Brownian loops and conformally invariant systems

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GTP Seminar

November 23, 2022

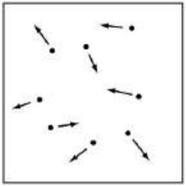
Outline

1. Conformal field theories
    - Statistical mechanics
    - Conformal invariance
    - Examples
2. Brownian loop soup
    - Primary operator spectrum
    - Correlation functions
3. Percolation
4. Overview and summary

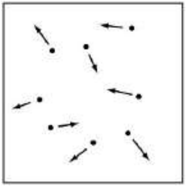
Statistical mechanics

Configuration space Ω with Gibbs measure for $\omega \in \Omega$

$$P(\omega) = \frac{1}{Z} e^{-\beta E(\omega)}$$



Statistical mechanics



Configuration space Ω with Gibbs measure for $\omega \in \Omega$

$$P(\omega) = \frac{1}{Z} e^{-\beta E(\omega)}$$

Partition function

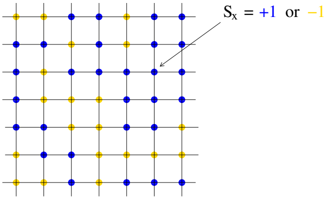
$$Z = \sum_{\omega \in \Omega} e^{-\beta E(\omega)}$$

Correlation functions

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\dots\rangle = \frac{1}{Z}\sum_{\omega\in\Omega} \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\dots e^{-\beta E(\omega)}$$

Example: Ising Model

$$\Omega = \{-1, 1\}^D$$
$$E(\omega) = -\sum_{\langle i,j \rangle} S_i S_j$$



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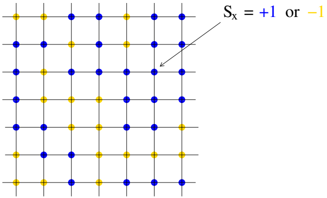
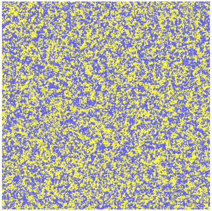
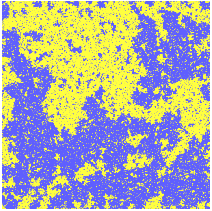


$\beta < \beta_{\text{critical}}$



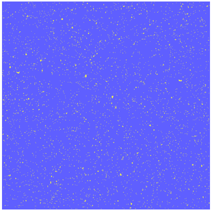
Disordered phase

β_{critical}



Critical point

$\beta > \beta_{\text{critical}}$



Collective behavior

Conformal transformations

Conformal field theories are quantum field theories that are invariant under angle-preserving transformations

$$x \to x' = x + \varepsilon(x)$$
$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x)$$

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Conformal transformations form the conformal group.

For $d \geq 3$

$$SO(d+1,1)$$

with $\frac{1}{2}(d+1)(d+2)$ generators:
translations, dilations, rotations, SCT

Conformal transformations

$d = 2$ (Euclidean field theory) is special. Conformal transformations are given by Cauchy–Riemann equations.

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All holomorphic (locally analytic) functions

$$z \to f(z)$$
$$\bar{z} \to \bar{f}(\bar{z})$$

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generate conformal transformations.

The (quantum) generators form the Virasoro algebra

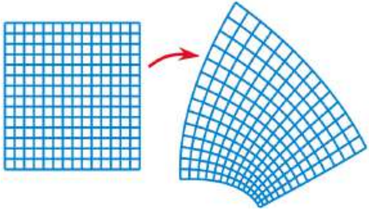
$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}$$
$$[L_n, \bar{L}_m] = 0$$

The global conformal transformations L_{-1}, L_0, L_1 form a sub-algebra

Conformal transformations

Any analytic function with nonzero derivative is a conformal map

$$z \to f(z)$$



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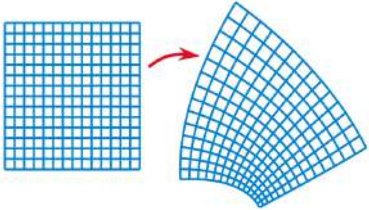

(Quasi-)primary operators behave like

$$\Phi(f(z), \bar{f}(\bar{z})) = \left(\frac{\partial f}{\partial z}\right)^{-\Delta} \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{-\bar{\Delta}} \Phi(z, \bar{z})$$

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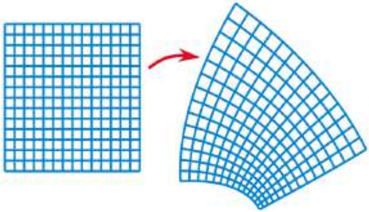

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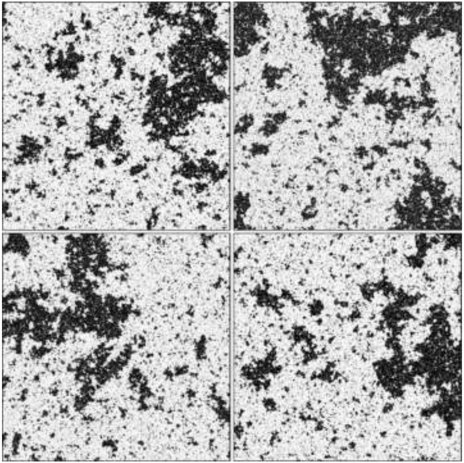
From here on $\langle \Phi_1(z_1, \bar{z}_1) \ldots \rangle \equiv \langle \Phi_1(z_1) \ldots \rangle$

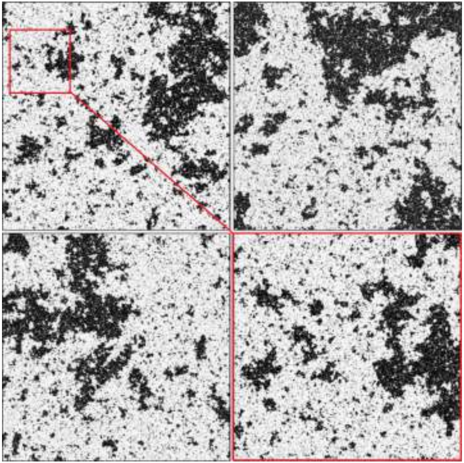
Self-similarity: Ising model



$$E = -\sum_{\langle i,j \rangle} S_i S_j, \quad S_i = \pm 1 \text{ at criticality}$$

Douglas Ashton, Critical Ising model on 10^{10} lattice sites

Self-similarity: Ising model



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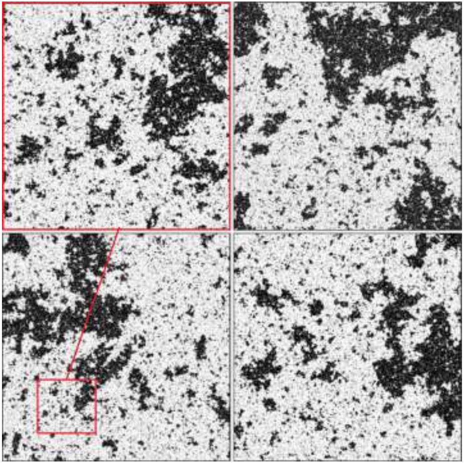

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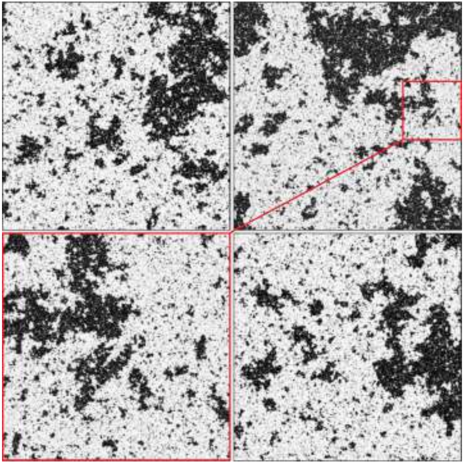

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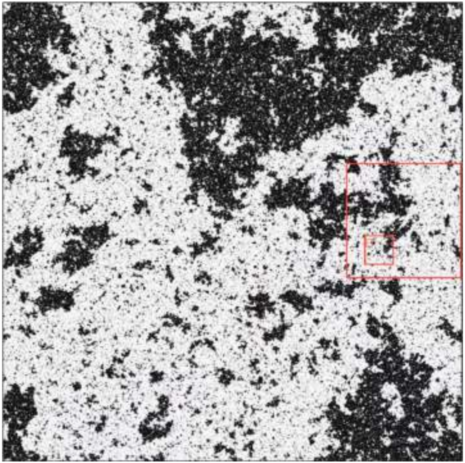

In the continuum: $\Phi(\lambda z) = \lambda^{-\Delta}\Phi(z)$

Douglas Ashton, Critical Ising model on 10^{10} lattice sites

Correlation functions

Every global conformal symmetry corresponds to one Ward identity:

$$\langle \Phi(z) \rangle$$
$$\langle \Phi(z_1)\Phi(z_2) \rangle$$
$$\langle \Phi_1(z_1)\Phi_2(z_2)\Phi_3(z_3) \rangle$$

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The four-point function

$$\langle \Phi_1(z_1)\Phi_2(z_2)\Phi_3(z_3)\Phi_4(z_4) \rangle = G(x) \prod_{i<j}^{4} z_{ij}^{c_{ij}} \bar{z}_{ij}^{\bar{c}_{ij}}$$

with $z_{ij} = z_i - z_j$ and cross-ratio $x = \frac{z_{12} z_{34}}{z_{13} z_{24}}$ contains non-trivial information

CFTs appear in

- Statistical physics
- Condensed matter physics
- String theory
- AdS/CFT
- Stochastic systems

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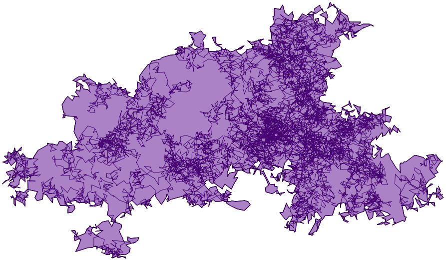
Let's cook a Brownian loop soup!

Brownian loop soup

The BLS is the Poissonian ensemble of Brownian loops in the plane



The outer boundary of a random walk is
a self-avoiding random walk

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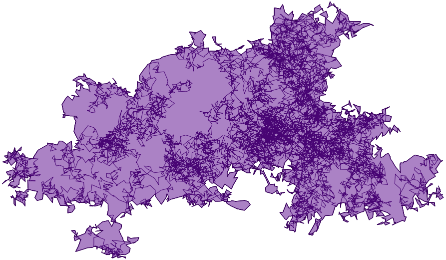
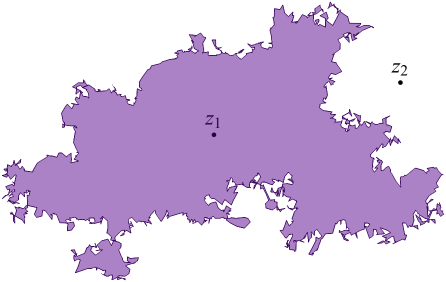




The layering number

The outer boundary of a random walk is a self-avoiding random walk

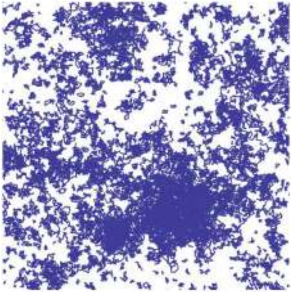
$$N(z_1) = 1$$

$$N(z_2) = 0$$

Brownian loop soup

The BLS in domain D with intensity λ

- is a Poisson point process
- of unrooted loops
- with unique measure $\lambda\mu^{\text{loop}}$



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Lawler, Werner 2004

Brownian loop soup

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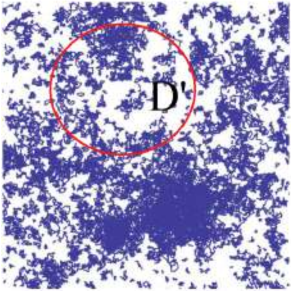
The **restriction** of D to $D' \in D$ is a BLS in D'



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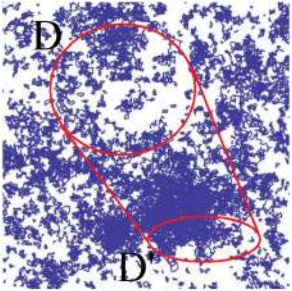

The **restriction** of D to $D' \in D$ is a BLS in D'

The image of $D \to D'$ under **conformal map** is BLS in D'

Lawler, Werner 2004

The layering vertex operators

The partition function

$$Z = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\lambda \mu^{\text{loop}}\right)^n$$

Loop measure

$$\mu^{\text{loop}} = \int_D \int_0^{\infty} \frac{1}{2\pi t^2} \mu_{z,t}^{\text{br}} \ dt \ dA(z)$$

Central charge

$$c = 2\lambda$$

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Primary operators

$$e^{i\beta N(z)}$$

with $N(z) = N_+(z) - N_-(z)$ have conformal dimension

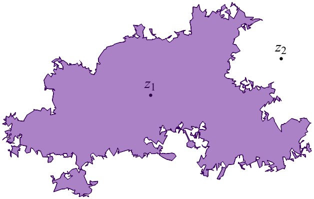
$$\boxed{\Delta = \bar{\Delta} = \frac{\lambda}{10}(1 - \cos\beta)}$$

Correlation functions

Example: The two-point function

$$\left\langle e^{i\beta N(z_1)} e^{-i\beta N(z_2)} \right\rangle = \begin{cases} |z_1 - z_2|^{-4\Delta} & \text{CFTs} \\ \exp\left[-\lambda(1-\cos\beta)(\alpha(z_1|z_2)+\alpha(z_2|z_1))\right] & \text{BLS} \end{cases}$$

$$\alpha(z_1|z_2) = \mu^{\text{loop}}(\text{loops covering } z_1 \text{ but not } z_2)$$



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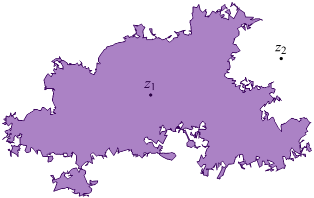


Compute the four-point function

$$\boxed{\left\langle \prod_{i=1}^{4} e^{i\beta_i N(z_i)} \right\rangle = \exp[-\lambda(\text{sum of weights})]}$$

These αs appear in the $O(n)$ model.
$n \to 0$ gives the ensemble of single, self-avoiding loops

The $O(n)$ model

Random n-dimensional vectors:

- $n = 1$ Ising
- $n = 2$ XY (Berezinskii–Kosterlitz–Thouless)
- $n \to 0$ single, self-avoiding loops

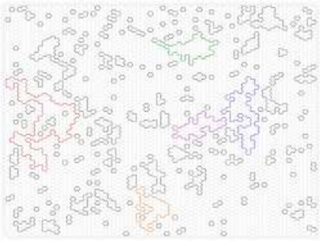
The $O(n)$ model

Random n-dimensional vectors:

- $n = 1$ Ising
- $n = 2$ XY (Berezinskii–Kosterlitz–Thouless)
- $n \to 0$ single, self-avoiding loops

$$Z(n) = \int_{\mathbf{S}} d\mathbf{S} \prod_{\langle i,j \rangle} (1 + k\mathbf{S}_i \cdot \mathbf{S}_j)$$
$$= \sum k^E n^L$$
$$= 1 + n(\ldots) + O(n^2)$$

Sum over self-avoiding loop configurations of L loops and E edges



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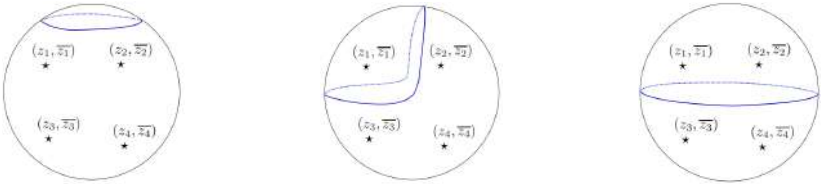

The BLS for $\lambda \to 0$ becomes the $O(0)$ model. Study the critical $O(n)$ model as a CFT!

Gamsa, Cardy 2006

A four-point function

All weights are determined

$$\alpha_{\mathbb{S}}(z_1|z_2,z_3,z_4) = \frac{1}{5}\left(\log\left|\frac{z_{12}z_{14}}{z_{24}}\right| + A(x)\right)$$

$$\alpha_{\mathbb{S}}(z_1,z_2|z_3,z_4) = -\frac{1}{5}\left(\log|x| + A(x)\right)$$

$$\vdots$$

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$$\vdots$$

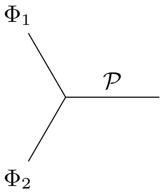
$$\boxed{\left\langle \prod_{i=1}^{4} e^{i\beta_i N(z_i)} \right\rangle = \exp\left[\tilde{\Delta} A(x)\right] \prod_{i<j}^{4} |z_{ij}|^{c_{ij}}}$$

$$A(x) = \frac{1}{4}\left[x\ {}_3F_2(x) + \overline{x}\ {}_3F_2(\overline{x})\right] - \text{const} \cdot |x(1-x)|^{\frac{2}{3}}\ |{}_2F_1(x)|^2$$

Conformal block expansion

Operator Product Expansion (fusion rules)

$$\Phi_1(z+\varepsilon)\Phi_2(z) = \sum_{\mathcal{P}} \varepsilon^{\delta_{\mathcal{P}}} \bar{\varepsilon}^{\bar{\delta}_{\mathcal{P}}} C_{12}^{\mathcal{P}} \mathcal{P}(z)$$



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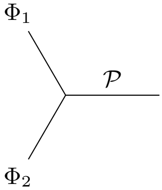
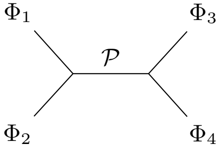


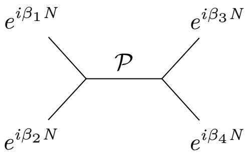
Apply to four-point function



Conformal block expansion

The Virasoro conformal block expansion

$$G(x) = \sum_{\mathcal{P}} C_{34}^{\mathcal{P}} C_{12}^{\mathcal{P}} \mathcal{F}(\mathcal{P}|x)\bar{\mathcal{F}}(\mathcal{P}|\bar{x})$$



$$\Delta_{\mathcal{P}} = \Delta_{12} + \frac{p}{3}$$

$$\bar{\Delta}_{\mathcal{P}} = \Delta_{12} + \frac{p'}{3}$$

$$\Delta - \bar{\Delta} \in \mathbb{Z}$$

$$\Delta_{12} = \frac{\lambda}{10}(1 - \cos(\beta_1 + \beta_2))$$

Conformal block expansion

The Virasoro conformal block expansion

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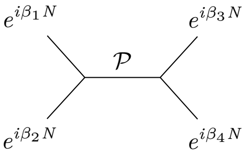

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$$\Delta - \bar{\Delta} \in \mathbb{Z}$$
$$\Delta_{12} = \frac{\lambda}{10}(1 - \cos(\beta_1 + \beta_2))$$

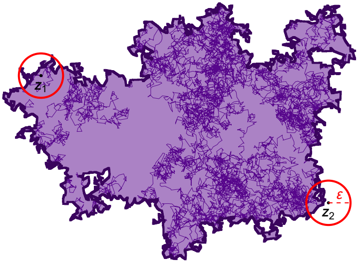
contains information about the entire CFT

- Central charge c
- Primary operator spectrum $\Delta_{\mathcal{P}}$
- Three-point function coefficients $C_{ij}^{\mathcal{P}}$

Edge counting operators

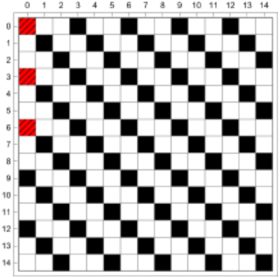
What are the new operators in terms of loops? Define

$$E_\epsilon(z) \coloneqq \text{Centered number of loops whose outer boundaries come } \epsilon\text{-close to } z$$



$$\Delta_{\mathcal{E}} = \bar{\Delta}_{\mathcal{E}} = \frac{1}{3}$$

Extended Symmetry?



$$(\Delta_{\mathcal{P}}, \bar{\Delta}_{\mathcal{P}}) = (p/3, p'/3)$$
$$\Delta_{\mathcal{P}} - \bar{\Delta}_{\mathcal{P}} = \mathbb{Z}$$

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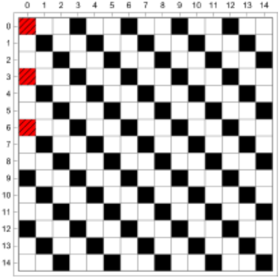
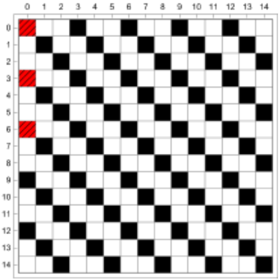


$$(\Delta_{\mathcal{P}}, \bar{\Delta}_{\mathcal{P}}) = (p/3, p'/3)$$
$$\Delta_{\mathcal{P}} - \bar{\Delta}_{\mathcal{P}} = \mathbb{Z}$$

$$T(z) = (L_{-2}\mathbb{1})(z)$$
$$J(z) = (J_{-1}\mathbb{1})(z)$$
$$[J] = (1, 0)$$

$$\psi = J_{-m_1} J_{-m_2} \ldots \phi$$
$$L_0 \phi = \Delta \phi$$
$$L_0 \psi = (\Delta + m_1 + m_2 + \ldots)\psi$$

Extended Symmetry?



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$$(\Delta_{\mathcal{P}}, \bar{\Delta}_{\mathcal{P}}) = (p/3, p'/3)$$
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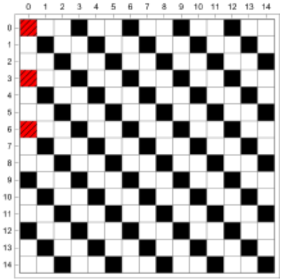
$$T(z)\phi_i(w) = \frac{\Delta_i}{(z-w)^2}\phi_i(w) + \frac{1}{z-w}\partial_w \phi_i(w)$$

Extended Symmetry?



$$T(z) = (L_{-2}\mathbb{1})(z)$$
$$J(z) = (J_{-1}\mathbb{1})(z)$$
$$[J] = (1, 0)$$

$$\psi = J_{-m_1} J_{-m_2} \dots \phi$$
$$L_0 \phi = \Delta \phi$$
$$L_0 \psi = (\Delta + m_1 + m_2 + \dots)\psi$$

$$(\Delta_{\mathcal{P}}, \bar{\Delta}_{\mathcal{P}}) = (p/3, p'/3)$$
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$$J(z)\phi_i(w) = \frac{t_i}{z-w}\phi_i(w)?$$

Some comments

- The BLS cannot be unitary, at least for $c < 1$
- We expect unitarity for $\lambda = 1/2$ or $c = 1$ (free Boson)
- The BLS can be generalized to domains with boundary
- What is the full symmetry algebra / operator spectrum? (Kac-Moody)
- Where does the BLS lie in the class of CFTs? (minimal models, logarithmic CFTs)

□

Percolation

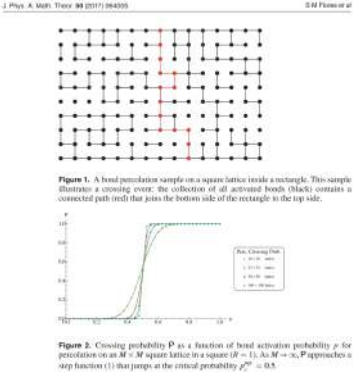
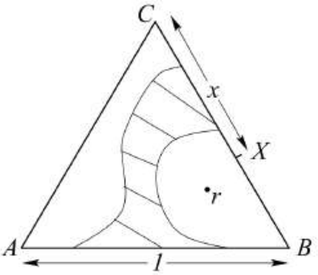


Figure 1. A bond percolation sample on a square lattice inside a rectangle. This sample illustrates a crossing event: the collection of all activated bonds (black) contains a connected path (red) that joins the bottom side of the rectangle to the top side.

Figure 2. Crossing probability P as a function of bond activation probability p for percolation on an $M \times M$ square lattice in a square ($R = 1$). As $M \to \infty$, P approaches a step function (1) that jumps at the critical probability $p_c^{sq} = 0.5$.



Cardy's connection formula

$$P(\text{a cluster connects AB and CX}) = x$$

Conformal probability

Modern probability theory studies random fractal objects that are conformally invariant in distribution. (Continuum scaling limit at criticality of percolation, Ising, Potts, $O(n)$, loop-erased random walk, self-avoiding walk, ...)

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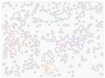
(Schramm)

Conformal loop ensembles
(Camia, Newman; Sheffield)



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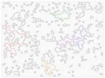


Brownian loop soup
(Lawler, Werner)

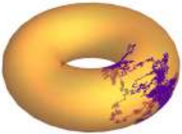
Gaussian free field and GMC
(Mandelbrot; Kahane, Peyrière)

Summary

Conformal probability provides a link between 2D statistical mechanics and conformal field theory (rigorous approach to CFT).

The Brownian loop soup

- allows a microscopic interpretation of the field theory objects
- has connections to the free Boson, $O(n)$ model, SLE, CLE, ...
- is a rich model with new features



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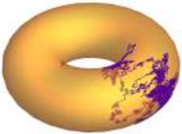

Based on
`arXiv:1912.00973, 2109.12116, 2112.00074, TBA`
with Federico Camia, Alberto Gandolfi, Matthew Kleban

Thanks!