CHAPTER III: Green Functions and Propagators

3. 1. Introduction: Scalar field

- Lagrangian with real scalar field $\phi(x)$ and "potential" $U(\phi)$.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 - U(\phi)$$

  – Field equations

$$(\Box + m^2)\phi + \frac{\partial U}{\partial\phi} = 0$$

$$\Rightarrow (\Box + m^2)\phi(x) = -\mathcal{J}(x) \qquad \text{with } \mathcal{J}(x) = \frac{\partial U}{\partial\phi}$$

  – Free field equation: Klein-Gordon equation

$$(\Box + m^2)\phi(x) = 0$$

  – Solution:

$$\phi(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3\, 2\omega_k}\Big(a_k\, e^{-ik\cdot x} + a_k^\dagger\, e^{ik\cdot x}\Big)$$

  – Commutation relations for a_k and $a_k^\dagger$:

$$\left[a_k,\, a_{k'}^\dagger\right] = (2\pi)^3\, 2\omega_k\, \delta^3\big(\vec{k} - \vec{k}'\big)$$

$$\left[a_k,\, a_{k'}\right] = \left[a_k^\dagger,\, a_{k'}^\dagger\right] = 0$$

- Definition of time-ordered product:

$$\mathcal{T}\,\phi(x')\phi(x) = \theta(t'-t)\,\phi(x')\phi(x) + \theta(t-t')\,\phi(x)\phi(x')$$

- Correlation function:

$$i\Delta_F(x'-x) = \langle 0|\mathcal{T}\,\phi(x')\phi(x)|0\rangle$$

  Feynman propagator: Green's function ↔ time-ordered correlation function

- Apply $\Box_{x'} + m^2$ to Δ_F:

  where

$$\Box_{x'} \equiv \frac{\partial^2}{\partial t'^2} - \vec{\nabla}^2_{\vec{x}'}$$

$$\frac{\partial}{\partial t'}\mathcal{T}\,\phi(x')\phi(x) = \frac{\partial}{\partial t'}\left[\theta(t'-t)\phi(x')\phi(x) + \theta(t-t')\,\phi(x)\phi(x')\right]$$
$$= \mathcal{T}\,\frac{\partial\phi(x')}{\partial t'}\,\phi(x) + \underbrace{\delta(t'-t)\phi(x')\phi(x) - \delta(t'-t)\phi(x)\phi(x')}_{\delta(t'-t)\,[\phi(x'),\,\phi(x)]=0}$$

$$\frac{\partial^2}{\partial t'^2}\mathcal{T}\,\phi(x')\phi(x) = \mathcal{T}\,\frac{\partial^2\phi(x')}{\partial t'^2}\,\phi(x) + \underbrace{\delta(t'-t)\left[\frac{\partial\phi(x')}{\partial t'},\,\phi(x)\right]}_{-i\delta^4(x'-x)}$$
$$= \mathcal{T}\,\big(\vec{\nabla}^2_{\vec{x}'} - m^2\big)\phi(x')\phi(x) - i\delta^4(x'-x)$$

$$\Rightarrow \qquad (\Box_{x'} + m^2)\mathcal{T}\,\phi(x')\phi(x) = -i\delta^4(x'-x)$$

$$\Rightarrow \qquad \boxed{(\Box_{x'} + m^2)\Delta_F(x'-x) = -\delta^4(x'-x)}$$

- Solution of inhomogeneous wave equation:

$$\phi(x) = \int \mathrm{d}^4x'\,\Delta_F(x'-x)\,\mathcal{J}(x')$$

- Propagator Δ_F is the Green's function of the Klein-Gordon equation.

- Fourier representation:

$$\boxed{\Delta_F(x'-x) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{e^{-ik\cdot(x'-x)}}{k^2 - m^2 + i\epsilon}}$$

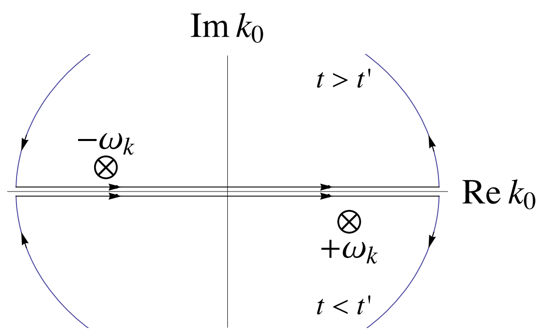
$$(\Box_{x'} + m^2)\Delta_F(x'-x) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{(\Box_{x'} + m^2)e^{-ik\cdot(x'-x)}}{k^2 - m^2(+i\epsilon)}$$
$$= \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{(-k^2 + m^2)e^{-ik\cdot(x'-x)}}{k^2 - m^2(+i\epsilon)}$$
$$= -\int \frac{\mathrm{d}^4k}{(2\pi)^4}\, e^{-ik\cdot(x'-x)}$$
$$= -\delta^4(x'-x)$$

- Poles at $k^2 = m^2$:

$$\omega_k = \sqrt{\vec{k}^{\,2} + m^2} \;\Rightarrow\; k_0 = \pm\omega_k$$

- Cauchy's integral formula:

$$\Delta_F(x'-x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{\mathrm{d}k_0}{2\pi} \frac{e^{-ik_0(t'-t)}}{k_0^2 - \vec{k}^2 - m^2 + i\epsilon} \; e^{i\vec{k}\cdot(\vec{x}'-\vec{x})}$$



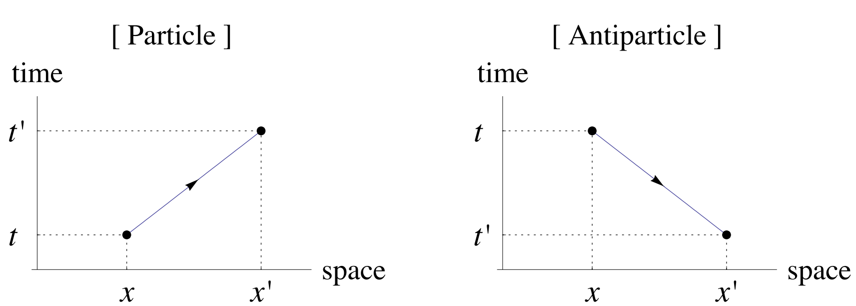
Using $\oint \frac{f(z)}{z-a} = 2\pi i\, f(a)$

$$\omega_k = \sqrt{\vec{k}^2 + m^2} - i\delta$$

- $t' > t$; $\Delta_F(x'-x) = -i \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{e^{-i\omega_k(t'-t)}}{2\omega_k} e^{i\vec{k}\cdot(\vec{x}'-\vec{x})}$
- $t' < t$; $\Delta_F(x'-x) = i \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{e^{i\omega_k(t'-t)}}{-2\omega_k} e^{i\vec{k}\cdot(\vec{x}'-\vec{x})}$
- For all t, t':

$$\Delta_F(x'-x) = -i \int \frac{\mathrm{d}^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}'-\vec{x})} \frac{1}{2\omega_k} \left[\theta(t'-t)e^{-i\omega_k(t'-t)} + \theta(t-t')e^{i\omega_k(t'-t)}\right] \quad (3.15)$$

  - The first term on the *r.h.s.* of (3.15) describes a particle running forward time with positive energy ω_k and $t'-t > 0$.
  - The second term describes a "particle" running backward time with negative ω_k and $t - t' > 0$: antiparticle.



3. 2. Dirac propagator

- Time ordered product of Dirac fields: $\mathcal{T}\,\psi_\alpha(x')\bar{\psi}_\beta(x)$

▷ Dirac propagator:

$$iS_F(x'-x)_{\alpha\beta} = \langle 0|\mathcal{T}\,\psi_\alpha(x')\bar{\psi}_\beta(0)|0\rangle$$

This is the Green's function of the free Dirac equation:

$$(i\gamma_\mu\partial^\mu_{x'} - m)S_F(x'-x)_{\alpha\beta} = \delta^4(x'-x)\delta_{\alpha\beta}$$

- Fourier representation

$$S_F(x'-x) = \int \frac{\mathrm{d}^4p}{(2\pi)^3} e^{-ip\cdot(x'-x)} \frac{\gamma_\mu p^\mu + m}{p^2 - m^2 + i\epsilon}$$

$$\begin{aligned}(i\gamma_\mu\partial^\mu_{x'} - m)S_F(x'-x) &= \int \frac{\mathrm{d}^4p}{(2\pi)^4} (i\gamma_\mu\partial^\mu_{x'} - m)\, e^{-ip\cdot(x'-x)} \frac{\gamma_\nu p^\nu + m}{p^2 - m^2 + i\epsilon} \\ &= \int \frac{\mathrm{d}^4p}{(2\pi)^4} \frac{(\gamma_\mu p^\mu - m)(\gamma_\nu p^\nu + m)}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x'-x)} \\ &= \int \frac{\mathrm{d}^4p}{(2\pi)^4} \frac{p^2 - m^2}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x'-x)} \\ &= \delta^4(x'-x)\end{aligned}$$

- Feynman propagator of spin-$\frac{1}{2}$ particle in momentum space:

$$\begin{aligned}\tilde{S}_F(p) &= \int \mathrm{d}^4x\, e^{ip\cdot(x'-x)}\, S_F(x'-x) \\ &= \frac{\not{p} + m}{p^2 - m^2 + i\epsilon} \qquad (\not{p} \equiv \gamma_\mu p^\mu)\end{aligned}$$

3. 3. Free gluon propagator

- Free gluon Green's function:

$$iD^{\mu\nu}_{ab}(x'-x) = \langle 0|\mathcal{T}\,A^\mu_a(x')A^\nu_b(x)|0\rangle$$

$$D^{\mu\nu}_{ab}(x'-x) = \delta_{ab} \int \frac{\mathrm{d}^4q}{(2\pi)^4} \frac{d^{\mu\nu}(q)}{q^2}$$

with $d^{\mu\nu}(q) = -g^{\mu\nu} + (1-\xi)\,\frac{q^\mu q^\nu}{q^2 + i\epsilon}$

$$\begin{cases} \xi = 1 \;:\; \text{Feynman gauge} \\ \xi = 0 \;:\; \text{Landau gauge} \end{cases}$$

CHAPTER IV: S-Matrix and Feynman Rules

4.1. Definition: S-matrix, T-matrix and cross section

- S-matrix

$$S_{BA} = \langle B,\, t\to\infty | A,\, t\to -\infty\rangle$$

$|A\rangle$ and $|B\rangle$ are asymptotic states:

$$|A,t\rangle = e^{iHt}|A,t=0\rangle$$

- T-matrix

$$\langle B|S|A\rangle \equiv S_{BA} = \delta_{BA} + i(2\pi)^4\delta^4(p_A - p_B)T_{BA}$$
$$\langle B|T|A\rangle \equiv T_{BA} = -\mathcal{M}_{BA}$$

- Differential cross section for $A \to B$
  - Prototype: two particles colliding in initial state: $A = a_1 + a_2$

$$\mathrm{d}\sigma(a_1 + a_2 \to B) = \frac{\mathcal{W}(a_1 + a_2 \to B)}{J_A}\,\mathrm{d}N_B$$

  - $\mathcal{W}(a_1 + a_2 \to B)$: Transition probability for $A \to B$ per unit time.
  - $\mathrm{d}N_B$: Phase space element in the final state B.
  - J_A: Flux of incoming particles in state A.

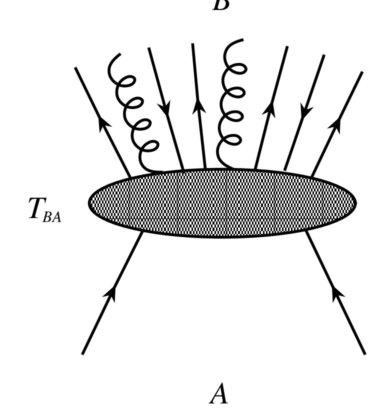
$$J_A = \frac{\text{number of particles}}{\text{time} \times \text{unit area}}$$

- Assume n particles in final state:

$$\mathrm{d}N_B = \prod_{i=1}^{n} \frac{\mathrm{d}^3 p_i}{(2\pi)^3\, 2E_i} \qquad \left(E_i = \sqrt{\vec{p}_i^{\,2} + m_i^2}\right)$$

4.2. Feynman rules (for the calculation of invariant amplitude T_{BA})

- Factors to be applied for each external lines



a) Incoming quark lines $\begin{cases} \text{in } A: \ u(p,s) \\ \text{in } B: \ v(p,s) \end{cases}$

b) Outgoing quark lines $\begin{cases} \text{in } A: \ \bar{v}(p,s) \\ \text{in } B: \ \bar{u}(p,s) \end{cases}$

c) External gluon lines: Polarization vector: ϵ^μ

- Remember QCD Lagrangian

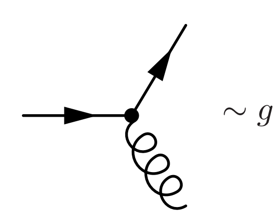
$$\mathcal{L}_{\text{QCD}} = \bar{\psi}(x)\left(i\gamma_\mu\left(\partial^\mu - ig\frac{\lambda_a}{2}A_a^\mu(x)\right) - \mathbf{m}\right)\psi(x)$$
$$-\frac{1}{4}G_a^{\mu\nu}(x)G^a_{\mu\nu}(x)$$
$$-\frac{1}{2\xi}(\partial_\mu A_a^\mu(x))^2$$

$$G_a^{\mu\nu}(x) = \partial^\mu A_a^\nu(x) - \partial^\nu A_a^\mu(x) + g f_{abc} A_b^\mu(x) A_c^\nu(x)$$

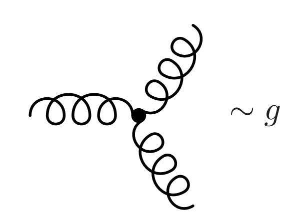
$\mathbf{m}$: quark mass matrix

$$\mathbf{m} = \begin{pmatrix} m_u & 0 & 0 & 0 & 0 & 0 \\ 0 & m_d & 0 & 0 & 0 & 0 \\ 0 & 0 & m_s & 0 & 0 & 0 \\ 0 & 0 & 0 & m_c & 0 & 0 \\ 0 & 0 & 0 & 0 & m_b & 0 \\ 0 & 0 & 0 & 0 & 0 & m_t \end{pmatrix}$$

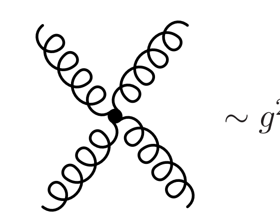
- Interaction vertices



Quark-gluon vertex



3-gluon vertex



4-gluon vertex

- Consider QCD in its perturbative domain ("Perturbative QCD"):

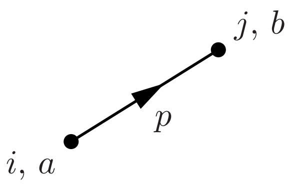
$\alpha_s = \dfrac{g^2}{4\pi} \ll 1 \Rightarrow$ Perturbative expansion of observables in powers of α_s.

m_u	m_d	m_s	m_c	m_b	m_t
$4\,(\pm 2)$ MeV	$7\,(\pm 2)$ MeV	$120\,(\pm 5)$ MeV	$1.3\,(\pm 0.1)$ GeV	$4.3\,(\pm 0.1)$ GeV	$174\,(\pm 5)$ GeV

TABLE 4.1: Values of quark masses

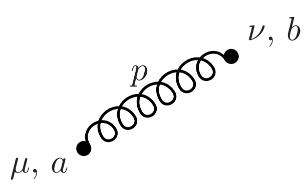
- Internal lines

a) Quark: (a, b: color indices; i, j: flavor indices)



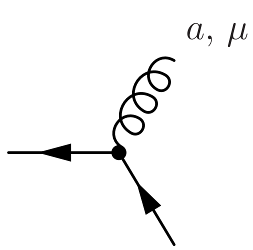
$$[iS_F(p)]^{ij}_{ab} = \delta_{ab}\,\delta^{ij}\,\frac{i}{\not p - m + i\epsilon} = \delta_{ab}\,\delta^{ij}\,\frac{i(\not p + m)}{p^2 - m^2 + i\epsilon}$$

b) Gluon: (a, b: color indices; μ, ν: Lorentz indices)



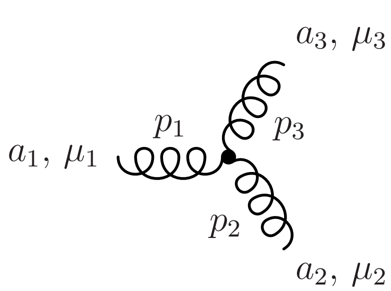
$$= \delta_{ab}\Big[- g^{\mu\nu} + (1-\xi)\,\frac{p^\mu p^\nu}{p^2}\Big]\,\frac{i}{p^2 + i\epsilon}$$

c) Quark-gluon vertex:



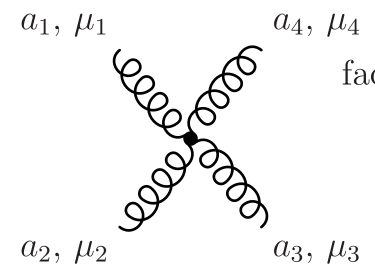
factor: $ig\gamma^\mu t_a$

d) 3-gluon vertex:



factor: $g f_{a_1 a_2 a_3}\big[\, g^{\mu_1\mu_2}(p_1 - p_2)^{\mu_3} + g^{\mu_2\mu_3}(p_2 - p_3)^{\mu_1} + g^{\mu_3\mu_1}(p_3 - p_1)^{\mu_2}\,\big]$

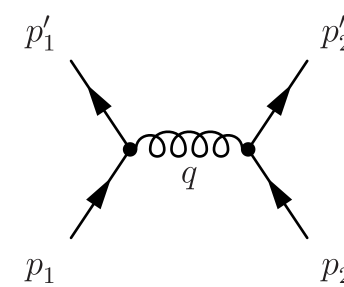
e) 4-gluon vertex:



factor: $-\,g^2\big[\, f_{a_1a_2a}f_{a_3a_4a}(g^{\mu_1\mu_3}g^{\mu_2\mu_4} - g^{\mu_1\mu_4}g^{\mu_2\mu_3}) + f_{a_1a_3a}f_{a_2a_4a}(g^{\mu_1\mu_2}g^{\mu_3\mu_4} - g^{\mu_1\mu_4}g^{\mu_2\mu_3}) + f_{a_1a_4a}f_{a_2a_3a}(g^{\mu_1\mu_2}g^{\mu_3\mu_4} - g^{\mu_1\mu_3}g^{\mu_2\mu_4})$

4. 3. Examples: Quark-quark and quark-antiquark scattering in one gluon exchange approximation

a) T-matrix for qq-scattering:

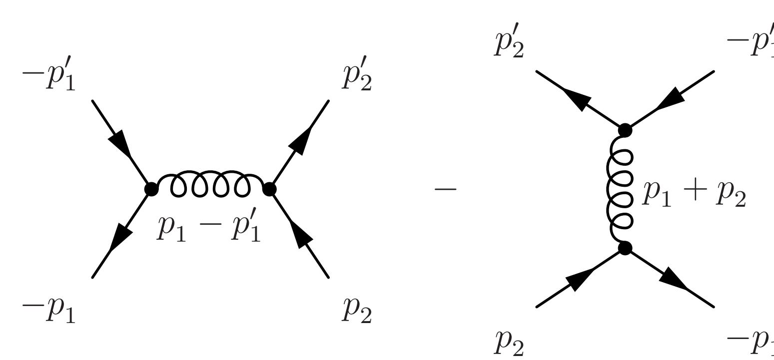


Feynman gauge ($\xi = 1$)

$q = p_1 - p_1' = p_2' - p_2$

$$iT = (ig)^2\big[\,\bar u(p_1')\,\gamma_\mu t_a\,u(p_1)\,\big]\,\frac{-ig^{\mu\nu}\delta_{ab}}{q^2 + i\epsilon}\,\big[\,\bar u(p_2')\,\gamma_\nu t_b\,u(p_2)\,\big]$$

b) T-matrix for $q\bar q$-scattering to order α_s:



$$iT = (ig)^2\big[\,\bar v(p_1)\,\gamma_\mu t_a\,v(p_1')\,\big]\,\frac{-ig^{\mu\nu}\delta_{ab}}{(p_1 - p_1')^2 + i\epsilon}\,\big[\,\bar u(p_2')\,\gamma_\nu t_b\,u(p_2)\,\big] - (ig)^2\big[\,\bar u(p_2')\,\gamma_\mu t_a\,v(p_1')\,\big]\,\frac{-ig^{\mu\nu}\delta_{ab}}{(p_1 + p_2)^2 + i\epsilon}\,\big[\,\bar v(p_1)\,\gamma_\nu t_b\,u(p_2)\,\big]$$

4.4. Sketch of path integrals (Functional integrals)

Systematic method for derivation of Feynman rules

- Illustration: example of scalar field theory

$$\mathcal{L}(\phi,\, \partial^\mu\phi) = \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2) - V(\phi)$$

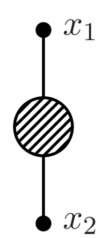
- Action functional:

$$S = \int \mathrm{d}^4x\, \mathcal{L}(\phi,\, \partial^\mu\phi) = S[\phi,\, \partial^\mu\phi]$$

- Basic relation for calculating n-point Green's functions (correlation function)

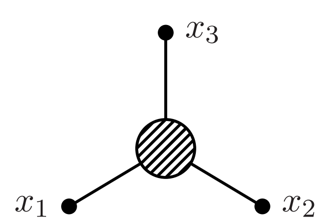
$$\langle 0|\mathcal{T}\,\phi(x_1)\phi(x_2)\cdots\phi(x_n)|0\rangle = \frac{\int \mathcal{D}\phi\,\phi(x_1)\cdots\phi(x_n)\,e^{iS[\phi,\partial^\mu\phi]}}{\int \mathcal{D}\phi\, e^{iS[\phi,\partial^\mu\phi]}}$$

For $n = 2$: "2-point function"



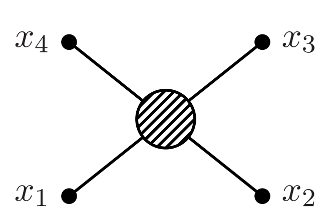
"Propagator"

For $n = 3$: "3-point function"



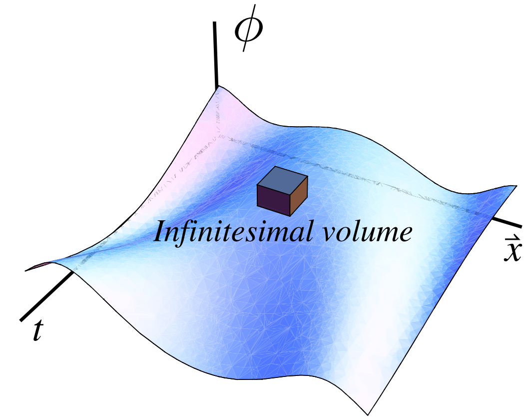
"Vertex"

For $n = 4$: "4-point function"



"Scattering amplitude"

- Definition of functional (path) integral:



Consider infinitesimal volume in space-time

$$\Delta v = \delta x_i\, \delta y_j\, \delta z_k\, \delta t_l$$

attached to a point $(x_i,\, y_j,\, z_k,\, t_l)$ with field $\phi(x_i,\, y_j,\, z_k,\, t_l)$ and its differential $\mathrm{d}\phi$ defined at that point.

$$\int \mathcal{D}\phi = \lim_{\Delta v\to 0} \prod_{ijkl} \int_{-\infty}^{+\infty} \mathrm{d}\phi(x_i\, y_j\, z_k\, t_l)$$

- Starting point: Generating functional

$$\mathcal{Z}[J] = \int \mathcal{D}\phi\, e^{iS}\, e^{i\int \mathrm{d}^4x\,\phi(x)J(x)}$$

$J(x)$: auxiliary source function.

Then the n-point function becomes

$$\begin{aligned}\mathcal{G}^{(n)}(x_1 x_2\cdots x_n) &\equiv \langle 0|\mathcal{T}\,\phi(x_1)\phi(x_2)\cdots\phi(x_n)|0\rangle \\ &= \frac{(-i)^n}{\mathcal{Z}[0]}\, \frac{\delta\mathcal{Z}[J]}{\delta J(x_1)\cdots\delta J(x_n)}\bigg|_{J=0}\end{aligned}$$

with the functional derivative:

$$\frac{\delta\mathcal{Z}[J(x)]}{\delta J(y)} = \lim_{\epsilon\to 0^+} \frac{\mathcal{Z}[J(x) + \epsilon\delta^4(x-y)] - \mathcal{Z}[J(x)]}{\epsilon}$$

$$\left(\text{in particular: } \frac{\delta J(x)}{\delta J(y)} = \delta^4(x-y)\right)$$

Example. Free scalar field

$$\mathcal{L}_0 = \frac{1}{2}\left(\partial_\mu\phi\,\partial^\mu\phi - m^2\phi^2\right)$$

$$\mathcal{Z}_0[J] = \int \mathcal{D}\phi\ \exp\left[i\int \mathrm{d}^4x\,\left(\mathcal{L}_0 + \phi(x)J(x)\right)\right]$$

Using

$$\int \mathrm{d}^4x\,\partial_\mu\phi\partial^\mu\phi = \underbrace{\int \mathrm{d}^4x\,\partial_\mu(\phi\partial^\mu\phi)}_{\text{surface integral }=0} - \int \mathrm{d}^4x\,\phi\Box\phi$$

it follows that

$$\Rightarrow\ \mathcal{Z}_0[J] = \int \mathcal{D}\phi\ \exp\left[-i\int \mathrm{d}^4x\left[\frac{1}{2}\phi(\Box + m^2)\phi - J\phi\right]\right]$$

equation of motion: $(\Box + m^2)\phi(x) = -J(x)$

$$\phi(x) = -\int \mathrm{d}^4y\,\Delta_F(x-y)J(y)$$

$$\text{with } \Delta_F(x-y) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\,\frac{e^{-ik\cdot(x-y)}}{k^2 - m^2 + i\epsilon}$$

$$\Rightarrow\ \mathcal{Z}_0[J] = \exp\left[-\frac{i}{2}\int \mathrm{d}^4x\int \mathrm{d}^4y\,J(x)\,\Delta_F(x-y)\,J(y)\right]$$
$$\times\int \mathcal{D}\phi\ \exp\left[-\frac{i}{2}\int \mathrm{d}^4x\,\phi(x)(\Box + m^2)\phi(x)\right]$$

Now calculate 2-point function as example:

$$\mathcal{G}^{(2)}(x_1,x_2) = \langle 0|\mathcal{T}\,\phi(x_1)\phi(x_2)|0\rangle$$
$$= -\frac{1}{\mathcal{Z}_0[0]}\,\frac{\delta^2\mathcal{Z}_0[J]}{\delta J(x_1)\delta J(x_2)}\bigg|_{J=0}$$
$$= i\Delta_F(x_1 - x_2)$$

Analogous procedures for n-point functions $\Rightarrow$ Feynman rules for scalar field theory.

4. 5. Appendix: Useful relations

When dealing with path integrals, some basic formulae:

(1) Important matrix identity: let $\mathbf{M}$ be a diagonalizable matrix

$$\ln\det\mathbf{M} = \mathrm{tr}\ln\mathbf{M}$$
$$\Rightarrow \det\mathbf{M} = \exp\left[\,\mathrm{tr}\ln\mathbf{M}\,\right]$$

(2) Gaussian integral:

$$\int_{-\infty}^{+\infty}\frac{\mathrm{d}x}{\sqrt{2\pi}}\,\exp\left[-\frac{1}{2}ax^2\right] = \frac{1}{\sqrt{a}}$$

- Let $\mathbf{M}$ be real, symmetric $N\times N$ matrix and $\mathbf{X}^T = (x_1,\cdots,x_N)$

  $\Rightarrow$ Generalization of Gaussian integral

$$\int_{-\infty}^{+\infty}\frac{\mathrm{d}x_1}{\sqrt{2\pi}}\cdots\int_{-\infty}^{+\infty}\frac{\mathrm{d}x_N}{\sqrt{2\pi}}\,\exp\left[-\frac{1}{2}\mathbf{X}^T\mathbf{M}\mathbf{X}\right] = \frac{1}{\sqrt{\det\mathbf{M}}} \tag{4.24}$$

- In functional integrals: often encounter

$$\int \mathcal{D}\phi\ \exp\left[-\frac{1}{2}\int \mathrm{d}^4x\int \mathrm{d}^4x'\,\phi(x')M(x',x)\phi(x)\right]$$

Approximate $\int \mathrm{d}^4x \to \sum_i \Delta v_i$ by sum over finite number $N = \left(\frac{L}{\epsilon}\right)^4$ of little cubes and use Eq. (4.24):

$$\int \mathcal{D}\phi\ \exp\left[-\frac{1}{2}\int \mathrm{d}^4x'\int \mathrm{d}^4x\,\phi(x')M(x',x)\phi(x)\right] = \frac{1}{\sqrt{\det M}}$$

- Complex scalar fields

$$\int \mathcal{D}\phi\int \mathcal{D}\phi^*\ \exp\left[-\frac{1}{2}\int \mathrm{d}^4x\int \mathrm{d}^4x'\,\phi^*(x')M(x',x)\phi(x)\right] = \frac{1}{\det M}$$
$$= \exp\left[-\,\mathrm{tr}\ln M\right]$$

4.6. Fermion fields

$$\psi(x) = \sum_{s=\pm\frac{1}{2}} \int \frac{d^3p}{(2\pi)^3\, 2E_p} \Big[a_{p,s} u_s(p) e^{-ip\cdot x} + b^*_{p,s} v_s(p) e^{ip\cdot x} \Big]$$

▷ Grassmann-Algebra:

$$\{a_i,\, a_j\} = \{b_i,\, b_j\} = \cdots = 0\,; \quad a_i^n = 0,\, n > 1$$

▷ Most general form of function of two Grassmann variables

$$f(a_1, a_2) = c_0 + c_1 a_1 + c_2 a_2 + c_3 a_1 a_2$$
$$= c_0 + c_1 a_1 + c_2 a_2 - c_3 a_2 a_1$$

- Derivative:

$$\frac{\partial f}{\partial a_1} = c_1 + c_3 a_2$$
$$\frac{\partial f}{\partial a_2} = c_2 - c_3 a_1$$
$$\frac{\partial^2}{\partial a_1 \partial a_2} = -\frac{\partial^2}{\partial a_2 \partial a_1}$$

- Integration:

$$\int da_1 da_2\, F \equiv \int da_1 \Big(\int da_2\, F \Big)$$

$$\int da = 0 \quad \text{because} \quad \Big(\int da \Big)^2 = -\Big(\int da \Big)^2 = 0$$

Definition: $\int da\, a = 1$ as a normalization

- Path integrals with fermion fields:

  Given an antisymmetric matrix A with $a^T = (a_1, \cdots, a_N)$

$$\int da_1 \cdots \int da_N \exp\Big[-\frac{1}{2}\, a^T A\, a \Big] = \sqrt{\det A}$$

$a_i^n = 0$ for $n > 1$, $\int da_i = 0$; $\int da_i\, a_i = 1$

- Complex fermion fields:

$$\int da_1 \int da_1^* \cdots \int da_N \int da_N^* \exp\Big[-\frac{1}{2}\, a^\dagger A\, a \Big] = \det A$$

- Functional integrals involving fermion fields:

$$\int \mathcal{D}\psi \int \mathcal{D}\psi^* \exp\Big[-\int d^4x d^4x'\, \psi^*(x') A(x', x) \psi(x) \Big] = \det A$$
$$= \exp\big[\mathrm{tr} \ln A \big]$$

4.7. Generating functional of QCD

▷ Lagrangian density (without gauge fixing):

$$\mathcal{L}_{\text{QCD}} = \bar{\psi}\big[i\gamma_\mu D^\mu - m \big]\psi - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a$$

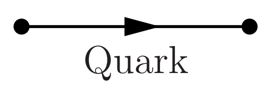
▷ Generating functional:

$$\mathcal{Z}_{\text{QCD}}[J, \eta, \bar{\eta}] = \int \mathcal{D}A \int \mathcal{D}\psi \int \mathcal{D}\bar{\psi}$$
$$\times \exp\Big[i \int d^4x \big(\mathcal{L}_{\text{QCD}}(x) + A^a_\mu(x) J^\mu_a(x) + \bar{\psi}(x)\eta(x) + \bar{\eta}(x)\psi(x) \big) \Big]$$

- Generate n-point functions by taking functional derivatives with respect to source fields $J(x)$, $\eta(x)$ and $\bar{\eta}(x)$.

Example-1. 2-point functions:

- Quark propagator: $\left.\dfrac{\delta^2}{\delta\eta(x)\delta\bar{\eta}(x)}\mathcal{Z}_{\rm QCD}\right|_{\eta,\,\bar{\eta}=0}$



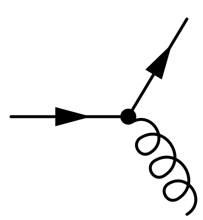
$$= \langle 0|\mathcal{T}\,\psi(x)\bar{\psi}(y)|0\rangle = iS_F(y-x)$$
$$= i\int \frac{{\rm d}^4p}{(2\pi)^4}\, e^{-ip\cdot(y-x)}\,\frac{\not{p}+m}{p^2-m^2+i\epsilon}$$

- Gluon propagator: $\left.\dfrac{\delta^2}{\delta J^a_\mu(x)\delta J^b_\nu(y)}\mathcal{Z}_{\rm QCD}\right|_{J=0}$

$$= \langle 0|\mathcal{T}\,A^\mu_a(x)A^\nu_b(y)|0\rangle = iD^{\mu\nu}_{ab}(y-x)$$
$$= i\delta_{ab}\int \frac{{\rm d}^4q}{(2\pi)^4}\,\frac{d^{\mu\nu}(q)}{q^2+i\epsilon}$$

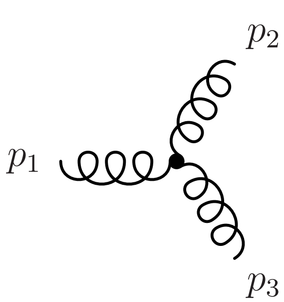
Example-2. 3-point functions:

- Quark-gluon vertex: $\left.\dfrac{\delta^3}{\delta J^a_\mu(x)\,\delta\eta(x)\,\delta\bar{\eta}(x)}\mathcal{Z}_{\rm QCD}\right|_{\eta,\bar{\eta},J=0}$



$$= ig\gamma^\mu t_a$$

- 3-gluon vertex: $\left.\dfrac{\delta^3}{\delta J^a_\mu(x)\,\delta J^b_\nu(x)\,\delta J^c_\lambda(x)}\mathcal{Z}_{\rm QCD}\right|_{J=0}$



$$= g f_{abc}\big[g^{\mu\nu}(p_1-p_2)^\lambda + cycl.perm.\big]$$

4. 8. Gauge invariance and gauge fixing (Sketch)

- Pure gluon theory:
$$\mathcal{L}_{\rm G} = -\frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a$$
where $G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc}A^b_\mu A^c_\nu$.

- Action functional:
$$S_{\rm G}[A,\partial A] = \int {\rm d}^4x\,\mathcal{L}_{\rm G}(A,\partial A)$$

- Generating functional:
$$\mathcal{Z}_{\rm G}[J] = \int \mathcal{D}A\,\exp\left[i\int {\rm d}^4x\,\big(\mathcal{L}_{\rm G} + A^\mu_a J^a_\mu\big)\right]$$

- Functional integral covers arbitrarily many gauge-equivalent field configurations.
$$\tilde{A}^\mu = U\Big[A^\mu - \frac{i}{g}U^\dagger\partial^\mu U\Big]U^\dagger \qquad \begin{cases} A^\mu \equiv A^\mu_a\dfrac{\lambda_a}{2} \\ U = \exp\Big(-i\theta_a(x)\dfrac{\lambda_a}{2}\Big)\end{cases}$$

- Gauge fixing needs a constraint.
$$\boxed{\partial^\mu A^a_\mu(x) = B^a(x)}$$
(In particular Lorenz condition $B^a(x)\equiv 0$)

- Insert "unity":
$$\mathbb{1} = \det\mathcal{M}\prod_{a=1}^{8}\int \mathcal{D}\theta_a\,\delta\big(\partial^\mu A^a_\mu(x) - B^a(x)\big)$$
with Jacobian of gauge transformation:
$$\mathcal{M}^{ab}(x,\,y) = \frac{\delta\big(\partial^\mu A^a_\mu(x)\big)}{\delta\theta^b(y)}$$

- Problem: to calculate Jacobian $\det\mathcal{M}$

4. 9. Faddeev-Popov method (sketch)

- Introduce a set of (unphysical) auxiliary fields: $\chi^a(x)$, $\chi^{*\,a}(x)$: anticommuting Bose fields ("*ghost fields*")

$$\det \mathcal{M} = i \int \mathcal{D}\chi \int \mathcal{D}\chi^* \, \exp\left[i \int \mathrm{d}^4x \, \partial^\mu \chi_a^*(x) D_\mu^{ab} \chi_b(y) \right]$$

with gauge covariant derivative:

$$D_\mu^{ab} = \delta^{ab}\partial_\mu - g f^{abc} A_\mu^c$$

- Result: Gauge fixing condition ⇒ extra term in Lagrangian density.

▷ QCD Lagrangian including gauge fixing:

$$\mathcal{L}_{\mathrm{QCD}} = -\frac{1}{4} G_{\mu\nu}^a(x) G_a^{\mu\nu}(x) - \frac{1}{2\xi}\left(\partial^\mu A_\mu^a\right)^2 + \mathcal{L}_{\mathrm{FP}}$$

with Faddeev-Popov term

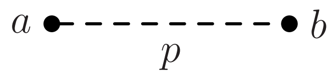
$$\mathcal{L}_{\mathrm{FP}} = \partial^\mu \chi^{*\,a}(x) D_\mu^{ab} \chi^b(x)$$

4. 10. Complete generating functional of QCD (including gauge fixing)

$$\mathcal{Z}_{QCD}[J, \eta, \bar{\eta}; j, j^*] = \int \mathcal{D}A \int \mathcal{D}\psi \int \mathcal{D}\bar{\psi} \int \mathcal{D}\chi \int \mathcal{D}\chi^* \times \exp\left[i \int \mathrm{d}^4x \left(\mathcal{L}_{\mathrm{QCD}} + A_\mu^a J_a^\mu + \bar{\psi}\eta + \bar{\eta}\psi + \chi^{*\,a} j_a + j_a^* \chi^a \right) \right]$$

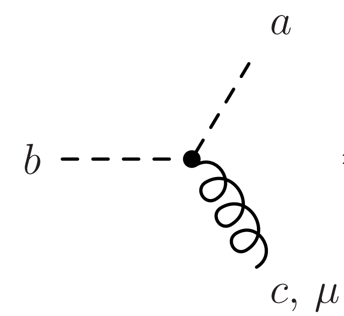
with $\mathcal{L}_{\mathrm{QCD}} = \bar{\psi}\left[i\gamma_\mu D^\mu - m\right]\psi - \frac{1}{4} G_{\mu\nu}^a G_a^{\mu\nu} - \frac{1}{2\xi}\left(\partial_\mu A_a^\mu\right)^2 + \mathcal{L}_{\mathrm{FP}}$

- Additional Feynman rules associated with *ghost*:
  - Ghost propagator in momentum space:



$$= \frac{i\delta_{ab}}{p^2 + i\epsilon}$$

  - Ghost-gluon vertex:



$$= g f_{abc}\, p^\mu$$