Yang-Mills Theory and Geometry

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In this first section we attempt to give a brief overview of mathematical work related to Yang-Mills (at least as it seeems from the authors perspective). We do not go into any technical details or definitions here: most of these are well represented in the literature, for example [17]. (We also mention a survey article [15] of the author, in a somewhat similar vein, which contains more detail.) In the following two sections we take up some of the ideas again, at a slightly more technical level.

Yang-Mills theory had a profound effect on the development of differential and algebraic geometry over the last quarter of the twentieth century, and it is now clear that this should be seen as part of a larger phenomenon: novel and profound interactions between theoretical physics and pure mathematics. The focus of mathematicians interest in Yang-Mills theory evolved over this period. To begin with, the emphasis was on the theory over four-dimensional Euclidean space, and its compactification the four-sphere (as well as the related theory of monopoles in three-space). The main lines of work were:

- Questions in the calculus of variations associated with the Yang-Mills functional. Here the emphasis was on analytical aspects, notably the foundational results of Uhlenbeck [47], and differential geometric aspects, as for example in the stability results of Bourguignon and Lawson [5].
- Algebro-geometric aspects, involving Ward's description of the Yang-Mills instantons in terms of holomorphic bundles over Penrose's twistor space, leading to the description of solutions via the ADHM construction [2].

It is important to ephasise however that, while can distinguish these different lines of work—involving very different mathematical techniques—the activities were closely related. Thus one notable feature of the impact of Yang-Mills theory within mathematics has been to increase the unity of the subject, throwing bridges between mathematical areas (e.g. PDE theory, vector bundles over complex projective space) which might have been seen before as having little connection. Another important strand was furnished by topology, notably in the paper of Atiyah and Jones [4]. The issue which this brought to the fore was that the Yang-Mills functional can be regarded as a function defined on a space with a rich topology (the space of connections on a fixed bundle, modulo gauge), so one could hope that this would be reflected (on the lines of Morse Theory) in the topology of the critical set. On the other hand this critical set consists of the moduli space of instantons (the absolute minimum) and higher critical points, so leading to a circle of questions involving the homotopy groups of the instanton moduli spaces and the existence of higher critical points (and their Morse indices). These topological aspects were also important in connection with "anomalies" and Atiyah-Singer index theory.

From the early 1980's the centre of activity evolved in roughly the following ways.

- Taubes introduced novel and deep techniques to attack the questions in the calculus of variations sketched above [43]. In a different direction, Taubes took the natural but critical step of studying Yang-Mills instantons over general Riemannian four-manifolds [42] (in contrast to earlier work which had concentrated on special classes of Riemannian manifolds such as symmetric spaces or "self-dual" manifolds [3]). In both cases, the work centred on the fact that one can have small, highly concentrated "bubble-like" instantons; related to the conformal invariance of Yang-Mills theory in four dimensions.

- The instanton equations, and the moduli spaces of their solutions, were applied to solve outstanding problems in four-manifold topology. The two themes here were first to show that certain intersection forms could not be realised by smooth four-manifolds and second, complementary, to define new invariants distinguishing smooth manifolds with the same intersection forms. Fortuitously, this development occurred at almost the same time as, coming from a completely different direction, Freedman produced his theory of *topological* four-manifolds.

- The Hitchin-Kobayshi conjecture (which was established in different forms in [10], [11], [48]) set out a very general relation between Yang-Mills theory over complex Kahler manifolds and holomorphic bundles, specifically Mumford's theory of "stability".

It is important to emphasise again that these different strands were tightly interwoven. For example, Taubes' techniques for constructing solutions lead to the description of the "boundary" of the instanton moduli space which was the key to the first differential-topological applications, and the solution of the Hitchin-Kobayshi conjecture allowed the first calculations of the new invariants, in the case of algebraic surfaces. There was an exciting period (in the late 1980's and early 1990's) when developments in the analysis of the instanton equations (as for example in the work of Mrowka [27]) and in the algebraic geometry of holomorphic bundles (as for example in the work of Friedman [24]) fed directly into new results in four-manifold topology.

In parallel with this activity in the mathematical aspects of Yang-Mills theory, another prominent development in mathematics arose in the field of *symplectic geometry*. This emerged as a major area over much the same period, seminal influences coming from Arnold, and in particular the "Arnold conjecture", and Gromov. This contemporaneous development was tightly bound up with that in Yang-Mills theory, in two different ways. On the one hand, there is a detailed analogy between Yang-Mills theory over 4-manifolds and the geometry of maps from a Riemann surface to a symplectic manifold. The Yang-Mills functional is analogous to the harmonic maps "energy functional" and the Yang-Mills instantons to the pesudo-holomorphic maps (defined after a choice of a compatible almost-complex structure on the symplectic manifold). On the other hand, symplectic geometry and the idea of a "moment map" provide an important motivation for the Kobayshi-Hitchin conjecture. This goes back to Atiyah and Bott [1] who pointed out, in particular, that the moduli spaces of flat connections over surfaces have natural symplectic structures. In the first direction, the analogy with harmonic maps was prominent in Uhlenbeck's work and also in Atiyah and Jones' formulation of their conjecture on the stabilistion of the homotopy groups of the instanton moduli spaces for large Pontrayagin class (analogous to a theorem of Segal for holomorphic maps). Gromov's celebrated paper [28] began the use of pseudo-holomorphic maps as a tool in symplectic geometry in analogy with the use of instantons in four-manifold theory. This was followed by the work of Floer, who introduced the Floer cohomology groups in both the the symplectic and Yang-Mills settings [22], [23]. In the second direction, the "moment map" point of view served as a guide to the introduction of many extensions of the Hitchin-Kobayshi conjecture to "augmented bundles": holomorphic bundles with some additional structure such as a section (or Higgs field)[7]. The symplectic structure on moduli spaces of flat connections was a vital feature of Witten's work on 3-manifold and knot invariants [49].

It is less easy to summarise developments over, roughly, the past decade. Partly, perhaps, this is because the author has been less actively involved in the area over this period. Partly, because the activity which could be immediately classified as "the mathematics of Yang-Mills theory" has, perhaps, abated somewhat. Some of the motivating questions sketched above have been answered: for example, in the proof of the Atiyah-Jones conjecture [6], and the existence of non-minimal solutions [39]. The theory of four-manifold invariants was revolutionised in 1994 with the work of Seiberg and Witten[50] (and the earlier work of Kronheimer and Mrowka [33]). Thus the theory of these new invariants for closed four-manifolds appears now in rather complete shape: we have a collection of distinguished "basic classes" in the two dimensional cohomology of the four-manifold, with associated multiplicities. Moreover these invariants can be calculated in many cases of interest. The basic classes can be thought of, very roughly, as generalisations of the first Chern class, in the case of a complex surface. For four-dimensional symplectic manifolds the invariants coincide with the Gromov-Witten pseudo-holomorphic curve invariants, by the renowned results of Taubes [44]. Many of the most interesting recent developments have involved the Floer Theory and interactions with three-manifold topology. A highlight here is the proof by Kronheimer and Mrowka of "Property P"[34]; involving a wonderful synthesis of instanton theory, Seiberg-Witten theory, symplectic and contact geometry.

The remarks above are not meant to suggest that the impact of Yang-Mills theory on mathematics has been a transient phenomenon. Rather, the ideas have diffused to stimulate a whole variety of fields and have thus merged with other strains, as opposed to forming a clearly defined research area. For one example, Hitchin's work on bundles with Higgs fields over Riemann surfaces [29] is important in Drinfeld's Geometric Langlands correspondence. For another, analogues of the instanton invariants for connections over manifolds with special holonomy seem to be coming into prominence in the context of M-theory.

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In this section we discuss some ideas which have been particularly important in the mathematical work on Yang-Mills theory, and which have subsequently had a wider influence in geometry.

2.1 Gauge invariance

One point to make is that Yang-Mills theory provides a relatively simple testing ground for various constructions. At the most naive level one can say that the Yang-Mills equations themselves can be written down in a comparatively compact shape. For example, the Yang-Mills instanton equations over $\mathbf{R}^4$ are:

$$F_{12} = F_{34}\ ,\ F_{13} = F_{42}\ ,\ F_{14} = F_{23},$$

where

$$F_{ij} = \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} - [A_i, A_j],$$

is the curvature expressed in terms of the (local) connection form A_i. Compare this with the complexity of the Einstein equations, say, for a Riemannian metric. (One important line of work, which we have not mentioned above, is that which casts most of the known integrable PDE as special reduced forms of these elegant instanton equations [35].) A more significant point is that we encounter gauge symmetry in the Yang-Mills equations

$$A \mapsto gAg^{-1} - (dg)g^{-1}.$$

Globally, we construct the "moduli space" of instantons (say) as a subset of a quotient $\mathcal{A}/\mathcal{G}$ where $\mathcal{A}$ is the affine space of all connections and $\mathcal{G}$ is the gauge group and (modulo technical details) we model this quotient on local slices using the Coulomb constraint

$$\{A_0 + a : d^*_{A_0} a = 0, \|a\| < \epsilon\}.$$

Locally, working with connections over a ball, the existence of a representative in the gauge group orbit which satisfies the Coulomb condition is at the core of Uhlenbeck's basic result [47]. All of these ideas are now well known and have been widely used. The point we wish to bring out is the comparatively simple nature of the gauge group, when compared with other infinite dimensional groups such as diffeomorphism groups. Yang-Mills theory thus provides perhaps the simplest step beyond linear theory (such as the Hodge theory to which Yang-Mills theory reduces in the abelian case). Similar ideas have since been applied in the context of Riemannian geometry and diffeomorphism groups: for example in the study of compactness of moduli spaces of Einstein four-manifolds, where the existence of harmonic coordinate systems takes the place of Uhlenbeck's Coulomb gauge fixing [36]. This is not to suggest that such ideas had not entered geometry before—for example, many occur in the Kodaira-Spencer theory of deformations of complex structures—but their success and imporatnce in Yang-Mills theory has certainly made them more of a standard tool.

2.2 Fredholm differential topology

The invariants of a compact four-manifold X defined by the Yang-Mills instanton equations can be put in the following conceptual picture. The self-dual part of the curvature furnishes a map

$$F_+ : \mathcal{A} \to \Omega^2_+(\mathrm{ad}P),$$

from the space $\mathcal{A}$ of connections to the self-dual 2-forms with values in the bundle associated to the adjoint representation. Passing down to the quotient $\mathcal{A}/\mathcal{G}$, as above (and again ignoring some technicalities) we get a section f_+ of an infinite-dimensional vector bundle say $\mathcal{E} \to \mathcal{A}/\mathcal{G}$. The invariant we have is then, at least formally, the *Euler class* of this infinite-dimensional vector bundle, taking values in the homology of $\mathcal{A}/\mathcal{G}$. The reason that this makes any kind of sense is that f_+ is a "Fredholm section". That is to say, the derivative of f_+ is a Fredholm linear map from the tangent space $\mathcal{A}/\mathcal{G}$ to the fibre of $\mathcal{E}$. This boils down to the fact that the differential operator

$$d^*_A \oplus d^+_A : \Omega^1(\mathrm{ad}P) \to \Omega^0(\mathrm{ad}P) \oplus \Omega^2_+(\mathrm{ad}P),$$

over X is elliptic, hence a Fredholm operator between suitable Sobolev spaces. (Here d^+ denotes the self-dual part of the exterior derivative.) It has been known for a long time (as least as far back as Smale's paper [40]) that some of the familiar constructions of differential topology can be transferred to infinite dimensions in the presence of such a Fredholm condition. Thus one can attempt to define the Euler class by perturbing f_+ (if necessary) to a suitably generic section which will will vanish along a finite-dimensional manifold Z in $\mathcal{A}/\mathcal{G}$, of dimension equal to the index of the operator above. Then, provided Z is compact, the Euler class is defined to be the fundamental class of Z, just as in finite dimensional differential topology. In the case of the instanton equation compactness is a crucial technical issue, but the point we want to bring out here is that while the general context for this construction has been around for some time, Yang-Mills instantons provided perhaps the first decisive application of the ideas. Nowadays, these ideas have entered the mainstream and become familiar. A parallel discussion applies to Gromov-Witten invariants defined by holomorphic maps into symplectic manifolds and this has grown into an enormous research area. The ideas are intimately connected with classical problems of enumerative geometry. The ideas have also been applied to, for example, Special Lagrangian submanifolds [31].

We would like to make two remarks about the applications of these notions of "Fredholm differential topology". The first is that their use in Yang-Mills theory required a significant change in viewpoint. The theory of deformations of Yang-Mills instantons was developed by Atiyah, Hitchin and Singer in analogy with the Kodaira-Spencer theory of deformation of complex structures, in turn reaching back into the general notion of moduli in complex geometry. In traditional moduli problems one quite often obtains a moduli space which has very "nongeneric" properties; for example, with singularities, or of a dimension greater than that predicted by the index calculation: in other words transversality fails for the original map defining the geometric problem. Nevertheless, in the classical setting, the moduli space is what it is, singularities and all. From the Fredholm differential topology point of view one wants to think rather differently: discarding the original moduli space and considering instead the solutions of a perturbed problem. Nowadays, this all seems familiar. In fact an extensive theory has been developed of "virtual moduli cycles" which associates a class μ in the homology of the original moduli space M to the infinitesimal data governing the deformation problem, in such a way that (in the Yang-Mills setting above) the homology class $[Z]$ of the zero set of a generic perturbed section is the image of μ under the inclusion $M \to \mathcal{A}/\mathcal{G}$. For example, in a simple case when the moduli space M is smooth but of dimension greater than that predicted by the index calculation the class μ will be the Poincaré dual of the Euler class of a finite dimensional vector bundle over M defined by the cokernel of the linearised operator.

The second remark concerns the general context in which these ideas can be applied in geometric problems. Consider, for example, the theory of holomorphic vector bundles over a compact complex manifold Y. Whatever the dimension of Y there is a deformation theory, describing (roughly speaking) a neighbourhood of a point $[E]$ in the moduli space of all holomorphic bundles in terms of data involving the cohomology groups $H^i(\mathrm{End}(E))$ for $i = 0, 1, 2$. When $i = 1$ we have the space of infinitesimal deformations and when $i = 2$ the space of obstructions. In the case when Y is a complex surface this moduli space can be interpreted (roughly speaking) as a moduli space of instantons, and one can imagine generic perturbations as described above. The fundamental reason why this works (ignoring issues of compactness etc.) is that—for trivial dimensional reasons—there are no other cohomology groups $H^i(\mathrm{End}(E))$ for $i > 2$. Similarly for moduli spaces of maps from one complex manifold U to another V: the deformation theory about a given map $f : U \to V$ is governed by the cohomology groups $H^i(U; f^*(TV))$ for $i = 0, 1$. (Now $i = 0$ gives the space of infinitesimal deformations and $i = 1$ the space of obstructions.) When U is a complex curve there are again no higher cohomology groups, for trivial dimensional reasons. In general one can hope to apply the "Fredholm differential topology" idea (leading to virtual moduli cycles etc.) in any deformation problem where *the higher cohomology groups, beyond the obstruction space* are zero. An example of this is Thomas' theory of "counting" holomorphic bundles over Calabi-Yau threefolds [46]. The general setting of that theory applies equally well to bundles over threefolds where the anticanonical bundle K^{-1} has a holomorphic section [18], since in that case the higher cohomology group $H^3(\mathrm{End}(E))$ vanishes for a stable bundle E.

These remarks suggest two questions.

- Is there some useful way of extending the ideas to problems in which higher cohomology groups enter in the deformation theory (for example to holomorphic bundles over complex manifolds of dimension bigger than 3)?

- Are there other interesting applications of the idea, to cases where the higher cohomology vanishes? For example in the case of moduli spaces of complex surfaces the deformations are governed by the cohomology groups of the tangent bundle, and the desired vanishing holds for dimensional reasons.(In other words, the theory is described by a two step elliptic complex.) So one could hope to define a virtual moduli cycle in that case, but it is not clear what this could be good for.

2.3 Glueing techniques

Probably the most influential idea on the PDE and analysis side of the work on Yang-Mills theory in differential geometry, is Taubes' approach to the construction of solutions by "glueing". The general strategy is to construct an explicit approximate solution, built out of standard models and appropriate cut-off functions, and then to deform this to a true solution by means of an implicit function theorem. The main technical work frequently comes down to estimates for the inverse of the differential operator defining the linearised problem. Taubes applied this strategy to a variety of problems in Yang-Mills theory (instantons on four-manifolds, monopoles on $\mathbf{R}^3$, the Seiberg-Witten equations). Since then the same kind of strategy has also been applied to other differential geometric problems, for example constant mean curvature surfaces [32] and manifolds of exceptional holonomy [30]. Quite possibly the same general strategy may have been used by other, earlier, writers in other problems, but in any case the work of Taubes in Yang-Mills theory has made this into something close to a standard tool in differential geometry. Of course the success of the method depends on having an appropriate approximate solution and this is connected to the "particle like" nature of the standard instantons and monopoles. (In the applications of this strategy there is also typically a real parameter in the problem— for example, monopole separation or instanton size.)

The remark we would like to make here is that Taubes' gluing technique gives a new point of view on results such as the classical Runge Theorem in complex analysis (approximation of holomorphic functions on a domain by rational functions). This is explained in [13]; written in response to questions of Gromov. The analogue of the Runge theorem for instantons is that any solution of the instanton equations on a domain Ω in S^4 can be approximated by global solutions (where the Pontrayagin number will usually need to tend to infinity as the approximation becomes better and better). The method of proof is first to choose an arbitrary extension of the instanton then to modify this to an aproximate solution by gluing in many small standard "bubbles" in $S^4 \setminus \Omega$ and finally to deform to a global solution. This approach should be applicable to other problems: for example it suggests that there may be a "Runge Theorem" for holomorphic maps into a complex manifold V provided that all tangent vectors in V arise as tangents to holomorphic spheres.

2.4 Moment maps and stability

The link between the Yang-Mills equations and the algebro-geometric notion of "stability" which appears in the Kobayshi-Hitchin conjecture has become influential in complex differential geometry as a whole. (Of course the Kobayashi-Hitchin conjecture was predated by the work of Narasimhan and Seshadri [38], which covers the special case of bundles over complex curves, but the wider significance of the ideas did not emerge until later.) It leads in to a general setting for many important links between complex geometry and "metrical" structures. In finite dimensions this is the "Kempf-Ness principle": if a group—say $GL(n, \mathbf{C})$—acts linearly on $\mathbf{C}^N$ then a criterion for picking out a preferred representative in each orbit is to choose a vector of *minimal length.* If the orbit is closed (or "stable") such a representaive exists and is unique up to the action of the unitary group $U(n)$. In the projective space $\mathbf{CP}^{N-1} = \mathbf{P}(\mathbf{C}^N)$ these representatives yield zeros of the "moment map" for the $U(n)$ action. Even in this simple setting, these preferred representives often have geometric interest. For example, the basic fact that any semisimple complex Lie group has a compact real form can be proved in this way. However the real impact of these ideas has arisen from the realisation (going back to Atiyah and Bott [1]) that the same notions can usefully be applied in cases where the finite-dimensional Lie groups $GL(n, \mathbf{C}), U(n)$ are replaced by infinite-dimensional groups. In the case of the Kobayashi-Hitchin conjecture, the analogue of the group $U(n)$ is the group $\mathcal{G}$ of gauge transformations of a unitary bundle over a compact Kahler manifold. The analogue of the projective space $\mathbf{CP}^{N-1}$ is the space $\mathcal{A}$ of connections, and the vanishing of the moment map is precisely the Yang-Mills equation ocurring in the Kobayashi-Hitchin conjecture. The analogue of the vector space $\mathbf{C}^N$ is more esoteric: it is essentially the total space of a "determinant line bundle". The functional which corresponds to the length of vectors in $\mathbf{C}^N$ is the Quillen norm in this determinant line bundle, defined via zeta-function regularisation of the determinants of Laplace operators. Thus there is a significant contact with the Arakelov theory of Bismut, Gillet, Soulé *et al* [41], which was developed—from quite a different direction—at about the same time.

This picture is essentially a formal one, and does not by itself lead directly to a proof of the Kobayashi-Hitchin conjecture. Such proofs require some input from PDE theory: elliptic estimates etc. However the formal picture is an important guide. In the past decade there has been considerable activity developing the analogue of these ideas for Kahler metrics on complex manifolds, rather than on auxiliary bundles as in the Yang-Mills case. For example, the fact that any compact Riemann surface has a metric of constant curvature appears as another instance of this general picture [16]. For the Kahler metric problem, the gauge group is replaced by the group of symplectomorphisms of the manifold (in line with the remarks in (2.1) above). The moment map condition is that a metric has constant scalar curvature: a condition discussed extensively by Calabi [8] (also motivated in part by Yang-Mills theory, although following a different route to the one we have discussed here). The analogue of the Kobayashi-Hitchin conjecture has not been proved yet, largely because of the formidable nature of the nonlinear, fourth order, PDE involved, but this is an important goal in current research.

3

In this section we will discuss a selection of problems involving mathematical aspects of Yang-Mills theory.

3.1 Four-manifolds and invariants

The classification of smooth four-manifolds

This is a very large problem–not to be taken literally. It is also one which may have little to do with Yang-Mills theory. The current state of the theory is illustrated well by the paper [21] of Fintushel and Stern. They construct a large family of four-manifolds—one for each knot in $\mathbf{R}^3$—all homeomorphic to the standard complex K3 surface. Many of these can be distinguished by their Seiberg-Witten invariants, which reduce in this case to the Alexander polynomials of the knots (and presumeably the Yang-Mills instanton invariants contain just the same information). On the other hand, there are many different knots with same Alexander polynomials, and for these it is completely unclear whether the correspoding four-manifolds are diffeomorphic. There are at present *no techniques* to either distinguish the manifolds or to establish the existence of diffeomorphisms between them (beyond, in the latter case, some inspired explicit construction). It is thus impossible to prediuct how the theory will develop in the future, but in any case since the known invariants (instanton, Seiberg-Witten) come from Yang-Mills theory, it is reasonable to hope that any developments in the understanding of four-manifolds will have some bearing on that theory.

Aside from the overall classification problem, there are various questions having more to do with the internal structure of the invariants for general four-manifolds. For example

Do all compact four-manifolds have "simple type" in the sense of Witten [50] and Kronheimer and Mrowka [33]?

An obvious outstanding problem is:

Prove Witten's conjecture [50] relating the instanton and Seiberg-Witten invariants.

There is little doubt that the result is true and a proof under some additional hypotheses has been given by Feehan and Leness [20]. The development of Feehan and Leness' approach would seem to require advances in the technique for describing the compactification of instanton moduli spaces. The description of these compactifications, in a form in which one can carry out the appropriate topological calculations (*i.e* evaluation of cohomology classes on the "links" of points at infinity) has been a long-running issue and it seems that new ideas are required to cut through the complexity of the phenomena that occur when the Pontrayagin class becomes large. Perhaps more significant than the actual proof of Witten's conjecture is the search for a mathematical understanding of the structural relationship between the theories: the appearance of elliptic curves etc. The work of Nakajima and Yoshioka [37] seems to have gone furthest in this direction.

3.2 Floer Theory

Understand the relation between the instanton and Seiberg-Witten Floer theories for three-manifolds

The Floer theories for three-manifolds, connected with the instanton and Seiberg-Witten equations over four-manifolds with boundaries, seem to be considerably more intricate and subtle than the theories for closed 4-manifolds. It is not clear how the two theories are related. In this direction we note the following. Let Y be a compact oriented three-manifold and consider the question of what negative-definite intersection forms can be realised by 4-manifolds X with boundary Y. One obtains restrictions on the intersection forms in either the instanton or Seiberg-Witten Floer theories (see [25], [14]). The general shape of these restrictions is that certain elements of the Floer group $HF^*(Y)$, determined by X, must be nonzero if the intersection form of X satisfies appropriate conditions (for example having no elements of length -1). In particular the Floer group itself must be nonzero. A natural question is whether the restrictions on the intersection forms obtained in the two theories are equivalent. This might give a guide to the general relation between the Floer theories.

A moduli space $M(S)$ of flat connections over a compact oriented surface S has a canonical sympletic form ω. Thus the mapping class group Γ_S of the surface acts by symplectomorphisms on $M(S)$. Dynamical aspects of this action have been studied by Goldman [26]. The action is also an important ingredient in Witten's theory of knot and 3-manifold invariants. One can ask about the composite map from Γ_S to the group $\Gamma(M(S), \omega))$ of symplectomorphisms of $M(S)$ modulo symplectic isotopy.

Does this map give an injection, or even an isomorphism from Γ_S to $\Gamma(M(S), \omega)$?

Floer theory provides one tool for attacking this question. By the result of Dostoglou and Salamon [19] the symplectic Floer groups—which are invariants of classes in $\Gamma(M(S), \omega)$—coincide with the instanton Floer groups of the three-manifolds obtained as mapping tori. Some results in this direction were obtained in the early 1990's by M. Callahan, but unfortunately Callahan left mathematics before writing up his work.

3.3 Gauge theory in higher dimensions

There are analogues of the instanton equation for connections on bundles over manifolds of dimension 7 or 8 with the exceptional holonomy groups G_2, Spin(7) respectively [18], together (potentially) with the attendant Floer theories etc. The overriding problem in developing a complete theory (*i.e* a theory of invariants in the Fredholm differential topology setting discussed in (2.2) above), is that of compactness. There are important results of Tian [45] here, but it seems that the theory depends on a corresponding (and currently absent) theory for the relevant codimension-4 calibrated submanifolds. Nevertheless there are many interesting questions which should be accessible such as

Construct non-trivial solutions of the G_2 and Spin(7) *instanton equations over the manifolds constructed by Joyce and Kovalev.*

The obvious approach here is to use some variant of Taubes' gluing strategy.

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