Tot Primer

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Contents

1 Introduction

The purpose of this primer is to lay out the current definition of Totalization of a cosimplicial space and provide an alternate, equivalent definition of Totalization that is more intuitive and easier to compute examples with.

2 Simplicial and Cosimplicial Spaces

Recall that Δ (or Ord) is the category of finite totally ordered sets (e.g. $[m] = \{0, 1, \ldots, m\}$) with monotone maps. A simplicial object of a category $\mathscr{C}$ is a covariant functor Δ^{op} to $\mathscr{C}$ and a cosimplicial object of a category $\mathscr{C}$ is a covariant functor Δ to $\mathscr{C}$.

Example. Common examples:

- $\mathscr{C} =$ Sets : Simplicial sets, Cosimplicial sets
- $\mathscr{C} = \mathrm{Top}_{CW}$: Simplicial Spaces, Cosimplicial Spaces
- $\mathscr{C} =$ Simplicial Sets: Bisimplicial Sets, Cosimplicial x Simplicial sets.
- $\mathscr{C}$ = Ab: Simplicial Abelian Groups (chain complexes), Cosimplicial Abelian Groups.

2.1 Some Formal Comments

An important construction to know about is geometric realization, $|-| : \mathfrak{C}^{\Delta^{op}} \to \mathfrak{C}$. If one is familiar with Kan extensions, this can be defined as a left Kan extension, $|X| = X \otimes_{\mathfrak{C}} \Delta$.

There is a remark in [GJ99] that Totalization is the dual or opposite to realization, which, if you're comfortable with Kan extensions is just saying that Totalization is a right Kan extension. The formal expression in [GJ99] is the following:

$$\mathrm{Tot} = |\ |^{op} : (s(S^{op}))^{op} \to (S^{op})^{op}$$

Using the coend definition of realization, this says that

$$\begin{tikzcd}
\text{Realization} & & \text{Realization}^{op} \\
\coprod\limits_{\substack{n,m \\ \Delta(n,m)}} X_m \times \Delta^n \arrow[d, shift left] \arrow[d, shift right] & & \prod\limits_{\substack{m,n \\ \Delta(n,m)}} \mathrm{Hom}_{Sset}(\Delta^n, X_m) \\
\coprod\limits_{m} X_m \times \Delta^m \arrow[d] \arrow[rr, "op"] & & \prod\limits_{m} \mathrm{Hom}_{Sset}(\Delta^m, X_m) \arrow[u, shift left] \arrow[u, shift right] \\
{|X|} & & \mathrm{Tot}(X) \arrow[u]
\end{tikzcd}$$

3 Defining and working with Tot

3.1 Examining the innards

Jim McClure [MS04] makes a remark about Tot that we'll use here. A point α in Tot $X^\bullet_\bullet$, i.e. an element of $\mathrm{Tot}(X^\bullet_\bullet)_{[0]}$, will be a sequence of maps $(\alpha^0, \alpha^1, \ldots)$ making the following diagram commute:

$$\begin{tikzcd}
\Delta^0 \arrow[r, shift left, "{d^0, d^1}"] \arrow[d, "\alpha^0"] & \Delta^1 \arrow[l, shift left, "s^0"] \arrow[r, shift left, "{d^0, d^1, d^2}"] \arrow[d, "\alpha^1"] & \Delta^2 \arrow[l, shift left, "{s^0, s^1}"] \arrow[d, "\alpha^2"] & \cdots \\
X^0 \arrow[r, shift left, "{d^0, d^1}"] & X^1 \arrow[l, shift left, "s^0"] \arrow[r, shift left, "{d^0, d^1, d^2}"] & X^2 \arrow[l, shift left, "{s^0, s^1}"] & \cdots
\end{tikzcd}$$

The dimension-increasing maps are the coface maps and the dimension-decreasing maps are the codegeneracy maps.

More generally, an element of $\mathrm{Tot}(X^\bullet_\bullet)_{[k]}$ for arbitrary k will be a sequence of maps making the following commute:

$$\begin{tikzcd}
\Delta^0 \times \Delta^k \arrow[r, shift left, "{d^0, d^1}"] \arrow[d, "\alpha^0_k"] & \Delta^1 \times \Delta^k \arrow[l, shift left, "s^0"] \arrow[r, shift left, "{d^0, d^1, d^2}"] \arrow[d, "\alpha^1_k"] & \Delta^2 \times \Delta^k \arrow[l, shift left, "{s^0, s^1}"] \arrow[d, "\alpha^2_k"] & \cdots \\
X^0 \arrow[r, shift left, "{d^0, d^1}"] & X^1 \arrow[l, shift left, "s^0"] \arrow[r, shift left, "{d^0, d^1, d^2}"] & X^2 \arrow[l, shift left, "{s^0, s^1}"] & \cdots
\end{tikzcd}$$

Where now the d^i and s^j are $d^i \times id$ and $s^j \times id$ on the top row.

However, since the the coface and codegeneracy maps of the Δ^k factor for a fixed k are trivial, we can view our k-level diagram as being, up to homotopy data, of the form of the 0th-level diagrams.

Formally,

Definition 3.1. $\mathrm{Tot}(X^\bullet_\bullet) = \mathrm{hom}_{cS}(\Delta \times \Delta, X^\bullet_\bullet) := \mathrm{Hom}_{cS}(\Delta, X^\bullet_\bullet)$

Where cS is the category of cosimplicial simplicial sets, and $\mathrm{Hom}_{cS}(-,-)$ is notational shorthand for $\mathrm{hom}_{cS}(- \times \Delta, -)$. Notice that $\mathrm{Tot} : cS \to sS$ where sS is the category of simplicial sets.

It can be advantageous to worry about a topological analogue of our resultanat simplicial space, so an alternate definition is

Definition 3.2. When considering cosimplicial *spaces* ,

$$\mathrm{Tot}(X^\bullet) = \mathrm{Map}_{cts}(|\Delta|^\bullet, X^\bullet)$$

Remark 3.3. Notice that $|-|$ and Sing are a Quillen pair between Simplicial Sets and Top. Since Δ is small, Sing and $|-|$ induce a Quillen pair on (Simplicial Sets)$^{\Delta}$ and Top$^{\Delta}$. $X_\bullet^\bullet$ is weakly equivalent to Sing$|X_\bullet^\bullet|$ (where these are interpreted as the sort of "levelwise" or "pointwise" applications), so (since Δ is cofibrant) when $X_\bullet^\bullet$ is fibrant, we have

$$\mathrm{Hom}_{cS}(\Delta, X_\bullet^\bullet) \sim \mathrm{Hom}_{\mathrm{Top}^\Delta}(|\Delta|, |X^\bullet|)$$

Where $\mathrm{Hom}_{\mathrm{Top}^\Delta}(|\Delta|, |X^\bullet|) \cong \mathrm{Sing}(\mathrm{Map}_{\mathrm{Top}^\Delta}(|\Delta|, |X^\bullet|)$ so that we get

$$\begin{aligned} \mathrm{Map}_{\mathrm{Top}^\Delta}(|\Delta|, |X^\bullet|) &\overset{w.e.}{\sim} |\mathrm{Sing}(\mathrm{Map}_{\mathrm{Top}^\Delta}(|\Delta|, |X^\bullet|)| \\ &\overset{w.e.}{\sim} |\,\mathrm{Hom}_{\mathrm{Top}^\Delta}(|\Delta|, |X^\bullet|)| \\ &\overset{w.e.}{\sim} |\,\mathrm{Hom}_{cS}(\Delta, X_\bullet^\bullet)| \end{aligned}$$

Thus we can view these two definitions of Tot as as "compatible" definitions.

3.2 Cosimplicial model for $\Omega B_\bullet$

McClure gives in his paper [MS04], for $B_\bullet$ a simplicial set, a cosimplicial simplicial set whose totalization is $\Omega B_\bullet$. In other words, he has given is a cosimplicial model of a loop space. Given $B_\bullet$, let $B_\bullet^\star$ be

$$\begin{tikzcd} * \arrow[r, shift left, "{d^0, d^1}"] & B_\bullet \arrow[l, shift left, "s^0"] \arrow[r, shift left, "{d^0, d^1, d^2}"] & B_\bullet \times B_\bullet \cdots \arrow[l, shift left, "{s^0, s^1}"] \end{tikzcd}$$

i.e. $B_\bullet^n := \underbrace{B_\bullet \times B_\bullet \times \cdots \times B_\bullet}_{n\ times}$ for $n > 0$ and $B_\bullet^0 = *$.

The coface and codegeneracy maps are defined by

$$d^i(b_1, \ldots, b_n) \;\; = \begin{cases} (*, b_1, \ldots, b_n) & \text{for i=1} \\ (b_1, \ldots, b_n, *) & \text{for } 1 < i < n \\ (b_1, \ldots, b_i, b_i, \ldots b_n, *) & \text{for i=n} \end{cases}$$

$$s^i(b_1, \ldots, b_n) \;\; = (b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n)$$

Let's examine elements of Tot $B_{[k]}^\star$:

$$\begin{tikzcd} \Delta^0 \arrow[r, shift left, "{d^0, d^1}"] \arrow[d, "\alpha^0"] & \Delta^1 \arrow[l, shift left, "s^0"] \arrow[r, shift left, "{d^0, d^1, d^2}"] \arrow[d, "\alpha^1"] & \Delta^2 \arrow[l, shift left, "{s^0, s^1}"] \arrow[d, "\alpha^2"] & \cdots \\ * \arrow[r, shift left, "{d^0, d^1}"] & B_\bullet \arrow[l, shift left, "s^0"] \arrow[r, shift left, "{d^0, d^1, d^2}"] & B_\bullet \times B_\bullet \arrow[l, shift left, "{s^0, s^1}"] & \cdots \end{tikzcd}$$

Notice that α^i for $i > 1$ is a map into a product. Maps into products are determined by maps into each of the factors. That, coupled with our diagram, forces

$$\alpha^2 = (\alpha^1 s^0, \alpha^1 s^1)$$

i.e. degeneracy of α^2 and an analogous condition for all subsequent α^i so that they are also degenerate.

This tells us that Tot of this cosimplicial simplicial set is determined entirely by the 1st square,

$$\begin{tikzcd}
\Delta^0 \arrow[r, shift left, "{d^0, d^1}"] \arrow[d] & \Delta^1 \arrow[l, shift left, "s^0"] \arrow[d] \\
* \arrow[r, shift left, "{d^0, d^1}"] & B_\bullet \arrow[l, shift left, "s^0"]
\end{tikzcd}$$

i.e. simplicial maps $\alpha^1 : \Delta^1 \to B$ that satisfy the relations enforced by commutivity:

$$\begin{aligned}
s^0 \circ \alpha 1 &= * \circ s^0 = * \\
\alpha^1 \circ d^0 &= d^0 \circ * = * \\
\alpha^1 \circ d^1 &= d^1 \circ * = *
\end{aligned}$$

This construction can just as easily be made for B a topological space. Then, maps from Δ^1 into B should look comfortingly like 1-cells and the above conditions force the two ends of the one cells to be equal to the basepoint since $\alpha^1 \circ d^0 = \alpha^1(0)$, i.e.it's 0th endpoint, or source and v.v for $\alpha^1 \circ d^1$.

3.3 Cosimplicial model for homotopy pullback

Given a diagram of spaces

$$\begin{tikzcd}
 & Y \arrow[d, "g"] \\
X \arrow[r, "f"] & B
\end{tikzcd}$$

we know that the pullback of this diagram looks like

$$X \times_B Y = \{(a, b) | f(a) = g(b)\}$$

and that the homotopy pullback "is"

$$X \times_B^{ho} Y = \{(a, b) | \exists h_{f,g}^{a,b} : f(a) \simeq g(b)\}$$

or, better yet,

$$X \times \{\rho \in \mathrm{Map}(\Delta^1, B) | \rho(0) = f(x), \rho(1) = g(y)\} \times Y$$

If we would like to build a cosimplicial space whose Tot is a model for the homotopy pullback, perhaps we can take our inspiration from the example of the Tot of the Bar construction on a space X producing ΩX. Recall that what forced the Tot to be a loopspace was the $*$ in dimension 0. Adding something to dimension 0 should relax constraints to get paths.

Naively, we can take our maps f and g and say that perhaps the 0^{th} and 1^{st} levels of our "cosimplicial space" should be

$$\begin{tikzcd} Z \\ X \times Y \arrow[u, shift left, "f"] \arrow[u, shift right, "g"'] \end{tikzcd}$$

i.e. that $d^0 = f$ and $d^1 = g$. The immediate problem with this is – how would you define $s^0(z)$ for $z \in Z$ (namely, so that $s^0 d^0 = \mathrm{id} = s^0 d^1$)? If we "carry" X and Y to the 1^{st} level, we have

$$\begin{tikzcd} X \times Z \times Y \arrow[d, "s^0"] \\ X \times Y \arrow[u, shift left=2, "d^0"] \arrow[u, shift right=2, "d^1"'] \end{tikzcd}$$

We can define

$$\begin{aligned} d^0(x, y) &= (x, f(x), y) \\ d^1(x, y) &= (x, g(x), y) \\ s^0(x, z, y) &= (x, y) \end{aligned}$$

We would like Tot of the cosimplicial space we build to collapse to be first square, so we can force the degenerateness of higher level stuff by mimicing the bar construction, so that

$$X \times Y \rightrightarrows X \times Z \times Y \; \substack{\rightarrow \\ \rightarrow \\ \rightarrow} \; X \times Z \times Z \times Y \; \substack{\rightarrow \\ \rightarrow \\ \rightarrow \\ \rightarrow} \cdots$$

The cosimplicial space that is the model for the homotopy pullback of our diagram is denoted $(X \times_B Y)$, where

$$(X \times_B Y)_n = X \times \underbrace{B \times \cdots \times B}_{n \text{ times}} \times Y$$

The general coface and codegeneracy maps are defined as follows:

$$d^i(x, b_1, b_2, \ldots, b_n, y) = \begin{cases} (x, f(x), b_1, b_2, \ldots, b_n, y) & \text{for } i = 0 \\ (x, b_1, \ldots b^i, b^i, \ldots, b_n, y) & \text{for } 0 < i < n \\ (x, b_1, b_2, \ldots, b_n, g(y), y) & \text{for } i = n \end{cases}$$

$$s^i(x, b_1, \ldots, b_n, y) = (x, b_1, \ldots b_{i-1}, b_{i+1}, b_n, y) \; \forall i \in [1, n]$$

Our model defined levelwise as above and with the given coface and codegeneracy maps is called the **geometric cobar construction** on X, Y over B.

3.3.1 Working through the Totalization

It can be noted that the original source of the geometric cobar construction above is a paper of Rector, [Rec70].

Here is our Tot-diagram from the geometric cobar construction above:

$$\begin{tikzcd}
\Delta^0 \arrow[r, shift left, "d^0, d^1"] \arrow[d, "\alpha^0"] & \Delta^1 \arrow[l, shift left, "s^0"] \arrow[r, shift left, "d^0, d^1, d^2"] \arrow[d, "\alpha^1"] & \Delta^2 \arrow[l, shift left, "s^0, s^1"] \arrow[d, "\alpha^2"] & \cdots \\
X \times Y \arrow[r, shift left, "d^0, d^1"] & X \times B \times Y \arrow[l, shift left, "s^0"] \arrow[r, shift left, "d^0, d^1, d^2"] & X \times B^2 \times Y \cdots \arrow[l, shift left, "s^0, s^1"] &
\end{tikzcd}$$

Commitivity forces the α^i to be fixed on X & Y by what α^0 does, so we can think of each α^i as being maps $\Delta^i \to B^i$. This reduces us to the same situation we handled in the model for ΩB case, i.e. that all of our α^i will be degenerate for $i > 1$.

This again tells us that Tot of this cosimplicial space is determined entirely by the 1st square,

$$\begin{tikzcd}
\Delta^0 \arrow[r, shift left, "d^0, d^1"] \arrow[d, "\alpha^0"] & \Delta^1 \arrow[l, shift left, "s^0"] \arrow[d, "\alpha^1"] \\
X \times Y \arrow[r, shift left, "d^0, d^1"] & X \times B \times Y \arrow[l, shift left, "s^0"]
\end{tikzcd}$$

Let α_x^0 denote α^0 followed by projection to X, v.v. for α_y^0. Then the relations we obtain from the square are:

(1) $\alpha^0 s^0 = s^0 \alpha^1$
(2) $\alpha^1 \circ d^0 = d^0 \circ \alpha^0 = (\alpha_x^0, f(\alpha_x^0), \alpha_y^0)$
(3) $\alpha^1 \circ d^1 = d^1 \circ \alpha^0 = (\alpha_x^0, g(\alpha_y^0), \alpha_y^0)$

Since s^0 is projection onto $X \times Y$, (1) tells us that α^1 is determined on $X \times Y$ precisely by α^0 . Then (2) & (3) imply that α^1 are paths in B between the images of f and g. Since we can recover α^0 from α^1 via $s^0 \circ \alpha^1 \circ d^{1/0} = \alpha^0$, the data from this square reduces to the α^1 maps. Since $\mathrm{Map}(\Delta^0, X) = X$ for X a space, we can see that the space of all α^0 will give us $X \times Y$ back again, so we have

$$X \times \{\rho \in \mathrm{Map}(\Delta^1, B) | \rho(0) = f(x), \rho(1) = g(y)\} \times Y$$

i.e. that Tot of our geometric cobar construction is the homotopy pullback of our original diagram.

4 Tot_n definition and fiber sequence

In algebra, we understand more about a group by considering a filtration of the group by subgroups, e.g. its lower central series. We can similarly filter Δ by the Δ^n:

$$\Delta^0 \subset \Delta^1 \subset \Delta^2 \subset \cdots$$

For a cosimplicial or simplicial object, we have a similar filtration by coskeleta or skeleta. For a discussion of (co)skeleta and matching/latching spaces, see appendix A.

4.1 $\mathrm{Tot}_n(X^\bullet_\bullet)$ standard definition

Definition 4.1. $\mathrm{Tot}_n(X^\bullet_\bullet) := \mathrm{Hom}_{cS}(\mathrm{sk}_n\Delta, X^\bullet_\bullet)$

Proposition 4.2. $\mathrm{Tot}_n(X^\bullet_\bullet) = \mathrm{Tot}(\mathrm{cosk}_n X^\bullet_\bullet)$

Proof.

$$\begin{array}{rcll}
\mathrm{Tot}_n(X^\bullet_\bullet) & = & \mathrm{Hom}_{cS}(\mathrm{sk}_n\Delta, X^\bullet_\bullet) & \\
& = & \mathrm{hom}_{cS}(\mathrm{sk}_n\Delta \times \Delta, X^\bullet_\bullet) & \\
& = & \mathrm{hom}_{cS}(\Delta, \mathrm{hom}_{cS}(\mathrm{sk}_n\Delta, X^\bullet_\bullet)) & \text{adjointness of hom and } \times \\
& = & \mathrm{hom}_{cS}(\Delta, \mathrm{hom}_{cS}(\Delta, \mathrm{cosk}_n X^\bullet_\bullet)) & \text{adjointness of cosk and sk} \\
& = & \mathrm{hom}_{cS}(\Delta \times \Delta, \mathrm{cosk}_n X^\bullet_\bullet) & \text{adjointness of hom and } \times \\
& = & \mathrm{Hom}_{cS}(\Delta, \mathrm{cosk}_n X^\bullet_\bullet) &
\end{array}$$

∎

Adjointness of cosk and sk is discussed in Remark A.1

Proposition 4.3. $\mathrm{Tot}_n(X^\bullet_\bullet) \to \mathrm{Tot}_{n-1}(X^\bullet_\bullet)$ is a fibration when $X^\bullet_\bullet$ is fibrant.

Remark 4.4. cS is a simplicial monoidal category; namely, it satisfies SM7,i.e. we have for $X^\bullet_\bullet$ fibrant and $A \to B$ a cofibration, $\mathrm{Hom}_{cS}(, X^\bullet_\bullet)$ applied to $A \to B$ yields a fibration with the arrow point in the other direction,

$$\mathrm{Hom}_{cS}(A, X^\bullet_\bullet) \leftarrow \mathrm{Hom}_{cS}(B, X^\bullet_\bullet)$$

Proof of Proposition 4.3. By definition, $\mathrm{Tot}_n(X^\bullet_\bullet) = \mathrm{Hom}_{cS}(\Delta, X^\bullet_\bullet)$. The map Tot_n to Tot_{n-1} is then

$$\mathrm{Hom}_{cS}(\mathrm{sk}_n\,\Delta, X^\bullet_\bullet) \to \mathrm{Hom}_{cS}(\mathrm{sk}_{n-1}\,\Delta, X^\bullet_\bullet)$$

Since Δ is cofibrant, the inclusion $\mathrm{sk}_n\Delta \subset \mathrm{sk}_{n+1}\Delta$ is a cofibration (if it wasn't cofibrant, an extra condition would be required. See the section on Reedy Model Structure for details). By Remark 4.4, we know that this is a fibration. ∎

Proposition 4.5. For $X^\bullet_\bullet$ a cosimplicial simplicial set,

$$\mathrm{Tot}\, X^\bullet_\bullet = \varprojlim \mathrm{Tot}_s\, X^\bullet_\bullet$$

Proof.

$$
\begin{aligned}
\varprojlim \mathrm{Tot}_s X &= \varprojlim_s \mathrm{Hom}_{cS}(\Delta, \mathrm{cosk}_s X^\bullet_\bullet) \\
&= \mathrm{Hom}_{cS}(\Delta, \varprojlim_s \mathrm{cosk}_s X^\bullet_\bullet) \\
&= \mathrm{Hom}_{cS}(\Delta, X^\bullet_\bullet)
\end{aligned}
$$

□

Note that hom (and thus Hom = hom$(- \times \Delta, -)$) commutes with inverse limits in the second/covariant variable. Also, as s increases, the amount of truncation before extending in taking coskeleton decreases, i.e.

$$
\begin{aligned}
\varprojlim_s (\mathrm{cosk}_s X^\bullet_\bullet)_m &= \varprojlim_s (\mathrm{cosk}_s X^m_\cdot) \\
&= \varprojlim_s \varprojlim_{k \to m} (tr_s X^m_\cdot) \\
&= \varprojlim_{k \to m} \varprojlim_s (tr_s X^m_\cdot) \\
&= \varprojlim_{k \to m} X^m_\cdot \\
&= X^\bullet_\bullet
\end{aligned}
$$

Note: we have $k < m$ (i.e. this is the same inverse limit as that giving in the definition of G & J).

Remark 4.6. This argument can also be made for $X^\bullet_\bullet$ a cosimplicial *space*, but topology issues have to be taken into account.

4.2 Fiber of $\mathrm{Tot}_n(X^\bullet_\bullet) \to \mathrm{Tot}_{n-1} X^\bullet_\bullet$ is $\Omega^n N^n X$

Recall $s = \prod s^i : X^n \to M^n$;$M^n X$ is the n^{th} matching space of X.
$N^n X := \ker(X^n \xrightarrow{s} M^n X) = X^n \cap \ker s^0 \cap \cdots \cap \ker s^n$

We have the following pullback square from [GJ99]):

$$
\begin{tikzcd}
\mathrm{Tot}_n X^\bullet_\bullet \arrow[r] \arrow[d] & \mathrm{Hom}_{cS}(\Delta^n, X^\bullet_\bullet) \arrow[d] \\
\mathrm{Tot}_{n-1} X^\bullet_\bullet \arrow[r] & Y
\end{tikzcd}
$$

Where Y is a pullback itself:

$$Y := \mathrm{Hom}_{cS}(\partial\Delta^n, X^\bullet_\bullet) \times_{\mathrm{Hom}_{cS}(\partial\Delta^n, M^n X^\bullet_\bullet)} \mathrm{Hom}_{cS}(\Delta^n, M^n X^\bullet_\bullet)$$

Its elements are commutative squares of the form

$$
\begin{tikzcd}
\partial\Delta^n \arrow[r, hook] \arrow[d] & \Delta^n \arrow[d] \\
X^\bullet_\bullet \arrow[r] & M^n X^\bullet_\bullet
\end{tikzcd}
$$

A map $\varphi \in \mathrm{Hom}_{cS}(\Delta^n, X^\bullet_\bullet)$ gves a commutative square (this is the description of our righthand vertical map in the first diagram) by pre and post composing by s, i (where i is the inclusion $\partial\Delta^n \hookrightarrow \Delta^n$).

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, hook, "i"] \arrow[d, "\varphi i"'] & \Delta^n \arrow[d, "s\varphi"] \arrow[dl, "\varphi"'] \\
X^\bullet_\bullet \arrow[r, "s"] & M^n X^\bullet_\bullet
\end{tikzcd}$$

Note: For $X^\bullet_\bullet$ fibrant, we get that our righthand vertical arrow of the original square is a fibration. So if we show that the fibre of that map is $\Omega^n N^n X$, then we have that the fiber of our Tot fibration is homeomorphic to $\Omega^n N^n X$. This is because pulling back a fibration preserves the fiber up to homeomorphism.

The kernel of the map $\mathrm{Hom}_{cS}(\Delta^n, X^\bullet_\bullet) \to Y$ consists of maps that determine trivial squares, i.e. squares where the two vertical maps are trivial. The lefthand vertical map is restriction to the boundary, so any map out of Δ^n that is trivial when restricted to the boundary is the same as a map from S^n. The righthand vertical map is post-composition with s, so for it to be trivial it's a map whose image in X is trivial under s, i.e. its image lands in $N^n X = ker(s)$. Thus, the kernel of the map is $\mathrm{Hom}_{cS}(S^n, N^n X) = \Omega^n N^n X \quad \square$.

5 New Tot_n Definition

Notice that in our above examples, we reduced our analysis to a single square and then to maps that fit into the rightmost vertical arrow-position of the diagram. This reoccuring reduction suggests that perhaps our definition of Tot_n can be similarly reduced or rephrased.

5.1 Motivation using Tot_1

Recall that $\mathrm{Tot}_1 X^\bullet_\bullet = \mathrm{Map}_{\mathrm{Top}^\Delta}(|\Delta|, \mathrm{cosk}_1 X^\bullet_\bullet)$ where $\mathrm{cosk}_n X$ can be thought of as having the data of X up to and including the nth level and then extended by degeneracies. As we saw with our examples of ΩB and homotopy pullback models, Tot-data reduces to the squares which are non-degerate, e.g. Tot_1-data for a space X corresponds to diagrams:

$$\begin{tikzcd}
\Delta^0 \arrow[r, shift left, "{d^0, d^1}"] \arrow[d, "\rho^0"] & \Delta^1 \arrow[l, shift left, "s^0"] \arrow[d, "\rho^1"] \\
X^0 \arrow[r, shift left, "{d^0, d^1}"] & X^1 \arrow[l, shift left, "s^0"]
\end{tikzcd}$$

In our two examples, we were able to reduce this even further, to conditions on maps that fit into the ρ^1-slot. It stands to reason that perhaps this can be done in general.

Commutivity of the above squares produces the following relations:

$$\begin{array}{rlcl} \text{(I)} & \rho^1 d^0 & = & d^0\rho^0 \text{ and } \rho^1 d^1 = d^1\rho^0 \\ \text{(II)} & \rho^0 & = & s^0\rho^1 d^0 = s^0\rho^1 d^1 \\ \text{(III)} & \rho^0 s^0 & = & s^0\rho^1 \end{array}$$

Claim: These relations reduce to just equation (I), restated as

$$\begin{array}{rlcl} \text{(I)}' & \rho^1 d^0 & = & d^0(s^0\rho^1 d^0) \\ & \rho^1 d^1 & = & d^1(s^0\rho^1 d^0) \end{array}$$

Justification. We'll show (I)' $\Rightarrow$ (II) $\Rightarrow$ (III)
(I)' $\Rightarrow$ **(II)** Assume (I)' and apply s^0 to both equations.
(II) $\Rightarrow$ **(III)** Rewrite (III) with the substitutions from (II) and we get

$$\text{(III)}' \ (s^0\rho^1 d^0)s^0 \quad = \quad s^0\rho^1$$

Assume (II) and apply d^i to the expression on the LHS of the "=" of (III)' and separately to the expression on the RHS.

$$\begin{array}{rcl} LHS & = & (s^0\rho^1 d^0)s^0 d^0 = s^0\rho^1 d^0 \\ RHS & = & s^0\rho^1 d^0 \end{array}$$

Since result of application of d^0 are equal, then (III)' holds since the d^i are injective.

□

So, we get

Definition 5.1. $\mathrm{Tot}_1 X^\bullet_\bullet = \left\{ \rho \in \mathrm{Hom}_{cS}(\Delta^1, X^1) \,\middle|\, \begin{array}{l} \rho d^0 = d^0(s^0\rho d^1) \\ \rho d^1 = d^1(s^0\rho d^0) \end{array} \right\}$

Remark 5.2. The definition is equivalent to one where we replace $s^0\rho d^1$ with $s^0\rho d^0$.

What does this definition tell us? That elements of Tot_1 can be seen as paths between two images of a vertex (under d^0 and d^1).

The "natural map" $\mathrm{Tot}_1 X^\bullet_\bullet \to \mathrm{Tot}_0 X^\bullet_\bullet$ is $\rho \mapsto s^0\rho d^0$. So, this new description has already told me something new, that the map $\mathrm{Tot}_1 \to \mathrm{Tot}_0$ factors as 2 maps.

5.2 Tot_2 and generalizing to Tot_n

Based on our Tot_1 result, we might immediately think that Tot_2 should satisfy the same equations + 1 more since 1-cells have 2 ends to control and 2-cells have 3 edges (one more "end" than a 1-cell).

Definition 5.3. $\mathrm{Tot}_2 X^{\bullet}_{\bullet} = \left\{ \rho \in \mathrm{Hom}_{cS}(\Delta^2, X^2) \;\middle|\; \begin{array}{l} \rho d^0 = d^0(s^0 \rho d^2) \\ \rho d^1 = d^1(s^0 \rho d^0) \\ \rho d^2 = d^2(s^0 \rho d^1) \end{array} \right\}$

Justification of definition. As above, we'll show (I)' $\Rightarrow$ (II) $\Rightarrow$ (III), where now (I)' is the 3 constraints in the definition above and (II) and (III) are now

$$\begin{array}{l} \text{(II)} s^j \rho d^i \text{ all equal, for } i \in \{0,1,2\}, j \in \{0,1\} \\ \text{(III)} (s^j \rho d^i) s^k = s^k \rho \end{array}$$

(II) $\Rightarrow$ **(III)** The exact argument as in the Tot_1 case still holds and generalizes immediately for all n.

(I)' $\Rightarrow$ **(II)**: Let us denote

$$\begin{array}{rcl} (0) \rho d^0 & = & d^0(s^0 \rho d^2) \\ (1) \rho d^1 & = & d^1(s^0 \rho d^0) \\ (2) \rho d^1 & = & d^1(s^0 \rho d^1) \end{array}$$

We would like to show (II), i.e. that $s^j \rho d^i$ all equal for $j \in [0,1]$ and $i \in [0,2]$. We get

$$\begin{array}{lcl} s^0 \text{ applied to...} & & \\ (0) \text{ gives } s^0 \rho d^0 & = & s^0 \rho d^2 (\text{ because } s^j d^i = id \; i = j, j+1) \\ (1) \text{ gives } s^0 \rho d^1 & = & s^0 \rho d^0 (\text{ because } s^j d^i = id \; i = j, j+1) \end{array}$$

i.e. now we have that all of the $s^0 \rho d^k$ are equal for all k and

$$\begin{array}{lcl} s^1 \text{ applied to...} & & \\ (1) \text{ gives } s^1 \rho d^1 & = & s^0 \rho d^0 (\text{ because } s^j d^i = id \; i = j, j+1) \\ (2) \text{ gives } s^1 \rho d^2 & = & s^0 \rho d^1 (\text{ because } s^j d^i = id \; i = j, j+1) \\ (0) \text{ gives } s^1 \rho d^0 & = & s^1 d^0 (s^0 \rho d^2) \\ & = & s^1 d^0 (s^0 \rho d^0) \text{ (because all } s^0 \rho d^k \text{ equal } \forall k) \\ & & (\Rightarrow d^0 s^0 = id) \\ & = & s^1 d^0 (s^0 \rho d^1) \text{ (because all } s^0 \rho d^k \text{ equal } \forall k) \\ & = & s^1 \rho d^1 \end{array}$$

This shows that all of the $s^j \rho d^i$ all equal for $j \in [0,1]$ and $i \in [0,2]$ ☐

Then, the natural generalization would be:

Definition 5.4. $\mathrm{Tot}_n(X^{\bullet}_{\bullet}) = \left\{ \rho \in \mathrm{Hom}_{cS}(\Delta^n, X^n) \;\middle|\; \begin{array}{rcl} (0)\; \rho d^0 & = & d^0(s^0 \rho d^n) \\ (1)\; \rho d^1 & = & d^1(s^0 \rho d^0) \\ (2)\; \rho d^2 & = & d^2(s^0 \rho d^1) \\ \vdots & & \\ (n)\; \rho d^n & = & d^n(s^0 \rho d^{n-1}) \end{array} \right\}$

Justification of definition. The proof for (II) $\Rightarrow$ (III) readily generalizes.

We need now to show that (I)' $\Rightarrow$ (II) holds in general. Note that

(I)' will be the n-1 equations labeled (0) through (n) above
and (II) is that all of the $s^j\rho d^i$ are equal $\forall j \in [0, n-1], k \in [0, n]$.

We can treat the cases of Tot_1 and Tot_2 as our base cases and prove by induction.

Our *inductive hypothesis* is that (I)' is assumed and $s^j\rho d^i$ are known to be equal $\forall j \in [0, n-2]\ , i \in [0, n-1]$.

Notice that $s^{n-1}\rho d^n = s^0\rho d^{n-1}$, so we need to show

(a) $s^{n-1}\rho d^i$ all equal for $i \in [0, n]$(and equal to the other $s^j\rho d^i$)
(b) $s^j\rho d^n$ all equal for $j \in [0, n-1]$(and equal to the other $s^j\rho d^i$)

Proof of (a):

$$
\begin{array}{rcll}
s^{n-1}\rho d^i & = & s^{n-1}d^i(s^0\rho d^{i-1}) & i \in [0, n-1] \\
& = & s^{n-1}d^i(s^{i-2}\rho d^k) & \text{for any } k \in [0, n-1] \text{ by induction hyp} \\
& = & s^{n-1}s^{i-1}d^i\rho d^k & (\text{ because } d^i s^{j-1} = s^j d^i\ i < j) \\
& = & s^{n-1}\rho d^k & (\text{ because } s^j d^i = id\ i = j, j+1) \\
& = & s^{n-1}\rho d^{n-1} & \text{(namely, since any k works)} \\
& = & s^0\rho d^{n-2} & \text{ by induction hyp}
\end{array}
$$

So, $s^{n-1}\rho d^i = s^0\rho d^{n-2} = s^j\rho d^i\ \forall j \in [0, n-2],\ i \in [0, n-1]$

Proof of (b):

$$
\begin{array}{rcll}
s^j\rho d^n & = & s^j d^n(s^0\rho d^{n-1}) & j \in [0, n-1] \\
& = & s^j d^n(s^{n-2}\rho d^k) & (\text{ for any } k \in [0, n-1] \text{ by induction hyp}) \\
& = & s^j s^{n-1}d^n\rho d^k & (\text{ because } d^i s^{j-1} = s^j d^i\ i < j) \\
& = & s^j\rho d^k & (\text{ because } s^j d^i = id\ i = j, j+1) \\
& = & s^j\rho d^n - 1 & \text{(namely, since any k works)} \\
& = & s^0\rho d^{n-2} & \text{ by induction hyp}
\end{array}
$$

So, $s^j\rho d^n = s^j\rho d^i\ \forall j \in [0, n-1],\ i \in [0, n]$ $\square$

6 Examining the map $\rho \mapsto s^0\rho d^0 = s^i\rho d^j$

Now that we are perhaps convinced that this is a valid change of definition, we should examine the natural map that arises as our map between Tot_n and Tot_{n-1}. Namely, it needs to be shown that it is a fibration and the fiber is what we expect it to be.

$$
\begin{tikzcd}
\rho \in \mathrm{Tot}_{n+1} = \mathrm{Hom}_{cS}(\Delta^{n+1}, X^{n+1}|\cdots) \arrow[r, "s^0\rho d^0"] \arrow[d, "d^0"] & \mathrm{Tot}_n = \mathrm{Hom}_{cS}(\Delta^n, X^n|\cdots) \arrow[r, "s^0\rho d^0"] \arrow[d, "d^0"] & \mathrm{Hom}_{cS}(\Delta^{n-1}, X^{n-1}) \\
\rho d^0 \in \mathrm{Hom}_{cS}(\Delta^n, X^{n+1}|\cdots) \arrow[ur, "s^0"] & \mathrm{Hom}_{cS}(\Delta^{n-1}, X^n|\cdots) \arrow[ur, "s^0"] &
\end{tikzcd}
$$

6.1 $s^0\cdot : \mathrm{im}(\cdot d^0) \to \mathrm{Tot}_n$ is an isomorphism

We are considering the following situation

$$\begin{tikzcd}
\rho \in \mathrm{Tot}_{n+1} \arrow[r, "\cdot d^0"] \arrow[rr, bend left, "s^0\rho d^0"] & \mathrm{im}(\cdot d^0) \arrow[r, "s^0\cdot"] & \mathrm{Tot}_n \arrow[l, dashed, bend left, "?\ d^0\cdot"]
\end{tikzcd}$$

where the dashed arrow labeled ? $d^0\cdot$ is speculative and needs to be shown to be well defined as well as the other piece of $s^0\cdot$ being an isomorphism.

Proposition 6.1. $s^0\cdot : \mathrm{im}(\cdot d^0) \to Tot_n$, $d^0\cdot : \mathrm{Tot}_n \to \mathrm{im}(\cdot d^0)$ are the two pieces of an isomorphism.

Lemma 6.2. $d^0\cdot : \mathrm{Tot}_n \to \mathrm{im}(\cdot d^0)$ and is well-defined.

Argument. For $\gamma \in \mathrm{Tot}_n$, recall that even though our data reduces to a finite diagram of adjacent squares, that it is actually infinite and that we have a $\tilde{\gamma}$ as in the following:

$$\begin{tikzcd}
\Delta^0 \arrow[d] & \cdots & \Delta^{n-1} \arrow[r, shift left, "d^0,\dots,d^n"] \arrow[d, "s^0\gamma d^0"'] & \Delta^n \arrow[l, shift left, "s^0,\dots,s^{n-1}"] \arrow[r, shift left, "d^0,\dots,d^{n+1}"] \arrow[d, "\gamma"'] & \Delta^{n+1} \arrow[l, shift left, "s^0,\dots,s^n"] \arrow[d, "\tilde{\gamma}"'] \\
X^0 & \cdots & X^{n-1} \arrow[r, shift left, "d^0,\dots,d^n"] & X^n \arrow[l, shift left, "s^0,\dots,s^{n-1}"] \arrow[r, shift left, "d^0,\dots,d^{n+1}"] & X^{n+1} \arrow[l, shift left, "s^0,\dots,s^n"]
\end{tikzcd}$$

Where commutivity of the rightmost displayed square gives us that $d^0\gamma = \tilde{\gamma}d^0$, i.e. that d^0 takes us to $\mathrm{im}(\cdot d^0)$. □

Proof of Proposition 6.1. Given the above lemma, we just need to show the two equalities $s^0 d^0 = id_{\mathrm{Tot}_n}$ and $d^0 s^0 = id_{\mathrm{im}(\cdot d^0)}$:

- $s^0 d^0 = id_{\mathrm{Tot}_n}$: Our cosimplicial relations imply that $s^0 d^0 = id_{\mathrm{Tot}_n}$.
- $d^0 s^0 = id_{\mathrm{im}(\cdot d^0)}$: Notice that $\rho d^0 \overset{s^0\cdot}{\mapsto} s^0\rho d^0 \overset{d^0\cdot}{\mapsto} d^0 s^0 \rho d^0 = \rho d^0$
  The last equality is because $\rho \in \mathrm{Tot}_{n+1}$ and we can conclude that $d^0 s^0 = id_{\mathrm{im}(\cdot d^0)}$.

□

6.2 $\mathrm{Tot}_n X^\bullet_\bullet \to \mathrm{Tot}_{n-1} X^\bullet_\bullet$ is a fibration for $X^\bullet_\bullet$ fibrant

It is enough to first show the equivalence of the two definitions and then show that the fibration (when $X^\bullet_\bullet$ is fibrant) in the standard definition can be viewed as $\rho \mapsto \rho d^0$, i.e. the same as our induced map between Tot_n in the new definition.

For the purposes of distinguishing instances of the new definition, I will in this section denote the "standard" Tot by Tot^{std} and our proposed new definition by Tot^{new}.

Proposition 6.3. $\mathrm{Tot}_n^{std}(X_\bullet^\bullet) \cong \mathrm{Tot}_n^{new}(X_\bullet^\bullet)$

Proof. In the construction of our new definition, we implicitly created a map

$$\mathrm{Tot}_n^{std}(X^\bullet) \xrightarrow{\varphi} \mathrm{Tot}_n^{new}(X^\bullet)$$

$$\rho = (\rho^0, \dots, \rho^n) \mapsto \rho^n$$

This projection to the last coordinate clearly commutes with coface and codegeneracy maps.

We should construct a (simplicial) map in the other direction, ψ and show both compositions are identity.

Define

$$\psi(\rho) = (s^0 \underbrace{\cdots}_{n\ times} s^0 \rho d^0 \underbrace{\cdots}_{n\ times} d^0, \dots, s^o \rho d^0, \rho)$$

By the constraints on ρ (which correspond exactly to what is needed to have a commuting square), we get that our map commutes with coface and codegeneracies and is levelwise a map of simplicial sets.

Note that our finite square extends to an infinite one which is just the finite one extended by degeneracies.

Then it is clear that $\varphi\psi = id$.

Also, commutivity of squares gives us for $\alpha = (\alpha^0, \dots, \alpha^n) \in \mathrm{Tot}_n^{std}$,

$$\alpha^{n-1} = s^0 \alpha^n d^0 = s^i \alpha^n d^l \text{ for all valid } i, j$$

So it follows that $\psi\varphi = id$. □

Lemma 6.4. $\mathrm{Tot}_n^{std} \to \mathrm{Tot}_{n-1}^{std}$ can be seen as $\rho \mapsto \rho d^0$ (i.e. is the same map as in the new situation).

Proof. The map in our standard situation is the induced map by inclusion of $\mathrm{sk}_{n-1} \Delta \subset \mathrm{sk}_n \Delta$. We can take this inclusion to be d^0 and then the map $\mathrm{Tot}_n^{std} \to \mathrm{Tot}_{n-1}^{std}$ is $\rho \mapsto \rho d^0$ □

We know in the standard setting that for $X^\bullet$ fibrant, the map $\mathrm{Tot}_n^{std} X^\bullet \to \mathrm{Tot}_{n-1}^{std} X^\bullet$ is a fibration, so with the proposition and lemma above, we can port this to the new setting and gain the same result. In other words, we have just shown

Theorem 6.5. $\mathrm{Tot}_n^{new} X_\bullet^\bullet \to \mathrm{Tot}_{n-1}^{new} X_\bullet^\bullet$ is a fibration for $X_\bullet^\bullet$ fibrant

6.3 Fiber of $\rho \mapsto s^0 \rho d^0 : \mathrm{Tot}_n \to \mathrm{Tot}_{n-1}$ is isomorphic to $\Omega^n N X^n$

Remark 6.6. We are assuming that $X_\bullet^\bullet$ is basepointed and that our maps are basepoint-preserving.

Note that this is saying that the fiber in our new situation is isomorphic to that in the original situation, using the new definition instead of relying on equivalence of definitions. It turns out that the proof is more natural in this new setting.

The fiber of $\rho \mapsto s^0\rho d^0 : \mathrm{Tot}_n X^{\bullet}_{\bullet} \to \mathrm{Tot}_{n-1X^{\bullet}_{\bullet}}$ will be the space

$$\left\{ \rho \in \mathrm{Hom}_{cS}(\Delta^n \times \Delta, X^n) \;\middle|\; \begin{matrix} \rho d^0 = d^0(s^0\rho d^0) \\ \vdots \\ \rho d^n = d^n(s^0 \rho d^{n-1}) \end{matrix} \quad \& \; s^0\rho d^0 = * \right\}$$

6.3.1 Looks like a loopspace:

This description of the fiber gives us the following equalities:

$$\begin{array}{llll} \rho d^0 & = d^0 s^0 \rho d^n & = d^n * & = * \\ \rho d^1 & = d^1 s^0 \rho d^0 & = d^0 * & = * \\ \vdots & & & \\ \rho d^n & = d^n s^0 \rho d^{n-1} & = d^{n-1} * & = * \end{array}$$

These tell us that the maps comprising our fiber are actually maps out of S^n instead of just Δ^n, since the equlaties we obtained tell us that the maps of Δ^n collapse all of its faces to a point.

6.3.2 The loopspace is $\Omega^n N X^n$:

Recall that when we reduced to this new definition, we had a set of equalities that followed from those of the form $\rho d^i = d^i(s^j \rho d^k)$. Those which were labeled as (III) were:

$$\begin{array}{ll} s^0 \rho & = (s^0 \rho d^0) s^0 \\ \vdots & \\ s^{n-1}\rho & = (s^0 \rho d^0) s^{n-1} \end{array}$$

In our fiber, we have that $s^0\rho d^0 = *$, so each of these equalties reduces to $*s^i = *$, i.e. our fiber is in $\{\ker s^i\}_{i=0}^{n-1} \cap X^n$

This tells us that our fiber is maps $S^n \to \{\ker s^i\}_{i=0}^{n-1} \cap X^n$, which is exactly the definition of $\Omega^n N X^n$

A Skeleta and Coskeleta

Recall that for a category $\mathcal{C}$ that a simplicial object in $\mathcal{C}$ is an element of $\mathcal{C}^{\Delta^{op}}$ and a cosimplicial object is an element of $\mathcal{C}^{\Delta}$. We can define the subcategory $\Delta_n \overset{i_n}{\subset} \Delta$ consisting of all $[k]$ for $k < n$. This gives induced truncation functors

$$\begin{array}{rl} & (i_n)^* = tr_n : \mathcal{C}^{\Delta^{op}} \to \mathcal{C}^{\Delta_n^{op}} \\ \text{and} & (i_n)^* = tr_n : \mathcal{C}^{\Delta} \to \mathcal{C}^{\Delta_n} \end{array}$$

In both situations, the truncation functor has both a left and a right adjoint, given by left and right Kan extensions along i_n.

Let $X^\bullet$ be a cosimplicial object and $Y_\bullet$ be a simplicial object and let r^n, l^n be the left and right adjoints of cosimplicial truncation and r_n, l_n dually for simplicial, defined as follows (for $k \leq n$):

$$\begin{array}{rclrcl} (l^n X^\bullet)^m & = & \operatorname{colim}_{k \hookrightarrow m} X^k & (r^n X^\bullet)^m & = & \lim_{m \twoheadrightarrow k} X^k \\ (l_n Y_\bullet)_m & = & \operatorname{colim}_{m \twoheadrightarrow k} Y_k & (r_n Y_\bullet)_m & = & \lim_{k \hookrightarrow m} Y_k \end{array}$$

In either the cosimplicial or simplicial case, the skeleton will be truncation followed by a left kan extension and the coskeleton will be truncation followed by a right kan extension. Definitions for coskeleton and skeleton are (for $k \leq n$)

$$\begin{array}{rclrcl} \operatorname{cosk}_n X^k & = & r^n(tr_n X^\bullet) & \operatorname{sk}_n X^k & = & l^n(tr_n X^\bullet) \\ \operatorname{cosk}_n Y_k & = & r_n(tr_n Y_\bullet) & \operatorname{sk}_n Y_k & = & l_n(tr_n Y_\bullet) \end{array}$$

Remark A.1. It is casually trivial to the obvious observer that coskeleton and skeleton will be adjoint functors in each situation.

A nicer way to view at least the simplicial skeleta and coskeleta comes from the unpublished paper of Reedy [Ree74], wherein he gives cosk as a pullback and sk as a pushout:

(a) The following is pushout, and definition of sk_n:

$$\begin{tikzcd} \coprod_{\alpha:n\to k} \operatorname{sk}_{n-1}(X)_n \arrow[r] \arrow[d] & \coprod_{\alpha:n\to k} X_n \arrow[d] \\ \operatorname{sk}_{n-1}(X)_k \arrow[r] & \operatorname{sk}_n(X)_k \end{tikzcd}$$

where sums are over degeneracies n to k and the vertical maps are sums of degeneracies in $sk_n(X)$.

(b) The following is pullback, and definition of cosk_n:

$$\begin{tikzcd} \operatorname{cosk}_n(X)_k \arrow[r] \arrow[d] & \operatorname{cosk}_{n-1}(X)_k \arrow[d] \\ \prod_{\alpha:k\to n} X_n \arrow[r] & \prod_{\alpha:k\to n} \operatorname{cosk}_{n-1} X_n \end{tikzcd}$$

where products are over degeneracies from n to k and the vertical maps are products of the face maps of $cosk_n(X)$.

Remark A.2. For X a simplicial object, $(\operatorname{cosk}_n X)_k \cong X_k$ and $(\operatorname{sk}_n X)_k \cong X_k$ for $k \leq n$. It is also true that for Y a cosimplicial object that we have $(\operatorname{cosk}_n Y)^k \cong Y^k$ and $(\operatorname{sk}_n Y)^k \cong Y^k$ for $k \leq n$.

Lemma A.3. The cosimplicial skeleton of Δ, i.e. $\mathrm{sk}_n \Delta$, is the functor $s \mapsto \mathrm{sk}_n \Delta^s$

Proof. By definition of the simplicial skeleton, we get that for $s \leq n, \mathrm{sk}_n \Delta^s \cong \Delta^s$. So, our levelwise description agrees with what the cosimplicial coskeleton should look like for the levels where it should just look like cosimplicial truncation.

Then for $s > n$, $\mathrm{sk}_n \Delta^s = \underset{k \hookrightarrow s}{\mathrm{colim}} (\mathrm{tr}_n \Delta)^k$. This looks like the injections $n \hookrightarrow s$ subject to the relations inforced by the relations of the lower dimensional injections. These injections are precisely the n-cells on Δ^s where the lower-dimensional relations tie them together to be the n-skeleto, which agrees with our levelwise description. ☐

Remark A.4. $\mathrm{sk}_n\Delta \subset \mathrm{sk}_{n+1}\Delta$

Notice that cosk and sk can both be interpreted as reducing the information we were given to what is known for truncation to the nth level and then filling in by degeneracies.

A.1 Matching and Latching spaces

This is the most natural place for this section to occur in terms of required definitions that need to be fresh, but it is not perhaps optimal in terms of usage of the objects.

Definition A.5. Given a simplicial object $X_\bullet$ and a cosimplicial object $Y^\bullet$, the *nth* matching and latching objects are given by:

$$\mathrm{M}_n X_\bullet := \lim_{\substack{k \hookrightarrow n \\ k<n}} X_k \qquad \mathrm{L}_n X_\bullet := \underset{\substack{n \twoheadrightarrow k \\ k<n}}{\mathrm{colim}} X_k$$
$$\mathrm{M}^n Y^\bullet := \lim_{\substack{n \twoheadrightarrow k \\ k<n}} Y^k \qquad \mathrm{L}^n Y^\bullet := \underset{\substack{k \hookrightarrow n \\ k<n}}{\mathrm{colim}} Y^k$$

These differ from those of Goerss and Jardine by a degree shift. Our L^n and M^n are their L^{n-1} and M^{n-1}

B Reedy Model Structure

The information from this section is taken nearly wholesale from Philip Hirschorn's book on Model Categories. The proposition and example about fibrancy are from Bausfield & Kan.

Definition B.1. A *Reedy Category* is a category $\mathscr{C}$ with two subcategories $\overleftarrow{\mathscr{C}}$ (inverse subcategory) and $\overrightarrow{\mathscr{C}}$ (direct subcategory) and an assignment of degree to each object of $\mathscr{C}$. Both subcategories contain all the objects of $\mathscr{C}$. All non-identity morphisms of the inverse subcategory lower degree and those of the direct subcategory raise degree. Additionally, every morphism $g \in \mathscr{C}$ can be factored as a composition of morphisms $\overrightarrow{g}\overleftarrow{g} = g$ where $\overleftarrow{g} \in \overleftarrow{\mathscr{C}}$ and $\overrightarrow{g} \in \overrightarrow{\mathscr{C}}$.

Example. Δ the category with objects $[n]$ for $n \geq 0$ and morphisms the weakly monotone functions $[n] \to [k]$ is a Reedy category where the direct subcategory's maps are the coface maps d^i and the inverse subcategory's maps are the codegeneracy maps s^j.

Definition B.2. X a cosimplicial set, the nth latching object, $L_n X$ is naturally isomorphic to the subcomplex of X^n which is simplices that lie in the image of the coface operator.

Definition B.3. Let $\mathscr{C}$ be a Reedy category and $\mathscr{M}$ a model category and $\mathbf{X}, \mathbf{Y}, \mathscr{C}-$ diagrams in $\mathscr{M}$. Then the Reedy model structure on $\mathscr{M}^{\mathscr{C}}$ is defined as, for $f : \mathbf{X} \to \mathbf{Y}$ a map of diagrams:

(i) f is a Reedy weak equivalence if for every object $c \in \mathscr{C}$, the map $f_c : X_c \to Y_c$ is a weak equivalence in $\mathscr{M}$.

(ii) f is a Reedy cofibration if for every object $c \in \mathscr{C}$, the relative latching map $X_c \coprod_{L_c X} L_c Y \to Y_c$ is a cofibration in $\mathscr{M}$

(iii) f is a Reedy fibration if for every $c \in \mathscr{C}$, the relative matching map $X_c \to Y \times_{M_c Y} M_c X$ is a fibration in $\mathscr{M}$

Proposition B.4. Cosimplicial simplicial sets has a model structure as in B.3, where $\mathscr{C} = \Delta$ is a Reedy category and $\mathscr{M} = \textbf{Ssets}$ has a model category structure were fibrations are Kan fibrations, cofibrations are monomorphisms and weak equivalences afre maps whos realization is a weak equivalence of topological spaces.

Note that the weak equivalences in cosimplicial simplicial sets are levelwise. A more convenient condition for a map to be a cofibration will follow.

Definition B.5. For X a cosimplicial simplicial set, a *maximal augmentation* of X is a the simplicial set which is the equalizer of the diagram

$$X^0 \underset{d^1}{\overset{d^0}{\rightrightarrows}} X^1$$

i.e. an n-simplex of the maximal augmentation of X is an n-simplex σ of X^0 such that $d^0\sigma = d^1\sigma$.

Then

Theorem B.6. If $f : X \to Y$ is a cosimplicial simplicial set map, then f is a cofibration if and only if it is a monomorphism that takes the maximal augmentation of X onto that of Y.

Heuristically, this is true because the Reedy structure says that f is a cofibration if it's a levelwise cofibration and if the induced maps on Latching objects are cofibrations. Monomorphisms are the cofibrations in simplicial sets and the condition of latching objects is satisfied by f be a levelwise monomorphism and correcting at the 0th level with the maximal augmentation condition. For a real proof, see Hirschorn.

Corollary B.7. A cosimplicial simplicial set is Reedy cofibrant if and only if its maximal augmentation is empty.

Example. The standard cosimplicial simplicial set Δ is Reedy cofibrant.

Proposition B.8. X a cosimplicial simplicial set is Reedy fibrant if maps $s : X^n \to M^n X$ for $n \geq 0$ are all fibrations (i.e. Kan fibrations).

Example. All cosimplicial groups (called in DK "grouplike cosimplicial objects") are Reedy fibrant. [BK72]

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