

GROMOV'S CONTRIBUTION TO THE OKA PRINCIPLE

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ABSTRACT. We describe Gromov's seminal contribution [9] to the development of modern *Oka principle*, a form of homotopy principle in complex analysis which includes the classical *Oka-Grauert principle*. Recently this led to the introduction of a new class of complex manifolds and holomorphic maps, the *Oka manifolds* and *Oka maps*. For more information on this subject see the monograph [6] and the survey [7].

1. THE OKA-GRAUERT PRINCIPLE

The homotopy principle in complex analysis is commonly known as the *Oka Principle* after Kiyoshi Oka (1901-78). In his series of papers during 1936–1953, Oka invented new methods of constructing global analytic objects from local ones. The Oka principle first appeared in his 1939 paper [12] where he showed that a holomorphic line bundle on a domain of holomorphy is holomorphically trivial if (and only if) it is topologically trivial.

Domains of holomorphy form a subclass of the class of *Stein manifolds* that were introduced by Karl Stein in 1951. During 1950s, Hans Grauert and Reinhold Remmert studied *Stein spaces*, complex spaces that are holomorphically convex and on which holomorphic functions separate points. In 1958, Grauert [8] extended Oka's theorem to principal fiber bundles with arbitrary complex Lie group fibers over Stein spaces, showing that the holomorphic classification of such bundles coincides with their topological classification. More precisely, if X is a Stein space and G is a complex Lie group, then the inclusion $\mathcal{O}_X^G \hookrightarrow \mathcal{C}_X^G$ of the sheaf of germs of holomorphic maps $X \rightarrow G$ into the sheaf of germs of continuous maps induces an isomorphism $H^1(X; \mathcal{O}_X^G) \cong H^1(X; \mathcal{C}_X^G)$ of the 1st Čech cohomology groups. Oka's theorem corresponds to the case when $G = \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, and it says that the Picard group $\text{Pic}(X) = H^1(X; \mathcal{O}^*)$ is isomorphic to the topological Picard group, and hence (via the first Chern class map) to $H^2(X; \mathbb{Z})$.

Grauert's result also pertains to fiber bundles with complex homogeneous fibers; in particular, to complex vector bundles. Interesting generalizations and applications were found by Forster and Ramspott, Henkin and Leiterer, and others (see Chapter 7 in [6]). This led to the formulation of the following heuristic principle:

Oka Principle: Analytic problems on Stein spaces which can be cohomologically formulated have only topological obstructions.

Oka's original theorem is proved by looking at the exact cohomology sequence associated to the short exact sequence $0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O} \xrightarrow{\sigma} \mathcal{O}^* \rightarrow 1$, where $\sigma(f) = \exp(2\pi i f)$. This cohomological proof fails for nonabelian Lie groups, and in particular for $GL_n(\mathbb{C})$ when $n > 1$. Grauert reduced the proof to the problem of constructing holomorphic sections of an associated holomorphic fiber bundle $E \rightarrow X$ with fiber G whose transition maps are left and right multiplications by elements of G . The method is similar to the construction of global sections of coherent analytic sheaves over Stein spaces (Cartan's Theorem A). The key step consists in gluing a pair of holomorphic sections over a suitable geometric configuration (A, B) in X , called a *Cartan pair*. This is accomplished by the *Cartan lemma* on splitting a holomorphic map $f: A \cap B \rightarrow G$ to a complex Lie group G into a product $f = f_A f_B$ of two holomorphic maps $f_A: A \rightarrow G$, $f_B: B \rightarrow G$, each one defined on one of the larger sets A, B in our configuration.

The classical Oka-Grauert principle is limited to fiber bundles with complex homogeneous fibers. Challenging new problems in Stein geometry called for a more general Oka principle. One such case was Forster's result from 1970 [2] on the existence of proper holomorphic embeddings of a Stein manifold X^n into \mathbb{C}^N for values of N well below the classical result $N = 2n + 1$ of Remmert, Bishop and Narasimhan. Forster conjectured that for $n > 1$ one can take $N = \lceil \frac{3n}{2} \rceil + 1$, the smallest number for which there are no topological obstructions. Forster's conjecture was only confirmed two decades later by Eliashberg and Gromov [1], using Gromov's pioneering work on Oka principle presented in the next section.

2. GROMOV'S OKA PRINCIPLE

The modern development of the Oka principle started with Gromov's seminal paper of 1989 [9] in which the emphasis moved from the cohomological to the homotopy-theoretic aspect. Grauert's proof uses the exponential map on a complex Lie group for two purposes: to prove a Runge-type approximation theorem for holomorphic maps to a complex Lie group, and to linearize the gluing problem for holomorphic sections. Gromov introduced a much more flexible concept of a *dominating (holomorphic) spray* on a complex manifold Y : A triple (E, π, s) consisting of a holomorphic vector bundle $\pi: E \rightarrow Y$ and a holomorphic map $s: E \rightarrow Y$ such that for each point $y \in Y$ we have $s(0_y) = y$ and the differential $ds_{0_y}: T_{0_y}E \rightarrow T_yY$ maps the vertical subspace E_y of the tangent space $T_{0_y}E$ surjectively onto the tangent space T_yY . A complex manifold is said to be *elliptic* if it admits a dominating spray. Gromov's first main result is the following:

Theorem 2.1. (M. Gromov, [9]) *Maps $X \rightarrow Y$ from a Stein manifold X to an elliptic manifold Y satisfy all forms of the Oka principle. The same holds for sections $f: X \rightarrow E$ of any holomorphic fiber bundle $\pi: E \rightarrow X$ with a Stein base X and an elliptic fiber Y .*

This means that every continuous map (resp. section) is homotopic to a holomorphic one, with uniform approximation on compact holomorphically convex subsets of X and with interpolation on closed complex subvarieties of X . The analogous result holds with continuous dependence on parameters. In particular, the inclusion $\mathcal{O}(X, Y) \hookrightarrow \mathcal{C}(X, Y)$ of the space of holomorphic maps into the space of continuous sections is a weak homotopy equivalence.

Here are a few examples that were pointed out by Gromov:

(A) If a complex Lie group G acts transitively on Y by holomorphic automorphisms, we obtain a spray $s: E = Y \times \mathfrak{g} \rightarrow Y$ by taking $s(y, v) = \exp(v)y$. Here, $\mathfrak{g} = T_1G \cong \mathbb{C}^k$ ($k = \dim G$) is the Lie algebra of G .

(B) Let V_1, \dots, V_k be complete holomorphic vector fields on a complex manifold Y ; that is, the flow $\phi_t^j(y)$ of V_j exists for all complex values of time t , starting at any point $y \in Y$. If the vectors $V_j(y)$ span the tangent space T_yY at each point $y \in Y$, then we get a dominating spray $s: Y \times \mathbb{C}^k \rightarrow Y$ by the formula $s(y, t_1, \dots, t_k) = \phi_{t_1}^1 \circ \phi_{t_2}^2 \circ \dots \circ \phi_{t_k}^k(y)$.

(C) A dominating spray of type (B) exists on $\mathbb{C}^n \setminus A$, where A is an algebraic subvariety of \mathbb{C} which does not contain any hypersurfaces.

Dominating sprays are used by Gromov in essentially the same way as sprays of type (A) in Grauert's construction; however, the details are considerably more involved. Theorem 2.1 also holds for the ostensibly larger class of *subelliptic* manifolds: A complex manifold Y with a finite family of holomorphic sprays (E_j, π_j, s_j) which together dominate at every point $y \in Y$, meaning that T_yY is spanned by the vector subspaces $(ds_j)_{0_y}(E_{j,y})$. For example, if $A \subset \mathbb{C}\mathbb{P}^n$ is a projective subvariety of codimension > 1 , then $\mathbb{C}\mathbb{P}^n \setminus A$ is subelliptic, but is not known to be elliptic.

Gromov considered the Oka principle in the more general context of sections of holomorphic submersions over Stein manifolds. A surjective holomorphic submersion $\pi: Z \rightarrow X$ is said to be *elliptic* if each point $x_0 \in X$ admits an open neighborhood $U \subset X$ and a family of dominating sprays s_x on the fibers Z_x , depending holomorphically on the base point $x \in U$. Similarly one defines a *subelliptic submersion*. We introduce a *stratified (sub-) elliptic submersion* by asking that the base X , which may now be a complex space with singularities, is stratified by a descending chain of closed complex subvarieties $X = X_0 \supset X_1 \supset \dots \supset X_m = \emptyset$, with smooth differences $S_j = X_j \setminus X_{j+1}$, such that the restriction of π to any stratum (a connected component of a difference S_j) is a (sub-) elliptic submersion. Gromov's main theorem [9, Main Theorem 4.5] is included in the following result from [5].

Theorem 2.2. *If X is a Stein space and $\pi: Z \rightarrow X$ is a stratified (sub-) elliptic submersion, then section $X \rightarrow Z$ of π satisfy the Oka principle.*

Example 2.3. Let $\pi: E \rightarrow X$ be a holomorphic vector bundle of rank $n > 1$, and let $\Sigma \subset E$ be a complex subvariety with affine algebraic fibers $\Sigma_x = \Sigma \cap E_x \subset E_x \cong \mathbb{C}^n$ ($x \in X$) of codimension > 1 . If Σ is locally

uniformly tame (a condition concerning its behavior at infinity), then the restricted submersion $\pi: E \setminus \Sigma \rightarrow X$ is elliptic. If X is Stein, it follows that the Oka principle holds for sections $X \rightarrow E \setminus \Sigma$.

The Oka principle in the above example is used to construct proper holomorphic immersions and embeddings of Stein manifolds of dimension > 1 into Euclidean spaces of minimal dimension; see [1] and Chapter 8 in [6]. The problem of embedding open Riemann surfaces properly holomorphically into \mathbb{C}^2 is still very much open since the Oka principle does not apply in this case (see Sect. 8.9 in [6] for results in this direction).

An interesting recent application of Theorem 2.2 was found by Ivarsson and Kutzschebauch [10] who solved the following *Gromov-Vaserstein problem*: Every null-homotopic holomorphic map $X \rightarrow SL_n(\mathbb{C})$ from a finite dimensional Stein space X to a special linear group can be factored into a finite product of upper- and lower triangular holomorphic maps into $SL_n(\mathbb{C})$.

A detailed exposition of Theorems 2.1 and 2.2 can be found in [3, 4, 5], and also in Chapters 5 and 6 of [6].

3. FROM ELLIPTIC MANIFOLDS TO OKA MANIFOLDS AND OKA MAPS

Gromov asked in [9] whether the Oka principle for maps $X \rightarrow Y$ from Stein manifolds X to a given complex manifold Y could be characterized by a Runge approximation property for entire maps $\mathbb{C}^n \rightarrow Y$ from Euclidean spaces to Y . This conjecture was confirmed in 2006 by Forstnerič who showed that it suffices to ask for Runge approximation on a special class of compact (geometrically!) convex sets in Euclidean spaces. This condition, called CAP (the *Convex Approximation Property*), is equivalent to some dozen ostensibly different Oka properties; a complex manifold satisfying these equivalent properties is called an *Oka manifold*. (See [7] and Chapter 5 in [6] for more information.) The simple characterization of Oka manifolds by CAP paved the way to prove some functorial properties which are unknown in the class of elliptic manifolds. For example, if E and B are complex manifolds and $E \rightarrow B$ is a holomorphic fibre bundle whose fiber is an Oka manifold, then B is Oka if and only if E is Oka.

By Gromov's Theorem 2.1, every elliptic manifold is an Oka manifold. A partial converse, due to F. Lárusson, pertains to 'good' complex manifolds; this class includes all Stein manifolds and all quasi-projective manifolds. A good manifold Y is Oka if and only if there exists an affine holomorphic bundle $E \rightarrow Y$ whose total space E is Oka and Stein, hence elliptic.

Any natural property of objects in a given category should induce a corresponding property of morphisms. Following this philosophy, a holomorphic map $\pi: E \rightarrow B$ is said to be an *Oka map* if it is a Serre fibration and it enjoys the parametric Oka property. The latter is a parametric version of the basic Oka property of π which pertains to the possibility of deforming

any continuous π -lifting $F_0: X \rightarrow E$ of a given holomorphic map $f: X \rightarrow B$ from a Stein space X into a holomorphic lifting $F_1: X \rightarrow E$ of f .

Finnur Lárússon explained how Oka manifolds and Oka maps naturally fit into an abstract homotopy-theoretic framework. The category of complex manifolds and holomorphic maps can be embedded into a model category such that: (a) a holomorphic map is acyclic (as a map in the ambient model category) if and only if it is a homotopy equivalence in the usual topological sense; (b) a holomorphic map is a fibration if and only if it is an Oka map. In particular, a complex manifold is fibrant if and only if it is Oka; (c) a complex manifold is cofibrant if and only if it is Stein; (d) a Stein inclusion is a cofibration. (See [11] and Sect. 7 in [7] for more information.)

A central problem is to determine the place of Oka manifolds in the classification of complex manifolds. This is well understood only in dimension one: a Riemann surface is Oka if and only if it is not Kobayashi hyperbolic. In particular, the compact Riemann surfaces that are Oka are the Riemann sphere and all elliptic curves. Already for complex surfaces the problem is difficult and to a large extent open. Whether the Oka property is preserved by modifications such as blowing up and blowing down is a closely related problem. In particular, we do not know whether an Oka manifold of dimension > 1 blown up at a point, or punctured at one point, is still Oka.

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