Morse homotopy,

A^{∞}-category,

and

Floer homologies.

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This paper is a mixture of a survey article and a research announcement. Chapters 2 and 3 are survey of results by Gromov, Floer, Ruan, and others. Many of the material in the other chapters are new. However these chapters also include several parts which survey earlier works or restate them in a bit different way. Also there may be a possible overlaps of the results with one by other authors. (See remarks in those chapters.)

Here are rough summary of each chapter.

Chapter one is devoted to a construction which detects some information of homotopy types of manifolds using Morse functions. The result of this chapter is a "toy model" of the construction we will perform in later chapters.

Chapter two is a rough summary of Floer's idea on Arnold conjecture. In this chapter we assume rather restrictive hypothesis and try to discuss the basic points without studying various difficulties.

Those troubles we meet are discussed in Chapter three. But in this chapter we rather discuss the case of pseudo-holomorphic sphere. (While one needs to study pseudo-holomorphic disk for Arnold conjecture.) We apply the result on pseudo-holomorphic sphere to define Gromov-Ruan's [R] invariant which justify several constructions in topological σ-model.

In Chapter four, first we join the ideas in Chapters 2 and 3 to Maslov index and Novikov ring and define Floer homology for Lagrangian intersection in pseudo Einstein symplectic manifold of nonnegative curvature. Then we combine the construction of Floer homology to one in Chapter one and define an A^{∞}–category.

In Chapter five we first recall the definition of Floer homology of 3-manifold and the results by Dostoglou-Salamon [DS] and Yoshida [Y2] which relate it to symplectic Floer theory. Then, using the result of the last section, we define the Floer homology for 3-manifolds with boundary and discuss its properties.

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Chapter 1 Morse Homotopy

§1 Witten Complex

This chapter consists of some ideas which may be useful to analyse homotopy types of manifold using Morse theory. This argument may be regarded as a dual to De-Rham homotopy theory.

First let us recall how we find a homology group of manifold using a Morse function. ([W1])

Let $f: M \to \mathbf{R}$ be a function of C^∞-class. We say that it is a Morse function if, for each $p \in M$, we find its neighborhood and a coordinate there, such that f, with respect to that coordinate, is written as either $f(x_1, \cdots, x_n) = x_1$ or $f(x_1, \cdots, x_n) = \sum_{i=1}^{k} -x_i^2 + \sum_{i=k+1}^{n} x_i^2$.

We say that p is a critical point if $df(p) = 0$. Let $Cr(f)$ be the set of all critical point of f. In other words, p is a critical point if $f(x_1, \cdots, x_n) = \sum_{i=1}^{k} -x_i^2 + \sum_{i=k+1}^{n} x_i^2$ in a neighborhood of p. The number k is called the *Morse index* of f at p and is denoted by $\mu(p)$.

Next, we pick up a Riemannian metric on M. Then, using it, we get a gradient vector field $-\mathrm{grad}\ f$ of f.

We consider following "moduli space" of gradient lines of f. Namely for $p, q \in Cr(f)$ we put :

$$\mathcal{M}(p,q) = \left\{ \ell : \mathbf{R} \to M \;\middle|\; \begin{array}{l} \dfrac{d\ell}{dt} = -\mathrm{grad}\ f, \\ \lim\limits_{t\to-\infty} \ell(t) = p, \ \lim\limits_{t\to+\infty} \ell(t) = q \end{array} \right\}.$$

(Hereafter we write $\ell(-\infty) = p$ in place of $\lim\limits_{t\to-\infty} \ell(t) = p$.)

We define an equivalence relation on $\mathcal{M}(p,q)$ by $\ell \sim \ell_C$, $\ell_C(t) = \ell(C+t)$, and put $\overline{\mathcal{M}}(p,q) = \mathcal{M}(p,q)/\sim$.

In general, $\mathcal{M}(p,q)$ is not a manifold, but, by perturbing it a bit, we can always assume that it is a manifold. (Hereafter we write " if f is generic **L** " in case **L** is satisfied after perturbing f a bit.)

Lemma 1.1 : *If* f *is generic then*

$$\dim \mathcal{M}(p,q) = \mu(p) - \mu(q).$$

Let us give an example (in place of proving the lemma.) We consider $M = S^2$ and the map f given by the following figure

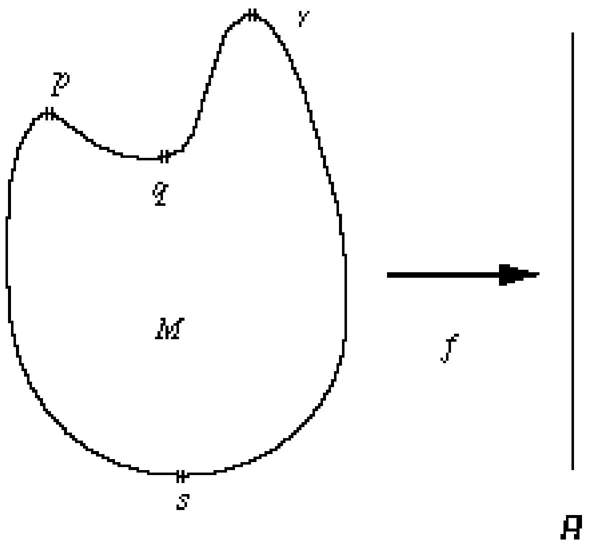
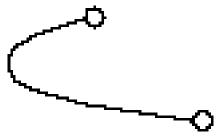


Figure 1.2

Then $Cr(f) = \{p,q,r,s\}$. $\mu(p) = \mu(r) = 2$, $\mu(q) = 1$, $\mu(s) = 0$. $\overline{\mathcal{M}}(p,q)$ is one point represented by a unique line joining p to q. Hence $\dim \overline{\mathcal{M}}(p,q) = 0 = \mu(p) - \mu(q) - 1$. Similarly $\overline{\mathcal{M}}(r,q)$ is one point. Also $\overline{\mathcal{M}}(q,r) = \text{two points}$. On the other hand, $\overline{\mathcal{M}}(p,s)$ is one dimensional manifold :  . Hence $\dim \overline{\mathcal{M}}(p,s) = \mu(p) - \mu(s) - 1 = 1$.

We do not prove the following :

Fact 1.3 : $\overline{\mathcal{M}}(p,q)$ *is orientable.*

Now we define the complex $(C_*(M,f),\partial)$ as follows :

Definition 1.4 :

(1) $C_k(M,f) = \bigoplus_{\mu(p)=k} \mathbf{Z} \cdot [p]$. (Free abelian group, whose generator is identified with the set of critical point of Morse index k.)

(2) $\partial : C_k(M,f) \to C_{k-1}(M,f)$ is defined by

$$\partial[p] = \sum_{\mu(q)} < \partial p, q > [q],$$

$$< \partial p, q > = \# \overline{\mathcal{M}}(p,q), \quad \text{(counted with sign.)}$$

Theorem 1.5 (Morse-Thom-Smale-Witten [W1] etc.) **:**

(1) $\partial \circ \partial = 0$.

(2) $H_*(C_*(X,f)) = H_*(X,\mathbf{Z})$.

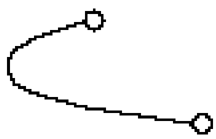
Sketch of Proof

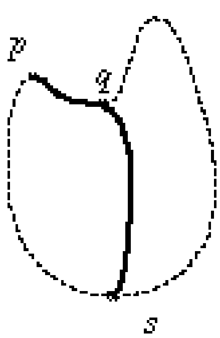
(1) We want to use the following lemma. Assume $\mu(p) - \mu(s) = 2$. (Then $\overline{\mathcal{M}}(p,s)$ is one dimensional. (Put $k = \mu(s) = \mu(p) - 2$.)

Lemma 1.6 : $\overline{\mathcal{M}}(p,s)$ *is compactified such that*

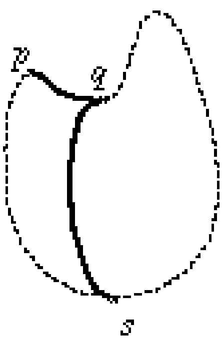
$$\partial \overline{\mathcal{M}}(p,s) = \bigsqcup_{\eta(q)=k+1} \overline{\mathcal{M}}(p,q) \times \overline{\mathcal{M}}(q,s),$$

as oriented manifolds.

In place of proving the lemma, let us consider the case of Figure 1.2 we discussed before. As we explained $\overline{\mathcal{M}}(p,s)$ =  = arc. The boundary of it consists of two points, which corresponds to that are



and



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Thus $\partial \overline{\mathcal{M}}(p,q) = \overline{\mathcal{M}}(p,r) \times \overline{\mathcal{M}}(r,s)$, here $\overline{\mathcal{M}}(p,r) = \text{one point}$, $\overline{\mathcal{M}}(r,s) = \text{two points}$. The lemma follows.

By lemma we have

$$\sum_{\mu(q)=k-1} <\partial p, q><\partial q, s> = 0$$

for each $p,s \in Cr(f)$, $\mu(p) = \mu(s) + 2$. It follows immediately $\partial\partial = 0$.

In place of proving (2) of Theorem 1.5 (the proof is a good exercise of Morse theory) , let us compute $H_*(C_*(M,f))$ for our example in Figure 1.2. We have $\overline{\mathcal{M}}(p,q) = \overline{\mathcal{M}}(r,q) = \text{one points}$, $\overline{\mathcal{M}}(q,s) = \text{two points}$. Hence $<\partial p, q> = <\partial r, q> = 1$. And if one defines the orientation appropriately, one finds the contributions from two points of $\overline{\mathcal{M}}(q,r)$ cancel to each other, hence $<\partial q, r> = 0$. Thus we have :

$$\begin{cases} \partial[p] = \partial[q] = [r], \\ \partial[q] = 0, \\ \partial[s] = 0. \end{cases}$$

We conclude :

$$H_*(C_*(M,f)) = \begin{cases} Z & k = 2, \\ 0 & k = 1, \\ Z & k = 0. \end{cases}$$

This coincides with $H_*(S^2;Z)$.

§2 Cup product

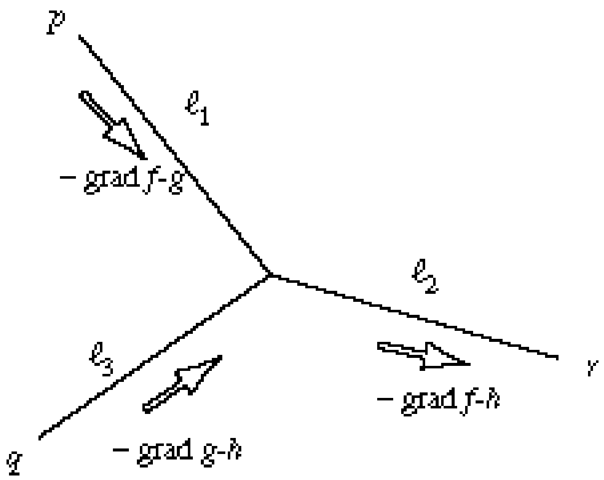
Next we consider three functions f, g, h on M, (which we assume to be generic.) We put $C_*(f,g) = C_*(M, f-g)$ etc. We are going to define a map

$$C_*(f,g) \otimes C_*(g,h) \to C_*(f,h).$$

Let $p \in Cr(f-g)$, $q \in Cr(g-h)$, $r \in Cr(f-h)$. We consider the following moduli space $\mathcal{M}(p,q,r)$.

$$\mathcal{M}(p,q,r) = \left\{ (\ell_1,\ell_2,\ell_3) \;\middle|\; \begin{array}{l} \ell_1 : (-\infty,0] \to M,\ \ell_2 : (-\infty,0] \to M, \\ \ell_3 : [0,\infty) \to M, \\ \dfrac{d\ell_1}{dt} = -\mathrm{grad}(f-g), \dfrac{d\ell_2}{dt} = -\mathrm{grad}(g-h) \\ \dfrac{d\ell_3}{dt} = -\mathrm{grad}(f-h) \\ \ell_1(0) = \ell_2(0) = \ell_3(0) \\ \ell_1(-\infty) = p, \ell_2(-\infty) = q, \ell_3(\infty) = r \end{array} \right\}.$$

Namely we consider the moduli space of the following configurations[2] :



Lemma 1.7 : *If f, g and h are generic, then $\mathcal{M}(p,q,r)$ is a manifold such that*

$$\dim \mathcal{M}(p,q,r) = \mu(p) + \mu(q) - \mu(r).$$

Using the moduli space $\mathcal{M}(p,q,r)$, we defines a map $\eta_2 : C_*(f,g) \otimes C_*(g,h) \to C_*(f,h)$ as follows.

Definition 1.8 :

[2] **Feymann diagram.**

$$\eta_2([p]\otimes[q]) = \sum_r \eta_2(p,q,r)[r],$$
$$\eta_2(p,q,r) = \#\mathcal{M}(p,q,r).$$

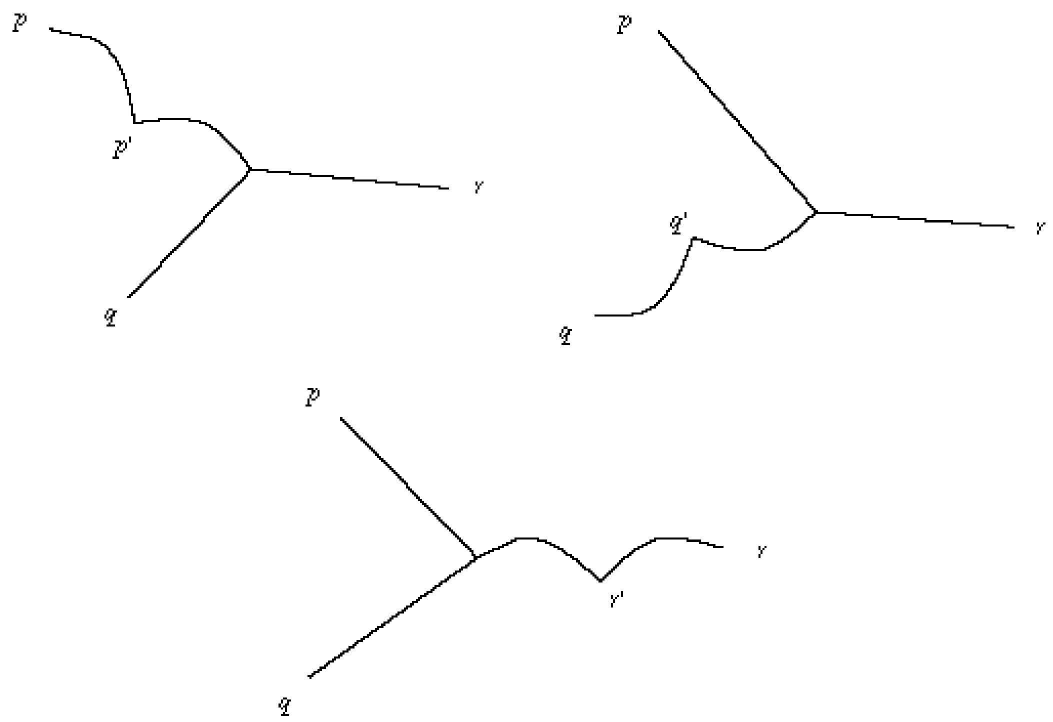
Proposition 1.9 : η_2 *is a chain map.*

Proof Proof is quite similar to one for $\partial\partial = 0$. We use the following lemma.

Lemma 1.10 : $\mathcal{M}(p,q,r)$ *is compactified such that*

$$\partial\mathcal{M}(p,q,r) = \bigcup_{p'} \overline{\mathcal{M}}(p,p')\times\mathcal{M}(p',q,r)\cup\bigcup_{q'} \overline{\mathcal{M}}(q,q')\times\mathcal{M}(p,q',r)$$
$$\cup\bigcup_{r'} \overline{\mathcal{M}}(r',r)\times\mathcal{M}(p,q,r').$$

The elements of right hand side is written as



Now let us suppose $\mu(p)+\mu(q)-\mu(r)=1$. Then $\mathcal{M}(p,q,r)$ is one dimensional.

By Lemma 1.7 we have

$$\begin{aligned} 0 \quad &= \pm\sum_{p'} < \partial p, p' > \eta_2(p', q, r) \\ &\pm\sum_{q'} < \partial q, q' > \eta_2(p, q', r) \\ &\pm\sum_{r'} < \partial r', r > \eta_2(p, q, r'). \end{aligned}$$

It follows that

$$\eta_2(\partial[q]\otimes[r]) \pm \eta_2([q]\otimes\partial[r]) \pm \partial\eta_2([q]\otimes[r]) = 0.$$

This proves the proposition.

Thus η_2 induces a map

$$(\eta_2)_* : H_*(C_*(f,g))\otimes H_*(C_*(g,h)) \to H_*(C_*(f,h)).$$

By Theorem 1.5, all these three homology groups are isometric to $H_*(M;Z)$. Hence we have

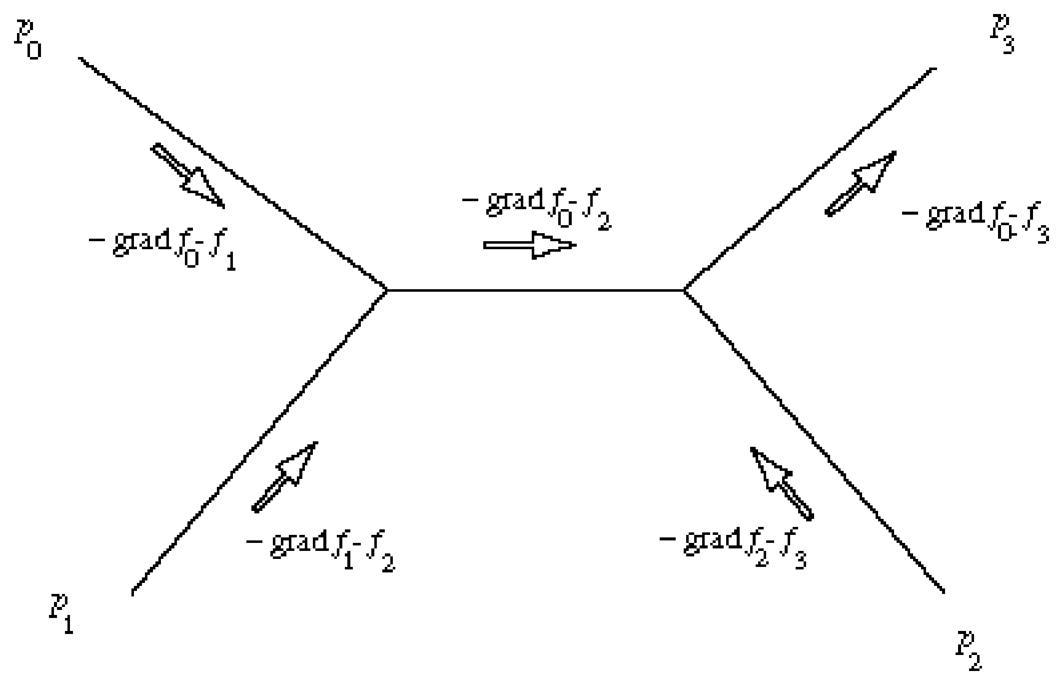
$$(\eta_2)_* : H_*(M;Z)\otimes H_*(M;Z) \to H_*(M;Z).$$

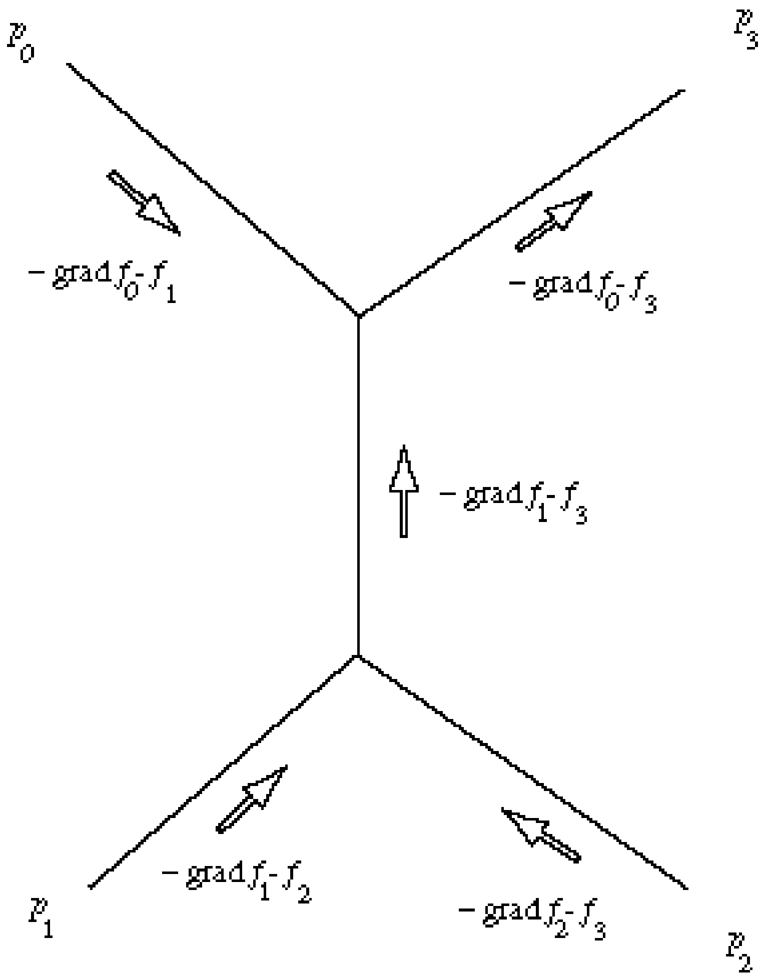
Proposition 1.10 : $(\eta_2)_*(x\otimes y) = PD((PD\,x)\cup(PD y))$, *where* PD *is Poincaré duality.*

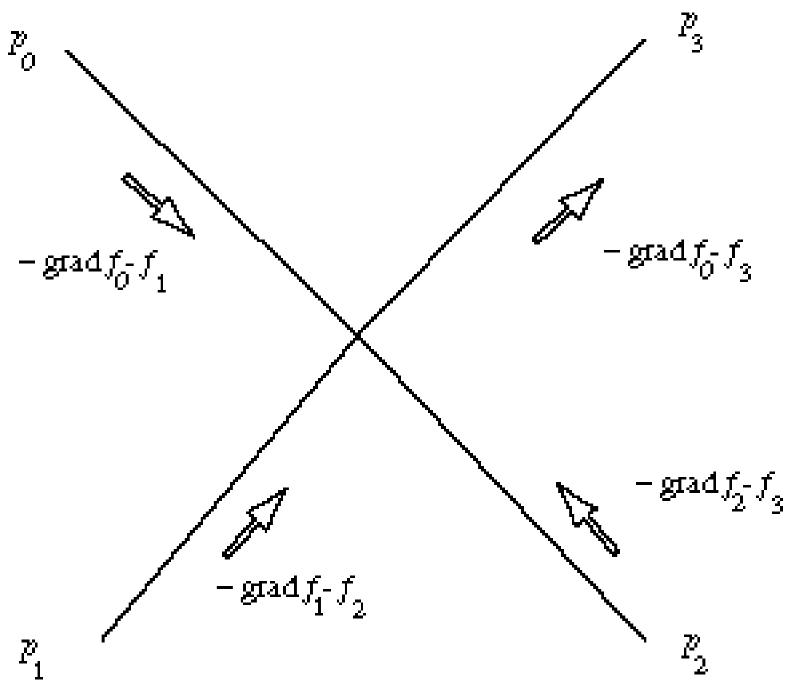
The proof is again a good exercise of Morse theory.

§3 Massey product

Next we discuss what happens when we have 4 functions, f_i, $i = 0,1,2,3$. We put $C_*(f_i, f_j) = C_*(M; f_i - f_j)$. Let $p_i \in Cr(f_i - f_{i+1})$. (We put $f_4 = f_0$.) We define the moduli space $\mathcal{M}(p_0, p_1, p_2, p_3)$ as the union of the spaces of the following three configurations.


p_0
p_3
$-\text{grad}\, f_0 - f_1$
$-\text{grad}\, f_0 - f_2$
$-\text{grad}\, f_0 - f_3$
$-\text{grad}\, f_1 - f_2$
$-\text{grad}\, f_2 - f_3$
p_1
p_2

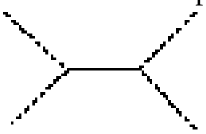




We can prove the following :

Proposition 1.11 : *For generic* f_i, *our moduli space* $\mathcal{M}(p_0,p_1,p_2,p_3)$ *is a smooth manifold of dimension* $\mu(p_0)+\mu(p_1)+\mu(p_2)-\mu(p_3)+1$.

We remark that the moduli space of each of the first two configurations are manifolds of dimension $\mu(p_0)+\mu(p_1)+\mu(p_2)-\mu(p_3)+1$. And the moduli of the third configurations is manifold of dimension $\mu(p_0)+\mu(p_1)+\mu(p_2)-\mu(p_3)$. The first two moduli spaces are patched together along the third one.

We also remark that in our moduli space we include the "moduli parameter" that is the length of the real line in  .

The proof of Proposition 1.11 is a standard application of the transversality argument.

Now we use this moduli space to define a map

$$\eta_3 : \quad C_k(f_0,f_1)\otimes C_\ell(f_1,f_2)\otimes C_m(f_2,f_3)\to C_{k+\ell+m+1}(f_0,f_3),$$

as follows :

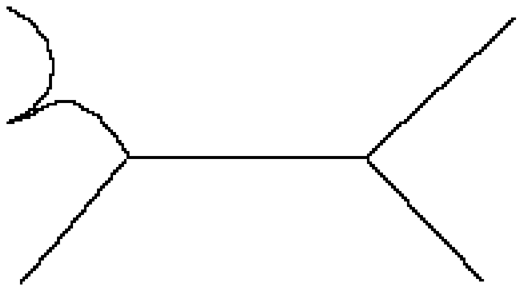
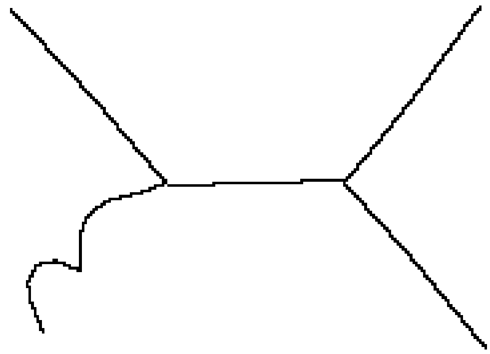
Definition 1.12 :

$$\begin{aligned}\eta_3([p_0]\otimes[p_1]\otimes[p_2]) &= \sum_{p_3}\eta_3(p_0,p_1,p_2,p_3)\cdot[p_3]\\ \eta_3(p_0,p_1,p_2,p_3) &= \#\mathcal{M}(p_0,p_1,p_2,p_3)\end{aligned}.$$

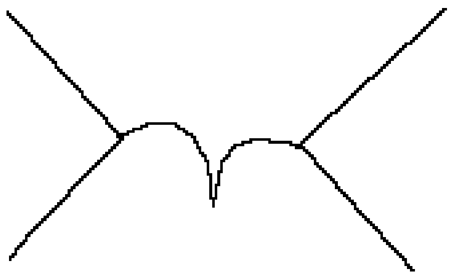
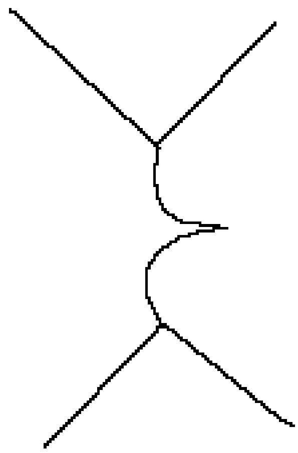
Proposition 1.13 :

$$\begin{aligned}&\partial(\eta_3(a\otimes b\otimes c))\pm\eta_3(\partial a\otimes b\otimes c)\pm\eta_3(a\otimes\partial b\otimes c)\pm\eta_3(a\otimes b\otimes\partial c)\\ &=\eta_2(\eta_2(a\otimes b)\otimes c)\pm\eta_2(a\otimes\eta_2(b\otimes c))\end{aligned}.$$

Sketch of the proof As in the previous section, we consider the case when $\dim\mathcal{M}(p_0,p_1,p_2,p_3)=1$ and study the boundary of this moduli space $\mathcal{M}(p_0,p_1,p_2,p_3)$. The boundary is described by one of the figures we describe below. There are 10 types of such a figures. First 8 figures are

and similar ones. (There are 6 more.) They correspond to the left hand side of the formula. There are two different types of figures namely :

The left figure above corresponds $\eta_2(\eta_2(a\otimes b)\otimes c)$ and the right figure above corresponds $\eta_2(a\otimes \eta_2(b\otimes c))$.

Thus we can complete the proof in a similar way as in the proof of Proposition 1.9.

Now we are in the position to discuss the output of our propositions.

(1) We can prove that the product defined by η_2 is associative. In fact suppose $u_0 \in C_*(f_0,f_1), u_1 \in C_*(f_1,f_2), u_2 \in C_*(f_2,f_3)$ such that $\partial u_0 = \partial u_1 = \partial u_2 = 0$. Then Proposition 1.13 implies that

$$\eta_2(\eta_2(u_0\otimes u_1)\otimes u_2) \pm \eta_2(u_0\otimes \eta_2(u_1\otimes u_2)) = \partial\eta_3(u_0\otimes u_1\otimes u_2).$$

The associativity (in homology level) follows.

(2) Secondly, we can define Massey product. Let us first recall the definition of Massey product using differential form. Let u, v, w be closed forms. We assume that the De-Rham cohomology classes of $u \wedge v$ and $v \wedge w$ are 0. Hence there exists x, y such that $dx = u \wedge v$ and $dy = v \wedge w$. We then consider the form $x \wedge w \pm u \wedge y$. We calculate

$$d(x \wedge w \pm u \wedge y) = dx \wedge w - u \wedge dy = 0.$$

We thus get a De-Rham cohomology class

$$[x \wedge w \pm u \wedge y] \in H^*_{DR}(M; \mathbf{R}).$$

This class depends on the choice of x, y. But its image in $\dfrac{H^*_{DR}(M;\mathbf{R})}{[u] \cup H^*_{DR}(M;\mathbf{R}) + H^*_{DR}(M;\mathbf{R}) \cup [w]}$ is well defined. We call this element the *Massey triple product* $\{u, v, w\}$.

Massey product and its higher analogue is used extensively in the works by Quillen and Sullivan etc. They constructed a theory which describe rational homotopy type using De-Rham theory. (Sullivan used forms over $\mathbf{Q}$.)

Now we define Morse theory version of Massey product using our maps η_2, η_3.

Let $u \in C_*(f_0, f_1)$, $v \in C_*(f_1, f_2)$, $w \in C_*(f_2, f_3)$ such that $\partial u = \partial v = \partial w = 0$. We assume

$$[\eta_2(u \otimes v)] = 0 \quad \text{in} \quad H_*(C_*(f_0, f_2))$$
$$[\eta_2(v \otimes w)] = 0 \quad \text{in} \quad H_*(C_*(f_1, f_3)).$$

Take $x \in C_*(f_0, f_2)$, $y \in C_*(f_1, f_3)$ such that $\partial x = \eta_2(u \otimes v)$, $\partial y = \eta_2(v \otimes w)$. Now we calculate

$$\begin{aligned}
&\partial(\eta_2(x \otimes w) \pm \eta_2(u \otimes y) \pm \eta_3(u \otimes v \otimes w)) \\
&\quad = \eta_2(\partial x \otimes w) \pm \eta_2(u \otimes \partial y) \pm \partial \eta_3(u \otimes v \otimes w) \\
&\quad = \eta_2(\eta_2(u \otimes v) \otimes w) \pm \eta_2(u \otimes \eta_2(v \otimes w)) \pm \partial \eta_3(u \otimes v \otimes w) \\
&\quad = 0.
\end{aligned}$$

by Proposition 1.13. We thus obtain the following :

Definition 1.14 :

$$\{[u],[v],[w]\} = [\eta_2(x \otimes w) \pm \eta_2(u \otimes y) \pm \eta_3(u \otimes v \otimes w)].$$

(This element is well defined modulo $[\eta_2(x \otimes \bullet)]$ and $[\eta_2(\bullet \otimes y)]$.)

4 A^∞-Category

One can continue and play the same game with arbitrary many functions. Then we get a complicated system of chain complexes and maps which satisfy some complicated relations. To understand what we obtained in that way, we need some notions to describe them.

In fact such a notion has been known for more than 30 years, in the study of loop spaces. Stasheff [St] defined a notion of A^∞-algebra, we slightly modify this notion and define :

Definition 1.15 : An A^∞-category consists of a set, (the set of objects), which we write Ob, the set of morphisms $C(a,b)$ for each $a,b \in Ob$, and a map η_k, the (higher) composition, such that

(1) $C_*(a,b)$ is a chain complex.

(2) For $a_0,\cdots,a_k \in Ob$

$$\eta_k : C_*(a_0,a_1) \otimes \cdots \otimes C_*(a_{k-1},a_k) \to C_*(a_0,a_k),$$

is a linear map. (We do not specify coefficient ring here. In this chapter it is $\boldsymbol{Z}$ and later it will be the Laurant polynomial ring $\boldsymbol{Z}[T]][[T^{-1}]]$ or $\boldsymbol{Z}[T^1]][[T]]$.)

They satisfy the following :

(3) η_2 is a chain map.

(4) $(\eta_3)(a\otimes b\otimes c) = \pm\eta_2(\eta_2(a\otimes b)\otimes c)\pm\eta_2(a\otimes\eta_2(b\otimes c))$. (Here and here after $\partial\eta_3 = \partial\circ\eta_3 \pm \eta_3\circ\partial$.)

(5) $$(\partial\eta_k)(x_1\otimes\cdots\otimes x_k) = \sum_{1\le i<j\le k} \pm\eta_{k-j+i}(x_1\otimes\cdots\otimes\eta_{j-i}(x_i\otimes\cdots x_j)\otimes\cdots\otimes x_k).$$

We remark that A^∞-category is not a category in usual sense, since $\eta_2(\eta_2(a\otimes b)\otimes c) = \eta_2(a\otimes\eta_2(b\otimes c))$ does *not* hold in general. But the way how it fails to hold is controlled by higher compositions η_k, $k\ge 3$.

Now let us give an example of A^∞-category, which goes back to 30 years ago when the notion A^∞-algebra was introduced.

Let X be an arbitrary topological space. For each $p,q\in X$, we put

$$\Omega(p,q) = \{\ell:[0,1]\to X \mid \ell(0)=p, \ell(1)=q\}.$$

There is an obvious map : $\Omega(p,q)\times\Omega(q,r)\to\Omega(p,r)$ defined by loop sum.

We define an A^∞-category ΩX as follows. Its object is a point of X. For $p,q\in X$, the set of morphisms $C_*(p,q)$ in our A^∞-category ΩX is the singular chain complex $S_*(\Omega(p,q))$. Then using the map : $\Omega(p,q)\times\Omega(q,r)\to\Omega(p,r)$, we get a chain map, $\eta_2: C_*(p,q)\otimes C_*(q,r)\to C_*(p,r)$.

Now we ask whether $\eta_2(\eta_2(a\otimes b)\otimes c) = \eta_2(a\otimes\eta_2(b\otimes c))$ hold, or not.

It holds almost, but not quite, since the loop sum $\Omega(p,q)\times\Omega(q,r)\to\Omega(p,r)$ is not associative but is homotopy associative. And the homotopy is quite canonical. Using them we get :

$$\eta_3: \quad C_k(p,q)\otimes C_\ell(q,r)\otimes C_m(r,s)\to C_{k+\ell+m+1}(p,s).$$

For detail see for example, [Ad].

Now we return to our discussion of Morse homotopy and define an A^∞-category.

Let M be a compact manifold and we take a Riemannian metric. Then our A^∞-category is as follows :

(1) Its object is a smooth function on M. Namely $Ob = C^\infty(M)$.

(2) The space $C_*(f_1, f_2)$ of morphisms between two elements $f_1, f_2 \in C^\infty(M) = Ob$ is the Witten complex $C_*(f_1, f_2) = C_*(M, f_1 - f_2)$ we discussed in § 1.

(3) The (higher) compositions are as we defined as in §§ 2 and 3.

The discussions in §§ 2,3 imply that they satisfy the axiom of A^∞-category. We will write it $\mathrm{Ms}(M)$.

We remark here that in $\mathrm{Ms}(M)$ the set of morphisms are not defined for all pair of objects but only a dense subset of it. Also the (higher) composition is defined only in a dense subset. One can define a notion of topological A^∞-category to analyses such a situation. We omit the detail.

Problem 1.16 : How much of the homotopy type of M is determined by $\mathrm{Ms}(M)$?

Using the relation of higher compositions and Massey product it might be possible to show that $\mathrm{Ms}(M)$ determines the rational homotopy type of M.

As we mentioned before, our A^∞-category is parallel to the construction of De-Rham homotopy by Sullivan etc ([Q],[S]) . One disadvantage of our approach is that it is more complicated than differential graded algebra used in De-Rham homotopy. But it has an advantage that is it works over integer.

Problem 1.17 : How can one involve Steenrod square in our story ?

In A^∞-category, the structures related to the asssociativity of cup product are involved but not the commutativity. To study this problem, one need to analyse how we can understand commutativity in our setting.

Remark : As was mentioned in introduction, this article is essentially an enriched version of author's lectures in Seoul national university. After Soul the author participated Geogia International Congress on Topology. There he got two informations related to this chapter, which we describe below.

(1) Marty Betz mentioned in Geogia that he did a similar work as this chapter independently. Moreover he said he found how to involve Steenrod square in the story. His

main idea is to use the automorphism of the graph.[3] These results will be the contents of his PHD Thesis. ([BC])

(2) V.A. Smirnov told the author that he had used the notion essentially the same as A^∞-Category and A^∞-functor in his paper [Sm1]. Also in his recent work related to stable homotopy group of spheres he found the following. ([Sm2]) Let us consider the stable homotopy category SH. Namely its object is the spectrum (X_i, φ_i) where X_i is the spaces and φ_i is a map from the suspension ΣX_i of X_i to X_{i+1}. The set of morphisms $C_*((X_i,\varphi_i),(Y_i,\psi_i))$ is a graded Abelian groups such that $C_k((X_i,\varphi_i),(Y_i,\psi_i)) = \operatorname{ind}\lim_{i\to\infty}\{X_{i+j},Y_i\}$. (Here $\{X,Y\}$ denotes the set of based homotopy class of maps from X to Y.) We regard them as chain complexes with trivial boundary operator. Then Smirnov constructed higher composition and proved that they satisfy the axiom of A^∞-category.

His construction suggests that it is natural to regard stable homotopy category as an A^∞-Category. Then every operators and secondary operators between generalized cohomology theories can be described in that way.

In the theory of Floer homology, its periodicity etc. is similar to the objects in stable homotopy category. Several people is trying to realize "Floer homotopy type" as an objects in stable homotopy category. (See [CJS].) In regard with the result of this article, it seems to the author that it is natural to find something like "quatized stable homotopy category" and realize Floer homotopy type there. In fact in later chapters we quantize the contents of this chapter.

[3] **On the other hand it seems that the moduli parameter is not included in his story.**

Chapter 2 Symplectic Floer theory

§ 1 Symplectic manifolds

First let us recall several basic facts on symplectic geometry.

Definition 2.1 : A *symplectic manifold* (X,ω) is a $2n$-dimensional manifold X with a 2-form ω on it such that

(1) $d\omega = 0$,

(2) $\omega^n \neq 0$, everywhere.

Definition 2.2 : An n-dimensional submanifold Λ^n of (X^{2n},ω) is called a Lagrangian submanifold of it if $\omega|_\Lambda$, the restriction of ω to Λ^n, vanishes.

Example 2.3 :

(1) $\mathbf{C}P^n$ has a natural symplectic structure. (Kähler form of Fubini-Study metric.) Every complex submanifold of $\mathbf{C}P^n$ has also symplectic structure.

(2) Let M be a manifold, then its cotangent bundle T^*M has a natural symplectic structure as we describe below.

Let $p \in M$, choose a coordinate $x^1,\cdots,x^n$ in a neighborhood U of p. Then an element of T^*U (the cotangent bundle of U) is given by (q,u) where $q \in U$, $u \in T_q^* M$. We put $u = \sum p^i\, dx^i$ and let $(x^1,\cdots,x^n)$ be the coordinate of q. Then $(x^1,\cdots,x^n,p^1,\cdots,p^n)$ will be the coordinate of (q,u). With respect to this coordinate system, our symplectic form ω is given as

$$\omega = \sum dp^i \wedge dx^i .$$

There is more intrinsic way to describe this symplectic structure. Let $(q,u) \in T^* M$. Let $\pi : T^*M \to M$ be the projection. Its differential gives $\pi_* : T_{(q,u)}(T^*M) \to T_q(M)$. This map is surjective and its kernel is identified to $T_q^* M$. Hence we have an exact sequence :

$$0 \to T_q^* M \to T_{(q,u)} T^* M \xrightarrow{\pi_*} T_q M \to 0$$

We defines one form θ on $T^* M$ by $\theta_{(q,u)}(V) = u(\pi_* V)$. ($V \in T_{(q,u)} T^* M$.) In our local coordinate $\theta = \sum p^i dx^i$. Then we have $\omega = d\theta$. (This second description implies that ω is globally well defined.)

One important notion in symplectic geometry is a Hamiltonian vector field which we will now describe.

Let (X^{2n}, ω) be a symplectic manifold and $f : X \to \mathbf{R}$ be a smooth function on it. Then the Hamiltonian vector field H_f is one which satisfy

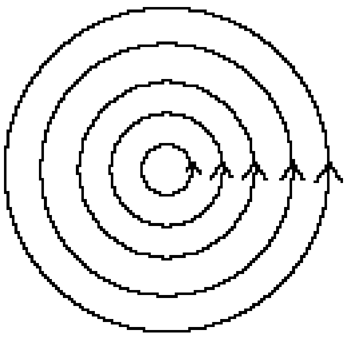
$$\omega(H_f, V) = df(V).$$

for arbitrary $V \in TX$.

If, for example, $(X, \omega) = (\mathbf{R}^{2n}, \sum dp^i \wedge dx^i)$, then

$$H_f = \sum \frac{\partial f}{\partial p^i} \frac{\partial}{\partial x^i} - \sum \frac{\partial f}{\partial x^i} \frac{\partial}{\partial p^i}.$$

If $n = 1$, and $f = x^2 + p^2$ then H_f is described by the following figure :



Now we consider the ordinary differential equation :

ǂ $$\frac{d\ell}{dt}(t) = H_f(\ell(t)).$$

We are interested in its periodic orbit. Namely the solution of ǂ which satisfies

$$\ast\ast \qquad \ell(t+1) = \ell(t),$$

in addition. Then we have :

Theorem 2.4 (Eliashberg) **:** *Let (X,ω) be a compact symplectic manifold and f be a Morse function. Then the number of solutions of $\ast$ satisfying $\ast\ast$ is not smaller than*

$$\sum \text{rank } H_i(X;\mathbf{R}).$$

Proof First we remark that if $p \in Cr(f)$ then the curve $\ell(t) \equiv p$ satisfies $\ast, \ast\ast$, since $H_f(p) = 0$. Hence we are only to show

$$\# Cr(f) \geq \sum \text{rank } H_i(X;\mathbf{R}).$$

(This is the classical Morse inequality.) To prove it, we recall the construction of the last section. There, we defined a chain complex $(C_*(X,f),\partial)$ such that

$$\# Cr(f) = \sum \text{rank } C_i(X,f)$$

and

$$H_k(X;\mathbf{Z}) \cong H_k(C_*(X,f)).$$

It follows immediately

$$\# Cr(f) \geq \sum \text{rank } H_i(X;\mathbf{R}),$$

as required.

Now we want to generalize our equation $\ast$ to one where f depends on t.

Namely, suppose we have a family of functions f_t such that $f_{t+1} = f_t$. We then consider the equation

✝' $$\frac{d\ell}{dt}(t) = H_{f_t}(\ell(t))$$

with condition ✝✝. Arnold [Ar] proposed the following :

Conjecture 2.5 : Let all the solutions of ✝' satisfying ✝✝ are nondegenerate[4]. Then the number of such solutions is not smaller than

$$\sum \operatorname{rank} H_i(X;\boldsymbol{R}).$$

This conjecture was proved by various mathematicians under various additional assumptions. One remarkable result is due to Floer, who proved it under the assumption that (X,ω) is monotone[5].

§2 Symplectic diffeomorphism

Before discussing Floer's result, let us change our point of view a bit. That is, let us count the number of fixed point of a symplectic diffeomorphisms in place of counting the number of periodic orbits.

Given a family of functions f_t, we define a map $\varphi: X \to X$ as follows. Let $p \in X$, we consider a curve $\ell:[0,1] \to X$ such that

$$\begin{cases} \dfrac{d\ell}{dt}(t) & = H_{f_t}(\ell(t)) \\ \ell(0) & = p \end{cases}.$$

There is always a unique solution of this equation. We put $\varphi(p) = \ell(1)$. Thus we get a diffeomorphism $\varphi: X \to X$.

One can easily verify that $\varphi^*\omega = \omega$. We say that φ is a symplectic diffeomorphism if $\varphi^*\omega = \omega$. We say that φ is an exact symplectic diffeomorphism if there exists f_t such that φ is obtained from it in the way we described above. Let $Diff(X,\omega)$ be the set of all symplectic diffeomorphisms and $Diff_0(X,\omega)$ the set of all exact symplectic diffeomorphisms.

[4]This notion will be explained later.

[5]The definition of it will be explained later.

Namely

$$Diff(X,\omega) = \left\{\varphi : X \to X \,\middle|\, \varphi^*\omega = \omega\right\}$$
$$Diff_0(X,\omega) = \left\{\varphi : X \to X \,\middle|\, \exists f_t\ \varphi(p) = \ell(1) \text{ where } \ell \text{ satisfies } \ast. \right\}$$

In fact, $Diff_0(X,\omega)$ is the connected component of $Diff(X,\omega)$ if $H_1(X;\mathbf{R}) \cong 0$. Then Arnold conjecture is restated as follows.

Conjecture 2.6 : Let φ be an exact symplectic diffeomorphism of X to itself. Assume that every fixed point of φ is nondegenerate[6]. Then the number of fixed point of φ is not smaller than $\sum$ rank $H_i(X;\mathbf{R})$.

Now we rewrite this conjecture a bit more using the notion of Lagrangian submanifold. (Definition 2.2)

We consider the manifold $Y = X \times X$ with symplectic form $\omega \oplus -\omega$. Then for each symplectic diffeomorphism φ its graph $\Lambda_\varphi = \{(x,\varphi(x)) \mid x \in X\}$ is a Lagrangian submanifold. In particular the diagonal $\Delta = \{(x,x) \mid x \in X\}$ is a Lagrangian submanifold.

We remark that the set of fixed point of φ is identified to the intersection $\Delta \cap \Lambda_\varphi$. We define that a fixed point p of φ is *nondegenerate* if Λ_φ and Δ are transversal to each other at (p,p).

Arnold conjecture is generalized as follows :

Conjecture 2.7 : Let Λ be a Lagrangian submanifold in a symplectic manifold (Y,ω) and $\varphi : Y \to Y$ be an exact symplectic diffeomorphism. Suppose that $\varphi(\Lambda)$ is transversal to Λ. Then

$$\#(\varphi(\Lambda) \cap \Lambda) \geq \sum \text{rank } H_i(\Lambda;\mathbf{R}).$$

§3 Floer homology for Lagrangian intersection

Now we are going to explain the ideas employed by Floer to prove this conjecture under additional assumptions. In this section we consider the case when $\pi_1(\Lambda) = \pi_2(Y) = 1$. (This is the case which Floer studied in [Fl1]. Later he dealed with more general case.

[6] The definition wil be given soon.

([Fl3])) In fact this assumption is rather restrictive. But the basic idea appeared in a most simply way in this case. In next chapter we consider what kind of troubles arise when we study more general case and how one can handle those troubles.

The idea employed by Floer to this problem is Morse theory in infinite dimension. Let us take two Lagrangian submanifolds Λ_1, Λ_2 of Y and assume that $\pi_1(\Lambda_1) = \pi_1(\Lambda_2) = \pi_2(Y) = 1$. We study the following infinite dimensional manifold.

$$\Omega(\Lambda_1, \Lambda_2; Y) = \{\ell : [0,1] \to Y \mid \ell(0) \in \Lambda_1, \ell(1) \in \Lambda_2\}.$$

(Here we do not mention how much differentiability of ℓ we assume. As usual, to develop necessary analysis to justify this kinds of arguments, one needs to fix a function space. But we do not try to do it here.)

We define a function σ on it as follows. First we define $\sigma(\ell_0, \ell_1)$ for each two elements ℓ_0, ℓ_1 of $\Omega(\Lambda_1, \Lambda_2; Y)$. For this purpose, we first take a path ℓ_t, $t \in [0,1]$ in $\Omega(\Lambda_1, \Lambda_2; Y)$ joining ℓ_0 and ℓ_1.[7] Now we put

$$\sigma(\ell_0, \ell_1) = \int_{[0,1]\times[0,1]} \ell^* \omega \ .$$

Here $\ell : [0,1] \times [0,1] \to Y$ is defined by $\ell(s,t) = \ell_s(t)$. Now we prove

Lemma 2.8 : *$\sigma(\ell_0, \ell_1)$ is independent of the choice on the homotopy ℓ_t, $t \in [0,1]$ joining ℓ_0 and ℓ_1.*

Proof Let ℓ'_t be the another homotopy. Then patching ℓ_t and ℓ'_t we find a map $h : [0,1] \times S^1 \to Y$ such that

$$\int_{[0,1]\times S^1} h^* \omega = \int_{[0,1]\times[0,1]} \ell^* \omega - \int_{[0,1]\times[0,1]} \ell'^* \omega \ .$$

and that

[7] Since we do not assume that $\pi_1(Y) = 1$, the space $\Omega(\Lambda_1, \Lambda_2; Y)$ may not be connected. So, in fact, we consider the function on a component of it, which contains a constant path.

$$h(\{0\} \times S^1) \subset \Lambda_1$$
$$h(\{1\} \times S^1) \subset \Lambda_2.$$

Since $\pi_1(\Lambda_1) = \pi_1(\Lambda_2) = 1$, we can find disks $D_i \subset \Lambda_i$ such that $\partial D_i = h(\{i\} \times S^1)$. We can patch disks $D_i \subset \Lambda_i$ with $h: [0,1] \times S^1 \to Y$ and obtain $\hat{h}: S^2 \to Y$. Since Λ_i are Lagrangian it follows that $\int_{D_i} \omega = 0$. Therefore

$$\int_{[0,1]\times S^1} h^*\omega = \int_{S^2} \hat{h}^*\omega .$$

On the other hand, since $\pi_2(Y) = 1$, there exists D^3 which bounds $\hat{h}(S^2)$. Hence by Stokes' theorem, we have :

$$\int_{S^2} \hat{h}^*\omega = \int_{\partial D^3} \hat{h}^*\omega = \int_{D^3} d(\hat{h}^*\omega) = 0 .$$

Thus we conclude

$$0 = \int_{[0,1]\times[0,1]} \ell^*\omega - \int_{[0,1]\times[0,1]} \ell'^*\omega ,$$

as required.

It is easy to verify

$$\sigma(\ell_0,\ell_1) + \sigma(\ell_1,\ell_2) = \sigma(\ell_0,\ell_2).$$

Thus we can define $\sigma : \Omega(\Lambda_1,\Lambda_2;Y) \to \boldsymbol{R}$ by $\sigma(\ell) = \sigma(\ell_0,\ell)$. Where ℓ_0 is a point of $\Omega(\Lambda_1,\Lambda_2;Y)$. ($\sigma$ is independent of ℓ_0 up to constant.)

We want to discuss Morse theory of this function on $\Omega(\Lambda_1,\Lambda_2;Y)$, an infinite dimensional manifold.

To follow the argument of the last chapter, we need to study the gradient vector field of this function. So let us calculate it. For this purpose we need a Riemannian metric on $\Omega(\Lambda_1,\Lambda_2;Y)$, which we now define.

First consider a linear map $J: TY \to TY$. We say that J is an almost complex structure if $J^2 = -1$. (Given such J, the real vector bundle TY will be a complex one such that $J = \times\sqrt{-1}$.)

Definition 2.9 : The symplectic structure ω is said to be compatible with an almost complex structure J if

(1) $\omega(X, J(X)) \geq 0$, the equality holds only if $X = 0$.

(2) $\omega(JX, JX) = \omega(X, X)$.

It is known that each symplectic structure has compatible almost complex structure and the space of compatible almost complex structures are connected. ([Gr])

Now, given a symplectic manifold (Y, ω) and a compatible almost complex structure J, we define a Riemannian metric g on Y by

$$g(V, W) = \omega(V, JW).$$

We have

$$g(W, V) = \omega(W, JV) = \omega(JW, JJV) = -\omega(JW, V) = \omega(V, JW) = g(V, W),$$

by Condition (2). Similarly g is nondegenerate by Condition (1). Then g is a Riemannian metric on M.

In fact, three structures on Y, symplectic structure ω, almost complex structure J, and Riemannian metric g are closely related to each other in this situation and each two of them determine the third structure. If the almost complex structure J is integrable then the metric g is a Kähler metric.

Now we use such structures g, J to define Riemannian metric on $\Omega(\Lambda_1, \Lambda_2; Y)$. We remark that

$$T_{\ell}(\Omega(\Lambda_1, \Lambda_2; Y)) = \Gamma(\ell^* TY).$$

For $U, V \in \Gamma(\ell^* TY)$, we define metric by

$$\langle U, V \rangle = \int_0^1 g(U(t), V(t))\, dt.$$

Lemma 2.10 : *The gradient vector* grad σ *is given by*

$$\mathrm{grad}_\ell \sigma = J\left(\frac{d\ell}{dt}\right) \in \Gamma(\ell^* TY).$$

Proof : We calculate

$$\begin{aligned}
\left.\frac{d\sigma(\ell_s)}{ds}\right|_{s=0} &= \frac{d}{ds}\int_0^s ds \int_0^1 dt \quad \omega\left(\ell_*\left(\frac{\partial}{\partial s}\right), \ell_*\left(\frac{\partial}{\partial t}\right)\right)\Bigg|_{s=0} \\
&= \int_0^1 dt \; g\left(\ell_*\left(\frac{\partial}{\partial s}\right), J\ell_*\left(\frac{\partial}{\partial t}\right)\right)\Bigg|_{s=0} \\
&= \left\langle \ell_*\left(\frac{\partial}{\partial s}\right), J\ell_*\left(\frac{\partial}{\partial t}\right)\right\rangle\Bigg|_{s=0}
\end{aligned}$$

Here we put $\ell(s,t) = \ell_s(t)$. We recall that $\ell_*\left(\frac{\partial}{\partial s}\right)$ is the tangent vector at $s = 0$ of the path ℓ_s in $\Omega(\Lambda_1, \Lambda_2; Y)$. The lemma follows immediately.

Thus we find the gradient vector field. The lemma immediately implies the following :

Corollary 2.11 : $\ell \in \Omega(\Lambda_1, \Lambda_2; Y)$ *is a critical point of* σ *only if* $\ell =$ constant.

In fact, $\mathrm{grad}_\ell \sigma = 0$ implies $0 = \frac{d\ell}{dt}$.

When $\ell \in \Omega(\Lambda_1, \Lambda_2; Y)$ is constant we have $\ell(0) = \ell(1) \in \Lambda_1 \cap \Lambda_2$. Therefore, we have

$$Cr(\sigma) \cong \Lambda_1 \cap \Lambda_2.$$

In this way, we can find a relation between $\Lambda_1 \cap \Lambda_2$ and the critical point of σ, (which is related to the topology of $\Omega(\Lambda_1, \Lambda_2; Y)$).

We next remark that for $p \in \Lambda_1 \cap \Lambda_2$, the element $\ell_p \equiv p$ of $\Omega(\Lambda_1, \Lambda_2; Y)$ is a nondegenerate critical point of σ if and only if Λ_1 and Λ_2 are transversal at p.

Now we study the moduli space of gradient lines of σ. We put $E = \boldsymbol{R} \times [0,1]$ and regard it as a subset of complex plain $\boldsymbol{C}$ with complex coordinate $z = s + \sqrt{-1}t$. (In other words $\dfrac{\partial}{\partial t} = J\dfrac{\partial}{\partial s}$.) Now let ℓ_s $s \in \boldsymbol{R}$ be a path in $\Omega(\Lambda_1,\Lambda_2;Y)$. We put $h(s,t) = \ell_s(t)$.

Lemma 2.12 : *h is a holomorphic map : $E \to Y$ if and only if ℓ_s is a gradient line of σ. In other words*

$$h_* J = J h_* \quad \Leftrightarrow \quad \frac{\partial \ell}{\partial s} = -\ \mathrm{grad}_{\ell_s} \sigma \ .$$

This lemma is also an immediate consequence of Lemma 2.10.

The study of symplectic manifold using holomorphic map from complex plain is initiated by Gromov. We discuss a part of it in next chapter.

We put , for each $p,q \in \Lambda_1 \cap \Lambda_2$,

$$\begin{aligned} \mathcal{M}_{\mathrm{symp}}(p,q;\Lambda_1,\Lambda_2) &= \left\{ \ell_s : \boldsymbol{R} \to \Omega(\Lambda_1,\Lambda_2;Y) \,\middle|\, \frac{\partial \ell}{\partial s} = -\ \mathrm{grad}_{\ell_s}\sigma \right\} \\ &= \left\{ h : E \to Y \,|\, h_* J = J h_* \right\} \end{aligned}$$

We write $\mathcal{M}_{\mathrm{symp}}(p,q)$ in case no confusion can occur.

Theorem 2.13 (Floer) **:** *Suppose* $\pi_1(\Lambda_1) = \pi_1(\Lambda_2) = \pi_2(Y) = 1$ *and* Λ_1 *is transversal to* Λ_2. *Then there exists a map* $\mu : \Lambda_1 \cap \Lambda_2 \to \boldsymbol{Z}$, *such that, for each* $p,q \in \Lambda_1 \cap \Lambda_2$, *the space* $\mathcal{M}_{\mathrm{symp}}(p,q)$ *is a smooth manifold of dimension* $\mu(p) - \mu(q)$.

(In fact Theorem 2.13 holds only after appropriate perturbation. We omit the discussion about it.)

This theorem is parallel to Chapter 1 Lemma 1.1. But one essential difference is that the Morse index in usual sense, (that is the number of negative eigenvalues of the Hessian), is infinite in our situation. Hence the map $\mu : \Lambda_1 \cap \Lambda_2 \to \boldsymbol{Z}$ is a "renormalized" Morse index, namely usual Morse index minus $\infty/2$.

We take an equivalence relation , $h(s,t) \sim h(s+s_0,t)$ on $\mathcal{M}_{\mathrm{symp}}(p,q)$ and let $\overline{\mathcal{M}}_{\mathrm{symp}}(p,q)$ be the quotient space with respect to this action. We can also prove the following analogy of Lemma 1.6.

Theorem 2.14 (Floer) **:** *In the situation of* Theorem 2.13, *we can compactify*

$\overline{\mathcal{M}}_{\mathrm{symp}}(p,q)$ *such that*

$$\partial \overline{\mathcal{M}}_{\mathrm{symp}}(p,q) = \bigsqcup_r \overline{\mathcal{M}}_{\mathrm{symp}}(p,r) \times \overline{\mathcal{M}}_{\mathrm{symp}}(r,q).$$

Now we define Floer complex of Lagrangian intersection as follows.

Definition 2.15 :

(1)

$$CF_k(\Lambda_1,\Lambda_2) = \bigoplus_{\substack{p\in\Lambda_1\cap\Lambda_2 \\ \eta(p)=k}} \mathbf{Z}\cdot[p],$$

(2) $\partial : CF_k(\Lambda_1,\Lambda_2) \to CF_{k-1}(\Lambda_1,\Lambda_2)$ is defined by

$$\partial[p] = \sum_{\eta(q)} <\partial p, q>[q],$$

$$<\partial p,q> = \#\overline{\mathcal{M}}_{\mathrm{symp}}(p,q). \quad \text{(counted with sign.)}$$

Then we can use Theorems 2.13, 2.14 in the same way as the proof of Theorem 1.5 Chapter 1 to prove that $\partial\partial = 0$. So we define Floer homology of the Lagrangian intersection by

$$HF_k(\Lambda_1,\Lambda_2) = H_k(CF_*(\Lambda_1,\Lambda_2)).$$

Floer proved that it is independent of the choice of almost structure J, perturbation etc. and is an invariant of symplectic manifold Y and its Lagrangians.

§4 Arnold conjecture and Floer homology

Now we want to discuss the way Floer homology is applied to study Arnold conjecture. Let Y be a symplectic manifold and $\Lambda \subset Y$ is a Lagrangian submanifold. Let $\varphi : Y \to Y$ be an exact symplectic diffeomorphism. We assume that $\pi_1(\Lambda) = \pi_2(Y) = 1$ and Λ is transversal to $\varphi(\Lambda)$. In this case, Conjecture 2.7 asserts $\#(\varphi(\Lambda)\cap\Lambda) \ge \sum \operatorname{rank} H_i(\Lambda;\mathbf{R})$, which we want to prove. (In fact our assumption $\pi_1(\Lambda) = \pi_2(Y) = 1$ is quite restrictive and hence our case does not imply nontrivial result to Conjecture 2.5 but does have an application to 2.7.)

We remark that

$$\sum_{k} \text{rank } CF_k(\Lambda_1, \Lambda_2) = \#(\Lambda_1 \cap \Lambda_2).$$

Hence to show $\#(\varphi(\Lambda) \cap \Lambda) \geq \sum \text{rank } H_i(\Lambda; \boldsymbol{R})$ one only needs to prove

(2.16) $$H_k(\Lambda; \boldsymbol{Z}) \cong HF_k(\Lambda, \varphi(\Lambda)),$$

in our situation. Floer proved this in the following two steps.

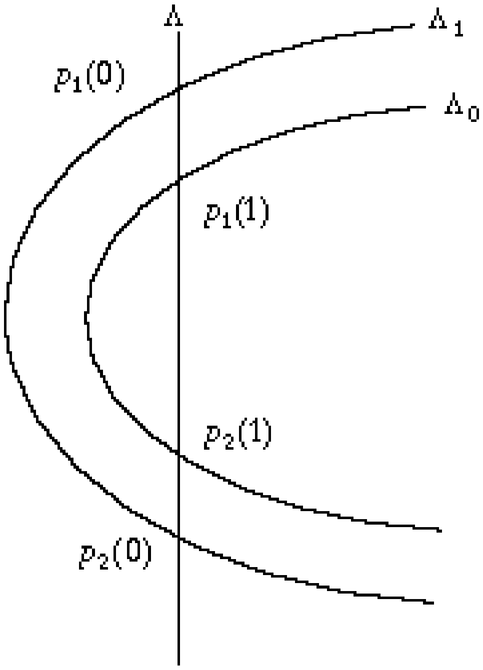
(A) If φ is C^2-close to identity then (2.16) holds.

(B) Let $\varphi_t : Y \to Y$ be a one parameter family of exact diffeomorphisms. Assume that $\varphi_t(\Lambda)$ is transversal to Λ for $t = 0,1$. Then,

$$HF_k(\Lambda, \varphi_0(\Lambda)) \cong HF_k(\Lambda, \varphi_1(\Lambda)).$$

<u>Sketch of Proof of (B)</u>

We put $\Lambda_t = \varphi_t(\Lambda)$. We first assume that Λ_t is transversal to Λ for arbitrary $t \in [0,1]$. Then $\#(\Lambda_t \cap \Lambda)$ is independent of $t \in [0,1]$ and there exists smooth maps $p_i : [0,1] \to Y$ $i = 1, \cdots, k$ such that $\Lambda_t \cap \Lambda = \{p_1(t), \cdots, p_k(t)\}$.



We consider the space

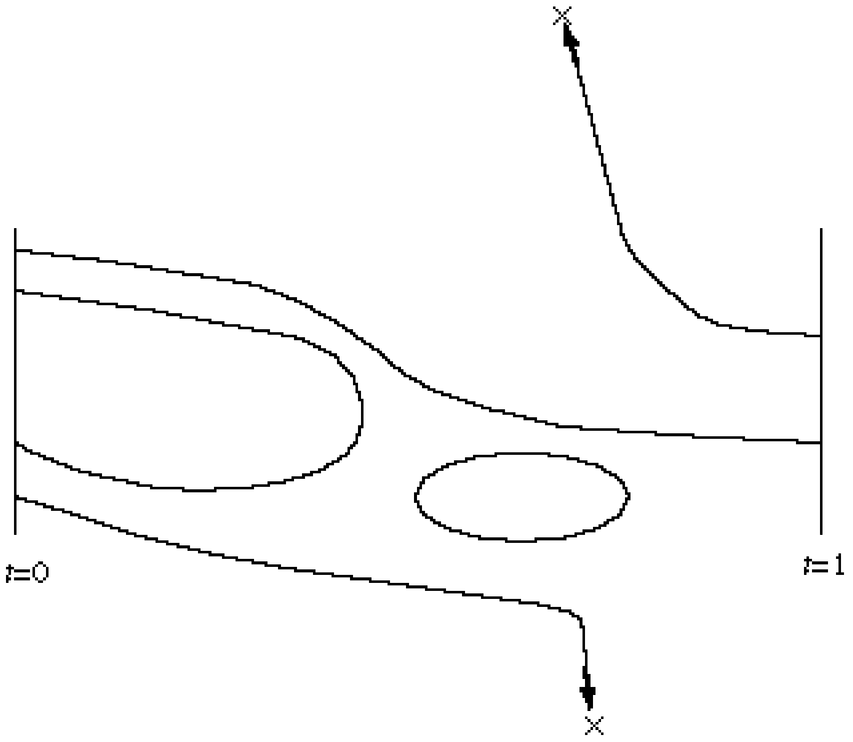
$$\mathcal{M}(i,j) = \bigcup_{t \in [0,1]} \overline{\mathcal{M}}_{\text{symp}}(p_i(t), p_j(t); \Lambda, \Lambda_t).$$

Lemma 2.17 : *The space $\mathcal{M}(i,j)$ is a manifold of dimension $\eta(p_i) - \eta(p_j)$. It can be compactified such that*

$$\begin{aligned}\partial\mathcal{M}(i,j) &= \bigcup_{t\in[0,1]}\bigcup_{k} \overline{\mathcal{M}}_{\text{symp}}(p_i(t), p_k(t); \Lambda, \Lambda_t) \times \overline{\mathcal{M}}_{\text{symp}}(p_k(t), p_j(t); \Lambda, \Lambda_t) \\ &\quad \cup \overline{\mathcal{M}}_{\text{symp}}(p_i(0), p_j(0); \Lambda, \Lambda_0) \cup -\overline{\mathcal{M}}_{\text{symp}}(p_i(1), p_j(1); \Lambda, \Lambda_1).\end{aligned}$$

We omit the proof. Let us consider the case when $\eta(p_i) - \eta(p_j) = 1$. Then by Lemma 2.17 we have

$$\begin{aligned}&\#\overline{\mathcal{M}}_{\text{symp}}(p_i(0), p_j(0); \Lambda, \Lambda_0) - \#\overline{\mathcal{M}}_{\text{symp}}(p_i(1), p_j(1); \Lambda, \Lambda_1) \\ &= \#\bigcup_{t\in[0,1]}\bigcup_{k} \overline{\mathcal{M}}_{\text{symp}}(p_i(t), p_k(t); \Lambda, \Lambda_t) \times \overline{\mathcal{M}}_{\text{symp}}(p_k(t), p_j(t); \Lambda, \Lambda_t).\end{aligned} \tag{2.18}$$



× = (point in $\overline{\mathcal{M}}_{\text{symp}}(p_i(t), p_k(t);\Lambda,\Lambda_t)\times\overline{\mathcal{M}}_{\text{symp}}(p_k(t), p_j(t);\Lambda,\Lambda_t)$)

Let us remark that the virtual dimension of

$$\overline{\mathcal{M}}_{\text{symp}}(p_i(t), p_k(t);\Lambda,\Lambda_t)\times\overline{\mathcal{M}}_{\text{symp}}(p_k(t), p_j(t);\Lambda,\Lambda_t)$$

is

$$(\mu(p_i)-\mu(p_k)-1)+(\mu(p_k)-\mu(p_j)-1)=-1.$$

(We remark that $\eta(p_n(t))$ is independent of t. Hence we write it $\eta(p_n)$.)

It follows that $\overline{\mathcal{M}}_{\text{symp}}(p_i(t), p_k(t);\Lambda,\Lambda_t)\times\overline{\mathcal{M}}_{\text{symp}}(p_k(t), p_j(t);\Lambda,\Lambda_t)$ is empty for generic t. But there may be finitely many t for which it is not empty but is a finitely many point. In other words, 1-dimensional family of -1-dimensional spaces is 0-dimensional. Thus the right hand side of (2.18) makes sense.

So one needs to study the moduli space $\bigcup_{t\in[0,1]}\overline{\mathcal{M}}_{\text{symp}}(p_n(t), p_m(t);\Lambda,\Lambda_t)$ in case $\mu(p_n)-\mu(p_m)=0$. By subdividing the interval if necessary, we may assume that there is a number $t_0 \in [0,1]$ and a pair $n \neq m$ such that if $\mu(p_i) - \mu(p_j) = 0$ then the set $\overline{\mathcal{M}}_{\text{symp}}(p_i(t), p_j(t); \Lambda, \Lambda_t)$ is nonempty only if $t = t_0$ and $(i,j) = (m,n)$. We may assume also $\# \overline{\mathcal{M}}_{\text{symp}}(p_m(t), p_n(t); \Lambda, \Lambda_{t_0}) = 1$.[8] Hence (2.18) implies that for i,j with $\mu(p_i) - \mu(p_j) = 1$ we have:

$$
\text{(2.19)} \quad
\begin{aligned}
<\partial p_i(0), p_j(0)> &= <\partial p_i(1), p_j(1)> \qquad \text{if } i \neq n,\ j \neq m, \\
<\partial p_n(0), p_j(0)> &= <\partial p_n(1), p_j(1)> \pm <\partial p_m(1), p_j(1)>, \\
<\partial p_i(0), p_m(0)> &= <\partial p_i(1), p_m(1)> \pm <\partial p_i(0), p_n(0)>.
\end{aligned}
$$

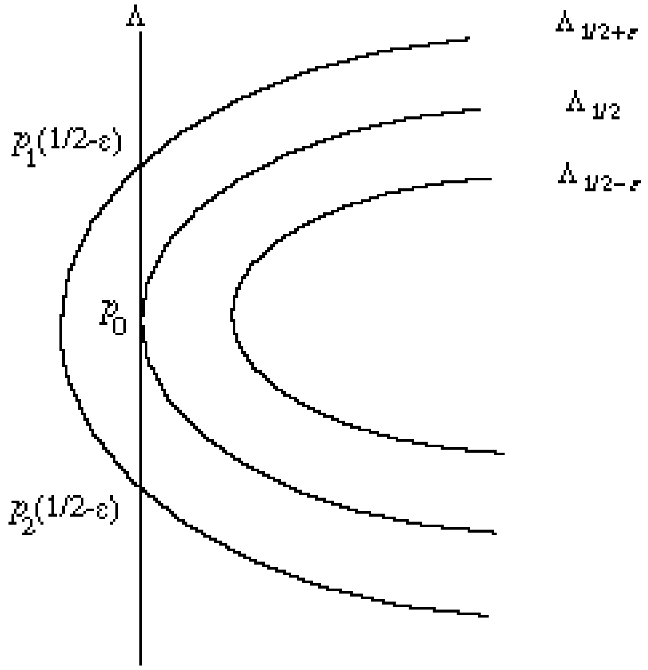
We define $I: CF_*(\Lambda, \Lambda_0) \to CF_*(\Lambda, \Lambda_1)$ by

$$
\begin{aligned}
I([p_n(0)]) &= [p_n(1)] \pm [p_m(1)], \\
I([p_i(0)]) &= [p_i(1)] \qquad \text{if } i \neq n.
\end{aligned}
$$

Then (2.19) implies that I is a chain map and it gives chain homotopy equivalence between $CF_*(\Lambda, \Lambda_0)$ and $CF_*(\Lambda, \Lambda_1)$.

Next we consider the case when Λ_t is not necessary transversal to Λ at some t. By subdividing the interval $[0,1]$ and perturbing the family Λ_t we may assume that Λ_t is transversal to Λ for $t \neq 1/2$, and that for $t = 1/2$ Λ_t is transversal to Λ outside one point say p_0. We may also assume that there exists $p_i(t): [0,1] \to Y$ $i = 3, 4, \cdots, k$ and $p_i(t): [0,1/2) \to Y$ for $i = 1,2$ such that $\Lambda \cap \Lambda_t = \{p_1(t), \cdots, p_k(t)\}$ for $t \in [1,1/2)$ and $\Lambda \cap \Lambda_t = \{p_3(t), \cdots, p_k(t)\}$ for $t \in (1/2, 1]$. (See figure below.)

[8] **By taking a perturbation if necessary.**

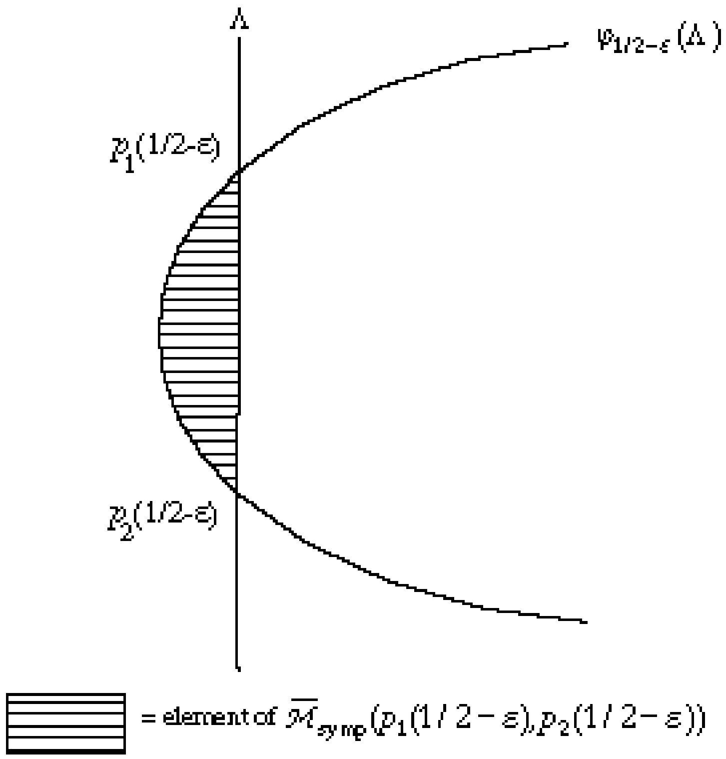


Then to show that $HF_*(\Lambda,\Lambda_{1/2-\varepsilon}) \cong HF_*(\Lambda,\Lambda_{1/2+\varepsilon})$, we are only to show the following lemma. (In fact the lemma implies that $p_1(1/2-\varepsilon)$ and $p_2(1/2-\varepsilon)$ cancels.)

Lemma 2.21 :

$$< \partial p_i(1/2-\varepsilon), p_j(1/2-\varepsilon) > \quad = \quad < \partial p_i(1/2+\varepsilon), p_j(1/2+\varepsilon) > \qquad i, j > 3$$
$$< \partial p_1(1/2-\varepsilon), p_2(1/2-\varepsilon) > \quad = \quad 1.$$

The first formula in Lemma 2.21 follows immediately from a lemma similar to 2.17. The number 1, in the second formula corresponds to the following element of $\mathcal{M}_{\text{symp}}(p_1(1/2-\varepsilon), p_2(1/2-\varepsilon))$.



We omit the detail.

Sketch of Proof of (A)

Let Λ_1, Λ_2 be two Lagrangian submanifolds which are C^2-close to each other. We want to prove

(2.21) $$HF_*(\Lambda_1, \Lambda_2) \cong H_*(\Lambda_1; Z).$$

Let us take the cotangent bundle $T^*\Lambda_1$ of Λ_1. As we explained at the beginning of this chapter, $T^*\Lambda_1$ has a canonical symplectic structure. We identify Λ_1 to the zero section of $T^*\Lambda_1$. Given a Riemannian metric on Λ_1, we define an almost complex structure on $T^*\Lambda_1$ as follows. The Riemannian metric on Λ_1 induces an isomorphism $I: T\Lambda_1 \to T^*\Lambda_1$. We have $T_{(q,u)}(T^*\Lambda_1) = T_q^*\Lambda_1 \oplus T_q\Lambda_1$. Then we put $J(X, Y) = (I(Y), -I^{-1}(X))$. One can prove the following :

Lemma 2.22 : *There exists a neighborhood U of Λ_1 in $T^*\Lambda_1$ and V of Λ_1 in Y*

such that (U,Λ_1) *is symplectic diffeomorphic to* (V,Λ_1). *The symplectic diffeomorphism preserves almost complex structures at* Λ_1.

We may assume that Λ_2 is contained in V. Hence we may consider that Λ_2 is a submanifold of $T^*\Lambda_1$. Also, since the tangent space of Λ_2 is close to one of Λ_1, it follows that Λ_2 is transversal to the fibre of $T^*\Lambda_1$. Therefore there exists a 1-form u on Λ_1 such that Λ_2 is a graph of u. Here we use the following :

Lemma 2.23 : *Let* M *be a manifold and* u *be a 1-form on it. Then the graph of* u *is a Lagrangian submanifold of* T^*M *if and only if* $du = 0$.

The proof is an easy calculation. Now we use the assumption $H^1(\Lambda_1;\boldsymbol{R}) = 0$ to show that there exists a function $f:\Lambda_1 \to \boldsymbol{R}$ such that $u = df$. We consider the Witten complex we discussed at the beginning of Chapter 1. Here we write the moduli space of gradient line as $\mathcal{M}_{\text{Morse}}(p,q)$. ($p,q \in Cr(f)$), in order to distinguish them from the moduli space of pseudo-holomorphic maps. We remark that $\Lambda_1 \cap \Lambda_2 = \{x \in \Lambda_1 \mid du(x) = 0\} = Cr(f)$. Hence to prove (2.21) we are only to show the following :

Theorem 2.24 : *If* Λ_1 *is* C^1*-close to* Λ_2 *then* $\mathcal{M}_{\text{Morse}}(p,q)$ *is homeomorphic to* $\mathcal{M}_{\text{symp}}(p,q)$ *for each* $p,q \in Cr(f)$.

Sketch of the proof.

Let $\ell:\boldsymbol{R} \to \Lambda_1$ be a curve. We define $h_\ell: E \to T^*\Lambda_1$ by

$$h_\ell(s,t) = s \cdot df(\ell(t)).$$

(We recall $E = [0,1] \times \boldsymbol{R}$.) Then we have

Lemma 2.25 : $h_* J = J h_*$ *at* $\{0\} \times \boldsymbol{R}$ *if and only if* $\dfrac{d\ell}{dt} = -\text{grad } f$.

Proof Let $X \in T_{\ell(t)}\Lambda_1 \subset T_{(\ell(t),0)}(T^*\Lambda_1)$ and $Y \in T^*_{\ell(t)}\Lambda_1 \subset T_{(\ell(t),0)}(T^*\Lambda_1)$. (We recall $T_{(\ell(t),0)}(T^*\Lambda_1) = T^*_{\ell(t)}\Lambda_1 \oplus T_{\ell(t)}\Lambda_1$.) Then we calculate at $s = 0$

$$\begin{aligned}\left\langle \frac{\partial h_\ell}{\partial s}, X \oplus Y \right\rangle &= \langle df(\ell(t)), Y \rangle \\ &= g(Y, \text{grad } f) \\ &= -\omega(Y, J(\text{grad } f)).\end{aligned}$$

By the definition of the almost complex structure on cotangent bundle we have

$$-\omega(X, J(\text{grad } f)) = -I^{-1}(X)(\text{grad } f).$$

On the other hand, we have

$$\left\langle J\left(\frac{\partial h}{\partial t}\right), X \oplus Y \right\rangle = \left\langle \frac{\partial h}{\partial t}, (-I(Y), I^{-1}(X)) \right\rangle = I^{-1}(X)\left(\frac{d\ell}{dt}\right).$$

The lemma follows.

At (s,t), $s \neq 0$, the equivalence $hJ = Jh \Leftrightarrow \dfrac{d\ell}{dt} = -\text{grad } f$ does not hold strictly. But this formula gives a first approximation.

So, given an element ℓ of $\mathcal{M}_{Morse}(p,q)$, a gradient line of f joining p and q, we find that h_ℓ is an approximate solution of $h_* J = J h_*$. Hence using implicit function theorem, one can find an element of $\mathcal{M}_{\text{symp}}(p,q)$ in its neighborhood. Thus we can construct a map $\mathcal{M}_{\text{Morse}}(p,q) \to \mathcal{M}_{\text{symp}}(p,q)$. By reversing this procedure we can prove that it is a diffeomorphism. The details are omitted.

This is an outline of the proof by Floer of Arnold conjecture with additional assumptions.

Chapter 3 Pseudo holomorphic curve, and topological σ- model

§1 Two troubles in the definition of the Floer homology

In the last chapter, we discussed the definition of the Floer homology of Lagrangian intersection under the assumption $\pi_1(\Lambda) = \pi_2(X) = 1$. Here X is a symplectic manifold and Λ is a Lagrangian submanifold of it. We discuss first what happens if we remove these two assumptions. We show two examples.

Example 3.1 :

We first consider $X = T^2$ the two dimensional torus. We consider circles in it as Lagrangians. (Figure 3.2)

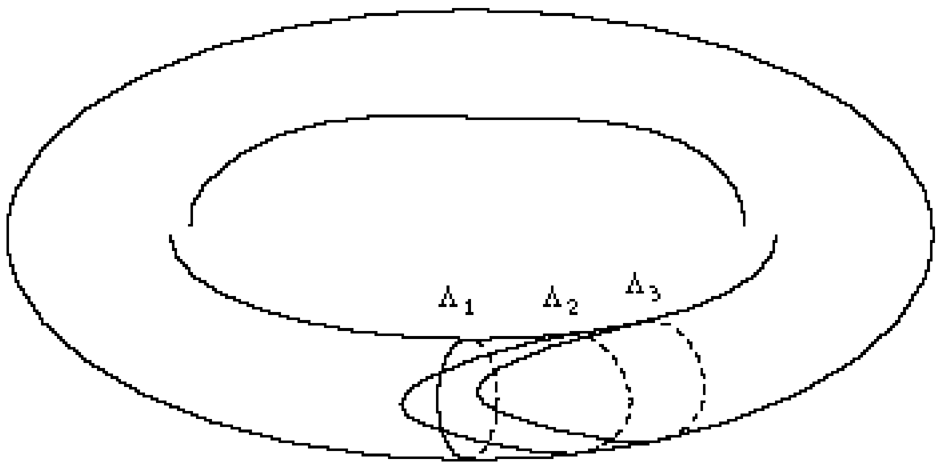


Figure 3.2

The two circles Λ_1 and Λ_2 intersect to each other at two points p, q and $\Lambda_1 \cap \Lambda_3 = \emptyset$. Then we recall Lemma 2.21 in last section and may imagine that $\partial[p] = [q]$. It would then follow that $HF(\Lambda_1, \Lambda_2) \cong HF(\Lambda_1, \Lambda_3)$. But, this is not correct. In fact, one finds that $\partial[p] = 0$. Because there are two elements of $\overline{\mathcal{M}}(p,q)$ as in Figure 3.3.

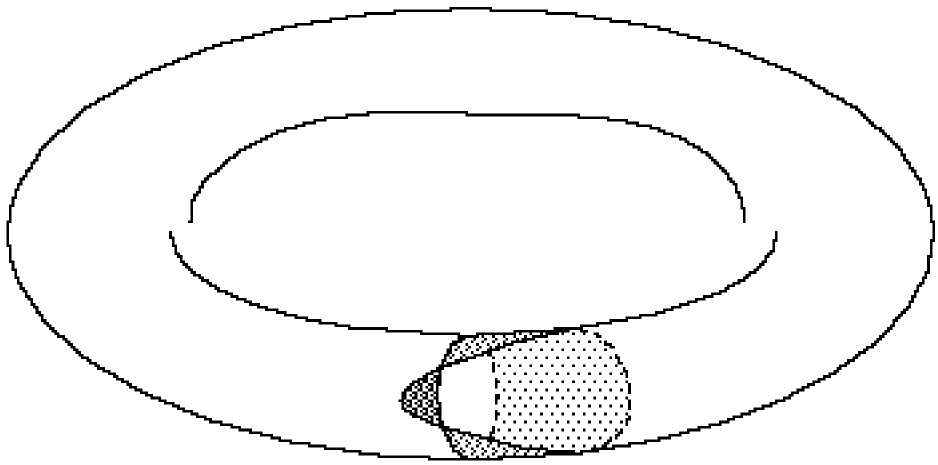


Figure 3.3

Thus we have $HF(\Lambda_1,\Lambda_2) \neq HF(\Lambda_1,\Lambda_3)$. Namely Floer homology is not an invariant of the deformation of the Lagrangian in this case.

Let us look at this example in a bit more detail. In case when p and q are sufficiently close to each other, we see that one of the element of $\overline{\mathcal{M}}(p,q)$ has very small volume but the symplectic area of the other element is not so small.

If we examine the proof of Lemma 2.21, one finds that the proof is roughly speaking to show that there is only one element in $\overline{\mathcal{M}}(p,q)$ which has very small symplectic area. In the case when $\pi_1(\Lambda) = \pi_2(X) = 1$ we find that the symplectic area of the element of $\overline{\mathcal{M}}(p,q)$ is independent of the element and depends only on p and q. (See Lemma 2.8.) $\langle \partial p, q \rangle = 1$ follows. But in our case when the assumption $\pi_1(\Lambda) = 1$ is not satisfied, this is not the case. The proof of Lemma 2.8 does not work and in fact the boundary curve of the two element of $\overline{\mathcal{M}}(p,q)$ is not homotopic to each other. Thus we found an example where the Floer homology is not a deformation invariant.

Example 3.4 :

We next consider the case when $X = S^2 = \mathbf{C} \cup \{\infty\}$ and the Lagrangians are circles. (See Figure 3.5.) In this case there are at least two elements A, B of $\overline{\mathcal{M}}(p,q)$ which are shown in Figure 3.5.

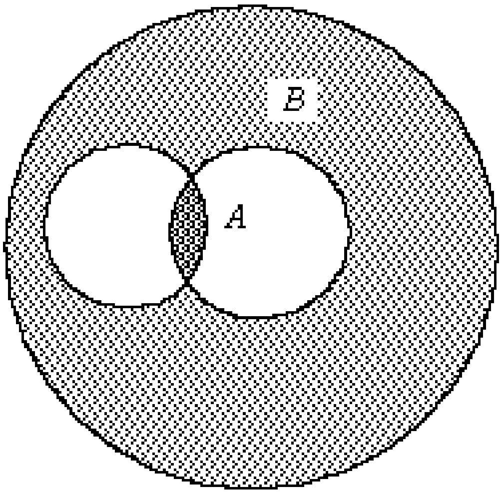


Figure 3.5

(These two elements do not have the same volume since our Lagrangians are not simply connected. But this point is not of our main interest here.)

The point we want to discuss here is that there are infinitely many components of $\overline{\mathcal{M}}(p,q)$ whose dimensions depend on the component. So let us find another element of $\overline{\mathcal{M}}(p,q)$. Let us consider a holomorphic map $z \mapsto z^2$. We choose an element φ of $PSL(2;\mathbf{C}) = \mathrm{Auto}(\mathbf{C}P^1)$ such that $\varphi(-1)^2 = p$, $\varphi(1)^2 = q$ and $\varphi(0),\varphi(\infty) \in A$. We put $f(z) = \varphi(z)^2$. Then $f^{-1}(A)$ is connected and is an annulus and $f^{-1}(B)$ has two components $f^{-1}(B)_1, f^{-1}(B)_2$ each of which is diffeomorphic to a disk. (See Figure 3.6.) The complement of $f^{-1}(A)$ in $\mathbf{C}P^1$ consists of 2 component. Let D_1, D_2 be those component containing $f^{-1}(B)_1$, $f^{-1}(B)_2$, respectively.

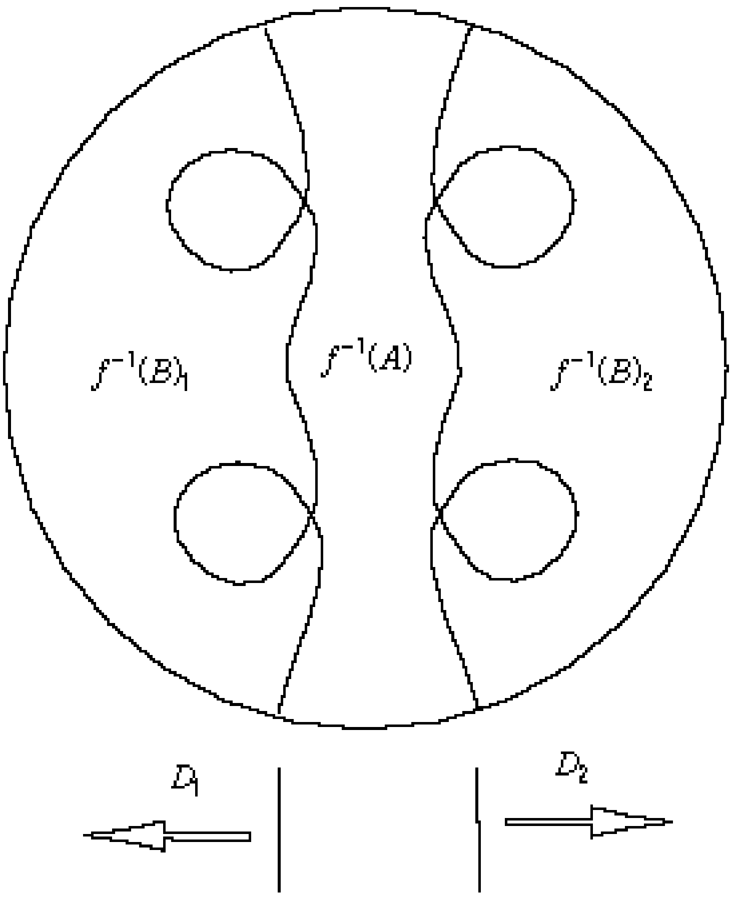


Figure 3.6

We take an element ψ of $PSL(2;\mathbf{C})$ such that $\psi(D^2) = D_1 \cup f^{-1}(A)$, $\psi(\pm 1) = \pm 1$. (Here we put $D^2 = \{z \in \mathbf{C} \mid |z| < 1\}$.) Now we set $h = f \circ \psi : D^2 \to S^2$. Then h is a holomorphic map and belongs to $\overline{\mathcal{M}}(p,q)$.

We will study the component of $\overline{\mathcal{M}}(p,q)$ containing h. We perturb the map $z \mapsto z^2$, to, for example, $z \mapsto z^2 + az + b$, then we can perform the same construction to obtain an element of $\overline{\mathcal{M}}(p,q)$. Thus the component of $\overline{\mathcal{M}}(p,q)$ containing h contains a set which is diffeomorphic to the neighborhood of $z \mapsto z^2$ in the moduli space of rational map of degree 2. Here we define degree of the rational map as follows.

Definition 3.7 : Let $\Phi : \mathbf{C}P^1 \to \mathbf{C}P^1$ be a holomorphic map. We say that Φ is a *rational map of degree* k, if there exists a polynomials P, Q such that $\Phi(z) = \dfrac{P(z)}{Q(z)}$, $\max\{\deg P, \deg Q\} = k$ and that P and Q are prime to each other.

We remark that the real dimension of the moduli space of rational maps of degree k is $4k+2$, since in general such an element are written as

$$\frac{a_0 z^k + \cdots + a_{k-1} z + a_k}{b_0 z^k + \cdots + b_{k-1} z + b_k},$$

and two such elements are equal if and only if $(a_0, \cdots, b_k) = \lambda(a'_0, \cdots, b'_k)$.

The construction we described above gives an element of $\overline{\mathcal{M}}(p,q)$ to each rational map Φ close to $z \to z^k$. The two rational maps Φ and Φ' gives the same element of $\overline{\mathcal{M}}(p,q)$ if there exists $\varphi \in PSL(2;C)$ such that $\Phi' = \Phi \circ \varphi$. Therefore the elements of $\overline{\mathcal{M}}(p,q)$ constructed from rational map of degree k consists of $4k-4$ dimensional family.

Thus we find that $\overline{\mathcal{M}}(p,q)$ has infinitely many components each of which has different dimension. Hence the definition of the boundary operator has a lot of trouble. This phenomenon is related to the fact that $\pi_2(S^2) \neq 0$. In fact each of the components of $\overline{\mathcal{M}}(p,q)$ corresponds to a homotopy class of the map.

We thus discussed two main troubles which arises when we drop the assumption $\pi_1(\Lambda) = \pi_2(X) = 1$. They are related to the study of moduli space of pseudo holomorphic maps. So, in this chapter, we are going to discuss those points.

§2 Pseudo holomorphic sphere in symplectic manifold

In this section, we discuss the case of holomorphic map from 2-sphere. For a symplectic manifold X with almost complex structure J, we put

$$\mathcal{M}(X,J) = \{h: S^2 \to X \mid Jh_* = h_* J\}. \tag{3.8}$$

In order to imitate the argument in Chapter 1, the basic facts we need for this moduli space is its compactification and the formula to give its dimension.

We first discuss its dimension. For this purpose we study the linearization of the equation $Jh_* = h_* J$, which is given as follows. Let $h_t: S^2 \to X$ be a family of maps such that $h_{t*}J = Jh_{t*}$, and $h_0 = h$. Then the differential $\frac{dh_t}{dt}$ can be regarded as a section of $h^* TX$. The bundle $h^* TX$ is a complex vector bundle. By differentiating the equation

$h_{t*}J = Jh_{t*}$, we find that $\dfrac{dh_t}{dt}$ is a holomorphic section of h^*TX. We calculate the dimension of $O(h^*TX)$, the set of holomorphic sections of h^*TX, using Rieman-Roch theorem and obtain the dimension of the moduli space of $\mathcal{M}(X,J)$. But in case when $\pi_2(X) \neq 0$, the dimension depends on the component. So we put

$$\mathcal{M}_k(X,J) = \left\{ h \in \mathcal{M}(X,J) \,\middle|\, h^*(c^1(TX,J)) \cap [S^2] = k \right\}.$$

Then we get

Theorem 3.9 (Gromov) **:**

$$\dim_{\mathbf{R}} \mathcal{M}_k(X,J) = 2\dim_{\mathbf{C}} X + 2k.$$

In place of proving Theorem 2.2 let us verify it in the simplest case, that is $X = \mathbf{C}P^1$. In this case $\mathcal{M}_k(\mathbf{C}P^1,J)$ is the moduli space of rational maps of degree $k/2$. (Recall $c^1(T\mathbf{C}P^1) \cap [\mathbf{C}P^1] = 2$.) Thus as we discussed before $\dim_{\mathbf{R}} \mathcal{M}_k(\mathbf{C}P^1,J) = 2k+2$, as asserted in Theorem 3.9.

Before discussing the compactification of the moduli space we want to remark that there is an action of $\mathrm{Aut}(\mathbf{C}P^1) = PSL(2;\mathbf{C})$ on $\mathcal{M}_k(\mathbf{C}P^1,J)$. Namely by $\varphi \cdot h = h \circ \varphi$. Let $\overline{\mathcal{M}}_k(\mathbf{C}P^1,J)$ be the quotient space. Hence, in case the action is free, we have

$$\dim_{\mathbf{R}} \overline{\mathcal{M}}_k(X,J) = 2\dim_{\mathbf{C}} X + 2k - 6.$$

It is natural to consider $\overline{\mathcal{M}}_k(\mathbf{C}P^1,J)$ rather than $\mathcal{M}_k(\mathbf{C}P^1,J)$. But there may be a trouble since the action is not necessary free. In that case the moduli space $\overline{\mathcal{M}}_k(\mathbf{C}P^1,J)$ is singular. We will discuss this point a bit more later.

We now discuss the compactification of the moduli space $\overline{\mathcal{M}}_k(\mathbf{C}P^1,J)$. The basic results on it are established by Gromov [Gr]. His result is an analogy of the results by Sacks-Uhlenbeck [SU] on Harmonic maps. First we have :

Theorem 3.10 : *Let $h_i \in \mathcal{M}(X,J)$ be a sequence of pseudo holomorphic spheres such that $\int_{S^2} h_i^*\omega$ is bounded. Then there exists a subsequences (which we denote by the same symbol h_i) and a finitely many points $p_1,\cdots,p_k$ in $\mathbf{C}P^1$ such that the restriction of h_i to*

$CP^1 - \{p_1, \cdots, p_k\}$ converges in C^∞-topology.

Let us recall the case when $X = CP^1$. In this case $\int_{S^2} h_i^* \omega = \frac{1}{2}\int_{S^2} h_i^* c^1(TX, J)$. Hence by its boundedness we may assume that h_i is a rational map of degree k for some number k independent of i. We recall that the set of rational map of degree k is identified to a subset of $CP^{2k-1} = \{[a_0, \cdots, a_k, b_0, \cdots, b_k] \mid a_i, b_i \in C\}$, where $[a_0, \cdots, a_k, b_0, \cdots, b_k]$ corresponds to

$$z \mapsto \frac{a_0 z^k + \cdots + a_{k-1} z + a_k}{b_0 z^k + \cdots + b_{k-1} z + b_k}.$$

Now we regard $h_i \in CP^{2k-1}$. Then it has a convergent subsequence in CP^{2k-1}. Let $[a_0, \cdots, a_k, b_0, \cdots, b_k]$ be its limit in CP^{2k-1}. If $P = a_0 z^k + \cdots + a_{k-1} z + a_k$ and $Q = b_0 z^k + \cdots + b_{k-1} z + b_k$ are prime to each other then $[a_0, \cdots, a_k, b_0, \cdots, b_k]$ represents an element of $\mathcal{M}(X, J)$. In this case we can prove that h_i converges in $\mathcal{M}(X, J)$ to $\frac{P(z)}{Q(z)}$ in C^∞-topology. Hence the conclusion of the Theorem 3.10 holds for $k = 0$.

We next suppose that $P = \prod_{i=1}^{\ell} (z - \rho_i) \cdot P'$, $Q = \prod_{i=1}^{\ell} (z - \rho_i) \cdot Q'$ and that P' and Q' are prime to each other. Then we can prove easily that, on $S^2 - \{\rho_1, \cdots, \rho_\ell\}$, h_i converges to $\frac{P'(z)}{Q'(z)}$ in C^∞-topology. Thus Theorem 3.10 holds in this case also. We remark that in the later case the limit $\frac{P'(z)}{Q'(z)}$ does not belong to the same component as h_i. (Namely $\frac{P'(z)}{Q'(z)} \in \mathcal{M}_{k-\ell}(X, J)$ while $h_i \in \mathcal{M}_k(X, J)$.) We next state the following :

Theorem 3.11 : *Let $h: S^2 - \{p_1, \cdots, p_k\} \to (X, \omega, J)$ be a pseudo holomorphic map such that $\int_{S^2 - \{p_1, \cdots, p_k\}} h^* \omega$ is finite. Then h can be extended to a pseudo holomorphic map from S^2.*

We remark that in case when $X = CP^1$, this theorem is Picard's theorem. Theorem 3.11 can be proved in essentially the same way as a Picard's theorem namely by using Schwartz inequality.

Theorem 3.11 implies in particular that the limit of h_i in Theorem 3.10 can be extended to an element of $\mathcal{M}(X, J)$.

Let us emphasis here the role played by the assumption of the finiteness of symplectic volume in Theorems 3.10 and 3.11. This is the basic point and the point where we can not work with an almost complex manifold but a symplectic manifold.

Now let h_i be as in Theorem 3.10. We assume that it converges to h_∞ on $CP^1 - \{p_1, \cdots, p_k\}$. We are trying to find what happens in neighborhoods of p_i. We may assume that $p_1 = 0 \in C \subset CP^1$. By Rieman's removable singularity theorem, h_i converges also to h_∞ at 0 if $|h_i|$ is bounded. Hence we may assume that $T_i = |h_i|$ converges to infinity. We put $\hat{h}_i(z) = h_i(z/T_i)$. (We have $\hat{h}_i'(0) = 1$.) By applying Theorem 3.10 we obtain a subsequence such that $\hat{h}_i$ converges to $\hat{h}_\infty$ in C^∞-topology on $CP^1 - \{q_1, \cdots, q_\ell\}$. ($0 \neq q_i$) We recall $T_i \to \infty$. Hence $\hat{h}_\infty$ is nothing with h_∞. The situation in case when $k = 0, \ell = 0$ is illustrated in Figure 3.12 below. (In this case $q_1 = \infty$.)

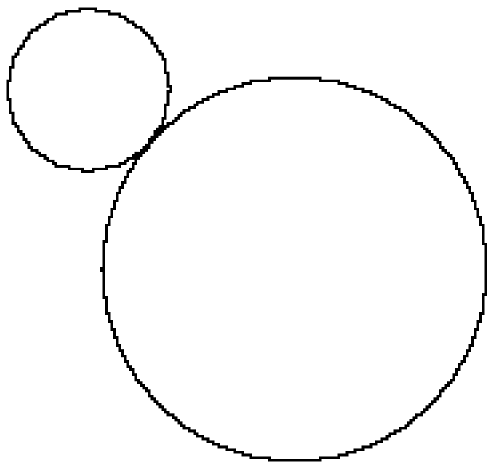


Figure 3.12

In general we can repeat the construction above and can find a finite number of pseudo holomorphic spheres as in Figure 3.13 to which the sequence h_i converges.

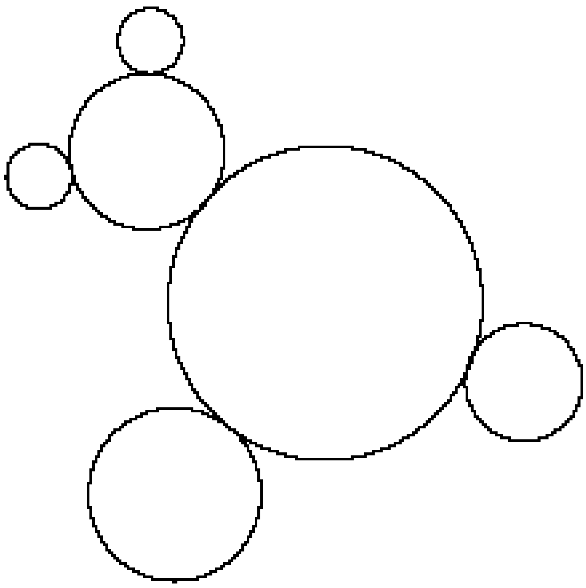


Figure 3.13

We state it precisely a bit later. Before doing it let us discuss the case when $X = \mathbf{C}P^1$.
We put, for example

$$h_i(z) = \frac{z(z+1)}{(z+1/i)}.$$

Outside $p_1 = 0$, our sequence of function h_i converges in C^∞-topology to $h_\infty(z) = z + 1$. To study the behavior of h_i in the neighborhood of $p_1 = 0$, let us follows the discussion above. We then get $T_i = h_i'(0) = i$. Therefore

$$\hat{h}_i(z) = \frac{z(z/i+1)}{(z+1)}.$$

Hence $\hat{h}_i$ converges to $\hat{h}_\theta(z) = \dfrac{z}{(z+1)}$ outside $q_1 = \infty$. Thus, in this example, a sequence h_i of maps of degree two rational converges to the union of two rational maps of degree 1.

Now we are going to describe a compactification of $\mathcal{M}(X,J)$.

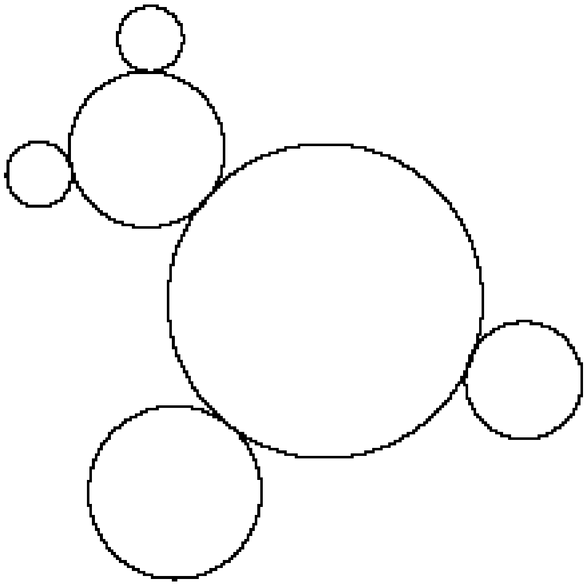
We say that $\Sigma = \bigcup S_i$ is a *cusp curve of genus* 0 if

(1) $S_i \cong \mathbf{C}P^1$,

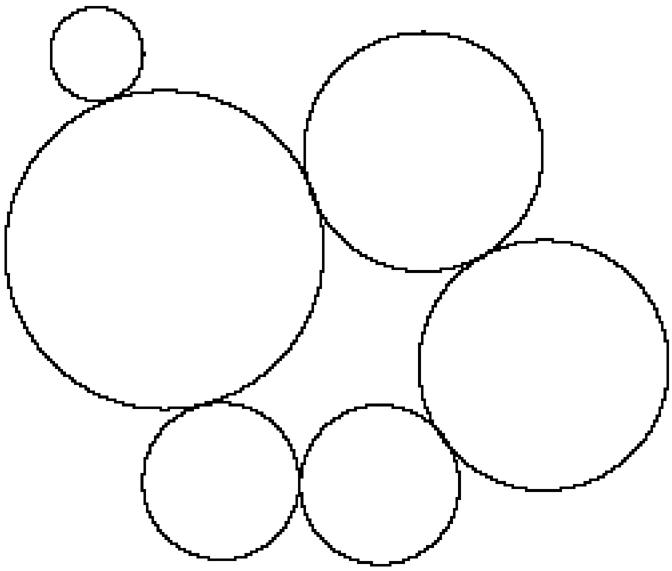
(2) $\#(S_i \cap S_j) \le 1$,

(3) Σ is connected and simply connected.

Namely



is a cusp curve of genus 0 but



is not a cusp curve of genus 0. (Since it is not simply connected.)

Let $\Sigma = \bigcup S_i$ be a cusp curve of genus 0. We say a map $h: \Sigma \to (X, \omega, J)$ to be a pseudo holomorphic map if

(1) h is continuous,

(2) the restriction of h to each S_i is pseudo holomorphic.

Now we state Gromov-Ruan's result on the compactification of $\mathcal{M}(X,J)$.

Theorem 3.14 : *Let* $h_i \in \mathcal{M}(X,J)$ *be as in* Theorem 3.10. *Then there exists* $\Sigma = \bigcup S_i$, *and cusp curve of genus* 0, $h : \Sigma \to X$ *an pseudo holomorphic map, and a subsequence* h_{i_k}, *such that*

$$\lim_{k\to\infty} h_{i_k}(\mathbf{C}P^1) = h(\Sigma).$$

There are various ways to state the precise meaning of the convergence $\lim_{k\to\infty} h_{i_k}(\mathbf{C}P^1) = h(\Sigma)$. But we do not try to do it here. We also need arguments about perturbation and transversality, which is also omitted.

§3 Topological σ–model and Gromov-Ruan invariant

To apply the argument of § 2 to generalize the discussion of Chapter 2, we need to consider its generalization to the case of pseudo holomorphic maps from a disk $D^2 = \{z \in \mathbf{C} \mid |z| < 1\}$. We are going to discuss it in next section. In this section we describe a construction of symplectic invariant using the result of the last section. This result is due to Y. Ruan [R].

First let us consider the moduli space of the configurations of n-points in $\mathbf{C}P^1$. Namely we put

$$\mathcal{T}_{n,0} = \left\{(z_1,\cdots,z_n) \,\middle|\, z_i \in \mathbf{C}P^1, z_i \neq z_j\right\}.$$

The group $PSL(2;\mathbf{C}) = \mathrm{Aut}(\mathbf{C}P^1)$ acts on this space in an obvious way. Then we consider the space

$$\mathcal{M}(X,J) \times_{SL(2;\mathbf{C})} \mathcal{T}_{n,0} = \frac{\mathcal{M}(X,J) \times \mathcal{T}_{n,0}}{PSL(2;\mathbf{C})}.$$

We define

$$ev : \mathcal{M}(X,J) \times_{SL(2;\mathbf{C})} \mathcal{T}_{n,0} \to X^n,$$

by

$$ev(h,(z_1,\cdots,z_n)) = (h(z_1),\cdots,h(z_n)).$$

We calculate the dimension as

$$\dim \mathcal{M}_k(X,J) \times_{PSL(2;\mathbf{C})} \mathcal{T}_{n,0} = 2k + 2\dim X + 2n - 6.$$

There is a natural projection $\pi : \mathcal{M}_k(X,J) \times_{SL(2;\mathbf{C})} \mathcal{T}_{n,0} \to \bar{\mathcal{T}}_{n,0}$. Here we put $\bar{\mathcal{T}}_{n,0} = \mathcal{T}_{n,0}/PSL(2;\mathbf{C})$. Thus if we forget all the troubles which may occur, we will get an element

$$ev_*\left(PD\left(\pi^*([\bar{\mathcal{T}}_{n,0}])\right)\right) = H_{2k+2\dim X}(X^n;\mathbf{Z}). \tag{3.15}$$

This coincides to what physicists calls topological sigma model of genus 0, and is established rigorously in a mathematical sense by Ruan, (based on Gromov's results on pseudo holomorphic curves). (This invariant is one Ruan defined in Theorem B of [R].)

Let us describe this invariant in a bit more precise way. We first need to specify the homology class $u \in H_2(X;\mathbf{Z})$ such that $u \cap c^1(TX) = k$. We put

$$\mathcal{M}_u(X,J) = \left\{ h \in \mathcal{M}_k(X,J) \,\middle|\, h_*[S^2] = u \right\}.$$

For simplicity we consider the case when n, the number of points we take on $\mathbf{C}P^1$, is three. (In fact one can prove that the invariant for general n is determined by the case $n = 3$.) In this case we rewrite the definition (3.15) as follows. We remark that $PSL(2;\mathbf{C})$ acts freely on $\mathcal{T}_{3,0}$ and the quotient $\bar{\mathcal{T}}_{3,0} = \mathcal{T}_{3,0}/PSL(2;\mathbf{C})$ is exactly one point. Thus we can identify $\mathcal{M}_u(X,J) \times_{PSL(2;\mathbf{C})} \mathcal{T}_{3,0} = \mathcal{M}_u(X,J)$. Now we take three cohomology classes $\alpha_i \in H^{\ell_i}(X;\mathbf{Z})$, $i = 1,2,3$. Choose cycles $Z_i \subseteq X$ representing Poincaré dual to α_i. Then Z_i is a subspace of codimension ≥ 2 singularity and of dimension $\dim X - \ell_i$. We choose ℓ_i and k such that $k + \ell_1 + \ell_2 + \ell_3 = \dim X$. Then $\dim \mathcal{M}_u(X,J) + \dim(Z_1 \times Z_2 \times Z_3) = \dim X^3$. Hence if we assume transversality we can define

$$\bar{\Phi}_X(\alpha_1,\alpha_2,\alpha_3) = [\{(h(0),h(1),h(\infty)) \mid h \in \mathcal{M}_u(X,J)\}] \cap [Z_1 \times Z_2 \times Z_3] \in \mathbf{Z}. \tag{3.16}$$

(Here we take three points $0,1,\infty$. This choice is quite arbitrary and any choice will give the same result.)

To justify Formula 3.16 we meet two troubles. One of them is the singularity of $\overline{\mathcal{M}_u(X,J)}$, the other is its compactification.

First we consider the problem of the singularity. We embed the space $\overline{\mathcal{M}_u(X,J)}$ to $Map_k(X)=\left\{ f: \mathbf{C}P^1 \to X \;\middle|\; f^* c^1(TX)\cap [\mathbf{C}P^1]=k \right\}$. On $Map_k(X)$ there is an action of the group $\mathrm{Diff}(\mathbf{C}P^1)$, given by $(\varphi, f)\mapsto f\circ\varphi$. The singularity occurs at the fixed point of this action. Namely if $f\in\mathcal{M}_u(X,J)$ satisfies $f\circ\varphi=f$ for some nontrivial element $\varphi\in\mathrm{Diff}(\mathbf{C}P^1)$. Then φ is automatically an element of $PSL(2;\mathbf{C})$. Hence $f\circ\varphi=f$ is satisfied if there exists $\pi:\mathbf{C}P^1\to\mathbf{C}P^1$ such that $f=\bar{f}\circ\pi$ for some $\bar{f}:\mathbf{C}P^1\to X$. In other words it occurs in the case when the curve $f(\mathbf{C}P^1)$ is reduced. The trouble caused by the singularity is that the naive dimension counting argument do not work there. This problem combined with the second problem causes serious trouble. We will describe it a bit later.

Secondly we discuss the problem about the compactification. We mentioned it already and we found that in the limit the elements of $\overline{\mathcal{M}_u(X,J)}$ may split into cusp curve of genus one. Roughly speaking we can define an invariant like (3.13) if the boundary appear is codimension ≥ 2. First we give quite naive argument which in fact is not correct. We then point out some troubles.

Let us consider a sequence of elements f_i of $\mathcal{M}_u(X,J)$. We remark that the symplectic volume of elements of $\mathcal{M}_u(X,J)$ depends only on our cohomology class u. Hence symplectic volume of f_i is bounded. Therefore, we can apply Theorem 3.14 and may assume that f_i converges to a cusp curve of genus 0. For simplicity, we assume that this sequence diverges and the limit in the sense of Theorem 3.14 is $f:\Sigma\to X$ where Σ is a union of two S^2's attached at one point. (The case when f_i converges to the map from the union of more than 2 spheres can be handled similarly.) In the limit, our three points $0,1,\infty=p_1,p_2,p_3$ will be situated on S_1^2 or S_2^2. Essentially there are two cases namely $p_1,p_2\in S_1^2$, $p_3\in S_2^2$ or $p_1,p_2,p_3\in S_1^2$.

Let $f_{(i)}$ be the restriction of f to S_i^2. We put $k_i=\int_{S_i^2} f_{(i)}^*(\omega)$. We have $k=k_1+k_2$.

Now we consider the first case, our space $(\Sigma,(p_1,p_2,p_3))$ is :

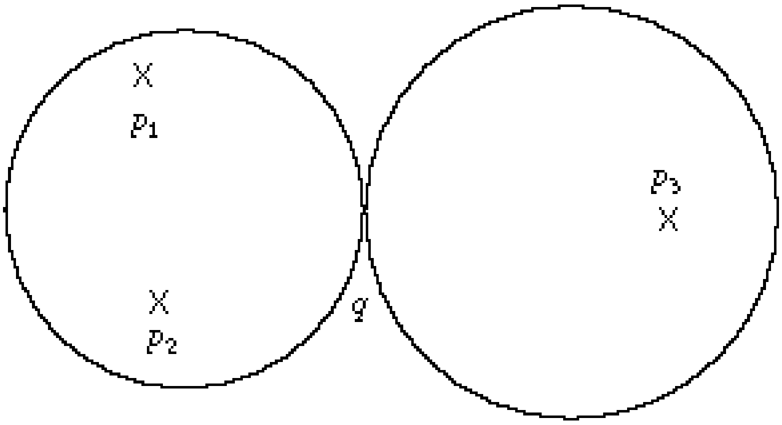


Figure 3.17

The biholomorphic automorphism of this figure is $\mathbf{C}^*$ which acts on S_2^2. The moduli space of holomorphic structure of this figure is 0-dimensional. The set pseudo holomorphic maps from S_1^2 consists of $2k_1 + \dim X$ -dimensional family, while the pseudo holomorphic maps from S_2^2 consists of $2k_2 + \dim X$-dimensional family. There is one more constraint that is $f_{(1)}(q) = f_{(2)}(q)$. Thus we have $2k_1 + 2k_2 + \dim X = 2k + \dim X$ dimensional family of moduli spaces. After dividing the $\mathbf{C}^*$ action we have $2k + \dim X - 2$ dimensional submanifold in X^3. Since the space $\mathcal{M}_k(X, J)$ is $2k + \dim X$ dimensional our boundary corresponding to Figure 3.15 do not affect the well definedness of 3.14.

We next consider the case when $p_1, p_2, p_3 \in S_1^2$.

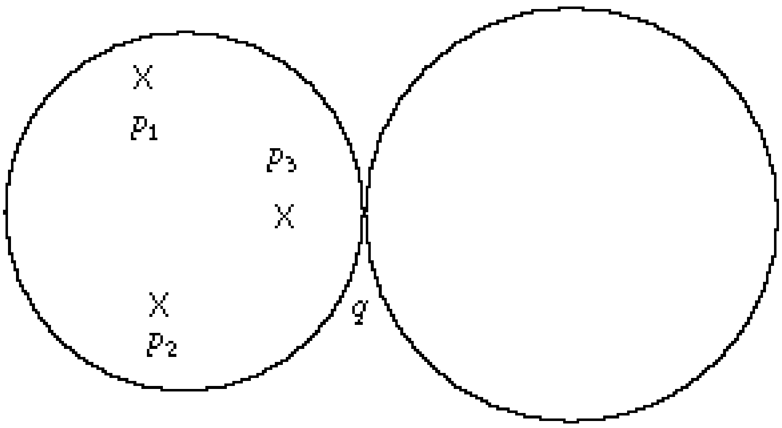


Figure 3.18

We can apply the same calculation and find again the codimension of the moduli space corresponding to this configuration is not smaller than 2. Thus this boundary do not affect the well definedness.

Thus we are done ? But in fact there are cheatings in the above argument.

Let us explain a trouble which may occur if we do not put additional assumption for (X,ω,J). For example let us consider the case of Figure 3.17. Let $\varphi_m : S^2_2 \to S^2$ be a rational map of degree m which branches at q and p_3. We consider a pseudo holomorphic map $h_0 : S^2 \to X$ such that $h_0^*(TX) \cap [S^2] = -s < 0$. We choose $\dim X - 2s - 6 \geq 0$. Then we consider moduli space of $f : \Sigma \to X$ such that $f_2 = f|_{S^2_2} = h_0 \circ \varphi_m$ and $f_1 = f|_{S^2_1} \in \mathcal{M}_{k+sm}(X,J)$. Now for large m the "virtual dimension" for f_2 is $\dim X - 2sm - 6 < 0$. But this set cannot perturbed to be empty. (Since there is no perturbation to remove $h_0 : S^2 \to X$.) (This point is related to the first trouble we mentioned.) On the other hand the dimension of $\mathcal{M}_{k+sm}(X,J)$ is $\dim X + k + 2sm - 6$, which can be very large. So naive dimension counting we did before do not work. This trouble is related to the stability of the curves which are known for a long time in algebraic geometry. (Namely S^2 with two points is *not* stable.)

In order to avoid this trouble we consider the following class of symplectic manifold.

Definition 3.19 : A symplectic manifold (X,ω,J) is called *semipositive* if for each $f : S^2 \to X$ we have $\int_{S^2} f^*\omega \leq 0$ if $6 - \dim X \leq 2 f^*(c^1(TX,J)) \cap [S^2] < 0$.

We remark that if $f : S^2 \to X$ is pseudo holomorphic (and is nontrivial) then its symplectic volume $\int_{S^2} f^*\omega$ is positive. Then in semipositive symplectic manifold we have $6 - \dim X > 2 f^*(c^1(TX,J)) \cap [S^2]$ or $f^*(c^1(TX,J)) \cap [S^2] \geq 0$. On the other hand Theorem 3.9 shows that the dimension of $\overline{\mathcal{M}}_k(X,J)$ is $2k + \dim X - 6$. In the case $6 - \dim X > 2 f^*(c^1(TX,J)) \cap [S^2]$ this number is negative, hence there is no such pseudo holomorphic curves[9]. Hence we conclude :

Lemma 3.20 : *For all pseudo holomorphic curve $f : S^2 \to X$ in semipositive symplectic manifold X, we have*

$$f^*(c^1(TX,J)) \cap [S^2] \geq 0.$$

[9]In fact we have to consider the possibility of singular point. But one finds that this does not occur either.

Using Lemma 3.20 we can prove that the trouble we mentioned above do not occur and we can justify the construction in the case when our symplectic manifold is semipositive. Of course there is a lot of things we have to do to make the argument rigorous. (For example the transversality.) But we do not mention them here. See [R].

Before closing this chapter, let us point out that there is a problem to generalize the construction of this chapter to the case of pseudo holomorphic curve of higher genus. (See [Ko3]) Namely let us put

$$\bar{\mathcal{T}}_{n,g} = \left\{ (\Sigma_g, J, (p_1,\cdots,p_n)) \,\middle|\, \begin{array}{l} J \text{ is a complex structure on } \Sigma \\ p_i \in \Sigma \end{array} \right\} \Big/ Diff(\Sigma).$$

Here the action of $Diff(\Sigma)$ is defined by

$$\varphi \cdot (\Sigma, J, (p_1,\cdots,p_n)) = (\Sigma, \varphi^* J, (\varphi(p_1),\cdots,\varphi(p_n))).$$

There is also a map :

$$ev : map(\Sigma, X) \times_{Diff(\Sigma)} \left\{ (\Sigma_g, J, (p_1,\cdots,p_n)) \,\middle|\, \begin{array}{l} J \text{ is a complex structure on } \Sigma \\ p_i \in \Sigma \end{array} \right\} \to X^n,$$

which might be used also to construct an invariant of a symplectic manifold X. In this situation one might be also able to construct cohomology classes of Teichmüller space $\bar{\mathcal{T}}_{n,g}$ from one on X.

But so far there are a lot of difficulty to compactify the moduli of pseudo holomorphic map of higher genus.

Chapter 4 Maslov index, Novikov ring, and Lagrangian homology

§1 Moduli space of pseudo holomorphic disks

In this section, we discuss the moduli of pseudo-holomorphic disks in symplectic manifold. Namely the space of maps h from $D^2 = \{z \in \mathbf{C} \mid |z| < 1\}$ to (X, ω, J) satisfying $h_* J = J h_*$.

To get a moduli space of finite dimension, we need to assume a boundary condition. A natural way to do so it to take some Lagrangian submanifold Λ of X and assume that $h(\partial D^2) \subseteq \Lambda$. Namely we consider

$$\mathcal{M}(X,J;\Lambda) = \left\{ h : D^2 \to X \;\middle|\; \begin{array}{l} Jh_* = h_* J \\ h(\partial D^2) \subseteq \Lambda \end{array} \right\}.$$

We discussed similar moduli space in Chapter 2. There we assumed $\pi_1(\Lambda) = \pi_2(X) = 1$. The presence of $\pi_2(X)$ has an effect that the dimension of each component of $\mathcal{M}(X,J;\Lambda)$

depends on the pull back of the 1st Chern class to D^2, as we discussed in the last chapter. On the other hand the presence of $\pi_1(\Lambda)$ implies that $[h(\partial D^2)] \in H_1(\Lambda;\mathbf{Z})$ has a contribution to the dimension of each component of $\mathcal{M}(X,J;\Lambda)$, as we will discuss soon. Usually it is difficult to separate the these two effects. In fact, in general, pull back of the 1st Chern class to D^2 is not well defined as an integral homology class. There is various way to take into account these two effects. And so far the author do not know what is the most natural way to do so in general case. So in this section, we put a bit restrictive assumption to both symplectic manifold and Lagrangian, so that we can isolate each of these two effects. These assumptions are satisfied in the case we need to study the relative (gauge theory) Floer homology.

Assumption 4.1 :

A symplectic manifold (X,ω) with almost complex structure J is said to be *pseudo-Einstein* if there exists an integer N such that $N\cdot[\omega] = [c^1]$ as De-Rham cohomology class.

Hereafter, for pseudo-Einstein symplectic manifold, we choose and fix a hermitian connection ∇ on TX such that $N\omega = c^1(TX)$ holds as forms. Hereafter we also assume that the De-Rham homology class $[\omega]$ is integral namely contained in the image of $H^*(X;\mathbf{Z})$ in $H^*_{\mathrm{Dr}}(X;\mathbf{C})$. Then as is well known there is a complex line bundle L on X such that $[c^1(L)] = [\omega]$. This bundle L is called prequantum bundle. We also choose and fix a connection ∇ of L such that $c^1(L) = \omega$ as forms.

Let Λ be a Lagrangian submanifold of X. Then by definition of Lagrangian submanifold $c^1(L)|_\Lambda = 0$. Hence (L,∇) is a flat bundle on Λ. If we assume furthermore that Λ is simply connected then it follows that (L,∇) is a trivial bundle (with trivial connection) on Λ. This was the case of Chapter 2. But we do not assume that Λ is simply connected in this chapter. We define :

Definition 4.2 : A Lagrangian Λ is said to be a *Borh-Sommerfert orbit* (abbreviated by BS-orbit hereafter) if the restriction of (L,∇) to it is a trivial bundle with trivial connection.

Remark 4.3 : BS-orbit has the following origin in quantum mechanics. Let us consider the classical phase space of one particle in two dimensional Euclidean space (that is $T^*\mathbf{R}^2$) and consider usual 2 body problem. Namely we consider the Hamiltonian $h = \sum p_i^2 - \frac{1}{\sqrt{\sum x_i^2}}$. We consider the orbit of Hamiltonian vector field H_h. The are two quantities invariant by this vector field, that are energy and angular momentum. Hence the level set of these two quantities are (in generic case) a 2 dimensional torus. The restriction of the integration curves of Hamiltonian vector field to each torus is the parallel circle. Thus the splitting the phase space into tori describes the behavior of our Hamiltonian vector field

Our tori is also a Lagrangian submanifold. We consider tori which are BS-orbit also. One can find that for each E there is only a finite number of tori which is BS-orbit and the energy is smaller than E. Borh-Sommerfert's quatization condition says that the dimension of quatum Hilbert space of this system of energy $< E$ is equal to the number of such BS-orbit.

Remark 4.4 : We also remark here the relation of BS-orbit to the exactness of symplectic diffeomorphisms. Let (X,ω) be a symplectic manifold and $\varphi_t : X \to X$ is a family of symplectic diffeomorphism such that $\varphi_0 = id$. The graph Λ_t of φ_t are Lagrangian of $(X \times X, \omega \oplus -\omega)$. One then can prove that Λ_t are BS-orbits if and only if φ_t are exact symplectic diffeomorphism.

In place of proving this assertion, we explain that the obstruction for a Lagrangian to be BS-orbit is the same as one for the symplectic diffeomorphism to be exact.

Let Λ_t be a family of Lagrangian such that Λ_0 is a BS-orbit. We consider the monodromy $h_t : \pi_1(\Lambda) \to U(1)$. (Since each member of the family Λ_t are diffeomorphic to each other we identify their fundamental groups and simply write it as $\pi_1(\Lambda)$.) Now since Λ_0 is a BS-orbit it follows that $h_0 \equiv 1$. Hence we may regard $\left.\frac{dh_t}{dt}\right|_{t=0}$ as an element of $H^1(\Lambda;\mathbf{R})$. (Note that $\mathbf{R}$ is Lie algebra of $U(1)$.) Thus the obstruction for Λ_t to be BS-orbit is given by an element of $H^1(\Lambda;\mathbf{R})$.

Next we consider the family of symplectic diffeomorphism φ_t such that $\varphi_0 = id$. We consider the vector field $X = \left.\frac{d\varphi_t}{dt}\right|_{t=0}$. We define the 1-form u by $u(Y) = \omega(X,Y)$. The infinitesimal version of the condition that φ_t is a family of symplectic diffeomorphism is equivalent to $du = 0$. On the other hand X is a Hamilton vector field if and only if $u = df$ for some function f. The condition that X is a Hamilton vector field is an infinitesimal version of the condition that φ_t are exact. Then the obstruction for φ_t to be exact lies in the De-Rham cohomology group $H^1(X;\mathbf{R})$.

By simply calculation we can find that the two obstructions (one for graph to be BS-orbit, the other for maps to be exact) are the same.

Now we define a map $m:\pi_1(\Lambda) \to \mathbf{Z}$, which we call *Maslov index*, for a BS-orbit Λ. First we put

$$Gr_{lag,n} = \left\{ E \subset \mathbf{C}^n \,\middle|\, \begin{array}{l} E \cong \mathbf{R}^n \\ \sum dx_i \wedge dy_i\big|_E = 0 \end{array} \right\},$$

This manifold is called the Lagrangian Grassmannian. We recall the following :

Lemma 4.5 : $\pi_1(Gr_{lag,n}) = \mathbf{Z}$.

The generator of $\pi_1(Gr_{lag,n}) = \mathbf{Z}$ is called the universal Maslov class.

Now we define $m:\pi_1(\Lambda) \to \mathbf{Z}$. Let $[\ell] \in \pi_1(\Lambda, p_0)$. Our assumption $N\omega = c^1(TX)$ implies that there exists a canonical isomorphism $\det TX \cong L$. Hence the bundle $\det TX$ together with its induced connection is trivial on our BS-orbit Λ. Therefore, the monodromy $h(\ell)$ of the tangent bundle TX along ℓ is contained in $SU(n)$. Take a path in $SU(n)$ which joins $h(\ell)$ to the unit. Then we have a trivialization in $\ell^*(TX)$. (Here $\ell^*(TX)$ is a vector bundle over S^1.) On this (trivial) bundle $\ell^*(TX) \cong S^1 \times \mathbf{C}^n$, there is a family of Lagrangian vector subspaces $T_{\ell(t)}\Lambda$. Thus we get a loop in $Gr_{lag,n}$. Then by Lemma 4.5 we get a number in $\pi_1(Gr_{lag,n}) = \mathbf{Z}$. We denote this number by $m([\ell])$. Using the fact that $\pi_1(SU(n)) = 1$, we find that $m([\ell])$ is independent of the path in $SU(n)$ which joins $h(\ell)$ to the unit.

Lemma 4.6 : $m([\ell])$ *is even if and only if* $[\ell] \in \pi_1(\Lambda, p_0)$ *respects the orientation of* Λ.

We omit the proof. Hereafter we always assume that our Lagrangian Λ is oriented. Hence our Maslov index $m([\ell])$ is always even.

The purpose of this section is to find a formula to give a dimension of moduli space of stable holomorphic disks. Two numbers are related to it. One is Maslov index we discussed. The other is relative Chern number, which we define now.

Let Λ be a BS-orbit and $h \in \mathcal{M}(X, J; \Lambda)$. We put

$$c^1(h) = \int_{D^2} h^*\left(c^1(TX)\right).$$

Here $c^1(TX)$ is the Chern form defined by using the connection ∇ on TX. We remark that by our assumption the form $c^1(TX)$ is trivial on Λ. Hence the number $c^1(h)$ is regarded as relative Chern number and is an integer. Now we put

$$\mathcal{M}_{k,m}(X, J; \Lambda) = \left\{ h \in \mathcal{M}(X, J; \Lambda) \;\middle|\; \begin{array}{l} c^1(h) = k \\ m(h([\ell])) = m \end{array} \right\}.$$

Theorem 4.7 (Gromov) **:** *In case everything are transversal we have*

$$\dim_{\mathbf{R}} \mathcal{M}_{k,m}(X, J; \Lambda) = n + 2k + m\,.$$

Here n is the complex dimension of X.

To sketch the proof of Theorem 4.7, let us recall the reflection principle in the theory of function of one complex variables. Namely let D be a domain in the complex plain which is invariant by complex conjugate. Put $D^+ = \{z \in D \mid \operatorname{Im} z \ge 0\}$ and $D^- = \overline{D}^+$, $D_{\mathbf{R}} = D \cap \mathbf{R}$. Let h be a holomorphic function on D^+ (which is continuous on the boundary) such that $h(D_{\mathbf{R}}) \subset \mathbf{R}$. Then for $z \in D^-$ we put $h(z) = \overline{h(\bar{z})}$ and then get a holomorphic function on D.

We use a variant of this in the following way. We consider $D^2 \subseteq \mathbf{C}P^1$. Let E be a complex vector bundle on D^2 and let $E_{\mathbf{R}}$ be a real subbundle of $E|_{\partial D^2}$ such that $E_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C} \cong E|_{D^2}$. Then there is a unique complex bundle $\tilde{E}$ on $\mathbf{C}P^1$ together with conjugate holomorphic isomorphism $conj : \tilde{E} \to \tilde{E}$ which cover $z \mapsto \bar{z}$ and the fixed point set of it is $E_{\mathbf{R}}$. Then, for each holomorphic section φ of E such that $\varphi(z) \in E_{\mathbf{R}}$

for each $z \in \partial D^2$, we can extend it to a holomorphic section in the same way. Thus we find

(4.8) $$\dim_{\mathbf{R}} \left\{ \varphi \in \Gamma(D^2, E) \,\middle|\, \begin{array}{l} \varphi \text{ is holomorphic} \\ \varphi(z) \in E_{\mathbf{R}} \ \text{ for } z \in \partial D^2 \end{array} \right\} = \dim_{\mathbf{C}} H^0(\mathbf{C}P^1, \mathcal{O}(\tilde{E})).$$

Now we are going to apply Formula (4.8) to prove Theorem 4.7. Let $h_t \in \mathcal{M}_{k,m}(X, J; \Lambda)$. Then $\left.\frac{dh_t}{dt}\right|_{t=0}$ is a holomorphic section of $h_0^*(TX)$ such that $\left.\frac{dh_t}{dt}\right|_{t=0}(z) \in h_0^*(T\Lambda)$ for $z \in \partial D^2$. We put $E = h_0^*(TX)$ and $E_{\mathbf{R}} = h_0^*(T\Lambda)$. Then (4.8) implies that in generic situation, we have :

(4.9) $$\dim_{\mathbf{R}} \mathcal{M}_{k,m}(X, J; \Lambda) = \dim_{\mathbf{C}} H^0(\mathbf{C}P^1, \mathcal{O}(\tilde{E})).$$

So we are going to calculate the Chern number of the bundle E. Roughly speaking E is a double of $h_0^*(TX)$. But it is *not* true that $c^1(\tilde{E}) = 2c^1(h_0^*(TX))$. Because there are two different trivializations we used for $h_0^*(TX)|_{\partial D^2}$. (More precisely the trivialization of $\det h_0^*(TX)|_{\partial D^2}$.) One trivialization comes from the equality $c^1(TX) = N\omega = Nc^1(L)$ and the triviality of L on Λ. This trivialization is used to define $c^1(h_0^*(TX))$. The other trivialization is induced by $h_0^*(TX)|_{\partial D^2} = h_0^*(T\Lambda)|_{\partial D^2} \otimes_{\mathbf{R}} \mathbf{C}$ [10]. This trivialization is used to define E. Then our definition of Maslov index exactly evaluate the difference between these two trivializations. Hence the correct formula is

$$c^1(\tilde{E}) = 2c^1(h_0^*(TX)) + m(h_0(\partial D^2)).$$

Theorem 4.7 follows from (4.9) and Rieman-Roch formula.

§2 Floer homology for Lagrangian intersection

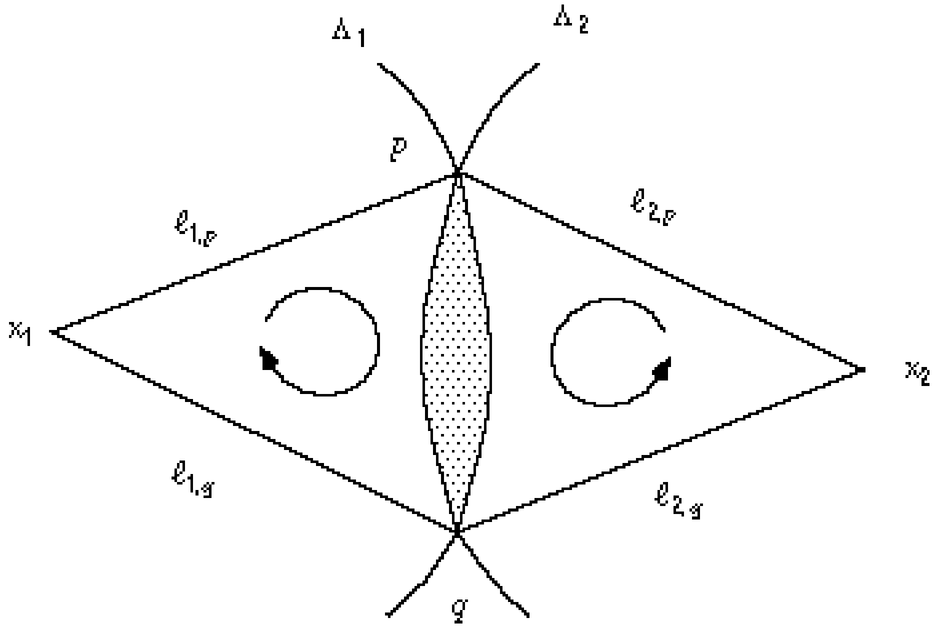
Now we consider pseudo-Einstein symplectic manifold (X, ω, J) such that $c^1(TX) = N\omega$ with $N \geq 0$. We consider two BS orbits Λ_1, Λ_2 on it. We assume that they are transversal to each other. For each $p, q \in \Lambda_1 \cap \Lambda_2$, we consider

[10] Since we assumed that Λ is oriented it follows that $h_0^*(T\Lambda)|_{\partial D^2}$ is trivial.

$$\mathcal{M}_{\text{symp}}(p,q;\Lambda_1,\Lambda_2) = \left\{ h : D^2 \to X \;\middle|\; \begin{array}{l} h_* J = J h_* \\ h(-1) = p, h(1) = q \\ h(z) \in \Lambda_1, \text{ for } z \in \partial D^2, \operatorname{Im} z > 0 \\ h(z) \in \Lambda_2, \text{ for } z \in \partial D^2, \operatorname{Im} z < 0 \end{array} \right\}.$$

In the case when $\pi_1(\Lambda_i) = \pi_2(X) = 1$, the dimension of each component of $\mathcal{M}_{\text{symp}}(p,q;\Lambda_1,\Lambda_2)$ is equal to $\mu(p) - \mu(q)$ for some map $\mu : \Lambda_1 \cap \Lambda_2 \to \mathbf{Z}$. Since we do not assume $\pi_1(\Lambda_i) = \pi_2(X) = 1$ the dimension of $\mathcal{M}_{\text{symp}}(p,q;\Lambda_1,\Lambda_2)$ depends on the component as in the case of pseudo holomorphic map from $\mathbf{C}P^1$. As we discussed in § 1 (where we considered the case we have one BS-orbit for boundary value) there are two factors which contribute the dimension of $\mathcal{M}_{\text{symp}}(p,q;\Lambda_1,\Lambda_2)$, that is the Maslov index and Chern number (or symplectic area.)

First we will define Maslov index. We fix base point $x_i \in \Lambda_i$, and for each $p \in \Lambda_1 \cap \Lambda_2$ we fix paths $\ell_{i,p}$ joining x_i to p in Λ_i.[11] Then for $h \in \mathcal{M}_{\text{symp}}(p,q;\Lambda_1,\Lambda_2)$ we define its Maslov index $m(h)$ as follows. We consider the union of three arcs $\ell_{1,p}$, $h(\partial_1 D^2)$, $\ell_{1,q}$. Here $\partial_1 D^2 = \{z \in \partial D^2 \mid \operatorname{Im} z > 0\}$. These three paths give a loop its homotopy class give an element of $\pi_1(\Lambda_1, p_1)$, let m_1 be its Maslov index. We define m_2 in a similar way from $\ell_{2,p}$, $h(\partial_2 D^2)$, $\ell_{2,q}$. Then we put $m(h) = m_1 - m_2$.



[11]The Maslov index depends on the homotopy type of the path but not the Floer homology definied.

We next define relative Chern number. There is a small trouble about it. Namely $\int_{D^2} h^*(c^1(TX))$ may not be an integer. So we modify this number and make it to an integer in the following way. For $p,q \in \Lambda_1 \cap \Lambda_2$, we consider a union of 4 paths $\ell_{1,p}$, $\ell_{1,q}$, $\ell_{2,p}$, $\ell_{2,q}$ and get a loop, $\ell_{p,q;1,2}$. Let $h_{p,q} \in U(1)$ be its monodromy of the prequantum bundle L along $\ell_{p,q;1,2}$. By definition we have $h_{p,q}h_{q,r} = h_{p,r}$. Regard $U(1) = \mathbf{R}/\mathbf{Z}$. Then, by a simple combinatorial argument, we can lift $h_{p,q} \in U(1)$ to $\tilde h_{p,q} \in \mathbf{R}$ such that

$$\tilde h_{p,q} + \tilde h_{q,r} = \tilde h_{p,r}. \tag{4.10}$$

(This lift is not unique.) Since Λ_i is BS-orbit, the holonomy of the prequantum bundle L along $h(\partial D^2)$ coincides with $h_{p,q}$ if $h \in \mathcal{M}_{\mathbf{symp}}(p,q;\Lambda_1,\Lambda_2)$. Hence by using our lift $\tilde h_{p,q}$ we can construct a trivialization of $h|^*_{\partial D^2}(\det(TX))$. (Note $\det(TX) = L^{\otimes N}$.) We define $c^1(h)$ to be the relative first Chern number of $h^*(TX)$ with respect to this trivialization. (We have $c^1(h) \in N\mathbf{Z}$.)

In other words we have

$$c^1(h) = \int_{D^2} h^*(c^1(TX)) - N\tilde h_{p,q}.$$

A consequence of (4.10) is as follows. Let $[h_i] \in \overline{\mathcal{M}}_{\mathbf{symp}}(p,q;\Lambda_1,\Lambda_2)$ be a sequence converging to $([h_1],[h_2]) \in \overline{\mathcal{M}}_{\mathrm{symp}}(p,r;\Lambda_1,\Lambda_2) \times \overline{\mathcal{M}}_{\mathrm{symp}}(r,q;\Lambda_1,\Lambda_2)$. Then $\lim c^1(h_i) = c^1(h_1) + c^1(h_2)$. (We note that $\lim m(h_i) = m(h_1) + m(h_2)$ also holds from definition.)

We put

$$\mathcal{M}^{symp}_{k,m}(p,q;\Lambda_1,\Lambda_2) = \left\{ h \in \mathcal{M}_{\mathbf{symp}}(p,q;\Lambda_1,\Lambda_2) \,\middle|\, \begin{matrix} m(h) = m \\ c^1(h) = k \end{matrix} \right\}.$$

Then, by the same argument as in Chapter 2, we have

$$\partial \overline{\mathcal{M}}^{symp}_{k,m}(p,q;\Lambda_1,\Lambda_2) = \bigsqcup_{r,k',m'} \overline{\mathcal{M}}^{symp}_{k',m'}(p,r;\Lambda_1,\Lambda_2) \times \overline{\mathcal{M}}^{symp}_{k-k',m-m'}(r,q;\Lambda_1,\Lambda_2) \tag{4.11}$$

We have also the following :

Theorem 4.12: *If everything is transversal then there exists* $\mu : \Lambda_1 \cap \Lambda_2 \to \mathbf{Z}$ *such that*

$$\dim \mathcal{M}_{k,m}^{symp}(p,q;\Lambda_1,\Lambda_2) = \mu(p) - \mu(q) + 2k + m.$$

<u>Sketch of the Proof.</u>

Let $[h_i] \in \overline{\mathcal{M}}_{\mathrm{symp}}(p,q;\Lambda_1,\Lambda_2)$ be a sequence converging to $([h_1],[h_2]) \in \overline{\mathcal{M}}_{\mathrm{symp}}(p,r;\Lambda_1,\Lambda_2) \times \overline{\mathcal{M}}_{\mathrm{symp}}(r,q;\Lambda_1,\Lambda_2)$. Let $\overline{\mathcal{M}}_{\mathrm{symp}}(p,q;\Lambda_1,\Lambda_2)_\alpha$, $\overline{\mathcal{M}}_{\mathrm{symp}}(p,r;\Lambda_1,\Lambda_2)_\beta$, $\overline{\mathcal{M}}_{\mathrm{symp}}(r,q;\Lambda_1,\Lambda_2)_\gamma$ be connected components of $\overline{\mathcal{M}}_{\mathrm{symp}}(p,q;\Lambda_1,\Lambda_2)$, $\overline{\mathcal{M}}_{\mathrm{symp}}(p,r;\Lambda_1,\Lambda_2)$, $\overline{\mathcal{M}}_{\mathrm{symp}}(r,q;\Lambda_1,\Lambda_2)$ containing $[h_i]$, $[h_1]$, $[h_2]$ respectively. Using excision property of Index of elliptic operator, we can prove that

$$\dim \overline{\mathcal{M}}_{\mathrm{symp}}(p,q;\Lambda_1,\Lambda_2)_\alpha = \dim \overline{\mathcal{M}}_{\mathrm{symp}}(p,r;\Lambda_1,\Lambda_2)_\beta + \dim \overline{\mathcal{M}}_{\mathrm{symp}}(r,q;\Lambda_1,\Lambda_2)_\gamma .$$

Then Theorem follows from $\lim c^1(h_i) = c^1(h_1) + c^1(h_2)$, $\lim m(h_i) = m(h_1) + m(h_2)$.

Thus to construct Floer homology we need to take into account the quantities k, m. The key idea for doing it was introduced by Novikov[12] [N], and used by Sikorov [Si], Hofer-Salamon [HS], Ono [On] and Ono-Vân [OV] in symplectic Floer theory. Before explaining it, we need one remark. Note that the integral $\int_{D^2} h^*(c^1(TX))$ is always nonnegative if h is pseudo holomorphic. Hence by definition $\mathcal{M}_{k,m}^{symp}(p,q;\Lambda_1,\Lambda_2)$ is nonempty only if $k > k_0(p,q)$ for some number $k_0(p,q)$ depending only on p,q. We consider two cases $N = 0$, $N > 0$ separately.

First we consider the case $N > 0$. Then Theorem 4.11 implies

$$\dim \mathcal{M}_{k,m}^{symp}(p,q;\Lambda_1,\Lambda_2) \equiv \mu(p) - \mu(q) + 2k \mod 2N.$$

So we consider chain complex with $\mathbf{Z}/2N\mathbf{Z}$-grading. To take into account the effect of m we take the Novikov ring $R = \mathbf{Z}[T][[T^{-1}]]$. We put

$$C_*(\Lambda_1,\Lambda_2) = \bigoplus_{p \in \Lambda_1 \cap \Lambda_2} R \cdot [p].$$

[12]Novikov found this construction to study Morse theorey for multivalued function.

The boundary operator is defined by

$$\textbf{(4.13.1)} \qquad \partial[p] = \sum_{\mu(p)=\mu(q)+m \mod 2N} <\partial p, T^m q> T^m[q],$$

$$\textbf{(4.13.2)} \qquad <\partial p, T^m q> = \# \overline{\mathcal{M}}_{k,m}^{symp}(p,q), \quad \text{(counted with sign.)}$$

Here we choose k such that $\dim \overline{\mathcal{M}}_{k,m}^{symp}(p,q)=0$. The right hand side of (4.13.1) is contained in $\bigoplus_{p\in\Lambda_1\cap\Lambda_2} R\cdot[p]$ since $\mathcal{M}_{k,m}^{symp}(p,q;\Lambda_1,\Lambda_2)$ is nonempty only for $k>k_0(p,q)$. (Namely $<\partial p, T^m q>$ is nonzero only for $m\le \mu(p)-\mu(q)-2Nk_0(p,q)$.) Then using the property (4.11), we have :

Theorem 4.14 : $\partial\partial=0$.

We put $HF_*(\Lambda_1,\Lambda_2)=H_*(C_*,\partial)$. We have also

Theorem 4.15 : $HF_*(\Lambda_1,\Lambda_2)$ *is invariant under the perturbation of the BS-orbits* Λ_1,Λ_2 .

We remark that we made some choices to define Maslov index and relative Chern number. But changing them corresponds to change the generators $[p]$ to $T^a[p]$ for some a. Since T is invertible in our Novikov ring R, this does not change the Floer homology $HF_*(\Lambda_1,\Lambda_2)$. Thus we constructed the Floer homology of intersection of BS-orbits in the case when $c^1(TX)=N\cdot\omega$, $N\in \mathbf{Z}_+$. The same remark can be applied to the case $N=0$ also.

We next consider the case when $N=0$. (This corresponds to Calabi-Yau manifold when our symplectic manifold (X,ω) is Kähler.) In this case Theorem 4.11 implies that symplectic volume do not affect the dimension of the moduli space. Therefore for each m there may be infinitely many components $\mathcal{M}_{*,m}^{symp}(p,q;\Lambda_1,\Lambda_2)$ of the same dimension. So we need to use again Novikov ring to construct Floer homology. In this case we take $R=\mathbf{Z}[T^{-1}][[T]]$. Then we put

$$C_*(\Lambda_1,\Lambda_2) = \bigoplus_{p\in\Lambda_1\cap\Lambda_2} R\cdot[p]$$

$$\partial[p] = \sum < \partial p, T^k q > T^k[q],$$

$$< \partial p, T^k q > = \#\overline{\mathcal{M}}^{symp}_{k,\mu(q)-\mu(p)-1}(p,q), \quad (\text{ conted with s}).$$

Then again we have $\partial\partial = 0$. We put $HF_*(\Lambda_1,\Lambda_2) = H_*(C_*,\partial)$. Theorem 4.15 holds in this case also. Thus we constructed Floer homology between two BS-orbit in the case when $c^1(TX) = 0$.

So far we considered only the case when (X,ω,J) is pseudo Einstein. Probably one needs to consider Novikov ring of several variables to deal with more general case.

Finally we remark that Oh [Oh] discussed Floer homology of Lagrangian intersection in a bit different way.

§3 Lagrangian homology

In this section we combine the ideas discussed in Chapters 1 and 4 and define a "quantum version" or Morse homotopy.

Let (X,ω,J) be a pseudo-Einstein symplectic manifold with $N \ge 0$. Let $\Lambda_1,\cdots,\Lambda_s$ be BS-orbits of X which are pairwise transversal. We put

$$\mathcal{T}_{\partial,0,s} = \left\{(z_1,\cdots,z_s) \,\middle|\, z_i \in \partial D^2,\ (z_1,\cdots,z_s) \text{ respects cyclic ordering}\right\}.$$

For $(z_1,\cdots,z_s) \in \mathcal{T}_{\partial,0,s}$, we let $C_i(z_1,\cdots,z_s)$ be the component of $S^2 - \{z_i, z_{i+1}\}$ which contains no other z_j's.

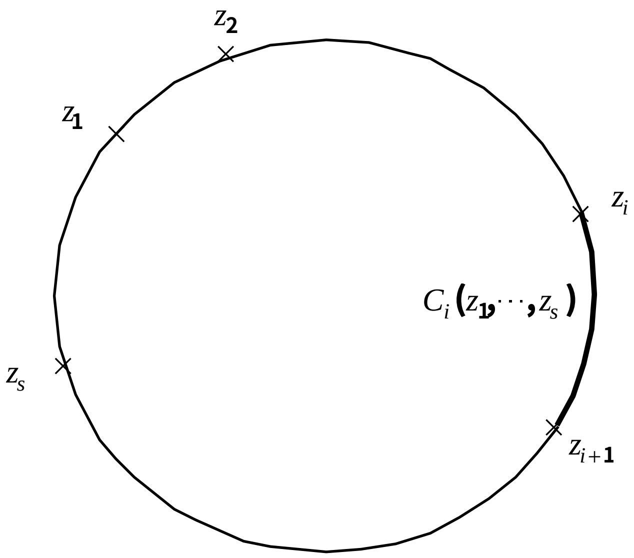


Figure 4.16

Let $p_i \in \Lambda_i \cap \Lambda_{i+1}$. (We put $\Lambda_{s+1} = \Lambda_1$.) Then we define the moduli space $\mathcal{M}_{symp}(p_1,\cdots,p_s;(X,\omega,J))$ by

$$\mathcal{M}_{symp}(p_1,\cdots,p_s;(X,\omega,J)) = \left\{ (h,(z_1,\cdots,z_s)) \;\middle|\; \begin{array}{l} (z_1,\cdots,z_s) \in \mathcal{T}_{\partial,0,s} \\ h: D^2 \to X, h_* J = J h_* \\ h(z_i) = p_i, h(C_i(z_1,\cdots,z_s)) \subseteq \Lambda_i \end{array} \right\}.$$

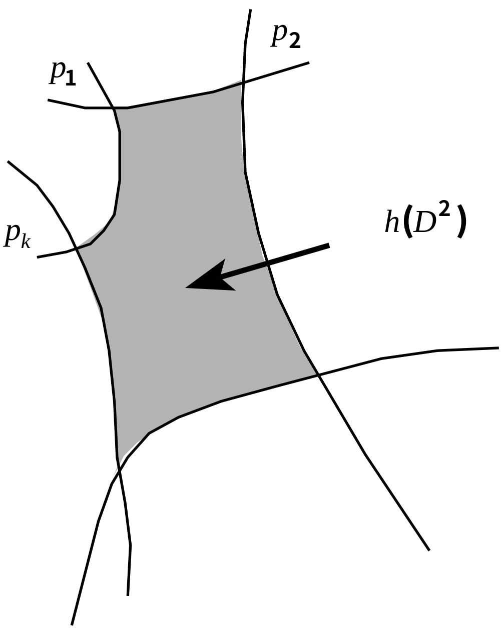


Figure 4.17

To study the dimension of this moduli space we again need to consider the Maslov index and Chern number (or equivalently symplectic volume).

We choose base points $x_i \in \Lambda_i$ and paths ℓ_{i,p_i} $\ell_{i,p_{i+1}}$ from x_i to p_i or p_{i+1} in Λ_i respectively. We obtain a loop $L(p_1,\cdots,p_s)$ by joining those $2s$ loops.

Let $h_{\Lambda_1\cdots\Lambda_s;p_1,\cdots p_s} \in U(1)$ be the holonomy of the bundle L along this loop. We have $h_{p_i,\cdots p_j,q}h_{q,p_{j+1},\cdots,p_s,p_1,\cdots,p_{i-1}} = h_{p_1,\cdots,p_s}$, and $h_{p_1',p_1}h_{p_1,\cdots,p_s} = h_{p_1',\cdots,p_s}$.

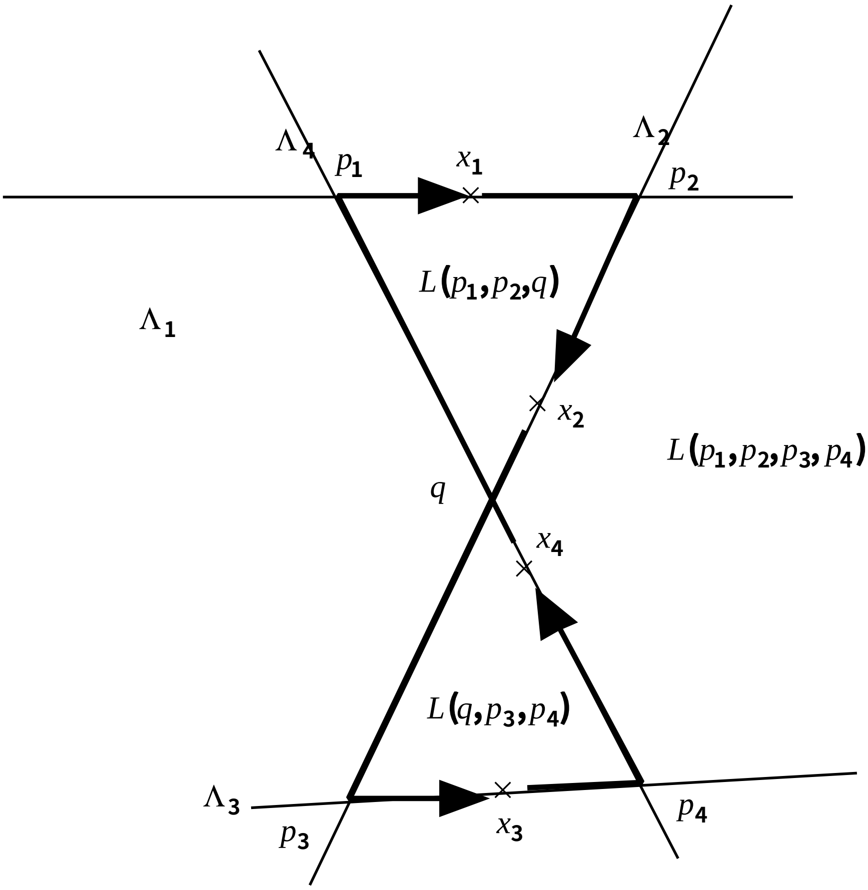


Figure 4.18

Then again by a simple combiantorial argument, we can lift them to $\tilde h_{\Lambda_1\cdots\Lambda_s;p_1,\cdots p_s} \in \boldsymbol{R}$ such that $\tilde h_{p_i,\cdots p_j,q} + \tilde h_{q,p_{j+1},\cdots,p_s,p_1,\cdots,p_{i-1}} = \tilde h_{p_1,\cdots,p_s}$. $\tilde h_{p_1',p_1} + \tilde h_{p_1,\cdots,p_s} = \tilde h_{p_1',\cdots,p_s}$. Now, for $h \in \mathcal{M}_{symp}(p_1,\cdots,p_s;(X,\omega,J))$, we put

$$c^1(h) = \int_{D^2} h^*(c^1(TX)) - N\tilde h_{p_1,\cdots,p_s}.$$

We have $c^1(h) \in \mathbf{Z}$. We also remark that $c^1(h) \ge k_0(p_1,\cdots,p_k)$ because of the positivity of symplectic volume of pseudo holomorphic disk.

We next define a Maslov index. For each $h \in \mathcal{M}_{symp}(p_1,\cdots,p_s;(X,\omega,J))$ and $1 \le i \le k$ the union of three arcs ℓ_{i,p_i} $\ell_{i,p_{i+1}}$, and $h(C_i(z_1,\cdots,z_s))$ gives an element of $\pi_1(\Lambda_i, x_i)$. We let $m(h)$ be the Maslov index of this loop.

We then put

$$\mathcal{M}^{symp}_{k,m}(p_1,\cdots,p_s;(X,\omega,J)) = \left\{ (h,(z_1,\cdots,z_s)) \in \mathcal{M}_{symp}(p_1,\cdots,p_s;(X,\omega,J)) \times \mathcal{T}_{\partial,0,s} \;\middle|\; \begin{matrix} c^1(h) = k \\ m(h) = m \end{matrix} \right\}.$$

The group $PSL(2;\mathbf{R}) = \mathrm{Aut}(D^2)$ acts on $\mathcal{M}^{symp}_{k,m}(p_1,\cdots,p_s;(X,\omega,J))$ by $\varphi \cdot (h,(z_1,\cdots,z_s)) = (h \circ \varphi,(\varphi(z_1),\cdots,\varphi(z_s)))$. Let $\overline{\mathcal{M}}^{symp}_{k,m}(p_1,\cdots,p_s;(X,\omega,J))$ be the quotient space of this action. Then we can generalize Theorem 4.12 as follows :

Theorem 4.19 : *If everything is transversal, then* $\overline{\mathcal{M}}^{symp}_{k,m}(p_1,\cdots,p_s;(X,\omega,J))$ *is a smooth manifold and that we have*

$$\dim \overline{\mathcal{M}}^{symp}_{k,m}(p_1,\cdots,p_s;(X,\omega,J)) = \sum_{i=1}^{s} \mu(p_i;\Lambda_i,\Lambda_{i+1}) + 2k + m + s - 3.$$

(We write $\mu(p_i;\Lambda_i,\Lambda_{i+1})$ in place of $\mu(p_i)$ to clarify the order of two Lagrangians which intersect at p_i. In fact we have $\mu(p_i;\Lambda_i,\Lambda_{i+1}) = -\mu(p_i;\Lambda_{i+1},\Lambda_i)$.)

Also we have:

Theorem 4.20[13] : *We can compactify* $\overline{\mathcal{M}}^{symp}_{k,m}(p_1,\cdots,p_s;(X,\omega,J))$ *such that*

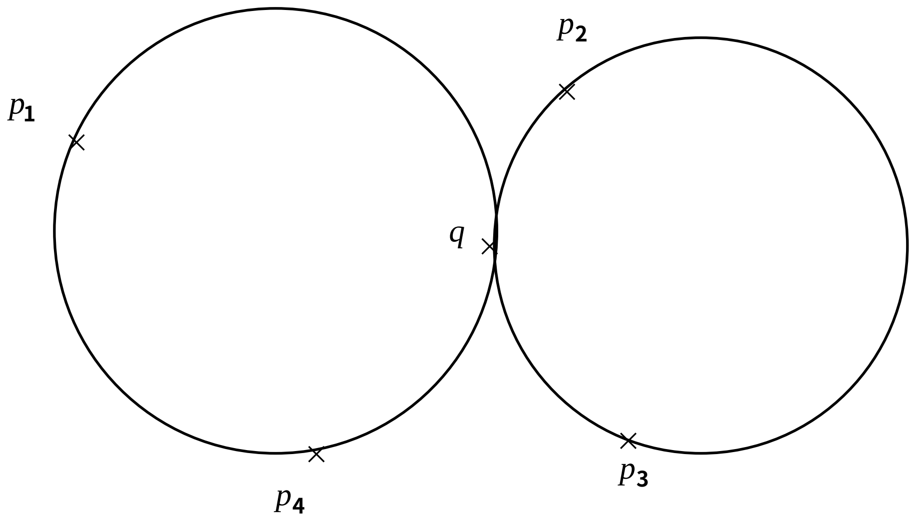
---

[13]This Theorem is stated in a bit unprecise way. In fact the is another end related to the bubling phenomenon. There are two kinds of bubling one is interior bubling and the other is boundary bubling. We do not go into detail.

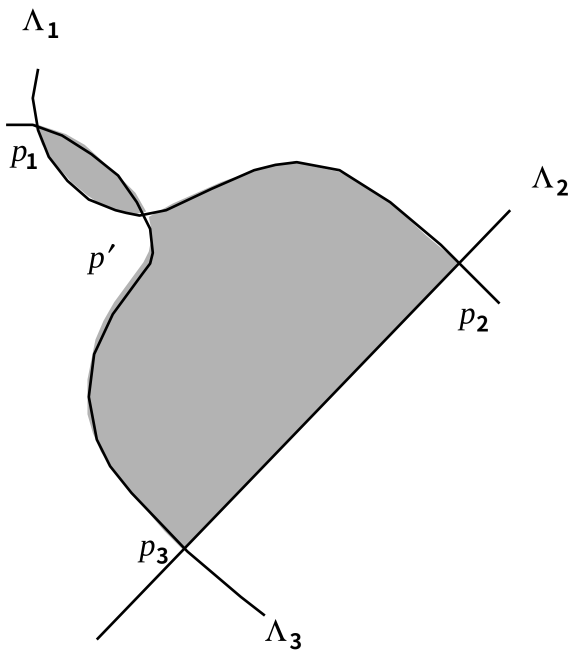
$$\partial \overline{\mathcal{M}}^{symp}_{k,m}(p_1,\cdots,p_s;(X,\omega,J))$$
$$= \bigsqcup_{i,k',j'} \bigsqcup_{p_i'\in\Lambda_i\cap\Lambda_{i+1}} \overline{\mathcal{M}}^{symp}_{k',m'}(p_i,p_i';\Lambda_i,\Lambda_{i+1})\times\overline{\mathcal{M}}^{symp}_{k-k',m-m'}(p_1,\cdots,p_i',\cdots,p_k;(X,\omega,J))$$
$$\cup \bigsqcup_{1\le i<j\le s,k',m'} \bigsqcup_{q\in\Lambda_i\cap\Lambda_j} \overline{\mathcal{M}}^{symp}_{k-k',m-m'}(p_1,\cdots,p_i,q,p_j,\cdots,p_s;(X,\omega,J))$$
$$\times\overline{\mathcal{M}}^{symp}_{k',m'}(q,p_{i+1},\cdots p_{j-1};(X,\omega,J))$$

The proof goes roughly the same as one for Theorem 4.12 and Formula 4.11.

For the proof of Theorem 4.20 we recall that the compactification of the moduli space $\bar{\mathcal{T}}_{\partial,k,0}=\mathcal{T}_{\partial,k,0}/PSL(2;\mathbf{R})$ is given by $\bar{\mathcal{T}}_{\partial,k,0}\cup\bigsqcup_{k'}\bar{\mathcal{T}}_{\partial,k',0}\times\bar{\mathcal{T}}_{\partial,k+2-k',0}\cup\bigsqcup_{k',k''}\bar{\mathcal{T}}_{\partial,k',0}\times\bar{\mathcal{T}}_{\partial,k'',0}\times\bar{\mathcal{T}}_{\partial,k+4-k'-k',0}\cup\cdots$.

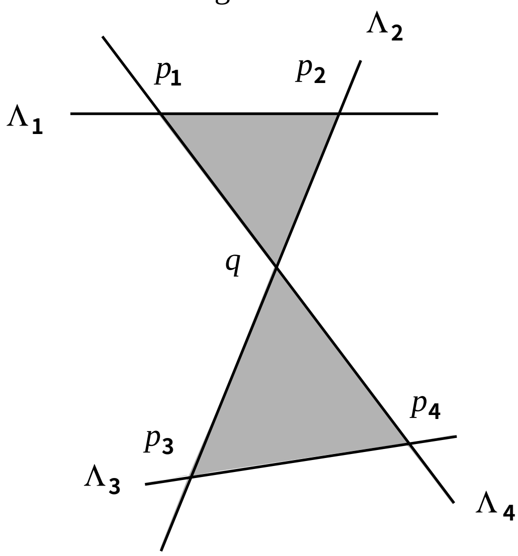


The second term of the formula corresponds to $\bigsqcup_{k'}\bar{\mathcal{T}}_{\partial,k',0}\times\bar{\mathcal{T}}_{\partial,k+2-k',0}$. Geometrically these terms corresponds to the splitting of pseudo holomorphic disks shown in the figures below.



This figure corresponds tot the first term.

Figure 4.21



This figure corresponds second term.

Figure 4.22

We mention a bit about the proof of the dimension formula in Theorem 4.19.

First by considering the excision property of index for the pseudo holomorphic disk in Figure 4.21, we find that there exists $k(\Lambda_1,\cdots,\Lambda_s)$ such that

$$\dim \overline{\mathcal{M}}_{k,m}^{symp}(p_1,\cdots,p_s;(X,\omega,J)) = \sum_{i=1}^{s}\mu(p_i;\Lambda_i,\Lambda_{i+1}) + 2k + m + s - 3 + k(\Lambda_1,\cdots,\Lambda_s).$$

(Here $k(\Lambda_1,\cdots,\Lambda_s)$ is independent of p_i.)

We remark here that the number $\mu(p_i;\Lambda_i,\Lambda_{i+1})$ in Theorem 4.14 has ambiguity. Namely we can replace it by $\mu(p_i;\Lambda_i,\Lambda_{i+1}) + c(\Lambda_i,\Lambda_{i+1})$. We need to adjust them in order to prove Theorem 4.16. In fact if we change $\mu(p_i;\Lambda_i,\Lambda_{i+1})$ to $\mu(p_i;\Lambda_i,\Lambda_{i+1}) + c(\Lambda_i,\Lambda_{i+1})$, then $k(\Lambda_1,\cdots,\Lambda_s)$ will change to $k(\Lambda_1,\cdots,\Lambda_s) + \sum c(\Lambda_i,\Lambda_{i+1})$.

We apply the excision property of index to the pseudo holomorphic curve of Figure 4.22 (and a similar curve where Λ_1 and Λ_3 intersects) we obtain

(4.23) $$k(\Lambda_1,\Lambda_2,\Lambda_4) + k(\Lambda_2,\Lambda_3,\Lambda_4) = k(\Lambda_1,\Lambda_3,\Lambda_4) + k(\Lambda_1,\Lambda_2,\Lambda_3).$$

Formula 4.23 implies that we can find $c(\Lambda_i,\Lambda_{i+1})$ such that $k(\Lambda_1,\Lambda_2,\Lambda_3) + \sum c(\Lambda_i,\Lambda_{i+1}) = 0$ for arbitrary $\Lambda_1,\Lambda_2,\Lambda_3$. (This follows from vanishing of appropriate second cohomology group of free $\mathbf{Z}$ -module.) Then using excision property of index we can prove that the formula holds in general.

By the above argument we find that $\mu(p_i;\Lambda_i,\Lambda_{i+1})$ is well defined modulo constant independent of Λ_i,Λ_{i+1}. One can determine these constant by considering the case when Λ_{i+1} is the perturbation of Λ_i. Thus the index is determined uniquely in our situation. (However they does depend on the choice of the paths $\ell_{1,p}$ and the lifts $\tilde h_{\Lambda_1\cdots\Lambda_s;p_1,\cdots p_s}$.)

Now, first in the case when $N > 0$, we define the (higher)-composition map $\eta_{s-1}: C_{\ell_1}(\Lambda_1,\Lambda_2)\otimes_R \cdots C_{\ell_{s-1}}(\Lambda_{s-1},\Lambda_s) \to C_{\sum \ell_i + s-3}(\Lambda_1,\Lambda_s)$, by

(4.24)
$$\begin{aligned}
\eta_{s-1}([p_1]\otimes\cdots\otimes[p_{s-1}]) &= \sum \eta_{s-1}(p_1,\cdots,p_s;m)\cdot T^m[p_s] \\
\eta_{s-1}(p_1,\cdots,p_s;m) &= \#\overline{\mathcal{M}}_{k,m}^{symp}(p_1,\cdots,p_s;(X,\omega,J))
\end{aligned}$$
.

Here k is chosen so that $\dim \overline{\mathcal{M}}_{k,m}^{symp}(p_1,\cdots,p_s;(X,\omega,J)) = 0$.

Then we have :

Theorem 4.25 : *For each pseudo Einstein symplectic manifold with* $N > 0$ *exists an* A^∞*-Category, whose object is BS-orbit, the set of morphisms are Floer homology of Lagrangian intersection and whose (higher) composition is given in* Formula 4.24.

The proof that the maps in (4.24) satisfies the axioms of A^∞-category is based on Theorem 4.20.

We can perform a similar construction and can prove the same result in the case when $N = 0$.

§4 Quantum ring

We now go back to the situation of the Chapter 2, where we consider a symplectic manifold (X,ω) and consider an exact symplectic diffeomorphism $\varphi:(X,\omega)\to(X,\omega)$. We assume that (X,ω) is pseudo Einstein. Let $Y = X\times X$. Then $(Y,\omega\oplus-\omega)$ is again pseudo Einstein. We consider the diagonal $\Delta\subseteq Y$ and graphs, G_{φ_i}, of exact symplectic diffeomorphisms φ_i. We find (as discussed before) that G_{φ_i} are BS-orbit. We assume that φ_i are C^1-close to identity. Then $G_{\varphi_i}\subseteq T^*X$ and G_{φ_i} can be identified to the graph of exact one form df_i. Hence $G_{\varphi_i}\cap G_{\varphi_j} = Cr(f_i - f_j)$.

Now the argument of Chapter 2 can be used to show that

$$\mathcal{M}_{Morse}(p,q)\cong\mathcal{M}_{0,0}^{symp}(p,q;(Y,J)). \tag{4.26}$$

Moreover we conjecture :

$$\mathcal{M}_{Morse}(p_1,\cdots,p_s)\cong\mathcal{M}_{0,0}^{symp}(p_1,\cdots,p_s;(Y,J)), \tag{4.27}$$

for $p_i\in Cr(f_i - f_{i+1})$. In the case when Maslov index of Y is trivial, we can prove (4.27) in the cohomology level.

Namely by putting $T = 0$ the construction of the last section reduces to the construction of Chapter one. We remark here that in Floer homology for time dependent Hamiltonian (as is discussed in [Fl3]) the contribution from other component vanishes in (4.26). (But not in (4.27). See the discussion in [Fl3], in the case of Projective spaces.)

Studying the case when $k,\ell\neq 0$ corresponds to including the quatum effect in the story. Then matrix element of our higher composition operator coincides to what is called correlation function in mathematical physics.

Here we leave T as free variable but from physical point of view it is natural to plug in $T=\exp(-1/\hbar)$. This construction is proposed M.Kontsevitch [Ko3]. In order to show that this converges we need to solve the following :

Conjecture 4.28 (Kontsevitch) **:** Let $N = 0$. Then

$$\#\overline{\mathcal{M}}_{k,m}^{symp}(p_1,\cdots,p_s;(X,\omega,J)) < Ce^{Ck},$$

for some constant C .

One can state a similar conjecture in case $N > 0$. If Conjecture 2.28 is proved then for sufficiently small $\hbar$ the boundary operator $\partial[p] = \sum \#\overline{\mathcal{M}}_{k,\mu(q)-\mu(p)-1}^{symp}(p,q)\exp(-k/\hbar)[q]$ is well defined. And, if we can prove a similar conjecture for other moduli spaces, $\overline{\mathcal{M}}_{k,m}^{symp}(p_1,\cdots,p_s;(X,\omega,J))$, we also have (higher) compositions which are defined over $\boldsymbol{R}$. Thus we get a quatum deformation of the rational homotopy type of Calabi-Yau manifold. The cohomology ring of this quantized Calabi-Yau manifold is exactly what is called A-model by Witten [W4]. But we have more structure than ring structure. That is A^∞-structure. (If we do not plug in $T = \exp(-1/\hbar)$ then we have also torsion structure, and probably cohomology operations. So it might be possible to define "quntum homotopy group" using "quntized Adams spectral sequence".) Kontsevitch conjectured that one can find a similar A^∞-structure on deformation space which are related to the structure discussed above by Mirror symmetry. See [Ko3].

The discussion so far uses Lagrangian intersection and is an "open string" version of "quitized Morse homotopy". There is also a closed string version of it. We will discuss it in [Fu2]. The author conjectures that the closed string version reduces to the case of diagonal and its perturbation of open string version. But the proof of it requires the solution of a delicate singular bifurcation problem, which is not yet settled.

Chapter 5 Floer homology for 3-manifolds with boundary

§1 A quick review of Gauge theory Floer homology

So far we have been discussing Floer homology of Lagrangian intersection. There is another kind of Floer homology, that is Floer homology of closed 3 manifold. First, in this section, we give a quick review of it and then in next section we describe its relation to symplectic Floer homology. The later is due to Dostoglou-Salamon [DS] and Yoshida [Y2]. Finally we combine them with the discussion we have done in last chapter to define Floer homology for 3-manifold with boundary.

Let M be a closed 3 manifold. We consider the set of all connections $\mathcal{A}(M)$ of trivial $SU(2)$ bundle on it. (In fact every $SU(2)$ bundle on M is trivial.) $\mathcal{A}(M)$ can be identified to the set of $su(2)$ valued one forms on M. On this space there exists an action of the group $\mathcal{G}(M) = Map(M, SU(2))$ given by $g^*a = g^{-1}dg + g^{-1}ag$. Let $\mathcal{B}(M)$ be the quotient space of this action. Floer homology for 3 manifold M is defined by studying Morse theory on this space $\mathcal{B}(M)$. We choose Chern-Simons invariant cs as Morse function. cs is defied by

$$cs(a) = \frac{1}{(4\pi)^2}\int_M Tr(a \wedge da + a \wedge a \wedge a).$$

In the case when M is a closed 3-manifold, we can prove that $cs(g \cdot a) - cs(a) \in \mathbf{Z}$ hence cs induces a function : $\mathcal{B}(M) \to S^1 = \mathbf{R}/\mathbf{Z}$. (We use the same symbol for his function.) The gradient vector field of it can be calculated as

$$\mathrm{grad}_a cs = *F_a$$

where F_a is the curvature and $*$ is the Hodge $*$-operator. Hence the critical point set $Cr(cs)$ is identified to $R(M) = \{[a] \in \mathcal{B}(M) \mid F_a = 0\}$, the set of flat connections. In other word, it is equal to the set of all conjugacy classes of the representation of $\pi_1(M)$ to

$SU(2)$. Let $[a],[b] \in Cr(cs)$. We consider the set $\mathcal{M}(a,b)$ of all gradient lines of the gradient vector field $\mathrm{grad}_a cs = *F_a$ of Chern-Simons functional. Then we find that

$$\mathcal{M}(a,b) = \left\{ A \;\middle|\; \begin{array}{l} \text{connections on } M \times \mathbf{R} \\ *F_A = -F_A \\ A|_{M\times\{-\infty\}} \sim a \\ A|_{M\times\{+\infty\}} \sim b \end{array} \right\} \Big/ Map(M \times \mathbf{R}, SU(2)),$$

Here $A|_{M\times\{-\infty\}} \sim a$ means that $\lim_{t\to-\infty} A|_{M\times\{t\}}$ converges and its limit is gauge equivalent to a.

$\mathcal{M}(a,b)$ has an $\mathbf{R}$-action induced by the translation along $\mathbf{R}$ of $M \times \mathbf{R}$. Let $\overline{\mathcal{M}}(a,b)$ be its quotient. We have[14] :

Theorem 5.1 (Floer [Fl2]) **:**

There exists $\mu : R(M) \to \mathbf{Z}_8$ such that $\mathcal{M}(a,b)$ is a smooth manifold and the dimension of each component is given by

$$\dim \mathcal{M}(a,b) \equiv \mu(a) - \mu(b) \mod 8.$$

The ambiguity of $\mu : R(M) \to \mathbf{Z}_8$ is by essentially the same reason as we discussed in the symplectic case. Now imitating the construction in Chapter 1, we put

(5.2)
$$\begin{cases} C_k(\Lambda_1, \Lambda_2) & = \bigoplus_{\substack{a \in R(M) \\ \mu(a) \equiv k}} \mathbf{Z} \cdot [a] \\ \partial[a] & = \sum \langle \partial a, b \rangle [b], \\ \langle \partial a, b \rangle & = \# \overline{\mathcal{M}}(a,b), \quad (\text{ counted with sign.}) \end{cases}.$$

Here, in the third formula, we consider only zero dimensional components. In fact, there is some trouble for these constructions, since our space $\mathcal{B}(M)$ is singular. One can

[14]**This theorem holds only after appropriate perturbation. We omit discussions about perturbation here. Also we need to consider the trouble coming from the reducible connections, the singularity of $\mathcal{B}(M)$. We will describe it soon.**

find that the intersection of the singular locus $\mathcal{B}(M)$ and $Cr(cs) = R(M)$ is the set of reducible flat connections that is the set of all conjugacy classes of flat connections the image of whose holonomy representation is abelian. In general the set of reducible flat connections is of positive dimension. So the Morse theory has serious trouble. Therefore, Floer consider the case when M is a homology 3-sphere. In that case, the intersection $\mathcal{B}(M) \cap R(M)$ consists of one element that is the trivial connection. So more precisely Theorem 5.1 holds for a homology 3-sphere M and $[a],[b] \ne [0]$. Thus, in Definition 5.2, we consider only nontrivial flat connections. Then Floer proved that $\partial\partial = 0$. So we obtain Floer homology $HF_*(M)$ of homology 3-sphere M. Floer proved also that it is invariant by the change of metric and the perturbation, (which we need in the case when Chern-Simons functional degenerate.)

So far we discussed the case of trivial $SU(2)$-bundle over homology 3-spheres. There is another case which we can discuss in essentially the same way. Let $E \to M$ be an $SO(3)$-bundle over 3-manifold M, (which is not necessary a homology 3-sphere). ([Fl4]) We assume that there is no reducible flat connection of $E \to M$. This assumption is satisfied if there is a surface Σ of M such that the restriction of $E \to M$ to Σ is nontrivial. (In other words $[\Sigma] \cap w^2(E) \ne 0$.) Let $\mathcal{A}(M;E)$ be the set of all connections of E and $\mathcal{G}(M;E)$ be the set of all bundle automorphisms of $E \to M$, in other words the set of all gauge transformations. Let $\mathcal{B}(M;E)$ be the quotient of $\mathcal{A}(M;E)$ by the action of $\mathcal{G}(M;E)$. We define Chern-Simons invariant cs on $\mathcal{A}(M;E)$. We have $s(g \cdot a) - cs(a) \in \frac{1}{2}\mathbf{Z}$. Hence again $cs : \mathcal{B}(M;E) \to \mathbf{R}\big/\frac{1}{2}\mathbf{Z}$ is well defined. The argument we outlined above works in the same way except the index $\mu : R(M) \to \mathbf{Z}_4$ takes value in $\mathbf{Z}_4$ rather than $\mathbf{Z}_8$. (This reflects the fact that $SU(2)$ is a double cover of $SO(3)$.) Thus we again obtain a Floer homology $HF_*(M;E)$.

§2 Symplectic versus Gauge theory in Floer homologies

It was conjectured by Atiyah and Floer that the Gauge theory Floer homology we outlined in § 1 and the symplectic Floer homology we discussed in Chapters 2,3 and 4, are closely related to each other. The rough (and imprecise) ideas behind it is as follows.

We consider a Rieman surface Σ and a bundle E on it. E may be either the trivial $SU(2)$-bundle or a nontrivial $SO(3)$-bundle on it. Let $\tilde{R}(\Sigma;E)$ be the set of all flat connections of E. We divide it by an action of Gauge transformation group and let $R(\Sigma;E)$ be the quotient space. We remark that in the case when E is a nontrivial $SO(3)$

-bundle, the space $R(\Sigma;E)$ is smooth manifold while in the case when E is the trivial $SU(2)$-bundle, the space $R(\Sigma;E)$ has a singularity.

Now we first recall that $R(\Sigma;E)$ has a symplectic structure. ([Go]) The symplectic structure is given as follows. Let $a \in \tilde{R}(\Sigma;E)$. Then the tangent space $T_{[a]}R(\Sigma;E)$ is identified to the cohomology group $H^1(\Sigma;su(2)^a)$. (Here $su(2)^a = so(3)^a$ is the local system associated to the flat bundle a by the adjoint representation.) Elements of $H^1(\Sigma;su(2)^a)$ are realized by harmonic one forms with $su(2)^a$-coefficient. Let $u,v \in H^1(\Sigma;su(2)^a)$ then we put

$$\omega(u,v) = \int_\Sigma Tr(u \wedge v) \ .$$

It is proved that ω gives a symplectic structure on $R(\Sigma;E)$. (In case $R(\Sigma;E)$ is singular ω defines a symplectic structure on the regular part of it.) If we fix a complex structure J of our Rieman surface Σ then it induces one on $R(\Sigma;E)$ since J preserves harmonic one form and hence induces a map on $H^1(\Sigma;su(2)^a)$.

Now, let D be a domain in $\mathbf{C}$. Let us take a map $h: D \to R(\Sigma;E)$. We lift it to $\tilde{h}: D \to \tilde{R}(\Sigma;E)$ then we have a family of connections on Σ parametrized by D. We can regard it as a connection on $D \times \Sigma$ and write it as A_h. (Until here the description is precise.) The following "Theorem" is ***not correct*** but is something similar to (and simpler than) the correct result.

"Theorem" 5.3. $*F_{A_h} = -F_{A_h}$ *if and only if* $h: D \to R(\Sigma;E)$ *is holomorphic.*

Proof ? Let $z = s + \sqrt{-1}t$ be the coordinate of D. Then we calculate

$$\begin{aligned} F_{\tilde{A}}(x,s,t) &= F_{\tilde{h}(s,t)}(x) + \frac{\partial \tilde{h}}{\partial t} \wedge dt + \frac{\partial \tilde{h}}{\partial s} \wedge ds \\ &= \frac{\partial \tilde{h}}{\partial t} \wedge dt + \frac{\partial \tilde{h}}{\partial s} \wedge ds. \end{aligned}$$

We recall that Hodge $*$ coincides with complex structure J on Rieman surface, (and in particular on D.) Hence we find

$$*F_{\tilde{A}}(x,s,t) = -J\frac{\partial \tilde{h}}{\partial t} \wedge ds + J\frac{\partial \tilde{h}}{\partial s} \wedge dt.$$

Hence $*F_{A_h} = -F_{A_h}$ is equivalent to $J\dfrac{\partial \tilde h}{\partial t} = -\dfrac{\partial \tilde h}{\partial s}$. Namely the holomorphicity of $\tilde h$. So we are done ?

In fact, the trouble is that we need to study the holomorphicity of h and not one for $\tilde h$. (There is no natural symplectic structure on $\tilde R(\Sigma;E)$.) And unfortunately the holomorphicity of $\tilde h$ is not equivalent to one of h. So the story is not so simple. But the general idea in this calculation can work in some case and imply interesting results.

There are two results of this kind one by Dostoglou-Salamon [DS] and the other by Yoshida [Y]. We state them without proof.

First we consider the following situation. Let $\overline{E} \to \Sigma$ be a nontrivial $SO(3)$ bundle over a Rieman surface Σ and $\phi : \Sigma \to \Sigma$ be an orientation preserving diffeomorphism which lifts to an isomorphism of $\overline{E}$. Then we get a 3-manifold M which is a fibre bundle over a circle with fibre Σ and monodromy $\phi : \Sigma \to \Sigma$. There is an $SO(3)$-bundle $E \to M$ over M obtained by the lift of ϕ. This bundle E satisfies our assumption since its restriction to a fibre is nontrivial. Hence we obtain a Gauge theory Floer homology $HF_*(M;E)$.

On the other hand, the diffeomorphism ϕ and its lift to $\overline{E}$ determines a symplectic diffeomorphism $\phi_* : R(M;\overline{E}) \to R(M;\overline{E})$. (This symplectic diffeomorphism is not necessary exact.) We consider its graph $G_\phi \subseteq R(M;\overline{E}) \times R(M;\overline{E})$. We can prove that it is a BS-orbit. Also one can prove that $R(\overline{E})$ is pseudo Einstein. In fact the Chern class is 2 times the generator of $H_2(R(M;\overline{E});\mathbf{Z})$. We multiply our symplectic form so that its De-Rham cohomology class is the generator. Hence we can take $N = 2$. Thus we can define Floer homology for Lagrangian intersection $HF_*(G_\phi,\Delta)$. Here Δ is the diagonal.

Furthermore one can prove that $R(M;\overline{E})$ and G_ϕ are simply connected. Therefore in this case we do not have to consider Maslov index and only consider the Chern number. (See the argument of the last chapter.) Hence the group $HF_*(G_\phi,\Delta)$ is $\mathbf{Z}$ coefficient and index by $* \in \mathbf{Z}_4$. (Thus the coefficient and the index coincide to one for Gauge theory Floer homology.)

Then we have :

Theorem 5.4 (Dostoglou-Salamon) : $HF_*(G_\phi,\Delta)$ *is isomorphic to* $HF_*(M;E)$.

In fact Dostoglou-Salamon state their result in a bit different way using periodic Hamiltonian. (See the discussion in Chapter 2.) But we can easily show that they are equivalent, to each other.

We remark that the intersection $G_\phi \cap \Delta$ can be identified to $R(M;E)$. Hence the chain complexes for the two Floer homology groups are isomorphic as abelian groups. To show that the boundary operators are the same to each other, one need to compare the moduli spaces used for their definitions. The one for Gauge theory Floer homology is the moduli space of connections satisfying $*F_A = -F_A$. The one for symplectic geometry Floer homology is the moduli space of peudo-holomorphic curves. (Both with appropriate boundary conditions.) Hence if we can prove a result similar to "Theorem" 5.3 then we can prove that the boundary operators coincide to each other. Dostoglou-Salamon proved that the moduli spaces are homeomorphic to each other.

We next consider the trivial $SU(2)$ connection on Rieman surface Σ of genus g. Let $H_{g,i}$, $i = 1,2$ be handle bodies which bounds Σ. We patch them and get a 3-manifold M. We assume that M is a homology 3-sphere. Then we can define (Gauge theory) Floer homology $HF_*(M)$ on M using trivial $SU(2)$ bundle. On the other hand, the set of (gauge equivalent class of) flat connections $R(H_{g,i})$ on $H_{g,i}$, $i = 1,2$ is embedded by restriction map to $R(\Sigma)$. And one can prove that its image is BS-orbit. However there is a trouble to define a Floer homology of Lagrangian intersection between them since the space $R(\Sigma)$ is singular. But Yoshida analyzed the moduli space of Lagrangian intersection in this case and proved necessary results[15] to define it. Then we have $HF_*(R(H_{g,1}),R(H_{g,2}))$.

Theorem 5.5 (Yoshida) **:** *$HF_*(R(H_{g,1}),R(H_{g,2}))$ is isomorphic to $HF_*(M)$.*

§3 Floer homology for 3-manifolds with boundary (1)

Theorem 5.5 suggests that one can consider the Lagrangian $R(H_{g,i})$ as the "invariant" of the handle body and patching those two "invariant" gives Floer homology of closed three manifold. For general 3-manifold M with boundary Σ and a bundle E on it, we can

[15] For example the smoothness of moduli space, the formula which gives its dimension, its compactifcation etc.

prove that after appropriate perturbation the space $R(M;E)$ is the immersed Lagrangian of $R(\Sigma)$. (In fact we can also prove that it is a BS-orbit.) So as S.Donaldson pointed out in his lecture at Warwick in 1992 July, the Lagrangian $R(M;E)$ is the first approximation of the Floer homology of 3-manifold M with boundary Σ.

But one can find easily that this Lagrangian itself does not have enough information to recover the Floer homology of 3-manifold when we patch M with another 3-manifold with the same boundary. To give an example of this we recall the following result stated by A.Floer and proved by P.Braam and S.Donaldson. Let M_i be closed 3-manifold and E_i be nontrivial $SO(3)$-bundle on it. Suppose that there is an embedded tori T_i^2 in M_i on which the bundles E_i are nontrivial. Then we consider $M_i - T_i^2$. After compactification they have two disjoint unions of tori as their boundaries. Then we patch $\overline{M_1 - T_1^2}$ and $\overline{M_2 - T_2^2}$ along their boundaries to get a closed 3-manifold M and an $SO(3)$-bundle on it. (See Figure 5.6)

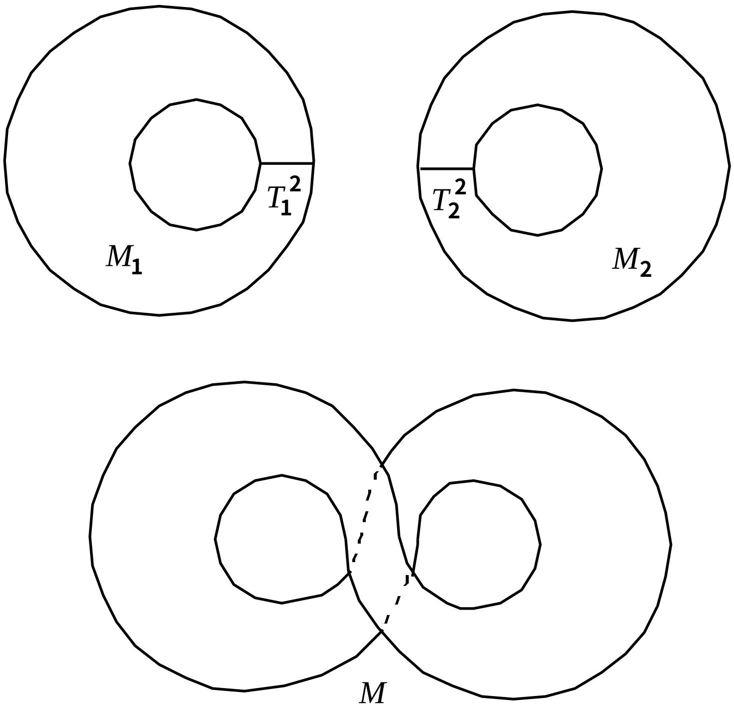


Figure 5.6

Braam-Donaldson [BD] proved that

(5.7) $$HF_*(M;Q) = HF_*(M_1;Q) \otimes HF_*(M_2;Q).$$

Now let M be a 3-manifold with boundary Σ and E be an $SO(3)$-bundle on it. Assume that there is an embedded torus $T^2 \subseteq M$ on which M is nontrivial. Suppose that there are two closed 3-manifolds M_1, M_2 and $SO(3)$-bundles on it such that there are embedded tori T_i^2 in M_i on which the bundles E_i are nontrivial. We assume furthermore that the set of flat connections on E_1 is equal to one on E_2 but the Floer homology are different to each other[16]. (Namely the boundary operator is different.) We then patch M with M_1 and M_2 respectively along the tori and obtain M_i'.

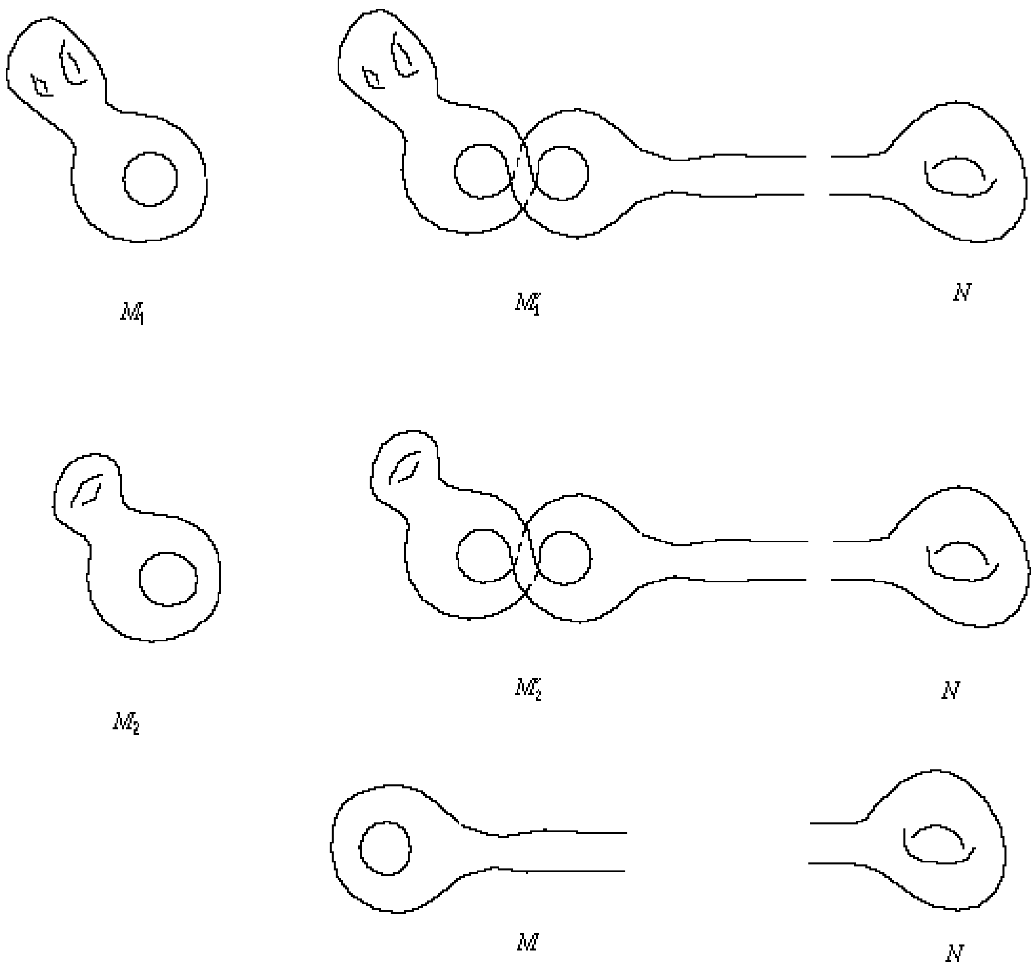


Figure 5.8

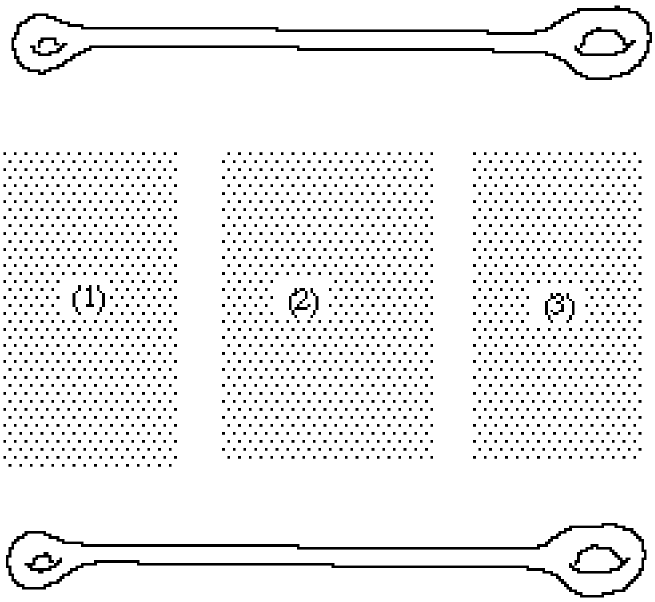
$R(M_1')$ and $R(M_2')$ are the same ad immersed Lagrangians. But (5.7) shows that after patching another 3-manifold N with boundary Σ, we have

[16] In fact the author do not know an explicite example of such pair, since it is usually very difficult to culculate the boundary operator of Floer homology. But it is almost sure that such an pair exists.

$$HF_*(M_1' \cup_\Sigma N) \neq HF_*(M_2' \cup_\Sigma N).$$

Thus $R(M_i') \subseteq R(\Sigma)$ do not have enough information to determine the Floer homology after patching.

We can explain this in the following way. Let us consider a closed 3-manifold $M = M_1 \cup_\Sigma M_2$. To study its Floer homology we need to study the equation $*F_A = -F_A$ on $M \times \boldsymbol{R}$. We put the metric on M such that it contains $\Sigma \times [-S,S]$ for large S. We then, roughly, divide the manifold $M \times \boldsymbol{R}$ into three parts, $M_1 \times \boldsymbol{R}$, $\Sigma \times [-T,T] \times \boldsymbol{R}$, $M_2 \times \boldsymbol{R}$.



Now we consider the solution of the equation of $*F_A = -F_A$. If the support of F_A is (roughly speaking) contained in the domain (2) above then it corresponds to the pseudo holomorphic curve on $R(\Sigma)$. But if A is a solution of $*F_A = -F_A$ for which the support of F_A lies in (1) or (3), one can not find its effect on Floer homology by studying pseudo holomorphic curve on $R(\Sigma)$.

Thus, roughly speaking, the Floer homology of 3-manifold with boundary is something which is a mixture of Lagrangian and chain complex. The notions we developed so far can be used to define such an object.

§3 Floer homology for 3-manifolds with boundary (2)

To describe an object which is a mixture of Lagrangian and chain complex we use the notion of A^∞-functor. We need it only in a special case here. First let us define

Definition 5.9 : The A^∞-category $\mathcal{Ch}$ is the category such that elements of $\mathcal{Ob}(\mathcal{Ch})$ is a chain complex, and for $A_1, A_2 \in \mathcal{Ob}(\mathcal{Ch})$ the set $C(A_1, A_2)$ of morphisms between them is given by $C(A_1, A_2) = Hom(A_1, A_2)$. Here Hom means module homomorphism (which is not necessary a chain homomorphism). $C(A_1, A_2)$ is a chain complex in an obvious way. The 2-composition is the composition of the homomorphisms in the usual sense. Higher compositions are all zero.

Definition 5.10 : Let $\mathcal{C}$ be an A^∞-category. An A^∞*-functor* from $\mathcal{C}$ to $\mathcal{Ch}$ is given as $(F_1; F_2, \cdots)$ such that

(1) $F_1 : \mathcal{Ob}(\mathcal{C}) \to \mathcal{Ob}(\mathcal{Ch})$,

(2) $F_2(A, B) : C(A, B) \to C(F_1(A), F_1(B))$ is a chain homomorphism,

(3) $F_3(A_1, A_2, A_3) : C(A_1, A_2) \otimes C(A_2, A_3) \to C(F_1(A_1), F_1(A_3))$ is a homomorphism such that

$$F_2(\eta_2(x \otimes y)) \pm \eta_2(F_2(x) \otimes F_2(y)) = \pm(\partial F_3)(x \otimes y \otimes z).$$

and so on.

Let $\mathcal{Func}(\mathcal{C}, Ch)$ be the set of all A^∞-functor from $\mathcal{C}$ to $\mathcal{Ch}$.

An important example of A^∞-functor from $\mathcal{C}$ to $\mathcal{Ch}$ is one which is representable (by an object of $\mathcal{C}$). Namely let $A \in \mathcal{Ob}(\mathcal{C})$. We define $F_A \in \mathcal{Func}(\mathcal{C}, Ch)$ as follows. We put $F_A(B) = C(A, B)$ for $B \in \mathcal{Ob}(\mathcal{C})$. Let $x \in C(B_1, B_2)$. We define $F_{A,2}(B_1, B_2)(x) \in Hom(C(A, B_1), C(A, B_2))$ by $F_{A,2}(B_1, B_2)(x)(y) = \eta_2(x \otimes y)$. $F_{A,3}(B_1, B_2, B_3)$ etc. is defined by using η_k in $\mathcal{C}$.

Suppose that C' is a fixed chain complex. Then $F_A \otimes C' : B \mapsto C(A, B) \otimes C'$ is again an A^∞-functor from $\mathcal{C}$ to $\mathcal{Ch}$. Thus elements of $\mathcal{Func}(\mathcal{C}, Ch)$ contains a mixture of the object of $\mathcal{C}$ and a chain complex.

We next define a natural transformation.

Definition 5.11 : Let $F^{(1)}, F^{(2)} \in \mathcal{Func}(\mathcal{C}, Ch)$. Then the *natural transformation* H between them is given by $(H(A))_{A\in \mathcal{O}b(C)}, (H_2(A_1,A_2), \overline{H}_2(A_1,A_2))_{A_1,A_2\in \mathcal{O}b(C)}, \cdots$ such that

(1) $H(A) \in C(F_1^{(1)}(A), F_1^{(2)}(A))$,

(2)

$$H_2(A_1,A_2): C(A_1,A_2) \to C(F_1^{(1)}(A_1), F_1^{(2)}(A_2))$$
$$\overline{H}_2(A_1,A_2): C(A_1,A_2) \to C(F_1^{(1)}(A_1), F_1^{(2)}(A_2))$$

such that

$$\partial(H_2(A_1,A_2)(x)) \pm H_2(A_1,A_2)(\partial x) \pm \overline{H}_2(A_1,A_2)(x) = \pm\eta_2(H(A_1), F_2^{(2)}(x)) \pm \eta_2(F_1^{(2)}(x), H(A_2)).$$

and so on.

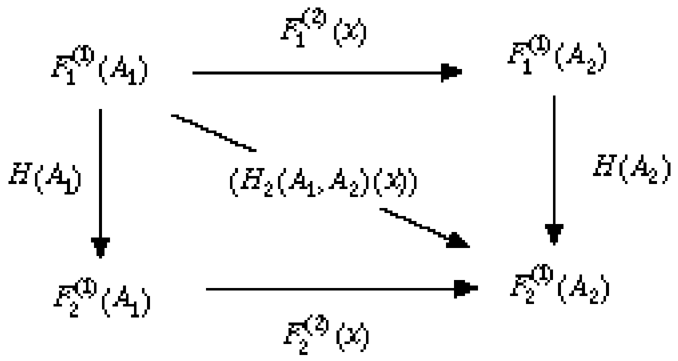


Diagram 5.12

If $\left((H(A))_{A\in \mathcal{O}b(C)}, (H_2(A_1,A_2), \overline{H}_2(A_1,A_2))_{A_1,A_2\in \mathcal{O}b(C)}, \cdots\right)$ is a natural transformation then $\left((\partial H(A))_{A\in \mathcal{O}b(C)}, (\overline{H}_2(A_1,A_2), 0)_{A_1,A_2\in \mathcal{O}b(C)}, \cdots\right)$ is again a natural transformation. Hence by putting

$$\begin{aligned}\partial((H(A))_{A\in \mathcal{O}b(C)}, (H_2(A_1,A_2), \overline{H}_2(A_1,A_2))_{A_1,A_2\in \mathcal{O}b(C)}, \cdots)\\ = ((\partial H(A))_{A\in \mathcal{O}b(C)}, (\overline{H}_2(A_1,A_2), 0)_{A_1,A_2\in \mathcal{O}b(C)}, \cdots),\end{aligned}$$

the set of all natural transformations is a chain complex.

Let $A, B \in \mathcal{O}b(C)$ and $x \in C(A,B)$. We define a natural transform H_x from F_B to F_A as follows. We put $H_C : C(B,C) \to C(A,C), y \mapsto \eta_2(x \otimes y)$, $H_2(C_1,C_2): C(C_1,C_2) \to Hom(C(B,C_1), C(A,C_2))$ by $H_2(C_1,C_2)(y)(z) = \eta_3(x \otimes z \otimes y)$ and $\overline{H}_2(C_1,C_2): C(C_1,C_2) \to Hom(C(B,C_1), C(A,C_2))$ by $\overline{H}_2(C_1,C_2)(y)(z) = \eta_3(\partial x \otimes z \otimes y)$, etc.

If we define a notion of contravariant A^∞-functor from C to $\mathcal{F}unc(C, Ch)$ in a similar way as Definition 5.8, we find that $A \mapsto F_A$ gives such an A^∞-functor.

All the above construction is an analogy of the corresponding construction in additive category.

Now we are going to state our result about relative Floer homology.

Let M be a 3-manifold which bounds Σ and E be an $SO(3)$ vector bundle on M. We assume that E is nontrivial on each connected component of Σ. (We remark that Σ is necessary to be disconnected in case such a bundle exists.)

As we discussed in § 2, we have a symplectic manifold $R(\Sigma, E)$ which is pseudo-Einstein with $N = 2$. Hence we get an A^∞-category $C(\Sigma, E)$.

Theorem 5.13 :

(1) *We can define an A^∞-functor $HF(M): C(\Sigma) \to Ch$ which is, up to chain homotopy, is an invariant of (M, E).*

(2) *If $\partial M_1 = \partial M_2 = \Sigma$, then there exists a chain map.*

$$\varphi : C_*(M_1 \# (-M_2)) \quad \to \quad C_*(HF(M_1), HF(M_2)),$$

Here $C_*(M_1 \# (-M_2))$ is a Floer homology of the closed 3 manifold $M_1 \# (-M_2)$ with an $SO(3)$ vector bundle E, and $C_*(HF(M_1), HF(M_2))$ is the set of all natural transformations between two functors, $HF(M_1), HF(M_2)$.

(3) *Suppose that $\partial M_1 = \partial M_2 = \partial M_3 = \Sigma$, then there exists a commutative diagram :*

$$\begin{array}{ccc}
C_*(M_1 \# (-M_2)) \otimes C_*(M_2 \# (-M_3)) & \longrightarrow & C_*(M_1 \# (-M_3)) \\
\varphi \Big\downarrow & & \Big\downarrow \varphi \otimes \varphi \\
C_*(HF(M_1), HF(M_2)) \otimes C_*(HF(M_2) \# HF(M_3)) & \longrightarrow & C_*(HF(M_1), HF(M_3))
\end{array}$$

Diagram 5.14

Here the first horizontal line of the Diagram 5.14 is the relative Donaldson Polynomial of the 4-manifold in Figure 5.15. (This map was mentioned by Donaldson in his Warwick lecture.)

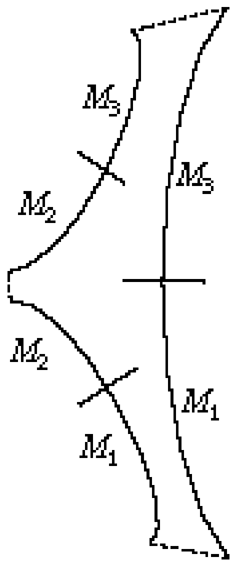


Figure 5.13.

The statement is closely related to the following program due to G.Segal.

For each surface Σ we associate a category $C(\Sigma)$. For each 3-manifold M which bounds Σ associate an object $HF(M)$ of $C(\Sigma)$. Let us call it the Floer homology of M. Let M' be another 3-manifold which bounds Σ. Then the Floer homology of the closed 3-manifold obtained by patching M and M' is equal to the set of all morphisms from $HF(M)$ to $HF(M')$.

Roughly speaking Theorem 5.11 says that for each BS orbit Λ in $R(\Sigma;E)$ we can associate a chain complex $HF(M)(\Lambda)$.

Now we explain the role played by the BS-condition. We assumed for the object of our A^{∞}-category $C(\Sigma,E)$. We are going to construct a chain complex using Gauge theory on M and symplectic geometry on $R(\Sigma;E)$. If one try to use an analogy of the construction of Chapter one we need some Morse function. If we try to use the similar argument as § 1 (where we considered the case when the 3-manifold is close), we take Chern-Simons functional as Morse function. But the trouble here is that the Chern-Simon functional is not Gauge invariant in the case when manifold has a boundary. Here we recall the result by Jefferey and Weitzmann [JW]. Let $\mathcal{A}(M;E)$ be the set of all connections on $E \to M$ and $\mathcal{G}(M;E)$ be the set of all Gauge transformations there. Let $\mathcal{G}(\Sigma;E)$ be the set of all Gauge transformations on Σ and $\tilde{R}(\Sigma;E)$ be the set of all flat connections there. ($\tilde{R}(\Sigma;E)/\mathcal{G}(\Sigma;E) = R(\Sigma;E)$.) Let $a \in \mathcal{A}(M;E)$ whose restriction is $\alpha \in \tilde{R}(\Sigma;E)$ and $g \in \mathcal{G}(M;E)$ whose restriction is $\bar{g} \in \mathcal{G}(\Sigma;E)$. Then there exists $\lambda(\alpha,\bar{g}) \in \mathbf{R}$ depending only on $\alpha,\bar{g}$ such that

(5.16) $$cs(g^*a) - cs(a) = \lambda(\alpha,\bar{g}).$$

Moreover we can prove

(5.17) $$\lambda(\alpha,\bar{g}_1\bar{g}_2) = \lambda(\alpha,\bar{g}_2) + \lambda(\bar{g}_2^*\alpha, \bar{g}_1).$$

We consider the direct product $\hat{R}(\Sigma;E) \times \mathbf{C}$ and define an action of $\mathcal{G}(\Sigma;E)$ on it by $\bar{g}^*(\alpha,c) = (\bar{g}^*\alpha, e^{2\pi\sqrt{-1}\lambda(\alpha,\bar{g})}c)$. By (5.17) this action is well defined. Hence dividing $\hat{R}(\Sigma;E) \times \mathbf{C}$ by this action we get a complex line bundle L on $R(\Sigma,E)$.

Now let $\hat{\mathcal{A}}(M;E)$ be the set of all elements of $\mathcal{A}(M;E)$ whose restriction to Σ is flat. We divide it by $\mathcal{G}(\Sigma;E)$ to obtain $\hat{\mathcal{B}}(M;E)$. There is a natural projection $\pi : \hat{\mathcal{B}}(M;E) \to R(\Sigma,E)$. We pull back the bundle L to $\hat{\mathcal{B}}(M;E)$ and denote it by the same symbol. (5.16) implies that $\exp(2\pi\sqrt{-1}cs(a))$ can be regarded as a section to this line bundle.

On other hand we can prove that the first Chern class $c^1(L) \in H^2(R(\Sigma,E)) = \mathbf{Z}$ is a generator and is equal to the symplectic form. Hence we can choose L as the prequatum bundle we used to define BS orbit. (The trivial connection on $\hat{R}(\Sigma;E) \times \mathbf{C}$ induces a connection on L.) As in proved [JW], $\exp(cs(a))$ is a flat section of this bundle. Suppose that Λ is a BS orbit in $R(\Sigma;E)$. Then by definition L is a trivial bundle there. Hence we find that $\exp(2\pi\sqrt{-1}cs(a))$ can be regarded as a function on $\pi^{-1}(\Lambda) \subseteq \hat{\mathcal{B}}(M;E)$. Thus one may develop a Morse theory on $\pi^{-1}(\Lambda)$ to get a Chain complex. If so we may take it as $HF(M)(\Lambda)$.

In fact since the space $\pi^{-1}(\Lambda)$ is of infinite dimension the construction is a bit more complicated and is described as follows.

We consider the equation

(5.18)
$$\frac{\partial A_t}{\partial t} = * F_{A_t}.$$

Naively, (5.18) with the condition $F_{A_t} \in \hat{\mathcal{B}}(M;E)$ should give the equation for the gradient line of cs on $\hat{\mathcal{B}}(M;E)$. But if we put the boundary condition $F_{A_t} \in \hat{\mathcal{B}}(M;E)$ to (5.18), we do not obtain a moduli space of finite dimension. Hence we have to change the construction a bit.

The linearized equation of (5.18) is :

(5.19)
$$\frac{\partial B_t}{\partial t} = D_{B_t},$$
$$D_{B_t} = \begin{pmatrix} *d_{A_t} & d_{A_t} \\ (d_{A_t})^* & 0 \end{pmatrix} \circlearrowright (\Lambda^1 \oplus \Lambda^0) \otimes su(2)(M).$$

We consider the this equation with one (moving) boundary condition :

(5.20)
$$B_t\big|_{\Sigma} \in T_{a_t}\Lambda \oplus P^+(a_t).$$

(Here a_t is a path in Λ.) (This is a "nonlinear version" of an idea due to Yoshida [Y1].)

Let us explain the notations in Formula (5.20). In a neighborhood of Σ, our manifold M is diffeomorphic to $\Sigma \times [0,\infty)$. Let s be the coordinate of the second parameter. Then we can split our operator D_{B_t} as

$$D_{B_t} = \sigma\left(\frac{\partial}{\partial s} + P\right)$$

Here

$$\sigma = \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$
$$P = \begin{pmatrix} 0 & d_{a_t} & *d_{a_t} \\ d_{a_t}^* & 0 & 0 \\ -*d_{a_t} & 0 & 0 \end{pmatrix}.$$

Let $P^{\pm}(a_t)$ be the positive eigenspace of P. One can identify the zero eigen space of P with $T_{a_t} R(\Sigma, E)$. The operator σ defines an almost complex structure on $(\Lambda^1 \oplus \Lambda^0) \otimes su(2)(M)$. And there is a natural compatible symplectic structure there. For this symplectic structure $T_{a_t} \Lambda \oplus P^+(a_t)$ is a Lagrangian subspace. Hence it gives an appropriate boundary condition.

(5.18) is the linearization of the following boundary condition for (5.16). We consider $\mathcal{B}(\Sigma, E)$ (the gauge equivalence class of all connections on Σ). We consider the neighborhood of $R(\Sigma, E)$ in $\mathcal{B}(\Sigma, E)$ and replace it by

$$\hat{\mathcal{B}}(\Sigma, E) = \left\{ (a,b) \left| \begin{array}{l} [a] \in R(\Sigma, E) \\ b \in P^+(a) \oplus P^-(a) \end{array} \right. \right\} \subseteq (\Lambda^1 \oplus \Lambda^0 \oplus \Lambda^0)(\Sigma) \otimes su(2).$$

This space has a natural symplectic structure induced by one on $R(\Sigma, E)$ and the operator σ, (which interchanges $P^+(a)$ and $P^-(a)$.) The subspace

$$\hat{\mathcal{B}}_+(\Sigma, E; \Lambda) = \left\{ (a,b) \left| \begin{array}{l} [a] \in \Lambda \\ b \in P^+(a) \end{array} \right. \right\},$$

is a Lagrangian submanifold of $\hat{\mathcal{B}}(\Sigma, E)$. We remark that $(\Lambda^1 \oplus \Lambda^0 \oplus \Lambda^0)(\Sigma) \otimes su(2) = \Gamma(\Sigma; \Lambda^1(M^3 \times \mathbf{R}))$. Now we consider the moduli space

$$\mathcal{M}'(M, \Lambda) = \left\{ A \left| \begin{array}{l} A \text{ is an ASD-connection on } M \times \mathbf{R}, \\ A|_{\Sigma \times \{t\}} \in \hat{\mathcal{B}}_+(\Sigma, E; \Lambda) \quad \text{for each } t \in \mathbf{R} \\ \|F_A\|_{L^2} < \infty \end{array} \right. \right\} \Big/ \{\text{Gauge transform}\}.$$

This should be the moduli space of gradient lines. But in fact we have to modify a bit since in the case the second factor b is large for $(a,b) \in \hat{\mathcal{B}}_+(\Sigma, E)$ one can not control the nonlinear effect. So we put

$$\hat{\mathcal{B}}_{+,\varepsilon}(\Sigma, E; \Lambda) = \left\{ (a,b) \left| \begin{array}{l} [a] \in \Lambda \\ b \in P^+(a) \\ |b| < \varepsilon \end{array} \right. \right\}.$$

We use this space in place of $\hat{\mathcal{B}}_+(\Sigma, E; \Lambda)$ and get $\mathcal{M}_\varepsilon(M, \Lambda)$. (Then there is another trouble since $\hat{\mathcal{B}}_{+,\varepsilon}(\Sigma, E; \Lambda)$ has another boundary $(|b| = \varepsilon.)$) More precisely we put the metric on M such that $\Sigma\times[0,S]\subseteq M$ and put $\mathcal{M}(M,\Lambda)=\mathcal{M}_{C/S}(M,\Lambda)$ for sufficiently large S. Then in the case when everything is transversal we can prove the following result.

Theorem 5.21 :

(1) $\mathcal{M}(M,\Lambda)=\bigcup_{a,b\in R(M,E)\cap\Lambda}\mathcal{M}(M,\Lambda;\alpha,\beta)$, *where* $\mathcal{M}(M,\Lambda;\alpha,\beta)$ *is the set of elements* A *of* $\mathcal{M}(M,\Lambda)$ *such that* $A(\pm\infty)=\alpha,\beta$.

(2) *We put*

$$\mathcal{M}(\alpha,\beta;M,\Lambda)=\bigcup_{\ell}\mathcal{M}_\ell(\alpha,\beta;M,\Lambda),$$

where $A\in\mathcal{M}(\alpha,\beta;M,\Lambda)$ *is contained in* $\mathcal{M}_\ell(\alpha,\beta;M,\Lambda)$ *if the Maslov index of* $\left[A|_{\Sigma\times\boldsymbol{R}}\right]\in\pi_1(\Lambda)$ *is* ℓ. *Then there exists a map* $\mu:\Lambda(M)\cap\Lambda\to\boldsymbol{Z}/4\boldsymbol{Z}$ *such that* $\mathcal{M}_\ell(\alpha,\beta;M,\Lambda)$ *is a manifold of dimension* $\mu(\beta)-\mu(\alpha)+\ell$.

(3) *Let* $\mu(\beta)-\mu(\alpha)+\ell=2$ *then the space* , $\overline{\mathcal{M}}_\ell(\alpha,\beta;M,\Lambda)=\mathcal{M}_\ell(\alpha,\beta;M,\Lambda)/\boldsymbol{R}$ *is compactified such that its boundary is :*

$$\partial\overline{\mathcal{M}}_\ell(\alpha,\beta;M,\Lambda)=\bigcup_{\substack{\mu(\gamma)=\mu(\alpha)+1\\ \ell'+\ell''=\ell}}\overline{\mathcal{M}}_{\ell'}(\alpha,\gamma;M,\Lambda)\times\overline{\mathcal{M}}_{\ell''}(\gamma,\beta;M,\Lambda).$$

(4) *Let* $\mu(\beta)-\mu(\alpha)+\ell=1$. *Then* $\overline{\mathcal{M}}_\ell(\alpha,\beta;M,\Lambda)$ *is a finite set. Furthermore there exists* $\varepsilon(S)>0$ *such that* $\varepsilon(S)\to 0$ *as* $S\to\infty$ *and that the following holds. Let* $A\in\mathcal{M}(\alpha,\beta;M,\Lambda)$, $A|_{\Sigma\times\boldsymbol{R}}=a_t+b_t$, *then :*

$$\|b_t\|<\varepsilon(S)/S.$$

The property (4) exclude the extra end we mentioned before. Using Theorem 5.21 one can imitate the construction of Chapter one and obtained a chain complex $HF(M)(\Lambda)$ (over Novikov ring.) Namely we put

$$HF_*(M)(\Lambda) = \bigoplus_{\alpha\in R(M;E)\cap\Lambda} \mathbf{Z}[T][[T^{-1}]]\cdot[p]$$

$$\partial[\alpha] = \sum <\partial\alpha, T^k\beta> T^k[\beta],$$

$$<\partial\alpha, T^k\beta> = \#\overline{\mathcal{M}}_k(\alpha,\beta;M,\Lambda), \quad \text{(counted with sign.)}$$

To prove Theorem 5.13 (1), we also need to construct a chain map $HF(\Lambda_1,\Lambda_2)_2 : C(\Lambda_1,\Lambda_2) \to Hom(HF(M)(\Lambda_1), HF(M)(\Lambda_2))$ for each BS-orbits Λ_1,Λ_2. For this purpose we consider the following moduli space

$$\mathcal{M}(M,\Lambda_1,\Lambda_2) = \left\{ (A,t_0) \;\middle|\; \begin{array}{l} A \text{ is an ASD-connection on } M\times\mathbf{R}, \\ A|_{\Sigma\times\{t\}} \in \hat{\mathcal{B}}_{+,\varepsilon}(\Sigma,E;\Lambda_1) \quad \text{for each } t\le t_0, \\ A|_{\Sigma\times\{t\}} \in \hat{\mathcal{B}}_{+,\varepsilon}(\Sigma,E;\Lambda_2) \quad \text{for each } t\ge t_0, \\ A|_{\Sigma\times\{t_0\}} \in \Lambda_1\cap\Lambda_2, \\ \|F_A\|_{L^2} < \infty \end{array} \right\} \Big/ \{\text{Gauge transform}\}.$$

(Here $\varepsilon = C/S$.)

This space is decomposed as

$$\mathcal{M}(M,\Lambda_1,\Lambda_2) = \bigcup_{\substack{x\in\Lambda_1\cap\Lambda_2 \\ \alpha\in R(M)\cap\Lambda_1 \\ \beta\in R(M)\cap\Lambda_2}} \mathcal{M}(M,\Lambda_1,\Lambda_2;x,\alpha,\beta).$$

Here $A|_{\Sigma\times\{t_0\}} = x$, $A|_{M\times\{-\infty\}} = \alpha$, $A|_{M\times\{+\infty\}} = \beta$ for $[A,t_0]\in\mathcal{M}(M,\Lambda_1,\Lambda_2;x,\alpha,\beta)$. We can prove a similar dimension formula as Theorem 5.19 (2). (Again $\mathcal{M}(M,\Lambda_1,\Lambda_2;x,\alpha,\beta)$ decomposes to $\bigcup_k \mathcal{M}_k(M,\Lambda_1,\Lambda_2;x,\alpha,\beta)$.) Also we can prove the following analogy of Theorem 5.19 (3).

(5.20)

$$\begin{aligned}\partial\overline{\mathcal{M}}(M,\Lambda_1,\Lambda_2;x,\alpha,\beta) = &\bigcup_{\gamma} \overline{\mathcal{M}}(M,\Lambda_1,\Lambda_2;x,\alpha,\gamma)\times\overline{\mathcal{M}}(M,\Lambda_1,\Lambda_2;x,\gamma,\beta) \\ &\bigcup_{y} \overline{\mathcal{M}}(M,\Lambda_1,\Lambda_2;y,\alpha,\beta)\times\mathcal{M}(\Lambda_1,\Lambda_2;y,x).\end{aligned}$$

We put

$$HF(M)_2(\Lambda_1,\Lambda_2)(x)(\alpha) = \sum_{\beta,k} \# \overline{\mathcal{M}}_k(M,\Lambda_1,\Lambda_2;x,\alpha,\beta)\cdot T^k \beta$$

Formula (3) of Definition 5.8 follows from (5.20). The construction of $HF(\Lambda_1,\Lambda_2)_k$ $(k \geq 3)$ is similar. This is an outline of the proof of Theorem 5.11 (1).

To prove Theorem 5.11 (2) we consider the following 4-manifold, X :

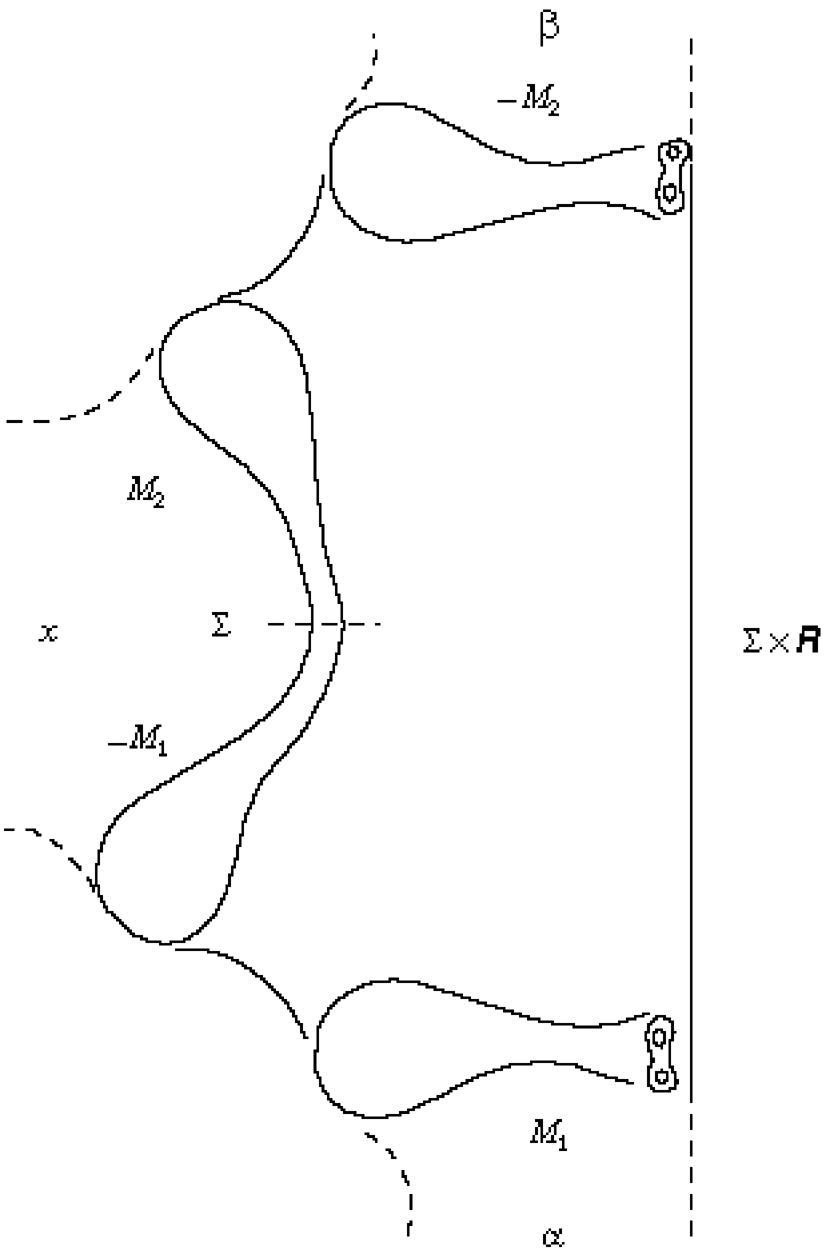


Figure 5.23

and consider the following moduli space :

$$\mathcal{M}(X;x,\alpha,\beta;\Lambda)=\left\{ A \,\middle|\, \begin{array}{l} A \text{ is an ASD - connection on } X, \\ A\big|_{-M_1\#M_2}=x, \\ A\big|_{M_1}=\alpha, \qquad A\big|_{M_2}=\beta, \\ A\big|_{\Sigma\times\{t\}}=\hat{\mathcal{B}}_{+,\varepsilon}(\Sigma,E;\Lambda). \end{array} \right\} \Big/ \{\text{Gauge transform}\}$$

(Here $x\in R(-M_1\#M_2;E)$, $\alpha\in R(M_1;E)\cap\Lambda$, $\beta\in R(M_2;E)\cap\Lambda$.) Again the space decompose as $\mathcal{M}(X;x,\alpha,\beta;,\Lambda)=\bigcup \mathcal{M}_\ell(X;x,\alpha,\beta;,\Lambda)$. Then we put

$$\varphi(x)_1(\Lambda)([\alpha])=\sum_{\beta,\ell}\#\mathcal{M}_\ell(X;x,\alpha,\beta;,\Lambda)\cdot T^\ell[\beta].$$

Hence $\varphi(x)_1(\Lambda)\in Hom(HF_*(M_1;E)(\Lambda),HF_*(M_2;E)(\Lambda))$. To define an A^∞-functor $\varphi(x)$ we need to define

$$\varphi(x)_2(\Lambda_1,\Lambda_2),\overline{\varphi(x)_2}(\Lambda_1,\Lambda_2)\in Hom(CF(\Lambda_1,\Lambda_2)\otimes HF_*(M_1;E)(\Lambda_1),HF_*(M_2;E)(\Lambda_2)).$$

For this purpose we consider the moduli space

$$\mathcal{M}(X;x,\alpha,\beta,\gamma;\Lambda_1,\Lambda_2)=\frac{\left\{(A,t_0) \,\middle|\, \begin{array}{l} A \text{ is an ASD - connection on } X, \\ A\big|_{-M_1\#M_2}=x, \\ A\big|_{M_1}=\alpha, \qquad A\big|_{M_2}=\beta, \\ A\big|_{\Sigma\times\{t\}}=\hat{\mathcal{B}}_{+,\varepsilon}(\Sigma,E;\Lambda_1) \quad \text{for } t<t_0, \\ A\big|_{\Sigma\times\{t\}}=\hat{\mathcal{B}}_{+,\varepsilon}(\Sigma,E;\Lambda_2) \quad \text{for } t>t_0 \\ A\big|_{\Sigma\times\{t_0\}}=\gamma \end{array} \right\}}{\{\text{Gauge transform}\}}$$

for $x\in R(-M_1\#M_2;E)$, $\alpha\in R(M_1;E)\cap\Lambda_1$, $\beta\in R(M_2;E)\cap\Lambda_2$, $\gamma\in\Lambda_1\cap\Lambda_2$ and decompose it to $\bigcup_k \mathcal{M}_k(X;x,\alpha,\beta,\gamma;,\Lambda_1,\Lambda_2)$. Then we put

$$\varphi(x)_2(\Lambda_1,\Lambda_2)([\gamma]\otimes[\alpha]) = \sum_{\beta,k} \#\mathcal{M}_k(X;x,\alpha,\beta,\gamma;,\Lambda_1,\Lambda_2)\cdot T^k[\beta].$$

$$\overline{\varphi(x)_2}(\Lambda_1,\Lambda_2) = \varphi(\partial x)_2(\Lambda_1,\Lambda_2).$$

$\varphi(x)_3$ etc. can be defined in a similar way. We thus constructed an A^∞- functor $\varphi(x)$.

We omit the discussion of the proof of Theorem 5.13 (3).

Conjecture 5.24 : The chain map in Theorem 5.13 (2) is a chain homotopy equivalence.

See [Fu2] for a discussion about this conjecture. There we also discussed Donaldson polynomial for 4 manifolds with corners.

§ 4 Formal analogy to Jones-Witten invariant and conformal field theory.

In this last section we try to explain that the Floer homology for 3-manifold with boundary is parallel to the Theory of Jones-Witten invariant (Chern-Simons Gauge theory). ([W3])

To see this (formal) analogy we need to study the invariant of the relative Floer homology under various choices.

Besides the perturbation, there are two choice we need in order to define $HF(M;E)$. ($\partial M = \Sigma$). One is the almost complex structure on $R(\Sigma;E)$ and the other is a Rieman metric on M. In fact they are related to each other. In fact if we fix a Riemannian metric on M, then we get one on Σ hence a conformal structure there. Namely we get a complex structure on our Rieman surface. It induces a complex (Kähler structure) on $R(\Sigma;E)$.

First we fix a conformal structure (complex structure) on Σ. Then we obtain an A^∞ -category $C(R(\Sigma;E),J)$. The two metrics on g_1 and g_2 on M (compatible with our conformal structure on the boundary) defines two functors $HF((M,g_1);E)$ and $HF((M,g_2);E)$. Choose a path g_t joining the two metrics. Then using a moduli spaces such as

$$\mathcal{M}_{para}(M,\Lambda) = \left\{ (A,t) \left| \begin{array}{l} A \text{ is an ASD - connection on } M\times\boldsymbol{R} \\ \quad \text{with respect to the metric } g_t, \\ A|_{\Sigma\times\{t\}} \in \hat{\mathcal{B}}_{+,C/S}(\Sigma,E;\Lambda) \quad \text{for each } t\in\boldsymbol{R} \\ \|F_A\|_{L^2} < \infty \end{array} \right. \right\} \Big/ \{\text{Gauge transform }\},$$

we can prove that the two functors $HF((M,g_1);E)$ and $HF((M,g_2);E)$ are chain homotopic to each other.[17] Hence relative Floer homology is invariant of the choice of the metric at the interior of M in that sense.

We now discuss what happens when we change the conformal structure of the boundary. In this case the complex structure of $R(\Sigma;E)$ changes hence the A^∞-category $C(R(\Sigma;E),J)$ changes as well. (They are chain homotopy equivalent to each other if one defines this notion appropriately.)

To see this more systematically, we consider the Teichmüller space $\mathcal{T}_g$. (The space consisting of complex structures on Σ_g, the Rieman surface of genus g.) Roughly speaking we assert that the family of A^∞-category $C(R(\Sigma;E),J)$ $J \in \mathcal{T}_g$ consists of the "flat bundle" over $\mathcal{T}_g$. To state it, let us recall the following Proposition. Let $Vect$ be the category whose object is the vector space (of finite dimension) and the morphism is a linear homomorphism. Since the set of morphisms are abelian groups in this case, we regard it as chain complexes with trivial boundary. Then $Vect$ is an A^∞-category with trivial higher compositions. Let X be a manifold. We consider the A^∞-category ΩX introduced in Chapter one § 4.

Proposition 5.25 : *There exists a one to one correspondence between a flat vector bundle on X and an A^∞-functor from ΩX to $Vect$.*

This proposition is quite obvious. In fact let F be an A^∞-functor from ΩX to $Vect$. We define a bundle E_F on X such that the fibre of it at $x \in X$ is $F(x)$. (Remark that the object of ΩX is a point of X.) We next define a holonomy $P_\ell : F(x) \to F(y)$ for each path ℓ joining x to y such that it depends only on the homotopy class of ℓ. We recall that $C(x,y) = S(\Omega(X,x,y))$, the singular chain complex of the space of path joining x to y . We can regard $[\ell] \in C_0(x,y)$. Hence $F([\ell]) \in Hom(F(x),F(y))$. Suppose that ℓ_1 is homotopic to ℓ_2. Then there is $[\sigma] \in C_1(x,y)$ such that $\partial[\sigma] = [\ell_1] - [\ell_2]$. Then, since F is a chain map it follows that $F([\ell_1]) - F([\ell_2]) = \partial F([\sigma]) = 0$. (Note that the boundary operators are zero in $Vect$.) Hence $F([\ell])$ depends only on the homotopy class of ℓ. We put $P_\ell = F([\ell])$. Thus we obtained a flat bundle on X.

The construction of the opposite direction is similar.

---

[17] **Here we do not define that two A^∞-functors to be chain homotopic. But one can define it in a straight forward way.**

By this proposition, the following statement is equivalent to the statement that the family of A^∞-category $C(R(\Sigma;E),J)$ $J \in \mathcal{T}_g$ consists of the "flat bundle" over $\mathcal{T}_g$.

Let $\mathcal{A}$ be an A^∞-category whose object is an A^∞-category and morphisms is an A^∞-functor.[18]

Theorem 5.26 : *There is an A^∞-functor from $\Omega\mathcal{T}_g$ to $\mathcal{A}$ such that $J \in \mathcal{T}_g = Ob(\Omega\mathcal{T}_g)$ is send to $C(R(\Sigma;E),J)$.*

Sketch of the proof Let $\sigma \in C(J_1,J_2)$. In other words $\sigma : \Delta^k \times [0,1] \to \mathcal{T}_g$. Then for $a,b \in \Lambda_1 \cap \Lambda_2$, we consider the moduli space

$$\bigcup_{(x,t)\in\Delta^k\times[0,1]} \mathcal{M}_{symp}(a,b;(\mathcal{T}_g,\sigma(x,t)).$$

Using this moduli space and its compactification we can construct a map

$$C_*(J_1,J_2) \otimes C(\Lambda_1,\Lambda_2;(\mathcal{T}_g,J_1)) \to C(\Lambda_1,\Lambda_2;(\mathcal{T}_g,J_2)).$$

This map defines an A^∞*-functor from* $\Omega\mathcal{T}_g$ *to* $\mathcal{A}$.

"Theorem" 5.27 : *Relative Floer homology is a "flat section" of the "flat bundle" obtained in* Theorem 5.26.

We do not try to make this statement precise in this article.

We recall that in the Theory of Jones-Witten invariant, Witten considered the conformal block, that is the flat bundle (projectively flat vector bundle) over Teichmüller space $\mathcal{T}_g$. In case when $\partial M^3 = \Sigma_g$, the relative Jones-Witten invariant is regarded as a flat section of this bundle. ([W3]) Theorems 5.26 and 5.27 are formally analogous to this.

We next consider the fundamental group of $\mathcal{T}_g$, (the mapping class group.) For each element $\gamma \in \pi_1\mathcal{T}_g$, we obtain a closed 3-manifold M_γ, with is a Σ_g bundle over circle with monodromy γ. In the case of Jones-Witten invariant, the invariant for M_γ is equal

[18] **There is a set theoretical trouble in this definition. But since there is an obvious way to remove this trouble in the case we need, we do not mind that trouble here.**

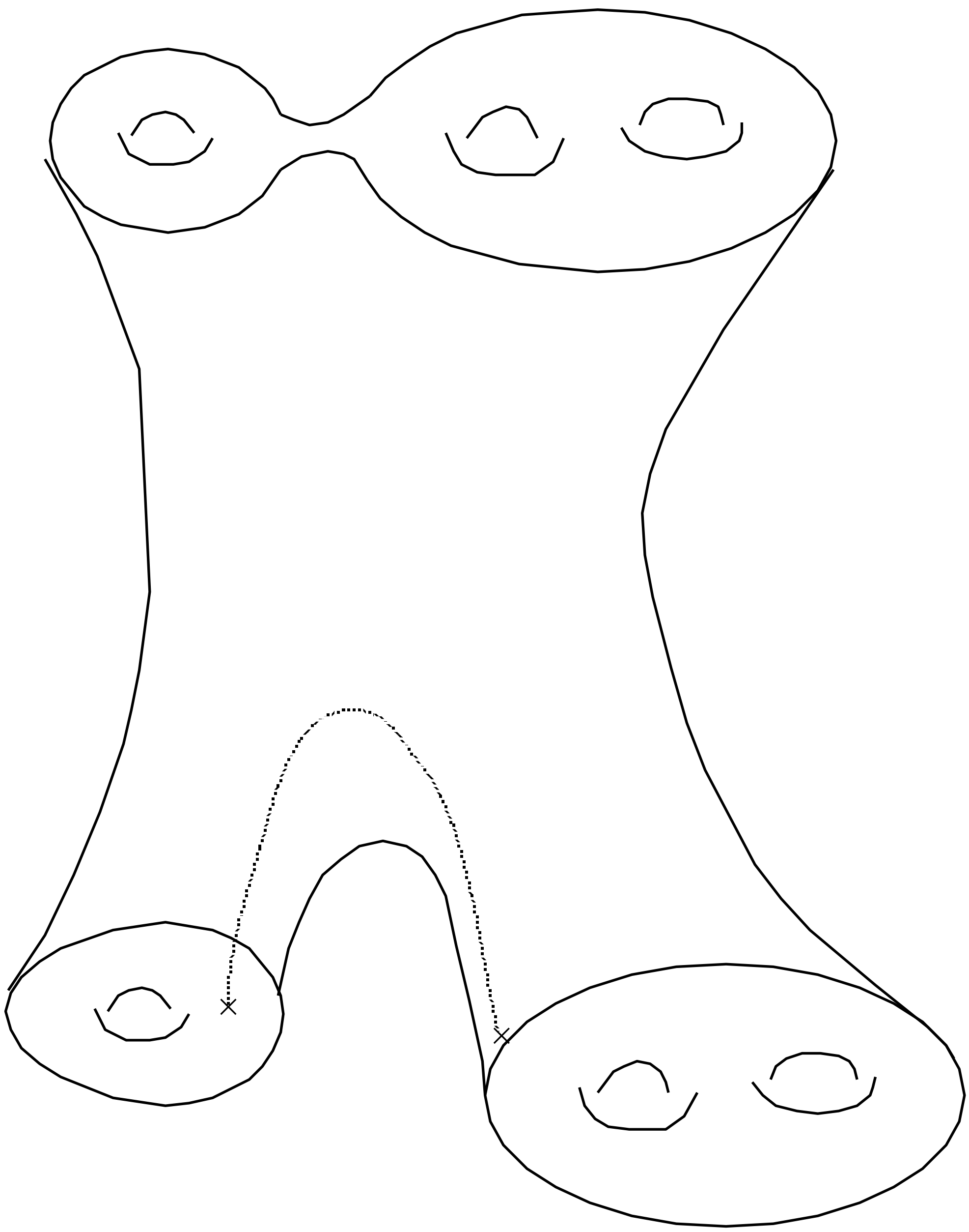
to the trace of the holonomy of conformal block with respect to this loop.

In our situation we have $F(\gamma): C(R(\Sigma_g;E)) \to C(R(\Sigma_g;E))$. (The "holonomy" of the "flat bundle" given by Theorem 5.26.) In fact this functor is described as follows. $\gamma \in \pi_1 \mathcal{T}_g$ is regarded as an isotopy class of self diffeomorphism of our surface. Hence it induces a symplectic self-diffeomorphism φ_γ of $R(\Sigma_g;E)$. This symplectic diffeomorphism induces an isomorphism $\varphi_{\gamma*}: C(R(\Sigma_g;E)) \to C(R(\Sigma_g;E))$, which is exactly equal to our functor.

What is the trace of it ? We recall that for a matrix A we have $Tr(A) = \langle I, A \rangle$, where I is the identity matrix and $\langle , \rangle$ is the invariant inner product.

Now $\varphi_{\gamma*} = F(\gamma): C(R(\Sigma_g;E)) \to C(R(\Sigma_g;E))$ may be identified to the graph G_{φ_γ} of our symplectic diffeomorphism $\varphi_\gamma: R(\Sigma_g;E) \to R(\Sigma_g;E)$. On the other hand the identity is identified to the diagonal Δ of $R(\Sigma_g;E) \times R(\Sigma_g;E)$. Thus $\text{Trace}(F_\gamma) \leftrightarrow \left\langle G_{\varphi_\gamma}, \Delta \right\rangle$, in our situation. Therefore by the theorem of Dostglou-Salamon $\text{Trace}(F_\gamma) = HF(M_\gamma)$. Again our analogy works.

Finally we discuss the "Fusion rule" or "Verlinde formula". In the case of conformal field theory, roughly speaking, they controls how the conformal block behave under the change of genus of Rieman surface or the degeneration of it. The analogy of it in Floer homology should be obtained to consider the relative Floer homology of the following "3-dimensional pants."



Here the dotted line is regarded as a Wilson line. ([W3]) Hence to consider the Floer homology of the 3-dimensional pants as above, we need to generalize our story to the case when 3-manifold has arcs I such that $\partial I = I \cap \partial M$.[19] Then our Rieman surface has marked points on it. In this case it is natural to take the moduli space of parabolic bundles (in place of $R(\Sigma, E)$) as our symplectic manifold.

Let us write it as $R((\Sigma,(x_1,\cdots,x_k),E)$. The first trouble we meet is that this symplectic manifold is not necessary pseudo-Einstein. So provably, we need to use more complicated Novikov ring to justify the discussion of Chapters 3,4, (which the author did not know how to do yet.) Then let us suppose that one can somehow find an A^∞-category,

[19] It may also be possible to involve (closed) circle.

$C(R((\Sigma,(x_1,\cdots,x_k),E))$. So in case $\partial(M,(I_1,\cdots,I_{k/2}))=(\Sigma,(x_1,\cdots,x_k))$, we need to find a functor $HF((M,(I_1,\cdots,I_{k/2})):C(R((\Sigma,(x_1,\cdots,x_k),E))\to \mathcal{C}h$. If we try to imitate the construction of this chapter one needs to study the moduli space of ASD-connection on $(M,(I_1,\cdots,I_{k/2}))\times \boldsymbol{R}$ with appropriate boundary condition. Here the submanifold $(I_1,\cdots,I_{k/2})\times \boldsymbol{R}$ plays a role as the support of the singularity of our ASD connection.

In fact such a moduli space is studied extensively by Sibner-Sibner [SS] and Kronheimer-Mrowka [KM]. What seems related to our problem is that Kronheimer found a construction which is quite similar to the Novikov ring there. ([Kr].) Provably we can join these two kinds of Novikov ring somehow.

But all these has still a lot of trouble to be settled. Hence to find what should be the Fusion rule in relative Floer homology is the problem left for future research.

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