Deformation classes of invertible field theories and the Freed–Hopkins conjecture

Daniel Grady

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- By Wick-rotating to Euclidean field theories, we can use Segal's axioms to model the low energy effective theory.
- Unitarity manifests itself as reflection positivity after Wick rotation.

Theorem (Freed–Hopkins)

Let $H = \operatorname{colim}_{d\to\infty} H_d$ be a stable symmetry type. There is a bijective correspondence

$$\left\{\begin{array}{l}\text{deformation classes of reflection}\\ \text{positive invertible } d\text{-dimensional}\\ \text{extended topological field theories}\\ \text{with symmetry type } (H_d, \rho_d)\end{array}\right\} \cong [MTH, \Sigma^{d+1} I_{\mathbb{Z}(1)}]_{\mathrm{tor}}$$

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Conjecture (Freed–Hopkins)

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- In the case of invertible field theories, the FT take values in $\mathrm{Line} \subset \mathrm{Vect}$.

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- The inclusion of isomorphism classes of topological field theories into all field theories is

$$\begin{array}{ccc} [X, \mathbf{B}(\mathbb{C}^\times)^\delta] = H^1(X; \mathbb{C}^\times) & \longrightarrow & [X, \mathbf{B}_\nabla \mathbb{C}^\times] \\ & \searrow^{\beta} & \downarrow \text{def. classes} \\ & & H^2(X; \mathbb{Z}) \end{array}$$

- The image of β is the torsion subgroup.

Reflection positivity

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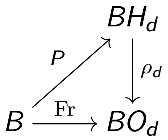
- An (H_d, ρ_d)-structure on a bordism an H_d-subbundle $P \to B$ of the bundle of orthonormal frames. Equivalently, it is a lift:



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- ϕ_σ induces an involution β on the bordism category, sending $(B, P \to B)$ to $(B, P' \to B)$. The bundle $P' \to B$ is the H_d-bundle obtained by precomposing the original H_d-action on P by the automorphism ϕ_σ.

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- Given a field theory with reflection structure Z, we have a hermitian form

  $$h : Z(B) \otimes \overline{Z(B)} \cong Z(B) \otimes Z(\beta B) \cong Z(B) \otimes Z(B^\vee) \to \mathbb{C}.$$

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- If h is positive definite, we say that the field theory is *positive*.

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Theorem (GMTW, Schommer-Pries)

The homotopy type of the fully-extended bordism category $\mathrm{Bord}_d^{H_d}$ is $\Sigma^d MTH_d$.

- A field theory $Z : \mathrm{Bord}_d^{H_d} \to \Sigma^{d+1} I_{\mathbb{Z}(1)}$ canonically factors as

$$\begin{array}{ccc} \mathrm{Bord}_d^{H_d} & \longrightarrow & \Sigma^{d+1} I_{\mathbb{Z}(1)} \\ \downarrow & \nearrow & \\ \Sigma^d MTH_d & & \end{array}$$

- The involution β induces an involution on $\Sigma^d MTH_d$.

- By definition, the Anderson dual $I_{\mathbb{Z}(1)}$ sits in a long homotopy fiber/cofiber sequence

$$\ldots \to \Sigma^d I_{\mathbb{Z}(1)} \to \Sigma^d I_{\mathbb{C}} \to \Sigma^d I_{\mathbb{C}^\times} \to \Sigma^{d+1} I_{\mathbb{Z}(1)} \to \ldots .$$

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- The space of $\mathbb{Z}/2$-actions on $I_{\mathbb{Z}(1)}$ that is compatible with the conjugation action is not contractible. Freed and Hopkins make a prefered choice of action γ.
- The deformation classes of (nontopological) reflection theories are conjectured correspond to maps of equivariant spectra

$$Z : (\Sigma^d MTH_d)^\beta \to (\Sigma^{d+1} I_{\mathbb{Z}(1)})^\gamma.$$

- I will not discuss positivity in the extended setting.

- The map $\Sigma^d I_{\mathbb{C}^\times} \to \Sigma^{d+1} I_{\mathbb{Z}(1)}$ in the long fiber/cofiber sequence induces a map

$$[\Sigma^d MTH_d, \Sigma^d I_{\mathbb{C}^\times}] \to [\Sigma^d MTH_d, \Sigma^{d+1} I_{\mathbb{Z}(1)}]$$

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- To prove the theorem, one must do the following:
  1. Enhance the H_d-structure on bordisms to a *differential* H_d-structure (including connections on bundles).
  2. Construct a geometric refinement $\Sigma^d I_{\mathbb{C}^\times_{\mathrm{sm}}}$ whose "deformation spectrum" is $\Sigma^{d+1} I_{\mathbb{Z}(1)}$. The refinement should see the smooth structure of $\mathbb{C}^\times$.

The main theorem

Theorem (G.)

The following spaces are isomorphic in the homotopy category of spaces

1. *Smooth deformations of field theories with smooth (H_d, ρ_d)-structure:* $I_d(\mathcal{H}_d) := \mathrm{Fun}^{\otimes}(\mathrm{Bord}_d^{\mathcal{H}_d}, \Sigma^d I_{\mathbb{C}^\times_{\mathrm{sm}}})$
2. *Smooth deformations of field theories with differential (H_d, ρ_d)-structure:* $I_d(\mathcal{H}_d^\nabla) := \mathrm{Fun}^{\otimes}(\mathrm{Bord}_d^{\mathcal{H}_d^\nabla}, \Sigma^d I_{\mathbb{C}^\times_{\mathrm{sm}}})$
3. *Smooth deformations of field theories with flat (H_d, ρ_d)-structure:* $I_d(\mathcal{H}_d^{\mathrm{fl}}) := \mathrm{Fun}^{\otimes}(\mathrm{Bord}_d^{\mathcal{H}_d^{\mathrm{fl}}}, \Sigma^d I_{\mathbb{C}^\times_{\mathrm{sm}}})$
4. *The space of morphisms of spectra:* $\mathrm{Map}(\Sigma^d MTH_d, \Sigma^{d+1} I_{\mathbb{Z}(1)})$.

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- The presheaf $\mathcal{H}_d^\nabla$ is given by the homotopy pullback

$$\begin{array}{ccc} \mathcal{H}_d^\nabla & \longrightarrow & \mathbf{B}_\nabla H_d \\ \downarrow & & \downarrow \\ \mathrm{Riem} & \xrightarrow{\mathrm{LC}} & \mathbf{B}_\nabla \mathrm{GL}_d \end{array}$$

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- A vertex in $\mathcal{H}_d^{\nabla}(M \to U)$ is a fiberwise principal H_d-bundle with connection, a Riemannian metric and a connection preserving isomorphism of bundles between the associated bundle with connection with the fiberwise tangent bundle, with the Levi-Civita connection.

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  where the hom is taken in sheaves of abelian groups.
- There is a $\mathbb{Z}/2$-action on $I_{\mathbb{C}^\times_{\mathrm{sm}}}$, compatible with complex conjugation.

- Recall the functor

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- Since $\mathbb{C}$ deformation retracts through group homomorphisms to 0, applying $\int$ gives an equivalence

$$\int \Sigma^d I_{\mathbb{C}^\times_{\mathrm{sm}}} \xrightarrow{\simeq} \Sigma^{d+1} I_{\mathbb{Z}(1)}$$

- The homotopy type of the bordism category $\mathrm{Bord}_d^{\mathcal{H}_d}$ is computed as the composition of two functors

$$C^\infty \mathrm{Cat}^{\otimes}_{\infty,d} \xrightarrow{|\cdot|} Sh(Cart; Sp) \xrightarrow{\int} Sp$$

Theorem (G., Pavlov)

Fix $d \geq 0$. We have an equivalence

$$\int |\mathrm{Bord}_d^{\mathcal{H}_d^{\mathrm{fl}}}| \simeq \Sigma^d MTH_d$$

- The canonical map $\mathcal{H}_d^{\mathrm{fl}} \to \mathcal{H}_d^{\nabla}$ of flat bundles into all bundles induces an equivalence

$$\int \mathrm{Bord}_d^{\mathcal{H}_d^{\mathrm{fl}}} \to \int \mathrm{Bord}_d^{\mathcal{H}_d^{\nabla}}$$

- Argument has an h-principle flavor.

- We have an equivalence

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and a map (taking deformations)

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The right side was computed as

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- Since $\Sigma^d I_{\mathbb{C}^\times}$ is homotopy invariant, we have also have

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The right side was computed as

$$\mathrm{Fun}^{\otimes}(\textstyle\int |\mathrm{Bord}_d^{\mathcal{H}_d^{\nabla}}|, \int \Sigma^d I_{\mathbb{C}^\times_{\mathrm{sm}}}) \simeq \mathrm{Map}(\Sigma^d MTH_d, \Sigma^{d+1} I_{\mathbb{Z}(1)})$$

- Since $\Sigma^d I_{\mathbb{C}^\times}$ is homotopy invariant, we have also have

$$\mathrm{Fun}^{\otimes}(\textstyle\int |\mathrm{Bord}_d^{\mathcal{H}_d^{\nabla}}|, \Sigma^d I_{\mathbb{C}^\times}) \simeq \mathrm{Map}(\Sigma^d MTH_d, \Sigma^d I_{\mathbb{C}^\times})$$

- The canonical inclusion $I_{\mathbb{C}^\times} \to I_{\mathbb{C}^\times_{\mathrm{sm}}}$ therefore induces a map

$$\mathrm{Map}(\Sigma^d MTH_d, \Sigma^d I_{\mathbb{C}^\times}) \to \mathrm{Map}(\Sigma^d MTH_d, \Sigma^{d+1} I_{\mathbb{Z}(1)})$$

- We have an equivalence

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_d^{\mathcal{H}_d^{\nabla}}, \Sigma^d I_{\mathbb{C}^\times_{\mathrm{sm}}}) \overset{\simeq}{\leftarrow} \mathrm{Fun}^{\otimes}(|\mathrm{Bord}_d^{\mathcal{H}_d^{\nabla}}|, \Sigma^d I_{\mathbb{C}^\times_{\mathrm{sm}}})$$

and a map (taking deformations)

$$\mathrm{Fun}^{\otimes}(|\mathrm{Bord}_d^{\mathcal{H}_d^{\nabla}}|, \Sigma^d I_{\mathbb{C}^\times_{\mathrm{sm}}}) \overset{\int}{\to} \mathrm{Fun}^{\otimes}(\textstyle\int |\mathrm{Bord}_d^{\mathcal{H}_d^{\nabla}}|, \int \Sigma^d I_{\mathbb{C}^\times_{\mathrm{sm}}}).$$

The right side was computed as

$$\mathrm{Fun}^{\otimes}(\textstyle\int |\mathrm{Bord}_d^{\mathcal{H}_d^{\nabla}}|, \int \Sigma^d I_{\mathbb{C}^\times_{\mathrm{sm}}}) \simeq \mathrm{Map}(\Sigma^d MTH_d, \Sigma^{d+1} I_{\mathbb{Z}(1)})$$

- Since $\Sigma^d I_{\mathbb{C}^\times}$ is homotopy invariant, we have also have

$$\mathrm{Fun}^{\otimes}(\textstyle\int |\mathrm{Bord}_d^{\mathcal{H}_d^{\nabla}}|, \Sigma^d I_{\mathbb{C}^\times}) \simeq \mathrm{Map}(\Sigma^d MTH_d, \Sigma^d I_{\mathbb{C}^\times})$$

- The canonical inclusion $I_{\mathbb{C}^\times} \to I_{\mathbb{C}^\times_{\mathrm{sm}}}$ therefore induces a map

$$\mathrm{Map}(\Sigma^d MTH_d, \Sigma^d I_{\mathbb{C}^\times}) \to \mathrm{Map}(\Sigma^d MTH_d, \Sigma^{d+1} I_{\mathbb{Z}(1)})$$

- Taking π_0 the image is the torsion subgroup of deformation classes of topological theories.

Thank you!