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Lectures on Representation Theory and Knizhnik-Zamolodchikov Equations

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This is closely related with the isomorphism between this homology space and spaces of intertwining operators for quantum groups, which we will discuss later (see Section 8.7).

7.5. Gauss-Manin connection.

To complete the analysis of the integral formulas for solutions of the KZ equations for $\mathfrak{sl}_2$, we need one more step. So far we have developed cohomology and homology theory for one local system. For the KZ equations, we need to consider families of local systems. Indeed, we have a bundle

$$\pi : X_{N+m} \to X_N \tag{7.17}$$

such that the fiber over a point $\mathbf{z} \in X_N$ is $Y_{\mathbf{z},m}$ (see Section 7.3). To construct a solution, we must choose a cycle C in every fiber (at least, locally, i.e. in a neighborhood of a point $\mathbf{z}$), and in order for the proofs of Lecture 4 to be valid these cycles must be chosen in some compatible way.

Here is the appropriate mathematical language. First of all, note that the function $\psi_m(\mathbf{z},\mathbf{t})$ defined by (7.14) gives rise to a local system $\mathcal{L}(\mu,\kappa)$ on X_{N+m}, and local systems on each $Y_{\mathbf{z},m}$ are obtained by restriction of this one. Next, the fibration (7.17) gives rise to the vector bundle $H\ (X_{N+m}/X_N,\mathcal{L}\ )$ whose fiber at a point $\mathbf{z}$ is equal to $H\ (Y_{\mathbf{z},m},\mathcal{L}\ )$.

It turns out that this vector bundle carries a natural flat connection, called the *Gauss-Manin connection*. In our case it can be defined quite easily. Namely, let us take some point $\mathbf{z} \in X_N$. Let $C \in H_i(Y_{\mathbf{z},m},\mathcal{L}\ )$ be a cycle. Considering all $Y_{\mathbf{z}',m}$ as subsets in $\mathbb{C}^m$, it is easy to see that in fact C defines a cycle in each $Y_{\mathbf{z}',m}$ for $\mathbf{z}^0$ close enough to $\mathbf{z}$ (this also requires that we have a local system on X_{N+m}, not only on each fiber); thus, we have a local section of the bundle $C_i(X_{N+m}/X_N,\mathcal{L}\ )$. Let us define a connection in $C\ (X_{N+m}/X_N,\mathcal{L}\ )$ by the condition that all local sections obtained in the above described manner are flat. One easily checks that this defines a flat connection in the homology bundle $H\ (X_{N+m}/X_N,\mathcal{L}\ )$. Similarly, we can pass to S_m-symmetric homology and define a flat connection in the bundle $H\ (X_{N+m}/X_N,\mathcal{L}\ )^{S_m}$.

Now we can summarize most of the results of the previous sections in the following theorem.

THEOREM 7.5.1. *For any μ_i and almost all κ, the map $C \mapsto \Psi_C$ establishes an isomorphism of the local systems*

$$H_m(X_{N+m}/X_N,\mathcal{L}\ (\mu,\kappa))^{S_m} \simeq W_{KZ}, \tag{7.18}$$

where $H_m(X_{N+m}/X_N,\mathcal{L}\ )^{S_m}$ is endowed with the Gauss-Manin connection and W_{KZ} is the trivial vector bundle over X_N with the fiber $W = (V^{\mathfrak{n}^-})^\lambda$, $\lambda = -\sum \mu_i + 2m$ (see (4.4)), endowed with the Knizhnik-Zamolodchikov connection.

In more elementary terms, this theorem can be formulated as follows.

THEOREM 7.5.2. *In the notation of the previous theorem, let $\mathbf{z} \in X_N$ and let us identify the homology spaces $H_m(Y_{\mathbf{z}',m})$ for $\mathbf{z}^0$ in some neighborhood of $\mathbf{z}$ as described above. Then for any μ_i and almost all κ, the map $C \mapsto \Psi_C$, defined by (7.15), is an isomorphism of $H_m(Y_{\mathbf{z},m})^{S_m}$ with the space of W-valued solutions of KZ equations in the neighborhood of $\mathbf{z}$.*