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Singular homology for amateurs

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In the first half of the twentieth century, 'algebraic topology' was remarkably successful in answering questions about continuous functions between elementary structures (balls, spheres and so on) in finite-dimensional Euclidean space. No pure mathematician can fail to be struck by the theorems, but the proofs are at the limit of what can be expected of an undergraduate course, and many of us have never felt we had time to grasp them fully. One of the most powerful techniques developed then was a method of associating continuous functions with homomorphisms between 'homology groups'. In this note I describe a version of homology theory strong enough to give the hairy ball theorem and the Brouwer fixed-point theorem, among others (see Chapter 4). It is designed for relatively mature mathematicians who are happy with high levels of abstraction, but does not demand much specialized knowledge. I hope it will be accessible to graduate students with a solid base in undergraduate analysis and some algebra and set theory, but with no prior acquaintance with algebraic topology.

1 Simplicial complexes

I begin with a long section setting up the algebraic framework of the theory.

1A Definition An (abstract) simplicial complex is a non-empty finite family $\mathcal{K}$ of sets ('unoriented simplexes') such that $\mathcal{P}K \subseteq \mathcal{K}$ for every $K \in \mathcal{K}$, that is, $K' \in \mathcal{K}$ whenever $K' \subseteq K \in \mathcal{K}$. In this case, of course, $\emptyset \in \mathcal{K}$ and every member of $\mathcal{K}$ must be finite.

In this context, an **n-simplex**, where $n \in \mathbb{N}$, will be a member of $\mathcal{K}$ with $n+1$ members; a **vertex** of $\mathcal{K}$ is a an element v of $\bigcup\mathcal{K}$, so that $\{v\}$ is a 0-simplex; the set $\bigcup\mathcal{K}$ is the **vertex set** of $\mathcal{K}$; it must be finite.

If $\mathcal{K}$ and $\mathcal{L}$ are simplicial complexes with vertex sets V and W, a **simplicial map** from $\mathcal{K}$ to $\mathcal{L}$ is a function $h : V \to W$ such that $h[K] = \{h(v) : v \in K\}$ belongs to $\mathcal{L}$ for every $K \in \mathcal{K}$. Observe that the composition of two simplicial maps is simplicial, just because $(gh)[K] = g[h[K]]$ for all functions g and h and all sets K. We say that two simplicial maps f_0, $f_1 : V \to W$ are **contiguous** if $f_0[K] \cup f_1[K] \in \mathcal{L}$ for every $K \in \mathcal{K}$.

1B Permutations and signatures I rather suppose that most readers will be familiar with the following ideas, either from a course in elementary group theory or from a course in linear algebra. However I run through the essential concepts to make sure that my notation will give no surprises.

I will identify each non-negative integer with the set of its predecessors, so that $0 = \emptyset$, $1 = \{0\}$, $2 = \{0, 1\}$, etc.; thus any $n \in \mathbb{N}$ is actually a set with n members. A permutation of n is just a bijection from n to itself; the set S_n of such permutations, with the operation of composition (so that $(\rho\sigma)(i) = \rho(\sigma(i))$ for $i < n$) is a group. When $n = 0 = \emptyset$, the only function from n to itself is the empty function; $S_0 = \{\emptyset\}$.[1]

If $\rho \in S_n$, we have an equivalence relation on n defined by saying that $i \sim j$ if there is a $k \in \mathbb{N}$ such that $j = \rho^k(i)$; the equivalence classes under this relation are the **orbits** of ρ. We say that ρ is **even** or **odd** according to whether it has an even or odd number of orbits of even size. Thus the identity permutation ι, for which all the orbits are singletons, is even. The **signature** of ρ, ϵ_ρ, is 1 for even ρ and -1 for odd ρ.

Next, a **transposition** is a permutation which exchanges two points and leaves all others fixed; it has just one orbit of even size, and is an odd permutation. If τ is a transposition then $\tau^2 = \iota$.

It can be shown that if we regard ι as a product of 0 transpositions, then any $\rho \in S_n$ can be expressed as a product of transpositions, and ρ is even or odd according to whether it is expressible as a product of an

[1] Perhaps I should say here that I identify a function with its graph, so that a function f is a set of ordered pairs such that whenever (x, y) and (x, y') belong to f then $y = y'$. In particular, I count the empty set as a function, the unique function with domain $\emptyset$.

even or an odd number of transpositions. Consequently $\epsilon_{\rho\sigma} = \epsilon_\rho\epsilon_\sigma$ for all ρ, $\sigma \in S_n$. Observe that $\epsilon_\iota = 1$, $\epsilon_{\rho^{-1}} = \epsilon_\rho$ for every $\rho \in S_n$, and $\epsilon_\tau = -1$ for every transposition τ.

1C Chain groups (a) Let $\mathcal{K}$ be a simplicial complex, with vertex set V, and $n \ge -1$. An **n-chain** in $\mathcal{K}$ is a function $x : V^{n+1} \to \mathbb{Z}$ such that

—— whenever $p \in V^{n+1}$ and $\sigma \in S_{n+1}$ then $x(p\sigma) = \epsilon_\sigma x(p)$,

—— if $p \in V^{n+1}$ and $p[n+1] = \{p(0), \dots , p(n)\}$ does not belong to $\mathcal{K}$, then $x(p) = 0$.

(b)(i) I think of V^{n+1} as the set of functions from $n+1$ to V. So if $p \in V^{n+1}$ and $\sigma \in S_{n+1}$ the composition $p\sigma$ is a function from $n+1$ to V and belongs to V^{n+1}.

When $n = -1$ the only function from 0 to V is the empty set (whether or not V itself is empty), so a (-1)-chain is a function from $V^0 = \{\emptyset\}$ to $\mathbb{Z}$.

(ii) Write $C_n(\mathcal{K})$ for the set of all n-chains in $\mathcal{K}$. Evidently $C_n(\mathcal{K})$ is closed under addition and subtraction and contains the zero function, that is, is a subgroup of the product group $\mathbb{Z}^{V^{n+1}}$, so in itself is a commutative group; we call it the n-dimensional **chain group** of $\mathcal{K}$. Note that $C_{-1}(\mathcal{K}) \cong \mathbb{Z}$.

(iii) If $n \ge 1$, $x \in C_n(\mathcal{K})$, $p \in V^{n+1}$ and $p(0), \dots , p(n)$ are not all different, that is, p is not an injective function, then $x(p) = 0$. **P** If $i < j$ are such that $p(i) = p(j)$, let $\tau \in S_{n+1}$ be the transposition exchanging i and j; then $p\tau = p$ so $x(p) = x(p\tau) = \epsilon_\tau x(p) = -x(p)$. **Q**

(iv) The simplest type of non-zero chain is as follows. Take an injective $p \in V^{n+1}$ such that $p[n+1] \in \mathcal{K}$. Then we have a function $e_p : V^{n+1} \to \mathbb{Z}$ defined by saying that if $q \in V^{n+1}$ then

$$\begin{aligned} e_p(q) &= \epsilon_\sigma \text{ if } \sigma \in S_{n+1} \text{ and } q = p\sigma, \\ &= 0 \text{ if there is no such } \sigma. \end{aligned}$$

Now $e_p \in C_n(\mathcal{K})$. **P** If $q \in V^{n+1}$ and $\rho \in S_{n+1}$, then

$$\begin{aligned} e_p(q) \ne 0 &\Longrightarrow q = p\sigma \text{ for some } \sigma \in S_{n+1} \\ &\Longrightarrow q\rho = (p\sigma)\circ\rho = p\circ(\sigma\rho) \end{aligned}$$

(because composition of functions is associative)

$$\begin{aligned} &\Longrightarrow e_p(q\rho) = \epsilon_{\sigma\rho} = \epsilon_\sigma\epsilon_\rho = \epsilon_\rho e_p(q), \\ e_p(q\rho) \ne 0 &\Longrightarrow q\rho = p\sigma \text{ for some } \sigma \in S_{n+1} \\ &\Longrightarrow q = p\sigma\rho^{-1} \\ &\Longrightarrow e_p(q) \ne 0, \end{aligned}$$

so we always have $e_p(q\rho) = \epsilon_\rho e_p(q)$. **Q**

Moreover, we see that $e_p = \epsilon_\sigma e_{p\sigma}$ for every $\sigma \in S_{n+1}$. **P** For $q \in V^{n+1}$,

$$e_{p\sigma}(q) \ne 0 \iff q[n+1] = p\sigma[n+1] = p[n+1] \iff e_p(q) \ne 0,$$

and if $e_{p\sigma}(q) \ne 0$ then there is a $\rho \in S_{n+1}$ such that $q = p\sigma\rho$, in which case

$$e_p(q) = \epsilon_{\sigma\rho} = \epsilon_\sigma\epsilon_\rho = \epsilon_\sigma e_{p\sigma}(q). \text{ } \mathbf{Q}$$

(v) Note that if V is not empty, then all the sets V^n, for $n \ge 0$, are disjoint (starting from $V^0 = \{\emptyset\}$), so elements of different chain groups have different domains and must be different functions, and the chain groups are disjoint. If $V = \emptyset$, that is, $\mathcal{K} = \{\emptyset\}$, then $V^n = \{\emptyset\}$ for every $n \ge 0$ and the chain groups are actually identical, all being the set of functions from $\{\emptyset\}$ to $\mathbb{Z}$.

1D Simplicial maps and chain groups (a) Suppose that $\mathcal{K}$ and $\mathcal{L}$ are simplicial complexes, with vertex sets V and W, and $h : V \to W$ is a simplicial map from $\mathcal{K}$ to $\mathcal{L}$. Again regarding a member p of V^{n+1} as a function from $n+1$ to V, the composition hp belongs to W^{n+1} whenever $n \ge -1$ and $p \in V^{n+1}$.

If $n \ge -1$ and $x \in C_n(\mathcal{K})$, we can define $h\bullet x : W^{n+1} \to \mathbb{Z}$ by setting

$$(h\bullet x)(q) = \sum_{p\in V^{n+1}, hp=q} x(p)$$

for every $q \in W^{n+1}$. Now $h\bullet x \in C_n(\mathcal{L})$. **P** If $p \in V^{n+1}$ and $\sigma \in S_{n+1}$, $h(p\sigma) = (hp)\sigma$. Turning this round, if $p \in V^{n+1}$, $q \in W^{n+1}$ and $\sigma \in S_{n+1}$,

$$\begin{aligned} hp = q\sigma &\Longrightarrow hp\sigma^{-1} = q\sigma\sigma^{-1} = q \\ &\Longrightarrow hp = hp\sigma^{-1}\sigma = q\sigma. \end{aligned}$$

So if $q \in W^{n+1}$ and $\sigma \in S_{n+1}$,

$$\begin{aligned} (h\bullet x)(q\sigma) &= \sum_{\substack{p\in V^{n+1}\\ hp=q\sigma}} x(p) = \sum_{\substack{p\in V^{n+1}\\ hp\sigma^{-1}=q}} x(p) \\ &= \sum_{\substack{p\in V^{n+1}\\ hp\sigma^{-1}=q}} \epsilon_\sigma\epsilon_{\sigma^{-1}} x(p) = \sum_{\substack{p\in V^{n+1}\\ hp\sigma^{-1}=q}} \epsilon_\sigma x(p\sigma^{-1}) = \sum_{\substack{p\in V^{n+1}\\ hp=q}} \epsilon_\sigma x(p) \end{aligned}$$

(because $p \mapsto p\sigma^{-1} : V^{n+1} \to V^{n+1}$ is a bijection)

$$= \epsilon_\sigma \sum_{\substack{p\in V^{n+1}\\ hp=q}} x(p) = \epsilon_\sigma (h\bullet x)(q).$$

Next, if $p \in V^{n+1}$ and $x(p) \neq 0$, then $p[n+1] \in \mathcal{K}$ so $(hp)[n+1] = h[p[n+1]] \in \mathcal{L}$; turning this round, if $q \in W^{n+1}$ and $q[n+1] \notin \mathcal{L}$, then $x(p) = 0$ whenever $p \in V^{n+1}$ and $hp = q$, so $(h\bullet x)(q) = 0$. **Q**

(b) The formula for $h\bullet x$ now tells us at once that $h\bullet(x+y) = h\bullet x + h\bullet y$ for all x, $y \in C_n(\mathcal{K})$, that is, that $x \mapsto h\bullet x : C_n(\mathcal{K}) \to C_n(\mathcal{L})$ is a group homomorphism.

(c) If $\mathcal{K}$, $\mathcal{L}$ and $\mathcal{M}$ are simplicial complexes with vertex sets U, V and W, and $g : U \to V$, $h : V \to W$ are simplicial maps, then $(hg)\bullet x = h\bullet(g\bullet x)$ whenever $n \geq -1$ and $x \in C_n(\mathcal{K})$. **P** I noted in 1A that $hg : U \to W$ is a simplicial map. So $(hg)\bullet x$ is defined; and of course $h\bullet(g\bullet x)$ is defined because $g\bullet x \in C_n(\mathcal{L})$. To see that they are equal, take $r \in (\bigcup\mathcal{M})^{n+1}$ and compute

$$\begin{aligned} (h\bullet(g\bullet x))(r) &= \sum_{\substack{q\in(\bigcup\mathcal{L})^{n+1}\\ hq=r}} (g\bullet x)(q) = \sum_{\substack{q\in(\bigcup\mathcal{L})^{n+1}\\ hq=r}} \sum_{\substack{p\in(\bigcup\mathcal{K})^{n+1}\\ gp=q}} x(p) \\ &= \sum_{\substack{p\in(\bigcup\mathcal{K})^{n+1}\\ q\in(\bigcup\mathcal{L})^{n+1}\\ gp=q\\ hq=r}} x(p) = \sum_{\substack{p\in(\bigcup\mathcal{K})^{n+1}\\ hgp=r}} x(p) = ((hg)\bullet x)(r). \text{ } \mathbf{Q} \end{aligned}$$

(d) I ought to remark that $\bullet$ here is not exactly a function. In order to interpret $h\bullet x$ correctly we need to know not only what the sets h and x are, and what $\mathcal{K}$ and $\mathcal{L}$ are, but also which power W^{n+1} to take as the domain of $h\bullet x$. Ordinarily we expect to be able to determine this from the domain of x. But in the rather special case in which $\mathcal{K} = \{\emptyset\}$ and $V = \emptyset$, the powers V^{n+1} are all $\{\emptyset\}$, as noted in 1C(b-v). This case is exceptional in other ways, and it is tempting to disallow it from the beginning, by amending Definition 1A. But the principle of the theory here is that we have sequences of operators, each acting on a space at a particular level; and while it is customary, and declutters the page, to allow the levels to be understood in many formulae (even when they change in the middle of the formulae), I think it better to regard such expressions as $h\bullet$ here, and ∂ in the next paragraph, as abbreviations for sequences $\langle h\bullet_n\rangle_{n\geq-1}$, $\langle\partial_n\rangle_{n\geq0}$ rather than as simple functions of the type suggested by the grammar of their definitions.

1E The boundary operation We come now to the second central concept of the theory. Let $\mathcal{K}$ be a simplicial complex, with vertex set V, and $n \geq -1$. If $p \in V^{n+1}$ and $v \in V$, define $p^\frown\!<\!v\!> \in V^{n+2}$ by saying that

$$(p^\frown<v>)(i) = p(i) \text{ if } i \le n,$$
$$= v \text{ if } i = n+1.$$

(In the formalism I am using here, $p^\frown<v> = p \cup \{(n+1, v)\}$. When $n = -1$, so that $p = \emptyset \in V^0$, then $<v> = \emptyset^\frown<v> \in V^1$ is the one-term sequence taking the value v at 0.)

(a) We have a function $\partial : C_{n+1}(\mathcal{K}) \to C_n(\mathcal{K})$ defined by setting

$$\partial(x)(p) = \sum_{v \in V} x(p^\frown<v>)$$

for $x \in C_{n+1}(\mathcal{K})$ and $p \in V^{n+1}$, and $\partial(x) \in C_n(\mathcal{K})$ for every $x \in C_{n+1}(\mathcal{K})$. **P** If $\rho \in S_{n+1}$, write σ for the permutation of $n+2$ defined by setting

$$\sigma(n+1) = n+1, \quad \sigma(i) = \rho(i) \text{ for } i \le n.$$

Then $\epsilon_\sigma = \epsilon_\rho$ (since the orbits of σ are the orbits of ρ, with one singleton orbit added). Now if $p \in V^{n+1}$ and $v \in V$,

$$(p^\frown<v>)\sigma = (p\rho)^\frown<v>.$$

So if $x \in C_n(\mathcal{K})$,

$$\begin{aligned}\partial(x)(p\rho) &= \sum_{v \in V} x(p\rho^\frown<v>) = \sum_{v \in V} x((p^\frown<v>)\sigma)\\ &= \sum_{v \in V} \epsilon_\sigma x(p^\frown<v>) = \epsilon_\rho \sum_{v \in V} x(p^\frown<v>) = \epsilon_\rho \partial(x)(p).\end{aligned}$$

Next, if $p \in V^{n+1}$ and $p[n+1] \notin \mathcal{K}$ then $(p^\frown<v>)[n+2] = p[n+1] \cup \{v\} \notin \mathcal{K}$ for every $v \in V$ (because subsets of members of $\mathcal{K}$ belong to $\mathcal{K}$), so $x(p^\frown<v>) = 0$ for every v and $\partial(x)(p) = 0$. So both the conditions of the definition in 1Ca are satisfied and $\partial(x) \in C_n(\mathcal{K})$.

(b) It is now obvious that $\partial : C_{n+1}(\mathcal{K}) \to C_n(\mathcal{K})$ is a group homomorphism. An element of the form $\partial(x)$, for some $x \in C_{n+1}(\mathcal{K})$, is called a **boundary**; if $n \ge 0$, an $x \in C_n(\mathcal{K})$ such that $\partial(x) = 0$ is called a **cycle**. Being the range and kernel of group homomorphisms, the sets $Z_n(\mathcal{K})$ of cycles (for $n \ge 0$) and $B_n(\mathcal{K})$ of boundaries (for $n \ge -1$) are subgroups of $C_n(\mathcal{K})$.

(c) Once again, I am using ∂ as a generic name for any of a sequence of functions. The situation is not exactly the same as in 1D, because in the confusing case, when $V = \emptyset$, all the $C_n(\mathcal{K})$ are the same, and the formula $\partial(x)(p) = \sum_{v \in V} x(p^\frown<v>)$ always asks for the empty sum, which we interpret as 0 in the common value of $C_n(\mathcal{K})$.

1F The Boundary Theorem Let $\mathcal{K}$ be a simplicial complex, and $n \ge 1$. Then $\partial^2 x = 0$ for every $x \in C_n(\mathcal{K})$.

Remark A stricter notation might be to write $\partial_{n-1}\partial_n x = 0_{n-2}$, where 0_{n-2} is the zero of the group $C_{n-2}(\mathcal{K})$. I am sure we are all very well accustomed to the meaning of the symbol 0 being different at different points in a sentence, and I am asking for a similar flexibility for some of the basic symbols used in this note.

proof If $x \in C_n(\mathcal{K})$ and $p \in V^{n-1}$, where V is the vertex set of $\mathcal{K}$, then

$$\begin{aligned}\partial^2 x(p) = \partial\partial(x)(p) &= \sum_{v \in V} \partial(x)(p^\frown<v>)\\ &= \sum_{v \in V}\sum_{w \in V} x(p^\frown<v>^\frown<w>) = -\sum_{v,w \in V} x(p^\frown<w>^\frown<v>)\end{aligned}$$

(because the transposition on $n+1$ which exchanges n and $n-1$ has signature -1)

$$= -\sum_{w \in V}\sum_{v \in V} x(p^\frown<w>^\frown<v>) = -\partial^2 x(p),$$

so $\partial^2 x(p) \in \mathbb{Z}$ must be 0. As p are arbitrary, $\partial^2 x = 0$ in $C_{n-2}(\mathcal{K})$.

Remark I hope it is obvious that $p^\frown{<}v{>}^\frown{<}w{>}$ is to be interpreted as $(p^\frown{<}v{>})^\frown{<}w{>}$. You might wish to consider what could be meant by $p^\frown({<}v{>}^\frown{<}w{>})$, or indeed $p^\frown q$.

1G Corollary In the language of 1Eb, $B_n(\mathcal{K}) \subseteq Z_n(\mathcal{K})$ for every $n \in \mathbb{N}$.

1H Simplicial maps and the boundary operation Let $\mathcal{K}$ and $\mathcal{L}$ be simplicial complexes, with vertex sets V and W, and $h : V \to W$ a simplicial map. If $n \ge -1$, then

$$\partial(h\bullet x) = h\bullet\partial(x)$$

for every $x \in C_{n+1}(\mathcal{K})$.

Remark In the formula here I am abusing the symbol ∂ in another way; on the left, it means one of the boundary operators associated with the simplicial complex $\mathcal{L}$, while on the right it means the corresponding boundary operator associated with the complex $\mathcal{K}$. A fuller notation would be

$$\partial^{(\mathcal{L})}_{n+1}(h\bullet_{n+1} x) = h\bullet_n \partial^{(\mathcal{K})}_{n+1}(x) \text{ for every } x \in C_{n+1}(\mathcal{K}).$$

proof For any $q \in W^{n+1}$,

$$\partial(h\bullet x)(q) = \sum_{w\in W} (h\bullet x)(q^\frown{<}w{>}) = \sum_{w\in W} \sum_{\substack{p\in V^{n+2}\\ hp=q^\frown{<}w{>}}} x(p)$$

$$= \sum_{w\in W} \sum_{v\in V} \sum_{\substack{p\in V^{n+1}\\ h(p^\frown{<}v{>})=q^\frown{<}w{>}}} x(p^\frown{<}v{>})$$

(because $(v, p) \mapsto p^\frown{<}v{>}$ is a bijection between $V \times V^{n+1}$ and V^{n+2})

$$= \sum_{v\in V} \sum_{\substack{p\in V^{n+1}\\ hp=q}} x(p^\frown{<}v{>})$$

(because $h(p^\frown{<}v{>}) = hp^\frown{<}h(v){>}$ for all v and p)

$$= \sum_{\substack{p\in V^{n+1}\\ hp=q}} \sum_{v\in V} x(p^\frown{<}v{>}) = \sum_{\substack{p\in V^{n+1}\\ hp=q}} \partial(x)(p) = (h\bullet\partial(x))(q).$$

1I Corollary If $\mathcal{K}$ and $\mathcal{L}$ are simplicial complexes with vertex sets V and W, and $h : V \to W$ is a simplicial map, then

$$h\bullet x \in Z_n(\mathcal{L}) \text{ for every } x \in Z_n(\mathcal{K})$$

for every $n \in \mathbb{N}$, and

$$h\bullet x \in B_n(\mathcal{L}) \text{ for every } x \in B_n(\mathcal{K})$$

for every $n \ge -1$.

1J Homology groups We are ready for the third pillar of the theory. Let $\mathcal{K}$ be a simplicial complex and $n \in \mathbb{N}$. Then $Z_n(\mathcal{K})$ and $B_n(\mathcal{K})$ are subgroups of the commutative group $C_n(\mathcal{K})$, and $B_n(\mathcal{K}) \subseteq Z_n(\mathcal{K})$. So $B_n(\mathcal{K})$ is a normal subgroup of $Z_n(\mathcal{K})$ and we can form the quotient group $H_n(\mathcal{K})$, the n**th** (**reduced**) **homology group** of $\mathcal{K}$.

1K Theorem (a) Let $\mathcal{K}$ and $\mathcal{L}$ be simplicial complexes with vertex sets U and V, and $g : U \to V$ a simplicial map. Then for each $n \in \mathbb{N}$ we have a homomorphism $g_\star : H_n(\mathcal{K}) \to H_n(\mathcal{L})$ defined by setting $g_\star(x^\bullet) = (g\bullet x)^\bullet$ for every $x \in Z_n(\mathcal{K})$.

(b) If now $\mathcal{M}$ is another simplicial complex with vertex set W, and $h : V \to W$ is a simplicial map, then $(hg)_\star = h_\star g_\star : H_n(\mathcal{K}) \to H_n(\mathcal{M})$.

Remark Here, and elsewhere, I write '$x^\bullet$' to denote the equivalence class containing x, under a relation which is supposed to be understood. In the formula above, $x^\bullet$ refers to the relation $\{(x, y) : x,\ y \in C_n(\mathcal{K}),\ x - y \in B_n(\mathcal{K})\}$, so that $x^\bullet$ is the coset $x + B_n(\mathcal{K}) \subseteq Z_n(\mathcal{K})$, while $(g\bullet x)^\bullet = (g\bullet x) + B_n(\mathcal{L}) \subseteq Z_n(\mathcal{L})$.

Once again, I am using the formulae $g_\star$, $h_\star$ as generic names for members of a sequence of functions. So the formula '$(hg)_\star = h_\star g_\star$' should be interpreted as '$(hg)_{\star n} = h_{\star n} g_{\star n}$ for every $n \ge 0$', in which ordinary composition of functions appears on both sides of the equation.

proof (a) As noted in 1D, $x \mapsto g\bullet x : C_n(\mathcal{K}) \to C_n(\mathcal{L})$ is a homomorphism. Since $g\bullet x \in Z_n(\mathcal{L})$ whenever $x \in Z_n(\mathcal{K})$ (1I), we have a homomorphism $x \mapsto g\bullet x : Z_n(\mathcal{K}) \to Z_n(\mathcal{L})$, and therefore a homomorphism $x \mapsto (g\bullet x)^\bullet : Z_n(\mathcal{K}) \to H_n(\mathcal{L})$. Because $g\bullet x \in B_n(\mathcal{L})$ whenever $x \in B_n(\mathcal{K})$ (1I again), we see that the kernal of $x \mapsto (g\bullet x)^\bullet$ includes $B_n(\mathcal{K})$. By the First Isomorphism Theorem, we have a factorization through $H_n(\mathcal{K})$ defined by the formula $x^\bullet \mapsto (g\bullet x)^\bullet$, and we may call this $g_\star$.

(b) If $x \in Z_n(\mathcal{K})$ then, using 1Dc,

$$h_\star(g_\star(x^\bullet)) = h_\star((g\bullet x)^\bullet) = (h\bullet(g\bullet x))^\bullet = (hg\bullet x)^\bullet = (hg)_\star(x^\bullet).$$

1L Identities and automorphisms Let $\mathcal{K}$ be a simplicial complex with vertex set V. If $f(v) = v$ for every $v \in V$ then f is a simplicial map from $\mathcal{K}$ to itself (because $f[K] = K \in \mathcal{K}$ whenever $K \in \mathcal{K}$), $f\bullet x = x$ whenever $n \ge -1$ and $x \in C_n(\mathcal{K})$, and $f_\star : H_n(\mathcal{K}) \to H_n(\mathcal{K})$ is the identity homomorphism on $H_n(\mathcal{K})$ for every $n \in \mathbb{N}$.

If $g : V \to V$ is an automorphism of the structure $(V, \mathcal{K})$, that is, $\mathcal{K} = \{g[K] : K \in \mathcal{K}\}$ (or, if you prefer, $\mathcal{K} = \{g^{-1}[K] : K \in \mathcal{K}\}$), then $g_\star : H_n(\mathcal{K}) \to H_n(\mathcal{K})$ is an automorphism for every $n \in \mathbb{N}$ (because $g_\star(g^{-1})_\star = (g^{-1})_\star g_\star$ is the identity on $H_n(\mathcal{K})$).

1M Supports (a) Definition Let $\mathcal{K}$ be a simplicial complex with vertex set V. If $n \ge -1$ and $y \in C_n(\mathcal{K})$, I will say that the **support** of y is

$$\operatorname{supp}(y) = \bigcup\{q[n+1] : q \in V^{n+1},\ y(q) \ne 0\}.$$

I will say that a subset V' of V **supports** y if $\operatorname{supp}(y) \subseteq V'$.

(b) Note that the support of every member of $C_{-1}(\mathcal{K})$ is $\emptyset$, as $q[0] = \emptyset$ for every $q \in V^0$. If $y \in C_n(\mathcal{K})$, then $\operatorname{supp}(ry) = \operatorname{supp}(y)$ for every non-zero $r \in \mathbb{Z}$; if $y,\ y' \in C_n(\mathcal{K})$, then $\operatorname{supp}(y + y') \subseteq \operatorname{supp}(y) \cup \operatorname{supp}(y')$. If $y = 0$ in $C_n(\mathcal{K})$ then $\operatorname{supp}(y) = \emptyset$.

(c) If $n \ge 0$ and $y \in C_n(\mathcal{K})$ then $\operatorname{supp}\partial(y) \subseteq \operatorname{supp}(y)$. **P** If $q \in V^n$ and $\partial(y)(q) \ne 0$, there is a $v \in V$ such that $y(q^\frown{<}v{>}) \ne 0$, so $q[n] \subseteq (q^\frown{<}v{>})[n+1] \subseteq \operatorname{supp}(y)$; as q is arbitrary, $\operatorname{supp}\partial(y) \subseteq \operatorname{supp}(y)$. **Q**

(d) Now suppose that $\mathcal{L}$ is another simplicial complex, with vertex set W, and that $h : V \to W$ is a simplicial map from $\mathcal{K}$ to $\mathcal{L}$. If $x \in C_n(\mathcal{K})$ then $\operatorname{supp}(h\bullet x) \subseteq h[\operatorname{supp}(x)]$. **P** If $q \in W^{n+1}$ is such that $(h\bullet x)(q) \ne 0$, then there is a $p \in V^{n+1}$ such that $hp = q$ and $x(p) \ne 0$, in which case $q[n+1] = h[p[n+1]] \subseteq h[\operatorname{supp}(x)]$. As q is arbitrary, $\operatorname{supp}(h\bullet x) \subseteq h[\operatorname{supp}(x)]$. **Q**

1N Lemma Let $\mathcal{K}$ be a simplicial complex with vertex set V. Take $v \in V$, $n \ge -1$ and $y \in C_n(\mathcal{K})$. Then there can be at most one $x \in C_{n+1}(\mathcal{K})$ such that

$$y(q) = x(q^\frown{<}v{>}) \text{ whenever } q \in V^{n+1},$$

$$v \in p[n+2] \text{ whenever } p \in V^{n+2} \text{ and } x(p) \ne 0.$$

proof Suppose that $x,\ x' \in C_{n+2}(\mathcal{K})$ both have these two properties. Let $p \in V^{n+2}$ be such that $x(p) \ne 0$. Then $v \in p[n+2]$; say $v = p(j)$ where $j \le n+1$. Let $\sigma \in S_{n+1}$ be such that $\sigma(n+1) = j$. Then $v = p\sigma(n+1)$ so $p\sigma = q^\frown{<}v{>}$ where $q(i) = p\sigma(i)$ for $i \le n$. Now

$$\begin{aligned}\epsilon_\sigma x'(p) = x'(p\sigma) = x'(q^\frown{<}v{>}) &= 0 = x(q^\frown{<}v{>}) = x(p\sigma) = \epsilon_\sigma x(p) \text{ if } v \in q[n+1]\\ &= y(q) = x(p\sigma) = \epsilon_\sigma x(p) \text{ otherwise,}\end{aligned}$$

and $x'(p) = x(p)$. Similarly, $x(p) = x'(p)$ if $x'(p) \ne 0$. Thus in fact $x'(p) = x(p)$ for every $p \in V^{n+2}$ and $x = x'$, as claimed.

1O Definition Let $\mathcal{K}$ be a simplicial complex with vertex set V. Take $v \in V$, $n \ge -1$ and $y \in C_n(\mathcal{K})$. I will say that $y * v$ is defined, and equal to x, if $x \in C_{n+1}(\mathcal{K})$ and

$$y(q) = x(q^\frown\!<\!v\!>) \text{ whenever } q \in V^{n+1},$$

$$v \in p[n+2] \text{ whenever } p \in V^{n+2} \text{ and } x(p) \ne 0.$$

Remarks (a) By 1N, given v and y, there will be at most one x satisfying these requirements.

(b) If $v \in V$ and y, $y' \in C_n(\mathcal{K})$ are such that $y * v$ and $y * v'$ are both defined, then $(y + y') * v$ is defined and equal to $y * v + y * v'$. **P** We have

$$\begin{aligned}(y + y')(q) = y(q) + y'(q) &= (y * v)(q^\frown\!<\!v\!>) + (y * v')(q^\frown\!<\!v\!>)\\ &= (y * v + y * v')(q^\frown\!<\!v\!>)\end{aligned}$$

for every $q \in V^{n+1}$. If $p \in V^{n+2}$ is such that $(y * v + y * v')(p) \ne 0$ then at least one of $(y * v)(p)$, $(y * v')(p)$ is non-zero; in either case $v \in p[n+2]$. So $y * v + y * v'$ has the required properties. **Q**

1P Proposition Let $\mathcal{K}$ be a simplicial complex with vertex set V. Take $v \in V$.

(a) Suppose that $n \ge -1$ and $y \in C_n(\mathcal{K})$.

(i) $y * v$ is defined in $C_{n+1}(\mathcal{K})$ iff $v \notin \operatorname{supp}(y)$ and $\{v\} \cup q[n+1] \in \mathcal{K}$ whenever $q \in V^{n+1}$ and $y(q) \ne 0$.

(ii) In this case, $\{p[n+2] : p \in V^{n+2},\ (y * v)(p) \ne 0\} = \{\{v\} \cup q[n+1] : q \in V^{n+1},\ y(q) \ne 0\}$. So $\operatorname{supp}(y * v) = \operatorname{supp}(y) \cup \{v\}$ if $y \ne 0$.

(b) Suppose that $n \ge 0$ and $y \in C_n(\mathcal{K})$ are such that $y * v$ is defined in $C_{n+1}(\mathcal{K})$. Then $\partial(y) * v$ is defined and $\partial(y * v) = y - \partial(y) * v$.

(c) Suppose that $n \ge 0$ and $x \in C_n(\mathcal{K})$.

(i) x is uniquely expressible as $y * v + z$ where $y \in C_{n-1}(\mathcal{K})$, $z \in C_n(\mathcal{K})$ and $v \notin \operatorname{supp}(y) \cup \operatorname{supp}(z)$.

(ii) If x is a cycle then $y = -\partial(z)$ is a boundary.

(d) Now suppose that $\mathcal{L}$ is another simplicial complex with vertex set W, and that $h : V \to W$ is an injective simplicial map. If $n \ge -1$ and $y \in C_n(\mathcal{K})$ is such that $y * v$ is defined, then $(h\bullet y) * v$ is defined and equal to $h\bullet(y * v)$.

proof (a)(i)(α) If $x = y * v$ is defined, and $q \in V^{n+1}$ is such that $y(q) \ne 0$, then $x(q^\frown\!<\!v\!>) \ne 0$, so $\{v\} \cup q[n+1] = (q^\frown\!<\!v\!>)[n+2]$ belongs to $\mathcal{K}$; and as $q^\frown\!<\!v\!>$ must be injective, v cannot be a value of q. As q is arbitrary, $v \notin \operatorname{supp}(y)$.

(β) Now suppose that y satisfies the conditions. If $p \in V^{n+2}$ and σ, $\sigma' \in S_{n+2}$ and q, $q' \in V^{n+1}$ are such that $p\sigma = q^\frown\!<\!v\!>$ and $p\sigma' = q'^\frown\!<\!v\!>$ then $\epsilon_\sigma y(q) = \epsilon_{\sigma'} y(q')$. **P** If $y(q) = y(q') = 0$ this is trivial. If $y(q) \ne 0$ then v is not a value of q, while q is injective, so $p\sigma$ is injective; consequently p is injective. Since $p\sigma(n+1) = v = p\sigma'(n+1)$, $\sigma(n+1) = \sigma'(n+1)$ and $\sigma^{-1}\sigma'(n+1) = n+1$. We therefore have a permutation $\tau \in S_{n+1}$ given by setting $\tau(i) = \sigma^{-1}\sigma'(i)$ for $i \le n$, and $\epsilon_\tau = \epsilon_{\sigma^{-1}\sigma'}$ because the orbits of $\sigma^{-1}\sigma'$ are the orbits of τ with an extra singleton orbit $\{n+1\}$. Next, $q'(i) = p\sigma'(i) = p\sigma(\tau(i)) = q(\tau(i))$ for $i \le n$, that is, $q' = q\tau$. Accordingly

$$\epsilon_{\sigma'} y(q') = \epsilon_{\sigma'}\epsilon_\tau y(q) = \epsilon_{\sigma'}\epsilon_{\sigma^{-1}}\epsilon_{\sigma'} y(q) = \epsilon_\sigma y(q).$$

The same argument applies if $y(q') \ne 0$, so in all cases we have $\epsilon_\sigma y(q) = \epsilon_{\sigma'} y(q')$. **Q**

Accordingly we can define $x : V^{n+2} \to \mathbb{Z}$ by setting

$$\begin{aligned}x(p) &= \epsilon_\sigma y(q) \text{ if } \sigma \in S_{n+2},\ q \in V^{n+1} \text{ are such that } p\sigma = q^\frown\!<\!v\!>,\\ &= 0 \text{ if there is no such pair } \sigma,\ q.\end{aligned}$$

Observe that if $x(p) \ne 0$ the first clause in this definition must have applied, in which case $y(q) \ne 0$ and $p[n+2] = p\sigma[n+2] = \{v\} \cup q[n+1]$ belongs to $\mathcal{K}$; note that we also have $v \in p[n+2]$.

If $\tau \in S_{n+2}$ and $p \in V^{n+2}$ then $x(p\tau) = \epsilon_\tau x(p)$. **P** If $x(p) \neq 0$ then we have σ, q such that $p\sigma = q^\frown{<}v{>}$. Now $p\tau(\tau^{-1}\sigma) = q^\frown{<}v{>}$ so

$$x(p\tau) = \epsilon_{\tau^{-1}\sigma} y(q) = \epsilon_{\tau^{-1}}\epsilon_\sigma y(q) = \epsilon_\tau x(p).$$

Similarly, if $x(p\tau) \neq 0$ then $x(p) = x(p\tau\tau^{-1}) = \epsilon_{\tau^{-1}} x(p\tau)$ and again $x(p\tau) = \epsilon_\tau x(p)$. And of course $x(p\tau) = \epsilon_\tau x(p)$ if both $x(p)$ and $x(p\tau)$ are 0. **Q**

Thus $x \in C_{n+1}(\mathcal{K})$. We have already seen that if $x(p) \neq 0$ then v is a value of p. Finally, if $q \in V^{n+1}$ and $x = y^\frown{<}v{>}$ then $x(p) = \epsilon_\iota y(q) = y(q)$. So x satisfies the requirements of Definition 1O and $y * v = x$ is defined.

(ii) In the course of the construction in (i-β) above, I showed that if $p \in V^{n+2}$ and $(y*v)(p) \neq 0$ then $p[n+2] = \{v\} \cup q[n+1]$ for some $q \in V^{n+1}$ such that $y(q) = (y*v)(p) \neq 0$. Conversely, if $q \in V^{n+1}$ and $y(q) \neq 0$ then $(y*v)(p) \neq 0$ where $p = q^\frown{<}v{>}$ and $\{v\} \cup q[n+1] = p[n+2]$. Accordingly, if $y \neq 0$,

$$\operatorname{supp}(y*v) = \bigcup_{\substack{p \in V^{n+2} \\ (y*v)(p) \neq 0}} p[n+2] = \bigcup_{\substack{q \in V^{n+1} \\ y(q) \neq 0}} q[n+1] \cup \{v\} = \operatorname{supp}(y) \cup \{v\}.$$

(b) Set $x = y * v \in C_{n+1}(\mathcal{K})$, $z = y - \partial(x) \in C_n(\mathcal{K})$. If $q \in V^{n+1}$,

$$z(q) = y(q) - \textstyle\sum_{w \in V} x(q^\frown{<}w{>}) = -\sum_{w \in V, w \neq v} x(q^\frown{<}w{>}).$$

Now if $z(q) \neq 0$ there must be a $w \neq v$ such that $x(q^\frown{<}w{>}) \neq 0$ in which case v is a value of $q^\frown{<}w{>}$ and therefore of q. Next, if $r \in V^n$ and $v \notin r[n]$,

$$z(r^\frown{<}v{>}) = -\sum_{\substack{w \in V \\ w \neq v}} x(r^\frown{<}v{>}^\frown{<}w{>}) = -\sum_{w \in V} x(r^\frown{<}v{>}^\frown{<}w{>})$$

(because $x(r^\frown{<}v{>}^\frown{<}v{>})$ is surely 0)

$$= \sum_{w \in V} x(r^\frown{<}w{>}^\frown{<}v{>})$$

(because $r^\frown{<}v{>}^\frown{<}w{>} = (r^\frown{<}w{>}^\frown{<}v{>})\tau$ where $\tau \in S_{n+2}$ is the transposition exchanging n and $n+1$)

$$= \sum_{w \in V} y(r^\frown{<}w{>}) = \partial(y)(r).$$

So we see that $\partial(y) * v$ is defined and equal to z, and $\partial(x) = y - z = y - \partial(y) * v$.

(c)(i)($\boldsymbol{\alpha}$) Set $P_1 = \{p : p \in V^{n+1},\ v \in p[n+1]\}$, $P_0 = V^{n+1} \setminus P_1$. If $p \in P_1$ and $\sigma \in S_{n+1}$ then $v \in p[n+1] = p[\sigma[n+1]] = (p\sigma)[n+1]$ and $p\sigma \in P_1$. Similarly, $p\sigma \in P_0$ whenever $p \in P_0$ and $\sigma \in S_{n+1}$. So if we define x', $z : V^{n+1} \to \mathbb{Z}$ by setting

$$x'(p) = 0 \text{ and } z(p) = x(p) \text{ for every } p \in P_0,$$

$$x'(p) = x(p) \text{ and } z(p) = 0 \text{ for every } p \in P_1,$$

both x' and z will belong to $C_n(\mathcal{K})$, $v \notin \operatorname{supp}(z)$ and $x = x' + z$. Next, $v \in p[n+1]$ whenever $x'(p) \neq 0$. Define $y : V^n \to \mathbb{Z}$ by setting $y(q) = x'(q^\frown{<}v{>})$ for every $q \in V^n$. Then $y \in C_{n-1}(\mathcal{K})$. **P** If $q \in V^n$ and $\sigma \in S_n$ define $\sigma' \in S_{n+1}$ by setting $\sigma'(n) = n$ and $\sigma'(i) = \sigma'(i)$ for $i < n$. Then

$$\begin{aligned} y(q\sigma) &= x'(q\sigma^\frown{<}v{>}) = x'((q^\frown{<}v{>})\sigma') \\ &= \epsilon_{\sigma'} y(q) = \epsilon_\sigma y(q) \end{aligned}$$

because the orbits of σ' match the orbits of σ with one extra orbit of size 1. So q has the first property required in Definition 1Ca. As for the other, if $y(q) \neq 0$ then $x(q^\frown{<}v{>}) \neq 0$ so $\mathcal{K}$ contains $(q^\frown{<}v{>})[n+1] = \{v\} \cup q[n]$ and $q[n] \in \mathcal{K}$. **Q**

Since $v \in p[n+1]$ whenever $p \in V^{n+1}$ and $x'(p) \neq 0$, $y * v$ is defined and equal to x', and $x = y * v + z$. By (a-i) above, $v \notin \operatorname{supp}(y)$, so $v \notin \operatorname{supp}(y) \cup \operatorname{supp}(z)$.

(β) If we have an alternative expression $x = y' * v + z'$ where $v \notin \operatorname{supp}(y') \cup \operatorname{supp}(z')$, then observe that if $p \in P_0$ we must have $(y' * v)(p) = 0$ and $z'(p) = x(p) = z(p)$, while if $p \in P_1$ we must have $z'(p) = 0 = z(p)$. So $z' = z$ and $y * v = y' * v$. At the same time we have

$$y'(q) = (y' * v)(q^\frown\!<\!v\!>) = (y * v)(q^\frown\!<\!v\!>) = y(q)$$

for every $q \in V^n$ and $y' = y$. Thus the expression is unique, as claimed.

(ii) If x is a cycle, then

$$0 = \partial(x) = \partial(y * v) + \partial(z) = y - \partial(y) * v + \partial(z)$$

by (b). Take any $q \in V^n$. We know that $v \notin \operatorname{supp}(y) \cup \operatorname{supp}\partial(z)$ (1Pc). So

$$\partial(y)(q) = (\partial(y) * v)(q^\frown\!<\!v\!>) = (y + \partial(z))(q^\frown\!<\!v\!>) = 0.$$

As q is arbitrary, $\partial(y) = 0$ and $y = -\partial(z)$ is a boundary.

(d) Write x for $y * v$. If $q' \in W^{n+1}$ then

$$(h\bullet y)(q') = \sum_{\substack{p' \in V^{n+1} \\ hp' = q'}} y(p') = \sum_{\substack{p' \in V^{n+1} \\ hp' = q'}} x(p'^\frown\!<\!v\!>) = \sum_{\substack{p \in V^{n+2} \\ hp = q'^\frown<h(v)>}} x(p)$$

(because h is injective, so if $hp = q'^\frown\!<\!h(v)\!>$ then $hp(n+1) = h(v)$ and $p(n+1) = v$)

$$= (h\bullet x)(q'^\frown\!<\!h(v)\!>).$$

Moreover, if $(h\bullet x)(q) \neq 0$ there is a p such that $hp = q$ and $x(p) \neq 0$, in which case $v \in p[n+2]$ and $h(v) \in h[p[n+2]] = q[n+2]$. So $h\bullet(y * v) = h\bullet x$ satisfies both conditions in 1O and is equal to $(h\bullet y) * v$.

1Q The Contiguity Theorem Let $\mathcal{K}$ and $\mathcal{L}$ be simplicial complexes, with vertex sets V and W. Suppose that g, $h : V \to W$ are contiguous simplicial maps from $\mathcal{K}$ to $\mathcal{L}$. Then $g_\star : H_n(\mathcal{K}) \to H_n(\mathcal{L})$ and $h_\star : H_n(\mathcal{K}) \to H_n(\mathcal{L})$ coincide for every $n \in \mathbb{N}$.

proof (a) Consider first the case in which g and h are both injective and are equal except at just one point v of V. We need to show that $(g\bullet x)^\bullet = (h\bullet x)^\bullet$, that is, that $g\bullet x - h\bullet x$ is a boundary, for every cycle x.

Take $x \in Z_n(\mathcal{K})$. If $v \notin \operatorname{supp}(x)$ then $gp = hp$ whenever $x(p) \neq 0$ so

$$(g\bullet x)(q) = \textstyle\sum_{gp=q} x(p) = \sum_{hp=q} x(p) = (h\bullet x)(q)$$

for every $q \in W^{n+1}$ and $g\bullet x$ is actually equal to $h\bullet x$, so we can stop. Otherwise, x is expressible as $z - \partial(z) * v$ for some $z \in C_n(\mathcal{K})$ such that $v \notin \operatorname{supp}(z)$, by 1Pc. Now $v \notin \operatorname{supp}\partial(z)$ (1Mc) so we have $g\bullet z = h\bullet z$ and $g\bullet\partial(z) = h\bullet\partial(z)$ and

$$g\bullet x - h\bullet x = -g\bullet(\partial(z) * v) + h\bullet(\partial(z) * v) = -g\bullet(\partial(z)) * g(v) + h\bullet(\partial(z)) * h(v)$$

(1Pd)

$$= y * h(v) - y * g(v)$$

where $y = g\bullet(\partial(z)) \in C_{n-1}(\mathcal{L})$. Note that $\partial(y) = g\bullet(\partial^2 z) = 0$ by 1H. We are supposing that $g(v)$ and $h(v)$ are distinct. Next, we can form $w = (y * g(v)) * h(v) \in C_{n+1}(\mathcal{L})$. **P** Because g and h are injective, neither $g(v)$ nor $h(v)$ belongs to

$$h[\operatorname{supp}(z)] = g[\operatorname{supp}(z)] \supseteq g[\operatorname{supp}\partial(z)] \supseteq \operatorname{supp}(y).$$

If $q \in W^n$ is such that $y(q) \neq 0$, there is a $p \in V^n$ such that $gp = q$ and $\partial(z)(p) \neq 0$, in which case $\{v\} \cup p[n] \in \mathcal{K}$ so $\mathcal{L}$ contains

$$g[\{v\} \cup p[n]] \cup h[\{v\} \cup p[n]] = \{g(v)\} \cup \{h(v)\} \cup q[n].$$

In particular, we see that $\{g(v)\}\cup q[n]\in\mathcal{L}$ whenever $y(q)\ne 0$, while $g(v)\notin\operatorname{supp}(y)$, so $y*g(v)$ is defined in $C_n(\mathcal{L})$, by 1P(a-i). Next, $\operatorname{supp}(y*g(v))=\operatorname{supp}(y)\cup\{g(v)\}$ does not contain $h(v)$, and if $(y*g(v))(p)\ne 0$, there is a q such that $y(q)\ne 0$ and $p[n+1]=q[n]\cup\{g(v)\}\in\mathcal{L}$, so that $p[n]\cup\{h(v)\}=q(n)\cup\{g(v),h(v)\}\in\mathcal{L}$; now we can apply 1P(a-i) again to see that $w=(y*g(v))*h(v)$ is defined in $C_{n+1}(\mathcal{L})$. **Q**

Now we have

$$\begin{aligned}\partial(w)&=y*g(v)-\partial(y*g(v))*h(v)\\&=y*g(v)-y*h(v)+(\partial(y)*g(v))*h(v)=y*g(v)-y*h(v)\end{aligned}$$

so

$$g\bullet x-h\bullet x=y*h(v)-y*g(v)=-\partial(w)$$

is a boundary, as required.

(b) Secondly, consider the case in which g and h are equal except at just one point v of V, but are no longer required to be injective. Set $\mathcal{M}=\bigcup_{K\in\mathcal{K},L\in\mathcal{L}}\mathcal{P}(K\times L)$. Then $\mathcal{M}$ is a simplicial complex and its vertex set is $V\times W$. Define $\tilde g$, $\tilde h:V\to V\times W$ by setting

$$\tilde g(v')=(v',g(v')),\qquad \tilde h(v')=(v',h(v'))$$

for $v'\in V$. Then both $\tilde g$ and $\tilde h$ are injective, and they agree except at v. If $K\in\mathcal{K}$, then

$$\tilde g[K]\cup\tilde h[K]\subseteq(K\times g[K])\cup(K\times h[K])=K\times(g[K]\cup h[K])\in\mathcal{M},$$

so $\tilde g$ and $\tilde h$ are simplicial maps and satisfy the conditions of this theorem. By (a), $\tilde g_\star:H_n(\mathcal{K})\to H_n(\mathcal{M})$ and $\tilde h_\star:H_n(\mathcal{K})\to H_n(\mathcal{M})$ are equal for every $n\ge -1$. Next, define $f:V\times W\to W$ by setting $f(v,w)=w$ for $v\in V$ and $w\in W$. If $M\in\mathcal{M}$ there are $K\in\mathcal{K}$, $L\in\mathcal{L}$ such that $M\subseteq K\times L$ so $f[M]\subseteq L\in\mathcal{L}$. Accordingly f is a simplicial map. We now see that $g=f\tilde g$ and $h=f\tilde h$. So

$$g_\star=f_\star\tilde g_\star=f_\star\tilde h_\star=h_\star:H_n(\mathcal{K})\to H_n(\mathcal{L})$$

for every n, and we have the result in this case also.

(c) For the general case, given $n\in\mathbb{N}$ and $x\in Z_n(\mathcal{K})$, set

$$J_x=\{v:v\in\operatorname{supp}(x),\ g(v)\ne h(v)\}.$$

I seek to show by induction on $\#(J_x)$ that $(g\bullet x)^\bullet=(h\bullet x)^\bullet$. If J_x is empty then $gp=hp$ whenever $p\in V^{n+1}$ and $x(p)\ne 0$, so

$$(g\bullet x)(q)=\sum_{\substack{p\in V^{n+1}\\gp=q}}x(p)=\sum_{\substack{p\in V^{n+1}\\x(p)\ne0\\gp=q}}x(p)=\sum_{\substack{p\in V^{n+1}\\x(p)\ne0\\hp=q}}x(p)=(h\bullet x)(q)$$

for every $q\in W^{n+1}$, and $g\bullet x=h\bullet x$. For the inductive step to $\#(J_x)=k+1$, take any $v\in J_x$. Define $g_1:V\to W$ by setting $g(v)=h(v)$, $g_1(v')=g(v')$ for $v\in V\setminus\{v\}$. Then g_1 differs from g at v and nowhere else. Moreover, if $K\in\mathcal{K}$, then $g_1[K]\cup g[K]\cup h[K]=g[K]\cup h[K]$ belongs to $\mathcal{K}$. By (b), $g_{1\star}=g_\star:H_n(\mathcal{K})\to H_n(\mathcal{L})$ and $(g_1\bullet x)^\bullet=(g\bullet x)^\bullet$. Next,

$$J'_x=\{v':v'\in\operatorname{supp}(x),\ g_1(v')\ne h(v')\}=J_x\setminus\{v\}$$

and $\#(J'_x)=k$. By the inductive hypothesis, $(g_1\bullet x)^\bullet=(h\bullet x)^\bullet$, so $(g\bullet x)^\bullet=(h\bullet x)^\bullet$ and the induction proceeds.

Thus we see that $(g\bullet x)^\bullet=(h\bullet x)^\bullet$ for every $x\in Z_n(\mathcal{K})$ and $g_\star$ and $h_\star$ agree on $H_n(\mathcal{K})$, as claimed.

1R Elementary subdivisions Let $\mathcal{K}$ be a simplicial complex with vertex set V. An **elementary subdivision** of $\mathcal{K}$ is a simplicial complex $\mathcal{L}$ constructed by the following method. Take an edge (that is, a 1-simplex) $I=\{v_0,v_1\}$ of $\mathcal{K}$ and an object $\hat v$ not in V. Set

$$\begin{aligned}\mathcal{L} = \{K : K \in \mathcal{K}, I \not\subseteq K\} \\ \cup \bigcup_{I \subseteq K \in \mathcal{K}} \{(K \setminus \{v_0\}) \cup \{\hat{v}\}, (K \setminus \{v_1\}) \cup \{\hat{v}\}, (K \setminus I) \cup \{\hat{v}\}\} \\ = \{K : K \in \mathcal{K}, I \not\subseteq K\} \cup \{(K \setminus J) \cup \{\hat{v}\} : I \subseteq K \in \mathcal{K}, \emptyset \ne J \subseteq I\}.\end{aligned}$$

Then $\mathcal{L}$ is a simplicial complex. **P** Of course $\mathcal{L}$ is finite. Suppose that $w \in L \in \mathcal{L}$; set $L' = L \setminus \{w\}$. If $L \in \mathcal{K}$ and $I \not\subseteq K$ then $L' \in \mathcal{K}$ and $I \not\subseteq L'$ so $L' \in \mathcal{L}$. If $L = (K \setminus J) \cup \{\hat{v}\}$ where $I \subseteq K \in \mathcal{K}$ and $\emptyset \ne J \subseteq I$, then

— if $w \in I$ then $\emptyset \ne J \cup \{w\} \subseteq I$ and $L' = (K \setminus (J \cup \{w\})) \cup \{\hat{v}\} \in \mathcal{K}$,
— if $w \in K \setminus I$ then $I \subseteq K \setminus \{w\} \in \mathcal{K}$ and $L' = ((K \setminus \{w\}) \setminus J) \cup \{\hat{v}\} \in \mathcal{L}$,
— if $w = \hat{v}$ then $I \not\subseteq K \setminus J \in \mathcal{K}$ and $L' = K \setminus J \in \mathcal{L}$.

This shows that $L \setminus \{w\} \in \mathcal{L}$ whenever $w \in L \in \mathcal{L}$; as every member of $\mathcal{L}$ is finite, it follows by induction on $\#(L \setminus L')$ that $L' \in \mathcal{L}$ whenever $L' \subseteq L \in \mathcal{L}$, and $\mathcal{L}$ is a simplicial complex. **Q**

Note that the vertex set of $\mathcal{L}$ is $V \cup \{\hat{v}\}$, that $\{v_0, v_1\} \notin \mathcal{L}$ and that $L \in \mathcal{K}$ whenever $L \in \mathcal{L}$ and $\hat{v} \notin L$.

1S The Subdivision Theorem Let $\mathcal{K}$ be a simplicial complex with a doubleton set $\{v_0, v_1\} \in \mathcal{K}$, $\hat{v}$ an object not in the vertex set V of $\mathcal{K}$, and $\mathcal{L}$ the corresponding elementary subdivision of $\mathcal{K}$ as constructed in 1R. Write W for $V \cup \{\hat{v}\}$, the vertex set of $\mathcal{L}$. Define $g_0 : W \to V$ by setting $g_0(\hat{v}) = v_0$ and $g_0(v) = v$ for $v \in V$. Then $g_{0\star} : H_n(\mathcal{L}) \to H_n(\mathcal{K})$ is an isomorphism for every $n \in \mathbb{N}$.

proof (a) We had better check that g_0 is a simplicial map from $\mathcal{L}$ to $\mathcal{K}$; but this is elementary in view of the definition of $\mathcal{L}$. Next, let $g_1 : W \to V$ be the companion simplicial map defined by setting $g_1(\hat{v}) = v_1$ and $g_1(v) = v$ for $v \in V$. I seek to define homomorphisms $\pi : C_n(\mathcal{K}) \to C_n(\mathcal{L})$ to provide inverses of the functions $g_{0\star} : H_n(\mathcal{L}) \to H_n(\mathcal{K})$. Given $n \ge -1$, $x \in C_n(\mathcal{K})$ and $q \in W^{n+1}$, then define

$$\begin{aligned}\pi(x)(q) &= x(q) \text{ if } q[n+1] \in \mathcal{L} \text{ and } \hat{v} \notin q[n+1], \\ &= x(g_0 q) \text{ if } q \text{ is injective and } \{\hat{v}, v_1\} \subseteq q[n+1] \in \mathcal{L}, \\ &= x(g_1 q) \text{ if } q \text{ is injective and } \{\hat{v}, v_0\} \subseteq q[n+1] \in \mathcal{L}, \\ &= 0 \text{ otherwise.}\end{aligned}$$

There are some consistency checks to be made here, of course. As $\{v_0, v_1\} \notin \mathcal{L}$ we cannot simultaneously have $\{\hat{v}, v_0\} \subseteq q[n+1] \in \mathcal{L}$ and $\{\hat{v}, v_1\} \subseteq q[n+1] \in \mathcal{L}$, so the formula defines $\pi(x)(q)$ unambiguously. And if $\hat{v} \notin q[n+1]$ and $q[n+1] \notin \mathcal{L}$ then $q[n+1] \notin \mathcal{K}$ and $x(q) = 0$, while of course $x(q) = 0$ if q is not injective; so $\pi(x)(q) = 0$ if either q is not injective or $q[n+1] \notin \mathcal{L}$. As for $q\sigma$, where $\sigma \in S_{n+1}$, $q\sigma[n+1] = q[n+1]$, so we shall have used the same formula to calculate $\pi(x)(q)$ and $\pi(x)(q\sigma)$, and either

$$\pi(x)(q\sigma) = x(q\sigma) = \epsilon_\sigma x(q) = \epsilon_\sigma \pi(x)(q)$$

or

$$\pi(x)(q\sigma) = x(g_0 q\sigma) = \epsilon_\sigma x(g_0 q) = \epsilon_\sigma \pi(x)(q)$$

or

$$\pi(x)(q\sigma) = x(g_1 q\sigma) = \epsilon_\sigma x(g_1 q) = \epsilon_\sigma \pi(x)(q)$$

or

$$\pi(x)(q\sigma) = 0 = \epsilon_\sigma \pi(x)(q).$$

Thus $\pi(x) \in C_n(\mathcal{L})$ whenever $x \in C_n(\mathcal{K})$. Note also that as $x \mapsto \pi(x)(q) : C_n(\mathcal{K}) \to \mathbb{Z}$ is a group homomorphism for every $q \in W^{n+1}$, $\pi : C_n(\mathcal{K}) \to C_n(\mathcal{L})$ is a homomorphism.

(b) The key fact is that, for any $n \ge 0$, $\partial\pi = \pi\partial : C_n(\mathcal{K}) \to C_{n-1}(\mathcal{L})$. **P** The verification is laborious but essentially elementary. Starting from $x \in C_n(\mathcal{K})$ and $q \in W^n$, I need to show that

$$\partial\pi(x)(q) = \pi\partial(x)(q).$$

As in (a), this demands an analysis depending on the form of q. Since $\partial\pi(x)$ and $\pi\partial(x)$ both belong to $C_{n-1}(\mathcal{L})$, we can suppose that q is injective and $q[n]\in\mathcal{L}$; in particular, v_0 and v_1 cannot both belong to $q[n]$. Of course we always have

$$\partial\pi(x)(q)=\textstyle\sum_{w\in W}\pi(x)(q^\frown{<}w{>})=\pi(x)(q^\frown{<}\hat v{>})+\sum_{v\in V}\pi(x)(q^\frown{<}v{>}).$$

Write J for $\{v_0,\hat v,v_1\}$.

case 1a Suppose that $J\cap q[n]=\emptyset$. Then $J\cap(q^\frown{<}\hat v{>})[n+1]=\{\hat v\}$, so $\pi(x)(q^\frown{<}\hat v{>})=0$ and

$$\pi\partial(x)(q)=\partial(x)(q)=\sum_{v\in V}x(q^\frown{<}v{>})=\sum_{v\in V}\pi(x)(q^\frown{<}v{>})$$

(because $\hat v\notin(q^\frown{<}v{>})[n+1]$ for any $v\in V$)

$$=\sum_{w\in W}\pi(x)(q^\frown{<}w{>})=\partial\pi(x)(q).$$

case 1b Suppose that $J\cap q[n]=\{v_0\}$. Then

$$\pi(x)(q^\frown{<}\hat v{>})=x(g_1\circ(q^\frown{<}\hat v{>}))=x(q^\frown{<}v_1{>})=\pi(x)(q^\frown{<}v_1{>})=0$$

because $g_1(q^\frown{<}\hat v{>})=q^\frown{<}v_1{>}$, while $(q^\frown{<}v_1{>})[n+1]$ contains both v_0 and v_1, so does not belong to $\mathcal{L}$, and $x(q^\frown{<}v_1{>})$, $\pi(x)(q^\frown{<}v_1{>})$ must both be 0. Now

$$\begin{aligned}\pi\partial(x)(q)=\partial(x)(q)&=x(q^\frown{<}v_1{>})+\sum_{\substack{v\in V\\ v\ne v_1}}x(q^\frown{<}v{>})\\&=\pi(x)(q^\frown{<}\hat v{>})+\pi(x)(q^\frown{<}v_1{>})+\sum_{\substack{v\in V\\ v\ne v_1}}\pi(x)(q^\frown{<}v{>})\\&=\sum_{w\in W}\pi(x)(q^\frown{<}w{>})=\partial\pi(x)(q).\end{aligned}$$

case 1c Similarly, $\partial\pi(x)(q)=\pi\partial(x)(q)$ if $J\cap q[n]=\{v_1\}$. As $q[n]$ cannot contain both v_0 and v_1, this deals with all the cases in which $\hat v\notin q[n]$.

case 2 Suppose that $\hat v\in q[n]$. Then $q^\frown{<}\hat v{>}$ is not injective, so $\pi(x)(q^\frown{<}\hat v{>})=0$ and $\partial\pi(x)(q)=\sum_{v\in V}\pi(x)(q^\frown{<}v{>})$.

case 2a If $J\cap q[n]=\{\hat v\}$, then $\pi\partial(x)(q)=0$; at the same time, if $v\in V$, then $J\cap(q^\frown{<}v{>})[n+1]=\{\hat v\}$, so $\pi(x)(q^\frown{<}v{>})=0$. Summing, $\partial\pi(x)(q)=0=\pi\partial(x)(q)$.

case 2b Suppose that $J\cap q[n]=\{\hat v,v_0\}$. Then $q^\frown{<}v_0{>}$ is not injective and $(q^\frown{<}v_1{>})[n+1]\notin\mathcal{L}$ so $\partial\pi(x)(q)=\sum_{v\in V,v\ne v_0,v_1}\pi(x)(q^\frown{<}v{>})$. If $v\in V\setminus\{v_0,v_1\}$ then $J\cap(q^\frown{<}v{>})[n+1]=\{\hat v,v_0\}$ so

$$\pi(x)(q^\frown{<}v{>})=x(g_1(q^\frown{<}v{>}))=x((g_1q)^\frown{<}g_1(v){>})=x((g_1q)^\frown{<}v{>}).$$

Accordingly

$$\partial\pi(x)(q)=\sum_{\substack{v\in V\\ v\ne v_0,v_1}}x((g_1q)^\frown{<}v{>}))=\sum_{v\in V}x((g_1q)^\frown{<}v{>}))$$

(because $\{v_0,v_1\}\subseteq g_1q[n]$)

$$=\partial(x)(g_1q)=\pi\partial(x)(q).$$

case 2c Similarly, $\partial\pi(x)(q)=\pi\partial(x)(q)$ if $J\cap q[n]=\{\hat v,v_1\}$. So $\partial\pi(x)(q)=\pi\partial(x)(q)$ for every $q\in W^n$ and $\partial\pi(x)=\pi\partial(x)$, as claimed. **Q**

(c) Just as in Theorem 1K, it follows that we have a homomorphism $\pi_\star:H_n(\mathcal{K})\to H_n(\mathcal{L})$ for each $n\ge0$.

Now we find that $g_0{\bullet}\pi(x) = x$ whenever $n \ge -1$ and $x \in C_n(\mathcal{K})$. **P** Take $p \in V^{n+1}$. If either p is not injective or $p[n+1] \notin \mathcal{K}$ then $(g_0{\bullet}\pi(x))(p) = 0 = x(p)$. So suppose that p is injective and $p[n+1] \in \mathcal{K}$.

case 1 If $v_0 \notin p[n+1]$ then the only $q \in W^{n+1}$ such that $g_0q = p$ is p itself; accordingly

$$(g_0{\bullet}\pi(x))(p) = \sum_{\substack{q\in W^{n+1}\\ g_0q=p}} \pi(x)(q) = \pi(x)(p) = x(p)$$

since of course $\hat{v} \notin p[n+1]$.

case 2 If $v_0 \in p[n+1]$, let $i \le n$ be such that $p(i) = v_0$. Then $\{q : q \in W^{n+1},\ g_0q = p\} = \{p, p'\}$ where $p'(i) = \hat{v}$ and $p'(j) = p(j)$ for $j \ne i$.

If $v_1 \in p[n+1]$ then $p[n+1] \notin \mathcal{L}$, so $\pi(x)(p) = 0$ and

$$\begin{aligned}(g_0{\bullet}\pi(x))(p) &= \sum_{\substack{q\in W^{n+1}\\ g_0q=p}} \pi(x)(q) = \pi(x)(p') = x(g_0p') \\ &\text{(because } \{\hat{v}, v_1\} \subseteq p'[n+1] = (p[n+1] \setminus \{v_0\}) \cup \{\hat{v}\} \in \mathcal{L}) \\ &= x(p).\end{aligned}$$

If $v_1 \notin p[n+1]$ then $p[n+1] \in \mathcal{L}$, neither v_0 nor v_1 belongs to $p'[n+1]$ and $\pi(x)(p') = 0$, so

$$(g_0{\bullet}\pi(x))(p) = \sum_{\substack{q\in W^{n+1}\\ g_0q=p}} \pi(x)(q) = \pi(x)(p) = x(p).$$

Thus $(g_0{\bullet}\pi(x))(p) = x(p)$ for every p and $g_0{\bullet}\pi(x) = x$. **Q**

It follows at once that if $n \ge 0$ then

$$g_{0\star}\pi_\star(x^\bullet) = g_{0\star}(\pi(x)^\bullet) = (g_0{\bullet}\pi(x))^\bullet = x^\bullet$$

for every $x \in Z_n(\mathcal{K})$ and $g_{0\star}\pi_\star$ is the identity on $H_n(\mathcal{K})$.

(d)(i) In the other direction we need a more sophisticated approach. The key fact is the following. As in (c), set $J = \{v_0, \hat{v}, v_1\}$. Now if $n \ge 0$, $y \in Z_n(\mathcal{L})$ and $y(q) = 0$ whenever $q \in W^{n+1}$ and $J \cap q[n+1] = \{\hat{v}\}$, then $\pi(g_0{\bullet}y)(q) = y(q)$ whenever $q \in W^{n+1}$ is injective and $q[n+1] \in \mathcal{L}$.

P case 1 Suppose that $J \cap q[n+1]$ is either $\emptyset$ or $\{v_1\}$. Then

$$\pi(g_0{\bullet}y)(q) = (g_0{\bullet}y)(q) = \sum_{\substack{q_1\in W^{n+1}\\ g_0q_1=q}} y(q_1) = y(q)$$

because if $g_0q_1 = q$ then $\hat{v} \notin q_1[n+1]$ and $q_1 = g_0q_1 = q$.

case 2 Suppose that $J \cap q[n+1] = \{v_0\}$. For $i \le n$ set $q'(i) = \hat{v}$ if $q(i) = v_0$, $q(i)$ otherwise. If $q_1 \in W^{n+1}$ and $g_0q_1 = q$ then q_1 must be either q or q'. So

$$\pi(g_0{\bullet}y)(q) = (g_0{\bullet}y)(q) = \sum_{\substack{q_1\in W^{n+1}\\ g_0q_1=q}} y(q_1) = y(q) + y(q') = y(q)$$

because $J \cap q'[n+1] = \{\hat{v}\}$.

case 3 Suppose that $J \cap q[n+1] = \{\hat{v}\}$. Then $\pi(g_0{\bullet}y)(q) = 0 = y(q)$.

case 4 Suppose that $J \cap q[n+1] = \{\hat{v}, v_1\}$. If $q_1 \in W^{n+1}$ and $g_0q_1 = g_0q$ then q_1 must be either q or g_0q. So

$$\pi(g_0{\bullet}y)(q) = (g_0{\bullet}y)(g_0q) = y(q) + y(g_0q) = y(q)$$

because v_0, v_1 both belong to $g_0q[n+1]$.

case 5 Suppose that $J\cap q[n+1]=\{\hat v,v_0\}$.

(α) Consider first the case in which $q(n-1)=\hat v$ and $q(n)=v_0$. Set $p=q{\restriction}n-1$ so that $p\in V^{n-1}$ is injective, $J\cap p[n-1]=\emptyset$ and $q=p^\frown{<}\hat v{>}^\frown{<}v_0{>}$. Then $\pi(g_0{\bullet}y)(q)=(g_0{\bullet}y)(g_1q)$ and $g_1q=p^\frown{<}v_1{>}^\frown{<}v_0{>}$. Now

$$(g_0{\bullet}y)(g_1q)=\sum_{\substack{q_1\in W^{n+1}\\ g_0q_1=g_1q}}y(q_1)=y(p^\frown{<}v_1{>}^\frown{<}\hat v{>})+y(p^\frown{<}v_1{>}^\frown{<}v_0{>})=y(p^\frown{<}v_1{>}^\frown{<}\hat v{>}).$$

At this point, we need to recall that y is supposed to be a cycle, so $\partial(y)=0$ and we have

$$0=\partial(y)(p^\frown{<}\hat v{>})=\sum_{w\in W}y(p^\frown{<}\hat v{>}^\frown{<}w{>})=y(p^\frown{<}\hat v{>}^\frown{<}v_0{>})+y(p^\frown{<}\hat v{>}^\frown{<}v_1{>})$$

(because if $w\in W\setminus\{v_0,v_1\}$ then $J\cap(p^\frown{<}\hat v{>}^\frown{<}w{>})[n+1]=\{\hat v\}$)

$$=y(q)-y(p^\frown{<}v_1{>}^\frown{<}\hat v{>})$$

and

$$(g_0{\bullet}y)(g_1q)=y(p^\frown{<}v_1{>}^\frown{<}\hat v{>})=y(q).$$

(β) In the general case, since $q[n+1]\supseteq\{\hat v,v_0\}$, there is a $\sigma\in S_{n+1}$ such that $q\sigma(n-1)=\hat v$ and $q\sigma(n)=v_0$. We still have an injective $q\sigma\in W^{n+1}$ such that $q\sigma[n+1]\in\mathcal{L}$ and $J\cap q\sigma[n+1]=\{\hat v,v_0\}$, so

$$\pi(g_0{\bullet}y)(q)=\epsilon_\sigma\pi(g_0{\bullet}y)(q\sigma)=\epsilon_\sigma y(q\sigma)=y(q).\ \mathbf{Q}$$

(ii) Thus we see that $\pi(g_0{\bullet}y)=y$ whenever $n\ge 0$, $y\in Z_n(\mathcal{L})$ and $y(q)=0$ for every $q\in W^{n+1}$ and $J\cap q[n+1]=\{\hat v\}$.

For general $y\in Z_n(\mathcal{L})$, set

$$D_y=\{q:q\in W^{n+1},\ J\cap q[n+1]=\{\hat v\},\ y(q)\ne 0\}.$$

Now $\pi(g_0{\bullet}y)-y\in B_n(\mathcal{L})$. $\mathbf{P}$ Induce on $\#(D_y)$. If D_y is empty, (i) just above tells us that $\pi(g_0{\bullet}y)-y=0\in B_n(\mathcal{L})$. For the inductive step to $\#(D_y)=m+1>0$, take $q_0\in D_y$ and write $r=y(q_0)\in\mathbb{Z}\setminus\{0\}$. Then $q_0[n+1]$ is of the form $(K\setminus\{v_0,v_1\})\cup\{\hat v\}$ where $\{v_0,v_1\}\subseteq K\in\mathcal{K}$. So $q_0[n+1]\cup\{v_0\}\in\mathcal{L}$. Set $p=q_0^\frown{<}v_0{>}$ and consider $e_p\in C_{n+1}(\mathcal{L})$. Then

$$\partial(e_p)(q_0)=\sum_{w\in W}e_p(q_0^\frown{<}w{>})=e_p(q_0^\frown{<}v_0{>})=1.$$

(If $w\in W\setminus\{v_0\}$ then $v_0\notin(q_0^\frown{<}w{>})[n+2]$ and $p(q_0^\frown{<}w{>})=0$.) Consider $y'=y-r\partial(e_p)$. Then $y'(q_0)=0$. But if $q\in W^{n+1}$, $J\cap q[n+1]=\{\hat v\}$ and $\partial(e_p)(q)\ne 0$, we have

$$0\ne\partial(e_p)(q)=\sum_{w\in W}e_p(q^\frown{<}w{>})=e_p(q^\frown{<}v_0{>})$$

and $q[n+1]\cup\{v_0\}=q_0[n+1]\cup\{v_0\}$, which has $n+2$ members, so that $q[n+1]=q_0[n+1]$ and $q=q_0\sigma$ for some $\sigma\in S_{n+1}$; in which case $y'(q)=0$. Turning this round, we see that if $q\in W^{n+1}$, $J\cap q[n+1]=\{\hat v\}$ and $y'(q)\ne 0$ then $\partial(e_p)(q)=0$ and $y(q)\ne 0$. Thus $D_{y'}\subseteq D_y$; but also $q_0\in D_y\setminus D_{y'}$, so $\#(D_{y'})\le r$. By the inductive hypothesis, $\pi(g_0{\bullet}y')-y'\in B_n(\mathcal{L})$. On the other hand,

$$\pi(g_0{\bullet}\partial e_p)-\partial e_p=\pi\partial(g_0{\bullet}e_p)-\partial e_p=\partial\pi(g_0{\bullet}e_p)-\partial e_p=\partial(\pi(g_0{\bullet}e_p)-e_p)\in B_n(\mathcal{L})$$

so

$$\pi(g_0{\bullet}y)-y=\pi(g_0{\bullet}y')-y'-r\pi(g_0{\bullet}\partial e_p)+r\partial e_p\in B_n(\mathcal{L}).$$

(iii) We know by the arguments in part (c) of this proof that if $y\in Z_n(\mathcal{L})$ then $g_0{\bullet}y\in Z_n(\mathcal{K})$, $\pi(g_0{\bullet}y)\in Z_n(\mathcal{L})$ and

$$(\pi(g_0{\bullet}y))^\bullet=\pi_\star(g_0{\bullet}y)^\bullet=\pi_\star g_{0\star}y^\bullet$$

in $H_n(\mathcal{L})$. Since we now know also that $(\pi(g_0{\bullet}y))^\bullet=y^\bullet$ for every $y\in Z_n(\mathcal{L})$, $\pi_\star g_{0\star}$ is the identity on $H_n(\mathcal{L})$. And we saw in (c) that $g_{0\star}\pi_\star$ is the identity on $H_n(\mathcal{K})$. So $g_{0\star}$ and $\pi_\star$ are the two halves of an isomorphism between $H_n(\mathcal{K})$ and $H_n(\mathcal{L})$.

Notes and comments I should have liked to call these notes 'singular homology for dummies', but I fear that such a title would have been misleading. The presentation above has been ruthlessly stripped of geometric interpretation or other supports for intuition, and only those who can cope with very abstract approaches will be happy with it. The argument is reduced to detailed manipulation of complex algebraic structures, with correspondingly lengthy calculations in 1P and 1Q, culminating in a tour de force in 1S. I will attempt in §E1 below to flesh the some of the ideas out with an indication of how they may be considered in easy cases. The applications in two and three dimensions are already non-trivial. In my defence, let me point out that applications in higher dimensions are an essential part of the target, and that the geometric interpretation of 'oriented simplex' is difficult in three dimensions and downright obscure in four or more, so that while there are alternative approaches they too necessarily rely on formal manipulations.

I ought to repeat that 1J is the definition of 'reduced' homology groups. Most authors prefer not to discuss the chain groups C_{-1} or define the boundary of a 0-chain, and instead declare that $Z_0(\mathcal{K}) = C_0(\mathcal{K})$. Consequently you will find that in most texts $H_0(\mathcal{K})$, now being $C_0(\mathcal{K})/B_0(\mathcal{K})$, is larger than the H_0 of this note. For $n \geq 1$ I think everyone agrees on what (up to isomorphism) $H_n(\mathcal{K})$ should be. The reason for my choice is that (in the formalism I am using) we have a natural interpretation of $\partial : C_0(\mathcal{K}) \to C_{-1}(\mathcal{K})$, and if we use this we find that the arguments dealing with H_n, for $n \geq 1$, can be applied unchanged to H_0. The information carried by H_0 (see E1F below) is essentially the same on either definition.

In 1L I gave a sketch of an argument, quoting Theorem 1K, to support my assertion that if $g : V \to V$ is an automorphism of $(V, \mathcal{K})$ then $g_\star : H_n(\mathcal{K}) \to H_n(\mathcal{K})$ will be an automorphism of $H_n(\mathcal{K})$. But really it goes deeper than that; the point is that the group $H_n(\mathcal{K})$ is built by a definite process from the set $\mathcal{K}$, and a rearrangement of $\mathcal{K}$ (provided it respects the relation $\subseteq$) must correspond to a rearrangement of $H_n(\mathcal{K})$ respecting its group structure.

This chapter has been exclusively devoted to 'integer' homology. It is in fact possible to do everything above with an arbitrary commutative group F in the place of $\mathbb{Z}$, so that an n-chain becomes a function from V^{n+1} to F. For completely general commutative groups F some serious renegotiation is needed at the very beginning, because in the formula $x(p\sigma) = \epsilon_\sigma x(p)$ (1Ca) we need to think of ϵ_σ as an automorphism of F ($\epsilon_\sigma r = r$ if σ is even, $-r$ if σ is odd) rather than as $+1$ or -1. The definition of the basic chains e_p (1C(b-iv)) must also be re-examined if F has no obvious generator; we may have to look at a wider family of objects e_{pr} where $p \in V^{n+1}$ and $r \in F$. A further, less fundamental, complication is that if $F = \mathbb{Z}_2$, for instance, then we can have $r = -r$ for non-zero r; this will mean that we have to add a clause '$x(p) = 0$ whenever $p \in V^{n+1}$ is not injective' as part of the definition of n-chain, instead of deducing it as in 1C(b-iii). There are other points where this makes a difference (in the proof of the Boundary Theorem, for instance), but small adaptations in proofs can deal with these.

While I do not believe that time spent searching for generalizations is ever completely wasted in pure mathematics, I don't think that readers new to this topic should put much thought into this question now, unless they are particularly attracted to it. In Chapter 6, however, I will ask you to re-work this chapter with $\mathbb{Q}$ in place of $\mathbb{Z}$. The very special properties of the additive group $(\mathbb{Q}, +)$ mean, I think, that every word of the arguments above can be kept, and you just need to replace every $\mathbb{Z}$ with $\mathbb{Q}$.

E1 Examples for Chapter 1

E1A A fundamental calculation Let $\mathcal{K}$ be a simplicial complex with vertex set V, and $n \in \mathbb{N}$. Suppose that $p \in V^{n+1}$ is injective and $p[n+1] \in \mathcal{K}$, so that e_p is defined in $C_n(\mathcal{K})$ (1C(b-iv)). For $j \leq n$ define $p^{\uparrow j} \in V^n$ by saying

$$\begin{aligned} p^{\uparrow j}(i) &= p(i) \text{ if } i < j, \\ &= p(i+1) \text{ if } j \leq i < n. \end{aligned}$$

Then $p^{\uparrow j}$ is injective and $(p^{\uparrow j})[n] = p[n+1] \setminus \{p(j)\} \in \mathcal{K}$, so $e_{p^{\uparrow j}}$ is defined in $C_{n-1}(\mathcal{K})$. Now

$$\partial(e_p) = \sum_{j=0}^{n} (-1)^{n-j} e_{p^{\uparrow j}}.$$

proof (a) I begin by calculating $\partial(e_p)(p^{\uparrow k})$ where $k \leq n$. Let $\sigma \in S_{n+1}$ be the permutation such that

$$\begin{aligned}\sigma(i) &= i \text{ if } i < k,\\ &= i+1 \text{ if } k \le i < n,\\ &= k \text{ if } i = n.\end{aligned}$$

Then the set of orbits of σ is $\{\{i\} : i < k\} \cup \{\{i : k \le i \le n\}\}$, and the number of orbits of even size is 1 if $n+1-k$ is even, 0 otherwise; so $\epsilon_\sigma = (-1)^{n-k}$. Now

$$\begin{aligned}p\sigma(i) &= p(i) = p^{\uparrow k}(i) = ((p^{\uparrow k})^\frown{<}p(k){>})(i) \text{ if } i < k,\\ &= p(i+1) = p^{\uparrow k}(i) = ((p^{\uparrow k})^\frown{<}p(k){>})(i) \text{ if } k \le i < n,\\ &= p(k) = ((p^{\uparrow k})^\frown{<}p(k){>})(n) \text{ if } i = n.\end{aligned}$$

So

$$e_p((p^{\uparrow k})^\frown{<}p(k){>}) = e_p(p\sigma) = \epsilon_\sigma = (-1)^{n-k}.$$

But if $v \in V \setminus \{p(k)\}$ then $p(k) \notin ((p^{\uparrow k})^\frown{<}v{>})[n+1]$ and $e_p((p^{\uparrow k})^\frown{<}v{>}) = 0$. Accordingly

$$\partial(e_p)(p^{\uparrow k}) = \textstyle\sum_{v\in V} e_p((p^{\uparrow k})^\frown{<}v{>}) = e_p((p^{\uparrow k})^\frown{<}p(k){>}) = (-1)^{n-k}.$$

(b) Now turn to general $q \in V^n$. If either q is not injective or $q[n] \not\subseteq p[n+1]$, $e_p(q^\frown{<}v{>}) = 0 = e_{p^{\uparrow j}}(q)$ for every $v \in V$ and $j \le n$, so certainly $\partial(e_p)(q) = \sum_{j=0}^n (-1)^{n-j} e_{p^{\uparrow j}}(q)$. Otherwise, there is a $k \le n$ such that

$$q[n] = p[n+1] \setminus \{p(k)\} = p^{\uparrow k}[n].$$

If $j \le n$ and $j \ne k$ then $p(j) \in q[n] \setminus p^{\uparrow j}[n]$ and $e_{p^{\uparrow j}}(q) = 0$. And there is a $\rho \in S_n$ such that $q = p^{\uparrow k}\rho$, so

$$\begin{aligned}\partial(e_p)(q) &= \epsilon_\rho \partial(e_p)(p^{\uparrow k}) = (-1)^{n-k}\epsilon_\rho\\ &= (-1)^{n-k} e_{p^{\uparrow k}}(q) = \sum_{j=0}^n (-1)^{n-j} e_{p^{\uparrow j}}(q).\end{aligned}$$

As q is arbitrary, we have the result.

E1B An alternative description of chain groups The importance of the basic chains e_p comes from the following.

Proposition Let $\mathcal{K}$ be a simplicial complex, with vertex set V, and $n \ge -1$. Then any $x \in C_n(\mathcal{K})$ is either 0 or can be expressed as a sum $\sum_{j=0}^m r_j e_{p_j}$ where $r_j \in \mathbb{Z}$, $p_j \in V^{n+1}$ is injective and $p_j[n+1] \in \mathcal{K}$ for each $j \le m$.

proof Induce on $l = \#(\{q : q \in V^{n+1},\ x(q) \ne 0\})$. If $l = 0$ then $x = 0$ and we can stop. Otherwise, there is a $p \in V^{n+1}$ such that $r = x(p)$ is non-zero. Consider $x' = x - re_p$. Then $x'(p) = 0$. **?** If $q \in V^{n+1}$ is such that $x(q) = 0 \ne x'(q)$ then $e_p(q) \ne 0$, so there is a $\sigma \in S_{n+1}$ such that $q = p\sigma$; now

$$x'(q) = x(p\sigma) - re_p(p\sigma) = \epsilon_\sigma(x(p) - r) = 0. \ \mathbf{X}$$

So $\#(\{q : q \in V^{n+1},\ x'(q) \ne 0\}) < l$. By the inductive hypothesis, x' is either zero or expressible in the required form; now $x = x' + re_p$ is also in the required form.

E1C An alternative description of the action of a simplicial map Suppose that $\mathcal{K}$ and $\mathcal{L}$ are simplicial complexes, with vertex sets V and W, and $h : V \to W$ is a simplicial map. Suppose that $n \ge -1$, that $p \in V^{n+1}$ is injective and that $p[n+1] \in \mathcal{K}$, so that e_p is defined in $C_n(\mathcal{K})$. Then

$$\begin{aligned}h{\bullet}e_p &= e_{hp} \text{ if } hp \in W^{n+1} \text{ is injective},\\ &= 0 \text{ otherwise}.\end{aligned}$$

proof (a) If hp is injective, then $hp[n+1] = h[p[n+1]] \in \mathcal{L}$, because h is a simplicial map, so e_{hp} is defined in $C_n(\mathcal{L})$. If $q = hp$, then $(h{\bullet}e_p)(q) = \sum_{p'\in V^{n+1}, hp'=hp} e_p(p')$. But if $p' \in V^{n+1}$ and $p' \ne p$, then either $p'[n+1] \ne p[n+1]$ and $e_p(p') = 0$, or $p'[n+1] = p[n+1]$ and $hp' \ne hp$ because h is injective on $p[n+1]$. So

$$(h\bullet e_p)(hp) = e_p(p) = 1 = e_{hp}(hp).$$

If $q \in W^{n+1}$ and $q[n+1] = hp[n+1]$, there is a $\sigma \in S_{n+1}$ such that $q = hp\sigma$, so that

$$(h\bullet e_p)(q) = \epsilon_\sigma (h\bullet e_p)(hp) = \epsilon_\sigma e_{hp}(hp) = e_{hp}(q).$$

If $q \in W^{n+1}$ and $q[n+1] \neq hp[n+1]$ then $p'[n+1] \neq p[n+1]$ whenever $hp' = q$ so

$$(h\bullet e_p)(q) = 0 = e_{hp}(q).$$

Thus $e_{hp} = h\bullet e_p$ if hp is injective.

(b) If hp is not injective and $q \in W^{n+1}$, then $hp\sigma$ is not injective for any $\sigma \in S_{n+1}$, so hp' is not injective for any p' such that $e_p(p') \neq 0$. But now

$$(h\bullet e_p)(q) = \sum_{p' \in V^{n+1}, hp'=q} e_p(p') = 0$$

whenever $q \in W^{n+1}$ is injective; as it is certainly zero whenever q is not injective, $h\bullet e_p = 0$, as claimed.

E1D Homology groups: the most trivial case Suppose that $\mathcal{K} = \{\emptyset\}$. Then $V = \emptyset$. $V^0 = \{\emptyset\}$, $V^n = \emptyset$ for $n > 0$; there are no n-simplices for any $n \geq 0$. For $n \geq 0$, $C_n(\mathcal{K}) = \{\emptyset\}$ is a group with one element so $Z_n(\mathcal{K}) = B_n(\mathcal{K}) = \{0\}$ and $H_n(\mathcal{K}) = \{0\}$.

E1E The next most trivial case Suppose that V is a non-empty finite set and $\mathcal{K} = \{\{v\} : v \in V\} \cup \{\emptyset\}$. The 0-simplices are the singleton subsets of $V \cong V^1$. $C_0(\mathcal{K})$ is the set of all functions from V^1 to $\mathbb{Z}$; for $n \geq 1$, the only member of $C_n(\mathcal{K})$ is the zero function from V^{n+1} to $\mathbb{Z}$, since no member of V^{n+1} can be injective. So $H_n(\mathcal{K}) = \{0\}$ for $n \geq 1$. As for $n = 0$, $B_0(\mathcal{K}) = \{0\}$ because $C_1(\mathcal{K}) = \{0\}$. If $x \in C_0(\mathcal{K})$ then $\partial(x)(\emptyset) = \sum_{v \in V} x(<v>)$, so $Z_0(\mathcal{K}) = \{x : x \in C_0(\mathcal{K}),\ \sum_{v \in V} x(<v>) = 0\}$. If we fix a member v_0 of V, then

$$H_0(\mathcal{K}) = Z_0(\mathcal{K})/\{0\} \cong Z_0(\mathcal{K}) = \{x : x(<v_0>) = -\sum_{v \in V \setminus \{v_0\}} x(<v>)\}$$

is isomorphic to $\mathbb{Z}^k$ where $k = \#(V) - 1$. So the group $H_0(\mathcal{K})$ tells us the size of V (excep that it cannot distinguish $V = \emptyset$ from $V = \{v_0\}$).

E1F The group $H_0(\mathcal{K})$ We have a general approach to the lowest homology group, as follows.

(a) Let $\mathcal{K}$ be any simplicial complex other than $\{\emptyset\}$, and V its vertex set. For v, $v' \in V$ say that $v \sim v'$ if there are $v_0, \dots, v_m \in V$ such that $v = v_0$, $v' = v_m$ and $\{v_i, v_{i+1}\} \in \mathcal{K}$ for $i < m$. Then $\sim$ is an equivalence relation. Let $V_0, \dots, V_k$ be the equivalence classes under $\sim$; these are the **components** of $\mathcal{K}$. If $\mathcal{K}$ has only one component, we say that it is **connected**.

As in E1E, $\partial(x)(\emptyset) = \sum_{v \in V} x(<v>)$ for $x \in C_0(\mathcal{K})$, so $Z_0(\mathcal{K}) = \{x : \sum_{v \in V} x(<v>) = 0\}$.

(b) As for $B_0(\mathcal{K})$, we have the following.

(i) If $y \in B_0(\mathcal{K})$ and $i \leq k$ then there is an $x \in C_1(\mathcal{K})$ such that $y = \partial(x)$, so

$$\sum_{v \in V_i} y(<v>) = \sum_{v \in V_i} \sum_{v' \in V} x(<v>\frown<v'>) = \sum_{v \in V_i} \sum_{v' \in V_i \setminus \{v\}} x(<v>\frown<v'>)$$

(because if $v \in V_i$ and $x(<v>\frown<v'>) \neq 0$ then $v' \neq v$ and $\{v, v'\} \in \mathcal{K}$, so $v' \in V_i$)

$$= -\sum_{v \in V_i} \sum_{v' \in V_i \setminus \{v\}} x(<v'>\frown<v>) = -\sum_{v' \in V_i} \sum_{v \in V_i \setminus \{v'\}} x(<v'>\frown<v>)$$

$$= -\sum_{v' \in V_i} y(<v'>) = -\sum_{v \in V_i} y(<v>)$$

and $\sum_{v \in V_i} y(<v>) = 0$.

(ii) In the other direction, I show by induction on m that if $y \in C_0(\mathcal{K})$, $\#(\{v : v \in V,\ y(<v>) \neq 0\}) = m$ and $\sum_{v \in V_i} y(<v>) = 0$ for every i, then $y \in B_0(\mathcal{K})$. **P** If $m = 0$ this is trivial, as $y = 0$. For the inductive step to $m+1$, there is a $v' \in V$ such that $y(<v'>) \neq 0$. Let $l \leq k$ be such that $v' \in V_l$; as $\sum_{v \in V_l} y(<v>) = 0$,

$V_l \neq \{v'\}$ and there is a $v'' \neq v'$ such that $\{v', v''\} \in \mathcal{K}$. Consider $x = e_{<v'>^\frown<v''>} \in C_1(\mathcal{K})$. Then $\partial(x) = e_{<v'>} - e_{<v''>}$ belongs to $B_0(\mathcal{K})$, and we have

$$\partial(x)(<v'>) = 1, \quad \partial(x)(<v''>) = -1, \quad \partial(x)(<v>) = 0 \text{ if } v \in V \setminus \{v', v''\}.$$

Set $r = y(<v'>)$ and $y' = y - rx$. Then $y' \in C_0(\mathcal{K})$ and $\sum_{v\in V_i} y'(<v>) = 0$ for every i; also $\{v : y'(<v>) \neq 0\} \subseteq \{v : y(<v>) \neq 0\} \setminus \{v'\}$ has at most m members. By the inductive hypothesis, $y' \in B_0(\mathcal{K})$ so $y = y' + rx \in B_0(\mathcal{K})$. Thus the induction continues. **Q**

(c) Thus we see that

$$\begin{aligned} B_0(\mathcal{K}) &= \{y : y \in C_0(\mathcal{K}), \sum_{v\in V_i} y(<v>) = 0 \text{ for every } i \le k\} \\ &= \{y : y \in Z_0(\mathcal{K}), \sum_{v\in V_i} y(<v>) = 0 \text{ for every } i < k\}. \end{aligned}$$

So this time the quotient $H_0(\mathcal{K}) = Z_0(\mathcal{K})/B_0(\mathcal{K})$ is isomorphic to $\mathbb{Z}^k$ where $\mathcal{K}$ has $k+1$ components.

(d) (a)-(c) above were all written on the assumption that $\mathcal{K} \neq \{\emptyset\}$. But I will also say that the trivial simplicial complex with empty vertex set is connected, with no components. On this convention $H_0(\mathcal{K}) = \{0\}$ iff $\mathcal{K}$ is connected, whether with one component or none.

E1G Constant simplicial maps Let $\mathcal{K}$ and $\mathcal{L}$ be simplicial complexes, with vertex sets V and W, and $g : V \to W$ a constant map. Then $g_\star : H_n(\mathcal{K}) \to H_n(\mathcal{L})$ is the zero homomorphism for every $n \in \mathbb{N}$.

proof (a) I had better deal at once with the case $V = \emptyset$. In this case $g[V] = \emptyset \in \mathcal{L}$ so g is a simplicial map, and $H_n(\mathcal{K}) = \{0\}$ for every n, as noted in E1D, so $g_\star$ is necessarily trivial.

(b) If V is not empty, then W is not empty and there is a $w \in W$ such that $g(v) = w$ for every $v \in V$. As $g[K] \subseteq \{w\} \in \mathcal{L}$ for every $K \in \mathcal{K}$, g is a simplicial map. Consider the simplicial complex $\mathcal{M} = \mathcal{P}(\{w\})$. We can express g as a composition fh, where $h : V \to \{w\}$ is constant and $f : \{w\} \to W$ takes w to itself, so that they are simplicial maps from $\mathcal{K}$ to $\mathcal{M}$ and from $\mathcal{M}$ to $\mathcal{L}$. Now $g_\star = f_\star h_\star : H_n(\mathcal{K}) \to H_n(\mathcal{L})$. But $H_n(\mathcal{M}) = \{0\}$, by E1E with $k = 1$. So $f_\star$ and $h_\star$ are both trivial, and their composition $g_\star$ is trivial too.

E1H Simple simplices: Proposition Suppose that V is a finite set and that $\mathcal{K} = \mathcal{P}V$. Then $H_n(\mathcal{K}) = \{0\}$ for every $n \in \mathbb{N}$.

proof If $V = \emptyset$ this is immediate from E1D. So suppose that $V \neq \emptyset$. Take $\hat{v} \in V$. Then the constant function $g : V \to \{\hat{v}\}$ is a simplicial map from $\mathcal{K}$ to itself. At the same time, the identity function $f : V \to V$ is simplicial. But $f[K] \cup g[K] = K \cup \{\hat{v}\}$ belongs to $\mathcal{K}$ for every $K \in \mathcal{K}$, so f and g are contiguous. By the Contiguity Theorem (1Q), $f_\star$ and $g_\star$ agree on $H_n(\mathcal{K})$. But of course $f_\star$ is just the identity homomorphism. And by E1G, $g_\star = f_\star$ is the zero homomorphism. So $H_n(\mathcal{K})$ must be $\{0\}$.

E1I Evacuated simplices Now suppose that V is a finite set with $\#(V) = m + 2 \ge 2$, and that $\mathcal{K} = \mathcal{P}V \setminus \{V\}$. Then $H_m(\mathcal{K}) \cong \mathbb{Z}$ and $H_n(\mathcal{K}) = \{0\}$ if $n \in \mathbb{N} \setminus \{m\}$.

proof Note first that for $n > m$ there is no injective $p \in V^{n+1}$ such that $p[n+1] \in \mathcal{K}$, so $C_n(\mathcal{K}) = \{0\}$ and $H_n(\mathcal{K}) = \{0\}$. Let $\mathcal{L}$ be the simplicial complex $\mathcal{P}V$. Then for $n \le m$ we have $C_n(\mathcal{K}) = C_n(\mathcal{L})$ (because $\mathcal{L}$ and $\mathcal{K}$ contain the same sets of size $n+1$), and the boundary operator ∂ acts in the same way on $C_n(\mathcal{K})$ and $C_n(\mathcal{L})$ because $\bigcup \mathcal{L} = V$. So (if $n \ge 0$) $Z_n(\mathcal{K}) = Z_n(\mathcal{L})$ and (if $0 \le n < m$) $B_n(\mathcal{K}) = B_n(\mathcal{L})$. Accordingly $H_n(\mathcal{K}) = H_n(\mathcal{L}) = \{0\}$ for $0 \le n < m$, by E1H.

As for the case $n = m$, we have $Z_m(\mathcal{K}) = Z_m(\mathcal{L}) = B_m(\mathcal{L})$, while $B_m(\mathcal{K}) = \{0\}$ because $C_{m+1}(\mathcal{K}) = \{0\}$. So $H_m(\mathcal{K}) \cong B_m(\mathcal{L})$. Fix on an injective $p \in V^{m+2}$. Then every injective $q \in V^{m+2}$ is of the form $p\sigma$ for some $\sigma \in S_{m+2}$. So if $x \in C_{m+1}(\mathcal{L})$, $x = re_p$ where $r = x(p)$. Accordingly $B_m(\mathcal{L}) = \{r\partial(e_p) : r \in \mathbb{Z}\}$. Of course $\partial(e_p) \neq 0$, since $\partial(e_p)(q) = 1$ where $q(i) = p(i)$ for $i \le m$, so $H_m(\mathcal{K}) \cong B_m(\mathcal{L}) \cong \mathbb{Z}$, as asserted.

Remark Observe that this matches E1F in the case $m = 0$, as then the components of $\mathcal{K}$ are the two singleton subsets of V.

E1J A simplicial map Continuing the example E1I, take distinct v_0, $v_1 \in V$ and let $f : V \to V$ be the transposition exchanging v_0 and v_1. Then f is a simplicial isomorphism both for $\mathcal{K}$ and for $\mathcal{L}$. Any injective $p \in V^{m+2}$ must be bijective, so there are i, $j \le m+1$ such that $p(i) = v_0$ and $p(j) = v_1$; now $fp = p\tau$ where τ is the transposition exchanging i and j. Accordingly $x(fp) = -x(p)$ for every $x \in C_{m+1}(\mathcal{L})$. This means that if $x \in C_{m+1}(\mathcal{L})$ then $(f\bullet x)(q) = \sum_{fp=q} x(p) = x(fq) = -x(q)$ for every injective $q \in V^{m+2}$ and $f\bullet x = -x$.

If $y \in Z_m(\mathcal{K}) = B_m(\mathcal{L})$ there is an $x \in C_{m+1}(\mathcal{L})$ such that $y = \partial(x)$ and

$$f\bullet y = f\bullet\partial(x) = \partial(f\bullet x) = -\partial(x) = -y.$$

So $f_\star(a) = -a$ for every $a \in H_m(\mathcal{K})$. In particular, $f_\star$ is not the identity on $H_m(\mathcal{K})$.

Notes and comments If you look at any other treatment of this subject, you may well notice that my version has a second eccentricity, after the choice of 'reduced' homology groups H_0. The formula corresponding to that in E1A is usually different, commonly becoming $\partial(e_p) = \sum_{j=0}^{n}(-1)^j e_{p\uparrow j}$ once you have found the proper translation of my formula e_p. In effect, my version of the boundary operator has a different sign when n is odd. I hope it is clear that this makes no difference to the theory here. I chose the particular rule

$$\partial(x)(q) = \sum_{v\in V} x(q^\frown{<}v{>})$$

(rather than $\sum_{v\in V} x({<}v{>}^\frown q)$, for instance) because it makes some formulae run more smoothly.

In E3A below I will come to geometric interpretations of the 'simple simplices' and 'evacuated simplices' in E1H-E1I.

2 Čech homology

Given a topological space, we need to find a way of associating it with simplicial complexes to which we can apply the theory of Chapter 1.

2A Definitions Let X be a topological space and V a proper finite open cover of X, that is, a finite family of non-empty open subsets of X such that $\bigcup V = X$. Set $\mathcal{K}_V = \{K : K \subseteq V, \bigcap K \neq \emptyset\}$; then $\mathcal{K}_V$ is a simplicial complex, and $\bigcup \mathcal{K}_V = V$. [Note that $\bigcap \emptyset = \{t : t \in K$ whenever $K \in \emptyset\}$ is the universal class and is never empty, so $\mathcal{K}_V$ will certainly contain the empty set.]

If W is another proper finite open cover of X, say that $V \preccurlyeq W$ if V **refines** W, that is, every member of V is included in some member of W. Of course $V \preccurlyeq V$ for every V, and if $V \preccurlyeq V' \preccurlyeq V''$ then $V \preccurlyeq V''$. Note that if V and V' are proper finite open covers of X then $W = \{v \cap v' : v_0 \in V,\ v' \in V'\} \setminus \{\emptyset\}$ is a proper finite open cover of X refining both V and V'.

Remark I am talking here about arbitrary topological spaces. But if you have not encountered this concept in its general form, let me assure you that for the purposes of this note you can take every topological space to be a metric space, and nearly all are subspaces of finite-dimensional Euclidean spaces $\mathbb{R}^r$. Most of the general topology which is essential to us is sketched in §§2A2-2A3 of Fremlin 01.

2B Lemma Suppose that X is a topological space, V and W are proper finite open covers of X and $V \preccurlyeq W$. There are functions $g : V \to W$ such that $v \subseteq g(v)$ for every $v \in V$. If g, h are any two such functions, they are simplicial maps from $\mathcal{K}_V$ to $\mathcal{K}_W$, and the corresponding homomorphisms $g_\star : H_n(\mathcal{K}_V) \to H_n(\mathcal{K}_W)$, $h_\star : H_n(\mathcal{K}_V) \to H_n(\mathcal{K}_W)$ are equal for every $n \in \mathbb{N}$.

proof The existence of these functions is just the definition of $\preccurlyeq$. If g, $h : V \to W$ have the declared property and $K \in \mathcal{K}_V$, there is a point $t \in \bigcap K$; now $t \in v \subseteq g(v) \cap h(v)$ for every $v \in K$, so $t \in w$ for every $w \in g[K] \cup h[K]$, and $g[K] \cup h[K] \in \mathcal{K}_W$. Thus g and h are contiguous simplicial maps, and $g_\star = h_\star$ by the Contiguity Theorem (1Q).

2C Definition Suppose that X is a topological space, V and W are proper finite open covers of X such that $V \preccurlyeq W$, and $n \in \mathbb{N}$. Write $\pi_{WV} : H_n(\mathcal{K}_V) \to H_n(\mathcal{K}_W)$ for the common value of $g_\star : H_n(\mathcal{K}_V) \to H_n(\mathcal{K}_W)$ for any function $g : V \to W$ such that $v \subseteq g(v)$ for every $v \in V$.

2D Lemma Let X be a topological space and V, V' and V'' proper finite open covers of X such that $V \preccurlyeq V' \preccurlyeq V''$. Then $\pi_{V''V} = \pi_{V''V'}\pi_{V'V} : H_n(\mathcal{K}_V) \to H_n(\mathcal{K}_{V''})$ for every $n \in \mathbb{N}$.

proof Let $g : V \to V'$ and $h : V' \to V''$ be such that $v \subseteq g(v)$ for every $v \in V''$ and $w \subseteq h(w)$ for every $w \in V'$. Then $v \subseteq hg(v) \in V$ for every $v \in V''$ and

$$\pi_{V''V} = (hg)_\star = h_\star g_\star$$

(1Kb)

$$= \pi_{V''V'}\pi_{V'V}.$$

2E Lemma Let X and Y be topological spaces and $f : X \to Y$ a continuous function. For a proper finite open cover W of Y, write $f^{-1}[[W]]$ for $\{f^{-1}[w] : w \in W\} \setminus \{\emptyset\}$.

(a) $V = f^{-1}[[W]]$ is a proper finite open cover of X.

(b) For each $n \in \mathbb{N}$ we have a homomorphism $\phi_W : H_n(\mathcal{K}_V) \to H_n(\mathcal{K}_W)$ defined by saying that $\phi_W = g_\star$ whenever $g : V \to W$ is such that $v = f^{-1}[g(v)]$ for every $v \in V$.

(c) If V_0 is a proper finite open cover of X with $V_0 \preccurlyeq V$, and $h : V_0 \to W$ is such that $v \subseteq f^{-1}[h(v)]$ for every $v \in V$, then $\phi_W \pi_{VV_0} = h_\star : H_n(\mathcal{K}_{V_0}) \to H_n(\mathcal{K}_W)$ for every $n \in \mathbb{N}$.

(d) If W' is another proper finite open cover of Y, $W \preccurlyeq W'$ and $V' = f^{-1}[[W']]$, then $V \preccurlyeq V'$ and $\phi_{W'}\pi_{V'V} = \pi_{W'W}\phi_W$.

proof (a) By the definition of 'continuous function', $f^{-1}[w] = \{t : t \in X,\ f(t) \in w\}$ is an open set for every $w \in W$, so $V = f^{-1}[[W]] \setminus \{\emptyset\}$ is a finite family of non-empty open subsets of X. If $t \in X$, then $f(t) \in Y$ so there is a $w \in W$ such that $f(t) \in w$, in which case $t \in f^{-1}[w] \in V$. Thus V covers X.

(b) By the definition of V, there must be a function $g : V \to W$ such that $v = f^{-1}[g(v)]$ for every $v \in V$. If $h : V \to W$ also has this property and $K \in \mathcal{K}_V$, then there is a $t \in \bigcap \mathcal{K}$, in which case $t \in f^{-1}[g(v)] \cap f^{-1}[h(v)]$, that is, $f(t) \in g(v) \cap h(v)$, for every $v \in K$, and $g[K] \cup h[K] \in \mathcal{K}_W$. But this means that g and h are contiguous simplicial maps and $g_\star$, $h_\star$ agree on $H_n(\mathcal{K}_V)$, by 1Q again. We can therefore take this common value for $\phi_W : H_n(\mathcal{K}_V) \to H_n(\mathcal{K}_W)$.

(c) We can express π_{VV_0} as $k_\star$ where $k : V_0 \to V$ is such that $v \subseteq k(v)$ for every $v \in V_0$, and π_W as $g_\star$ where $g : V \to W$ is such that $v = f^{-1}[g(v)]$ for $v \in V$. So $v \subseteq k(v) \subseteq f^{-1}[gk(v)]$ for every $v \in V$. But this means that if $K \in \mathcal{K}_V$, so that we have a point $t \in \bigcap K$, then $f(t) \in gk(v)$ for every $v \in K$, while also $f(t) \in h(v)$ for every $v \in K$. So $f(t) \in w$ for every $w \in gk[K] \cup h[K]$ and $gk[K] \cup h[K] \in \mathcal{K}_W$. Thus h and gk are contiguous simplicial maps and

$$\pi_W \pi_{VV_0} = g_\star k_\star = (gk)_\star = h_\star,$$

using 1Q once more.

(d) If $v \in V$ there is a $w \in W$ such that $v = f^{-1}[w]$; now there is a $w' \in W'$ such that $w \subseteq w'$, and $v \subseteq f^{-1}[w'] \in V'$. Thus $V \preccurlyeq V'$.

To see that $\phi_{W'}\pi_{V'V} = \pi_{W'W}\phi_W$, choose functions $g : V \to W$ and $h : W \to W'$ such that $v \subseteq f^{-1}[g(v)]$ and $w \subseteq h(w)$ for all $v \in V$, and $w \in W$. Then

$$v \subseteq f^{-1}[g(v)] \subseteq f^{-1}[h(g(v))] = f^{-1}[hg(v)]$$

for every $v \in V$. By (c),

$$\phi_{W'}\pi_{V'V} = (hg)_\star = h_\star g_\star = \pi_{W'W}\phi_W.$$

2F Definitions Let X be a topological space, $\mathbb{V}$ the set of proper finite open covers of X and $n \in \mathbb{N}$.

(a) Write $H_n(X)$ for the set of functions a with domain $\mathbb{V}$ such that $a(V) \in H_n(\mathcal{K}_V)$ for every $V \in \mathbb{V}$ and $\pi_{V'V}(a(V) = a(V'))$ whenever $V \preccurlyeq V'$ in $\mathbb{V}$.

(b) If Y is another topological space, $\mathbb{W}$ is the set of proper finite open covers of Y and $f : X \to Y$ is continuous, set

$$f_\star(a) = \langle \phi_W(a(f^{-1}[[W]])) \rangle_{W \in \mathbb{W}}$$

for $a \in H_n(X)$.

2G Proposition Let X, Y and Z be topological spaces, $\mathbb{V}$, $\mathbb{W}$ and $\mathbb{U}$ the corresponding sets of proper finite open covers, $f : X \to Y$ and $g : Y \to Z$ continuous functions, and $n \in \mathbb{N}$.

(a) $H_n(X)$ is a subgroup of the product group $\prod_{V \in \mathbb{V}} H_n(\mathcal{K}_V)$.

(b) $f_\star$ is a homomorphism from $H_n(X)$ to $H_n(Y)$.

(c) $(gf)_\star = g_\star f_\star : H_n(X) \to H_n(Z)$.

proof (a) We have only to note that if a, $b \in H_n(X)$, then $a + b = \langle a(V) + b(V) \rangle_{V \in \mathbb{V}}$ and

$$\phi_{V'V}(a(V) + b(V)) = \pi_{V'V}(a(V)) + \pi_{V'V}(b(V)) = a(V') + b(V')$$

whenever $V \preccurlyeq V'$, so $a + b \in H_n(X)$.

(b) By 2Ea, $f_\star$ is well-defined as a function from $H_n(X)$ to $\prod_{W \in \mathbb{W}} H_n(\mathcal{K}_W)$. To see that $f_\star(a) \in H_n(Y)$ for every $a \in H_n(X)$, write $b(W) = \phi_W(a(f^{-1}[[W]]))$ for $W \in \mathcal{W}$. Suppose that $W \preccurlyeq W'$ in $\mathbb{W}$ and set $V = f^{-1}[[W]]$, $V' = f^{-1}[[W']]$. By 2Ed, $V \preccurlyeq V'$ and $\phi_{W'}\pi_{V'V} = \pi_{W'W}\phi_W$. But this means that

$$\pi_{W'W} b(W) = \phi_{W'W}\phi_W(a(V)) = \phi_{W'}\pi_{V'V}(a(V)) = \phi_{W'}(a(V')) = b(W').$$

As W and W' are arbitrary, $b \in H_n(Y)$, as required.

Now $f_\star$ is a homomorphism just because $a \mapsto a(V)$ is a homomorphism for every $V \in \mathbb{V}$, so $a \mapsto \phi_W(a(f^{-1}[[W]]))$ is a homomorphism for every $W \in \mathbb{W}$.

(c) If $a \in H_n(X)$ and $U \in \mathbb{U}$, set $W = g^{-1}[[U]]$ and $V = f^{-1}[[W]]$. Then

$$\begin{aligned}(gf)^{-1}[[U]] &= \{(gf)^{-1}[u] : u \in U\} \setminus \{\emptyset\} = \{f^{-1}[g^{-1}[u] : u \in U\} \setminus \{\emptyset\} \\ &= f^{-1}[[g^{-1}[u] : u \in U\} \setminus \{\emptyset\}]] = f^{-1}[[g^{-1}[[U]]\,]] = f^{-1}[[W]] = V\end{aligned}$$

and

$$((gf)_\star(a))(U) = a(V) = (f_\star(a))(W) = (g_\star(f_\star(a)))(U).$$

As U is arbitrary, $(gf)_\star(a) = g_\star f_\star(a)$; as a is arbitrary, $(gf)_\star = g_\star f_\star$.

2H Identities and homeomorphisms If X is a topological space and $f : X \to X$ is the identity function on X, then $f_\star : H_n(X) \to H_n(X)$ is the identity on $H_n(X)$ for every $n \in \mathbb{N}$. As in 1L, you can prove this by tracing through the definitions; but as in 1L, the real point is just that $H_n(X)$ is built by a definite (if lengthy and abstract) process from $(X, \mathfrak{T})$ where $\mathfrak{T}$ is the set of open subsets of X.

Similarly, if $(X, \mathfrak{T})$ and $(Y, \mathfrak{S})$ are topological spaces and $f : X \to Y$ is a homeomorphism, so that $\mathfrak{S} = \{f[v] : v \in \mathfrak{T}\}$ and $\mathfrak{T} = \{f^{-1}[w] : w \in \mathfrak{S}\}$, then $f_\star : H_n(X) \to H_n(Y)$ will be an isomorphism for every $n \in \mathbb{N}$.

2I Definition Let X and Y be topological spaces and f_0, f_1 continuous functions from X to Y. Then f_0 is **homotopic** to f_1 if there is a continuous function $f : [0,1] \times X \to Y$ such that $f(0,t) = f_0(t)$ and $f(1,t) = f_1(t)$ for every $t \in X$.

Note that homotopy is an equivalence relation on the set $C(X;Y)$ of continuous functions from X to Y. **P** If $f_0 \in C(X;Y)$ then $(\alpha, t) \mapsto f_0(t)$ is a homotopy from f_0 to itself. If f is a homotopy from f_0 to f_1, then $(\alpha, t) \mapsto f(1-\alpha, t)$ is a homotopy from f_1 to f_0. If f' is a homotopy from f_0 to f_1 and f'' is a homotopy from f_1 to f_2 set

$$\begin{aligned} f(\alpha,t) &= f'(2\alpha,t) \text{ for } \alpha \in [0,\tfrac{1}{2}],\ t \in X, \\ &= f''(2\alpha-1,t) \text{ for } \alpha \in [\tfrac{1}{2},1],\ t \in X; \end{aligned}$$

then f is a homotopy from f_0 to f_1. **Q**

You must remember that the definition here depends on using the same codomain Y throughout; we can easily have a subspace Y_0 of Y and two continuous functions f_0, $f_1 : X \to Y_0$ such that f_0 and f_1 are homotopic in $C(X;Y)$ but not in $C(X;Y_0)$.

2J Lemma Suppose that X is a compact topological space, Y is any topological space, W is an open cover of Y, and $\hat{f}_0$, $\hat{f}_1 : X \to Y$ are homotopic continuous functions. Then there are a finite open cover V of X and a finite string $f_0, \dots, f_m$ of continuous functions from X to Y such that $f_0 = \hat{f}_0$, $f_m = \hat{f}_1$ and for every $i < m$ and $v \in V$ there is a $w \in W$ such that $v \subseteq f_i^{-1}[w] \cap f_{i+1}^{-1}[w]$.

proof Let $\hat{f} : [0,1] \times X \to Y$ be a continuous function such that $\hat{f}(0,t) = \hat{f}_0(t)$ and $\hat{f}(1,t) = \hat{f}_1(t)$ for every $t \in X$. Let $\mathcal{J}$ be the family of non-empty sub-intervals of $[0,1]$ which are relatively open in $[0,1]$, that is, are of one of the forms

$$[0,\beta[, \quad]\alpha,\beta[, \quad]\alpha,1]$$

where $0 < \alpha < \beta < 1$. Then, writing $\overline{J}$ for the closure of J,

$$\mathcal{U} = \bigcup_{w \in W} \{J \times v : J \in \mathcal{J},\ v \subseteq X \text{ is open and not empty, } \overline{J} \times v \subseteq \hat{f}^{-1}[w]\}$$

is an open cover of $[0,1] \times X$. Because X is compact, so is $[0,1] \times X$ (Engelking 89, 3.2.4[2]), and there is a finite set $\mathcal{U}_0 \subseteq \mathcal{U}$ which covers $[0,1] \times X$. Set $\mathcal{J}_0 = \{J : J \times v \in \mathcal{U}_0\}$, $\mathcal{V}_0 = \{v : J \times v \in \mathcal{U}_0\}$; then $\mathcal{J}_0$ and $\mathcal{V}_0$ are finite. Set $D = \{\inf J : J \in \mathcal{J}_0\} \cup \{\sup J : J \in \mathcal{J}_0\} \cup \{0,1\}$, so that D is a finite set with

[2]For general topology beyond the basic elements reviewed in Fremlin 01, I will use Engelking 89 as my standard reference. I ought however to remark that this wonderful book is not really the place to start from if you need support in the elementary material needed for the present note.

least element 0 and greatest element 1, and enumerate D in ascending order as $(\alpha_0, \ldots, \alpha_m)$. For $t \in X$ set $v_t = \bigcap\{v : t \in v \in V_0\}$, so that $V = \{v_t : t \in X\}$ is a finite open cover of X (with $\#(V) \le 2^{\#(V_0)}$).

Set $f_i(t) = \hat{f}(\alpha_i, t)$ for $0 \le i \le m$ and $t \in X$. Then f_i is a continuous function from X to Y for each i, $f_0 = \hat{f}_0$ and $f_m = \hat{f}_1$.

Take $i < m$ and $v \in V$. Then there is a $t \in X$ such that $v = v_t$. Consider (α_i, t). There is a member U of $\mathcal{U}_0$ such that $(\alpha_i, t) \subseteq U$; express U as $J \times v'$ where $J \in \mathcal{J}_0$ and $v' \in V_0$. Let $w \in W$ be such that $\overline{J} \times v' \subseteq \hat{f}^{-1}[w]$. As $\alpha_i \in J$ and $\alpha_i < 1$, $\alpha_i < \sup J = \max \overline{J} \in D$ and $\alpha_{i+1} \in \overline{J}$, while of course $\alpha_i \in \overline{J}$. Next, $t \in v'$ so $v = v_t \subseteq v'$. So (α_i, t') and (α_{i+1}, t') belong to $\hat{f}^{-1}[w]$ for every $t' \in v$; consequently $f_i(t')$ and $f_{i+1}(t')$ belong to w for every $t' \in v$, and $v \subseteq f_i^{-1}[w] \cap f_{i+1}^{-1}[w]$, as required.

2K The Homotopy Theorem Suppose that X is a compact topological space, Y is any topological space, and $\hat{f}^{(0)}$, $\hat{f}^{(1)} : X \to Y$ are homotopic continuous functions, then $\hat{f}^{(0)}_\star$, $\hat{f}^{(1)}_\star : H_n(X) \to H_n(Y)$ are equal for every $n \in \mathbb{N}$.

proof Take $a \in H_n(X)$ and a proper finite open cover W of Y. By 2J, we have a string $f^{(0)}, \ldots, f^{(m)}$ of continuous functions from X to Y and a finite open cover V of X such that for every $v \in V$ and $i < m$ there is a $w \in W$ such that $f^{(i)}[v] \cup f^{(i+1)}[v] \subseteq w$, while $f^{(0)} = \hat{f}^{(0)}$ and $f^{(m)} = \hat{f}^{(1)}$.

I claim that $(f^{(i)}_\star(a))(W) = (f^{(i+1)}_\star(a))(W)$ for every $i < m$. **P** Choose $h : V \to W$ such that $f^{(i)}[v] \cup f^{(i+1)}[v] \subseteq h(v)$ for every $v \in V$. Set $V' = (f^{(i)})^{-1}[[W]]$, and define $\phi'_W : H_n(\mathcal{K}_{V'}) \to H_n(\mathcal{K}_W)$ from $f^{(i)} : X \to Y$ as in Lemma 2Eb. As $V \preccurlyeq V'$,

$$(f^{(i)}_\star(a))(W) = \phi'_W(a(V')) = \phi'_W \pi_{V'V}(a(V)) = h_\star(a(V))$$

by 2Ec. But now the same argument tells us that

$$(f^{(i+1)}_\star(a))(W) = h_\star(a(V)) = (f^{(i)}_\star(a))(W). \text{ } \mathbf{Q}$$

As i is arbitrary, $(\hat{f}^{(0)}_\star(a))(W) = (\hat{f}^{(1)}_\star(a))(W)$. As W is arbitrary, $\hat{f}^{(0)}_\star(a) = \hat{f}^{(1)}_\star(a)$; as a is arbitrary, $\hat{f}^{(0)}_\star = \hat{f}^{(1)}_\star$, as claimed.

Notes and comments In this section we find ourselves in territory which is commonly thought of as 'analysis' rather than 'algebra'. But down to Proposition 2G we are really still doing algebra, even though I speak of topological spaces. The point I wish to make is that even though we have a second-order structure with formulae like '$t \in v \in V$', there is no hint of a limit operation. (For instance, in 2A I use the rule that the intersection of two open sets is open, but nowhere do I use the rule that any union of open sets is open, which requires quantifying over subsets of the set of open sets.) The definition of continuity, '$f^{-1}[w]$ is open for every open w', quantifies only over a set of open sets. So all the theory in this part could be carried out for arbitrary lattices with greatest and least members in place of lattices of open subsets of topological spaces, replacing the operators $w \mapsto f^{-1}[w]$, for continuous functions f, with appropriate lattice homomorphisms.

Note that $H_n(X)$, as defined in 2F, is the 'inverse limit' of the system $(\langle H_n(\mathcal{K}_V)\rangle_{V \in \mathbb{V}}, \langle \pi_{V'V}\rangle_{V \preccurlyeq V'})$. Naturally, there is a general theory of such limits; but we shall be able to avoid any appeal to results from that theory, at the cost of some repetition of calculations (compare E2Cd just below with part (a-ii) of the proof of 3H later on).

In 2J we enter analysis proper, with an appeal to the basic properties of compactness, and in particular to the theorem that a product of compact sets is compact. The Homotopy Theorem (2K), a dramatic elaboration of the Contiguity Theorem (1Q, used in Lemmas 2B and 2E), is the point at which we get a really powerful insight into the nature of continuous functions. Its application will of course depend on being able to calculate the homology groups of some interesting topological spaces. This will be the subject of Chapters 3 and 4.

E2 Examples for Chapter 2

E2A The most trivial cases (a) As in E1D, I take the trouble to spell out what happens for the simplest of all topological spaces, the empty set. In this case, since every open set is empty, there is exactly one proper finite open cover, which is the empty cover $V_0 = \emptyset$. Now $\mathcal{K}_{V_0} = \mathcal{P}V_0 = \{\emptyset\}$ and $H_n(\mathcal{K}_{V_0}) = \{0\}$ for every $n \in \mathbb{N}$, by E1D. As the set $\mathbb{V}$ of all proper finite open covers of X is $\{V_0\}$, the product $\prod_{V\in\mathbb{V}} H_n(\mathcal{K}_V)$ is isomorphic to $H_n(\mathcal{K}_{V_0})$ and is $\{0\}$. So $H_n(X)$, which is a quotient of a subgroup of $\prod_{V\in\mathbb{V}} H_n(\mathcal{K}_V)$, is also $\{0\}$, for every $n \in \mathbb{N}$.

(b) Similarly, if X is a singleton, there is only one proper finite open cover of X, this time being $V_0 = \{X\}$. As in (a), $H_n(X) \cong H_n(\mathcal{K}_{V_0})$; but by E1E, $H_n(\mathcal{K}_{V_0}) = \{0\}$.

E2B Constant functions Let X and Y be topological spaces and $f : X \to Y$ a constant function. Then $f_\star : H_n(X) \to H_n(Y)$ is the zero homomorphism for every $n \in \mathbb{N}$.

proof (a) By E2A, this is certainly true if X is empty.

(b) Otherwise, f can be expressed as gh where $h : X \to \{t\}$ is constant and $g : \{t\} \to Y$ takes t to itself. But $H_n(\{t\}) = 0$ for every n, by E2Ab. So $h_\star$, $g_\star$ and $f_\star = g_\star h_\star$ must all be trivial. (Compare E1G.)

E2C The group $H_0(X)$ (a) In E1F, I showed that if $\mathcal{K}$ is an abstract simplicial complex then $H_0(\mathcal{K})$ is determined by the number of components of $\mathcal{K}$, and that except for the special case $K = \emptyset$, we can determine the number of components from the group $H_0(\mathcal{K})$. For topological spaces things can be more complicated, but we have a similar pattern. First, we need to know something about connectedness. A topological space X is **connected** if the only subsets of X which are both open and closed are X and $\emptyset$. If two subsets A and B of X are connected and $A \cap B \ne \emptyset$, then $A \cup B$ is connected (Engelking 89, 6.1.10. We therefore have an equivalence relation $\sim$ on X defined by saying that $s \sim t$ if there is a connected subset of X containing both s and t. The equivalence classes under this relation are the **components** of X, which are connected and closed (Engelking 89, 6.1.10-6.1.11). If X has only finitely many components, they are themselves open as well as closed.

(b) From now on, down to the end of (e), suppose that X has just k components where $k \ge 1$. Write $\mathbb{V}$ for the set of proper finite open covers of X. Then the set W of components of X belongs to $\mathbb{V}$. For any $V \in \mathbb{V}$ which refines W, we have a unique function $h_V : V \to W$ such that $v \subseteq h_V(v)$ for every $v \in V$. Now the set of components of $\mathcal{K}_V$, as described in E1Fa, is just $\{h_V^{-1}[\{w\}] : w \in W\}$. **P** If v, $v' \in V$ and $v \cap v' \ne \emptyset$ then $h_V(v)$ meets $h_V(v')$ and $h_V(v) = h_V(v')$. So if v and v' belong to the same component of $\mathcal{K}_V$ then $h_V(v) = h_V(v')$. On the other hand, if V' is a component of $\mathcal{K}_V$ then $v' \cap v = \emptyset$ whenever $v' \in V'$ and $v \in V \setminus V'$, so $\bigcup V'$ and $X \setminus \bigcup V' = \bigcup(V \setminus V')$ are open; now $\bigcup V'$ is included in the common value w of $h_V(v')$ for $v' \in V'$, and w is connected, so $w \setminus \bigcup V'$ is empty, $w = \bigcup V'$ and $V' = h_V^{-1}[\{w\}]$. Thus we see that for every $v' \in V$, the component of $\mathcal{K}_V$ containing v' is $h_V^{-1}[\{h_V(v')\}]$. Finally, if $w \in W$, $w \ne \emptyset$ so there is a $v \in V$ such that $v \cap w \ne \emptyset$ and $h_V(v) = w$. So the set of components of $\mathcal{K}_V$ is $\{h_V^{-1}[\{w\}] : w \in W\}$ precisely. **Q**

(c) We saw also, in E1Fb, that if $V \preccurlyeq W$ and y is a 0-chain in $\mathcal{K}_V$ then y is a boundary iff $\sum_{v\in V'} y(<v>) = 0$ for every component V' of $\mathcal{K}_V$. But in view of (b) this means that y is a boundary iff $h_V{\bullet}y = 0$. As W is a disjoint family of sets, $\mathcal{K}_W$ has no 1-simplices and $B_0(\mathcal{K}_W) = \{0\}$. So, for $y \in C_0(\mathcal{K}_V)$, $y \in B_0(\mathcal{K}_V)$ iff $h_V{\bullet}y \in B_0(\mathcal{K}_W)$. We also know that

$$\begin{aligned} y \in Z_0(\mathcal{K}_V) &\iff \sum_{v\in V} y(<v>) = 0 \\ &\iff \sum_{w\in W} \sum_{\substack{v\in V \\ h_V(v)=w}} y(<v>) = 0 \\ &\iff \sum_{w\in W} (h_V{\bullet}y)(<w>) = 0 \iff h_V{\bullet}y \in Z_0(\mathcal{K}_W). \end{aligned}$$

Also $h_V[C_0(\mathcal{K}_V)] = C_0(\mathcal{K}_W)$. **P** If $y \in C_0(\mathcal{K}_W)$, choose for each $w \in W$ a $t_w \in W$ and a $v_w \in V$ such that $t_w \in v_w$; then $h(v_w) \cap w \ne \emptyset$ so $h(v_w) = w$. Of course $w \mapsto v_w$ is injective. Define $x \in C_0(\mathcal{K}_V)$ by setting $x(<v_w>) = y(<w>)$ for every $w \in W$; then $h{\bullet}x = y$. **Q**

Accordingly $h_V[Z_0(\mathcal{K}_V)] = Z_0(\mathcal{W})$ and $\pi_{WV} = h_{V\star} : H_0(\mathcal{K}_V) \to H_0(\mathcal{K}_W)$ is an isomorphism.

(d) It follows that $a \mapsto a(W) : H_0(X) \to H_0(\mathcal{K}_W)$ is an isomorphism.

P(i) If a, $b \in H_0(X)$ and $a \ne b$, there is a $V' \in \mathbb{V}$ such that $a(V') \ne b(V')$. Now there is a $V \in \mathbb{V}$ refining both V' and W, in which case $\pi_{V'V}(a(V)) = a(V') \ne b(V') = \pi_{V'V}(b(V))$ and $a(V) \ne b(V)$. Since π_{WV} is injective, $a(W) = \pi_{WV}(a(V)) \ne \pi_{WV}(b(V)) = b(W)$. This shows that $a \mapsto a(W)$ is injective.

(ii) Now suppose that $c \in H_0(\mathcal{K}_W)$. Define $a \in \prod_{V\in\mathbb{V}} H_0(\mathcal{K}_V)$ as follows. Given $V \in \mathbb{V}$, write $V \wedge W$ for $\{v \cap w : v \in V,\, w \in W\} \setminus \{\emptyset\}$; then $V \wedge W \in \mathbb{V}$ refines both V and W. Set

$$a(V) = \pi_{V,V\wedge W}\pi^{-1}_{W,V\wedge W}c.$$

If $V \preccurlyeq V'$ in $\mathbb{V}$, then $V \wedge W \preccurlyeq V' \wedge W \preccurlyeq W$ and

$$\pi_{V'V}\pi_{V,V\wedge W} = \pi_{V',V\wedge W} = \pi_{V',V'\wedge W}\pi_{V'\wedge W,V\wedge W}$$

while

$$\pi_{W,V'\wedge W}\pi_{V'\wedge W,V\wedge W} = \pi_{W,V\wedge W}, \quad \pi_{V'\wedge W,V\wedge W}\pi^{-1}_{W,V\wedge W} = \pi^{-1}_{W,V'\wedge W}.$$

Accordingly

$$\begin{aligned}\pi_{V'V}(a(V)) &= \pi_{V'V}\pi_{V,V\wedge W}\pi^{-1}_{W,V\wedge W}(c) = \pi_{V',V'\wedge W}\pi_{V'\wedge W,V\wedge W}\pi^{-1}_{W,V\wedge W}(c)\\ &= \pi_{V',V'\wedge W}\pi^{-1}_{W,V'\wedge W}(c) = a(V').\end{aligned}$$

As V and V' are arbitrary, $a \in H_n(X)$ and of course $a(W) = c$. So we see that $a \mapsto a(W)$ is surjective; with (i), this shows that it is bijective; of course it is a homomorphism; so it is an isomorphism. **Q**

(e) Since W is disjoint, $\mathcal{K}_W = \{\emptyset\} \cup \{\{w\} : w \in W\}$ and the number of components of $\mathcal{K}_W$ is $\#(W) = k$. So

$$H_0(X) \cong H_0(\mathcal{K}_W) \cong \mathbb{Z}^{k-1}$$

by E1F. Thus for a non-empty topological space X with finitely many components, the group $H_0(X)$ tells us the number of components of X.

(f) All this work has been done on the assumption that X has a component. But if X is empty, then we know from E2A that $H_0(X) = \{0\}$.

3 Geometric complexes

Chapter 2 offered a way of building simplicial complexes from topological spaces, and hence defining homology groups for an arbitrary topological space, with homomorphisms derived from continuous functions. But in order to calculate the homology groups of even the most familiar topological spaces, we need to have a way of building a topological space from a simplicial complex.

3A Definitions (a) Let $\mathcal{K}$ be a finite simplicial complex such that its vertex set V is a subset of Euclidean r-dimensional space $\mathbb{R}^r$, where $r \in \mathbb{N}$. Write $M(\mathcal{K})$ for the set of functions $\alpha : V \to [0,1]$ such that $\sum_{v\in V} \alpha(v) = 1$ and $\{v : \alpha(v) > 0\} \in \mathcal{K}$. For $\alpha \in M(\mathcal{K})$, set $\operatorname{brc}(\alpha) = \sum_{v\in V} \alpha(v)v \in \mathbb{R}^r$, the **barycenter** of α. I will say that $\mathcal{K}$ is **geometrically realizable** if $\operatorname{brc} : M(\mathcal{K}) \to \mathbb{R}^r$ is injective.

(b) I will say that a **geometric complex** is a pair $(X,\mathcal{K})$ where $\mathcal{K}$ is a geometrically realizable simplicial complex with vertex set included in $\mathbb{R}^r$ for some $r \in \mathbb{N}$, and $X = \{\operatorname{brc}(\alpha) : \alpha \in M(\mathcal{K})\}$. In this case I will call X the **carrier** of $\mathcal{K}$.

(c) A topological space is **triangulable** if it is homeomorphic to the carrier of some some geometrically realizable simplicial complex.

3B Remarks As usual, I am putting conciseness and logical rigour well ahead of intuitive clarity. We shall not be able to move forward at all without some elementary geometric facts. Let $\mathcal{K}$ be a geometrically realizable finite simplicial complex in $\mathbb{R}^r$ where $r \in \mathbb{N}$, and X its carrier.

(a)(i) For any set $D \subseteq \mathbb{R}^r$, write $\Gamma(D)$ for its **convex hull**, that is, the smallest convex subset of $\mathbb{R}^r$ including D, the set

$$\{\textstyle\sum_{i=0}^n \alpha_i d_i : d_0, \dots , d_n \in D,\ \alpha_0, \dots , \alpha_n \in [0,1],\ \sum_{i=0}^n \alpha_i = 1\}.$$

(ii) If D is finite then $\Gamma(D)$ is compact. **P** If D is empty, so is $\Gamma(D)$ and we can stop. Otherwise, set $M(D) = \{\alpha : \alpha \in [0,1]^D,\ \sum_{d\in D} \alpha(d) = 1\}$; then $M(D)$ is closed and bounded, therefore compact, and the map $\alpha \mapsto \operatorname{brc}(\alpha) = \sum_{d\in D} \alpha(d)d : M(D) \to \mathbb{R}^r$ is a continuous surjection from $M(D)$ onto $\Gamma(D)$, so $\Gamma(D)$ is compact. **Q**

(iii) If D is bounded then $\operatorname{diam}\Gamma(D) = \operatorname{diam}(D)$, where $\operatorname{diam}(D) = \sup_{t,t'\in D} \|t - t'\|$ is the **diameter** of D. (In this formula, interpret $\sup\emptyset$ as 0 if D is empty.) **P** As $D \subseteq \Gamma(D)$, $\operatorname{diam}(D) \le \operatorname{diam}\Gamma(D)$. In the other direction, write $B(t,\gamma) = \{t' : \|t' - t\| \le \gamma\}$ for $t \in \mathbb{R}^r$ and $\gamma \ge 0$. Then $B(t,\gamma) = t + \gamma\boldsymbol{B}_r$ is always convex, and $D \subseteq B(t, \operatorname{diam}(D))$ for every $t \in D$, by the definition of 'diameter'. But this implies that $\Gamma(D) \subseteq B(t, \operatorname{diam}(D))$ for every $t \in D$, that is, that $t' \in B(t, \operatorname{diam}(D))$ whenever t', $t \in \Gamma(D)$, that is, $\operatorname{diam}\Gamma(D) \le \operatorname{diam}(D)$. **Q**

(b) $X = \bigcup_{K\in\mathcal{K}} \Gamma(K)$. **P** If $K \in \mathcal{K}$ and $t \in \Gamma(K)$, then $t = \operatorname{brc}(\alpha)$ for some $\alpha : K \to [0,1]$ such that $\sum_{v\in K} \alpha(v) = 1$; setting $\alpha'(v) = \alpha(v)$ for $v \in K$, 0 for $v \in V \setminus K$, we get a member of $M(\mathcal{K})$ and

$$t = \operatorname{brc}(\alpha) = \operatorname{brc}(\alpha') \in X.$$

Thus $\bigcup_{K\in\mathcal{K}} \Gamma(K) \subseteq X$. Conversely, if $t \in X$, there is an $\alpha \in M(\mathcal{K})$ such that $t = \operatorname{brc}\alpha$ and $K = \{v : \alpha(v) > 0\}$ belongs to $\mathcal{K}$; in which case $t = \sum_{v\in K} \alpha(v)v$ belongs to $\Gamma(K)$. As t is arbitrary, $X \subseteq \bigcup_{K\in\mathcal{K}} \Gamma(K)$ and we have equality. **Q**

(c) If $K \in \mathcal{K}$, then $V \cap \Gamma(K) = K$. **P** If $v_0 \in K$ then surely $v_0 \in V \cap \Gamma(K)$. If $v_0 \in V \cap \Gamma(K)$ then we can express v_0 simultaneously as $\operatorname{brc}(\alpha)$ and as $\operatorname{brc}(\alpha')$ where

$$\alpha(v_0) = 1,\ \alpha(v) = 0 \text{ for } v \in V \setminus \{v_0\},$$

$$\alpha'(v) \ge 0 \text{ for } v \in K, \quad \textstyle\sum_{v\in K} \alpha'_v = 1. \quad \alpha(v) = 0 \text{ for } v \in V \setminus K.$$

By the definition of geometric coherence, $\alpha' = \alpha$ and in particular $\alpha'(v_0) > 0$ so $v_0 \in K$. **Q**

(d) If we think of $M(\mathcal{K})$ as a subset of

$$\{\alpha : \alpha \in [0,1]^V,\ \textstyle\sum_{v\in V} \alpha(v) = 1\},$$

it is a closed bounded set in $\mathbb{R}^V$ and is therefore compact. Next, brc : $M(\mathcal{K}) \to \mathbb{R}^r$ is continuous, so $X = \mathrm{brc}[M(\mathcal{K})]$ is compact; and because brc is one-to-one, it is actually a homeomorphism between $M(\mathcal{K})$ and X.

3C Simple covers (a) Let $\mathcal{K}$ be a geometrically realizable simplicial complex in $\mathbb{R}^r$, and X its carrier. For $v \in V$, write

$$\widehat{v} = \{\mathrm{brc}(\alpha) : \alpha \in M(\mathcal{K}),\, \alpha(v) > 0\} \subseteq X.$$

Then $\widehat{v} = X \setminus \{\mathrm{brc}(\alpha) : \alpha \in M(\mathcal{K}),\, \alpha(v) = 0\}$. Since $\{\alpha : \alpha \in M(\mathcal{K}),\, \alpha(v) = 0\}$ is a closed subset of $M(\mathcal{K}) \subseteq [0,1]^V$, therefore compact, and brc is a continuous function, $\{\mathrm{brc}(\alpha) : \alpha \in M(\mathcal{K}),\, \alpha(v) = 0\}$ is compact, therefore closed, and $\widehat{v}$ is open for the subspace topology of X. Note that $V \cap \widehat{v} = \{v\}$, because $v = \mathrm{brc}(\alpha)$ where $\alpha(v) = 1$, $\alpha(v') = 0$ for $v' \in V \setminus \{v\}$. So $\widehat{v} \neq \widehat{v'}$ if v, $v' \in V$ are distinct.

If $\alpha \in M(\mathcal{K})$, $\sum_{v \in V} \alpha(v) = 1$, so there is surely a $v \in V$ such that $\alpha(v) > 0$, and $\mathrm{brc}(\alpha) \in \widehat{v}$. Thus $\widehat{V} = \{\widehat{v} : v \in V\}$ is a proper open cover of X.

For $K \subseteq V$, $K \in \mathcal{K}$ iff $\bigcap\{\widehat{v} : v \in K\} \neq \emptyset$. **P** If $K = \emptyset$, then $K \in \mathcal{K}$ and $\bigcap\{\widehat{v} : v \in K\} = \bigcap\emptyset$ is non-empty. If $K \neq \emptyset$ and $K \in \mathcal{K}$, then we have an $\alpha \in M(\mathcal{K})$ such that $\alpha(v) = \frac{1}{\#(K)}$ for every $v \in K$, and now $\mathrm{brc}(\alpha) \in \bigcap_{v \in K} \widehat{v}$, so $\bigcap_{v \in K} \widehat{v}$ is non-empty. If $K \neq \emptyset$ and $t \in \bigcap_{v \in K} \widehat{v}$, then $t \in X$ so there is just one $\alpha \in M(\mathcal{K})$ such that $t = \mathrm{brc}(\alpha)$; now $\alpha(v) > 0$ for every $v \in K$, so $K \subseteq \{v : \alpha(v) > 0\} \in \mathcal{K}$ and $K \in \mathcal{K}$. **Q**

What this means is that the bijection $v \mapsto \widehat{v}$ defines an isomorphism between $\mathcal{K}$ and $\mathcal{K}_{\widehat{V}}$.

(b) Now suppose that $\mathcal{L}$ is a geometrically realizable simplicial complex in $\mathbb{R}^s$, where $s \in \mathbb{N}$, and Y its carrier. Write $\mathrm{brc}_{\mathcal{K}} : M(\mathcal{K}) \to X$ and $\mathrm{brc}_{\mathcal{L}} : M(\mathcal{L}) \to Y$ for the barycenter maps involved.

(i) Let $g : V \to W$ be a simplicial map from $\mathcal{K}$ to $\mathcal{L}$. Then we have a continuous function $\bar{g} : X \to Y$ defined by setting

$$\bar{g}(\mathrm{brc}_{\mathcal{K}}(\alpha)) = \mathrm{brc}_{\mathcal{L}}(\beta)$$

whenever $\alpha \in M(\mathcal{K})$, $\beta \in M(\mathcal{L})$ and $\beta(w) = \sum_{v \in V, g(v) = w} \alpha(v)$ for $w \in W$. **P** For $\alpha \in M(\mathcal{K})$, we can define $h(\alpha) \in [0, \infty[^W$ by setting $h(\alpha)(w) = \sum_{v \in V, g(v) = w} \alpha(v)$ for $w \in W$, and $\sum_{w \in W} h(\alpha)(w) = \sum_{v \in V} \alpha(v) = 1$. Also $\{w : w \in W,\, h(\alpha)(w) > 0\} = \{g(v) : v \in V,\, \alpha(v) > 0\}$ is of the form $g[K]$ for some $K \in \mathcal{K}$, so belongs to $\mathcal{L}$. Thus h is a function from $M(\mathcal{K})$ to $M(\mathcal{L})$. Moreover, for any $w \in W$, $\alpha \mapsto h(\alpha)(w) = \sum_{v \in V, g(v) = w} \alpha(v)$ is continuous, so h is continuous when regarded as a function from $M(\mathcal{K})$ either to $\mathbb{R}^W$ or to $M(\mathcal{L})$.

Since $\bar{g} = \mathrm{brc}_{\mathcal{L}}\, h\, \mathrm{brc}_{\mathcal{K}}^{-1}$, and both $\mathrm{brc}_{\mathcal{L}}$ and $\mathrm{brc}_{\mathcal{K}}^{-1}$ are continuous (3Bd), $\bar{g}$ is continuous.

(ii) If $v \in V$ then $\widehat{v} \subseteq \bar{g}^{-1}[\widehat{g(v)}]$. **P** Take $t \in \widehat{v}$. Set $\alpha = \mathrm{brc}_{\mathcal{K}}^{-1}(t)$; then $\alpha(v) > 0$. So $h(\alpha)(g(v)) = \sum_{v' \in V, g(v') = g(v)} \alpha(v')$ is greater than 0 and

$$\bar{g}(t) = \mathrm{brc}_{\mathcal{L}}\, h(\alpha) \in \widehat{g(v)},$$

that is, $t \in \bar{g}^{-1}[\widehat{g(v)}]$. **Q**

(c) I hope that using the expression $\widehat{v}$ to represent a set and the expression $\widehat{V}$ to represent a family of sets will not be intolerably confusing. Note in particular that they are defined in terms of a well-hidden variable, a geometrically realizable simplicial complex $\mathcal{K}$ with vertex set V containing v.

3D Subdivisions: Proposition Let $\mathcal{K}$ be a geometrically realizable simplicial complex in $\mathbb{R}^r$. Let $I = \{v_0, v_1\}$ be an edge of $\mathcal{K}$ and set $\hat{v} = \frac{1}{2}(v_0 + v_1)$. Let $\mathcal{L}$ be the elementary subdivision of $\mathcal{K}$ based on I and $\hat{v}$, as described in 1R. Write V and $W = V \cup \{\hat{v}\}$ for the vertex sets of $\mathcal{K}$, $\mathcal{L}$ respectively.

(a) $\mathcal{L}$ is a geometrically realizable simplicial complex in $\mathbb{R}^r$.

(b) $\mathcal{K}$ and $\mathcal{L}$ have the same carrier.

(c) Writing $\widehat{V} = \{\widehat{v}_{\mathcal{K}} : v \in V\}$ and $\widehat{W} = \{\widehat{w}_{\mathcal{L}} : w \in W\}$, $\widehat{W}$ refines $\widehat{V}$. If $n \in \mathbb{N}$, then $\pi_{\widehat{V}\widehat{W}} : H_n(\mathcal{K}_{\widehat{W}}) \to H_n(\mathcal{K}_{\widehat{V}})$, as defined in 2C, is an isomorphism.

proof (a)(i) I ought to note at once that $\hat{v}$ belongs to $\Gamma(I) \setminus I$ so cannot belong to V, by 3Bc. We can therefore use it in the way envisaged in 1R.

(ii) We have a function $\theta : M(\mathcal{K}) \to M(\mathcal{L})$ defined by setting $\theta(\alpha) = \beta$ where

$$\beta(v) = 2\delta \text{ if } v = \hat{v},$$

where $\delta = \min(\alpha(v_0), \alpha(v_1))$

$$= \alpha(v) - \delta \text{ if } v \in I,$$
$$= \alpha(v) \text{ if } v \in V \setminus I.$$

P Of course β is a function from $V \cup \{\hat{v}\}$ to $[0, \infty[$ and $\sum_{v \in V \cup \{\hat{v}\}} \beta(v) = \sum_{v \in V} \alpha(v) - 2\delta + 2\delta = 1$. Also, if $K = \{v : v \in V, \alpha(v) > 0\}$ then

$$\begin{aligned}\{v : v \in V \cup \{\hat{v}\}, \beta(v) > 0\} &= K \text{ if } I \not\subseteq K,\\ &= (K \setminus \{v_0\}) \cup \{\hat{v}\} \text{ if } I \subseteq K \text{ and } \alpha(v_0) < \alpha(v_1),\\ &= (K \setminus I) \cup \{\hat{v}\} \text{ if } I \subseteq K \text{ and } \alpha(v_0) = \alpha(v_1),\\ &= (K \setminus \{v_1\}) \cup \{\hat{v}\} \text{ if } I \subseteq K \text{ and } \alpha(v_1) < \alpha(v_0),\end{aligned}$$

and in any case belongs to $\mathcal{L}$. So $\beta \in M(\mathcal{L})$. **Q**

(iii) In the other direction, we have a function $\theta' : M(\mathcal{L}) \to M(\mathcal{K})$ defined by setting $\theta'(\beta) = \alpha$ where

$$\begin{aligned}\alpha(v) &= \beta(v) + \frac{1}{2}\beta(\hat{v}) \text{ if } v \in I,\\ &= \beta(v) \text{ if } v \in V \setminus I.\end{aligned}$$

P Again, α is a function from V to $[0, \infty[$ and $\sum_{v \in V} \alpha(v) = \sum_{v \in V} \beta(v) + \beta(\hat{v}) = 1$. And setting $L = \{v : v \in V \cup \{\hat{v}\}, \beta(v) > 0\}$ we have $L \in \mathcal{L}$ and

$$\begin{aligned}\{v : v \in V, \alpha(v) > 0\} &= (L \setminus \{\hat{v}\}) \cup I \text{ if } \hat{v} \in L,\\ &= L \text{ otherwise},\end{aligned}$$

and in either case belongs to $\mathcal{K}$. So $\alpha \in M(\mathcal{K})$. **Q**

(iv)($\boldsymbol{\alpha}$) Now if $\alpha \in M(\mathcal{K})$ and $\beta = \theta(\alpha)$, $\theta'(\beta) = \alpha$. **P** Writing δ for $\min(\alpha(v_0), \alpha(v_1))$ and α' for $\theta'(\beta)$, we have

$$\begin{aligned}\alpha'(v) &= \beta(v) + \frac{1}{2}\beta(\hat{v}) = \beta(v) + \delta = \alpha(v) \text{ if } v \in I,\\ &= \beta(v) = \alpha(v) \text{ if } v \in V \setminus I. \textbf{ Q}\end{aligned}$$

($\boldsymbol{\beta}$) Similarly, if $\beta \in M(\mathcal{L})$ and $\alpha = \theta'(\beta)$, then $\theta(\alpha) = \beta$. **P** Note that

$$\min(\alpha(v_0), \alpha(v_1)) = \min(\beta(v_0) + \frac{1}{2}\beta(\hat{v}), \beta(v_1) + \frac{1}{2}\beta(\hat{v})) = \frac{1}{2}\beta(\hat{v})$$

because $\{v_0, v_1\} \notin \mathcal{L}$ so $\min(\beta(v_0), \min \beta(v_1)) = 0$. Now, writing β' for $\theta(\alpha)$,

$$\begin{aligned}\beta'(v) &= 2\min(\alpha(v_0), \alpha(v_1)) = \beta(\hat{v}) \text{ if } v = \hat{v},\\ &= \alpha(v) - \frac{1}{2}\beta(\hat{v}) = \beta(v) \text{ if } v \in I,\\ &= \alpha(v) = \beta(v) \text{ if } v \in V \setminus I.\end{aligned}$$

So $\beta' = \beta$. **Q**

(v) Accordingly θ and θ' are the two halves of a one-to-one correspondence between $M(\mathcal{K})$ and $M(\mathcal{L})$. Now observe that if $\beta \in M(\mathcal{L})$ and $\alpha = \theta'(\beta)$,

$$\operatorname{brc}(\alpha) = \sum_{v \in V} \alpha(v)v = \sum_{v \in V} \beta(v)v + \frac{1}{2}\beta(\hat{v})(v_0 + v_1) = \sum_{w \in V \cup \{\hat{v}\}} \beta(w)w = \operatorname{brc}(\beta).$$

If β, β' are distinct points of $M(\mathcal{L})$, $\theta'(\beta) \neq \theta'(\beta')$ so

$$\mathrm{brc}(\beta) = \mathrm{brc}(\theta'(\beta)) \neq \mathrm{brc}(\theta'(\beta')) = \mathrm{brc}(\beta')$$

and brc is injective on $M(\mathcal{L})$. Thus $\mathcal{L}$ is geometrically realizable in $\mathbb{R}^r$.

(b) The formulae of (a-iv) and (a-v) now show that if X is the carrier of $\mathcal{K}$ then

$$X = \{\mathrm{brc}(\alpha) : \alpha \in M(\mathcal{K})\} = \{\mathrm{brc}(\theta'(\beta)) : \beta \in M(\mathcal{L})\} = \{\mathrm{brc}(\beta) : \beta \in M(\mathcal{L})\}$$

is the carrier of $\mathcal{L}$.

(c) I write $\widehat{v}_{\mathcal{K}}$ and $\widehat{w}_{\mathcal{L}}$ emphasize that we need to look carefully at what happens when $v \in V$ is regarded as a member of the vertex set of $\mathcal{L}$.

(i) We know that $\theta : M(\mathcal{K}) \to M(\mathcal{L})$ is a bijection and that $\mathrm{brc}(\theta(\alpha)) = \mathrm{brc}(\alpha)$ for every $\alpha \in M(\mathcal{K})$. For $v \in V \setminus I$,

$$\begin{aligned}\widehat{v}_{\mathcal{L}} &= \{\mathrm{brc}(\beta) : \beta \in M(\mathcal{L}),\, \beta(v) > 0\} = \{\mathrm{brc}(\alpha) : \alpha \in M(\mathcal{K}),\, (\theta(\alpha))(v) > 0\} \\ &= \{\mathrm{brc}(\alpha) : \alpha \in M(\mathcal{K}),\, \alpha(v) > 0\} = \widehat{v}_{\mathcal{K}}.\end{aligned}$$

For $v \in I$,

$$\begin{aligned}\widehat{v}_{\mathcal{L}} &= \{\mathrm{brc}(\alpha) : \alpha \in M(\mathcal{K}),\, (\theta(\alpha))(v) > 0\} \\ &\subseteq \{\mathrm{brc}(\alpha) : \alpha \in M(\mathcal{K}),\, \alpha(v) > 0\} = \widehat{v}_{\mathcal{K}},\end{aligned}$$

while

$$\begin{aligned}\widehat{\hat{v}}_{\mathcal{L}} &= \{\mathrm{brc}(\alpha) : \alpha \in M(\mathcal{K}),\, (\theta(\alpha))(\hat{v}) > 0\} \\ &= \{\mathrm{brc}(\alpha) : \alpha \in M(\mathcal{K}),\, \min(\alpha(v_0), \alpha(v_1)) > 0\} = \widehat{v}_{0\mathcal{K}} \cap \widehat{v}_{1\mathcal{K}}.\end{aligned}$$

Thus in every case, if $w \in W$, there is a $v \in V$ such that $\widehat{w}_{\mathcal{L}} \subseteq \widehat{v}_{\mathcal{K}}$, and $\widehat{W}$ refines $\widehat{V}$.

(ii) Now take any $n \geq 0$. Writing f for the identity function on X, we see that $\widehat{V} = f^{-1}[[\widehat{V}]]$. In the language of 2E, the homomorphism $\phi_{\widehat{V}} : H_n(\mathcal{K}_{f^{-1}[[\widehat{V}]]}) \to H_n(\mathcal{K}_{\widehat{V}})$ is actually the identity. Moreover, if we define $h : \widehat{W} \to \widehat{V}$ by setting $h(\widehat{v}_{\mathcal{L}}) = \widehat{v}_{\mathcal{K}}$ for $v \in V$, while $h(\widehat{\hat{v}}_{\mathcal{L}}) = \widehat{v}_{0\mathcal{K}}$, we shall have a homomorphism $h_\star : H_n(\mathcal{K}_{\widehat{W}}) \to H_n(\mathcal{K}_{\widehat{V}})$, and 2Ec tells us that $h_\star = \phi_{\widehat{V}} \pi_{\widehat{V}\widehat{W}} = \pi_{\widehat{V}\widehat{W}}$.

(iii) At this point observe that, under the natural bijections $V \leftrightarrow \widehat{V}$ and $W \leftrightarrow \widehat{W}$, $h : \widehat{W} \to \widehat{V}$ corresponds to $k : W \to V$ where $k(v) = v$ for $v \in V$ and $k(\hat{v}) = v_0$. And because the same bijections map $\mathcal{K}_V$ onto $\mathcal{K}_{\widehat{V}}$ and $\mathcal{K}_W$ onto $\mathcal{K}_{\widehat{W}}$, $h_\star : H_n(\mathcal{K}_{\widehat{W}}) \to H_n(\mathcal{K}_{\widehat{V}})$ must match $k_\star : H_n(\mathcal{K}_W) \to H_n(\mathcal{K}_V)$. But the Subdivision Theorem (1S) tells us that $k_\star$ is an isomorphism. So $\pi_{\widehat{V}\widehat{W}} = h_\star$ also is an isomorphism.

3E Lemma Let $r \in \mathbb{N}$ be an integer. On $\mathbb{R}^r$ let $\| \;\|$ be the Euclidean norm, so that $\|t\| = \sqrt{\sum_{i=0}^{r-1} t(i)^2}$ for $t \in \mathbb{R}^r$. (For the case $r = 0$, I count the empty sum $\sum_{i=0}^{-1}$ as 0. The ordinary Euclidean distance between two points t_0, t_1 is $\|t_1 - t_0\|$.)

Suppose that v_0, v_1, $v \in \mathbb{R}^r$ are such that $\gamma = \|v_1 - v_0\| \geq \max(\|v - v_0\|, \|v - v_1\|)$. Set $\hat{v} = \frac{1}{2}(v_0 + v_1)$. Then $\|v - \hat{v}\| \leq \frac{\sqrt{3}}{2}\|v_1 - v_0\|$.

proof Set $t_0 = v - v_0$, $t_1 = v - v_1$. Then $v - \hat{v} = \frac{1}{2}(t_0 + t_1)$ while $\|t_1 - t_0\| \geq \max(\|t_0\|, \|t_1\|)$. Writing $(t|t') = \sum_{i=0}^{r-1} t(i)t'(i)$ for t, $t' \in \mathbb{R}^r$, we have

$$\begin{aligned}\|t_1 + t_0\|^2 + \|t_1 - t_0\|^2 &= (t_1 + t_0|t_1 + t_0) + (t_1 - t_0|t_1 - t_0) + (t_1|t_1) - 2(t_1|t_0) + (t_0|t_0) \\ &= 2(\|t_1\|^2 + \|t_0\|^2) \leq 4\|t_1 - t_0\|^2,\end{aligned}$$

so

$$\|v - \hat{v}\|^2 = \|\frac{1}{2}(t_1 + t_0)\|^2 = \frac{1}{4}\|t_1 + t_0\|^2 \leq \frac{3}{4}\|t_1 - t_0\|^2 = \frac{3}{4}\|v_1 - v_0\|^2$$

and $\|v-\hat{v}\| \le \frac{\sqrt{3}}{2}\|v_1-v_0\|$.

3F Definition Let $\mathcal{K}$ be a simplicial complex with vertex set $V \subseteq \mathbb{R}^r$ for some $r \ge 0$. Set

$$\Delta(\mathcal{K}) = \max\{\|v-w\| : \{v,w\} \in \mathcal{K}\},$$

interpreting $\sup\emptyset$ as 0 if $\mathcal{K} = \{\emptyset\}$.

3G Repeated subdivisions: Construction Let $\mathcal{K}$ be a non-empty geometrically realizable simplicial complex in $\mathbb{R}^r$ and X its carrier. Then we can choose a sequence $\langle\mathcal{K}_m\rangle_{m\in\mathbb{N}}$ of geometrically realizable simplicial complexes in $\mathbb{R}^r$, as follows. Start with $\mathcal{K}_0 = \mathcal{K}$. If $\Delta(\mathcal{K}_m) = 0$, set $\mathcal{K}_{m+1} = \mathcal{K}_m$. Otherwise, take an edge I_m of $\mathcal{K}_m$ of length $\Delta(\mathcal{K}_m)$, and let $\mathcal{K}_{m+1}$ be the elementary subdivision of $\mathcal{K}_m$ obtained from I_m by the method of 3D, so that $\bigcup\mathcal{K}_{m+1} = \bigcup\mathcal{K}_m \cup \{\hat{v}_m\}$ where $\hat{v}_m$ is the midpoint of I_m.

(a) $\lim_{m\to\infty}\Delta(\mathcal{K}_m) = 0$.

(b) For each m, write V_m for the vertex set of $\mathcal{K}_m$ and $\widehat{V_m}$ for the corresponding open cover of X (recall that X will always be the carrier of $\mathcal{K}_m$, by 3Db). If W is any open cover of X, there is an m such that $\widehat{V_m}$ refines W.

proof (a) If $\Delta(\mathcal{K}_m) = 0$ for some m this is trivial. Otherwise, note that if $m \in \mathbb{N}$ every edge of $\mathcal{K}_{m+1}$ is either an edge of $\mathcal{K}_m$, or one half of the edge I_m, or $\{\hat{v}_m, v\}$ for some $v \in V_m \setminus I_m$ such that $\{v,v'\} \in \mathcal{K}_m$ for both $v' \in I_m$. So all the new edges have lengths at most $\frac{\sqrt{3}}{2}\Delta(\mathcal{K}_m)$, by 3E. In particular, $\Delta(\mathcal{K}_{m+1}) \le \Delta(\mathcal{K}_m)$. As this is true for every m, $\gamma = \lim_{m\to\infty}\Delta(\mathcal{K}_m)$ is defined. **?** If $\gamma > 0$, let m be such that $\Delta(\mathcal{K}_m) < \frac{2}{\sqrt{3}}\gamma$, and set $\mathcal{I}_l = \{I : I \text{ is an edge of } \mathcal{K}_l \text{ of length greater than } \gamma\}$ for $l \ge m$. For any $l \ge m$, $\gamma \le \Delta(\mathcal{K}_l) \le \Delta(\mathcal{K}_m)$, so I_l has length less than $\frac{2}{\sqrt{3}}\gamma$ and none of the new edges of $\mathcal{K}_{l+1}$ can belong to $\mathcal{I}_{l+1}$, while also $I_l \in \mathcal{I}_l \setminus \mathcal{I}_{l+1}$. Thus $\mathcal{I}_{l+1} = \mathcal{I}_l \setminus \{I_l\}$ is a proper subset of $\mathcal{I}_l$. However $\mathcal{I}_m$ is finite because $\mathcal{K}_m$ is finite, so this is impossible. **X**

(b) Because X is compact, the Lebesgue Covering Lemma (Engelking 89, 4.3.31) tells us that there is a $\delta > 0$ such that for every $v \in X$ there is a $w \in W$ including the convex set $U_v = \{t : t \in X, \|t-v\| \le \delta\}$. Now there is an m such that $\Delta(\mathcal{K}_m) \le \delta$. Take any $v \in V_m$. Let $w \in W$ be such that $U_v \subseteq w$. If $t \in \widehat{v}_{\mathcal{K}_m}$, there is a $K \in \mathcal{K}_m$ such that $v \in K$ and $t \in \Gamma(K)$. Now $\|v'-v\| \le \Delta(\mathcal{K}_m)$ and $v' \in U_v$ for every $v' \in K$; accordingly $\Gamma(K) \subseteq U_v \subseteq w$ and $t \in w$. Thus $\widehat{v}_{\mathcal{K}_m} \subseteq w$. As v is arbitrary, $\widehat{V_m} \preccurlyeq W$.

3H The Geometric Homology Theorem Let $\mathcal{K}$ be a geometrically realizable simplicial complex in $\mathbb{R}^r$, X its carrier, and $\widehat{V}$ the simple cover of X defined from $\mathcal{K}$. Take any $n \in \mathbb{N}$.

(a) The map $a \mapsto a(\widehat{V})$ is an isomorphism between $H_n(X)$ and $H_n(\mathcal{K}_{\widehat{V}})$, so $H_n(X) \cong H_n(\mathcal{K})$.

(b) Let $\mathcal{L}$ be another geometrically realizable simplicial complex, this time in $\mathbb{R}^s$ where $s \in \mathbb{N}$, (Y,W) the associated geometric complex, $g : V \to W$ a simplicial map from $\mathcal{K}$ to $\mathcal{L}$ and $\bar{g} : X \to Y$ the corresponding continuous function, as in 3Cb. Then $\bar{g}_\star : H_n(X) \to H_n(Y)$ corresponds to $g_\star : H_n(\mathcal{K}) \to H_n(\mathcal{L})$.

proof (a) Let $\mathbb{V}$ be the set of all proper finite open covers of X.

(i) Choose a sequence $\langle\mathcal{K}_m\rangle_{m\in\mathbb{N}}$ as in 3G. For each m, let V_m be the vertex set of $\mathcal{K}_m$ and $\widehat{V_m}$ the corresponding open cover of X. By 3Dc, all the homomorphisms $\pi_{\widehat{V_m}\widehat{V_{m+1}}} : H_n(\mathcal{K}_{\widehat{V_{m+1}}}) \to H_n(\mathcal{K}_{\widehat{V_m}})$ are isomorphisms. It follows that $\pi_{\widehat{V_k}\widehat{V_m}} : H_n(\mathcal{K}_{\widehat{V_m}}) \to H_n(\mathcal{K}_{\widehat{V_k}})$ is an isomorphism whenever $k \le m$ (induce on m, using the fact that $\pi_{\widehat{V_k}\widehat{V_{m+1}}} = \pi_{\widehat{V_k}\widehat{V_m}}\pi_{\widehat{V_m}\widehat{V_{m+1}}}$ for every $m \ge k$, by Lemma 2D). In particular, $\pi_{\widehat{V}\widehat{V_m}} = \pi_{\widehat{V_0}\widehat{V_m}}$ is an isomorphism for every m.

(ii) Now consider the homomorphism $\psi : H_n(X) \to H_n(\mathcal{K}_{\widehat{V}})$ defined by setting $\psi(a) = a(\widehat{V})$ for $a \in H_n(X)$. For any $a \in H_n(X)$ and $m \in \mathbb{N}$, $a(\widehat{V}) = \pi_{\widehat{V}\widehat{V_m}}(a(\widehat{V_m}))$ so

$$a(\widehat{V_m}) = \pi^{-1}_{\widehat{V}\widehat{V_m}}(a(\widehat{V})) = \pi^{-1}_{\widehat{V}\widehat{V_m}}\psi(a).$$

(α) ψ is injective. **P** Suppose that $a,b\in H_n(X)$ and $a\ne b$. Then there is a proper finite open cover W of X such that $a(W)\ne b(W)$. Now there is an $m\in\mathbb{N}$ such that $\widehat{V_m}\preccurlyeq W$, so that

$$\begin{aligned}\pi_{W\widehat{V_m}}\pi^{-1}_{\widehat{V}\widehat{V_m}}\psi(a)&=\pi_{W\widehat{V_m}}(a(\widehat{V_m}))=a(W)\ne b(W)\\&=\pi_{W\widehat{V_m}}\pi^{-1}_{\widehat{V}\widehat{V_m}}\psi(b)\end{aligned}$$

and $\psi(a)\ne\psi(b)$. **Q**

(β) ψ is surjective. **P** Given $c\in H_n(\mathcal{K}_{\widehat{V}})$ define $a\in\prod_{W\in\mathbb{V}}H_n(\mathcal{K}_W)$ as follows. For $W\in\mathbb{V}$ let $k\in\mathbb{N}$ be minimal such that $\widehat{V_k}\preccurlyeq W$, and set

$$a(W)=\pi_{W\widehat{V_k}}\pi^{-1}_{\widehat{V}\widehat{V_k}}(c).$$

If $W\preccurlyeq W'$ in $\mathbb{V}$, then we have

$$a(W)=\pi_{W\widehat{V_k}}\pi^{-1}_{\widehat{V}\widehat{V_k}}(c),\quad a(W')=\pi_{W'\widehat{V_{k'}}}\pi^{-1}_{\widehat{V}\widehat{V_{k'}}}(c)$$

where $k'\le k$. Now

$$\pi_{\widehat{V}\widehat{V_k}}=\pi_{\widehat{V}\widehat{V_{k'}}}\pi_{\widehat{V_{k'}}\widehat{V_k}}$$

so

$$\pi^{-1}_{\widehat{V}\widehat{V_k}}=\pi^{-1}_{\widehat{V_{k'}}\widehat{V_k}}\pi^{-1}_{\widehat{V}\widehat{V_{k'}}}.$$

As $\widehat{V_k}\preccurlyeq\widehat{V_{k'}}\preccurlyeq W'$ and $\widehat{V_k}\preccurlyeq W\preccurlyeq W'$,

$$\begin{aligned}\pi_{W'W}(a(W))&=\pi_{W'W}\pi_{W\widehat{V_k}}\pi^{-1}_{\widehat{V}\widehat{V_k}}(c)=\pi_{W'\widehat{V_k}}\pi^{-1}_{\widehat{V_{k'}}\widehat{V_k}}\pi^{-1}_{\widehat{V}\widehat{V_{k'}}}(c)\\&=\pi_{W'\widehat{V_{k'}}}\pi_{\widehat{V_{k'}}\widehat{V_k}}\pi^{-1}_{\widehat{V_{k'}}\widehat{V_k}}\pi^{-1}_{\widehat{V}\widehat{V_{k'}}}(c)=\pi_{W'\widehat{V_{k'}}}\pi^{-1}_{\widehat{V}\widehat{V_{k'}}}(c)=a(W').\end{aligned}$$

This shows that $a\in H_n(X)$, and of course

$$a(\widehat{V})=\pi_{\widehat{V}\widehat{V_0}}\pi^{-1}_{\widehat{V}\widehat{V_0}}(c)=c.$$

As c is arbitrary, ψ is surjective. **Q**

(iii) Thus ψ is an isomorphism between $H_n(X)$ and $H_n(\mathcal{K}_{\widehat{V}})$. As $\mathcal{K}_{\widehat{V}}\cong\mathcal{K}$ (3Ca), we surely have $H_n(\mathcal{K}_{\widehat{V}})\cong H_n(\mathcal{K})$ and $H_n(X)\cong H_n(\mathcal{K})$.

(b) Set $\psi'(b)=b(\widehat{W})$ for $b\in H_n(Y)$, so that $\psi':H_n(Y)\to H_n(\mathcal{K}_{\widehat{W}})$ is an isomorphism. Because $v\mapsto\widehat{v}:V\to\widehat{V}$ and $w\mapsto\widehat{w}:W\to\widehat{W}$ are bijective, we have a function $f:\widehat{V}\to\widehat{W}$ defined by setting $f(\widehat{v})=\widehat{g(v)}$ for every $v\in V$. Because $v\mapsto\widehat{v}:V\to\widehat{V}$ and $w\mapsto\widehat{w}:W\to\widehat{W}$ correspond respectively to isomorphisms between $\mathcal{K}$ and $\mathcal{K}_{\widehat{V}}$ and $\mathcal{L}$ and $\mathcal{K}_{\widehat{W}}$, f is a simplicial map and $g_\star:H_n(\mathcal{K})\to H_n(\mathcal{L})$ corresponds to $f_\star:H_n(\mathcal{K}_{\widehat{V}})\to H_n(\mathcal{K}_{\widehat{W}})$. If $v\in V$, then 3C(b-ii) tells us that

$$\widehat{v}\subseteq\bar{g}^{-1}[\widehat{g(v)}]=\bar{g}^{-1}[f(\widehat{v})].$$

In particular, $\widehat{V}$ refines $\bar{g}^{-1}[[\widehat{W}]]$; and Lemma 2Ec tells us that $\phi_{\widehat{W}}\pi_{\bar{g}^{-1}[[\widehat{W}]]\widehat{V}}=f_\star$, where $\phi_{\widehat{W}}$ is defined from f as in 2Eb.

Take any $a\in H_n(X)$. By the definition in 2Fb,

$$\begin{aligned}\psi'\bar{g}_\star(a)&=\bar{g}_\star(a)(\widehat{W})=\phi_{\widehat{W}}(a(\bar{g}^{-1}[[\widehat{W}]]))\\&=\phi_{\widehat{W}}\pi_{\bar{g}^{-1}[[\widehat{W}]]\widehat{V}}(a(\widehat{V}))=f_\star(a(\widehat{V}))=f_\star\psi(a).\end{aligned}$$

So $\bar{g}_\star$ corresponds to $f_\star$ through the canonical isomorphisms ψ and ψ', and hence to $g_\star$.

3I Corollary Let $\mathcal{K}$ be a simplicial complex. Then $H_n(M(\mathcal{K}))\cong H_n(\mathcal{K})$ for every $n\in\mathbb{N}$.

proof Writing V for the vertex set of $\mathcal{K}$, we have a natural basis $\langle d_v\rangle_{v\in V}$ for the Euclidean space $\mathbb{R}^V$, setting

$$d_v(v)=1,\quad d_v(w)=0 \text{ for } w\in V\setminus\{v\}.$$

Setting $K'=\{d_v:v\in K\}$ for $K\in\mathcal{K}$ and $\mathcal{K}'=\{K':K\in\mathcal{K}\}$, $\mathcal{K}'$ is geometrically realizable, because if α, $\beta\in M(\mathcal{K}')$ and $\operatorname{brc}(\alpha)=\operatorname{brc}(\beta)$ then

$$\alpha(d_v)=\sum_{w\in V}\alpha(d_w)d_w(v)=\operatorname{brc}(\alpha)(v)=\operatorname{brc}(\beta)(v)=\beta(d_v)$$

for every $v\in V$, and $\alpha=\beta$. The same formula shows that the carrier of $\mathcal{K}'$ is just $\bigcup_{K'\in\mathcal{K}'}\Gamma(K')=M(\mathcal{K})$. Also, of course, $\mathcal{K}$ and $\mathcal{K}'$ are isomorphic. So 3H tells us that

$$H_n(M(\mathcal{K}))\cong H_n(\mathcal{K}')\cong H_n(\mathcal{K})$$

for every n.

Remark In 5I-5K I will offer an example of a sequence of topological spaces $\boldsymbol{P}_r$ for most of which it seems easier to find a simplicial complex $\mathcal{K}$ for which we can prove that $\boldsymbol{P}_r$ is homeomorphic to $M(\mathcal{K})$ than to present a natural embedding of $\boldsymbol{P}_r$ into a Euclidean space.

3J The Approximation Theorem Let $\mathcal{K}$, $\mathcal{L}$ be geometrically realizable simplicial complexes, with vertex sets $V\subseteq\mathbb{R}^r$ and $W\subseteq\mathbb{R}^s$, and carriers X and Y. Let $g:X\to Y$ be a continuous function. Let $\langle\mathcal{K}_m\rangle_{m\in\mathbb{N}}$ be a sequence of subdivisions of $\mathcal{K}$ as in 3G; write V_m for the vertex set of $\mathcal{K}_m$ for each m. Then there are an $m\in\mathbb{N}$ and a simplicial map $f:V_m\to W$ from $\mathcal{K}_m$ to $\mathcal{L}$ such that

$$g(\widehat{v})\subseteq\widehat{f(v)} \text{ for every } v\in V_m$$

(calculating $\widehat{v}$ with respect to $\mathcal{K}_m$ and $\widehat{f(v)}$ with respect to $\mathcal{L}$, of course),

$$g \text{ and } \bar f:X\to Y \text{ are homotopic},\qquad \|g(t)-\bar f(t)\|\le\Delta(\mathcal{L}) \text{ for every } t\in X.$$

proof $\{g^{-1}[\widehat{w}]:w\in W\}$ is an open cover of X; by 3Gb, there is an $m\in\mathbb{N}$ such that $\widehat{V_m}$ refines it. For each $v\in V_m$, choose $f(v)\in W$ such that $\widehat{v}\subseteq g^{-1}[\widehat{f(v)}]$, that is, $g[\widehat{v}]\subseteq\widehat{f(v)}$. If $K\in\mathcal{K}\setminus\{\emptyset\}$, $t=\frac{1}{\#(K)}\sum_{v\in K}v$ belongs to $\bigcap_{v\in K}\widehat{v}$, so $g(t)\in\bigcap_{v\in K}\widehat{f(v)}$ and $f[K]$ must belong to $\mathcal{L}$. Thus f is a simplicial map from $\mathcal{K}_m$ to $\mathcal{K}$.

Write $\theta:X\to M(\mathcal{K}_m)$ for the inverse of the homeomorphism $\operatorname{brc}_{\mathcal{K}_m}:M(\mathcal{K}_m)\to X$, and define $h:M(\mathcal{K}_m)\to M(\mathcal{L})$ by setting $h(\alpha)(w)=\sum_{v\in V_m,f(v)=w}\alpha(v)$ for $\alpha\in M(\mathcal{K}_m)$ and $w\in W$, as in 3Cb; disentangling the definitions there, we see that $\bar f=\operatorname{brc}_{\mathcal{L}}h\theta$. If $t\in X$ and $K=\{v:\theta(t)(v)>0\}$, then $f[K]=\{w:h\theta(t)(w)>0\}$ and $\bar f(t)\in\Gamma(f[K])$. At the same time, $t\in\bigcap_{v\in K}\widehat{v}$ so

$$g(t)\in\bigcap_{v\in K}\widehat{f(v)}=\bigcap_{w\in f[K]}\widehat{w}\subseteq\Gamma(f[K]).$$

But this means that if we set

$$\tilde g(\gamma,t)=\gamma\bar f(t)+(1-\gamma)g(t)$$

for $\gamma\in[0,1]$, $\tilde g(\gamma,t)\in\Gamma(f[K])\subseteq Y$ for every γ. Now $\tilde g:[0,1]\times X\to Y$ is continuous, so is a homotopy from g to $\bar f$. We see at the same time that

$$\|g(t)-\bar f(t)\|\le\operatorname{diam}\Gamma(f[K])=\operatorname{diam}(f[K])\le\Delta(\mathcal{L})$$

by 3B(a-iii).

Notes and comments Any presentation of the ideas of singular homology has to deal with several genuinely difficult points. Typically the combinatoric side (here covered in Chapter 1) is manageable and becomes seriously obscure only when we come to apply it to topological (or metric) spaces and continuous functions. I have broken the hard step into two parts. In Chapter 2 I showed (at the cost of moving to the notion of general finite open cover) that we could at least develop an abstract homology theory for topological spaces. That section ignored the problem of actually calculating any homology groups other than H_0 (E2C). At some point we have to start thinking geometrically, as in the present section. It is here that we get to the real point of the elementary subdivisions of 1R-1S; they correspond to a special kind of refinement which simultaneously leaves the carrier (of a geometric complex) and the homology groups (of an abstract complex) unchanged. A little bit of analysis (3G) shows that elementary subdivision, repeated often enough, can reach far enough along the inverse-limit structures of Chapter 2 to give us control over topological homology groups. In Chapter 4 I will use this method to describe the homology groups of the simplest objects, starting with balls and spheres. The Approximation Theorem (3J) shows that, quite apart from homology groups, the algebra of simplicial complexes and simplicial maps will tell us a great deal about the possible natures of continuous functions between the carriers of geometrically realizable simplicial complexes.

I should probably apologise again for the notation of this chapter. The problem is that while 'simplicial complex' itself (1A) is a straightforward idea, we have to develop a necessarily complex structure to define its homology groups (1J). This part of the work, at least, can be done on the assumption that the vertices of a simplicial complex are abstract points with no structure. However, Čech homology is based on the idea of using open sets in topological spaces as the vertices of simplicial complexes. In Chapter 2 this was bearable; we just had to remember that the symbol v now denoted a set rather than a point, and if we have survived indoctrination into Zermelo-Frankel set theory we shall have no ideological objection to the shift. But in the present chapter we have to handle simultaneously a simplicial complex $\mathcal{K}$ with vertex set $V \subseteq \mathbb{R}^r$ (that is to say, a vertex of $\mathcal{K}$ is a function from $r = \{0, \dots, r-1\}$ to $\mathbb{R}$), its carrier $X \subseteq \mathbb{R}^r$, relatively open sets $\widehat{v} \subseteq X$, the open cover $\widehat{V} = \{\widehat{v} : v \in V\}$ and the homology groups of the associated simplicial complex $\mathcal{K}_{\widehat{V}}$. The ordinary convention that our structures should be described with 'types' in Russell's sense, an object of type $n+1$ being a set of objects of type n, and that each type should be associated with a different alphabet (as in $v \in V \in \mathbb{V}$), becomes unmanageable because we don't have enough alphabets. So I use the superscript $\widehat{\ }$ to give myself a couple more.

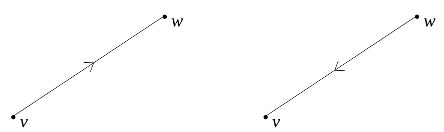
E3 Examples for Chapter 3

E3A Oriented simplices Let $\mathcal{K}$ be a simplicial complex with vertex set V.

(a) In 1C I introduced the elementary chains e_p where $p \in V^{n+1}$ is injective and $p[n+1] \in \mathcal{K}$, and showed that if $\sigma \in S_{n+1}$ then $e_{p\sigma} = \epsilon_\sigma e_p$. This means that if $K \in \mathcal{K}$ and $\#(K) = n+1 \geq 2$, there are just two such elementary chains based on K, one of them being the additive inverse of the other. These are the two orientations of the unoriented simplex K. (If $n = 0$ and K is a singleton, or $n = -1$ and $K = \emptyset$, then S_{n+1} is the identity alone and K has only one orientation.)

(b) If $V \subseteq \mathbb{R}^3$ and $\mathcal{K}$ is geometrically realizable, then at least for dimensions 1 and 2 we have a geometric interpretation of orientation.

(i) If $K \in \mathcal{K}$ is a doubleton set $\{v, w\}$ then the two elementary chains are $e_{<v>\frown<w>}$ and $e_{<w>\frown<v>}$. I will declare $e_{<v>\frown<w>}$ to be the line segment 'from v to w', while $e_{<w>\frown<v>}$ is the same line segment in the reverse direction.

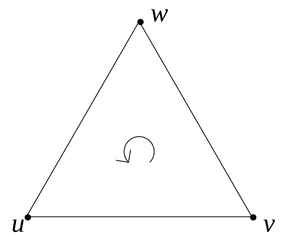


The difference shows up in the boundaries; using the formula in E1A, we have $\partial(e_{<v>\frown<w>}) = e_{<v>} - e_{<w>}$ while $\partial(e_{<w>\frown<v>}) = e_{<w>} - e_{<v>}$.

(ii) If $K = \{u, v, w\}$ is a set with three elements, then $\Gamma(K)$ is a triangle. In this case

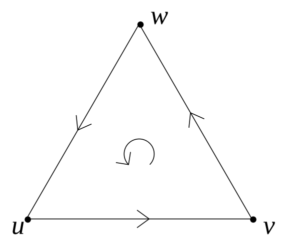
$$e_{<u>\frown<v>\frown<w>} = e_{<v>\frown<w>\frown<u>} = e_{<w>\frown<u>\frown<v>}$$

can be thought of as imposing a direction of rotation of the vertices in the sense $u \to v \to w \to u$



which transfers to the three edges of the triangle when we take the boundary

$$\begin{aligned}\partial(e_{<u>\frown<v>\frown<w>}) &= e_{<u>\frown<v>} - e_{<u>\frown<w>} + e_{<v>\frown<w>}\\ &= e_{<u>\frown<v>} + e_{<v>\frown<w>} + e_{<w>\frown<u>}.\end{aligned}$$

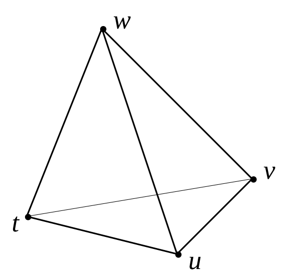


Note that this represents the boundary as a simple closed curve with a definite orientation, which makes the word 'cycle' reasonable.

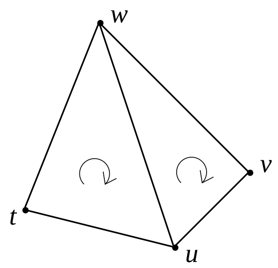
(c) For a 3-simplex the situation is less transparent but we can still do something if we look once more at its boundary. Still supposing that $\mathcal{K}$ is geometrically realizable and that $K \in \mathcal{K}$ is a four-element set $\{t, u, v, w\}$, this must be a tetrahedron (so r must be at least 3). Now

$$\begin{aligned}&\partial(e_{<t>\frown<u>\frown<v>\frown<w>})\\ &\quad = e_{<t>\frown<u>\frown<v>} - e_{<t>\frown<u>\frown<w>} + e_{<t>\frown<v>\frown<w>} - e_{<u>\frown<v>\frown<w>}\\ &\quad = e_{<t>\frown<u>\frown<v>} + e_{<t>\frown<w>\frown<u>} + e_{<t>\frown<v>\frown<w>} + e_{<u>\frown<w>\frown<v>}\end{aligned}$$

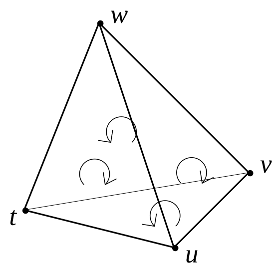
and drawing the tetrahedron as



this corresponds to orientations of the triangles as

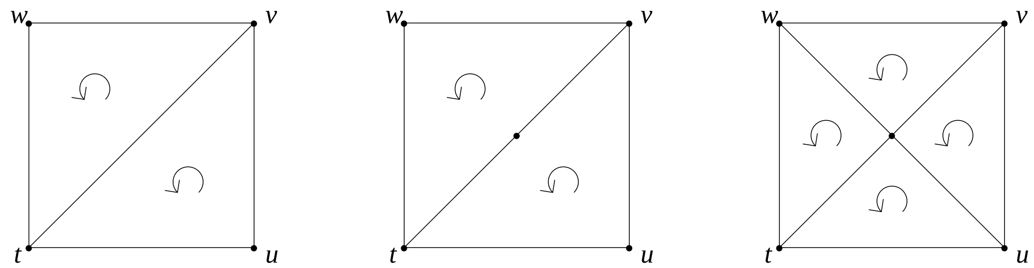


or, including the hidden faces,



Note that for each of the six edges of the tetrahedron the two triangles to which it belongs have orientations which lend opposite directions to the edge; this corresponds to the fact that that $\partial^2 e_{<t>^\frown<u>^\frown<v>^\frown<w>} = 0$. The picture assumes that $r = 3$, in which case we have a notion of 'inside' and 'outside' (the interior and complement of $\Gamma(K)$), and in the boundary each of the four triangular faces is oriented clockwise when seen from outside. If we started from the other orientation of K then the faces in the boundary would become oriented clockwise when seen from inside.

E3B Elementary subdivisions Again, in one and two dimensions these are straightforward. The following gives the idea.



Starting with the left-hand geometric complex with two 2-simplices sharing an edge, we can take a point of that edge, and divide both the edge and the two triangles into two. I have marked orientations for the triangles because it is an essential element of the construction that these are preserved; in the proof of the Subdivision Theorem 1S, this fact appears in the formula $\partial\pi = \pi\partial$.

E3C Finite sets Suppose that $r \in \mathbb{N}$ and that $X \subseteq \mathbb{R}^r$ is finite and not empty. Then $H_0(X) \cong \mathbb{Z}^{\#(X)-1}$ and $H_n(X) = \{0\}$ for $n > 0$. **P** We can regard X as the carrier of the geometrically realizable simplicial complex $\mathcal{K} = \{\{v\} : v \in X\} \cup \{\emptyset\}$. By the Geometric Homology Theorem 3Ha, $H_n(X) \cong H_n(\mathcal{K})$ for every n, and E1E gives the result. **Q**

Notes and comments Part of my motivation for the very abstract algebra in Chapter 1 was that I wanted to avoid coming to grips with the notion of 'oriented simplex'. The difficulty is that while in an individual picture we may feel that we can distinguish 'positively' from 'negatively' oriented simplices, there is no way to make this convincing in the general theory. For instance, we can endow a vertex set with a total ordering, and say that an oriented n-simplex e_p is 'positively' oriented if $p(0) < p(1) < \ldots < p(n)$. This has the advantage that e_p and $e_{p\uparrow j}$ will have the 'same' orientation for every $j \le n$. But the vertex sets of the geometric complexes we are primarily interested in are not naturally ordered, and the assignment is bound to be arbitrary. I therefore prefer a formulation which clearly distinguishes the two orientations of an unoriented simplex without attempting to give primacy to either.

4 The main theorems

I come to the natural questions in the topology of Euclidean space which the theory here was created to answer.

4A Balls, spheres and regular simplices Some of the arguments below will run more smoothly if I spell out a little elementary geometry in multidimensional Euclidean space.

(a) If $r \in \mathbb{N}$, I will write $\boldsymbol{B}_r$ for the unit ball

$$\{t : t \in \mathbb{R}^r,\ \|t\| \le 1\}$$

of $\mathbb{R}^r$, and $\boldsymbol{S}_r$ for the unit sphere

$$\{t : t \in \mathbb{R}^{r+1},\ \|t\| = 1\}$$

in $\mathbb{R}^{r+1}$, the topological boundary of $\boldsymbol{B}_{r+1}$. It will be useful to know that for every $r \ge 1$ there is a 'regular' simplex of dimension r in $\mathbb{R}^r$, with vertices $v_0, \dots , v_r$ such that $\|v_j - v_i\| = \gamma$ whenever i, $j \le r$ are distinct, for some $\gamma > 0$. For definiteness, I show by induction on r how to construct such a simplex with the additional properties that

$$\|v_i\| = 1 \text{ for every } i \le r, \quad (v_i|v_j) = -\frac{1}{r} \text{ whenever } i < j \le r, \quad \textstyle\sum_{i=0}^r v_i = 0.$$

(b) Construction Start with $r = 1$ and $v_0^{(1)} = <1>$, $v_0^{(1)} = <-1> \in \mathbb{R}^1 \cong \mathbb{R}$. Given that $v_i^{(r)} \in \mathbb{R}^r$ has been defined for $i \le r$, with $\|v_i^{(r)}\| = 1$ for each i, $\sum_{i=0}^r v_i^{(r)} = 0$ and $(v_i^{(r)}|v_j^{(r)}) = -\frac{1}{r}$ whenever $i < j \le r$, define $v_0^{(r+1)}, \dots , v_{r+1}^{(r+1)}$ in $\mathbb{R}^{r+1}$ by setting

$$\begin{aligned} v_i^{(r+1)}(k) &= \frac{\sqrt{r(r+2)}}{r+1} v_i^{(r)}(k) \text{ if } i \le r \text{ and } k < r, \\ &= -\frac{1}{r+1} \text{ if } i \le r \text{ and } k = r, \\ &= 0 \text{ if } i = r + 1 \text{ and } k < r, \\ &= 1 \text{ if } i = r + 1 \text{ and } k = r. \end{aligned}$$

If $i \le r$, then

$$\|v_i^{(r+1)}\|^2 = \sum_{k=0}^{r} v_i^{(r+1)}(k)^2 = \frac{r(r+2)}{(r+1)^2} \sum_{k=0}^{r-1} v_i^{(r)}(k)^2 + \frac{1}{(r+1)^2} = \frac{r(r+2)}{(r+1)^2} + \frac{1}{(r+1)^2} = 1,$$

so $\|v_i^{(r+1)}\| = 1$, while of course $\|v_{r+1}^{(r+1)}\| = 1$. Next,

$$\begin{aligned} \sum_{i=0}^{r+1} v_i^{(r+1)}(k) &= \frac{\sqrt{r(r+2)}}{r+1} \sum_{i=0}^{r} v_i^{(r)}(k) = 0 \text{ if } k < r, \\ &= -\frac{r+1}{r+1} + 1 = 0 \text{ if } k = r, \end{aligned}$$

so $\sum_{i=0}^{r+1} v_i^{(r+1)} = 0$ in $\mathbb{R}^{r+1}$. Also

$$\begin{aligned} (v_i^{(r+1)}|v_j^{(r+1)}) &= \sum_{k=0}^{r} v_i^{(r+1)}(k) v_j^{(r+1)}(k) \\ &= \frac{r(r+2)}{(r+1)^2}(v_i^{(r)}|v_j^{(r)}) + \frac{1}{(r+1)^2} = -\frac{r(r+2)}{r(r+1)^2} + \frac{1}{(r+1)^2} = -\frac{1}{r+1} \text{ if } i < j \le r \\ &= -\frac{1}{r+1} \text{ if } i \le r \text{ and } j = r + 1. \end{aligned}$$

So the induction proceeds.

We see now that, for any $r \geq 1$ and distinct i, $j \leq r$,

$$\|v_i^{(r)} - v_j^{(r)}\|^2 = \|v_i^{(r)}\|^2 - 2(v_i^{(r)}|v_j^{(r)}) + \|v_j^{(r)}\|^2 = 2 + \frac{2}{r} = \frac{2(r+1)}{r}$$

and $\|v_i^{(r)} - v_j^{(r)}\| = \sqrt{\frac{2(r+1)}{r}}$. So $\{v_0^{(r)}, \dots, v_r^{(r)}\}$ is a regular simplex as described.

(c) Suppose that $r \geq 1$ and that $v_0, \dots, v_r$ are the vertices of a regular simplex in $\mathbb{R}^r$ constructed as in (b).

(i) Every point t of $\mathbb{R}^r$ is uniquely expressible as $t = \sum_{i=0}^r \alpha(i)v_i$ where $\alpha \in \mathbb{R}^{r+1}$ and $\sum_{i=0}^r \alpha(i) = 1$. In this case $(t|v_j) = \frac{r+1}{r}\alpha(j) - \frac{1}{r}$ for every $j \leq r$. **P** Define $T : \mathbb{R}^{r+1} \to \mathbb{R}^r$ by saying that $T\beta = \sum_{i=0}^r \beta(i)v_i$ for every $\beta \in \mathbb{R}^{r+1}$. Then T is linear and

$$(T\beta|v_j) = \sum_{i=0}^r \beta(i)(v_i|v_j) = \beta(j) - \frac{1}{r}\sum_{i \neq j} \beta(i) = \frac{r+1}{r}\beta(j) - \frac{1}{r}\sum_{i=0}^r \beta(i)$$

for each j. Consequently the kernel $\{\beta : T\beta = 0\}$ of T is included in

$$\{\beta : \beta(j) = \frac{1}{r+1}\sum_{i=0}^r \beta(i) \text{ for every } j \leq r\} = \{\beta : \beta(i) = \beta(0) \text{ for every } i \leq r\}$$

which is one-dimensional. It follows that $T[\mathbb{R}^{r+1}]$ is an r-dimensional linear subspace of $\mathbb{R}^r$ and must be the whole of $\mathbb{R}^r$. Next, given any $\beta \in \mathbb{R}^{r+1}$, we can set $\alpha(j) = \beta(j) + \frac{1}{r+1}(1 - \sum_{i=0}^r \beta(i))$ for each j and see that

$$\sum_{j=0}^r \alpha(j) = 1, \quad T\alpha = T\beta$$

because $\sum_{i=0}^r v_i = 0$ in $\mathbb{R}^r$. We now have

$$\frac{r+1}{r}\alpha(j) - \frac{1}{r} = (T\alpha|v_j)$$

for every $j \leq r$. This shows that if $t \in \mathbb{R}^r$ there is exactly one $\alpha \in \mathbb{R}^{r+1}$ such that $t = \sum_{i=0}^r \alpha(i)v_i$ and $\sum_{i=0}^r \alpha(i) = 1$. **Q**

(ii) Set $V = \{v_0, \dots, v_r\}$. Let X be $\Gamma(V)$, the set of points expressible in the form $t = \sum_{i=0}^r \alpha(i)v_i$ where $\sum_{i=0}^r \alpha(i) = 1$ and $\alpha_i \geq 0$ for every $i \leq r$. By (i) just above, we can restate this as

> X is the set of points expressible in the form $t = \sum_{i=0}^r \alpha(i)v_i$ where $\sum_{i=0}^r \alpha(i) = 1$ and $(t|v_j) \geq -\frac{1}{r}$ for $j \leq r$,

that is,

> X is the set of points $t \in \mathbb{R}^r$ such that $(t|v_j) \geq -\frac{1}{r}$ for $j \leq r$.

(iii) Consider $\mathcal{L} = \mathcal{P}V \setminus \{V\}$. Then $\mathcal{L}$ is geometrically realizable and its carrier $Y \subseteq X$ is

$$\bigcup_{j \leq r}\{\sum_{i=0}^r \alpha(i)v_i : \sum_{i=0}^r \alpha(i) = 1,\ \alpha(i) \geq 0 \text{ for every } i \leq r \text{ and } \alpha(j) = 0\}$$

$$= \{t : \min_{j \leq r}(t|v_j) = \frac{1}{r}\}.$$

The last expression shows that if $t \in Y$ then $\delta t \notin Y$ for any $\delta \in [0, \infty[\setminus \{1\}$; we see also, if we were in any doubt, that $0 \notin Y$.

(iv) For $t \in \boldsymbol{S}_{r-1}$, $\min_{j \leq r}(t|v_j) \leq -\frac{1}{r}$. **P?** Otherwise, there would be a $\delta > 1$ such that $(\delta t|v_j) \geq -\frac{1}{r}$ for every j and $\delta t \in \Gamma(V)$. But $V \subseteq \boldsymbol{B}_r$ and $\boldsymbol{B}_r$ is convex, so $\Gamma(V) \subseteq \boldsymbol{B}_r$ and $\delta = \|\delta t\| \leq 1$. **XQ**

We therefore have a function $q : \boldsymbol{S}_{r-1} \to]0, 1]$ defined by saying that

$$q(t) = -\frac{1}{r \min_{j \leq r}(t|v_j)},$$

and q is continuous, while $\min_{j\le r}(q(t)t|v_j) = -\frac{1}{r}$, that is, $q(t)t \in Y$ for every $t \in \boldsymbol{S}_{r-1}$. So setting $g(t) = q(t)t$ for $t \in \boldsymbol{S}_{r-1}$, $g : \boldsymbol{S}_{r-1} \to Y$ is continuous. Of course it is injective, because $t = \frac{1}{\|q(t)t\|}q(t)t$ for every $t \in \boldsymbol{S}_{r-1}$. It is also surjective, because if $t \in Y$ then $t' = \frac{1}{\|t\|}t$ belongs to $\boldsymbol{S}_{r-1}$, and $q(t')t'$ is a multiple of t belonging to Y; by the last sentence of (iii) above, $t = q(t')t' = g(t')$ is a value of g.

(v) Now define $f : \boldsymbol{B}_r \to X$ by setting

$$\begin{aligned} f(t) &= 0 \text{ if } t = 0,\\ &= q(\frac{1}{\|t\|}t)t \text{ otherwise.} \end{aligned}$$

Then f is a continuous bijection, therefore a homeomorphism, and $f[\boldsymbol{S}_{r-1}] = Y$.

P **(α)** f is continuous at 0 because $f(0) = 0$ and $\|f(t) - 0\| = \|f(t)\| \le \|t\| = \|t - 0\|$ for every $t \in \boldsymbol{B}_r$. Elsewhere f is continuous because q is continuous.

(β) f is injective because if $t \in \boldsymbol{B}_r \setminus \{0\}$ then $q(\frac{1}{\|t\|}t) > 0$ and $f(t) \ne 0 = f(0)$, while if t, $t' \in \boldsymbol{B}_r \setminus \{0\}$ and $f(t) = f(t')$ then t and t' are both positive multiples of the common value $f(t) = f(t')$, so $\frac{1}{\|t\|}t = \frac{1}{\|t'\|}t'$ and

$$t = q(\frac{1}{\|t\|}t)^{-1}f(t) = q(\frac{1}{\|t'\|}t')^{-1}f(t') = t'.$$

(γ) f is surjective because $f(0) = 0$ and if $s \in X \setminus \{0\}$ then $\min_{j\le r}(s|v_j) \ge -\frac{1}{r}$. There is therefore a $\delta \ge 1$ such that $\min_{j\le r}(\delta s|v_j) = -\frac{1}{r}$. Now $\delta s \in Y$ so there is a $t' \in \boldsymbol{S}_{r-1}$ such that $q(t')t' = \delta s$, by (iv). In this case $t = \frac{1}{\delta}t'$ belongs to $\boldsymbol{B}_r$, $\frac{1}{\|t\|}t = t'$ and $f(t) = q(t')t = s$.

(δ) Because $\boldsymbol{B}_r$ is compact (and X is metrizable), f is a homeomorphism (Engelking 3.1.13). **Q**

(d) Recapitulating the essential facts, we see that for any $r \ge 1$ we have a geometrically realizable simplex $\mathcal{K} = \mathcal{P}V$, where $V = \{v_0, \dots , v_r\} \subseteq \mathbb{R}^r$, with carrier

$$X = \Gamma(V) = \{t : t \in \mathbb{R}^r,\ (t|v_j) \ge -\frac{1}{r} \text{ for every } j \le r\},$$

and that if $\mathcal{L} = \mathcal{K} \setminus \{V\}$ then the carrier of $\mathcal{L}$ is

$$Y = \{t : t \in \mathbb{R}^r,\ \min_{j\le r}(t|v_j) = -\frac{1}{r}\}.$$

The primary point of the construction is that we have a homeomorphism $f : \boldsymbol{B}_r \to X$ such that $f[\boldsymbol{S}_{r-1}] = Y$. But it will be useful later to be able to appeal to further features arising from the specific process used in choosing the points v_i and the function f.

(e) For instance, an easy induction on r shows that, in the language of (b),

$$v_1^{(r)}(0) = -v_0^{(r)}(0), \quad v_1^{(r)}(k) = v_0^{(r)}(k) \text{ for } 0 < k < r, \quad v_i^{(r)}(0) = 0 \text{ for } 1 < i \le r$$

for every $r \ge 1$. So in the language of (c)-(d),

$$v_1(0) = -v_0(0), \quad v_1(k) = v_0(k) \text{ for } 0 < k < r, \quad v_i(0) = 0 \text{ for } 1 < i \le r.$$

Next, if we define a reflection $T : \mathbb{R}^r \to \mathbb{R}^r$ by setting

$$T(t)(0) = -t(0), \quad T(t)(i) = t(i) \text{ for } 1 \le i < r$$

for $t \in \mathbb{R}^r$, $\|T(t)\| = \|t\|$ for every t and T^2 is the identity on $\mathbb{R}^r$, so $T[\boldsymbol{S}_{r-1}] = \boldsymbol{S}_{r-1}$. We have $T(v_0) = v_1$, $T(v_1) = v_0$ and $T(v_i) = v_i$ for $1 < i \le r$. For any $t \in \mathbb{R}^r$,

$$\min_{j\le r}(t|v_j) = \min_{j\le r}(T(t)|T(v_j)) = \min_{j\le r}(T(t)|v_j),$$

so $T(t) \in X$ iff $t \in X$ and $T(t) \in Y$ iff $t \in Y$, while $qT(t) = q(t)$ if $t \in \boldsymbol{S}_{r-1}$. It follows that

$$fT(t) = q(\frac{1}{\|T(t)\|}T(t))T(t) = q(\frac{1}{\|t\|}t)T(t) = T(q(\frac{1}{\|t\|}t)t) = Tf(t)$$

for every $t \in \boldsymbol{B}_r$.

4B Theorem Let $r \in \mathbb{N}$. Then $H_n(\boldsymbol{B}_r) = \{0\}$ for every $n \in \mathbb{N}$.

proof If $r = 0$ then $\boldsymbol{B}_r = \{0\}$ so the result is immediate from E2Ab. Otherwise, let $V \subseteq \mathbb{R}^r$ and $X = \Gamma(V)$ be as in 4A. Then, for any $n \in \mathbb{N}$,

$$\begin{aligned} H_n(\boldsymbol{B}_r) &\cong H_n(X) \quad \text{because } \boldsymbol{B}_r \text{ and } X \text{ are homeomorphic} \\ &\cong H_n(\mathcal{P}V) \quad \text{by the Geometric Homology Theorem (3Ha)} \\ &= \{0\} \end{aligned}$$

by E1H.

4C Theorem Let $r \in \mathbb{N}$. Then $H_r(\boldsymbol{S}_r) \cong \mathbb{Z}$ and $H_n(\boldsymbol{S}_r) = \{0\}$ for other $n \in \mathbb{N}$.

proof This time, if $r = 0$, we have $\boldsymbol{S}_0 = \{-1, 1\} \subseteq \mathbb{R}$, so E3C gives the result. If $r \geq 1$ then, taking $V \subseteq \mathbb{R}^{r+1}$, $X = \Gamma(V)$ and Y as in 4A but one dimension higher, Y is the carrier of $\mathcal{L} = \mathcal{P}V \setminus \{V\}$ and is homeomorphic to $\boldsymbol{S}_r$. So we have

$$\begin{aligned} H_n(\boldsymbol{S}_r) \cong H_n(\mathcal{L}) &\cong \mathbb{Z} \text{ if } n = r, \\ &= \{0\} \text{ otherwise} \end{aligned}$$

by E1I.

4D Theorem For any $r \in \mathbb{N}$, the identity function on on $\boldsymbol{S}_r$ is not homotopic to a constant function.

proof Write f for the identity function on $\boldsymbol{S}_r$ and let $g : \boldsymbol{S}_r \to \boldsymbol{S}_r$ be a constant function. Then $f_\star : H_r(\boldsymbol{S}_r) \to H_r(\boldsymbol{S}_r)$ is the identity homomorphism while $g_\star : H_r(\boldsymbol{S}_r) \to H_r(\boldsymbol{S}_r)$ is the zero homomorphism, by E2B. By Theorem 4C, $H_r(\boldsymbol{S}_r)$ is nontrivial, so $f_\star \neq g_\star$. By the Homotopy Theorem (2K), f and g cannot be homotopic.

4E Theorem Let $r \geq 1$ be an integer. Then there is no continuous function $g : \boldsymbol{B}_r \to \boldsymbol{S}_{r-1}$ such that $g\restriction\boldsymbol{S}_{r-1}$ is homotopic to the identity function on S_{r-1}.

proof Of course the statement makes sense because $\boldsymbol{S}_{r-1}$ is included in $\boldsymbol{B}_r$.

? Suppose, if possible, that there were such a function. Then we should have a continuous function $f : [0,1] \times \boldsymbol{S}_{r-1} \to \boldsymbol{S}_{r-1}$ defined by setting $f(\alpha, t) = g(\alpha t)$ for every $\alpha \in [0,1]$ and $t \in \boldsymbol{S}_{r-1}$. Now $f(0,t) = g(0)$ and $f(1,t) = g(t)$ for every $t \in \boldsymbol{S}_{r-1}$. So the constant function on $\boldsymbol{S}_{r-1}$ with value $g(0)$ is homotopic to $g\restriction\boldsymbol{S}_{r-1}$ and therefore to the identity function on $\boldsymbol{S}_{r-1}$. But we saw in Theorem 4D that this could not be so. **X**

4F The Brouwer Fixed Point Theorem Let $r \geq 1$ be an integer. Then every continuous function from $\boldsymbol{B}_r$ to itself has a fixed point.

proof ? Otherwise, let $f : \boldsymbol{B}_r \to \boldsymbol{B}_r$ be a function with no fixed point. Define $g : \boldsymbol{B}_r \to \boldsymbol{S}_{r-1}$ by saying that if $t \in \boldsymbol{B}_r$ then $g(t)$ is to be that point of $\boldsymbol{S}_{r-1}$ such that $t \in \Gamma(\{g(t), f(t)\})$; that is, the point where the line segment from $f(t)$ to t, extended beyond t, meets the topological boundary $\boldsymbol{S}_{r-1}$ of $\boldsymbol{B}_r$. Then g is continuous[3]. But $g(t) = t$ for every $t \in \boldsymbol{S}_{r-1}$, so this is impossible, by Theorem 4E. **X**

4G Reflections: Theorem Suppose that $r \in \mathbb{N}$, $U \subseteq \mathbb{R}^{r+1}$ is an r-dimensional subspace and $T : \mathbb{R}^{r+1} \to \mathbb{R}^{r+1}$ is the reflection in U. Then $T[\boldsymbol{S}_r] = \boldsymbol{S}_r$ and $T\restriction\boldsymbol{S}_r$ is not homotopic to the identity function on $\boldsymbol{S}_r$.

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[3] This is supposed to be 'easy to see'. The idea is that wiggling t a little bit will wiggle $f(t)$ by a little bit and the line from $f(t)$ to t also can't move much so will hit $\boldsymbol{S}_{r-1}$ in nearly the same place. A more formal argument depends on understanding the compact set $\{(t, s, \alpha) : t \in \boldsymbol{B}_r,\ s \in \boldsymbol{S}_{r-1},\ \alpha \in [0,1],\ \alpha s + (1-\alpha)f(t) = t\}$ and showing that its projection onto $\boldsymbol{B}_r$ is bijective, therefore a homeomorphism.

proof (a) Consider first the case in which $U = \{t : t \in \mathbb{R}^{r+1},\ t(0) = 0\}$, so that

$$T(t)(0) = -t(0), \quad T(t)(i) = t(i) \text{ for } 1 \le i \le r$$

for $t \in \mathbb{R}^{r+1}$. Then 4Ae tells us that if $v_0, \dots , v_{r+1} \in \mathbb{R}^{r+1}$, X, Y and $f : \boldsymbol{B}_{r+1} \to X$ are constructed as in 4A, one dimension higher, then

$$T(v_0) = v_1, \quad T(v_1) = v_0, \quad T(v_i) = v_i \text{ for } 1 < i \le r + 1,$$

while $T[\boldsymbol{S}_r] = \boldsymbol{S}_r$, $T[Y] = Y$ and $Tf = fT$.

Consider $T{\restriction}Y$. This is the continuous function corresponding to the function $h : V \to V$ such that $h(v_0) = v_1$, $h(v_1) = v_0$ and $h(v_i) = v_i$ for $1 < i \le r$. Consider the homomorphism $h_\star : H_r(\mathcal{L}) \to H_r(\mathcal{L})$ where $\mathcal{L} = \mathcal{P}V \setminus \{V\}$. We know from E1J that $h_\star$ is not the identity. So $(T{\restriction}Y)_\star : H_r(Y) \to H_r(Y)$ is not the identity, by 3Hb.

Now $f{\restriction}\boldsymbol{S}_r$ is a homeomorphism from $\boldsymbol{S}_r$ to Y. So we have a corresponding isomorphism $(f{\restriction}\boldsymbol{S}_r)_\star : H_r(\boldsymbol{S}_r) \to H_r(Y)$. Since $Tf{\restriction}\boldsymbol{S}_r = fT{\restriction}\boldsymbol{S}_r$,

$$(f{\restriction}\boldsymbol{S}_r)(T{\restriction}\boldsymbol{S}_r)(f{\restriction}\boldsymbol{S}_r)^{-1} = T{\restriction}Y$$

and

$$\begin{aligned}(f{\restriction}\boldsymbol{S}_r)_\star(T{\restriction}\boldsymbol{S}_r)_\star((f{\restriction}\boldsymbol{S}_r)_\star)^{-1} &= (f{\restriction}\boldsymbol{S}_r)_\star(T{\restriction}\boldsymbol{S}_r)_\star((f{\restriction}\boldsymbol{S}_r)^{-1})_\star \\ &= (T{\restriction}Y)_\star : H_r(Y) \to H_r(Y)\end{aligned}$$

is not the identity. But this means that $(T{\restriction}\boldsymbol{S}_r)_\star$ cannot be the identity on $H_r(\boldsymbol{S}_r)$ and $T{\restriction}\boldsymbol{S}_r$ is not homotopic to the identity on $\boldsymbol{S}_r$.

(b) In general, there are a unit vector u_0 orthogonal to the subspace U and an orthogonal transformation $R : \mathbb{R}^{r+1} \to \mathbb{R}^{r+1}$ such that $R(u_0)(0) = 1$ and $R(u_0)(i) = 0$ for $i \ne 0$. In this case, RTR^{-1} will be the reflection reversing the first coordinate, so that $RTR^{-1}{\restriction}\boldsymbol{S}_r$ is not homotopic to the identity on $\boldsymbol{S}_r$. But as $R{\restriction}\boldsymbol{S}_r$ is a homeomorphism from $\boldsymbol{S}_r$ to itself, this means that $T{\restriction}\boldsymbol{S}_r$ cannot be homotopic to the identity on $\boldsymbol{S}_r$, as claimed.

4H Antipodeal maps: Theorem Let $r \in \mathbb{N}$. Set $h(t) = -t$ for $t \in \boldsymbol{S}_r$.

(a) $h_\star(a) = (-1)^{r+1}a$ for every $a \in H_r(\boldsymbol{S}_r)$.

(b) h is homotopic to the identity function on $\boldsymbol{S}_r$ iff r is odd.

proof (a)(i) We need a triangulation of $\boldsymbol{S}_r$ which can represent h; the simplest is the following. Let $d_0, \dots , d_r$ be the usual orthogonal basis of $\mathbb{R}^{r+1}$. Set $V = \{d_0, \dots , d_r, -d_0, \dots , -d_r\}$ and $\mathcal{K} = \{K : K \subseteq V$, $\{d_i, d_{i+1}\} \not\subseteq K$ for every $i \le r\}$. Then the maximal members of $\mathcal{K}$ are the 2^{r+1} r-simplices of the form $\{\pm d_0, \dots , \pm d_r\}$, and $\mathcal{K}$ is geometrically realizable, with carrier $X = \{t : t \in \mathbb{R}^{r+1},\ \|t\|_1 = \sum_{i=0}^{r} |t(i)| = 1\}$. The maps $t \mapsto \frac{1}{\|t\|}t : X \to \boldsymbol{S}_r$ and $t \mapsto \frac{1}{\|t\|_1}t : \boldsymbol{S}_r \to X$ are the two halves of a homeomorphism between X and $\boldsymbol{S}_r$. Setting $g(v) = -v$ for $v \in V$, $g : V \to V$ is an automorphism of $(V, \mathcal{K})$ and the corresponding function $\bar g : X \to X$, as defined in 3Cb, has $\bar g(t) = -t$ for every $t \in X$, so matches the antipodeal map on $\boldsymbol{S}_r$. Accordingly $g_\star : H_r(\mathcal{K}) \to H_r(\mathcal{K})$ will correspond to $h_\star : H_r(\boldsymbol{S}_r) \to H_r(\boldsymbol{S}_r)$.

(ii) As we know that $C_{r+1}(\mathcal{K}) = \{0\}$ (because $\mathcal{K}$ has no $(r + 1)$-simplices), therefore $B_r(\mathcal{K}) = \{0\}$, and that $H_r(\mathcal{K}) \cong H_r(\boldsymbol{S}_r) \cong \mathbb{Z}$, we know there is a non-zero cycle in $C_r(\mathcal{K})$. I can give a formula for such a cycle. Let P be the set of those $p \in V^{r+1}$ such that $p(i) = \pm d_i$ for each $i \le r$, and for $j \le r$ let Q_j be the set of those $q \in V^r$ such that $q(i) = \pm d_i$ for $i < j$, $q(i) = \pm d_{i+1}$ for $j \le i < r$. Note that if $p \in P$ then p is injective and $p[r + 1] \in \mathcal{K}$ and $p^{\uparrow j}$, as defined in E1A, belongs to Q_j for every $j \le r$. For $p \in P$, set $I_p = \{i : p(i) = -d_i\}$ and $\delta_p = (-1)^{I_p}$. Consider $x = \sum_{p\in P} \delta_p e_p$ (definition: 1C(b-iv)). Then

$$\partial(x) = \sum_{p\in P} \delta_p \partial(e_p) = \sum_{p\in P} \sum_{j=0}^{r} (-1)^{r-j} \delta_p e_{p^{\uparrow j}}$$

(E1A)

$$= \sum_{j=0}^{r}(-1)^r \sum_{q\in Q_j} \sum_{p\in P, p^{\uparrow j}=q} \delta_p e_q.$$

But for any $j \le r$ and $q \in Q_j$ there are just two members p of P with $p^{\uparrow j} = q$, and these differ only at the coordinate j, where one takes the value d_j and the other takes the value $-d_j$; so δ takes opposite values on them, and $\sum_{p\in P,p^{\uparrow j}=q} \delta_p = 0$. It follows that $\partial(x) = 0$.

(iii) Since $x \ne 0$, indeed $x(p) = \delta_p$ for every $p \in P$, $x^\bullet$ is a non-zero member of $H_r(\mathcal{K})$. Now consider $g_\bullet e_p$, for $p \in P$. Then $gp \in P$ and $g_\bullet e_p = e_{gp}$ (E1C); moreover, $I_{gp} = (r + 1) \setminus I_p$, so

$$\delta_{gp} = (-1)^{r+1-\#(I_p)} = (-1)^{r+1}\delta_p.$$

Now

$$g_\bullet x = \sum_{p\in P} \delta_p g_\bullet e_p = \sum_{p\in P} \delta_p e_{gp} = (-1)^{r+1} \sum_{p\in P} \delta_{gp} e_{gp} = (-1)^{r+1} \sum_{p\in P} \delta_p e_p$$

(because $p \mapsto gp : P \to P$ is a bijection)

$$= (-1)^{r+1}x$$

and $g_\star(x^\bullet) = (-1)^{r+1}x^\bullet$.

(iv) Since we know that $H_r(\mathcal{K}) \cong \mathbb{Z}$, the subgroup $\{a : g_\star(a) = (-1)^{r+1}a\}$ must be either $\{0\}$ or the whole of $H_r(\mathcal{K})$, and the former is ruled out by (iii). So $g_\star(a) = (-1)^{r+1}a$ for every $a \in H_r(\mathcal{K})$ and $h_\star(a) = (-1)^{r+1}a$ for every $a \in H_r(\boldsymbol{S}_r)$.

(b) If r is even, then (a) tells us that $h_\star$ is not the identity on $H_r(\boldsymbol{S}_r)$, so h cannot be homotopic to the identity, by the Homotopy Theorem. Now suppose that r is odd, so that $r+1$ is even; set $k = (r+1)/2 \ge 1$. Define $f : [0, 1] \times \boldsymbol{S}_r \to \boldsymbol{S}_r$ by setting

$$f(\alpha, t)(2i) = t(2i)\cos(\pi\alpha) - t(2i + 1)\sin(\pi\alpha),$$

$$f(\alpha, t)(2i + 1) = t(2i + 1)\cos(\pi\alpha) + t(2i)\sin(\pi\alpha)$$

for $i < k$, $t \in \boldsymbol{S}_r$ and $\alpha \in [0, 1]$. Then f is continuous and $f(0, t) = t$, $f(1, t) = h(t)$ for $t \in \boldsymbol{S}_r$. Thus f is a homotopy between the identity and h.

4I The Hairy Ball Theorem Suppose that $r \in \mathbb{N}$ is even. Then there is no continuous function $g : \boldsymbol{S}_r \to \boldsymbol{S}_r$ such that $(g(t)|t) = 0$ for every $t \in \boldsymbol{S}_r$.

proof ? If there were such a function, we should be able to define a continuous function $f : [0, 1] \times \boldsymbol{S}_r \to \boldsymbol{S}_r$ by setting

$$f(\alpha, t) = \cos(\pi\alpha)t + \sin(\pi\alpha)g(t)$$

for $\alpha \in [0, 1]$ and $t \in \boldsymbol{S}_r$; since $f(0, t) = t$ and $f(1, t) = -t$ for every $t \in \boldsymbol{S}_r$, the antipodeal map would be homotopic to the identity, and we know from 4H that this is not so. **X**

Why 'hairy ball'? Imagine the ball $\boldsymbol{B}_{r+1}$ with hairs sprouting perpendicularly from every point. Can they be combed flat along the surface $\boldsymbol{S}_r$ of the ball in a continuous way, with no whorls or spikes? If r is odd, the formula of the proof of 4Ha tells us what to do. But if r is even, it's impossible.

4J Corollary Suppose that $r \in \mathbb{N}$ is even. Then for every continuous $g : \boldsymbol{S}_r \to \boldsymbol{S}_r$ there is a $t \in \boldsymbol{S}_r$ such that $g(t)$ is either t or $-t$.

proof ? If $g(t) \notin \{t, -t\}$ for every t, then $g(t)$ is never a multiple of t, so we can form

$$f(t) = g(t) - (g(t)|t)t, \quad h(t) = \frac{1}{\|f(t)\|} f(t)$$

for every $t \in \boldsymbol{S}_r$. These functions f and h are continuous, while $h(t) \in \boldsymbol{S}_r$ and $(h(t)|t) = 0$ for every $t \in \boldsymbol{S}_r$; but the Hairy Ball Theorem tells us that this is impossible. **X**

4K Definition Let X be a topological space and $\mathbb{V}$ the set of open covers of X. Set

$$\dim^+(X) = \sup_{W\in\mathbb{V}} \min_{V\in\mathbb{V},V\preccurlyeq W} \sup_{t\in X} \#(\{v : t \in v \in V\}),$$

counting $\sup\emptyset$ as 0 if $X = \emptyset$, $\mathbb{V} = \{\emptyset, \{\emptyset\}\}$. For $r \geq -1$, say that $\dim X$, the **Lebesgue covering dimension** of X, is equal to r if $\dim^+(X) = r + 1$.

4L The Dimension Theorem For any $r \in \mathbb{N}$, $\dim \boldsymbol{B}_r = \dim \mathbb{R}^r = r$.

proof (a) Take $V \subseteq \mathbb{R}^r$, $X = \Gamma(V)$, $Y = \bigcup_{v\in V} \Gamma(V \setminus \{v\})$ and $f : \boldsymbol{B}_r \to X$ as as in 4A, so that $f\restriction\boldsymbol{S}_r$ is a homeomorphism from $\boldsymbol{S}_r$ to Y. As in the proof of 4B, set $\mathcal{K} = \mathcal{P}V$, so that the carrier of $\mathcal{K}$ is X.

(i) Let W be an open cover of X for the subspace topology on X. By 3G, there is a geometrically realizable simplicial complex $\mathcal{K}_m$, with carrier X and set of vertices V_m, such that $\widehat{V_m} = \{\widehat{v} : v \in V_m\}$ refines W, where $\widehat{V_m}$ is the 'simple cover' of X defined from $\mathcal{K}_m$ as in in 3C. Because $\mathcal{K}_m$ is geometrically realizable in $\mathbb{R}^r$, or otherwise, $\mathcal{K}_m$ has no $(r+1)$-simplices and any point t of X belongs to $\Gamma(K)$ for some $K \subseteq V_m$ with at most $r+1$ vertices. Now $\{\widehat{v} : v \in V_m,\, t \in \widehat{v}\} \subseteq \{\widehat{v} : v \in K,\, t \in \widehat{v}\}$ has at most $r+1$ members. As W is arbitrary, $\dim^+(X) \leq r + 1$.

(ii)($\boldsymbol{\alpha}$) Set $\widehat{V} = \{\widehat{v} : v \in V\}$, the simple cover defined from $\mathcal{K}$, so that $\widehat{V}$ is an open cover of X for the subspace topology on X. Let W be an open cover of X refining $\widehat{V}$. For each $w \in W$ let $h(w) \in V$ be such that $w \subseteq \widehat{h(w)}$. For $v \in V$, set $\widehat{v}^\dagger = \bigcup\{w : w \in W,\, h(w) = v\}$. Then $\{\widehat{v}^\dagger : v \in V\}$ is an open cover of X.

Now let $\mathcal{K}_m$ and V_m be as in (i) just above, but such that $\delta = \Delta(\mathcal{K}_m)$ is so small that for every $t \in X$ there is a $v \in V$ such that $\{s : \|s - t\| \leq \delta\} \subseteq \widehat{v}^\dagger$. For $v \in V_m$ let $g(v) \in V$ be such that $\{s : \|s - v\| \leq \delta\} \subseteq g(v)^\dagger$. Then g is a simplicial map, and we have a corresponding continuous function $\bar{g} : X \to X$ defined as in 3Cb.

($\boldsymbol{\beta}$) At this point observe that whenever $v \in V$ and $K = V \setminus \{v\}$ then $\Gamma(K) \cap \widehat{v} = \emptyset$ and $\Gamma(K) \cap \widehat{v}^\dagger$ must be empty. But this means that $g(v') \in K$ whenever $v' \in V_m \cap \Gamma(K)$ and $\bar{g}(t) \in K$ whenever $t \in K$. Consequently $\bar{g}(t) \in Y$ for every $t \in Y$. Moreover, $\bar{g}\restriction Y$ is homotopic to the identity on Y. **P** For every $t \in Y$, there is a K such that t, $\bar{g}(t)$ both belong to $\Gamma(K)$ and $\Gamma(K) \subseteq Y$. Now $\Gamma(\{t, \bar{g}(t)\}) \subseteq Y$. So we can define a continuous function $\hat{g} : [0, 1] \times Y \to Y$ by setting $\hat{g}(\alpha, t) = \alpha t + (1 - \alpha)\bar{g}(t)$ for $\alpha \in [0, 1]$ and $t \in Y$, and $\hat{g}$ will be a homotopy from $\bar{g}$ to the identity. **Q**

($\boldsymbol{\gamma}$) Since we have a homeomorphism between $\boldsymbol{B}_r$ and X under which $\boldsymbol{S}_{r-1}$ corresponds to Y, we can apply 4E to the pair (X, Y) in place of $(\boldsymbol{B}_r, \boldsymbol{S}_{r-1})$ to see that $\bar{g}[X] \not\subseteq Y$. Let $t \in X$ be such that $\bar{g}(t) \notin Y$, and $K \in \mathcal{K}_m$ such that $t \in \Gamma(K)$. Since $g[K]$ cannot be included in $V \setminus \{v\}$ for any $v \in V$, $g[K] = V$. But if $v \in K$ then $\|t - v\| \leq \delta$ so $t \in g(v)^\dagger$. So $t \in \bigcap_{v\in V} \widehat{v}^\dagger$. Returning to the open cover W, $\#(\{w : t \in w \in W\}) \geq \#(\{v : t \in \widehat{v}^\dagger\}) = r + 1$. As W is arbitrary, $\dim^+(X) \geq r + 1$.

(iii) Putting (i) and (ii) together, we see that $\dim^+(X) = r + 1$ and $r = \dim X = \dim \boldsymbol{B}_r$.

(b) Now let us turn to the dimension of $\mathbb{R}^r$.

(i) $\dim^+ \mathbb{R}^r \geq r + 1$. **P** Let V be a cover of $\boldsymbol{B}_r$ consisting of sets which are open for the subspace topology of $\boldsymbol{B}_r$. Let W be

$$\{w : w \subseteq \mathbb{R}^r \text{ is open},\ w \cap \boldsymbol{B}_r \in V\} \cup \{\mathbb{R}^r \setminus \boldsymbol{B}_r\}.$$

Because $\boldsymbol{B}_r$ is a closed subset of $\mathbb{R}^r$, W is a family of open subsets of $\mathbb{R}^r$. Because every $v \in V$ can be expressed as $w \cap \boldsymbol{B}_r$ for some open $w \subseteq \mathbb{R}^r$,

$$\bigcup W \supseteq \bigcup V \cup (\mathbb{R}^r \setminus \boldsymbol{B}_r) = \mathbb{R}^r.$$

So there is an open cover W' of $\mathbb{R}^r$ such that W' refines W and $\#(\{w : t \in w \in W'\}) \leq \dim^+(\mathbb{R}^r)$ for every $t \in \mathbb{R}^r$. For $w \in W'$ let $v_w \in V \cup \{\emptyset\}$ be such that $w \cap \boldsymbol{B}_r \subseteq v_w$; for $v \in V$, set $v' = \bigcup\{w \cap \boldsymbol{B}_r : w \in W',\ v_w = v\}$; set $V' = \{v' : v \in V\}$. Then V' is a family of relatively open subsets of $\boldsymbol{B}_r$, and

$$\bigcup V' = \bigcup\{w : w \in W',\, v_w \in V\} \supseteq \bigcup\{w \cap \boldsymbol{B}_r : w \in W'\} = \boldsymbol{B}_r \cap \bigcup W' = \boldsymbol{B}_r.$$

As $v' \subseteq v$ for every $v \in V$, V' refines V. And for any $t \in X$,

$$\#(\{v' : v' \in V',\, t \in v'\}) \leq \#(\{w : w \in W',\, t \in w\}) \leq \dim^+ \mathbb{R}^r.$$

As V is arbitrary, $\dim^+ \boldsymbol{B}_r \le \dim^+ \mathbb{R}^r$. But we know that $\dim^+ \boldsymbol{B}_r = r+1$, so $\dim^+ \mathbb{R}^r \ge r+1$. **Q**

(ii) In the other direction, we have rather more work to do. For $m \in \mathbb{N}$ set $G_m = \{t : t \in \mathbb{R}^r, \|t\| < m\}$ and $F_m = \{t : t \in \mathbb{R}^r, \|t\| \le m\}$. Then G_m is open and F_m is closed. $G_0 = \emptyset$ and $G_m \subseteq F_m$ for every m. If $m \ge 1$ then $F_m = \{mt : t \in \boldsymbol{B}_r\}$ is homeomorphic to $\boldsymbol{B}_r$ so $\dim^+ F_m = \dim^+ \boldsymbol{B}_r = r+1$.

($\boldsymbol{\alpha}$) Let W be an open cover of $\mathbb{R}^r$ and $m \ge 1$. Then there is an open cover W' of $\mathbb{R}^r$, refining W, such that $\#(\{w : t \in w \in W'\}) \le r+1$ for every $t \in F_m$. **P** $V = \{w \cap F_m : w \in W\}$ is a cover of F_m by sets which are relatively open in F_m. So there is a refinement V' of V, covering F_m, consisting of sets which are relatively open in F_m, and such that no point belongs to more than $r+1$ members of V'. For each $v \in V'$ choose $w_v \in W$ such that $v \subseteq w_v$; for $w \in W$ set

$$w' = (w \setminus F_m) \cup \bigcup\{v : v \in V', w_v = w\};$$

set $W' = \{w' : w \in W\}$. If $w \in W$ and $v \in V'$ is such that $v \subseteq w$, then $v \cup (\mathbb{R}^r \setminus F_m)$ is open in $\mathbb{R}^r$, so $v \cup (w \setminus F_m) = w \cap (v \cup (\mathbb{R}^r \setminus F_m))$ is open in $\mathbb{R}^r$. It follows that

$$w' = (w \setminus F_m) \cup \bigcup\{v : v \in V', w_v = w\} = \bigcup\{v \cup (w \setminus F_m) : v \in V', w_v = w\}$$

is open in $\mathbb{R}^r$, for every $w \in W$, and W' is an open cover of $\mathbb{R}^r$. And if $t \in F_m$,

$$\#(\{w : w \in W', t \in w\}) \le \#(\{v : v \in V', t \in v\} \le r+1,$$

as required. **Q**

($\boldsymbol{\beta}$) Suppose that an open cover W of $\mathbb{R}^r$ and $m \in \mathbb{N}$ are such that $\#(\{v : t \in v \in W\}) \le r+1$ for every $t \in G_m$. Then there is an open refinement W'' of W, covering $\mathbb{R}^r$, such that $\#(\{w : t \in w \in W''\}) \le r+1$ whenever $t \in G_{m+1}$, and W'' contains w whenever $w \in W$ and $w \subseteq G_m$. **P** By (α), there is an open refinement W' of W, covering $\mathbb{R}^r$, such that $\#(\{w : t \in w \in W'\}) \le r+1$ for every $t \in F_{m+1}$. For $w \in W'$ choose $v_w \in W$ such that $w \subseteq v_w$. For $v \in W$ set $v' = (v \cap G_m) \cup \bigcup\{w : w \in W', v_w = v\}$. Set $W'' = \{v' : v \in W\}$. Then W'' is a family of open sets in $\mathbb{R}^r$ and $\bigcup W'' \supseteq \bigcup W' = \mathbb{R}^r$. If $t \in G_{m+1}$ then

$$\begin{aligned}\#(\{w : w \in W'', t \in w\}) \le \#(\{v : v \in W, t \in v'\}) &\le \#(\{v : v \in W, t \in v\}) \le r+1 \\ &\qquad \text{if } t \in G_m, \\ &\le \#(\{w : w \in W', t \in w\}) \le r+1 \text{ otherwise.}\end{aligned}$$

If $v \in W$ and $v \subseteq G_m$ then $v = v \cap G_m = v' \in W''$. So W'' serves. **Q**

($\boldsymbol{\gamma}$) Now take any open cover W of $\mathbb{R}^r$. Choose a sequence $\langle W_m \rangle_{m \in \mathbb{N}}$ of open covers of $\mathbb{R}^r$ as follows. $W_0 = \{w \cap \{t : \|t - t_0\| < \frac{1}{2}\} : w \in W, t_0 \in \mathbb{R}^r\}$, so that W_0 is an open cover refining W and every member of W_0 is included in a ball of radius $\frac{1}{2}$. Given that W_m is an open cover of $\mathbb{R}^r$ such that $\#(\{w : t \in w \in W_m\}) \le r+1$ for every $t \in G_m$, use (β) to find an open cover W_{m+1} of $\mathbb{R}^r$, refining W_m, containing w whenever $w \in W_m$ and $w \subseteq G_m$, and such that $\#(\{w : t \in w \in W_{m+1}\}) \le r+1$ for every $t \in G_{m+1}$.

At the end of the induction, $\langle W_m \rangle_{m \in \mathbb{N}}$ is such that $\ldots \preccurlyeq W_2 \preccurlyeq W_1 \preccurlyeq W_0 \preccurlyeq W$. Set

$$W' = \bigcup_{k \in \mathbb{N}} \bigcap_{m \ge k} W_m.$$

Then W' is a family of open subsets of $\mathbb{R}^r$ and $W' \preccurlyeq W$.

Take any $t \in \mathbb{R}^r$. Then there is a $k \in \mathbb{N}$ such that $t \in G_k$. Let $w \in W_{k+1}$ be such that $t \in w$. There is a $t_0 \in \mathbb{R}^r$ such that $w \subseteq \{t' : \|t' - t_0\| < \frac{1}{2}\}$. So if $t' \in w$ we have

$$\|t'\| \le \|t' - t_0\| + \|t_0\| \le \|t' - t_0\| + \|t_0 - t\| + \|t\| < \tfrac{1}{2} + \tfrac{1}{2} + k = k+1.$$

This shows that $w \subseteq G_{k+1}$. Consequently $w \in G_{k+2}$. Repeating the argument, we see that $w \in G_m$ for every $m \ge k+1$ and $w \in W'$. As t is arbitrary, W' is an open cover of $\mathbb{R}^r$ refining W.

? Suppose, if possible, that there is a $t \in \mathbb{R}^r$ such that $\#(\{w : t \in w \in W'\}) > r+1$. Choose distinct $w_0, \ldots, w_{r+1} \in W'$ such that $t \in w_i$ for every $i \le r+1$. There is a $k \in \mathbb{N}$ such that $t \in G_k$, and now there is an $m \ge k$ such that $w_i \in W_m$ for every $i \le r+1$. But now $t \in G_m$ and $\#(\{w : t \in w \in W_m\}) > r+1$. **X**

Thus we have an open cover W' of $\mathbb{R}^r$, refining W, such that no point belongs to more than $r+1$ members of W'. As W is arbitrary, $\dim^+ \mathbb{R}^r \le r+1$.

(iii) Putting (i) and (ii) together, $\dim^+ \mathbb{R}^r = r + 1$ and $\dim \mathbb{R}^r = r$. This completes the proof.

Remark Modifying the arguments of (b) it is not hard to show that $\dim G = r + 1$ for every non-empty open subset G of $\mathbb{R}^r$.

4M Corollary If r, $s \in \mathbb{N}$ are different, no two of the spaces $\mathbb{R}^r$, $\boldsymbol{B}_r$, $\boldsymbol{S}_r$, $\mathbb{R}^s$, $\boldsymbol{B}_s$ and $\boldsymbol{S}_s$ are homeomorphic.

proof $\mathbb{R}^r$ and $\mathbb{R}^s$ are not homeomorphic because they have different Lebesgue covering dimensions. Neither is homeomorphic to any of the other four because $\boldsymbol{B}_r$, $\boldsymbol{S}_r$, $\boldsymbol{B}_s$ and $\boldsymbol{S}_s$ are compact but $\mathbb{R}^r$ and $\mathbb{R}^s$ are not. $\boldsymbol{B}_r$ and $\boldsymbol{B}_s$ are not homeomorphic because they have different covering dimensions. Finally, $\boldsymbol{S}_r$ and $\boldsymbol{S}_s$ are distinguished from each other and from the balls by their homology groups (4B-4C).

4N More generally, the covering dimension is directly related to the question of whether a space has infinitely many non-trivial homology groups.

Proposition Let X be a topological space with Lebesgue covering dimension $r \geq -1$. Then $H_n(X) = \{0\}$ whenever $n > r$.

proof I return to the notation of Chapter 2. If $a \in H_n(X)$ then $a(V) \in H_n(\mathcal{K}_V)$ for every open cover V of X (2F). If W is an open cover of X there is an open cover V of X, refining W, such that no point of X belongs to more than n members of V, that is, $\mathcal{K}_V$ has no n-simplices and $H_n(\mathcal{K}_V) = \{0\}$. So $a(V) = 0$; but $a(W) = \pi_{WV}(a(V))$, so $a(W) = 0$. As W is arbitrary, $a = 0$.

Notes and comments It is not customary to make such a mouthful of the ideas in 4A, and if you wonder why I thought it necessary you will not be alone. But we do have to believe the claims in 4Ad, and while it is natural to rely on our geometric intuition for dimensions 1, 2 and 3, and to take it for granted that we can extrapolate to higher dimensions, I am not convinced that this is quite safe.

I ought perhaps to remind you, in regard to 4B-4C, that I am speaking throughout of 'reduced' homology groups, so that $H_0(\boldsymbol{B}_r)$ and $H_0(\boldsymbol{S}_r)$ (for $r \geq 1$) are isomorphic to $\mathbb{Z}^0$ rather than $\mathbb{Z}$.

I have tried to remember to say 'covering dimension' throughout the paragraphs 4K-4N because there are other definitions of dimension (inductive dimension, Hausdorff dimension) which in their own realms are interesting and important and are potentially very different, though they can be expected to agree on the basic cases. In particular, the phrase 'zero-dimensional' is commonly used to mean something other than 'with Lebesgue covering dimension zero'. In addition, even with 'covering dimension', you will find many minor variations on the formulation in 4K (e.g., in Engelking 89, p. 385).

The proof that $\dim \mathbb{R}^r = \dim \boldsymbol{B}_r$ in 4L has nothing to do with homology. I include it because it really seems important that the topological dimension of $\mathbb{R}^r$ is equal to its linear space dimension.

The theorems of this section have been exhaustively studied for a hundred years now, and many alternative proofs have been devised. Indeed, I think that for each of the principal results (the Brouwer Fixed Point Theorem, the Hairy Ball Theorem, the Dimension Theorem) there is a proof which most analysts, at least, will find significantly easier than the arguments spelt out in Chapters 1-3 of this note. (See Engelking 89, 7.3.18-7.3.19 for the Brouwer Fixed Point Theorem and the Dimension Theorem, and Milnor 78 for the Hairy Ball Theorem.) I do not recall seeing a direct approach to the reflection theorem 4G, but I should be surprised if there isn't one. Of course Milnor's proof of the Hairy Ball Theorem uses enough analysis to be quite demanding for the general pure mathematician, though I think that Fremlin 01 covers what is needed. However I hope you will agree that it is worth a bit of effort to acquire a technique which can get proofs of all these results into half a dozen pages.

5 Torus, Klein bottle, real projective spaces

With a stretch (Lemma 5C), we can reach the homology groups of some simple classical spaces (5D-5H, 5L).

5A Vertex-restriction subcomplexes Let $\mathcal{K}$ be a simplicial complex, with vertex set V, and V_1 a subset of V. Then $\mathcal{K}_1 = \mathcal{K} \cap \mathcal{P}V_1$ is a simplicial complex with vertex set $\{v : \{v\} \in \mathcal{K}_1\} = V_1$.

(a) Set $h(v) = v$ for $v \in V_1$. Then h, regarded as a function from V_1 to V, is a simplicial map from $\mathcal{K}_1$ to $\mathcal{K}$. If $n \geq -1$ and $y \in C_n(\mathcal{K}_1)$, then

$$(h\bullet y)(p) = \sum_{\substack{q \in V_1^{n+1} \\ hq=p}} y(q) = 0 \text{ if } p[n+1] \not\subseteq V_1,$$

$$= y(p) \text{ if } p[n+1] \subseteq V_1, \text{ that is, } p \in V_1^{n+1}.$$

So $\operatorname{supp}(h\bullet y) = \operatorname{supp}(y) \subseteq V_1$.

(b) In the other direction, we have for each $n \geq -1$ a function $\phi_n : C_n(\mathcal{K}) \to \mathbb{Z}^{V_1^{n+1}}$ defined by setting $\phi_n(x) = x\restriction V_1^{n+1}$ for $x \in C_n(\mathcal{K})$. Now $\phi_n(x) \in C_n(\mathcal{K}_1)$ for $x \in C_n(\mathcal{K})$. **P** If $p \in V_1^{n+1}$ and $\sigma \in S_{n+1}$,

$$\phi_n(x)(p\sigma) = x(p\sigma) = \epsilon_\sigma x(p) = \epsilon_\sigma \phi_n(x)(p),$$

and if $\phi_n(x)(p) \neq 0$ then $p[n+1] \in \mathcal{K} \cap \mathcal{P}V_1 = \mathcal{K}_1$. **Q**

Next, if $n \geq -1$ and $y \in C_n(\mathcal{K}_1)$, $\phi_n(h\bullet y) = y$, while if $x \in C_n(\mathcal{K})$,

$$(h\bullet\phi_n(x))(p) = x(p) \text{ if } p \in V_1^{n+1},$$

$$= 0 \text{ if } p \in V^{n+1} \setminus V_1^{n+1}.$$

So if $x \in C_n(\mathcal{K})$ and $\operatorname{supp}(x) \subseteq V_1$, $h\bullet\phi_n(x) = x$. Of course both $\phi_n : C_n(\mathcal{K}) \to C_n(\mathcal{K}_1)$ and $y \mapsto h\bullet y : C_n(\mathcal{K}_1) \to C_n(\mathcal{K})$ are group homomorphisms. Accordingly $y \mapsto h\bullet y$ and $x \mapsto x\restriction V_1^{n+1}$ are the two halves of an isomorphism between $C_n(\mathcal{K}_1)$ and $\{x : x \in C_n(\mathcal{K}), \operatorname{supp}(x) \subseteq V_1\}$.

(c) For $n \geq 0$ and $y \in C_n(\mathcal{K}_1)$, $h\bullet y \in Z_n(\mathcal{K})$ whenever $y \in Z_n(\mathcal{L})$ and $h\bullet y \in B_n(\mathcal{K})$ whenever $y \in B_n(\mathcal{L})$, by 1I. And conversely, if $y \in C_n(\mathcal{K}_1)$ and $h\bullet y \in Z_n(\mathcal{K})$, then $h\bullet\partial(y) = \partial(h\bullet y) = 0$, so $\partial(y) = 0$ and $y \in Z_n(\mathcal{K}_1)$. But it is not always the case that $y \in B_n(\mathcal{K}_1)$ whenever $h\bullet y \in B_n(\mathcal{K})$.

(d) If $n \geq 0$, $x \in C_n(\mathcal{K})$ and $\operatorname{supp}(x) \subseteq V_1$, then $\phi_{n-1}\partial(x) = \partial\phi_n(x)$. **P** Setting $y = \phi_n(x)$, $x = h\bullet y$ (by (b)) so

$$\phi_{n-1}\partial(x) = \phi_{n-1}(h\bullet\partial(y)) = \partial(y) = \partial\phi_n(x). \quad \mathbf{Q}$$

It follows that if $z \in Z_n(\mathcal{K})$ and $\operatorname{supp}(z) \subseteq V_1$ then $z\restriction V_1^{n+1} \in Z_n(\mathcal{K}_1)$.

(e) Now let be $\mathcal{L}$ another simplicial complex, with vertex set W, and $g : V \to W$ a simplicial map from $\mathcal{K}$ to $\mathcal{L}$. Suppose that $W_1 \subseteq W$ is such that $V_1 = g^{-1}[W_1]$; set $\mathcal{L}_1 = \mathcal{L} \cap \mathcal{P}W_1$.

(i) $g_1 = g\restriction V_1$ is a simplicial map from $\mathcal{K}_1$ to $\mathcal{L}_1$, because if $K \in \mathcal{K}_1$ then $g_1[K] = g[K] \in \mathcal{L} \cap \mathcal{P}W_1$.

(ii) If $n \geq -1$ and $x \in C_n(\mathcal{K})$, $g_1\bullet(x\restriction V_1^{n+1}) = (g\bullet x)\restriction W_1^{n+1}$. **P** If $q \in W_1^{n+1}$ then

$$(g_1\bullet(x\restriction V_1^{n+1}))(q) = \sum_{\substack{p \in V_1^{n+1} \\ g_1 p = q}} x(p) = \sum_{\substack{p \in V_1^{n+1} \\ gp=q}} x(p) = \sum_{\substack{p \in V^{n+1} \\ gp=q}} x(p)$$

(because if $p \in V^{n+1}$ and $gp = q$, then $gp[n+1] \subseteq W_1$ so $p[n+1] \subseteq g^{-1}[W_1] = V_1$)

$$= (g\bullet x)(q). \quad \mathbf{Q}$$

5B Image complexes: Lemma Suppose that $\mathcal{K}$ is a simplicial complex with vertex set V, W is a set, and $f : V \to W$ is a surjective function.

(a) $\mathcal{L} = \{f[K] : K \in \mathcal{K}\}$ is a simplicial complex with vertex set W, and f is a simplicial map from $\mathcal{K}$ to $\mathcal{L}$.

(b) If $n \geq -1$ and $y \in C_n(\mathcal{L})$ there is an $x \in C_n(\mathcal{K})$ such that $y = f\bullet x$.

(c) If $\mathcal{K}$ is connected then $\mathcal{L}$ is connected.

proof (a) If $L' \subseteq L \in \mathcal{L}$, there is a $K \in \mathcal{K}$ such that $L = f[K] \subseteq f[V] = W$. Now $K \cap f^{-1}[L'] \in \mathcal{K}$ and $L' = f[K \cap f^{-1}[L']] \in \mathcal{L}$. Thus $\mathcal{L}$ is a simplicial complex. And if $w \in W$ then there is a $v \in V$ such that $f(v) = w$ and $\{w\} = f[\{v\}] \in L$, so W is the vertex set of $\mathcal{L}$. Obviously $f[K] \in \mathcal{L}$ for every $K \in \mathcal{K}$, so f is a simplicial map.

(b) By Proposition E1B, it is enough to prove the result when $y = e_q$ for some injective $q \in W^{n+1}$ such that $q[n+1] \in \mathcal{L}$. In this case there is a $K \in \mathcal{K}$ such that $f[K] = q[n+1]$. For each $i \leq n$ take $p(i) \in K$ such that $fp(i) = q(i)$; then $p \in V^{n+1}$ is injective, $p[n+1] \in \mathcal{K}$ and $fp = q$. So we have e_p defined in $C_n(\mathcal{K})$, and $e_q = e_{fp} = f\bullet e_p$ (E1C) is of the required form.

(c) ? Otherwise, W has a component $W' \neq W$. Set $V' = f^{-1}[W'] \subset V$. If $v \in V'$ and $v' \in V \setminus V'$, $\{f(v), f(v')\} \notin \mathcal{L}$ so $\{v, v'\} \notin \mathcal{K}$. So if v_0 is any member of V', the component of $\mathcal{K}$ containing v_0 is included in V' and is not equal to V; but this means that $\mathcal{K}$ is not connected. **X**

5C Lemma Let $\mathcal{K}$ be a simplicial complex with non-empty vertex set V, W a set, $f : V \to W$ a surjection, W_1 a subset of W and $m \geq 0$. Write W_0 for $W \setminus W_1$, V_1 for $f^{-1}[W_1]$, V_0 for $f^{-1}[W_0] = V \setminus V_0$, $\mathcal{L}$ for $\{f[K] : K \in \mathcal{K}\}$, $\mathcal{K}_1$ for $\mathcal{K} \cap \mathcal{P}V_1$ and $\mathcal{L}_1$ for $\mathcal{L} \cap \mathcal{P}W_1$. Suppose that

(α) $H_n(\mathcal{K}) = \{0\}$ for every $n \in \mathbb{N}$, $H_m(\mathcal{K}_1) \cong \mathbb{Z}$, $H_n(\mathcal{K}_1) = \{0\}$ for every $n \neq m$;

(β) $\mathcal{K}$ has no $(m+2)$-simplices, $\mathcal{K}_1$ has no $(m+1)$-simplices;

(γ) $f[V_0] \cap W_1 = \emptyset$, $f{\restriction}V_0$ is injective and $f{\restriction}\bigcup\{K : v \in K \in \mathcal{K}\}$ is injective for every $v \in V_0$.

Then

(a) $\mathcal{L}$ is a simplicial complex with vertex set W and no $(m+2)$-simplices. $\mathcal{L}_1 = \{f[K] : K \in \mathcal{K}_1\}$ is a simplicial complex with vertex set W_1 and no $(m+1)$-simplices. $f : V \to W$ is a simplicial map from $\mathcal{K}$ to $\mathcal{L}$ and $f{\restriction}V_1 : V_1 \to W_1$ is a simplicial map from $\mathcal{K}_1$ to $\mathcal{L}_1$.

(b) $H_n(\mathcal{L}) \cong H_n(\mathcal{L}_1)$ for $0 \leq n < m$.

(c) Set $A = \{x : x \in C_{m+1}(\mathcal{K}),\ \operatorname{supp}\partial(x) \subseteq V_1,\ (\partial(x){\restriction}V_1^{m+1})^\bullet$ generates $H_m(\mathcal{K}_1)\}$. Then $A \neq \emptyset$. If $x \in A$, $x' \in C_{m+1}(\mathcal{K})$ and $\operatorname{supp}\partial(x') \subseteq V_1$, then x' is an integer multiple of x. If $x, x' \in A$ then $x' = \pm x$.

(d) Take any $\hat{x} \in A$. Then $H_m(\mathcal{L}) \cong Z_m(\mathcal{L}_1)/G$ where G is the set of multiples of $(f\bullet\partial(\hat{x})){\restriction}W_1^{m+1}$. So if $f\bullet\partial(\hat{x}) = 0$ then $H_m(\mathcal{L}) \cong H_m(\mathcal{L}_1)$.

(e)(i) If $\hat{x} \in A$ and $f\bullet\partial(\hat{x}) = 0$ then $Z_{m+1}(\mathcal{L})$ is the set of multiples of $f\bullet\hat{x}$ and $H_{m+1}(\mathcal{L}) \cong \mathbb{Z}$.

(ii) If $\hat{x} \in A$ and $f\bullet\partial(\hat{x}) \neq 0$ then $H_{m+1}(\mathcal{L}) = \{0\}$.

(f) $H_n(\mathcal{L}) = \{0\}$ if $n > m+1$.

proof (a)(i) We know from 5B that $\mathcal{L}$ is a simplicial complex with vertex set W and that $f : V \to W$ is a simplicial map from $\mathcal{K}$ to $\mathcal{L}$. If $L \in \mathcal{L}$ there is a $K \in \mathcal{K}$ such that $L = f[K]$, in which case $\#(L) \leq \#(K) \leq m+2$; accordingly $\mathcal{L}$ has no $(m+2)$-simplices.

(ii) As noted in 5A, $\mathcal{K}_1$ and $\mathcal{L}_1$ are simplicial complexes with vertex sets V_1 and W_1 respectively.

(iii) If $L \in \mathcal{L}_1$ let $K \in \mathcal{K}$ be such that $f[K] = L$; then (because W_0 is disjoint from W_1) $K \subseteq V_1$ so $K \in \mathcal{K}_1$. Thus $\mathcal{L}_1 \subseteq \{f[K] : K \in \mathcal{K}_1\} = \{(f{\restriction}V_1)[K] : K \in \mathcal{K}_1\}$. On the other hand, if $K \in \mathcal{K}_1$, then $f[K] = (f{\restriction}V_1)[K]$ surely belongs to $\mathcal{L}_1$. So $\mathcal{L}_1 = \{f[K] : K \in \mathcal{K}_1\}$ and $f{\restriction}V_1$ is a simplicial map from $\mathcal{K}_1$ to $\mathcal{L}_1$, while $\mathcal{L}_1$ has no $(m+1)$-simplices because $\mathcal{K}_1$ has none.

(b)(i) If $n \geq -1$ and $y \in C_n(\mathcal{L})$ there is an $x \in C_n(\mathcal{K})$ such that $y = f\bullet x$, by 5Bb.

(ii) If $n \geq -1$ and $x \in C_n(\mathcal{K})$ then $\operatorname{supp}(f\bullet x) \subseteq W_1$ iff $\operatorname{supp}(x) \subseteq V_1$.

P(α) If $\operatorname{supp}(x) \subseteq V_1$ then $\operatorname{supp}(f\bullet x) \subseteq f[V_1] = W_1$ by 1Md.

(β) If $\operatorname{supp}(x) \not\subseteq V_1$, take $v \in V_0 \cap \operatorname{supp}(x)$ and $p \in V^{n+1}$ such that $v \in p[n+1]$ and $x(p) \neq 0$. If $p' \in V^{n+1}$ is such that $p'[n+1] \in \mathcal{K}$ and $fp' = fp$, we have $f(v) \in fp'[n+1]$, and there is a $v' \in p'[n+1]$ such that $f(v) = f(v')$. Since $f[V_0] \cap f[V_1] = \emptyset$, $v' \in V_0$; since $f{\restriction}V_0$ is injective, $v = v'$ and $v \in p[n+1] \cap p'[n+1]$. By the hypothesis ($\gamma$), $f{\restriction}p[n+1] \cup p'[n+1]$ is injective and $p = p'$.

So we see that

$$0 \neq x(p) = \sum_{\substack{p' \in V^{n+1} \\ fp' = fp}} x(p') = (f\bullet x)(fp)$$

and $f(v) \in W_0 \cap fp[n+1] \subseteq W_0 \cap \operatorname{supp}(f\bullet x)$, so $\operatorname{supp}(f\bullet x) \not\subseteq W_1$. **Q**

(iii) If $n \in \mathbb{N}$ and $y \in Z_n(\mathcal{L})$ there is an $x \in C_n(\mathcal{K})$ such that $f\bullet x = y$ and $\operatorname{supp}\partial(x) \subseteq V_1$. **P** By (i) just above, we have an $x \in C_n(\mathcal{K})$ such that $y = f\bullet x$. Now $f\bullet\partial(x) = \partial(y) = 0$ (1H) and $\operatorname{supp}(f\bullet\partial(x)) = \emptyset \subseteq W_1$. So (ii) tells us that $\operatorname{supp}\partial(x) \subseteq V_1$. **Q**

(iv) If $0 < n \le m$ and $y \in Z_n(\mathcal{L})$ there is a $y' \in Z_n(\mathcal{L})$ such that $\operatorname{supp}(y') \subseteq W_1$ and $y - y' \in B_n(\mathcal{L})$. **P** By (iii), there is an $x \in C_n(\mathcal{K})$ such that $f\bullet x = y$ and $\operatorname{supp}\partial(x) \subseteq V_1$. Now 5Ad tells us that

$$\partial(\partial(x)\restriction V_1^n) = (\partial\partial(x))\restriction V_1^{n-1} = 0$$

so

$$\partial(x)\restriction V_1^n \in Z_{n-1}(\mathcal{K}_1) = B_{n-1}(\mathcal{K}_1)$$

(because $H_{n-1}(\mathcal{K}_1) = \{0\}$) and there is a $z \in C_n(\mathcal{K}_1)$ such that $\partial(x)\restriction V_1^n = \partial(z)$. By 5Aa, there is a (unique) $x' \in C_n(\mathcal{K})$ such that $\operatorname{supp}(x') \subseteq V_1$ and $z = x'\restriction V_1^{n+1}$. In this case, $\operatorname{supp}\partial(x') \subseteq V_1$ and $\partial(x')\restriction V_1^n = \partial(z)$ (see 5Ac), so $\partial(x') = \partial(x)$ and $x - x' \in Z_n(\mathcal{K}) = B_n(\mathcal{K})$; setting $y' = f\bullet x'$, $y - y' = f\bullet(x - x') \in B_n(\mathcal{L})$ (1I), while $\operatorname{supp}(y') \subseteq W_1$, and of course $y' \in Z_n(\mathcal{L})$. **Q**

(v) If $n \ge -1$ and $x \in C_n(\mathcal{K})$ then $(f\bullet x)\restriction W_1^{n+1} = (f\restriction V_1)\bullet(x\restriction V_1^{n+1})$. **P** We saw in 5Ab that $x\restriction V_1^{n+1} \in C_n(\mathcal{K}_1)$, so we can speak of $(f\restriction V_1)\bullet(x\restriction V_1^{n+1}) \in C_n(\mathcal{L}_1)$. Now if $p \in V^{n+1} \setminus V_1^{n+1}$ there is a $v \in V_0 \cap p[n+1]$, in which case $f(v) \in W_0 \cap fp[n+1]$ and $fp \notin W_1^{n+1}$. Accordingly, for any $q \in W_1^{n+1}$,

$$(f\bullet x)(q) = \sum_{\substack{p \in V^{n+1} \\ fp = q}} x(p) = \sum_{\substack{p \in V_1^{n+1} \\ fp = q}} x(p) = (f\restriction V_1)\bullet(x\restriction V_1^{n+1})(q). \textbf{ Q}$$

(vi) If $0 < n < m$, $y \in B_n(\mathcal{L})$ and $\operatorname{supp}(y) \subseteq W_1$ then $y\restriction W_1^{n+1} \in B_n(\mathcal{L}_1)$. **P** There is an $x \in C_{n+1}(\mathcal{K})$ such that $y = \partial(f\bullet x) = f\bullet\partial(x)$. So $\operatorname{supp}\partial(x) \subseteq V_1$, by (ii). Consequently

$$\partial(\partial(x)\restriction V_1^{n+1}) = (\partial\partial(x))\restriction V_1^n = 0$$

by 5Ad, and

$$\partial(x)\restriction V_1^{n+1} \in Z_n(\mathcal{K}_1) = B_n(\mathcal{K}_1).$$

We therefore have a $z \in C_{n+1}(\mathcal{K}_1)$ such that $\partial(x)\restriction V_1^{n+1} = \partial(z)$. Now

$$\partial((f\restriction V_1)\bullet z) = (f\restriction V_1)\bullet\partial(z) = (f\restriction V_1)\bullet(\partial(x)\restriction V_1^{n+1}) = (f\bullet\partial(x))\restriction W_1^{n+1}$$

(by (v) above)

$$= y\restriction W_1^{n+1}$$

and $y\restriction W_1^{n+1} \in B_n(\mathcal{L}_1)$. **Q**

(vii) If $0 < n < m$ then $H_n(\mathcal{L}) \cong H_n(\mathcal{L}_1)$. **P** Setting $h(w) = w$ for $w \in W_1$, h is a simplicial map from $\mathcal{L}_1$ to $\mathcal{L}$, and we have a corresponding homomorphism $h_\star : H_n(\mathcal{L}_1) \to H_n(\mathcal{L})$. If $y \in Z_n(\mathcal{L}_1) \setminus B_n(\mathcal{L}_1)$ then $y = (h\bullet y)\restriction W_1^{n+1}$, while $\operatorname{supp}(h\bullet y) \subseteq W_1$ so $h\bullet y \notin B_n(\mathcal{L})$, by (vi). This means that $h_\star$ is injective on $H_n(\mathcal{L}_1)$. If $y \in Z_n(\mathcal{L})$ then (iv) tells us that there is a $y_1 \in Z_n(\mathcal{L})$ such that $\operatorname{supp}(y_1) \subseteq W_1$ and $y - y_1 \in B_n(\mathcal{L})$. But now

$$h_\star((y_1\restriction W_1^{n+1})^\bullet) = (h\bullet(y_1\restriction W_1^{n+1}))^\bullet = y_1^\bullet = y^\bullet$$

in $H_n(\mathcal{L})$. So $h_\star$ is surjective and is an isomorphism. **Q**

(viii) I have been passing over the case $n = 0$. But here we know that $H_0(\mathcal{K}) = \{0\}$ and $H_0(\mathcal{K}_1) = \{0\}$, so $\mathcal{K}$ and $\mathcal{K}_1$ are connected (E1F). By 5Bc and (a) above, $\mathcal{L}$ and $\mathcal{L}_1$ are connected, so by E1F in the other direction, $H_0(\mathcal{L}) = \{0\}$ and $H_0(\mathcal{L}_1) = \{0\}$.

(c)(i) As $H_m(\mathcal{K}_1)$ is isomorphic to $\mathbb{Z}$, it is generated by $y^\bullet$ for some $y \in Z_m(\mathcal{K}_1) \setminus B_m(\mathcal{K}_1)$. By 5Aa, as always, there is a $z \in C_m(\mathcal{K})$ such that $\operatorname{supp}(z) \subseteq V_1$ and $y = z\restriction V_1^{m+1}$. $\operatorname{supp}\partial(z) \subseteq V_1$ and $\partial(z)\restriction V_1^{m+1} = \partial(y) = 0$ so $\partial(z) = 0$; but $H_m(\mathcal{K}) = \{0\}$ so $z \in B_m(\mathcal{K})$ and $z = \partial(x)$ for some $x \in C_{m+1}(\mathcal{K})$. Now $\partial(x)\restriction V_1^{m+1} = y$ so $(\partial(x)\restriction V_1^{m+1})^\bullet$ generates $H_m(\mathcal{K}_1)$ and $x \in A$.

(ii) Suppose that $x \in A$, $x' \in C_{m+1}(\mathcal{K})$ and $\operatorname{supp}\partial(x') \subseteq V_1$. By 5Ad, $\partial(x')\restriction V_1^{m+1} \in Z_m(\mathcal{K}_1)$. We are supposing that $(\partial(x)\restriction V_1^{m+1})^\bullet$ generates $H_m(\mathcal{K}_1) \cong \mathbb{Z}$, so there is a $k \in \mathbb{Z}$ such that $(\partial(x')\restriction V_1^{m+1})^\bullet = k(\partial(x)\restriction V_1^{m+1})^\bullet$. Because $C_{m+1}(\mathcal{K}_1) = \{0\}$, $B_m(\mathcal{K}_1) = \{0\}$ and the map $y \mapsto y^\bullet : C_m(\mathcal{K}_1) \to H_m(\mathcal{K}_1)$ is an isomorphism; accordingly $\partial(x')\restriction V_1^{m+1} = k\partial(x)\restriction V_1^{m+1}$. As $\operatorname{supp}\partial(x)$ and $\operatorname{supp}\partial(x')$ are both included in V_1, $\partial(x') = k\partial(x)$.

Thus $x' - kx \in Z_{m+1}(\mathcal{K})$. But $H_{m+1}(\mathcal{K}) = \{0\}$, so $Z_{m+1}(\mathcal{K}) = B_{m+1}(\mathcal{K})$, which is $\{0\}$ because $\mathcal{K}$ has no $(m+2)$-simplices. We conclude that $x' = kx$ is a multiple of x.

If x and x' both belong to A, then each is a multiple of the other, so $x' = \pm x$.

(d) If $y \in Z_m(\mathcal{L})$ there is a $y' \in Z_m(\mathcal{L})$ such that $\operatorname{supp}(y') \subseteq W_1$ and $y - y' \in B_m(\mathcal{L})$, by (b-iv). So

$$H_m(\mathcal{L}) = \{y^\bullet : y \in Z_m(\mathcal{L}),\ \operatorname{supp}(y) \subseteq W_1\}.$$

Next, if $y \in Z_m(\mathcal{L})$ and $\operatorname{supp}(y) \subseteq W_1$ then y is of the form $h\bullet z$ where $z \in C_m(\mathcal{L}_1)$ and $h : W_1 \to W$ is the identity map, as in (b-vii). Now $h\bullet\partial(z) = \partial(y) = 0$ so $\partial(z) = 0$ (because the action of h on $C_m(\mathcal{L}_1)$ is injective, by 5Aa), and $z \in Z_m(\mathcal{L}_1)$. Thus

$$H_m(\mathcal{L}) = \{(h\bullet z)^\bullet : z \in Z_m(\mathcal{L}_1)\}.$$

For $z \in Z_m(\mathcal{L}_1)$, $h\bullet z \in B_m(\mathcal{L})$ iff it is expressible as $\partial(f\bullet x) = f\bullet\partial(x)$ for some $x \in C_{m+1}(\mathcal{K})$ (using (b-i) again). In this case, $\operatorname{supp}(f\bullet\partial(x)) \subseteq W_1$, so $\operatorname{supp}\partial(x) \subseteq V_1$, by (b-ii). By (c), x must be a multiple of $\hat{x}$, so $h\bullet z$ is a multiple of $f\bullet\partial(\hat{x})$ and z is a multiple of $(f\bullet\partial(\hat{x}))\restriction W_1^{m+1}$.

Thus the kernel of the homomorphism

$$z \mapsto (h\bullet z)^\bullet : Z_m(\mathcal{L}_1) \to H_m(\mathcal{L})$$

is the subgroup G generated by $(f\bullet\partial(\hat{x}))\restriction W_1^{m+1}$, and $H_m(\mathcal{L}) \cong Z_m(\mathcal{L}_1)/G$. If $f\bullet\partial(\hat{x}) = 0$ then $G = \{0\}$ and $H_m(\mathcal{L}) \cong Z_m(\mathcal{L}_1) \cong H_m(\mathcal{L}_1)$ because $\mathcal{L}_1$ has no $(m+1)$-simplices.

(e) If $y \in Z_{m+1}(\mathcal{L})$, take $x \in C_{m+1}(\mathcal{K})$ such that $f\bullet x = y$. Then $f\bullet\partial(x) = 0$. By (b-ii), $\operatorname{supp}\partial(x) \subseteq V_1$. So x is a multiple of $\hat{x}$, by (c), and y is a multiple of $f\bullet\hat{x}$. Now $\hat{x}$ is non-zero, $\operatorname{supp}(\hat{x})$ must meet V_0 (as every $(m+1)$-simplex of $\mathcal{K}$ meets V_0), $\operatorname{supp}(f\bullet\hat{x}) \not\subseteq W_1$ (by (b-ii)) and $f\bullet\hat{x} \ne 0$.

(i) If $0 = f\bullet\partial(\hat{x}) = \partial(f\bullet\hat{x})$, $f\bullet\hat{x}$ does belong to $Z_{m+1}(\mathcal{L})$. So $Z_{m+1}(\mathcal{L}) \cong \mathbb{Z}$; as $\mathcal{L}$ has no $(m+2)$-simplices, $H_{m+1}(\mathcal{L}) \cong Z_{m+1}(\mathcal{L}) \cong \mathbb{Z}$.

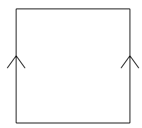
(ii) On the other hand, if $\partial(f\bullet\hat{x}) \ne 0$, then $\partial(k \cdot f\bullet\hat{x}) \ne 0$ for every non-zero $k \in \mathbb{Z}$. In this case, $k \cdot f\bullet\hat{x} \notin Z_{m+1}(\mathcal{L})$ for any non-zero k and $Z_{m+1}(\mathcal{L}) = \{0\}$. So in this case $H_{m+1}(\mathcal{L}) = \{0\}$.

(f) Finally, since $\mathcal{L}$ has no $(m+2)$-simplices, $C_n(\mathcal{L})$ and $H_n(\mathcal{L})$ are both trivial for $n > m+1$.

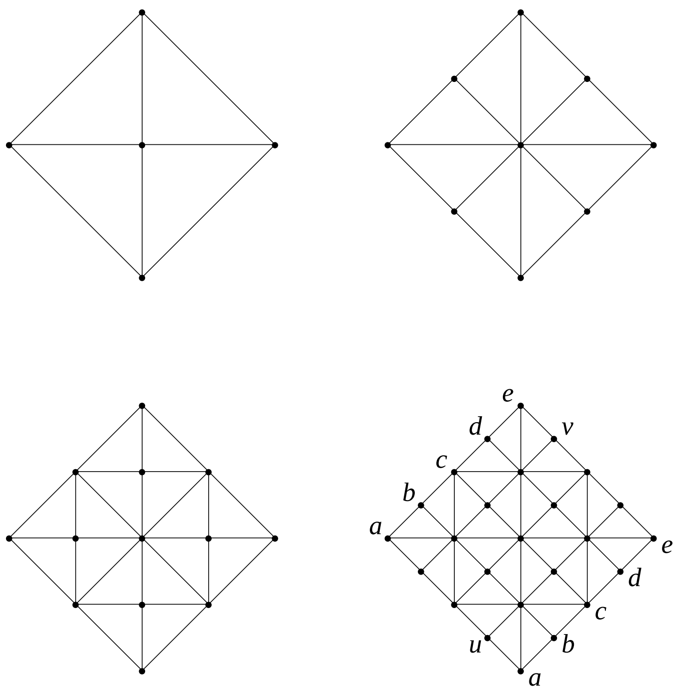
This completes the proof of the lemma.

Remark In the statment of part (d) of the lemma we see the cycle group $Z_m(\mathcal{L}_1)$, while elsewhere the hypotheses and results are stated in terms of homology groups. In this context, because $\mathcal{L}_1$ has no $(m+1)$-simplices, $Z_m(\mathcal{L}_1) \cong H_m(\mathcal{L}_1)$. However we can expect applications to depend on identifying the group G not only up to isomorphism (it will be either $\{0\}$ or isomorphic to $\mathbb{Z}$) but as a subgroup of $Z_m(\mathcal{L}_1)$.

5D The cylinder There are other, and more instructive, ways to calculate the homology groups of the cylinder, but I will present it as my first example of the use of Lemma 5C. We can form a cylinder by starting with a square and curling it round to match one edge with the opposite one. We can represent this in two dimensions with the diagram



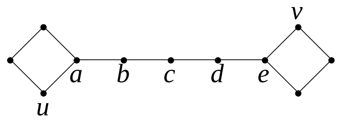
in which the arrows at the left and right indicate an instruction to join these sides together. If we take a suitable triangulation $\mathcal{K}$ of the square this will correspond to a version of 5C in which $\mathcal{K}_1$ consists of the simplices on its edges and f collapses corresponding points of the joined edges. V_0 will be the set of interior vertices; declaring that $\mathcal{K}_1 = \mathcal{K} \cap \mathcal{P}V_1$ forces every triangle of $\mathcal{K}$ to meet V_0; requiring f to be injective on the union of all the triangles sharing any vertex in V_0 is a further limitation. There are quite simple diagrams which will fulfil the requirements in this particular case, but as I wish to deal with some more elaborate identifications, and also to suggest what analogous constructions in higher dimensions could look like, I will go through a subdivision process starting with the representation described in the proof of 4H, based on the unit vectors in $\mathbb{R}^2$.



You will see that at each stage I have performed a string of elementary subdivisions bisecting all the edges of maximal length in the previous stage.

The final version of the diagram represents the complex $\mathcal{K}$ to which I mean to apply the lemma; V_0 is the set of nine interior vertices and V_1 the set of sixteen vertices on the boundary; the 1-simplices of $\mathcal{K}_1$ are the sixteen edges along the boundary; the $\hat{x}$ of parts (d) and (e) of the lemma will have $\hat{x}(p) = 1$ whenever $p \in V^3$ enumerates the vertices of a 2-simplex of $\mathcal{K}$ in an anticlockwise manner, so that $\partial(\hat{x})(q) = 1$ whenever q enumerates the vertices of a 1-simplex on the boundary in a manner consistent with an anticlockwise path around the boundary. (The labels on the boundary vertices correspond to the labels their images in W_1 will be given in the next diagram.) We can be sure of the homology requirements in condition (α) of 5C because the carriers of the geometrically realizable complexes $\mathcal{K}$ and $\mathcal{K}'$ are homeomorphic to $\boldsymbol{B}_2$ and $\boldsymbol{S}_1$ respectively, as observed in part (b-i) of the proof of 4H. The dimension requirements in (β) of 5C, and the requirements on the function f in (γ) of 5C are also evident from the diagram; I chose it as the first stage in the process where vertices in V_0 would be adequately isolated from the identifications imposed by f.

$\mathcal{L}_1$ can now be represented as



with a single path from a to e and with loops $e \to \ldots \to v \to e$ and $a \to u \to \ldots \to a$ at the ends. So $Z_1(\mathcal{L}_1) \cong \mathbb{Z}^2$, since if $q = {<}a{>}{}^\frown{<}u{>}$ and $q' = {<}e{>}{}^\frown{<}v{>}$ then a 1-cycle y is determined by the values $y(q)$ and $y(q') \in \mathbb{Z}$. If we think of z_r, z_l as the 1-cycles corresponding to taking the right and left circuits in the diagram anticlockwise, $y = y(q)z_r + y(q')z_l$.

When we look at $(f{\bullet}\partial(\hat{x}))\restriction W_1^2$ in the diagram for $\mathcal{L}_1$, this is a circuit

$$a \to b \to c \to d \to e \to v \to \ldots \to e \to d \to c \to b \to a \to u \to \ldots \to a.$$

The sections $a \to b \to c \to d \to e$ and $e \to d \to c \to b \to a$ cancel out (remember that $(f{\bullet}\partial(\hat{x}))(q) = \sum_{fp=q} \partial(\hat{x})(p)$, and that $\partial(\hat{x})({<}b{>}{}^\frown{<}a{>}) = -\partial(\hat{x})({<}a{>}{}^\frown{<}b{>})$), so $(f{\bullet}\partial(\hat{x}))\restriction W_1^2 = z_r + z_l$. Thus the group $G \le Z_1(\mathcal{L}_1)$, as described in (d) of 5C, is the set of multiples of $z_r + z_l$, and $Z_1(\mathcal{L}_1)/G \cong \mathbb{Z}$, being generated either by $z_r^\bullet$ or by $z_l^\bullet$.

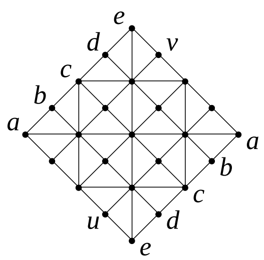
Reading off the results from 5C, we see that

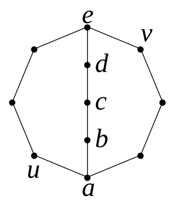
$$H_0(\mathcal{L}) = \{0\}, \quad H_1(\mathcal{L}) \cong \mathbb{Z}, \quad H_2(\mathcal{L}) = \{0\}, \quad H_n(\mathcal{L}) = \{0\} \text{ for } n > 2;$$

that is, $\mathcal{L}$ has the same homology as $\boldsymbol{S}_1$. Looking for a generator for $H_1(\mathcal{L})$, we see that $z_r^\bullet$ corresponds to the edge $e \to \ldots \to v \to e$ in the diagram for $\mathcal{K}$, and that $-z_l^\bullet$ corresponds to the parallel edge $a \to \ldots \to u \to a$.

5E The Möbius strip Very much the same methods are effective with the Möbius strip. Here, instead of smoothly wrapping a square round to put two edges together, we stretch it into a thin rectangle and give it a half-twist, so that the matched edges are in opposite directions.

Transferring this into diagrams for $\mathcal{K}$ and $\mathcal{L}_1$ as in 5D, we get





$\mathcal{L}_1$ now has three paths from a to e, one from each of the sides of the square on which f is injective, and one for the two sides which have been put together. We see that once again $Z_1(\mathcal{L}_1) \cong \mathbb{Z}^2$, generated by cycles z_r, z_l corresponding to the circuits $a \to b \to c \to d \to e \to v \to \ldots \to a$ and $a \to u \to \ldots \to e \to d \to c \to b \to a$. This time, $(f{\bullet}\partial(\hat{x}))\restriction W_1^2$ corresponds to a circuit

$$a \to b \to c \to d \to e \to v \to \ldots \to a \to b \to c \to d \to e \to u \to \ldots \to a.$$

So we no longer have a cancellation and $(f{\bullet}\partial(\hat{x}))\restriction W_1^2 = z_r - z_l$. G is now the subgroup of $Z_1(\mathcal{L}_1)$ generated by $z_r - z_l$; but the quotient is still generated by either $z_r^\bullet$ or $z_l^\bullet$ and is still isomorphic to $\mathbb{Z}$. So here again we have

$$H_0(\mathcal{L}) = \{0\}, \quad H_1(\mathcal{L}) \cong \mathbb{Z}, \quad H_2(\mathcal{L}) = \{0\}, \quad H_n(\mathcal{L}) = \{0\} \text{ for } n > 2;$$

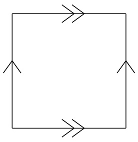
and $\mathcal{L}$ has the same homology as $\boldsymbol{S}_1$.

Looking again for a generator of $H_1(\mathcal{L})$, we see that z_r can also be thought of as representing the circuit

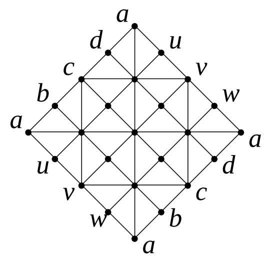
$$c \to d \to e \to v \to \ldots \to a \to b \to c$$

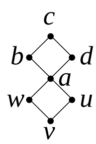
in the diagram of $\mathcal{L}_1$; while in the diagram of $\mathcal{K}$ this is homotopic to the direct path from the north-western c to the south-eastern c. Translated into our picture of the twisted strip, this last path is the median circuit in $\mathcal{L}$. In contrast, $(f\bullet\partial(\hat{x}))\restriction W_1^2$ corresponds to the single edge of $\mathcal{L}$, winding twice around the hole.

5F The torus To get a torus, one method is to start from a cylinder, stretched out into a hose, and bend the ends around to meet each other. Starting from a square, and allowing for the stretching step, we see that we can think of this as wrapping it into a cylinder and then joining the other two edges of the square, as in



In terms of diagrams for $\mathcal{K}$ and $\mathcal{L}_1$, we get





Once again, $Z_1(\mathcal{L}_1) \cong \mathbb{Z}^2$, with generating cycles from the circuits $a \to b \to c \to d \to a$ and $a \to u \to v \to w \to a$.

Looking at $(f\bullet\partial(\hat{x}))\restriction W_1^2$ in the same way as before, we find ourselves with the circuit

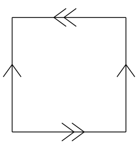
$$a \to b \to c \to d \to a \to u \to v \to w \to a \to d \to c \to b \to a \to w \to v \to u \to a.$$

But this time we get a double cancellation of $a \to b \to c \to d \to a$ with $a \to d \to c \to b \to a$ and of $a \to u \to v \to w \to a$ with $a \to w \to v \to u \to a$. So $(f\bullet\partial(\hat{x}))\restriction W_1^2 = 0$. Accordingly we have the other pattern envisaged in (d)-(e) of 5C, and

$$H_0(\mathcal{L}) = \{0\}, \quad H_1(\mathcal{L}) \cong \mathbb{Z}^2, \quad H_2(\mathcal{L}) \cong \mathbb{Z}, \quad H_n(\mathcal{L}) = \{0\} \text{ for } n > 2.$$

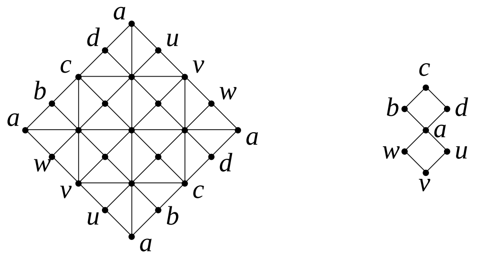
Observe that both the torus and the sphere $\boldsymbol{S}_2$ have the second homology group H_2 isomorphic to $\mathbb{Z}$. This of course means that they have 2-cycles which are not boundaries, and is associated with the fact that they are **orientable**. Geometrically, no edge is shared by more than two 2-simplices, and the 2-simplices can be assigned orientations in such a way that if two share an edge they give it opposite directions. For objects like the torus and 2-sphere, which can be embedded in $\mathbb{R}^3$, these consistent orientations are like that considered in E3Ac for the faces of a 3-simplex, being consistently clockwise or anticlockwise when viewed from outside the object. In the algebraic terms of Chapter 1, this means that there is a cycle x in $C_2(\mathcal{K})$ such that $x(p) = \pm 1$ whenever $p \in V^3$ is injective and $p[3] \in \mathcal{K}$.

5G The Klein bottle Suppose we reverse an arrow in the basic diagram I offered for the torus,



getting a sort of cross between a Möbius strip and a torus. You can build it by taking two Möbius strips and (in a four-dimensional space) stitching their edges together. Alternatively, from a cylinder we can try the same idea as with a torus, but somehow reflecting or reversing one of the ends of our hose before joining it to the other. It's not good enough just to twist it longitudinally – that will merely get a different representation of a torus; we need to go into a new dimension (as we did for the Möbius strip itself). We can get a representation in three dimensions if we allow one end of our hose to magically pass through the side of the hose and approach the other end from inside rather than from outside. (See `https://plus.maths.org/content/imaging-maths-inside-klein-bottle` for some elegant images of this approach.)

Algebraically, if not geometrically, the methods developed here are fully adequate. The $\mathcal{K}$–$\mathcal{L}_1$ diagram becomes



hardly changed from the version in 5F. But there is a difference in the circuit representing $(f\bullet\partial(\hat{x}))\restriction W_1^2$, which is now

$$a \to b \to c \to d \to a \to u \to v \to w \to a \to d \to c \to b \to a \to u \to v \to w \to a.$$

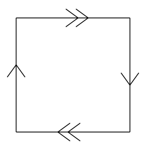
The segments $a \to b \to c \to d \to a$ and $a \to d \to c \to b \to a$ still cancel, but we are then left with $a \to u \to v \to w \to a$ twice. If, as in 5C and 5E, we give names to the obvious generators of $Z_1(\mathcal{L}_1)$, so that z_u is derived from the upper circuit $a \to b \to c \to d \to a$ and z_l from the lower circuit $a \to u \to v \to w \to a$, $(f\bullet\partial(\hat{x}))\restriction W_1^2$ becomes $2z_l$. The quotient group $Z_1(\mathcal{L})/G$ therefore becomes isomorphic to $\mathbb{Z} \times \mathbb{Z}_2$, an isomorphism being given by the formula $(k, i) \mapsto kz_u^\bullet + iz_l^\bullet$ if we think of $\mathbb{Z}_2$ as $\{0, 1\}$ with addition mod 2.

So this time we get

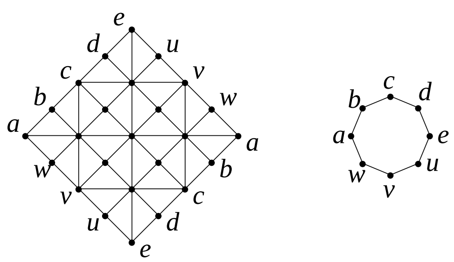
$$H_0(\mathcal{L}) = \{0\}, \quad H_1(\mathcal{L}) \cong \mathbb{Z} \times \mathbb{Z}_2, \quad H_2(\mathcal{L}) = \{0\}, \quad H_n(\mathcal{L}) = \{0\} \text{ for } n > 2.$$

In particular, we see for the first time a homology group which isn't a free group.

5H The real projective plane Yet another variation on the theme appears if we have contrary arrows in both dimensions:



I postpone discussion of the geometrical idea behind this, but algebraically there is no new problem. The $\mathcal{K}$–$\mathcal{L}_1$ diagram is now



This time, $Z_1(\mathcal{L}_1)$ is isomorphic to $\mathbb{Z}$, with a generator z corresponding to the circuit $a \to b \to c \to d \to e \to u \to v \to w \to a$. For $(f\bullet\partial(\hat{x}))\restriction W_1^2$ we have the circuit

$$a \to b \to c \to d \to e \to u \to v \to w \to a \to b \to c \to d \to e \to u \to v \to w \to a,$$

so $(f\bullet\partial(\hat{x}))\restriction W_1^2 = 2z$. Accordingly $Z_1(\mathcal{L})/G \cong \mathbb{Z}_2$ and

$$H_0(\mathcal{L}) = \{0\}, \quad H_1(\mathcal{L}) \cong \mathbb{Z}_2, \quad H_2(\mathcal{L}) = \{0\}, \quad H_n(\mathcal{L}) = \{0\} \text{ for } n > 2.$$

5I Real projective spaces Having run through enough two-dimensional examples to demonstrate some of the phenomena we can expect, I embark on something more ambitious.

(a) For any $r \in \mathbb{N}$, write $\boldsymbol{P}_r$ for the **real projective space** of dimension r, the set of lines through 0 in $\mathbb{R}^{r+1}$. The topology of $\boldsymbol{P}_r$ is in fact that induced by the Fell topology on the set of closed subsets of $\mathbb{R}^{r+1}$ (Fremlin 03, 4A2T); but I expect most readers would prefer a definition based on the following idea. We have a function $g : \boldsymbol{S}_r \to \boldsymbol{P}_r$ defined by saying that, for each $t \in \boldsymbol{S}_r$, $g(t)$ is to be the line through 0 and t. g is surjective and, for t, $t' \in \boldsymbol{S}_r$, $g(t) = g(t')$ iff $t = \pm t'$. It is easy to see that we have a metric ρ on $\boldsymbol{P}_r$ defined by saying that $\rho(g(s), g(t)) = \min(\|s - t\|, \|s + t\|)$ for all s, $t \in \boldsymbol{S}_r$; the topology of $\boldsymbol{P}_r$ is the topology defined by this metric. Observe that $g : \boldsymbol{S}_r \to \boldsymbol{P}_r$ is continuous. So $\boldsymbol{P}_r$ is a compact metrizable space.

(b) If $G \subseteq \boldsymbol{P}_r$ then G is open iff $g^{-1}[G]$ is an open subset of $\boldsymbol{S}_r$. **P** If G is open then $g^{-1}[G]$ is open because g is continuous. If $g^{-1}[G]$ is open and $s \in G$, let $t \in \boldsymbol{S}_r$ be such that $g(t) = s$. Then there is a $\delta > 0$ such that $t' \in g^{-1}[G]$ whenever $t' \in \boldsymbol{S}_r$ and $\|t - t'\| \le \delta$. Suppose that $s' \in \boldsymbol{P}_r$ is such that $\rho(s, s') \le \delta$. Then $s' = g(t')$ for some t' such that $\min(\|t - t'\|, \|t + t'\|) \le \delta$; but as s' is also equal to $g(-t')$, we can suppose that $\|t - t'\| \le \delta$, in which case $t' \in g^{-1}[G]$ so $s' \in G$. As s was arbitrary, G is open. **Q**

(c) It follows that if (Z, ρ') is any metric space and $h : \boldsymbol{S}_r \to Z$ is a continuous surjection such that, for t, $t' \in \boldsymbol{S}_r$, $h(t) = h(t')$ iff $t = \pm t'$, then Z is homeomorphic to $\boldsymbol{P}_r$. **P** Because $h(t) = h(-t)$ for every $t \in \boldsymbol{S}_r$, we have a function $f : \boldsymbol{P}_r \to Z$ defined by saying that $f(s) = h(t)$ whenever $g(t) = s$. Now if $G \subseteq Z$ is open,

$$f^{-1}[G] = \{s : h(t) \in G \text{ for some } t \text{ such that } g(t) = s\} = g[h^{-1}[G]],$$

so

$$g^{-1}[f^{-1}[G]] = g^{-1}[g[h^{-1}[G]]] = \{t : t \in h^{-1}[G]\} \cup \{-t : t \in h^{-1}[G]\}$$

is open because $h^{-1}[G]$ is open and $t \mapsto -t : \boldsymbol{S}_r \to \boldsymbol{S}_r$ is an autohomeomorphism. By (b), $f^{-1}[G] \subseteq \boldsymbol{P}_r$ is open; as G is arbitrary, f is continuous.

Because $h = fg$ is surjective, so is f. If s, $s' \in \boldsymbol{P}_r$ and $f(s) = f(s')$, there are t, $t' \in \boldsymbol{S}_r$ such that $s = g(t)$, $s' = g(t')$ and $h(t) = h(t')$. But now $t = \pm t'$ so $g(t) = g(t')$ and $s = s'$. Thus f is injective. Accordingly f is a continuous bijection from the compact metric space $(\boldsymbol{P}_r, \rho)$ to the metric space (Z, ρ'), But this forces f to be a homeomorphism (Engelking 89, 3.1.13), so Z is homeomorphic to $\boldsymbol{P}_r$. **Q**

5J Symmetric subdivisions The next step will be to find triangulations for each $\boldsymbol{P}_r$; that is to say, geometrically realizable simplicial complexes with carriers homeomorphic to $\boldsymbol{P}_r$. We do this in an organised way, so that we can relate them to triangulations of $\boldsymbol{B}_r$, $\boldsymbol{S}_r$ and $\boldsymbol{P}_{r-1}$, with enough control to give us a chance of applying Lemma 5C.

My starting point will be a multidimensional version of the construction in 5D, as follows.

(a) Fix $r \in \mathbb{N}$. Let $\mathcal{K}^{(0)}$ be the simplicial complex described in part (a-i) of the proof of 4H, so that the vertices of $\mathcal{K}^{(0)}$ are just $\pm d_i$ where $(d_0, \dots, d_r)$ is the usual orthonormal basis for $\mathbb{R}^{r+1}$, and the simplices of $\mathcal{K}^{(0)}$ are just those subsets of $\bigcup \mathcal{K}^{(0)}$ which do not contain both d_i and $-d_i$ for any $i \le r$. Then the carrier X of $\mathcal{K}^{(0)}$ is homeomorphic to $\boldsymbol{S}_r$, and $\mathcal{K}^{(0)}$ is **antipodeally symmetric** in the sense that whenever $K \in \mathcal{K}^{(0)}$ then $-K = \{-v : v \in K\}$ belongs to $\mathcal{K}^{(0)}$.

(b) Now 3G gives us a method of constructing sequences $\langle \mathcal{K}^{(m)} \rangle_{m \in \mathbb{N}}$ of progressively finer subdivisions of $\mathcal{K}^{(0)}$. The point is that we can arrange for these to be antipodeally symmetric for even m. **P** We know that $\mathcal{K}^{(0)}$ is antipodeally symmetric. Given that $\mathcal{K}^{(m)}$ is antipodeally symmetric, where $m \in \mathbb{N}$ is even, then we take an arbitrary edge $I = I_m$ of $\mathcal{K}^{(m)}$ of maximal length $\Delta(\mathcal{K}^{(m)})$ and use this to form the elementary subdivision $\mathcal{K}^{(m+1)}$ of $\mathcal{K}^{(m)}$. At this point note that $-I$ must also be an edge of $\mathcal{K}^{(m)}$, and as the carrier of $\mathcal{K}^{(m)}$ is equal to X and does not contain 0, $-I$ cannot be equal to I, and indeed they have no endpoint in common. Accordingly $-I$ is an edge of $\mathcal{K}^{(m+1)}$. Since $\Delta(\mathcal{K}^{(m+1)}) \le \Delta(\mathcal{K}^{(m)})$, and $-I$ has the same length as I, $-I$ is an edge of maximal length in $\mathcal{K}^{(m+1)}$, and we can use it in forming the next elementary subdivision $\mathcal{K}^{(m+2)}$.

If $I = \{v_0, v_1\}$ and $\hat{v} = \frac{1}{2}(v_0 + v_1)$ is its midpoint, then $-I = \{-v_0, -v_1\}$ has midpoint $-\hat{v}$. Reviewing the definition in 1R, we see that

$$\mathcal{K}^{(m+1)} = \{K : K \in \mathcal{K}^{(m)},\ I \not\subseteq K\} \cup \{(K \setminus J) \cup \{\hat{v}\} : K \in \mathcal{K}^{(m)},\ I \subseteq K, J \subseteq I,\ J \ne \emptyset\}$$

and

$$\begin{aligned}\mathcal{K}^{(m+2)} = \{K &: K \in \mathcal{K}^{(m)},\ I \not\subseteq K,\ -I \not\subseteq K\} \\ &\cup \bigcup_{\substack{K \in \mathcal{K}^{(m)} \\ I \subseteq K, -I \not\subseteq K}} \{(K \setminus J) \cup \{\hat{v}\} : \emptyset \ne J \subseteq I\} \\ &\cup \bigcup_{\substack{K \in \mathcal{K}^{(m)} \\ I \not\subseteq K, -I \subseteq K}} \{(K \setminus J) \cup \{-\hat{v}\} : \emptyset \ne J \subseteq -I\} \\ &\cup \bigcup_{\substack{K \in \mathcal{K}^{(m)} \\ I \subseteq K, -I \subseteq K}} \{(K \setminus J) \cup \{\hat{v}, -\hat{v}\} : J \in \mathcal{J}\}\end{aligned}$$

where $\mathcal{J} = \{J : J \subseteq I \cup (-I)\},\ J \cap I \ne \emptyset,\ J \cap (-I) \ne \emptyset\}$. From this description it is clear that $\mathcal{K}^{(m+2)}$ is antipodeally symmetric and the induction continues. **Q**

(c) A further special property of $\mathcal{K}^{(0)}$ is that if $K \in \mathcal{K}^{(0)}$ and $i \le r$ then either $v(i) \ge 0$ for every $v \in K$ or $v(i) \le 0$ for every $v \in K$. Evidently this is inherited by all the subdivisions $\mathcal{K}^{(m)}$, since if $L \in \mathcal{K}^{(m)}$ there is a $K \in \mathcal{K}^{(0)}$ such that $L \subseteq \Gamma(K)$. Since $0 \notin X$, $\{v, -v\} \notin \mathcal{K}^{(m)}$ for any vertex v of $\mathcal{K}^{(m)}$. Moreover, if we set

$$\mathcal{K}_k^{(m)} = \{K : K \in \mathcal{K}^{(m)},\ v(i) = 0 \text{ whenever } v \in K \text{ and } k < i \le r\}$$

for $k \le r$ and $m \in \mathbb{N}$, then for each m

— if the edge I_m used in the construction of $\mathcal{K}^{(m)}$ is included in $\mathcal{K}_k^{(m)}$, $\mathcal{K}_k^{(m+1)}$ is the elementary subdivision of $\mathcal{K}_k^{(m)}$ determined by I_m,

— otherwise, the new vertex of $\mathcal{K}^{(m+1)}$ does not belong to $\bigcup \mathcal{K}_k^{(m+1)}$ and $\mathcal{K}_k^{(m+1)} = \mathcal{K}_k^{(m)}$.

Accordingly the carrier of $\mathcal{K}_k^{(m)}$ is always equal to the carrier of $\mathcal{K}_k^{(0)}$, which is $\{t : t \in \mathbb{R}^{r+1},\ \|t\| = 1,\ t(i) = 0$ for $k < i \le r\}$ (see part (a-i) of the proof of 4H). We see also that as $\mathcal{K}_k^{(0)}$ has no $(k+1)$-simplices, nor has $\mathcal{K}_k^{(m)}$ for any m.

Of course $\mathcal{K}_k^{(m)}$ will be antipodeally symmetric for every even m.

(d) The carrier of $\mathcal{K}^{(0)}$ is $X = \{t : t \in \mathbb{R}^{r+1},\ \|t\|_1 = 1\}$, so this is also the carrier of $\mathcal{K}^{(m)}$ for every m. It is convenient to have an explicit lower bound on $\|t\|$ for $t \in X$; this is $\frac{1}{\sqrt{r+1}}$ because $\|t\|_1 = \sum_{i=0}^r |t(i)| \le \|t\|\sqrt{r+1}$ for every $t \in \mathbb{R}^{r+1}$, by Cauchy's inequality.

If $r \ge 1$ and $k \le r$, then the maximal simplices of $\mathcal{K}_k^{(0)}$ are the $(k+1)$-simplices of the form $\{\pm d_0, \dots, \pm d_k\}$, so the carrier of $\mathcal{K}_k^{(0)}$ is $X_k = \{t : t \in \mathbb{R}^{r+1},\ \|t\|_1 = 1,\ t(i) = 0$ for $k < i \le r\}$, and this is also the carrier of every subsequent $\mathcal{K}_k^{(m)}$. Observe that just as $t \mapsto \frac{1}{\|t\|_1}t$ is a homeomorphism from $\boldsymbol{S}_r$ to X, $t \mapsto \frac{1}{\|t\|_1}t^\frown{<}0{>}^\frown \dots ^\frown{<}0{>}$ is a homeomorphism from $\boldsymbol{S}_k$ to X_k.

5K A triangulation of $\boldsymbol{P}_r$: Proposition Let $r \in \mathbb{N}$. Take the set $X \subseteq \mathbb{R}^{r+1}$ and the sequence $\langle \mathcal{K}^{(m)} \rangle_{m \in \mathbb{N}}$ from 5J. Fix on an even m such that $\Delta(\mathcal{K}^{(m)}) < \frac{1}{\sqrt{r+1}}$. Write V for the vertex set of $\mathcal{K}^{(m)}$. Set

$W = \{\{t, -t\} : t \in V\}$, $f(t) = \{t, -t\}$ for $t \in V$, $\mathcal{L} = \{f[K] : K \in \mathcal{K}^{(m)}\}$. Then $M(\mathcal{L})$, as defined in 3A, is homeomorphic to $\boldsymbol{P}_r$.

proof (a) Write $\mathcal{K}$ for $\mathcal{K}^{(m)}$. Note first that $V = -V$, because m is even so $-K \in \mathcal{K}$ for every $K \in \mathcal{K}$. Next, if K, $K' \in \mathcal{K}$ and $K \cap K' \neq \emptyset$, $(-K) \cap K' = \emptyset$. **P** If $t \in K$ and $t' \in -K$ then $-t' \in K$ so

$$\|t + t'\| = \|t - (-t')\| \leq \Delta(\mathcal{K}), \quad \frac{2}{\sqrt{r+1}} \leq 2\|t\| \leq \|t - t'\| + \|t + t'\|,$$

$$\|t - t'\| \geq \frac{2}{\sqrt{r+1}} - \Delta(\mathcal{K}) > \Delta(\mathcal{K})$$

and no member of $\mathcal{K}$ can contain both t and t'. **Q**

In particular, there can be no $t \in K$ such that $-t \in K$, and $f[K] \cap f[-K] = \emptyset$.

(b) Define $h : M(\mathcal{K}) \to \mathbb{R}^W$ by saying that

$$h(\alpha)(w) = \sum_{v \in V, f(v)=w} \alpha(v)$$

for every $w \in W$. Then $h(\alpha)(w) \geq 0$ for every w and $\sum_{w \in W} h(\alpha)(w) = \sum_{v \in V} \alpha(v) = 1$. Also

$$\{w : w \in W,\, h(\alpha)(w) > 0\} = \{f(v) : v \in V,\, \alpha(v) > 0\} \in \mathcal{L}$$

because $f[K] \in \mathcal{L}$ for every $K \in \mathcal{K}$. So $h(\alpha) \in M(\mathcal{L})$ for every $\alpha \in M(\mathcal{K})$.

(c) Let $g : X \to M(\mathcal{K})$ be the inverse of the barycenter map $\operatorname{brc} : M(\mathcal{K}) \to X$; then g is a homeomorphism. If $t \in X$, then $hg(t) = hg(-t)$. **P** Set $\alpha = g(t)$, and set $\beta(v) = \alpha(-v)$ for every $v \in V$; then

$$\{v : \beta(v) > 0\} = -\{v : \alpha(v) > 0\} \in \mathcal{K}$$

(and of course $\beta(v) \geq 0$ for every v, while $\sum_{v \in V} \beta(v) = 1$), so $\beta \in M(\mathcal{K})$. Now

$$\operatorname{brc}(\beta) = \sum_{v \in V} \beta(v)v = \sum_{v \in V} \alpha(-v)v = \sum_{v \in V} \alpha(v)(-v) = -\operatorname{brc}(\alpha) = -t,$$

so $\beta = g(-t)$. Next,

$$h(\alpha)(w) = \sum_{\substack{v \in V \\ f(v)=w}} \alpha(v) = \sum_{\substack{v \in V \\ f(-v)=w}} \alpha(v)$$

(because $f(-v) = f(v)$ for every v)

$$= \sum_{\substack{v \in V \\ f(v)=w}} \alpha(-v) = \sum_{\substack{v \in V \\ f(v)=w}} \beta(v) = h(\beta)(w)$$

for every $w \in W$, and $hg(t) = hg(-t)$. **Q**

(d) If t, $t' \in X$ and $hg(t) = hg(t')$, then $t = \pm t'$. **P** Write α for $g(t)$ and α' for $g(t')$; set $K = \{v : \alpha(v) > 0\}$, $K' = \{v : \alpha'(v) > 0\}$. If $v \in K \cup K'$ then

$$0 < \alpha(v) + \alpha(-v) = h(\alpha)(v) = h(\alpha')(v) = \alpha'(v) + \alpha'(-v)$$

so $v \in K' \cup (-K')$; thus $K \cup (-K) \subseteq K' \cup (-K')$. Similarly, of course, $K' \cup (-K') \subseteq K \cup (-K)$ and we have equality. But as K' cannot meet both K and $-K$, it must be included in one of them, while $-K'$ is included in the other; and K' is either K or $-K$.

If $K' = K$, then, for every $v \in K$, $\alpha(-v) = \alpha'(-v) = 0$ so

$$\alpha(v) = \alpha(v) + \alpha(-v) = h(\alpha)(v) = h(\alpha')(v) = \alpha'(v).$$

So in this case $\alpha = \alpha'$ and $t = t'$. Otherwise, $K' = -K$, so, for $v \in K$, $\alpha(-v) = \alpha'(v) = 0$ and

$$\alpha(v) = \alpha(v) + \alpha(-v) = h(\alpha)(v) = h(\alpha')(v) = \alpha'(-v),$$

while $\alpha(v) = 0 = \alpha'(-v)$ for $v \in V \setminus K$. We now have

$$t = \sum_{v \in V} \alpha(v)v = \sum_{v \in V} \alpha'(-v)v = -\sum_{v \in V} \alpha'(v)v = -t',$$

so the result is true in this case also. **Q**

(e) If $s \in M(\mathcal{L})$ there is a $t \in X$ such that $hg(t) = s$. **P** $L = \{w : s(w) > 0\}$ belongs to $\mathcal{L}$ so there is a $K \in \mathcal{K}$ such that $f[K] = L$. For each $w \in L$ choose $v_w \in K$ such that $f(v_w) = w$, and define $\alpha \in M(\mathcal{K})$ by setting

$$\alpha(v) = s(w) \text{ if } w \in L \text{ and } v = v_w, \quad \alpha(v) = 0 \text{ if there is no such } w.$$

If $v \in K$ then $-v \notin K$ so $\alpha(-v) = 0$; accordingly

$$h(\alpha)(w) = \sum_{\substack{v\in V\\ f(v)=w}} \alpha(v) = \alpha(v_w) = s(w)$$

for every $w \in L$, and $h(\alpha) = s$. (Of course $h(\alpha)(w) = 0$ for $w \in W \setminus L$ because $\sum_{w\in W} h(\alpha)(w) = 1 = \sum_{w\in L} s(w)$.) Setting $t = \mathrm{brc}(\alpha) \in X$, $s = hg(t)$. **Q**

(f) Thus $hg : X \to M(\mathcal{L})$ is a continuous surjection such that $hg(t) = hg(t')$ iff $t = \pm t'$. Finally, recall that we have a homeomorphism $t \mapsto \frac{1}{\|t\|_1} t : \boldsymbol{S}_r \to X$ which takes antipodeal points to antipodeal points. So setting

$$h_1(t) = hg(\frac{1}{\|t\|_1} t)$$

for $t \in \boldsymbol{S}_r$, $h_1 : \boldsymbol{S}_r \to M(\mathcal{K})$ is a continuous surjection such that $h_1(t) = h_1(t')$ iff $t = \pm t'$. By 5Ic, $M(\mathcal{L})$ is homeomorphic to $\boldsymbol{P}_r$.

5L Theorem For r, $n \in \mathbb{N}$

$$\begin{aligned} H_n(\boldsymbol{P}_r) &= \{0\} \text{ if } n \le r \text{ is even,}\\ &\cong \mathbb{Z}_2 \text{ if } n < r \text{ is odd,}\\ &\cong \mathbb{Z} \text{ if } n = r \text{ is odd,}\\ &= \{0\} \text{ if } n > r. \end{aligned}$$

proof Take $r \in \mathbb{N}$ and the family $\langle \mathcal{K}_k^{(m)} \rangle_{k \le r, m\in\mathbb{N}}$ from 5Jc, and let m be even and so large that $\Delta(\mathcal{K}_r^{(m)}) < \frac{1}{\sqrt{r+1}}$.

(a) Suppose that $0 \le k \le r$.

(i) Write $\mathcal{K}_k^\dagger$ for $\mathcal{K}_k^{(m)}$ and $X_k^\dagger$ for $\{t : t \in \mathbb{R}^{r+1}, \|t\|_1 = 1, t(i) = 0 \text{ for } k < i \le r\}$, so that $X_k^\dagger$ is the carrier of $\mathcal{K}_k^\dagger$ (5Jd). Let $V_k^\dagger$ be the vertex set of $\mathcal{K}_k^\dagger$ and W_k the set of doubletons $\{\{v, -v\} : v \in V_k^\dagger\}$. Note that because m is even, $\mathcal{K}_k^\dagger$ is antipodeally symmetric, so that $V_k^\dagger = -V_k^\dagger$. Define $f_k^\dagger : V_k^\dagger \to W_k$ by setting $f_k^\dagger(v) = \{v, -v\}$ for $v \in V_k^\dagger$, and set $\mathcal{L}_k = \{f_k^\dagger[K] : K \in \mathcal{K}_k^\dagger\}$; then the vertex set of $\mathcal{L}_k$ is W_k.

(ii) Recall from 5Jc that if $K \in \mathcal{K}_k^\dagger$ then either $v(k) \ge 0$ for every $v \in K$ or $v(k) \le 0$ for every $v \in K$. Let $\mathcal{K}_k$ be $\{K : K \in \mathcal{K}_k^\dagger, v(k) \ge 0 \text{ for every } v \in K\}$, V_k the vertex set of $\mathcal{K}_k$ and X_k the carrier of $\mathcal{K}_k$, that is, $V_k = \{v : v \in V_k^\dagger, v(k) \ge 0\}$ and $X_k = \{t : t \in X_k^\dagger, t(k) \ge 0\}$. Then $\mathcal{K}_k = \mathcal{K}_k^\dagger \cap \mathcal{P}V_k$. Note that $f_k^\dagger[V_k] = W_k$, because if $v \in V_k^\dagger$ at least one of v, $-v$ belongs to V_k. Set $f_k = f_k^\dagger \restriction V_k$. Then

$$\mathcal{L}_k = \{f_k^\dagger[K] : K \in \mathcal{K}_k^\dagger\} = \{f_k^\dagger[K] : K \in \mathcal{K}_k\} = \{f_k[K] : K \in \mathcal{K}_k\}$$

because if $K \in \mathcal{K}_k^\dagger$ then $f_k^\dagger[K] = f_k^\dagger[-K]$ and at least one of K, $-K$ is included in V_k and belongs to $\mathcal{K}_k$. So f_k is a simplicial map from $\mathcal{K}_k$ to $\mathcal{L}_k$.

Set

$$V_k^+ = \{v : v \in V_k, v(k) > 0\} = V_k \setminus (-V_k) = V_k^\dagger \setminus (-V_k);$$

then $f_k \restriction V_k^+$ is injective.

(iii) $X_k^\dagger$ is homeomorphic to $\boldsymbol{S}_k$ and X_k is homeomorphic to $\boldsymbol{B}_k$. **P** For $t \in X_k^\dagger$, set $\theta_k(t) = \frac{1}{\|t\|}(t\restriction(k+1)) \in \boldsymbol{S}_k$. Then θ_k is a homeomorphism between $X_k^\dagger$ and $\boldsymbol{S}_k$ (see 5Jd), and $\theta_k(-t) = -\theta_k(t)$ for every $t \in X_k^\dagger$. Now $\theta_k[X_k] = \{t : t \in \boldsymbol{S}_k,\, t(k) \ge 0\}$ and $t \mapsto t\restriction k$ is a homeomorphism between $\theta_k[X_k]$ and $\boldsymbol{B}_k$. So X_k, $\theta_k[X_k]$ and $\boldsymbol{B}_k$ are all homeomorphic. **Q**

Accordingly $H_n(\mathcal{K}_k) = \{0\}$ for every $n \in \mathbb{N}$, while $H_k(\mathcal{K}_k^\dagger) \cong \mathbb{Z}$ and $H_n(\mathcal{K}_k^\dagger) = \{0\}$ for every $n \ne k$. As noted in 5Jc, $\mathcal{K}_k^\dagger$ and $\mathcal{K}_k$ have no $(k+1)$-simplices.

(iv) If $v_0 \in V_k^+$ then f_k is injective on $\bigcup\{K : v_0 \in K \in \mathcal{K}_k\}$. **P** If K, $K' \in \mathcal{K}_k$ both contain v_0, $v \in K$ and $v' \in K'$, then

$$\|v - v'\| \le \|v - v_0\| + \|v' - v_0\| \le 2\Delta(\mathcal{K}_k) < \frac{2}{\sqrt{r+1}} \le 2\|v\|,$$

so $v' \ne -v$ and $f_k(v') \ne f_k(v)$. **Q**

(v) Define $h_k : V_k^\dagger \to V_k^\dagger$ by setting $h_k(v) = -v$ for $v \in V_k^\dagger$. Then $h_k\bullet z = (-1)^{k+1}z$ for every $z \in Z_k(\mathcal{K}_k^\dagger)$. **P** Writing $\bar h_k$ for the associated function from $X_k^\dagger$ to itself (3Cb), we have $\bar h_k(t) = -t$ for every $t \in X_k^\dagger$. So $\theta_k\bar h_k\theta_k^{-1}(t) = -t$ for every $t \in \boldsymbol{S}_k$. By 4Ha, $(\theta_k\bar h_k\theta_k^{-1})_\star(a) = (-1)^{k+1}a$ for every $a \in H_k(\boldsymbol{S}_k)$. But $\theta_{k\star} : H_k(X_k^\dagger) \to H_k(\boldsymbol{S}_k)$ is an isomorphism with inverse $(\theta_k^{-1})_\star$, so $\bar h_{k\star}(a) = (-1)^{k+1}a$ for every $a \in H_k(X_k^\dagger)$ and $h_{k\star}(a) = (-1)^{k+1}a$ for every $a \in H_k(\mathcal{K}_k^\dagger)$ (3Hb), that is,

$$(h_k\bullet z)^\bullet = h_{k\star}(z^\bullet) = (-1)^{k+1}z^\bullet$$

for every $z \in Z_k(\mathcal{K}_k^\dagger)$. But as $\mathcal{K}_k^\dagger$ has no $(k+1)$-simplices, $B_k(\mathcal{K}_k^\dagger) = \{0\}$ and $h_k\bullet z = (-1)^{k+1}z$ for every $z \in Z_k(\mathcal{K}_k^\dagger)$. **Q**

(vi) Define $g_k : V_k \to V_k^\dagger$ by setting $g_k(v) = v$ for $v \in V_k$. Then g_k is a simplicial map from $\mathcal{K}_k$ to $\mathcal{K}_k^\dagger$, by the definition of $\mathcal{K}_k$. But also $f_k^\dagger$ is a simplicial map from $\mathcal{K}_k^\dagger$ to $\mathcal{L}_k$. **P** If $K \in \mathcal{K}_k^\dagger$ then either $v(k) \ge 0$ for every $v \in K$ or $v(k) \le 0$ for every $v \in K$. In the former case $K \in \mathcal{K}_k$ and $f_k^\dagger[K] = f_k[K] \in \mathcal{L}_k$. In the latter case, $-K \subseteq V_k$ and $-K \in \mathcal{K}_k^\dagger$, so $-K \in \mathcal{K}_k$ and $f_k^\dagger[K] = f_k^\dagger[-K] = f[-K] \in \mathcal{L}_k$. So we always have $f_k^\dagger[K] \in \mathcal{L}_k$ and $f_k^\dagger$ is simplicial. **Q**

Observe that $f_k^\dagger h_k = f_k^\dagger$ and $f_k = f_k^\dagger g_k$. So if k is even, $f_k^\dagger\bullet z = 0$ for every $z \in Z_k(\mathcal{K}_k^\dagger)$ and $f_k\bullet z = 0$ for every $z \in Z_k(\mathcal{K}_k)$. **P** If $z \in Z_k(\mathcal{K}_k^\dagger)$ then $h_k\bullet z = -z$ and

$$f_k^\dagger\bullet z = (f_k^\dagger h_k)\bullet z = f_k^\dagger\bullet(h_k\bullet z) = -f_k^\dagger\bullet z, \quad f_k^\dagger\bullet z = 0,$$

and if $z \in Z_k(\mathcal{K}_k)$ then

$$f_k\bullet z = f_k^\dagger\bullet(g_k\bullet z) = 0. \text{ **Q**}$$

(b) Suppose that $k < r$.

(i)

$$V_k \subseteq V_k^\dagger \subseteq V_{k+1} \subseteq V_{k+1}^\dagger, \quad V_{k+1}^+ = V_{k+1} \setminus V_k^\dagger, \quad W_k \subseteq W_{k+1},$$

$$\mathcal{K}_k^\dagger = \mathcal{K}_{k+1} \cap \mathcal{P}V_k^\dagger = \mathcal{K}_{k+1}^\dagger \cap \mathcal{P}V_k^\dagger, \quad \mathcal{K}_k = \mathcal{K}_{k+1} \cap \mathcal{P}V_k,$$

$$f_k^\dagger = f_{k+1}\restriction V_k^\dagger, \quad f_k = f_k^\dagger\restriction V_k = f_{k+1}\restriction V_k, \quad f_{k+1}^{-1}[W_k] = V_k^\dagger.$$

We find also that $\mathcal{L}_k = \mathcal{L}_{k+1} \cap \mathcal{P}W_k$. **P** Of course $\mathcal{L}_k \subseteq \mathcal{P}W_k$ and also

$$\mathcal{L}_k = \{f_k[K] : K \in \mathcal{K}_k\} = \{f_{k+1}[K] : K \in \mathcal{K}_k\} \subseteq \{f_{k+1}[K] : K \in \mathcal{K}_{k+1}\} = \mathcal{L}_{k+1}.$$

In the other direction, take $L \in \mathcal{L}_{k+1} \cap \mathcal{P}W_k$. Then there is a $K \in \mathcal{K}_{k+1}$ such that $L = f_{k+1}[K]$. Set $K' = K \cap V_k^\dagger$; then $K' \in \mathcal{K}_k^\dagger$ and $L = f_{k+1}[K] \cap W_k = f_{k+1}[K']$ because $f_{k+1}^{-1}[W_k] = V_k^\dagger$. Now either $v(k) \ge 0$ for every $v \in K'$ or $v(k) \le 0$ for every $v \in K'$, that is, either $K' \in \mathcal{K}_k$ or $-K' \in \mathcal{K}_k$. Since $f_{k+1}[-K'] = f_{k+1}[K'] = L$, in either case we have a $K'' \in \mathcal{K}_k$ such that $L = f_{k+1}[K''] = f_k[K'']$ and $L \in \mathcal{L}_k$. **Q**

(ii) I noted in (a-iii) that $H_k(\mathcal{K}_k^\dagger)\cong\mathbb{Z}$, while $\mathcal{K}_k^\dagger$ has no $(k+1)$-simplices, so $B_k(\mathcal{K}_k^\dagger)=\{0\}$ and $H_k(\mathcal{K}_k^\dagger)\cong Z_k(\mathcal{K}_k^\dagger)$. There is therefore a non-zero generator z_k of $Z_k(\mathcal{K}_k^\dagger)$. Now $\mathcal{K}_k^\dagger=\mathcal{K}_{k+1}\cap\mathcal{P}V_k^\dagger$, so we have a chain $z'_k\in C_k(\mathcal{K}_{k+1})$ defined by setting

$$z'_k(p)=z_k(p)\text{ for }p\in(V_k^\dagger)^{k+1},\ 0\text{ for other }p\in V_{k+1}^{k+1}$$

(5Aa), and now $\partial(z'_k)=0$, as noted in 5Ac. Thus $z'_k\in Z_k(\mathcal{K}_{k+1})$, while $\operatorname{supp}(z'_k)\subseteq V_k^\dagger$. But $H_k(\mathcal{K}_k)=\{0\}$, so z'_k must be a boundary. Let $\hat{x}_k\in C_{k+1}(\mathcal{K}_{k+1})$ be such that $z'_k=\partial(\hat{x}_k)$; then $z_k=(\partial(\hat{x}_k))\restriction(V_k^\dagger)^{k+1}$. Note that z_k, z'_k and $\hat{x}_k$ are all non-zero.

(iii) We know that $\operatorname{supp}(z'_k)=\operatorname{supp}\partial(\hat{x}_k)\subseteq V_k^\dagger$, that $V_k^\dagger=f_{k+1}^{-1}[W_k]$ and that $f_k^\dagger=f_{k+1}\restriction V_k^\dagger$, so

$$\begin{aligned}(f_{k+1}\bullet\partial(\hat{x}_k))\restriction W_k^{k+1}&=(f_{k+1}\bullet z'_k)\restriction W_k^{k+1}=f_k^\dagger\bullet(z'_k\restriction(V_k^\dagger)^{k+1})\\&=f_k^\dagger\bullet z_k.\end{aligned}\tag{5Ae}$$

(c) Again suppose that $k<r$. Consider the hypotheses of Lemma 5C interpreted with

$$\mathcal{K}:=\mathcal{K}_{k+1},\quad V:=V_{k+1},\quad W:=W_{k+1},\quad f:=f_{k+1},\quad V_1:=V_k^\dagger,$$

$$m:=k,\quad V_0:=V_{k+1}^+,\quad \mathcal{L}:=\mathcal{L}_{k+1},\quad W_1:=W_k,\quad \mathcal{K}_1:=\mathcal{K}_k^\dagger,\quad \mathcal{L}_1:=\mathcal{L}_k$$

writing := to indicate assignments. Apart from the facts in (b-i), we need to know that

$H_n(\mathcal{K}_{k+1})=\{0\}$ for every n, $\quad H_k(\mathcal{K}_k^\dagger)\cong\mathbb{Z}$, $\quad H_n(\mathcal{K}_k^\dagger)=\{0\}$ for every $n\ne k$,

$\mathcal{K}_{k+1}$ has no $(k+2)$-simplices, $\quad\mathcal{K}_k^\dagger$ has no $(k+1)$-simplices,

$f_{k+1}[V_{k+1}^+]\cap W_k=\emptyset$, $\quad f_{k+1}\restriction V_{k+1}^+$ is injective, $\quad f_{k+1}\restriction\bigcup\{K:v\in K\in\mathcal{K}_{k+1}\}$ is injective for every $v\in V_{k+1}^+$,

all of which have been dealt with above ((a-ii), (a-iii), (a-iv)).

Now consider $\hat{x}_k$ as defined in (b-ii). We have

$$\hat{x}_k\in C_{k+1}(\mathcal{K}_{k+1}),$$

$$\operatorname{supp}\partial(\hat{x}_k)=\operatorname{supp}(z'_k)\subseteq V_k^\dagger,$$

$$(\partial(\hat{x}_k))\restriction(V_k^\dagger)^{k+1}=z_k\text{ generates }Z_k(\mathcal{K}_k^\dagger),$$

so $((\partial(\hat{x}_k))\restriction(V_k^\dagger)^{k+1})^\bullet$ generates $H_k(\mathcal{K}_k^\dagger)$. We can therefore use $\hat{x}_k$ for $\hat{x}$ in parts (d) and (e) of 5C, and get

$H_n(\mathcal{L}_{k+1})=H_n(\mathcal{L}_k)$ for $0\le n<k$,

$H_k(\mathcal{L}_{k+1})\cong Z_k(\mathcal{L}_k)/G$ where G is the set of multiples of $(f_{k+1}\bullet\partial(\hat{x}_k))\restriction W_k^{k+1}$,

if $f_{k+1}\bullet\partial(\hat{x}_k)=0$ then $H_k(\mathcal{L}_{k+1})\cong H_k(\mathcal{L}_k)$ and $H_{k+1}(\mathcal{L}_{k+1})\cong\mathbb{Z}$, while $Z_{k+1}(\mathcal{L}_{k+1})$ is generated by $f_{k+1}\bullet\hat{x}_k$,

if $f_{k+1}\bullet\partial(\hat{x}_k)\ne 0$ then $H_{k+1}(\mathcal{L}_{k+1})=\{0\}$,

$H_n(\mathcal{L}_{k+1})=\{0\}$ for $n>k+1$.

(d) Suppose that $k<r$ is even. Then $f_{k+1}\bullet\partial(\hat{x}_k)=0$. **P**

$$(f_{k+1}\bullet\partial(\hat{x}_k))\restriction W_k^{k+1}=f_k^\dagger\bullet z_k=0$$

by (b-iii) and (a-vi), since $z_k\in Z_k(\mathcal{K}_k^\dagger)$. As

$$\operatorname{supp}(f_{k+1}\bullet\partial(\hat{x}_k))\subseteq f_{k+1}[\operatorname{supp}\partial(\hat{x}_k)]\subseteq W_k,$$

$f_{k+1}\bullet\partial(\hat{x}_k)=0$. **Q**

So in this case $H_k(\mathcal{L}_{k+1})\cong H_k(\mathcal{L}_k)$, $H_{k+1}(\mathcal{L}_{k+1})\cong\mathbb{Z}$ and $Z_{k+1}(\mathcal{L}_{k+1})$ is generated by $f_{k+1}\bullet\hat{x}_k$.

(e) Now suppose that $k<r$ is odd.

(i) Consider

$$z = g_k \bullet \hat{x}_{k-1} + h_k \bullet (g_k \bullet \hat{x}_{k-1}) \in C_k(\mathcal{K}_k^\dagger).$$

Then $\partial(z) = 0$. **P**

$$\partial(g_k \bullet \hat{x}_{k-1}) = g_k \bullet \partial \hat{x}_{k-1} = g_k \bullet z'_{k-1}$$

and $\operatorname{supp}(z'_{k-1}) \subseteq V_{k-1}^\dagger = g_k^{-1}[V_{k-1}^\dagger]$, while $z'_{k-1} \restriction (V_{k-1}^\dagger)^k = z_{k-1}$. But this means that $(g_k \bullet z'_{k-1}) \restriction (V_{k-1}^\dagger)^k = z_{k-1}$ and moreover that $(h_k \bullet (g_k \bullet z'_{k-1})) \restriction (V_{k-1}^\dagger)^k = h_{k-1} z_{k-1}$ (5A(e-ii), because $h_{k-1} = h_k \restriction V_{k-1}^\dagger$).

So we see that $\partial(z)$ is supported by $V_{k-1}^\dagger$ and

$$\partial(z) \restriction (V_{k-1}^\dagger)^k = z_{k-1} + h_{k-1} z_{k-1} = z_{k-1} + (-1)^k z_{k-1} = 0$$

by (b-v), because $z_{k-1} \in Z_{k-1}(\mathcal{K}_{k-1}^\dagger)$. So $\partial(z) = 0$. **Q**

Consequently $z \in Z_k(\mathcal{K}_k^\dagger)$ is a multiple of z_k, by the choice of z_k.

(ii) Let $l \in \mathbb{Z}$ be such that $z = l z_k$.

(α) Note first that if $p \in V_k^{k+1}$ and $\hat{x}_{k-1}(p) \neq 0$ then $z(p) = \hat{x}_{k-1}(p)$. **P** $p[k+1] \in \mathcal{K}_k$ is a k-simplex, because p must be injective. So $p[k+1] \notin \mathcal{K}_{k-1}^\dagger$, $p[k+1] \not\subseteq V_{k-1}^\dagger$ and there is a $v \in p[k+1] \cap V_k^+$. But in this case $h_k(v) \notin V_k$ and

$$((h_k g_k) \bullet \hat{x}_{k-1})(p) = (g_k \bullet \hat{x}_{k-1})(h_k^{-1} p)$$

(because h_k is bijective)

$$= (g_k \bullet \hat{x}_{k-1})(h_k p) = 0$$

because $h_k p \neq g_k q$ for any $q \in V_k^{k+1}$. So $z(p) = (g_k \bullet \hat{x}_{k-1})(p) = \hat{x}_{k-1}(p)$, since $g_k q = p$ iff $q = p$. **Q**

(β) I noted in (b-ii) that $\hat{x}_{k-1} \neq 0$, so $z \neq 0$ and $l \neq 0$. Accordingly $z(p) = l z_k^\dagger(p)$ is an integer multiple of l for every $p \in (V_k^\dagger)^{k+1}$, and now ($\alpha$) tells us that $\hat{x}_{k-1}(p)$ is a multiple of l for every $p \in V_k^{k+1}$. Working backwards through (b-ii), $z'_{k-1}(q) = \sum_{v \in V_k} \hat{x}_{k-1}(q^\frown\langle v\rangle)$ is a multiple of l for every $q \in V_k^k$ and $z_{k-1}(q)$ is a multiple of l for every $q \in (V_{k-1}^\dagger)^k$. Accordingly we can define $\tilde{z} : (V_{k-1}^\dagger)^k \to \mathbb{Z}$ by setting $\tilde{z}(q) = \frac{1}{l} z_{k-1}(q)$ for $q \in (V_{k-1}^\dagger)^k$. It is elementary to check that $\tilde{z} \in C_{k-1}(\mathcal{K}_{k-1}^\dagger)$, and now $0 = \partial(z_{k-1}) = l \partial \tilde{z}$, so $\partial \tilde{z} = 0$, $\tilde{z} \in Z_{k-1}(\mathcal{K}_{k-1}^\dagger)$ and $\tilde{z}$ is a multiple of $z_{k-1} = l \tilde{z}$. This is possible only if $l = \pm 1$.

(γ) We conclude that actually $z = \pm z_k$.

(iii)

$$(f_{k+1} \bullet \partial(\hat{x}_k)) \restriction W_k^{k+1} = f_k^\dagger \bullet z_k$$

((b-iii) again)

$$= \pm f_k^\dagger \bullet z$$

((ii) just above),

$$\begin{aligned} f_k^\dagger \bullet z &= f_k^\dagger (g_k \bullet \hat{x}_{k-1} + h_k \bullet (g_k \bullet \hat{x}_{k-1}) \\ &= (f_k^\dagger g_k) \bullet \hat{x}_{k-1} + (f_k^\dagger h_k g_k) \bullet \hat{x}_{k-1} = 2 f_k \bullet \hat{x}_{k-1} \end{aligned}$$

because $f_k^\dagger h_k = f_k^\dagger$ and $f_k^\dagger g_k = f_k$. But $k-1$ is even, so (d) tells us that $f_k \bullet \hat{x}_{k-1}$ is a generator of $Z_k(\mathcal{L}_k)$. Accordingly the subgroup G of (c) above is the set of multiples of twice a generator of the infinite cyclic group $Z_k(\mathcal{L}_k)$, and $H_k(\mathcal{L}_{k+1}) \cong Z_k(\mathcal{L}_k)/G \cong \mathbb{Z}_2$.

(iv) At the same time, we see that $f_{k+1} \bullet \partial(\hat{x}_k) \neq \{0\}$, so that $H_{k+1}(\mathcal{L}_{k+1}) = \{0\}$.

(f) We now have all the components for a proof by induction that if $k \leq r$ and $n \in \mathbb{N}$

$$
\begin{aligned}
H_n(\mathcal{L}_k) &= \{0\} \text{ if } n \le k \text{ is even,}\\
&\cong \mathbb{Z}_2 \text{ if } n < k \text{ is odd,}\\
&\cong \mathbb{Z} \text{ if } n = k \text{ is odd,}\\
&= \{0\} \text{ if } n > k.
\end{aligned}
$$

P If $k = 0$, then (in the language of 5J) $V_0 = \{d_0, -d_0\}$, $\mathcal{K}_0 = \{\emptyset, \{d_0\}, \{-d_0\}\}$, $W_0 = \{\{d_0, -d_0\}\}$, $\mathcal{L}_0 = \mathcal{P}W_0$ and $H_n(\mathcal{L}_0) = \{0\}$ for every n. For the inductive step to $k+1 \le r$,

$$
\begin{aligned}
H_n(\mathcal{L}_{k+1}) &\cong H_n(\mathcal{L}_k) = \{0\} \text{ if } n < k \text{ is even} && \text{(c) above}\\
&= H_k(\mathcal{L}_{k+1}) \cong H_k(\mathcal{L}_k) = \{0\} \text{ if } n = k \text{ is even} && \text{(d)}\\
&= \{0\} \text{ if } n = k+1 \text{ is even} && \text{(e-iv), as } k \text{ is odd}\\
&\cong H_n(\mathcal{L}_k) \cong \mathbb{Z}_2 \text{ if } n < k \text{ is odd} && \text{(c) again}\\
&\cong \mathbb{Z}_2 \text{ if } n = k \text{ is odd} && \text{(e-iii)}\\
&\cong \mathbb{Z} \text{ if } n = k+1 \text{ is odd} && \text{(d) again}\\
&= \{0\} \text{ if } n > k+1 && \text{(c) once more. } \mathbf{Q}
\end{aligned}
$$

(g) Finally, to convert this into the statement of the theorem, use 5K. Since $\mathcal{K}^{(m)} = \mathcal{K}_r^{(m)}$ in 5K has become $\mathcal{K}_r^\dagger$ here, $\mathcal{L}_r$ here corresponds to $\mathcal{L}$ in 5K, and $\boldsymbol{P}_r$ is homeomorphic to $M(\mathcal{L}_r)$. Consequently

$$H_n(\mathcal{P}_r) \cong H_n(M(\mathcal{L}_r)) \cong H_n(\mathcal{L}_r)$$

for every n, by 3I. So the homology groups of $\boldsymbol{P}_r$ are given by the case $k = r$ of (f) above.

Notes and comments Lemma 5C is a kind of covering theorem; it can also be thought of as a very primitive kind of 'block welding theorem'. The idea is that $(\mathcal{K}, \mathcal{K}_1)$ looks like $(\boldsymbol{B}_{m+1}, \boldsymbol{S}_m)$ and that we wish to compute the homology groups of $\mathcal{L}$ from those of $\mathcal{L}_1$. In parts (d) and (e) of 5C we see that while the formula is not as simple as we might like, it can be reduced to a question about $f{\bullet}\hat{x}$ where we can hope to identify a suitable $\hat{x}$.

There is an alarming complexity in the proof of Theorem 5L. One of the problems is that a full set of names

$$\mathcal{K}_k^\dagger,\ X_k^\dagger,\ V_k^\dagger,\ W_k,\ f_k^\dagger,\ \mathcal{K}_k,\ \mathcal{L}_k,\ V_k,\ X_k,\ f_k,\ V_k^+,\ \theta_k$$

is assembled for every stage in the induction, rather than simply working in adjacent levels $r-1$, r. The trouble here is that since the key formula

$$H_k(\mathcal{L}_{k+1}) \cong Z_k(\mathcal{L}_k)/G$$

depends on understanding a generator of the cycle group $Z_k(\mathcal{L}_k)$, we cannot simply use the inductive hypothesis that $H_k(\mathcal{L}_k) \cong \mathbb{Z}$. And there is a step earlier on, when confirming that f_k is adequately injective ((a-iv) of the proof of 5L) which demands that the subdivision $\mathcal{K}_k$ should be sufficiently fine; the preliminary run of the ideas in 5H showed that how far we have to go depends on the dimension, and that if we don't set up a method of doing the whole induction (up to r) on a single structure, we are going to need an inductive hypothesis which allows us to subdivide further.

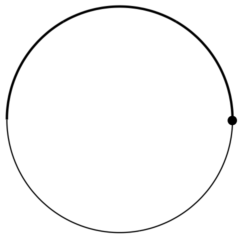
You will I hope be impatient with a formalism which leads to such complexity. The key pattern

$$V_k \subseteq V_k^\dagger \subseteq V_{k+1}$$

in the proof of 5H corresponds to a step

$$\boldsymbol{B}_k \longrightarrow \boldsymbol{S}_k \subseteq \boldsymbol{B}_{k+1}$$

in which the ball $\boldsymbol{B}_k$ is mapped onto half the sphere $\boldsymbol{S}_k$. The diagram in 5L attempts to show the step $\boldsymbol{S}_1 \subseteq \boldsymbol{B}_2 \to \boldsymbol{S}_2$. A more elegant form would be



Here I have tried to show the fragments of $\boldsymbol{P}_2$ associated with $f_1\bullet\hat{x}_0$ (a semicircle) and $f_2\bullet\hat{x}_1$ (a hemisphere). We observe that $\boldsymbol{P}_0$ is a singleton, $\boldsymbol{P}_1$ is homeomorphic to $\boldsymbol{S}_1$ and $\boldsymbol{P}_2$ is something new. $\boldsymbol{P}_1$ has a 1-cycle, as the bullet-point lies at both ends of the curve, and $H_1(\boldsymbol{P}_1) \cong \mathbb{Z}$. $\boldsymbol{P}_2$ does not have a non-zero 2-cycle, just as $\boldsymbol{B}_2$ doesn't, and the boundary of the hemisphere is the whole circle, which corresponds to two copies of $\boldsymbol{P}_1$. What is interesting is that the induction takes a different shape at odd and even stages, associated with the different natures of the antipodeal map on $\boldsymbol{S}_r$ for odd and even r (4H).

The method I am using to calculate homology groups necessarily demands a geometrically realizable complex with a suitable carrier. In the cases of $\boldsymbol{B}_r$ and $\boldsymbol{S}_r$, I could use complexes with carriers obviously homeomorphic to balls and spheres. For the examples in 5D-5H, I used more abstract carriers, but at least they were realizable in a sensible number of dimensions. In 5K-5L, however, I chose a carrier $M(\mathcal{L})$ in a large number of dimensions not related directly to the (covering or inductive or Hausdorff) dimension r of $\boldsymbol{P}_r$. Now the metric suggested in 5I immediately directs our attention to a natural differential structure on $\boldsymbol{P}_r$ (locally identical to that of $\boldsymbol{S}_r$) and to a notion of surface measure (corresponding to half of $\boldsymbol{S}_r$). I do not know whether these can be replicated by any expression of $\boldsymbol{P}_r$ as a subspace of a Euclidean or Hilbert space.

6 Rational homology groups and the Lefschetz-Hopf fixed point theorem

It is a remarkable fact that real projective spaces of *even* dimension have the fixed-point property. For this we need to develop some more machinery (6A, 6H).

6A Rational homology (a) I remarked in 1F that one can replace $\mathbb{Z}$ in the theory there by another commutative group. In particular we can replace $\mathbb{Z}$ by a field ($\mathbb{Z}_2$ or $\mathbb{Q}$, for instance), which gives us some new tools to use in discussing homomorphisms between the groups C_n, Z_n and H_n.

For definiteness, I will take it in this section that we are replacing $\mathbb{Z}$ by $\mathbb{Q}$. I have to ask you to re-read essentially the whole of Chapters 1-5 with this substitution. You will find that you can very nearly simply overwrite every $\mathbb{Z}$ with a $\mathbb{Q}$ and that almost every argument is completely unaffected. (In particular, you can keep the phrasing of the definition of chain groups in 1C and the proof of the Boundary Theorem in 1F.) To avoid any confusion, I will write $C_n(\mathcal{K},\mathbb{Q})$, $Z_n(\mathcal{K},\mathbb{Q})$, $B_n(\mathcal{K},\mathbb{Q})$, $H_n(\mathcal{K},\mathbb{Q})$ for the new groups associated with a simplicial complex, so that in the adjusted version of 1C I say that

> $C_n(\mathcal{K},\mathbb{Q})$ is the set of all rational n-chains in $\mathcal{K}$, where a rational n-chain is a function $x : V^{n+1} \to \mathbb{Q}$ such that
>
> — whenever $p \in V^{n+1}$ and $\sigma \in S_{n+1}$ then $x(p\sigma) = \epsilon_\sigma x(p)$,
>
> — if $p \in V^{n+1}$ and $p[n+1] = \{p(0), \dots, p(n)\}$ does not belong to $\mathcal{K}$, then $x(p) = 0$.

The formula

$$\partial(x)(p) = \sum_{v\in V} x(p^\frown\!<\!v\!>) \text{ for every } p \in V^{n+1}$$

(1E) is unchanged, but of course we are now thinking of x as belonging to $\mathbb{Q}^{V^{n+2}}$.

It is important to note that our groups $C_n(\mathcal{K},\mathbb{Q})$, etc., are now linear spaces over $\mathbb{Q}$, because $\mathbb{Q}^{V^{n+1}}$ is. Moreover, the homomorphisms $x \mapsto h\bullet x : C_n(\mathcal{K},\mathbb{Q}) \to C_n(\mathcal{L},\mathbb{Q})$ of 1D, or $\partial : C_{n+1}(\mathcal{K},\mathbb{Q}) \to C_n(\mathcal{K},\mathbb{Q})$ are not only group homomorphisms but also linear transformations. (Generally, if C, C' are linear spaces over $\mathbb{Q}$ and $\phi : C \to C'$ is additive, that is, $\phi(x + y) = \phi(x) + \phi(y)$ for every $x \in C$, then $\phi(qx) = q\phi(x)$ for every $x \in C$ and $q \in \mathbb{Q}$, because if $m \in \mathbb{N} \setminus \{0\}$ is such that $mq \in \mathbb{Z}$, then

$$q(\phi x) = \frac{1}{m} \cdot mq\phi(x) = \frac{1}{m}\phi((mq)x) = \frac{1}{m}\phi(m(qx)) = \frac{1}{m} \cdot m\phi(qx) = \phi(qx).)$$

So in this matter we can be casual, most of the time. But we do need to remember that it means that the kernel $Z_n(\mathcal{K},\mathbb{Q})$ and the image $B_{n-1}(\mathcal{K},\mathbb{Q})$ of $\partial : C_n(\mathcal{K},\mathbb{Q}) \to C_{n-1}(\mathcal{K},\mathbb{Q})$ are linear subspaces, not just subgroups, of the chain groups involved. Consequently the homology groups $H_n(\mathcal{K},\mathbb{Q})$ can also be thought of as linear spaces over $\mathbb{Q}$. We can go clear through to 4B-4C and get

$$H_n(\boldsymbol{B}_r,\mathbb{Q}) = \{0\} \text{ for every } n \in \mathbb{N},$$

$$H_r(\boldsymbol{S}_r,\mathbb{Q}) \cong \mathbb{Q}, \quad H_n(\boldsymbol{S}_r,\mathbb{Q}) = \{0\} \text{ for } n \neq r.$$

The rest of the section works equally well with rational homology instead of integer homology, but since we are using this mostly to determine topological properties, it's not necessary to examine the adaptations, except in 4Ha.

(b) The point at which we have to think, as opposed to copy, comes in Lemma 5C and its corollaries. In the statement of 5C itself, I wrote

> (c) Set $A = \{x : x \in C_{m+1}(\mathcal{K}),\ \operatorname{supp}\partial(x) \subseteq V_1,\ (\partial(x)\restriction V_1^{m+1})^\bullet$ generates $H_m(\mathcal{K}_1)\}$. Then $A \neq \emptyset$. If $x \in A$, $x' \in C_{m+1}(\mathcal{K})$ and $\operatorname{supp}\partial(x') \subseteq V_1$, then x' is an integer multiple of x. If x, $x' \in A$ then $x' = \pm x$.

I expected you to interpret '$(\partial(x)\restriction V_1^{m+1})^\bullet$ generates $H_m(\mathcal{K}_1)$' as '$H_m(\mathcal{K}_1) = \{k(\partial(x)\restriction V_1^{m+1})^\bullet : k \in \mathbb{Z}\}$'. In the reinterpretation we need to speak of rational multiples, not just integer multiples; and the final sentence has to be changed as well, leading to

> (c) Set $A = \{x : x \in C_{m+1}(\mathcal{K},\mathbb{Q}),\ \operatorname{supp}\partial(x) \subseteq V_1,\ H_m(\mathcal{K}_1,\mathbb{Q}) = \{q(\partial(x)\restriction V_1^{m+1})^\bullet : q \in \mathbb{Q}\}\}$. Then $A \neq \emptyset$. If $x \in A$, $x' \in C_{m+1}(\mathcal{K})$ and $\operatorname{supp}\partial(x') \subseteq V_1$, then x' is a rational multiple of x. If x, $x' \in A$ then x' is a non-zero rational multiple of x.

We come next to

> (d) If $\hat{x} \in A$ then $H_m(\mathcal{L}) \cong Z_m(\mathcal{L}_1)/G$ where G is the set of multiples of $(f\bullet\partial(\hat{x}))\restriction W_1^{m+1}$. So if $f\bullet\partial(\hat{x}) = 0$ then $H_m(\mathcal{L}) \cong H_m(\mathcal{L}_1)$.

This must now be read as

(d) If $\hat{x} \in A$ then $H_m(\mathcal{L},\mathbb{Q}) \cong Z_m(\mathcal{L}_1,\mathbb{Q})/G$ where G is the set of rational multiples of $(f{\bullet}\partial(\hat{x}))\restriction W_1^{m+1}$. So if $f{\bullet}\partial(\hat{x})=0$ then $H_m(\mathcal{L},\mathbb{Q}) \cong H_m(\mathcal{L}_1,\mathbb{Q})$.

(c) The same issue, the interpretation of the group G, arises in the applications in 5D-5H. I speak indifferently of 'the set of multiples of' or 'the subgroup generated by'; but these should now rather be taken to be 'the set of rational multiples of' or 'the linear subspace generated by'. In 5D-5F there is no great change, and we get

$$H_0(\mathcal{L},\mathbb{Q})=\{0\},\quad H_1(\mathcal{L},\mathbb{Q})\cong\mathbb{Q},\quad H_2(\mathcal{L},\mathbb{Q})=\{0\},\quad H_n(\mathcal{L},\mathbb{Q})=\{0\}\text{ for }n>2;$$

for the cylinder or the Möbius strip (or $\boldsymbol{S}_1$, of course), and

$$H_0(\mathcal{L},\mathbb{Q})=\{0\},\quad H_1(\mathcal{L},\mathbb{Q})\cong\mathbb{Q}^2,\quad H_2(\mathcal{L},\mathbb{Q})\cong\mathbb{Q},\quad H_n(\mathcal{L},\mathbb{Q})=\{0\}\text{ for }n>2$$

for the torus.

Things get more interesting with the Klein bottle and the projective plane. In 5G we have $(f{\bullet}\partial(\hat{x}))\restriction W_1^2 = 2z_l$. Over the rationals, this means that $G=\{q(f{\bullet}\partial(\hat{x}))\restriction W_1^2 : q\in\mathbb{Q}\}=\{qz_l : q\in\mathbb{Q}\}$. Since we still have $Z_1(\mathcal{L},\mathbb{Q})\cong\mathbb{Q}^2$, generated by z_r and z_l, the quotient $Z_1(\mathcal{L},\mathbb{Q})/G$ is now isomorphic just to $\mathbb{Q}$, and we have

$$H_0(\mathcal{L},\mathbb{Q})=\{0\},\quad H_1(\mathcal{L},\mathbb{Q})\cong\mathbb{Q},\quad H_2(\mathcal{L},\mathbb{Q})=\{0\},\quad H_n(\mathcal{L},\mathbb{Q})=\{0\}\text{ for }n>2.$$

In the case of $\boldsymbol{P}_2$ the same thing happens and now we find that every homology group is trivial, as with $\boldsymbol{B}_2$.

(d) The same phenomenon arises in Theorem 5L. Once again, in (e-i) of the proof, we find that z is a (rational) multiple of z_k; as in (e-ii-β), we observe that $z\ne 0$; so we can now conclude that z and z_k generate the same linear space $Z_k(\mathcal{K}_k^\dagger,\mathbb{Q})$. And when, as in (e-iii), we discover that $f_k^\dagger{\bullet}z = 2f_k{\bullet}\hat{x}_{k-1}$, while $Z_k(\mathcal{L}_k,\mathbb{Q})$ is the linear space generated by $f_k{\bullet}\hat{x}_{k-1}$, we see that $(f_{k+1}{\bullet}\partial(\hat{x})_k)\restriction W_k^{k+1} = f_k^\dagger{\bullet}z_k$ also generates $Z_k(\mathcal{L}_k,\mathbb{Q})$, and $H_k(\mathcal{L}_{k+1},\mathbb{Q})=\{0\}$.

So the inductive procedure in (f) of that proof, and the translation in (g), now give

$$H_n(\boldsymbol{P}_r,\mathbb{Q})\cong\mathbb{Q}\text{ if }n=r\text{ is odd, and is }\{0\}\text{ in all other cases.}$$

6B The trace of a matrix (a) If $A=\langle\alpha_{ij}\rangle_{i,j<m}$ is an $m\times m$ matrix, its **trace** $\operatorname{tr}(A)$ is $\sum_{i=0}^{m-1}\alpha_{ii}$.

(b) If A and B are $m\times m$-matrices, $\operatorname{tr}(AB)=\operatorname{tr}(BA)$. **P** If $A=\langle\alpha_{ij}\rangle_{i,j<m}$ and $B=\langle\beta_{ij}\rangle_{i,j<m}$ then

$$\operatorname{tr}(AB)=\sum_{i=0}^{m-1}\sum_{j=0}^{m-1}\alpha_{ij}\beta_{ji}=\sum_{j=0}^{m-1}\sum_{i=0}^{m-1}\beta_{ji}\alpha_{ij}=\operatorname{tr}(BA).\ \textbf{Q}$$

(c) If A and B are $m\times m$-matrices and B is invertible, $\operatorname{tr}(B^{-1}AB)=\operatorname{tr}(BB^{-1}A)=\operatorname{tr}(A)$.

6C The trace of a linear operator (a) Let U be a finite-dimensional linear space and $T:U\to U$ a linear operator. Suppose that $\langle u_i\rangle_{i<m}$ and $\langle v_i\rangle_{i<m}$ are bases of U and that $A=\langle\alpha_{ij}\rangle_{i,j<m}$, $B=\langle\beta_{ij}\rangle_{i,j<m}$ are the corresponding matrices representing T, so that

$$T(u_i)=\sum_{j=0}^{m-1}\alpha_{ij}u_j,\quad T(v_i)=\sum_{j=0}^{m-1}\beta_{ij}v_j$$

for $i<n$. Then $\operatorname{tr}(A)=\operatorname{tr}(B)$. **P** Let $C=\langle\gamma_{ij}\rangle_{i,j<m}$ and $D=\langle\delta_{ij}\rangle_{i,j<m}$ be the change-of-basis matrices for $\langle u_i\rangle_{i<m}$ and $\langle v_i\rangle_{i<m}$, so that

$$v_i=\sum_{j=0}^{m-1}\gamma_{ij}u_j,\quad u_i=\sum_{j=0}^{m-1}\delta_{ij}v_j$$

for $i<m$. Then $A=DBC$ and $CD=DC$ is the identity, so $\operatorname{tr}(A)=\operatorname{tr}(B)$ by 6Bc. **Q**

(b) We may therefore define the **trace** $\operatorname{tr}(T)$ of the linear operator T to be the common value of the traces of the matrices representing T.

The formulae of (a) are not quite explicit about the case $U=\{0\}$, $m=0$, which seems to ask for an empty matrix. But if we allow this, and say that its trace is 0, the same result is trivially true, and we find ourselves saying that $\operatorname{tr}(T)=0$ if T is an isomorphism of a zero-dimensional space.

6D The quotient theorem for traces Let U be a finite-dimensional linear space and $T:U\to U$ a linear operator. Let V be a linear subspace of U such that $T[V]\subseteq V$, and U/V the quotient linear space.

(a) We have a linear operator $\tilde{T} : U/V \to U/V$ defined by the formula

$$\tilde{T}(u^\bullet) = (Tu)^\bullet \text{ for every } u \in U.$$

(b) $\operatorname{tr}(T) = \operatorname{tr}(T{\restriction}V) + \operatorname{tr}(\tilde{T})$.

proof (a) To see that $\tilde{T}$ is well-defined as a function from U/V to U/V, we need to know only that $(Tu)^\bullet = (Tv)^\bullet$ whenever u, $v \in U$ and $u^\bullet = v^\bullet$; but in this case $u - v \in V$ so $Tu - Tv = T(u-v) \in V$ and $(Tu)^\bullet = (Tv)^\bullet$. Now it is elementary to check that $\tilde{T}$ is linear.

(b)(i) Let $\langle u_i\rangle_{i<r}$ be a basis for V and $\langle u_i\rangle_{i<s}$ an extension to a basis for U. Let $A = \langle \alpha_{ij}\rangle_{i,j<s}$ be the matrix of T with respect to $\langle u_i\rangle_{i<s}$. For $i < r$, $Tu_i \in V$ so Tu_i is a linear combination of $\langle u_j\rangle_{j<r}$ and $\alpha_{ij} = 0$ for $r \le j$; accordingly $\langle \alpha_{ij}\rangle_{i,j<r}$ is the matrix of $T{\restriction}V$ with respect to $\langle u_i\rangle_{i<r}$. Next, $\langle u_i^\bullet\rangle_{r\le i<s}$ is a basis for U/V. $\mathbf{P}(\alpha)$ If $u \in U$, we can express u as $\sum_{i<s}\beta_i u_i$ for some $\langle \beta_i\rangle_{i<s}$; now

$$u^\bullet = \textstyle\sum_{i<s}\beta_i u_i^\bullet = \sum_{r\le i<s}\beta_i u_i^\bullet$$

because $u_i \in V$ and $u_i^\bullet = 0$ for $i < r$. Thus $\langle u_i^\bullet\rangle_{r\le i<s}$ spans U/V. (β) If $\langle \beta_i\rangle_{r\le i<s}$ is such that

$$0 = \textstyle\sum_{r\le i<s}\beta_i u_i^\bullet = (\sum_{r\le i<s}\beta_i u_i)^\bullet$$

in U/V, then $\sum_{r\le i<s}\beta_i u_i$ and $-\sum_{r\le i<s}\beta_i u_i$ belong to V, so there is a family $\langle \beta_i\rangle_{i<r}$ such that

$$\textstyle\sum_{i<r}\beta_i u_i = -\sum_{r\le i<s}\beta_i u_i, \quad \sum_{i<s}\beta_i u_i = 0,$$

But $\langle u_i\rangle_{i<s}$ is linearly independent so $\beta_i = 0$ for every $i < s$, and in particular whenever $r \le i < s$. As $\langle \beta_i\rangle_{r\le i<s}$ is arbitrary, $\langle u_i^\bullet\rangle_{r\le i<s}$ is linearly independent, therefore a basis for U/V. $\mathbf{Q}$

(ii) For $r \le i < s$,

$$\tilde{T}u_i^\bullet = (Tu_i)^\bullet = (\textstyle\sum_{j<s}\alpha_{ij}u_j)^\bullet = \sum_{j<s}\alpha_{ij}u_j^\bullet = \sum_{r\le j<s}\alpha_{ij}u_j^\bullet$$

because $u_j^\bullet = 0$ for $j < r$. So the matrix of $\tilde{T}$ with respect to $\langle u_i^\bullet\rangle_{r\le i<s}$ is $\langle \alpha_{ij}\rangle_{r\le i,j<s}$. Accordingly

$$\operatorname{tr}(T) = \textstyle\sum_{i<s}\alpha_{ii} = \sum_{i<r}\alpha_{ii} + \sum_{r\le i<s}\alpha_{ii} = \operatorname{tr}(T{\restriction}V) + \operatorname{tr}(\tilde{T})$$

as claimed.

6E Corollary Suppose that we have a sequence $\langle C_n\rangle_{n\ge -1}$ of finite-dimensional linear spaces, all but finitely many of them zero-dimensional, for each $n \ge 0$ a linear operator $\partial_n : C_n \to C_{n-1}$, and for each $n \ge -1$ a linear operator $T_n : C_n \to C_n$. Suppose that $\partial_n\partial_{n+1} = 0$ and $T_{n-1}\partial_n = \partial_n T_n$ for every $n \in \mathbb{N}$. Set $Z_n = \{x : x \in C_n, \partial_n x = 0\}$, and $B_{n-1} = \partial_n[C_n]$ for each $n \in \mathbb{N}$.

(a) For each $n \in \mathbb{N}$, $B_n \subseteq Z_n$. Write H_n for the quotient Z_n/B_n.

(b) For each $n \in \mathbb{N}$, $T_n[Z_n] \subseteq Z_n$, $T_{n-1}[B_{n-1}] \subseteq B_{n-1}$ and we have a linear operator $T_{\star n} : H_n \to H_n$ defined by saying that $T_{\star n}(x^\bullet) = (T_n(x))^\bullet$ for $x \in Z_n$. Now $\operatorname{tr}(T_{\star n}) = \operatorname{tr}(T_{n-1}{\restriction}B_{n-1})$.

(c) $\sum_{n=0}^\infty (-1)^n \operatorname{tr}(T_n) = \operatorname{tr}(T_{-1}{\restriction}B_{-1}) + \sum_{n=0}^\infty (-1)^n \operatorname{tr}(T_{\star n})$.

proof (a) $B_n \subseteq Z_n$ because $\partial_n\partial_{n+1} = 0$.

(b) $T_n[Z_n] \subseteq Z_n$ and $T_{n-1}[B_{n-1}] \subseteq B_{n-1}$ because $\partial_n T_n = T_{n-1}\partial_n$. So $T_n{\restriction}Z_n$ is a linear operator from Z_n to itself, and gives rise to an operator $T_{\star n} : H_n \to H_n$ as in 6D because $T_n[B_n] \subseteq B_n$.

At the same time, the First Isomorphism Theorem tells us that we have an isomorphism $\tilde{\partial}_n : C_n/Z_n \to B_{n-1}$ such that $\tilde{\partial}_n(x^\bullet) = \partial_n(x)$ for every $x \in C_n$. Now if we take $\tilde{T}_n : C_n/Z_n \to C_n/Z_n$ to be the operator defined by saying that $\tilde{T}_n(x^\bullet) = (T_n(x))^\bullet$ for $x \in C_n$, as in 6D,

$$\tilde{\partial}_n\tilde{T}_n(x^\bullet) = \tilde{\partial}_n(T_n(x)^\bullet) = \partial_n T_n(x) = T_{n-1}\partial_n(x) = T_{n-1}\tilde{\partial}_n(x^\bullet)$$

for every $x \in C_n$, so $\tilde{\partial}_n\tilde{T}_n = T_{n-1}\tilde{\partial}_n : C_n/Z_n \to B_{n-1}$ and $T_{n-1}{\restriction}B_{n-1} = \tilde{\partial}_n^{-1}\tilde{T}_n\tilde{\partial}_n$. Consequently $\operatorname{tr}(T_{n-1}{\restriction}B_{n-1}) = \operatorname{tr}(\tilde{T}_n)$.

(c) Since all but finitely many of the C_n, and therefore all but finitely many of the H_n, are trivial, all but finitely many of the traces $\operatorname{tr}(T_n)$, $\operatorname{tr}(\tilde{T}_n)$ are zero, and the sums are well-defined. Now, applying 6D repeatedly, we see that

$$\begin{aligned}
\sum_{n=0}^{\infty}(-1)^n \operatorname{tr}(T_n) &= \sum_{n=0}^{\infty}(-1)^n(\operatorname{tr}(T_n\restriction Z_n)+\operatorname{tr}(\tilde{T}_n))\\
&= \sum_{n=0}^{\infty}(-1)^n(\operatorname{tr}(T_{\star n})+\operatorname{tr}(T_n\restriction B_n)+\operatorname{tr}(T_{n-1}\restriction B_{n-1}))\\
&= \sum_{n=0}^{\infty}(-1)^n(\operatorname{tr}(T_{\star n})+\operatorname{tr}(T_n\restriction B_n))+\sum_{n=-1}^{\infty}(-1)^{n+1}\operatorname{tr}(T_n\restriction B_n)\\
&= \operatorname{tr}(T_{-1}\restriction B_{-1})+\sum_{n=0}^{\infty}(-1)^n \operatorname{tr}(T_{\star n}).
\end{aligned}$$

6F A note on subdivisions For the next theorem, we need a new construction, based on an argument in the proof of the Subdivision Theorem (1S).

(a)(i) Let $\mathcal{K}$ be a geometrically realizable simplicial complex, with vertex set $V \subseteq \mathbb{R}^r$ and carrier X. Suppose that $I \in \mathcal{K}$ is a doubleton set and $\mathcal{L}$ is the elementary subdivision of $\mathcal{K}$ based on I. Taking the new vertex $\hat{v}$ of $\mathcal{L}$ to be the average of the two vertices in I, as in 3D, we know that the vertex set of $\mathcal{L}$ is $V \cup \{\hat{v}\}$ and that X is still the carrier of L.

In part (a) of the proof of 1S, I introduced a linear operator $\pi : C_n(\mathcal{K},\mathbb{Q}) \to C_n(\mathcal{L},\mathbb{Q})$ for each $n \geq -1$, and in part (b) I showed that $\pi\partial = \partial\pi$. (I am moving directly to the expression in terms of rational homology. Of course the argument so far works just as well for integral homology.)

(ii) Looking again at the formula for π,

$$\begin{aligned}
(\pi(x))(q) &= x(q) \text{ if } q[n+1] \in \mathcal{L} \text{ and } \hat{v} \notin q[n+1],\\
&= x(g_0 q) \text{ if } q \text{ is injective and } \{\hat{v}, v_1\} \subseteq q[n+1] \in \mathcal{L},\\
&= x(g_1 q) \text{ if } q \text{ is injective and } \{\hat{v}, v_0\} \subseteq q[n+1] \in \mathcal{L},\\
&= 0 \text{ otherwise}
\end{aligned}$$

for $n \geq -1$, $x \in C_n(\mathcal{K},\mathbb{Q})$ and $q \in (V \cup \{\hat{v}\})^{n+1}$, we see that

$$\operatorname{supp}(\pi(x)) \subseteq \operatorname{supp}(x) \cup \{\hat{v}\} \text{ for every } x \in C_n(\mathcal{K},\mathbb{Q}),$$

$$\operatorname{supp}(\pi(x)) = \operatorname{supp}(x) \text{ if } x \in C_n(\mathcal{K},\mathbb{Q}) \text{ and } I \not\subseteq \operatorname{supp}(x)$$

where the support $\operatorname{supp}(x)$ is defined in 1M. But the geometrical form of the subdivision I have chosen means that

$$\operatorname{supp}(\pi(x)) \subseteq \Gamma(\operatorname{supp}(x))$$

for every x.

(iii) Recall also that the point of introducing π in 1S was that the induced homomorphisms $\pi_\star : H_n(\mathcal{K},\mathbb{Q}) \to H_n(\mathcal{L},\mathbb{Q})$, defined by saying that $\pi_\star(x^\bullet) = (\pi(x))^\bullet$ for $x \in C_n(\mathcal{K},\mathbb{Q})$, are isomorphisms.

(b) Now suppose that we have a sequence $\langle \mathcal{K}^{(m)} \rangle_{m\in\mathbb{N}}$ of successive subdivisions of our geometrically realizable complex $\mathcal{K} = \mathcal{K}_0$, all using the method above, as in 3G-3H. Then we can construct inductively a family $\langle \pi^{(m)} \rangle_{m\in\mathbb{N}}$ such that each $\pi^{(m)}$ gives linear operators from $C_n(\mathcal{K},\mathbb{Q})$ to $C_n(\mathcal{K}^{(m)},\mathbb{Q})$, the associated operators $\pi_\star^{(m)} : H_n(\mathcal{K},\mathbb{Q}) \to H_n(\mathcal{K}_n^{(m)},\mathbb{Q})$ are isomorphisms, and $\operatorname{supp}(\pi^{(m)}(x))$ is always included in $\Gamma(\operatorname{supp}(x))$.

6G Definition If X is a triangulable topological space and $g : X \to X$ is a continuous function, the **Lefschetz number** $\Lambda(g)$ of g is $1 + \sum_{n=0}^{\infty} \operatorname{tr}(g_{n\star})$, where here I write $g_{n\star}(a) = g_\star(a)$ for $a \in H_n(X,\mathbb{Q})$.

(b) The formula here is not the one most commonly presented because it is expressed in terms of reduced homology groups, so that $H_0(\mathcal{K},\mathbb{Q})$ is zero- rather than one-dimensional if $\mathcal{K}$ is connected (and not $\{\emptyset\}$).

6H The Lefschetz Fixed Point Theorem Let X be a non-empty triangulable topological space and $g : X \to X$ a continuous function with no fixed point. Then $\Lambda(g) = 0$.

proof (a) The hypothesis is that X is homeomorphic to the carrier of a geometrically realizable simplicial complex; of course we may suppose that X actually is the carrier of a complex $\mathcal{K}$ with vertex set included in $\mathbb{R}^r$. Because X is compact and non-empty, $\delta = \frac{1}{3}\min_{t\in X}\|g(t) - t\|$ is greater than 0. Let $\langle\mathcal{K}_m\rangle_{m\in\mathbb{N}}$ be a sequence of repeated subdivisions of $\mathcal{K}$ as in 3G. Then there is a $k \in \mathbb{N}$ such that $\Delta(\mathcal{K}_k) \le \delta$. Now the Approximation Theorem (3J) tells us that there are an $m \ge k$ and a simplicial map $f : V_m \to V_k$ from $\mathcal{K}_m$ to $\mathcal{K}_k$ such that

g and $\bar f$ are homotopic functions from X to itself,

$\|g(t) - \bar f(t)\| \le \Delta(\mathcal{K}_m) \le \delta$ for every $t \in X$,

$g[\widehat{v}] \subseteq \widehat{f(v)}$ for everu $v \in V_m$, where $\widehat{v} \subseteq X$ is calculated in terms of V_m and $\widehat{f(v)}$ is calculated in terms of V_k.

From 6F we know that for each $n \ge -1$ we have a linear operator $\pi : C_n(\mathcal{K}_k, \mathbb{Q}) \to C_n(\mathcal{K}_m, \mathbb{Q})$ such that there is an induced isomorphism $\pi_\star : H_n(\mathcal{K}_k, \mathbb{Q}) \to H_n(\mathcal{K}_m, \mathbb{Q})$ and $\mathrm{supp}(\pi(x))$ is included in $\Gamma(\mathrm{supp}(x))$ for every $x \in C_n(\mathcal{K}_k, \mathbb{Q})$. Define $T_n : C_n(\mathcal{K}_m, \mathbb{Q}) \to C_n(\mathcal{K}_m, \mathbb{Q})$ by setting $T_n(x) = \pi(f\bullet x)$ for each $x \in C_n(\mathcal{K}_m, \mathbb{Q})$.

(b) The trace of T_n is zero for every $n \ge 0$. **P** Let $Q \subseteq V_m^{n+1}$ be the set of those injective $q : n+1 \to V_m$ such that $q[n+1] \in \mathcal{K}_m$. For $q, q' \in Q$ say that $q \sim q'$ if $q[n+1] = q'[n+1]$, that is, if there is a $\sigma \in S_{n+1}$ such that $q' = q\sigma$; let $P \subseteq Q$ be a selector for the equivalence relation $\sim$. If $x \in C_n(\mathcal{K}_m, \mathbb{Q})$ then $x = \sum_{p\in P} x(p)e_p$, so $\langle e_p\rangle_{p\in P}$ is a basis for the linear space $C_n(\mathcal{K}_m, \mathbb{Q})$ (see E1B). Note that $\mathrm{supp}(x) = \bigcup_{p\in P, x(p)\ne0} p[n+1]$.

We are concerned with the matrix $\langle T_n(e_p)(q)\rangle_{p,q\in P}$ and with the diagonal entries $T_n(e_p)(p)$. Take $p \in P$. If $f\bullet e_p = 0$ then $T_n(e_p) = 0$ and $T_n(e_p)(p) = 0$. Otherwise, $f\bullet e_p = e_{fp}$ (E1C). Set $t = \frac{1}{n+1}\sum_{i=0}^n p(i)$; then $t \in \bigcap_{i\le n}\widehat{p(i)}$ so

$$g(t) \in \textstyle\bigcap_{i\le n} g[\widehat{p(i)}] \subseteq \bigcap_{i\le n}\widehat{fp(i)}.$$

Now $\widehat{w} \subseteq \{u : \|u - w\| \le \Delta(\mathcal{K}_k)\}$ for every $w \in V_k$, so $\|fp(i) - g(t)\| \le \delta$ for every $i \le n$ and $\mathrm{supp}(e_{fp}) = fp[n+1]$ is included in the convex set $B = \{u : u \in \mathbb{R}^s,\ \|u - g(t)\| \le \delta\}$. By 6Fb, $\mathrm{supp}(\pi(e_{fp})) \subseteq B$. Of course we also have $\widehat{p(0)} \subseteq \{u : \|u - p(0)\| \le \Delta(\mathcal{K}_m)\}$, so $\|p(0) - g(t)\| \ge \|t - g(t)\| - \|p(0) - t\| \ge 2\delta$ and $p(0) \notin \mathrm{supp}(\pi(e_{fp}))$. So here again we must have $T_n(e_p)(p) = \pi(e_{fp})(p) = 0$.

Accordingly every diagonal element of the matrix of T_n is zero, and $\mathrm{tr}(T_n) = 0$. **Q**

(c) I wish to apply the formula in 6Ec, for which I need to decide what $T_{-1}\restriction B_{-1}(\mathcal{K}_m, \mathbb{Q})$ must be. I have not mentioned B_{-1} specifically before. But in E1F I remarked that

$$Z_0(\mathcal{K}_m, \mathbb{Q}) = \{x : x \in C_0(\mathcal{K}_m, \mathbb{Q}),\ \textstyle\sum_{v\in V} x(<v>) = 0\},$$

so $B_{-1}(\mathcal{K}_m, \mathbb{Q}) \cong C_0(\mathcal{K}_m, \mathbb{Q})/Z_0(\mathcal{K}_m, \mathbb{Q})$ is not $\{0\}$ and must be the whole of the one-dimensional space $C_{-1}(\mathcal{K}_m, \mathbb{Q})$ (see 1C(b-i)). This is where we need to know that X is not empty, so that $\mathcal{K} \ne \{\emptyset\}$ and $V_m \ne \emptyset$.

Tracing through the definitions of $\bullet$ in 1D and π in 6F, we see that if $e \in C_{-1}(\mathcal{K}_m, \mathbb{Q})$ is such that $e(\emptyset) = 1$, then $(f\bullet e)(\emptyset) = 1$ and $T_{-1}(e)(\emptyset) = (\pi(f\bullet e))(\emptyset) = 1$ and $T_{-1}(e) = e$. So $\mathrm{tr}(T_{-1}\restriction B_{-1}(\mathcal{K}_m, \mathbb{Q})) = 1$.

(d) Since g and $\bar f$ are homotopic, we see that $g_\star$ and $\bar f_\star$ agree on the homology groups of X, by the Homotopy Theorem (2K). Next, $\bar f_\star : H_n(X, \mathbb{Q}) \to H_n(X, \mathbb{Q})$ corresponds to $f_\star : H_n(\mathcal{K}_m, \mathbb{Q}) \to H_n(\mathcal{K}_k, \mathbb{Q})$, by the Geometric Homology Theorem (3Hb); and $\pi_\star f_{\star n}(a) = T_{\star n}(a)$ for every $a \in H_n(\mathcal{K}_m, \mathbb{Q})$. Since $\pi_\star : H_n(\mathcal{K}_k, \mathbb{Q}) \to H_n(\mathcal{K}_m, \mathbb{Q})$ is an isomorphism, we have $\mathrm{tr}(g_{\star n}) = \mathrm{tr}(f_{\star n}) = \mathrm{tr}(T_{\star n})$ for every n. Consequently

$$\begin{aligned}0 &= \sum_{n=0}^\infty (-1)^n\,\mathrm{tr}(T_n) = \mathrm{tr}(T_{-1}\restriction B_{-1}(\mathcal{K}_m, \mathbb{Q})) + \sum_{n=0}^\infty (-1)^n\,\mathrm{tr}(T_{\star n})\\ &= 1 + \sum_{n=0}^\infty (-1)^n\,\mathrm{tr}(g_{\star n}) = \Lambda(g),\end{aligned}$$

as claimed.

6I Corollary If X is a triangulable topological space, not $\emptyset$, such that $H_n(X,\mathbb{Q})=\{0\}$ for every $n\in\mathbb{N}$, then every continuous function from X to itself has a fixed point.

proof If $g:X\to X$ is continuous then, in the language of 6G, $\mathrm{tr}(g_{n\star})$ must be zero for every n. So $\Lambda(g)=1$ is not zero and g has a fixed point, by 6H.

6J Corollary If r is even, then every continuous function from $\boldsymbol{P}_r$ to itself has a fixed point.

proof Put 6Ad and 6I together.

Notes and comments The target of this section – the fixed-point property of $\boldsymbol{P}_2$, etc. (6J), is innocent enough, and a very natural question following the Brouwer Fixed Point Theorem (4F). It is easy to see that the cylinder and the Möbius strip and the torus and the Klein bottle do not have the fixed-point property – just look at the 'basic diagrams' in 5D-5G) – and nor does $\boldsymbol{P}_r$ for odd r; the idea of 4Hb works for this as well. But to reach the positive result for $\boldsymbol{P}_2$ we seem to need at least two quite new ideas. First, we need the concept of 'trace' of an operator. In fact there is a workable definition of 'trace' for any homomorphism from a finitely generated commutative group to itself, leading to theorems corresponding to 6D-6E (Schubert 68, IV.3.6). However, I do not think that many of my readers will have encountered it in an undergraduate course, and while it is not especially difficult by the standards of the present note, it takes a while to explain. So I have chosen instead to use the relatively accessible notion of 'trace' for linear operators on finite-dimensional linear spaces, at the price of asking you to review almost the whole theory described so far (6A). I have cast this section in terms of the field $\mathbb{Q}$ of rational numbers, but it works just as well for any field which is not of characteristic 2. You will see that there is an apparently straightforward relationship between integral and rational homology groups: you just throw torsion groups away (6Ac-6Ad). It is not quite as simple as that, of course, but it is true that (for any group F) the groups $H_n(\mathcal{K},F)$ are determined by the groups $H_n(\mathcal{K},\mathbb{Z})$, which is why we always reach first for the latter, as being potentially more informative.

The second new idea is in 6Fb. The simplices of the original simplex $\mathcal{K}$ correspond to chains in the subdivided simplices $\mathcal{K}_m$ and the homomorphisms π_m keep track, as in 1S. Because they commute with the boundary operator, the π_m act on the homology groups and (in the proof of 6H) enable us to look at a simplicial map from a finer subdivision $\mathcal{K}_m$ to a coarser subdivision $\mathcal{K}_k$ in terms of actions on the chain groups of $\mathcal{K}_m$. Consequently we have linear maps from the finite-dimensional linear spaces $C_n(\mathcal{K}_m,\mathbb{Q})$ and $H_n(\mathcal{K}_m,\mathbb{Q})$ to themselves, and these linear maps have traces. Nearly-elementary linear algebra leads to 6Ec, and because the support of an $x\in C_n(\mathcal{K}_m,\mathbb{Q})$ has a topological interpretation in the carrier of $\mathcal{K}_m$, we have a chance of relating these traces to topological properties of simplicial maps from $\mathcal{K}_m$ to $\mathcal{K}_k$.

7 The Borsuk-Ulam Theorem

I now turn to a further result (7E) which clearly belongs with those of Chapter 4 except that, as far as I know, it does not have a proof based on the elementary homology theory of Chapters 1-3. However one of the standard proofs uses geometrisable simplicial complexes and elementary subdivisions, so we have a head start.

7A Lemma Let $\mathcal{K}$ be a simplicial complex with vertex set V, $I=\{v_0,v_1\}\in\mathcal{K}$ a 1-simplex, $\hat{v}$ something not in V, and $\mathcal{L}$ the elementary subdivision of $\mathcal{K}$ constructed by the method of 1R from I and $\hat{v}$. Write V' for the vertex set $V\cup\{\hat{v}\}$ of $\mathcal{L}$. For $K\in\mathcal{K}$ and $L\in\mathcal{L}$ set

$$A_K=\{v:v\in V\setminus K,\,K\cup\{v\}\in\mathcal{K}\},\quad A'_L=\{v:v\in V'\setminus L,\,L\cup\{v\}\in\mathcal{L}\},$$

Suppose that $k\ge 1$ is such that

$\mathcal{K}$ has no $(k+1)$-simplices and $\#(A_K)=2$ for every $(k-1)$-simplex of $\mathcal{K}$.

Then $\mathcal{L}$ has no $(k+1)$-simplices and for every $(k-1)$-simplex $L\in\mathcal{L}$

either $L\in\mathcal{K}$ and $A'_L=A_L$,

or $v_0\in L$, $A'_L=\{\hat{v},v\}$ where $v\in V\setminus I$ and there is a $(k-1)$-simplex $K\in\mathcal{K}$ such that $A_K=\{v_1,v\}$,

or $v_1\in L$, $A'_L=\{\hat{v},v\}$ where $v\in V\setminus I$ and there is a $(k-1)$-simplex $K\in\mathcal{K}$ such that $A_K=\{v_0,v\}$,

or $\hat{v}\in L$, $A'_L=I$ and $(L\cup I)\setminus\{\hat{v}\}\in\mathcal{K}$.

So if $\#(A_K)=2$ for every $(k-1)$-simplex $K\in\mathcal{K}$ then $\#(A'_L)=2$ for every $(k-1)$-simplex $L\in\mathcal{L}$.

proof (a) If $L\in\mathcal{L}$ then there is a $K\in\mathcal{K}$ such that L is a proper subset of $K\cup\{\hat{v}\}$. So $\#(L)\le\#(K)\le k+1$ and L cannot be a $(k+1)$-simplex.

(b) Now suppose that $L\in\mathcal{L}$ is a $(k-1)$-simplex, that is, $\#(L)=k$. Write J for $\{v_0,\hat{v},v_1\}$.

($\boldsymbol{\alpha}$) Suppose that $L\cap J=\emptyset$. Then $L\in\mathcal{K}$. $\#(L\cup I)=k+2$ so $L\cup I\notin\mathcal{K}$ and $L\cup\{\hat{v}\}\notin\mathcal{L}$. On the other hand, if $v\in V\setminus L$ then $I\not\subseteq L\cup\{v\}$. So

$$\begin{aligned}A'_L&=\{v:v\in V'\setminus L,\,L\cup\{v\}\in\mathcal{L}\}\\&=\{v:v\in V\setminus L,\,L\cup\{v\}\in\mathcal{L}\}=A_K.\end{aligned}$$

($\boldsymbol{\beta}$) Suppose that $L\cap J=\{v_0\}$. Then $L\in\mathcal{K}$ and $L\cup\{\hat{v}\}\in\mathcal{L}$ iff $L\cup\{v_1\}\in\mathcal{K}$, while $L\cup\{v_1\}\notin\mathcal{L}$ and of course $L\cup\{\hat{v}\}\notin\mathcal{K}$. On the other hand, for $v\in V\setminus(L\cup J)=V'\setminus(L\cup J)$, $L\cup\{v\}\in\mathcal{L}$ iff $L\cup\{v\}\in\mathcal{K}$. So A'_L is either A_L or $\{\hat{v},v\}$ where $A_L=\{v_1,v\}$.

($\boldsymbol{\gamma}$) The same argument applies if $L\cap J=\{v_1\}$.

($\boldsymbol{\delta}$) Suppose that $L\cap J=\{\hat{v}\}$. Then $L=(K\setminus I)\cup\{\hat{v}\}$ where $K=(L\cup I)\setminus\{\hat{v}\}$ belongs to $\mathcal{K}$, and $L\cup\{v_0\}=(K\setminus\{v_1\})\cup\{\hat{v}\}$, $L\cup\{v_1\}=(K\setminus\{v_0\})\cup\{\hat{v}\}$ belong to $\mathcal{L}$. On the other hand, if $v\in V\setminus(L\cup\{v_0,v_1\})$ then $(L\cup\{v\}\cup I)\setminus\{\hat{v}\}$ has $k+2$ members and cannot belong to $\mathcal{K}$, so $L\cup\{v\}\notin\mathcal{L}$. Thus $\{v:v\in V'\setminus L,$ $L\cup\{v\}\in\mathcal{L}\}=I$.

($\boldsymbol{\epsilon}$) Suppose that $L\cap J=\{v_0,\hat{v}\}$. Then $K=L\triangle\{v_1,\hat{v}\}$ belongs to $\mathcal{K}$. Now $L\cup\{v_1\}\notin\mathcal{L}$ and $K\cup\{\hat{v}\}\notin\mathcal{K}$, while for $v\in V\setminus(K\cup J)=V'\setminus(L\cup J)$ we have

$$\begin{aligned}K\cup\{v\}\in\mathcal{K}&\Longrightarrow L\cup\{v\}=(K\cup\{v\})\triangle\{v_1,\hat{v}\}\in\mathcal{L}\\&\Longrightarrow K\cup\{v\}=(L\cup\{v\})\triangle\{v_1,\hat{v}\}\in\mathcal{K}.\end{aligned}$$

So here $A'_L=A_K$.

($\boldsymbol{\zeta}$) The same argument applies if $L\cap J=\{v_1,\hat{v}\}$; and as v_0, v_1 cannot both belong to L, this exhausts the possibilities and we have the result claimed.

7B Symmetric subdivisions again We need to look again at the construction used in 5J when setting up a framework for the homology of projective spaces. I continue that analysis in the same notation.

(a) If $m \in \mathbb{N}$ and $k \le r$ there are no $(k+1)$-simplices in $\mathcal{K}_k^{(m)}$ and every $(k-1)$-simplex in $\mathcal{K}_k^{(m)}$ is included in just two k-simplices in $\mathcal{K}_k^{(m)}$. **P** If $m = 0$ then $\bigcup \mathcal{K}_k^{(0)} = \{d_0, \dots, d_k, -d_0, \dots, -d_k\}$ and any simplex of $\mathcal{K}_k^{(0)}$ is included in one of the form $\{\pm d_0, \dots, \pm d_k\}$, with at most $k+1$ vertices. A $(k-1)$-simplex of $\mathcal{K}_k^{(0)}$ is of the form $L = \{\pm d_0, \dots, \pm d_k\} \setminus \{d_j, -d_j\}$ for some $j \le k$ and the k-simplices of $\mathcal{K}_k^{(0)}$ including L are $L \cup \{d_j\}$ and $L \cup \{-d_j\}$. If $k = 0$ then $\mathcal{K}_0^{(m)} = \mathcal{K}_0^{(0)}$ for every m so we can stop. Otherwise, induction on m, using Lemma 7A at the inductive step, shows that $\mathcal{K}_k^{(m)}$ has the required property for every m. **Q**

(b) We shall need a refinement of (a), as follows. Suppose that $m \in \mathbb{N}$, $k < r$ and that $L \in \mathcal{K}_k^{(m)}$ is a k-simplex. Then L is also a k-simplex in $\mathcal{K}_{k+1}^{(m)}$, so has just two extensions $L \cup \{v\}$, $L \cup \{v'\}$ to simplices in $\mathcal{K}_{k+1}^{(m)}$, by (a). The point is that $v(k+1)v'(k+1) < 0$. **P** If $m = 0$, L must be of the form $\{\pm d_0, \dots, \pm d_k\}$ and its two extensions in $\mathcal{K}_{k+1}^{(0)}$ are $L \cup \{d_{k+1}\}$, $L \cup \{-d_{k+1}\}$ so $v(k+1)v'(k+1) = -1$. Assuming the result for m, look at the list in 7A, thinking of $L \in \mathcal{K}_k^{(m)}$ as a k-simplex in the elementary subdivision $\mathcal{K}_{k+1}^{(m+1)}$ of $\mathcal{K}_{k+1}^{(m)}$. In the language there, $\{v, v'\} = A'_L$ and $I_m = \{v_0, v_1\}$.

? Suppose, if possible, that $\hat{v} \in L$ and $(L \cup I_m) \setminus \{\hat{v}\} \in \mathcal{K}_{k+1}^{(m)}$. Then $\hat{v}(k+1) = 0$. Since $v_0(k+1)v_1(k+1) \ge 0$, by 5Jc, and $\hat{v}(k+1) = \frac{1}{2}(v_0(k+1) + v_1(k+1))$, both $v_0(k+1)$ and $v_1(k+1)$ are zero and the $(k+1)$-simplex $(L \cup I_m) \setminus \{\hat{v}\}$ belongs to $\mathcal{K}_k^{(m)}$, which is impossible. **X**

We must therefore be in one of the first three cases listed in the penultimate sentence of 7A. If $\hat{v} \notin \mathcal{L}$ then $L \in \mathcal{K}_{k+1}^{(m)}$, $A'_L = A_L$ and $v(k+1)v'(k+1) < 0$ by the inductive hypothesis. Otherwise $\hat{v} \in L$ and $L \cap I_m \ne \emptyset$ and $A'_L = \{\hat{v}, w\}$, where one of $\{v_0, w\}$, $\{v_1, w\}$ is A_K for some k-simplex $K \in \mathcal{K}_{k+1}^{(m)}$. By the inductive hypothesis, one of $v_0(k+1)w(k+1)$, $v_1(k+1)w(k+1)$ is strictly negative. But as $v_0(k+1)v_1(k+1) \ge 0$, neither $v_0(k+1)w(k+1)$ nor $v_1(k+1)w(k+1)$ can be strictly positive, and $\hat{v}(k+1)w(k+1) < 0$, as required. **Q**

7C Definition Let V be a set and $q : V \to \mathbb{N}$, $q' : V \to \{-1, 1\}$ two functions. I will say that a non-empty finite subset K of V is

— (q, q')**-positive** if $q \restriction K$ is injective and $q'(v) = (-1)^{\#(\{v' : v' \in K, q(v') < q(v)\})}$ for every $v \in K$,

— (q, q')**-negative** if $q \restriction K$ is injective and $q'(v) = -(-1)^{\#(\{v' : v' \in K, q(v') < q(v)\})}$ for every $v \in K$.

7D Lemma Let $\mathcal{K}$ be a simplicial complex with vertex set V, and $q : V \to \mathbb{N}$, $q' : V \to \{-1, 1\}$ functions such that $q'(v) = q'(v')$ whenever $\{v, v'\} \in \mathcal{K}$ and $q(v) = q(v')$. Write $\mathcal{E}$ for the set of $(k+1)$-simplices in $\mathcal{K}$ and $\mathcal{E}_1$ for the set of k-simplices in $\mathcal{K}$ which are included in an odd number of members of $\mathcal{E}$. Then

$$\begin{aligned} &\#(\{K : K \in \mathcal{E} \text{ is either } (q, q')\text{-positive or } (q, q')\text{-negative}\}) \\ &\qquad \equiv_2 \#(\{K : K \in \mathcal{E}_1 \text{ is } (q, q')\text{-positive}\}), \end{aligned}$$

where I say that $m \equiv_2 n$ if $m - n$ is even.

proof Consider the set R of pairs (v, K) where $K \in \mathcal{E}$ and $v \in K$. For each $K \in \mathcal{E}$, write $A(K)$ for $\{v : (v, K) \in R,\ K \setminus \{v\} \text{ is positive}\}$.

(a)(i) If $K \in \mathcal{E}$ and K is either positive or negative, then $A(K)$ has just one member. **P** $q \restriction K$ is injective, and if v_0 is the point of K at which q is least and v_1 is the point of K at which q is greatest, one of $K \setminus \{v_0\}$, $K \setminus \{v_1\}$ is positive and the other is negative, while $K \setminus \{v\}$ is neither positive nor negative for any other $v \in K$. So $A(K)$ is either $\{v_0\}$ or $\{v_1\}$. **Q**

(ii) If $K \in \mathcal{E}$, $q \restriction K$ is injective and K is neither positive nor negative, then $A(K)$ is either empty or a doubleton. **P** There must be a pair (v_0, v_1) of points of K, adjacent for the ordering of K induced by q, such that $q'(v_0) = q'(v_1)$. Now $K \setminus \{v\}$ cannot be positive for any $v \in K \setminus \{v_0, v_1\}$, so if there is any $v \in A(K)$, it must be either v_0 or v_1, in which case the other also belongs to $A(K)$. **Q**

(iii) If $K \in \mathcal{E}$ and $q \restriction K$ is not injective, then $A(K)$ is either empty or a doubleton. **P** If $v \in A(K)$, then there must be just one $v' \in K \setminus \{v\}$ such that $q(v') = q(v)$, and now $q'(v') = q'(v)$, so $K \setminus \{v'\}$ is positive and $A(K) = \{v, v'\}$. **Q**

(b) From (a) we see that

$$
\begin{aligned}
\#(\{K : K \in \mathcal{E} & \text{ is either positive or negative}\}) \\
&\equiv_2 \sum_{K\in\mathcal{E}} \#(A(K)) \\
&= \#(\{(v,K) : v \in K \in \mathcal{E},\ K \setminus \{v\} \text{ is positive}\} \\
&= \sum_{m=0}^{\infty} m\#(\{K' : K' \in \mathcal{K} \text{ is positive}, \#(K') = k+1 \\
&\qquad\qquad \text{and } K' \text{ is included in just } m \text{ members of } \mathcal{E}\}) \\
&\equiv_2 \#(\{K' : K' \in \mathcal{E}_1 \text{ is positive}\}).
\end{aligned}
$$

7E The Borsuk-Ulam Theorem Suppose that s, $r \in \mathbb{N}$ and $s < r$. Then there is no continuous function $g : \boldsymbol{S}_r \to \boldsymbol{S}_s$ such that $g(-t) = -g(t)$ for every $t \in \boldsymbol{S}_r$.

proof (a) I will assume, on the contrary, that we have a continuous function $g : \boldsymbol{S}_r \to \boldsymbol{S}_s$ such that $g(-t) = -g(t)$ for every $t \in \boldsymbol{S}_r$, and work towards a contradiction. Since there is certainly a continuous $h : \boldsymbol{S}_s \to \boldsymbol{S}_{r-1}$ such that $h(-t) = -h(t)$ for every $t \in \boldsymbol{S}_r$, we can take it that $s = r - 1$.

(b) Let $\mathcal{K} = \mathcal{K}_r^{(0)}$ be the simplicial complex of 5J and 7B, with vertices of the form $\pm d_0, \dots, \pm d_r$ where $(d_0, \dots, d_r)$ is the usual basis of $\mathbb{R}^{r+1}$, and $X \subseteq \mathbb{R}^{r+1}$ its carrier. Then we have a homeomorphism $t \mapsto \frac{1}{\|t\|} : X \to \boldsymbol{S}_r$ which respects antipodes, so if we set $g_1(t) = g(\frac{1}{\|t\|}t)$ for $t \in X$ then $g_1 : X \to \boldsymbol{S}_{r-1}$ is a continuous map such that $g_1(-t) = -g_1(t)$ for every $t \in X$. Define $q : X \to r$, $q' : X \to \{-1, 1\}$ by the rule

$$q(t) = j \text{ if } j = \min\{i : i < r,\ |g_1(t)(i)| \ge \frac{1}{\sqrt{r}}|,$$

$$q'(t) = \operatorname{sgn} g_1(t)(q(t))$$

for every $t \in X$. (Because $\sum_{i=0}^{r-1} g_1(t)(i)^2 = 1$, $q(t)$ and $q'(t)$ are always defined.) Observe that

$$q(-t) = q(t), \quad q'(-t) = -q'(t)$$

for every $t \in X$.

(c) Because g_1 is continuous and X is compact, g_1 is uniformly continuous (Engelking 89, 4.3.32) and there is a $\delta > 0$ such that $\|g_1(t) - g_1(t')\| < \frac{2}{\sqrt{r}}$ whenever t, $t' \in X$ and $\|t - t'\| \le \delta$. It follows that if t, $t' \in X$, $\|t - t'\| \le \delta$ and $q(t) = q(t')$, then $q'(t) = q'(t')$. **P** Writing i for the common value of $q(t)$ and $q(t')$, we have

$$|g_1(t)(i)| \ge \frac{1}{\sqrt{r}}, \quad |g_1(t')(i)| \ge \frac{1}{\sqrt{r}}, \quad |g_1(t)(i) - g_1(t')(i)| \le \|g_1(t) - g_1(t')\| < \frac{2}{\sqrt{r}};$$

it follows that

$$q'(t) = \operatorname{sgn} g_1(t)(i) = \operatorname{sgn} g_1(t')(i) = q'(t'). \ \textbf{Q}$$

Define 'positive' and 'negative' finite subsets of X from the pair (q, q') as in 7C. Observe that if $K \subseteq X$ is a non-empty finite subset and we write $-K$ for $\{-v : v \in K\}$, then K is positive iff $-K$ is negative, and vice versa.

(d) Let $\langle \mathcal{K}^{(m)} \rangle_{m\in\mathbb{N}} = \langle \mathcal{K}_r^{(m)} \rangle_{m\in\mathbb{N}}$ be a sequence constructed as in 5J. As in the proof of 5L, fix an *even* $m \in \mathbb{N}$ such that $\Delta(\mathcal{K}^{(m)}) \le \delta$. Then $\mathcal{K}^{(m)}$ will be antipodeally symmetric and whenever $K \in \mathcal{K}^{(m)}$, v, $v' \in K$ and $q(v) = q(v')$, we shall have $q'(v) = q'(v')$.

Again as in the proof of 5L, write $\mathcal{K}_k^{\dagger}$ for

$$\mathcal{K}_k^{(m)} = \{K : K \in \mathcal{K}^{(m)},\ v(i) = 0 \text{ whenever } v \in K \text{ and } k < i \le r\}.$$

Then we see from 7B that there are no $(k+1)$-simplices in $\mathcal{K}_k^\dagger$, that every $(k-1)$-simplex in $\mathcal{K}_k^\dagger$ is included in just two k-simplices in $\mathcal{K}_k^\dagger$, and that if $k<r$ and $L\in\mathcal{K}_k^\dagger$ is a k-simplex, then the two $(k+1)$-simplices in $\mathcal{K}_{k+1}^\dagger$ including L are of the form $L\cup\{v\}$, $L\cup\{v'\}$ where $v(k+1)<0$ and $v'(k+1)>0$.

(e) (The key.) For every $k\le r$, the number of positive k-simplices in $\mathcal{K}_k^\dagger$ is odd. **P** Induce on k.

(i) If $k=0$, then $\mathcal{K}_0^\dagger=\{\emptyset,\{d_0\},\{-d_0\}\}$ because d_0 and $-d_0$ are the only two points v of X such that $v(i)=0$ for $0<i\le r$. Now $q(d_0)=q(-d_0)=0$, $q'(d_0)=1$ and $q'(-d_0)=-1$, So the 0-simplices of $\mathcal{K}_0^\dagger$ are $\{d_0\}$, which is positive, and $\{-d_0\}$, which is negative.

(ii) Suppose that $0\le k<r$ and $\mathcal{K}_k^\dagger$ has an odd number of positive k-simplices. Set

$$\mathcal{K}_{k+1}=\{K:K\in\mathcal{K}_{k+1}^\dagger,\ v(k+1)\ge 0\text{ for every }v\in K\},$$

$$\mathcal{K}_{k+1}^-=\{K:K\in\mathcal{K}_{k+1}^\dagger,\ v(k+1)\le 0\text{ for every }v\in K\}.$$

Then $\mathcal{K}_{k+1}$ and $\mathcal{K}_{k+1}^-$ are simplicial complexes, and their intersection is $\mathcal{K}_k^\dagger$. By 5Jc, $\mathcal{K}_{k+1}\cup\mathcal{K}_{k+1}^-=\mathcal{K}_{k+1}^\dagger$.

If $K\in\mathcal{K}_{k+1}$ is a k-simplex, then it is included in exactly two $(k+1)$-simplices in $\mathcal{K}_{k+1}^\dagger$, by 7Ba. If $K\in\mathcal{K}_k^\dagger$, then one of these does not belong to $\mathcal{K}_{k+1}$ and the other does not belong to $\mathcal{K}_{k+1}^-$, by 7Bb; so just one of them belongs to $\mathcal{K}_{k+1}$. While if $K\notin\mathcal{K}_k^\dagger$, then there is a $v\in K$ such that $v(k+1)\ne 0$, in which case $v(k+1)>0$ and neither of the extensions of K can belong to $\mathcal{K}_{k+1}^-$. So K is included in just two $(k+1)$-simplices in $\mathcal{K}_{k+1}$.

We can therefore apply Lemma 7D to see that

$$\{K:K\in\mathcal{K}_{k+1}\text{ is a }(k+1)\text{-simplex and is either positive or negative}\}$$

has an odd number of members. Next, because $\mathcal{K}_{k+1}^\dagger$ is antipodeally symmetric, $\mathcal{K}_{k+1}^-=\{-K:K\in\mathcal{K}_{k+1}\}$; also, because $\mathcal{K}_k^\dagger$ has no $(k+1)$-simplices, every $(k+1)$-simplex of $\mathcal{K}_{k+1}^\dagger$ belongs to just one of $\mathcal{K}_{k+1}$, $\mathcal{K}_{k+1}^-$. Now a simplex $K\in\mathcal{K}_{k+1}^-$ is positive iff $-K\in\mathcal{K}_{k+1}$ is negative. So

$$\begin{aligned}&\#(\{K:K\in\mathcal{K}_{k+1}^\dagger\text{ is a positive }(k+1)\text{-simplex}\})\\ &\quad=\#(\{K:K\in\mathcal{K}_{k+1}\text{ is a positive }(k+1)\text{-simplex}\})\\ &\qquad\qquad+\#(\{K:K\in\mathcal{K}_{k+1}^-\text{ is a positive }(k+1)\text{-simplex}\})\\ &\quad=\#(\{K:K\in\mathcal{K}_{k+1}\text{ is a positive }(k+1)\text{-simplex}\})\\ &\qquad\qquad+\#(\{K:K\in\mathcal{K}_{k+1}\text{ is a negative }(k+1)\text{-simplex}\})\end{aligned}$$

is odd, and the induction proceeds. **Q**

(f) At the end of the induction, we see that there is a positive r-simplex $K\in\mathcal{K}_r^\dagger$. But now $q:K\to r$ is injective, while K has $r+1$ points. So we have the hoped-for contradiction, and the theorem is proved.

7F Corollary Suppose that $r\in\mathbb{N}$.

(a) If $f:\boldsymbol{S}_r\to\mathbb{R}^r$ is continuous and $f(-t)=-f(t)$ for every $t\in\boldsymbol{S}_r$, then there is a $t\in\boldsymbol{S}_r$ such that $f(t)=0$.

(b) If $f:\boldsymbol{S}_r\to\mathbb{R}^r$ is continuous, there is a $t\in\boldsymbol{S}_r$ such that $f(-t)=f(t)$.

proof (a) ? Otherwise, $\mathbb{R}^r\ne\{0\}$ and $r\ge 1$. Set $g(t)=\frac{1}{\|f(t)\|}f(t)$ for $t\in\boldsymbol{S}_r$; then g is a continuous function from $\boldsymbol{S}_r$ to $\boldsymbol{S}_{r-1}$ and $g(-t)=-g(t)$ for every t. But this is impossible, by 7E. **X**

(b) Apply (a) to the function $t\mapsto f(t)-f(-t)$.

7G For the next result I allow myself to use a fragment of measure theory from Fremlin 01, as well as some slightly less basic general topology.

Definitions Let $r\ge 0$ be an integer.

(a) For $t\in\boldsymbol{S}_r$ and $\gamma\in\mathbb{R}$ set

$$H_{t,>\gamma} = \{t' : (t'|t) > \gamma\}, \quad H_{t\gamma} = \{t' : (t'|t) = \gamma\}, \quad H_{t,<\gamma} = \{t' : (t'|t) < \gamma\}$$

in $\mathbb{R}^{r+1}$. Note that $H_{t,>\gamma}$ and $H_{t,<\gamma}$ are open sets.

(b) If ν is a totally finite topological measure on $\mathbb{R}^{r+1}$, I will say that a **median hyperplane** for ν is a hyperplane of the form $H_{t\gamma}$, where $t \in \boldsymbol{S}_r$, $\gamma \in \mathbb{R}$ and

$$\nu H_{t,>\gamma} \le \frac{1}{2}\nu\mathbb{R}^{r+1}, \quad \nu H_{t,<\gamma} \le \frac{1}{2}\nu\mathbb{R}^{r+1}.$$

7H Lemma Let $r \in \mathbb{N}$ be an integer, $\langle\nu_m\rangle_{m\in\mathbb{N}}$ a sequence of Radon probability measures on $\mathbb{R}^{r+1}$, $\langle t_m\rangle_{m\in\mathbb{N}}$ a sequence in $\boldsymbol{S}_r$ and $\langle\gamma_m\rangle_{m\in\mathbb{N}}$ a sequence in $\mathbb{R}$ such that $H_{t_m\gamma_m}$ is a median hyperplane for ν_m for every m. Suppose that ν is a Radon probability measure on $\mathbb{R}^{r+1}$ such that $\langle\nu_m\rangle_{m\in\mathbb{N}} \to \nu$ uniformly, in the sense that

$$\lim_{m\to\infty} \sup_{E\subseteq\mathbb{R}^r \text{ is Borel}} |\nu E - \nu_m E| = 0.$$

If $t = \lim_{m\to\infty} t_m$ and $\gamma = \lim_{m\to\infty} \gamma_m$ in $\boldsymbol{S}_r$ and $\mathbb{R}$ respectively, then $H_{t\gamma}$ is a median hyperplane for ν.

proof (a) $H_{t,>\gamma} \subseteq \bigcup_{m\in\mathbb{N}}\bigcap_{k\ge m} H_{t_k,>\gamma_k}$. **P** If $(t'|t) > \gamma$, let $m \in \mathbb{N}$ be such that $\|t_k - t\|\|t'\| < (t'|t) - \gamma_k$ for every $k \ge m$. Then, for $k \ge m$, $(t'|t_k) \ge (t'|t) - \|t_k - t\|\|t'\| > \gamma_k$, so $t' \in \bigcap_{k\ge m} H_{t_k,>\gamma_k}$. **Q**

Consequently $\nu H_{t,>\gamma} \le \liminf_{m\to\infty} \nu H_{t_m,>\gamma_m}$. Now setting $\delta_m = \sup_{E\subseteq\mathbb{R}^r \text{ is Borel}} |\nu E - \nu_m E|$, we are supposing that $\lim_{m\to\infty} \delta_m = 0$, so

$$\nu H_{t,>\gamma} \le \liminf_{m\to\infty} \nu H_{t_m,>\gamma_m} - \delta_m \le \liminf_{m\to\infty} \nu_m H_{t_m,>\gamma_m} \le \frac{1}{2}.$$

Similarly, $\nu H_{t,<\gamma} \le \frac{1}{2}$ and $H_{t\gamma}$ is a median hyperplane for ν.

7I Lemma Let $r \in \mathbb{N}$ be an integer and ν a Radon probability measure on $\mathbb{R}^{r+1}$ with bounded support. Then there is a continuous function $h : \boldsymbol{S}_r \to \mathbb{R}$ such that $H_{t,h(t)}$ is a median hyperplane and $h(-t) = -h(t)$ for every $t \in \boldsymbol{S}_r$.

proof (a) For $t \in \boldsymbol{S}_r$ we have

$$\lim_{\gamma\to\infty} \nu H_{t,>\gamma} = 0, \quad \lim_{\gamma\to-\infty} \nu H_{t,>\gamma} = 1,$$

so $g_1(t) = \sup\{\gamma : \nu H_{t,>\gamma} \ge \frac{1}{2}\}$ is defined in $\mathbb{R}$. Now $g_1 : \boldsymbol{S}_r \to \mathbb{R}$ is lower semi-continuous. **P** Suppose that $t_0 \in \boldsymbol{S}_r$, $\alpha \in \mathbb{R}$ and $g_1(t_0) > \alpha$. Take β, β' such that $\alpha < \beta < \beta' < g_1(t_0)$. Then $\nu H_{t_0,>\beta'} \ge \frac{1}{2}$. Let F be the support of ν; we are supposing that it is bounded. Suppose that $t \in \boldsymbol{S}_r$ is such that $\|t - t_0\| \le \frac{\beta'-\beta}{\sup_{t'\in F}\|t'\|}$. If $t' \in F \cap H_{t_0,>\beta'}$, then

$$(t'|t) \ge (t'|t_0) - \|t'\|\|t - t_0\| > \beta' - (\beta' - \beta) = \beta.$$

So $H_{t,>\beta} \supseteq \nu(F \cap H_{t,>\beta'})$ and

$$\nu H_{t,>\beta} \ge \nu(F \cap H_{t,>\beta'}) = \nu H_{t,>\beta'} \ge \tfrac{1}{2}$$

and $g_1(t) \ge \beta$. As t_0 is arbitrary, $\{t : t \in \boldsymbol{S}_r, g_1(t) > \alpha\}$ is open; as α is arbitrary, g_1 is lower semi-continuous. **Q**

Note that $\nu H_{t,<g_1(t)} \le \frac{1}{2}$. **P** Let $\langle\gamma_m\rangle_{m\in\mathbb{N}}$ be a strictly increasing sequence with limit $g_1(t)$. Then $\nu H_{t,>\gamma_m} \ge \frac{1}{2}$ so $\nu H_{t,<\gamma_m} \le \frac{1}{2}$ for every m. But $\langle H_{t,<\gamma_m}\rangle_{m\in\mathbb{N}}$ is a non-decreasing sequence of measurable sets with union $H_{t,<g_1(t)}$, so $\nu H_{t,<g_1(t)} = \sup_{m\in\mathbb{N}} \nu H_{t,<\gamma_m} \le \frac{1}{2}$. **Q**

(b) Similarly, g_0 is upper semi-continuous, where

$$g_0(t) = \inf\{\gamma : \nu H_{t,<\gamma} \ge \tfrac{1}{2}\}$$

for $t \in \boldsymbol{S}_r$, and $\nu H_{t,>g_0(t)} \le \frac{1}{2}$ for every $t \in \boldsymbol{S}_r$. Observe that if $\gamma > g_1(t)$ then there is a $\beta \in]g_1(t), \gamma[$ for which $\nu H_{t,>\beta} \le \frac{1}{2}$ while $H_{t,>\beta} \cup H_{t,<\gamma} = \mathbb{R}^{r+1}$ so $\nu H_{t,<\gamma} \ge \frac{1}{2}$ and $\gamma \ge g_0(t)$. So $g_0(t) \le g_1(t)$ for every $t \in \boldsymbol{S}_r$.

(c) Because $\boldsymbol{S}_r$ is metrizable it is a normal topological space (Engelking 89, 4.1.13). There is therefore a continuous function $g : \boldsymbol{S}_r \to \mathbb{R}$ such that $g_0(t) \le g(t) \le g_1(t)$ for every $t \in \boldsymbol{S}_r$ (Engelking 89, 1.7.15). Now $H_{tg(t)}$ is a median hyperplane for ν, for every $t \in \boldsymbol{S}_r$. **P** As $g(t) \ge g_0(t)$,

$$\nu H_{t,>g(t)} \le \nu H_{t,>g_0(t)} \le \frac{1}{2} \text{ because } g(t) \ge g_0(t),$$

$$\nu H_{t,<g(t)} \le \nu H_{t,<g_1(t)} \le \frac{1}{2} \text{ because } g(t) \le g_1(t). \text{ \textbf{Q}}$$

(d) Finally, set $h(t) = \frac{1}{2}(g(t) - g(-t))$ for $t \in \boldsymbol{S}_r$. Then h is continuous and $h(-t) = -h(t)$ for every t. For any t, $h(t)$ lies between $g(t)$ and $-g(-t)$, so $H_{t,>h(t)}$ lies between $H_{t,>g(t)}$ and $H_{t,>-g(-t)}$. But

$$H_{t,>-g(-t)} = \{t' : (t'|t) > -g(-t)\} = \{t' : (t'|-t) < g(-t)\} = H_{-t,<g(-t)},$$

so $\nu H_{t,>-g(-t)} \le \frac{1}{2}$; as $\nu H_{t,>g(t)}$ is also at most $\frac{1}{2}$, so is $\nu H_{t,>h(t)}$. Similarly, $\nu H_{t,<h(t)} \le \frac{1}{2}$. Thus $H_{th(t)}$ is a median hyperplane for every t, as required.

7J The Ham Sandwich Theorem Suppose that $r \in \mathbb{N}$ is an integer, and $\nu_0, \dots, \nu_r$ are Radon probability measures on $\mathbb{R}^{r+1}$. Then they have a common median hyperplane.

proof (a) Consider first the case in which every ν_i has bounded support. By Lemma 7I, we have for each $i \le r$ a continuous function $h_i : \boldsymbol{S}_r \to \mathbb{R}$ such that $h_i(-t) = -h_i(t)$ and $H_{th_i(t)}$ is a median hyperplane for ν_i, for every $t \in \boldsymbol{S}_r$. Set $f(t) = \langle h_i(t) - h_r(t)\rangle_{t<r}$ for $t \in \boldsymbol{S}_r$. Then $f : \boldsymbol{S}_r \to \mathbb{R}^r$ is continuous and $f(-t) = -f(t)$ for every t. By Corollary 7Fa, there is a $t \in \boldsymbol{S}_r$ such that $f(t) = 0$, that is, $h_i(t) = h_r(t)$ for every $i < r$. So $H_{th_r(t)}$ is a median hyperplane for every ν_i.

(b) For the general case, define ν_{im}, for $i \le r$ and $m \in \mathbb{N}$, by setting $D_m = m\boldsymbol{B}_{r+1}$ for each m and, for $E \subseteq \mathbb{R}^{r+1}$,

$$\begin{aligned}\nu_{im}E &= \nu_i(E \cap D_m) \text{ if } E \cap D_m \text{ is measured by } \nu_i \text{ and } 0 \notin E,\\ &= \nu_i(E \cap D_m) + \nu_i(\mathbb{R}^r \setminus D_m) \text{ if } E \cap D_m \text{ is measured by } \nu_i \text{ and } 0 \in E.\end{aligned}$$

Then every ν_{im} is a Radon probability measure on $\mathbb{R}^{r+1}$ and its support is included in D_m, so is bounded.

By (a), we have for each m a pair $(t_m, \gamma_m) \in \boldsymbol{S}_r \times \mathbb{R}$ such that $H_{t_m\gamma_m}$ is a median hyperplane for ν_{im} for every $i \le r$. Now $\langle \gamma_m \rangle_{m\in\mathbb{N}}$ is bounded. **P** There is a $k \in \mathbb{N}$ such that $\nu_0 D_k \ge \frac{3}{4}$. Take any $m \in \mathbb{N}$. Then

$$\nu_{0m} H_{t,<\gamma} = \nu_0((H_{t,<\gamma} \cap D_m) \cup (\mathbb{R}^{r+1} \setminus D_m)) \ge \nu_0 D_k \ge \tfrac{3}{4}$$

whenever $t \in \boldsymbol{S}_r$ and $\gamma > k$, while

$$\nu_{0m} H_{t,>\gamma} = \nu_0((H_{t,>\gamma} \cap D_m) \cup (\mathbb{R}^{r+1} \setminus D_m)) \ge \nu_0 D_k \ge \tfrac{3}{4}$$

if $t \in \boldsymbol{S}_r$ and $\gamma < -k$. As $H_{t_m\gamma_m}$ is a median hyperplane for ν_{0m}, $|\gamma_m| \le k$. **Q**

Since $\boldsymbol{S}_r$ is compact, there is a strictly increasing sequence $\langle m_k \rangle_{k\in\mathbb{N}}$ such that $t = \lim_{k\to\infty} t_{m_k}$ and $\gamma = \lim_{k\to\infty} \gamma_{m_k}$ are defined in $\boldsymbol{S}_r$, $\mathbb{R}$ respectively. Note next that if $i \le r$, $m \in \mathbb{N}$ and $E \subseteq \mathbb{R}^{r+1}$ is Borel, then $\nu_i E$ and $\nu_{im} E$ both lie between $\nu_i(E \cap D_m)$ and $\nu_i(E_i \cap D_m) + \nu(\mathbb{R}^{r+1} \setminus D_m)$, so

$$\lim_{m\to\infty} \sup_{E\subseteq\mathbb{R}^{r+1} \text{ is Borel}} |\nu_i E - \nu_{im} E| \le \lim_{m\to\infty} \nu_i(\mathbb{R}^{r+1} \setminus D_m) = 0.$$

By Lemma 7H, $H_{t\gamma}$ is a median hyperplane for ν_i, for every $i \le r$. So the theorem is proved.

7K Corollary Suppose that $r \in \mathbb{N}$. Write μ for Lebesgue measure on $\mathbb{R}^{r+1}$. If $E_0, \dots, E_r \subseteq \mathbb{R}^{r+1}$ are measurable subsets of finite measure, there are $t \in \boldsymbol{S}_r$ and $\gamma \in \mathbb{R}$ such that

$$\mu(E_i \cap H_{t,>\gamma}) = \mu(E_i \cap H_{t,<\gamma}) = \frac{1}{2}\mu E_i$$

for every $i \le r$.

proof For $i \le r$, set $F_i = E_i$ if $\mu E_i > 0$, $\boldsymbol{B}_{r+1}$ otherwise, and define ν_i by saying that

$$\nu_i E = \frac{1}{\mu F_i}\mu(E \cap F_i) \text{ if } E \subseteq \mathbb{R}^{r+1} \text{ and } E \cap F_i \text{ is measured by } \mu.$$

Then ν_i is a Radon probability measure on $\mathbb{R}^{r+1}$. By 7J, we can find t, γ such that

$$\nu_i H_{t,>\gamma} \le \frac{1}{2}, \quad \nu_i H_{t,<\gamma} \le \frac{1}{2}$$

for every $i \le r$, that is,

$$\mu(F_i \cap H_{t,>\gamma}) \le \frac{1}{2}\mu F_i, \quad \mu(F_i \cap H_{t,<\gamma}) \le \frac{1}{2}\mu F_i$$

for every i. Of course it follows at once that

$$\mu(E_i \cap H_{t,>\gamma}) \le \frac{1}{2}\mu E_i, \quad \mu(E_i \cap H_{t,<\gamma}) \le \frac{1}{2}\mu E_i$$

for every i, since this corollary is formulated in a way which makes any hyperplane acceptable to a negligible set E_i. But since $\mu H_{t\gamma} = 0$,

$$\mu(E_i \cap H_{t,>\gamma}) = \frac{1}{2}\mu E_i, \quad \mu(E_i \cap H_{t,<\gamma}) = \frac{1}{2}\mu E_i$$

for every $i \le r$, as required.

Notes and comments I ought to say that I took the idea of Lemma 7D from Wikipedia, where a version of it is called the Ky Fan Lemma.

In 7B (as in 4A) I give more time than most authors feel is necessary in justifying assertions about geometry in multidimensional Euclidean space. As in Chapter 4, the natural generalization of the one- and two-dimensional cases works perfectly, and the reduction of geometry to algebra obscures the intuitions on which the arguments to follow are really based. But, as in Chapter 4, I feel more secure when they are supported by combinatorial formulations.

I have given a version of the Ham Sandwich Theorem (7J) which goes farther than most presentations trouble to do. But I think it is of interest, in 7K, that we can get on very well with unbounded sets (provided they have finite measure), and in 7J that we do not need absolute continuity with respect to Lebesgue measure (provided we fix on the right target in Definition 7Gb).

Acknowledgement Correspondence with C.P.Rourke.

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