ON THE PERIODS OF INTEGRALS ON ALGEBRAIC MANIFOLDS*

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This is a summary of some results in the transcendental theory of algebraic varieties. The problem is to analyze the periods, as functions of the parameters, in an algebraic family of algebraic manifolds. The following is a brief outline of this work.

Given an algebraic manifold V in a projective space $\mathbf{P}_N$, we construct the period matrix space $D = D_q$ of all possible period matrices for the primitive harmonic q-forms on manifolds homeomorphic to V. It turns out that D is an open homogeneous complex manifold, which is sometimes a Cartan domain, and on which there is a p-convex polarization giving rise to automorphic cohomology but, in general, not automorphic forms. In case $q = 2m + 1$ is odd, D_{2m+1} is a parameter space for complex tori with a certain r-convex polarization ($r = h^{2m,1} + h^{2m-2,3} + \cdots$). The torus $H^{2m+1}(V, \mathbf{R})/H^{2m+1}(V, \mathbf{Z}) = T_m(V)$ is naturally a complex torus with r-convex polarization and the period matrix of V gives the same point in D_{2m+1} as $T_m(V)$. ($T_m(V)$ is generally not Weil's higher Jacobian.) Finally, there is naturally defined a properly discontinuous group Γ of analytic automorphisms of D such that $M = D/\Gamma$ is an analytic space and then V defines a unique point $\Phi(V) \in M$.

Now let $\{V_t\}_{t \in B}$ be an algebraic family of varieties $V_t \subset \mathbf{P}_N$ such that V belongs to the family. The period matrix mapping $\Phi : B \to M$ is defined by $\Phi(t) = \Phi(V_t)$, provided V_t is non-singular. It is proved that, even though periods of (p, q) forms are involved, Φ is holomorphic. In partcular, $T_m(V_t)$ varies analytically with t. The rank of the Jacobian matrix of Φ is computed cohomologically and it is shown, e.g., that the periods give local moduli for a wide class of varieties.

To study the global nature of Φ, it is shown that the periods on V_t may be given by periods of algebraic integrals This involves a generalized residue calculus.

Once we have shown that the periods are represented by algebraic integrals,

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we introduce the differential equations of the periods. Having discussed the formal properties of these differential equations, we show that the periods satisfy an equation with regular singular points. This gives an asymptotic estimate on the periods of V_t as $V_t \to V_0$ where V_0 is singular. It is shown that the group of the differential equation is generated by (essentially) unipotent matrices.

Applying these results to the period mapping $\Phi(t)$, we find that, as $V_t \to V_0$, $\Phi(t)$ tends to a unique point $\Phi(0)$ in $\bar{D}$ modulo Γ. Furthermore, $\Phi(0)$ is a fixed point of a rational unipotent element in the automorphism group of D.

As an application of the continuity theorem above, we see that, in certain cases, Φ gives either a holomorphic or rational mapping into a compactification M^* of M. This gives then information on moduli of curves, $K3$ surfaces, cubic threefolds, etc.

Other applications (not discussed below) are to mapping problems in several complex variables (generalizing Picard's theorem), differentials of the second kind, and the cohomology of affine varieties.

1. *Construction of the Period Matrix and Modular Varieties*

Let V_∞ be a differentiable $2n$-manifold and $\omega \in H^2(V_\infty, \mathbf{Z})$ a cohomology class. We suppose that there is a *polarized algebraic manifold* V whose underlying C^∞ manifold is V_∞ and whose polarizing cycle is ω; that is to say, over V there is a *positive holomorphic line bundle* $L \to V$ and the characteristic class of L is ω.

For $0 < q \leqq n$ the *primitive cohomology* $H^q(V)_0 = \{\phi \in H^q(V, \mathbf{C})$ such that $\omega^{n-q+1}\phi = 0\}$ is a vector space defined over the rationals $\mathbf{Q}$ and $H^p(V, \mathbf{C}) = \Sigma\, \omega^{2r} H^{p-2r}(V)_0$ (*Lefschetz decomposition*), so that we can recover $H^*(V, \mathbf{C})$ from the $H^q(V)_0$. On $H^q(V)_0$ there is a non-singular bilinear form: $Q(\phi, \psi) = c_q \int_V \omega^{n-q}\phi\psi$, where c_q is a suitable constant.

The complex structure on V gives the *Hodge decomposition*:

$$H^q(V)_0 = \sum H^{q-r,r}(V)_0, \qquad (\overline{H^{q-r,r}(V)_0} = H^{r,q-r}(V)_0),$$

where $H^{q-r,r}(V)_0 = H_0^{q-r,r}$ are the primitive classes of type $(q-r, r)$. We have that $Q(H_0^{q-r,r}, H_0^{s,q-s}) = 0$ for $s \neq r$ and $Q(H_0^{q-r,r}, \bar{H}_0^{q-r,r})$ is either positive or negative definite, written $Q(H_0^{q-r,r}, \bar{H}_0^{q-r,r}) > 0$, depending on q and r.

Let now $S^r = H_0^{q,0} + \cdots + H_0^{q-r,r}$ so $S^0 \subset S^1 \subset \cdots \subset S^t \subset H^q(V)_0$ $\left(t = \left[\frac{q-1}{2}\right]\right)$. Then we have defined a point $\Omega = [S^0, S^1, \cdots, S^t]$ in a flag manifold $\mathscr{F} = \mathscr{F}(V_\infty, \omega)$ and each polarized algebraic structure (V, L) on (V_∞, ω) defines a (non-unique) point $\Omega(V)$ in $\mathscr{F}$.

The flags $\Omega(V)$ are subject to two *bilinear relations*:

(i) $Q(S^r, S^r) = 0 \quad (r \neq q/2)$;

(ii) $Q(S^r/S^{r-1}, \bar{S}^r/\bar{S}^{r-1}) > 0$.

In (ii), for example, S^2/S^1 is the subspace of S^2 defined by $S^2/S^1 = \{\phi \in S^2 \text{ such that } Q(\phi, \bar{S}^1) = 0\}$.

We let X be the flags Ω satisfying (i) and D the flags satisfying (i) and (ii).

Theorem. *Let G be the group of real linear transformations of $H^q(V)_0$ which preserve Q. Then G acts transitively on D with compact isotropy group so that $D = G/H$ where G is a real simple Lie group and H is the centralizer of a torus in G.*

Remarks. It is also shown that the complex group $\tilde{G}$ of all linear transformations on $H^q(V)_0$ which preserve Q acts transitively on X, and thus D is an open G-orbit on a homogeneous algebraic manifold.

It may be seen that D is a coordinate free manner of describing all possible period matrices for the primitive q-forms on polarized algebraic manifolds (V, L) with underlying (V_∞, ω).

There is naturally defined an arithmetic subgroup $\Gamma \subset G$, where Γ contains all automorphisms of $H^q(V)_0$ induced by homeomorphisms of V_∞ which preserve ω. The complex manifold D is called the *period matrix space* and the analytic variety $M = D/\Gamma$ is the *modular variety*. The polarized algebraic manifold (V, L) with underlying (V_∞, ω) defines a unique point $\Phi(V) \in M$.

Theorem. *Given (V, L) and (V', L') with the same underlying (V_∞, ω), $\Phi(V) = \Phi(V')$ if, and only if, there is a polarization preserving homeomorphism $f: V \to V'$ such that the graph $G_f \in H_{2n}(V \times V', \mathbf{Z})$ is of type (n, n).*

For $q = 1$, D is a *Siegel upper half space* and $\Phi(V)$ is the point determined by the *Jacobian variety* $T_0(V)$. This generalizes as follows: Let D_q be the period matrix space for $H^q(V)_0$ and $D_{(m)} = D_1 \times \cdots \times D_{2m+1}$. We introduce the notion of a *p-convex polarization* to be a complex manifold W together with a line bundle $E \to W$ whose characteristic class ω is locally $\omega = \frac{\sqrt{-1}}{2} \sum_{\alpha,\beta} h_{\alpha\beta} dz^\alpha \wedge d\bar{z}^\beta$ where $H = (h_{\alpha\beta})$ is a non-singular Hermitian matrix with p negative eigenvalues; thus ${}^t A H \bar{A} = \begin{bmatrix} 1 & & & & & \bigcirc \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ \bigcirc & & & & & -1 \end{bmatrix}$ with p entries -1.

for suitable A. Concerning the real torus $H^{2m+1}(V_\infty, \mathbf{R})/H^{2m+1}(V_\infty, \mathbf{Z}) = T_m(V_\infty)$, we have:

Theorem. *Given (V, L) with underlying (V_∞, ω), there exists a complex structure $T_m(V)$ on $T_m(V_\infty)$ such that*: (i) *$T_m(V)$ has a translation-invariant p-convex polarization for some p (in fact, $p = h^{2m,1} + h^{2m-2,3} + \cdots$)*; (ii) *$T_m(V)$ with its polarization is functorial*; (iii) *$T_m(V)$ varies analytically with V.*

Remarks. $T_m(V)$ is generally *not* Weil's higher Jacobian $J_m(V)$; $J_m(V)$ does not vary analytically with V. For example, if $m = 1$, $T_m(V) = J_m(V)$ if, and only if, $H^{3,0} = 0$ or $H^{2,1} = 0$.

Theorem. *$D_{(m)}$ is naturally a parameter space for complex tori with p-convex polarization, and $\Omega(V) = (\Omega_1(V), \cdots, \Omega_{2m+1}(V))$ is the point in $D_{(m)}$ corresponding to $T_m(V)$.*

Remarks. A zero-convex polarization on $T_m(V)$ is an ordinary polarization; in this case we have *theta-functions* (sections of L), *modular forms* derived from theta nullwerte, etc. For a p-convex polarization, we have *theta cohomology* $H^p(T_m(V), L)$ and *modular cohomology*.

For $q = 2$, $D = D_2$ is the space of period matrices for the holomorphic 2-forms on V. If $h = h^{2,0}$ and $k = h_0^{1,1}$ $(= h^{1,1} - 1)$, then:

$$D \cong SO(2h, k)/U(h) \times O(k) = G/H,$$

where $SO(2h, k)$ is the real group of $Q = \begin{pmatrix} I_{2k} & 0 \\ 0 & -I_k \end{pmatrix}$. We observe that D is bounded symmetric domain if, and only if, $h = h^{2,0} = 1$.

Theorem. *D has a natural G-invariant p-convex polarization where L is the* canonical bundle *(canonical factor of automorphy) and $p = (h^2 - h)/2$.*

Remarks. Thus, instead of automorphic forms for D under Γ, we find, at least when D/Γ is compact, *automorphic cohomology* in dimension p. It seems quite likely, although it is not proved yet, that there will be automorphic cohomology on D/Γ in general.

In fact, this brings up one of the central questions our study has raised: In two instances above, we have started with an algebraic object (polarized algebraic manifold), performed natural constructions, and ended up with p-convex polarized complex manifolds which have cohomology in dimension p instead of sections. Now what is the meaning of this cohomology to the geometric problems which gave rise to it?

For example, $M = D/\Gamma$ can be compactified to a Hausdorff topological space M^*, and there is some evidence that M^* is an analytic space (but not an algebraic variety) in which M is Zariski open. However, it will certainly be necessary to understand the modular cohomology in order to make sense out of this.

To be specific for a moment, let H_n be the Siegel upper space in genus n and Γ_n the modular group. Then $M_1 = H_n/\Gamma_n$ is the modular variety for the periods of the holomorphic 1-forms on normally polarized abelian varieties. The modular variety for the periods of the holomorphic 2-forms is:

$$M_2 = \Gamma\backslash SO(n(n-1), n^2-1) / U(n(n-1)) \times O(n^2-1);$$

M_2 has a p-convex polarization where $p > 0$ if $n \geqq 3$. Now there is an embedding $M_1 \to M_2$ (since the periods of the 2-forms determine the periods of the 1-forms); we may ask what is the relation of the modular cohomology on M_2 to the modular forms on M_1.

2. *Local Study of the Period Mapping*

Suppose now that we have an *algebraic family* $\{V_t\}_{t\in B}$ of polarized algebraic manifolds; to be precise, suppose that we are given irreducible, complete projective varieties $\mathbf{V}$, B and a regular mapping $\mathbf{V} \xrightarrow{\pi} B$ such that $V_t = \pi^{-1}(t)$ is a non-singular algebraic manifold for t a general point on B. For our purposes, we may assume that $\mathbf{V}, B$ are non-singular and we let $S \subset B$ be those $t \in B$ such that π does not have maximal rank along V_t. Then $\{V_t\}_{t\in B-S}$ is an algebraic family, which is topologically a fibre bundle. We let (V, L) be the polarized algebraic manifold which is a fixed general member of $\{V_t\}_{t\in B-S}$.

Let $M = D/\Gamma$ be the modular variety associated to the periods of the primitive q-forms on V. There is the period mapping:

$$\Phi\colon B - S \to M,$$

given by: $\Phi(t) = \Phi(V_t)$ is the period matrix of the primitive q-forms on V_t.

Now $\Phi(t)$ involves the periods of the general $(q-r, r)$ forms on V_t; these periods themselves are not analytic functions of t, but we can show:

Theorem. *The period mapping Φ is holomorphic.*

Remarks. More precisely, let $\{V_t\}_{t\in\Delta}$(Δ = polycylinder) be a *differentiable family* of algebraic manifolds in the sense of Kodaira-Spencer. If $t_0 \in \Delta$ and $t^1, \cdots, t^m$ are local holomorphic coordinates around t_0, there is defined the *Kodaira-Spencer mapping*:

$$\rho\colon T_{t_0}(\Delta) \to H^1(V_{t_0}, \Theta^0{}_{V_{t_0}}).$$

Then, if $\rho\left(\frac{\partial}{\partial \bar{t}^\alpha}\right) = 0$ for all t_0, it follows that Φ is holomorphic.

The Kodaira-Spencer mapping ρ measures to what extent the family $\{V_t\}_{t\in\Delta}$ contains analytically distinct varieties; e.g., if ρ is injective, then the family is *effectively parametrized.*

Suppose now that $V = V_{t_0}$ as above. The contraction $\Theta \otimes \Omega^{q-r} \to \Omega^{q-r-1}$ induces a *cup product*:

$$H^1(V,\Theta) \otimes H^r(V,\Omega^{q-r}) \to H^{r+1}(V,\Omega^{q-r-1}).$$

Using the *Dolbeault isomorphism*, we get:

$$H^1(V,\Theta) \otimes H^{q-r,r} \to H^{q-r-1,r+1},$$

and, because we have a polarized family, we have

$$(\#) \qquad H^1(V,\Theta) \otimes H_0^{q-r,r} \to H_0^{q-r-1,r+1}.$$

Theorem. *There is a natural isomorphism*

$$T_{\Phi(V)}(M) \cong \sum_{0\leqq r\leqq[(q-1)/2]} \operatorname{Hom}(H_0^{q-r,r}, H_0^{q-r-1,r+1}).$$

For $\tau \in T_{t_0}(\Delta)$ and $\psi \in H_0^{q-r,r}$,

$$\Phi_*(\tau)(\psi) = \rho(\tau)\psi \in H_0^{q-r-1,r+1},$$

where the latter symbol is the cup product $(\#)$.

Remark. This theorem gives a cohomological way of telling to what extent the *periods give local moduli* for the variety V (assuming ρ is injective).

Corollary. *If the cup product μ given by*

$$(\#) \qquad \begin{array}{c} \displaystyle\sum_{0\leqq r\leqq[(q-1)/2]} H^{n-r-1}(V,\Omega^{n-q+r+1}) \otimes H^r(V,\Omega^{q-r}) \\ \Big\downarrow \mu \\ H^{n-1}(V,\Omega^1 \otimes \Omega^n) \end{array}$$

is onto, then the periods of the q-forms give local moduli for V.

Examples. (i) In case $n = \dim V = 1$, the cup product $(\#)$ reduces to $H^0(\Omega^1) \otimes H^0(\Omega^1) \to H^0(\Omega^1 \otimes \Omega^1)$; this is the *Theorem of Rauch* that the periods give local coordinates in the Teichmüller space if the quadratic differentials are generated by abelian differentials (which happens when V is non-hyperelliptic).

(ii) If the canonical bundle of V is trivial, then the periods give local moduli.

(iii) The periods give local moduli in the following cases: $V \subset \mathbf{P}_3$ is a non-singular surface of degree $d \geqq 4$; $V \subset \mathbf{P}_3$ is a non-singular threefold of degree $d \geqq 3$; V is a hypersurface of degree $d \geqq 3$ on a three-dimensional abelian variety; $V \subset \mathbf{P}_4$ is a complete intersection of surfaces of sufficiently high degree; V is a multiple plane with sufficiently general branch curve.

We are led to conjecture that, if V is a surface whose *canonical series* is *sufficiently ample*, then the periods give local moduli.

(iv) The periods *fail* to give local moduli in the following cases: V is a surface having biregular, but not birational, moduli (e.g., cubic surface in $\mathbf{P}_3$); V is the *Enriques surface.*

A final local result on the period mapping Φ is of a geometric nature and describes the position of the image $\Phi(B-S)$ in M. We recall that the period matrix space D is a a homogeneous complex manifold G/H where G is a real, non-compact Lie group and $H \subset G$ is a compact subgroup. If $K \subset G$ is the maximal compact subgroup, then the K-orbits of maximal dimension give a family $\{Y_\lambda\}$ of compact, complex analytic submanifolds of D.

Example. Suppose we are looking at the Hodge decompositions of $H^2(V_\infty, \mathbf{C})_0 = H^2(V)_0$. Let $\Omega = [H^{2,0}, H^{2,0} + H_0^{1,1}]$ be one such point in D and let Y_Ω be the flags $[S^0, S^1]$ in D with $S^0 + \bar{S}^0 = H^{2,0} + H^{0,2}$. Then $Y_\Omega \subset D$ is a compact analytic subvariety which is the K-orbit of Ω.

Observe that $Y_\Omega \cong SO(2h)/U(h)$ where $h = h^{2,0}$.

Theorem. *The image $\Phi(B-S)$ is transverse to the compact subvarieties Y_λ.*

Remarks. In the example just above, we observe that $\dim Y_\Omega = \dfrac{h^2-h}{2}$, while D has a p-convex polarization with $p = \dfrac{h^2-h}{2}$. In fact, the canonical bundle $L \to D$, which gives this p-convex polarization, is negative on Y_Ω.

Concerning the geometry of these period matrix domains D, we have several results, many of which are due to W. Schmid or H. Wu. We list these as:

Geometric results on D. (i) There exists an *exhaustion function* ψ on D whose *E. E. Levi form* $L(\psi) = \partial\bar{\partial}\psi$ has everywhere $m-p$ positive eigenvalues ($m = \dim D$).

Using the *vanishing theorem* of Andreotti-Grauert, we get from (i) that: (ii) $H^q(D, S) = 0$ for $q > p$ and S any coherent sheaf on D.

Let $Y \subset D$ be the compact submanifold which is the K-orbit of the origin. If $I \subset O_D$ is the ideal sheaf of Y, $N \to Y$ the *normal bundle*, and $N^{*\mu} \to Y$ the μth symmetric power of the dual bundle, then we have exact sheaf sequences $0 \to I^{\mu+1}L^k \to I^\mu L^k \to O_Y(N^{*\mu}L^k) \to 0$. From the exact cohomology sequences and $H^{p+1}(D, I^\mu \cdot L^k) = 0$, we find, for all $\mu \geqq 0$, $H^p(D, I^{\mu+1}L^k) \to H^p(D, I^\mu L^k) \to H^p(Y, O(N^{*\mu}L^k)) \to 0$. This gives: (iii) $H^p(D, L^k)$ has a decreasing *filtration* whose *associated grading is* $\sum_{\mu \geqq 0} H^p(Y, O(N^{*\mu}L^k))$.

From the *Borel-Weil theorem* on Y, we find: (iv) $\dim H^p(D, L^k) = \infty$ for $k > 0$. In fact, for $k \gg 0$, $H^p(D, L^k)$ is *sufficiently ample*, and $H^p(D, L^k)$ can be expanded in a formal power series around Y.

Now $D = G/H$ is an open complex manifold and the fibering $G/H \to G/K$ gives a fibering of D by a family $\{Y_\lambda\}$ of compact, complex submanifolds. We have: (v) There exists a unique G-invariant splitting of the complex tangent bundle $T(D) = V \oplus H$ where, for $\Omega \in D$, $T_\Omega(D) = V_\Omega \oplus H_\Omega$ and V_Ω is the tangent space to the unique compact submanifold Y_λ passing through Ω.

It can be shown then that D is *hyperbolic* in the following sense: (vi) Let Δ be a complex manifold and $\{\phi_n\}$ a sequence of holomorphic mappings $\phi_n : \Delta \to D$ such that $(\phi_n)_* T_z(\Delta) \subset H_{\Phi_n(z)}$. Then, either $\{\phi_n\}$ diverges or else there exists a subsequence $\{\phi_{l(n)}\}$ of $\{\phi_n\}$ which is uniformly convergent on compact sets.

That this result is applicable to moduli follows from the following generalization of the transversality theorem above:

Theorem. $\Phi : B - S \to M$ *is transverse to the compact subvarieties and, in fact,* $\Phi_* T_\Omega(B - S) \subset H_{\Phi(\Omega)}$.

As a corollary we find using (i) and (vi) above:

Corollary. *Let* $\{V_t\}_{t \in B}$ *be a family of non-singular varieties where either* B *is compact and simply connected or* $B = \mathbf{C}^l$. *Then the periods of* $\{V_t\}_{t \in B}$ *are all constant.*

The algebraic-geometric meaning of the transversality theorem is that there are *additional period relations* to Riemann bilinear relations. For example, if Ω is the period matrix of the holomorphic 2 forms on a (generic) algebraic surface V, then ${}^t\Omega Q \Omega = 0$, ${}^t\Omega Q \overline{\Omega} > 0$ (Riemann relations), and ${}^t d\Omega Q \Omega = 0$ (transversality relation).

Proposition. *The canonical bundle of* M *is* positive *on* $\Phi(B - S)$.

3. *Representation of the Periods by Algebraic Integrals*

Our central problem is to study the period mapping $\Phi : B - S \to M$. We would like to prove theorems of the following sort:

(i) There exists an analytic compactification M^* of M such that Φ extends to meromorphic (or holomorphic) mapping $\Phi: B \to M^*$;

(ii) There exists a subfield $\mathscr{F}$ of the field $\mathscr{F}(B)$ of rational functions such that $\mathscr{F}$ determines the same equivalence relation on B as Φ (the level sets of $\mathscr{F}$ are the fibres of Φ).

This involves first giving an asymptotic analysis of the periods on V_t as V_t becomes singular, and then constructing M^*. We are able to give the asymptotic behavior of the periods and this involves representing the periods of $(q-r, r)$ forms as periods of rational integrals, to which we now turn.

The idea is to use a *generalized residue calculus*. Let V be an algebraic n-manifold and $S \subset V$ a non-singular subvariety. Denote by $A^q(kS)$ the vector space of closed, rational q-forms on V with poles of order less than or equal to k along S. Each $\phi \in A^q(kS)$ defines a cohomology class $\mathbf{R}(\phi) \in H^{q-1}(S)$ by *integration over the fibre* in the normal bundle: If $\delta \in H_{q-1}(S, Z)$ is a $(q-1)$-cycle on S, we let $\tau(\delta)$ be a tube in $V - S$ lying over δ; $\tau(\delta) \in H_q(V - S)$. Then $\langle \mathbf{R}(\phi), \delta \rangle = \dfrac{1}{2\pi\sqrt{-1}} \displaystyle\int_{\tau(\delta)} \phi$.

For $q = n$, $k = 1$, $\mathbf{R}(\phi)$ is a holomorphic $(n-1)$-form on S given explicitly by the Poincaré *residue operator*. For general q and $k = 1$, we get the residue operator of Leray (who uses C^∞ forms).

Generalizing this, we let $A^{q-1}(S)$ be the vector space of rational $(q-1)$-forms on S and prove:

Theorem. *There is a linear operator* (residue operator): $R: A^q(kS) \to A^{q-1}(S)$ *with the following properties*:

(i) *R is well-defined if $k = 1$ and defined modulo exact forms if $k > 1$*;

(ii) *R takes closed (exact) forms into closed (exact) forms*;

(iii) *For $q = n$, $k = 1$, R is the Poincaré residue operator*;

(iv) *If $\phi \in A^q(S)$, $R(\phi)$ is holomorphic on S*;

(v) *If $\phi \in A^q(kS)$, $R(\phi)$ is of 2nd kind on S*;

(vi) *If $\delta \in H_{q-1}(X, Z)$ lies outside a suitable subvariety $S' \subset S$, then*

$$\int_\delta R(\phi) = \frac{1}{2\pi\sqrt{-1}} \int_{\tau(\delta)} \phi.$$

Concerning the operator $\mathbf{R}$, we have:

Theorem. (i) $\mathbf{R}: A^q(kS) \to H^{q-1,0}(S) + \cdots + H^{q-k,k-1}(S)$ *so that, in particular,*

$$\mathbf{R}: A^q(kS)/A^q((k-1)S) \to H^{q-k,k-1}(S);$$

(ii) *If $S \subset V$ is a positive subvariety, then*

$$H^{q-k,k-1}(S) = H^{q-k,k-1}(V) + \mathbf{R}\{A^q(kS)/A^q((k-1)S)\};$$

(iii) *If $\delta \in H_{q-1}(S, \mathbf{Z})$ lies outside $S' \subset S$, then $\langle \mathbf{R}(\phi), \delta \rangle = \int_\delta R(\phi)$.*

Remarks. Part (i) of the second theorem tells us that the *type* of $\mathbf{R}(\phi)$ in the Hodge decomposition of $H^{q-1}(S, \mathbf{C})$ can be read off from the *order of the pole* which ϕ has on S; part (ii) tells us that, if $S \subset V$ is positive, then all of the cohomology of S comes by restriction or by residues. In particular, if S lies in an algebraic family $\{S_t\}$ of hypersurfaces, then the periods of the forms restricted from V are constant while the periods of the cohomology classes $R(\phi)$ $(\phi \in A^q(kS))$ are obviously holomorphic. Thus part (i) of this theorem explains the motivation for using flags in the construction of period matrix varieties while (ii) makes plausible the fact that the period mapping is holomorphic. Finally, because of the first theorem and (iii) of the second theorem, the possibility of representing the periods of $H^{q-k,k+1}(S)$ by periods of algebraic integrals on S is certainly raised.

To carry through the representation of the cohomology of V by algebraic integrals, we take a fixed projective embedding $V \subset \mathbf{P}_N$ and let $H \subset V$ be a general hyperplane section. Denote by $A^q(K) = A^q_V(kH)$ the vector space of closed, rational q-forms on V, having poles of order $k+1$ on H, and reduced modulo exact forms.

Theorem. *There exists subspaces $\mathscr{H}^q(k) \subset A^q_V(kH)$ and linear isomorphisms $\{H^{q,0}(V) + \cdots + H_0^{q-k,k}(V)\} \xrightarrow{r} \mathscr{H}^q(k)$ such that:*

(i) *$\mathscr{H}^q(k-1) \subset \mathscr{H}^q(k)$ and the forms in $\mathscr{H}^q(k)$ have no residues on H;*

(ii) *If $\boldsymbol{\delta} \in H_q(V, \mathbf{Z})_0$ is a rational primitive cycle, then $\boldsymbol{\delta}$ is homologous to a cycle δ in $V - H$ and, for any*

$$\phi \in H^{q,0}(V) + \cdots + H_0^{q-k,k}(V), \int_{\boldsymbol{\delta}} \phi = \int_\delta r(\phi).$$

Remarks. The mapping r is constructed as follows: Let Ω^s_c be the sheaf of *closed*, holomorphic s-forms on V. Then we prove:

(α) There is a natural isomorphism:

$$H^k(V, \Omega_c^{q-k}) = H^{q,0}(V) + \cdots + H^{q-k,k}(V).$$

(β) Next, by generalizing Weil's proof of the deRham theorem, we show that there is a natural mapping

$$H^k(V, \Omega_c^{q-k}) \xrightarrow{\theta} A^q(k).$$

(γ) Then we show that, under the isomorphism in (α), the kernel of θ is

$$\{\omega \wedge H^{q-2}(V, \mathbf{C})\} \cap \{H^{q,0}(V) + \cdots + H^{q-k,k}(V)\},$$

where $\omega \in H^2(V, \mathbf{C})$ is dual to H.

It follows that $\theta\{H^k(V, \Omega_c^{q-k})\} = \mathscr{H}^q(k)$ is our desired subspace.

Example. Let $V \subset \mathbf{P}_3$ be the non-singular surface given in affine coordinates by $x^4 + y^4 + z^4 = 1$. Then $\mathscr{H}^2(0)$ is generated by $\dfrac{dxdy}{z^3}$; $\mathscr{H}^2(1)$ is generated by $\mathscr{H}^2(0)$ plus the forms $x^\alpha y^\beta z^\gamma \dfrac{dxdy}{z^3}$ with $2 \leqq \alpha + \beta + \gamma \leqq 4$ and $\alpha, \beta, \gamma \leqq 3$; and $\mathscr{H}^2(2)$ is generated by $\mathscr{H}^2(1)$ plus the form $x^2y^2z^2\dfrac{dxdy}{z^3}$. Observe that $\dim \mathscr{H}^2(1)/\mathscr{H}^2(0) = 19$ so that $\dim H^2(V, \mathbf{C})_0 = 21$, which checks with the fact that V is a $K3$ *surface*.

We observe that $\mathscr{H}^2(k) = \mathscr{H}^2(k+1) = \cdots = \mathscr{H}^2(*)$ for $k > q$.

4. *Differential Equations of the Periods*

Suppose now that we consider, as in the beginning of §3, an algebraic family $\{V_t\}_{t \in B}$ given by $\mathbf{V} \xrightarrow{\pi} B$. We take a sufficiently general projective embedding $\mathbf{V} \subset \mathbf{P}_N$ and choose a generic hyperplane section $\boldsymbol{H}$ of $\mathbf{V}$ such that:

(i) The hyperplane section $H_t = \boldsymbol{H} \cdot V_t$ is a general plane section of V_t for $t \in B - S$;

(ii) The polarization induced by H_t on V_t is the given polarization; and

(iii) If we let $C^q(k) = C^q_{\mathbf{V}}(k\boldsymbol{H})$ be the vector space of rational q-forms ψ such that ψ has a pole of order $k+1$ on $\boldsymbol{H}$ and $\psi \mid V_t$ is closed for a general point t, then the mappings $C^q(k) \to A^q_{V_t}(kH_t) \to 0$ are onto for $t \in B - S$, $k > 0$.

Remarks. In general, a form $\psi \in A^q_V(kH) = A^q(k)$ (V is a general V_t, $H = H_t$) will *not* be the restriction to V of a *closed* form ψ on $\mathbf{V}$; in particular, if ψ is holomorphic, it will not be the restriction of a holomorphic form on $\mathbf{V}$.

Let $\mathscr{E}(B)$ be the algebra of rational differential operators on B and let $\mathscr{H}^q_t(k) \subset A^q_{V_t}(kH_t)$ be the subspace representing the primitive cohomology as given in the theorem of §4. We set $\mathscr{H}^q(k) = \mathscr{H}^q_V(kH)$ where V is a general V_t, $\mathscr{H}^q(*) = \mathscr{H}^q_V(kH)$ for $k > q$, etc.

Theorem. (i) $\mathscr{E}(B)$ *is an algebra of operators on* $\mathscr{H}^q(*)$. *If* $\mathscr{E}_l(B) \subset \mathscr{E}(B)$ *is the subspace of operators of degree* l, *then*

$$\mathscr{E}_l(B)\cdot\mathscr{H}^q(k)\subset\mathscr{H}^q(k+l)\qquad (k>0)$$

$$\mathscr{E}_l(B)\cdot\mathscr{H}^q(0)\subset\mathscr{H}^q(l+2)$$

(ii) *If* ψ_t *is a rational family of sections of* $\mathscr{H}^q_t(*)$, *if* $\delta_t\in H_q(V_t-H_t)$ *is a family of primitive cycles, and if* $D\in\mathscr{E}(B)$, *then*

$$D\left(\int_{\delta_t}\psi_t\right)=\int_{\delta_t}(D\cdot\psi_t),$$

provided both sides make sense.

Remarks. This result forms the algebraic basis for studying the differential equations of the periods.

To get an idea of how $\mathscr{E}(B)$ acts on $\mathscr{H}^q(*)$, we suppose that $B=\mathbf{P}_1$ with coordinate t. Let ψ_t be a rational family of q-forms such that ψ_t is in $A^q_{V_t}(*)$; i.e., ψ_t is a closed q-form on V_t with poles on H_t. By (iii) we can choose a rational q-form ψ on $\mathbf{V}$ such that ψ has poles on $\boldsymbol{H}$ and $\psi\,|\,V_t=\psi_t$. Now $d\psi\,|\,V_t=0$ since ψ_t is closed, and so we can write: $d\psi=\phi\wedge dt$, where ϕ is a q-form with poles on $\boldsymbol{H}$. We set $\partial/\partial t(\psi_t)=\phi_t=\phi\,|\,V_t$. If we had another $\tilde{\phi}$ with $d\psi=\tilde{\phi}\wedge dt$, then $\phi_t-\tilde{\phi}_t$ will be exact, so that $\partial/\partial t(\psi_t)$ is defined modulo exact forms. Also, if $\psi_t=d\xi_t$, then we would find that $\partial/\partial t(\psi_t)$ is exact, so that $\partial/\partial t$ operates on $\mathscr{H}^q_{V_t(*)}$.

The proof of $\dfrac{\partial}{\partial t}\left(\displaystyle\int_{\delta_t}\psi_t\right)=\displaystyle\int_{\delta_t}\left(\dfrac{\partial\psi_t}{\partial t}\right)$ is done by Stokes's Theorem.

To define the differential equations, we let ${}_1\psi_t,\cdots,{}_m\psi_t$ be a family of rational forms giving a basis for $\mathscr{H}^q_{V_t}(k)$. A *differential equation* (*of level* k) is given by $(D_1,\cdots,D_m)$ where $D_\alpha\in\mathscr{E}(B)$ and $\sum_{\alpha=1}^m D_\alpha\cdot{}_\alpha\psi_t=0$ in $\mathscr{H}^q_{V_t}(*)$. We let $\mathscr{L}_k\subset\underbrace{\mathscr{E}(B)+\cdots+\mathscr{E}(B)}_{m}$ be the $\mathscr{E}(B)$-module of differential equations of level k.

Theorem. (i) $\mathscr{L}_k$ *is a finitely generated* $\mathscr{E}(B)$-*module.*

(ii) *If* $\dim B=1$ (*i.e.,* B *is a curve*), *and if* $\xi_1(t),\cdots,\xi_m(t)$ *are local analytic functions on* B *such that* $\sum_{\alpha=1}^m D_\alpha\xi_\alpha(t)=0$ *for every* $(D_1,\cdots,D_m)\in\mathscr{L}_k$, *then there exists a primitive cycle* $\delta_t\in H_q(V_t-H_t)$ *such that*

$$\xi_\alpha(t)=\int_{\delta_t}{}_\alpha\psi_t.$$

Remarks. To prove (ii), one writes down explicitly generators for $\mathscr{L}_k$. In doing this, the following numbers appear naturally:

$$m_l=\dim\{\mathscr{E}_l(B)\cdot\mathscr{H}^q(k)\}\quad\text{where}\quad\mathscr{E}_l(B)\cdot\mathscr{H}^q(k)\subset\mathscr{H}^q(k+l).$$

By the results in §2, these numbers have a cohomological interpretation, and by pursuing this, we can prove (ii).

For $\dim V_t = 1$, (ii) was the key result in Manin's treatment of rational points on curves defined over function fields.

These results are formal algebraic results on the nature of the module of differential equations. To obtain our main analytic theorem, we restrict $\{V_t\}_{t\in B}$ to a disk $\{V_t\}_{t\in\Delta}$ such that V_t is non-singular for $t\neq 0$. We let $\{\psi_t\}_{t\in B}$ be a family of differentials in $A^q_{V_t}(*)$ such that ψ_t is a closed, rational q-form on V_t for $t\neq 0$. We set $\psi_t^{(\alpha)} = \partial^\alpha\psi_t/\partial t^\alpha$ and let m be the least integer such that we have $\sum_{\alpha=0}^{m} r_\alpha(t)\psi_t^{(\alpha)} \equiv 0$ (exact forms), $r_m(t)\equiv 1$.

Theorem. (i) *$r_\alpha(t)$ is a single-valued analytic function with a pole of order $m-\alpha$ at $t=0$.*

(ii) *If $\delta\in H_q(V_t - H_t)$ is a primitive q-cycle, then the* period $\pi(t) = \int_{\delta_t}\psi_t$ *satisfies the differential equation,*

$$\sum_{\alpha=0}^{m} r_\alpha(t)\,\frac{\partial^\alpha\pi(t)}{\partial t^\alpha} = 0,$$

which is an ordinary D.E. having a regular singular point *at $t=0$.*

Remarks. This is the main analytic result on the asymptotic behavior of the period $\pi(t)$. For example, having chosen a branch of the multiple-valued analytic function $\pi(t)$, it follows that:

$$|\pi(t)| \leqq c\,|t|^{-N},$$

so that $\pi(t)$ does *not* have an essential singularity at $t=0$.

Corollary. *The generators of $\mathscr{L}_k$, the module of differential equations of level k, are each* Fuchsian differential equations *on B.*

Remarks. A Fuchsian D.E. on B is a rational differential operator θ on B which has the local form

$$\theta(t) = \sum_{\alpha=0}^{m}\theta_\alpha(t)\left(t\frac{\partial}{\partial t}\right)^\alpha, \quad \theta_m(0)\neq 0,$$

where the $\theta_\alpha(t)$ are holomorphic functions of t. This is a purely algebraic statement for which I know of no algebraic proof.

Example. Let $V_t\subset \mathbf{P}_3$ be the surface with equation $x^4+y^4+z^4+1 = 4txyz$. This is a family of $K3$ *surfaces* and V_t is non-singular for $t^4\neq 1$, $t\neq\infty$. We let $\psi_t = \dfrac{dx\,dy}{z^3 - txy}$ be the holomorphic 2-form on V_t. Then the D.E. of the periods of ψ_t is:

$$\frac{\partial^3\pi}{\partial t^3} + \left(\frac{6t^3}{t^4 - 1}\right)\frac{\partial^2\pi}{\partial t^2} + \left(\frac{7t^2}{t^4 - 1}\right)\frac{\partial\pi}{\partial t} + \left(\frac{t}{t^4 - 1}\right)\pi = 0.$$

5. *Global Study of the Period Mapping*

We consider an algebraic family $\{V_t\}_{t\in B}$, as in §2, and we suppose that $\{V_t\}_{t\in B-S}$ is a fibre space where $S \subset B$ is a non-singular subvariety. We want to discuss the global nature of the period mapping $\Phi:(B - S) \to M$.

To describe Φ explicitly, we recall that $M = D/\Gamma$ where the period matrix space D is an open set on an algebraic variety X. Now X is a submanifold of a flag manifold $\mathscr{F}$, the points $\Omega[S^0, S^1, \cdots, S^t]$ in $\mathscr{F}$ being subspaces $S^0 \subset S^1 \subset \cdots \subset S^t \subset H^q(V,\mathbf{C})_0$ where $V = V_{t_0}$ is a general member of $\{V_t\}_{t\in B}$. There is a natural embedding of $\mathscr{F}$ into a product of *Grassmann manifolds* which sends $\mathbf{\Omega}$ into $(S^0, S^1, \cdots, S^t)$, the subspace S^k giving a point $\mathbf{\Omega}_k$ in a Grassmann manifold $\mathscr{G}_k$ of linear subspaces through the origin in $H^q(V,\mathbf{C})_0$.

Thus the mapping $\Phi(t)$ is given by a multiple-valued mapping $\mathbf{\Omega}(t)$ of $B - S$ into $\mathscr{F}$, and $\mathbf{\Omega}(t)$ is in turn given by multiple-valued mappings $\mathbf{\Omega}_k(t)$ into Grassmann manifolds $\mathscr{G}_k$.

To give $\mathbf{\Omega}_k(t)$ explicitly, we choose forms $\psi^1, \cdots, \psi^k$ in $C^q_V(kH)$ such that $\psi^1 \,|\, V_t, \cdots, \psi^h \,|\, V_t$ give a basis for $\mathscr{H}^q_{V_t}(k)$ $(t \in B - S)$. We then choose cycles $\delta_1(t), \cdots, \delta_b(t)$ which give a basis for $H_q(V_t, \mathbf{Q})_0$, the primitive rational q-cycles. Then we form the *period matrix*:

$$\Omega_k(t) = (\pi_{\alpha\rho}(t)),$$

where $\pi_{\alpha\rho}(t) = \int_{\delta_\rho(t)}\psi^\alpha$. The matrix $\Omega_k(t)$ is of rank h and the space spanned by the rows of $\Omega_k(t)$ gives the linear subspace $\mathbf{\Omega}_k(t)$ in $\mathscr{G}_k$. In other words, $\mathbf{\Omega}_k(t)$ is given the *Plücker coordinates* of $\Omega_k(t)$.

Now $\Omega_k(t)$ is *not* single-valued on $B - S$ because the fundamental group $\pi_1(B - S)$ acts on $H_q(V, \mathbf{Q})_0$; given a closed loop λ in $B - S$, continuation of $\delta_1(t), \cdots, \delta_b(t)$ around λ leads to a substitution $\delta_\rho(t) \to \sum_{\sigma=1}^{b} \lambda_{\sigma\rho}\delta_\rho(t)$. Letting $T = (\lambda_{\sigma\rho})$, analytic continuation sends $\Omega_k(t)$ into $\Omega_k(t)T$.

We are interested in the behavior of $\Phi(t)$ as t approaches S, and for this we let λ be a loop surrounding a branch of S. In fact, there is no essential loss of generality if we restrict $\{V_t\}_{t\in B}$ to a family $\{V_t\}_{t\in\Delta}$ over the disk Δ and where $B \cap \Delta = \{0\}$. Then λ is a loop around zero in the t-disk, and we may symbolically write: $\Omega_k(e^{2\pi\sqrt{-1}}t) = \Omega_k(t)T$.

The question of finding T is a purely topological problem; in the simplest case when the singular variety V_0 has an ordinary double point, T is given by the *Picard-Lefschetz* theorem. We can show in general that:

Theorem. *All eigenvalues of T are roots of one.*

Actually, much stronger results have been found by Landman in his Berkeley thesis. Using the explicit methods of Lefschetz explained in the Borel tract, Landman proves:

Theorem. *There exists an integer N such that $(T^N - I)^{q+1} = 0$ where $T: H_q(V) \to H_q(V)$ is the* Picard-Lefschetz transformation.

More precisely even, suppose we resolve the singularities of V_0 so that $\tilde{V}_0 = m_1 V_0^1 + \cdots + m_l V_0^l$ where the V_0^α are hypersurfaces in general position with respect to each other. Then,

Theorem. *$T: H_n(V) \to H_n(V)$ has the property that each eigenvalue of T is an $m_j{}^{th}$ root of one, and $(T^m - I)^r = 0$ where $m = l.c.m(m_1, \cdots, m_l)$ and r is the largest number such that $V_0^{\alpha_1} \cap \cdots \cap V_0^{\alpha_r} \neq 0$.*

Theorem. *By replacing t by t^N, we have,*

$$\Omega_k(t) = \Omega_{k,0}(t) + (\log t)\Omega_{k,1}(t) + \cdots + (\log t)^q \Omega_{k,q}(t),$$

where each $\Omega_{k,j}(t)$ is a single-valued meromorphic matrix.

This asymptotic formula for the periods allows us to analyze the mapping $\Phi: B - S \to M$ and to prove,

Theorem. *Let $t_0 \in S$ and consider $\Phi(t)$ as a mapping of $B - S$ into D modulo Γ. Then*

(i) *$\lim_{t \to t_0} \Phi(t) = \Phi(t_0)$ exists and is a unique point in $\bar{D}$* modulo Γ;

(ii) *$\Phi(t_0)$ depends holomorphically on $t_0 \in S$; and*

(iii) *the point $\Phi(t_0) \in \bar{D}$ is a fixed point of $\gamma \in \Gamma$, where γ is a rational element of G such that γ^N is unipotent.*

Remarks. This is our main result on the asymptotic nature of the period mapping Φ. It is not as strong as we desire because we should like to show:

(i) $\Phi(t_0)$ belongs to a *rational boundary component* of D;

(ii) $\Phi(t)$ converges to $\Phi(t_0)$ in the sense of Satake and Borel-Baily.

We know of no example where (i) and (ii) fail, but we are unable to prove these statements except in several special cases. If (i) and (ii) hold, then it follows that, if a compactification M^* of M exists (with the properties found by Borel and Baily where D is a Cartan domain), then Φ extends to a holomorphic mapping $\Phi: B \to M^*$.

By explicit computation, we can show:

Theorem. *Suppose that V_{t_0} is irreducible and has* ordinary singularities without multiple components. *Then $\Phi(t_0)$ lies in a rational boundary component $D_1 \subset \bar{D}$. This boundary component corresponds to the period matrix space for the* normalization *$\tilde{V}_{t_0}$ of V_{t_0}.*

Examples. (i) There exists a family $\{V_t\}_{t\in B}$ of *curves* with *ordinary double points* where B is irreducible and such that every non-singular curve of genus p is in the family (Severi). The modular variety M has a compactification to a normal algebraic variety M^* (Satake, Baily), and $\Phi(B) \subset M^*$ is an algebraic subvariety. In particular, this implies that the set of Jacobian varieties among normally polarized abelian varieties is an algebraic subvariety (Baily).

(ii) Let $\{V_t\}_{t\in B}$ be a family of polarized $K3$ *surfaces* (e.g., double planes with a non-singular sextic branch curve). The modular variety M has a compactification to a normal algebraic variety M^* (Borel-Baily) and, at least for the classical families of $K3$ surfaces, we have that $\Phi: B \to M^*$ is a rational map. In particular, Φ is surjective so that all points on M^* are periods of (possibly degenerated) $K3$ surfaces. The points on the boundary $M^* - M$ can be identified as periods of $K3$ surfaces having acquired a double curve.

(iii) Let $\{V_t\}_{t\in B}$ be the family of *cubic threefolds* in $\mathbf{P}_4$. Then D is a *Siegel upper half-space* and M has a compactification M^* as above. The period mapping $\Phi: B \to M^*$ is rational, so that the intermediate Jacobians of cubic threefolds form an algebraic family. In this case also, the general boundary point can be identified as corresponding to certain non-singular threefolds whose "genus" has dropped by one.

Numerous other examples along these lines can be given.

6. *Complex Tori, Algebraic Cycles, and Holomorphic Vector Bundles*

We return to the tori of §1 and, with a slight shift in notation, set $T_q(V) = H^{2q-1}(V, \mathbf{R})/H^{2q-1}(V, Z)$. We may identify the invariant complex-valued 1-forms on $T_q(V)$ with $H^{2n-2q+1}(V, \mathbf{C})$; letting $S = H^{2n-2q+1,0} + \cdots + H^{n-q+1,n-q}$, we have $S + \bar{S} = H^{2n-2q+1}(V, \mathbf{C})$, $S \cap \bar{S} = 0$. This gives a complex structure, with p-convex polarization, to $T_q(V)$, the holomorphic 1-forms on $T_q(V)$ simply being S. The torus $T_q(V)$ with this complex structure is the one given in §1.

We may give S intrinsically as follows (cf. remark (α) in §3): For a C^∞ form ϕ of degree r on V, write $\phi = \sum_{s+t=r} \phi_{s,t}$ where $\phi_{s,t}$ is the (s,t) part of ϕ. Let $A^{r,s}$ be the C^∞ forms ϕ of degree r and with $\phi_{t,u} = 0$ for $t < s$ (i.e., ϕ has at least s dz's), and $A_c^{r,s}$ the closed forms. Then we have that:

Proposition. *There is a natural isomorphism*

$$S \cong \frac{A_c^{2n-2q+1,n-q+1}}{dA^{2n-2q,n-q+1}}.$$

Let now $\Sigma_q(V)$ be the algebraic cycles of codimension q on V which are homologous to zero. If $Z \in \Sigma_q(V)$, $Z = \partial C$ for some $2n-2q+1$ chain C on V. Let $\omega^1, \cdots, \omega^m$ be a basis for S and define $\phi: \Sigma_q(V) \to T_q(V)$ by $\phi(Z) = \begin{bmatrix} \vdots \\ \int_C \omega^\alpha \\ \vdots \end{bmatrix}$ (modulo periods). If $n = q = 1$, this is the classical mapping in the theory of curves.

Theorem. *ϕ is well-defined and is holomorphic.*

Remarks. Using the isomorphism in the proposition above, we may replace ω^α by $\omega^\alpha + d\eta^\alpha$ where $\eta^\alpha \in A^{2n-2q,n-q+1}$. Then $\int_C d\eta^\alpha = \int_Z \eta^\alpha = 0$, which proves that ϕ is well-defined. The fact that ϕ is holomorphic means holomorphic in the weak sense: if $\{Z_\lambda\}_{\lambda\in\Delta}$ is an analytic family of q-codimensional algebraic cycles, then $\phi(Z_\lambda - Z_{\lambda_0})$ is holomorphic in λ.

The analogue of the transversality theorem in §2 is:

Theorem. *$\phi_*: T_Z(\Sigma_q(V)) \to H^{q-1,q}(V)_{\phi(Z)}$. In particular, the polarizing line bundle on $T_q(V)$ is positive on $\phi(\Sigma_q(V))$.*

Remarks. In this case we can show how, by an integration over the fibre, the *theta cohomology* $H^p(T_q(V), L)$ gives holomorphic sections of $L \mid \phi(\Sigma_q(V))$.

To compute ϕ_*, we select a point $Z \in \Sigma_q(V)$ such that $Z = \sum_{\alpha=1}^k m_\alpha Z_\alpha$ with the Z_α submanifolds. Let $N \to Z$ be the normal bundle; then $T_Z(\Sigma_q(V)) \cong H^0(Z, N)$ (Kodaira). Thus $\phi_*: H^0(Z,N) \to H^{q-1,q}(V)$.

Theorem. *The adjoint of ϕ_* is the* Poincaré residue operator $R: H^{n-q+1,n-q}(V) \to H^{n-q,n-q}(Z, N^*)$.

Remarks. To give R, we let $\omega \in H^{n-q+1,n-q}(V)$ and let $f_1 = 0, \cdots, f_q = 0$ be local defining equations for Z. Then $\omega = \sum_{j=1}^q \omega_j \wedge df_j$ and $R(\omega) = \begin{pmatrix} \vdots \\ \omega_j \\ \vdots \end{pmatrix} \Big| Z$.

Let now $E \to V$ be a holomorphic vector bundle with fibre $\mathbf{C}^k$ and $Z_1(E), \cdots, Z_k(E)$ the *Chern cycles* of $E \to V$. Thus $Z_q(E)$ is an algebraic cycle of codimension q, defined up to rational equivalence. For example, if $E \to V$ has a holomorphic cross-section σ, then $Z_k(E) = \{\sigma = 0\}$.

Suppose now that $E_0 \to V$ is a fixed holomorphic vector bundle and let $\Sigma(E_0)$ be the set of holomorphic vector bundles $E \to V$ which are topologically isomorphic to $E_0 \to V$. For $E \in \Sigma(E_0)$, define $\phi_q(E) \in T_q(V)$ by $\phi_q(E) = \phi(Z_q(E) - Z_q(E_0))$. The images $\phi_q(\Sigma(E_0))$ as E_0 runs over all bundles generate $\phi(\Sigma_q(V))$; thus, we want to calculate $\phi_q(\Sigma(E_0))$. We will do this infinitesimally. Since $T_E(\Sigma(E_0)) \cong H^{0,1}(V, \mathrm{Hom}(E,E))$ (Kuranishi), we must have $(\phi_q)_*: H^{0,1}(V, \mathrm{Hom}(E,E)) \to H^{q-1,q}(V)$. For a $k \times k$ matrix A, write $\sum_{q=0}^{k} P_q(A) t^{k-q} = \det((i/2\pi)A + tI)$, and let $P_q(A_1, \cdots, A_q)$ be the symmetric multilinear form obtained by polarizing $P_q(A)$. Let $\Theta \in H^{1,1}(V, \mathrm{Hom}(E,E))$ be a *curvature* in E (Atiyah).

Theorem. $(\phi_q)_*(\eta) = qP_q(\underbrace{\Theta, \cdots, \Theta}_{q-1}, \eta) \ (\eta \in H^{0,1}(V, \mathrm{Hom}(E,E)))$.

Remarks. For $q = 1$, this formula is equivalent to *Abel's Theorem*. The general proof requires the development of a residue calculus along q-codimensional subvarieties which are not complete intersections; it is in the development of this residue calculus that the Chern polynomials $P_q(A)$ appear.

To close, we want to give two functional properties of the tori $T_q(V)$ and maps $\phi: \Sigma_q(V) \to T_q(V)$. Let $W = W_{n-r}$ be a non-singular subvariety of codimension r in $V = V_n$. Then the inclusion $i: W \to V$ induces:

$$\begin{cases} i^*: T_q(V) \to T_q(W) & \text{from } i^*: H^{2q-1}(V) \to H^{2q-1}(W); \\ i_*: T_q(W) \to T_{q+r} & \text{from the } \textit{Gysin homomorphism} \\ & \qquad i_*: H^{2q-1}(W) \to H^{2q+2r-1}(V). \end{cases}$$

On the other hand, we have $\Sigma_q(W) \to \Sigma_{q+r}(V)$ (inclusion) and $\Sigma_q(V) \to \Sigma_q(W)$ (intersection with W). Relating these, we have:

Theorem. *The following diagrams commute*:

$$\begin{array}{ccc} \Sigma_q(V) & \xrightarrow{\phi} & T_q(V) \\ \downarrow & & \downarrow i^* \\ \Sigma_q(W) & \xrightarrow{\phi} & T_q(W) \end{array} \qquad \begin{array}{ccc} \Sigma_q(W) & \longrightarrow & T_q(W) \\ \downarrow & & \downarrow i_* \\ \Sigma_{q+r}(V) & \longrightarrow & T_{q+r}(V) \end{array}$$

Remark. Several interesting and similar properties of the Weil Jacobians have been found by D. Lieberman.

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