Platonic 2-Groups

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Abstract

This paper will construct examples of 2-groups that are grounded in Euclidean geometry. The paper will introduce a notion of 2-group and discuss how to classify them using the more familiar algebraic objects that are groups, modules and cocycles. It concludes by explicitly constructing a cocycle that can be used to construct a 2-group.

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1 Preliminaries

1.1 Acknowledgements

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1.2 Introduction

2-groups are fundamental objects in higher dimensional algebra. While intricate in itself, higher dimensional algebra has gained much interest due to its applications in physics. Much of the motivation for this comes from sting theory where point particles are superseded by paths (or strings). Just as the representation theory of Lie groups has important applications in particle physics, the 2-representation theory of Lie 2-groups plays an important role in string theory.[1] Similarly just as ordinary groups play a role in the representation theory of Lie groups, it is anticipated that 2-groups play a role in 2-representation theory. As such this paper aims to construct particularly

[1]See Baez and Huerta [2].

concrete example of 2-groups that are based on symmetries of geometric objects, namely the Platonic solids.

The first substantive section will analyse in detail the tetrahedral group and show explicitly how its binary extension sits inside the group of unit quaternions, $Sp(1)$. The second section will list all the finite subgroups of $Sp(1)$ that may be obtained in a similar manner. In the third section 2-groups will be introduced and discussed culminating in a classification of a certain kind of 2-group. In the 3rd section, the co-cycle from the classification will be put in context and in the final section such a cocycle will be explicitly constructed, giving sufficient data to form a 2-group.

2 The Tetrahedral Group

The tetrahedral group, T, is the group of rotational symmetries of a tetrahedron in Euclidean 3-space. It may be embedded into the symmetric group on four elements, S_4, as every rotational symmetry permutes the four vertices of the tetrahedron.

Proposition 2.0.1. *This embedding identifies T with the the alternating group on four elements, A_4 which, in cycle notation, consists of the elements:*

$$A_4 = \{(1), (12)(34), (13)(24), (14)(23), (123),$$
$$(132), (124), (142), (134), (143), (234), (243)\}$$

Proof. See Appendix. □

Henceforth, we will refer to elements of T using the notation of permutations in A_4.

Proposition 2.0.2. *The group T has presentation $\langle a, b \mid a^3, b^3, (ab)^2\rangle$*

Proof. See Appendix. □

The binary tetrahedral group, $\tilde{T}$ is given by the presentation:

$$\tilde{T} = \left\langle e, f \mid e^3 = f^3 = (ef)^2 \right\rangle$$

Proposition 2.0.3. *The sequence:*

$$1 \to \{\pm 1\} \hookrightarrow \tilde{T} \xrightarrow{\pi} T \to 1$$

Is short exact and a central extension. Where the map π is defined as:

$$\pi : \tilde{T} \to T$$
$$e \mapsto a$$
$$f \mapsto b$$

Proof. See Appendix. $\square$

The group $\tilde{T}$ may be embedded into the group of unit norm quaternions, $Sp(1)$, which is a compact Lie group with its underlying manifold being the 3-sphere, $\mathbb{S}^3$. On the other hand T may be embedded into a special orthogonal group, $SO(3)$ which we identify with the group of rotations about an axis in 3 dimensions in the canonical manner.

The second embedding is the tetrahedral representation of T with the tetrahedron embedded into $\mathbb{R}^3$ as follows: Let the verticies 1, 2, 3 and 4 be at the points (-1,-1,1), (1,-1,-1), (-1,1,-1) and (1,1,1) respectively. Since each element of T is a rotation of a tetrahedron in 3-space, it is mapped to the element of $SO(3)$ as the rotation of $\mathbb{R}^3$ that achieves the same permutation on the subset consisting of the verticies.

The first embedding may be described as follows:

$$\begin{aligned} r : \tilde{T} &\to Sp(1) \\ e &\mapsto \frac{1}{2}(1 - \mathbb{i} - \mathbb{j} - \mathbb{k}) \\ f &\mapsto \frac{1}{2}(1 - \mathbb{i} + \mathbb{j} - \mathbb{k}) \end{aligned}$$

It will be proved below that it is injective.

Proposition 2.0.4. *The group $SO(3)$ is doubly covered by $Sp(1)$ via a map*

$$\begin{aligned} p : Sp(1) &\to SO(3) \\ q &\mapsto \{T : r \mapsto qrq^{-1}\} \end{aligned}$$

Proof. See Appendix. □

The proof suggests an alternative rule for the map p. Let $q = a + b\mathbb{i} + c\mathbb{j} + d\mathbb{k}$ where $a^2 + b^2 + c^2 + d^2 = 1$, then $p(q)$ is a rotation of $2\cos^{-1}(a)$ clockwise about an axis of (b, c, d).

Thus we have the following diagram:

$$\begin{tikzcd} \tilde{T} \arrow[r, "r"] \arrow[d, two heads, "\pi"] & Sp(1) \arrow[d, two heads, "p"] \\ T \arrow[r, tail, "\rho"] & SO(3) \end{tikzcd}$$

Proposition 2.0.5. *The above diagram commutes.*

Proof. Consider the generator $e \in \tilde{T}$.

$$\pi(e) = a = (132)$$

And its image under ρ is a $\frac{4\pi}{3}$ clockwise rotation about the axis $(1, 1, 1)$.

$$r(e) = \tfrac{1}{2}(1 - \mathbb{i} - \mathbb{j} - \mathbb{k})$$

And $p(\frac{1}{2}(1-\mathbb{i}+\mathbb{j}-\mathbb{k}))$ is a $\frac{2\pi}{3}$ clockwise rotation about the axis $(-1, -1, -1)$.

Which is equivalent to a $\frac{4\pi}{3}$ clockwise rotation about the axis $(1, 1, 1)$.

Also $\pi(f) = b = (143)$.

And $\rho((143))$ is a $\frac{4\pi}{3}$ clockwise rotation about the axis $(-1, 1, -1)$.

While $r(f) = \frac{1}{2}(1 - \mathbb{i} - \mathbb{j} - \mathbb{k})$.

And $p(\frac{1}{2}(1 - \mathbb{i} + \mathbb{j} - \mathbb{k}))$ is a $\frac{2\pi}{3}$ clockwise rotation about the axis $(1, -1, 1)$.

Which is equivalent to a $\frac{4\pi}{3}$ clockwise rotation about the axis $(-1, 1, -1)$.

Thus $\rho(\pi(e)) = p(r(e))$ and $\rho(\pi(f)) = p(r(f))$

Since e and f generate $\tilde{T}$, $\rho \circ \pi = p \circ r$. ☐

Finally it is required to prove that r is injective.

We may note that the kernel of $ker(p)$ gives rise to the following diagram:

$$\begin{tikzcd}
ker(\pi) \arrow[r, dashed, "\varphi"] \arrow[d, tail, "i_1"] & ker(p) \arrow[d, tail, "i_2"] \\
\tilde{T} \arrow[r, "r"] \arrow[d, two heads, "\pi"] & Sp(1) \arrow[d, two heads, "p"] \\
T \arrow[r, tail, "\rho"] & SO(3)
\end{tikzcd}$$

The homomorphism φ exists and is unique by the universal property of i_2

as:

$$(p \circ r \circ i_1)(ker(\pi)) = (\rho \circ \pi \circ i_1)(ker(\pi)) = \{1\}$$

Thus:

$$\begin{aligned} i_2(\varphi(-1)) &= r(i_1(-1)) \\ &= r(g) = r(e^3) \\ &= \left(\frac{1}{2}(1 - \mathbb{i} - \mathbb{j} - \mathbb{k})\right)^3 \\ &= -1 \end{aligned}$$

Thus $i_2 \circ \varphi$ is injective, but as $r \circ i_1 = i_2 \circ \varphi$ and i_1 is injective, r must be injective. Thus the final form of the diagram is:

$$\begin{tikzcd} \tilde{T} \arrow[r, tail, "r"] \arrow[d, two heads, "\pi"] & Sp(1) \arrow[d, two heads, "p"] \\ T \arrow[r, tail, "\rho"] & SO(3) \end{tikzcd}$$

It should be noted that such a diagram may be constructed with any finite subgroup of $SO(3)$ taking the place of T, which leads to the next section.

3 Binary Polyhedral Groups

In the previous section there we constructed a finite subgroups of $Sp(1)$ from a finite subgroup of $SO(3)$.

Up to isomorphism there is a classification of such groups, (See [12, 27]) they are:

- The finite cyclic groups C_n;
- The finite dihedral groups D_{2n};
- The tetrahedral group T;
- The octahedral group O; and,
- The icosahedral group I.

As one may have guessed these are all possible rotational symmetry groups for polygons and polyhedra in Eucledian 3-space. (Note that although the dihedral group is traditionally thought of consisting of both rotations and reflections, the same permutations of verticies may be achieved by purely by rotations in 3-space.)

The following is a table of polyhedral groups their presentations, a presentation of their $\mathbb{Z}/2$ extension in $Sp(1)$ and the order of the extension.

G	Presentation	$\tilde{G}$	$\lvert\tilde{G}\rvert$
C_n	$\langle a \mid a^n \rangle$	$\langle e \mid e^{2n} \rangle$	$2n$
D_{2n}	$\langle a, b \mid a^n, b^2, (ab)^2 \rangle$	$\langle e, f \mid e^n = f^2 = (ef)^2 \rangle$	$4n$
T	$\langle a, b \mid a^3, b^3, (ab)^2 \rangle$	$\langle e, f \mid e^3 = f^3 = (ef)^2 \rangle$	24
O	$\langle a, b \mid a^4, b^3, (ab)^2 \rangle$	$\langle e, f \mid e^4 = f^3 = (ef)^2 \rangle$	48
I	$\langle a, b \mid a^5, b^3, (ab)^2 \rangle$	$\langle e, f \mid e^5 = f^3 = (ef)^2 \rangle$	120

4 2-Groups

Our discussion now turns to a catergorfied concept of groups known as 2-groups. We build by the definition of a 2-group from stages assuming that the definitions of category, functors and natural transformations are known.

Definition 4.0.1. *A* ***weak monoidal category*** *is a (small) category, $\mathcal{M}$ together with:*

- *A bifunctor $t = - \otimes - : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$*
- *A distinguished object $1 \in Ob(\mathcal{M})$*
- *Natural isomorphisms*

$$a : t \circ (t \times id_{\mathcal{M}}) \Rightarrow t \circ (id_{\mathcal{M}} \times t)$$

$$l : t|_{\mathbf{1} \times \mathcal{M}} \Rightarrow id_{\mathcal{M}}$$

$$r : t|_{\mathcal{M} \times \mathbf{1}} \Rightarrow id_{\mathcal{M}}$$

Here $\mathbf{1}$ is the full subcategory generated by the object 1.

In addition to naturality, the component isomorphisms,

$$a_{x,y,z} : (x \otimes y) \otimes z \xrightarrow{\cong} x \otimes (y \otimes z)$$

$$l_x : 1 \otimes x \xrightarrow{\cong} x$$

$$r_x : x \otimes 1 \xrightarrow{\cong} x$$

Must satisfy the following diagrams:

$$\begin{tikzcd}
 & & (w \otimes x) \otimes (y \otimes z) \arrow[drr, "a_{w,x,y \otimes z}"] & & \\
((w \otimes x) \otimes y) \otimes z \arrow[urr, "a_{w \otimes x,y,z}"] \arrow[dr, "a_{w,x,y} \otimes id_z"'] & & & & w \otimes (x \otimes (y \otimes z)) \\
 & (w \otimes (x \otimes y)) \otimes z \arrow[rr, "a_{w,x \otimes y,z}"'] & & w \otimes ((x \otimes y) \otimes z) \arrow[ur, "id_w \otimes a_{x,y,z}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
(x \otimes 1) \otimes y \arrow[rr, "a_{x,1,y}"] \arrow[dr, "r_x \otimes id_y"'] & & x \otimes (1 \otimes y) \\
 & x \otimes y \arrow[ur, "id_x \otimes l_y"'] &
\end{tikzcd}$$

The component isomorphisms $a_{x,y,z}$, l_x and r_x are often referred to as the associator, left unit and right unit respectively, since if they were identities instead of isomorphisms they would correspond exactly to the axioms of multiplication in a monoid. As expected, a 2-group refines this notion by requiring invertability of objects and morphisms, in the following sense:

Definition 4.0.2. *A **weak 2-group** is a weak monoidal category, $\mathcal{G}$, such that:*

- *If $x \in Ob(\mathcal{G})$ then $\exists\, y \in Ob(\mathcal{G})$ s.t. $x \otimes y \cong 1$ and $y \otimes x \cong 1$; and*
- *If $f : x \to y$ is a morphism in $\mathcal{G}$ then $\exists\, g : y \to x$ in $\mathcal{G}$ s.t. $f \circ g = id_y$ and $g \circ f = id_x$.*

As expected we may define homomorphisms of groups to be weak monoidal functors, but we refer a reader interested in such notions to Baez and Lauda [3].

It follows from the definition that the automorphisms of any object form a group under composition. Furthermore given any automorphism of 1, we may identify it with an automorphism of any $x \in Ob(\mathcal{G})$ as follows:
Let $f \in Aut(1)$, define:

$$\begin{aligned} \varphi_x : Aut(1) &\longrightarrow Aut(x) \\ f &\longmapsto l_x \circ (f \otimes id_x) \circ l_x^{-1} \end{aligned}$$

The map φ_x is a homomorphism since:

$$\begin{aligned} \varphi_x(f \circ g) &= l_x \circ ((f \circ g) \otimes id_x) \circ l_x^{-1} \\ &= l_x \circ (f \otimes id_x) \circ (g \otimes id_x) \circ l_x^{-1} && \text{Since } \otimes \text{ is a bifunctor.} \\ &= l_x \circ (f \otimes id_x) \circ l_x^{-1} \circ l_x \circ (g \otimes id_x) \circ l_x^{-1} \\ &= \varphi_x(f) \circ \varphi_x(g) \end{aligned}$$

We use this homomorphism to argue that the automorphism groups are abelian, although we cannot quite achieve this in a weak 2-group.
If $\mathcal{G}$ is a weak 2-group, and $x, y \in Ob(\mathcal{G})$, and $f \in Aut(1)$ and $g \in Hom(x, y)$, then the top face of the diagram below commutes as both direction are equal to $g \otimes f$.

$$
\begin{tikzcd}
& x \otimes 1 \arrow[rr, "id_x \otimes f"] \arrow[dl, "g \otimes id_1"'] \arrow[dd, "r_x"'] & & x \otimes 1 \arrow[dl, "g \otimes id_1"] \arrow[dd, "r_x"] \\
y \otimes 1 \arrow[rr, "id_y \otimes f"] \arrow[dd, "r_x"'] & & y \otimes 1 \arrow[dd, "r_x"] & \\
& x \arrow[rr, "\varphi_x(f)"'] \arrow[dl, "g"] & & x \arrow[dl, "g"] \\
y \arrow[rr, "\varphi_y(f)"'] & & y &
\end{tikzcd}
$$

Since r is a natural isomorphism, the vertical arrows are invertible. Thus

$$
\begin{aligned}
g \circ \varphi_x(f) &= r_x \circ (g \otimes id_1) \circ (id_x \otimes f) \circ r_x^{-1} \\
&= r_x \circ (id_y \otimes f) \circ (g \otimes id_1) \circ r_x^{-1} \\
&= \varphi_y(f) \circ g
\end{aligned}
$$

Thus f may be said to "commute" with g. To have proper commutativity, we obviously need $x = y = 1$. However we also need to change the type of 2-group we are dealing with as:

1. Inverses are only unique up to isomorphism (we will need this later).
2. The right unit law is not an identity but merely an isomorphism.

The first point may be addressed by taking the **skeleton** of $\mathcal{G}$ which is obtained by replacing its objects by isomorphism classes. As the isomorphism classes of objects in a weak 2-group form an ordinary group (under the tensor product), the objects of the skeleton are a group.

Definition 4.0.3. *A* ***skeletal category*** *is a category in which every pair of isomorphic objects are identical.*

The second is addressed by the following definition:

Definition 4.0.4. *A* ***special 2-group*** *is a weak 2-group for which the underlying category is skeletal and the maps induced by the natural transformations l and r are identities.*

Note that the associator need not be an identity morphism as the automorphisms of an object in a skeletal category need not be trivial, however the triangular diagram gives $a_{x,1,y} = id_{x\otimes 1\otimes y} = id_{x\otimes y}$.

Now if f and $g \in Aut(1)$ in a special 2-group we have the following diagram, which is a special case of the previous commutative cube:

$$\begin{tikzcd}
 & 1\otimes 1 \arrow[rr, "id_1\otimes f"] \arrow[dl, "g\otimes id_1"'] \arrow[dd, equal] & & 1\otimes 1 \arrow[dl, "g\otimes id_1"] \arrow[dd, equal] \\
1\otimes 1 \arrow[rr, "id_1\otimes f"] \arrow[dd, equal] & & 1\otimes 1 \arrow[dd, equal] & \\
 & 1 \arrow[rr, "f"'] \arrow[dl, "g"] & & 1 \arrow[dl, "g"] \\
1 \arrow[rr, "f"'] & & 1 &
\end{tikzcd}$$

Not only do we have $g \circ f = f \circ g$, but also $g \circ f = g \otimes f$, i.e. $Aut(1)$ is an abelian group under composition which is equivalent to the tensor product of morphisms.

Lemma 4.0.5. *Let $\mathcal{G}$ be a special 2-group and $x \in Ob(\mathcal{G})$, then the homomorphism*

$$\varphi_x : Aut(1) \to Aut(x)$$

is an isomorphism.

Proof. Note that as $l_x = id_x$, we have

$$\begin{aligned} \varphi_x(f) &= id_x \circ (f \otimes id_x) \circ id_x^{-1} \\ &= f \otimes id_x \end{aligned}$$

Define, as the inverse:

$$\begin{aligned} \psi_x : Aut(x) &\longrightarrow Aut(1) \\ g &\longmapsto g \otimes id_{x^{-1}} \end{aligned}$$

Where x^{-1} is the (unique) inverse of x. The map ψ_x is is well defined though

$$\begin{aligned} g \otimes id_{x^{-1}} &\in Aut(x \otimes x^{-1}) \\ x \otimes x^{-1} &= 1 && \text{As } \mathcal{G} \text{ is special.} \end{aligned}$$

The map ψ is a homomorphism by a similar argument as that used for φ_x.

They are isomorphisms as:

$$\begin{aligned}
\varphi_x(\psi_x(g)) &= (g \otimes id_{x^{-1}}) \otimes id_x \\
&= g \otimes (id_{x^{-1}} \otimes id_x) \\
&= g \otimes id_{x^{-1} \otimes x} \\
&= g \otimes id_1 \\
&= g
\end{aligned}$$

With the reverse direction being formally identical. $\square$

In order to simplify things slightly, we shall identify each $Aut(x)$ with $Aut(1)$ via the isomorphism ψ_x.

This enables us to define a group action:

$$\begin{aligned}
\cdot : Ob(\mathcal{G}) \times Aut(1) \to\ & Aut(x) \to Aut(1) \\
(x, f) \mapsto\ & id_x \otimes f \mapsto \psi_x(id_x \otimes f) \\
&= id_x \otimes f \otimes id_{x^{-1}} \\
&= id_x \otimes f \qquad \text{under the identification.}
\end{aligned}$$

Let $x, x_1, x_2 \in Ob(\mathcal{G})$ and $f, f_1, f_2 \in Aut(1)$

$$\begin{aligned}
1 \cdot f &= id_1 \otimes f \\
&= l_1 \circ f \circ l_1^{-1} \\
&= f && \text{as } l_1 = id_1 \\
x \cdot (f_1 \circ f_2) &= id_x \otimes (f_1 \circ f_2) \\
&= id_x \otimes f_1 \circ id_x \otimes f_2 \\
&= x \cdot f_1 \circ x \cdot f_2 \\
(x_1 + x_2) \cdot f &:= x_1 \cdot f \circ x_2 \cdot f \\
(x_1 \otimes x_2) \cdot f &= id_{x_1 \otimes x_2} \otimes f \\
&= id_{x_1} \otimes id_{x_2} \otimes f \\
&= x_1 \cdot (x_2 \cdot f)
\end{aligned}$$

Thus in a skeletal weak 2-group, $\mathcal{G}$, $Aut(1)$ is a left $\mathbb{Z}[Ob(\mathcal{G})]$-module.

Furthermore, as $\mathcal{G}$ is skeletal, every object in the pentagonal diagram above is in fact the same object and every morphism is an automorphism of that object, which identified with an element of the abelian group $Aut(1)$. As is conventional for abelian groups, we write composition ($\circ$) as addition to obtain:

$$a_{w,x,y} \otimes id_z + a_{w,x\otimes y,z} + id_w \otimes a_{x,y,z} - a_{w\otimes x,y,z} - a_{w,x,y\otimes z} = 0$$

Now $id_w \otimes a_{x,y,z} = w \cdot a_{x,y,z}$ while $a_{w,x,y} \otimes id_z$ is a right action

$$\begin{aligned}\psi_z(a_{w,x,y} \otimes id_z) &= a_{w,x,y} \otimes id_z \otimes id_{z^{-1}} \\ &= a_{w,x,y}\end{aligned}$$

So the right action is trivial, which gives us:

$$a_{w,x,y} + a_{w,x\otimes y,z} + w \cdot a_{x,y,z} - a_{w\otimes x,y,z} - a_{w,x,y\otimes z} = 0$$

It may be observed that this is exactly the 3-cocycle condition in the context of a group, G, acting on an abelian group, H, and $\phi : G \times G \times G \to H$:

$$\begin{aligned}\phi(g_0, g_1, g_2) + \phi(g_0, g_1g_2, g_3) &+ g_0.\phi(g_1, g_2, g_3) \\ &- \phi(g_0g_1, g_2, g_3) - \phi(g_0, g_1, g_2g_3) = 0\end{aligned}$$

We have show that every special 2-group may be contains the the data of group, a module of the group ring and a 3-cocyle from the group to the module, in fact this is a classification of special 2-groups.

Theorem 4.0.6 (See Baez and Lauda [3]). *Special 2-groups are classified by triples (G, H, a) where:*

- *G is a group*
- *H is a (left) $\mathbb{Z}G$-module*
- *a is a normalised 3-cocycle, $a : G^3 \to H$*

Moreover, 2-groups that are equivalent, in the sense that there exists some notion of a 2-group isomorphism between them, are classified by isomorphic groups and modules and cohomologous cocycles [3]. As, at this stage, we have not yet defined what a cocycle is, this may seem vague. So in the next section we shed some light on what the cocycle is.

5 Setting the Scene for the Co-cycle

As their name suggests, the natural place to look for cocycles is in some cohomology. Thus in this section we will define group cohomology. We will also relate this to the cohomology of a topological space that is constructed from the group known as the classifying space.

5.1 G-Bundles and the Classifying Space

In this subsection, the notation (E, p, B, F) will denote a fibre bundle with total space E, base space B, bundle projection $p : E \to B$ and fibre F. Where it is obvious what the fibre is, the last term in the quadruple will often be omitted. Our discussion will largely follow that of Benson [4].

Definition 5.1.1. *Let $\xi = (E, p, B, F)$ and $\xi' = (E', p', B', F')$ be fibre bundles.*

*A **bundle morphism** from ξ to ξ' or consists of two continuous maps, $\tilde{f} : E \to E'$ and $f : B \to B'$ such that the following diagram commutes:*

$$\begin{array}{ccc} E & \xrightarrow{\tilde{f}} & E' \\ \Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle p'} \\ B & \xrightarrow{f} & B' \end{array}$$

Definition 5.1.2. *Let $\xi = (E, p, B, F)$ and $\xi' = (E', p', B, F')$ be fibre bundles.*

*A **bundle morphism over** B is a bundle morphism ξ to ξ', i.e. a morphisms between two bundles with identical base space.*

Definition 5.1.3. *Let $\xi = (E, p, B, F)$ be a fibre bundle and $f : B' \to B$ a continuous map.*

The ***pullback bundle*** *of ξ, $f!(\xi)$ is a fibre bundle (E', p', B', F') where*

$$E' = \{(x, y) \in B' \times E \mid f(x) = p(y)\}$$

with projection

$$\begin{aligned} p : E' &\longrightarrow B' \\ (x, y) &\longmapsto x \end{aligned}$$

Pullback bundles fit in the pullback square:

$$\begin{tikzcd} E' \arrow[r, "\tilde{f}"] \arrow[d, "p'"'] & E \arrow[d, "p"] \\ B' \arrow[r, "f"] & B \end{tikzcd}$$

Like all pullbacks, any bundle which fits into a similar pullback square is isomorphic to $f!(\xi)$, also the fibres F and F' are homeomorphic.

Definition 5.1.4. *Let G be a group with topology.*

A ***principal*** *G**-bundle**, is a fibre bundle (E, p, B, F), where F is homeomorphic to G, E is a (left) G-space and there exists an open cover $\{U_\alpha\}$ of B with homeomorphisms $\phi_\alpha : G \times U_\alpha \to p^{-1}(U_\alpha)$ such that the following diagram commutes:*

$$\begin{tikzcd}
G \times U_\alpha \arrow[rr, "\phi_\alpha"] \arrow[dr, "\pi_2"'] & & p^{-1}(U_\alpha) \arrow[dl, "p"] \\
& U_\alpha &
\end{tikzcd}$$

Where π_2 is the projection onto the second factor.

As this definition is rather cumbersome, we will use the following theorem when, in a later section, we need to show that things are principle G-bundles.

Theorem 5.1.5 (Chevelley see [4])**.** *If H is a Lie group and G is a closed subgroup of H, then $(H, p, G\backslash H)$ is a principal G-bundle where $p : H \to G\backslash H$ is the natural map into the coset space.* □

Proposition 5.1.6. *If G is a group with topology, B' is a paracompact space, $f, g : B' \to B$ are homotopic maps and $\xi = (E, p, B)$ is a principal G-bundle, then the two pullback bundles are isomorphic as principal G-bundles over B', i.e. $f^!(\xi) \cong g^!(\xi)$.*

Proof. See [4, pp 37-8]. □

We need not divulge what a paracompact space is as all spaces we are dealing with are compact.

Definition 5.1.7. *Let G be a group, let $\xi = (E, p, B)$ be a principal G-bundle*

.

*A **universal** G**-bundle**, ξG, is a principal G-bundle (EG, π, BG) such that* *for any paracompact space B' the map:*

$$Hom_{\mathit{frTop}}(B', BG) \longrightarrow {Princ_G(X)}\big/_{\cong}$$
$$f \longmapsto f^!(\xi)$$

is a bijection, i.e. the homotopy classes of maps into BG are in 1-1 correspondence with the isomorphisms classes of principal G-bundles.

The base space of a universal G-bundle is referred to as a **classifying space** of G.

Lemma 5.1.8. *If $\xi_1 G$ and $\xi_2 G$ are two universal bundles, then there exists a homotopy equivalence $f_1 : B_1G \to B_2G$ with pullback $f_1^!(\xi_2 G) = \xi_1 G$.*

Proof. Let $f_1 : B_1G \to B_1G$ and $f_2 : B_2G \to B_1G$ be the images of $\xi_1 G$ and $\xi_2 G$ under the bijection mentioned above.
Then $f_1 \circ f_2$ is the base map of a bundle morphism over B_1G.
All bundle morphism over a common base are isomorphisms (Husemoller, [8, p 43]).
Therefore $f_1 \circ f_2 : B_1G \to B_1G$ along with $\tilde{f}_1 \circ \tilde{f}_2 : E_1G \to E_1G$ give a bundle isomorphism.
Hence by the property in Definition 5.1.7, $f_1 \circ f_2 \simeq id_{B_1G}$.
Similarly, $f_2 \circ f_1 \simeq id_{B_2G}$. □

Hence we may refer to any classifying space of G as the classifying space of G. Furthermore, there is a theorem of Milnor which states that a classifying space always exists for a given group (see [4, pp 38-40]).

As the groups we are dealing with are somewhat more simple than the more general groups that the theorem encompasses, the following simplified construction suffices.

Theorem 5.1.9 (Segal [11]). *If G is a discrete group, and $\hat{G}$ is the category with one object whose endomorphisms are all automorphisms and whose automorphism group (under composition) is G, then geometric realisation of the nerve of $\hat{G}$ is a classifying space for G., i.e.*

$$BG \simeq |N_*(\hat{G})|$$

$\square$

Notice that the definition required the group to be discrete. The following theorem may be used for construction of classifying spaces of groups with (non-discrete) topology.

Theorem 5.1.10. *If G is a topological group, and EG is a contractible paracompact topological space on which G acts continuously (from the left) , then $(EG, \pi, EG/G)$ is a universal G-bundle where*

$$\pi : EG \longrightarrow EG/G$$

is the natural projection onto the quotient space. $\square$

Corollary 5.1.11. *If G and EG are as above, then*

$$BG \simeq EG/G \qquad \square$$

Example 5.1.12. *The classifying space for the Lie group $Sp(1)$ is homotopy equivalent to the infinite quaternion projective space, i.e.*

$$B(Sp(1)) \simeq \mathbb{H}P^{\infty}$$

Proof. Consider the action of the unit quaternions on

$$\mathbb{S}^{4n+3} = \left\{ (q_1, q_2, ..., q_{n+1}) \in \mathbb{H}^{n+1} \;\middle|\; \sum_{i=1}^{n} q_i \overline{q_i} = 1 \right\}$$

By co-ordinatewise left multiplication, i.e. for $q \in Sp(1)$

$$q \cdot (q_1, q_2, ..., q_n) = (qq_1, qq_2, ..., qq_n)$$

The action is well defined, as:

$$\begin{aligned}
\sum_{i=1}^{n} qq_i \overline{qq_i} &= \sum_{i=1}^{n} qq_i \overline{q}_i \overline{q} \\
&= q\overline{q} \sum_{i=1}^{n} q_i \overline{q}_i \\
&= 1 \times 1 = 1
\end{aligned}$$

Furthermore, the action is clearly continuous and free.

We may build the space $\mathbb{S}^{\infty}$ as follows

$$\begin{aligned}
\mathbb{S}^{\infty} &:= \varinjlim \mathbb{S}^{4n-1} \\
&= \left(\coprod_{n=1}^{\infty} \mathbb{S}^{4n+3} \right) \Big/ \sim
\end{aligned}$$

With maps $f_{ij} : \mathbb{S}^{4i-1} \hookrightarrow \mathbb{S}^{4j-1}$ for $i \leq j$ being inclusions into the first i co-ordinates. By this construction, $\mathbb{S}^\infty$ is the unit sphere in the infinite dimensional space $\mathbb{H}^\infty$ which is defined analogously to the be set of infinite tuples of quaternions with only finitely may non-zero components.

The action is carried through the direct limit as the action commutes with the maps f_{ij}:

$$\begin{aligned} q \cdot f_{ij}((q_1, q_2, ..., q_i)) &= q \cdot (q_1, q_2, ..., q_i, 0, ..., 0) \\ &= (qq_1, qq_2, ..., qq_i, 0, ..., 0) \\ &= f_{ij}(q \cdot (q_1, q_2, ..., q_i)) \end{aligned}$$

The space $\mathbb{S}^\infty$ is contractible since, given $i \in \mathbb{Z}_{\geq 0}$ pick n such that $i < 4n+3$. Then by cellular approximation:

$$\begin{aligned} \pi_i(\mathbb{S}^\infty) &\cong \pi_i(\mathbb{S}^{4n+3}) \\ &\cong \{1\} \end{aligned}$$

Therefore, the constant map $g : \mathbb{S}^\infty \to *$ must induce isomorphisms between all homotopy groups. Therefore by Whitehead's Theorem [6], g is a homotopy equivalence, i.e. $\mathbb{S}^\infty$ is contractible.

Thus $\mathbb{S}^\infty = E(Sp(1))$ and by Theorem 5.1.10, $B(Sp(1)) \simeq \mathbb{S}^\infty / Sp(1)$.

This is the projective space $\mathbb{H}P^\infty$. $\square$

5.2 The Nerve of a Category

Definition 5.2.1. *Let $\mathcal{C}$ be a category, $n \in \mathbb{Z}_{\geq 0}$.*
*An n-**morphism chain** in $\mathcal{C}$ is an $(n+1)$-tuple of objects from $\mathcal{C}$ with an n-tuple of morphisms, one for each pair of adjacent objects.*
e.g. if $x, y, z \in Ob(\mathcal{C})$ and $f \in Hom_{\mathcal{C}}(x, y)$ and $g \in Hom_{\mathcal{C}}(y, z)$ then

$$x \xrightarrow{f} y \xrightarrow{g} z$$

is a 3-morphism chain.

Definition 5.2.2. *Let $\mathcal{C}$ be a (small) category, $n \in \mathbb{Z}_{\geq 0}$.*
*The **nerve** of $\mathcal{C}$ is a simplicial set (See Alligretti [1]), $N_*(\mathcal{C})$, constructed as follows:*

- *0-simplicies are the objects, $Ob(\mathcal{C})$*
- *1-simplicies are 1-morphism chains in $\mathcal{C}$ and in general,*
- *n-simplicies are n-morphism chains in $\mathcal{C}$*

The face maps of an n-simplex are given by a deleting an object and composing the morphisms that have it as a source with the one that has it as a target, but for the objects at either end of the chain, the morphism is simply dropped *along with the object:*

$$\partial_i(x_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} x_n) = \begin{cases} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} x_n & i = 0, \\ x_0 \xrightarrow{f_1} \cdots x_{i-1} \xrightarrow{f_i \circ f_{i+1}} x_{i+1} \cdots \xrightarrow{f_n} x_n & i = 1, ..., n-1, \\ x_0 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} x_{n-1} & i = n. \end{cases}$$

The degeneracy maps are simply composition with an identity morphism at the appropriate point:

$$\varsigma_i(x_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} x_n) = x_0 \xrightarrow{f_1} \cdots x_i \xrightarrow{id} x_i \cdots \xrightarrow{f_n} x_n$$

As we want to turn this combinatorial data into a topological space, it is natural to consider gluing simplicies along their faces according to the face and degeneracy maps to obtain a Δ-complex.

Let $\Delta^n := \left\{ (t_0, ..., t_n) \in \mathbb{R}^{n+1} \,\middle|\, t_i \geq 0, \sum_{i=0}^{n} t_i = 1 \right\}$ be the standard topological n-simplex i.e. it has the subspace topology.

For each $0 \leq i \leq n$, define maps:

$$\begin{aligned} \delta_i : \Delta^{n-1} &\to \Delta^n \\ (t_0, ..., t_{n-1}) &\mapsto (t_0, ..., t_{i-1}, 0, t_i, ..., t_{n-1}) \\ \sigma_i : \Delta^{n+1} &\to \Delta^n \\ (t_0, ..., t_{n+1}) &\mapsto (t_0, ..., t_{i-1}, t_i + t_{i+1}, t_{i+2}, ..., t_{n+1}) \end{aligned}$$

Definition 5.2.3 (From Alligretti [1]). *Let X_* be a simplicial set. The*

geometric realisation *of* X_* *is a is a topological space,* $|X_*|$ *defined as*

$$|X_*| := \left(\coprod_{n=0}^{\infty} \Delta^n \times X_n \right) \Big/ \sim$$

Where $\sim$ *is the equivalence relation generated by:*

If $x \in X_n, u \in \Delta^{n-1}, v \in \Delta^{n+1}$ *and* $i = 0, 1, ..., n,$ *then*

$$(\delta_i(u), x) \sim (u, \partial_i(x))$$
$$(\sigma_i(v), x) \sim (v, \varsigma_i(x))$$

5.3 Group Cohomology

This construction of the classifying space is insufficient for the purposes of calculating group cohomology as it does not take into account the action of the group on the coefficient module. So in this section we give the formal definition of group cohomology and relate it to the classifying space.

Definition 5.3.1. *Let* R *be a ring and* M *be an* R*-module. A* ***resolution*** *of* M *is an exact sequence of* R*-modules:*

$$\cdots \xrightarrow{\phi_2} N_1 \xrightarrow{\phi_1} N_0 \xrightarrow{\varepsilon} M \to 0$$

The resolution will often be abbreviated as $N_*(M)$*.*

Note that we will treat this as a chain complex with M having position -1.

If the modules are all of a certain type, the resolution is referred as having that type, eg the type of resolution we will be interested in are projective resolutions which are comprised of projective modules. Note that as all free modules are projective, it suffices to find a free resolution. A given module will always have free resolutions (and hence projective resolutions) as if $M \cong \langle X \mid S \rangle$ as an R-module then

$$\cdots \to 0 \to R[S] \hookrightarrow R[X] \twoheadrightarrow R[X]/R[S] \to 0$$

is a free resolution of M. There are numerous such resolutions and indeed we will construct a different free resolution below.

Given an R-module, N, a projective resolution, $P_*(M)$, may be dualised by applying the contravariant functor $Hom_R(-, N)$ to the resolution to obtain:

$$\cdots \xleftarrow{Hom_R(\phi_1,N)} Hom_R(P_0, N) \xleftarrow{Hom_R(\varepsilon,N)} Hom_R(M, N) \leftarrow 0$$

We denote this sequence as $Hom_R(P_*(M), N)$, also the homomorphisms $Hom(\phi_i, N)$ and $Hom_R(\varepsilon, N)$ are abbreviated to ϕ_i^* and ε^*.
The sequence $Hom_R(P_*(M), N)$ is a cochain complex by the following proposition:

Proposition 5.3.2. *If C_* is a chain complex, then $Hom_R(C_*, N)$ is a cochain complex.*

Proof. Suppose

$$\cdots \to P_{i+1} \xrightarrow{\varphi_{i+1}} P_i \xrightarrow{\varphi_i} P_{i-1} \to \cdots$$

is a chain complex. Therefore $\varphi_i \circ \varphi_{i+1} = 0$. Dualising gives:

$$\cdots \leftarrow Hom_R(P_{i+1}, N) \xleftarrow{\varphi_{i+1}^*} Hom_R(P_i, N) \xleftarrow{\varphi_i^*} Hom_R(P_{i-1}, N) \leftarrow \cdots$$

Suppose $\alpha \in Hom_R(P_{i-1}, N)$, then

$$\begin{aligned}
\varphi_i^*(\alpha) &= \alpha \circ \varphi_i \\
\varphi_{i+1}^* \circ \varphi_i^*(\alpha) &= \varphi_{i+1}^*(\alpha \circ \varphi_i) \\
&= \alpha \circ \varphi_i \circ \varphi_{i+1} \\
&= \alpha \circ 0 = 0 \\
\varphi_{i+1}^* \circ \varphi_i^* &= 0
\end{aligned}$$

□

Thus the exactness of $P_*(M)$ ensures that $Hom_R(P_*(M), N)$ is a cochain complex, however it is not necessarily exact in turn, which allows for the following definition.

Definition 5.3.3. *Let R be a ring and N a left R-module, $n \geq 0$.*
Then Ext_R^n is a contravariant functor:

$$Ext_R^n(-, N) : {}_RMod \to Ab$$
$$Ext_R^n(M, N) = H^n(Hom_R(P_*(M), N))$$

where $P_*(M)$ is any projective resolution of M.

This is independent of choice of resolution see [6, pp 194-5]

Definition 5.3.4. *Let G be a group and M be a $\mathbb{Z}G$-module.*

The ***group cohomology*** *of G with coefficients in M is defined as $\mathbb{Z}G$-modules by:*

$$H^n(G; M) := Ext^n_{\mathbb{Z}G}(\mathbb{Z}, M)$$

where $\mathbb{Z}$ is considered to be a trivial $\mathbb{Z}G$-module.

We will construct a projective resolution for $\mathbb{Z}$ in order to calculate the cohomology directly.

We now define the cochain complex of a simplicial set, after which its homology and cohomology may be calculated using the usual definitions.

Definition 5.3.5. *Let S_* be a simplicial set.*

The ***cochain complex of*** *S_* is the sequence of abelian groups:*

$$C_0(S_*) \xrightarrow{d_1} C_1(S_*) \xrightarrow{d_2} \cdots$$

with chain groups

$$C_i(S_*) = \mathbb{Z}[S_i]$$

and boundary homomorphisms

$$d_i = \sum_{j=0}^{i} \partial_i$$

It is relativity simple to prove that this is a chain complex and a proof may be found in [6] albeit with different terminology.

Let G be a discrete group and let $\overline{G}$ be the category with:

- $Ob(\overline{G}) = G$
- $\forall\, g, h \in G, Hom(g, h) = \{f_{(g,h)}\}$

Then the n-simplicies in $N_*(\overline{G})$ have the form

$$g_0 \xrightarrow{f_{(g_0,g_1)}} g_1 \xrightarrow{f_{(g_1,g_2)}} \cdots \xrightarrow{f_{(g_{n-1},g_n)}} g_n$$

Since the morphisms are uniquely determined by the adjacent objects, there is one such simplex for each element of G^{n+1}. Thus the simplicial chain complex of $N_*(\overline{G})$ consists of the abelian groups $C_n(N_*(\overline{G})) = \mathbb{Z}[G^{n+1}]$ with boundary homomorphism:

$$\begin{aligned} d_{n+1} : C_{n+1}(N_*(\overline{G})) &\to C_n(N_*(\overline{G})) \\ (g_0, ..., g_n) &\mapsto \sum_{i=0}^{n} (-1)^i (g_0, ..., \hat{g}_i, ..., g_n) \end{aligned}$$

Since G acts on the $\mathbb{Z}$-bases by left multiplication, each abelian group in the chain complex is in fact a (left) $\mathbb{Z}G$-module, and the boundary homomorphisms are clearly G-equivariant, so $C_*(N_*(\overline{G}))$ is chain complex of $\mathbb{Z}G$-modules.

The topological space $|N_*(\overline{G})|$ is contractible and in fact it is a model for the total space EG as both are contractible and have a free G action [11]. However, at this stage we only require that $C_*(N(\overline{G}))$ is exact, which we may prove by use of the abelian group homomorphisms:

$$\Sigma_n : C_n(N_*(\overline{G})) \to C_{n+1}(N_*(\overline{G}))$$
$$(g_0, ..., g_n) \mapsto (1, g_0, ..., g_n)$$

This is a contracting homotopy as

$$\begin{aligned}
&(d_{n+1}\Sigma_n + \Sigma_{n-1}d_n)(g_0, ..., g_{n-1}) \\
&= (g_0, ..., g_{n-1}) + \sum_{i=1}^{n}(-1)^i(1, g_0, ..., \hat{g}_{i-1}, ..., g_{n-1}) + (-1)^{n+1}(1, g_0, ..., g_{n-2}) \\
&\quad + (1, g_1, ..., g_{n-1}) + \sum_{i=1}^{n-1}(-1)^{i+1}(1, g_0, ..., \hat{g}_i, ..., g_{n-1}) + (-1)^n(g_0, ..., g_{n-2}) \\
&= (g_0, ..., g_{n-1}) + \sum_{i=0}^{n-1}(-1)^i(1, g_0, ..., \hat{g}_i, ..., g_{n-1}) + (-1)^{n+1}(1, g_0, ..., g_{n-2}) \\
&\quad + \sum_{i=0}^{n-1}(-1)^i(1, g_0, ..., \hat{g}_i, ..., g_{n-1}) + (-1)^n(g_0, ..., g_{n-2}) \\
&= (g_0, ..., g_{n-1})
\end{aligned}$$

In other words, $d_{n+1}\Sigma_n + \Sigma_{n-1}d_n = 1$, therefore Σ is a contracting homotopy. This implies that $C_*(N_*(\overline{G}))$ is exact as a chain complex of abelian groups, which in turn implies that it is exact as chain complex of $\mathbb{Z}G$-modules (see [7]).

For the final homomorphism, define

$$\varepsilon : C_0(N_*(\overline{G})) \to \mathbb{Z}$$
$$g \mapsto 1$$

Since $C_0(N_*(\overline{G}))$ is a free abelian group, ε is surjective. Hence,

$$C_*(N_*(\overline{G})) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

is a free resolution for $\mathbb{Z}$ as $\mathbb{Z}G$ modules and thus,

$$H^*(G; M) = H^*(Hom_{\mathbb{Z}G}(C_*(N_*(\overline{G})), M)$$

In particular, the cochain complex is:

$$Hom_{\mathbb{Z}G}(C_*(N_*(\overline{G})), M) \xleftarrow{\varepsilon^*} Hom_{\mathbb{Z}G}(\mathbb{Z}, M) \leftarrow 0$$

Observe that $\{(1, g_1, g_2, ..., g_n) \mid g_i \in G\}$ is a basis for $C_n(N_*(\overline{G}))$ as a free $\mathbb{Z}G$-module. This motivates us to seek a basis for the modules with basis G^n. This may be achieved by keeping track of the morphisms instead of the objects in each morphism chain, and this is the purpose of the **bar notation**:

$$\begin{aligned} [g_1|g_2|...|g_n] &:= 1 \xrightarrow{g_1} g_1 \xrightarrow{g_2} g_1g_2 \to \cdots \xrightarrow{g_n} g_1g_2...g_n \\ &= (1, g_1, g_1g_2, ..., g_1g_2...g_n) \end{aligned}$$

Note that the 0-simplicies are represented by the empty brackets: [] which equals 1. Thus as a $\mathbb{Z}G$-module,

$$\begin{aligned} C_n(N_*(\overline{G})) &= \mathbb{Z}G[\{[g_1|...|g_n] \mid g_i \in G\}] \\ &= \mathbb{Z}G[G^n] \end{aligned}$$

Note that the action is now

$$g \cdot [g_1|g_2|...|g_n] = (g, gg_1, gg_1g_2, ..., gg_1g_2...g_n)$$

The face maps are now:

$$\partial_i([g_1|g_2|...|g_n]) = \begin{cases} g_1 \cdot [g_2|...|g_n] & i = 0, \\ [g_1|g_2|...|g_{i-1}|g_ig_{i+1}|g_{i+2}|...|g_n] & i = 1, ..., n-1, \\ [g_1|g_2|...|g_{n-1}] & i = n. \end{cases}$$

While the degeneracy maps are:

$$\varsigma_i([g_1|g_2|...|g_n]) = [g_1|...|g_{i-1}|1|g_i|...|g_n]$$

Since the boundary homomorphisms are G-equivariant, they are uniquely determined by their mapping of the $\mathbb{Z}G$-basis, G^n:

$$\begin{aligned} d_n([g_1|g_2|...|g_n]) = g_1 \cdot [g_2|...|g_n] + \sum_{i=1}^{n-1}(-1)^i[g_1|g_2|...|g_{i-1}|g_ig_{i+1}|g_{i+2}|...|g_n] \\ + (-1)^n[g_1|g_2|...|g_{n-1}] \end{aligned}$$

Recall that an element of a cochain module is a cocycle if it is in the kernel of a coboundary homomorphism. Thus letting ϕ be a generator of $C^n(G; H)$, the condition is:

$$\begin{aligned}
0 &= \delta^{n+1}(\phi)([g_1|g_2|...|g_{n+1}]) \\
&= \phi \circ d_{n+1}([g_1|g_2|...|g_{n+1}]) \\
&= \phi \left(\begin{array}{l} g_1 \cdot [g_2|...|g_{n+1}] + \sum_{i=1}^{n-1} (-1)^i [g_1|g_2|...|g_{i-1}|g_i g_{i+1}|g_{i+2}|...|g_{n+1}] \\ + (-1)^n [g_1|g_2|...|g_n] \end{array} \right) \\
&= g_1 \cdot \phi([g_2|...|g_{n+1}]) + \sum_{i=1}^{n-1} (-1)^i \phi([g_1|g_2|...|g_{i-1}|g_i g_{i+1}|g_{i+2}|...|g_{n+1}]) \\
&\quad + (-1)^n \phi([g_1|g_2|...|g_n])
\end{aligned}$$

In particular, the 3-cocycle condition is:

$$\begin{aligned}
g_1 \cdot \phi([g_2|g_3|g_4]) - \phi([g_1 g_2|g_3|g_4]) + \phi([g_1|g_2 g_3|g_4]) \\
- \phi([g_1|g_2|g_3 g_4]) + \phi([g_1|g_2|g_3]) \quad &= 0
\end{aligned}$$

Just as with the boundary homomorphisms, ϕ is uniquely determined by a map $G^3 \to H$ and so we have precisely achieved the cocycle condition presented above.

The cocycle we are looking for is a normalised cocycle and it resides in the normalised cochain complex. This is constructed by removing degenerate simplicies from $N_*(\overline{G})$, namely the ones which have an identity in their morphism chains. For each $n \in \mathbb{N}$ the degenerate objects form the subcomplex:

$$\begin{aligned}
D_n &= \mathbb{Z}[\{(g_0, ...g_n) \mid \exists\, i \in \{0, ..., n-1\} \text{ s.t. } g_i = g_{i+1}\}] \\
&= \mathbb{Z}G[\{[g_1|...|g_n] \mid \exists\, i \in \{1, ..., n\} \text{ s.t. } g_i = 1\}]
\end{aligned}$$

For each n, $D_n \leq C_n|N_*(\overline{G}))$, so we can define as abelian groups:

$$\overline{C}_* := C_*(N_*(\overline{G}))/D_*$$

We check that the boundary homomorphisms respect the degenerate subcomplex. Let $I = \{1, 2, ..., j-1, j+2, ..., n\}$, then

$$\begin{aligned} d_n(g_0, ..., g_j, g_j, ..., g_n) &= \sum_{i \in I} (-1)^i (g_0, ..., \hat{g}_i, ..., g_n) \\ &\quad + (-1)^j (g_0, ..., g_{j-1}, \hat{g}_j, g_j, g_{j+2}, ..., g_n) \\ &\quad + (-1)^{j+1} (g_0, ..., g_{j-1}, g_j, \hat{g}_j, g_{j+2}, ..., g_n) \\ &= \sum_{i \in I} (-1)^i (g_0, ..., \hat{g}_i, ..., g_n) \in D_{n-1} \end{aligned}$$

Since $d_n(D_n) \leq D_{n-1}$ the boundary homomorphisms in $C_*(N_*(\overline{G}))$ induce boundary homomorphisms in $\overline{C}_*$ making it a chain complex. Furthermore as $\Sigma_n(D_n) \subseteq D_{n+1}$, the homomorphism Σ is a contracting homotopy on $\overline{C}_*$ making it exact.

Thus if we define $\overline{\varepsilon}$ accordingly,

$$\overline{C}_* \xrightarrow{\overline{\varepsilon}} \mathbb{Z} \to 0$$

is also a free resolution for $\mathbb{Z}$ as $\mathbb{Z}G$-modules. As the group cohomology is independent of the choice of resolution,

$$H^*(G; M) \cong H^*(Hom_{\mathbb{Z}G}(\overline{C}_*, M))$$

Now we return to the setting of the un-normalised cochain complex. Let us suppose that the action of the group G on the coefficient module M is trivial. Now it was observed above that the Δ-complex $|N_*(\overline{G})|$ is a contractible space that G acts upon freely. Hence $|N_*(\overline{G})| = EG$, also recall that $BG = |N_*(\hat{G})|$. The simplicial complex of BG will now be constructed for comparison with that of EG.

The simplicial set $N_*(\hat{G})$ is comprised of:

- 0-simplicies: the sole object 1
- 1-simplicies: $g \, \forall \, g \in G$
- n-simplicies: $g_1 g_2 ... g_n \, \forall \, (g_1, g_2, ..., g_n) \in G^n$

With boundary maps:

$$\partial_i(g_1 g_2 ... g_n) = g_1 g_2 ... \hat{g}_i ... g_n$$

Thus the chain groups are $C_n^{\Delta}(BG) = \mathbb{Z}[G^n]$ for $n = 0, 1, ...$ with boundary homomorphisms

$$d_n(g_1 g_2 ... g_n) = \sum_{i=1}^{n} (-1)^i g_1 ... \hat{g}_i ... g_n$$

There is also an explicit construction of the projection map due to Segal [11].

There is a functor:

$$P : \overline{G} \to \hat{G}$$

$$g \mapsto 1$$

$$f_{(g,h)} \mapsto g^{-1}h$$

This functor induces a map (See Hatcher [6, 165]) which is the projection map in the bundle:

$$\pi : |N_*(\overline{G})| \to |N_*(\hat{G})| = BG$$

Intuitively, the map is just what one would expect it to be, it maps all the verticies of the simplicies to a point etc.

Let the G-**equivariant cohomology** of a G-space be the cohomology of the cochain complex which is formed by applying the functor $Hom_G(-, M)$ functor, which gives the G-equivariant homomorphisms into M, to the (simplicial or singular) chain complex of the space i.e.

$$H^*_G(X; M) := H^*(Hom_G(C_*(X), M))$$

Theorem 5.3.6. *Let G be a group, M be a $\mathbb{Z}G$-module, then the group cohomology of G with coefficients in M is the G-equivariant simplicial cohomology of EG with coefficients in M, i.e.*

$$H^*(G; M) = H^*_G(EG; M)$$

Proof. Observe that it is very simple to triangulate EG back to the simplicial set $N_*(\overline{G})$. The identifies the simplicial chain complex of EG with the chain complex of $N_*(\overline{G})$.

$$C_*(N_*(\overline{G})) = \Delta_*(EG)$$

The $\mathbb{Z}G$-module homomorphisms with source in $C_n(N_*(\overline{G}))$ are precisely the G-equivariant homomorphisms from $C_n(N_*(\overline{G}))$ as an abelian group. Thus

$$Hom_{\mathbb{Z}G}(C_*(N_*(\overline{G}), M) = Hom_G(\Delta_*(EG), M)$$

is an identity of chain complexes. Hence,

$$H^*(G; M) = H^*_G(EG; M)$$

□

Corollary 5.3.7. *Suppose M is a trivial G-module, i.e. $g \cdot m = m$ is the G action, then the group cohomology is isomorphic to the cohomology of the classifying space as modules.*

$$H^*(G; M) \cong H^*(BG; M)$$

Proof. As the action is trivial, every abelian group homomorphism is a G-equivariant homomorphism, i.e.

$$Hom_G(\Delta_n(EG), M) = Hom_{\mathcal{Ab}}(\Delta_n(EG), M)$$

where $\mathcal{Ab}$ is the category of abelian groups.

The orbit set $G\backslash\Delta_n(EG)$ is no longer a G-module.

However since the G action commutes with the $\mathbb{Z}$ action on $\Delta_n(EG)$, it is nevertheless an abelian group.

The basis of $G\backslash\Delta_n(EG)$ are the n-morphisms chains commencing in 1, the set of which is in bijection with G^n.

This, in turn, is in bijection with the basis of $\Delta_n(G\backslash EG)$. Hence,

$$G\backslash\Delta_n(EG) \cong \Delta_n(BG)$$

Let $\phi : \Delta_n(EG) \to M$, be a G-equivariant homomorphism then

If $g \in G$, $\sigma \in \Delta_n(EG)$,

$$\begin{aligned} \phi(g \cdot \sigma) &= g \cdot \phi(\sigma) \\ &= \phi(\sigma) \end{aligned}$$

Since ϕ is constant along the orbits, it descends to a unique abelian group homomorphism $\phi' : G\backslash\Delta_n(EG) \to M$.

But this assignment is surjective as the inverse, composition with the quotient map, works for every $\psi : G\backslash\Delta_n(EG) \to M$.

$$\begin{tikzcd} \Delta_n(EG) \arrow[r, "\phi"] \arrow[d, "q"'] & M \\ G\backslash\Delta_n(EG) \arrow[ur, "\phi'"'] & \end{tikzcd}$$

Hence there is a one-one correspondence:

$$Hom_G(\Delta_n(EG), M) \longleftrightarrow Hom_{\mathcal{Ab}}(\Delta_n(BG), M)$$

Thus

$$H^n(G; M) \cong H^n(BG; M)$$

$\square$

6 Constructing the Co-cycle

The cocycle we seek is a G-equivariant homomorphism from 3rd simplicial chain group of EG into some G-module. Since this description is rather discouraging when searching for explicit construction, in this section we will instead maps the 3-skeleton of EG to the 3-sphere, where as we will later see, there is a rather explicit method of constructing cocycles.

As different choices of G that were enumerated in Section 2 were finite subgroups of $Sp(1) = \mathbb{S}^3$, the group G acts on $\mathbb{S}^3$ by left multiplication. As G is finite, it is closed in the Lie group $\mathbb{S}^3$ and hence by Theorem 5.1.5, the map $\mathbb{S}^3 \xrightarrow{p} G\backslash\mathbb{S}^3$ is the projection of a principal G-bundle. Thus by the correspondence in Definition 5.1.7, the following pullback square exists.

$$\begin{tikzcd}
\mathbb{S}^3 \arrow[r, "\tilde{f}_p"] \arrow[d, "p"'] & EG \arrow[d, "\pi"] \\
G\backslash\mathbb{S}^3 \arrow[r, "f_p"] & BG
\end{tikzcd}$$

We start by constructing a G-equivariant map $s : EG^{(3)} \to \mathbb{S}^3$.

By $EG^{(3)}$, we mean 3-skeleton of EG which is defined as follows:

Definition 6.0.1. *Let X_* be a simplicial set.*

*The k-**skeleton of** $|X_*|$ is a is a topological space, $|X_*|^{(k)}$ defined as*

$$|X_*|^{(k)} := \left(\coprod_{n=0}^{k} \Delta^n \times X_n \right) \Big/ \sim$$

Where $\sim$ is the equivalence relation generated by:

If $x \in X_n, u \in \Delta^{n-1}, v \in \Delta^{n+1}$ *and* $i = 0, 1, ..., n$*, then*

$$(\delta_i(u), x) \sim (u, \partial_i(x))$$

$$(\sigma_i(v), x) \sim (v, \varsigma_i(x))$$

Where δ, ∂, σ *and* ς *are as defined in Section 5.1.*

Assuming such an s exists, the following diagram will commute up to G-equivariant homotopy.

$$\begin{tikzcd}
\mathbb{S}^3 \arrow[r, "\tilde{f}_p"] & EG \\
& EG^{(3)} \arrow[u, hook, "\iota"'] \arrow[ul, dashed, "s"]
\end{tikzcd}$$

Proof. Let $\xi = (EG, p, BG)$ be the universal principal bundle. A free and properly discontinuous action of G on $EG^{(3)}$ is inherited from EG.
Thus we have the principal G-bundle, $\xi^{(3)} : EG^{(3)} \xrightarrow{\pi^{(3)}} G\backslash EG^{(3)}$.
Since both ι and $\tilde{f} \circ s$ are G-equivariant maps $EG^{(3)} \to EG$, they descend to maps $G\backslash EG^{(3)} \to BG$, giving the following bundle morphisms

$$\begin{tikzcd}
EG^{(3)} \arrow[r, "\tilde{f}_p \circ s"] \arrow[d, "\pi^{(3)}"'] & EG \arrow[d, "\pi"] \\
G\backslash EG^{(3)} \arrow[r, "\underline{(\tilde{f}_p \circ s)}"] & BG
\end{tikzcd}
\qquad
\begin{tikzcd}
EG^{(3)} \arrow[r, "\iota"] \arrow[d, "\pi^{(3)}"'] & EG \arrow[d, "\pi"] \\
G\backslash EG^{(3)} \arrow[r, "\underline{\iota}"] & BG
\end{tikzcd}$$

By a theorem in Husemoller [8, p 44], each pullback bundle is isomorphic to

$\xi^{(3)}$. Hence they are isomorphic to each other.

$$(\underline{\iota})^!(\xi) \cong \left(\underline{(\tilde{f}_p \circ s)}\right)^! (\xi)$$

Thus by the universal property in Definition 5.1.7, we have:

$$\underline{\iota} \simeq \underline{(\tilde{f}_p \circ s)}$$

This induces a G-equivariant homotopy between the top rows of bundles morphisms above. $\square$

6.1 Construction of the map s

The simplicies in EG are of the form $\Delta^n \times g[g_1|...|g_n]$ where $g, g_i \in G$ and $0 \leq n \leq 3$. If we define the map s on the simplicies of $EG^{(3)}$, we may obtain a map from $EG^{(3)}$ to $\mathbb{S}^3$. As we want a G-equivariant map, we instead define in on $\Delta^n \times [g_1|...|g_n]$ and extend it G-equivariantly. For reasons that will be revealed later, we must choose s so that its restriction to any particular simplex is a smooth map.

Note that by Definition 6.1.1, the only simplicies in $EG^{(3)}$ of dimension greater than 3 are degeneracies of simplicies of dimension 3 or less. In all dimensions, we ensure that s restricted to the degenerate simplicies, is a degenerate simplex in $\mathbb{S}^3$ which is identical to s restricted to any simplex that it is a degeneracy of, more precisely:
Suppose the map s is well defined on non-degenerate simplicies.

If $\Delta^n \times [g_1|...|g_n]$ is an degenerate n-simplex then there exists $[h_1|...|h_{n-1}]$ and $i \in \{1,...,n\}$ such that:

$$\begin{aligned}\Delta^n \times [g_1|...|g_n] &= \Delta^n \times \varsigma_i([h_1|...|h_{n-1}]) \\ &\sim \sigma_i(\Delta^n) \times [h_1|...|h_{n-1}]\end{aligned}$$

We set

$$s|_{\Delta^n \times [g_1|...|g_n]} = s|_{\sigma_i(\Delta^n) \times [h_1|...|h_{n-1}]}$$

Now there is unique non-degenerate simplex obtained by applying ς^{-1} to $[g_1|...|g_n]$ in all possible ways. It is obtained by deleting all the g_j that are equal to 1. Thus the map is well defined on degenerate simplicies.

Thus in the construction below, we may assume that the simplicies are non-degenerate.

We map the simplex $\Delta^0 \times []$ to the point 1 in $\mathbb{S}^3$.

The 1-simplicies are mapped as smooth paths, so if $\Delta^1 \times [g]$ is a 1-simplex in $EG^{(3)}$, then $s(\Delta^1 \times [g])$ is a smooth path from 1 to g in $\mathbb{S}^3$.
Such path exists as $\mathbb{S}^3$ is path connected. So for each 1-simplex, fix such a path.
A consequence of this construction is that:

$$s|_{\partial(\Delta^1 \times [g])} = \partial s|_{(\Delta^1 \times [g])} \tag{6.1}$$

If $\Delta^2 \times [g|h]$ is a 2-simplex in $EG^{(3)}$ then we wish to map it to a smooth simplex with boundary $s|_{\partial(\Delta^2 \times [g|h])}$.

Such a surface exists as:

$$\begin{aligned} \partial s|_{\partial(\Delta^2 \times [g|h])} &= s|_{\partial\partial(\Delta^2 \times [g|h])} && \text{By (6.1)} \\ &= \varnothing \end{aligned}$$

Now $s|_{\Delta^2 \times [g|h]}$ is a simplicial chain of $\mathbb{S}^3$, and the previous equation shows that it is in fact a cycle.

As $H_1(\mathbb{S}^3) = 0$, it is also a boundary, i.e., there exists a smooth simplex $\tau_2 : \Delta^2 \to \mathbb{S}^3$ such that:

$$s|_{\partial(\Delta^2 \times [g|h])} = \partial \tau_2$$

We will see by Theorem 6.1.3 below that τ_2 may be taken to be smooth simplex.

We choose one such τ_2 to define s on the appropriate 2-simplex, i.e.

$$s|_{\Delta^2 \times [g|h]} := \tau_2$$

Then we again have:

$$s|_{\partial(\Delta^2 \times [g|h])} = \partial s|_{\Delta^2 \times [g|h]}$$

Now as $H^2(\mathbb{S}^3) = 0$, we may repeat this procedure to define

$$s|_{\Delta^2 \times [g|h|k]} := \tau_3$$

Where $\tau_3 : \Delta^3 \to \mathbb{S}^3$ is a smooth map. Hence,

$$s|_{\partial(\Delta^3 \times [g|h|k])} = \partial s|_{\Delta^3 \times [g|h|k]}$$

As there are no more non-degenerate simplicies, we have finished constructing s.

Definition 6.1.1. *Let X be a smooth manifold.*
*The **smooth singular chain complex of** X $C^S_*(X)$ is the subcomplex of the (continuous) singular chain complex consisting of free abelian groups with a basis of smooth maps $\Delta^n \to X$.*

Definition 6.1.2. *Let R be a ring, M be an R-module and X be a smooth manifold.*
*The **smooth singular cochain complex of** X is defined as*

$$C^*_S(X; M) := Hom(C^S_*(X), M)$$

With smooth homology and cohomology defined as usual.

Theorem 6.1.3 (See Bredon [5])**.** *If X is a smooth manifold, R is a ring and M is an R-module then the smooth (co)homology is isomorphic to the* *(co)homology.*

$$H^S_*(X) \cong H_*(X)$$

$$H^*_S(X;M) \cong H^*(X;M)$$

6.2 On the Level of Chains...

From the construction it follows that s induces a chain map between the simplicial complex of $EG^{(3)}$ and the smooth singular complex of $\mathbb{S}^3$.

$$\begin{tikzcd}
C^S_0(\mathbb{S}^3) & C^S_1(\mathbb{S}^3) \arrow[l, "d"'] & C^S_2(\mathbb{S}^3) \arrow[l, "d"'] & C^S_3(\mathbb{S}^3) \arrow[l, "d"'] & C^S_4(\mathbb{S}^3) \arrow[l, "d"'] \\
\Delta_0(EG^{(3)}) \arrow[u, "s_\sharp"] \arrow[d, equal] & \Delta_1(EG^{(3)}) \arrow[l, "d"'] \arrow[u, "s_\sharp"] \arrow[d, equal] & \Delta_2(EG^{(3)}) \arrow[l, "d"'] \arrow[u, "s_\sharp"] \arrow[d, equal] & \Delta_3(EG^{(3)}) \arrow[l, "d"'] \arrow[u, "s_\sharp"] \arrow[d, equal] & \Delta_4(EG^{(3)}) \arrow[l, "d"'] \arrow[u, "s_\sharp"] \arrow[d] \\
\Delta_0(EG^{(3)}) & \Delta_1(EG^{(3)}) \arrow[l, "d"'] & \Delta_2(EG) \arrow[l, "d"'] & \Delta_3(EG) \arrow[l, "d"'] & \Delta_4(EG) \arrow[l, "d"']
\end{tikzcd}$$

Lemma 6.2.1. *If $s, s' : EG^{(3)} \to \mathbb{S}^3$ are two alternative choices for the map described above, then there exists a chain map, $s''_\sharp$ with the properties.*

1. *$s''_\sharp$ is chain homotopic to $s_\sharp$; and,*

2. *$s''_n = s'_n$ for $n \neq 3$.*

Proof. We will attempt to construct a chain homotopy Σ as depicted in the following diagram.

$$\begin{tikzcd}
C_0^S(\mathbb{S}^3) & C_1^S(\mathbb{S}^3) \arrow[l, "d_1"'] & C_2^S(\mathbb{S}^3) \arrow[l, "d_2"'] & C_3^S(\mathbb{S}^3) \arrow[l, "d_3"'] & C_4^S(\mathbb{S}^3) \arrow[l, "d_4"'] \\
\Delta_0(EG^{(3)}) \arrow[u, "s_0\text{-}s_0'"] \arrow[ur, "\Sigma_0"] & \Delta_1(EG^{(3)}) \arrow[l, "d_1"'] \arrow[u, "s_1\text{-}s_1'"] \arrow[ur, "\Sigma_1"] & \Delta_2(EG^{(3)}) \arrow[l, "d_2"'] \arrow[u, "s_2\text{-}s_2'"] \arrow[ur, "\Sigma_2"] & \Delta_3(EG^{(3)}) \arrow[l, "d_3"'] \arrow[u, "s_3\text{-}s_3'"] \arrow[ur, "\Sigma_3"] & \Delta_4(EG^{(3)}) \arrow[l, "d_4"'] \arrow[u, "s_4\text{-}s_4'"']
\end{tikzcd}$$

This will work up until degree 3, but the error will dictate how to define s_3''. In degree 0 we are forced to take $\Sigma_0 =$ since the other terms in the following equation are zero.

$$s_0 - s_0' = d_1 \circ \Sigma_0 + \Sigma_{-1} \circ d_1$$

Thus we must define Σ_1 such that:

$$s_1 - s_1' = d_2 \circ \Sigma_1 \tag{6.2}$$

Let $\sigma_1 \in \Delta_1(EG^{(3)})$.

$$\begin{aligned}
d_1 \circ (s_1 - s_1')(\sigma_1) &= (s_0 - s_0')(d_1(\sigma_1)) \\
&= 0 && \text{since } s_0 = s_0'
\end{aligned}$$

Thus $(s_1 - s_1')(\sigma_1)$ is a 1-cycle in $C_1^S(\mathbb{S}^3)$.
Since $H_1^S(\mathbb{S}^3) = 0$, there exists $\tau_2 \in C_2^S(\mathbb{S}^3)$ such that:

$$(s_1 - s_1')(\sigma_1) = d_2(\tau_2)$$

Thus we may construct Σ_1 by choosing appropriate τ_2:

$$\Sigma_1(\sigma_1) := \tau_2$$

For degree 2 we have:

$$\begin{aligned}
d_2 \circ (s_2 - s_2' - \Sigma_1 \circ d_2) &= d_2 \circ (s_2 - s_2') - d_2 \circ \Sigma_1 \circ d_2 \\
&= d_2 \circ (s_2 - s_2') - (s_1 - s_1') \circ d_2 && \text{By (6.2)} \\
&= 0 && \text{As } s_\sharp \text{ is a chain map}
\end{aligned}$$

Hence if $\sigma_2 \in \Delta_2(EG^{(3)})$, then $(s_2 - s_2' - \Sigma_1 \circ d_2)(\sigma_2)$ is a 2-cycle in $C_2^S(\mathbb{S}^3)$.
Since $H_2^S(\mathbb{S}^3) = 0$, it is also a 2-boundary.
Thus we may choose $\tau_3 \in C_3^S(\mathbb{S}^3)$ so that:

$$(s_2 - s_2')(\sigma_2) = (\Sigma_1 \circ d_2)(\sigma_2) + d_3(\tau_3)$$

Thus if we define:

$$\Sigma_2(\sigma_2) := \tau_3$$

We have:

$$s_2 - s_2' = d_3 \circ \Sigma_2 + \Sigma_1 \circ d_2 \tag{6.3}$$

In degree 3 we have:

$$\begin{aligned}
d_3 \circ (s_3 - s_3' - \Sigma_2 \circ d_3) &= d_3 \circ (s_3 - s_3') - d_3 \circ \Sigma_2 \circ d_3 \\
&= d_3 \circ (s_3 - s_3') - (s_2 - s_2' - \Sigma_1 \circ d_2) \circ d_3 && \text{By (6.3)} \\
&= d_3 \circ (s_3 - s_3') - (s_2 - s_2') \circ d_3 && \text{As } d^2 = 0 \\
&= 0 && \text{As } s_\sharp \text{ is a chain map}
\end{aligned}$$

Hence if $\sigma_3 \in \Delta_3(EG^{(3)})$, then $(s_3 - s_3' - \Sigma_2 \circ d_3)(\sigma_3)$ is a 3-cycle in $C_3^S(\mathbb{S}^3)$. However, since $H_3^S(\mathbb{S}^3) \cong \mathbb{Z}$, there exists $m \in \mathbb{Z}$ and $\tau_4 \in C_4^S(\mathbb{S}^3)$ such that:

$$(s_3 - s_3' - \Sigma_2 \circ d_3)(\sigma_3) + m\varsigma = d_4(\tau_4)$$

Where $\varsigma \in [\mathbb{S}^3]$, the fundamental class of $\mathbb{S}^3$. Thus if we define:

$$\Sigma_3(\sigma_3) := \tau_4$$

We have:

$$(s_3 - s_3')(\sigma_3) + m\varsigma = (d_4 \circ \Sigma_3 + \Sigma_2 \circ d_3)(\sigma_3)$$

Although Σ is not a chain homotopy between $s_\sharp$ and $s_\sharp'$, it will be a chain homotopy (up to degree 3) between $s_\sharp$ and $s_\sharp' - t_\sharp$ where $t_\sharp$ is a chain map such that:

$$t_n(\sigma_n) = \begin{cases} m\varsigma & n = 3, \\ 0 & n \neq 3. \end{cases}$$

With m is as it is in Lemma [6.2.1]. The homomorphisms $t_\sharp$ form a chain map as $d_n \circ t_n = t_{n-1} \circ d_n$ trivially for $n \neq 3$, while:

$$\begin{aligned} d_3 \circ t_3 &= d_3(m\varsigma) \\ &= 0 \\ &= t_2 \circ d_3 \end{aligned}$$

Define a chain map $s''_\sharp$ so that:

$$s''_n = s'_n + t_\sharp$$

We now continue the chain homotopy for higher degrees but this time with $s''_\sharp$ In degree 3 we have:

$$s_3 - s''_3 = d_4 \circ \Sigma_3 + \Sigma_2 \circ d_3$$

As $H^S_4(\mathbb{S}^3) = 0$, we may apply the same reasoning as in degree 3 to obtain:

$$\Sigma_4 := \tau_4$$

And

$$s_4 - s''_4 = d_5 \circ \Sigma_4 + \Sigma_3 \circ d_4$$

And since $H^S_n(\mathbb{S}^3) = 0$ for $n \geq 4$, it is easy to see that the required chain homotopy exists by induction.

The second property is obvious from the definition of $s''_\sharp$. $\square$

Suppose σ is a 3-chain in $\Delta_3(EG)$ and s, s' are as in the lemma and $s''_\sharp$ is as in its proof, then:

$$
\begin{aligned}
d_3(s''_3(\sigma)) &= d_3(s'_3(\sigma)) \\
d_3(s''_3(\sigma) - s'_3(\sigma)) &= 0
\end{aligned}
$$

Again we fix a representative $\varsigma \in [\mathbb{S}^3]$ to obtain:

$$
s''_3(\sigma) - s'_3(\sigma) - m\varsigma = d_4(\tau) \tag{6.4}
$$

Where m is dependant on σ, s' and s'' as in the proof of Lemma [6.2.1].
Let $\phi : C_3(\mathbb{S}^3) \to \mathbb{R}$ be a cocycle (for $\mathbb{S}^3$).
Since $H^3_S(\mathbb{S}^3; \mathbb{R}) = \mathbb{R}$, we may normalise ϕ so that $\phi(\varsigma) = 1$.
Applying it to (6.4) gives:

$$
\begin{aligned}
\phi(s''_3(\sigma) - s'_3(\sigma) - m\varsigma) &= \phi(d_4(\tau)) \\
\phi(s''_3(\sigma)) - \phi(s'_3(\sigma)) - \phi(m\varsigma) &= 0 && \text{since } \phi \text{ is a cocycle.} \\
(\phi \circ s''_3)(\sigma)) - (\phi \circ s'_3)(\sigma)) - \phi(m\varsigma) &= 0
\end{aligned}
$$

Now if we compose ϕ with the quotient homomorphism $q : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$, we have:

$$\begin{aligned}(q \circ \phi \circ s_3'')(\sigma) - (q \circ \phi \circ s_3')(\sigma) - q(\phi(m\varsigma)) &= 0 \\ (q \circ \phi \circ s_3'')(\sigma) - (q \circ \phi \circ s_3')(\sigma) &= q(m \cdot 1) \\ &= 0 \\ (q \circ \phi \circ s_3'')(\sigma) &= (q \circ \phi \circ s_3')(\sigma)\end{aligned}$$

By the lemma there exists a chain homotopy, Σ, i.e.

$$s_n - s_n'' = d_{n+1} \circ \Sigma_n + \Sigma_{n-1} \circ d_n$$

As $Hom_{\mathbb{Z}G}(-, \mathbb{R})$ is an additive functor, there exits a cochain homotopy, S, between the dual chain maps:

$$s^n - (s'')^n = S^n \circ \delta^{n+1} + \delta^n \circ S^{n-1}$$

Hence $s^\sharp(\phi)$ and $(s'')^\sharp(\phi)$ are cohomologous in $\Delta^n(EG)$.
Hence $(q \circ s^\sharp)(\phi)$ and $(q \circ (s')^\sharp)(\phi)$ are cohomologous in $\Delta^n(EG)$.
Thus any two choices of the map s will give cohomologous homomorphisms $(q \circ s^3)(\phi)$.

We need to check that $(q \circ s^3)(\phi)$ is a cocycle for EG, i.e., if $\rho \in \Delta_4(EG)$, then

$$\delta^4((q \circ s^3)(\phi))[\rho] = (q \circ \phi \circ s_3 \circ d_4)(\rho) = 0$$

We have:

$$\begin{aligned} d_3(s_3(d_4(\rho))) &= (s_3 \circ d_3 \circ d_4)(\rho) \\ &= 0 \end{aligned}$$

Hence $(s_3 \circ d_4)(\rho)$ is a 3-cycle in $C_3^S(\mathbb{S}^3)$.

As before we may conclude that there exists $\tau \in \Delta_4(\mathbb{S}^3)$ and $m \in \mathbb{Z}$ such that:

$$\begin{aligned} s_3(d_4(\rho)) - m\varsigma &= d_4(\tau) \\ \phi(s_3(d_4(\rho) - m\varsigma)) &= \phi(d_4(\tau)) = 0 && \text{since } \phi \text{ is a cocycle.} \\ \phi(s_3(d_4(\rho)) &= \phi(m\varsigma) \\ q(\phi(s_3(d_4(\rho)))) &= q(m\phi(\varsigma)) \\ (q \circ \phi \circ s_3 \circ d_4)(\rho) &= q(m) \\ &= 0 \end{aligned}$$

Hence $(q \circ s^3)(\phi) = q \circ \phi \circ s_3$ is a 3-cocycle in $\Delta^3(EG; \mathbb{R}/\mathbb{Z})$.

6.3 Choosing the Cocycle by Integration

Now we may describe a choice of a 3-cocycle ϕ in $C^3(\mathbb{S}^3; \mathbb{R})$. This can be achieved via integration of differential forms on the smooth manifold $\mathbb{S}^3$.

More precisely, let υ be the normalised volume form on $\mathbb{S}^3$ such that:

$$\int_{\mathbb{S}^3} \upsilon = 1$$

As $\mathbb{S}^3$ is a 3-manifold, at every point $x \in X$ the cotangent space is of dimension three. Thus the 4-th exterior power of $T_x^*(\mathbb{S}^3)$ is trivial, and every 4-from is trivial. The form υ is a 3-form, therefore $d\upsilon = 0$, i.e., υ is closed.

As per de Rham's theorem if X is a smooth manifold, the following homomorphisms define a chain map which induces an isomorphism on the level of cohomology:

$$\begin{aligned} i : \Omega^n(X) &\to C^n_S(X; \mathbb{R}) \\ \omega &\mapsto \phi(\sigma) = \int_{\Delta^n} \sigma^* \omega \qquad\qquad \text{for } \sigma \in C^S_n(X). \end{aligned}$$

As any chain map sends cocycles to cocycles, the image of υ under this homomorphism is a cocycle.
More explicitly, let $\tau \in C_4^S(\mathbb{S}^3)$, then

$$\begin{aligned} \delta^4(i(\upsilon))(\tau) &= (i(\upsilon) \circ d_4)(\tau) \\ &= \int_{\Delta^3} (d_4(\tau))^* \upsilon \\ &= \int_{\partial\Delta^4} \tau^* \upsilon && \text{as this is a sum of} \\ & && \text{integrals over 3-simplicies.} \\ &= \int_{\Delta^4} \tau^* d\upsilon && \text{by Stoke's theorem.} \\ &= 0 && \text{since } \upsilon \text{ is a closed 3-form.} \end{aligned}$$

Hence $i(\upsilon)$ is a 3-cocycle in $C^3_S(\mathbb{S}^3;\mathbb{R})$ and it is the one we choose for our construction, i.e. $\phi = i(\upsilon)$.

Our cocycle must also be G-equivariant, but this follows from the fact that G acts on $\mathbb{S}^3$ by isometries and thus $g^*\upsilon = \upsilon$. Thus if $\sigma \in C^S_3(\mathbb{S}^3)$ then

$$\begin{aligned}
\phi(g\cdot\sigma) &= \int_{\Delta^3} (g\cdot\sigma)^*\upsilon \\
&= \int_{\Delta^3} \sigma^* g^* \upsilon \\
&= \int_{\Delta^3} \sigma^* \upsilon \\
&= \phi(\sigma) \\
&= g\cdot\phi(\sigma) && \text{Since } \mathbb{R} \text{ is a trivial } G\text{-module.}
\end{aligned}$$

Finally, we need that it is a normalised cocycle, but this follows as the map s sent degenerate simplicies in EG to degenerate simplicies in $\mathbb{S}^3$, the images of which have zero volume.
As our cocycle, $q \circ i(\upsilon) \circ s_3 = 0$ on the degenerate subgroup, it descends to a normalised cocycle ψ. This is the cocycle that we set out to construct.

Now we may construct the special 2-group, $\mathcal{G}$. Its set of objects will be G, with $Aut(g) = \mathbb{R}/\mathbb{Z}$.

7 Appendix

7.1 Proof of 2.0.1

T consists of:

- The identity;
- Two $\dfrac{2\pi}{3}$ rotations about each of the four axes that passes through a vertex and the centroid of the opposite face; and
- One π rotation about each of the three axes that pass through the centre of an edge and the unique edge that does not touch either of the vertices that bound the first edge

This gives 12 elements in total.

It may be observed that 2 three cycles such as (132) and (143) generate A_4.

However, (132) and (143) are images of rotations under the aforementioned embedding, $\phi : T \rightarrowtail S_4$.

Therefore, $A_4 \leq \phi(T)$, but as $|A_4| = |T| = 12$, $A_4 = \phi(T)$ and hence $A_4 \cong T$. $\square$

7.2 Proof of 2.0.2

We may define a homomorphism:

$$\varphi : \left\langle a, b \mid a^3, b^3, (ab)^2 \right\rangle \to T, \text{ where}$$

$$a \mapsto (132)$$

$$b \mapsto (143)$$

The homomorphism φ is well defined as $(132)^3 = (143)^3 = ((132)(143))^2 = 1$.

It is surjective as generators go to generators

It may be shown using the Todd-Coxeter Process [10] or otherwise that

$$\left| \left\langle a, b \mid a^3, b^3, (ab)^2 \right\rangle \right| = 12$$

As noted above, $|T| = 12$, therefore φ is injective and hence an isomorphism.

$\square$

7.3 Proof of 2.0.3

The homomorphism π is clerly surjective so it remains to show that $ker(\pi) = \{\pm 1\}$.

There is an isomorphism $\langle e, f | - \rangle \cong \langle a, b | - \rangle$ that fits into the following commutative diagram.

$$
\begin{tikzcd}
\langle e, f \mid - \rangle \arrow[r, "\cong", "\iota"'] \arrow[d, two heads, "q_1"] & \langle a, b \mid - \rangle \arrow[d, two heads, "q_2"] \\
\tilde{T} \arrow[r, two heads, "\pi"] & T
\end{tikzcd}
$$

Where q_1 and q_2 are the natural surjections.

The map π as defined fits into the central extension.

Proof.

$$ker(q_2) = \left\langle a^3, b^3, (ab)^2 \right\rangle^{\langle a,b \rangle}$$

Where H^G is the the normal closure of H in G. Therefore,

$$ker(q_2 \circ \iota) = \left\langle e^3, f^3, (ef)^2 \right\rangle^{\langle e,f \rangle}$$

In $\tilde{T}$, let $e^3 = f^3 = (ef)^2$ be denoted by g.
Then $q_1(ker(q_2 \circ \iota)) \subseteq \langle g \rangle^{\tilde{T}}$ and $\langle g \rangle \subseteq q_1(ker(q_2 \circ \iota))$.
But $gf = f^4 = fg$ and $ge = e^4 = eg$, therefore $g \in Z(\tilde{T})$, the centre of T.
Hence,

$$\langle g \rangle = \langle g \rangle^{\tilde{T}} \text{ and } q_1(ker(q_2 \circ \iota)) = \langle g \rangle .$$

Since q_1 is surjective, $ker(\pi) = \langle g \rangle$ Note that

$$f^2 = efe \tag{7.1}$$

$$e^2 = fef \tag{7.2}$$

We may use this to show that $g^2 = 1$.

$$\begin{aligned}
e^{10} &= ef^9 \\
&= efe^6f^2 \\
e^7f^3 &= f^2e^5f^2 \\
&= f^2ef^3ef \\
e^7f^2 &= fe^2fe^2 \\
e^8fe &= fe^2fe^2 \\
e^8f &= fe^2fe \\
e^5fe^3 &= fe^2fe \\
e^5fe^2 &= fe^2f \\
e^5ffef &= fe^2f \\
e^5f^2 &= fe \\
e^6f^2 &= efe \\
e^6f^2 &= f^2 \\
e^6 &= 1
\end{aligned}$$

□

Hence $ker(\pi) = \{1, g\} \cong \{\pm 1\}$. $\square$

7.4 Proof of 2.0.4

The map p will first be defined into $Aut(\mathbb{R}^3) = GL_3(\mathbb{R})$ before showing that the image is in $O(3)$ and then $SO(3)$ recalling that

$$SO(3) = \{A \in Aut(\mathbb{R}^3) \mid AA^t = 1, det(A) = 1\}$$

Identify $\mathbb{R}^3$ with the orthogonal complement of the real axis in $\mathbb{H}$, i.e. $\mathbb{R}^3 = \{q \in \mathbb{H} \mid Re(q) = 0\}$

$$\begin{aligned} \text{Define: } \cdot : Sp(1) \times \mathbb{R}^3 &\to \mathbb{R}^3 \\ \text{where } (q, r) &\mapsto qrq^{-1} \end{aligned}$$

To see the action is well defined:

$$\begin{aligned} \text{Let } q &= a + b\mathbb{i} + c\mathbb{j} + d\mathbb{k}, r = x\mathbb{i} + y\mathbb{j} + z\mathbb{k}, \text{ then} \\ Re(q \cdot r) &= d(-cx + by + az) + c(dx + ay - bz) \\ &\quad + b(ax - dy + cz) + a(-bx - cy - dz) \\ &= 0 \end{aligned}$$

The action is linear in the second argument as quaternion multiplication is distributive. Thus we may define:

$$p : Sp(1) \to GL_3(\mathbb{R})$$
$$q \mapsto \{T : r \mapsto qrq^{-1}\}$$

In order to show that $p(Sp(1)) \subseteq O(3)$, it will be shown that elements in the image of p preserve the standard inner product on $\mathbb{R}^3$.

Let $p(q) = T : \mathbb{R}^3 \to \mathbb{R}^3$

T is clearly invertible with $T^{-1}s = q^{-1}sq$

Let $r, s \in \mathbb{R}^3 \subset \mathbb{H}$ note that $rs = -\langle r, s\rangle + r \times s$

Thus $\langle r, s\rangle = -Re(rs)$

So $\left\langle qrq^{-1}, s\right\rangle = -Re(qrq^{-1}s) = -Re(rq^{-1}sq) = \left\langle r, q^{-1}sq\right\rangle$

The central equality may be explained by noticing that quaternions commute in the real part:

Let $q = a + b\mathbb{i} + c\mathbb{j} + d\mathbb{k}$, and $t = e + f\mathbb{i} + g\mathbb{j} + h\mathbb{k}$. Then

$$\begin{aligned} Re(qt) &= ae - bf - cg - dh \\ &= ea - fb - gc - hd && \text{Since } \mathbb{R} \text{ is commutative.} \\ &= Re(tq) \end{aligned}$$

Thus the transpose is $A^t s = q^{-1}sq$ and we may see that $A^tA = AA^t = Id$ and hence A is orthogonal.

It will now be shown that $det(A) = 1$ and hence $p(Sp(1)) \subseteq SO(3)$.

$$q\mathbb{i}q^{-1} = (a^2 + b^2 - c^2 - d^2)\mathbb{i} + (2bc + 2ad)\mathbb{j} + (-2ac + 2bd)\mathbb{k}$$

$$q\mathbb{j}q^{-1} = (2bc - 2ad)\mathbb{i} + (a^2 - b^2 + c^2 - d^2)\mathbb{j} + (2ab + 2cd)\mathbb{k}$$

$$q\mathbb{k}q^{-1} = (2ac + 2bd)\mathbb{i} + (-2ab + 2cd)\mathbb{j} + (a^2 - b^2 - c^2 + d^2)\mathbb{k}$$

Then $det(A) = (a^2 + b^2 + c^2 + d^2)^3 = 1$ as $q \in Sp(1)$

It will now be shown that $SO(3) \subseteq p(Sp(1))$ and hence π is surjective.

Every rotation in 3-space is a rotation about an axis, $u = (m, n, o) \in \mathbb{S}^2 \subset \mathbb{R}^3$ at an angle of 2θ for some $\theta \in [0, \pi]$.

Note that $u = m\mathbb{i} + n\mathbb{j} + o\mathbb{k} \in \mathbb{H}$.

$$
\begin{aligned}
\text{Let } v &= \cos(\theta) + \sin(\theta)u \\
v &\in Sp(1) \text{ as } \langle v, v\rangle = \cos^2(\theta) + \sin^2(\theta)(m^2 + n^2 + o^2) = 1 \\
vrv^{-1} &= (\cos(\theta) + \sin(\theta)u)r(\cos(\theta) - \sin(\theta)u) \\
&= \cos^2(\theta)r + \sin(\theta)\cos(\theta)(ur - ru) - \sin^2(\theta)uru \\
&= \cos^2(\theta)r + \sin(\theta)\cos(\theta)(-\langle u, r\rangle + u \times r + \langle r, u\rangle - r \times u) \\
&\quad - \sin^2(\theta)(-\langle u, r\rangle + u \times r)u \\
&= \cos^2(\theta)r + 2\sin(\theta)\cos(\theta)(u \times r) \\
&\quad - \sin^2(\theta)(-\langle u, r\rangle u - \langle u \times r, u\rangle + (u \times r) \times u) \\
&= \cos^2(\theta)r + 2\sin(\theta)\cos(\theta)(u \times r) \\
&\quad - \sin^2(\theta)(-\langle u, r\rangle u - 0 - \langle u, r\rangle u + - \langle u, u\rangle r) \\
&= \cos^2(\theta)r + 2\sin(\theta)\cos(\theta)(u \times r) - \sin^2(\theta)(\langle u, u\rangle r - 2\langle u, r\rangle r) \\
&= (\cos^2(\theta) - \sin^2(\theta))r + 2\sin(\theta)\cos(\theta)(u \times r) + (2\sin^2(\theta))\langle u, r\rangle u \\
&= \cos(2\theta)r + \sin(2\theta)(u \times r) + (1 - \cos(2\theta))\langle u, r\rangle r \\
&= \cos(2\theta)(r - \langle u, r\rangle u) + \sin 2\theta(u \times r) + \langle u, r\rangle u
\end{aligned}
$$

Which is the Rodrigues rotation formula [9] for such a rotation.

Hence, $S0(3) \subseteq p(Sp(1))$ and p is surjective. $\square$

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