the category $\mathbf{Fun}([1], \mathscr{C})$.*
4. *$\mathbf{L}$ is closed under pushouts of morphisms in $\mathscr{C}$ and $\mathbf{R}$ is closed under pullbacks of morphisms in $\mathscr{C}$.*
5. *$\mathbf{L}$ is closed under coproducts in $\mathbf{Fun}([1], \mathscr{C})$ and $\mathbf{R}$ is closed under products in $\mathbf{Fun}([1], \mathscr{C})$.*

Proof. 1. Consider a commutative solid arrow diagram in $\mathscr{C}$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"'] \arrow[ur, dotted, "f \circ i^{-1}"] & Y \end{tikzcd}$$

with i an isomorphism. We can construct a lift (the dotted arrow in the diagram). In particular there is a lift if $p \in \mathbf{R}$ and hence $i \in \mathbf{L}$. The other case is dual. For the identity we just adapt the very same diagram.

2. Consider the following commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "g\in\mathbf{L}"'] & B \arrow[dd, "p\in\mathbf{R}"] \\
C \arrow[d, "h\in\mathbf{L}"'] & \\
D \arrow[r] & E
\end{tikzcd}$$

which we rewrite as the commuting solid arrow diagram

$$\begin{tikzcd}
A \arrow[rr] \arrow[d, "g\in\mathbf{L}"'] & & B \arrow[d, "p\in\mathbf{R}"] \\
C \arrow[r, "h\in\mathbf{L}"'] \arrow[urr, dotted, "l"] & D \arrow[r] & E.
\end{tikzcd}$$

Since we are dealing with a WFS there exists a lift l (dotted arrow) in the above diagram. Rearranging the diagram gives us a commuting solid arrow diagram

$$\begin{tikzcd}
C \arrow[r, "l"] \arrow[d, "h\in\mathbf{L}"'] & B \arrow[d, "p\in\mathbf{R}"] \\
D \arrow[r] \arrow[ur, dotted, "k"] & E.
\end{tikzcd}$$

The same property of a WFS as before gives us another lift k (dotted arrow) making the diagram commute. This gives us the commuting diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "g\in\mathbf{L}"'] & B \arrow[dd, "p\in\mathbf{R}"] \\
C \arrow[d, "h\in\mathbf{L}"'] & \\
D \arrow[r] \arrow[uur, "k"] & E
\end{tikzcd}$$

i.e. $g \circ h$ has the LLP against $\mathbf{R}$ with respect to p i.e. $g \circ h \in \mathbf{L}$. The other case is dual. It follows that $\mathbf{L}$ is closed under transfinite composition since it is given by colimits of sequential composition and successive lifts against the underlying sequence. As above this constitutes a cocone, the existence of the lift follows by its universal property.

3. Consider a commutative diagram

$$\begin{tikzcd}
id_A : A \arrow[r] \arrow[d, "j"'] & C \arrow[r] \arrow[d, "i\in\mathbf{L}"] & A \arrow[d, "j"] \\
id_B : B \arrow[r] & D \arrow[r] & B
\end{tikzcd}$$

we paste the commuting diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "j"'] & X \arrow[d, "f\in\mathbf{R}"] \\
B \arrow[r] & Y
\end{tikzcd}$$

to it resulting in a solid commuting arrow diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, "\in\mathbf{L}"] & A \arrow[r] & X \arrow[d, "\in\mathbf{R}"] \\ B \arrow[r, "i"'] & D \arrow[r] \arrow[urr, dotted, "l"] & B \arrow[r] & Y \end{tikzcd}$$

from the properties of a WFS we get a lift l (dotted arrow) making the above diagram commutative. But then $l \circ i$ is a lift for the following diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "j"'] & X \arrow[d, "\in\mathbf{R}"] \\ B \arrow[r] \arrow[ur] & Y \end{tikzcd}$$

in other words, this means that j has the LLP against $\mathbf{R}$ but then $j \in \mathbf{L}$. The other case is as usual formally dual.

4. Let $p \in \mathbf{R}$ and consider a pullback diagram in $\mathscr{C}$

$$\begin{tikzcd} Z \prod X \arrow[r] \arrow[d, "f^*p"'] & X \arrow[d, "p"] \\ Z \arrow[r, "f"'] & Y. \end{tikzcd}$$

We need to show, that f^*p has the RLP with respect to all $i \in \mathbf{L}$. Consider now the commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & Z \prod X \arrow[d, "f^*p"] \\ B \arrow[r, "g"'] & Z. \end{tikzcd}$$

We paste to get the commutative solid arrow diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i\in\mathbf{L}"'] & Z \prod X \arrow[r] & X \arrow[d, "p\in\mathbf{R}"] \\ B \arrow[r, "g"'] \arrow[urr, dotted, "l"] & Z \arrow[r, "f"'] & Y \end{tikzcd}$$

again since we are dealing with a WFS there is a lift l in the above diagram (dotted arrow). We get a diagram

$$\begin{tikzcd} B \arrow[dr, dotted, "\hat{g}"] \arrow[drr, bend left, "l"] \arrow[ddr, bend right, "g"'] & & \\ & Z \prod X \arrow[r] \arrow[d] & X \arrow[d] \\ & Z \arrow[r] & Y \end{tikzcd}$$

and by the universal property of pullbacks there is a unique map $\hat{g} : B \to Z$. We are left to show that the upper triangle of

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & Z \prod X \arrow[d, "f^*p"] \\ B \arrow[r] \arrow[ur, dotted, "g"] & Z \end{tikzcd}$$

commutes. We have cones given by $A \to Z \prod X \to X$ and $A \xrightarrow{i} B \xrightarrow{g} Z$. Which gives a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, dotted] & Z \prod X \arrow[dr] & \\ B \arrow[dr] & Z \prod X \arrow[r] \arrow[d] & X \arrow[d] \\ & Z \arrow[r] & Y \end{tikzcd}$$

where the dotted arrow is unique by the universal property of pullbacks. On the other hand we get from $A \to B \to Z \prod X \to X$ and $A \to B \to Z$ a diagram

$$\begin{tikzcd} A \arrow[dr] & & & \\ & B \arrow[r] \arrow[dr, dotted] \arrow[ddr, bend right] & Z \prod X \arrow[dr] & \\ & & Z \prod X \arrow[r] \arrow[d] & X \arrow[d] \\ & & Z \arrow[r] & Y. \end{tikzcd}$$

Again by the universal property of pullbacks we get a unique dotted map in the above diagram. By uniqueness it now follows, that the maps $A \to Z \prod X$ and $A \to B \to Z \prod X$ are equal, hence the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & Z \prod X \arrow[d, "f^*p"] \\ B \arrow[r] \arrow[ur, dotted, "g"] & Z \end{tikzcd}$$

commutes. Therefore f^*p has the RLP with respect to all $i \in \mathbf{L}$. The other case is dual.

5. Consider $\{(A_s \xrightarrow{i_s} B_s) \in \mathbf{L}\}_{s \in S}$ a set of elements in $\mathbf{L}$. Colimits in $\mathbf{Fun}([1], \mathscr{C})$ are computed componentwise. This product in $\mathbf{Fun}([1], \mathscr{C})$ is the universal morphism out of the coproduct of objects $\coprod_{s \in S} A_s$ induced via the universal property by the set of morphisms $(i_s)_{s \in S} : \coprod_{s \in S} A_s \to \coprod_{s \in S} B_s$. Consider

$$\begin{tikzcd} \coprod_{s \in S} A_s \arrow[r] \arrow[d, "(i_s)_{s \in S}"'] & X \arrow[d, "f \in \mathbf{R}"] \\ \coprod_{s \in S} B_s \arrow[r] & Y. \end{tikzcd}$$

This is the cocone under the coproduct of objects, hence by the universal property of the coproduct, we get an equivalence to the collection of solid arrow diagrams

$$\begin{tikzcd} A_s \arrow[r] \arrow[d, "i_s \in \mathbf{L}"'] & X \arrow[d, "f \in \mathbf{R}"] \\ B_s \arrow[r] \arrow[ur, dotted, "l_s"] & Y \end{tikzcd}$$

for every $s \in S$. From WFS we get a list $l_s : B_s \to X$ for every $s \in S$ (dotted arrow). Now the above collection with the lifts is itself a cocone again by the universal property of coproducts equivalent to a lift $(l_s)_{s\in S}$ in

$$\begin{tikzcd} \coprod_{s\in S} A_s \arrow[r] \arrow[d, "(i_s)_{i\in S}"'] & X \arrow[d, "f \in \mathbf{R}"] \\ \coprod_{s\in S} B_s \arrow[r] \arrow[ur, "(l_s)_{s\in S}"] & S. \end{tikzcd}$$

Now the coproduct of the $(l_s)_{s\in S}$ has the LLP against $f \in \mathbf{R}$ and hence $(i_s)_{s\in S} \in \mathbf{L}$. The other case is again dual.

■

We state a remark which will be important for later, namely when we argue the model structures on topological spaces and simplicial sets.

Remark 2.6. *This result also holds if we consider a cocomplete category $\mathscr{C}$ and choose a class of morphisms $\mathcal{S}$ in $\mathscr{C}$ and further $\mathbf{L} = \mathcal{S}$-proj and $\mathbf{R} = \mathcal{S}$-inj. That is to say, that the proof is carried out in the exact same way as the original one about the closure properties. When we speak of the closure properties we refer to both these versions depending on the situation.* ♦

This remark and the above theorem now have the following consequence.

Corollary 2.7. *Let $\mathscr{C}$ be a cocomplete category and $\mathcal{S}$ a class of morphisms in $\mathscr{C}$. Then every $\mathcal{S}$-inj morphism has the RLP with respect to all relative $\mathcal{S}$-cell complexes and their retracts.*

Proof. Theorem 2.5 and Remark 2.6. ■

Lemma 2.8 (Retract Argument)**.** *Consider a composite morphism $f : X \xrightarrow{i} A \xrightarrow{p} Y$. Then the following hold.*

1. *If f has the LLP against p, then f is a retract of i.*
2. *If f has the RLP against i, then f is a retract of p.*

Proof. From the composite morphism we consider the following solid arrow diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & A \arrow[d, "p"] \\ Y \arrow[r, equal] \arrow[ur, dotted, "l"] & Y \end{tikzcd}$$

where the dotted arrow exists by assumption and we have $p \circ l = id_Y$. We then consider the diagram

$$\begin{tikzcd}
id_X : X \arrow[r, equal] \arrow[d, "f"'] & X \arrow[r, equal] \arrow[d, "i"] & X \arrow[d, "f"] \\
id_Y : Y \arrow[r, "l"'] & A \arrow[r, "p"'] & Y
\end{tikzcd}$$

i.e. f is a retract of i. The second part is formally dual. ■

2.2 Functorial Weak Factorisation Systems

First we introduce functorial factorisations via a functor construction to encapsulate the most important properties of such a factorisation. It is indeed inspired by the definitions of [ERie09] and [nLab].

Definition 2.9 (Functorial Factorisation)**.** *For $\mathscr{C}$ a category, a* ***functorial factorisation*** *of the morphisms in $\mathscr{C}$ is a functor*

$$F : \mathbf{Fun}([1], \mathscr{C}) \to \mathbf{Fun}([2], \mathscr{C})$$

which is a section of the composition functor $\mathrm{D}^1 : \mathbf{Fun}([2], \mathscr{C}) \to \mathbf{Fun}([1], \mathscr{C})$ i.e. such that $\mathrm{D}^1 \circ F = id_{\mathbf{Fun}([1],\mathscr{C})}$.

The following remark will help to clarify this definition.

Remark 2.10 (A Remark on Functorial Factorisations)**.** *The arrow category $Arr(\mathscr{C})$ is eqivalent to the functor category $\mathbf{Fun}([1], \mathscr{C})$, while $\mathbf{Fun}([2], \mathscr{C})$ has as objects pairs of composable morphisms in $\mathscr{C}$.*

There are 3 injective functors $d^i : [1] \to [2]$ (d^i omits the index i in its image). By precomposition, this induces a functor $\mathrm{D}^i : \mathbf{Fun}([2], \mathscr{C}) \to \mathbf{Fun}([1], \mathscr{C})$. Here, D^1 sends a pair of composable morphisms to their composition.

More detailed, we have

$$\begin{gathered} d^i : [1] \to [2] \\ \{0 \to 1\} \mapsto \{0 \to 1 \to 2\}, \end{gathered}$$

where

$$d^0 \colon \{1 \to 2\} \longmapsto \begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd}$$

$$d^2 \colon \{0 \to 1\} \longmapsto \begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd}$$

$$d^1\colon \{0 \to 2\} \longmapsto \begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd}$$

So D^1 *sends maps* $f : A \to B$, $g : B \to C$ *in* $\mathscr{C}$ *to* $g \circ f : A \to C$ *i.e.*

$$\mathrm{D}^1 : \{A \xrightarrow{f} B \xrightarrow{g} C\} \longmapsto \{A \xrightarrow{g\circ f} C\}$$

where F *considers*

$$\{A \xrightarrow{g\circ f} C\} \longmapsto \begin{tikzcd} & B \arrow[dr, "g"] & \\ A \arrow[ur, "f"] \arrow[rr, "g\circ f"'] & & C \end{tikzcd}$$

i.e. F *is a section for* D^1. ⧫

Definition 2.11 (Functorial WFS - FWFS)**.** *A weak factorisation system on a category* $\mathscr{C}$ *is called* ***functorial (FWFS)****, if every morphism of* $\mathscr{C}$ *comes from a functorial factorisation as described in Definition 2.9.*

2.3 The Model Structure

Definition 2.12 (Two-out-of-Three)**.** *Let* $\mathscr{C}$ *be a category and* $\mathcal{W}$ *a class of morphisms of* $\mathscr{C}$*. For any two composable morphisms* f, g *of* $\mathscr{C}$*, if two of* $f, g, g \circ f$ *are in* $\mathcal{W}$ *then so is the third.*

Definition 2.13 (Category of Weak Equivalences)**.** *A* ***category with weak equivalences*** *is a category* $\mathscr{C}$ *equipped with a subcategory* $\mathcal{W} \subseteq \mathscr{C}$*, which contains all isomorphisms of* $\mathscr{C}$ *and satisfies the two-out-of-three property.*

Definition 2.14 (Model Structure)**.** *A* ***model structure*** *on a category* $\mathscr{C}$ *is a triple* $(\mathcal{W}, \mathcal{C}, \mathcal{F})$ *of classes of morphisms in* $\mathscr{C}$*, subject to*

1. $\mathcal{W}$ *turns* $\mathscr{C}$ *into a category with weak equivalences.*
2. $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ *and* $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ *are two functorial weak factorisation systems on* $\mathscr{C}$*.*

Remark 2.15. *We often denote the arrows in* $\mathcal{W}$ *by* $\xrightarrow{\sim}$ *and call them weak equivalences, the ones in* $\mathcal{C}$ *by* $\hookrightarrow$ *and call them cofibrations, finally the ones in* $\mathcal{F}$ *by* $\twoheadrightarrow$ *and call them fibrations.* ⧫

Finally we are ready to state the amazing definition of a model category (in our case with the addition of functorial factorisations).

Definition 2.16 (Model Category)**.** *A* ***model category*** *is a bicomplete category* $\mathscr{C}$ *equipped with a model structure.*

When we will work with different model categories later on, we often use a subscript with the respective classes of morphisms to better distinguish between them. For the above general model category $\mathscr{C}$, we would write $\mathcal{W}_{\mathscr{C}}$, $\mathcal{C}_{\mathscr{C}}$ and $\mathcal{F}_{\mathscr{C}}$ for the respective classes of morphisms.

Remark 2.17. *The bicompleteness condition yields the existence of an initial object, usually denoted by $\emptyset$, and terminal object, usually denoted by $*$, for a model category $\mathscr{C}$.* ⧫

A trivial but still important example is the following one.

Example 2.18. *Any category can be endowed with the trivial model category structure, that is the weak equivalences are the isomorphisms in this category and the fibrations and cofibrations consist of any map in the category. If we consider a category $\mathscr{C}$ we usually denote this model category structure by $\mathscr{C}_T$.* ◀

The following lemma is from [ERie09]. A slightly more general statement of the lemma, not relying on functorial factorisations, may also be found in [ERie09].

Lemma 2.19 ([ERie09])**.** *Let $\mathscr{C}$ be a model category, then the class $\mathcal{W}$ is closed under retracts.*

Proof. Let $w \in \mathcal{W}$ be a weak equivalence and suppose that we have a retract diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & C \arrow[r] \arrow[d, "w"] & A \arrow[d, "f"] \\ B \arrow[r] & D \arrow[r] & B. \end{tikzcd}$$

Applying the functorial factorisation from $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ to this diagram gives a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "u"] & A \arrow[d, "g"] \\ V \arrow[r] \arrow[d, "h"'] & W \arrow[r] \arrow[d, "v"] & V \arrow[d, "h"] \\ B \arrow[r] & D \arrow[r] & B \end{tikzcd}$$

with $u, g \in \mathcal{C} \cap \mathcal{W}$ and $v, h \in \mathcal{F}$ such that $w = v \circ u$ and $f = h \circ g$. The horizontal composites of the first diagram are identities. Since we are dealing with functorial factorisations, the middle horizontal composite of the above diagram is also an identity.

Hence, h is a retract of v. By the 2-3 property v is a weak equivalence, so $v \in \mathcal{F} \cap \mathcal{W}$ and hence $h \in \mathcal{F} \cap \mathcal{W}$ since both classes of FWFS are closed under retracts. Thus $f = h \circ g \in \mathcal{W}$ by the 2-3 property. ■

With this lemma and all the work we did so far, we showed, that Definition 2.16 actually coincides with the usual definition of a model category, where usual means a model category with functorial factorisations, see for instance [MHov91]. Even if not, the above proof can be slightly modified to not use functorial factorisations (see for instance [ERie09]) showing that Definition 2.16 without using FWFS but just WFS is equivalent to the usual definition of a model category as stated for example in [DwSp95], this definition can also be found in Appendix B of the present work.

Definition 2.20. *1. A map of a model category $\mathscr{C}$ which is a fibration and a weak equivalence is called an* ***acyclic fibration*** *or* ***trivial fibration****, similarly a map which is a cofibration and a weak equivalence is called an* ***acyclic cofibration*** *or* ***trivial cofibration****.*

2. We denote by $\emptyset$ the initial object and by $*$ the terminal object in $\mathscr{C}$.

3. An object A in $\mathscr{C}$ is called ***cofibrant****, if the unique map $\emptyset \to A$ is a cofibration and we call the class of such objects $C(\mathscr{C})$.*

4. An object A in $\mathscr{C}$ is called ***fibrant****, if the unique map $A \to *$ is a fibration and we call the class of such objects $F(\mathscr{C})$.*

Notice that the initial and terminal object in the above definition always exist, this follows from the bicompleteness property of model categories.

The proof of the following result will already use material which we introduce in the next sections. It is intentionally stated here as a nice termination of the discussion about model categories. The proposition and proof is from Appendix E in [AnJo].

Proposition 2.21 ([AnJo]). *A model category $\mathscr{C}$ is determined by its class of cofibrations $\mathcal{C}$ together with its class of fibrant objects $F(\mathscr{C})$.*

Proof. It is enough to show that the class $\mathcal{W}$ is determined by $\mathcal{C}$ and $F(\mathscr{C})$. The class of acyclic cofibrations is determined by $\mathcal{C}$, since the right class of a weak factorisation system is determined by its left class. For any map $u : A \to B$, there exists a commutative square

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "u'"'] & A \arrow[d, "u"] \\ B' \arrow[r] & B \end{tikzcd}$$

in which the horizontal maps are acyclic fibrations and the objects A' and B' are cofibrants. The map u is acyclic iff the map u' is acyclic. Hence it suffices to show that the class $\mathcal{W} \cap \mathcal{C}$ is determined by $\mathcal{C}$ and $F(\mathscr{C})$. If A and B are two objects of $\mathscr{C}$, we denote by $h(A, B)$ the set of maps $A \to B$ in the homotopy category $\mathrm{Ho}(\mathscr{C})$. A map between two cofibrant objects $u : A \to B$ belongs to $\mathcal{W}$ iff the map $h(u, X) : h(B, X) \to h(A, X)$ is bijective for every object $X \in F(\mathcal{W})$. If A is cofibrant in $\mathscr{C}$ and $X \in F(\mathcal{W})$, then the set $h(A, X)$ is a quotient of the set $\mathscr{C}(A, X)$ by the left homotopy relation. Let us factor the codiagonal map $A \coprod A \to A$ as a cofibration $(i_0, i_1) : A \coprod A \to \mathrm{Cyl}(A)$ followed by an acyclic fibration $\mathrm{Cyl}(A) \to A$. The construction of the cylinder object only depends on $\mathcal{C}$. It follows that the left homotopy relation on the set $\mathscr{C}(A, X)$ only depends on $\mathcal{C}$. Hence also the set $h(A, X)$. It follows that $\mathcal{W}$ is determined by $\mathcal{C}$ and $F(\mathscr{C})$. ■

2.4 The Small Object Argument

We will now state one of the more important results, a tool which will be used to show that a certain category is a model category. It will be especially useful for the categories **Top** of topological spaces and **sSet** of simplicial sets, though we will use it in a form of another theorem (the recognition theorem for cofibrantly generated model categories) which makes it somehow easier to check our desired properties.

We use material from [PHir03] and [MHov91].

Definition 2.22 (Permit Small Object Argument)**.** *Let $\mathscr{C}$ be a model category and I a set of morphisms in $\mathscr{C}$. We say that I **permits the small object argument**, if the domains of the elements of I are small relative to I-cell.*

The next result is a very important one, it can be found in this form in [MHov91] or [PHir03], the proof follows [MHov91].

Theorem 2.23 (The Small Object Argument)**.** *Let $\mathscr{C}$ be a cocomplete category and I a set of morphisms in $\mathscr{C}$. Suppose that I permits the small object argument, then any morphism $f \in \mathscr{C}$ may be factored as*

$$f : X \xrightarrow{\in\ I\text{-cell}} E_f \xrightarrow{\in\ I\text{-inj}} Y$$

in a functorial way.

Proof. Consider a cardinal κ such that every domain of I is κ-small relative to I-cell and λ a κ-filtered ordinal. Given $f : X \to Y$ we will define a functorial λ-sequence $Z^f : \lambda \to \mathscr{C}$ such that $Z_0^f = X$ and a natural transformation $Z^f \xrightarrow{\rho^f} Y$ factoring f. Each map $Z_\beta^f \to Z_{\beta+1}^f$ will be a pushout of a coproduct of maps of I. Then we will define γf to be the composition of Z^f and δf to be the map $E_f = \operatorname{colim} Z^f \to Y$ induced by ρ^f. γ and δ will depend on the choice of colimit functor as well. It follows from the previous two lemmata that γf is a relative I-cell complex.

Z^f and $\rho^f : Z^f \to Y$ will be defined by transfinite induction. We begin with $Z_0^f = X$ and $\rho_0^f = f$. Assume we have defined Z_α^f and ρ_α^f for all $\alpha < \beta$ for some limit ordinal β. Define $Z_\beta^f = \operatorname{colim}_{\alpha<\beta} Z_\alpha^f$ and define ρ_α^f to be the map induced by the ρ_α^f. From Z_β^f and ρ_β^f we define $Z_{\beta+1}^f$ and $\rho_{\beta+1}^f$ in the following way.

Let S be the set of all commutative squares

$$\begin{tikzcd} A \arrow[r] \arrow[d, "g"'] & Z_\beta^f \arrow[d, "\rho_\beta^f"] \\ B \arrow[r] & Y \end{tikzcd}$$

where $g \in I$. For $s \in S$ let $g_s : A_s \to B_s$ denote the corresponding map of I. Define $Z_{\beta+1}^f$ to be the pushout in the diagram

$$\begin{tikzcd} \coprod_{s\in S} A_s \arrow[r] \arrow[d, "\coprod g_s"'] & Z^f_\beta \arrow[d] \\ \coprod_{s\in S} B_s \arrow[r] & Z^f_{\beta+1}. \end{tikzcd}$$

Define $\rho^f_{\beta+1}$ to be the map induced by ρ^f_β.

we are left to show that $\delta f = \operatorname{colim} \rho^f_\beta$. $E_f = \operatorname{colim} Z^f_\beta \to Y$ has the RLP with respect to I. Indeed, assume we have a commutative square

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "g"'] & E_f \arrow[d, "\delta f"] \\ B \arrow[r, "k"'] & Y \end{tikzcd}$$

where g is a map of I. Since the domains of the maps of I are κ-small relative to I-cell there is a $\beta < \lambda$ such that h is the composite $A \xrightarrow{h_\beta} Z^f_\beta \to E_f$. By construction there is a map $B \xrightarrow{k_\beta} Z^f_{\beta+1}$ such that $k_\beta g = i h_\beta$ and $k = \rho_{\beta+1 k_\beta}$ where i is the map $Z^f_\beta \to Z^f_{\beta+1}$. The composition $B \xrightarrow{k_\beta} Z^f_{\beta+1} \to E_f$ is the required lift in our diagram. ■

Corollary 2.24 ([MHov91])**.** *Let I be a set of morphisms in a cocomplete category $\mathscr{C}$. Suppose also that I permits the small object argument. Then given $f : A \to B$ in I-cof, there is a map $g : A \to C$ in I-cell such that f is a retract of g by a map which fixes A.*

Proof. By the small object argument we have a factorisation $f = p \circ g$, where $g \in I$-cell and $p \in I$-inj. $f \in I$-cof implies that f has the LLP with respect to p and so we conclude by the retract argument. ■

Proposition 2.25 ([MHov91])**.** *Let I be a set of maps in a cocomplete category $\mathscr{C}$. Suppose that I permits the small object argument and A is some object which is small relative to I-cell. Then A is in fact small relative to I-cof.*

Proof. Assume A is κ-small relative to I-cell. Suppose that λ is a κ-filtered ordinal and $X : \lambda \to \mathscr{C}$ is a λ-sequence of I-cofibrations. We construct a λ-sequence Y of relative I-cell complexes and natural transformations $i : X \to Y$ and $r : Y \to X$ with $ri = 1$ by transfinite induction. Define $Y_0 = X_0$ and i_0 and r_0 to be the identity map. Having defined Y_β, i_β and r_β apply functorial factorisations of the small object argument to the composite $Y_\beta \xrightarrow{r_\beta} X_\beta \xrightarrow{X_{\beta+1}}$ to obtain $g_\beta : Y_\beta \to Y_{\beta+1}$ and $r_{\beta+1} : Y_{\beta+1} \to X_{\beta+1}$ with $g_\beta \in I$-cell, $r_{\beta+1} \in I$-inj and $r_{\beta+1} g_\beta = f_\beta r_\beta$. Then we have a solid arrow commutative diagram

$$\begin{tikzcd} X_\beta \arrow[r, "g_\beta \circ i_\beta"] \arrow[d, "f_\beta"'] & Y_{\beta+1} \arrow[d, "r_{\beta+1}"] \\ X_{\beta+1} \arrow[ur, dotted, "i_{\beta+1}"] \arrow[r, equal] & X_{\beta+1}. \end{tikzcd}$$

Since $f_\beta \in I$-cof and $r_{\beta+1} \in I$-inj there is a lift in the above diagram (dotted arrow). For limit ordinals β we define $Y_\beta = \operatorname{colim}_{\alpha<\beta} i_\beta = \operatorname{colim}_{\alpha<\beta} i_\alpha$ and $r_\beta = \operatorname{colim}_{\alpha<\beta} r_\alpha$. Now the map $\operatorname{colim}\mathscr{C}(W, X_\beta) \to \mathscr{C}(W, \operatorname{colim} X_\beta)$ is a retract of the corresponding map for Y. Since W is κ-small relative to I-cell the corresponding map for Y is an isomorphism. Therefore the map X must also be an isomorphism and so W is κ-small relative to I-cof as well. ■

The reason for the above proposition is, that some authors use I-cof instead of I-cell in the definition of permitting the small object argument.

2.5 Cofibrantly Generated Model Categories

Cofibrantly generated model categories are particularly nice to work with, as they provide the small object argument. Most likely the notion is derived from an attempt to generalise the concept observed in the specific cases of topological spaces and simplicial sets. These two examples of cofibrantly generated model categories will be discussed later. Another helpful fact is that most categories used in practice, turn out to be cofibrantly generated.

Furthermore this kind of model category turns out to be very helpful in order to establish Quillen pairs between certain categories, but more on that later.

It is also an important notion in order to define the model structures which will follow, these include for example cellular model categories in which we are particularly interested.

We will follow [MHov91], [PHir03] and [nLab].

Definition 2.26. *Let $\mathscr{C}$ be a model category. We say that a triple $(\mathscr{C}, I, J)$ is a* ***cofibrantly generated model category*** *for sets I and J of morphisms of $\mathscr{C}$ if*

1. *I and J permit the small object argument.*
2. *The class of fibrations is J-inj.*
3. *The class of acyclic fibrations is I-inj.*

Notation 2.27. *We refer to I as the set of* ***generating cofibrations*** *and to J as the set of* ***generating acyclic cofibrations****. Sometimes we may put a subscript to emphasise the category we are considering i.e. $I_{\mathscr{C}}$ or $J_{\mathscr{C}}$.*

We summarise some properties about cofibrantly generated model categories.

Proposition 2.28 ([PHir03])**.** *Let $(\mathscr{C}, I, J)$ be a cofibrantly generated model category.*

1. *The cofibrations form the class I-cof.*
2. *Every cofibration is a retract of a relative I-cell complex.*
3. *The domains of I are small relative to the cofibrations.*
4. *The acyclic cofibrations form the class of J-cof.*
5. *Every acyclic cofibration is a retract of a relative J-cell complex.*
6. *The domains of J are small relative to the acyclic cofibrations.*

Proof. Immediate consequence of Corollary 2.24 and Proposition 2.25. ■

Therefore we also have the following corollary.

Corollary 2.29. *Let $(\mathscr{C}, I, J)$ be a cofibrantly generated model category. Then we have*

1. *I-cof $= LLP(RLP(I))$.*
2. *J-cof $= LLP(RLP(J))$.*

Proposition 2.28 and Corollary 2.29 really give us an idea, why cofibrantly generated model categories are called the way they are. Indeed, the proposition tells us, that I-cof is the class of cofibrations and J-cof is the class of acyclic cofibrations. The corollary says, that any of these can be created with the left and right lifting properties from the sets I and J respectively.

Remark 2.30. *The functorial factorisations in a cofibrantly generated model category need not be given by the small object argument, but those factorisations are always available. Even though some authors define cofibrantly generated model categories in such a way, that these functorial factorisations are choosen. This is for instance the case in [PHir03], however we followed the definition of [MHov91].* ♦

2.5.1 The Recognition Theorem

It follows one of the more important results, which gives a way to determine if a category admits the structure of a cofibrantly generated model category. This theorem was due to D. M. Kan and can be found in this form in [MHov91] or [PHir03], the proof will follow [MHov91].

Theorem 2.31 (Recognition Theorem for Cofibrantly Generated Model Categories)**.** *Let $\mathscr{C}$ be a bicomplete category. Suppose $\mathcal{W}$ is a subcategory of $\mathscr{C}$ and I and J are sets of maps of $\mathscr{C}$. Then there is a cofibrantly generated model structure $(\mathscr{C}, I, J)$ on $\mathscr{C}$, with $\mathcal{W}$ as the subcategory of weak equivalences iff the following conditions hold.*

1. *The subcategory $\mathcal{W}$ has the 2-3 property and is closed under retracts.*
2. *I and J permit the small object argument.*
3. *J-cof $\subseteq \mathcal{W} \cap I$-cof and I-inj $\subseteq \mathcal{W} \cap J$-inj.*
4. *$\mathcal{W} \cap I$-cof $\subseteq J$-cof or $\mathcal{W} \cap J$-inj $\subseteq I$-inj.*

Proof. The conditions hold for a cofibrantly generated category. For the other implication suppose that we have a category $\mathscr{C}$ with a subcategory $\mathcal{W}$ and sets of maps I and J satisfying the hypotheses of the theorem. Define a map to be a fibration iff it is in J-inj and define a map to be a cofibration iff it is in I-cof. Then the fibrations and cofibrations are closed under retracts. It follows from the hypotheses that every map in J-cell is an acyclic cofibration and also that every map in I-cell is an acyclic fibration.

We define functorial factorisations $f = \beta(f) \circ \alpha(f) = \delta(f) \circ \gamma(f)$ by using the small object argument on I and J respectively by choosing colimit functors and appropriate cardinals. Thus $\alpha(f)$ is in I-cell and is hence a cofibration, $\beta(f)$ is in I-inj and hence an acyclic fibration $\gamma(f)$ is in J-cell and hence an acyclic cofibration and $\delta(f)$ is in J-inj and hence a fibration.

We use the last hypotheses to conclude. Assume $\mathcal{W} \cap I$-cof $\subseteq J$-cof, then every acyclic cofibration is in J-cof and so has the LLP with respect to the fibrations which form the class J-inj. Given an acyclic fibration $p : X \to Y$ we need to show that p has the RLP with respect to all cofibrations or equivalently with respect to I. We can factor $p = \beta(p) \circ \alpha(p)$ where $\alpha(p)$ is a cofibration and $\beta(p) \in I$-inj. Since $\mathcal{W}$ has the 2-3 property $\alpha(p)$ is an acyclic cofibration. Hence p has the RLP with respect to $\alpha(p)$. The retract argument gives us that p is a retract of $\beta(p)$ so $p \in I$-inj as required.

A similar argument concludes the case for $\mathcal{W} \cap J$-inj $\subseteq I$-inj. ■

We will now state a very nice result which was originally due to D. M. Kan. It basically sais that under certain conditions, there is an induced cofibrantly generated model structure on a category if there is an adjuncion between the categories. This version and also the proof of the statement can be found as Theorem 11.3.2 in [PHir03].

Theorem 2.32 ([PHir03])**.** *Let $(\mathscr{C}, I, J)$ be a cofibrantly generated model category. Let $\mathscr{D}$ be a bicomplete category, and let $F : \mathscr{C} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathscr{D} : U$ be a pair of adjoint functors. If we let $FI = \{Fu \mid u \in I\}$ and $FJ = \{Fv \mid v \in J\}$ and if in addition*

1. *FI and FJ permit the small object argument*
2. *U takes relative FJ-cell complexes to weak equivalences*

then there is a cofibrantly generated model category structure $(\mathscr{D}, FI, FJ)$ on $\mathscr{D}$ and the weak equivalences are the maps that U takes into a weak equivalence in $\mathscr{C}$.

The main advantage of working with cofibrantly generated model categories is, that it will be easier to check that functors are Quillen functors. Even though Quillen pair will be defined later, it is best to state this result already here.

Lemma 2.33 ([MHov91])**.** *Let $(\mathscr{C}, I, J)$ be a cofibrantly generated model category and $\mathscr{D}$ a model category. Assume that $F : \mathscr{C} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathscr{D} : U$ is an adjunction pair. Then $F : \mathscr{C} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathscr{D} : U$ is a Quillen pair iff Ff is a cofibration for all $f \in I$ and Ff is an acyclic cofibration for all $f \in J$.*

Proof. The right implication is clear. For the other one Lemma 1.34 says that $F(I\text{-cof}) \subset FI\text{-cof}$. Let $\mathcal{C}$ be the class of cofibrations in $\mathscr{D}$. Then by hypothesis $FI \subset \mathcal{C}$ and so $FI\text{-cof} \subset \mathcal{C}\text{-cof}$. But the definition of a model category implies that $\mathcal{C}\text{-cof} = \mathcal{C}$. Therefore $F(I\text{-cof}) \subset \mathcal{C}$ and so F preserves cofibrations. A similar argument shows that F preserves acyclic cofibrations and so F is a left Quillen functor. ■

Remark 2.34. *The combination of the above theorem and lemma, implies that if we are able to apply Theorem 2.32, then we automatically get a Quillen pair between the two categories. Since we are most interested about Quillen equivalences, this is a great step in the right direction. As we will see in Part III, it is enough to only apply the lemma to give us some desired Quillen pairs.* ♦

2.6 Cellular Model Categories

Here we will give the definition of one of the most important model structures. As far as I know they were first introduced by P. Hirschhorn in [PHir03]. They are based on the cofibrantly generated model structure. This class of model structure turns out to be one of the largest classes, where Bousfield localisations exist for any class of morphisms from the underlying category, under the assumption that we are also left respectively right proper.

This section follows [PHir03] and will be needed for the existence of left Bousfield localisations.

Definition 2.35 (Regular Monomorphism)**.** *A **regular monomorphism** is a morphism $f : A \to B$ in some category $\mathscr{C}$ which occurs as the equaliser of some pair of parallel morphisms $D \rightrightarrows E$ i.e. for which a diagram of the form*

$$C \xrightarrow{f} D \rightrightarrows E$$

exists.

Definition 2.36 (Effective Monomorphism)**.** *Let $\mathscr{C}$ be a category that is closed under pushouts. The map $f : A \to B$ in $\mathscr{C}$ is an **effective monomorphism** if f is the equaliser of the pair of natural inclusions*

$$B \rightrightarrows B \coprod_A B.$$

Example 2.37. *In* **Set** *the class of effective monomorphisms is the class of injective maps. But then again in* **Set** *effective epimorphisms are also the class of surjective maps, so at least in this case we can not see what makes them "effective" morphisms. Still as an example this should be good enough.* ◀

Definition 2.38 (Cellular Model Category)**.** *A **cellular model category** is a cofibrantly generated model category $\mathscr{C}$ such that*

1. *the domains and codomains of the elements of I are compact.*
2. *the domains of the elements of J are small relative to I.*
3. *the cofibrations are effective monomorphisms.*

The following results will be helpful for later, when we show that certain categories are cellular ones.

Proposition 2.39 ([PHir03])**.** *If $\mathscr{C}$ is a category that is closed under pushouts, then a map is an effective monomorphism iff it is a regular monomorphism.*

Proof. If $f : A \to B$ is an effective monomorphism it is defined to be the equaliser of a particular pair of maps. Conversely if $f : A \to B$ is the equaliser of the maps $\begin{tikzcd} B \arrow[r, bend left, "g"] \arrow[r, bend right, "h"'] & W \end{tikzcd}$ then g and h factor as

$$\begin{tikzcd} B \arrow[r, bend left, "i_0"] \arrow[r, bend right, "i_1"'] & B \coprod_A B \arrow[r, "g \coprod h"] & W \end{tikzcd}$$

and we must show that f is the equaliser of i_0 and i_1. Since $(g \coprod h)i_0 = g$ and $(g \coprod h)i_1 = h$ this follows by definition. ■

Proposition 2.40 ([PHir03])**.** *If $\mathscr{C}$ is a category that is closed under pushouts, then an effective monomorphism is a monomorphism.*

Proof. Let $f : A \to B$ be an effective monomorphism and let $g : W \to A$ and $h : W \to A$ be maps such that $fg = fh$. If i_0 and i_1 are natural maps from B to $B \coprod_A B$, then $i_0 f = i_1 f$ and so $i_0 fg = i_1 fg$ and $i_0 fh = i_1 fh$. The universal property of the equaliser now implies $g = h$. ■

Proposition 2.41 ([PHir03])**.** *If $\mathscr{C}$ is a category that is closed under pushouts, then the class of effective monomorphisms is closed under retracts.*

Proof. If $f : A \to B$ is a retract of $g : C \to D$, then we have a diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] & C \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\
B \arrow[r] \arrow[d, bend right, "i_0"'] \arrow[d, bend left, "i_1"] & D \arrow[r] \arrow[d, bend right, "j_0"'] \arrow[d, bend left, "j_1"] & B \arrow[d, bend right, "i_0"'] \arrow[d, bend left, "i_1"] \\
B \coprod_A B \arrow[r] & D \coprod_C D \arrow[r] & B \coprod_A B
\end{tikzcd}$$

in which all of the horizontal compositions are identity maps. If g is an effective monomorphism then g is the equaliser of j_0 and j_1. By a diagram chase argument it follows that f is the equaliser of i_0 and i_1. ■

Proposition 2.42 ([PHir03])**.** *Let $(\mathscr{C}, I, J)$ be a cofibrantly generated model category. If relative I-cell complexes are effective monomorphisms then all cofibrations are effective monomorphisms.*

Proof. This follows from Proposition 2.28 and Proposition 2.41. ■

2.6.1 The Recognition Theorem

We state the recognition theorem for cellular model categories. This theorem was due to P. Hirschhorn and appears in [PHir03].

Theorem 2.43 (Recognition Theorem for Cellular Model Categories)**.** *Let $\mathscr{C}$ be a model category, then $\mathscr{C}$ is a cellular model category if there are sets I and J of maps in $\mathscr{C}$ such that:*

1. *a map is an acyclic fibration iff it has the RLP with respect to every element of I*
2. *a map is a fibration iff it has the RLP with respect to every element of J*
3. *the domains and codomains of the elements of I are compact relative to I*
4. *the domains of the elements of J are small relative to I*
5. *relative I-cell complexes are effective monomorphisms.*

Proof. This is Theorem 12.1.9 in [PHir03]. One shows that I permits the small object argument therefore I is the set of generating cofibrations for $\mathscr{C}$. One then shows that J is the set of generating acyclic cofibrations. Finally one concludes by Proposition 2.42. ■

Remark 2.44. *It is worth pointing out, that if we already have a cofibrantly generated model category and want to show that it is a cellular model category, it is indeed enough to only verify conditions 3. and 5. of the above theorem. We will make use of this observation later on.* ♦

2.7 Combinatorial Model Categories

This section is more for completion, since combinatorial model categories yield another large class of categories, where the Bousfield localisation exists for any class of morphisms in the underlying category, under the assumption that we are also left respectively right proper.

We will follow [ClBa] and [nLab].

Definition 2.45 (Locally Presentable)**.** *A category $\mathscr{C}$ is **locally presentable** if it satisfies the following conditions:*

1. *$\mathscr{C}$ is cocomplete.*
2. *There is a small set S of objects in $\mathscr{C}$ which generates $\mathscr{C}$ Under colimits i.e. every object of $\mathscr{C}$ may be obtained as the colimit of a small diagram taking values in S.*
3. *Every object in $\mathscr{C}$ is small.*
4. *$\mathscr{C}$ is a locally small category.*

Definition 2.46 (Combinatorial)**.** *A model category $\mathscr{C}$ is **combinatorial** if it is locally presentable as a category and cofibrantly generated as a model category.*

Definition 2.47 (Accessible)**.** *A locally small category $\mathscr{C}$ is κ-**accessible** for a regular cardinal κ if*

1. *the category has κ-filtered colimits*
2. *there is a set of κ-compact objects that generate the category under κ-filtered colimits.*

*If there is such a κ for a category $\mathscr{C}$, then $\mathscr{C}$ is called an **accessible category**.*

2.7.1 The Recognition Theorem

The theorem and the concept of this type of categories was originally due to Smith. The result appears in a similar form with proof as Proposition 1.7 in [ClBa], the form stated here originates from [nLab].

Theorem 2.48 (Recognition Theorem for Combinatorial Model Categories)**.** *If $\mathscr{C}$ is a locally presentable category, $\mathrm{Arr}_{\mathcal{W}}(\mathscr{C}) \subseteq \mathrm{Arr}(\mathscr{C})$ an accessible full subcategory of the arrow category $\mathrm{Arr}(\mathscr{C})$ on a class $\mathcal{W} \subseteq \mathrm{Mor}(\mathscr{C})$, $I \subseteq \mathrm{Mor}(\mathscr{C})$ a proper set of morphisms in $\mathscr{C}$ such that*

1. *$\mathcal{W}$ has the 2-3 property*
2. *I-inj $\subset \mathcal{W}$*
3. *$\mathcal{C}(I) \cap \mathcal{W}$ closed under pushouts and transfinite composition*

then $\mathscr{C}$ is a combinatorial model category with weak eqivalences $\mathcal{W}$ and cofibrations $\mathcal{C}(I)$ and fibrations $inj(\mathcal{W} \cap \mathcal{C}(I))$.

2.8 Simplicial Model Categories

This section is also mainly for completion and also since this type of model structure is a very important one for a lot of applications.

Definition 2.49 (Simplicial Category)**.** *A **simplicial category** $\mathscr{C}$ is a category together with the following data.*

1. *(Function Complex) for every $X, Y \in \mathrm{ob}(\mathscr{C})$ a simplicial set $\mathrm{Map}(X, Y)$*
2. *(Composition Rule) for every $X, Y, Z \in \mathrm{ob}(\mathscr{C})$ a map of simplicial sets*
$$c_{X,Y,Z} : \mathrm{Map}(Y, Z) \times \mathrm{Map}(X, Y) \to \mathrm{Map}(X, Z)$$
3. *for every $X \in \mathrm{ob}(\mathscr{C})$ a map of simplicial sets $i_* : * \to \mathrm{Map}(X, X)$, here $*$ denotes the simplicial set with a single point*
4. *for every $X, Y \in \mathrm{ob}(\mathscr{C})$ an isomorphism*
$$\mathrm{Map}(X, Y)_0 \cong \mathscr{C}(X, Y)$$
commuting with the composition rule

such that for all $W, X, Y, Z \in \mathrm{ob}(\mathscr{C})$ the following diagrams commute

1. *(Associativity)*
$$\begin{tikzcd}
(\mathrm{Map}(Y,Z) \times \mathrm{Map}(X,Y)) \times \mathrm{Map}(W,X) \arrow[r, "c_{X,Y,Z} \times 1_{\mathrm{Map}(W,X)}"] \arrow[d, "\cong"'] & \mathrm{Map}(X,Z) \times \mathrm{Map}(W,X) \arrow[dd, "c_{W,X,Z}"] \\
\mathrm{Map}(Y,Z) \times (\mathrm{Map}(X,Y) \times \mathrm{Map}(W,X)) \arrow[d, "1_{\mathrm{Map}(Y,Z)} \times c_{W,X,Y}"'] & \\
\mathrm{Map}(Y,Z) \times \mathrm{Map}(W,Y) \arrow[r, "c_{W,Y,Z}"'] & \mathrm{Map}(W,Z)
\end{tikzcd}$$
2. *(Left Unit)*
$$\begin{tikzcd}
* \times \mathrm{Map}(X,Y) \arrow[rr, "i_Y \times 1_{\mathrm{Map}(X,Y)}"] \arrow[dr, "\cong"'] & & \mathrm{Map}(Y,Y) \times \mathrm{Map}(X,Y) \arrow[dl, "c_{X,Y,Y}"] \\
& \mathrm{Map}(X,Y) &
\end{tikzcd}$$

3. *(Right Unit)*

$$\begin{tikzcd} \mathrm{Map}(X,Y) \times * \arrow[rr, "i_Y \times 1_{\mathrm{Map}(X,Y)}"] \arrow[dr, "\cong"'] & & \mathrm{Map}(Y,Y) \times \mathrm{Map}(X,Y). \arrow[dl, "c_{X,X,Y}"] \\ & \mathrm{Map}(X,Y) & \end{tikzcd}$$

Definition 2.50 (Simplicial Model Category)**.** *A **simplicial model category** is a model category $\mathscr{C}$ that is also a simplicial category such that the following hold.*

1. *For any $X, Y \in \mathrm{ob}(\mathscr{C})$ and for simplicial set K there are objects $X \otimes K$ and Y^K of $\mathscr{C}$ such that there are isomorphisms of sets*
$$\mathrm{Map}(X \otimes K, Y) \cong \mathrm{Map}(K, \mathrm{Map}(X,Y)) \cong \mathrm{Map}(X, Y^K)$$
natural in X, Y and K.
2. *If $i : A \to B$ is a cofibration and $p : X \to Y$ is a fibration then the map of simplicial sets*
$$\mathrm{Map}(B,X) \xrightarrow{i^* \times p_*} \mathrm{Map}(A,X) \times_{\mathrm{Map}(A,Y)} \mathrm{Map}(B,Y)$$
is a fibration that is an acyclic fibration if either i or p is a weak equivalence.

Remark 2.51. *Basically, a simplicial model category is a model category enriched over* $\mathbf{sSet}_Q$ *in a compatible way, but the Quillen model structure on* **sSet** *will be discussed in a later section.* ♦

2.9 Proper Model Categories

We introduce the last type of categories needed in order to permit the existence of Bousfield localisations with respect to any class of morphismsm of a given category. We will see later that both the category of topological spaces and the one of simplicial sets are left and right proper.

Definition 2.52 (Proper Categories)**.** *Let $\mathscr{C}$ be a model category.*

1. *$\mathscr{C}$ will be called **left proper** if every pushout of a weak equivalence along a cofibration is a weak equivalence.*
2. *$\mathscr{C}$ is called **right proper** if every pullback of a weak equivalence along a fibration is a weak equivalence.*
3. *$\mathscr{C}$ is called **proper** if it is both left proper and right proper.*

The following result is especially powerful, though also not at all easy to prove. Anyways it has a very nice consequence in the form of the following proposition. It can be found as Proposition 13.1.2 in [PHir03].

Proposition 2.53 (C. L. Reedy, [PHir03]). *Let $\mathscr{C}$ be a model category.*

1. *Every pushout of a weak equivalence between cofibrant objects along a cofibration is a weak equivalence.*
2. *Every pullback of a weak equivalence between fibrant objects alsong a fibration is a weak equivalence.*

Corollary 2.54 ([PHir03]). *Let $\mathscr{C}$ be a model category.*

1. *If every object of $\mathscr{C}$ is cofibrant then $\mathscr{C}$ is left proper.*
2. *If every object of $\mathscr{C}$ is fibrant then $\mathscr{C}$ is right proper.*
3. *If every object of $\mathscr{C}$ is both cofibrant and fibrant then $\mathscr{C}$ is proper.*

Proof. Consequence of Proposition 2.53. ■

3 Homotopy

Let $\mathscr{C}$ be a model category and fix two objects A and X in $\mathscr{C}$. The goal is to construct some reasonable homotopy relations on $\mathscr{C}(A, X)$. This relation will indeed turn out to be an equivalence relation, the main reason for defining this is to be able to define the category of fibrant cofibrant objects modulo this equivalence relation. The existence of this category will depend upon this chapter. This category will turn out to have nice properties, especially it remains locally small if the initial category was already locally small.

When we define the homotopy category of a model category this construction will help us to show that the definition of the homotopy category, which will be a localisation, preserves a lot of good properties from the original category. That is, our main concern that we will remain locally small should not be a problem at all anymore. We will achieve this by showing that the homotopy category is equivalent to the category of fibrant cofibrant objects modulo the above equivalence relation.

We will follow [DwSp95] with some inspiration from [nLab] other references include [PHir03] and [MHov91]. That is, most of the statements are from [DwSp95] but we provide a lot more detailed proofs to all these statements.

3.1 Homotopy Relations on Maps

3.1.1 Cylinder and Path Objects

Definition 3.1 (Cylinder and Path Object)**.** *Let $\mathscr{C}$ be a model category and A, X two objects in $\mathscr{C}$.*

*A **cylinder object** for A is an object $\mathrm{Cyl}(A)$ of $\mathscr{C}$ together with a diagram*

$$\Delta_A : A \coprod A \xrightarrow{i=(i_0,i_1)} \mathrm{Cyl}(A) \xrightarrow{\sim} A$$

which is a factorisation of the codiagonal map (or folding map)

$$\Delta_A : A \coprod A \to A.$$

We call a cylinder object $\mathrm{Cyl}(A)$

1. *a **good cylinder object** if $A \coprod A \to \mathrm{Cyl}(A)$ is a cofibration.*
2. *a **very good cylinder object** if it is a good cylinder object and if in addition the map $\mathrm{Cyl}(A) \to A$ is an (acyclic) fibration.*

*Dually we define a **path object** for X to be an object $\mathrm{Path}(X)$ of $\mathscr{C}$ together with a diagram*

$$\nabla_X : X \xrightarrow{\sim} \mathrm{Path}(X) \xrightarrow{p=(p_0,p_1)} X \prod X$$

which is a factorisation of the diagonal map

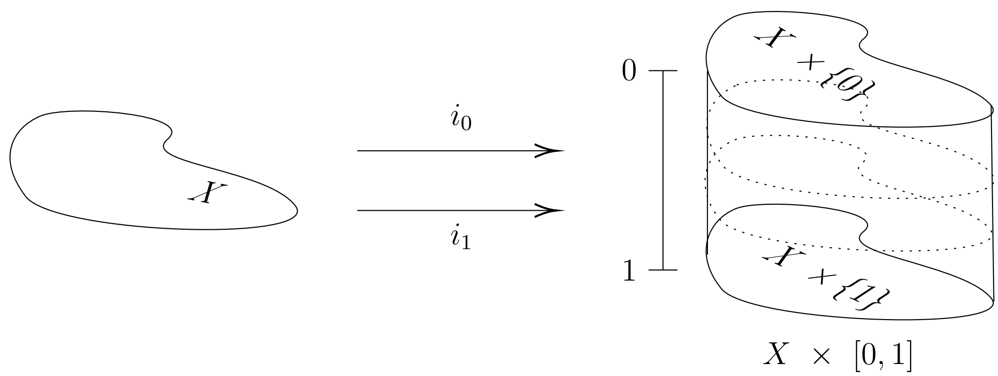
$$\nabla_X : X \to X \prod X.$$

We call a path object $\mathrm{Path}(X)$

1. *a* ***good path object*** *if* $\mathrm{Path}(X) \to X \prod X$ *is a fibration.*
2. *a* ***very good path object*** *if it is a good path object and if in addition the map* $X \to \mathrm{Path}(X)$ *is an (acyclic) cofibration.*

The following example is inspired by [nLab].

Example 3.2. *In the category* **Top** *of topological spaces, with the Quillen model structure (which we will define later) a cylinder object for some object* X *in* **Top** *would be the standard cylinder object* $X \times [0,1]$*. In the model structure of Quillen a sufficient condition for* $X \times [0,1]$ *to be good, is that* X *is a CW complex. From this example one really sees, why such an object is called cylinder object.*



◀

Remark 3.3. *Notice, that there is always the trivial cylinder and path object for every object in a model category* $\mathscr{C}$*. Just choose* $\mathrm{Cyl}(A) := A$ *or* $\mathrm{Path}(X) := X$ *for some objects* A *and* X *in* $\mathscr{C}$*. Also, as we will see in the next lemma, since we may factor* Δ *and* ∇*, there will always exist one specific cylinder and path object for some object in* $\mathscr{C}$*.* ◆

Lemma 3.4. *Let* $\mathscr{C}$ *be a model category and let* A *and* X *be objects in* $\mathscr{C}$*. Then the following holds:*

1. *There is at least one very good cylinder object for* A*.*
2. *There is at least one very good path object for* X*.*

Proof. Consider the codiagonal map $\Delta_A : A \coprod A \to A$, which we may factor as

$$\Delta_A : A \coprod A \xhookrightarrow{(i_0, i_1)} B \underset{p}{\overset{\sim}{\twoheadrightarrow}} A.$$

Define $B := \mathrm{Cyl}(A)$, which is now a very good cylinder object for A. This concludes the first part, the second follows by duality. ■

Lemma 3.5 ([DwSp95])**.** *Let* $\mathscr{C}$ *be a model category.*

1. *If* A *is a cofibrant object in* $\mathscr{C}$ *and* $\mathrm{Cyl}(A)$ *is a good cylinder object for* A*, then the maps* $i_0, i_1 : A \to \mathrm{Cyl}(A)$ *are acyclic cofibrations.*
2. *If* X *is a fibrant object in* $\mathscr{C}$ *and* $\mathrm{Path}(X)$ *is a good path object for* X*, then the maps* $p_0, p_1 : \mathrm{Path}(X) \to X$ *are acyclic fibrations.*

Proof. Consider the identity map $id_A : A \to A$, by Definition 3.1 we get

$$id_A : A \xrightarrow{i_0} \mathrm{Cyl}(A) \xrightarrow[p]{\sim} A \quad \text{and} \quad id_A : A \xrightarrow{i_1} \mathrm{Cyl}(A) \xrightarrow[p]{\sim} A$$

id_A and p are weak equivalences and hence, by the 2-3 property we get that i_0 and i_1 must be weak equivalences.
Since A is cofibrant we have by definition a map $\emptyset \to A$ and $A \coprod A$ is defined via the pushout diagram

$$\begin{tikzcd} \emptyset \arrow[r, "t"] \arrow[d, "l"'] & A \arrow[d, "in_0"] \\ A \arrow[r, "in_1"'] & A \coprod A. \end{tikzcd}$$

Notice that this diagram is well defined and commutes, since l is unique and hence we must have equality $in_0 \circ l = in_1 \circ l$.

in_0 is the pushout of l which is a cofibration and in_1 is the pushout of t which is also a cofibration since A is cofibrant.

Hence in_0 and in_1 are both cofibrations. We have $i_0 = i \circ in_0$ and $i_1 = i \circ in_1$ for $i : A \coprod A \to \mathrm{Cyl}(A)$.

Since $\mathrm{Cyl}(A)$ is a good cylinder object it follows that $A \coprod A \xrightarrow{i} \mathrm{Cyl}(A)$ is a cofibration. Consider

$$i_0 : A \xrightarrow{in_0} A \coprod A \xrightarrow{i} \mathrm{Cyl}(A) \quad \text{and} \quad i_1 : A \xrightarrow{in_1} A \coprod A \xrightarrow{i} \mathrm{Cyl}(A).$$

in_0, in_1 and i are all cofibrations then so are the composites i_0 and i_1. The second property follows by a dual argument. ■

3.1.2 Left and Right Homotopy

Next we define the notions of left and right homotopy.

Definition 3.6 (Left and Right Homotopy)**.** *Let $\mathscr{C}$ be a model category and $f, g : A \to X$ two parallel morphisms in $\mathscr{C}$.*

*A **left homotopy** $\eta : f \Rightarrow_L g$ from f to g is a morphism $\eta : \mathrm{Cyl}(A) \to X$ from a cylinder object of A such that the following diagram commutes*

$$\begin{tikzcd} A \arrow[r, "i_0"] \arrow[dr, "f"'] & \mathrm{Cyl}(A) \arrow[d, "\eta"] & A. \arrow[l, "i_1"'] \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

A left homotopy is called

1. *a **good left homotopy** if $\mathrm{Cyl}(A)$ is a good cylinder object.*
2. *a **very good left homotopy** if $\mathrm{Cyl}(A)$ is a very good cylinder object.*

Dually, a ***right homotopy*** $\eta : f \Rightarrow_R g$ *from* f *to* g *is a morphism* $\eta : A \to \mathrm{Path}(X)$ *to some path object of* X *such that the following diagram commutes*

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[d, "\eta"] \arrow[dr, "g"] & \\ X & \mathrm{Path}(X) \arrow[l, "p_0"] \arrow[r, "p_1"'] & X. \end{tikzcd}$$

A right homotopy is called

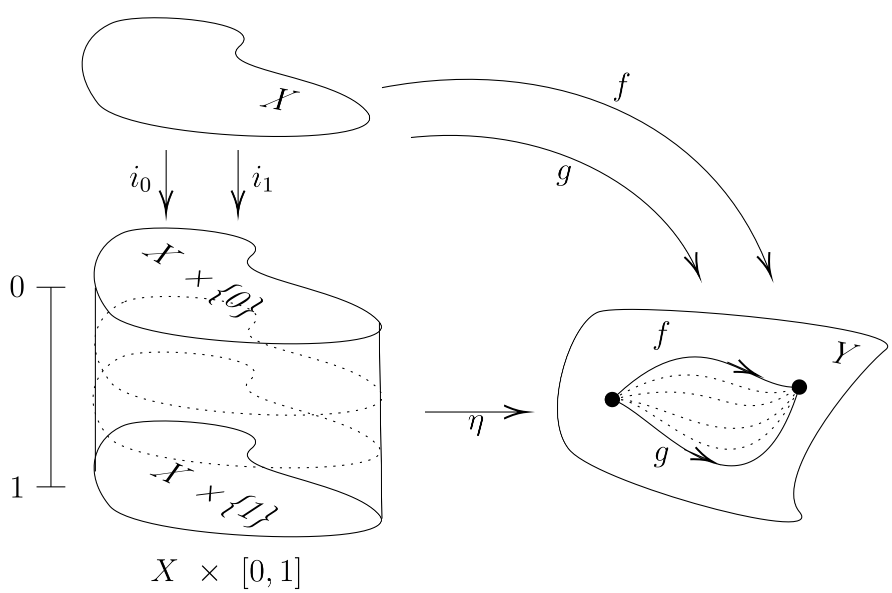
1. *a* ***good right homotopy*** *if* $\mathrm{Path}(X)$ *is a good path object.*
2. *a* ***very good right homotopy*** *if* $\mathrm{Path}(X)$ *is a very good path object.*

The following example is inspired from [nLab].

Example 3.7. *In* **Top** *the category of topological spaces, consider two maps* $f, g : X \to Y$*. A left homotopy* $\eta : F \Rightarrow_L g$ *is a continuous function* $\eta : X \times [0,1] \to Y$ *such that it fits in a commutative diagram of the form*

$$\begin{tikzcd} X \arrow[r, "i_0"] \arrow[dr, "f"'] & X \times [0,1] \arrow[d, "\eta"] & X. \arrow[l, "i_1"'] \arrow[dl, "g"] \\ & Y & \end{tikzcd}$$

In words, η *deforms* f *into* g *in a continuous way. As a picture, one may think of it in the following way*



◀

Lemma 3.8 ([DwSp95]). *Let $\mathscr{C}$ be a model category and $f, g : A \to X$ two morphisms in $\mathscr{C}$.*

1. *If $f \Rightarrow_L g : A \to X$ is a left homotopy η then there exists a good left homotopy from f to g.*
2. *If $f \Rightarrow_L g : A \to X$ is a left homotopy η and X is in addition fibrant then there exists a very good left homotopy from f to g.*
3. *If $f \Rightarrow_R g : A \to X$ is a right homotopy η then there exists a good right homotopy from f to g.*
4. *If $f \Rightarrow_R g : A \to X$ is a right homotopy η and A is in addition cofibrant then there exists a very good right homotopy from f to g.*

Proof. 1. Consider the map

$$A \coprod A \to \mathrm{Cyl}(A) \to A,$$

which we get from our left homotopy $\eta : f \Rightarrow_L g$.
Now we apply functorial factorisations to $A \coprod A \to \mathrm{Cyl}(A)$, hence there exists an object $\mathrm{Cyl}'(A)$ such that

$$A \coprod A \hookrightarrow \mathrm{Cyl}'(A) \xrightarrow{\sim} \mathrel{\mkern-14mu}\rightarrow \mathrm{Cyl}(A).$$

Therefore

$$A \coprod A \hookrightarrow \mathrm{Cyl}'(A) \xrightarrow{\sim} \mathrel{\mkern-14mu}\rightarrow \mathrm{Cyl}(A) \xrightarrow{\sim} A.$$

Finally this gives us

$$A \coprod A \hookrightarrow \mathrm{Cyl}'(A) \xrightarrow{\sim} A.$$

Hence $\mathrm{Cyl}'(A)$ is a good cylinder object, which gives us a good left homotopy $\eta' : f \Rightarrow_L g$.

2. By 1. we have that there exists a good left homotopy for f and g, say $\eta : f \Rightarrow_L g : A \to X$. Therefore we know the existence of a good cylinder object $\mathrm{Cyl}(A)$ i.e.

$$A \coprod A \hookrightarrow \mathrm{Cyl}(A) \xrightarrow{\sim} A.$$

We factor the map $\mathrm{Cyl}(A) \xrightarrow{\sim} A$ by applying functorial factorisations and the 2-3 property as

$$\mathrm{Cyl}(A) \overset{\sim}{\hookrightarrow} \mathrm{Cyl}'(A) \xrightarrow{\sim} \mathrel{\mkern-14mu}\rightarrow A.$$

We have

$$A \coprod A \hookrightarrow \mathrm{Cyl}(A) \overset{\sim}{\hookrightarrow} \mathrm{Cyl}'(A) \xrightarrow{\sim} \mathrel{\mkern-14mu}\rightarrow A$$

and hence

$$A \coprod A \hookrightarrow \mathrm{Cyl}'(A) \xrightarrow{\sim} \mathrel{\mkern-14mu}\rightarrow A$$

meaning that $\mathrm{Cyl}'(A)$ is a very good cylinder object.
Therefore we have that

$$\mathrm{Cyl}(A) \overset{\sim}{\hookrightarrow} \mathrm{Cyl}'(A)$$

is a weak equivalence.
Next we will use that X is fibrant, hence we get a diagram (since $*$ is our terminal object)

$$\begin{tikzcd} \mathrm{Cyl}(A) \arrow[r, "\eta"] \arrow[d, "a"'] & X \arrow[d, "b"] \\ \mathrm{Cyl}'(A) \arrow[r] & *. \end{tikzcd}$$

Since a is an acyclic cofibration and b is a fibration we may apply properties of FWFS to this diagram to get a lift $\eta' : \mathrm{Cyl}'(A) \to X$

$$\begin{tikzcd}
\mathrm{Cyl}(A) \arrow[r, "\eta"] \arrow[d, "a"'] & X \arrow[d, "b"] \\
\mathrm{Cyl}'(A) \arrow[r] \arrow[ur, "\eta'"] & *
\end{tikzcd}$$

which gives us a left homotopy $\eta' : \mathrm{Cyl}'(A) \to X$ from f to g. η' is a very good left homotopy from f to g, since $\mathrm{Cyl}'(A)$ is a very good cylinder object.

The remaining properties follow by duality of the first two. ∎

Lemma 3.9 ([DwSp95])**.** *Let $\mathscr{C}$ be a model category and let $f, g : A \to X$ be two morphisms in $\mathscr{C}$.*

1. *Let A be cofibrant and $\eta : f \Rightarrow_L g$ a left homotopy from f to g. Then f is a weak equivalence iff g is a weak equivalence.*
2. *Let X be fibrant and $\eta : f \Rightarrow_R g$ a right homotopy from f to g. Then f is a weak equivalence iff g is a weak equivalence.*

Proof. By Lemma 3.8 1. we may choose a good left homotopy. Consider the diagram

$$\begin{tikzcd}
& A \coprod A \arrow[d, "i"] & \\
A \arrow[ur, "in_0"] \arrow[r, "i_0"] \arrow[dr, "f"'] & \mathrm{Cyl}(A) \arrow[d, "\eta"] & A. \arrow[ul, "in_1"'] \arrow[l, "i_1"'] \arrow[dl, "g"] \\
& X &
\end{tikzcd}$$

Now, since A is fibrant, we have by Lemma 3.5 that i, i_0 and i_1 are acyclic cofibrations, meaning that they are weak equivalences.
From the above diagram we get

$$f = \eta \circ i_0 \quad \text{and} \quad g = \eta \circ i_1.$$

" $\Rightarrow$ " : We assume that f is a weak equivalence, but then since $f = \eta \circ i_0$ it follows by the 2-3 property that η is also a weak equivalence. From $g = \eta \circ i_1$ and another application of the 2-3 property we get that g is a weak equivalence.

" $\Leftarrow$ " : On the other hand, assume now, that g is a weak equivalence, since $g = \eta \circ i_1$ and by the 2-3 property we have that η must be a weak equivalence. Again, since $f = \eta \circ i_0$ by the 2-3 property it follows immediately, that f itself is a weak equivalence. The second part is the dual statement. ∎

Lemma 3.10 ([DwSp95])**.** *Let $\mathscr{C}$ be a model category.*

1. *If $A \in \mathscr{C}$ is cofibrant, then $\Rightarrow_L$ is an equivalence relation on $\mathscr{C}(A, X)$.*
2. *If $X \in \mathscr{C}$ is fibrant, then $\Rightarrow_R$ is an equivalence relation on $\mathscr{C}(A, X)$.*

Proof. As usual we need to show reflexivity, symmetry and transitivity.

Reflexivity: Just consider $A = \mathrm{Cyl}(A)$, since $A \coprod A \to A \to A$ fulfills Definition 3.1. Therefore we may consider the homotopy $f \Rightarrow_L f$ from f to f, which is well defined.

Symmetrie: We define a switching map. Let $\eta : f \Rightarrow_L g$ be a left homotopy from f to g i.e.

$$\begin{tikzcd}
 & A\coprod A \arrow[d, "i"] & \\
A \arrow[ur, "in_0"] \arrow[r, "i_0"] \arrow[dr, "f"'] & \mathrm{Cyl}(A) \arrow[d, "\eta"] & A. \arrow[ul, "in_1"'] \arrow[l, "i_1"'] \arrow[dl, "g"] \\
 & X &
\end{tikzcd}$$

Relabeling the diagram gives us

$$\begin{tikzcd}
 & A\coprod A \arrow[d, "i"] & \\
A \arrow[ur, "in_1"] \arrow[r, "i_1"] \arrow[dr, "g"'] & \mathrm{Cyl}(A) \arrow[d, "\eta"] & A. \arrow[ul, "in_0"'] \arrow[l, "i_0"'] \arrow[dl, "f"] \\
 & X &
\end{tikzcd}$$

Which means that we have a left homotopy $\eta : g \Rightarrow_L f$ from g to f

Transitivity: Suppose that we have a left homotopy from f to g, say $\eta : f \Rightarrow_L g$ and a left homotopy from g to h, say $\nu : g \Rightarrow_L h$.
Since η is a left homotopy, there exists a good left homotopy by part one of Lemma 3.8 $\eta' : \mathrm{Cyl}(A) \to X$ from f to g and from the same lemma, we get the existence of a good left homotopy $\nu' : \mathrm{Cyl}'(A) \to X$. Therefore

$$A \coprod A \hookrightarrow \mathrm{Cyl}(A) \xrightarrow{\sim} A \quad \text{and} \quad A \coprod A \hookrightarrow \mathrm{Cyl}'(A) \xrightarrow{\sim} A. \tag{1}$$

We use the following diagram to conclude

$$\begin{tikzcd}
 & A\coprod A \arrow[d, "i"] & & A\coprod A \arrow[d, "i'"] & \\
A \arrow[ur, "in_0"] \arrow[r, "i_0"] \arrow[dr, "f"'] & \mathrm{Cyl}(A) \arrow[d, "\eta'"] & A \arrow[ul, "in_1"'] \arrow[l, "i_1"'] \arrow[dl, "g"] \arrow[ur, "in_0'"] \arrow[r, "i_0'"] \arrow[dr, "g"'] & \mathrm{Cyl}'(A) \arrow[d, "\nu'"] & A. \arrow[ul, "in_1'"'] \arrow[l, "i_1'"'] \arrow[dl, "h"] \\
 & X & & X &
\end{tikzcd}$$

Since A is cofibrant, we know from Lemma 3.5 part 1. that i_0, i_0', i_1 and i_1' are all acyclic cofibrations, hence

$$\mathrm{Cyl}(A) \overset{i_1}{\underset{\sim}{\hookleftarrow}} A \overset{i_0'}{\underset{\sim}{\hookrightarrow}} \mathrm{Cyl}'(A).$$

From this we construct a pushout and call it $\mathrm{Cyl}''(A)$

$$\begin{tikzcd} A \arrow[r, "i_0'"] \arrow[d, "i_1"'] & \mathrm{Cyl}'(A) \arrow[d] \\ \mathrm{Cyl}(A) \arrow[r] & \mathrm{Cyl}''(A). \end{tikzcd}$$

Notice that this diagram must exist due to bicompleteness. We need to show, that $\mathrm{Cyl}''(A)$ is indeed a cylinder object for A.
We use the universal property of the above pushout to get the following diagram

$$\begin{tikzcd} A \arrow[r, "i_0'"] \arrow[d, "i_1"'] & \mathrm{Cyl}'(A) \arrow[d, "a_0"] \arrow[ddr, bend left, "\sim"] & \\ \mathrm{Cyl}(A) \arrow[r, "a_1"'] \arrow[drr, bend right, "\sim"'] & \mathrm{Cyl}''(A) \arrow[dr, dotted, "p"] & \\ & & A \end{tikzcd}$$

therefore there is a unique map $p : \mathrm{Cyl}''(A) \to A$. Furthermore we get

$$A \coprod A \xrightarrow{(a_0 \circ i_0', a_1 \circ i_1)} \mathrm{Cyl}''(A) \underset{p}{\xrightarrow{\sim}} A.$$

a_0 and a_1 are weak equivalences since weak equivalences are stable under pushouts meaning that p itself is a weak equivalence by the 2-3 property.

Hence we get that that $\mathrm{Cyl}''(A)$ is indeed a cylinder object for A.
We will use the same trick for the missing left homotopy, consider the pushout

$$\begin{tikzcd} A \arrow[r, "i_0'"] \arrow[d, "i_1"'] & \mathrm{Cyl}'(A) \arrow[d, "a_0"] \arrow[ddr, bend left, "\nu'"] & \\ \mathrm{Cyl}(A) \arrow[r, "a_1"'] \arrow[drr, bend right, "\eta'"'] & \mathrm{Cyl}''(A) \arrow[dr, dotted, "\eta''"] & \\ & & X. \end{tikzcd}$$

Once more by the universal property of pushouts we get a unique map $\eta'' : \mathrm{Cyl}''(A) \to X$, which is the desired left homotopy $\eta'' : f \Rightarrow_L h$ from f to h. The second part follows from duality. ■

Let $\pi^L(A, X)$ denote the set of equivalence classes of $\mathscr{C}(A, X)$ under the equivalence relation generated by $\Rightarrow_L$. Furthermore, let $\pi^R(A, X)$ denote the set of equivalence classes of $\mathscr{C}(A, X)$ under the equivalence relation generated by $\Rightarrow_R$.

Lemma 3.11 ([DwSp95]). *Let $\mathscr{C}$ be a model category A, X objects in $\mathscr{C}$ and $p : Y \to X$, $i : A \to B$ morphisms in $\mathscr{C}$.*

1. *Let A be cofibrant and consider the map $p : Y \to X$ to be an acyclic fibration, then composition with p induces a bijection*
$$\begin{aligned} p_* : \pi^L(A, Y) &\to \pi^L(A, X) \\ [f] &\mapsto [p \circ f] \end{aligned}$$

2. *Let X be fibrant and consider the map $i : A \to B$ to be an acyclic cofibration, then composition with i induces a bijection*
$$\begin{aligned} i^* : \pi^R(B, X) &\to \pi^R(A, X) \\ [f] &\mapsto [f \circ i] \end{aligned}$$

Proof. We show that p_* is well-defined, injective and surjective.

Well defined: Let $f, g : A \to Y$ be two parallel maps in $\mathscr{C}$ and let $\eta : f \Rightarrow_L g$ be a left homotopy from f to g. Then $p \circ \eta : p \circ f \Rightarrow_L p \circ g$ is a left homotopy from $p \circ f$ to $p \circ g$. Indeed we have a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[dr, "f"'] \arrow[ddr, bend right, "p\circ f"'] & \mathrm{Cyl}(A) \arrow[d, "\eta"] & A \arrow[l] \arrow[dl, "g"] \arrow[ddl, bend left, "p\circ g"] \\ & Y \arrow[d, "p"] & \\ & X & \end{tikzcd}$$

giving the desired left homotopy.

Surjectivity: Let $[f] \in \pi^L(A, X)$. A is cofibrant and $p : Y \to X$ an acyclic fibration, we consider the diagram

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d, "i"'] & Y \arrow[d, "p"] \\ A \arrow[r, "f"'] & X. \end{tikzcd}$$

By the properties of FWFS (since i is a cofibration and p an acyclic fibration), there exists a lift $g : A \to Y$

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d, "i"'] & Y \arrow[d, "p"] \\ A \arrow[r, "f"'] \arrow[ur, "g"] & X. \end{tikzcd}$$

We then have, by the commutativity of the above diagram

$$p_*(g) \overset{def.}{=} [p \circ g] = [f].$$

Now, since $[f] \in \pi^L(A, X)$ was arbitrary, we are done.

Injectivity: Let $f, g : A \to Y$ be two parallel maps in $\mathscr{C}$ and say that

$$p \circ f \Rightarrow_L p \circ g : A \to X$$

is a left homotopy from $p \circ f$ to $p \circ g$. Therefore, by Lemma 3.8 we have a good left homotopy, say $\eta : \mathrm{Cyl}(A) \to X$ and hence

$$A \coprod A \overset{i}{\hookrightarrow} \mathrm{Cyl}(A) \overset{\sim}{\to} A,$$

where $\mathrm{Cyl}(A)$ is a good cylinder object.
Now consider the diagram

$$\begin{tikzcd}
A \coprod A \arrow[r, "{(f,g)}"] \arrow[d, "i"'] & Y \arrow[d, "p"] \\
\mathrm{Cyl}(A) \arrow[r, "\eta"'] & X.
\end{tikzcd}$$

Then there exists a lift $\eta' : \mathrm{Cyl}(A) \to Y$ in the above diagram, by the properties of FWFS. But this lift is precisely the map defining our desired left homotopy

$$\eta' : f \Rightarrow_L g.$$

Therefore if $p \circ f \Rightarrow_L p \circ g$ then $f \Rightarrow_L g$ due to the following diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[dr, "f"'] \arrow[ddr, bend right, "p\circ f"'] & \mathrm{Cyl}(A) \arrow[d, "\eta"] & A. \arrow[l] \arrow[dl, "g"] \arrow[ddl, bend left, "p\circ g"] \\
 & Y \arrow[d, "p"] & \\
 & X &
\end{tikzcd}$$

The rest holds by a dual argument. $\blacksquare$

Lemma 3.12 ([DwSp95])**.** *Let $\mathscr{C}$ be a model category A, X objects in $\mathscr{C}$ and $f, g : A \to X$ morphisms in $\mathscr{C}$.*

1. *Let X be fibrant and consider a left homotopy from f to g i.e. $\eta : f \Rightarrow_L g$. For $h : A' \to A$ in $\mathscr{C}$, there exists a left homotopy*

$$f \circ h \Rightarrow_L g \circ h.$$

2. *Let A be cofibrant and consider a right homotopy from f to g i.e. $\eta : f \Rightarrow_R g$. For $h : X \to Y$ in $\mathscr{C}$, there exists a right homotopy*

$$h \circ f \Rightarrow_R h \circ g.$$

Proof. From the left homotopy $\eta : f \Rightarrow_L g$ and that X is fibrant we get by Lemma 3.8 that there must exist a very good homotopy from f to g

$$\eta' : \mathrm{Cyl}(A) \to X,$$

with

$$A \coprod A \hookrightarrow \mathrm{Cyl}(A) \underset{m}{\overset{\sim}{\twoheadrightarrow}} A,$$

for $\mathrm{Cyl}(A)$ a very good cylinder object.
Let us choose a good cylinder object $\mathrm{Cyl}(A')$ for A' i.e.

$$A' \coprod A' \overset{j}{\hookrightarrow} \mathrm{Cyl}(A') \underset{l}{\overset{\sim}{\to}} A'.$$

Consider the diagram

$$\begin{tikzcd} A' \coprod A' \arrow[r, "{(h,h)}"] \arrow[d, "j"'] & A \coprod A \arrow[r, "i"] & \mathrm{Cyl}(A) \arrow[d, "\sim"] \\ \mathrm{Cyl}(A') \arrow[r, "l"'] & A' \arrow[r, "h"'] & A \end{tikzcd} \quad \text{i.e.} \quad \begin{tikzcd} A' \coprod A' \arrow[r, "{i \circ (h,h)}"] \arrow[d, "j"'] & \mathrm{Cyl}(A) \arrow[d, "\sim"] \\ \mathrm{Cyl}(A') \arrow[r, "h \circ l"'] & A. \end{tikzcd}$$

Now, by the properties of FWFS, there must exist a lift $k : \mathrm{Cyl}(A') \to \mathrm{Cyl}(A)$ in the above diagram.
Hence, we have

$$A' \coprod A' \to \mathrm{Cyl}(A') \xrightarrow{k} \mathrm{Cyl}(A) \xrightarrow{\eta'} X.$$

Therefore we have a left homotopy

$$\eta' \circ k : f \circ h \Rightarrow_L g \circ h.$$

The second part is the dual argument. ∎

Lemma 3.13 ([DwSp95])**.** *Let $\mathscr{C}$ be a model category and A, X objects in $\mathscr{C}$.*

1. *Let X be fibrant, then the composition in $\mathscr{C}$ induces a map*

$$\begin{aligned} \pi^L(A', A) \times \pi^L(A, X) &\to \pi^L(A', X) \\ ([h], [f]) &\mapsto [f \circ h]. \end{aligned}$$

2. *Let A be cofibrant, then the composition in $\mathscr{C}$ induces a map*

$$\begin{aligned} \pi^R(A, X) \times \pi^R(X, Y) &\to \pi^R(A, Y) \\ ([h], [f]) &\mapsto [f \circ h]. \end{aligned}$$

Proof. We want to show that if $h \Rightarrow_L k : A' \to A$ and $f \Rightarrow_L g : A \to X$, then

$$f \circ h \quad \text{and} \quad g \circ k$$

represent the same element in $\pi^L(A', X)$.
For this, we may show that

$$\eta : f \circ h \Rightarrow_L g \circ h : A' \to X \quad \text{and} \quad \nu : g \circ h \Rightarrow_L g \circ k : A' \to X$$

are left homotopies.

From Lemma 3.12 we immediately get the homotopy η.

For ν consider the left homotopy $\nu' : h \Rightarrow_L k$ i.e.

$$A' \coprod A' \hookrightarrow \mathrm{Cyl}(A') \xrightarrow{\sim} A', \quad A' \coprod A' \hookrightarrow \mathrm{Cyl}(A') \xrightarrow{\nu'} A.$$

Now, for $g : A \to X$ we have

$$A' \coprod A' \to \mathrm{Cyl}(A') \xrightarrow{\nu'} A \xrightarrow{g} X \quad \text{i.e.} \quad A' \coprod A' \to \mathrm{Cyl}(A') \xrightarrow{g \circ \nu'} X.$$

$$\nu = g \circ \nu' : g \circ h \Rightarrow_L g \circ k$$

is the desired left homotopy. The rest is the dual statement. ■

3.2 Relationship between Left and Right Homotopy

The left and right homotopy relations agree in $\mathscr{C}(A, X)$ if A is cofibrant and X is fibrant.

Lemma 3.14 ([DwSp95])**.** *Let $\mathscr{C}$ be a model category and $f, g : A \to X$ two morphisms in $\mathscr{C}$.*

1. *If A is cofibrant and $f \Rightarrow_L g$, then $f \Rightarrow_R g$.*
2. *If X is fibrant and $f \Rightarrow_R g$, then $f \Rightarrow_L g$.*

Proof. Consider $f \Rightarrow_L g$ we know, that there exists a good left homotopy for f to g by Lemma 3.8. Therefore, we have a good cylinder object for A say

$$A \coprod A \xhookrightarrow{(i_0, i_1)} \mathrm{Cyl}(A) \underset{j}{\xrightarrow{\sim}} A$$

and a homotopy $\eta : \mathrm{Cyl}(A) \to X$ from f to g.

Since A is cofibrant and $\mathrm{Cyl}(A)$ is a good cylinder object we get by Lemma 3.5 that the maps

$$i_0, i_1 : A \overset{\sim}{\hookrightarrow} \mathrm{Cyl}(A)$$

are acyclic cofibrations.

Now we choose a good path object for X i.e.

$$X \underset{q}{\xrightarrow{\sim}} \mathrm{Path}(X) \xrightarrow{(p_0, p_1)} X \times X.$$

Consider the diagram

$$\begin{tikzcd} A \arrow[r, "i_0"] \arrow[d, "q \circ f"'] \arrow[dr, "f"] & \mathrm{Cyl}(A) \arrow[d, "\eta"] \\ \mathrm{Path}(X) \arrow[r, "p_1"'] & X. \end{tikzcd}$$

It commutes, since $p_1 \circ q = id_X$. Similarly we consider the diagram

$$\begin{tikzcd} A \arrow[r, "i_0"] \arrow[d, "q\circ f"'] \arrow[dr] & \mathrm{Cyl}(A) \arrow[d, "f\circ j"] \\ \mathrm{Path}(X) \arrow[r, "p_0"'] & X. \end{tikzcd}$$

It commutes, since $j \circ i_0 = id_A$. Combining these diagrams yields a commutative diagram

$$\begin{tikzcd} A \arrow[r, "q\circ f"] \arrow[d, "i_0"'] & \mathrm{Path}(X) \arrow[d, "(p_0,p_1)"] \\ \mathrm{Cyl}(A) \arrow[r, "(f\circ j,\eta)"'] & X \times X. \end{tikzcd}$$

Now by properties of FWFS (since i_0 is an acyclic cofibration and (p_0, p_1) is a fibration), there exists a lift $\nu : \mathrm{Cyl}(A) \to \mathrm{Path}(X)$ in the above diagram.
Consider

$$\begin{tikzcd} & A \arrow[ddl, bend right, "f"'] \arrow[d, "i_0"] \arrow[ddr, bend left, "g"] & \\ & \mathrm{Cyl}(A) \arrow[dl, "\eta"'] \arrow[d, "\nu"] \arrow[dr, "\eta"] & \\ X & \mathrm{Path}(X) \arrow[l] \arrow[r] & X. \end{tikzcd}$$

Now the upper two triangles commute because of our left homotopy and the lower two since ν is a lift. It follows, that $\nu \circ i_0$ is a right homotopy from f to g. The second statement follows from duality. ■

If $\mathscr{C}$ is a model category and we are in the situation where A is a cofibrant object and X is a fibrant object in $\mathscr{C}$, we then denote the set of equivalence relations on $\mathscr{C}(A, X)$

$$\pi(A, X) = \pi^L(A, X) = \pi^R(A, X).$$

Notice, that these equalities hold due to Lemma 3.14.

We will now use the symbol

$$\sim$$

to denote the equivalence of elements in $\pi(A, X)$, instead of $\Rightarrow_L$ and $\Rightarrow_R$.

Corollary 3.15 ([DwSp95])**.** *Let $\mathscr{C}$ be a model category, A a cofibrant and X a fibrant object of $\mathscr{C}$. Fix a good cylinder object* $\mathrm{Cyl}(A)$ *for A and a good path object* $\mathrm{Path}(X)$ *for X. Let $f, g : A \to X$ be maps in $\mathscr{C}$. Then*

1. *$f \sim g$ iff $f \Rightarrow_R g$ via the fixed path object* $\mathrm{Path}(X)$*.*
2. *$f \sim g$ iff $f \Rightarrow_L g$ via the fixed cylinder object* $\mathrm{Cyl}(A)$*.*

Proof. " $\Leftarrow$ " : trivial.

" $\Rightarrow$ " : Let $f \sim g$, then by Lemma 3.14, there exists a left homotopy $f \Rightarrow_L g$. From the proof of Lemma 3.14 there exists a good path object for X. Especially we may choose the given path object $\mathrm{Path}(X)$ from the proof. The other case is the dual statement. ■

Definition 3.16 (Homotopy Inverse)**.** *Let $\mathscr{C}$ be a model category and $f : A \to X$ a morphism in $\mathscr{C}$. We say that f has a* ***homotopy inverse****, if there exists some map $g : X \to A$ in $\mathscr{C}$ such that*

$$g \circ f \sim id_A \quad \textit{and} \quad f \circ g \sim id_X.$$

Proposition 3.17 ([DwSp95])**.** *Let $\mathscr{C}$ be a model category and suppose that $f : A \to X \in \mathscr{C}(A, X)$ and that A and X are both fibrant and cofibrant objects in $\mathscr{C}$. Then f is a weak equivalence iff f has a homotopy inverse.*

Remark 3.18. *We will see later, that at least for a model category, every isomorphism arises in this particular way, i.e. the weak equivalences are the preimages of the isomoprhisms. Of course this is generally not true if we do not consider a model category.* ♦

Proof. " $\Rightarrow$ " : Let $f : A \to X$ in $\mathscr{C}$ be a weak equivalence. Then by the FWFS we get

$$f : A \underset{i}{\overset{\sim}{\hookrightarrow}} B \underset{p}{\twoheadrightarrow} X$$

for some B in $\mathscr{C}$.

Now by the 2-3 property, since f and i are weak equivalences, p is also a weak equivalence. Consider the diagram

$$\begin{tikzcd} A \arrow[r, "id_A"] \arrow[d, "i"'] & A \arrow[d] \\ B \arrow[r] & *. \end{tikzcd}$$

By FWFS, since i is an acyclic cofibration and $A \to *$ is a fibration (since A is fibrant), there is a lift $r : B \to A$ in the above diagram i.e.

$$\begin{tikzcd} A \arrow[r, "id_A"] \arrow[d, "i"'] & A \arrow[d] \\ B \arrow[r] \arrow[ur, "r"] & *. \end{tikzcd}$$

From the commutativity of the diagram we get

$$r \circ i = id_A$$

i.e. i has a left inverse r.

From Lemma 3.11 i induces a bijection

$$\begin{aligned} i^* : \pi^R(B, B) &\to \pi^R(A, B) \\ [g] &\mapsto [g \circ i]. \end{aligned}$$

Now by

$$i^*([i \circ r]) = [i \circ r \circ i] = [i \circ id_A] = [i] = [id_B \circ i] = i^*([id_B])$$

we have that

$$i \circ r \Rightarrow_R id_B.$$

Therefore, r is a right inverse of i up to right homotopy. Hence

$$r \circ i = id_A \quad \text{and} \quad i \circ r \sim id_B.$$

Now we have a two sided right homotopy inverse of i. Since left and right homotopy coincide, they are homotopy inverses.

" $\Leftarrow$ " : Suppose now that f has a homotopy inverse. Since f is in $\mathscr{C}$, there exists maps p and q in $\mathscr{C}$ with $f = p \circ q$ due to functorial factorisations, such that for $f : A \to X$ we get

$$f : A \overset{\sim}{\underset{q}{\hookrightarrow}} B \underset{p}{\twoheadrightarrow} X.$$

We claim, that B is fibrant and cofibrant.

Indeed, since A and X are both fibrant and cofibrant we get a diagram

$$\begin{tikzcd}
& \emptyset \arrow[dl, "a"'] \arrow[d, "k"] \arrow[dr, "b"] & \\
A \arrow[r, "q"] \arrow[dr, "c"'] & B \arrow[r, "p"] \arrow[d, "l"] & X \arrow[dl, "d"] \\
& * &
\end{tikzcd}$$

where a, b are cofibrations, q is an acyclic fibration and c, d, p are fibrations.
Therefore, since $a = q \circ k$ we have that k is also a cofibration and hence

$$\emptyset \to B$$

is a cofibration i.e. B is cofibrant.

From $l = d \circ p$ we have that l also a fibration and hence

$$B \to *$$

is a fibration i.e. B is fibrant. Hence B is fibrant and cofibrant.

We want to show that f is a weak equivalence. Since

$$f : A \overset{\sim}{\underset{q}{\hookrightarrow}} B \underset{p}{\twoheadrightarrow} X$$

by the 2-3 property, it is enough to show, that p is a weak equivalence.

Let $g : X \to A$ be a homotopy inverse for f s.t.

$$f \circ g \sim id_X \quad \text{and} \quad g \circ f \sim id_A$$

and let $\eta : \mathrm{Cyl}(X) \to X$ be a homotopy between $f \circ g$ and id_X

$$X \coprod X \xrightarrow{(i_0, i_1)} \mathrm{Cyl}(X) \xrightarrow{\eta} X.$$

By Lemma 3.5, (i_0, i_1) are acyclic cofibrations (since X, A are fibrant and cofibrant). Consider the diagram

$$\begin{tikzcd} X \arrow[r, "q \circ g"] \arrow[d, "i_0"'] & B \arrow[d, "p"] \\ \mathrm{Cyl}(X) \arrow[r, "\eta"'] & X. \end{tikzcd}$$

There exists a lift $\eta' : \mathrm{Cyl}(X) \to B$ in the above diagram, by the properties of FWFS.

Define $s := \eta' \circ i_1$ i.e.

$$s : X \xrightarrow{i_1} \mathrm{Cyl}(X) \xrightarrow{\eta'} B.$$

Therefore

$$X \xrightarrow{i_1} \mathrm{Cyl}(X) \xrightarrow{\eta'} B \overset{p}{\twoheadrightarrow} X$$

i.e. $p \circ s = id_X$. Notice, that this is well defined and makes sense, since $\eta' \circ p = \eta$ and $p \circ \eta' \circ i_1 = \eta \circ i_1 = id_X$, since η is a homotopy for $f \circ g$ and id_X.

Now q is a weak equivalence, we then know that q has a homotopy inverse (from the last lemma), call it r i.e. $q \circ r \sim id_B$ and $r \circ q \sim id_A$.

Consider $f = p \circ q$ then by Lemma 3.13 there exists a map

$$\begin{aligned} \pi^L(B, B) \times \pi^L(B, X) &\to \pi^L(B, X) \\ ([id_B], [p]) &\mapsto [p \circ id_B] \\ ([q \circ r], [p]) &\mapsto [p \circ q \circ r] = [f \circ r] \end{aligned}$$

hence $[p \circ id_B] = [f \circ r]$ giving $p \circ id_B \sim f \circ r$ hence $p \sim f \circ r$.
Next we claim, that $s \sim q \circ g$. Indeed, consider the diagram

$$\begin{tikzcd} X \arrow[r, "q \circ g"] \arrow[d, "i_0"'] & B \\ \mathrm{Cyl}(X) \arrow[ur, "\eta'"'] & \end{tikzcd}$$

i.e. $\eta' \circ i_0 = q \circ g$.
By the diagram and (i_0, i_1) we have

$$s = \eta' \circ i_1 \sim \eta' \circ i_0 = q \circ g.$$

Giving us $s \sim q \circ g$.

From the above reasoning and the fact that q and r are homotopy inverses we have

$$s \circ p \sim q \circ g \circ p \sim q \circ g \circ f \circ r \sim q \circ r \sim id_B.$$

Now by Lemma 3.9, since id_B is a weak equivalence, A cofibrant, X fibrant and $s \circ p \sim id_B$, $s \circ p$ must be a weak equivalence.

Consider the commutative diagram

$$\begin{tikzcd} B \arrow[r, "id_B"] \arrow[d, "p"'] & B \arrow[r, "id_B"] \arrow[d, "s \circ p"] & B \arrow[d, "p"] \\ X \arrow[r, "s"'] & B \arrow[r, "p"'] & X. \end{tikzcd}$$

Since $id_B \circ id_B = id_B$ and $p \circ s = id_X$ it follows, that p is a retract of $s \circ p$ and hence by the retract argument, that p is a weak equivalence. ■

4 The Homtopy Category, Quillen Functors and Derived Functors

In this section we define the notion of homotopy category for a model category. We first give the definition which uses localisations of categories. After the definition we give a careful discussion about localisations and which problems may arise from such a construction.

The basic result is that the localisation $\mathrm{Ho}(\mathscr{C}) := \mathscr{C}[\mathcal{W}^{-1}]$ of a model category $\mathscr{C}$ obtained by inverting the weak equivalences is equivalent (as categories) to the quotient category $\mathscr{C}_{cf}/_\sim$ of the cofibrant and fibrant objects by the homotopy relation, defined in the last section.

It is very important to point out that these categories are not the same, they are just equivalent.

Finally we introduce some important machinery, namely the one of Quillen equivalences which seems to be the right notion for an equivalence of model categories. The notion is indeed weaker than for instance the notion of an equivalence of categories. One may think of it as a way more general concept of homotopy equivalence.

We use the material from [Stac17], [MHov91], [TFri11], [GaZi67], [SMcl97], [PHir03] and inspiration from [nLab].

4.1 The Homotopy Category of a Model Category

4.1.1 The Definition of the Homotopy Category $\mathrm{Ho}(\mathscr{C})$

Definition 4.1 (Homotopy Category)**.** *We define the* ***homotopy category*** $\mathrm{Ho}(\mathscr{C})$ *to be the localisation of a model category* $\mathscr{C}$*, where we localise with respect to the weak equivalences i.e.*

$$\mathrm{Ho}(\mathscr{C}) := \mathscr{C}[\mathcal{W}^{-1}].$$

Before we define the localisation of categories we give some motivation in the case of commutative unital rings. We also need to check if this definition is well defined and behaves as we want, that is no size issues should occur. We will see that under the localisation the result does not need to be locally small if we started with a locally small category, we will also see that this does hold in the case of model categories. All of this will be established in the following sections.

4.1.2 Localisation of Commutative Unital Rings

This section follows the definitions and structures given in [Stac17], with some additional proofs.

For the whole section, a ring will always be a commutative unital ring.

Definition 4.2. *Let A be a ring, S a subset of A. We say that S is a* ***multiplicative subset*** *of A if $1 \in S$ and S is closed under multiplication i.e. $s, s' \in S$ implies that $ss' \in S$.*

Given a ring A and a multiplicative subset S of A, we define a relation on $A \times S$ as follows:

$$(x, s) \sim (y, t) \Leftrightarrow \exists\ u \in S \ \text{ s.t. } \ (xt - ys)u = 0.$$

Lemma 4.3. *The relation $\sim$ is an equivalence relation.*

Proof. The relation is clearly reflexive and symmetric. If $(x, s) \sim (y, t)$ and $(y, t) \sim (z, l)$, then

$$(xt - ys)u = 0 \quad \text{and} \quad (yl - zt)v = 0 \qquad \text{for some } u, v \in S.$$

Multiplying the first equation by lv and the second by su and adding them gives

$$(xl - zs)tuv = 0.$$

Since S is closed under multiplication, $(x, s) \sim (z, l)$ and hence $\sim$ is transitive. ■

Let x/s be the equivalence class of (x, s) and $A[S^{-1}]$ be the set of all equivalence classes. We define addition and multiplication in the following way.

$$x/s + y/t = (xt + ys)/st, \quad x/s \cdot y/t = xy/st$$

Lemma 4.4. *$(A[S^{-1}], +, \cdot)$ is a ring.*

Proof. Addition is well defined. Let $(x, s) \sim (x', s')$ and $(y, t) \sim (y', t')$. Then there exists $u, v \in S$ such that

$$(xs' - x's)u = 0 \quad \text{and} \quad (yt' - y't)v = 0.$$

However, $(xt+ys)s't' - (x't'+y's')st = xts't' + yss't' - x't'st - y's'st = (xs'-x's)tt' + (yt'-y't)ss'$, multiplication by uv gives us 0. Hence $(xt + ys)/st = (x't' + y's')/s't'$. Multiplication follows in a similar way. It is not hard to check, that those operations turn $A[S^{-1}]$ into a commutative ring with identity $1/1 = s/s$ for all $s \in S$ and zero element $0/1 = 0/s$ for all $s \in S$. ■

Definition 4.5. *The ring $A[S^{-1}]$ is called the **localisation** of A with respect to S.*

There is a natural ring map from A to its localisation $A[S^{-1}]$ defined as

$$\begin{aligned} \varphi : A &\to A[S^{-1}] \\ x &\mapsto x/1. \end{aligned}$$

Definition 4.6. *The map φ is called **localisation map**.*

Remark 4.7. *In general this map is not injective, unless S contains no zerodivisors. Indeed, if $x/1 = 0$, then there exists some $u \in S$ such that $xu = 0$ in A but this means that $x = 0$, since S contains no zerodivisors.* ♦

Universal Property

The localisation has the following universal property:

Proposition 4.8. *Let $f : A \to B$ be a ring homomorphism that sends every element in S to a unit of B. Then there exists a unique ring homomorphism $g : A[S^{-1}] \to B$ such that the diagram*

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[dr, "f"'] & A[S^{-1}] \arrow[d, "g"] \\ & B \end{tikzcd}$$

commutes.

Proof. **Existence:** We define a map $g : A[S^{-1}] \to B$ as follows:
for $x/s \in A[S^{-1}]$ let $g(x/s) = f(x)f(s)^{-1} \in B$. This homomorphism is a well-defined ring homomorphism and makes our diagram commute. Indeed,

- **well-defined:** Say $(x, s) \sim (y, t)$, then we have $(xt - ys)u = 0$ for some $u \in S$. This gives us

$$(f(xt) - f(ys))f(u) = 0 \quad \text{in } S.$$

Hence we have
$$f(xt) - f(ys) = 0 \ \text{ since } f(u) \text{ is a unit in } S.$$
This yields
$$f(x)f(s)^{-1} = f(y)f(t)^{-1}$$
and finally
$$g(x/s) = g(y/t).$$

- **ring-homomorphism** Let $1/1, x/s, y/t \in A[S^{-1}]$.

  - $g(1/1) = f(1)f(1)^{-1} = 1$.
  - $g(x/s + y/t) = g((xt + ys)/st) = f(xt + ys)f(st)^{-1}1 = (f(xt) + f(ys))f(st)^{-1} = f(xt)f(st)^{-1} + f(ys)f(st)^{-1} = f(x)f(s)^{-1} + f(y)f(t)^{-1} = g(x/s) + g(y/t)$.
  - $g(x/s \cdot y/t) = g(xy/st) = f(xy)f(st)^{-1} = f(x)f(s)^{-1} \cdot f(y)f(t)^{-1} = g(x/s) \cdot g(y/t)$.

- **commutativity:** We get that $g \circ \varphi = f$, take $a \in A$ and consider $g(\varphi(a)) = g(a/1) = f(a)f(1)^{-1} = f(a)$.

Uniqueness: Let $g' : A[S^{-1}] \to B$ be a homomorphism such that $g'(x/1) = f(x)$. Hence $f(s) = g'(s/1)$ for $s \in S$ by commutativity of the diagram. But then $g'(x/s) = g'(x/1)g'(1/s) = g'(x/1)g'(s/1)^{-1} = f(x)f(s)^{-1} = g(x/s)$, meaning that $g' = g$. ■

Lemma 4.9. *The localisation $A[S^{-1}]$ is the 0 ring iff $0 \in S$.*

Proof. If $0 \in S$ then $(a, s) \sim (0, 1)$ for any pair by definition. If $0 \notin S$, then $1/1 \neq 0/1$ in $A[S^{-1}]$. ■

Example 4.10.
- *Let A be a ring and S the multiplicative set of nonzero divisors of A. $A[S^{-1}]$ is called the total quotient ring of A. For A an integral domain, $A[S^{-1}]$ will be the fraction field of A.*

- *Let* $A = \mathbb{Z}$ *and* $S = \{2^n \mid n \in \mathbb{N}\}$, *then* $A[S^{-1}] = \mathbb{Z}\left[\frac{1}{2}\right]$. *Indeed, consider*

$$\begin{aligned} \hat{\varphi} : \mathbb{Z}\left[\frac{1}{2}\right] &\to A[S^{-1}] \\ \frac{x}{2^n} &\mapsto \left[\frac{x}{2^n}\right]. \end{aligned}$$

- *(Localising at a prime) Let* P *be a prime ideal in any ring* A *and let* $S = A \backslash P$. *By definition of prime ideal,* S *is multiplicatively closed. Passing to the ring* $A[S^{-1}]$ *in this case is localising* A *at* P *and the ring* $A[S^{-1}]$ *is denoted by* A_P. *Every element of* A *not in* P *becomes a unit in* A_P.

  For example: let $A = \mathbb{Z}$ *and* $P = (p)$ *be a prime ideal, for some prime* p, *then*

$$\mathbb{Z}_{(p)} = \left\{ \frac{a}{b} \;\middle|\; p \nmid b \right\} \subset \mathbb{Q}$$

  and every integer b *not divisible by* p *is a unit. [DuFo04]*

◀

4.1.3 Localisation of Categories

It may happen, depending on the context, that one has to deal with a category which is not well behaved for the desired purpose. In such a situation it is often helpful to find some different category and a connection between the two of them, in which those certain properties behave in a better way. One way to achieve this, is the concept of localisations, in which one will loose some properties but the most important concepts may still be valid. This is also seen in the idea that we will use, i.e. we use the quotient category in order to define the localisation of categories.

Another issue with localisations, as we will see, are size issues. If one starts with a locally small category the localisation may result in a large category (which tends to happen most of the time). This is not the case when localising model categories with respect to the weak equivalences. It is also true that there are no size issues if one is dealing with small categories, which we will also prove in this section.

We will use the idea in [TFri11], with slightly different proofs, notation and some other additional examples and remarks. The inspiration for the construction and notation in the proof of the main theorem is taken from [nLab] and [SMcl97], the original idea seems to be from [GaZi67], though they give a more refined version of the whole concept called a calculus of fractions.

Quotient of Categories

Say $\sim$ is an equivalence relation on every morphism set $\mathscr{C}(A, B)$ which is preserved under composition, meaning that

$$(f_1 \sim f_2) \quad \Rightarrow \quad (f_1 \circ g \sim f_2 \circ g) \quad \text{and} \quad (h \circ f_1 \sim h \circ f_2) \quad \forall\ f_1, f_2, g, h \in \mathrm{mor}(\mathscr{C}),$$

whenever the compositions are defined.

Then composition of equivalence classes is well-defined and defines the **quotient category** $\mathscr{C}/_\sim$ together with a canonical projection functor $\mathscr{C} \to \mathscr{C}/_\sim$. The objects of $\mathscr{C}/_\sim$ are the same as the objects of $\mathscr{C}$.

Example 4.11. *Monoids and groups may be regarded as categories with one object. In this case the quotient category coincides with the notion of a quotient monoid or a quotient group.* ◀

Localisation of Categories

Let $\mathscr{C}$ be a category and let $\mathcal{W} \subseteq \mathrm{mor}(\mathscr{C})$ be a subclass ($\mathcal{W}$ for "weak equivalence"). We will turn all morphisms of $\mathcal{W}$ into isomorphisms by adjoining formal inverses of them.

We are looking for a category $\mathscr{C}[\mathcal{W}^{-1}]$ equipped with a localisation functor

$$\mathrm{Loc} : \mathscr{C} \to \mathscr{C}[\mathcal{W}^{-1}]$$

with the following universal property.

1. $\mathrm{Loc}(\omega)$ is an isomorphism for all $\omega \in \mathcal{W}$.
2. If $F : \mathscr{C} \to \mathscr{D}$ is any functor which maps $\mathcal{W}$ to isomorphisms, then there exists a unique functor $L : \mathscr{C}[\mathcal{W}^{-1}] \to \mathscr{D}$ such that the diagram

$$\begin{tikzcd} \mathscr{C} \arrow[r, "\mathrm{Loc}"] \arrow[dr, "F"'] & \mathscr{C}[\mathcal{W}^{-1}] \arrow[d, "\exists ! L"] \\ & \mathscr{D} \end{tikzcd}$$

commutes.

Definition 4.12. *If such a functor* Loc *exists, the category* $\mathscr{C}[\mathcal{W}^{-1}]$ *is called the* ***localisation*** *of* $\mathscr{C}$ *with respect to* $\mathcal{W}$.

Remark 4.13.
- *Notice, that even though* Loc *sends elements of* $\mathcal{W} \subseteq \mathrm{Mor}(\mathscr{C})$ *to isomorphisms in* $\mathscr{C}[\mathcal{W}^{-1}]$, *not every isomorphisms in* $\mathscr{C}[\mathcal{W}^{-1}]$ *is forced to come from an element of* $\mathcal{W}$.
- *If we localise a category, this category will most likely grow in size. For this consider the following.*

  We define a category $\mathscr{C}$ *such that*

  $$\mathrm{ob}(\mathscr{C}) := \{*, \bullet\}$$

  and

  $$\mathrm{Mor}(\mathscr{C}) := \{id_*, id_\bullet, * \xrightarrow{a} \bullet, * \xrightarrow{b} \bullet\}.$$

  It looks something like this (without the identities):

  $$\mathscr{C} : \qquad \begin{tikzcd} * \arrow[r, bend left, "a"] \arrow[r, bend right, "b"'] & \bullet \end{tikzcd}$$

We choose $\mathcal{W} := \{* \xrightarrow{a} \bullet\}$ *and localise with respect to it i.e.*

$$\mathscr{C}[\mathcal{W}^{-1}]: \qquad \begin{tikzcd} * \arrow[r, bend left, "a"] \arrow[r, bend right, "b"'] & \bullet \arrow[l, squiggly, "a^{-1}"'] \end{tikzcd}$$

The objects will still be the same

$$\mathrm{ob}(\mathscr{C}[\mathcal{W}^{-1}]) = \mathrm{ob}(\mathscr{C}) = \{*, \bullet\},$$

but since we want $\mathscr{C}[\mathcal{W}^{-1}]$ *to be a category we gain a lot more morphisms than before (due to composition), we get*

$$\mathrm{mor}(\mathscr{C}[\mathcal{W}^{-1}]) = \{(a^{-1})^m(ba^{-1})^n(b)^l \mid m, l = 0 \vee 1 \textit{ and } n \in \mathbb{N}\} \cup \{a, id_*, id_\bullet\}.$$

♦

Theorem 4.14. $\mathscr{C}[\mathcal{W}^{-1}]$ *and* Loc *always exist and are unique up to unique isomorphism.*

Proof. Let $\mathscr{C}$ be a category and choose $\mathcal{W}$ in $\mathrm{Mor}(\mathscr{C})$. Let $\mathcal{W}^{op}$ in $\mathscr{C}^{op}$ be the corresponding class to $\mathcal{W}$ in $\mathscr{C}$.

Existence: The pair $(\mathrm{Loc}, \mathscr{C}[\mathcal{W}^{-1}])$ exists.

We define a **directed graph** $\mathscr{G}_{[\mathscr{C},\mathcal{W}^{-1}]}$ as:

1. the vertices of $\mathscr{G}_{[\mathscr{C},\mathcal{W}^{-1}]}$ -call them $\mathrm{obj}(\mathscr{G}_{[\mathscr{C},\mathcal{W}^{-1}]})$- are the objects of $\mathscr{C}$ and
2. arrows of $\mathscr{G}_{[\mathscr{C},\mathcal{W}^{-1}]}$ between vertices x and y are given by $\mathscr{C}(x,y) \sqcup \mathcal{W}^{op}(x,y)$.

The arrows in $\mathcal{W}^{op}$ are written as ω^{-1} for $\omega \in \mathcal{W}(y,x)$.

Let $\mathscr{P}(\mathscr{G}_{[\mathscr{C},\mathcal{W}^{-1}]})$ be the **path category** on $\mathscr{G}_{[\mathscr{C},\mathcal{W}^{-1}]}$ such that:

1. $\mathrm{obj}(\mathscr{P}(\mathscr{G}_{[\mathscr{C},\mathcal{W}^{-1}]})) = \mathrm{ob}(\mathscr{G}_{[\mathscr{C},\mathcal{W}^{-1}]}) = \mathrm{obj}(\mathscr{C})$
2. a morphism from a to b is a string $\langle a_n, f_n, a_{n-1}, \dots, a_1, f_1, a_0\rangle$ $n \geq 0$ with $a_i \in \mathrm{ob}(\mathscr{C})$, $a = a_0$, $b = a_n$ and $\forall\ 0 < i \leq n$ $f_i : a_{i-1} \to a_i$ is an edge from a_{i-1} to a_i in $\mathscr{G}_{[\mathscr{C},\mathcal{W}^{-1}]}$.
3. composition is given by concatenation:
   $\langle a_n, f_n, a_{n-1}, \dots, a_1, f_1, a_0\rangle * \langle b_m, g_m, b_{m-1}, \dots, b_1, g_1, b_0\rangle :=$
   $$\langle a_n, f_n, a_{n-1}, \dots, a_1, f_1, a_0 = b_m, g_m, b_{m-1}, \dots, b_1, g_1, b_0\rangle$$
   whenever $a_0 = b_m$.
4. $\mathrm{dom}(\langle a_n, f_n, a_{n-1}, \dots, a_1, f_1, a_0\rangle) = a_0$ and $\mathrm{codom}(\langle a_n, f_n, a_{n-1}, \dots, a_1, f_1, a_0\rangle) = a_n$.
5. the identity is given by the empty string $\langle a\rangle$ $\forall\ a \in \mathrm{ob}(\mathscr{C})$.

We will use the notation $\langle f_n, \ldots, f_1\rangle$ for $\langle a_n, f_n, a_{n-1}, \ldots, a_1, f_1, a_0\rangle$.

Remark that $\langle f_n, \ldots, f_1\rangle * \langle g_m, \ldots, g_1\rangle = \langle f_n, \ldots, f_1, g_m, \ldots, g_1\rangle = \langle g_1\rangle \circ \cdots \circ \langle g_m\rangle \circ \langle f_1\rangle \circ \cdots \circ \langle f_n\rangle$.

We get a canonical map $\mathscr{C} \to \mathscr{P}(\mathscr{G}_{[\mathscr{C},\mathcal{W}^{-1}]})$ which is the identity on objects and maps every morphism f of $\mathscr{C}$ to a corresponding single-literal $\langle f\rangle$. This map is a good candidate for the desired map Loc. However, neither is this map a functor nor does it map $\mathcal{W}$ to isomorphisms.

To fix this, we introduce the equivalence relation $\sim$:

1. $\langle id_x\rangle \sim \langle x\rangle \qquad \forall\ x \in \mathrm{ob}(\mathscr{C})$
2. $\langle f, g\rangle \sim \langle g \circ f\rangle \qquad \forall\ f : x \to y,\ g : y \to z \in \mathrm{Mor}(\mathscr{C})$
3. $\langle \omega, \omega^{-1}\rangle \sim \langle id_x\rangle$ and $\langle \omega^{-1}, \omega\rangle \sim \langle id_y\rangle \qquad \forall\ \omega : x \to y$ in $\mathcal{W}$

This gives us the **quotient category** $\mathscr{P}(\mathscr{G}_{[\mathscr{C},\mathcal{W}^{-1}]})/_\sim$ and an induced map

$$\mathrm{Loc} : \mathscr{C} \to \mathscr{P}(\mathscr{G}_{[\mathscr{C},\mathcal{W}^{-1}]})/_\sim.$$

With the relation $\sim$ the map Loc clearly is a functor that maps $\mathcal{W}$ to isomorphisms. Indeed,

- $\mathrm{Loc}(f \circ g) = \langle f \circ g\rangle/\!\sim\, = \langle g, f\rangle/\!\sim\, = \langle g\rangle/\!\sim * \langle f\rangle/\!\sim\, = \langle f\rangle/\!\sim \circ \langle g\rangle/\!\sim\, = \mathrm{Loc}(f) \circ \mathrm{Loc}(g)$
- $\mathrm{Loc}(id_a) = \langle id_a\rangle/\!\sim\, = \langle a\rangle/\!\sim\, = id_a = id_{\mathrm{Loc}(a)}$
- Let $\omega : x \to y$ in $\mathcal{W}$.
  $\mathrm{Loc}(\omega \circ \omega^{-1}) = \langle \omega \circ \omega^{-1}\rangle/\!\sim\, = \langle \omega^{-1}, \omega\rangle/\!\sim\, = \langle \omega^{-1}\rangle/\!\sim * \langle \omega\rangle/\!\sim\, = \langle \omega\rangle/\!\sim \circ \langle \omega^{-1}\rangle/\!\sim\, = \mathrm{Loc}(\omega) \circ \mathrm{Loc}(\omega^{-1})$. Since $\langle \omega, \omega^{-1}\rangle \sim \langle id_x\rangle \sim \langle x\rangle$ we have

  $$\mathrm{Loc}(\omega) \circ \mathrm{Loc}(\omega^{-1}) = id_x = id_{\mathrm{Loc}(x)}.$$

  Similarly, $\mathrm{Loc}(\omega^{-1} \circ \omega) = \mathrm{Loc}(\omega^{-1}) \circ \mathrm{Loc}(\omega)$ and from $\langle \omega^{-1}, \omega\rangle \sim \langle id_y\rangle \sim \langle y\rangle$ we get

  $$\mathrm{Loc}(\omega^{-1}) \circ \mathrm{Loc}(\omega) = id_y = id_{\mathrm{Loc}(y)}.$$

  Hence, $\mathrm{Loc}(\omega)$ and $\mathrm{Loc}(\omega^{-1})$ are isomorphisms in $\mathscr{P}(\mathscr{G}_{[\mathscr{C},\mathcal{W}^{-1}]})/_\sim$ and thus Loc maps $\mathcal{W}$ to isomorphisms.

Construction of the unique functor $L : \mathscr{P}(\mathscr{G}_{[\mathscr{C},\mathcal{W}^{-1}]})/_\sim \to \mathscr{D}$:

• Existence of L:

Let $\langle l_n, \ldots, l_1\rangle/\!\sim$ be a string in $\mathscr{P}(\mathscr{G}_{[\mathscr{C},\mathcal{W}^{-1}]})/_\sim$. Then we define a map L as follows

$$L(\langle l_1 \circ \cdots \circ l_n\rangle/\!\sim) := L(\langle l_1\rangle/\!\sim) \circ \cdots \circ L(\langle l_n\rangle/\!\sim),$$

where

$$L(\langle l_i\rangle/\!\sim) = \begin{cases} F(l_i) & \text{if } l_i \in \mathrm{Mor}(\mathscr{C}) \\ F(l_i)^{-1} & \text{if } l_i \in \mathcal{W}^{op} \end{cases}$$

and

$$L(\langle a\rangle/\sim) := id_a \text{ for all object } a \in \mathscr{C}.$$

This clearly is a functor (by construction).

Let $\omega : x \to y$ in $\mathcal{W}$, it follows
$L(\langle\omega\rangle/\sim) \circ L(\langle\omega^{-1}\rangle/\sim) = L(\langle\omega \circ \omega^{-1}\rangle/\sim) = L(\langle\omega^{-1}, \omega\rangle/\sim) = L(\langle id_y\rangle/\sim) = L(\langle y\rangle/\sim) = id_{L(y)}$
and
$L(\langle\omega^{-1}\rangle/\sim) \circ L(\langle\omega\rangle/\sim) = L(\langle\omega^{-1} \circ \omega\rangle/\sim) = L(\langle\omega, \omega^{-1}\rangle/\sim) = L(\langle id_x\rangle/\sim) = L(\langle x\rangle/\sim) = id_{L(x)}$.

From the diagram

$$\begin{tikzcd} \mathscr{C} \arrow[r, "\text{Loc}"] \arrow[dr, "F"'] & \mathscr{P}(\mathscr{G}_{[\mathscr{C},\mathcal{W}^{-1}]})/_\sim \arrow[d, "L"] \\ & \mathscr{D} \end{tikzcd}$$

and the above computation, we have that $L(\langle\omega\rangle/\sim) = F(\omega)$. Which gives us

$$L(\langle\omega^{-1}\rangle/\sim) = F(\omega)^{-1}.$$

Hence $L \circ \text{Loc}$ maps elements of $\mathcal{W}$ to isomorphisms in $\mathscr{D}$.

L is well-defined, since we have that:

- $L(\langle a\rangle/\sim) = id_a = L(\langle id_a\rangle/\sim)$ for all object $A \in \mathscr{C}$.
- $F(f \circ g) = L(\text{Loc}(f \circ g)) = L(\langle f \circ g\rangle/\sim) = L(\langle g, f\rangle/\sim) = L(\langle g\rangle/\sim * \langle f\rangle/\sim) = L(\langle f\rangle/\sim \circ \langle g\rangle/\sim) = L(\langle f\rangle/\sim) \circ L(\langle g\rangle/\sim) = L(\text{Loc}(f)) \circ L(\text{Loc}(g)) = F(f) \circ F(g)$ for f, g morphisms in $\mathscr{C}$.
- Let $\omega : x \to y$ in $\mathcal{W}$.
  $L(\langle\omega^{-1} \circ \omega\rangle/\sim) = L(\langle\omega^{-1}\rangle/\sim) \circ L(\langle\omega\rangle/\sim) = F(\omega)^{-1} \circ F(\omega) = id_x = L(\langle id_x\rangle/\sim) = L(\langle x\rangle/\sim)$.

By construction this functor clearly renders the above diagram commutative. This finally gives us the desired map L.

• **Uniqueness of L:**

Let $L' : \mathscr{P}(\mathscr{G}_{[\mathscr{C},\mathcal{W}^{-1}]})/_\sim \to \mathscr{D}$ be another functor such that the desired diagram commutes. We then have $L'(\langle\omega\rangle/\sim) = F(\omega)$, hence

$$L'(\langle\omega^{-1}\rangle/\sim) = F(\omega)^{-1} \text{ for } \omega \in \mathcal{W}.$$

By commutativity of the diagram $L'(\langle l\rangle/\sim) = F(l)$ for $l \in \mathscr{C}$. Then

$$L'(\langle l_i\rangle/\sim) = F(l_i) = L(\langle l_i\rangle/\sim),$$

for $l_i \in \text{Mor}(\mathscr{C}) \cup \mathcal{W}^{-1}$.

Let $a, b \in \mathrm{ob}(\mathscr{C})$ and consider $\langle l_n, \dots, l_1\rangle/\sim$ in $\mathscr{P}(\mathscr{G}_{[\mathscr{C},\mathcal{W}^{-1}]})/_\sim$. We have

$$L'(\langle l_n, \dots, l_1\rangle/\sim) = L'(\langle l_1 \circ \dots \circ l_n\rangle/\sim) = L'(\langle l_1\rangle/\sim) \circ \dots \circ L'(\langle l_n\rangle/\sim) =$$

$$F(l_1) \circ \dots \circ F((l_n)) = L(\langle l_1\rangle/\sim \circ \dots \circ \langle l_n\rangle/\sim) = L(\langle l_1 \circ \dots \circ l_n\rangle/\sim) = L(\langle l_n, \dots, l_1\rangle/\sim).$$

Hence L is unique. We define $\mathscr{C}[\mathcal{W}^{-1}] := \mathscr{P}(\mathscr{G}_{[\mathscr{C},\mathcal{W}^{-1}]})/_\sim$

Uniqueness: The pair $(\mathrm{Loc}, \mathscr{C}[\mathcal{W}^{-1}])$ is unique.

We have seen, that there exists a pair $(\mathrm{Loc}, \mathscr{C}[\mathcal{W}^{-1}])$ having the universal property

$$\begin{tikzcd} \mathscr{C} \arrow[r, "\mathrm{Loc}"] \arrow[dr, "F"'] & \mathscr{C}[\mathcal{W}^{-1}] \arrow[d, "\exists! L"] \\ & \mathscr{D}. \end{tikzcd}$$

Now let $(\mathrm{Loc}', \mathscr{C}[\mathcal{W}^{-1}]')$ be another such pair having the universal property

$$\begin{tikzcd} \mathscr{C} \arrow[r, "\mathrm{Loc}'"] \arrow[dr, "F"'] & \mathscr{C}[\mathcal{W}^{-1}]' \arrow[d, "\exists! L'"] \\ & \mathscr{D}. \end{tikzcd}$$

By universality of Loc and Loc' we have that

$$\begin{tikzcd} & \mathscr{C}[\mathcal{W}^{-1}] \arrow[d, "\exists! L"] \\ \mathscr{C} \arrow[ur, "\mathrm{Loc}"] \arrow[r, "\mathrm{Loc}'"] \arrow[dr, "\mathrm{Loc}"] \arrow[ddr, "\mathrm{Loc}'"'] & \mathscr{C}[\mathcal{W}^{-1}]' \arrow[d, "\exists! L'"] \\ & \mathscr{C}[\mathcal{W}^{-1}] \arrow[d, "\exists! L"] \\ & \mathscr{C}[\mathcal{W}^{-1}]' \end{tikzcd}$$

such that $L \circ \mathrm{Loc} = \mathrm{Loc}'$ and $L' \circ \mathrm{Loc}' = \mathrm{Loc}$. From the above diagram we get

$$\mathrm{Loc} = (L' \circ L) \circ \mathrm{Loc}$$

but we also have that

$$\mathrm{Loc} = id_{\mathscr{C}[\mathcal{W}^{-1}]} \circ \mathrm{Loc}$$

by uniqueness this yields

$$L' \circ L = id_{\mathscr{C}[\mathcal{W}^{-1}]}.$$

Similarly we have

$$\mathrm{Loc}' = (L \circ L') \circ \mathrm{Loc}'$$

again

$$\mathrm{Loc}' = id_{\mathscr{C}[\mathcal{W}^{-1}]'} \circ \mathrm{Loc}'$$

uniqueness of L' then gives

$$L \circ L' = id_{\mathscr{C}[\mathcal{W}^{-1}]'}.$$

Hence L and L' are isomorphisms, which gives us that the pair

$$(\mathrm{Loc}, \mathscr{C}[\mathcal{W}^{-1}])$$

is unique up to unique isomorphism. This finally concludes the proof.

■

Example 4.15. *Let $\mathscr{C}$ be a category and consider $\mathcal{W} = \mathrm{mor}(\mathscr{C})$, then $\mathscr{C}[\mathcal{W}^{-1}]$ is a groupoid.* ◄

4.1.4 Size Issues of Localisations

As already mentioned the localisation of a locally small category need not be locally small again and it usually results in a large category. Showing that a category remains locally small is not always an easy task.

What follows is an example for size issues in a localisation of a locally small category.

Later we will see that there are no size issues for small categories.

Example 4.16. *Let S be a class and not a set. We define the following category $\mathscr{C}$. The objects of $\mathscr{C}$ will be*

$$\mathrm{ob}(\mathscr{C}) = \{*, \bullet\} \cup S,$$

for some elements $*, \bullet \notin S$.

There will be the identity morphisms, unique morphisms from $*$ to all elements of S (one for each element), and unique morphisms from $\bullet$ to elements of S (one for each element), i.e.

$$\mathrm{mor}(\mathscr{C}) := \{id_*, id_\bullet, id_s, * \xrightarrow{\exists!} s, \bullet \xrightarrow{\exists!} s \mid \text{for all } s \in S\},$$

which is a class.

A graphic representation would be (without the identity morphisms):

$$\mathscr{C} : \quad \begin{tikzcd}
 & & * \arrow[dl] \arrow[d] \arrow[dr] & & \\
\dots & s' & \dots & s & \dots \\
 & & \bullet \arrow[ul] \arrow[u] \arrow[ur] & &
\end{tikzcd}$$

for $s, s' \in S$.

This constructed category $\mathscr{C}$ is a locally small category. It is not hard to see that it is actually a category and indeed locally small since the hom-sets are empty or singletons i.e.

$$\mathscr{C}(*, s) = \{* \to_{s*} s \mid \textit{where } \to_{s*} \textit{ is the unique arrow from } * \textit{ to } s.\}$$

$$\mathscr{C}(\bullet, s) = \{\bullet \to_{s\bullet} s \mid \textit{where } \to_{s\bullet} \textit{ is the unique arrow from } \bullet \textit{ to } s.\}$$

both contain exactly one element for every $s \in S$, the other cases include the empty set and the identity morphisms.

We now choose

$$\mathcal{W} := \{* \to s \mid \textit{for all } s \in S\}$$

and localise.

The resulting category looks like this (without identity morphisms)

$$\mathscr{C}[\mathcal{W}^{-1}] : \qquad \begin{tikzcd}
 & & * \arrow[dl] \arrow[d] \arrow[dr] & & \\
\dots & s' \arrow[ur, squiggly] & \dots \arrow[u, squiggly] & s \arrow[ul, squiggly] & \dots \\
 & & \bullet \arrow[ul] \arrow[u] \arrow[ur] & &
\end{tikzcd}$$

for $s, s' \in S$.

But now, there exist morphisms $\bullet \to *$ namely for every element of S at least one of them. But this implies, that $\mathscr{C}[\mathcal{W}^{-1}](\bullet, *)$ is not a set but a class.

Hence, since $\mathscr{C}[\mathcal{W}^{-1}](\bullet, *)$ is a proper subclass of $\mathrm{mor}(\mathscr{C}[\mathcal{W}^{-1}])$ our category $\mathscr{C}[\mathcal{W}^{-1}]$ can not be locally small, even though $\mathscr{C}$ was. ◀

If we are in the case of a small category, then there are no size issues, as we will see.

Theorem 4.17. *Let $\mathscr{C}$ be a small category and let $\mathcal{W} \subseteq \mathrm{mor}(\mathscr{C})$, then the path category $\mathscr{P}(\mathscr{G}_{[\mathscr{C},\mathcal{W}^{-1}]})$ of $\mathscr{C}$ is also a small category.*

Proof. Let $\mathscr{C}$ be a small category and choose $\mathcal{W} \subseteq \mathrm{mor}(\mathscr{C})$. By the definition of a small category we have that $\mathrm{ob}(\mathscr{C})$ and $\mathrm{mor}(\mathscr{C})$ are sets. By construction of the localisation in Theorem 4.14 we have

$$\mathrm{ob}(\mathscr{C}) = \mathrm{ob}(\mathscr{P}(\mathscr{G}_{[\mathscr{C},\mathcal{W}^{-1}]}))$$

and thus a set. Note that $\mathrm{Mor}(\mathscr{P}(\mathscr{G}_{[\mathscr{C},\mathcal{W}^{-1}]}))$ can be identified with

$$\mathrm{ob}(\mathscr{C}) \cup \bigcup_{n\in\mathbb{N}} \{f : \{1,\dots,n\} \to \mathrm{mor}(\mathscr{C}) \sqcup \mathcal{W}^{op}\} \cong \mathrm{ob}(\mathscr{C}) \cup \bigcup_{n\in\mathbb{N}} (\{1,\dots,n\} \times (\mathrm{mor}(\mathscr{C}) \sqcup \mathcal{W}^{op})),$$

where $\mathrm{ob}(\mathscr{C})$ represents the empty strings (identities) and $\{f : \{1,\dots,n\} \to \mathrm{mor}(\mathscr{C}) \sqcup \mathcal{W}^{op}\}$ are the strings of length n. Hence $\mathrm{mor}(\mathscr{P}(\mathscr{G}_{[\mathscr{C},\mathcal{W}^{-1}]}))$ is a set.

Finally, we get that the path category $\mathscr{P}(\mathscr{G}_{[\mathscr{C},\mathcal{W}^{-1}]})$ is a small category. ■

A direct consequence of this theorem is the following corollary.

Corollary 4.18. *Let $\mathscr{C}$ be a small category and $\mathcal{W} \subseteq \mathrm{mor}(\mathscr{C})$, then the localisation $\mathscr{C}[\mathcal{W}^{-1}]$ of $\mathscr{C}$ is also a small category.*

Proof. $\mathscr{C}$ small $\overset{Thm.3.2}{\Rightarrow}$ $\mathscr{P}(\mathscr{G}_{[\mathscr{C},\mathcal{W}^{-1}]})$ small $\Rightarrow$ $\mathscr{P}(\mathscr{G}_{[\mathscr{C},\mathcal{W}^{-1}]})/_{\sim}$ small $\Rightarrow$ $\mathscr{C}[\mathcal{W}^{-1}]$ small. ∎

Let us now argue that we do not run into trouble with this construction. We have seen that the localisation always exists, that is true for a general category with weak equivalences. But we also want to claim, that this category (localisation) is a locally small category if we consider a model category. It could for example be, that $\mathrm{Ho}(\mathscr{C}(A, B))$ may not not be a set (as we have seen in the above examples). So for $\mathrm{Ho}(\mathscr{C})$ to exist we would need to pass to a higher universe.

Let us now argue that there is no trouble with $\mathrm{Ho}(\mathscr{C})$.

4.1.5 The Equivalence of $\mathrm{Ho}(\mathscr{C})$ and $\mathrm{Ho}(\mathscr{C}_{cf})$

Definition 4.19.
1. *Let $\mathscr{C}_c$ be the full subcategory of $\mathscr{C}$ generated by the cofibrant objects in $\mathscr{C}$.*
2. *Let $\mathscr{C}_f$ be the full subcategory of $\mathscr{C}$ generated by the fibrant objects in $\mathscr{C}$.*
3. *Let $\mathscr{C}_{cf}$ be the full subcategory of $\mathscr{C}$ generated by objects of $\mathscr{C}$ which are fibrant and cofibrant.*

4.1.6 Fibrant and Cofibrant Approximations

Definition 4.20 (Fibrant/Cofibrant Approximations)**.** *Let $\mathscr{C}$ be a model category.*

1. (a) *A **cofibrant approximation** to an object X is a pair (QX, i) where QX is a cofibrant object and $i : QX \to X$ is a weak equivalence.*
   (b) *A **fibrant cofibrant approximation** to X is a cofibrant approximation (QX, i) such that $i : QX \to X$ is an acyclic fibration.*
2. (a) *A **fibrant approximation** to an object X is a pair (RX, j) where RX is a fibrant object and $j : X \to RX$ is a weak equivalence.*
   (b) *A **cofibrant fibrant approximation** to X is a fibrant approximation (RX, j) such that $j : X \to RX$ is an acyclic cofibration.*

Remark 4.21. *A fibrant cofibrant approximation is often also referred to as a **fibrant replacement**. Similarly a cofibrant fibrant approximation is sometimes referred to as a **cofibrant replacement**.*

We sometimes use cofibrant approximation to refer to the object QX without explicitely mentioning the map i. Similarly, we sometimes use fibrant approximation to refer to the object RX without explicitely mentioning the map j. ♦

Lemma 4.22. *Let $\mathscr{C}$ be a model category and let $X \in \mathscr{C}$ be an object. We then have an acyclic fibration $p_X : QX \overset{\sim}{\twoheadrightarrow} X$ with QX a cofibrant element in $\mathscr{C}$ and an acyclic cofibration $i_X : X \overset{\sim}{\hookrightarrow} RX$ with RX fibrant object in $\mathscr{C}$.*

Proof. By the bicompleteness of a model category we have the existence of an initial object $\emptyset \in \mathscr{C}$ and a terminal object $* \in \mathscr{C}$ such that

$$\emptyset \to X \quad \text{and} \quad X \to *$$

exist. We now apply functorial factorisations to the above maps to get

$$\emptyset \underset{i_X}{\hookrightarrow} QX \underset{p_X}{\twoheadrightarrow} X \quad \text{and} \quad X \underset{j_X}{\hookrightarrow} RX \underset{q_X}{\twoheadrightarrow} *$$

for some QX, RX in $\mathscr{C}$.
Notice, that QX is cofibrant and RX is fibrant in $\mathscr{C}$. ■

Remark 4.23. *If X is already cofibrant, we just choose $p_X = id_x$ and if X is already fibrant we choose $q_X = id_X$.* ♦

The next proposition shows us that $\mathrm{Ho}(\mathscr{C}_{cf})$ is equivalent to $\mathrm{Ho}(\mathscr{C})$ as categories, the proof follows [MHov91].

Proposition 4.24. *Let $\mathscr{C}$ be a model category and consider the inclusion functors*

$$\begin{tikzcd} & \mathscr{C}_c \arrow[dr] & \\ \mathscr{C} \arrow[ur] \arrow[dr] & & \mathscr{C}_{cf}. \\ & \mathscr{C}_f \arrow[ur] & \end{tikzcd}$$

They induce equivalences of categories

$$\mathrm{Ho}(\mathscr{C}_{cf}) \to \mathrm{Ho}(\mathscr{C}_c) \to \mathrm{Ho}(\mathscr{C}) \quad \textit{and} \quad \mathrm{Ho}(\mathscr{C}_{cf}) \to \mathrm{Ho}(\mathscr{C}_f) \to \mathrm{Ho}(\mathscr{C}).$$

Proof. We show that $\mathrm{Ho}(\mathscr{C}_c) \to \mathrm{Ho}(\mathscr{C})$ is an equivalence. $\mathscr{C}_c \xrightarrow{i} \mathscr{C}$ preserves weak equivalences and so does induce a functor $\mathrm{Ho}(i) : \mathrm{Ho}(\mathscr{C}_c) \to \mathrm{Ho}(\mathscr{C})$. The inverse is induced by the cofibrant replacemcent functor Q. Recall that QX is cofibrant and there is a natural acyclic fibration $QX \xrightarrow{p_X} X$. In particular, Q preserves weak equivalences and so induces a functor $\mathrm{Ho}(Q) : \mathrm{Ho}(\mathscr{C}) \to \mathrm{Ho}(\mathscr{C}_c)$. The natural transformation p can be thought of as a natural weak equivalence $Q \circ i \to 1_{\mathscr{C}_c}$ or $i \circ Q \to 1_{\mathscr{C}}$. On the homotopy category, $\mathrm{Ho}(p)$ is therefore a natural isomorphism $\mathrm{Ho}(i) \circ \mathrm{Ho}(Q) \to 1_{\mathrm{Ho}(\mathscr{C}_c)}$ and natural isomorphism $\mathrm{Ho}(Q) \circ \mathrm{Ho}(i) \to 1_{\mathrm{Ho}(\mathscr{C})}$, so $\mathrm{Ho}(Q)$ and $\mathrm{Ho}(i)$ are inverse equivalences of categories. A very similar argument holds for the equivalence $\mathrm{Ho}(\mathscr{C}_{cd}) \to \mathrm{Ho}(\mathscr{C}_c)$ where we use the fibrant replacement and conclude by a similar argument as above. Finally the last chain of equivalences is also very similar to the above discussion. ■

4.1.7 The Category $\mathscr{C}_{cf}/_{\sim}$

With the machinery introduced in the section before we are now ready to construct a category $\mathscr{C}_{cf}/_{\sim}$ which remains locally small by construction if $\mathscr{C}$ was. This is particularly important since in the next section we will define the homotopy category of a model category. The definition will use localisations which may not turn out to preserve locally samllness as discussed in the next section. It will however turn out that the homotopy category of $\mathscr{C}_{cf}$ is equivalent to the category $\mathscr{C}_{cf/\sim}$ and further we prove that the homotopy category of a model category $\mathscr{C}$ is equivalent to the homotopy category of $\mathscr{C}_{cf}$ which gives us the desired smallness properties for the localisation.

Proposition 4.25. *The category $\mathscr{C}_{cf}/_{\sim}$ exists.*

Proof. Indeed this follows immediately from the last section especially Lemma 3.10, Lemma 3.11 and Lemma 3.12. ■

We have the following corollary from Proposition 3.17.

Corollary 4.26 ([MHov91])**.** *Let $\mathscr{C}$ be a model category. Let $\gamma : \mathscr{C}_{cf} \to \mathrm{Ho}(\mathscr{C}_{cf})$ and $\delta : \mathscr{C}_{cf} \to \mathscr{C}_{cf}/_{\sim}$ be the canonical functors. Then there is a unique isomorphism of categories $j : \mathscr{C}_{cf}/_{\sim} \to \mathrm{Ho}(\mathscr{C}_{cf})$ such that $j \circ \delta = \gamma$. Furthermore j is the identity on objects.*

Proof. We show that $\mathscr{C}_{cf}/_{\sim}$ has the same universal property that $\mathrm{Ho}(\mathscr{C}_{cf})$ has. The functor δ takes homotopy equivalences to isomorphisms and hence takes weak equivalences to isomorphisms by Proposition 3.17. Suppose that $F : \mathscr{C}_{cf} \to \mathscr{D}$ is a functor that takes weak equivalences to isomorphisms. Let $A \coprod A \xrightarrow{(i_0,i_1)} A' \xrightarrow{s} A$ be a cyclinder object for A. Then $s \circ i_0 = s \circ i_1 = 1_A$ and so since s is a weak equivalence we have $F \circ i_0 = F \circ i_1$. Thus if $H : A' \to B$ is a left homotopy between f and g, we have $Ff = (FH)(Fi_0) = (FH)(Fi_1) = Fg$ and so F identifies left homotopic maps, dually right homotopic maps. Thus there is a unique functor $G : \mathscr{C}_{cf}/_{\sim} \to \mathscr{D}$ such that $G \circ \delta = F$. Indeed, G is the identity on objects and takes the equivalence class of a map f to Ff. The properties of localisations conclude the proof. ■

4.1.8 The Equivalence of $\mathscr{C}_{cf}/_{\sim}$ and $\mathrm{Ho}(\mathscr{C}_{cf})$

The following theorem is often referred to as the fundamental theorem of model categories, as it is called in that way for example in [MHov91]. Either way this theorem gives us the last missing step in concluding the discussion about the homotopy category of a model category, finally yielding the amazing fact that model categories remain locally small under localisation.

Consider a model category $\mathscr{C}$, even though we argued in an earlier section, that not every isomorphism in $\mathrm{Ho}(\mathscr{C})$ comes from a weak equivalence of $\mathscr{C}$ this does indeed hold for model categories, as we will see in the next result.

Theorem 4.27 (Fundamental Theorem of Model Categories, [MHov91])**.** *Let $\mathscr{C}$ be a model category. Let $\gamma : \mathscr{C} \to \mathrm{Ho}(\mathscr{C})$ denote the canonical functor, and let Q denote the cofibrant replacement functor of $\mathscr{C}$ and R denote the fibrant replacement functor.*

1. *The inclusion $\mathscr{C}_{cf} \to \mathscr{C}$ induces an equivalence of categories $\mathscr{C}_{cf}/_{\sim} \xrightarrow{\sim} \mathrm{Ho}(\mathscr{C}_{cf}) \to \mathrm{Ho}(\mathscr{C})$.*
2. *There are natural isomorphisms*
$$\mathscr{C}(QRX, QRY)/_{\sim} \cong \mathrm{Ho}(\mathscr{C}(\gamma X, \gamma Y)) \cong \mathscr{C}(RQX, RQY)/_{\sim}.$$
In addition, there is a natural isomorphism $\mathrm{Ho}(\mathscr{C}(\gamma X, \gamma Y)) \cong \mathscr{C}(QX, RY)/_{\sim}$ and if X is cofibrant and Y is fibrant there is a natural isomorphism $\mathrm{Ho}(\mathscr{C}(\gamma X, \gamma Y)) \cong \mathscr{C}(X, Y)/_{\sim}$. In particular, $\mathrm{Ho}(\mathscr{C})$ is a category without moving to a higher universe.
3. *The functor $\gamma : \mathscr{C} \to \mathrm{Ho}(\mathscr{C})$ identifies left or right homotopic maps.*
4. *If $f : A \to B$ is a map in $\mathscr{C}$ such that γf is an isomorphism in $\mathrm{Ho}(\mathscr{C})$, then f is a weak equivalence.*

Proof.
1. This is Proposition 4.24 and Corollary 4.26.
2. The inverse of the equivalence $\mathrm{Ho}(\mathscr{C}_{cf}) \to \mathrm{Ho}(\mathscr{C})$ is given by $\mathrm{Ho}(Q) \circ \mathrm{Ho}(R)$ or $\mathrm{Ho}(R) \circ \mathrm{Ho}(Q)$. This gives us the natural isomorphism $\mathscr{C}(QRX, QRY)/_{\sim} \cong \mathrm{Ho}(\mathscr{C}(\gamma X, \gamma Y)) \cong \mathscr{C}(RQX, RQY)$. The rest follows from Lemma 2.8, Lemma 2.9 Lemma 2.10 and the fact in the proof of Proposition 1.12 and the natural equivalences $QX \to X \to RX$.
3. This is basically Corollary 4.26 and part 1 of this theorem.
4. Suppose $f : A \to B$ is a map in $\mathscr{C}$ such that γf is an isomorphism in $\mathrm{Ho}(\mathscr{C})$. Then QRf is an isomorphism in $\mathscr{C}_{cf}/_{\sim}$, from which it follows that QRf is a homotopy equivalence. By Proposition 3.17 we see that QRf is a weak equivalence. Then using the fact that both the natural transformations $QX \to X$ and $X \to RX$ are weak equivalences we find that f must be a weak equivalence.

■

Example 4.28. *Consider a model category with the trivial model structure, then the homotopy category is the category itself.* ◀

4.2 Quillen Functors and Derived Functors

The main goal of this section is to introduce the concept of Quillen equivalences. This section follows [MHov91].

4.2.1 Quillen Functors

Definition 4.29 (Quillen Functors)**.** *Let $\mathscr{C}$ and $\mathscr{D}$ be model categories.*

1. *We call a functor $F : \mathscr{C} \to \mathscr{D}$ a* ***left Quillen functor*** *if F is a left adjoint and preserves cofibrations and acyclic cofibrations.*
2. *We call a functor $U : \mathscr{D} \to \mathscr{C}$ a* ***right Quillen functor*** *if U is a right adjoint and preserves fibrations and acyclic fibrations.*
3. *Suppose $F : \mathscr{C} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathscr{D} : U$ is an adjunction. We call $F : \mathscr{C} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathscr{D} : U$ a* ***Quillen adjunction*** *or a* ***Quillen pair*** *if F is a left Quillen functor and denote it by $(F, U)_Q$.*

We could also reformulate condition 3. in the sense that we want U to be a right Quillen functor instead of F to be a left Quillen functor, as the following lemma shows.

Lemma 4.30. *Let $F : \mathscr{C} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathscr{D} : G$ be an adjunction, for $\mathscr{C}$ and $\mathscr{D}$ two model categories. Then $F : \mathscr{C} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathscr{D} : G$ is a Quillen adjunction iff U is a right Quillen functor.*

Proof. One uses adjointness to show that Ff has the LLP with respect to p iff f has the LLP with respect to Up. Then we use the characterisation of cofibrations, acyclic cofibrations, fibrations and acyclic fibrations by lifting properties. ■

Here comes Ken Brown's lemma, it is a mighty tool indeed. Sometimes even a life saver, as we will see in Part III.

Lemma 4.31 (Ken Brown's Lemma)**.** *Let $\mathscr{C}$ be a model category and $(\mathscr{D}, \mathcal{W})$ a category of weak equivalences.*

Assume that $F : \mathscr{C} \to \mathscr{D}$ is a functor sending acyclic cofibrations between cofibrant objects to weak equivalences. Then F sends all weak equivalences between cofibrant objects to weak equivalences.

Dually, if F sends acyclic fibrations between fibrant objects to weak equivalences, then F sends all weak equivalences between fibrant objects to weak equivalences.

Proof. Let A, B be cofibrant objects and $f : A \to B \in \mathcal{W}$. We factor the map $(f, id_B) : A \coprod B \to B$ as $(f, id_B) : A \coprod B \xrightarrow{q \in \mathcal{C}} C \xrightarrow{p \in \mathcal{W} \cap \mathcal{F}} B$. With the pushout diagram

$$\begin{tikzcd}
\emptyset \arrow[r, "\in \mathcal{C}"] \arrow[d, "\in \mathcal{C}"'] & A \arrow[d, "i_0"] \\
B \arrow[r, "i_1"'] & A \coprod B
\end{tikzcd}$$

we get that i_0 and i_1 are cofibrations. By the two out of three property, both $q \circ i_0$ and $q \circ i_1$ are weak equivalences, hence acyclic cofibrations of cofibrant objects.

By hypothesis of F, we then have that both $F(q \circ i_0)$ and $F(q \circ i_1)$ are weak equivalences. Since $F(p) \circ F(q \circ Fi_1) = F(p \circ q \circ i_1) = F(id_B) = id_{F(B)}$ is also a weak equivalence, the two out of three property yields that $F(p)$ is a weak equivalence, the exact same argument shows that $F(f) = F(p \circ q \circ i_0)$ is a weak equivalence. The dual argument follows. ■

Remark 4.32. *K. Brown's lemma implies that every left Quillen functor preserves weak equivalences between cofibrant objects, and that every right Quillen functor preserves weak equivalences between fibrant objects. This is a very helpful fact that we should keep in mind, it will be used in Part III of the thesis.* ♦

Notation 4.33. *Given a Quillen adjunction $(F, U)_Q$, we usually denote the unit map $X \to UFX$ by η and the counit map $FUX \to X$ by ϵ.*

Lemma 4.34 ([MHov91])**.** *Left (right) Quillen functors are stable under composition. Also Quillen adjunctions are stable under composition.*

Proof. Indeed, let $F : \mathscr{C} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathscr{D} : U$ and $F' : \mathscr{D} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathscr{E} : U'$ be adjunctions. We can define their composition to be the adjunction

$$F' \circ F : \mathscr{C} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathscr{E} : U \circ U'.$$

If we assume $\varphi : \mathscr{D}(FA, B) \to \mathscr{C}(A, UB)$ to be a natural isomorphism expressing U as a right adjoint of F and $\varphi' : \mathscr{E}(F'A, B) \to \mathscr{D}(A, U'B)$ to be the natural isomorphism expressing U' as a right adjoint of F' then we may consider their composition $\varphi \circ \varphi'$ as the composite

$$\mathscr{E}(F'FA, B) \to \mathscr{D}(FA, U'B) \to \mathscr{C}(A, UU'B).$$

Composition of adjunctions is associative and has identities. The identity adjunction of a category $\mathscr{C}$ is the identity functor together with identity adjointness isomorphism. The composition of Quillen adjunctions is a Quillen adjunction. ■

Therefore it is possible to define several different notions of a category of model categories using as morphisms left Quillen functors, right Quillen functors or Quillen adjunctions. In the spirit of [MHov91] we will consider Quillen adjunctions. However independent of this choice such a morphism never has to preserve the functorial factorisations.

If we consider a model category $\mathscr{C}$ and use the same category, cofibrations, fibrations and weak equivalences but different functorial factorisations to form a new model category $\mathscr{C}'$ the identity functor will be an isomorphism of model categories between them. Meaning that the choice of functorial factorisations has no effect on the isomorphism class of the model category. Further remarks may be found in [MHov91].

4.2.2 Derived Functors and Naturality

We will now study the functors on the homotopy category induced by Quillen functors.

Definition 4.35 (Total Derived Functors)**.** *1. If $F : \mathscr{C} \to \mathscr{D}$ is a left Quillen functor, define the* ***total left derived Quillen functor*** $\mathbb{L}F : \mathrm{Ho}(\mathscr{C}) \to \mathrm{Ho}(\mathscr{D})$ *to be the composite*

$$\mathrm{Ho}(\mathscr{C}) \xrightarrow{\mathrm{Ho}(Q)} \mathrm{Ho}(\mathscr{C}_c) \xrightarrow{\mathrm{Ho}(F)} \mathrm{Ho}(\mathscr{D}).$$

Given a natural transformation $\tau : F \to F'$ of left Quillen functors, define the ***total derived natural transformation*** $\mathbb{L}\tau$ *to be* $\mathrm{Ho}(\tau) \circ \mathrm{Ho}(Q)$*, so that* $(\mathbb{L}\tau)_X = \tau_{QX}$*.*

2. If $U : \mathscr{D} \to \mathscr{C}$ is a right Quillen functor, define the ***total right derived functor*** $\mathbb{R}U : \mathrm{Ho}(\mathscr{D}) \to \mathrm{Ho}(\mathscr{C})$ *of U to be the composite*

$$\mathrm{Ho}(\mathscr{D}) \xrightarrow{\mathrm{Ho}(R)} \mathrm{Ho}(\mathscr{D}_f) \xrightarrow{\mathrm{Ho}(U)} \mathrm{Ho}(\mathscr{C}).$$

Given a natural transformation $\tau : U \to U'$ of right Quillen functors, define the ***total derived natural transformation*** $\mathbb{R}\tau$ *to be* $\mathrm{Ho}(\tau) \circ \mathrm{Ho}(R)$*, so that* $R_{\tau X} = \tau_{RX}$*.*

This definition is the reason we have assumed that the functorial factorizations are part of the structure of a model category. Otherwise, in order to define $\mathbb{L}F$, we would have to choose a functorial cofibrant replacement, so we would not be able to define $\mathbb{L}F$ in a way that depends only on the model category $\mathscr{C}$.

Note that we can define $\mathbb{L}F$ even if F is not a left Quillen functor, but just a functor that takes weak equivalences between cofibrant objects to weak equivalences. Dually, we can define $\mathbb{R}U$ if U is any functor that takes weak equivalences between fibrant objects to weak equivalences.

Lemma 4.36 ([MHov91])**.** *The total derived natural transformation is functorial.*

Proof. Indeed, if $\tau : F \to F'$ and $\tau' : F' \to F''$ are natural transformations between weak left Quillen functors, then $\mathbb{L}(\tau' \circ \tau) = (\mathbb{L}\tau') \circ (\mathbb{L}\tau)$, and of course $\mathbb{L}(1_F) = 1_{\mathbb{L}F}$. We have a dual statement for natural transformations between right Quillen functors. ■

The following theorem can be found as Theorem 1.3.7 in [MHov91].

Theorem 4.37 ([MHov91])**.** *Let $\mathscr{C}$ be a model category. There is a natural isomorphism $\alpha : \mathbb{L}(1_{\mathscr{C}}) \to 1_{\mathrm{Ho}(\mathscr{C})}$. Also, for every pair of left Quillen functors $F : \mathscr{C} \to \mathscr{D}$ and $F' : \mathscr{D} \to \mathscr{E}$, there is a natural isomorphism $m := M_{F'F} : \mathbb{L}F' \circ \mathbb{L}F \to \mathbb{L}(F' \circ F)$. These natural isomorphisms satisfy the following properties.*

1. *An associativity coherence diagram commutes. That is, if $F : \mathscr{C} \to \mathscr{C}'$, $F' : \mathscr{C}' \to \mathscr{C}''$ and $F'' : \mathscr{C}'' \to \mathscr{C}'''$ are left Quillen functors, then the following diagram commutes.*

$$\begin{tikzcd}
(\mathbb{L}F'' \circ \mathbb{L}F') \circ \mathbb{L}F \arrow[r, "m_{F''F'} \circ \mathbb{L}F"] \arrow[d, equal] & \mathbb{L}(F'' \circ F') \circ \mathbb{L}F \arrow[r, "m_{(F'' \circ F')F}"] & \mathbb{L}((F'' \circ F') \circ F) \arrow[d, equal] \\
\mathbb{L}F'' \circ (\mathbb{L}F' \circ \mathbb{L}F) \arrow[r, "\mathbb{L}F'' \circ m_{F'F}"'] & \mathbb{L}F'' \circ \mathbb{L}(F' \circ F) \arrow[r, "m_{F''(F' \circ F)}"'] & L(F'' \circ (F' \circ F)).
\end{tikzcd}$$

2. *A left unit coherence diagram commutes. That is, if $F : \mathscr{C} \to \mathscr{D}$ is a left Quillen functor, then the following diagram commutes.*

$$\begin{tikzcd}
\mathbb{L}1_{\mathscr{D}} \circ \mathbb{L}F \arrow[r, "m"] \arrow[d, "\alpha\circ\mathbb{L}F"'] & L(1_{\mathscr{D}} \circ F) \arrow[d, equal] \\
1_{\mathrm{Ho}(\mathscr{D})} \circ \mathbb{L}F \arrow[r, equal] & \mathbb{L}F.
\end{tikzcd}$$

3. *A right unit coherence diagram commutes. That is, if $F : \mathscr{C} \to \mathscr{D}$ is a left Quillen functor, then the following diagram commutes.*

$$\begin{tikzcd}
\mathbb{L}F \circ \mathbb{L}1_{\mathscr{C}} \arrow[r, "m"] \arrow[d, "\mathbb{L}F\circ\alpha"'] & \mathbb{L}(F \circ 1_{\mathscr{C}}) \arrow[d, equal] \\
\mathbb{L}F \circ 1_{\mathrm{Ho}(\mathscr{C})} \arrow[r, equal] & \mathbb{L}F.
\end{tikzcd}$$

Definition 4.38 (Horizontal Composition)**.** *Suppose $\sigma : F \Rightarrow G$ is a natural transformation of functors $F, G : \mathscr{C} \to \mathscr{D}$ and $\tau : F' \Rightarrow G'$ is a natural transformation of functors $F', G' : \mathscr{D} \to \mathscr{E}$. The* ***horizontal composition*** *$\tau * \sigma$ is the natural transformation $F' \circ F \Rightarrow G' \circ G$ given by $(\tau \circ \sigma)_X = \tau_{GX} \circ F'\sigma_X = G'\sigma_X \circ \tau_{FX}$.*

Lemma 4.39 ([MHov91])**.** *Suppose $\sigma : F \Rightarrow G$ is a natural transformation of weak left Quillen functors $F, G : \mathscr{C} \to \mathscr{D}$ and $\tau : F' \Rightarrow G'$ is a natural transformation of weak left Quillen functors $F', G' : \mathscr{D} \to \mathscr{E}$. Let m be the composition isomorphism of Theorem 4.37. Then the following diagram commutes.*

$$\begin{tikzcd}
\mathbb{L}F' \circ \mathbb{L}F \arrow[r, "m"] \arrow[d, "\mathbb{L}_\tau * \mathbb{L}_\sigma"'] & \mathbb{L}(F' \circ F) \arrow[d, "\mathbb{L}(\tau*\sigma)"] \\
\mathbb{L}G' \circ \mathbb{L}G \arrow[r, "m"'] & \mathbb{L}(G' \circ G).
\end{tikzcd}$$

Proof. The map $\mathbb{L}(\tau*\sigma)\circ m : F'QFQX \to G'GQX$ is the composite $\tau_{GQX} \circ F'\sigma_{QX} \circ F'Q\sigma_{QX}$. By the naturality of q this may be rewriten as $\tau_{GQX} \circ F'q_{GQX} \circ F'Q\sigma_{QX}$. Then by the naturality of τ we further rewrite this as $G'q_{GQX} \circ \tau_{QGQX} \circ F'Q\sigma_{QX}$ which is the definition of $m \circ (\mathbb{L}\tau * \mathbb{L}\sigma)$. ∎

We would also like the same claim for adjunctions, for that matter we have to show that the total derived functor preserves adjunctions.

Lemma 4.40 ([MHov91])**.** *Suppose $\mathscr{C}$ and $\mathscr{D}$ are model categories and $(F, U)_Q : \mathscr{C} \to \mathscr{D}$ is a Quillen adjunction. Then $\mathbb{L}F$ and $\mathbb{R}U$ are part of the adjunction $(\mathbb{L}F, \mathbb{R}U)$, which we call the derived adjunction.*

Proof. Consider the natural isomorphism $\varphi : \mathscr{D}(FX, Y) \to \mathscr{C}(X, UY)$ coming from the given adjunction, but then $R\varphi : \mathrm{Ho}(\mathscr{D}(FQX, Y)) \to \mathrm{Ho}(\mathscr{C}(X, URY))$ must be a natural isomorphism. As we have seen, there are natural isomorphisms $\mathrm{Ho}(\mathscr{D}(FQX, Y)) \cong \mathscr{D}(FQX, RY)/_\sim$ and $\mathrm{Ho}(\mathscr{C}(X, URY)) \cong \mathscr{C}(QX, URY)/_\sim$. Therefore we must show, that φ respects the homotopy relation. Let $A \in \mathscr{C}$ be cofibrant and $B \in \mathscr{D}$ fibrant and assume that $f, g : FA \to B$ are homotopic. Then there is a path object $\mathrm{Path}(B)$ for B and a right homotopy $H : FA \to \mathrm{Path}(B)$ from f to g. Since U preserves products, fibrations and weak equivalences between fibrant objects, $U\,\mathrm{Path}(B)$ is a path object for UB. But this means that $\varphi H : A \to U\,\mathrm{Path}(B)$ is a right homotopy from φf to φg. Conversely let φf and φg be homotopic. Then there is a cylinder object $\mathrm{Cyl}(A)$ for A and a left homotopy $H : \mathrm{Cyl}(A) \to UB$ from φf to φg. Since F preserves coproducts, cofibrations and weak equivalences between cofibrant objects, $F\,\mathrm{Cyl}(A)$ is a cylinder object for FA. Finally $\varphi^{-1} : F\,\mathrm{Cyl}(A) \to B$ is a left homotopy from $\varphi^{-1}\varphi f = f$ to g. ■

4.2.3 Quillen Equivalences

It may happen, that the derived adjunction $(\mathbb{L}F, \mathbb{R}U)$ is an equivalence of categories but (F, U) is not. In the following subsection we will discuss this behavior. This will lead us to the definition of a Quillen equivalence.

From now on if we talk about an adjunction (F, U) i.e. $F : \mathscr{C} \underset{\perp}{\rightleftarrows} \mathscr{D} : U$ we consider a natural isomorphism $\varphi : \mathscr{D}(FA, B) \to \mathscr{C}(A, UB)$ expressing U as a right adjoint of F. We use the notation (F, U, φ) for the adjunction and if it happens to be a Quillen adjunction we use $(F, U, \varphi)_Q$.

Definition 4.41. *A Quillen adjunction $(F, U, \varphi)_Q$ between $\mathscr{C}$ and $\mathscr{D}$ is called a* ***Quillen equivalence*** *iff for all cofibrant X in $\mathscr{C}$ and fibrant Y in $\mathscr{D}$, a map $f : FX \to Y$ is a weak equivalence in $\mathscr{D}$ iff $\varphi(f) : X \to UY$ is a weak equivalence in $\mathscr{C}$.*

Proposition 4.42 ([MHov91])**.** *Let $(F, U, \varphi)_Q : \mathscr{C} \to \mathscr{D}$ be a Quillen adjunction. Then the following are equivalent.*

1. *$(F, U)_Q$ is a Quillen equivalence.*
2. *The composite $X \xrightarrow{\eta} UFX \xrightarrow{Ur_{FX}} URFX$ is a weak equivalence for all cofibrant X and the composite $FQUX \xrightarrow{Fq_{UX}} FQX \xrightarrow{\epsilon}$ is a weak equivalence for all fibrant X.*
3. *$(\mathbb{L}F, \mathbb{R}U)_Q$ is an adjoint equivalence of categories.*

Proof. "1. $\Rightarrow$ 2." : Let $(F, U, \varphi)_Q$ be a Quillen equivalence and X cofibrant then $\varphi r_{FX} : X \to URFX \in \mathcal{W}$ adjoint to $r_{FX} : FX \to RFX \in \mathcal{W}$. In terms of the unit η of φ we have $\varphi r_{FX} = Ur_{FX} \circ \eta$.

Similarly if X is fibrant so $Fq_{UX} = \varphi^{-1} q_{UX} \in \mathcal{W}$ adjoint to $q_{UX} : QUX \to UX$.

"2. $\Rightarrow$ 1." : Assume (F, U, φ) satisfies 2. Let $f : FX \to Y \in \mathcal{W}$ for X cofibrant ad Y fibrant $\varphi f : X \xrightarrow{\eta} URX \xrightarrow{Uf} UY$ we have a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "\eta"] \arrow[d, equal] & UFX \arrow[r, "Uf"] \arrow[d, "Ur_{FX}"] & UY \arrow[d, "Ur_Y"] \\
X \arrow[r] & UR_{FX} \arrow[r, "URf"'] & URY.
\end{tikzcd}$$

$f \in \mathcal{W}$ implies $RF \in \mathcal{W}$. $URf \in \mathcal{W}$ since U preserves weak equivalences between fibrant objects therefore the bottom composition above is a weak equivalence and also the rightmost vertical map is a weak equivalence. Therefore the top composition $\varphi f \in \mathcal{W}$.

If $\varphi f : X \to UY \in \mathcal{W}$ then we have a commutative diagram

$$\begin{tikzcd} FQX \arrow[r, "FQ(\varphi f)"] \arrow[d, "F_{qX}"'] & FQUY \arrow[r] \arrow[d, "F_{qUY}"] & Y \arrow[d, equal] \\ FX \arrow[r, "F(\varphi f)"'] & FUY \arrow[r, "\epsilon"'] & Y. \end{tikzcd}$$

The bottom composition is f. The top composition and the leftmost vertical map are weak equivalences and hence $f \in \mathcal{W}$. "2. $\Leftrightarrow$ 3." : The unit of $R\varphi$ is the map $X \xrightarrow{q_X^{-1}} QX \xrightarrow{Ur_{FQX} \circ \eta} URFQX$. Hence by Theorem 4.27 the unit of $R\varphi$ is an isomorphism iff $Ur_{FQX}\eta$ is a weak equivalence for all cofibrant X. The proof uses the fact that F preserves weak equivalences between cofibrant objects, the fact that U preserves weak equivalences between fibrant objects and the commutative diagram

$$\begin{tikzcd} QX \arrow[r, "\eta"] \arrow[d, "q_X"'] & UFQX \arrow[r, "Ur_{FQX}"] \arrow[d, "UF_{qX}"] & URFQX \arrow[d, "URF_{qX}"] \\ X \arrow[r, "\eta"'] & UFX \arrow[r, "Ur_X"'] & URFX. \end{tikzcd}$$

Dually, the counit $R\varphi$ is an isomorphism iff $\epsilon \circ F_{qUX}$ is a weak equivalence for all fibrant X. ■

Going back to the remark in the introduction of this subsection, we now see that if $(\mathbb{L}F, \mathbb{R}U)$ is an equivalence of categories, then we say that (F, U) is a Quillen equivalence, and the other way around. This provides us a tool to compare different homotopy theories in the setting of model categories.

We state some useful corollaries which will not be proven but the proofs can be found in [MHov91].

Corollary 4.43 ([MHov91])**.** *Suppose (F, U, φ) and (F, U', φ') are Quillen adjunctions from $\mathscr{C}$ to $\mathscr{D}$. Then (F, U, φ) is a Quillen equivalence iff (F, U', φ') is so. Dually if $(F', U, \varphi'')_Q$ is another Quillen adjunction then $(F, U, \varphi)_Q$ is a Quillen equivalence iff $(F', U, \varphi'')_Q$ is so.*

The next result says, that Quillen equivalences have the 2-3 property.

Corollary 4.44 ([MHov91])**.** *Suppose $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{E}$ are left (resp. right) Quillen functors. Then if two out of three of F, G and GF are Quillen equivalences then so is the third.*

Corollary 4.45 ([MHov91]). *Suppose $(F, U, \varphi)_Q : \mathscr{C} \to \mathscr{D}$ is a Quillen adjunction. The following are equivalent.*

1. *(F, U, φ) is a Quillen equivalence.*
2. *F reflects weak equivalences between cofibrant objects and for every fibrant Y the map $FQUY \to Y$ is a weak equivalence.*
3. *U reflects weak equivalences between fibrant objects and for every cofibrant X the map $X \to URFX$ is a weak equivalence.*

Lemma 4.46 ([MHov91]). *Suppose $\tau : F \Rightarrow G$ is a natural transformation between left (resp. right) Quillen functors. Then L_τ (resp. R_τ) is a natural isomorphism iff τ_X is a weak equivalence for all cofibrant (resp. fibrant) X.*

Proof. Assume that F and G are left Quillen functors. Then $(L\tau)_X = \tau_{QX}$ so $L\tau$ is a natural isomorphism iff τ_{QX} is a weak equivalence for all X. Since F and G preserve weak equivalences between cofibrant objects this is true iff τ_X is a weak equivalence for all cofibrant X. The dual statement is similar. ∎

5 Bousfield Localisations

An important tool in order to prove our main results in Part III will be the machinery of Bousfield localisations, especially the case of left Bousfield localisations. They will also be used to formalise truncations of simplicial sets in the last part of the thesis.

The idea of Bousfield localisations is rather simple though if one actually wants to set up everything in a formal way it will become tough quite fast. Anyway, the idea of Bousfield localisations is roughly to add more weak equivalences to a model category but keep the same cofibrations, the result will be - if certain conditions are met - a new model category structure on the underlying category. This new model structure also has a lot of the nice properties the old one had. We will for example see, that a large class - where the localisation will exist for any set of morphisms from the respective category - exists. This class is given by cofibrantly generated, cellular and left proper model categories in the case of left Bousfield localisations. The existence theorem, which will be at the end of the section, will then imply that there is a new model category structure on the underlying category with more weak equivalences as before and this new model structure is in addition also cofibrantly generated, cellular and left proper. Furthermore, if the initial category was a simplicial one then the localisation also carries the structure of a siplicial model category.

There is also another large class, where the left Bousfield localisation exists for any set of morphisms of the respective category. That is, if we have a left proper, simplicial and combinatorial model category then the theorem implies that there is a new model category structure i.e. the left Bousfield localisation, on the underlying category. This result will only be stated for completeness of the discussion, in order to give the theorems for both cases. They give the largest known classes, where the left Bousfield localisation exists with respect to any set of morphisms. The results can be found in [PHir03] and [ClBa] respectively.

It was Bousfield who introduced the concept of such localisations, but it was Hirschhorn who further developed this concept and gave a lot of very useful properties about it. This can all be found in his book [PHir03].

We will only discuss left Bousfield localisations, since we will only deal with them. Furthermore they are easier to handle than the right ones. One more reason for doing so is that they seem to arise more naturally. Anyways, there are always similar results for the case of right Bousfield localisations, a discussion about them may also be found in [PHir03].

It is no surprise, that we will follow the book [PHir03] for this whole section and in addition, for the second theorem about the existence, we also follow [ClBa].

5.1 The Reedy Model Category Structure

We introduce some concepts needed, in order to be able to define Bousfield localisations. This theory can be found in [PHir03].

5.1.1 Reedy Categories

Definition 5.1 (Reedy Category). *A **Reedy category** is a quadruple $(\mathscr{C}, \mathscr{C}_+, \mathscr{C}_-, \deg)$, consisting of a small category $\mathscr{C}$ together with two subcategories $\mathscr{C}_+$ and $\mathscr{C}_-$ both containing all objects of $\mathscr{C}$ and in which every object can be assigned a **degree** $\deg : \mathrm{ob}(\mathscr{C}) \to \mathbb{Z}_{\geq 0}$, such that*

1. *Every non-identity morphism of $\mathscr{C}_+$ raises degree.*
2. *Every non-identity morphism of $\mathscr{C}_-$ lowers degree.*
3. *Every morphism g in $\mathscr{C}$ has a unique factorisation $g = g_+ \circ g_-$ with $g_+ \in \mathscr{C}_+$ and $g_- \in \mathscr{C}_-$.*

Remark 5.2. *The category $\mathscr{C}_+$ is called **direct subcategory** and the category $\mathscr{C}_-$ is called **inverse subcategory**.* ⧫

Example 5.3. *We already have an example of a Reedy category in section 1. Indeed, the category of finite ordinals is such an example. We just have to choose $\mathscr{C}_+ := \mathbf{\Delta}_+$ and $\mathscr{C}_- := \mathbf{\Delta}_-$ with degree*

$$\begin{aligned} \mathrm{ob}(\mathbf{\Delta}) &\to \mathbb{Z}_{\geq 0} \\ [n] &\mapsto n. \end{aligned}$$

◀

Proposition 5.4. *If $\mathscr{C}$ is a Reedy category, then $\mathscr{C}^{op}$ is a Reedy category in which $\mathscr{C}_+^{op} = (\mathscr{C}_+)^{op}$ and $\mathscr{C}_-^{op} = (\mathscr{C}_-)^{op}$.*

Proof. Just choose a degree function for $\mathscr{C}$, the same function will work as a degree function for $\mathscr{C}^{op}$. ■

Proposition 5.5. *If $\mathscr{C}$ and $\mathscr{D}$ are Reedy categories, then $\mathscr{C} \times \mathscr{D}$ is a Reedy category with $(\mathscr{C} \times \mathscr{D})_+ = \mathscr{C}_+ \times \mathscr{D}_+$ and $(\mathscr{C} \times \mathscr{D})_- = \mathscr{C}_- \times \mathscr{D}_-$.*

Proof. If we have chosen degree functions for $\mathscr{C}$ and $\mathscr{D}$, we define a degree function for $\mathscr{C} \times \mathscr{D}$ by $\deg(X \times Y) = \deg(X) + \deg(Y)$. The required uniqueness follows from the given uniqueness for $\mathscr{C}$ and $\mathscr{D}$. ■

5.1.2 The Reedy Model Category Structure

Definition 5.6 (Latching and Matching Category)**.** *Let $\mathscr{C}$ be a Reedy category and let $\alpha \in \mathscr{C}$ be an object.*

1. *The* ***latching category*** $\partial(\mathscr{C}_+ \downarrow \alpha)$ *of $\mathscr{C}$ at α is the full subcategory of $(\mathscr{C}_+ \downarrow \alpha)$ containing all objects except the identity map of α.*
2. *The* ***matching category*** $\partial(\alpha \downarrow \mathscr{C}_-)$ *of $\mathscr{C}$ at α is the full subcategory of $(\alpha \downarrow \mathscr{C}_-)$ containing all objects except the identity map of α.*

Remark 5.7. *Let $\mathscr{C}$ be a Reedy category and let α be an object of $\mathscr{C}$. Then the opposite of the latching category of $\mathscr{C}$ at α is naturally isomorphic to the matching category of $\mathscr{C}^{op}$ at α. Similarly the opposite of the matching category of $\mathscr{C}$ at α is naturally isomorphic to the latching category of $\mathscr{C}^{op}$ at α.* ♦

Definition 5.8. *Let $\mathscr{C}$ be a category and $\mathscr{D}$ a model category.*

1. *If $\mathbf{X}$ is a $\mathscr{C}$-diagram in $\mathscr{D}$ then $\mathbf{X}$ is*
   (a) ***objectwise cofibrant*** *if $\mathbf{X}_\alpha$ is a cofibrant object of $\mathscr{D}$ for every $\alpha \in \mathrm{ob}(\mathscr{C})$.*
   (b) ***objectwise fibrant*** *if $\mathbf{X}_\alpha$ is a fibrant object in $\mathscr{D}$ for every $\alpha \in \mathrm{ob}(\mathscr{C})$.*
2. *If $\mathbf{X}$ and $\mathbf{Y}$ are $\mathscr{C}$-diagrams in $\mathscr{D}$, then a map of diagrams $f : \mathbf{X} \to \mathbf{Y}$ is*
   (a) *an* ***objectwise cofibration*** *if $f_\alpha : \mathbf{X}_\alpha \to \mathbf{Y}_\alpha$ is a cofibration for all $\alpha \in \mathrm{ob}(\mathscr{C})$.*
   (b) *an* ***objectwise fibration*** *if $f_\alpha : \mathbf{X}_\alpha \to \mathbf{Y}_\alpha$ is a fibration for all $\alpha \in \mathrm{ob}(\mathscr{C})$.*
   (c) *an* ***objectwise weak equivalence*** *if $f_\alpha : \mathbf{X}_\alpha \to \mathbf{Y}_\alpha$ is a weak equivalence for all $\alpha \in \mathrm{ob}(\mathscr{C})$.*

Definition 5.9. *Let $\mathscr{C}$ be a Reedy category and $\mathscr{D}$ a model category. Let $\mathbf{X}$ be a $\mathscr{C}$-diagram in $\mathscr{D}$ and $\alpha \in \mathrm{ob}(\mathscr{C})$. We use $\mathbf{X}$ to denote also the induced $\partial(\mathscr{C}_+ \downarrow \alpha)$-diagram, defined on objects by $\mathbf{X}_{(\beta \to \alpha)} = \mathbf{X}_\beta$ and the induced $\partial(\alpha \downarrow \mathscr{C}_-)$-diagram, defined on objects by $\mathbf{X}_{(\beta \to \alpha)} = \mathbf{X}_\beta$.*

1. *The* ***latching object*** *of $\mathbf{X}$ at α is $L_\alpha^{\mathscr{C}}\mathbf{X} = \mathrm{colim}_{\partial(\mathscr{C}_+ \downarrow \alpha)} \mathbf{X}$ and the* ***latching map*** *of $\mathbf{X}$ at α is the natural map $L_\alpha^{\mathscr{C}}\mathbf{X} \to \mathbf{X}_\alpha$.*
2. *The* ***matching object*** *of $\mathbf{X}$ at α is $M_\alpha^{\mathscr{C}}\mathbf{X} = \lim_{\partial(\alpha \downarrow \mathscr{C}_-)} \mathbf{X}$ and the* ***matching map*** *of $\mathbf{X}$ at α is the natural map $\mathbf{X}_\alpha \to M_\alpha^{\mathscr{C}}\mathbf{X}$.*

Definition 5.10. *Let $\mathscr{C}$ be a Reedy category and $\mathscr{D}$ a model category. Let $\mathbf{X}$ and $\mathbf{Y}$ be $\mathscr{C}$-diagrams in $\mathscr{D}$ and let $f : \mathbf{X} \to \mathbf{Y}$ be a map of $\mathscr{C}$-diagrams.*

1. *If α is an object in $\mathscr{C}$ then the* ***relative latching map of*** *f* ***at*** *α is the map $\mathbf{X}_\alpha \coprod_{L_\alpha^{\mathscr{C}}\mathbf{X}} L_\alpha^{\mathscr{C}}\mathbf{Y} \to \mathbf{Y}_\alpha$.*
2. *If α is an object in $\mathscr{C}$ then the* ***relative matching map of*** *f* ***at*** *α is the map $\mathbf{X}_\alpha \to \mathbf{Y}_\alpha \prod_{M_\alpha^{\mathscr{C}}\mathbf{Y}} M_\alpha^{\mathscr{C}}\mathbf{X}$.*

Example 5.11. *We consider latching and matching objects in* $\mathbf{sSet}$*, the category of simplicial sets which we will define later. Consider $S : \mathbf{\Delta}^{op} \to \mathbf{Set}$, where we consider $\mathbf{Set}$ with the trivial model structure. $L_{[n]}^{\mathbf{\Delta}^{op}} S = \mathrm{colim}_{\partial(\mathbf{\Delta}_+^{op} \downarrow [n])} S$. Consider $[m] \to [l] \in \mathbf{\Delta}^{op}$ which gives a factorisation*

$$\begin{tikzcd} & {[n]} \arrow[dl] \arrow[dr] & \\ {[m]} & & {[l]} \arrow[ll] \end{tikzcd}$$

in $\partial(\mathbf{\Delta}_+^{op} \downarrow [n])$*. Then*

$$\begin{tikzcd}
S_m \arrow[dr] \arrow[ddr] & & S_l \arrow[ll] \arrow[dl] \arrow[ddl] \\
& \operatorname{colim} S \arrow[d, dotted] & \\
& S_n &
\end{tikzcd}$$

where the dotted map is the natural latching map. On the other hand for $M_{[n]}^{\mathbf{\Delta}^{op}} S = \lim_{\partial([n]\downarrow \mathbf{\Delta}_-^{op})} S$ *we have*

$$\begin{tikzcd}
& S_n \arrow[d, dotted] \arrow[ddl] \arrow[ddr] & \\
& \lim S \arrow[dl] \arrow[dr] & \\
S_m & & S_l \arrow[ll]
\end{tikzcd}$$

where the dotted line is the natural matching map. We also used the fact that

$$\begin{aligned}
X : (\mathbf{\Delta}_+^{op} \downarrow [n]) &\to \mathscr{D} \\
[m] \to [n] &\mapsto X_{[m]} =: X_m.
\end{aligned}$$

Actually it is worth pointing out that the n-th latching object is the union of all degenerate n-simplices and the n-th matching object is the $(n-1)$ *skeleton. Therefore "cofibrations" of simplicial sets in a model category with respect to a Reedy model structure are similar to building a space by cell attachements and Reedy fibrations are closely related to looking at a Postnikov tower. Some of this concepts will be introduced later, but it is still best to state this observation right here.* ◀

Definition 5.12 (Reedy Model Structure)**.** *Let* $\mathscr{C}$ *be a Reedy category, let* $\mathscr{D}$ *be a model category and let* $\mathbf{X}, \mathbf{Y} : \mathscr{C} \to \mathscr{D}$ *be* $\mathscr{C}$*-diagrams in* $\mathscr{D}$*.*

1. *A map of diagrams* $f : \mathbf{X} \to \mathbf{Y}$ *is a* ***Reedy weak equivalence*** *if for every object* α *of* $\mathscr{C}$ *the map* $f_\alpha : \mathbf{X}_\alpha \to \mathbf{Y}_\alpha$ *is a weak equivalence in* $\mathscr{D}$*.*
2. *A map of diagrams* $f : \mathbf{X} \to \mathbf{Y}$ *is a* ***Reedy cofibration*** *if for every object* α *of* $\mathscr{C}$ *the relative latching map*
$$\mathbf{X}_\alpha \coprod_{L_\alpha^{\mathscr{C}}\mathbf{X}} L_\alpha^{\mathscr{C}}\mathbf{Y} \to \mathbf{Y}_\alpha$$
is a cofibration in $\mathscr{D}$*.*
3. *A map of diagrams* $f : \mathbf{X} \to \mathbf{Y}$ *is a* ***Reedy fibration*** *if for every object* α *of* $\mathscr{C}$ *the relative matching map*
$$\mathbf{X}_\alpha \to \mathbf{Y}_\alpha \prod_{M_\alpha^{\mathscr{C}}\mathbf{Y}} M_\alpha^{\mathscr{C}}\mathbf{X}$$
is a fibration in $\mathscr{D}$*.*

Example 5.13. *Let $\mathscr{C}$ be a model category. Then the category $\mathbf{Fun}(\mathbf{\Delta}^{op}, \mathscr{C})$ of simplicial objects has a model category structure from the Reedy category structure of $\mathbf{\Delta}^{op}$. Similarly, the category $\mathbf{Fun}(\mathbf{\Delta}, \mathscr{C})$ of cosimplicial objects has a model category structure from the Reedy category structure of $\mathbf{\Delta}$.* ◀

We have the following result, which can be found in this form as Theorem 15.3.4 in [PHir03] and was due to D. M. Kan. The proof can be found as section 15.3.16 in [PHir03].

Theorem 5.14 ([PHir03])**.** *Let $\mathscr{C}$ be a Reedy category and let $\mathscr{D}$ be a model category.*

1. *The category $\mathscr{D}^{\mathscr{C}}$ of $\mathscr{C}$-diagrams in $\mathscr{D}$ with the Reedy weak equivalences, Reedy cofibrations and Reedy fibrations is a model category.*
2. *If $\mathscr{D}$ is a left proper, right proper or proper model category, then the model category defined in 1. is respectively left proper, right proper or proper.*
3. *If $\mathscr{D}$ is a simplicial model category, then the model category defined in 1. is a simplicial model category.*

5.2 Left Bousfield Localisations

We can finally introduce our main machinery.

5.2.1 Resolutions

Notation 5.15. *Let $\mathscr{C}$ be a model category and let $X \in \mathscr{C}$ be an object. Then*

1. *the* ***constant cosimplicial object*** *at X will be denoted cc_*X.*
2. *the* ***constant simplicial object*** *at X will be denoted cs_*X.*

Definition 5.16 (Resolutions)**.** *Let $\mathscr{C}$ be a model category and $X \in \mathscr{C}$ an object.*

1. (a) *A* ***cosimplicial resolution*** *of X is a cofibrant approximation $\widetilde{\mathbf{X}} \to cc_*X$ to cc_*X in the Reedy model category structure on $\mathbf{Fun}(\mathbf{\Delta}, \mathscr{C})$.*
   (b) *A* ***fibrant cosimplicial resolution*** *is a cosimplicial resolution in which the weak equivalence $\widetilde{\mathbf{X}} \to cc_*X$ is a Reedy acyclic fibration.*
2. (a) *A* ***simplicial resolution*** *of X is a fibrant approximation $cs_*X \to \widehat{\mathbf{X}}$ to cs_*X in the Reedy model category structure on $\mathbf{Fun}(\mathbf{\Delta}^{op}, \mathscr{C})$.*
   (b) *A* ***cofibrant simplicial resolution*** *is a simplicial resolution in which the weak equivalence $cs_*X \to \widehat{\mathbf{X}}$ is a Reedy acyclic cofibration.*

Remark 5.17. *Sometimes we use the term cosimplicial resolution to refer to the object $\widetilde{\mathbf{X}}$ without explicitely mentioning the weak equivalence $\widetilde{\mathbf{X}} \to cc_*X$. Similarly, we use simplicial resolution to refer to the object $\widehat{\mathbf{X}}$ without explicitely mentioning the weak equivalence $cs_*X \to \widehat{\mathbf{X}}$.* ♦

The next result appears as Proposition 16.1.3 in [PHir03].

Proposition 5.18 ([PHir03]). *Let $\mathscr{C}$ be a simplicial model category.*

1. *If X is an object of $\mathscr{C}$ and $W \to X$ is a cofibrant approximation to X, then the cosimplicial object $\widetilde{\mathbf{W}}$ in which $\widetilde{\mathbf{W}}^n = W \otimes \Delta^n$ is a cosimplicial resolution of X.*
2. *If Y is an object in $\mathscr{C}$ and $Y \to Z$ is a fibrant approximation to Y, then the simplicial object $\widehat{\mathbf{Z}}$ in which $\widehat{\mathbf{Z}}_n = Z^{\Delta^n}$ is a simplicial resolution of Y.*

Corollary 5.19. *Let $\mathscr{C}$ be a simplicial model category.*

1. *If X is a cofibrant object of $\mathscr{C}$, then the cosimplicial object $\widetilde{\mathbf{X}}$ in which $\widetilde{\mathbf{X}}^n = X \otimes \Delta^n$ is a cosimplicial resolution of X.*
2. *If Y is a fibrant object of $\mathscr{C}$, then the simplicial object $\widehat{\mathbf{Y}}$ in which $\widehat{\mathbf{Y}}_n = Y^{\Delta^n}$ is a simplicial resolution of Y.*

Proof. Follows from Proposition 5.18. ■

Notation 5.20. *Let $\mathscr{C}$ be a model category.*

1. *If $\mathbf{X}$ is a cosimplicial object in $\mathscr{C}$ and $Y \in \mathscr{C}$ then $\mathscr{C}(\mathbf{X}, Y)$ will denote the simplicial set natural in both $\mathbf{X}$ and Y defined by*

$$\mathscr{C}(\mathbf{X}, Y)_n = \mathscr{C}(\mathbf{X}^n, Y)$$

   with face and degeneracy maps induced by the coface and codegeneracy maps in $\mathbf{X}$.
2. *If $X \in \mathscr{C}$ and $\mathbf{Y}$ is a simplicial object in $\mathscr{C}$ then $\mathscr{C}(X, \mathbf{Y})$ will denote the simplicial set natural in X and $\mathbf{Y}$ defined by*

$$\mathscr{C}(X, \mathbf{Y})_n = \mathscr{C}(X, \mathbf{Y}_n)$$

   with face and degeneracy maps induced by those in $\mathbf{Y}$.
3. *If $\mathbf{X}$ is a cosimplicial object in $\mathscr{C}$, $\mathbf{Y}$ is a simplicial object in $\mathscr{C}$ then $\mathscr{C}(\mathbf{X}, \mathbf{Y})$ will denote a bisimplicial set, natural in $\mathbf{X}$ and $\mathbf{Y}$, defined by*

$$\mathscr{C}(\mathbf{X}, \mathbf{Y})_{n,k} = \mathscr{C}(\mathbf{X}^k, \mathbf{Y}_n)$$

   with face and degeneracy maps induced by the coface and codegeneracy maps in $\mathbf{X}$ and the face and degeneracy maps in $\mathbf{Y}$.
4. *If $\mathbf{X}$ is a cosimplicial object in $\mathscr{C}$, $\mathbf{Y}$ a simplicial object in $\mathscr{C}$, then $diag\mathscr{C}(\mathbf{X}, \mathbf{Y})$ will denote the simplicial set, natural in $\mathbf{X}$ and $\mathbf{Y}$ defined by*

$$(diag\mathscr{C}(\mathbf{X}, \mathbf{Y}))_n = \mathscr{C}(\mathbf{X}^n, \mathbf{Y}_n)$$

   with face and degeneracy maps induced by coface and codegeneracy maps in $\mathbf{X}$ and face and degeneracy maps in $\mathbf{Y}$.

5.2.2 Homotopy Function Complexes

Definition 5.21 (Left Homotopy Function Complexes)**.** *If $\mathscr{C}$ is a model category and $X, Y \in \mathscr{C}$, then a* ***left homotopy function complex*** *from X to Y is a triple*

$$(\widetilde{\mathbf{X}}, RY, \mathscr{C}(\widetilde{\mathbf{X}}, RY))$$

where

1. *$\widetilde{\mathbf{X}}$ is a cosimplicial resolution of X*
2. *RY is a fibrant approximation to Y*
3. *$\mathscr{C}(\widetilde{\mathbf{X}}, RY)$ is the simplicial set in 5.20 1.*

Definition 5.22 (Right Homotopy Function Complexes)**.** *If $\mathscr{C}$ is a model category and $X, Y \in \mathscr{C}$ then a* ***right homotopy function complex*** *from X to Y is a triple*

$$(QX, \widehat{\mathbf{Y}}, \mathscr{C}(QX, \widehat{\mathbf{Y}}))$$

where

1. *QX is a cofibrant approximation to X*
2. *$\mathbf{RY}$ is a simplicial resolution of Y*
3. *$\mathscr{C}(QX, \mathbf{RY})$ is the simplicial set in 5.20 2.*

Definition 5.23 (Two-sided Homotopy Function Complex)**.** *If $\mathscr{C}$ is a model category $X, Y \in \mathscr{C}$ then a* ***two-sided homotopy function complex*** *from X to Y is a triple*

$$(\widetilde{\mathbf{X}}, \widehat{\mathbf{Y}}, diag\mathscr{C}(\widetilde{\mathbf{X}}, \widehat{\mathbf{Y}}))$$

where

1. *$\widetilde{\mathbf{X}}$ is the cosimplicial resolution of X*
2. *$\widehat{\mathbf{Y}}$ is the simplicial resolution of Y*
3. *$diag\mathscr{C}(\widetilde{\mathbf{X}}, \widehat{\mathbf{Y}})$ is the simplicial set in 5.20 4.*

Definition 5.24 (Homotopy Function Complex)**.** *Let $\mathscr{C}$ be a model category. A* ***homotopy function complex*** *from X to Y is either*

1. *a left homotopy function complex from X to X*
2. *a right homotopy function complex from X to Y*
3. *a two-sided homotopy function complex from X to Y.*

The respective homotopy function complex will be denoted by $\operatorname{map}(X, Y)$.

Remark 5.25. *Homotopy function complexes are independent of the choice of simplicial and cosimplicial resolutions respectively. For a proof of this fact see for instance Theorem 17.4.8 in [PHir03].* ♦

The following theorem is one of the main results of P. Hirschhorn in his book [PHir03]. As one may imagine the proof is rather involved and will therefore not be given here. The result can be found as Theorem 17.7.7 in [PHir03]. In the theorem $\operatorname{map}(-,-)$ is a fixed homotopy function complex from Definition 5.24.

Theorem 5.26 ([PHir03])**.** *Let $\mathscr{C}$ be a model category and $g : X \to Y$ a morphism in $\mathscr{C}$. The following are equivalent.*

1. *g is a weak equivalence.*
2. *For any object W in $\mathscr{C}$, the map g induces a weak equivalence of homotopy function complexes*
$$g_* : \mathrm{map}(W, X) \xrightarrow{\sim} \mathrm{map}(W, Y).$$
3. *For any cofibrant object in $\mathscr{C}$, the map g induces a weak equivalence of homotopy function complexes*
$$g_* : \mathrm{map}(W, X) \xrightarrow{\sim} \mathrm{map}(W, Y).$$
4. *For any object Z in $\mathscr{C}$, the map g induces a weak equivalence of homotopy function complexes*
$$g^* : \mathrm{map}(Y, Z) \xrightarrow{\sim} \mathrm{map}(X, Z).$$
5. *For any fibrant object Z in $\mathscr{C}$, the map g induces a weak equivalence of homotopy function complexes*
$$g^* : \mathrm{map}(Y, Z) \xrightarrow{\sim} \mathrm{map}(X, Z).$$

This theorem should remind us of some sort of weaker version of Yoneda's lemma.

Remark 5.27. *Homotopy function complexes actually have an interesting connection to the homotopy category of a given model category in the following sense.*

Let $\mathscr{C}$ be a model category, $X, Y \in \mathscr{C}$ and $\mathrm{map}(X, Y)$ a homotopy function complex, then $\pi_0(\mathrm{map}(X, Y))$ is naturally isomorphic to the set of maps from X to Y in $\mathrm{Ho}(\mathscr{C})$. For a proof of this statement, see for instance Theorem 17.7.2 in [PHir03]. ♦

5.2.3 Left Localisation of Model Categories

We will just very briefly give the definition of a left localisation of a model category, as stated in [PHir03].

Definition 5.28 (Left Localisation)**.** *Let $\mathscr{C}$ be a model category and let $\mathcal{S}$ be a class of morphisms in $\mathscr{C}$. A* ***left localisation of $\mathscr{C}$ with respect to $\mathcal{S}$*** *is a model category $L_{\mathcal{S}}(\mathscr{C})$ together with a left Quillen functor $j : \mathscr{C} \to L_{\mathcal{S}}(\mathscr{C})$ such that*

1. *The total left derived functor $\mathbb{L}j : \mathrm{Ho}(\mathscr{C}) \to \mathrm{Ho}(L_{\mathcal{S}}(\mathscr{C}))$ of j takes the images in $\mathrm{Ho}(\mathscr{C})$ of the elements of $\mathcal{S}$ into isomorphisms in $\mathrm{Ho}(L_{\mathcal{S}}(\mathscr{C}))$.*
2. *If $\mathscr{D}$ is a model category and $\varphi : \mathscr{C} \to \mathscr{D}$ is a left Quillen functor such that $\mathbb{L}\varphi : \mathrm{Ho}(\mathscr{C}) \to \mathrm{Ho}(\mathscr{D})$ takes the images in $\mathrm{Ho}(\mathscr{C})$ of the elements of $\mathcal{S}$ into isomorphisms in $\mathrm{Ho}(\mathscr{D})$, then there is a unique left Quillen functor $\delta : L_{\mathcal{S}}(\mathscr{C}) \to \mathscr{D}$ such that $\delta \circ j = \varphi$.*

Proposition 5.29. *Let $\mathscr{C}$ be a model category and let $\mathcal{S}$ be a class of maps in $\mathscr{C}$. If $L_{\mathcal{S}}(\mathscr{C})$ exists it is unique up to unique isomorphism.*

Proof. This follows directly from the universal property provided in Definition 5.28. ■

More details and properties about this kind of localisation may be found in Chapter 3 of [PHir03]. The only important thing for us is to know that the left Bousfield localisation, which we define in the next section, is also a left localisation of model categories.

This will be rather important in Part III, when we have to deal with a specific left Bousfield localisation, where it is very helpful to know the localisation functor.

5.2.4 Left Bousfield Localisations

We will restrict ourselves to the case of left Bousfield localisations, there are similar results for the case of right Bousfield localisations. In practice the left localisations are used more often than the right ones. All these results may be found in [PHir03].

Notation 5.30. *Let $\mathscr{C}$ be a model category, $X,\ Y \in \mathrm{ob}(\mathscr{C})$ objects. We use the notation $\mathrm{map}(X,Y)$ to denote a simplicial set that is some unspecified homotopy function complex from X to Y. Therefore $\mathrm{map}(X,Y)$ will denote either*

1. *$\mathbb{L}\mathscr{C}(X,Y) := \mathscr{C}(\widetilde{\mathbf{X}}, RY)$ that is part of a left homotopy function complex $(\widetilde{\mathbf{X}}, RY, \mathscr{C}(\widetilde{\mathbf{X}}, RY))$, which we often call **left derived hom-space***
2. *$\mathbb{R}\mathscr{C}(X,Y) := \mathscr{C}(QX, \widehat{\mathbf{Y}})$ which is part of a right homotopy function complex $\mathscr{C}(QX, \widehat{\mathbf{Y}}, \mathscr{C}(QX, \widehat{\mathbf{Y}}))$, which we often call **right derived hom-space***
3. *$\mathbb{D}\mathscr{C}(X,Y) := diag\mathscr{C}(\widetilde{\mathbf{X}}, \widehat{\mathbf{Y}})$ which is part of a two-sided homotopy function complex $(\widetilde{\mathbf{X}}, \widehat{\mathbf{Y}}, diag\mathscr{C}(\widetilde{\mathbf{X}}, \widehat{\mathbf{Y}}))$, which we often call **two-sided derived hom-space**.*

Definition 5.31 (Local)**.** *Let $\mathscr{C}$ be a model category and $\mathcal{S}$ a class of morphisms in $\mathscr{C}$.*

1. (a) *An object $W \in \mathscr{C}$ is **$\mathcal{S}$-local** if W is fibrant and for all $f : A \to B$ in $\mathcal{S}$ the induced map of homotopy function complexes*

   $$f^* : \mathrm{map}(B, W) \to \mathrm{map}(A, W)$$

   is a weak equivalence.

   (b) *If $\mathcal{S}$ consists of the single map $f : A \to B$ then an $\mathcal{S}$-local object will also be called **f-local**.*

   (c) *If $\mathcal{S}$ consists of the single map $\emptyset \to A$ then an $\mathcal{S}$-local object is also called **A-local**.*
2. (a) *A map $g : X \to Y$ in $\mathscr{C}$ is a **$\mathcal{S}$-local equivalence** if for all $\mathcal{S}$-local objects W the induced map of homotopy function complexes*

   $$g^* : \mathrm{map}(Y, W) \to \mathrm{map}(X, W)$$

   is a weak equivalence.

   (b) *If $\mathcal{S}$ consists of the single map $f : A \to B$ then an $\mathcal{S}$-local equivalence is called an **f-local equivalence**.*

   (c) *If $\mathcal{S}$ consists of the single map $\emptyset \to A$ then an $\mathcal{S}$-local equivalence will also be called an **A-local equivalence**.*

Definition 5.32 (Left Bousfield Localisation)**.** *Let $\mathscr{C}$ be a model category and let $\mathcal{S}$ be a class of maps in $\mathscr{C}$. The **left Bousfield localisation** of $\mathscr{C}$ with respect to $\mathcal{S}$ (if it exists) is a model category structure $L^B_{\mathcal{S}}(\mathscr{C})$ on the underlying category $\mathscr{C}$ such that*

1. *The class of weak equivalences of $L^B_{\mathcal{S}}(\mathscr{C})$ equals the class of $\mathcal{S}$-local equivalences of $\mathscr{C}$.*
2. *The class of cofibrations of $L^B_{\mathcal{S}}(\mathscr{C})$ equals the class of cofibrations of $\mathscr{C}$.*
3. *The class of fibrations of $L^B_{\mathcal{S}}(\mathscr{C})$ equals the class of maps with the RLP with respect to those maps that are both cofibrations and $\mathcal{S}$-local equivalences.*

As stated in the above definition, we do not yet know under which conditions such Bousfield localisations exist. In the next section we will address this problem. For now, assuming that they exist in the particular cases, we state some nice properties about Bosufield localisations, The results can be found in [PHir03] as Proposition 3.4.1, Proposition 3.3.3 and Proposition 3.3.18.

Proposition 5.33 ([PHir03])**.** *Let $\mathscr{C}$ be a model category and let $\mathcal{S}$ be a class of maps in $\mathscr{C}$.*

If $\mathscr{C}$ is a left proper model category and $L^B_{\mathcal{S}}\mathscr{C}$ is the left Bousfield localisation of $\mathscr{C}$ with respect to $\mathcal{S}$, then an object of $\mathscr{C}$ is $\mathcal{S}$-local iff it is fibrant in $\mathscr{C}$.

Proposition 5.34 ([PHir03])**.** *Let $\mathscr{C}$ be a model category and let $\mathcal{S}$ be a class of maps in $\mathscr{C}$. If $L^B_{\mathcal{S}}\mathscr{C}$ is the left Bousfield localisation of $\mathscr{C}$ with respect to $\mathcal{S}$, then*

1. *every weak equivalence of $\mathscr{C}$ is a weak equivalence of $L^B_{\mathcal{S}}\mathscr{C}$*
2. *the class of acyclic fibrations of $L^B_{\mathcal{S}}\mathscr{C}$ equals the class of acyclic fibrations in $\mathscr{C}$*
3. *every fibration of $L^B_{\mathcal{S}}\mathscr{C}$ is a fibration of $\mathscr{C}$*
4. *every acyclic cofibration of $\mathscr{C}$ is an acyclic cofibration of $L^B_{\mathcal{S}}\mathscr{C}$.*

Proposition 5.35 ([PHir03])**.** *Let $\mathscr{C}$ be a model category and let $\mathcal{S}$ be a class of morphisms in $\mathscr{C}$. If $L^B_{\mathcal{S}}(\mathscr{C})$ is the left Bousfield localisation of $\mathscr{C}$ with respect to $\mathcal{S}$, $\mathscr{D}$ is a model category, and $F : \mathscr{C} \to \mathscr{D}$ is a left Quillen functor that takes every cofibrant approximation to an element of $\mathcal{S}$ into a weak equivalence in $\mathscr{D}$, then F is a left Quillen functor when considered as a functor $L^B_{\mathcal{S}}(\mathscr{C}) \to \mathscr{D}$.*

Theorem 5.36 (Bousfield Localisation is a Localisation, [[PHir03])**.** *] Let $\mathscr{C}$ be a model category and let $\mathcal{S}$ be a class of maps in $\mathscr{C}$. If $L^B_{\mathcal{S}}(\mathscr{C})$ is the left Bousfield localisation of $\mathscr{C}$ with respect to $\mathcal{S}$, then the identity functor $\mathscr{C} \to L^B_{\mathcal{S}}(\mathscr{C})$ is a left localisation of $\mathscr{C}$ with respect to $\mathcal{S}$.*

Proof. Let $L_{\mathcal{S}}(\mathscr{C})$ be the left Bousfield localisation of $\mathscr{C}$ and $j : \mathscr{C} \to L_{\mathcal{S}}(\mathscr{C})$ be the identity functor. Let $F : \mathscr{C} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathscr{D} : U$ be a Quillen pair such that the total left derived functor $\mathbb{L}F : \mathrm{Ho}(\mathscr{C}) \to \mathrm{Ho}(\mathscr{D})$ takes the images in $\mathrm{Ho}(\mathscr{D})$ of the elements of $\mathscr{C}$ into isomorphisms in $\mathrm{Ho}(\mathscr{D})$. Since j is the identity functor, the functor $F : L^B_{\mathcal{S}}(\mathscr{C}) \to \mathscr{D}$ is the unique functor such that $F \circ j = F$, and Proposition 5.35 shows that $F : L^B_{\mathcal{S}}(\mathscr{C}) \to \mathscr{D}$ is a left Quillen functor. ■

As the title of the above statement already suggests, we have showed that every left Bousfield localisation is indeed a left localisation of model categories.

The next theorem will turn out to be useful later on in Part III, it can be found in this form as Theorem 3.3.20 in [PHir03].

Theorem 5.37 ([PHir03])**.** *Let $\mathscr{C}$ and $\mathscr{D}$ be model categories and let $F : \mathscr{C} \underset{\perp}{\rightleftarrows} \mathscr{D} : U$ be a Quillen pair.*

If $\mathcal{S}$ is a class of maps in $\mathscr{C}$, $L^B_{\mathcal{S}}(\mathscr{C})$ is the left Bousfield localisation of $\mathscr{C}$ with respect to $\mathcal{S}$ and $L^B_{F(Q\mathcal{S})}\mathscr{D}$ is the left Bousfield localisation of $\mathscr{D}$ with respect to $F(Q\mathcal{S})$ then

1. *(F, U) is also a Quillen pair when considered as functors*
$$F : L^B_{\mathcal{S}}(\mathscr{C}) \underset{\perp}{\rightleftarrows} L^B_{F(Q\mathcal{S})}(\mathscr{D}) : U.$$
2. *If (F, U) is a Quillen equivalence when considered as functors $F : \mathscr{C} \underset{\perp}{\rightleftarrows} \mathscr{D} : U$ then (F, U) is also a pair of Quillen equivalences when considered as functors*
$$F : L^B_{\mathcal{S}}(\mathscr{C}) \underset{\perp}{\rightleftarrows} L^B_{F(Q\mathcal{S})}(\mathscr{D}) : U.$$

5.3 Existence of Left Bousfield Localisations

We state the two most important theorems for the existence of left Bousfield localisations. There are two large known classes of model categories for which these localisations exist for every set of morphisms of the given category. As we will see left properness is important in both cases and in addition we want our category to either be cellular or simplicial and combinatorial.

Before we can state the main theorems we will need to introduce some further theory. We will only be interested in the case, where we Bousfield localise with respect to a single map. This is the case which we will consider throughout Part III. We will introduce the more general theory here for completeness.

Something important to know, are the classes of generating cofibrations and generating acyclic cofibrations for the model structure created by the left Bousfield localisation. If we consider a general set of maps and localise with respect to it, it will not be so easy to determine the set of generating acyclic cofibrations. The set of generating cofibrations will - to no surprise - be the same. This simply follows, since we have the same cofibrations, but since we add more weak equivalences one has to expect, that the set of generating acyclic cofibrations will change.

5.3.1 Horns

Definition 5.38. *Let $\mathscr{C}$ be a model category and $\mathcal{S}$ a class of morphisms in $\mathscr{C}$, then a **full class of horns on $\mathcal{S}$** is a class $\Lambda(\mathcal{S})$ of maps obtained by choosing, for every element $f : A \to B \in \mathcal{S}$, a cosimplicial resolution $\tilde{f} : \widetilde{\mathbf{A}} \to \widetilde{\mathbf{B}}$ of f such that $\tilde{f}$ is a Reedy cofibration and letting $\Lambda(\mathcal{S})$ be the class of maps*

$$\Lambda(\mathcal{S}) = \left\{ \widetilde{\mathbf{A}} \otimes \Delta^n \coprod_{\widetilde{\mathbf{A}} \otimes \partial\Delta^n} \widetilde{\mathbf{B}} \otimes \Delta^n \;\middle|\; f : A \to B \in \mathcal{S}, n \geq 0 \right\}.$$

Definition 5.39. *Let $(\mathscr{C}, I_{\mathscr{C}}, J_{\mathscr{C}})$ be a cofibrantly generated model category. If $\mathcal{S}$ is a set of maps of $\mathscr{C}$, then an **augmented set of $\mathcal{S}$-horns** is a set $\overline{\Lambda(\mathcal{S})}$ of maps*

$$\overline{\Lambda(\mathcal{S})} = \Lambda(\mathcal{S}) \cup J_{\mathscr{C}}.$$

These definitions depend on the choice of a cosimplicial resolution. We will always assume that such a resolution was chosen. The next result can be found as Proposition 4.2.5 in [PHir03].

Proposition 5.40 ([PHir03]). *Let $(\mathscr{C}, I_{\mathscr{C}}, J_{\mathscr{C}})$ be a left proper cellular model category and let $\mathcal{S}$ be a set of maps in $\mathscr{C}$, then there is a set $\widetilde{\Lambda(\mathcal{S})}$ of relative $I_{\mathscr{C}}$-cell complexes with cofibrant domains such that every element of $\widetilde{\Lambda(\mathcal{S})}$ is an $\mathcal{S}$-local equivalence and an object X of $\mathscr{C}$ is $\mathcal{S}$-local iff the map $X \to *$ is in $\widetilde{\Lambda(\mathcal{S})}$-inj.*

The next theorem looks very promising as a candidate for the generating acyclic cofibrations, it is Theorem 4.2.9 in [PHir03].

Theorem 5.41 ([PHir03]). *Let $(\mathscr{C}, I_{\mathscr{C}}, J_{\mathscr{C}})$ be a left proper cellular model category and $\mathcal{S}$ a set of morphisms in $\mathscr{C}$. Then every relative $\widetilde{\Lambda(\mathcal{S})}$-cell complex is both a cofibration and an $\mathcal{S}$-local equivalence.*

Proposition 5.42 ([PHir03]). *Let $(\mathscr{C}, I_{\mathscr{C}}, J_{\mathscr{C}})$ be a left proper cellular model category and let $\mathcal{S}$ be a set of morphisms in $\mathscr{C}$. If $j : X \to RX$ is a relative $\widetilde{\Lambda(\mathcal{S})}$-cell complex and RX is a $\widetilde{\Lambda(\mathcal{S})}$-inj, then the pair (RX, j) is a cofibrant $\mathcal{S}$-localisation of X.*

Proof. Follows from Theorem 5.41 and Proposition 5.40. ■

It would be really beautiful if we could now claim that we already found our set of generating acyclic cofibrations with the above definitions and results. Unfortunately, reality is a cruel place and so we provide an example why this is not true in general. The example was due to A. K. Bousfield and can also be found in [PHir03].

Example 5.43. *Consider the category* $\mathbf{Top}_*$ *of pointed topological spaces and let $n > 0$ and choose $f : \partial D^{n+1} \to D^{n+1}$. The path space fibration $p : PK(\mathbb{Z}, n) \to K(\mathbb{Z}, n)$ is a $\overline{\Lambda\{f\}}$-inj and so every $\overline{\Lambda\{f\}}$-cof has the the right lifting property with respect to p.*

The cofibration $* \to S^n$ does not have the lifting property with respect to p and hence is not a $\overline{\Lambda\{f\}}$-cof.

However, since one can show that the composition $* \to S^n \to D^{n+1}$ and f itself are f-local equivalences, the 2-3 property of f-local equivalences implies that the inclusion $* \to S^n$ is an f-local equivalence, but it is not a $\overline{\Lambda\{f\}}$-cof. ◀

5.3.2 The Bousfield-Smith Cardinality Argument

We will need something known as the Bousfield-Smith cardinality argument, stated in the form of the following proposition, which gives us generating acyclic cofibrations, this will be justified in the proof of Theorem 5.45. The proof of the next proposition can be found as section 4.5 in [PHir03].

Proposition 5.44 ([PHir03]). *Let $(\mathscr{C}, I_{\mathscr{C}}, J_{\mathscr{C}})$ be a left proper cellular model category and $\mathcal{S}$ a set of maps in $\mathscr{C}$, then there is a set $J_{\mathcal{S}}$ of inclusions of cell complexes such that the class of $J_{\mathcal{S}}$-cof equals the class of cofibrations that are also $\mathcal{S}$-local equivalences.*

This very brief preliminary work will be used to prove the main theorem of this section. For more details on this whole machinery one is highly encouraged to look it up in [PHir03].

5.3.3 The Main Theorems

Theorem 5.45 (Existence of Left Bousfield Localisations I, [PHir03])**.** *Let $(\mathscr{C}, I_{\mathscr{C}}, J_{\mathscr{C}})$ be a left proper cellular model category and let $\mathcal{S}$ be a set of maps in $\mathscr{C}$.*

1. *The left Bousfield localisation of $\mathscr{C}$ with respect to $\mathcal{S}$ exists. That is, there is a model category structure $L^B_{\mathcal{S}}(\mathscr{C})$ on the underlying category $\mathscr{C}$ in which*
   (a) $\mathcal{W}_{L^B_{\mathcal{S}}(\mathscr{C})} = \{\mathcal{S}$*-local equivalences of* $\mathscr{C}\}$.
   (b) $\mathcal{C}_{L^B_{\mathcal{S}}(\mathscr{C})} = \{$*cofibrations of* $\mathscr{C}\}$
   (c) $\mathcal{F}_{L^B_{\mathcal{S}}(\mathscr{C})} = \{$*maps with RLP with respect to maps which are cofibrations and* $\mathcal{S}$*-local equivalences* $\}$.
2. *The fibrant objects of $L^B_{\mathcal{S}}(\mathscr{C})$ are the $\mathcal{S}$-local objects of $\mathscr{C}$.*
3. *$L^B_{\mathcal{S}}(\mathscr{C})$ is a left proper cellular model category.*
4. *If $\mathscr{C}$ is a simplicial model category then that simplicial structure gives $L^B_{\mathcal{S}}(\mathscr{C})$ the structure of a simplicial model category.*

proof (sketch). This is section 4.6 in [PHir03]. We will give a short discussion about the first part of the theorem, since we only need that part later. We will use the recognition theorem for cofibrantly generated model categories, Theorem 2.31.

The class of $\mathcal{S}$-local equivalences satisfies the 2-3 property and it is also closed under retracts, these facts follow from Proposition 3.2.3 and Proposition 3.2.4 in [PHir03].

Now, let $J_{\mathcal{S}}$ be the set provided by Proposition 5.44, and let $I_{L^B_{\mathcal{S}}(\mathscr{C})} = I_{\mathscr{C}}$. Every element of $J_{\mathcal{S}}$ has a cofibrant domain and every cofibrant object in a cellular model category is small relative to the subcategory of cofibrations (see Theorem 12.4.3 in [PHir03]). Therefore, $I_{\mathscr{C}}$ and $J_{\mathcal{S}}$ both admit the small object argument.

The subcategory of $I_{\mathscr{C}}$-cof is the subcategory of cofibrations in the given model category structure on $\mathscr{C}$ and the $I_{\mathscr{C}}$-inj are the acyclic fibrations in that model category. Thus Proposition 5.44 implies that the first condition of part 3 of Theorem 2.31 is satisfied.

Since the $J_{\mathcal{S}}$-cof are a subcategory of the $I_{\mathscr{C}}$-cof, every $I_{\mathscr{C}}$-inj must be a $J_{\mathcal{S}}$-inj. Proposition 3.1.5 in [PHir03] implies that every $J_{\mathcal{S}}$-inj is an $\mathcal{S}$-local equivalence, this gives the second part of the second condition of part 3 in Theorem 2.31.

We conclude by Proposition 5.44, giving the last needed condition. Therefore the recognition theorem for cofibrantly generated model categories shows that there is a model category structure on $L^B_{\mathcal{S}}(\mathscr{C})$ i.e. $(L^B_{\mathcal{S}}(\mathscr{C}), I_{\mathscr{C}}, J_{\mathcal{S}})$ is a cofibrantly generated model category. ■

For completeness we state the following theroem, characterising the other large class of model categories, where the left Bosufield localisation exists with respect to any set of maps. Since we will not be using it, there will not be a proof. It can be found in this form as Theorem 2.11 in [ClBa].

Theorem 5.46 (Existence of Left Bousfield Localisations II, [ClBa]). *If $\mathscr{C}$ is a left proper, simplicial, combinatorial model category and $\mathcal{S}$ is a set of morphisms in $\mathscr{C}$, then the left Bousfield localisation $L^B_{\mathcal{S}}(\mathscr{C})$ does exist as a combinatorial model category.*

Moreover it satisfies the following.

1. *The fibrant objects of $L^B_{\mathcal{S}}(\mathscr{C})$ are precisely the S-local objects of $\mathscr{C}$ that are fibrant in $\mathscr{C}$.*
2. *$L^B_{\mathcal{S}}(\mathscr{C})$ is a left proper model category.*
3. *$L^B_{\mathcal{S}}(\mathscr{C})$ is a simplicial model category.*

6 The General Nerve and Realisation Construction

We will give a convenient characterisation when two model categories -if they fulfill certain conditions- give an adjunction pair.

Since it is possible to state this construction in a great generality we shall do so. It is a construction which will be used a lot to establish Quillen equivalences later on, since every Quillen equivalence has to be a Quillen pair, which on the other hand has to be an adjunction pair.

We follow [BuFaBl04] and [FoLo17].

For the following, let $\mathscr{D}$ be a locally small cocomplete category (really for later we deal with model categories which are even bicomplete), $\mathscr{C}$ a small category and a category $\mathscr{V}$ such that $\mathscr{D}$ is tensored over $\mathscr{V}$. For a better intuition one may replace $\mathscr{C}$ with $\mathbf{\Delta}$ and $\mathscr{V}$ with **Set**, actually we will mostly use it in this case anyway.

Definition 6.1 (Tensored)**.** *Let $\mathscr{V}$ be a category. A $\mathscr{V}$-enriched category $\mathscr{C}$ will be called **tensored** over $\mathscr{V}$, if there exists a functor*

$$\otimes : \mathscr{V} \times \mathscr{C} \to \mathscr{C}$$

such that there is a natural isomorphism

$$\mathscr{C}(V \otimes C, C') \xrightarrow{\cong} \mathscr{V}(V, \mathscr{C}(C, C'))$$

$\forall\ V \in \mathscr{V}$ and $\forall\ C, C' \in \mathscr{C}$.

6.1 The General Nerve

We now define the nerve according to [BuFaBl04]. Actually it is not really needed in this kind of generality later on, but I got a bit carried away.

Definition 6.2 (Geometric Nerve)**.** *Let $\mathscr{C}$ be a small category and $\mathscr{D}$ a locally small and cocomplete category tensored over $\mathscr{V}$ for some category $\mathscr{V}$. Given a functor $\mathbf{i} : \mathscr{C} \to \mathscr{D}$ and an object $C \in \mathscr{C}$ we define*

$$\mathcal{N}(A)_C = \mathscr{D}(\mathbf{i}(C), A).$$

One obtains a functor

$$\mathcal{N} : \mathscr{D} \to \mathbf{Fun}(\mathscr{C}^{op}, \mathscr{V})$$

defined on objects as $\mathcal{N}(A) = \mathscr{D}(\mathbf{i}(-), A)$ and on arrows $f : A \to B$ in $\mathscr{D}$ via composition $\mathcal{N}(f) = \overline{f}$ is defined as the map $\mathcal{N}(A)_C \xrightarrow{\overline{f}_C} \mathcal{N}(B)_C, \quad \overline{f}_C(\alpha) = f \circ \alpha.$

Remark 6.3. *If we would replace $\mathscr{C}$ with $\mathbf{\Delta}$ and $\mathscr{V}$ with **Set**. The above definition with the functor $\mathbf{i} : \mathbf{\Delta} \to \mathscr{D}$ one can define the n-simplices of an object A of $\mathscr{D}$ as the arrows from $\mathbf{i}([n])$ to A i.e.*

$$\mathcal{N}(A)_n = \mathscr{D}(\mathbf{i}([n]), A).$$

Now the functoriality of $\mathbf{i}$ provides face and degeneracy operators satisfying the simplicial identities so that $\mathcal{N}(A)$ becomes a simplpicial set. Thus one obtains a functor

$$\mathcal{N} : \mathscr{D} \to \mathbf{Fun}(\mathbf{\Delta}^{op}, \mathbf{Set}) = \mathbf{sSet}$$

defined as above. ♦

6.2 The General Realisation

Now we define the geometric realisation functor, which we will define as left Kan extension. Though this is a very abstract construction, it will turn out to be helpful nonetheless for the case of simplicial sets, where it is not too hard to see what this should be.

Definition 6.4 (Geometric Realsation)**.** *Let $\mathscr{C}$ be $\mathscr{V}$-enriched small category and $\mathscr{D}$ a locally small and cocomplete category tensored over $\mathscr{V}$ for some category $\mathscr{V}$. The* ***geometric realisation functor*** Π *is defined as left Kan extension of*

$$\begin{tikzcd} \mathscr{C} \arrow[r, "\mathbf{i}"] \arrow[d, "\mathcal{Y}"'] & \mathscr{D} \\ \mathbf{Fun}(\mathscr{C}^{op}, \mathscr{V}) \arrow[ur, dotted, "L_{\mathcal{Y}}\mathbf{i}"'] & \end{tikzcd}$$

i.e. $\Pi := L_{\mathcal{Y}}\mathbf{i}$, *where* $\mathcal{Y} : \mathscr{C} \to \mathbf{Fun}(\mathscr{C}^{op}, \mathscr{V}),\quad C \mapsto \mathscr{C}(-, C)$ *is the enriched Yoneda embedding.*

Remark 6.5. *The left Kan extension $L_{\mathcal{Y}\mathbf{i}}$ above may be represented as a coend in the following way*

$$\Pi(K) \cong \int^{C \in \mathscr{C}} \mathbf{i}(C) \otimes K(C)$$

for some $K \in \mathbf{Fun}(\mathscr{C}^{op}, \mathscr{V})$ (this is usually referred to as the coend formula). ♦

Cocompleteness in the above definition is a crucial condition since it is needed for the existence of the left Kan extension.

6.3 The Main Theorem

The following result will state that the above two functors are indeed an equivalence pair. Again this statement is stated in a great generality, but I could not resist to show it in all its glory as presented here. Either way, the result which we will need is the corollary which we conclude as a special case from the next Theorem.

Theorem 6.6. *Let $\mathscr{C}$ be a $\mathscr{V}$-enriched small category and $\mathscr{D}$ a $\mathscr{V}$-enriched cocomplete category tensored over $\mathscr{V}$ for some category $\mathscr{V}$. There is an adjunction*

$$\Pi : \mathbf{Fun}(\mathscr{C}^{op}, \mathscr{V}) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathscr{D} : \mathcal{N}\,.$$

Proof. Let K be some object in $\mathbf{Fun}(\mathscr{C}^{op}, \mathscr{V})$ and D some object in $\mathscr{D}$, then we have

$$\begin{aligned} \mathscr{D}(\Pi(K), D) &\cong \mathscr{D}\left(\int^{C \in \mathscr{C}} \mathbf{i}(C) \otimes K(C), D\right) \\ &\cong \int_{C \in \mathscr{C}} \mathscr{D}(\mathbf{i}(C) \otimes K(C), D) \\ &\cong \int_{C \in \mathscr{C}} \mathscr{V}(K(C), \mathscr{D}(\mathbf{i}(C), D)) \\ &= \int_{C \in \mathscr{C}} \mathscr{V}(K(C), \mathcal{N}(D)(C)) \\ &\cong \mathbf{Fun}(\mathscr{C}^{op}, \mathscr{V})(K, \mathcal{N}(D)). \end{aligned}$$

The first step uses the above remark, the second one uses properties of ends and coends, the next step uses the property of the tensoring. The equality is just by definition and the last step uses the co-Yoneda lemma. Also notice, that all isomorphisms are natural. ■

Here comes an interesting example of the above construction. I found it in the n-category café and it was provided by Tom Leinster, the original example was provided in [SM94].

Example 6.7. *Let $S \in \mathbf{Top}$. Denote with $\mathcal{O}(S)$ the poset of open subsets of S regarded as a category, where $\mathcal{O}(S)(X,Y)$ is either empty or contains exactly one element. Consider the category $(\mathbf{Top} \downarrow S)$. We have a canonical functor*

$$\begin{aligned} J : \mathcal{O}(S) &\to (\mathbf{Top} \downarrow S) \\ U &\mapsto (U \hookrightarrow S). \end{aligned}$$

Fix some $S \in \mathbf{Top}$. Since $\mathbf{Top}$ has all small comlimits (which we will see later), $(\mathbf{Top} \downarrow S)$ also has all small colimits. Now applying Theorem 6.6 to J, gives an adjunction

$$\begin{tikzcd} \mathbf{Fun}(\mathcal{O}(S)^{op}, \mathbf{Set}) \arrow[r, shift left] \arrow[r, phantom, "\perp"] & (\mathbf{Top} \downarrow S) \arrow[l, shift left] \end{tikzcd}.$$

If we go right and left in the adjunction, this gives the so called sheafification of a presheaf, going left and then right gives the "étalification" of a space over S.

Restricting this adjunction to presheaves over S and étale spaces over S gives an equivalence of categories

$$\begin{tikzcd} \mathbf{Sh}(S) \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathbf{Et}(S) \arrow[l, shift left] \end{tikzcd}.$$

◀

Part II
Applications

This part consists of two sections, discussing the model structures on the categories of topological spaces and simplicial sets respectively.

The first section deals with the case of topological spaces. First we show that **Top** carries a cofibrantly generated model structure. Later on we also deduce that $\mathbf{Top}_Q$ is indeed a cofibrantly generated, cellular, simplicial and proper model category and that in addition every object is fibrant in $\mathbf{Top}_Q$. From the previous part we then know that the left Bousfield localisation exists with respect to any class of maps in $\mathbf{Top}_Q$.

After that we give a short discussion about Kelley spaces, and show that they carry the same model structure as $\mathbf{Top}_Q$. The reason for introducing those spaces is simply, that they are the very nice and well behaved topological spaces. This will ease the proof of the classical homotopy hypothesis later on. We indeed show that the categories $\mathbf{Top}_Q$ and $\mathbf{K}_Q$ are Quillen equivalent.

The second section deals with the case of simplicial sets. First we will discuss some basic properties about simplicial sets. Next we define the singular functor and the geometric realisation and show that they are an adjunction with the help of the previous part. This adjunction is important to define the Quillen model structure $\mathbf{sSet}_Q$ on **sSet**, as the weak equivalences are defined using the geometric realisation.

After that we will begin the discussion, that $\mathbf{sSet}_Q$ is a cofibrantly generated model category. We will not prove every result needed as it is already very involved to verify. Especially since we will also define homotpy groups in $\mathbf{sSet}_Q$, other key concepts will include anodyne extensions and the machinery of minimal fibrations. Furthermore, we will also argue that $\mathbf{sSet}_Q$ is indeed a cofibrantly generated, cellular, simplicial, combinatorial and proper model category. Similarly as for $\mathbf{Top}_Q$ we can then conclude, that the left Bousfield localisation exists for any class of maps in $\mathbf{sSet}_Q$.

Throughout the discussion we also give some nice properties of the geometric realisation functor, which will be helpful for the proof of the classical homotopy hypothesis, stating that $\mathbf{sSet}_Q$ and $\mathbf{Top}_Q$ are Quillen equivalent.

7 The Quillen Model Structure on Topological Spaces

In this section we will define the model structure $\mathbf{Top}_Q$ on the category $\mathbf{Top}$ of topological spaces and show that it is a model category in the sense of Definition 2.16. Later on we also show, that $\mathbf{Top}$ carries in addition the structure of a cellular, proper and simplicial model category. The last part of the chapter will be dealing with the category of Kelley spaces. Those spaces are particularly nice topological spaces and will make it easier to show the classical version of the homotopy hypothesis, stating that there is a Quillen equivalence between simplicial sets and topological spaces. Furthermore this category will also be helpful in the next section where we discuss the Quillen model structure on the category of simplicial sets.

We will follow [nLab], [PHir03] and [MHov91].

First we may argue that the category of topological spaces $\mathbf{Top}$ is bicomplete. Remember, that the category $\mathbf{Top}$ is the category, where objects are topological spaces (here topological spaces should be chosen nice enough) and the morphisms are continuous maps between topological spaces.

In this section $\mathbb{R}$ will denote the real numbers.

Proposition 7.1. $\mathbf{Top}$ *is bicompete.*

proof (sketch). Let $\mathscr{I}$ be a small category and consider the functor $F : \mathscr{I} \to \mathbf{Top}$. Now a limit of F is obtained in the following way. We take the limit in the category $\mathbf{Set}$ and then topologise it as a subspace of the product $\prod F(i)$ for $i \in \mathscr{I}$. The product is given by the product topology.

A colimit of F is obtained in the following way. We take the colimit $\operatorname{colim} F$ in the category $\mathbf{Set}$ and we say that a set U in $\operatorname{colim} F$ is open iff $j_i^{-1}(U)$ is open in $F(i)$ for any $i \in \mathscr{I}$, where $j_i : F(i) \to \operatorname{colim} F$ is the structure map of the colimit.

The result now follows from the fact, that $\mathbf{Set}$ is a bicomplete category. ∎

We are ready to state the definitions for the different classes of maps needed for the model structure.

Definition 7.2. *Write*

$$I \coloneqq [0,1] \hookrightarrow \mathbb{R}$$

for the ***standard topological interval****, a compact connected topological subspace of the real line.*

Definition 7.3. *For* $n \in \mathbb{N}$ *write*

$$D^n := \{x \in \mathbb{R}^n \mid |x| \leq 1\} \hookrightarrow \mathbb{R}^n \text{ for the standard } \textbf{topological n-disk}.$$
$$S^{n-1} = \partial D^n := \{x \in \mathbb{R}^n \mid |x| = 1\} \hookrightarrow \mathbb{R}^n \text{ for the standard } \textbf{topological n-sphere}.$$

With the convetion that $D^0 = \{0\}$*,* $S^{-1} = \emptyset$ *and* $S^0 = * \coprod *$*.*

We define the classic model structure on topological spaces and prove that it will indeed be a model category.

In order to do so, we will use Theorem 2.31. First of all we will need to define the set of generating cofibrations and generating acyclic cofibrations.

Definition 7.4 (Generating Cofibrations)**.** *The set*

$$I_{\mathbf{Top}} := \{\partial D^n \to D^n \mid n \geq 0\}$$

will be called the set of ***generating cofibrations for* Top***.*

Definition 7.5 (Generating Acyclic Cofibrations)**.** *The set*

$$J_{\mathbf{Top}} := \{D^n \to \mathrm{Cyl}(D^n),\ (x) \mapsto (x, 0) \mid n \geq 0\}$$

will be called the set of ***generating acyclic cofibrations for* Top***.*

Remark 7.6. *In the setting of topological spaces, the cylinder object* $\mathrm{Cyl}(D^n)$ *of* D^n *is just* $D^n \times I$*.* ♦

We are now able to define the model structure.

Definition 7.7 (Model Category of **Top**)**.** *Let* $f : X \to Y$ *be a morphism in* **Top***. We say that* f *is*

1. *a* ***weak equivalence*** *if* $\pi_n(f, x) : \pi_n(X, x) \to \pi_n(Y, f(x))$ *is an isomorphism for all* $n \geq 0$ *and all* $x \in X$ *(for* $n = 0$ *this is an isomorphism of sets and for* $n \geq 1$ *an isomorphism of groups).*
2. *a* ***fibration*** *if it is in* $J_{\mathbf{Top}}$*-inj.*
3. *a* ***cofibration*** *if it is in* $I_{\mathbf{Top}}$*-cof.*

We denote this model structure by $\mathbf{Top}_Q$*.*

Remark 7.8. *The weak equivalences as defined above are called weak homotopy equivalences. An interesting fact, is that* π_n *is an abelian group for* $n \geq 2$*.* ♦

Definition 7.9 (Relative Cell Complex)**.** *A map in* $I_{\mathbf{Top}}$*-cell is called a* ***relative cell complex****.*

Remark 7.10. *A relative CW-complex is a special case of a relative cell complex, where the cells can be attached in order of their dimensions.*

The maps of $J_{\mathbf{Top}}$ *are relative CW-complexes hence are relative* $I_{\mathbf{Top}}$*-cell complexes. This can be seen if one considers the following pushout diagram*

$$\begin{tikzcd}
\partial D^n \arrow[r] \arrow[d] & D^n \arrow[d] \\
D^n \arrow[r] & \mathrm{Cyl}(D^n).
\end{tikzcd}$$

Thus $J_{\mathbf{Top}}$*-cof* $\subset I_{\mathbf{Top}}$*-cof.* ♦

It is now worth pointing out that the fibrations are Serre fibrations. For comparison, here is the usual definition of a Serre fibration.

Definition 7.11 (Serre Fibration). *Let $f : X \to Y$ be a continuous map between topological spaces. f is a **Serre fibration** if for every commutative square of solid continuous functions*

$$\begin{tikzcd} D^n \times \{0\} \arrow[r] \arrow[d, "{(id,0)}"'] & X \arrow[d, "f"] \\ \mathrm{Cyl}(D^n) \arrow[r] \arrow[ur, dotted, "h"] & Y \end{tikzcd}$$

there exists a continuous (dotted) function $h : D^n \to X$ making the above diagram commute.

One can show, that that if f and g are homotopic then f is a weak equivalence iff g is a weak equivalence. Therefore every homotopy equivalence is a weak equivalence.

Now we are ready to start verifying the axioms of a model category, as stated in section 2, starting with weak equivalences.

Proposition 7.12. *The class $\mathcal{W}_{\mathbf{Top}}$ satisfies the 2-3 property.*

Proof. Let $f : X \to Y \in \mathcal{W}_{\mathbf{Top}}$ but then f induces isomorphisms $f_i^* : \pi_i(X) \xrightarrow{\cong} \pi_i(Y)$ for any $i \geq 0$ by definition. We are now left to show that the f_i^* have the 2-3 property. So it is enough to show this fact for isomorphisms. Since isomorphisms are invertible it is really just necessary to claim that the composition of isomorphisms is an isomorphism. This is clear, since if f and g are isomorphisms then so are f^{-1} and g^{-1}. Hence $(f \circ g)$ (if it exists) has an inverse $(f \circ g)^{-1}$, so it is indeed an isomorphism. ■

The next main step is to show the existence of the two functorial weak factorisation systems. This involves that we first state some helpful and technical results. The next proposition can be found as Proposition 10.7.4 in [PHir03].

Proposition 7.13 ([PHir03]). *If $f : X \to Y$ is a relative $I_{\mathbf{Top}}$-cell complex in* **Top**, *then a compact subset of Y can intersect the interiors of only finitely many cells of $Y \backslash im(f)$.*

Corollary 7.14 ([PHir03]). *A compact subset of a cell complex in* **Top** *can intersect the interiors of only finitely many cells.*

Proof. Follows from Proposition 7.13. ■

Proposition 7.15 ([PHir03]). *Every cell of a cell complex in* **Top** *is contained in a finite subcomplex of the cell complex.*

Proof. Use Corollary 7.14. ■

Corollary 7.16 ([PHir03]). *A compact subset of a cell complex in* **Top** *is contained in a finite subcomplex of the cell complex.*

Proof. This follows from Proposition 7.15 and Corollary 7.14. ■

Lemma 7.17. *The elements of $J_{\mathbf{Top}}$ are finite relative cell complexes.*

Proof. This is clear, since we are dealing with CW-complexes. ∎

Lemma 7.18. *Every relative $J_{\mathbf{Top}}$-cell complex is in $\mathcal{W}_{\mathbf{Top}}$.*

Proof. Let $f : X \to Y$ be a relative $J_{\mathbf{Top}}$-cell complex i.e. Y is attained from X by a possibly infinite "gluing" process. Therefore we have the following diagram.

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X_1 \arrow[r] \arrow[dl] & X_2 \arrow[r] \arrow[dll] & \dots \arrow[r] & Y \arrow[dllll] \\ \lim X_\alpha \end{tikzcd}$$

Notice that, since we are working with $J_{\mathbf{Top}}$, every step $X_\alpha \to X_{\alpha+1}$ in the transfinite composition is a weak equivalence (since it is a retraction).

Then by Corollary 7.14 this process terminates after a finite number of steps i.e. $\lim_\alpha X_\alpha = Y$. But then we get an isomorphism

$$\pi_n(X) \xrightarrow{\cong} \pi_n(\lim_\alpha(X_\alpha)) = \pi_n(Y)$$

i.e. $f \in \mathcal{W}_{\mathbf{Top}_Q}$. ∎

Lemma 7.19. *$I_{\mathbf{Top}}$-inj $\subseteq \mathcal{W}_{\mathbf{Top}}$.*

Proof. Consider a map $p : Y \to x$ that has the RLP with respect to $I_{\mathbf{Top}}$ i.e. there is a diagram with a lift

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & Y \arrow[d, "p"] \\ D^n \arrow[r] \arrow[ur, dotted] & X. \end{tikzcd}$$

We will show that $\pi_k(Y) \to \pi_k(X)$ is an isomorphism $\forall\ k \geq 0$.

For the case $k = 0$, the existence of a lift in the above diagram, says that each point of X is in the image of p and therefore $\pi_0(Y) \to \pi_0(X)$ is surjective (from n=0). Consider $n = 1$ and the map $* \coprod * \to Y$ which chooses two connected components in Y. The existence of the lift above again yields that if the two points have the same image in $\pi_0(X)$, then they were already in the same connected component in Y, and hence $\pi_0(Y) \to \pi_0(X)$ is injective and hence bijective.

Now the case for $k \geq 1$. Let $S^k \to Y$ represent an element of $\pi_k(Y)$ such that it will be trivialised in $\pi_k(X)$. Consider the following diagram

$$\begin{tikzcd} S^{n} \arrow[r] \arrow[d] & Y \arrow[d, "p"] \\ D^{n+1} \arrow[r] \arrow[ur, dotted] & X. \end{tikzcd}$$

which yields that it represents the trivial element (since $S^n \to D^{n+1}$ is a trivial element in $\pi_k(D^{n+1})$ and by the commutativity of the upper triangle). Therefore $\pi_k(Y) \to \pi_k(X)$ has trivial kernel and hence it is injective.

Consider the commutative diagram

$$\begin{tikzcd}
S^{k-1} \arrow[r] \arrow[d] & * \arrow[r] \arrow[d] & Y \arrow[dl] \\
D^k \arrow[r] & X &
\end{tikzcd}$$

which is a representative for any element of $\pi_k(X)$. Indeed, this follows from the following diagram

$$\begin{tikzcd}
S^{k+1} \arrow[rr] \arrow[dd] & & * \arrow[dl] \arrow[dd] \\
& S^k \arrow[dr] & \\
D^k \arrow[ur] \arrow[rr] & & X.
\end{tikzcd}$$

The above diagram together with the lift $D^k \to Y$ yields the following diagram.

$$\begin{tikzcd}
S^{k+1} \arrow[rr] \arrow[dd] & & * \arrow[dl] \arrow[dd] \\
& S^k \arrow[dr] & \\
D^k \arrow[ur] \arrow[rr] & & Y.
\end{tikzcd}$$

This gives the existence of a preimage in $\pi_k(Y)$ hence $\pi_k(Y) \to \pi_k(X)$ is surjective and hence also bijective. ∎

Lemma 7.20. *$I_{\mathbf{Top}}$-inj $\subseteq \mathcal{F}_{\mathbf{Top}}$.*

Proof. By Corollary 2.7 it follows, that an $I_{\mathbf{Top}}$-inj map also has the RLP with respect to all relative cell complexes. Then by Lemma 7.17 it is also $J_{\mathbf{Top}}$-inj and hence in $\mathcal{F}_{\mathbf{Top}}$. ∎

The proof of the next result is inspired from [nLab].

Lemma 7.21. *$\mathcal{W}_{\mathbf{Top}} \cap \mathcal{F}_{\mathbf{Top}} \subseteq I_{\mathbf{Top}}$-inj.*

Proof. Consider a map $f : X \to Y \in \mathcal{W}_{\mathbf{Top}} \cap \mathcal{F}_{\mathbf{Top}}$. This means especially, that we have isomorphisms $\pi_n(X) \xrightarrow{\cong} \pi_n(Y)$ for any $n \geq 0$. For $n = 0$ this menas we have isomorphisms on connected components, which on the other hand yields a lift in any commutative solid arrow square of the form

$$\begin{tikzcd} S^{-1} \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ D^0 \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

(by the surjectivity of f), and another lift in every solid arrow square of the form

$$\begin{tikzcd} S^0 \arrow[r] \arrow[d] & X \arrow[d] \\ D^1 \arrow[r] & Y \end{tikzcd}$$

(by injectivity of f), where by convention S^{-1} is the empty set and D^0 is a point and D^1 is an interval.

Therefore, we are left to show for any $n \geq 2$, that there is a lift in any solid arrow diagram of the form

$$\begin{tikzcd} S^{n-1} \arrow[r, "\alpha"] \arrow[d] & X \arrow[d, "f"] \\ D^n \arrow[r, "\kappa"'] & Y. \end{tikzcd}$$

This will now be a bit more involved to show. We will try to construct this lift. Choose a basepoint on S^{n-1} and write x, y repsectively for the images in X and Y. But then by the above diagram, $f_*[\alpha] = 0 \in \pi_{n-1}(Y, y)$ and hence (since $f \in \mathcal{W}_{\mathbf{Top}} \cap \mathcal{F}_{\mathbf{Top}}$), $[\alpha] = 0$ in $\pi_{n-1}(X, x)$. This yields the existence of some map κ' such that the following diagram commutes

$$\begin{tikzcd} S^{n-1} \arrow[r, "\alpha"] \arrow[d] & X. \\ D^n \arrow[ur, "\kappa'"'] & \end{tikzcd}$$

It is left to show that the lower triangle commutes. We have a commutative square

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & D^n \arrow[d, "f\circ\kappa'"] \\ D^n \arrow[r, "\kappa"'] & Y. \end{tikzcd}$$

Notice further, that there is a pushout

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & D^n \arrow[d] \\ D^n \arrow[r] & S^n. \end{tikzcd}$$

From the two above diagrams we get (by the universal property of pushouts), a unique map $\varphi : S^n \to Y$. This map represents an element of the n-th homotopy group $[\varphi] \in \pi_n(Y, y)$. Since $f \in \mathcal{W}_{\mathbf{Top}}$, there is an element $[\rho] \in \pi_n(X, x)$ such that $f_*[\rho] = [\varphi]$, which yields a lift ρ in the following diagram which commutes only up to homotopy

$$\begin{tikzcd} & X \arrow[d, "f"] \\ S^n \arrow[ur, "\rho"] \arrow[r, "\varphi"'] & Y. \end{tikzcd}$$

We may choose this ρ in a more suitable way as a map $\rho' : D^n \to X$. Consider the unique map from the above pushout $(f \circ \rho', \kappa) : S^n \to Y$, but then we have the following

$$\begin{aligned} [(f \circ \rho', \kappa)] &= [(f \circ \rho', f \circ \kappa')] + [(f \circ \kappa', \kappa)] \\ &= f_*[(\rho', \kappa')] + [(f \circ \kappa', \kappa)] \\ &= [\rho] + [(f \circ \kappa', \kappa)] \\ &= [(\kappa, f \circ \kappa')] + [(f \circ \kappa', \kappa)] \\ &= 0. \end{aligned}$$

Therefore there is a homotopy $\Phi : f \circ \rho' \Rightarrow \kappa$, which fixes the boundary of the n-disk.

We need to show, that Φ may be lifted to a homotopy of just ρ' fixing the boundary, this will yield a homotopy ρ'' which will be the desired lift. For $\Phi : D^n \times I \to Y$ to fix the boundary of the n-disk means that it extends to a morphism

$$S^{n-1} \coprod_{S^{n-1} \times I} D^n \times I \xrightarrow{(f \circ \alpha, \Phi)} Y.$$

Out of the pushout that identifies in the cylinder over D^n all points lying over the boundary. Hence we need a lift in the following diagram

$$\begin{tikzcd} D^n \arrow[r, "\rho'"] \arrow[d] & X \arrow[d, "f"] \\ S^{n-1} \coprod_{S^{n-1} \times I} D^n \times I \arrow[r, "(f \circ \alpha, \Phi)"'] & Y. \end{tikzcd}$$

The left map is homeomorphic to $D^n \to D^n \times I$ and hence we have a lift in the desired diagram. ■

This yields the following theorem. This is really the main result in order to conclude that $\mathbf{Top}_Q$ is a model category.

Theorem 7.22. *Morphisms in* **Top** *which have the RLP with respect to* $I_{\mathbf{Top}}$ *are in* $\mathcal{W}_{\mathbf{Top}} \cap \mathcal{F}_{\mathbf{Top}}$.

Proof. Lemma 7.19, Lemma 7.20 and Lemma 7.21. ■

We state some further helpful results.

Lemma 7.23 ([MHov91]). *Let $p : X \to Y$ be a fibration and $i : \partial D^n \to D^n$ is the boundary inclusion. Then the map $Q(i,p) : X^{D^n} \to X^{D^n} \times_{Y^{D^n}} Y^{D^n}$ is a fibration.*

Proof. By the same adjointness argument as before it suffices to show that the map

$$(D^m \times S^{n-1} \times I) \coprod_{D^m \times S^{n-1} \times \{0\}} (D^m \times D^n \times \{0\}) \to D^m \times D^n \times I$$

is in $J_{\mathbf{Top}}$-cof for all $m, n \geq 0$. The pair (D^n, S^{n-1}) is homeomorphic to the pair $(I^n, \partial I^n)$ where I^n is the n-cube, and the boundary is the collection of points where at least one coordinate is 0 or 1. Therefore the map

$$f : (S^{n-1} \times I) \coprod_{S^{n-1} \times \{0\}} (D^n \times \{0\}) \to D^n \times I$$

is homeomorphic to the map

$$(\partial I^n \times I) \coprod_{\partial I^n \times \{0\}} (I^n \times \{0\}) \to I^n \times I$$

which is in turn homeomorphic to the map $D^n \times \{0\} \to D^n \times I$ by flattening out the sides of the box $(\partial I^n \times I) \cup I^n \times \{0\}$. Thus the map f is in $J_{\mathbf{Top}}$-cof by definition and the map $D^m \times f$ is homeomorphic to $D^{m+n} \times \{0\} \to D^{m+n} \times I$ so is also in $J_{\mathbf{Top}}$-cof. ∎

Corollary 7.24 ([MHov91]). *Every object in* **Top** *is fibrant. Hence the map $Y^{D^n} \to Y^{S^{n-1}}$ is a fibration for all $n \geq 0$.*

Proof. Every map of $J_{\mathbf{Top}}$ is an inclusion of a retract. Hence every map of the form $Y \to *$ has the RLP with respect to $J_{\mathbf{Top}}$ so is a fibration. It follows from Lemma 7.23 applied to the fibration $Y \to *$ that the map $Y^{D^n} \to Y^{S^{n-1}}$ is a fibration. ∎

7.1 The Factorisations

These results are inspired from [nLab].

Proposition 7.25 (The First Factorisation). *Every $f : X \to Y \in \mathbf{Top}_Q$ factors in the following sense*

$$f : X \xrightarrow{\in \mathcal{C}_{\mathbf{Top}}} X' \xrightarrow{\in \mathcal{W}_{\mathbf{Top}} \cap \mathcal{F}_{\mathbf{Top}}} Y.$$

Proof. As we have seen in Corollary 7.14 and from the fact, that D^n and ∂D^n are compact (Indeed, S^n is compact, it is clearly bounded. Consider the continuous map $f : \mathbb{R}^{n+1} \to \mathbb{R}, x \mapsto \|x\|^2$, now $S^n = f^{-1}(\{1\})$ which means that S^n is closed and hence compact), $I_{\mathbf{Top}}$ admits the small object argument.

By the small object argument, we have a factorisation

$$f : X \xrightarrow{\in \mathcal{C}_{\mathbf{Top}}} X' \xrightarrow{\in I_{Top}\text{-inj}} Y.$$

Now we just need to apply Theorem 7.22 and we are done. ∎

Proposition 7.26 (The Second Factorisation)**.** *Every $f : X \to Y \in \mathbf{Top}_Q$ factors in the following sense*

$$f : X \xrightarrow{\in \mathcal{W}_{\mathbf{Top}} \cap \mathcal{C}_{\mathbf{Top}}} X' \xrightarrow{\in \mathcal{F}_{\mathbf{Top}}} Y.$$

Proof. As we have seen in Corollary 7.14 and from the fact, that D^n is compact, we get that $J_{\mathbf{Top}}$ admits the small object argument. This means there is a factorisation

$$f : X \xrightarrow{\in J_{\mathbf{Top}}\text{-cell}} X' \xrightarrow{J_{\mathbf{Top}}\text{-inj}} Y.$$

But as pointed out earlier, $J_{\mathbf{Top}}$-inj are Serre fibrations, hence $X' \to Y \in \mathcal{F}_{\mathbf{Top}}$. By Lemma 7.17, a relative $J_{\mathbf{Top}}$-cell complex is a relative $I_{\mathbf{Top}}$-cell complex. This implies that $X \to X'$ is a cofibration. Finally, by Lemma 7.18 $X \to X'$ is a weak equivalence. ■

7.2 Liftings

This result is inspired from [nLab].

Proposition 7.27 (Lifting)**.** *Consider a commutative solid arrow diagram*

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "g \in \mathcal{C}_{\mathbf{Top}}"'] & \bullet \arrow[d, "f \in \mathcal{F}_{\mathbf{Top}}"] \\ \bullet \arrow[r] \arrow[ur, dotted, "h"] & \bullet \end{tikzcd}$$

in $\mathbf{Top}_Q$, then the lift h (dotted line) exists, as soon as either g or f are in $\mathcal{W}_{\mathbf{Top}}$.

Proof. For the first case, assume, that $f \in \mathcal{W}_{\mathbf{Top}} \cap \mathcal{F}_{\mathbf{Top}}$. Then we apply Theorem 7.22 and conclude by the closure properties, that g has the LLP against f.

For the second case, assume that, $g \in \mathcal{W}_{\mathbf{Top}} \cap \mathcal{C}_{\mathbf{Top}}$. We may factor g as

$$g : X \xrightarrow{\in \mathcal{W}_{\mathbf{Top}} \cap \mathcal{C}_{\mathbf{Top}}} X' \xrightarrow{\in \mathcal{F}_{\mathbf{Top}}} Y.$$

Since $g \in \mathcal{W}_{\mathbf{Top}}$ we may apply the 2-3 property to conclude that $X' \to Y \in \mathcal{W}_{\mathbf{Top}}$. The first case now yields, that $X' \to Y$ has the RLP against g. Applying the retract argument, yields that g is a relative $J_{\mathbf{Top}}$-cell complex. Again by the closure properties, f has the RLP against g. In both cases we have a lift in the desired diagram. ■

7.3 The FWFS

The next result is inspired from [nLab].

Proposition 7.28. *There are two FWFS $(\mathcal{C}_{\mathbf{Top}}, \mathcal{W}_{\mathbf{Top}} \cap \mathcal{F}_{\mathbf{Top}})$ and $(\mathcal{W}_{\mathbf{Top}} \cap \mathcal{C}_{\mathbf{Top}}, \mathcal{F}_{\mathbf{Top}})$ in $\mathbf{Top}_Q$.*

Proof. The factorisation part follows from Proposition 7.25 and Proposition 7.26. We are left to show, that the respective classes have the desired lifting properties against each other.

$\mathcal{F}_{\mathbf{Top}}$ has by definition the RLP against $\mathcal{W}_{\mathbf{Top}} \cap \mathcal{C}_{\mathbf{Top}}$, and $\mathcal{W}_{\mathbf{Top}} \cap \mathcal{F}_{\mathbf{Top}}$ has the RLP against $\mathcal{C}_{\mathbf{Top}}$ by Theorem 7.22. We show, that $\mathcal{C}_{\mathbf{Top}}$ has the LLP with respect to $\mathcal{W}_{\mathbf{Top}} \cap \mathcal{F}_{\mathbf{Top}}$. Consider a morphism $f : X \to Y$ in $I_{\mathbf{Top}}$-cof, if this turns out to be a relative cell complex, then we are done.

We apply the small object argument to get

$$f : X \to Y' \xrightarrow{\in I_{\mathbf{Top}}\text{-inj}} Y.$$

Now f has the LLP with respect to $Y' \to Y$ i.e. we have the following commuting diagram.

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & Y' \arrow[d] \\
Y \arrow[r, "id"'] \arrow[ur] & Y.
\end{tikzcd}$$

We conclude by the retract argument that f is a retract of $X \to Y'$ by considering the following diagram.

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "f"'] & X \arrow[r, equal] \arrow[d] & X \arrow[d, "f"] \\
Y \arrow[r] & Y' \arrow[r] & Y.
\end{tikzcd}$$

This is a retract by the commutativity of the diagram given by the LLP above.

If we apply the small object argument to $J_{\mathbf{Top}}$ and factor f, we can conclude the same result for $\mathcal{W}_{\mathbf{Top}} \cap \mathcal{C}_{\mathbf{Top}}$, showing that $\mathcal{C}_{\mathbf{Top}}$ has the LLP with respect to $\mathcal{W}_{\mathbf{Top}} \cap \mathcal{F}_{\mathbf{Top}}$ and $\mathcal{W}_{\mathbf{Top}} \cap \mathcal{C}_{\mathbf{Top}}$ has the LLP with respect to $\mathcal{F}_{\mathbf{Top}}$. ■

Finally we have the first mandatory result of this section.

Theorem 7.29. $(\mathbf{Top}_Q, I_{\mathbf{Top}}, J_{\mathbf{Top}})$ *is a cofibrantly generated model category. Every object in* **Top** *is fibrant.*

Proof. One has to show, that $\mathbf{Top}_Q$ is bicomplete and equipped with a model structure. Bicompleteness follows from Proposition 7.1. The model structure follows from Proposition 7.12, and Proposition 7.28. The last statement follows from Corollary 7.24. ■

We will now show, that the category **Top** is in addition a cellular, proper and simplicial model category.

The next proposition is a slight generalisation of a result from [nLab].

Proposition 7.30. *In* **Top** *the effective monomorphisms are the topological embeddings.*

Proof. With Proposition 2.39 we reduce to the case that regular monomorphisms are topological embeddings. The equalisers in **Top** are topological subspace inclusions.

Conversely let $i : X \to Y$ be a topological space embedding, we want to show that it is the equaliser of some pair of parallel morphisms. Consider the cokernel pair (i_0, i_1) by taking the pushout of i against itself. By Proposition 2.39 the equaliser of that pair is the set theoretic equaliser of that pair of functions endowed with the subspace topology. ■

Theorem 7.31. $\mathbf{Top}_Q$ *is a cellular model category.*

Proof. We use the recognition theorem for cellular model categories. Since we already know that $\mathbf{Top}_Q$ is a cofibrantly generated model category, we only need to verify conditions 3. and 5. of the theorem.

Condition 3. holds. For this we consider $I_{\mathbf{Top}}$ and Corollary 7.16, which implies that every finite cell complex is ω-compact relative to $I_{\mathbf{Top}}$, where ω is the countable ordinal. Similarly, if γ is an infinite cardinal and X is a cell complex of size γ, then the corollary implies that X is γ-compact relative to $I_{\mathbf{Top}}$. Which is precisely what we wanted to show.

Condition 5. holds. Indeed now with the help of Proposition 7.30 and Proposition 2.39 we get that if a map is a topological embedding then it is also an effective monomorphism. We conclude by noticing that relative $I_{\mathbf{Top}}$-cell complexes are topological embeddings by construction (transfinite composition) and hence an effective monomorphism. ■

The next result can also be found as Theorem 13.1.11 in [PHir03].

Theorem 7.32. $\mathbf{Top}_Q$ *is a proper model category.*

Proof. Since every object in **Top** is fibrant (Theorem 7.34) we get that $\mathbf{Top}_Q$ is a right proper model category. We are left to show, that $\mathbf{Top}_Q$ is left proper.

We will use the definition of left proper. Let $f : X \to Y$ be a weak equivalenced in $\mathbf{Top}_Q$, $s : X \to W$ cofibrant. Consider a pushout

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "f"'] & W \arrow[d, "g"] \\ Y \arrow[r] & Z \end{tikzcd}$$

we want to show, that g is a weak equivalence. s must be a retract of a relative cell complex $t : X \to U$ is a cofibration. If

$$\begin{tikzcd} X \arrow[r, "t"] \arrow[d, "f"'] & U \arrow[d, "h"] \\ Y \arrow[r] & V \end{tikzcd}$$

is a pushout, then g is a retract of h, so we are left to show that h is a weak equivalence. We write t as a transfinite composition of maps, each of which attaches a single cell. It is worth noticing, that indeed it is also possible to add several cells at a time. Then one can argue by a transfinite induction that h is a weak equivalence and hence g has to be a weak equivalence. ■

Theorem 7.33. $\mathbf{Top}_Q$ *is a simplicial model category.*

proof (sketch). We give $\mathbf{Top}_Q$ the structure of a simplicial category. If X and Y are objects in $\mathbf{Top}$, we let $\mathrm{Map}(X, Y)$ be the simplicial set that in degree n is the set of continuous maps from $X \times |\Delta^n|$ to Y with face and degeneracy maps induced by the standard maps between the Δ^n.

The cartesian product and the internal hom give $\mathbf{Top}_Q$ the structure of a simplicial model category. ■

7.4 The Main Theorem

Theorem 7.34. $(\mathbf{Top}_Q, I_{\mathbf{Top}}, J_{\mathbf{Top}})$ *is a cofibrantly generated, cellular, simplicial, proper model category. Furthermore, every object in* $\mathbf{Top}_Q$ *is fibrant.*

Proof. This follows from Theorem 7.29, Theorem 7.31, Theorem 7.32 and Theorem 7.33. ■

Proposition 7.35. $\mathbf{Top}_Q$ *is not a combinatorial model category.*

Proof. From the definition it is enough to show that $\mathbf{Top}_Q$ is not finitely presentable and we may further reduce to just show that not every object in $\mathbf{Top}_Q$ is small.

Indeed, consider for example the Sierpinski space. Take a limit ordinal λ, we put the order topology on $Y = \lambda \cup \{\lambda\}$. $X_\alpha := Y \times \{0, 1\}/(x, 0) \sim (x, 1)$ if $x < \alpha$. Then X_α is a λ-sequence in $\mathbf{Top}$. Then $X = \operatorname{colim} X_\alpha$ and $X = Y \cup \{(\lambda, 1)\}$ with the same neighborhoods at λ.

These two points define a continuous map from the Sierpinski space into X which does not factor continuously through any X_α. ■

The inspiration for the above result and the idea of the example provided can also be found in [MHov91].

A now immediate consequence is the existence of left Bousfield Localisations for $\mathbf{Top}_Q$.

Theorem 7.36. *The left Bousfield localisation of* $\mathbf{Top}_Q$ *exists, with respect to any class of morphisms in* $\mathbf{Top}$*.*

Proof. This is now an immediate consequence of Theorem 7.34 as it fulfills any needed condition for the existence of Bousfield localisations (Theorem 5.45). ■

7.5 The Model Structure on Kelley Spaces

With the help of Kelley spaces it will be easier to deal with the Quillen equivalence needed for the classical version of the homotopy hypothesis. Kelley spaces are particularly nice topological spaces, and to no surprise they carry the same model structure as topological spaces.

We will follow [MHov91] and [Lew78] for this discussion.

7.5.1 The Category of Kelley Spaces

Definition 7.37. *Let $X \in \mathbf{Top}$.*

1. *X is **weak Hausdorff** if for every continuous map $f : K \to X$, where K is compact Hausdorff the image $f(K)$ is closed in X.*
2. *A subset U of X is **compactly open** if for every continuous map $f : K \to X$, where K is compact Hausdorff, $f^{-1}(U)$ is open in K. Similarly U is **compactly closed** if for every such map f. $f^{-1}(U)$ is closed in K.*
3. *X is a **Kelly space** or a k-space if every compactly open subset is open, or equivalently if every compactly closed subset is closed. We denote the full subcategory of $\mathbf{Top}$ consisting of k-spaces by $\mathbf{K}$.*
4. *The k**-space topology** on X denoted kX is defined by letting U be open in kX iff U is compactly open in X.*

Some basic facts about k-spaces are contained in the following proposition found in the appendix of [Lew78], for example the next result formulated in [MHov91].

Proposition 7.38 ([MHov91])**.** 1. *The inclusion functor $i : \mathbf{K} \to \mathbf{Top}$ has a right adjoint j and left inverse $k : \mathbf{Top} \to \mathbf{K}$ that takes X to X with its k-space topology.*
2. *$\mathbf{K}$ has all small limits and colimits where colimits are taken in $\mathbf{Top}$ and limits are taken by applying k to the limit in $\mathbf{Top}$.*
3. *For $X, Y \in \mathbf{K}$ define $C(X, Y)$ to be the set of continuous maps from X to Y given the topology generated by the subbasis $S(f, U)$. Here U is an open set in Y. $f : K \to X$ is a continuous map from a compact Hausdorff space K into X and $S(f, U)$ is the set of all $g : X \to Y$ such that $(g \circ f)(K) \subset U$. Define $\mathrm{Hom}(X, Y)$ to be $kC(X, Y)$. Then we have a natural isomorphism $\mathbf{K}(k(X \times Y), Z) \to \mathbf{K}(X, \mathrm{Hom}(Y, Z))$ for all $X, Y, Z \in \mathbf{K}$.*

7.5.2 The Model Structure on Kelley Spaces

It might come with no surprise that the model structure on $\mathbf{K}$ will be the same as for $\mathbf{Top}$.

Theorem 7.39. *There is a cofibrantly generated model category structure $\mathbf{K}_Q$ on $\mathbf{K}$, where a map is a weak equivalence, cofibration or fibration iff it is so in $\mathbf{Top}_Q$.*

Proof. This is very similar to the proof for topological spaces i.e. Theorem 7.34. ■

Theorem 7.40. *There is a Quillen equivalence*

$$i : \mathbf{K}_Q \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Top}_Q : j$$

where i is the inclusion.

Proof. First we note that the inclusion functor reflects weak equivalences between cofibrant objects. With Corollary 4.45 we have only to show that the map $kX \to X$ is a weak equivalence. But if A is compact Hausdorff then $\mathbf{Top}(A, X) = \mathbf{Top}(A, kX)$. It follows that $kX \to X$ is a weak equivalence as required. ■

The strategy will be to show, that there is also a Quillen equivalence between simplicial sets and Kelley spaces. This will be done in the next section and in Part III.

8 The Quillen Model Structure on Simplicial Sets

In this section - similar to the spirit of the last one - we will introduce the category of simplicial sets. The first goal is to define the Quillen model structure on this category and show that it is indeed a model category. At the same time we argue that it is indeed a cofibrantly generated model category. Later on we further argue, that it is in addition also a cellular, combinatorial, simplicial and proper model category.

As one will see, it is indeed very hard to show that this particular category is a cofibrantly generated model category. There are a lot of involved results and technical finesse to it. Especially due to the fact, that we also have to introduce a great amount of homotopy theory for simplicial sets. The further we go through the attempt of proving the main result, the more we have to deal with special concepts developed for this specific purpose, some of this machinery involves anodyne extensions and the concept of minimal fibrations.

This section uses material from [MHov91], [PHir03], [GoJa09] and [nLab]. A lot of the material, at least the whole discussion about the cofibrantly generated model structure on **sSet** follows [MHov91] very closely.

8.1 The Category of Simplicial Sets

In a first step we define the category of simplicial sets, afterwards we discuss some of its properties.

Definition 8.1 (Category of Simplicial Sets)**.** *We define the* ***category of simplicial sets*** *to be*

$$\mathbf{sSet} := \mathbf{Fun}(\mathbf{\Delta}^{op}, \mathbf{Set}).$$

Definition 8.2 (Set of n-Simplices)**.** *Let $X_\bullet$ be a simplicial set, we denote $X[n]$ by X_n and refer to X_n as the* ***set of n-simplices*** *of $X_\bullet$. For $X \in X_n$, we say that n is the* ***dimesnion*** *of X.*

Definition 8.3 (Yoneda Embedding)**.** *There is a functor $\mathcal{Y} : \mathbf{\Delta} \to \mathbf{sSet}$, called the* ***Yoneda embedding****, defined by the functor $\mathbf{\Delta}(-,-) : \mathbf{\Delta}^{op} \times \mathbf{\Delta} \to \mathbf{sSet}$ i.e. the functor $\Delta^n : \mathbf{\Delta}^{op} \to \mathbf{sSet}$ which takes $[k]$ to $\mathbf{\Delta}([k],[n])$.*

Therefore, by the Yoneda embedding one may conclude that for $X_\bullet \in \mathbf{sSet}$, $X_n \cong \mathbf{sSet}(\Delta^n, X_\bullet)$.

Dually to Definition 1.48, we may define the following.

Definition 8.4 (Face Maps, Degeneracie Maps and Simplicial Identities)**.** *We have*

1. ***face maps*** *$d_i : X_n \to X_{n-1}$ for $n \geq 1$ and $0 \leq i \leq n$.*
2. ***degeneracy maps*** *$s_i : X_{n-1} \to X_n$ for $n \geq 1$ and $0 \leq i \leq n-1$.*

Subject to the ***simplicial identities****:*

$$\begin{aligned} d_i d_j &= d_{j-1} d_i & i &< j \\ d_i s_j &= s_{j-1} d_i & i &< j \\ &= id & i &= j, j+1 \\ &= s_j d_{i-1} & i &> j+1 \\ s_i s_j &= s_j s_{i-1} & i &> j \end{aligned}$$

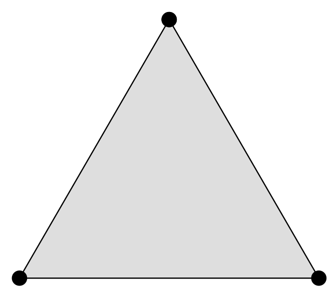
Remark 8.5. *1. A simplicial set $X_\bullet$ is equivalent to a collection of sets X_n and maps d_i and s_i as in Definition 8.4 satisfying the simplicial identities.*

2. A map of simplicial sets $f : X_\bullet \to Y_\bullet$ is equivalent to a collection of maps $f_n : X_n \to Y_n$ commuting with the face and degeneracy maps.

♦

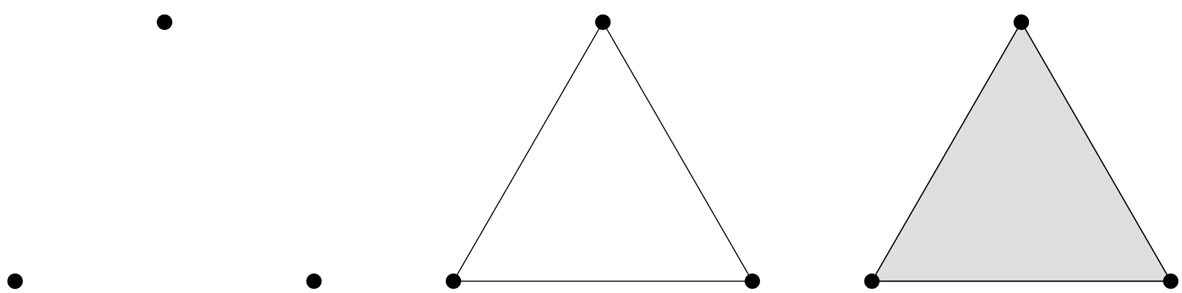
This is a rather nice remark, which we will use in several occasions throughout the present work. It may help to give some thought about a graphical interpretation of simplicial sets, especially in the terms of face and degeneracy maps. This especialy helps to clarify the above remark.

Example 8.6. *Consider $X_\bullet \in$ **sSet**. Assume that $X_\bullet = \Delta^2$ is the following simplicial set*



Δ^2

Then, X_0, X_1 and X_2 have the following graphical interpretation.

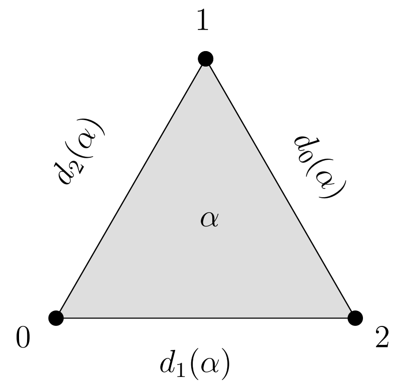


X_0 X_1 X_2

The picture is to be seen in the following way. X_0, X_1 and X_2 are all sets, X_0 consists of the three points depicted above, similarly X_1 is a set consisting of the three lines in the middle triangle. Finally the set X_2 consists of one element which is the filling in the last triangle. The picture shows how Δ^2 is build with the following face and degeneracy maps. These maps encapsulate the information how $X_\bullet = \Delta^2$ is build from X_0, X_1 and X_2.

$$\begin{tikzcd}
X_0 \arrow[r, "s_0" description] & X_1 \arrow[l, shift right=4, "d_0"'] \arrow[l, shift left=4, "d_1"] \arrow[r, shift left=2, "s_0" description] \arrow[r, shift right=6, "s_1" description] & X_2 \arrow[l, shift right=6, "d_0"'] \arrow[l, shift right=2, "d_1" description] \arrow[l, shift left=6, "d_2"]
\end{tikzcd}$$

that is if we consider d_0, d_1 and d_2 in the case of X_2 we have the following picture



This example should now clarify the above remark quite a bit. Now it should also be clear why morphisms of simplicial sets are determined by the respective sets of n-simplices. ◀

8.1.1 Properties of Simplicial Sets

We give some properties about simplicial sets.

Lemma 8.7 ([MHov91])**.** *Every object in* **sSet** *is small.*

Proof. Let $K_\bullet \in \mathbf{sSet}$ and the cardinality of the set of simplices of $K_\bullet$ is κ. κ is infinite. $K_\bullet$ is κ-small. Indeed, suppose λ is a κ-filtered ordinal and $X : \lambda \to \mathbf{sSet}$ is a λ-sequence. Give a map $f : K_\bullet \to \operatorname{colim} X_\alpha$ in **sSet** there is an $\alpha_n < \lambda$ such that f_n factors through X_{α_n}, hence the set K_n is κ-small. Since κ is infinite, there is an $\alpha < \lambda$ such that f factors through a map of sets $g : K_\bullet \to X_\alpha$. The map g may not be a map of simplicial sets. However for each pair (x, i), where x is a simplex of $K_\bullet$ and d_i is a face map applicable to x, there is a $\beta_{(x,i)}$ such that $g(d_i x)$ becomes equal to $d_i g x$ in $X_{\beta_{(x,i)}}$. There are κ such pairs (x, i), so there is a $\beta < \lambda$ and a factorisation of f through X_β compatible with the face maps. A similar argument shows that we can make the factorisation compatible with the degeneracy maps as well. This shows that the map $\operatorname{colim} \mathbf{sSet}(K_\bullet, X_\alpha) \to \mathbf{sSet}(K_\bullet, \operatorname{colim}_\alpha X_\alpha)$ is surjective. The κ-smallness of each K_n shows this map is injective as well. ■

Remark 8.8. **sSet** *is bicomplete, since limits and colimits can be constructed levelwise. So it is bicomplete, since* **Set** *is bicomplete.* ⧫

Definition 8.9 (Face and Degeneracy). *Let $X_\bullet$ be a simplicial set and X a simplex of $X_\bullet$. Any image of X under arbitrary iterations of face maps is called a* ***face*** *of X. Similarly any image of X under arbitrary iterations of degeneracy maps is called a* ***degeneracy*** *of X.*

Remark 8.10. *A special case is the one of 0 iterations where X is both a face and a degeneracy of itself.* ♦

Definition 8.11 (Non degenerate). *A simplex $X \in X_\bullet$ is called* ***nondegenerate*** *if it is a degeneracy only of itself.*

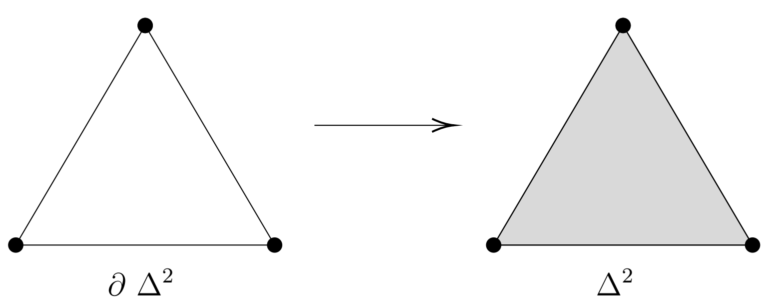
Definition 8.12 (Finite). *A simplicial set is called* ***finite*** *if it has only finitely many non-degenerate simplices.*

Remark 8.13. *Let X be any simplex of a simplicial set $X_\bullet$. Then there is a unique non-degenerate simplex Y of $X_\bullet$ such that X is a degeneracy of Y. Indeed, take Y to be a simplex of smallest dimension such that X is a degeneracy of Y. By the simplicial identities we have the uniqueness of such a simplex, also we have that every simplex Z such that X is a degeneracy of Z is in fact a degeneracy of Y.* ♦

Remark 8.14. *The simplicial set Δ^n has $\binom{n}{k}$ nondegenerate k-simplices corresponding to the injective order-preserving maps $[k] \to [n]$ and in particular one nondegenerate n-simplex i_n. There is a natural isomorphism $\mathbf{sSet}(\Delta^n, K_\bullet) \cong K_n$ (from the Yoneda embedding), which takes f to $f(i_n)$.* ♦

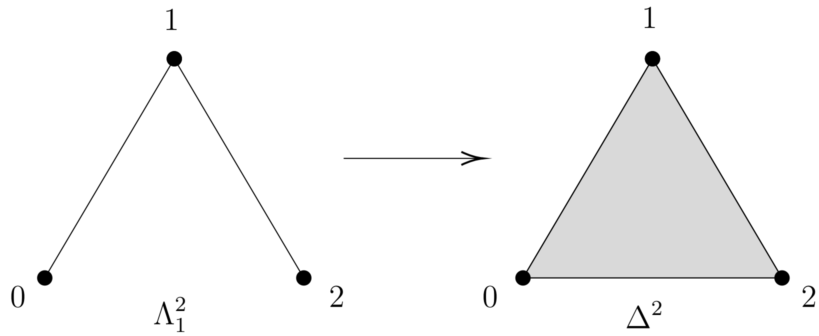
Definition 8.15 (Boundary). *We define $\partial\Delta^n$ to be the* ***boundary*** *of Δ^n whose nondegenerate k-simplices correspond to nonidentity injective order-preserving maps $[k] \to [n]$.*

Graphically, one may think of these in the following way.



Definition 8.16 (Horn). *By $\Lambda^n_k \subset \Delta^n$ we denote the* ***k-th horn****, obtained from the simplex Δ^n by deleting the interior and the face opposite to the k-th vertex.*

The next drawing really shows, why this definition is called a horn



To be honest, the case for Δ^3 would be better to see why we call this a horn, but the case of Δ^2 is easier to draw.

Remark 8.17. *Given some $0 \leq r \leq n$ the simplicial set Λ_r^n has nondegenerate k-simplices all injective order-preserving maps $[k] \to [n]$ except the identity and the injective order-preserving maps $[n-1] \xrightarrow{\mathrm{d}^r} [n]$ whose image does not contain r.*

Consider the category $\mathscr{D}$ whose objects are nonidentity injective order-preserving maps $[k] \to [n]$ whose image contains r and whose morphisms are commutative triangles. Then $\Lambda_r^n = \operatorname{colim}_{\mathscr{D}} \Delta^k$. ♦

Definition 8.18 (Category of Simplices)**.** *Let $X_\bullet \in \mathbf{sSet}$, then the category $(\mathbf{\Delta} \downarrow X_\bullet)$ will be called the* ***category of simplices****.*

Remark 8.19. *A map $X_\bullet \to Y_\bullet$ of simplicial sets induces an obvious functor $(\mathbf{\Delta} \downarrow X_\bullet) \to (\mathbf{\Delta} \downarrow Y_\bullet)$ so that this construction defines a functor from $\mathbf{sSet}$ to the category of small categories $\mathbf{Cat}$.* ♦

We will make use of the next lemma in Part III, since it gives a nice way to deal with simplicial sets in terms of Δ^n.

Lemma 8.20. *Let $X_\bullet \in \mathbf{sSet}$. Consider the functor*

$$\begin{aligned} F : (\mathbf{\Delta} \downarrow X_\bullet) &\to \mathbf{sSet} \\ (\Delta^n \to X_\bullet) &\mapsto \Delta^n. \end{aligned}$$

Then $\operatorname{colim}(F) = X_\bullet$.

Proof. This follows from the natural isomorphism $K_n \cong \mathbf{sSet}(\Delta^n, K_\bullet)$ provided by the Yoneda embedding. ■

What have we gained so far with this definitions. The advantage of this description of the category of simplices is, that it is functorial in the simplicial set $X_\bullet$. Often it is more useful to consider the nondegenerate simplices of a simplicial set. Anyway, this leads to a similar result as above.

8.2 Geometric Realisation and Singular Functor

In this section we will define the geometric realisation and singular functor. Not only are they needed to prove the classical version of the homotopy hypothesis, saying that there is a Quillen equivalence between the model category of simplicial sets and the model category of topological spaces, but it is also heavily used in the next section in order to define the model category structure on simplicial sets.

We follow [MHov91] and [nLab].

Definition 8.21 (Geometric Realisation)**.** *Let $X_\bullet \in$ **sSet**. The **geometric realisation** is defined as the functor*

$$\begin{aligned} |\ | : \mathbf{sSet} &\to \mathbf{Top} \\ X_\bullet \mapsto |X_\bullet| &:= \int^{[n]\in\mathbf{\Delta}} \Delta^n \times X_n. \end{aligned}$$

Definition 8.22 (Singular Functor)**.** *Let $X \in$ **Top**. The **singular functor** is defined as*

$$\begin{aligned} \mathrm{Sing} : \mathbf{Top} &\to \mathbf{sSet} \\ X \mapsto \mathrm{Sing}(X)_n &:= \mathbf{Top}(|\Delta^n|, X). \end{aligned}$$

Now we will apply Theorem 6.6 to show that these functors give an adjunction pair.

Lemma 8.23. *There is an adjunction*

$$|\ | : \mathbf{sSet} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Top} : \mathrm{Sing}\,.$$

Proof. Consequence of Theorem 6.6. ■

Remark 8.24. $|\Delta^n|$ *is a compact Hausdorff space so in particular is in* $\mathbf{K}$ *the category of k-spaces. Since* $\mathbf{K}$ *is closed under colimits in* $\mathbf{Top}$ *it follows that the adjunction* $(|\ |, \mathrm{Sing})$ *can be thought as of an adjunction* $\mathbf{sSet} \to \mathbf{K}$ *as well, indeed with the exact same functors.* ♦

Lemma 8.25. *The natural map*

$$|\Delta^m \times \Delta^n| \to |\Delta^m| \times |\Delta^n|$$

is a homeomorphism.

Proof. This is shown in the proof of Lemma 3.1.8 in [MHov91]. The prove is very combinatorial and technical, therefore there is no reason to replicate it here. ■

Lemma 8.26. $|\ | : \mathbf{sSet} \to \mathbf{K}$ *preserves finite products.*

Proof. Let $X_\bullet, Y_\bullet \in \mathbf{sSet}$, by the co-Yoneda lemma (sometimes also referred to as the Yoda lemma) we have

$$X_\bullet \cong \int^m X_m \times \mathbf{\Delta}(-, m), \qquad Y_\bullet \cong \int^n Y_n \times \mathbf{\Delta}(-, n).$$

Therefore,

$$\begin{aligned}
|X_\bullet \times Y_\bullet| &\cong \left| \left(\left(\int^m X_m \times \mathbf{\Delta}(-,m) \right) \times \left(\int^n Y_n \times \mathbf{\Delta}(-,n) \right) \right) \right| \\
&\cong \left| \left(\int^m \int^n X_m \times Y_n \times (\mathbf{\Delta}(-,m) \times \mathbf{\Delta}(-,n)) \right) \right| \\
&\cong \int^m \int^n X_m \times Y_n \times | \mathbf{\Delta}(-,m) \times \mathbf{\Delta}(-,n)| \\
&\cong \int^m \int^n X_m \times Y_n \times | \mathbf{\Delta}(-,m)| \times | \mathbf{\Delta}(-,n)| \\
&\cong \left(\int^m X_m \times | \mathbf{\Delta}(-,m)| \right) \times \left(\int^n Y_n \times | \mathbf{\Delta}(-,n)| \right) \\
&\cong \left| \int^m X_m \times \mathbf{\Delta}(-,m) \right| \times \left| \int^n Y_n \times \mathbf{\Delta}(-,n) \right| \\
&\cong |X_\bullet| \times |Y_\bullet|.
\end{aligned}$$

where we used, that the product preserves colimits (i.e. **K** is cartesian closed) and that $|\ |$ preserves colimits and Lemma 8.25. ■

8.3 The Quillen Model Structure on Simplicial Sets

In this section we will define the model category structure $\mathbf{sSet}_Q$ on the category **sSet** of simplicial sets and show that it is a model category in the sense of Definition 2.16. We will be following [MHov91] very closely as this discussion is rather involved.

We start with the definition of the classic model structure on simplicial sets, also called the Quillen model structure, and prove that it will indeed be a model category, even more we show that it is indeed a cofibrantly generated model structure.

Later on in the section we argue that **sSet** can be given the model structure of a simplicial, cofibrantly generated, cellular, combinatorial and proper model category. For this we include material from [MHov91] and [PHir03].

To show that the model category structure is cofibrantly generated, we will use Theorem 2.31, hence we will need to define the sets of generating cofibrations and generating acyclic cofibrations.

Definition 8.27 (Generating Cofibrations)**.** *The set*

$$I_{\mathbf{sSet}} := \{\partial\Delta^n \to \Delta^n \mid n \geq 0\}$$

*will be called the set of **generating cofibrations for*** **sSet***.*

Definition 8.28 (Generating Acyclic Cofibrations)**.** *The set*

$$J_{\mathbf{sSet}} := \{\Lambda^n_k \to \Delta^n \mid n \geq 0, n \geq k \geq 0\}$$

*will be called the set of **generating acyclic cofibrations for*** **sSet***.*

We will now give the definition of the model category structure for **sSet**.

Definition 8.29 (Model Category of $\mathbf{sSet}$)**.** *Let f be a morphism in $\mathbf{sSet}$. We say that f is*

1. *a* ***weak equivalence*** *iff $|f|$ is a weak equivalence in $\mathbf{Top}$.*
2. *a* ***fibration*** *iff it is in $J_{\mathbf{sSet}}$-inj.*
3. *a* ***cofibration*** *iff it is in $I_{\mathbf{sSet}}$-cof.*

We denote this model structure by $\mathbf{sSet}_Q$.

Remark 8.30. *Notice that the fibrations in the above definition are Kan-fibrations and the cofibrations are monomorphisms. This is a really nice fact to keep in mind (we will show this in the process).* ♦

We give the definition of Kan fibrations for comparison.

Definition 8.31 (Kan Fibration)**.** *A* ***Kan fibration*** *is a morphism $\pi : Y \to X$ in $\mathbf{sSet}$ with the lifting property for all horn inclusions i.e. for a commutative square of solid arrows*

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & Y \arrow[d, "\pi"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & X \end{tikzcd}$$

with $n \geq 1$ and $0 \leq k \leq n$, there always exists a lift (dotted arrow) making the above diagram commute.

Definition 8.32 (Kan Complex)**.** *Let $X_\bullet \in \mathbf{sSet}$. We say that $X_\bullet$ is a* ***Kan complex****, if the unique map $X_\bullet \to *$ is a Kan fibration, i.e. any solid arrow diagram of the form*

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & X_\bullet \\ \Delta^n \arrow[ur, dotted] & \end{tikzcd}$$

has a lift (dotted line) making the diagram commute.

Definition 8.33 (Anodyne Extension)**.** *The maps in $J_{\mathbf{sSet}}$ are called* ***anodyne extensions****.*

Here comes a little fun fact. I was not sure what the word anodyne means, so according to the Cambridge dictionary it means: "intended to avoid causing offence or disagreement, especially by not expressing strong feelings or opinions". Let us just hope that this holds true in this case as well.

Remark 8.34. *Consider a fibration $p : X_\bullet \to Y_\bullet$ and a vertex $v : \Delta^0 \to Y_\bullet$. We will often refer to the pullback $\Delta^0 \times_{Y_\bullet} X_\bullet$ as the fiber of p over v i.e. as a diagram*

$$\begin{tikzcd}
\Delta^0 \times_{Y_\bullet} X_\bullet \arrow[r] \arrow[d] & X_\bullet \arrow[d, "p"] \\
\Delta^0 \arrow[r, "v"'] & Y_\bullet.
\end{tikzcd}$$

♦

We now classify the cofibrations in **sSet**, they turn out to be the injective maps.

Proposition 8.35 ([MHov91]). *A map $f : X_\bullet \to Y_\bullet$ in* **sSet** *is in $\mathcal{C}_{\mathbf{sSet}}$ iff it is injective. In particular, every simplicial set is cofibrant. Furthermore, every cofibration is a relative $I_{\mathbf{sSet}}$-cell complex.*

Proof. The maps of $I_{\mathbf{sSet}}$ are injective. Injections are closed under pushouts, transfinite composition and retracts therefore every element of $I_{\mathbf{sSet}}$-cof is an injection as well. Conversely, suppose that $f : X_\bullet \to Y_\bullet$ is injective. We write f as a countable composition of pushouts of coproducts of maps of $I_{\mathbf{sSet}}$, thereby showing that $f \in I_{\mathbf{sSet}}$-cell. Define $A_0 = X_\bullet$. Having defined A_n and an inclusion $A_n \to Y_\bullet$ which is an isomorphism on simplices of dimension less than n, let S_n denote the set of n-simplices of $Y_\bullet$ not in the image of A_n. Each such simplex s is necessarily non-degenerate and corresponds to a map $\Delta^n \to Y_\bullet$. The restriction of s to $\partial\Delta^n$ factors uniquely through A_n. Define A_{n+1} as the pushout in the diagram

$$\begin{tikzcd}
\coprod_S \partial\Delta^n \arrow[r] \arrow[d] & A_n \arrow[d] \\
\coprod_S \Delta^n \arrow[r] & A_{n+1}.
\end{tikzcd}$$

Then the inclusion $A_n \to Y_\bullet$ extends to a map $A_{n+1} \to Y_\bullet$. This extension is surjective on simplices of dimension $\leq n$ by construction. It is also injective since we are only adding non-degenerate simplices. The map $f : X_\bullet \to Y_\bullet$ is a composition of the sequence A_n, so f is a relative $I_{\mathbf{Top}}$-cell complex. ■

Remark 8.36. *We have that $J_{\mathbf{sSet}}$-cof $\subseteq I_{\mathbf{sSet}}$-cof. Indeed this follows since the maps in $J_{\mathbf{sSet}}$ are injective and so we follow that $J_{\mathbf{sSet}} \subseteq J_{\mathbf{sSet}}$-cof which then gives the desired claim.* ♦

Remark 8.37. *$|\Delta^n|$ is homeomorphic to D^n and the homeomorphism takes $|\partial\Delta^n|$ to S^{n-1}. Furthermore D^n is homeomorphic to $D^{n-1} \times I$ and this homeomorphism can be chosen to take $|\Lambda^n_r|$ to D^{n-1}.* ♦

Proposition 8.38 ([MHov91]). *Every anodyne extension is an acyclic cofibration of simplicial sets.*

Proof. We use the above remark and Lemma 1.34. The lemma implies that $|I_{\mathbf{sSet}}$-cof$|$ consists of cofibrations of k-spaces. Furthermore, it implies that the singular functor takes fibrations of k-spaces to Kan fibrations and acyclic fibrations of k-spaces to maps of $I_{\mathbf{sSet}}$-inj. ■

Lemma 8.39 ([MHov91]). *The functor $|\ |: \mathbf{sSet} \to \mathbf{K}$ preserves all finite limits and in particular preserves pullbacks.*

Proof. In Lemma 8.26, we showed that $|\ |$ preserves finite products. We are now left to show, that it also preserves equalisers. Consider the equaliser diagram

$$\begin{tikzcd} K_\bullet \arrow[r] & L_\bullet \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & M_\bullet \end{tikzcd}$$

in **sSet**. Let Z be the corresponding equaliser in **Top**. The map $\emptyset \to M_\bullet$ is an injection and hence is in $I_{\mathbf{sSet}}$-cell by Proposition 8.35. Thus $|M_\bullet|$ is a cell complex.

It is in fact true that every cell complex is Hausdorff. This can be shown by transfinite induction using the fact that cells themselves are normal and that the inclusion of the boundary of a cell is a neighborhood deformation retract (for more details see the proof of Lemma 3.2.4 in [MHov91]).

It follows that Z is a closed subspace of $|L_\bullet|$. In particular Z is a k-space so is also the equaliser in $\mathbf{K}$. Now $|K_\bullet|$ is also homeomorphic to a closed subspace of $|L_\bullet|$. Indeed, $K_\bullet \to L_\bullet$ is an injection and so is in $I_{\mathbf{sSet}}$-cell by Proposition 8.35. Thus $|K_\bullet| \to |L_\bullet|$ is a relative cell complex in $\mathbf{K}$ and any such is a closed inclusion by Lemma 2.4.5 in [MHov91].

Since the image of $|K_\bullet|$ in $|L_\bullet|$ is contained in Z it suffices to show that every point of Z is in the image of $|K_\bullet|$. So take a $z \in Z$. The point z must be in the interior of an $|x|$ for a unique nondegenerate simplex x of $L_\bullet$, where we consider x as a simplicial set itself. By definition of the geometric realisation the only way for $|f|(z)$ to equal $|g|(z)$ is if $fx = gx$. Hence x is a necessarily nondegenerate simplex of $K_\bullet$ and so z is in the image of $|K_\bullet|$ as required. ■

The proof of the next two result is inspired by [MHov91].

Lemma 8.40. *Let $f: K_\bullet \to L_\bullet \in I_{\mathbf{sSet}}$-inj then $|f| \in \mathcal{F}_{\mathbf{Top}}$.*

Proof. Since f has the RLP with respect to $I_{\mathbf{sSet}}$, f has the RLP with respect to all inclusions of simplicial sets by Proposition 8.35. In particular we can find a lift (dotted line) in the following solid arrow diagram

$$\begin{tikzcd} K_\bullet \arrow[r, equal] \arrow[d, "(id,f)"'] & K_\bullet \arrow[d, "f"] \\ K_\bullet \times L_\bullet \arrow[r, "p_1"'] \arrow[ur, dotted] & L_\bullet \end{tikzcd}$$

where p_1 is the projection of the second coordinate. The lift (dotted arrow) makes f into a retract of p_1. Hence $|f|$ is a retract of $|p_1|$, which is a fibration since the geometric realisation preserves products by Lemma 8.26. Thus $|f| \in \mathcal{F}_{\mathbf{Top}}$. ■

Proposition 8.41. *Let $f: K_\bullet \to L_\bullet \in I_{\mathbf{sSet}}$-inj. Then $f \in \mathcal{W}_{\mathbf{sSet}} \cap \mathcal{F}_{\mathbf{sSet}}$.*

Proof. Since $J_{\mathbf{sSet}} \subseteq I_{\mathbf{sSet}}$ it follows that $f \in \mathcal{F}_{\mathbf{sSet}}$. We are therefore left to show, that $f \in \mathcal{W}_{\mathbf{sSet}}$.

Let $F_\bullet = f^{-1}(v)$ be the fiber of f over some vertex $v \in L_\bullet$ i.e. there is a pullback diagram

$$\begin{tikzcd} F_\bullet \arrow[r] \arrow[d] & K_\bullet \arrow[d, "f"] \\ \Delta^0 \arrow[r, "v"'] & L_\bullet. \end{tikzcd}$$

Since $|\ |$ preserves pullbacks and since $|f| \in \mathcal{F}_{\mathbf{Top}}$, $|f|$ has fiber $|F_\bullet|$.

Now the map $F_\bullet \to \Delta^0$ has the RLP with respect to $I_{\mathbf{sSet}}$ and hence with respect to all inclusions by Proposition 8.35.

In particular $F_\bullet$ is nonempty so we can find a 0-simplex w in $F_\bullet$. We denote the resulting map $F_\bullet \to \Delta^0 \xrightarrow{w} F_\bullet$ by w as well. We can then find a lift H in the following commutative solid arrow diagram

$$\begin{tikzcd} F_\bullet \times \partial\Delta^1 \arrow[r, "(id,w)"] \arrow[d] & F_\bullet \arrow[d] \\ F_\bullet \times \Delta^1 \arrow[r] \arrow[ur, dotted, "H"] & \Delta^0. \end{tikzcd}$$

Since the geometric realisation preserves products, $|H|$ is a homotopy between the identity map of $F_\bullet$ and a constant map so $|F_\bullet|$ is contractible.

By the long exact homotopy sequence of the fibration $|f|$, it is enough to show that $|f|$ is surjective on path components.

But any point in $|L_\bullet|$ is in the same path component as the realisation of some vertex x of $L_\bullet$. Since f has the RLP with respect to all inclusions, f is surjective and in particular surjective on vertices. Thus there is a vertex y of $K_\bullet$ such that $f(y) = x$ and so the path component containing y goes to the path component containing x. ■

We would now of course also like the converse of the above proposition, but in order to prove that we first have to develope some homotopy theory in the category **sSet**.

Anodyne Extensions

The main goal of this section is to prove the following theorem. We will again follow [MHov91], but only provide the proof of the main theorem here, for details the reader may have a look in the above mentioned work.

This result is needed for the next section, where we will deal with homotopy groups in **sSet**.

Theorem 8.42 ([MHov91])**.** *Suppose that $i : K_\bullet \to L_\bullet$ is an inclusion of simplicial sets and $p : X_\bullet \to Y_\bullet$ is a fibration of simlpicial sets. Then the induced map*

$$\mathrm{Map}_\square(i,p) : \mathrm{Map}(L_\bullet, X_\bullet) \to \mathrm{Map}(K_\bullet, X_\bullet) \times_{\mathrm{Map}(K_\bullet, Y_\bullet)} \mathrm{Map}(L_\bullet, Y_\bullet)$$

is a fibration.

Remark 8.43. *The simplicial sets in the above theorem are defined according to the description given in the proof of Theorem 8.83.*

If $X_\bullet$ is a fibrant simplicial set and $K_\bullet \to L_\bullet$ is an inclusion. The theorem gives us that the induced map $\mathrm{Map}(L_\bullet, X_\bullet) \to \mathrm{Map}(K_\bullet, X_\bullet)$ *is a fibration.* ⧫

In order to prove the above theorem we will show the following theorem, which is indeed equivalent to the above statement as pointed out in [MHov91].

Theorem 8.44 ([MHov91])**.** *For every anodyne extension $f : A_\bullet \to B_\bullet$ and inclusion $i : K_\bullet \to L_\bullet$ of simplicial sets, the induced map*

$$i\Box f : P(i,f) = (K_\bullet \times B_\bullet) \coprod_{K_\bullet \times A_\bullet} (L_\bullet \times A_\bullet) \to L_\bullet \times B_\bullet$$

is an anodyne extension.

We will construct some anodyne extensions in order to be able to prove the theorem. The following lemma can be found as Lemma 3.3.3 in [MHov91].

Lemma 8.45 ([MHov91])**.** *Let $i : \partial\Delta^n \to \Delta^n$ denote the boundary inclusion for $n \geq 0$ and let $f : \Lambda^1_\epsilon \to \Delta^1$ denote the obvious inclusion, for $\epsilon = 0, 1$. Then the map $i\Box f : P(i,f) = (\partial\Delta^n \times \Delta^1) \coprod_{\partial\Delta^n \times \Lambda^1_\epsilon} (\Delta^n \times \Lambda^1_\epsilon) \to \Delta^n \times \Delta^1$ is an anodyne extension.*

With this in mind we state the following result which can be found in this form as Proposition 3.3.4 in [MHov91].

Proposition 8.46 ([MHov91])**.** *Let $i : K_\bullet \to L_\bullet$ be an inclusion of simplicial sets and let $f : \Lambda^1_\epsilon \to \Delta^1$ be the usual inclusion where $\epsilon = 0, 1$. Then the map $i\Box f : P(i,f) \to L_\bullet \times \Delta^1$ is an anodyne extension.*

Definition 8.47. *Let $J'_{\mathbf{sSet}}$ denote the set of maps $J_{\mathbf{sSet}}\Box f$, where f is one of the maps $\Lambda^1_\epsilon \to \Delta^1$.*

With this definition we may give a different characterisation of an anodyne extension. This is Proposition 3.3.5 in [MHov91].

Proposition 8.48 ([MHov91])**.** *A map $g : K_\bullet \to L_\bullet$ of simplicial sets is an anodyne extension iff it is in $J'_{\mathbf{sSet}}$-cof.*

We now finally have all ingredients to show the main theorem of this section, the proof follows [MHov91].

Proof of Theorem 8.44. The goal is to prove that $i\Box f$ is an anodyne extension for all inclusions i and anodyne extensions f. First we show that $i\Box J'_{\mathbf{sSet}}$ consists of anodyne extensions. Indeed, we have $i\Box J'_{\mathbf{sSet}} = i\Box(J_{\mathbf{sSet}}\Box g)$, where g is one of the maps $\Lambda^1_\epsilon \to \Delta^1$ for $\epsilon = 0, 1$. A beautiful property of the box product is that it is associative (up to isomorphism at least), so we have $i\Box(J_{\mathbf{sSet}}\Box g) \cong (i\Box J_{\mathbf{sSet}})\Box g$. $i\Box J_{\mathbf{sSet}}$ consists of inclusions. Proposition 8.46 then implies that $(i\square J_{\mathbf{sSet}})\square g$ consists of anodyne extensions and hence that $i\square J'_{\mathbf{sSet}}$ consists of anodyne extensions.

We now show that this implies that $i\square f$ is an anodyne extension for all anodyne extensions f by a similar argument to Proposition 8.46. Indeed, we have just seen that the maps of $i\square J_{\mathbf{sSet}}$ have the LLP with respect to $J_{\mathbf{sSet}}$. Adjointness implies that the maps of $J'_{\mathbf{sSet}}$ have the LLP with respect to $\mathrm{Map}_{\square}(i, J_{\mathbf{sSet}}\text{-inj})$. But then the maps of $J'_{\mathbf{sSet}}$-cof must also have the LLP with respect to $\mathrm{Map}_{\square}(i, J_{\mathbf{sSet}}\text{-inj})$. Applying adjointness again, we find that $i\square f$ has the LLP with respect to $J_{\mathbf{sSet}}$-inj and is therefore an anodyne extension for all $f \in J'_{\mathbf{sSet}}$-cof. Since every anodyne extension is in $J'_{\mathbf{sSet}}$-cof by Proposition 8.48 we are done. ■

Homotopy Groups

We want to give a notion of homotopy groups in the category **sSet** of simplicial sets. Therefore we want to construct the homotopy groups of a fibrant object in $\mathbf{sSet}_Q$ using the results of the previous section.

One should keep in mind that at some point we would also be glad to show that an acyclic fibration has the RLP with respect to $I_{\mathbf{sSet}}$. In this section we will show something close to this desired version, namely that if $X_\bullet$ is a fibrant object in **sSet** with no nontrivial homotopy groups then the map $X_\bullet \to \Delta^0$ has the RLP with respect to $I_{\mathbf{sSet}}$.

We again follow [MHov91] very closely but will not provide every of proof here.

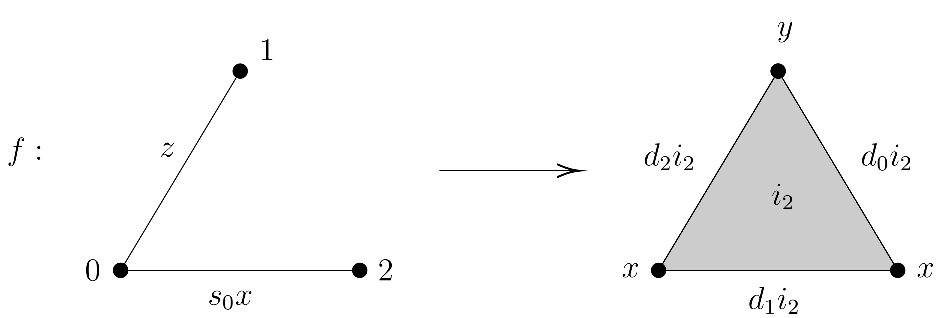
Definition 8.49 (Homotopic). *Suppose that $X_\bullet$ is a fibrant object in $\mathbf{sSet}_Q$ and $x, y \in X_0$ are 0-simplices. Define x to be **homotopic** to y, written $x \sim y$, by definition iff there is a 1-simplex $z \in X_1$ such that $d_1 z = x$ and $d_1 z = y$.*

The next lemma states that homotopy is an equivalence relation.

Lemma 8.50 ([MHov91]). *Suppose $X_\bullet$ is a fibrant object in $\mathbf{sSet}_Q$. Then homotopy of vertices is an equivalence relation. The set of equivalence classes is denoted by $\pi_0(X_\bullet)$*

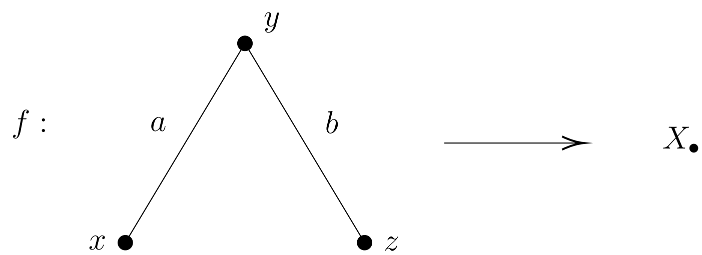
Proof. **Reflexivity:** This is obvious from the homotopy of vertices.

Symmetry: Say $x \sim y$, then we have a 1-simplex z such that $d_1 z = x$ and $d_0 z = y$. Therefore, there is a map $f : \Lambda^2_0 \to X_\bullet$ which is $s_0 x$ on $d_1 i_2$ and z on $d_2 i_2$, as a picture one may imagine the following.



$X_\bullet$ is fibrant and hence there exists an extension of f to a 2-simplex w of X_2. Then $d_0 w$ is the required homotopy from y to x, as we may see from the above picture, and hence $y \sim x$.

Transitivity: Let $x \sim y$ and $y \sim z$. We have 1-simplices a and b such that $d_1 a = x$, $d_0 a = d_1 b = y$ and $d_0 b = z$. Then a and b are defined by a map $f : \Lambda^2_1 \to X_\bullet$. Pictorially it looks as follows.



Since $X_\bullet$ is fibrant, there is an extension of g to a 2-simplex c of X_2. But then $d_1 c$ is the required homotopy from x to z and hence $x \sim z$. ■

Lemma 8.51 ([MHov91]). *Let $X_\bullet \in \mathbf{sSet}_Q$ be a fibrant object. Then there is a natural isomorphism $\pi_0(X_\bullet) \cong \pi_0(|X_\bullet|)$.*

Proof. Consider the natural map $\pi_0(X_\bullet) \to \pi_0(|X_\bullet|)$, taking a vertex v to the path component of $|X_\bullet|$ containing $|v|$.

Now as $|\Delta^n|$ is path connected for $n > 0$, the above map is surjective. This indeed holds, since every point of $|X_\bullet|$ is in the path component of a vertex.

On the other hand, define for $\alpha \in \pi_0(X_\bullet)$ the sub-simplicial set X_α of $X_\bullet$ to consist of all simplices x of $X_\bullet$ with vertex in α. Therefore $X_\bullet = \coprod_{\alpha \in \pi_0(X_\bullet)} X_\alpha$. As $|\ |$ preserves coproducts we are done, since coproducts in $\mathbf{Top}_Q$ are disjoint unions. That is, applying $|\ |$ to $X_\bullet$, tells us that if we consider two vertices α and β which coincide, they will be mapped to the same connected component. ■

The above lemma motivates the following definition.

Definition 8.52 (Path Components). *We call the elements of $\pi_0(X_\bullet)$ **path components**.*

Remark 8.53. *It is worth pointing out, that π_0 is a functor from fibrant simplicial sets to sets. If v is a vertex of a fibrant object $X_\bullet$ in $\mathbf{sSet}_Q$, $\pi_0(X_\bullet, v)$ is the pointed set $\pi_0(X_\bullet)$ with basepoint the equivalence class $[v]$ of v.* ♦

We will now extend the notion of homotopy.

Definition 8.54 (Homotopy Group). *Let $X_\bullet \in \mathbf{sSet}_Q$ be fibrant and $v \in X_0$ a vertex. For any $Y_\bullet \in \mathbf{sSet}_Q$ let us denote the map $Y_\bullet \to \Delta^0 \xrightarrow{v} X_\bullet$ by v as well and refer to it as the constant map at v.*

Let $F_\bullet$ denote the fiber over v of the fibration $\mathrm{Map}(\Delta^n, X_\bullet) \to \mathrm{Map}(\partial\Delta^n, X_\bullet)$. The map is a fibration by Theorem 8.44.

*Then we define the n**-th homotopy group** $\pi_n(X_\bullet, v)$ of $X_\bullet$ at v to be the pointed set $\pi_0(F_\bullet, v)$.*

It is certainly nice to call this the definition of homotopy groups but it is also interesting to note that at this point we do not even know if they really are groups. There will be no direct proof of this fact though. Instead we show that $\pi_n(X_\bullet, v) \cong \pi_n(|X_\bullet|, |v|)$ for a fibrant object $X_\bullet$ in $\mathbf{sSet}_Q$, i.e. $\pi_n(X_\bullet, v)$ is indeed a group for $n \geq 1$ and in fact abelian for $n \geq 2$.

Remark 8.55. *There is also another way of stating the above definition. $\pi_n(X_\bullet, v)$ is the set of equivalence classes $[\alpha]$ of n-simplices $\alpha : \Delta^n \to X_\bullet$ that send $\partial\Delta^n$ to v under the equivalence relation defined by $\alpha \sim \beta$ if there is a homotopy $H : \Delta^n \times \Delta^1 \to X_\bullet$ such that H is α on $\Delta^n \times \{0\}$, β on $\Delta^n \times \{1\}$ and the constant map v on $\partial\Delta^n \times \Delta^1$.*

Let $f : X_\bullet \to Y_\bullet \in \mathbf{sSet}_Q$ and v be a vertex of $X_\bullet$ there is an induced map $f_* : \pi_n(X_\bullet, v) \to \pi(Y_\bullet, f(v))$ making the homotopy groups functorial. (More about this may be found in [MHov91]).

♦

The following lemma gives an alternative characterisation of when an n-simplex is homotopic to the constant map.

Lemma 8.56 ([MHov91])**.** *Let $X_\bullet \in \mathbf{sSet}_Q$ be fibrant, v a vertex of $X_\bullet$ and $\alpha : \Delta^n \to X_\bullet$ an n-simplex of $X_\bullet$ such that $d_i\alpha = v$ for all i.*

Then $[\alpha] = [v] \in \pi_n(X_\bullet, v)$ iff there is an $(n+1)$-simplex x of $X_\bullet$ such that $d_{n+1}x = \alpha$ and $d_ix = v$ for $i \leq n$.

Proof. Assume that $[\alpha] = [v]$, i.e. there is a homotopy $H : \Delta^n \times \Delta^1 \to X_\bullet$ from α to v which is v on $\partial\Delta^n \times \Delta^1$.

Define a map $G : \partial\Delta^{n+1} \times \Delta^1 \to X_\bullet$ by

$$\begin{cases} G \circ (d^i \times 1) = v & \text{for } \; i < n+1 \\ G \circ (d^{n+1} \times 1) = H. \end{cases}$$

Then G is v on $\partial\Delta^{n+1} \times \{1\}$ and we have a commutative diagram

$$\begin{tikzcd} (\partial\Delta^{n+1} \times \Delta^1) \coprod_{\partial\Delta^{n+1}\times\{1\}} (\Delta^{n+1} \times \{1\}) \arrow[r, "G \coprod v"] \arrow[d] & X_\bullet \arrow[d] \\ \Delta^{n+1} \times \Delta^1 \arrow[r] & \Delta^0. \end{tikzcd}$$

Since $X_\bullet$ is fibrant, there is a lift $F : \Delta^{n+1} \times \Delta^1 \to X_\bullet$. The $(n+1)$-simplex $F(\Delta^{n+1} \times \{0\})$ is the desired x such that $d_{n+1}x = \alpha$ and $d_ix = v$ for $i \leq n$.

Conversely, let x be an $(n+1)$ simplex such that $d_{n+1}x = \alpha$ and $d_ix = v$ for $i \leq n$. Define a map

$$G : (\Lambda^{n+1}_{n+1} \times \Delta^1) \coprod_{\Lambda^{n+1}_{n+1} \times \partial\Delta^1} (\Delta^{n+1} \times \partial\Delta^1) \to X_\bullet$$

as v on $\Lambda^{n+1}_{n+1} \times \Delta^1$ and $\Delta^{n+1} \times \{1\}$; and x on $\Delta^{n+1} \times \{1\}$.

Since $X_\bullet$ is fibrant there is an extension of G to a map $F : \Delta^{n+1} \times \Delta^1 \to X_\bullet$. Let $H = F \circ (d^{n+1} \times 1)$. Then H is the desired homotopy between α and v. ■

Definition 8.57 (Homotopy). *Let $f, g : K_\bullet \to X_\bullet \in \mathbf{sSet}_Q$. We refer to a map $H : K_\bullet \times \Delta^1 \to X_\bullet$ such that H is f on $K_\bullet \times \{0\}$ and g on $K_\bullet \times \{1\}$ as a **homotopy** from f to g.*

The resulting homotopy relation will not always be an equivalence relation but it is one if $X_\bullet \in \mathbf{sSet}_Q$ is fibrant. Indeed, then f and g are homotopic iff they are homotopic as vertices of the fibrant simplicial set $\mathrm{Map}(K_\bullet, X_\bullet)$. One should now think back when we discussed homotopy for model categories in part I, where the homotopy relation is an equivalence relation if we considered fibrant and cofibrant objects.

The next lemma provides an important example of a homotopy, it can be found as Lemma 3.4.6 in [MHov91].

Lemma 8.58 ([MHov91]). *The vertex n is a deformation retract of Δ^n in the sense that there is a homotopy $H : \Delta^n \times \Delta^1 \to \Delta^n$ from the identity map to the constant map at n which sends $n \times \Delta^1$ to n. Furthermore, this homotopy restricts to a deformation retraction of Λ^n_n onto its vertex n.*

Corollary 8.59. *Δ^n is not fibrant in $\mathbf{sSet}_Q$.*

Proof. If we consider the homotopy provided by the above lemma, then there is no homotopy in the other direction. Indeed, it would have to be induced by a map of ordered sets that takes $(0, k)$ to n and $(k, 1)$ to k but there is no such map.

Therefore homotopy is not an equivalence relation on self-maps of Δ^n proving that Δ^n is not fibrant. ∎

We are now able to state the main result of this section.

Proposition 8.60 ([MHov91]). *Let $X_\bullet$ be a non-empty fibrant object in $\mathbf{sSet}_Q$ with no non-trivial homotopy groups. Then the map $X_\bullet \to \Delta^0$ is in $I_{\mathbf{sSet}}$-inj.*

Proof. We want to show, that any map $f : \partial\Delta^n \to X_\bullet$ has an extension to $\Delta^n \to X_\bullet$. Since $X_\bullet$ is non-empty we may assume, that $n > 0$.

If f and g are two homotopic maps, and g has in addition an extension $g' : \Delta^n \to X_\bullet$, then f may also be extended to $\Delta^n \to X_\bullet$. Indeed, there is a homotopy $H : \partial\Delta^n \times \Delta^1 \to X_\bullet$ between f and g. With the help of g' we may define a map

$$(\partial\Delta^n \times \Delta^1) \coprod_{\partial\Delta^n \times \{1\}} (\Delta^n \times \{1\}) \to X_\bullet.$$

$X_\bullet$ is fibrant and therefore there is an extension for this map to a homotopy $G : \Delta^n \times \Delta^1 \to X_\bullet$. Then $G(\Delta^n \times \{0\})$ is the desired extension for f.

Consider the composite

$$H' : \Lambda^n_n \times \Delta^1 \xrightarrow{H} \Lambda^n_n \xrightarrow{f} X_\bullet,$$

where H is the deformation retraction of Λ^n_n onto n as in Lemma 8.58 and f is the restriction of itself. Then H' and f define a map

$$(\Lambda^n_n \times \Delta^1) \coprod_{\Lambda^n_n \times \{0\}} (\partial\Delta^n \times \{0\}) \to X_\bullet.$$

Again since $X_\bullet$ is fibrant, there is an extension $G : \partial\Delta^n \times \Delta^1 \to X_\bullet$. This map G is a homotopy from f to g such that $g \circ d^i = f(n)$ for $i < n$. In particular, $g \circ d^n$ represents a class in $\pi_{n-1}(X_\bullet, f(n))$. Therefore we have by assumption $[g \circ d^n] = [f(n)]$. By Lemma 8.56, there is an n simplex g' such that $d_i g' = f(n)$ for $i < n$ and $d_n g' = g \circ d^n$. Thus g' is an extension of g, but then f is also an extension. ■

Minimal Fibrations

In a first step, we will point out that the main result in the last section provides us with a lifting result for some locally trivial fibrations (see for instance the next definition).

Finally, we show that any fibration is locally fiberwise homotopy equivalent to a locally trivial fibration. We want to point out what is needed for this locally fiberwise equivalence to actually be an isomorphism, this is where the notion of minimal fibrations show up.

In a last step we then show, that minimal fibrations are locally trivial and - yet more important - that any fibration is closely approximated by a minimal fibration.

To sum up, minimal fibrations provide us with a nice tool to handle fibrations.

This part is - as one may imagine - very technical which is why we will not provide every proof for all the results. Again we follow [MHov91] very closely and refer to his work for more details on proofs.

Definition 8.61 (Locally Trivial). *Let $p : X_\bullet \to Y_\bullet \in \mathcal{F}_{\mathbf{sSet}}$. We say that p is **locally trivial** if, for every simplex $\Delta^n \xrightarrow{y} Y_\bullet$ of $Y_\bullet$, the pullback fibration $y^*X_\bullet = \Delta^n \times_{Y_\bullet} X_\bullet \xrightarrow{y^*p} \Delta^n$ is isomorphic over Δ^n to a product fibration $\Delta^n \times F_{v\bullet} \xrightarrow{\pi_1} \Delta^n$.*

Corollary 8.62 ([MHov91]). *Let $p : X_\bullet \to Y_\bullet \in \mathbf{sSet}_Q$ be a locally trivial fibration such that every fiber of p is non-empty and has no non-trivial homotopy groups. Then $p \in I_{\mathbf{sSet}}$-inj.*

Proof. This is a consequence of Proposition 8.60. ■

The next result states, that any fibration over Δ^n is homotopy equivalent to a product fibration. The reason for this idea comes from the fact that Δ^n is contractible (in a simplicial way) onto its vertex n. This is Proposition 3.5.3 in [MHov91].

Proposition 8.63 ([MHov91]). *Let $p : X_\bullet \to Y_\bullet \in \mathbf{sSet}_Q$ and $f, g : K_\bullet \to Y_\bullet$ be maps such that there exists a homotopy from f to g. Then the pullback fibrations $f^*p : f^*X_\bullet \to K_\bullet$ and $g^*p : g^*X_\bullet \to K_\bullet$ are fiber homotopy equivalent.*

That is there are maps $\theta_* : f^*X_\bullet \to g^*X_\bullet$ and $\omega_* : g^*X_\bullet \to f^*X_\bullet$ such that $g^*p \circ \theta_* = f^*p$ and $f^*p \circ \omega_* = g^*p$ and there are homotopies from $\theta_*\omega_*$ to the identity of $g^*X_\bullet$ and from $\omega_*\theta_*$ to the identity of $f^*X_\bullet$ that cover the constant homotopy of $K_\bullet$.

Corollary 8.64 ([MHov91]). *Let $p : X_\bullet \to Y_\bullet$ be a fibration in $\mathbf{sSet}_Q$ and $\Delta^n \xrightarrow{y} Y_\bullet$ a simplex of $Y_\bullet$. Then the pullback $y^*X_\bullet \to \Delta^n$ is fiber homotopy equivalent to the product fibration $\Delta^n \times F_{n\bullet} \xrightarrow{\pi_1} \Delta^n$ where $F_{n\bullet}$ is the fiber of p over the vertex $y(n)$.*

Proof. Combine Lemma 8.58 and Proposition 8.63. ■

One could ask the question, when this homotopy equivalence could be an isomorphism and which restrictions one would need to put on the fibration p. We consider the following situation.

We consider two fibrations $p : X_\bullet \to Y_\bullet$ and $q : Z_\bullet \to Y_\bullet$ over the same base and two maps $f, g : X_\bullet \to Z_\bullet$ covering the identity map of $Y_\bullet$ such that g is an isomorphism. Further we assume them to be fiber homotopic so that there is a homotopy from f to g which covers the constant homotopy.

Given a fiber homotopy equivalence we would take g to be the identity map and f to be the composite of one of the maps with a homotopy inverse. We would like to conclude that f is an isomorphism.

In order to show that f is an isomorphism on vertices, let z be a vertex of $Z_\bullet$. Then there is a vertex x of $X_\bullet$ such that $gx = z$ since g is an isomorphism. From the homotopy we get a path from fx to gx which covers the constant path of $qgx = qfx$. Meaning that fx and gx are vertices of a fiber of q which are in the same path component of that fiber.

Now if we knew that every path component of every fiber of q has only one vertex, we could actually conclude that $fx = z$.

Similarly suppose that $fx = fy$, then the homotopy yields a path from fx to gx and from fy to gy in the fiber of q over qfx. Again if we would know that every path component of every fiber of q had only one vertex we could conclude that $gx = gy$ and so $x = y$.

One may try to extend this approach to n-simplices for positive n. Then we would like to assume that we had already proven the above.

This discussion motivates the following definitions.

Definition 8.65 (Minimal Fibration). *A fibration $p : X_\bullet \to Y_\bullet \in \mathbf{sSet}_Q$ is called a **minimal fibration** by definition iff for all $n \geq 0$ every path component of every fiber of the fibration $\mathrm{Map}_\square(i,p) : \mathrm{Map}(\Delta^n, X_\bullet) \to \mathrm{Map}(\partial\Delta^n, X_\bullet) \times_{\mathrm{Map}(\partial\Delta^n, Y_\bullet)} \mathrm{Map}(\Delta^n, Y_\bullet)$ has only one vertex.*

Definition 8.66 (p-related). *We define two n-simplices x and y of $X_\bullet \in \mathbf{sSet}_Q$ to be p-**related** if they represent vertices in the same path component of the same fiber of $\mathrm{Map}_\square(i,p)$. We write $x \sim_p y$ if x and y are p-related. Where for any n we consider the map $i : \partial\Delta^n \to \Delta^n$.*

The above relation is an equivalence relation due to Lemma 8.50. Further p is a minimal fibration iff $x \sim_p y$ implies $x = y$. Also, $x \sim_p y$ iff $p(x) = p(y)$, $d_i x = d_i y$ for all i such that $0 \leq i \leq n$ and there is a homotopy $\Delta^n \times \Delta^1 \xrightarrow{H} X$ from X to Y such that pH is the constant homotopy and H is constant on $\partial\Delta^n$. Therefore we have the next result, which is Lemma 3.5.6 in [MHov91].

Lemma 8.67 ([MHov91]). *Let $p : X_\bullet \to Y_\bullet, q : Z_\bullet \to Y_\bullet \in \mathcal{F}_{\mathbf{sSet}}$ such that q is a minimal fibration. Let $f, g : X_\bullet \to Z_\bullet$ be two maps such that $qf = qg = p$. Suppose $H : X \times \Delta^1 \to Z_\bullet$ is a homotopy from f to g such that $qH = p\pi_1$. Then if g is an isomorphism, so is f.*

Corollary 8.68 ([MHov91]). *Let $p : X_\bullet \to Y_\bullet$ be a minimal fibration in $\mathbf{sSet}_Q$. Then p is locally trivial.*

Proof. This follows from Corollary 8.64 and Lemma 8.67. ∎

The next result is Lemma 3.5.8 in [MHov91].

Lemma 8.69 ([MHov91]). *Let $p : X_\bullet \to Y_\bullet \in \mathcal{F}_{\mathbf{sSet}}$ and x and y are degenerate n-simplices of $X_\bullet$ such that $x \sim_p y$, then $x = y$.*

Now we can finally state the result, that any fibration is closely related to a minimal fibration in $\mathbf{sSet}_Q$. This is Theorem 3.5.9 in [MHov91].

Theorem 8.70 ([MHov91]). *Let $p : X_\bullet \to Y_\bullet \in \mathcal{F}_{\mathbf{sSet}}$. Then we can factor p as*

$$p : X_\bullet \xrightarrow{r} X'_\bullet \xrightarrow{p'} Y_\bullet,$$

where p' is a minimal fibration and r is a retract onto a subsimplicial set $X'_\bullet$ of $X_\bullet$ such that $r \in I_{\mathbf{sSet}}$-inj.

Corollary 8.71 ([MHov91]). *Let p be a fibration in* **sSet** *such that every fiber of p is non-empty and has no non-trivial homotopy groups. Then p is $I_{\mathbf{sSet}}$-inj.*

Proof. A consequence of Theorem 8.70, Corollary 8.68 and Corollary 8.62. ■

By the looks of the last corollary, we are now indeed very close to the desired result. Anyway, it does not imply that every acyclic fibration has the RLP with respect to $I_{\mathbf{sSet}}$. This will be worked out in the next section.

Fibrations and Geometric Realisation

We will finally conclude the proof that simplicial sets form a model category. In a first step we show that the geometric realisation preserves fibrations and use this fact to show that the homotopy groups of a fibrant object in $\mathbf{sSet}_Q$ are isomorphic to the homotopy groups of its geometric realisation i.e. in $\mathbf{Top}_Q$.

Furthermore we show that the geometric realisation is part of a Quillen equivalence from **sSet** to **Top**.

We again follow [MHov91] very closely. An equivalent formulation to next result can be found as Proposition 3.6.1 in [MHov91].

Proposition 8.72. *Let $p : X_\bullet \to Y_\bullet$ be a locally trivial fibration in $\mathbf{sSet}_Q$. Then $|p| \in \mathcal{F}_{\mathbf{Top}}$.*

Corollary 8.73. *Let $p \in \mathcal{F}_{\mathbf{sSet}}$. Then $|p|$ is a fibration of compactly generated topological spaces.*

Proof. A consequence of Theorem 8.70, Proposition 8.72 and Lemma 8.40. ■

The next result may be found as Proposition 3.6.3 in [MHov91].

Proposition 8.74 ([MHov91]). *Let $X_\bullet \in \mathbf{sSet}_Q$ and fibrant and v a vertex of $X_\bullet$. Then there is a natural isomorphism*

$$\pi_n(X_\bullet, v) \cong \pi_n(|X_\bullet|, |v|).$$

We now state our main result. It will also be the result which concludes the proof that $\mathbf{sSet}_Q$ is a cofibrantly generated model category.

The proof of the next theorem is inspired by [MHov91].

Theorem 8.75. *Let $p \in \mathcal{W}_{\mathbf{sSet}} \cap \mathcal{F}_{\mathbf{sSet}}$. Then p has the RLP with respect to $I_{\mathbf{sSet}}$.*

Proof. Corollary 8.71 tells us, that it suffices to show, that the fibers of p are non-empty and have no non-trivial homotopy groups.

Let $F_\bullet$ be a fiber of p over a vertex v. Then, by Corollary 8.73 $|F_\bullet|$ is the fiber of the fibration $|p|$ over $|v|$. Since $|p| \in \mathcal{W}_{\mathbf{Top}}$, $|F_\bullet|$ has no non-trivial homotopy groups and is non-empty.

Finally, Proposition 8.74 implies that $F_\bullet$ has no non-trivial homotopy groups and is non-empty. ■

With the help of this statement we conclude.

8.3.1 The Factorisations

Proposition 8.76 (The First Factorisation)**.** *Every $f : X_\bullet \to Y_\bullet \in \mathbf{sSet}_Q$ factors in the following sense*

$$f : X_\bullet \xrightarrow{\in \mathcal{C}_{\mathbf{sSet}}} X'_\bullet \xrightarrow{\in \mathcal{W}_{\mathbf{sSet}} \cap \mathcal{F}_{\mathbf{sSet}}} Y_\bullet.$$

Proof. $I_{\mathbf{sSet}}$ admits the small object argument. Therefore we have a factorisation

$$f : X_\bullet \xrightarrow{\in \mathcal{C}_{\mathbf{sSet}}} X'_\bullet \xrightarrow{\in I_{\mathbf{sSet}}\text{-inj}} Y_\bullet.$$

Now we apply Theorem 8.75 and we are done. ■

Proposition 8.77 (The Second Factorisation)**.** *Every $f : X_\bullet \to Y_\bullet \in \mathbf{sSet}_Q$ factors in the following sense*

$$f : X_\bullet \xrightarrow{\in \mathcal{W}_{\mathbf{sSet}} \cap \mathcal{C}_{\mathbf{sSet}}} X'_\bullet \xrightarrow{\in \mathcal{F}_{\mathbf{sSet}}} Y_\bullet.$$

Proof. $J_{\mathbf{sSet}}$ admits the small object argument. Therefore, there is a factorisation

$$f : X_\bullet \xrightarrow{\in J_{\mathbf{sSet}}\text{-cell}} X'_\bullet \xrightarrow{J_{\mathbf{sSet}}\text{-inj}} Y_\bullet.$$

But since the $J_{\mathbf{sSet}}$-inj are Kan fibrations we have $X'_\bullet \to Y_\bullet \in \mathcal{F}_{\mathbf{sSet}}$. Then since every relative $J_{\mathbf{sSet}}$-cell complex is a relative $I_{\mathbf{sSet}}$-cell complex, $X_\bullet \to X'_\bullet \in \mathcal{C}_{\mathbf{sSet}}$ and therefore it is also a weak equivalence. ■

8.3.2 Liftings

Proposition 8.78. *Consider a commutative solid arrow diagram*

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "g \in \mathcal{C}_{\mathbf{sSet}}"'] & \bullet \arrow[d, "f \in \mathcal{F}_{\mathbf{sSet}}"] \\ \bullet \arrow[r] \arrow[ur, dotted, "h"] & \bullet \end{tikzcd}$$

in $\mathbf{sSet}_Q$*, then the lift* h *(dotted line) exists, as soon as either* g *or* f *are in* $\mathcal{W}_{\mathbf{sSet}}$*.*

Proof. Assume, that $f \in \mathcal{W}_{\mathbf{sSet}} \cap \mathcal{F}_{\mathbf{sSet}}$. Then by Theorem 8.75 and the closure properties, we conclude that g has the LLP against f.

On the other hand, assume that $g \in \mathcal{W}_{\mathbf{sSet}} \cap \mathcal{C}_{\mathbf{sSet}}$. g can be factored as

$$g : X_\bullet \xrightarrow{\in \mathcal{W}_{\mathbf{Top}} \cap \mathcal{C}_{\mathbf{sSet}}} X'_\bullet \xrightarrow{\in \mathcal{F}_{\mathbf{sSet}}} Y_\bullet.$$

Since $g \in \mathcal{W}_{\mathbf{sSet}}$ we may apply the 2-3 property to conclude that $X'_\bullet \to Y_\bullet \in \mathcal{W}_{\mathbf{sSet}}$. The first case now yields, that $X'_\bullet \to Y_\bullet$ has the RLP against g. Applying the retract argument yields that g is a relative $J_{\mathbf{sSet}}$-cell complex. Again by the closure properties, f has the RLP against g. In both cases we have a lift in the desired diagram. ■

8.3.3 The FWFS

Proposition 8.79. *There are two FWFS* $(\mathcal{C}_{\mathbf{sSet}}, \mathcal{W}_{\mathbf{sSet}} \cap \mathcal{F}_{\mathbf{sSet}})$ *and* $(\mathcal{W}_{\mathbf{sSet}} \cap \mathcal{C}_{\mathbf{sSet}}, \mathcal{F}_{\mathbf{sSet}})$ *in* $\mathbf{sSet}_Q$*.*

Proof. The factorisation part follows from Proposition 8.76 and Proposition 8.77. We are left to show, that the respective classes have the desired lifting properties against each other.

$\mathcal{F}_{\mathbf{sSet}}$ has the RLP against $\mathcal{W}_{\mathbf{sSet}} \cap \mathcal{C}_{\mathbf{sSet}}$ by definition, and $\mathcal{W}_{\mathbf{sSet}} \cap \mathcal{F}_{\mathbf{sSet}}$ has the RLP against $\mathcal{C}_{\mathbf{sSet}}$ by Theorem 8.75. We show, that $\mathcal{C}_{\mathbf{sSet}}$ has the LLP with respect to $\mathcal{W}_{\mathbf{sSet}} \cap \mathcal{F}_{\mathbf{sSet}}$. Consider a morphism $f : X_\bullet \to Y_\bullet$ in $I_{\mathbf{sSet}}$-cof, if this turns out to be a relative cell complex, then we are done.

We apply the small object argument to get

$$f : X_\bullet \to Y'_\bullet \xrightarrow{\in I_{\mathbf{sSet}}\text{-inj}} Y_\bullet.$$

Now f has the LLP with respect to $Y''_\bullet \to Y_\bullet$, we conclude by the retract argument that f is a retract of $X_\bullet \to Y'_\bullet$.

If we apply the small object argument to $J_{\mathbf{sSet}}$ and factor f, we can conlcude the same result for $\mathcal{W}_{\mathbf{sSet}} \cap \mathcal{C}_{\mathbf{sSet}}$, showing that $\mathcal{C}_{\mathbf{sSet}}$ has the LLP with respect to $\mathcal{W}_{\mathbf{sSet}} \cap \mathcal{F}_{\mathbf{sSet}}$ and $\mathcal{W}_{\mathbf{sSet}} \cap \mathcal{C}_{\mathbf{sSet}}$ has the LLP with respect to $\mathcal{F}_{\mathbf{sSet}}$. ■

Now we can finally state the main result of the whole section.

Theorem 8.80. $(\mathbf{sSet}_Q, I_{\mathbf{sSet}}, J_{\mathbf{sSet}})$ *is a cofibrantly generated model category. Every object in* $\mathbf{sSet}_Q$ *is cofibrant.*

Proof. We have to show, that $\mathbf{sSet}_Q$ is bicomplete and equipped with a model structure. Bicompleteness follows from Remark 8.8. The model structure follows from the fact that $\mathcal{W}_{\mathbf{sSet}}$ has the 2-3 property and Proposition 8.79. The last statement follows from Proposition 8.35. ■

Theorem 8.81. $\mathbf{sSet}_Q$ *is a cellular model category.*

Proof. Since $\mathbf{sSet}_Q$ is a cofibrantly generated model categry (Theorem 8.85) we only need to verify condition 3. and 5. in the recognition theorem for cellular model categories.

Condition 3. holds. From the definition of simplicial sets, a compact subset of a cell complex in $\mathbf{sSet}_Q$ is contained in a finite subcomplex of the cell complex. From this fact every simplicial set is ω-compact relative to $I_{\mathbf{sSet}}$, where ω is the countable cardinal. Similarly, if γ is an infinite cardinal and X is a simplicial set of size γ, then by the above fact it follows that X is compact relative to $I_{\mathbf{sSet}}$, which is precisely what we wanted.

Condition 5. holds. This case is very similar as the case for topological spaces. In the sense that relative $I_{\mathbf{sSet}}$-cell complexes turn out to be embeddings and hence effective monomorphisms. ■

The next goal is to show properness, which really follows from properness of topological spaces due to all the work we have done before.

Theorem 8.82. $\mathbf{sSet}_Q$ *is a proper model category.*

Proof. Since every object in $\mathbf{sSet}_Q$ is cofibrant we automatically get that $\mathbf{sSet}_Q$ is left proper. We are left to show right properness. This follows from the right properness of **Top**, Corollary 8.73 and Lemma 8.39. ■

Theorem 8.83. $\mathbf{sSet}_Q$ *is a simplicial model category.*

proof (sketch). We give $\mathbf{sSet}_Q$ the structure of a simplicial model category by saying: If $X_\bullet$ and $Y_\bullet$ are objects in **sSet** we let $\mathrm{Map}(X_\bullet, Y_\bullet)$ be the simplicial set that in degree n is the set of maps of simplicial sets from $X_\bullet \times \Delta^n$ to $Y_\bullet$, with face and degeneracy maps induced by the standard maps between the Δ^n.

The cartesian product together with this internal hom give $\mathbf{sSet}_Q$ the structure of a simplicial model category. ■

Theorem 8.84. $\mathbf{sSet}_Q$ *is a combinatorial model category.*

proof (sketch). This follows, since $\mathbf{sSet}_Q$ is defined as a functor category via **Set** i.e. $\mathbf{sSet} = \mathbf{Fun}(\boldsymbol{\Delta}^{op}, \mathbf{Set})$. ■

8.4 The Main Theorem

Theorem 8.85. $(\mathbf{sSet}_Q, I_{\mathbf{sSet}}, J_{\mathbf{sSet}})$ *is a cofibrantly generated, cellular, simplicial, combinatorial, proper model category. Furthermore, every object in* $\mathbf{sSet}_Q$ *is cofibrant.*

Proof. This follows from Theorem 8.80, Theorem 8.81, Theorem 8.82, Theorem 8.83, Theorem 8.84 and Proposition 8.35. ■

A now immediate consequence is the existence of left Bousfield localisations.

Theorem 8.86. *The left Bousfield localisation of* $\mathbf{sSet}_Q$ *exists with respect to any class of morphisms in* $\mathbf{sSet}$*.*

Proof. This is now an immediate consequence of Theorem 8.85 as it fulfills any needed condition for the existence of Bousfield localisations. ∎

For completeness we state another very useful result, though we will not make use of it in the present work, it can be found as Proposition 3.6.8 in [MHov91].

Proposition 8.87. *Let* $\mathscr{C}$ *be a model category and* $F : \mathbf{sSet}_Q \to \mathscr{C}$ *a functor which preserves colimits and cofibrations. Then* F *preserves acyclic cofibrations (and weak equivalences) iff* $F(\Delta^n) \to F(\Delta^0)$ *is a weak equivalence for all* $n \geq 0$*.*

Part III

The Homotopy Hypothesis

This last part is really the core of the whole thesis. It consists of two sections descussing the homotpy hypothesis in different cases.

The first section is dedicated to the case of the classical homotopy hypothesis, which relies on the work done in the previous part. The main result will be, that there is a Quillen equivalence

$$| \, | : \mathbf{sSet}_Q \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Top}_Q : \mathrm{Sing}\,.$$

The second section is far more involved. We will introduce the concept of truncations for $\mathbf{Top}_Q$ and $\mathbf{sSet}_Q$ and give yet another Quillen equivalence between this truncated model categories in the sense that there is a Quillen euqivalence

$$| \, | : \mathbf{sSet}_Q^{\leq n} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Top}_Q^{\leq n} : \mathrm{Sing}\,.$$

The latter category may be seen as a model for $\infty\mathbf{Grpd}$, so that the Equivalence gives a way to relate $\infty\mathbf{Grpd}$ with the model category of n-truncated simplicial sets.

In a next step we will give an analysis about the category $\mathbf{sSet}_Q^{\leq n}$. This discussion uses a lot of rather involved theory. At the end we have a result which will tell us exactly how the localisation for this respective category behaves. It will use the concepts of fibrant replacements and the machinery of the coskeleton functor, which will also be introduced and discussed in this section.

The final part will now try to conclude three Quillen equivalences with three different settings. The first one, which is more of a warm-up exercise will yield that there is a Quillen equivalence

$$\Pi_0 : \mathbf{sSet}_Q^{\leq 0} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Set}_T : \mathcal{N}_0\,.$$

The next case is already more involved and will show, that there is a Quillen equivalence

$$\Pi_1 : \mathbf{sSet}_Q^{\leq 1} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Grpd}_{\mathcal{M}_1} : \mathcal{N}_1$$

which says, that the homotopy of $\mathbf{Grpd}_{\mathcal{M}_1}$ may be described with the homotopy of 1-truncated simplicial sets.

Finally, the last equivalence will state that there is a Quillen equivalence

$$\Pi_2 : \mathbf{sSet}_Q^{\leq 2} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{bi\text{-}Grpd}_{\mathrm{s}\mathcal{M}_2} : \mathcal{N}_2$$

which then says that 2-truncated simplicial sets are a good model for $\mathbf{bi\text{-}Grpd}_{\mathrm{s}\mathcal{M}_2}$ (bi-groupoids with strict 2-functors). A special case will also give a Quillen equivalence

$$\Pi_2 : \mathbf{sSet}_Q^{\leq 2} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} 2\text{-}\,\mathbf{Grpd}_{\mathcal{M}_2} : \mathcal{N}_2\,.$$

Furthermore, we will also argue, that the homotopy theories of 2- **Grpd** and **bi-Grpd** are closely related to each other and therefore also to the homotopy theory of $\mathbf{sSet}_Q^{\leq 2}$. Unfortunately this can not be done directly with a Quillen equivalence of the respective categories, as we will see.

9 The Homotopy Hypothesis

In this short section, we give the proof of the ”classical” version of the homotopy hypothesis, which makes for a nice introduction for this part. We use material from [MHov91] for the proof of the main theorem, there is also some inspiration from [nLab].

The model categories $\mathbf{Top}_Q$ and $\mathbf{sSet}_Q$ are Quillen equivalent and encapsulate much of “classical” homotopy theory. From a higher-categorical viewpoint, they can be regarded as models for ∞-groupoids (in terms of CW complexes or Kan complexes, respectively).

Remember that we already showed, that there is an adjunction

$$| \ | : \mathbf{sSet} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Top} : \mathrm{Sing},$$

which was the content of Lemma 8.23, of course with the exact same argument there is an adjunction between **sSet** and **K** with the exact same functors.

Proposition 9.1. *There is a Quillen pair*

$$| \ | : \mathbf{sSet}_Q \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Top}_Q : \mathrm{Sing}\,.$$

Proof. This follows from Lemma 2.33 and the discussions in part II. ∎

Theorem 9.2 (The Homotopy Hypothesis)**.** *There is a Quillen equivalence*

$$| \ | : \mathbf{sSet}_Q \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Top}_Q : \mathrm{Sing}\,.$$

Proof. We will first show, that there is a Quillen equivalence between the model categories $\mathbf{sSet}_Q$ and $\mathbf{K}_Q$. Proposition 9.1 with the same proof also yields a Quillen pair for $\mathbf{sSet}_Q$ and $\mathbf{K}_Q$.

By Corollary 4.45 it suffices to show that the map $|\operatorname{Sing}(X)| \to X$ is a weak equivalence for all $X \in \mathbf{K}$. By definition of weak equivalence it suffices to show that the map $\pi_i(|\operatorname{Sing}(X)|, |v|) \to \pi_i(X, v)$ is an isomorphism for every point v of X (since such points are in 1-1 correspondence with vertices of $\operatorname{Sing}(X)$ and every point of $|\operatorname{Sing}(X)|$ is in the same path component as a vertex).

Since $\operatorname{Sing}(X)$ is fibrant by Corollary 8.73, we have an isomorphism $\pi_i(\operatorname{Sing}(X), v) \to \pi_i(|\operatorname{Sing}(X)|, |v|)$ from Proposition 8.74. The composite map $\pi_i(\operatorname{Sing}(X), v) \to \pi_i(X, v)$ is the map induced by the adjunction that is, an element of $\pi_i(\operatorname{Sing}(X), v)$ is represented by a map $\Delta^i \to \operatorname{Sing}(X)$ sending $\partial\Delta^i$ to v. This map is adjoint to a map $D^i \cong |\Delta^i| \to X$ sending S^{i-1} to v and a map $S^i \to X$ sending the basepoint to v. This represents an element of $\pi_i(X, v)$. We can also run this adjunction backwards and we can apply it to homotopies as well. Thus the map $\pi_i(\operatorname{Sing}(X), v) \to \pi_i(X, v)$ is an isomorphism as required. This gives us a Quillen equivalence $| \ | : \mathbf{sSet}_Q \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{K}_Q : \mathrm{Sing}$.

To conclude the proof consider the following. From Lemma 4.34 we know that Quillen adjunctions are closed under composition and hence Quillen equivalences are closed under composition (it also follows from the fact that they satisfy the 2-3 property). Now we just need to combine Theorem 7.40 and Theorem 9.2. The adjunction will consist of the composition of the respective adjunctions which turns out to be the geometric realisation and the singular functor (see for instance Remark 8.24). ■

10 The n-Truncated Homotopy Hypothesis

We consider further cases of Quillen equivalences between certain categories. In any case the model category $\mathbf{sSet}_Q$ will play a central role. We will use a left Bousfield localisation with respect to a specific set and call this the n-truncated simplicial sets. To better grasp what will happen in this case we first give a discussion about topological spaces, where we give such quillen equivalences for any n. There the idea becomes clear, that an n-truncation kills off higher homotopy groups, precisely those greater than n. The other cases will include a discussion about sets followed by groupoids and bigroupoids, for the corresponding dimensions 0, 1 and 2 respectively.

10.1 The Case of Topological Spaces

We give the main theorem for this subsection.

Theorem 10.1. *Let $\mathcal{S} := \{\partial\Delta^{n+2} \to \Delta^{n+2}\}$. There is a Quillen equivalence*

$$|\ | : L^B_{\mathcal{S}}\,\mathbf{sSet}_Q \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} L^B_{|Q\mathcal{S}|}\,\mathbf{Top}_Q : \mathrm{Sing}\,.$$

Proof. First of all, we notice that the left Bousfield localisation of $\mathbf{Top}_Q$ and $\mathbf{sSet}_Q$ exist for any class of morphisms in $\mathbf{Top}$ and $\mathbf{sSet}$ respectively. This follows from Theorem 8.85 Theorem 7.34 and Theorem 5.45.

Now we localise $\mathbf{sSet}_Q$ with respect to $\mathcal{S}$ to get the model category $L^B_{\mathcal{S}}\,\mathbf{sSet}_Q$. Since $|\ | : \mathbf{sSet}_Q \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Top}_Q : \mathrm{Sing}$ is a Quillen equivalence an application of Theorem 5.37 yields the desired result. ■

We introduce some convenient notations for the following subsections.

Definition 10.2 (n-truncations)**.** *Let $\mathcal{S} := \{\partial\Delta^{n+2} \to \Delta^{n+2}\}$.*

1. *We will denote $L^B_{\mathcal{S}}\,\mathbf{sSet}_Q$ by $\mathbf{sSet}^{\leq n}_Q$, and call it* ***the model category of n-truncated simplicial sets****.*
2. *We will denote $L^B_{|Q\mathcal{S}|}\,\mathbf{Top}_Q$ by $\mathbf{Top}^{\leq n}_Q$, and call it* ***the model category of n-truncated topological spaces****.*

10.2 Analysis of the Bousfield Localisation $L_{\mathcal{S}}^{B}\,\mathbf{sSet}_Q$

We will discuss some properties about this category, which we will need in the discussion later on. We will give the localisation functor for this particular Bousfield localisation, which will turn out to be the coskeleton functor applied to the fibrant replacement of some simplicial set.

We start with the definition of the coskeleton functor and some properties about it.

We use material from [ERie14], [WDDK84] and [nLab].

Definition 10.3 (Category of Finite Ordered n-Ordinals)**.** *Denote the full subcategory of $\mathbf{\Delta}$ on the objects $[0], \dots, [n]$ by $\mathbf{\Delta}_{\le n}$ and call it the **category of finite ordered n-ordinals**.*

There is an inclusion functor $\mathbf{j_n} : \mathbf{\Delta}_{\le n} \to \mathbf{\Delta}$ which induces a functor

$$\mathrm{tr}_n : \mathbf{Fun}(\mathbf{\Delta}^{op}, \mathbf{Set}) \to \mathbf{Fun}(\mathbf{\Delta}_{\le n}^{op}, \mathbf{Set})$$

which we call the **truncation functor**. We denote the category $\mathbf{Fun}(\mathbf{\Delta}_{\le n}^{op}, \mathbf{Set})$ by $\mathbf{sSet}_{\le n}$.

Remark 10.4. *Notice the similarity in notation of $\mathbf{sSet}_{\le n}$ and $\mathbf{sSet}_Q^{\le n}$, but mathematically seen there is quite some difference as we will see in the process. Furthermore the index n does not correspond in the two notions. This comes from the fact of the map $\partial\Delta^{n+2} \to \Delta^{n+2}$, to which we localise.* ♦

Proposition 10.5. *The functor tr_n has a left adjoint ${\mathrm{tr}^{\mathrm{L}}}_n$ and a right adjoint ${\mathrm{tr}^{\mathrm{R}}}_n$ i.e.*

$${\mathrm{tr}^{\mathrm{L}}}_n : \mathbf{sSet}_{\le n} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{sSet} : \mathrm{tr}_n : \mathbf{sSet} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{sSet}_{\le n} : {\mathrm{tr}^{\mathrm{R}}}_n \,.$$

Proof. Since $\mathbf{Set}$ is a bicomplete category, we are allowed to apply Corollary A.27. This gives us a right and left adjoint to the truncation functor tr_n. The left adjoint ${\mathrm{tr}^{\mathrm{L}}}_n$ is given by the left Kan extension and ${\mathrm{tr}^{\mathrm{R}}}_n$ is given by the right Kan extension as described in Theorem A.26. ■

With the help of Theorem A.26, we are able to give a description of the left and right adjoint of the truncation functor tr_n. That is, they are defined as in the following diagrams.

$$\begin{tikzcd}
\mathbf{\Delta}_{\le n} \arrow[r, "\mathbf{j_n}"] \arrow[rr, bend left, "\mathbf{i_n}"] \arrow[d, "\mathcal{Y}_n"'] & \mathbf{\Delta} \arrow[r, "\mathcal{Y}"] & \mathbf{sSet} \\
\mathbf{Fun}(\mathbf{\Delta}_{\le n}^{op}, \mathbf{Set}) \arrow[rru, "{\mathrm{tr}^{\mathrm{L}}}_n"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{\Delta}_{\le n} \arrow[r, "\mathcal{Y}_n"] \arrow[d, "\mathbf{j_n}"] \arrow[dd, bend right, "\mathbf{i_n}"'] & \mathbf{Fun}(\mathbf{\Delta}_{\le n}^{op}, \mathbf{Set}) \arrow[ldd, "{\mathrm{tr}^{\mathrm{R}}}_n"] \\
\mathbf{\Delta} \arrow[d, "\mathcal{Y}"] & \\
\mathbf{sSet} &
\end{tikzcd}$$

Hence the functors are of the following form.

$${\mathrm{tr}^{\mathrm{L}}}_n = \int^{C \in \mathbf{\Delta}_{\le n}} \mathbf{i}(C) \times K(C), \qquad {\mathrm{tr}^{\mathrm{R}}}_n = \int_{C \in \mathbf{\Delta}_{\le n}} K(C)^{\mathbf{i}(C)}$$

with $K \in \mathbf{Fun}(\mathbf{\Delta}_{\le n}^{op}, \mathbf{Set})$.

Definition 10.6 (Skeleton and Coskeleton)**.** *We define* $\mathrm{sk}_n := \mathrm{tr^L}_n \circ \mathrm{tr}_n$ *(the composite comonad) and call it the* n***-skeleton functor****, similarly we let* $\mathrm{cosk}_n := \mathrm{tr^R}_n \circ \mathrm{tr}_n$ *(the composite monad) and call it the* n***-coskeleton functor****.*

Corollary 10.7. *There is an adjunction*

$$\mathrm{sk}_n : \mathbf{sSet} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{sSet} : \mathrm{cosk}_n\,.$$

Proof. Immediate from Proposition 10.5. ■

Remark 10.8. *We will give an intrinsic idea, of how the skeleton and coskeleton functor will behave. The description follows [WDDK84] as it gives a nice description.*

For every $X_\bullet \in \mathbf{sSet}$ *and every integer* $n \geq 0$*, the n-skeleton of* $X_\bullet$ *is the smallest subcomplex* $\mathrm{sk}_n(X_\bullet) \subset X_\bullet$ *containing all the n-simplices of* $X_\bullet$ *and the n-coskeleton of* $X_\bullet$ *is the simplicial set* $\mathrm{cosk}_n(X_\bullet)$ *which has as its j-simplices the maps* $\mathrm{sk}_n(\Delta^j) \to X_\bullet \in \mathbf{sSet}$*. The map* $X_\bullet \to \mathrm{cosk}_n(X_\bullet)$ *is a bijection in dimension* $\leq n$ *and hence induces isomorphisms of the homotopy groups in dimension* $< n$*.*

The k-th coskeleton for $X_\bullet \in \mathbf{sSet}$ *may also be given by*

$$\mathrm{cosk}_k(X_\bullet)_n := \mathbf{sSet}(\mathrm{sk}_k(\Delta^n), X_\bullet).$$

♦

Example 10.9. *The skeleton functor does what may be known from algebraic topology. That is to say, if we for an example consider a geometric cube* C*, then* $\mathrm{sk}_0(C) = 8$ *vertices,* $\mathrm{sk}_1(C) = 8$ *vertices and* 12 *edges,* $\mathrm{sk}_2(C) = 8$ *vertices and* 12 *edges and* 6 *faces.*

Another example may be attained from the 2*-sphere* S^2 *constructed from two points and two disks glued along the equator. Then we have* $\mathrm{sk}_0(S^2) = S^0$, $\mathrm{sk}_1(S^2) = S^1$, $\mathrm{sk}_2(S^2) = S^2$*.*

We give somewhat of an example for the coskeleton. We wand to consider $\mathrm{cosk}_2(X_\bullet)$ *and describe the* 3*-simplices. The* 3*-simplices of the* 2*-coskeleton are all four possible triangles in* $X_\bullet$ *that one can arrange in a tetrahedron shape.* ◀

Definition 10.10 (Coskeletal)**.** *Let* $X_\bullet \in \mathbf{sSet}$*.* $X_\bullet$ *is said to be* n***-coskeletal****, if there exists some* $Y_\bullet \in \mathbf{sSet}$ *such that* $X_\bullet \cong \mathrm{cosk}_n(Y_\bullet)$*.*

We will state two lemmata, which will turn out to be very useful.

Lemma 10.11. *Let $X_\bullet \in \mathbf{sSet}$. The following are equivalent.*

1. *$X_\bullet$ is n-coskeletal.*
2. *$X_\bullet \to \operatorname{cosk}_n(X_\bullet)$, the unit of the adjunction, is an isomorphism.*
3. *For $k > n$ and a solid arrow diagram*

$$\begin{tikzcd} \partial\Delta^k \arrow[r] \arrow[d] & X_\bullet \\ \Delta^k \arrow[ur, dotted] & \end{tikzcd}$$

there is a unique dotted arrow making the diagram commute.

Proof. We will first establish the equivalence of 1. and 2. Assume that $X_\bullet \in \mathbf{sSet}$ is n-coskeletal, but then there is a simplicial set $Y_\bullet$ such that $X_\bullet \cong \operatorname{cosk}_n(Y_\bullet)$. cosk_n is idempotent, indeed since

$$\mathbf{sSet}(\Delta^j, \operatorname{cosk}_n(\operatorname{cosk}_n(X_\bullet))) \cong \mathbf{sSet}(\operatorname{sk}_n(\Delta^j), \operatorname{cosk}_n(X_\bullet)) \cong \mathbf{sSet}(\Delta^j, \operatorname{cosk}_n(X_\bullet)).$$

But then we have that $\operatorname{cosk}_n(X_\bullet) \cong \operatorname{cosk}_n(\operatorname{cosk}_n(Y_\bullet)) \cong \operatorname{cosk}_n(Y_\bullet)$ and hence $X_\bullet \cong \operatorname{cosk}_n(X_\bullet)$. The other implication follows from the same argument.

Now we show the implication 2. implies 3. Let $k > n$, since

$$\mathbf{sSet}(\partial\Delta^k, \operatorname{cosk}_n(X_\bullet)) \cong \mathbf{sSet}(\operatorname{sk}_n(\partial\Delta^k), X_\bullet) \cong \mathbf{sSet}(\operatorname{sk}_n(\Delta^k), X_\bullet) \cong (\Delta^k, \operatorname{cosk}_n(X_\bullet)).$$

This gives an isomorphism

$$\phi : \mathbf{sSet}(\partial\Delta^k, \operatorname{cosk}_n(X_\bullet)) \to \mathbf{sSet}(\Delta^k, \operatorname{cosk}_n(X_\bullet)).$$

From this we get the following commutative diagram.

$$\begin{tikzcd} \partial\Delta^k \arrow[r] \arrow[d] & \operatorname{cosk}_n(X_\bullet) \\ \Delta^k \arrow[ur, dotted, "\exists!"'] & \end{tikzcd}$$

which concludes this implication.

The last step is to show, that 3. implies 2. Let $k > n$ and consider the diagram given in 3. For $k \leq n$ we have $\operatorname{sk}_n(\Delta^k) \cong \Delta^k$. For $k > n$, we have the following isomorphisms.

$$\mathbf{sSet}(\Delta^k, \operatorname{cosk}(X_\bullet)) \cong \mathbf{sSet}(\operatorname{sk}_n(\Delta^k), X_\bullet) \cong \mathbf{sSet}(\operatorname{sk}_n(\partial^{k-n}\Delta^k), X_\bullet)$$

$$\cong \mathbf{sSet}(\partial^{k-n}\Delta^k, X_\bullet) \cong \mathbf{sSet}(\Delta^k, X_\bullet).$$

Where we used the adjointness of sk_n and cosk_n, in the last step we assumed the condition given in the diagram in 3. Therefore, we have an isomorphism $X_\bullet \cong \operatorname{cosk}_n(X_\bullet)$ which then implies that $X_\bullet$ is n-coskeletal. Hence all equivalences are established. ■

Lemma 10.12. *The functor* cosk_n *preserves Kan complexes, i.e. preserves fibrant objects in* $\mathbf{sSet}_Q$*.*

Proof. We assume, that $X_\bullet \in \mathbf{sSet}_Q$ is a Kan complex, and will argue, that $\operatorname{cosk}_n(X_\bullet)$ is a Kan complex in that we will show, that it has the desired lifting properties. For this we consider the map

$$\Lambda^m_k \to \Delta^m$$

for different cases of m.

We first assume, that $m > n+1$, then the the morphism $\operatorname{sk}_n(\Lambda^m_k \to \Delta^m)$ is an isomorphism. Therefore, $\operatorname{cosk}_n(X_\bullet)$ has the RLP with respect to $\Lambda^n_k \to \Delta^m$ (here we also used that sk_n and cosk_n are adjoint functors). Interestingly enough, for this case it is not even necessary to assume that $X_\bullet$ is a Kan complex. For the next cases this will be crucial, though.

Next we address the case for $m \leq n$. Here, the map $\operatorname{sk}_n(\Lambda^m_k \to \Delta^m)$ is isomorphic to the map $\Lambda^n_k \to \Delta^m$ itself. Therefore, $\operatorname{cosk}_n(X_\bullet)$ has the RLP with respect to $\Lambda^m_k \to \Delta^m$ iff $X_\bullet$ has the RLP with respect to $\Lambda^n_k \to \Delta^m$. But since we assume, that $X_\bullet$ is a Kan complex, this is clearly satisfied.

The last case is the case, where $m = n+1$. Here we have that $\operatorname{sk}_n(\Lambda^m_k \to \Delta^m)$ is isomoprhic to the map $\Lambda^m_k \to \partial\Delta^m$. Therefore, we need to show that $\operatorname{cosk}_n(X_\bullet)$ has the RLP with respect to $\Lambda^m_k \to \Delta^m$ iff $X_\bullet$ has the RLP with respect to the map $\Lambda^m_k \to \partial\Delta^m$.

Since $X_\bullet$ is a Kan complex by assumption, it certainly has the RLP with respect to $\Lambda^m_k \to \Delta^m$. But we also have the sequence of inclusions

$$\Lambda^m_k \to \partial\Delta^m \to \Delta^m.$$

The lift in the diagram

$$\begin{tikzcd}
\Lambda^m_k \arrow[r] \arrow[d] & X_\bullet \\
\Delta^m \arrow[ur, dotted, "\exists"'] &
\end{tikzcd}$$

can be extended to the following diagram

$$\begin{tikzcd}
\Lambda^m_k \arrow[rr] \arrow[dd] & & X_\bullet \\
& \Delta^m \arrow[ur, dotted] & \\
\partial\Delta^m \arrow[ur] & &
\end{tikzcd}$$

But this means, that $X_\bullet$ has the RLP with respect to $\Lambda^m_k \to \partial\Delta^k$ and therefore (from the above argument), $\operatorname{cosk}_n(X_\bullet)$ musst have the RLP with respect to $\Lambda^m_k \to \Delta^m$.

The combination of all these cases yields, that $\operatorname{cosk}_n(X_\bullet)$ has the RLP with respect to $\Lambda^m_k \to \Delta^m$ iff $X_\bullet$ has the RLP with respect to the same maps for an arbitrary m. Finally this yields that $\operatorname{cosk}_n(X_\bullet)$ is indeed a Kan complex. ■

It is a fact (see for instance [WDDK84]) that if $\operatorname{cosk}_n(X_\bullet)$ is fibrant, then its homotopy groups are all trivial in dimension $\geq n$. Consequently the sequence

$$X_\bullet = \lim_i(\operatorname{cosk}_i(X_\bullet)) \to \cdots \to \operatorname{cosk}_{n+1}(X_\bullet) \to \operatorname{cosk}_n(X_\bullet) \to \dots$$

is, at least up to homotopy, a Postnikov decomposition of $X_\bullet$.

We want to determine a localisation functor for the category $\mathbf{sSet}_Q^{\leq n}$.

The next result is inspired from a discussion in [PHir03].

Proposition 10.13. *Consider the map $f : \partial\Delta^{n+2} \to \Delta^{n+2}$ in $\mathbf{sSet}$. $X_\bullet \in \mathbf{sSet}$ is f-local iff the map $X_\bullet \to *$ has the RLP with respect to every element of the set of augmented f-horns.*

Proof. Starting with a simplicial set $X_\bullet$, we want to construct an f-local space $\widehat{X}$ together with an f-local equivalence $X \to \widehat{X}$. For $\widehat{X}$ to be f-local it has to be fibrant, therefore the map $\widehat{X} \to *$ must have the RLP with respect to the inclusions $\Lambda^n_k \to \Delta^n$ for all $n > 0$ and $0 \leq k \leq n$. If $\widehat{X}$ is fibrant, then $f^* : \operatorname{Map}(\Delta^{n+2}, \widehat{X}) \to \operatorname{Map}(\partial\Delta^{n+2}, \widehat{X})$ is already a fibration of simplicial sets i.e. if $\widehat{X}$ is fibrant, then the assertion that $\widehat{X}$ is f-local is equivalent to the assertion that f^* is an acyclic fibration in $\mathbf{sSet}_Q$. Since a map in $\mathbf{sSet}_Q$ is an acyclic fibration iff it has the RLP with respect to the inclusion $\partial\Delta^n \to \Delta^n$ for $n \geq 0$, this implies that a fibrant space $\widehat{X}$ is f-local iff the dotted arrow exists in every solid arrow diagram of the form

$$\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[d] & \operatorname{Map}(\Delta^{n+2}, \widehat{X}) \arrow[d] \\
\Delta^n \arrow[r] \arrow[ur, dotted] & \operatorname{Map}(\partial\Delta^{n+2}, \widehat{X})
\end{tikzcd}$$

this is true iff (from the isomorphism $\mathbf{sSet}(X_\bullet \times K_\bullet, Y_\bullet) \cong \mathbf{sSet}(K_\bullet, \operatorname{Map}(X_\bullet, Y_\bullet))$) the dotted arrow exists in any solid arrow diagram of the following form

$$\begin{tikzcd}
\partial\Delta^{n+2} \times \Delta^n \coprod_{\partial\Delta^{n+2}\times\partial\Delta^n} \Delta^{n+2} \times \Delta^n \arrow[r] \arrow[d] & \widehat{X} \arrow[d] \\
\Delta^{n+2} \times \Delta^n \arrow[r] \arrow[ur, dotted] & *.
\end{tikzcd}$$

Thus, a space $\widehat{X}$ is f-local iff the map $\widehat{X} \to *$ has the RLP with respect to the maps $\Lambda^n_k \to \Delta^n$ for all $n > 0$ and $0 \leq k \leq n$ and the maps $\partial\Delta^{n+2} \times \coprod_{\partial\Delta^{n+2}\times\partial\Delta^n} \Delta^{n+2} \times \partial\Delta^n \to \Delta^{n+2} \times \Delta^n$ for all $n \geq 0$ ie.e the set of augmented f-horns. ∎

The next two results can be found for a different case in [PHir03].

Proposition 10.14. *Consider the map $f : \partial\Delta^{n+2} \to \Delta^{n+2}$ for $n \geq 0$. A fibrant $X_\bullet \in \mathbf{sSet}_Q$ is f-local iff $\pi_i(X_\bullet) = 0$ for $i > n$ and every choice of basepoint in $X_\bullet$.*

Proof. We prove this in the case of topological spaces, this is fine, since we are dealing with fibrant objects. If $k \geq 0$ then the inclusion $S^{n+1} \times \Delta^k \coprod_{S^{n+1} \times \partial\Delta^k} D^{n+2} \times \partial\Delta^k \to D^{n+2} \times \Delta^k$ is a relative CW complex that attaches a single cell of dimension $n+k+2$. Therefore any map $S^{n+1} \times \Delta^k \coprod_{S^{n+1} \times \partial\Delta^k} D^{n+2} \times \Delta^k \to X$ can be extended over $D^{n+2} \times \Delta^k$ iff $\pi_{n+k+1}(X) = 0$ for every choice of basepoint in X. The result follows from Proposition 10.13. ■

Proposition 10.15. *Consider the map $f : \partial\Delta^{n+2} \to \Delta^{n+2}$ for $n \geq 0$. If a map $g : X_\bullet \to Y_\bullet$, for $X_\bullet, Y_\bullet \in \mathbf{sSet}_Q$ fibrant, induces isomorphisms $g_* : \pi_i(X_\bullet) \cong \pi_i(Y_\bullet)$ for $i \leq n$ and every choice of basepoint in $X_\bullet$, then it is an f-local equivalence.*

Proof. As above, we will prove the statement in the case of topological spaces. If $g : X \to Y$ induces isomorphisms $g_* : \pi_i(X) \to \pi_i(Y)$ for $i \leq n$ we can choose a cofibrant approximation $\tilde{g} : \widetilde{X} \to \widetilde{Y}$ to g such that, $\widetilde{Y}$ is a CW-complex, $\tilde{g}$ is the inclusion of a subcomplex that contains the n-skeleton of $\widetilde{Y}$ and every $(n+1)$-cell of $\widetilde{Y} - \widetilde{X}$ is attached via a constant map of S^n.

If $k = 0$, then the map $\widetilde{X} \times \Delta^k \coprod_{\widetilde{X} \times \partial\Delta^k} \widetilde{Y} \times \partial\Delta^k \to \widetilde{Y} \times \Delta^k$ is just the map $\widetilde{X} \to \widetilde{Y}$, and if $k > 0$ it is the inclusion of a subcomplex that contains the $(n+k)$-skeleton. Thus, if Z is an f-local space, then Proposition 10.14 implies that every map $\widetilde{X} \times \Delta^k \coprod_{\widetilde{X} \times \partial\Delta^k} \widetilde{Y} \times \partial\Delta^k \to Z$ can be extended over $\widetilde{Y} \times \Delta^k$ and so g is an f-local equivalence (see for instance Proposition 9.3.10 in [PHir03]). ■

Finally we have the following result.

Theorem 10.16. *Let $\mathcal{S} = \{\partial\Delta^{n+2} \to \Delta^{n+2}\}$ and $X_\bullet \in \mathbf{sSet}_Q$. $f : X_\bullet \to \operatorname{cosk}_{n+1}(\widetilde{X_\bullet})$ is an $\mathcal{S}$-localisation map i.e. $\operatorname{cosk}_{n+1}(\widetilde{X_\bullet})$ is $\mathcal{S}$-local and f is an $\mathcal{S}$-local equivalence.*

Proof. First we notice, that we are able to take the fibrant replacement for some $X_\bullet \in \mathbf{sSet}_Q$. The result then follows from Proposition 10.14 and Proposition 10.15. The first result yields, that the n-th Postnikov approximation has the desired properties, but since the coskeleton functor, in this context, is up to homotopy itself a Postnikov approximation we are done. ■

In other words, this theorem says, that in the Bousfield localisation $\mathbf{sSet}_Q^{\leq n}$ of $\mathbf{sSet}_Q$, every object $X_\bullet \in \mathbf{sSet}_Q^{\leq n}$ is weakly equivalent to $\operatorname{cosk}_{n+1}(X_\bullet)$ i.e. $X_\bullet \sim \operatorname{cosk}_{n+1}(X_\bullet)$. This will turn out to be a key fact for the proof of the desired Quillen equivalences later on.

An immediate consequence of the above discussion is now the following Proposition.

Proposition 10.17. *The weak equivalences in $\mathbf{sSet}_Q^{\leq n}$ are precisely the maps which induce isomorphisms on π_k for all $0 \leq k \leq n$, for any basepoint.*

Proof. Let $\mathcal{S} = \{f : \partial\Delta^{n+2} \to \Delta^{n+2}\}$ and $g : X_\bullet \to Y_\bullet$ in $\mathbf{sSet}_Q$ be an $\mathcal{S}$-local equivalence, that is to say, a weak equivalence in $\mathbf{sSet}_Q^{\leq n}$. But then the induced map via the coskeleton functor $\operatorname{cosk}_{n+1}(g) : \operatorname{cosk}_{n+1}(\widetilde{X_\bullet}) \to \operatorname{cosk}_{n+1}(\widetilde{Y_\bullet})$ is a weak equivalence. Indeed, consider the diagram

$$\begin{tikzcd} X_\bullet \arrow[r, "\sim"] \arrow[d, "\sim"'] & Y_\bullet \arrow[d, "\sim"] \\ \operatorname{cosk}_{n+1}(X_\bullet^f) \arrow[r] & \operatorname{cosk}_{n+1}(Y_\bullet^f) \end{tikzcd}$$

the vertical maps are weak equivalences by Theorem 10.16 and the horizontal top map is a weak equivalence by assumption. We conclude by the 2-3 property for weak equivalences, that the horizontal bottom map is a weak equivalence (see also Theorem 1.2.29 in [PHir03]).

Then for any $0 \leq k \leq n$ and any choice of basepoint of $X_\bullet$ we get a diagram

$$\begin{tikzcd} \pi_k(X_\bullet) \arrow[r] \arrow[d, "\cong"'] & \pi_k(Y_\bullet) \arrow[d, "\cong"] \\ \pi_k(\operatorname{cosk}_{n+1}(\widetilde{X_\bullet})) \arrow[r, "\cong"'] & \pi_k(\operatorname{cosk}_{n+1}(\widetilde{Y_\bullet})). \end{tikzcd}$$

The horizontal maps are isomorphisms from the definiton of the coskeleton functor and the fact that we consider the case $k < n+1$. ■

10.2.1 A Case Analysis for Topological Spaces

This section is more like an alternative viewpoint of the above discussion. It is more for the understanding than anything else and can be skipped without regret.

Since $\mathbf{sSet}_Q$ is a simplicial model category, we have that

$$\mathbf{sSet}(X \times Y, Z) \cong \mathbf{sSet}(X, [Y, Z])$$

for $[X, Y]_n = \mathbf{sSet}(X \times \Delta_n, Y)$ in other words, there is an adjunction

$$X \times - : \mathbf{sSet} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{sSet} : [X, -].$$

Similarly, since $\mathbf{Top}_Q$ is a simplicial category, we have

$$\mathbf{Top}(X \times Y, Z) \cong \mathbf{Top}(X, \operatorname{Map}(Y, Z)).$$

Let X, Y be topological spaces, then from the adjunction $\operatorname{Sing}(X) \times - : \mathbf{sSet} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{sSet} : [\operatorname{Sing}(X), -]$ in combination with the adjunction $|\ | : \mathbf{sSet}_Q \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Top}_Q : \operatorname{Sing}$ we get an adjunction

$$X \times - : \mathbf{Top} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Top} : \operatorname{Map}(X, -).$$

Furthermore we have

$$|\operatorname{Sing}(X) \times \operatorname{Sing}(Y)| \cong |\operatorname{Sing}(X)| \times |\operatorname{Sing}(Y)| \xrightarrow{\sim} X \times Y$$

and therefore

$$\operatorname{Map}(X, Y) \xrightarrow{\sim} |[\operatorname{Sing}(X), \operatorname{Sing}(Y)]|$$

or equivalently, $\operatorname{Sing}(\operatorname{Map}(X, Y)) \xrightarrow{\sim} [\operatorname{Sing}(X), \operatorname{Sing}(Y)]$.

Let $\mathscr{C}$ be a simplicial model category. For X in $\mathscr{C}$ cofibrant it follows with Proposition 5.18 that the cosimplicial resolution of X is $\widetilde{X}^n := X \otimes \Delta^n$.

Now we consider $\mathscr{C} = \mathbf{sSet}$, we know that every object in $\mathbf{sSet}$ is cofibrant, hence the cosimplicial resolution of such an object X looks like $\widetilde{X}^n = X \times \Delta^n$. But then we have

$$\mathbb{L}\,\mathbf{sSet}(X,Y)_n = \mathbf{sSet}(\widetilde{X}^n, \widehat{Y}) = \mathbf{sSet}(X \times \Delta^n, \widehat{Y}) = [X, \widehat{Y}]_n$$

i.e. $\mathbb{L}\,\mathbf{sSet}(X,Y) = [X, \widehat{Y}]$.

From the Quillen pair $|\;| : \mathbf{sSet}_Q \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Top}_Q : \mathrm{Sing}$, we have

$$|\widetilde{X}|^n = |X \times \Delta^n| \cong |X| \times |\Delta^n|$$

also, in $\mathbf{Top}$ we have that $\widehat{Y} = Y$ since every object is fibrant. Therefore

$$\mathbb{L}\,\mathbf{Top}(|X|, Y)_n = \mathbf{Top}(|\widetilde{X}|^n, Y) = \mathbf{Top}(|X \times \Delta^n|, Y) \cong \mathbf{sSet}(X \times \Delta^n, \mathrm{Sing}(Y))$$

i.e. $\mathbb{L}\,\mathbf{Top}(|X|, Y) = [X, \mathrm{Sing}(Y)]$, hence

$$|\mathbb{L}\,\mathbf{Top}(|X|, Y)| = |[X, \mathrm{Sing}(Y)]| \xrightarrow{\sim} \mathrm{Map}(|X|, Y).$$

A very similar argument, shows a similar statement for the right homotopy function complex instead of the left homotopy function complex. One can also show that it does hold for the two-sided homotopy function complex.

This aside we will now show, that if we Bousfield localise $\mathbf{Top}_Q$ with respect to a specific map i.e. $|\partial\Delta^{n+2} \to \Delta^{n+2}|$, this map will kill off higher homotopy groups.

We define $\mathcal{S} := \{\partial D^{n+2} \to D^{n+2}\}$, further let Y be a topological space which in addition is $\mathcal{S}$-local and fibrant. Then by definition the induced map of $\partial D^{n+2} \to D^{n+2}$ of homotopy function complexes

$$\mathrm{map}(D^{n+2}, Y) \to \mathrm{map}(\partial D^{n+2}, Y)$$

is a weak equivalence in $\mathbf{sSet}_Q$, therefore we also have a weak equivalence

$$|\,\mathrm{map}(D^{n+2}, Y)| \to |\,\mathrm{map}(\partial D^{n+2}, Y)|$$

in $\mathbf{Top}_Q$, which follows from the Quillen equivalence between $\mathbf{sSet}_Q$ and $\mathbf{Top}_Q$.

Since $D^{n+2} \to *$ is a weak equivalence in $\mathbf{Top}_Q$ (since $D^{n+2} \sim_h *$) we have by Theorem 5.26 that

$$\mathrm{map}(*, Y) \to \mathrm{map}(D^{n+2}, Y)$$

is a weak equivalence in $\mathbf{sSet}_Q$ and with the same argument as before we therefore get a weak equivalence

$$|\,\mathrm{map}(*, Y)| \to |\,\mathrm{map}(D^{n+2}, Y)|$$

in $\mathbf{Top}_Q$.

We remember that $\mathrm{Sing}(X)_n = \mathbf{Top}(|\Delta^n|, X)$ and therefore (with the observations about inner Homs we did before)

$$\mathrm{map}(*, Y)_n = \mathbf{Top}(*, \mathrm{Map}(|\Delta^n|, Y)) \xrightarrow{\sim} \mathbf{Top}(|\Delta^n|, Y) = \mathrm{Sing}(Y)_n.$$

But then it follows
$$|\operatorname{map}(D^{n+2}, Y)| \xrightarrow{\sim} |\operatorname{Sing}(Y)| \xrightarrow{\sim} Y.$$
In a similar fashion we also deduce that $|\operatorname{map}(\partial D^{n+2}, Y)| \xrightarrow{\sim} \operatorname{Map}(\partial D^{n+2}, Y)$ in $\mathbf{Top}_Q$, indeed since Y is fibrant and $\mathcal{S}$-local in $\mathbf{sSet}_Q^{\leq n}$ we have that $\partial D^{n+2} \to D^{n+2}$ is a weak equivalence. Then by Theorem 5.26 we have a map $\operatorname{map}(D^{n+2}, Y) \xrightarrow{\sim} \operatorname{map}(\partial D^{n+2}, Y)$ and hence we get a weak equivalence
$$Y \xrightarrow{\sim} \operatorname{Map}(\partial D^{n+2}, Y)$$
in $\mathbf{Top}_Q$. We have a diagram

$$\begin{tikzcd} \Omega^{n+1}(Y) \arrow[r] & \operatorname{Map}(\partial D^{n+2}, Y) \arrow[d] \\ & Y. \end{tikzcd}$$

This gives rise to a long exact sequence in homotopy
$$\cdots \to \pi_k(\Omega^{n+1}(Y)) \to \pi_k(\operatorname{Map}(\partial D^{n+2}, Y)) \to \pi_k(Y) \to \dots$$
$$\cdots \to \pi_0(\Omega^{n+1}(Y)) \to \pi_0(\operatorname{Map}(\partial D^{n+2}, Y)) \to \pi_0(Y) \to 0.$$
Since we have a section, this long exact sequence splits, i.e.
$$\pi_k(\operatorname{Map}(\partial D^{n+2}, Y)) \cong \pi_k(\Omega^{n+1}(Y)) \oplus \pi_k(Y), \quad k \geq 2.$$
A case analysis for $k = 0, 1$ also yields desired isomorphisms for these cases respectively. Therefore we have that
$$\pi_k(\Omega^{n+1}(Y)) \cong 0 \quad \forall\ k \geq 0.$$
Since $\pi_k(\Omega^{n+1}(Y)) \cong \pi_{k+n+1}(Y)$ we have that $\pi_{k+n+1}(Y)$ has to be 0 for all $k \geq 0$ i.e. it is 0 iff $k \geq n+1$, which is precisely what we wanted to achieve.

As explained before, with the Bousfield localisation with respect to this specific map we are able to kill off homotopy in dimension greater than n.

10.3 The Quillen Equivalences

This subsection is really the core of the thesis. It is here, where we provide the promised Quillen equivalences using all the work we have done so far in the whole present paper. At the end we will have a bunch of Quillen equivalences, growing in difficulty to prove.

The first case will be the one of 0-truncations, where we will provide a Quillen equivalence between the model category $\mathbf{sSet}_Q^{\leq 0}$ and $\mathbf{Set}_T$.

The second case is the one of 1-truncations, where we give a Quillen equivalence between the model category $\mathbf{sSet}_Q^{\leq 1}$ and $\mathbf{Grpd}_{\mathcal{M}_1}$.

Finally the last case provided here discusses a Quillen equivalence between the model category $\mathbf{sSet}_Q^{\leq 2}$ and $\mathbf{bi\text{-}Grpd}_{s\mathcal{M}_2}$ and an equivalence of categories between the homotopy category $\operatorname{Ho}(\mathbf{sSet}_Q^{\leq 2})$ and the homotopy theory $\mathbf{bi\text{-}Grpd}[\mathcal{W}^{-1}]$, which can unfortunately not be done as elegantly as in the previous cases i.e. with a Quillen equivalence.

10.3.1 0-Truncations

As kind of a warm up exercise we discuss the Quillen equivalence between the model category of 0-truncated simplicial sets $\mathbf{sSet}_Q^{\leq 0}$ and the category $\mathbf{Set}_T$, which is the category $\mathbf{Set}$ with the trivial model category structure. Remember that the trivial model category for a category is the model structure in which the weak equivalences are the isomorphisms of the category and the fibrations and cofibrations consist of any morphism of the category.

We would first like to establish an adjunction

$$\begin{tikzcd} \mathbf{sSet} \arrow[r, shift left, "X_\bullet \mapsto \mathrm{coeq}(X_1 \rightrightarrows X_0)"] \arrow[r, phantom, "\perp"] & \mathbf{Set} \arrow[l, shift left, "\text{const.}"] \end{tikzcd},$$

where const. is the constant simplicial functor. To do so, we state some useful lemmata.

Lemma 10.18. *The coequaliser of the pair $X_1 \rightrightarrows X_0$ for a simplicial set $X_\bullet$ corresponds to $\pi_0(X_\bullet)$ the set of connected components of $X_\bullet$.*

Proof. First we notice that

$$\mathrm{coeq}\left(\begin{tikzcd} X_1 \arrow[r, bend left, "i_0"] \arrow[r, bend right, "i_1"'] & X_0 \end{tikzcd} \right) = X_0/_\sim$$

with $i_0 \sim i_1$ by definition (since we are in $\mathbf{Set}$).

Define a map

$$\begin{aligned} X_0/_\sim &\to \pi_0(|X_\bullet|) = \pi_0(X_\bullet) \\ [v] &\mapsto \alpha \end{aligned}$$

where α is a path connected component of $|X_\bullet|$ such that $v \in \alpha$.

We will show that this map is an isomorphism. First we argue that it is indeed well-defined. Let $[v] = [v']$ then there exists some $e \in X_1$ such that $i_0(e) = v$ and $i_1(e) = v'$. But then there is a path $\gamma : v \to v'$ and hence $v, v' \in \alpha$.

Next we argue injectivity. Assume that $\alpha = \beta$ for α, β some path connected components of $X_\bullet$. But then there is some path connecting α and β call it ξ. Such a path is of the form $|\xi| : |\Delta^1| \to |X_\bullet|$ i.e. $\xi : \Delta^1 \to X_\bullet$ which means that $\xi \in \mathbf{sSet}(\Delta^1, X_\bullet) = X_1$ (notice that this is not true in general but we only need this up to homotopy and in that case it holds).

Finally we show surjectivity. Every connected component must have a pre-image, since if not, they must have had the same identification in the coequaliser (by definition). ■

We can now show the adjunction. From now on we denote $\pi_0(X_\bullet)$ by $\Pi_0(X_\bullet)$ and const. by $\mathcal{N}_0$. These notations are only for aesthetical reasons, such that it fits better for the other cases. The functor const. should remind us of something like a trivial nerve functor (also called nerve of a discrete category) hence the notation.

Lemma 10.19. *There is an adjunction*

$$\Pi_0 : \mathbf{sSet} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Set} : \mathcal{N}_0 .$$

Proof. We want to show, that there is an isomorphism

$$\mathbf{Set}(\Pi_0(X), Y) \cong \mathbf{sSet}(X, \mathcal{N}_0(Y)).$$

Choose some morphism $f : \Pi_0(X) \to Y \in \mathbf{Set}(\Pi_0(X), Y)$, this morphism gets send to

$$(f : \Pi_0(X) \to Y) \mapsto \begin{cases} f_0 : X_0 \to \Pi_0(X) \xrightarrow{f} Y \quad \text{i.e.} \quad f_0 : X_0 \to Y \\ f_1 : X_1 \xrightarrow{i_0} X_0 \to \Pi_0(X) \xrightarrow{f} Y \quad \text{i.e.} \quad f_1 : X_1 \to Y \\ \vdots \\ f_n : X_n \to Y \end{cases}$$

the map consists of such a collection as above. Also notice that there is no choice involved here, i.e. it does not matter which map we choose (i_0 or i_1 etc..) since they will all be equalised.

To conclude we construct an inverse map for the above map. Let $f : X \to \mathcal{N}_0(Y)$ be a morphism in $\mathbf{sSet}(X, \mathcal{N}_0(Y))$ and we construct

$$(f : X \to \mathcal{N}_0(Y)) \mapsto \begin{tikzcd} f_1 : X_1 \arrow[r] \arrow[d, bend right] \arrow[d, bend left] & Y \arrow[d, "id"] \\ f_0 : X_0 \arrow[r] \arrow[d] & Y \arrow[d] \\ \Pi_0(X) \arrow[r] & Y = \Pi_0(\mathcal{N}_0(Y)) \end{tikzcd}$$

and everything commutes on the right hand side. This now establishes our isomomrphism, showing that there indeed is an adjunction. ■

Lemma 10.20. *There is a Quillen pair*

$$\Pi_0 : \mathbf{sSet}_Q \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Set}_T : \mathcal{N}_0 .$$

Proof. We will show, that $\Pi_0 : \mathbf{sSet}_Q \to \mathbf{Set}_T$ preserves cofibrations and acyclic cofibrations. Since any map in $\mathbf{Set}_T$ is a fibration and a cofibration (by definition of the trivial model structure), we are left to show that it preserves weak equivalences.

Let $f : X \to Y$ be a weak equivalence in $\mathbf{sSet}_Q$, but then we must have

$$\pi_n(|X|) \cong \pi_n(|Y|)$$

(by definition of weak equivalence in $\mathbf{Top}_Q$). But then we have as a special case, that $\pi_0(X) \cong \pi_0(Y)$ as isomorphism of sets, which is precisely a weak equivalence in $\mathbf{Set}_T$. This concludes the proof. ■

For the final step it is now crucial to work with 0-truncations.

Theorem 10.21. *There is a Quillen equivalence*

$$\Pi_0 : \mathbf{sSet}_Q^{\leq 0} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Set}_T : \mathcal{N}_0 \,.$$

Proof. From the above discussion, we have that $Y = \Pi_0(\mathcal{N}_0(Y))$, for any $Y \in \mathbf{Set}_T$. We are left to show, that $\mathcal{N}_0(\Pi_0(X_\bullet)) \xrightarrow{\sim} X_\bullet$ is a weak equivalence. First we notice, that $\pi_i(X_\bullet) = 0$ for all $i > 0$, since $X_\bullet \in \mathbf{sSet}_Q^{\leq 0}$. To be very precise it should actually say, that since every object $X_\bullet \in \mathbf{sSet}_Q^{\leq 0}$ is weakly equivalent to $\operatorname{cosk}_1(X_\bullet)$ this implies an isomorphism on homotopy groups and hence the above follows (see for instance Theorem 10.16 and Proposition 10.17).

We conclude by showing, that $|\mathcal{N}_0(\Pi_0(X_\bullet))| \to |X_\bullet|$ is a weak equivalence in $\mathbf{Top}_Q^{\leq 0}$. $\pi_i(|\mathcal{N}_0(\Pi_0(X_\bullet))|) = *$ for all $i > 0$ and hence we get that $\mathcal{N}_0(\Pi_0(X_\bullet)) \in \mathbf{sSet}_Q^{\leq 0}$. Furthermore

$$\Pi_0(|\mathcal{N}_0(\Pi_0(X_\bullet))|) = \Pi_0(X_\bullet) = \Pi_0(|X_\bullet|)$$

and so $\pi_i(X_\bullet) = *$ for all $i > 0$. This implies, that we indeed have a weak equivalence in $\mathbf{Top}_Q^{\leq 0}$ and hence a weak equivalence in $\mathbf{sSet}_Q^{\leq 0}$. This now establishes the Quillen equivalence we claimed. ■

10.3.2 1-Truncations

In this section, we aim to prove that there exists a Quillen equivalence between 1-truncated simplicial sets and groupoids. First we will introduce some basic definitions in order to set up the machinery we need. We will try to use the definitions, which we stated in the section about the general geometric nerve and general geometric realisation.

This section uses material from [GoJa09], [JLur09], [ShHo07], [NStr00], [DeCi] and [nLab].

Basic Definitions

Definition 10.22. *We define the category* **Cat** *to consist of all small categories and morphisms given by functors between such categories.*

Definition 10.23 (Groupoid). *A category $\mathscr{C} \in$* **Cat** *is called a* ***groupoid****, if every morphism in $\mathscr{C}$ is invertible. The category of all such categories is denoted* **Grpd***.*

Theorem 10.24. Cat *and* **Grpd** *are bicomplete.*

Proof. Let $\{\mathscr{C}_i\}_{i \in I}$ be a small diagram of categories. We define the limit $\mathscr{C}$ as the following category.

$$\operatorname{ob}(\mathscr{C}) := \lim_i \operatorname{ob}(\mathscr{C}_i)$$
$$\operatorname{mor}(\mathscr{C}) := \lim_i \operatorname{mor}(\mathscr{C}_i)$$

with the induced source, target and identity maps induced by the ones of the $\mathscr{C}_i$ and the functoriality of the limit. That is, composition is defined componentwise. Therefore $\mathscr{C}$ is a category and $\mathscr{C} \to \mathscr{C}_i$ satisfies the universal property of the limit and hence we have completeness.

Cocompleteness will be more involved. Consider the functor

$$\begin{aligned} \mathbf{Cat} &\to \mathbf{Set} \\ \mathscr{C} &\mapsto \lim_i \mathbf{Cat}(\mathscr{C}_i, \mathscr{C}). \end{aligned}$$

We want to show, that this functor is representable using Freyd's representability theorem (Theorem A.24). As seen above, **Cat** is complete and the functor is clearly continuous (since defined as a hom-set), therefore we are left to show the solution set condition. Consider the cardinal

$$\kappa = \aleph_0 \cdot \sum_{i \in I} \mathrm{card}(\mathrm{mor}(\mathscr{C}_i)).$$

Let S be the set of all categories whose object and morphism sets are subsets of κ. Therefore any category $\mathscr{C}$ with $\mathrm{card}(\mathrm{mor}(\mathscr{C})) \leq \kappa$ is isomorphic to some category in S.

Let $\{F_i : \mathscr{C}_i \to \mathscr{C}\}$ be a compatible family of functors. We define a subcategory $\mathscr{C}' \subseteq \mathscr{C}$ in the following way. The objects of $\mathscr{C}'$ are $F_i(X)$ with X object of $\mathscr{C}_i$ for some $i \in I$. A morphism in $\mathscr{C}'$ is a morphism in $\mathscr{C}$ which can be factored as $Y_0 \to Y_1 \to \cdots \to Y_n$, where each $Y_j \to Y_{j+1}$ lies in the image of some F_i. The case $n = 0$ will correspond to the identity morphism. Therefore $\mathscr{C}'$ is a subcategory of $\mathscr{C}$.

But then, $\mathrm{card}(\mathrm{mor}(\mathscr{C}')) \leq \sum_{n \in \mathbb{N}} \kappa^n = \kappa$. Hence $\mathscr{C}'$ is isomorphic to some object in S and we are done.

This last part will construct the cocones needed for the colimits, and the colimit is given by the representability i.e. from the isomorphism $\lim_i \mathbf{Cat}(\mathscr{C}_i, \mathscr{C}) \cong \mathbf{Cat}(\mathscr{D}, \mathscr{C})$. The proof for **Grpd** is similar. ∎

This proof is particularly nice, since it avoids the tiresome construction of coequalisers. It is inspired by a discussion from Stackexchange [2].

If we remember Definition 6.2 and Definition 6.4 about the general nerve and general realisation construction, we may adapt these for this specific case.

There is a natural choice of inclusion functor $\mathbf{i} : \mathbf{\Delta} \to \mathbf{Cat}$, sending objects to objects and adding composition of morphisms in **Cat** for strings of morphisms in $\mathbf{\Delta}$ (i.e. objects of $\mathbf{\Delta}$ are posets and posets are categories). Graphically one should have the following picture in mind.

$$\begin{tikzcd}
\mathbf{\Delta} \arrow[d] & 0 \arrow[r] & 1 \arrow[r] & 2 \arrow[r] & \cdots \arrow[r] & n \\
\mathbf{Cat} & 0 \arrow[r] \arrow[rr, bend left] \arrow[rrrr, bend left] & 1 \arrow[r] \arrow[rrr, bend right] & 2 \arrow[r] \arrow[rr, bend right] & \cdots \arrow[r] & n
\end{tikzcd}$$

Therefore, we define the **geometric** 1**-nerve** to be the functor $\mathcal{N}_1 : \mathbf{Cat} \to \mathbf{Fun}(\mathbf{\Delta}^{op}, \mathbf{Set}) = \mathbf{sSet}$, induced by the inclusion $\mathbf{i} : \mathbf{\Delta} \to \mathbf{Cat}$.

From this, define the **geometric** 1**-realisation** Π_1, to be the left Kan extension of the diagram

$$\begin{tikzcd} \mathbf{\Delta} \arrow[r, "\mathbf{i}"] \arrow[d, "y"'] & \mathbf{Cat}\,. \\ \mathbf{sSet} \arrow[ur, dotted, "\Pi_1"'] & \end{tikzcd}$$

Since $\mathbf{Cat}$ has all limits and colimits, i.e. is a bicomplete category, the left Kan extensions for the definition of the geometric 1-realisation always exists (actually only cocompleteness is needed here). This gives us the following adjunction.

Lemma 10.25. *There is an adjunction*

$$\Pi_1 : \mathbf{sSet} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Cat} : \mathcal{N}_1\,.$$

Proof. Immediate from Theorem 6.6. ∎

Notice, that one may obtain a groupoid from some category $\mathscr{C} \in \mathbf{Cat}$ by localising with respect to the class of morphisms $\mathrm{mor}(\mathscr{C})$ in $\mathscr{C}$. See for instance the section about the localisation of a category for more details. Hence there is a canonical functor $\mathrm{Loc} : \mathbf{Cat} \to \mathbf{Grpd}$. There is also the forgetful functor $U : \mathbf{Grpd} \to \mathbf{Cat}$. Not surprisingly, they form an adjunction.

Lemma 10.26. *There is an adjunction*

$$\mathrm{Loc} : \mathbf{Cat} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Grpd} : U.$$

Proof. We have to show that $\mathbf{Grpd}(\mathrm{Loc}(\mathscr{C}), \mathscr{G}) \cong \mathbf{Cat}(\mathscr{C}, U(\mathscr{G}))$. Consider a functor $\mathscr{C} \to U(\mathscr{G})$. Then by the universal property of the localisation and since every morphism of $\mathscr{C}$ is mapped to an isomorphism in $U(\mathscr{G})$ (hence also in $\mathscr{G}$) we have a unique map

$$\begin{tikzcd} \mathscr{C} \arrow[r] \arrow[d] & U(\mathscr{G}) \\ \mathrm{Loc}(\mathscr{C}) \arrow[ur, "\exists!"'] & \end{tikzcd}$$

and therefore also a unique map $\mathrm{Loc}(\mathscr{C}) \to \mathscr{G}$ which we claimed. ∎

Since composition of adjunctions gives again an adjunction, we have the following chain of adjunctions

$$\mathbf{sSet} \underset{\underset{\mathcal{N}_1}{\longleftarrow}}{\overset{\overset{\Pi_1}{\longrightarrow}}{\perp}} \mathbf{Cat} \underset{\underset{U}{\longleftarrow}}{\overset{\overset{\mathrm{Loc}}{\longrightarrow}}{\perp}} \mathbf{Grpd}\,.$$

By abuse of notation and more for aesthetic reasons, we will from now on denote with Π_1 the composition $\mathrm{Loc} \circ \Pi_1$ and similarly by $\mathcal{N}_1$ we mean $U \circ \mathcal{N}_1$ (which actually is $\mathcal{N}_1$ itself). Therefore, we have the following result.

Corollary 10.27. *There is an adjunction*

$$\Pi_1 : \mathbf{sSet} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Grpd} : \mathcal{N}_1\,.$$

Properties of the 1-Nerve

It actually turns out, that the 1-nerve is a very nice functor with good properties.

Remark 10.28. *Let $\mathscr{C}$ be a category, the geometric nerve $\mathcal{N}_1(\mathscr{C})$ is the following simplicial set.*

1. *Its vertices are the objects of $\mathscr{C}$.*
2. *1-simplices are the arrows $C_0 \xrightarrow{f} C_1$ in $\mathscr{C}$, with faces $d_0(f) = C_1$ and $d_1(f) = C_0$.*

$$0 \xrightarrow[f]{} 1$$

3. *2-simplices are the diagrams $\Delta = (g, h, f; \alpha)$ of the form*

$$\begin{tikzcd} & 1 \arrow[dr, "g"] & \\ 0 \arrow[ur, "f"] \arrow[rr, "h"'] \arrow[rr, phantom, bend left=20, "\alpha"] & & 2 \end{tikzcd} \qquad \alpha : g \circ f \ \Rightarrow \ h$$

   with α a 2-cell in $\mathscr{C}$, and faces which are the 1-simplices opposite to the indicated vertex, that is $d_0(\Delta) = g, d_1(\Delta) = h$ and $d_2(\Delta) = f$.
4. *At higher dimensions $\mathcal{N}_1(\mathscr{C})$ is 2-coskeletal.*

♦

Remark 10.29. *Let $F : \mathscr{C} \to \mathscr{D}$ be a functor between categories. Then $\mathcal{N}_1(F) : \mathcal{N}_1(\mathscr{C}) \to \mathcal{N}_1(\mathscr{D})$ is the simplicial map given by:*

1. *for 0-simplices $C \in \mathcal{N}_1(\mathscr{C})_0$, $\mathcal{N}_1(F)_0(C) = F(C)$*
2. *for 1-simplices $f \in \mathcal{N}_1(\mathscr{C})_1$, $\mathcal{N}_1(F)_1(f) = F(f)$*
3. *At higher dimensions $\mathcal{N}_1(F)$ is defined in the unique possible way, using the fact that in dimension 2 and above any simplex is determined by its faces (1-simplices).*

♦

Proposition 10.30. *Let $X_\bullet \in$ **sSet** and $\mathscr{G} \in$ **Grpd**. Then $\mathcal{N}_1(\mathscr{G}) \cong X_\bullet$ iff $X_\bullet$ is a Kan complex with unique horn fillers for $n > 1$ i.e. $X_n \to \mathbf{sSet}(\Lambda^n_i, X_\bullet)$ are bijective for all $n > 1$.*

Proof. Since $X_\bullet$ is a Kan complex, there exists a lift

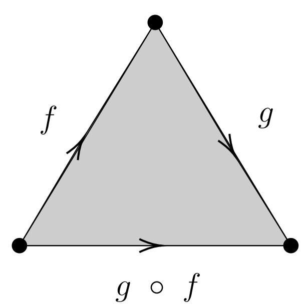
$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & X_\bullet \arrow[d] \\ \Lambda^n \arrow[r] \arrow[ur, dotted] & *. \end{tikzcd}$$

Define a groupoid $\mathscr{G}$ in the following way:

- $\mathrm{ob}(\mathscr{G}) = X_0$.
- For any $X, Y \in \mathscr{G}$, $\mathscr{G}(X,Y) = \{e : \Delta^1 \to X_\bullet \mid e|_{\{0\}} = X, e|_{\{1\}} = Y\}$.
- (Identities) For $X \in \mathscr{G}$ we have $id_X : \Delta^1 \to \Delta^0 \to X_\bullet$, which maps to X. Since Δ^0 is terminal, $\Delta^1 \to \Delta^0$ is unique.
- (Composition) Let f, g be composable morphisms in $\mathscr{G}$ i.e. $X \xrightarrow{f} Y \xrightarrow{g} Z \in \mathscr{C}$. Then $f : \Delta^1 \to X_\bullet$ and $g : \Delta^1 \to X_\bullet$, composition is defined (uniquely defined since we will be dealing with groupoids) according to

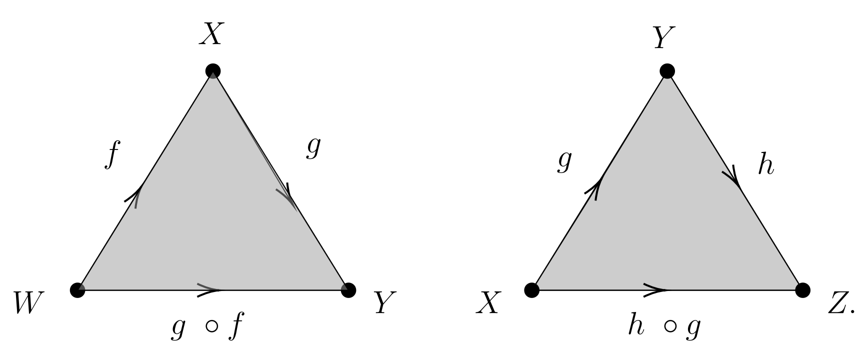
$$\begin{tikzcd}
\Lambda^2_1 \arrow[r] \arrow[d] & X_\bullet \\
\Delta^2 \arrow[ur, "\text{"}g\circ f\text{"}"'] &
\end{tikzcd}$$

  i.e. this corresponds to

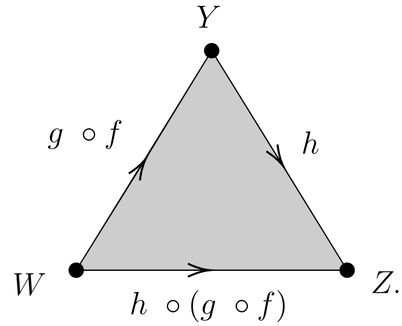


This indeed gives a category. Indeed, for any $Y \in \mathscr{G}$, the identity id_Y is a unit with respect to composition i.e. for any morphism $f : X \to Y \in \mathscr{G}$ and any morphism $g : Y \to Z \in \mathscr{G}$, we have $id_Y \circ f = f$ and $g \circ id_Y = g$. This equations are characterised by the 2-simplices $s_1(f), s_0(g) \in \mathbf{sSet}(\Delta^2, X_\bullet)$.

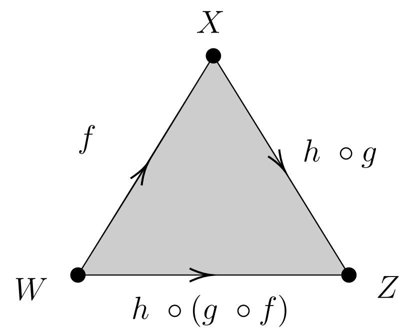
Composition is associative. Indeed, for any sequence of composable morphisms $W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$, we have $h \circ (g \circ f) = (h \circ g) \circ f$. To show this, consider the following 2-simplices



Next, we choose a 2-simplex of the form



These three 2-simplices define a map $\tau_0 : \Lambda^3_2 \to X_\bullet$. Since $X_\bullet$ is a Kan complex, we can extend τ_0 to a 3-simplex $\tau : \Delta^3 \to X_\bullet$. The composition $\Delta^2 \cong \Delta^{\{0,1,3\}} \subseteq \Delta^3 \xrightarrow{\tau} X_\bullet$, which corresponds to



This then gives the associativity axiom $h \circ (g \circ f) = (h \circ g) \circ f$.

We now argue, that we end up with a groupoid. Let $f \in \mathrm{mor}(\mathscr{G})$, $f : X \to Y$, $X, Y \in X_\bullet$. Consider the following map $\Lambda^2_0 \to X_\bullet$, where

$$\begin{tikzcd} 0 \arrow[r, "f"] \arrow[d, "id_X"'] & 1. \arrow[dl, dotted] \\ 2 & \end{tikzcd}$$

By the lifting property (since $X_\bullet$ is a Kan complex), there exists a map $\Delta^2 \xrightarrow{\sigma} X_\bullet$ such that f^{-1} is defined as $f^{-1} : \Delta^1 \cong \Delta^{\{0,1\}} \subseteq \Delta^2 \xrightarrow{\sigma} X_\bullet$. Hence $f \circ f^{-1} = id_X$. Similarly one may argue, that $f^{-1} \circ f = id_X$ i.e. every morphism in $\mathscr{G}$ is invertible.

We are left to show, that $\mathcal{N}_1(\mathscr{G}) \cong X_\bullet$. There is a canonical map $\Phi : X_\bullet \to \mathcal{N}_1(\mathscr{G})$ (defined levelwise and by construction of $\mathscr{G}$). We want that Φ is a bijection i.e. we need to show that $\Phi_n : X_n \to \mathcal{N}_1(\mathscr{G})_n$ is bijective for any $n \geq 0$. For $n = 0, 1$ this is by construction. Let $n \geq 2$. Consider the following diagram

$$\begin{tikzcd} X_n \cong \mathbf{sSet}(\Delta^n, X_\bullet) \arrow[r, "(**)"] \arrow[d, "\cong"'] & \mathbf{sSet}(\Delta^n, \mathcal{N}_1(\mathscr{G})) \arrow[d, "\cong"] \\ \mathbf{sSet}(\Lambda^n_k, X_\bullet) \arrow[r, "(*)"'] & \mathbf{sSet}(\Lambda^n_k, \mathcal{N}_1(\mathscr{G})). \end{tikzcd}$$

The vertical maps are isomorphisms because of the lift. $(*)$ is an isomorphism, since the horn consists only of $(n-1)$ simplices hence it follows by the induction hypothesis. Therefore, $(**)$ is an isomorphism and so we indeed have that $\mathcal{N}_1(\mathscr{G}) \cong X_\bullet$. ■

Proposition 10.31. *The functor $\mathcal{N}_1$ is 2-coskeletal. That is, for a category $\mathscr{C} \in$ **Cat** the simplicial set $\mathcal{N}_1(\mathscr{C})$ is 2-coskeletal.*

Proof. We want to show, that $\mathcal{N}_1(\mathscr{C})$ is 2-coskeletal for some $\mathscr{C} \in \mathbf{Cat}$. As usual we can restrict ourselves to the case of simplicial sets of the form Δ^n, since any simplicial set may be considered as a colimit of such simplices. Any simplicial map

$$f : \Delta^n \to \mathcal{N}_1(\mathscr{C})$$

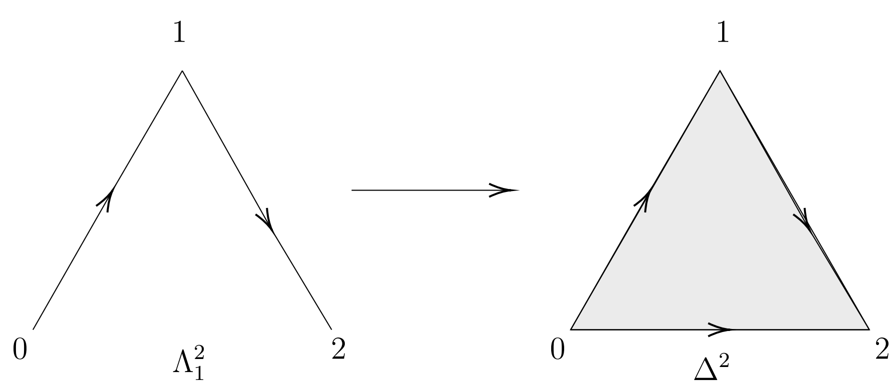
can be identified with a functor $f : [n] \to \mathscr{C}$. This functor is completely determined by its vertices (f_0), morphisms (f_1) and the requirement that f respects composition (f_2 and $d_i f_2 = f_1 d_i$). Hence, since $X_\bullet$ is the colim of some Δ^n, we have that $\mathcal{N}_1(\mathscr{C})$ is 2-coskeletal. ■

Proposition 10.32. *The nerve functor $\mathcal{N}_1$ is fully faithful.*

Proof. First we define the spine of a simplex Δ^n to be $sp^n := \bigcup_{0 \le i < n} \Delta^{\{i,i+1\}}$. We have a chain of natural inclusions $sp^n \to \partial\Delta^n \to \Delta^n$. If $X = \mathcal{N}_1(\mathscr{C})$ and $\mathscr{C} \in \mathbf{Cat}$, then the objects and morphisms of X are exactly the objects and diagrams of $\mathscr{C}$ respectively. Also commutative triangles in X are commutative triangles in $\mathscr{C}$. This corresponds to the following bijection

$$\mathbf{sSet}(\Delta^2, \mathcal{N}_1(\mathscr{C})) \cong \mathbf{sSet}(sp^2, \mathcal{N}_1(\mathscr{C})).$$

This actually also follows since we have unique fillers for $n > 1$ for horns and the spine is a particular kind of horn. So $sp^2 = \Lambda^2_1$, so pictorially this corresponds to



So this tells us that there is a unique arrow turning the left diagram into the right one.

In general, since an n-simplex of the nerve $\mathcal{N}_1(\mathscr{C})$ is a string of arrows of length n in $\mathscr{C}$, means that for any $n \geq 2$, the restriction along the inclusion $sp^n \to \Delta^n$ induces a bijective map

$$\mathbf{sSet}(\Delta^n, \mathcal{N}_1(\mathscr{C})) \cong \mathbf{sSet}(sp^n, \mathcal{N}_1(\mathscr{C})).$$

This actually also follows from the fact about the unique horn fillers. Therefore, for any $X_\bullet \in \mathbf{sSet}$ and any small category $\mathscr{C}$, a map from $X_\bullet$ to $\mathcal{N}_1(\mathscr{C})$ is completely determined by a map $f : X_1 \to \mathrm{mor}(\mathscr{C})$ such that

- For any $X \in X_\bullet$, $f(id_X)$ is the identity.
- For any commutative triangle $(g, h, l) \in X_\bullet$ we have $f(l) = f(h) \circ f(g)$.

We have $\mathcal{N}_1(\mathscr{C})_n = \mathbf{Fun}([n], \mathscr{C}) \cong \mathbf{sSet}(\Delta^n, \mathcal{N}_1(\mathscr{C})) \cong \mathbf{sSet}(sp^n, \mathcal{N}_1(\mathscr{C}))$.
Let $f \in \mathbf{sSet}(\mathcal{N}_1(\mathscr{C}), \mathcal{N}_1(\mathscr{D}))$ then we get maps

$$\begin{tikzcd}
f_n : \mathbf{sSet}(\Delta^n, \mathcal{N}_1(\mathscr{C})) \arrow[r] \arrow[d, "\cong"'] & \mathbf{sSet}(\Delta^n, \mathcal{N}_1(\mathscr{D})) \arrow[d, "\cong"] \\
\mathbf{sSet}(sp^n, \mathcal{N}_1(\mathscr{C})) \arrow[r] & \mathbf{sSet}(sp^n, \mathcal{N}_1(\mathscr{D})).
\end{tikzcd}$$

We construct a functor $F : \mathscr{C} \to \mathscr{D}$, The map f_n behaves like $(sp^n \to \mathcal{N}_1(\mathscr{C})) \mapsto (sp^n \to \mathcal{N}_1(\mathscr{C}) \xrightarrow{f} \mathcal{N}_1(\mathscr{D}))$. So on objects $(n = 0)$ we have

$$F : \mathscr{C} \to \mathscr{D}$$
$$(X : sp^0 \to \mathcal{N}_1(\mathscr{C})) \mapsto (f_0(X) : sp^0 \to \mathcal{N}_1(\mathscr{D}))$$

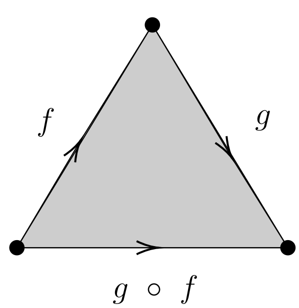
and on morphisms $(n = 1)$ we have

$$F : \mathscr{C} \to \mathscr{D}$$
$$((X \to Y) : \Delta^1 = sp^1 \to \mathcal{N}_1(\mathscr{C})) \mapsto ((f_1(X \to Y)) : \Delta^1 = sp^1 \to \mathcal{N}_1(\mathscr{D})).$$

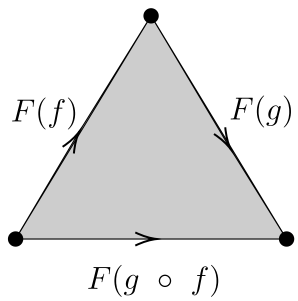
Now, we have to verify, that this construction indeed defines a functor. It preserves identities, since

$$(id_X : \Delta^1 \to \Delta^0 \to \mathcal{N}_1(\mathscr{C})) \mapsto (\Delta^1 \to \Delta^0 \to \mathcal{N}_1(\mathscr{C}) \to \mathcal{N}_0(\mathscr{D})) = id_{F(X)}.$$

We are left to show, that $F(f \circ g) = F(f) \circ F(g)$. Indeed, we have that $(f, g) : sp^2 \to \mathcal{N}_1(\mathscr{C})$, also $f \circ g : \Delta^2 \to \mathcal{N}_1(\mathscr{C})$, this corresponds to a commutative triangle



Consider $F(f \circ g) = f_2(f \circ g) : \Delta^2 \to \mathcal{N}_1(\mathscr{C}) \to \mathcal{N}_1(\mathscr{D})$, furthermore $F(f) : sp^1 \to \mathcal{N}_1(\mathscr{D})$ and $F(g) : sp^1 \to \mathcal{N}_1(\mathscr{D})$. Then $F(f) \circ F(g)$ corresponds to a diagram



Finally, we have a diagram

$$\begin{tikzcd}
\Delta^2 \arrow[r] \arrow[rr, bend left, "F(f\circ g)"] \arrow[drr, dotted, "F(f)\circ F(g)"] & \mathcal{N}_1(\mathscr{C}) \arrow[r] & \mathcal{N}_1(\mathscr{D}) \\
\Lambda^2_1 \arrow[r] \arrow[u] & \mathcal{N}_1(\mathscr{C}) \arrow[r] & \mathcal{N}_1(\mathscr{D}). \arrow[u, "id"']
\end{tikzcd}$$

Hence, $F(f \circ g) = F(f) \circ F(g)$ and therefore F is a full functor. Since F depends uniquely on f_n and it is determined by f_0 and f_1 we also have that F is faithful, which we wanted to show. ■

The Model Structure

As a preparation to eventually prove that there is a Quillen equivalence we first need to define a model category structure on the category of groupoids. It will turn out to be a cofibrantly generated one, which will make our live much easier in order to argue the Quillen pair and later on the Quillen equivalence.

Definition 10.33 (Generating Cofibrations)**.** *The set*

$$I_{\mathbf{Grpd}} := \{\Pi_1(\partial\Delta^n \to \Delta^n) \mid n \geq 0\}$$

*will be called the set of **generating cofibrations for** **Grpd**.*

Definition 10.34 (Generating Acyclic Cofibrations)**.** *The set*

$$J_{\mathbf{Grpd}} := \{\Pi_1(\Lambda^n_k \to \Delta^n) \mid n \geq 0, n \geq k \geq 0\}$$

*will be called the set of **generating acyclic cofibrations for** **Grpd**.*

Lemma 10.35. *The groupoids $\Pi_1(\partial\Delta^k)$ and $\Pi_1(\Delta^k)$ are isomorphic for $k \geq 3$. Similarly, the groupoids $\Pi_1(\Lambda^k_l)$ and $\Pi_1(\Delta^k)$ are isomorphic for $k \geq 2$.*

proof (sketch). This holds by the definition of a groupoid. Indeed, a groupoid only yields information about 0-cells and 1-cells, hence all the higher information is not taken into consideration i.e. under the functor Π_1, $\partial\Delta^k$ and Δ^k yield the same information for $k \geq 3$ and similarly for the horn inclusions. ■

In the next step we give the definition of the different classes of maps for a model structure.

Definition 10.36 (Isofibration)**.** *We call a morphism $F : \mathscr{G} \to \mathscr{H} \in$ **Grpd** an **isofibration**, if for any $A \in \mathscr{G}$ and for any $h : F(A) \to B \in \mathscr{H}$ there exists some $g : A \to A' \in \mathscr{G}$ such that $F(g) = h$.*

Definition 10.37 (Model Structure on **Grpd**)**.** *Let $F : \mathscr{G} \to \mathscr{H} \in$ **Grpd**. Then F is a*

1. *weak equivalence, if it is an equivalence of categories.*
2. *fibration, if it is an isofibration.*
3. *cofibration, if it is injective on objects.*

We denote this model structure by $\mathbf{Grpd}_{\mathcal{M}_1}$.

Remark 10.38. *One can see, that if we have an equivalence of categories, this functor is a fibration iff it is surjective i.e. elements in $\mathcal{W}_{\mathbf{Grpd}} \cap \mathcal{F}_{\mathbf{Grpd}}$ are surjective. More details can be found in [MaBe18].* ♦

Theorem 10.39. $\mathbf{Grpd}_{\mathcal{M}_1}$ *is a model category structure for* **Grpd** *turning it into a model category.*

Proof. A proof can be found in [NStr00]. One can also show this with Theorem 2.32. That is we transfer the cofibrantly generated model category structure from **sSet** to **Grpd**. ■

It turns out, that it is possible to turn the above model structure into a cofibrantly generated one. We will now attempt to argue this.

Lemma 10.40.
1. *$F \in \mathcal{F}_{\mathbf{Grpd}}$ iff F has the RLP with respect to $J_{\mathbf{Grpd}}$.*
2. *$F \in \mathcal{W}_{\mathbf{Grpd}} \cap \mathcal{F}_{\mathbf{Grpd}}$ iff F has the RLP with respect to $I_{\mathbf{Grpd}}$.*

Proof. 1. This is from the definition of isofibration. Consider the case $k, n = 1$

$$\begin{tikzcd}
\Pi_1(\Lambda^1_1) = [0] \arrow[r] \arrow[d] & \mathscr{G} \arrow[d, "F"] \\
\Pi_1(\Delta^1) = \Pi_1([1]) \arrow[r] \arrow[ur, dotted] & \mathscr{H}.
\end{tikzcd}$$

For $n \geq 2$ we have that $\Pi_1(\Lambda^n_k) \cong \Pi_1(\Delta^n)$ (basically since every morphism is invertible in a groupoid).

2. Assume that F has the RLP with respect to $I_{\mathbf{Grpd}}$. Then we have the following diagram

$$\begin{tikzcd}
\Pi_1(\partial\Delta^n) \arrow[r] \arrow[d] & \mathscr{G} \arrow[d, "F"] \\
\Pi_1(\Delta^n) \arrow[r] \arrow[ur, dotted] & \mathscr{H}.
\end{tikzcd}$$

For the case $n = 0$, this diagram tells us, that for any $B \in \mathscr{H}$, there is some $A \in \mathscr{G}$ such that $F(A) = B$, in other words F is essentially surjective (actually even surjective).

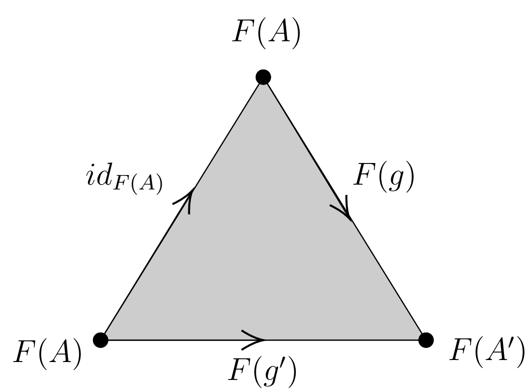
For $n = 1$ we have the following situation

$$\begin{tikzcd}
{[0]} \coprod [0] \arrow[r] \arrow[d] & \mathscr{G} \arrow[d, "F"] \\
\Pi_1([1]) \arrow[r, "h"'] \arrow[ur, dotted] & \mathscr{H}.
\end{tikzcd}$$

The lower commuting triangle says, that for any morphism in $\mathscr{H}$ there is one in $\mathscr{G}$, that is the functor F is full. The upper commuting triangle tells us that target and source get mapped via F (i.e. for any map $g : B \to B' \in \mathscr{H}$, there is a map $f : A \to A'$ such that $F(f) = g,\ F(A) = B$ and $F(A') = B'$). We consider the case $n = 2$. There is a diagram

$$\begin{tikzcd}
\Pi_1(\partial\Delta^2) \arrow[r, "(*)"] \arrow[d, "(**)"'] & \mathscr{G} \arrow[d, "F"] \\
\Pi_1(\Delta^2) = \Pi_1([2]) \arrow[r] \arrow[ur, dotted] & \mathscr{H}.
\end{tikzcd}$$

Consider further $\mathscr{G}(A, A') \to \mathscr{H}(F(A), F(A'))$ and assume that $F(g) = F(g')$ for some $g, g' \in \mathscr{G}(A, A')$. The morphism in $(*)$ corresponds to some diagram



On the other hand, the morphism $(**)$ corresponds to

$$\begin{tikzcd} & A \arrow[dr, "g"] & \\ A \arrow[ur, "id_A"] \arrow[rr, "g'"'] & & A' \end{tikzcd}$$

Now the lift in the original diagram assures us a way to fill this diagram (by the commutativity of the left triangle). Hence F is faithful.

Furthermore we have that $\Pi_1(\partial\Delta^k) \to \Pi_1(\Delta^k)$ are isomorphisms for $k \geq 3$ this concludes the other direction and concludes the proof.

■

Remark 10.41. *With the above proof, we are able to refine the definitions of the generating cofibrations and generating acyclic cofibrations for* **Grpd**. *That is*

$$\begin{aligned} I_{\mathbf{Grpd}} &:= \{\Pi_1(\partial\Delta^n \to \Delta^n) \mid n = 0,1,2\} \\ J_{\mathbf{Grpd}} &:= \{\Pi_1(\Delta^0 \to \Delta^1)\}. \end{aligned}$$

♦

Just for the sake of completeness we get the following corollary. This statement also appears in [ShHo07].

Corollary 10.42. $\mathbf{Grpd}_{\mathcal{M}_1}$ *is a cofibrantly generated model category structure on* **Grpd** *turning it into a model category.*

Groupoids and the Fundamental Groupoid

We may define the path category of a simplicial set in the following way.

Definition 10.43. *Let* $X_\bullet \in \mathbf{sSet}$, *the category generated by the graph* $X_1 \rightrightarrows X_0$ *of* 1*-simplices* $d_1(x) \to d_1(x)$ *subject to the relations* $d_1(\sigma) = d_0(\sigma) \circ d_2(\sigma)$ *given by the* 2*-simplices* σ *of* $X_\bullet$. *This category will be called the* ***path category of the simplicial set*** $X_\bullet$, *and is denoted by* $\mathcal{P}(X_\bullet)$.

Unwrapping the above definition should remind us of the pictures given by Remark 10.28.

Therefore, with this definition there is a natural bijection as described in the following lemma.

Lemma 10.44. *There is a bijection* $\mathbf{Cat}(\mathcal{P}(X_\bullet), \mathscr{C}) \cong \mathbf{sSet}(X_\bullet, \mathcal{N}_1(\mathscr{C}))$.

Proof. Consider the map $f : X_\bullet \to \mathcal{N}_1(\mathscr{C})$ in **sSet**, this associates to each n-simplex X of $X_\bullet$ a functor $f(X) : [n] \to \mathscr{C}$ which is completely determined by the 1-skeleton of $X_\bullet$ and the composition relations arising from 2-simplices of the form

$$\Delta^2 \to \Delta^n \xrightarrow{X} X_\bullet.$$

It follows, that f can be identified with the graph morphism

$$\begin{tikzcd}
X_1 \arrow[r, "f"] \arrow[d, shift right=2] \arrow[d, shift left=2] & \mathrm{mor}(\mathscr{C}) \arrow[d, shift right=2] \arrow[d, shift left=2] \\
X_0 \arrow[u] \arrow[r, "f"'] & \mathrm{ob}(\mathscr{C}) \arrow[u]
\end{tikzcd}$$

subject to the relations $f(d_1\sigma) = f(d_0\sigma) \circ f(d_2\sigma)$ arising from all 2-simplices σ of $X_\bullet$. Since $\mathcal{P}(X_\bullet)$ is the category freely associated to the graph

$$\begin{tikzcd}
X_1 \arrow[r, shift left=2] \arrow[r, shift right=2] & X_0 \arrow[l]
\end{tikzcd}$$

and the 2-simplex relations, there is an isomorphism $\mathbf{Cat}(\mathcal{P}(X_\bullet), \mathscr{C}) \cong \mathbf{sSet}(X_\bullet, \mathcal{N}_1(\mathscr{C}))$ as desired. ■

Hence, from the above lemma there is an adjunction

$$\mathcal{P} : \mathbf{sSet} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Cat} : \mathcal{N}_1 .$$

We are now able to turn the path category of a simplicial set into a groupoid for this specific simplicial set. We achieve this by localising the path category with respect to all morphisms. We denote the resulting category by $\mathcal{P}_{\mathcal{G}}(X_\bullet)$.

Similarly there is also an adjunction with this functor

$$\mathcal{P}_{\mathcal{G}} : \mathbf{sSet} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Grpd} : \mathcal{N}_1 .$$

Lemma A.14 now yields, that the functors Π_1 and $\mathcal{P}_{\mathcal{G}}$ are naturally isomorphic.

We introduce the fundamental groupoid for a topological space.

Definition 10.45 (Fundamental Groupoid)**.** *The **fundamental groupoid** of a topological space* $X \in$ **Top***, is a groupoid, where objects are the points of* X *and whose morphisms are paths in* X *identified up to endpoint-preserving homotopy. We denote this groupoid by* $\Pi_{\leq 1}(X)$.

A nice property is the following.

Remark 10.46. *For any point* $x \in X$ *of a topological space* X*, the fundamental group* $\pi_1(X, x)$ *arises as the automorphism group of* x *in* $\Pi_{\leq 1}(X)$ *i.e.*

$$\pi_1(X, x) = Aut_{\Pi_{\leq 1}(X)}(x).$$

♦

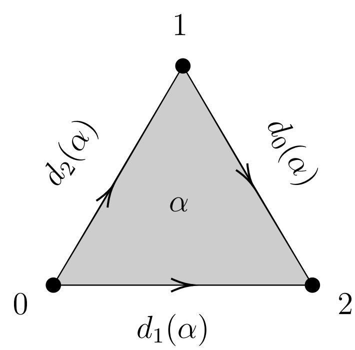
Unfortunately we will need this later for the proof of the Quillen adjunction, but for the setting of simplicial sets. But fear not, since Goerss and Jardine ([GoJa09]) provide help for such helpless situations. That is, in the form of the following theorem.

Theorem 10.47. *Let $X_\bullet \in \mathbf{sSet}_Q$. The groupoids $\Pi_{\leq 1}(|X_\bullet|)$ and $\mathcal{P}_{\mathcal{G}}(X_\bullet)$ are naturally equivalent as categories.*

Proof. First we notice, that the groupoids $\mathcal{P}_{\mathcal{G}}(\mathrm{Sing}(|X_\bullet|))$ and $\Pi_{\leq 1}(|X_\bullet|)$ are isomorphic. Indeed, let $X_\bullet \in \mathbf{sSet}_Q$ and consider the following map.

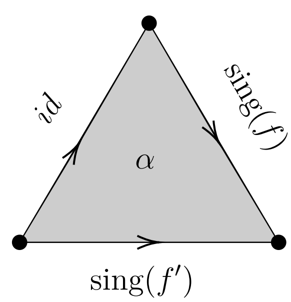
$$\begin{aligned} \gamma_{X_\bullet} : \Pi_{\leq 1}(|X_\bullet|) &\to \mathcal{P}_{\mathcal{G}}(\mathrm{Sing}(|X_\bullet|)) \\ [f : I \to |X_\bullet|] &\mapsto [\mathrm{Sing}(f) : \Delta^1 \to \mathrm{Sing}(|X_\bullet|)]. \end{aligned}$$

The relation on the left side is equivalence up to homotopy and the one on the right side comes from the following simplex



such that $[d_1\alpha] = [d_0\alpha] \cdot [d_2\alpha]$ for some 2-simplex $\alpha : |\Delta^2| \to |X_\bullet|$ of $\mathrm{Sing}(|X_\bullet|)$.

This map is indeed well defined. Let $f, f' \in [f : I \to |X_\bullet|]$ i.e. f and f' are homotopy equivalent to each other. Mapped under $\gamma_{X_\bullet}$, this corresponds to the following picture.



Hence, $\mathrm{Sing}(f) \sim \mathrm{Sing}(f')$. Therefore our map is indeed well defined.

This map is also an isomorphism, for this observe the following. If we are to map some morphism from the left hand side to the right hand side and then back again this is a composition which is related to the identity (equivalent to it). But for us to have an isomorphism this should actually be an equality. This can be fixed with the following Quillen equivalence already proven earlier in the thesis

$$|\ | : \mathbf{sSet}_Q \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Top}_Q : \mathrm{Sing}\,.$$

Since the composition we are interested in will be weakly equivalent to the identity which is in the equivalence class of the identity in $\Pi_{\leq 1}(|X_\bullet|)$. This shows that our map is indeed an isomorphism.

We have to show, that $\mathcal{P}_{\mathcal{G}}$ sends weak equivalences in $\mathbf{sSet}_Q$ to weak equivalences in $\mathbf{Grpd}_{\mathcal{M}_1}$. Let $f \in \mathcal{W}_{\mathbf{sSet}}$ then $f = q \circ j$ for some $q \in \mathcal{W}_{\mathbf{sSet}} \cap \mathcal{F}_{\mathbf{sSet}}$ and $j \in \mathcal{W}_{\mathbf{sSet}} \cap \mathcal{C}_{\mathbf{sSet}}$ (from the properties of FWFS and the 2-3 property). The map q is a left inverse to an acyclic cofibration and every acyclic cofibration is a retract of a map which is a filtered colimit of pushouts of maps of the form $\Lambda^n_k \to \Delta^n$. Therefore it suffices to show, by Ken Brown's lemma, that $\mathcal{P}_{\mathcal{G}}$ sends pushouts of maps $\Lambda^n_k \to \Delta^n$ to equivalences of categories.

The map $\mathcal{P}_{\mathcal{G}}(\Lambda^n_k) \to \mathcal{P}_{\mathcal{G}}(\Delta^n)$ is an isomorphism of groupoids for $n \geq 2$ and $\mathcal{P}_{\mathcal{G}}(\Lambda^1_i)$ is a strong deformation retract of $\mathcal{P}_{\mathcal{G}}(\Delta^1)$ for $i = 0, 1$. Indeed, this can be seen in the following way.

$$\Delta^1 \times \mathcal{P}_{\mathcal{G}}(\Delta^1) \xrightarrow{\quad H \quad} \mathcal{P}_{\mathcal{G}}(\Delta^1)$$

$$\begin{tikzcd}
(0,1) \arrow[r, "\simeq"] & (1,1) & & & \\
& & {} \arrow[r, mapsto] & 0 \arrow[r] & 1 \arrow[l, bend left] \\
(0,0) \arrow[uu, "\simeq"] \arrow[r, "\simeq"'] & (1,0) \arrow[uu, "\simeq"'] & & &
\end{tikzcd}$$

(colors will be mapped to colors) and with the morphisms

$$\begin{tikzcd}
* \arrow[r, shift left, "r"] & \mathcal{P}_{\mathcal{G}}(\Delta^1) \arrow[l, shift left, "p"]
\end{tikzcd}$$

we let $H|_0 = id$ and $H|_1 = r \circ p$. This gives us the desired strong deformation retractions.

Isomorphisms and strong deformation retracts of groupoids are closed under pushouts and so $\mathcal{P}_{\mathcal{G}}$ sends weak equivalences to weak equivalences in the respective categories. Finally, this yields that $\mathcal{P}_{\mathcal{G}}(X_\bullet)$ and $\Pi_{\leq 1}(|X_\bullet|)$ are naturally equivalent as categories. ■

This theorem, together with the fact that the functors $\mathcal{P}_{\mathcal{G}}$ and Π_1 are naturally isomorphic, give the following corollary.

Corollary 10.48. *Let $X_\bullet \in \mathbf{sSet}_Q$ be fibrant, then we have an isomorphism*

$$\pi_1(X_\bullet, x) \cong Aut_{\Pi_1(X_\bullet)}(x) \cong Aut_{\Pi_{\leq 1}(|X_\bullet|)} \cong \pi_1(|X_\bullet|, x).$$

Proof. We will show, that there is an isomorphism of groups

$$Aut_{\Pi_1(X_\bullet)} \cong Aut_{\Pi_{\leq 1}(|X_\bullet|)}.$$

First of all, since the above notions describe fundamental groups in the setting of simplicial sets and topological spaces respectively, a first condition to be satisfied is, that we consider a fibrant object. Therefore, let $X_\bullet \in \mathbf{sSet}_Q$ be fibrant.

The claimed isomorphism basically follows from the fact, that the groupoids $\mathcal{P}_\mathcal{G}(X_\bullet)$ and $\Pi_{\leq 1}(|X_\bullet|)$ are naturally equivalent as categories (see Theorem 10.47). In addition, we will use the fact, that the functors $\mathcal{P}_\mathcal{G}$ and Π_1 are naturally isomorphic.

Therefore we have a functor $F : \Pi_1(X_\bullet) \to \Pi_{\leq 1}(|X_\bullet|)$ which is fully faithful and essentially surjective. But if we restrict this to the groups $Aut_{\Pi_1(X_\bullet)}$ and $Aut_{\Pi_{\leq 1}(|X_\bullet|)}$ respectively, we indeed get the desired result. Surjectivity follows from the essential surjectivity of the functor and fully faithfulness implies injectivity. This gives us isomorphisms ■

Notice that the condition to be fibrant i.e. a Kan complex is indeed necessary here, as we can only relate the homotopy groups between simplicial sets and topological spaces if they are fibrant.

This observation will become crucial later on, in the proof of the main theorem.

The Quillen Equivalence

We are now ready to take on the final steps towards the desired Quillen equivalence.

Proposition 10.49. *There is a Quillen pair*

$$\Pi_1 : \mathbf{sSet}_Q \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Grpd}_{\mathcal{M}_1} : \mathcal{N}_1 \, .$$

Proof. Lemma 10.40 and Corollary 10.42 provide all needed conditions to apply Lemma 2.33, which then implies the result. ■

Furthermore, there is the following result i.e. the above Quillen pair still remains a Quillen pair in the truncated case.

Proposition 10.50. *There is a Quillen pair*

$$\Pi_1 : \mathbf{sSet}_Q^{\leq 1} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Grpd}_{\mathcal{M}_1} : \mathcal{N}_1 \, .$$

Proof. As we have seen , any object in $\mathbf{sSet}_Q$ is cofibrant. Proposition 5.35 now states, that if $\Pi_1 : \mathbf{sSet}_Q \to \mathbf{Grpd}_{\mathcal{M}_1}$ is a left Quillen functor, which holds by the above Proposition, and $\Pi_1(Q\mathcal{S}) \in \mathcal{W}_{\mathbf{Grpd}}$ for $\mathcal{S} = \{\partial\Delta^3 \to \Delta^3\}$, then $\Pi_1 : \mathbf{sSet}_Q^{\leq 1} \to \mathbf{Grpd}_{\mathcal{M}_1}$ is also a left Quillen functor.

In fact, $Q\mathcal{S}$ is just $\mathcal{S}$ itself, and as pointed out before, there is an isomorphism $\Pi_1(\partial\Delta^3) \to \Pi_1(\Delta^3)$ in $\mathbf{Grpd}_{\mathcal{M}_1}$ and hence also a weak equivalence. Therefore, the functor $\Pi_1 : \mathbf{sSet}_Q^{\leq 1} \to \mathbf{Grpd}_{\mathcal{M}_1}$ is indeed a left Quillen functor and therefore we get a Quillen pair

$$\Pi_1 : \mathbf{sSet}_Q^{\leq 1} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Grpd}_{\mathcal{M}_1} : \mathcal{N}_1 \, .$$

■

This is actually a very good example, why we want our model categories to be cofibrantly generated. They usually make it quite a bit easier to deal with Quillen pairs and Quillen equivalences.

Now is a good time to remember that we showed, that the functor $\mathcal{N}_1$ is fully faithful and combine it with Lemma A.15.

Finally, after a lot of work (pain?), we can state the main result.

Theorem 10.51. *There is a Quillen equivalence*

$$\Pi_1 : \mathbf{sSet}_Q^{\leq 1} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Grpd}_{\mathcal{M}_1} : \mathcal{N}_1 .$$

Proof. We have to show, that the derived unit and counit are weak equivalences, but since every object in $\mathbf{sSet}_Q$ is cofibrant and from the definition of isofibration one can deduce that every object in $\mathbf{Grpd}_{\mathcal{M}_1}$ is fibrant, it will be enough to show, that the maps $\epsilon : \Pi_1(\mathcal{N}_1) \to id_{\mathbf{Grpd}_{\mathcal{M}_1}}$ and $\eta : id_{\mathbf{sSet}_Q^{\leq 1}} \to \mathcal{N}_1(\Pi_1)$ are weak equivalences. Since the nerve $\mathcal{N}_1$ is fully faithful by Proposition 10.32 and hence by Lemma A.15 we have that $\Pi_1(\mathcal{N}_1) \to id_{\mathbf{Grpd}_{\mathcal{M}_1}}$ is an isomorphism and therefore also a weak equivalence.

We are left to show, that $\eta : id_{\mathbf{sSet}_Q^{\leq 1}} \to \mathcal{N}_1(\Pi_1)$ is a weak equivalence in $\mathbf{sSet}_Q^{\leq 1}$ i.e. that we have isomorphisms $\pi_i(X_\bullet) \cong \pi_i(\mathcal{N}_1(\Pi_1(X_\bullet)))$ for $i = 0, 1$. This condition is enough, according to Proposition 10.17.

Now for the next idea to work, we need to restrict ourselves to the case where we consider fibrant simplicial sets. Therefore we choose the fibrant replacement of the simplicial set $X_\bullet$ i.e. $X_\bullet \xrightarrow{\sim} X_\bullet^f$, which is a weak equivalence such that $X_\bullet^f$ is a Kan complex. We will use the notation $X_\bullet^f$ instead of $QX_\bullet$ for convenience.

Now the case for $i = 0$ follows by construction. The case $i = 1$ is a bit more involved. Since we work with the fibrant replacement we may apply Theorem 10.47. This yields an isomorphism

$$\pi_1(X_\bullet^f, x) \cong Aut_{\Pi_1(X_\bullet^f)}(x)$$

for any $x \in X_\bullet^f$.

But then we have the following isomorphisms

$$\begin{cases} \pi_1(X_\bullet^f, x) \cong Aut_{\Pi_1(X_\bullet)}(x) & \forall\, x \in X_\bullet^f \\ \pi_1(\mathcal{N}_1(\Pi_1(X_\bullet^f)), x) \cong Aut_{\Pi_1(\mathcal{N}_1(\Pi_1(X_\bullet^f)))}(x) & \forall\, x \in X_\bullet^f . \end{cases}$$

If we apply Π_1 to the isomorphism $\Pi_1 \mathcal{N}_1 \to id_{\mathbf{Grpd}_{\mathcal{M}_1}}$ we end up with an isomorphism

$$\Pi_1(X_\bullet^f) \xrightarrow{\cong} \Pi_1(\mathcal{N}_1(\Pi_1(X_\bullet^f))).$$

But this yields the following isomorphisms (by Corollary 10.48 and the fact, that $\mathcal{N}_1$ is fully faithful)

$$\pi_1(X_\bullet^f, x) \cong Aut_{\Pi_1(X_\bullet^f)}(x) \cong Aut_{\Pi_1(\mathcal{N}_1(\Pi_1(X_\bullet^f)))}(x) \cong \pi_1(\mathcal{N}_1(\Pi_1(X_\bullet^f)), x).$$

Therefore we end up with isomorphisms

$$\begin{cases} \pi_0(X_\bullet^f, x) \cong \pi_0(\mathcal{N}_1(\Pi_1(X_\bullet^f)), x) & \forall\ x \in X_\bullet^f \\ \pi_1(X_\bullet^f, x) \cong \pi_1(\mathcal{N}_1(\Pi_1(X_\bullet^f)), x) & \forall\ x \in X_\bullet^f. \end{cases}$$

Finally by Proposition 10.17, the map $X_\bullet^f \to \mathcal{N}_1(\Pi_1(X_\bullet^f))$ is a weak equivalence in $\mathbf{sSet}_Q^{\leq 1}$.

Time to address the problem with the fibrant replacement, since we want this to hold for any simplicial set. In such cases one applies Ken Brown.

First, notice that the weak equivalence $X_\bullet \xrightarrow{\sim} X_\bullet^f$ implies that $\Pi_1(X_\bullet) \to \Pi_1(X_\bullet^f)$ is also a weak equivalence, due to the fact that the first map is actually an acyclic cofibration and the left adjoint preserves acyclic cofibrations. Finally the map $\mathcal{N}_1(\Pi_1(X_\bullet)) \to \mathcal{N}_1(\Pi_1(X_\bullet^f))$ is also a weak equivalence due to Ken Brown's lemma.

The final step involves enjoying the following commutative diagram.

$$\begin{tikzcd} X_\bullet \arrow[r] \arrow[d, "\sim"'] & \mathcal{N}_1(\Pi_1(X_\bullet)) \arrow[d, "\sim"] \\ X_\bullet^f \arrow[r, "\sim"'] & \mathcal{N}_1(\Pi_1(X_\bullet^f)) \end{tikzcd}$$

Since weak equivalences have the 2-3 property the horizontal map must also be a weak equivalence, which concludes the proof. ■

10.3.3 2-Truncations

The goal of this section is to show -similarly to the previous section- that there is a Quillen equivalence between $\mathbf{sSet}_Q^{\leq 2}$ and $\mathbf{bi\text{-}Grpd}_{s\mathcal{M}_2}$, where the latter is the model category structure discussed in [MaBe18], the strict case can be found for example in [StLa], which will be stated here as a theorem. Furthermore, we want to show, that there is an equivalence of categories $\mathrm{Ho}(\mathbf{sSet}_Q^{\leq 2})$ and $\mathbf{bi\text{-}Grpd}[\mathcal{W}^{-1}]$. In the following we use material from [JoDu02], [BuFaBl04], [MaBe18], [IeMoJaSv93], [StLa06] and [StLa].

The desired nerve functor in this context is the so called Duskin nerve, developed in [JoDu02]. As pointed out in [BuFaBl04] this Duskin nerve is equivalent to the geometric nerve defined in [BuFaBl04].

But before we give the definition we need to clarify some notions and give additional definitions.

Basic Definitions

We will start with the definition of a bi-category and from this definition also give, as a special case, the definition of a 2-category, which is sometimes also referred to as a strict 2-category. After that we give the definition of a lax 2-functor and as special cases also the notion of weak 2-functor and the one of (strict) 2-functor. For these definitions we will follow [ToLe98], which are based on the ones given by Bénabou in [JeBe67] and is quite elegant but that is a question of personal taste.

Definition 10.52 (Bi-Category). *A* ***bi-category*** $\mathscr{B}$ *consists of the following data.*

1. *A set of objects* $\mathrm{ob}(\mathscr{B})$ *called* 0*-cells.*
2. *For any* $A, B \in \mathscr{B}$ *we have that* $\mathscr{B}(A, B) \in \mathbf{Cat}$*, called the category of morphisms.*
3. *For any* $A, B, C \in \mathscr{B}$ *there are functors*

$$\begin{aligned} C_{ABC} : \mathscr{B}(B, C) \times \mathscr{B}(A, B) &\to \mathscr{B}(A, C) \\ (g, f) &\mapsto g \circ f \\ (\beta, \alpha) &\mapsto \beta \circ_h \alpha \end{aligned}$$

(called horizontal composition operator) and for any $A \in \mathscr{B}$ *a map* $I_A : 1 \to \mathscr{B}(A, A)$*, thus* I_A *is a* 1*-cell* $A \to A$*.*

4. *For any* $A, B, C, D \in \mathscr{B}$ *natural isomorphisms*

$$\begin{tikzcd} \mathscr{B}(C, D) \times \mathscr{B}(B, C) \times \mathscr{B}(A, B) \arrow[r, "1 \times C_{ABC}"] \arrow[d, "C_{BCD} \times 1"'] & \mathscr{B}(C, D) \times \mathscr{B}(A, C) \arrow[d, "C_{ACD}"] \\ \mathscr{B}(B, D) \times \mathscr{B}(A, B) \arrow[r, "C_{ABD}"'] \arrow[ur, Rightarrow, "a_{ABCD}"] & \mathscr{B}(A, D) \end{tikzcd}$$

$$\begin{tikzcd} \mathscr{B}(A, B) \times 1 \arrow[d, "1 \times I_A"'] \arrow[dr, "\cong"] & \\ \mathscr{B}(A, B) \times \mathscr{B}(A, A) \arrow[r, "C_{AAB}"'] \arrow[ur, phantom, "\Rightarrow r_{AB}"] & \mathscr{B}(A, B) \end{tikzcd} \qquad \begin{tikzcd} 1 \times \mathscr{B}(A, B) \arrow[d, "I_B \times 1"'] \arrow[dr, "\cong"] & \\ \mathscr{B}(B, B) \times \mathscr{B}(A, B) \arrow[r, "C_{ABB}"'] \arrow[ur, phantom, "\Rightarrow l_{AB}"] & \mathscr{B}(A, B) \end{tikzcd}$$

and thus 2*-morphisms (called* 2*-cells)*

$$\begin{aligned} a_{hgf} &: (h \circ g) \circ f \xrightarrow{\cong} h \circ (g \circ f) \\ r_f &: f \circ I_A \xrightarrow{\cong} f \\ l_f &: I_B \circ f \xrightarrow{\cong} f. \end{aligned}$$

Such that the following axioms hold.

- *(Pentagon Law) The following diagram commutes.*

$$\begin{tikzcd} & ((k \circ h) \circ g) \circ f \arrow[r, "a \circ_h 1"] \arrow[dl, "a"'] & (k \circ (h \circ g)) \circ f \arrow[dr, "a"] & \\ (k \circ h) \circ (g \circ f) \arrow[drr, "a"'] & & & k \circ ((h \circ g) \circ f) \arrow[dl, "1 \circ_h a"] \\ & & k \circ (h \circ (g \circ f)) & \end{tikzcd}$$

- *(Triangle Identity) The following diagram commutes.*

$$\begin{tikzcd} (g \circ I) \circ f \arrow[rr, "a"] \arrow[dr, "r \circ_h 1"'] & & g \circ (I \circ f) \arrow[dl, "1 \circ_h l"] \\ & g \circ f & \end{tikzcd}$$

Remark 10.53. *The map a is usually referred to as the* ***associator*** *and the maps l and r are usually referred to as* ***unitors****, for more or less obvious reasons. Furthermore, the map I_A is often called* ***pseudo identity****.* ♦

Remark 10.54. *The definition of a bi-category $\mathscr{B}$ gives three types of cells:*

$$\begin{aligned} 0\text{-}\mathit{cells}: &\quad A \in \operatorname{ob}(\mathscr{B}) = 0\text{-}\operatorname{mor}(\mathscr{B}) \\ 1\text{-}\mathit{cells}: &\quad f : A \to B \in 1\text{-}\operatorname{mor}(\mathscr{B}) \\ 2\text{-}\mathit{cells}: &\quad \alpha : f \Rightarrow g \in 2\text{-}\operatorname{mor}(\mathscr{B}). \end{aligned}$$

So a 2-cell may be thought of as the following diagram

$$\begin{tikzcd} A \arrow[r, bend left=50, "f"{name=U}] \arrow[r, bend right=50, "g"'{name=D}] & B. \arrow[from=U, to=D, Rightarrow, shorten=2mm, "\alpha"] \end{tikzcd}$$

Furthermore, properties 2. and 3. give two kinds of composition for 2-cells, which as diagrams look the following way.

Vertical composition:

$$\begin{tikzcd} A \arrow[r, bend left=50, ""{name=U}] \arrow[r, ""{name=M}] \arrow[r, bend right=50, ""{name=D}] & B \arrow[from=U, to=M, Rightarrow, shorten=1mm, "\alpha"] \arrow[from=M, to=D, Rightarrow, shorten=1mm, "\beta"] \end{tikzcd} \quad \textit{becomes} \quad \begin{tikzcd} A \arrow[r, bend left=50] \arrow[r, bend right=50] \arrow[r, phantom, "\Downarrow \beta \circ_v \alpha"] & B. \end{tikzcd}$$

This is the composition that comes from the internal structure of the categories of morphisms (therefore it is also called internal composition).

Horizontal composition:

$$\begin{tikzcd} A \arrow[r, bend left=50] \arrow[r, bend right=50] \arrow[r, phantom, "\Downarrow \alpha"] & B \arrow[r, bend left=50] \arrow[r, bend right=50] \arrow[r, phantom, "\Downarrow \beta"] & C \end{tikzcd} \quad \textit{becomes} \quad \begin{tikzcd} A \arrow[r, bend left=50] \arrow[r, bend right=50] \arrow[r, phantom, "\Downarrow \beta \circ_h \alpha"] & C. \end{tikzcd}$$

This is also referred to as external composition, in contrast to the above composition. ♦

Notation 10.55 (Composition)**.** *We use the notation $- \circ_h -$ for* ***horizontal composition*** *and $- \circ_v -$ for* ***vertical composition*** *of 2-cells. The symbol $\circ$ will usually just denote standard composition i.e. composition of 1-cells.*

Definition 10.56 (2-category)**.** *A (strict) 2-**category** is a bi-category, in which a, l and r are identities i.e. $(h \circ g) \circ f = h \circ (g \circ f)$ and $I \circ f = f = f \circ I$ and similar for composition 2-cells.*

Actually one could just define a (strict) 2-category to be a category enriched over **Cat**, which is the same as the above definition.

Definition 10.57 (Equivalence)**.** *A* 1*-cell* $w : A \to B$ *in a bi-category* $\mathscr{B}$ *is called* ***equivalence*** *when there exists a* 1*-cell* $v : B \to A$ *and invertible* 2*-cells* $\eta : w \circ v \Rightarrow 1_B$ *and* $\epsilon : 1_A \Rightarrow v \circ w$ *satisfying the triangle identities. Such a* v *is called a* ***quasi inverse*** *of* w*.*

Next we give the definition of a bi-groupoid. This definition should correspond with the one provided in [MaBe18].

Definition 10.58 (bi-groupoid)**.** *A* ***bi-groupoid*** *is a bi-category in which every* 1*-morphism is an equivalence and every* 2*-morphism is an isomorphism.*

Definition 10.59 (2-groupoid)**.** *A* 2***-groupoid*** *is a* 2*-category in which every* 1*-morphism is an equivalence and every* 2*-morphism is an isomorphism.*

Since we like ridiculously long definitions here comes another one. We first define lax 2-functors and then weak 2-functors and (strict) 2-functors as special cases.

Definition 10.60 (Lax 2-Functors)**.** *A* ***lax*** 2***-functor*** $(F, \phi) : \mathscr{B} \to \mathscr{B}'$ *between bi-categories, consists of the following data.*

1. *A function* $F : \mathrm{ob}(\mathscr{B}) \to \mathrm{ob}(\mathscr{B}')$*.*
2. *For any* $A, B \in \mathrm{ob}(\mathscr{B})$*, functors* $F_{AB} : \mathscr{B}(A, B) \to \mathscr{B}'(FA, FB)$*.*
3. *For any* $A, B, C \in \mathrm{ob}(\mathscr{B})$ *natural transformations*

$$\begin{tikzcd} \mathscr{B}(B,C) \times \mathscr{B}(A,B) \arrow[r, "C"] \arrow[d, "F_{BC} \times F_{AB}"'] & \mathscr{B}(A,C) \arrow[d, "F_{AC}"] \\ \mathscr{B}'(FB,FC) \times \mathscr{B}'(FA,FB) \arrow[r, "C'"] \arrow[ur, Rightarrow, "\phi_{ABC}"] & \mathscr{B}'(FA,FC) \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, "I_A"] \arrow[d, equal] & \mathscr{B}(A,A) \arrow[d, "F_{AA}"] \\ 1 \arrow[r, "I'_{FA}"'] \arrow[ur, Rightarrow, "\phi_A"] & \mathscr{B}'(FA,FA) \end{tikzcd}$$

and thus 2*-cells*

$$\phi_{gf} : Fg \circ Ff \Rightarrow F(g \circ f)$$
$$\phi_A : I'_{FA} \Rightarrow FI_A.$$

Such that the following axioms are satisfied i.e. the following diagrams commute.

$$\begin{tikzcd} (Fh \circ Fg) \circ Ff \arrow[r, "\phi \circ_h 1"] \arrow[d, "a'"'] & F(h \circ g) \circ Ff \arrow[r, "\phi"] & F((h \circ g) \circ f) \arrow[d, "Fa"] \\ Fh \circ (Fg \circ Ff) \arrow[r, "1 \circ_h \phi"'] & Fh \circ (F(g \circ f)) \arrow[r, "\phi"'] & F(h \circ (g \circ f)). \end{tikzcd}$$

$$\begin{tikzcd} Ff \circ I'_{FA} \arrow[r, "1 \circ_h \phi"] \arrow[d, "r'"'] & Ff \circ FI_A \arrow[r, "\phi"] & F(f \circ I_A) \arrow[d, "Fr"] \\ Ff \arrow[rr, equal] & & Ff \end{tikzcd} \qquad \begin{tikzcd} I'_{FB} \circ Ff \arrow[r, "\phi \circ_h 1"] \arrow[d, "l'"'] & FI_B \circ Ff \arrow[r, "\phi"] & F(I_B \circ f) \arrow[d, "Fl"] \\ Ff \arrow[rr, equal] & & Ff \end{tikzcd}$$

Definition 10.61 (Weak 2-Functor)**.** *A* ***weak*** *2**-functor** is a lax 2-functor, such that* $Fg \circ Ff \cong F(g \circ f)$ *and* $FI \cong I'$.

Definition 10.62 (2-functor)**.** *A* ***(strict)*** *2**-functor** is a lax 2-functor, such that* $Fg \circ Ff = F(g \circ f)$ *and* $FI = I'$.

Remark 10.63. *From the above definitions we have that weak 2-functors have isomorphisms and strict 2-functors have identities. This is the important thing to keep in mind when we are working with these notions.*

Therefore, any (strict) 2-functor is a weak 2-functor and every weak 2-functor is a lax 2-functor. Sometimes a weak 2-functor is also called pseudo functor.

Some people also refer to a weak 2-functor as homomorphism of bi-categories and to a lax 2-functor as a morphism of bi-categories or strict homomorphism of bicategories. ♦

Notice, that bi-groupoids together with weak 2-functors form a category (see for instance Corollary 2.12 in [MaBe18]) and similar for bi-categories. We denote the category of all bi-categories together with weak 2-functors by **bi-Cat**, similarly we denote the category of all bi-groupoids and weak 2-functors by **bi-Grpd**. Furthermore, we denote by 2- **Cat** the strict two categories with strict 2 morphisms, similar for 2- **Grpd**.

If we speak of a strict 2 category with lax 2-functors we will often use the subscript "lax" i.e. $\mathbf{2\text{-}Cat}_{\mathrm{lax}}$ and if we speak about a bi-category with strict 2-functors we use a subscript "s" i.e. $\mathbf{bi\text{-}Cat}_{\mathrm{s}}$ etc. In fact we will use the latter one for quite some time throughout the following discussion, though for the case of bi-groupoids.

There is reason for the above discussion as we will see, the categories **bi-Cat** and **bi-Grpd** are neither complete nor cocomplete. Therefore we can not apply our machinery as we are used to (see for instance as in the last section). Hence, we will adapt everything for the case known to be working, that is the case of bi-groupoids with strict 2-morphisms, since this is a bicomplete category. Bicompleteness is very important for our constructions, as one may be convinced from the last section, it is needed in the definition of a model category and more important it is crucial for our definition of the geometric realisation (at least cocompleteness is needed there).

At the end of this section we will still relate the homotopy theories of 2-truncations and **bi-Grpd**, since this will still work, though not via a Quillen equivalence.

Proposition 10.64. bi-Cat *and* **bi-Grpd** *are neither complete nor cocomplete.*

Proof. We will actually show, that 2- $\mathbf{Cat}_{\mathrm{w}}$ does not have equalisers and coequalisers. This example can be adapted to the case of **bi-Grpd** and **bi-Cat** according to [MaBe18]. The example was provided in [StLa06].

We consider the ordered set $\mathbf{3} = \{0 < 1 < 2\}$ as a 2-category with no non-identity 2-cells. Consider a 2-category $\mathscr{K}$ with designated arrows $f : A \to B,\ g : B \to C$ and $h : A \to C$ and invertible 2-cell $\varphi : g \circ f \to h$. These determine a weak 2-functor $F : \mathbf{3} \to \mathscr{K}$ sending 0, 1 and 2 to A, B and C; sending $0 < 1$ to f, $1 < 2$ to g and $0 < 2$ to h; strictly preserving identities and with φ the pseudofunctoriality constraint. Suppose that there is a different $h' : A \to C$ and $\varphi' : g \circ f \to h'$ and write $F' : \mathbf{3} \to \mathscr{K}$ for the induced weak 2-functor. If F and F' had an equaliser it would have to contain the objects $0, 1$ and 2 and the arrows $0 < 1,\ 1 < 2$ but not $0 < 2$. This is impossible, therefore we do not have equalisers. Furthermore, the inclusions $\{1\} \to \{0 < 1\}$ and $\{1\} \to \{1 < 2\}$ have no pushout in $2\text{-}\mathbf{Cat}_{\mathrm{w}}$, and so there are also no coequalisers. Finally there is no bicompleteness. ■

In fact if we consider the proof, one can say that there is only hope for the case with strict functors, which turns out to be true. This result is well known and will not be proven.

Proposition 10.65. *$2\text{-}\mathbf{Cat}$ and $2\text{-}\mathbf{Grpd}$ are bicomplete.*

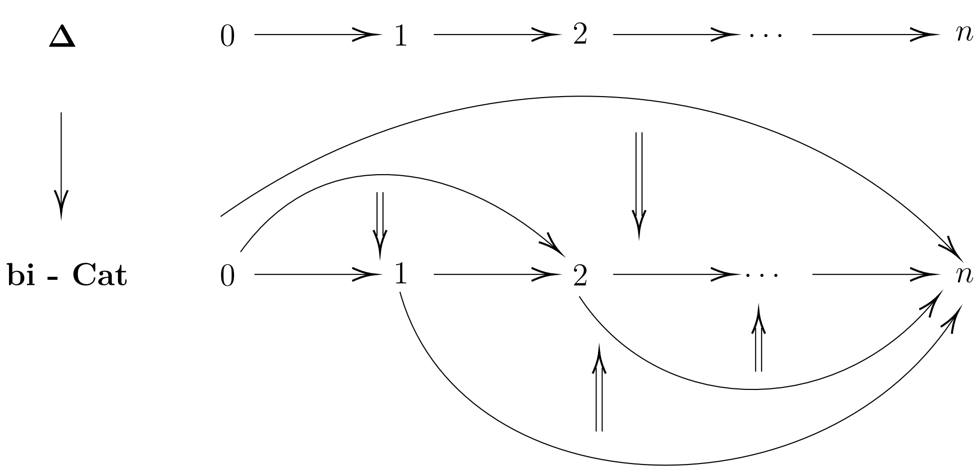
According to [StLa] we also have the following slightly more general result.

Proposition 10.66 ([StLa])**.** *$\mathbf{bi\text{-}Cat}_s$ and $\mathbf{bi\text{-}Grpd}_s$ are bicomplete.*

Finally we may give the definition of the geometric bi-nerve and geometric bi-realisation. We modify the definition of the nerve given in [BuFaBl04] for the case of **bi-Grpd**. According to them and as pointed out in [JoDu02], this turns out to be the same nerve as defined in [JoDu02], section 6. Furthermore, there is indeed also a remark in [JoDu02], which confirms this assumption.

As already done in the last section we consider a certain case of Definition 6.2 and Definition 6.4.

First we consider the inclusion functor $\mathbf{i} : \boldsymbol{\Delta} \to \mathbf{bi\text{-}Cat}$. First we include $\boldsymbol{\Delta}$ in $\mathbf{Cat}$ in the same way as described in the last section. Then we are able to turn the resulting category into a bi-category. Indeed, the only things we have to add for this to work are 2-cells. We add an identity 2-cell for between any composition of morphisms. It might be easier to see this as a picture to clarify what is actually going on.



The above idea can also be seen in the following way. Sets define discrete categories, therefore a category (here a set enriched category) determines a strict 2-category in a canonical way.

We define the **geometric bi-Nerve** to be the functor $\mathcal{N}_2 : \mathbf{bi\text{-}Cat} \to \mathbf{sSet}$, induced by the inclusion $\mathbf{i} : \mathbf{\Delta} \to \mathbf{bi\text{-}Cat}$, i.e. $\mathcal{N}_2(\mathscr{B}) = \mathbf{Fun}(\mathbf{i}(-), \mathscr{B})$.

The definition works in such generality, that one could use **bi-Cat** with **normal lax functors**, these are lax functors which strictly preserve identities. For our purpose, we will restrict ourselves to strict functors.

Therefore we may define the **geometric bi-Realisation** to be the functor Π_2 defined as the left Kan extension of

$$\begin{tikzcd} \mathbf{\Delta} \arrow[r, "\mathbf{i}"] \arrow[d, "\mathcal{Y}"'] & \mathbf{bi\text{-}Cat}_{\mathrm{s}} \\ \mathbf{sSet} \arrow[ur, dotted, "L_{\mathcal{Y}}\mathbf{i}"'] & \end{tikzcd}$$

i.e. $\Pi_2 := L_{\mathcal{Y}}\mathbf{i}$, where $\mathcal{Y}$ is the Yoneda embedding.

Of course this is only well-defined, if these Kan extensions actually exist in any case. This holds, since $\mathbf{bi\text{-}Cat}_{\mathrm{s}}$ is a bicomplete category (keep in mind that actually only cocompleteness is needed here).

The following question may come up now. Why do we define this in such generality, when we just need the special case of bi-groupoids? Well, of course this is preference but, as we will see, the bi-nerve as defined in such generality has some very good and helpful properties. It is very similar as in the setting of 1-truncations (but of course way more complicated). To spoil the fun, it will turn out that the bi-nerve is again fully faithful just as for the other case, which will again be an appreciated tool for the Quillen equivalence at the end of the section.

A first consequence is the following result.

Lemma 10.67. *There is an adjunction*

$$\Pi_2 : \mathbf{sSet} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{bi\text{-}Cat}_s : \mathcal{N}_2 .$$

Proof. This follows from Theorem 6.6. ■

As pointed out before, we need an adjunction between **sSet** and $\mathbf{bi\text{-}Grpd}_{\mathrm{s}}$. So instead of restricting definitions, we may use some similar argument as in the last section. Actually, there exists a generalisation of the idea of localising a category for this more general setting. This idea is introduced in [DoPr96]. We will just very briefly discuss the idea (since the argument is very similar as in the case discussed in the section about localisations. Already there these proofs were very technical so it will be even more involved for this case). The idea is a generalisation of the original work on such structures called "calculus of fractions" first introduced in [GaZi67], which treats the case for **Cat** (which we also discussed in the present work).

In order to localise with respect to a subclass of a bi-category, this subclass has to admit a right calculus of fractions, as defined in [DoPr96].

Anyway, since we want to localise with respect to any morphism of the bi-category all these conditions should be satisfied.

Here comes the main theorem.

Theorem 10.68 ([DoPr96])**.** *Let $\mathscr{C} \in$ **bi-Cat** and $\mathcal{W}$ be a class of 1-morphisms in $\mathscr{C}$ such that $\mathcal{W}$ admits a right calculus of fractions. There is a bi-category $\mathscr{C}[\mathcal{W}^{-1}]$ and a homomorphism* $\mathrm{Loc} : \mathscr{C} \to \mathscr{C}[\mathcal{W}^{-1}]$ *such that*

1. Loc *sends elements of $\mathcal{W}$ to equivalences (see Definition 10.57).*
2. Loc *is universal with this property i.e. composing with* Loc *gives an equivalence of bi-categories (see Definition 10.79)*

$$\mathbf{bi\text{-}Cat}(\mathscr{C}[\mathcal{W}^{-1}], \mathscr{D}) \to \mathbf{bi\text{-}Cat}_{\mathcal{W}}(\mathscr{C}, \mathscr{D})$$

where the latter is the sub-bi-category of those cells which send the elements of $\mathcal{W}$ to equivalences.

Proof. The idea is much like the case for **Cat** discussed earlier. Of course this is more involved as we have to deal with higher morphisms. A full proof may be found in [DoPr96] in section 2. ∎

There are a lot more useful properties about this construction in [DoPr96].

We will now give a way to turn a bi-category into a bi-groupoid using the above theorem and the localisation used to turn categories into groupoids. This will work in the following way.

Let $\mathscr{B} \in$ **bi-Cat** but then $\mathscr{B}(X,Y) \in$ **Cat** for any $X, Y \in \mathrm{ob}(\mathscr{B})$. Further let $\mathcal{W}^2_{X,Y} := \mathrm{mor}(\mathscr{B}(X,Y))$ and $\mathcal{W}^1 := 1\text{-}\mathrm{mor}(\mathscr{B})$. So the collection of all the $\mathcal{W}^2_{X,Y}$ are precisely the 2-cells of $\mathscr{B}$, we denote this whole collection by $\mathcal{W}^2$ i.e. $\mathcal{W}^2 = \bigcup_{X,Y \in \mathrm{ob}(\mathscr{B})} \mathcal{W}^2_{X,Y}$.

We will now apply the localisation given by the above theorem with respect to $\mathcal{W}^1$ and denote it by $\mathscr{B}[\mathcal{W}^{-1}]$. Notice that this is not yet a bi-groupoid. We indeed have that every 1-morphism is an equivalence but not every 2-morphism is invertible.

In a next step we apply the known localisation construction to $\mathscr{B}[\mathcal{W}^{-1}]$ in the following sense. For any $X, Y \in \mathrm{ob}(\mathscr{B}[\mathcal{W}^{-1}])$ we localise $\mathscr{B}[\mathcal{W}^{-1}](X,Y)$ with respect to $\mathcal{W}^2_{X,Y}$ i.e. $\mathscr{B}[\mathcal{W}^{-1}](X,Y)[(\mathcal{W}^2_{X,Y})^{-1}] \in$ **Grpd**. We do this for all objects in $\mathscr{B}[\mathcal{W}^{-1}]$ such that all $\mathscr{B}[\mathcal{W}^{-1}](-,-)[(\mathcal{W}^2_{-,-})^{-1}] \in$ **Grpd**. This now finally gives us a bi-groupoid, since all 1-morphisms are equivalences and every 2-morphism is invertible.

We denote this procedure of turning a bi-category into a bi-groupoid by $\mathscr{B}[(\mathcal{W}^2)^{-1}, (\mathcal{W}^1)^{-1}]$.

This gives a functor which we also denote by $\mathrm{Loc} :$ **bi-Cat** $\to$ **bi-Grpd**, we also have the forgetful functor $U :$ **bi-Grpd** $\to$ **bi-Cat**. As for the other case they also form an adjunction with the same argument (the argument is a bit more involved here as we are dealing with different localisations, but essentially they are the same).

Lemma 10.69. *There is an adjunction*

$$\mathrm{Loc} : \mathbf{bi\text{-}Cat} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{bi\text{-}Grpd} : U.$$

Proof. This comes from the universal properties of the localisations used above. ∎

And of course, this can be adapted for the strict case.

Since composition of adjunctions gives an adjunction, we have the following adjunctions

$$\mathbf{sSet} \underset{\mathcal{N}_2}{\overset{\Pi_2}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathbf{bi\text{-}Cat}_{\mathrm{s}} \underset{U}{\overset{\mathrm{Loc}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathbf{bi\text{-}Grpd}_{\mathrm{s}}\,.$$

By abuse of notation and more for aesthetic reasons, we will from now on denote with Π_2 the composition $\mathrm{Loc} \circ \Pi_2$ and similar by $\mathcal{N}_2$ we mean $U \circ \mathcal{N}_2$ (which is already the same functor). Finally this yields the following result.

Corollary 10.70. *There is an adjunction*

$$\Pi_2 : \mathbf{sSet} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{bi\text{-}Grpd}_s : \mathcal{N}_2\,.$$

Properties of the bi-Nerve

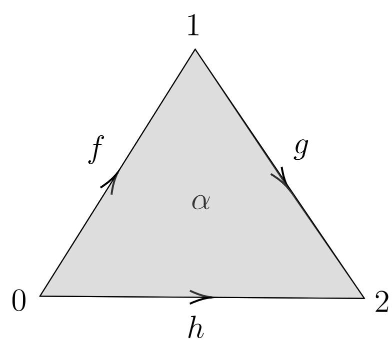
We give some nice properties about the bi-nerve. According to [BuFaBl04] one may give an explicit description of the nerve in the following way. This whole section holds for the more general case of **bi-Cat** or **bi-Grpd**.

Remark 10.71. *Let $\mathscr{C}$ be a bi-category, the geometric bi-nerve $\mathcal{N}_2(\mathscr{C})$ is the following simplicial set.*

1. *Its vertices are the objects of $\mathscr{C}$.*
2. *1-simplices are the arrows $C_0 \xrightarrow{f} C_1$ in $\mathscr{C}$, with faces $d_0(f) = C_1$ and $d_1(f) = C_0$.*

$$0 \xrightarrow{f} 1$$

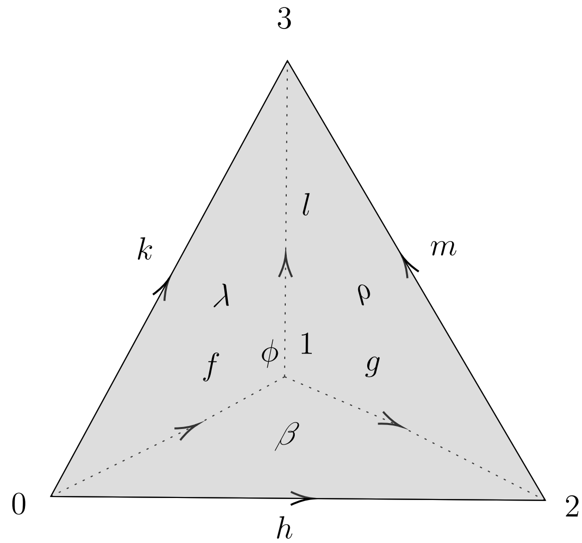
3. *2-simplices are the diagrams $\Delta = (g, h, f; \alpha)$ of the form*



$$\alpha : h \Rightarrow g \circ f$$

with $\alpha : h \Rightarrow g \circ f$ a 2-cell in $\mathscr{C}$, and faces which are the 1-simplices opposite to the indicated vertex, that is $d_0(\Delta) = g, d_1(\Delta) = h$ and $d_2(\Delta) = f$.

4. *3-simplices are "commutative" tetrahedral Θ of the form*



$$\begin{aligned} \phi : k &\Rightarrow m \circ h && \textit{(front face)} \\ \beta : h &\Rightarrow g \circ f && \textit{(lower face)} \\ \lambda : k &\Rightarrow l \circ f && \textit{(left face)} \\ \rho : l &\Rightarrow m \circ g && \textit{(right face)} \end{aligned}$$

where by commutativity of Θ *we mean that the following square of* 2*-cells commutes*

$$\begin{tikzcd} k \arrow[r, "\phi"] \arrow[d, "\lambda"'] & mh \arrow[d, "m\beta"] \\ lf \arrow[r, "\rho f"'] & mgf. \end{tikzcd}$$

The face operators for such Θ *are, as in the case of a* 2*-simplex, the* 2*-simplices opposite to the vertex indicated by the operator, i.e.*

$$\begin{aligned} d_0(\Theta) &= \Delta_0 \text{ is the right face } (\rho) \\ d_1(\Theta) &= \Delta_1 \text{ is the front face } (\phi) \\ d_2(\Theta) &= \Delta_2 \text{ is the left face } (\lambda) \\ d_3(\Theta) &= \Delta_3 \text{ is the lower face } (\beta) \end{aligned}$$

5. *At higher dimensions* $\mathcal{N}_2(\mathscr{C})$ *is* 3*-coskeletal (we will argue this later).*

♦

The above description also corresponds to the way that the bi-nerve is defined for example in [JoDu02] adapted for the case of bi-categories or bi-groupoids.

We may also give a description of the simplicial map associated to a weak functor by the binerve $\mathcal{N}_2$.

Remark 10.72. *Let* $F : \mathscr{C} \to \mathscr{D}$ *be a weak* 2*-functor between bi-categories, with structure map* σ*. Then* $\mathcal{N}_2(F) : \mathcal{N}_2(\mathscr{C}) \to \mathcal{N}_2(\mathscr{D})$ *is the simplicial map given by:*

1. *for* 0*-simplices* $C \in \mathcal{N}_2(\mathscr{C})_0,\ \mathcal{N}_2(F)_0(C) = F(C)$
2. *for* 1*-simplices* $f \in \mathcal{N}_2(\mathscr{C})_1,\ \mathcal{N}_2(F)_1(f) = F(f)$
3. *for* 2*-simplices* $\Delta \in \mathcal{N}_2(\mathscr{C})_2$*, as above,* $\mathcal{N}_2(F)_2(\Delta)$ *is obtained from the diagram*

$$\begin{tikzcd}
 & F(C_1) \arrow[dr, "F(g)"] & \\
F(C_0) \arrow[ur, "F(f)"] \arrow[rr, bend left, "F(gf)"] \arrow[rr, bend right, "F(h)"'] & & F(C_2)
\end{tikzcd}$$

by the vertical composite of deformations: $\sigma_{(f,g)}F(\alpha)$*, that is* $\mathcal{N}_2(\mathscr{C})_2(g,h,f;\alpha) = (F(g), F(h), F(f); \sigma_{(f,g)}F(\alpha))$.

4. *At higher dimensions* $\mathcal{N}_2(F)$ *is defined in the unique possible way, using the fact that in dimension* 3 *and above any simplex is determined by its faces.*

♦

When we work with bi-groupoids, we actually want the two cells described in the above two remarks to go the other way. This is not a problem since in that case they will all be invertible, and hence we will just choose the inverse of the respective two cell.

The next notion is just a more compact way to address a certain important property which will appear a lot in this context, so it is just for convenience.

Definition 10.73 (Hypergroupoid)**.** *An* n***-hypergroupoid*** *is a Kan complex* $K_\bullet$ *in which the horn fillers are unique in dimension greater than* n *i.e. there is a unique dotted arrow in the following diagram*

$$\begin{tikzcd}
\Lambda_i^k \arrow[r] \arrow[d] & K_\bullet \\
\Delta^k \arrow[ur, dotted, "\exists!"'] &
\end{tikzcd}$$

making it commute for $k > n$.

Remark 10.74. *So a special case of the above definition, say for* $n = 1$ *they are nerves of groupoids. Equivalently the above definition reads in the following way. The* n*-hypergroupoids are those Kan complexes which are* $(n+1)$*-coskeletal and such that* $(n+1)$*-horns and* $(n+2)$*-horns have unique fillers.* ♦

The first two properties of the following propositions are taken from Theorem 8.6 in [JoDu02]. This is the main theorem of the paper and working it out is very involved, therefore we will not give a proof.

Proposition 10.75. *The bi-nerve has the following properties.*

1. *Let $X_\bullet \in$ **sSet**. $X_\bullet$ is the nerve of a bi-groupoid iff $X_\bullet$ is a 2-hypergroupoid.*
2. *The functor $\mathcal{N}_2$ is* 3*-coskeletal. That is, for a bi-category $\mathscr{C}$, the simplicial set $\mathcal{N}_2(\mathscr{C})$ is* 3*-coskeletal.*
3. *The functor $\mathcal{N}_2 : \textbf{bi- Cat} \to \textbf{sSet}$ is fully faithful.*

Proof. This proof works essentially in the same way as discussed in the last section. The arguments will be the same, we just need to take care of one more dimension. The main idea is again to use the idea of the spine to deduce fully faithfulness and also that it is indeed 3-coskeletal.

For more details, the first two properties can be found as part of Theorem 8.6 in [JoDu02]. The last one is for instance argued as Proposition 4.3 in [BuFaBl04]. ■

The Model Structure

Now it is time to introduce the model category structure on **bi-Grpd** following [MaBe18]. We will conclude that there is indeed a model structure, further with the same technique as in the previous section, we will show that this model structure is also cofibrantly generated. This will then again help with the Quillen pair. Finally we show the main result of the present thesis, which is the Quillen equivalence between 2-truncated simplicial sets and bi-groupoids with their respective model category structures.

If we speak about bi-categories and bi-groupoids in the following sections we usually mean -unless otherwise claimed- elements of $\textbf{bi-Cat}_\text{s}$ and $\textbf{bi-Grpd}_\text{s}$ respectively.

Definition 10.76 (Generating Cofibrations)**.** *The set*

$$I_{\textbf{bi-Grpd}} := \{\Pi_2(\partial\Delta^n \to \Delta^n) \mid n \geq 0\}$$

will be called the set of ***generating cofibrations for*** $\textbf{bi-Grpd}_s$*.*

Definition 10.77 (Generating Acyclic Cofibrations)**.** *The set*

$$J_{\textbf{bi-Grpd}} := \{\Pi_2(\Lambda^n_k \to \Delta^n) \mid n \geq 0, n \geq k \geq 0\}$$

will be called the set of ***generating acyclic cofibrations for*** $\textbf{bi-Grpd}_s$*.*

Lemma 10.78. *The bi-groupoids $\Pi_2(\partial\Delta^k)$ and $\Pi_2(\Delta^k)$ correspond up to isomorphism for $k \geq 4$. Similarly, the bi-groupoids $\Pi_2(\Lambda^k_l)$ and $\Pi_2(\Delta^k)$ correspond up to isomorphism for $k \geq 3$.*

proof (sketch). This holds by the definition of a bi-groupoid. Indeed, a bi-groupoid only yields information about 0-cells, 1-cells and 2-cells, hence all the higher information is not taken in consideration i.e. under the functor Π_2, $\partial\Delta^k$ and Δ^k yield the same information for $k \geq 4$ and similarly for the horn inclusions. ■

In the next step we will define the different classes of morphisms needed for the model structure, as given in [MaBe18].

The □ in the next definition is a placeholder for the different categories, which we will use for our purpose. It can stand for **bi-Cat**, **bi-Grpd**, $\textbf{bi-Cat}_\text{s}$, $\textbf{bi-Grpd}_\text{s}$, 2- **Cat** or 2- **Grpd**.

Definition 10.79 (Model Structure on $\square$)**.** *A morphism $F : \mathscr{A} \to \mathscr{B}$ is called a*

1. ***weak equivalence****, if the following conditions are satisfies.*
   (a) *For every 0-cell $B \in \mathscr{B}$, there is a 0-cell $A' \in \mathscr{A}$ and a 1-cell $b : B \to FA'$ in $\mathscr{B}$.*
   (b) *For any 0-cells A, A' in $\mathscr{A}$, the functor $F_{AA'} : \mathscr{A}(A, A') \to \mathscr{B}(FA, FA')$ is an equivalence of categories.*

   This morphisms are usually referred to as equivalences of bi-categories.
2. ***cofibration****, if the following conditions are satisfied.*
   (a) *The function $F : \mathscr{A}_0 \to \mathscr{B}_0$ is injective.*
   (b) *For any 0-cells A, A' in $\mathscr{A}$, the functor $F_{AA'} : \mathscr{A}(A, A') \to \mathscr{B}(FA, FA')$ is injective on objects.*
3. ***fibration****, if the following conditions are satisfied.*
   (a) *For any 0-cell A' in $\mathscr{A}$ and every 1-cell $b : B \to FA'$ in $\mathscr{B}$, there is a 1-cell $a : A \to A'$ in $\mathscr{A}$ such that $FA = B$ and $Fa = b$.*
   (b) *For any 1-cell $a' : A \to A'$ in $\mathscr{A}$ and every 2-cell $\beta : b \Rightarrow Fa'$ in $\mathscr{B}$, there is a 2-cell $\alpha : a \Rightarrow a'$ in $\mathscr{A}$ such that $Fa = b$ and $F\alpha = \beta$.*

We denote this model structure by $\square_{\mathcal{M}_2}$.

Remark 10.80. *The above model structure is really a generalisation of the one for* **Grpd***, provided in the last section. This is seen by the property (b) for any of the three classes of morphisms of the above definition.*

To sum up the conditions a bit, we say that a morphism is a cofibration, if it is injective on 0-cells and locally injective on 1-cells, it is a fibration, if it lifts 1-cells and 2-cells and finally it is a weak equivalence if it is a biequivalence.

It is also worth pointing out, that the acyclic fibrations may be seen as those weak equivalences, which are surjective on 0-cells and locally surjective on 1-cells. ⧫

The next result is Theorem 3.5 in [MaBe18].

Theorem 10.81 ([MaBe18])**.** $\textbf{bi-Grpd}_{\mathcal{M}_2}$ *is a model category structure for* **bi-Grpd**.

The proof provided in [MaBe18] is elementary in the sense, that no heavy machinery (like the small onject argument) is used in order to verify all the conditions needed for a model category. Though elementary, the proof is still involved and quite long.

Notice, that we do not claim that the resulting model structure is a model category (this would be wrong, since we do not have bicompleteness in this case).

The next result can for example be found in [IeMoJaSv93] and [StLa]. The case for 2-**Grpd** can also be deduced from Theorem 10.81.

Theorem 10.82. $2\text{-}\mathbf{Cat}_{\mathcal{M}_2}$ *is a model category structure for* 2-**Cat** *turning it into a model category, and similar for* 2-**Grpd**.

Luckily, this can also be generalised slightly in the following way, it is also due to [StLa]. The case for $\textbf{bi-Grpd}_{\mathrm{s}}$ can also be deduced from Theorem 10.81.

Theorem 10.83. $\textbf{bi-Cat}_{s\mathcal{M}_2}$ *is a model category structure for* $\textbf{bi-Cat}_s$ *turning it into a model category, and similar for* $\textbf{bi-Grpd}_s$.

The next result is very useful, as it will help us with the verification of the Quillen pair needed later on.

Lemma 10.84. *1.* $F \in \mathcal{F}_{\textbf{bi-Grpd}_s}$ *iff* F *has the RLP with respect to* $J_{\textbf{bi-Grpd}}$.
2. $F \in \mathcal{W}_{\textbf{bi-Grpd}_s} \cap \mathcal{F}_{\textbf{bi-Grpd}_s}$ *iff* F *has the RLP with respect to* $I_{\textbf{bi-Grpd}}$.

Proof. This is very similar as for the case discussed in the last section. The first equivalence is established by the definition of a fibration in $\textbf{bi-Grpd}_{\mathrm{s}\mathcal{M}_2}$ and Lemma 10.78. The right direction of the second equivalence is also by definition of acyclic fibration and Lemma 10.78.

The other direction needs to consider the liftings for $n = 0, 1, 2, 3$ of the following commutative diagrams

$$\begin{tikzcd} \Pi_2(\partial\Delta^n) \arrow[r] \arrow[d] & \mathscr{C} \arrow[d, "F"] \\ \Pi_2(\Delta^n) \arrow[r] \arrow[ur, dotted] & \mathscr{D}. \end{tikzcd}$$

The case $n = 0$, corresponds to surjectivity on 0-cells. $n = 1$ gives locally surjectivity on 1-cells. Finally the cases $n = 2, 3$ give the property of being a weak equivalence in $\textbf{bi-Grpd}_{\mathrm{s}\mathcal{M}_2}$. That is, the functors $F_{X,Y} : \mathscr{C}(X, Y) \to \mathscr{D}(F(X), F(Y))$ are fully faithful. Furthermore if we consider the functor $(F_{X,Y})_{f,g} : \mathscr{C}(X, Y)(f, g) \to \mathscr{D}(F(X), F(Y)) \to \mathscr{D}(F(X), F(Y))(F(f), F(g))$ and choose some α and α' in $\mathscr{C}(X, Y)(f, g)$ such that $F(\alpha) = F(\alpha') : F(f) \Rightarrow F(g)$, then we may conclude ($n = 3$) that $\alpha = \alpha'$. And hence $F \in \mathcal{W}_{\textbf{bi-Grpd}_{\mathrm{s}}} \cap \mathcal{F}_{\textbf{bi-Grpd}_{\mathrm{s}}}$. ■

Remark 10.85. *With the above proof, we are able to refine the definitions of the generating cofibrations and generating acyclic cofibrations for* $\textbf{bi-Grpd}_s$ *respectively. That is*

$$\begin{aligned} I_{\textbf{bi-Grpd}} &:= \{\Pi_2(\partial\Delta^n \to \Delta^n) \mid n = 0, 1, 2, 3)\} \\ J_{\textbf{bi-Grpd}} &:= \{\Pi_2(\Lambda^n_k \to \Delta^n) \mid n = 0, 1; n \geq k \geq 0\}. \end{aligned}$$

♦

Hence we get the following result.

Corollary 10.86. $(\textbf{bi-Grpd}_{s\mathcal{M}_2}, I_{\textbf{bi-Grpd}}, J_{\textbf{bi-Grpd}})$ *is a cofibrantly generated model category structure for* $\textbf{bi-Grpd}_s$ *turning it into a model category. A similar statement can be made for the case of* 2-$\textbf{Grpd}$.

The statement above can be shown essentially in the same way as Theorem 3 and Theorem 4 in [StLa]. It uses Lemma 10.84 and one can further show that the generating cofibrations and generating acyclic cofibrations provide the samll object argument. Since we do not really need this result but only Lemma 10.84, the proof is left as an exercise for the interested reader.

Bi-groupoids and the Fundamental Bi-groupoid

Definition 10.87 (Fundamental Bi-groupoid)**.** *We define the* ***fundamental bi-groupoid*** *for a topological space X in the following way. The objects are the points in the space X. The 1-morphisms are continuous paths in X and the 2-morphisms are homotopy classes between continuous paths up to higher homotopy. There exist two notions of homotopy in this category, homotopy between continuous paths and homotopy between 2-morphisms. We denote this bi-groupoid by $\Pi_{\leq 2}(X)$.*

Remark 10.88. *The above category consists of the following kinds of homotopies. Homotopies between 1-morphisms are the usual ones i.e. $[f] = [f']$ per definition iff there exists a 2-morphism $\alpha : I \times I \to X$ such that $\alpha(0, t) = f(t)$ and $\alpha(1, t) = f'(t)$ for any $t \in I$. The homotopies between the two cells work in the following way. $[\alpha] = [\alpha']$ per definition iff there is a 3-cell $H : I \times (I \times I) \to X$ such that $H(0, s, t) = \alpha(s, t)$ and $H(1, s, t) = \alpha'(s, t)$ for any $(s, t) \in I \times I$.* ♦

Let $\mathscr{B} \in \mathbf{bi\text{-}Grpd}_{\mathrm{s}}$ and $X, Y \in \mathbf{bi\text{-}Grpd}_{\mathrm{s}}$, then we can define an equivalence relation in the following way.

$$X \sim Y \ :\Leftrightarrow \ \mathscr{B}(X, Y) \neq \emptyset.$$

The equivalence class $[X]$ is called **path component** of $\mathscr{B}$ at X. We call $\pi_0(\mathscr{B})$ **the class of all equivalence classes of objects in** $\mathscr{B}$.

Definition 10.89 (Groupoid of 1-automorphisms)**.** *Let $\mathscr{B} \in \mathbf{bi\text{-}Grpd}_s$ and $X \in \mathrm{ob}(\mathscr{B})$. Then we denote by $Aut_{\mathscr{B}}(X) := \mathscr{B}(X, X)$ the* ***groupoid of 1-automorphisms of $\mathscr{B}$ at*** *X.*

Definition 10.90 (Automorphism Group of 2-morphisms)**.** *Let $\mathscr{B} \in \mathbf{bi\text{-}Grpd}_s$ and $f \in \mathrm{mor}(\mathscr{B})$ such that $f : X \to Y$. Then we denote by $Aut_{\mathscr{B}(X,Y)}(f) := \mathscr{B}(X, Y)(f, f)$ the* ***automorphism group of 2-morphisms of $\mathscr{B}$ at*** *$f : X \to Y$.*

Definition 10.91 (Group of Isomorphism Classes)**.** *Let $\mathscr{B} \in \mathbf{bi\text{-}Grpd}_s$ and $X \in \mathrm{ob}(\mathscr{B})$. We will denote the* ***group of isomorphism classes of*** *$Aut_{\mathscr{B}}(X)$ by $[Aut_{\mathscr{B}}(X)]$. This carries a group structure via concatenation of 1-morphisms.*

We often write $Aut_{\mathscr{B}}(f)$, instead of $Aut_{\mathscr{B}(X,Y)}(f)$ to have a somewhat shorter way to write down everything.

The following remark, tells us, that with the above definitions we can retrieve information about the first and second homotopy groups and π_0, if we consider a topological space.

Remark 10.92. *For $X \in \mathbf{Top}$ and some $x_0 \in X$ we have that $Aut_{\Pi_{\leq 2}(X)}(x_0) \cong \Pi_{\leq 1}(\Omega_{x_0}(X))$. Let $c_{x_0} \in \Omega_{x_0}(X)$ be the constant path at x_0, then*

$$Aut_{\Pi_{\leq 2}(X)}(c_{x_0}) \cong \pi_1(\Omega_{x_0}(X), x_0) \cong \pi_2(X, x_0),$$

since $[S^1, \Omega_{x_0}(X)] \cong [S^2, X]_{x_0} = \pi_2(X, x_0)$.

Therefore $\Pi_{\leq 2}(X)$ provides information about π_0, π_1 and π_2. ♦

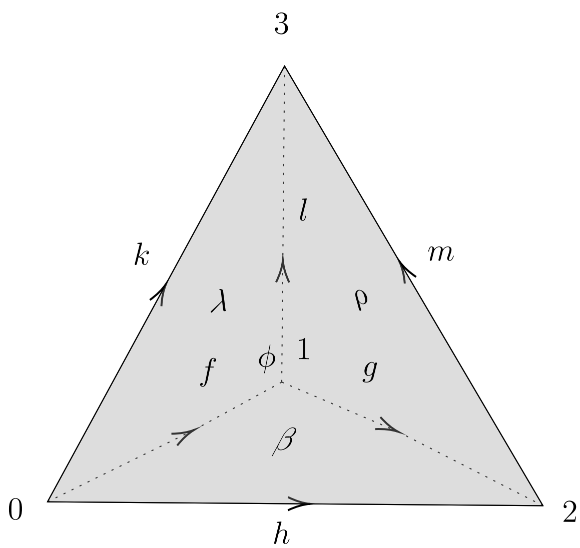
One may construct a bicategory from a simplicial set and call it the **bi-path category of a simplicial set** in the following sense (it is just an alternative construction for Π_2). Let $X_\bullet \in \mathbf{sSet}$. The bi-category will be generated from the following graphs.

$$\begin{tikzcd} X_3 \arrow[r, shift left=6, "d_0^3"] \arrow[r, shift left=2, "d_1^3"] \arrow[r, shift right=2, "d_2^3"] \arrow[r, shift right=6, "d_3^3"] & X_2 \arrow[r, shift left=4, "d_0^2"] \arrow[r, "d_1^2"] \arrow[r, shift right=4, "d_2^2"] & X_1 \arrow[r, shift left=2, "d_0^1"] \arrow[r, shift right=2, "d_1^1"] & X_0 \end{tikzcd}$$

such that

$$(d_1^2(d_0^3(\theta))d_3^3(\theta)) \circ d_1^3(\theta) = d_0^3(\theta) \circ (d_2^2(d_2^3(\theta))d_2^3(\theta))$$

for some 3-cell θ of $X_\bullet$. This ugly expression should encapsulate commutativity of the following graphical interpretation



$$\begin{aligned} \phi : k &\Rightarrow m \circ h && \text{(front face)} \\ \beta : h &\Rightarrow g \circ f && \text{(lower face)} \\ \lambda : k &\Rightarrow l \circ f && \text{(left face)} \\ \rho : l &\Rightarrow m \circ g && \text{(right face)} \end{aligned}$$

where by commutativity of θ we mean that the following square of 2-cells commutes

$$\begin{tikzcd} k \arrow[r, "\phi"] \arrow[d, "\lambda"'] & mh \arrow[d, "m\beta"] \\ lf \arrow[r, "\rho f"'] & mgf. \end{tikzcd}$$

And every triangle in the above diagram has to satisfy the following condition (adapted for the specific case of any triangle)

$$\begin{tikzcd} X_2 \arrow[r, shift left=4, "d_0^2"] \arrow[r, "d_1^2"] \arrow[r, shift right=4, "d_2^2"] & X_1 \arrow[r, shift left=2, "d_0^1"] \arrow[r, shift right=2, "d_1^1"] & X_0 \end{tikzcd}$$

such that

$$d_1^2(\alpha) = d_0^2(\alpha) \circ d_2^2(\alpha)$$

for some 2-cell α of $X_\bullet$. We denote this construction by $\mathcal{P}^2(X_\bullet)$.

Notice, that this construction yields a picture very similar to the one given in Remark 10.71. Therefore, we have the following adjunction. The proof is very much in the same spirit as the one given in the last section.

Lemma 10.93. *There is an adjunction*

$$\mathcal{P}^2 : \mathbf{sSet} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{bi\text{-}Cat}_s : \mathcal{N}_2 \,.$$

We can turn the above construction into a bi-groupoid. We will call this functor $\mathcal{P}^2_{\mathcal{G}}$, hence we have the following corollary.

Corollary 10.94. *There is an adjunction*

$$\mathcal{P}^2_{\mathcal{G}} : \mathbf{sSet} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{bi\text{-}Grpd}_s : \mathcal{N}_2 \,.$$

The proof is very similar as in the last section and will not be provided in detail here, the only difference will be more degeneracy maps encapsulating the same information about morphisms of simplicial sets.

The functor $\mathcal{P}^2_{\mathcal{G}}$ is naturally isomorphic to the functor Π_2.

Theorem 10.95. *Let $X_\bullet \in \mathbf{sSet}_Q$. The bi-groupoids $\Pi_{\leq 2}(|X_\bullet|)$ and $\mathcal{P}^2_{\mathcal{G}}(X_\bullet)$ are biequivalent as bi-categories.*

Proof. First we notice, that the bi-groupoids $\mathcal{P}^2_{\mathcal{G}}(\mathrm{Sing}(|X_\bullet|))$ and $\Pi_{\leq 2}(|X_\bullet|)$ are naturally biequivalent as bi-categories. Indeed, let $X_\bullet \in \mathbf{sSet}_Q$ and consider the following map.

$$\begin{aligned} \gamma_{X_\bullet} : \Pi_{\leq 2}(|X_\bullet|) &\to \mathcal{P}^2_{\mathcal{G}}(\mathrm{Sing}(|X_\bullet|)) \\ f : I \to |X_\bullet| &\mapsto \mathrm{Sing}(f) : \Delta^1 \to \mathrm{Sing}(|X_\bullet|) \\ [\alpha : I \times I \to |X_\bullet|] &\mapsto [\mathrm{Sing}(\alpha) : \Delta^2 \to \mathrm{Sing}(|X_\bullet|)]. \end{aligned}$$

As for the last case, the relation on the left side is equivalence up to homotopy (for the case of the fundamental bi-groupoid) and the right side comes from the relation given by the commutativity of the tetrahedron i.e. $(d_1^2(d_0^3(\theta))d_3^3(\theta)) \circ d_1^3(\theta) = d_0^3(\theta) \circ (d_2^2(d_2^3(\theta))d_2^3(\theta))$ for some 3-simplex $\theta : |\Delta^3| \to |X_\bullet|$ of $\mathrm{Sing}(|X_\bullet|)$. Notice, that in the case of 1-morphisms if we map a 1-morphism under $\gamma_{X_\bullet}$ and then back again we will end up with something weakly equivalent to the identity. The above map is an isomorphism for 2-morphisms, the argument here is very similar as in the last section. It is indeed also well defined. Choose some α and α' in $[\alpha : I \times I \to |X_\bullet|]$, that is α and α' are homotopy equivalent. Mapped under $\gamma_{X_\bullet}$, this can be arranged in a tetrahedron, such that we may relate those two faces of the tetrahedron via identities. Therefore $\alpha \sim \alpha'$ and the map is indeed well defined.

Next we argue that the above defined map is an isomorphism in the case of 2-morphisms. If we are to map some 2-morphism from the left hand side to the right hand side and then back again this is a composition which is related to the identity (equivalent to it). But for us to haven an isomorphism this should actually be an equality. This can be fixed with the following Quillen equivalence already used in the proof for the statement in the last section

$$| \ | : \mathbf{sSet}_Q \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Top}_Q : \mathrm{Sing}\,.$$

Since the composition we are interested in will be weakly equivalent to the identity which is in the equivalence class of the identity in $\Pi_{\leq 2}(|X_\bullet|)$. This shows that our map is indeed an isomorphism.

We have to show, that $\mathcal{P}^2_{\mathcal{G}}$ sends weak equivalences in $\mathbf{sSet}_Q$ to weak equivalences in $\mathbf{bi\text{-}Grpd}_{s\mathcal{M}_2}$. Let $f \in \mathcal{W}_{\mathbf{sSet}}$ then $f = q \circ j$ for some $q \in \mathcal{W}_{\mathbf{sSet}} \cap \mathcal{F}_{\mathbf{sSet}}$ and some $j \in \mathcal{W}_{\mathbf{sSet}} \cap \mathcal{C}_{\mathbf{sSet}}$ (from the properties of FWFS and the 2-3 property).

The map q is inverse to an acyclic cofibration and every acyclic cofibration is a retract of a map which is a filtered colimit of pushouts of maps of the form $\Lambda^n_k \to \Delta^n$. Therefore it suffices to show, by Ken Brown's lemma, that $\mathcal{P}^2_{\mathcal{G}}$ takes pushouts of maps $\Lambda^n_k \to \Delta^n$ to biequivalences of bi-categories.

The map $\mathcal{P}^2_{\mathcal{G}}(\Lambda^n_k) \to \mathcal{P}^2_{\mathcal{G}}(\Delta^n)$ is an isomorphism of bi-groupoids if $n \geq 3$. Unfortunately the argument with strong deformation retracts, which we used in the last section, will not work here. But there is still a way to show the desired result. We are able to give pushouts of the following form for $i = 0, 1, 2$.

$$\begin{tikzcd}
\mathcal{P}^2_{\mathcal{G}}(\Lambda^2_i) \arrow[r] \arrow[d] & \mathcal{P}^2_{\mathcal{G}}(X_\bullet) \arrow[d] \\
\mathcal{P}^2_{\mathcal{G}}(\Delta^2) \arrow[r] & \mathcal{P}^2_{\mathcal{G}}(X_\bullet).
\end{tikzcd}$$

Indeed, Since $\mathcal{P}^2_{\mathcal{G}}(X_\bullet)$ provides the information about the composition for $\mathcal{P}^2_{\mathcal{G}}(\Lambda^2_i)$ such that $\mathcal{P}^2_{\mathcal{G}}(\Delta^2)$ gets glued onto the image of $\mathcal{P}^2_{\mathcal{G}}(\Lambda^2_i)$. Therefore such a pushout may always be constructed, if we consider the identity on the right hand side.

Isomorphisms of bi-groupoids are closed under pushouts, therefore $\mathcal{P}^2_{\mathcal{G}}$ takes weak equivalences to weak equivalences. Hence we especially have that $\mathcal{P}^2_{\mathcal{G}}(X_\bullet)$ and $\Pi_{\leq 2}(|X_\bullet|)$ are naturally biequivalent as bi-categories, i.e. $X_\bullet \to \mathrm{Sing}(|X_\bullet|)$ is a weak equivalence and hence sent to a biequivalence of bi-groupoids. ■

From a similar argument as in the last section, adapted for this slightly more complicated case (it is just more complicated, because there is more notation, the argument is exactly the same) we have the following corollary.

Corollary 10.96. *Let $X_\bullet \in \mathbf{sSet}_Q$ be fibrant, then we have the following isomorphisms*

$$\pi_1(X_\bullet, x) \cong [Aut_{\Pi_2(X_\bullet)}(x)] \cong [Aut_{\Pi_{\leq 2}(|X_\bullet|)}(x)] \cong \pi_1(|X_\bullet|, x)$$
$$\pi_2(X_\bullet, x) \cong Aut_{\Pi_2(X_\bullet)}(c_x) \cong Aut_{\Pi_{\leq 2}(|X_\bullet|)}(c_x) \cong \pi_2(|X_\bullet|, x).$$

Proof. We first establish the isomoprhism of groups

$$[Aut_{\Pi_2(X_\bullet)}(x)] \cong [Aut_{\Pi_{\leq 2}(|X_\bullet|)}(x)].$$

By Theorem 10.95, the bi-groupoids $\mathcal{P}^2_{\mathcal{G}}(X_\bullet)$ and $\Pi_{\leq 2}(|X_\bullet|)$ are biequivalent as bi-categories, denote this bifunctor with F. Furthermore, since we want to establish isomorphisms of homotopy groups in the settings of simplicial sets and topological spaces with one another, we want $X_\bullet$ to be a fibrant simplicial set. We also use the fact, that the functors $\mathcal{P}^2_{\mathcal{G}}$ and Π_2 are naturally isomorphic.

From the above biequivalence we get an equivalence of categories between the following groupoids

$$G : Aut_{\Pi_2(X_\bullet)}(x) \simeq Aut_{\Pi_{\leq 2}(|X_\bullet|)}(x).$$

Then, there is also a map between groups

$$f : [Aut_{\Pi_2(X_\bullet)}(x)] \to [Aut_{\Pi_{\leq 2}(|X_\bullet|)}(x)].$$

We will now argue, why f is actually an isomorphism. Surjectivity follows directly from the essentially surjectivity of G, since essentially surjective is considered up to isomorphism. Injectivity follows from the fully faithfulness of G and the fact, that we work with strict 2-functors. Therefore

$$[Aut_{\Pi_2(X_\bullet)}(x)] \cong [Aut_{\Pi_{\leq 2}(|X_\bullet|)}(x)].$$

Next we establish the isomorphism of groups

$$Aut_{\Pi_2(X_\bullet)}(c_x) \cong Aut_{\Pi_{\leq 2}(|X_\bullet|)}(c_x).$$

This is now very similar as in the proof given in the last section. Again we may conclude by the equivalence of categories $G : Aut_{\Pi_2(X_\bullet)}(x) \simeq Aut_{\Pi_{\leq 2}(|X_\bullet|)}(x)$. Indeed, our case is just a special case for the constant path c_x. That is, $Aut_{\Pi_2(X_\bullet)}(c_x)$ and $Aut_{\Pi_{\leq 2}(|X_\bullet|)}(c_x)$ are the sets of morphisms with respect to c_x respectively. The key point is now, that our bifunctor sends identities to identities, but since we work with strict 2-functors this is indeed the case. Therefore, the isomorphism follows from the fully faithulness and essentially surjectivity of our bifunctor and hence

$$Aut_{\Pi_2(X_\bullet)}(c_x) \cong Aut_{\Pi_{\leq 2}(|X_\bullet|)}(c_x).$$

■

The Quillen Equivalence

As it turns out, we can now already show that there is a Quillen pair. Again, as in the last section, we use Lemma 2.33 to conclude.

Proposition 10.97. *There is a Quillen pair*

$$\Pi_2 : \mathbf{sSet}_Q \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \textbf{bi-Grpd}_{s\mathcal{M}_2} : \mathcal{N}_2 \,.$$

Proof. The above lemma and corollary provide all needed ingredients to apply Lemma 2.33, which then implies the result. ■

A last step before we show the main result is, to argue that this also yields a Quillen pair for the truncated case.

Proposition 10.98. *There is a Quillen pair*

$$\Pi_2 : \mathbf{sSet}_Q^{\leq 2} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \textbf{bi-Grpd}_{s\mathcal{M}_2} : \mathcal{N}_2 \,.$$

Proof. As we have seen , any object in $\mathbf{sSet}_Q$ is cofibrant. Proposition 5.35 now states, that if $\Pi_2 : \mathbf{sSet}_Q \rightarrow_{\mathcal{M}_2}$ is a left Quillen functor, which holds by the above Proposition, and $\Pi_2(Q\mathcal{S}) \in \mathcal{W}_{\textbf{bi-Grpd}_\mathrm{s}}$ for $\mathcal{S} = \{\partial\Delta^4 \to \Delta^4\}$, then $\Pi_2 : \mathbf{sSet}_Q^{\leq 2} \to \textbf{bi-Grpd}_{\mathrm{s}\mathcal{M}_2}$ is also a left Quillen functor. In fact, $Q\mathcal{S}$ is just $\mathcal{S}$ itself, and as pointed out before, there is an isomorphism $\Pi_2(\partial\Delta^4) \to \Pi_2(\Delta^4)$ in $\textbf{bi-Grpd}_{\mathrm{s}\mathcal{M}_2}$ and hence also a weak equivalence. Therefore, the functor $\Pi_2 : \mathbf{sSet}_Q^{\leq 2} \to \textbf{bi-Grpd}_{\mathrm{s}\mathcal{M}_2}$ is indeed a left Quillen functor and therefore we get a Quillen pair

$$\Pi_2 : \mathbf{sSet}_Q^{\leq 2} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \textbf{bi-Grpd}_{\mathrm{s}\mathcal{M}_2} : \mathcal{N}_2 \,.$$

■

Behold, it follows one of the main theorems of the thesis.

Theorem 10.99. *There is a Quillen equivalence*

$$\Pi_2 : \mathbf{sSet}_Q^{\leq 2} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \textbf{bi-Grpd}_{s\mathcal{M}_2} : \mathcal{N}_2 \,.$$

Proof. We have to show, that the derived unit and counit are weak equivalences, but since every object in $\mathbf{sSet}_Q$ is cofibrant and with the definition of fibration for 2-groupoids, every object in $\textbf{bi-Grpd}_{\mathrm{s}\mathcal{M}_2}$ is fibrant and therefore it suffices to show, that the maps $\epsilon : \Pi_2(\mathcal{N}_2) \to id_{\textbf{bi-Grpd}_{\mathrm{s}\mathcal{M}_2}}$ and $\eta : id_{\mathbf{sSet}_Q^{\leq 2}} \to \mathcal{N}_2(\Pi_2)$ are weak equivalences. Since the nerve $\mathcal{N}_2$ is fully faithful by Proposition 10.75 and hence by Lemma A.15, we have that $\Pi_2(\mathcal{N}_2) \to id_{\textbf{bi-Grpd}_{\mathrm{s}\mathcal{M}_2}}$ is an isomorphism and hence also a weak equivalence.

We are left to show, that $\eta : id_{\mathbf{sSet}_Q^{\leq 2}} \to \mathcal{N}_2(\Pi_2)$ is a weak equivalence in $\mathbf{sSet}_Q^{\leq 2}$ i.e. that we have isomorphisms $\pi_i(X_\bullet) \cong \pi_i(\mathcal{N}_2(\Pi_2(X_\bullet)))$ for $i = 0, 1, 2$. This condition is enough, according to Proposition 10.17.

Now for the next idea to work, we need to restrict ourselves to the case, where we consider fibrant simplicial sets. Therefore we choose a fibrant replacement of the simplicial set $X_\bullet$ i.e. $X_\bullet \xrightarrow{\sim} X_\bullet^f$, which is a weak equivalence such that $X_\bullet^f$ is a Kan complex. We will use the notation $X_\bullet^f$ instead of $QX_\bullet$ for convenience.

Now the case for $i = 0$ follows by construction. The cases $i = 1, 2$ are a bit more involved. Since we work with the fibrant replacement we may apply Theorem 10.95. This yields isomorphisms

$$\pi_1(X_\bullet^f, x) \cong [Aut_{\Pi_2(X_\bullet^f)}(x)]$$
$$\pi_2(X_\bullet^f, x) \cong Aut_{\Pi_2(X_\bullet^f)}(c_x)$$

But then we have the following isomorphisms

$$\begin{cases} \pi_1(X_\bullet^f, x) \cong [Aut_{\Pi_2(X_\bullet)}(x)] & \forall\ x \in X_\bullet^f \\ \pi_1(\mathcal{N}_2(\Pi_2(X_\bullet^f)), x) \cong [Aut_{\Pi_2(\mathcal{N}_2(\Pi_2(X_\bullet^f)))}(x)] & \forall\ x \in X_\bullet^f \\ \pi_2(X_\bullet^f, x) \cong Aut_{\Pi_2(X_\bullet)}(c_x) & \forall\ x \in X_\bullet^f \\ \pi_2(\mathcal{N}_2(\Pi_2(X_\bullet^f)), x) \cong Aut_{\Pi_2(\mathcal{N}_2(\Pi_2(X_\bullet^f)))}(c_x) & \forall\ x \in X_\bullet^f. \end{cases}$$

If we apply Π_2 to the isomorphism $\Pi_2\mathcal{N}_2 \to id_{\textbf{bi-Grpd}_{s\mathcal{M}_2}}$ we end up with an isomorphism

$$\Pi_2(X_\bullet^f) \xrightarrow{\cong} \Pi_2(\mathcal{N}_2(\Pi_2(X_\bullet^f))).$$

But this yields the following isomorphisms (by Corollary 10.96, and the fact that $\mathcal{N}_2$ is fully faithful)

$$\pi_1(X_\bullet^f, x) \cong [Aut_{\Pi_2(X_\bullet^f)}(x)] \cong [Aut_{\Pi_2(\mathcal{N}_2(\Pi_2(X_\bullet^f)))}(x)] \cong \pi_1(\mathcal{N}_2(\Pi_2(X_\bullet^f)), x)$$
$$\pi_2(X_\bullet^f, x) \cong Aut_{\Pi_2(X_\bullet^f)}(c_x) \cong Aut_{\Pi_2(\mathcal{N}_2(\Pi_2(X_\bullet^f)))}(c_x) \cong \pi_2(\mathcal{N}_2(\Pi_2(X_\bullet^f)), x).$$

Therefore we end up with

$$\begin{cases} \pi_0(X_\bullet^f, x) \cong \pi_0(\mathcal{N}_2(\Pi_2(X_\bullet^f)), x) & \forall\ x \in X_\bullet^f \\ \pi_1(X_\bullet^f, x) \cong \pi_1(\mathcal{N}_2(\Pi_2(X_\bullet^f)), x) & \forall\ x \in X_\bullet^f \\ \pi_2(X_\bullet^f, x) \cong \pi_2(\mathcal{N}_2(\Pi_2(X_\bullet^f)), x) & \forall\ x \in X_\bullet^f. \end{cases}$$

Finally by Proposition 10.17, the map $X_\bullet^f \to \mathcal{N}_2(\Pi_2(X_\bullet^f))$ is a weak equivalence in $\textbf{sSet}_Q^{\leq 2}$.

Time to address the problem with the fibrant replacement, since we want this to hold for any simplicial set. In such cases one applies Ken Brown.

First, notice that the weak equivalence $X_\bullet \xrightarrow{\sim} X_\bullet^f$ implies that $\Pi_2(X_\bullet) \to \Pi_2(X_\bullet^f)$ is also a weak equivalence, due to the fact that the first map is actually an acyclic cofibration and the left adjoint preserves acyclic cofibrations. Finally the map $\mathcal{N}_2(\Pi_2(X_\bullet)) \to \mathcal{N}_2(\Pi_2(X_\bullet^f))$ is also a weak equivalence due to Ken Brown's lemma (since all bi-groupoids are fibrant).

The final step involves enjoying the following commutative diagram.

$$\begin{tikzcd}
X_\bullet \arrow[r] \arrow[d, "\sim"'] & \mathcal{N}_2(\Pi_2(X_\bullet)) \arrow[d, "\sim"] \\
X_\bullet^f \arrow[r, "\sim"'] & \mathcal{N}_2(\Pi_2(X_\bullet^f))
\end{tikzcd}$$

Since weak equivalences have the 2-3 property the horizontal map must also be a weak equivalence, which concludes the proof. ■

Of course essentially the same arguments also give us the following Quillen equivalence for the more restricted case. It can for instance also be found in [IeMoJaSv93], but with a different approach and a different proof.

Theorem 10.100. *There is a Quillen equivalence*

$$\Pi_2 : \mathbf{sSet}_Q^{\leq 2} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} 2\text{-}\mathbf{Grpd}_{\mathcal{M}_2} : \mathcal{N}_2 \,.$$

Theorem 10.99 gives an equivalence of categories between the homotopy category $\mathrm{Ho}(\mathbf{sSet}_Q^{\leq 2})$ and the homotopy category $\mathrm{Ho}(\textbf{bi-Grpd}_{s\mathcal{M}_2})$, the remark after the theorem relates the two homotopy categories $\mathrm{Ho}(\textbf{bi-Grpd}_{s\mathcal{M}_2})$ and $\mathrm{Ho}(2\text{-}\mathbf{Grpd}_{\mathcal{M}_2})$. Following [StLa06] there is also an equivalence of categories between the homotopy category $\textbf{bi-Grpd}[\mathcal{W}^{-1}]$ and $\mathrm{Ho}(2\text{-}\mathbf{Grpd})$.

Proposition 10.101 ([StLa06])**.** *The inclusions* $2\text{-}\mathbf{Cat} \to 2\text{-}\mathbf{Cat}_w$ *and* $2\text{-}\mathbf{Cat} \to \textbf{bi-Cat}$ *have left adjoints. Furthermore, the inclusions* $2\text{-}\mathbf{Grpd} \to 2\text{-}\mathbf{Grpd}_w$ *and* $2\text{-}\mathbf{Grpd} \to \textbf{bi-Grpd}$ *have left adjoints.*

Therefore, there are adjoint equivalences of categories $\mathrm{Ho}(2\text{-}\boldsymbol{Cat}) \simeq \mathrm{Ho}(2\text{-}\boldsymbol{Cat}_w)$ *and* $\mathrm{Ho}(2\text{-}\boldsymbol{Cat}) \simeq \textbf{bi-Cat}[\mathcal{W}^{-1}]$*. Furthermore there are adjoint equivalences of categories* $\mathrm{Ho}(2\text{-}\mathbf{Grpd}) \simeq \mathrm{Ho}(2\text{-}\mathbf{Grpd}_w)$ *and* $\mathrm{Ho}(2\text{-}\mathbf{Grpd}) \simeq \textbf{bi-Grpd}[\mathcal{W}^{-1}]$*.*

Notice, that even though **bi-Grpd** is not a model category, we can still consider its homotopy category - though it may not at all be as well behaved in this case - $\textbf{bi-Grpd}[\mathcal{W}^{-}1]$. The category is created via a localisation with respect to all weak equivalences where by a weak equivalence we mean one of the model category structure $\mathcal{M}_2$. I will not use the notation $\mathrm{Ho}(\textbf{bi-Cat}_{\mathcal{M}_2})$ on purpose to avoid confusion, since **bi-Grpd** is not a model category. With this in mind we are able to give a possible way of comparing to compare the desired homotopy theories.

Still, the situation is not too bad though. The localisation adjunction is actually a Quillen equivalence in the sense that the derived unit and counit are equivalences of categories. The lack of bicompleteness is not so much of a problem in the end, since all other properties of being a model category are satisfied.

Furthermore the lack of limits and colimits can not really be carried out in the weak setting but they can be carried out in the strict setting which is just as good. Also we still have finite products and coproducts in the weak case.

This then yields the following concluding result.

Theorem 10.102. *There is an equivalence of categories*

$$\mathrm{Ho}(\mathbf{sSet}_Q^{\leq 2}) \simeq \mathbf{bi\text{-}Grpd}[\mathcal{W}^{-1}].$$

Proof. Combine Theorem 10.99, Theorem 10.100 and Proposition 10.101. ∎

11 Conclusion

We actually managed to show, that there are some nice Quillen equivalences relating the homotopy theories of these categories and giving us nice models for such theories. In the end we also argued, that the homotopy theory for the 2-truncated simplicial sets is somewhat related to the one for bi-groupoids.

All in all some pretty nice results indeed. Unfortunately it is not possible to to give the last result in the form of a Quillen equivalence, i.e. it would have been really beautiful to give and prove the following Quillen equivalence

$$\Pi_2 : \mathbf{sSet}_Q^{\leq 2} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{bi\text{-}Grpd}_{\mathcal{M}_2} : \mathcal{N}_2 \,.$$

But unfortunately this is simply not possible, and it is actually amazing that it fails because of bicompleteness. As we have seen, there is no hope that the category **bi-Grpd** is complete and cocomplete.

Nonetheless, the main goal - to relate the homotopy theories of 2-truncated simplicial sets and the homotopy theory given by bi-groupoids - was successful. Of course the other way would have been a lot more elegant, but nice things do not always happen in real life.

Anyway, this should teach a very valuable lesson about intuition and that it is often very important to work out claims and ideas, even if they seem very obvious or make a lot of sense on first or even second glance.

But it also teaches us that sometimes, there can be different ways of achieving ones goals. Maybe just not as one has expected something to work, but at least it shows that the intuition was correct up to some extend.

Since we apparently like to generalise things, let me say the following. From the experience gathered in this present work, I personally do not think, that this approach can be generalised for higher cases. For instance, in the case $n = 3$ it is not possible to relate the strict and weak cases in such a way (i.e. it is known that strict tricategories and weak tricategories are not always equivalent), which saved us this time for $n = 2$. For such ideas, one would need to come up with a different approach or theory.

Finally, let me say that I could enjoy myself quite a bit working with this theory. I was able to accumulate a lot of new knowledge about very interesting topics in mathematics. However frustrating it got at times, I am still thankful for the chance that I was able to work with this theory.

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Last but not least I also want to thank all my friends who were always ready to listen to my mathematical nonsense.

I conclude this work with the following quote.

> *'It's a dangerous business, Frodo, going out your door. You step onto the road, and if you don't keep your feet, there's no knowing where you might be swept off to.'*
>
> -J.R.R. Tolkien, The Fellowship of the Ring

A Category Theory

A nice work on category theory can be found in [SMcl97], [ToLe16] and [ERie14]. We follow all of them and give the most important concepts and ideas, this will be very brief and for more details one is asked to read one of the above citations.

A.1 Categories

We start with the definition of a category.

Definition A.1 (Category). *A* ***category*** $\mathscr{C}$ *consists of the following data.*

1. *A collection of objects* $\mathrm{ob}(\mathscr{C})$*.*
2. *For each* $A, B \in \mathrm{ob}(\mathscr{C})$*, a collection* $\mathscr{C}(A, B)$ *of maps or arrows or morphisms from* A *to* B*.*
3. *For each* $A, B, C \in \mathrm{ob}(\mathscr{C})$*, a function*

$$\begin{aligned} \mathscr{C}(B, C) \times \mathscr{C}(A, B) &\to \mathscr{C}(A, C) \\ (g, f) &\mapsto g \circ f, \end{aligned}$$

   called composition.
4. *For each* $A \in \mathrm{ob}(\mathscr{C})$*, an element* 1_A *of* $\mathscr{C}(A, A)$ *called the identity of* A*.*

Such that the following axioms are satisfied.

1. *(Associativity) For each* $f \in \mathscr{C}(A, B)$*,* $g \in \mathscr{C}(B, C)$ *and* $h \in \mathscr{C}(C, D)$ *we have*

$$(h \circ g) \circ f = h \circ (g \circ f).$$

2. *(Identity Laws) For each* $f \in \mathscr{C}(A, B)$*, we have*

$$f \circ 1_A = f = 1_B \circ f.$$

For convenience, one often writes $A \in \mathscr{C}$ for $A \in \mathrm{ob}(\mathscr{C})$ and similarly $f : A \to B$ for $f \in \mathscr{C}(A, B)$.

Every category $\mathscr{C}$ also has a **dual category** denoted $\mathscr{C}^{op}$, in which the arrows are reversed.

There is also the principle of duality in category, which is quite important. Basically it states that, every categorical definition, theorem and proof has a dual statement, obtained by reversing all the arrows.

A category $\mathscr{C}$ is called **locally small** if the classes $\mathscr{C}(A, B)$ are proper sets for any $A, B \in \mathscr{C}$. Furthermore, we call a category $\mathscr{C}$ **small** if it is locally small and in addition $\mathrm{ob}(\mathscr{C})$ is a proper set rather than a class.

A.2 Functors

We will now define functors, which are the "maps" between categories.

Definition A.2 (Functor)**.** *Let $\mathscr{C}$ and $\mathscr{D}$ be categories. A **functor** $F : \mathscr{C} \to \mathscr{D}$ consists of the following data.*

1. *A function $\mathrm{ob}(\mathscr{C}) \to \mathrm{ob}(\mathscr{D})$, written as $A \mapsto F(A)$.*
2. *For each $A, A' \in \mathscr{C}$, a function $\mathscr{C}(A, A') \to \mathscr{D}(F(A), F)(A'))$, written as $f \mapsto F(f)$.*

Satisfying the following axioms.

1. *$F(f' \circ f) = F(f') \circ F(f)$, whenever $A \xrightarrow{f} A' \xrightarrow{f'} A'' \in \mathscr{C}$.*
2. *$F(1_A) = 1_{F(A)}$, whenever $A \in \mathscr{C}$.*

Definition A.3. *Let $\mathscr{C}$ and $\mathscr{D}$ be categories. A **contravariant functor** from $\mathscr{C}$ to $\mathscr{D}$ is a functor $\mathscr{C}^{op}$ to $\mathscr{D}$.*

Definition A.4 (Full, Faithful)**.** *A functor $F : \mathscr{C} \to \mathscr{D}$ is **faithful/full**, if for any $A, A' \in \mathscr{C}$, the function*

$$\begin{aligned} \mathscr{C}(A, A') &\to \mathscr{D}(F(A), F(A')) \\ f &\mapsto F(f) \end{aligned}$$

is injective/surjective.

If a functor is full and faithful we often call it **fully faithful**.

Definition A.5 (Subcategory)**.** *Let $\mathscr{C}$ be a category. A **subcategory** $\mathscr{A}$ of $\mathscr{C}$ consists of a subclass $\mathrm{ob}(\mathscr{A})$ of $\mathrm{ob}(\mathscr{C})$, and for each $A, A' \in \mathrm{ob}(\mathscr{A})$ a subclass $\mathscr{A}(A, A')$ of $\mathscr{C}(A, A')$ such that $\mathscr{A}$ is closed under composition and identities.*

*It is called **full subcategory**, if in addition $\mathscr{A}(A, A') = \mathscr{C}(A, A')$ for any $A, A' \in \mathrm{ob}(\mathscr{A})$.*

Since this may arise, when one thinks about functors. It is not true, that the image of a functor is a subcategory in general.

A.3 Natural Transformations

Next comes the concept of natural transformations.

Definition A.6 (Natural Transformation)**.** *Let $\mathscr{C}$ and $\mathscr{D}$ be categories and let*

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] & \mathscr{D} \end{tikzcd}$$

*be functors. A **natura transformation** $\alpha : F \Rightarrow G$ is a family $(F(A) \xrightarrow{\sigma_A} G(A))_{A \in \mathscr{C}}$ of maps in $\mathscr{D}$ such that for every map $f : A \to A' \in \mathscr{C}$, there is a commuting square*

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha_A"'] & F(A') \arrow[d, "\alpha_{A'}"] \\ G(A) \arrow[r, "G(f)"'] & G(A'). \end{tikzcd}$$

*The maps α_A are called the **components** of α.*

For two categories $\mathscr{C}$ and $\mathscr{D}$ we may build a new category, in which the objects are functors between $\mathscr{C}$ and $\mathscr{D}$ and the morphisms are the natural transformations between functors. It is left as an exercise to show that this is indeed a category. We will call this category **functor category** and denote it with $\mathbf{Fun}(\mathscr{C}, \mathscr{D})$.

Definition A.7 (Natural Isomorphism)**.** *Let $\mathscr{C}$ and $\mathscr{D}$ be categories. A **natural isomorphism** between functors from $\mathscr{C}$ to $\mathscr{D}$ is an isomorphism in $\mathbf{Fun}(\mathscr{C}, \mathscr{D})$.*

So if we are given functors $F, G : \mathscr{C} \to \mathscr{D}$, we say that $F(A) \cong G(A)$ **naturally in** A if F and G are naturally isomorphic.

Definition A.8 (Equivalence)**.** *An **equivalence** between categories $\mathscr{C}$ and $\mathscr{D}$ consists of a pair of functors $F : \mathscr{C} \rightleftarrows \mathscr{D} : G$ together with natural isomorphisms*

$$\eta : 1_{\mathscr{C}} \to G \circ F \qquad \epsilon : F \circ G \to 1_{\mathscr{D}}.$$

Definition A.9 (Essentially Surjective)**.** *A functor $F : \mathscr{C} \to \mathscr{D}$ is called **essentially surjective** if for all $D \in \mathscr{D}$, there exists a $C \in \mathscr{C}$ such that $F(C) \cong D$.*

A nice result is the following.

Proposition A.10. *A functor is an equivalence iff it is fully faithful and essentially surjective.*

A.4 Adjoints

We introduce the concept of adjointness.

Definition A.11 (Adjoint)**.** *Consider some categories and functors $F : \mathscr{C} \rightleftarrows \mathscr{D} : G$. We say that F is **left adjoint** to G, and G is **right adjoint** to F, and write*

$$F : \mathscr{C} \underset{}{\overset{}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathscr{D}$$

if

$$\mathscr{D}(F(A), B) \cong \mathscr{C}(A, G(B))$$

naturally in $A \in \mathscr{C}$ and $B \in \mathscr{D}$.

For what naturally exactly means here one is referred to read p. 41,42 in [ToLe16].

Adjunctions can be composed to again get an adjunction.

A.5 The Yoneda Lemma

The following is one of the most important results in category theory.

Theorem A.12 (Yoneda). *Let $\mathscr{C}$ be a locally small category. Then*

$$\mathbf{Fun}(\mathscr{C}^{op}, \mathbf{Set})(\mathscr{C}(-, A), X) \cong X(A)$$

naturally in $A \in \mathscr{C}$ and $X \in \mathbf{Fun}(\mathscr{C}^{op}, \mathbf{Set})$.

Corollary A.13. *For any locally small category $\mathscr{C}$, the Yoneda embedding*

$$H_\bullet : \mathscr{C} \to \mathbf{Fun}(\mathscr{C}^{op}, \mathbf{Set})$$

if fully faithful.

Proposition A.14. *If $F \dashv G, G'$, then G and G' are naturally isomorphic. Furthermore, if $F, F' \dashv G$, then F and F' are naturally isomorphic.*

Proof. Assume, there are two adjunctions

$$F : \mathscr{C} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathscr{D} : G \quad \text{and} \quad F : \mathscr{C} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathscr{D} : G'.$$

But then, for any $X \in \mathscr{D}$ and $Y \in \mathscr{D}$ we have $\mathscr{C}(F(X), Y) \cong \mathscr{D}(X, G(Y))$ and $\mathscr{C}(F(X), Y) \cong \mathscr{D}(X, G'(Y))$. Then $\mathscr{D}(X, G(Y)) \cong \mathscr{D}(X, G'(Y))$. We conclude by the Yoneda lemma. ■

Lemma A.15. *Let $F : \mathscr{C} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathscr{D} : G$ be an adjunction. Then G is fully faithful iff the counit is an isomorphism. A similar result holds if F is fully faithful.*

Proof. Consider the natural isomorphism of the adjunction

$$\phi : \mathscr{D}(F(X), A) \cong \mathscr{C}(X, G(A)).$$

Also, the isomorphism

$$\mathscr{D}(A, B) \xrightarrow{G} \mathscr{C}(G(A), G(B)) \xrightarrow{\phi^{-1}} \mathscr{D}(F(G(A)), B)$$

associates to each $f \in \mathscr{D}(A, B)$ the morphism $\epsilon_B \circ F(G(f)) \in \mathscr{D}(F(G(A)), B)$. Indeed, since $\phi^{-1}(g) = \epsilon \circ F(g)$.

By naturality of ϵ we get $\epsilon_B \circ F(G(f)) = f \circ \epsilon_A$ i.e. the natural isomorphism is given by precompositon with ϵ_A i.e. it is representes by ϵ_A via the Yoneda embedding. The Yoneda embedding is actually an embedding and since embeddings reflect isomorphisms, we have that ϵ_A is an isomorphism. ■

A.6 Limits and Colimits

We will discuss limits in a category $\mathscr{C}$. A colimit will be a limit in $\mathscr{C}^{op}$. First we introduce three important special cases of limits.

We start with the definition of products and coproducts.

Definition A.16. *Product, Coproduct Let $\mathscr{C}$ be a category, I a set and $(X_i)_{i\in I}$ a family of objects of $\mathscr{C}$. A **product** of $(X_i)_{i\in I}$ consists of an object P and a family of maps*

$$(P \xrightarrow{p_i} X_i)_{i\in I}$$

with the property, that for all objects A and families of maps in $\mathscr{C}$

$$(A \xrightarrow{f_i} X_i)_{i\in I}$$

there exists a unique map $\overline{f} : A \to P$ such that $p_i \circ \overline{f} = f_i$ for all $i \in I$.

*A product in $\mathscr{C}^{op}$ is called **coproduct** in $\mathscr{C}$.*

Next we define equalisers and coequalisers. A **fork** in a category consists of objects and morphisms

$$\begin{tikzcd} A \arrow[r, "f"] & X \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] & Y \end{tikzcd}$$

such that $s \circ f = t \circ f$. There is also the dual concept called a **cofork**.

Definition A.17 (Equalisers, Coequalisers)**.** *Let $\mathscr{C}$ be a category and let $s, t : X \rightrightarrows Y$ be objects and morphisms in $\mathscr{C}$. An **equaliser** of s and t is an object E together with a morphism $E \xrightarrow{i} X$ in $\mathscr{C}$ such that*

$$\begin{tikzcd} E \arrow[r, "i"] & X \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] & Y \end{tikzcd}$$

is a fork and with the property that for any fork

$$\begin{tikzcd} A \arrow[r, "f"] & X \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] & Y \end{tikzcd}$$

in $\mathscr{C}$, there exists a unique map $\overline{f} : A \to E$ such that

$$\begin{tikzcd} A \arrow[d, dotted, "\overline{f}"'] \arrow[dr, "f"] & \\ E \arrow[r, "i"'] & X \end{tikzcd}$$

commutes.

A ***coequaliser*** *in $\mathscr{C}$ is an equaliser in $\mathscr{C}^{op}$.*

Finally we define pullbacks and pushouts.

Definition A.18 (Pullbacks and Pushouts)**.** *Let $\mathscr{C}$ be a category and consider the objects ans morphisms*

$$X \xrightarrow{s} Z \xleftarrow{t} Y$$

in $\mathscr{C}$. A ***pullback*** *of this diagram is an object P in $\mathscr{C}$ together with maps $p_1 : P \to X$ and $p_2 : P \to Y$ such that*

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "t"] \\ X \arrow[r, "s"'] & Z \end{tikzcd}$$

commutes and with the property that for any commutative square

$$\begin{tikzcd} A \arrow[r, "f_2"] \arrow[d, "f_1"'] & Y \arrow[d, "t"] \\ X \arrow[r, "s"'] & Z \end{tikzcd}$$

in $\mathscr{C}$, there exists a unique morphism $\overline{f} : A \to P$ such that the following diagram commutes

$$\begin{tikzcd} A \arrow[dr, dotted, "\overline{f}"] \arrow[drr, bend left, "f_2"] \arrow[ddr, bend right, "f_1"'] & & \\ & P \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "t"] \\ & X \arrow[r, "s"'] & Z. \end{tikzcd}$$

A ***pushout*** *in $\mathscr{C}$ is a pullback in $\mathscr{C}^{op}$.*

From the discussion so far, we will now be able to state the definition of limits and colimits.

Definition A.19 (Limit, Colimit)**.** *Let $\mathscr{C}$ be a category, $\mathscr{I}$ a small category and $D : \mathscr{I} \to \mathscr{C}$ a diagram in $\mathscr{C}$.*

1. *A* ***cone*** *on D is an object A in $\mathscr{C}$ (the* ***vertex*** *of the cone) together with a family*

   $$(A \xrightarrow{f_i} D(I))_{I \in \mathscr{I}}$$

   of morphisms in $\mathscr{C}$ such that for all maps $u : I \to J$ in $\mathscr{I}$, the trinagle

$$\begin{tikzcd} & A \arrow[dl, "f_I"'] \arrow[dr, "f_J"] & \\ D(I) \arrow[rr, "Du"'] & & D(J) \end{tikzcd}$$

commutes.

*A **cocone** in $\mathscr{C}$ is a cone in $\mathscr{C}^{op}$.*

2. *A **limit** of D is a cone $(L \xrightarrow{p_I} D(I))_{I\in\mathscr{I}}$ with the property that for any cone on D (see 1.), there exists a unique map $\overline{f} : A \to L$ such that $p_I \circ \overline{f} = f_I$ for all I in $\mathscr{I}$. The maps p_I are called **projections** of the limit.*

   *A **colimit** in $\mathscr{C}$ is a limit in $\mathscr{C}^{op}$.*

A nice result is the following.

Proposition A.20. *Let $\mathscr{C}$ be a category. If $\mathscr{C}$ has all products and equalisers, then $\mathscr{C}$ has all limits. Of course the dual statement also holds.*

Definition A.21. *Let $\mathscr{C}$ be a category. $\mathscr{C}$ is called **complete**, if $\mathscr{C}$ has all limits and $\mathscr{C}$ is called **cocomplete** if $\mathscr{C}$ has all colimits. Finally, $\mathscr{C}$ is called **bicomplete** if it is complete and cocomplete.*

It should be clear from the context and the definitions, but since it is rather important we will still point it out. Just because a category is complete does not meant that it is immediately bicomplete. One can not argue by duality here, duality merely implies, that $\mathscr{C}^{op}$ is cocomplete which exactly means that $\mathscr{C}$ is complete but does not say anything about the cocompleteness of $\mathscr{C}$.

Usually it iss easier to verify that a category is complete. Cocompleteness is often a challenge, especially the construction of coequalisers is a tough one.

Definition A.22 (Monic, Epic)**.** *Let $\mathscr{C}$ be a category. A morphism $f : X \to Y$ in $\mathscr{C}$ is **monic** (or a **monomorphism**) if for all objects A and morphisms $x, x' : A \rightrightarrows X$,*

$$f \circ x = f \circ x' \;\Rightarrow\; x = x'.$$

*The dual concept is called **epic** (or **epimorphism**).*

A.7 Adjoint Functor Theorems and Representability

The following can be fount in part V of [SMcl97].

Theorem A.23 (Genreal Adjoint Functor Theorem (GAFT))**.** *Let $\mathscr{C}$ be a small-complete category with small hom-sets, a functor $G : A \to X$ has a left adjoint iff it preserves all small limits and satisfies the following solution set condition.*

For any object $x \in X$ there is a small set I and an I-indexed family of arrows $f_i : x \to Ga_i$ such that every arrow $h : x \to Ga$ can be written as a composition $h = Gt \circ f_i$ for some index i and some $t : a_i \to a$.

The above theorem is often also called Freyd adjoint functor theorem.

Theorem A.24 (Representability Theorem)**.** *Let $\mathscr{D}$ be a small complete category with small hom-sets. A functor $K : \mathscr{D} \to \mathbf{Set}$ is representable iff K preserves all small limits and satisfies the following solution set condition.*

There exists a small set S of objects of $\mathscr{D}$ such that for any object $D \in \mathscr{D}$ and any element $X \in KD$, there exists an $s \in S$, an element $Y \in Ks$ and an arrow $f : s \to D$ with $(Kf)Y = X$.

The above theorem is also known as Freyd's representability theorem.

Theorem A.25 (Special Adjoint Functor Theorem (SAFT))**.** *Let $\mathscr{C}$ be a small complete category with small hom-sets and a cogenerating set Q, while every set of subobjects of an object $a \in \mathscr{C}$ has a pullback. Let the category $\mathscr{X}$ have all small hom-sets. Then a functor $G : \mathscr{C} \to \mathscr{X}$ has a left adjoint iff G preserves all small limits and all pullbacks of families of monics.*

A.8 Kan Extensions

The following is from [ERie14], originally it was stated in [SMcl97].

Theorem A.26. *Given a functor $F : \mathscr{C} \to \mathscr{E}$ and $K : \mathscr{C} \to \mathscr{D}$, if for every $d \in \mathscr{D}$ the colimit*

$$\mathrm{Lan}_K F(d) := \mathrm{colim}(K \downarrow d \xrightarrow{\Pi^d} \mathscr{C} \xrightarrow{F} \mathscr{E})$$

exists, then they define the left Kan extension $\mathrm{Lan}_K F : \mathscr{D} \to \mathscr{E}$, in which case the unit transformation $\eta : F \Rightarrow \mathrm{Lan}_K F \cdot K$ can be extracted from the colimit cone. Dually, if for every $d \in \mathscr{D}$ the limit

$$\mathrm{Ran}_K F(d) := \lim(d \downarrow K \xrightarrow{\Pi^d} \mathscr{C} \xrightarrow{F} \mathscr{E})$$

exists, then they define the right Kan extension $\mathrm{Ran}_K F : \mathscr{D} \to \mathscr{E}$, in which case the counit transformation $\epsilon : \mathrm{Ran}_K F \cdot K \Rightarrow F$ can be extracted from the limit cone.

Recall, that $K \downarrow d$ is the category of elements of the functor $\mathscr{D}(K-, d) : \mathscr{C}^{op} \to \mathbf{Set}$, and as such comes with a canonical projection functor $\Pi^d : K \downarrow d \to \mathscr{C}$.

This theorem is particularly nice, especially the following corollary which may be deduced from it. Also it gives a possibility to determine the left and right adjoints given in the next corollary.

Corollary A.27. *If $K : \mathscr{C} \to \mathscr{D}$ is a functor so that $\mathscr{C}$ is small and $\mathscr{D}$ is locally small then we have the following.*

1. *If $\mathscr{E}$ is cocomplete, then the left Kan extensions*

$$\mathrm{Lan}_K : \mathbf{Fun}(\mathscr{D}, \mathscr{E}) \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} \mathbf{Fun}(\mathscr{C}, \mathscr{E}) : K^*$$

   exist and are given by the colimit formula in Theorem A.26.

2. *If $\mathscr{E}$ is cocomplete, then right Kan extensions*

$$K^* : \mathbf{Fun}(\mathscr{D}, \mathscr{E}) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Fun}(\mathscr{C}, \mathscr{E}) : \mathrm{Ran}_K$$

   exist and are given by the limit formula in Theorem A.26.

B The Classical Axiomatic Definition of a Model Category

The following is the ”classic” axiomatic definition of a model category as it is for example stated in [DwSp95].

Definition B.1 (Model Category). *A* ***model category*** *is a category $\mathscr{C}$ with three classes of maps*

1. ***weak equivalences*** $\xrightarrow{\sim}$,
2. ***fibrations*** $\twoheadrightarrow$,
3. ***cofibrations*** $\hookrightarrow$.

Each class is closed under composition and contains all identity maps.

A map which is a fibration and a weak equivalence is called an ***acyclic fibration*** *or* ***trivial fibration****, similarly a map which is a cofibration and a weak equivalence is called an* ***acyclic cofibration*** *or* ***trivial cofibration****.*

Furthermore the following axioms have to be satisfied.

Axioms:

MC1: *All small limits and colimits exist in $\mathscr{C}$.*

MC2: *(2 out of 3 property) Let $f, g \in \mathscr{C}$ such that $g \circ f \in \mathscr{C}$ is defined. If two of f, g or $g \circ f$ are weak equivalences, then so is the third.*

MC3: *(Retract property) If $f \in \mathscr{C}$ is a retract of $g \in \mathscr{C}$ and g is a fibration, cofibration or weak equivalence, then so is f.*

MC4: *(Lifting property) Given a commutative diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & B \arrow[d, "p"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

in $\mathscr{C}$, a lift $h : C \to B$ exists in either of the following situations:

1. *i is a cofibration and p is an acyclic fibration.*
2. *i is an acyclic cofibration and p is a fibration.*

MC5: *(Factorization property) Any $f \in \mathscr{C}$ can be factorized in two ways:*

1. *$f = p \circ i$ for i a cofibration and p an acyclic fibration,*
2. *$f = p \circ i$ for i an acyclic cofibration and p a fibration.*

C Suggested Reading

Here I shall provide a list of material, which helped me to accumulate knowledge about the different topics and theories discussed in the present thesis. In my opinion it is especially helpful for people who are new to a certain kind of theory to be provided with enough material in order to learn something in an efficient way.

For an introduction about category theory and some of its important results, my suggestion would be to read [ToLe16] and [SMcl97].

When it comes to the topics about model categories and homotopy theory and even more interesting topics building on top of it, one should really start with [MHov91]. In my opinion there is no other book which provides so much detail about everything. After that one can give [GoJa09] a try. Though slightly more involved, it gives a lot of properties about simplicial sets and in general simplicial homotopy theory.

As soon as one feels safe with all this theory one may try to take on [PHir03]. It will be hard at the beginning but it is clearly one of the best books about the whole theory of localisations of model categories, especially about Bousfield localisations. Though, one should expect an overwhelming effort and addiction to cross referencing, like *really obsessive* cross referencing.

This is my suggestion to learn the basic and also a lot more advanced theory in this field, I hope that someone may take advantage of this list.

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