STRING TOPOLOGY OF CLASSIFYING SPACES

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Appendix A

A proof of $LBG \simeq Ad(EG)$

Proposition A.0.4. *Let G be a topological group having the homotopy type of a CW complex. Then LBG is homotopy equivalent to $Ad(EG)$ as fiberwise monoids over BG. That is, there exists a fiberwise monoid $\widetilde{LBG}/G$ over BG and maps*

$$LBG \xleftarrow{\xi} \widetilde{LBG}/G \xrightarrow{\psi} Ad(EG)$$

such that ξ and ψ are both morphisms of fiberwise monoids over BG and homotopy equivalences.

Remark A.0.5. That $LBG \simeq Ad(EG)$ is a "well-known fact" which suffers from a lack of good references. The reader may see [10] for a similar but simpler proof in the case that G is a discrete group.

Proof. Let $G \to EG \xrightarrow{p} BG$ be a universal principal G-bundle. Define

$$\widetilde{LBG} = \{\alpha : I \to EG \mid p(\alpha(0)) = p(\alpha(1))\}$$

and give $\widetilde{LBG}$ the compact-open topology. $\widetilde{LBG}$ has a free right action of G^I by pointwise multiplication, and hence also a free right action of G (by embedding $G \hookrightarrow G^I$ as the constant maps). In particular there is a commutative diagram where both columns are principal bundles:

$$\begin{array}{ccc} G & \longrightarrow & G^I \\ \downarrow & & \downarrow \\ \widetilde{LBG} & = & \widetilde{LBG} \\ \downarrow & & \downarrow \\ \widetilde{LBG}/G & \xrightarrow{\xi} & \widetilde{LBG}/G^I = LBG. \end{array}$$

Since the inclusion $G \hookrightarrow G^I$ is a homotopy equivalence and $\widetilde{LBG}/G$ and LBG both have the homotopy type of a CW complex, we see that ξ is a homotopy equivalence by Whitehead's Theorem. Furthermore, $\widetilde{LBG}/G$ and LBG both have fiberwise monoid structures over BG given by concatenation of paths and ξ is clearly a morphism of fiberwise monoids over BG.

Now $Ad(EG)$ is defined as $(EG \times G)/G$ where G acts on $EG \times G$ by $(x, g)h = (xh, h^{-1}gh)$. Define $\tilde{\psi} : \widetilde{LBG} \to EG \times G$ by $\psi(\alpha) = (\alpha(1), g)$ where $\alpha(0) = \alpha(1)g$. Then ψ induces a morphism of principal G-bundles:

$$\begin{array}{ccc} G & = & G \\ \downarrow & & \downarrow \\ \widetilde{LBG} & \xrightarrow{\tilde{\psi}} & EG \times G \\ \downarrow & & \downarrow \\ \widetilde{LBG}/G & \xrightarrow{\psi} & Ad(EG) \end{array}$$

To see that ψ is G-equivariant, take $\alpha \in \widetilde{LBG}$ and $h \in G$, and let $\tilde{\psi}(\alpha) = (\alpha(1), g)$. Then $\psi(\alpha h) = (\alpha(1)h, k)$ where $\alpha(1)hk = \alpha(0)h = \alpha(1)gh$. Hence $k = h^{-1}gh$ and $\psi(\alpha h) = \psi(\alpha)h$.

Let us first check that ψ is a morphism of fiberwise monoids over BG. Suppose that α, $\beta \in \widetilde{LBG}/G$ are in the same fiber over BG. Then there exist (non-unique) representatives $\tilde{\alpha}$, $\tilde{\beta} \in \widetilde{LBG}$ of α and β, respectively, such that $\tilde{\beta}(0) = \tilde{\alpha}(1)$. Write

$$\begin{aligned} \tilde{\psi}(\tilde{\alpha}) &= (\tilde{\alpha}(1), g) \\ \tilde{\psi}(\tilde{\beta}) &= (\tilde{\beta}(1), h). \end{aligned}$$

Then from Diagram A.1 one sees that $\tilde{\psi}(\tilde{\alpha}{\cdot}\tilde{\beta}) = (\tilde{\beta}(1), hg)$.

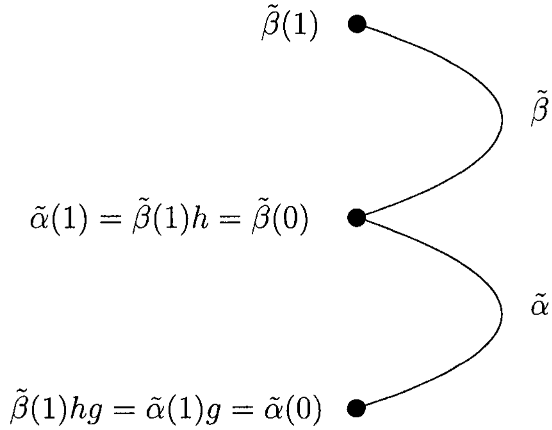

Figure A.1: Multiplication in $\widetilde{LBG}/G$

Notice that the fiberwise multiplication in $\widetilde{LBG}/G$ is well-defined independent of the choices of $\tilde{\alpha}$ and $\tilde{\beta}$. Hence $\psi(\alpha \cdot \beta) = [\tilde{\beta}(1), hg]$. On the other hand, in $Ad(EG)$ we have

$$[\tilde{\alpha}(1), g] \cdot [\tilde{\beta}(1), h] = [\tilde{\beta}(1)h, g] \cdot [\tilde{\beta}(1), h] = \\ [\tilde{\beta}(1), hgh^{-1}] \cdot [\tilde{\beta}(1), h] = [\tilde{\beta}(1), hgh^{-1}h] = [\tilde{\beta}(1), hg].$$

Hence $\psi(\alpha \cdot \beta) = \psi(\alpha) \cdot \psi(\beta)$ so ψ is a morphism of fiberwise monoids over BG.

To see that ψ is a homotopy equivalence, it is enough to see that $\tilde{\psi}$ is. Fix a contraction $F : EG \times I \to EG$ of EG to a point y_0. This gives a canonical path $\gamma_y : I \to EG$ from y to y_0 for any $y \in EG$, by $\gamma_y(t) = F(y, t)$. Define $\phi : EG \times G \to \widetilde{LBG}$

by:

$$\phi(z,g)(t) = \begin{cases} \gamma_{zg}(2t) & 0 \le t \le \frac{1}{2} \\ \gamma_z(2-2t) & \frac{1}{2} \le t \le 1. \end{cases}$$

Then $(\phi \circ \tilde{\psi})(\alpha)$ is the path from $\alpha(0)$ to $\alpha(1)$ that traverses $\gamma_{\alpha(0)}$ in time $\frac{1}{2}$ and then traverses $\gamma_{\alpha(1)}$ backwards in time $\frac{1}{2}$. $\phi \circ \tilde{\psi}$ is homotopic to the identity map on $\widetilde{LBG}$ by the homotopy $G : \widetilde{LBG} \times I \to \widetilde{LBG}$,

$$G(\alpha,s)(t) = \begin{cases} F(\alpha(0), 2t) & 0 \le t \le 1 - \frac{s}{2} \\ F(\alpha(\frac{1}{1-s}(t - \frac{s}{2})), s) & \frac{s}{2} < t < 1 - \frac{s}{2} \\ F(\alpha(1), 2-2t) & 1 - \frac{s}{2} \le t \le 1. \end{cases} \tag{A.1}$$

On the other hand, $(\tilde{\psi} \circ \phi)(z,g) = (z,g)$. Hence $\tilde{\psi}$ is a homotopy equivalence. Then ψ is also, so

$$LBG \simeq \widetilde{LBG}/G \simeq Ad(EG).$$

□