An algebraic theory of tricategories

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Abstract

The original definition of tricategory given by Gordon, Power, and Street is only partially algebraic. The definition is not fully algebraic since certain transformations are required to be weakly invertible as 1-cells of a functor bicategory, but no weak inverse is required as part of the data. We rectify this by replacing these equivalences with adjoint equivalences. We then prove coherence by providing a Yoneda embedding for a restricted class of tricategories in which the target of this embedding is a functor tricategory that is shown to be a **Gray**-category; in particular, this strategy avoids the use of the prerepresentations in the work of Gordon, Power, and Street.

Using the fact that the new definition of tricategory is algebraic, we compare the free tricategory on a category-enriched 2-graph with the free **Gray**-category on the same data and show that the natural comparison functor is a strict triequivalence. This is another statement of coherence, and also gives a proof that a large class of diagrams of constraint 3-cells commute in any tricategory. We then produce, from any tricategory T, a **Gray**-category $\mathrm{Gr}T$ and a triequivalence $\mathrm{Gr}T \to T$. A similar strategy applied to functors yields a coherence theorem for functors, and we then produce from any functor $F : S \to T$ between tricategories a **Gray**-functor $\mathrm{Gr}F : \mathrm{Gr}S \to \mathrm{Gr}T$.

Contents

Chapter 1

Introduction

The study of weakened higher dimensional structures in category theory began with the notion of bicategory, defined by Benabou in 1967 [5]. The study of bicategories now has two equally important components: one is as a tool to organize and generalize theorems from category theory, and another is the study of bicategories as interesting algebraic objects in their own right. An application of the first kind is the study of monads in a general 2-category [35], and an example of a theorem of the second kind is the coherence theorem for bicategories which states that every bicategory can be made strict in a precise sense [41], [17]. There are also important applications of this theory in physics, topology, and representation theory.

The intense focus on understanding structures of dimension $n > 2$ is a relatively recent phenomenon – the first paper to even hint at a possible definition of weak ω-category (a type of weak category-structure with cells of every dimension n for $n \geq 0$) is Street's *The algebra of oriented simplexes* in 1987 [39]. Since then, there have been many definitions of weak n- or ω-category proposed by a number of different authors. The survey [29] by Leinster provides a good account of many of these proposed definitions.

There is an important distinction to be made between weak and strict structures. In a strict n-category, all possible axioms hold, including those for cells that are not of the top dimension. This is not the case for weak n-categories, where we only have axioms governing cells of the top dimension and the old axioms for lower dimensional cells are replaced by invertible or weakly invertible cells subject to their own laws. An example of this phenomenon occurs in the definition of a bicategory, where 1-cells are only required to compose associatively up to 2-cell isomorphisms that are then required to satisfy a new axiom, the Mac Lane pentagon.

But even the basics of higher dimensional category theory are far from established. The project of comparing these different definitions is most likely years from completion, and for most definitions few, if any, significant applications have been produced. There have been a few applications to topology, with some success in using ideas from higher category theory to study n-fold loop spaces and homotopy n-types [4], [6], [7], [44].

The definitions alluded to above all have a general nature to them. They are intended to describe weak n-categories for arbitrary n, sometimes including weak ω-categories. Some of these definitions are inductive, but some also start by defining weak ω-category and then specializing to finite n. None of these definitions are what one would call "hands-on", though. They do not explicitly formulate the axioms involved, instead relying on complicated techniques to efficiently encode all of the axioms at once, usually in the form of the structure of an algebra over a suitably chosen monad or by requiring that certain "horn-filling" conditions hold.

There is a hands-on definition of weak 3-categories, which are called tricategories, defined by Gordon, Power, and Street in their 1995 Memoir [17]. This definition is a monumental achievement, and as such is long and complicated if not viewed from the proper perspective. To understand the complexities of the definition, it is necessary to think about the general philosophy of categorification and the coherence theory for bicategories.

Categorification is the term used to describe the general procedure of taking a definition involving sets, functions, and equations between them, and creating a new definition involving categories, functors, natural isomorphisms, and equations between those. The basic philosophy of categorification is to replace the old axioms with new pieces of data, and then to construct the appropriate axioms that this new data is to satisfy.

There are three important steps in the categorification process involved in the definition of tricategory. The first is categorifying the notion of isomorphism. Isomorphism is already the categorified version of equality, and the categorified notion of isomorphism present in [17] is that of equivalence. The second important aspect of the definition of tricategory is the introduction of two new pieces of data, denoted λ and ρ, that do not arise as the categorified versions of old axioms. This is somewhat misleading, as these new pieces of data are categorified versions of important results used in the proof of the coherence theorem for bicategories [22]. The third important step in this categorification process is finding the correct axioms that tricategories should satisfy. The associativity axiom for tricategories is recognizable as an incarnation of the fifth associahedron of Stasheff [43] or the fifth oriental of Street [39]. The two unit axioms are more mysterious, however, and in general the unit conditions for higher categories are not as well understood as the associativity conditions.

The work of Gordon, Power, and Street has the primary goal of proving a relevant coherence theorem for tricategories. The coherence problem for bicategories has a straightforward answer: every bicategory is biequivalent to a strict 2-category. Thus all of the "weakness" in a bicategory can be removed by replacing the bicategory in question with a biequivalent one. This is not the case for tricategories – not every tricategory is triequivalent to a strict 3-category, nor can this be true for any reasonable definition of tricategory and triequivalence as we shall see. Thus the coherence theorem for tricategories is more interesting because of the inherent complications that arise from going up a dimension.

The reason that tricategories cannot all be triequivalent to strict 3-categories is a consequence of the topology of homotopy 3-types. In his famous letter *Pursuing Stacks* [20], Grothendieck outlined some desiderata for a good theory of higher dimensional groupoids. In particular, he suggested that weak n-groupoids should be a model for homotopy n-types. This gives some insight into the structure of higher categories, as many topologists have studied the problem of finding algebraic models for homotopy n-types. For example, homotopy 2-types are modelled by monoidal categories in which each morphism is invertible and each object x has a tensor pseudoinverse, that is an object y for which $x \otimes y \cong I$ and $y \otimes x \cong I$. This is just a manifestation, in categorical language, of the fact that a connected homotopy 2-type is determined by its homotopy groups and the action of π_1 on π_2.

In dimension 3, the situation becomes slightly different. Connected, simply-connected homotopy 3-types are classified by their homotopy groups and their Whitehead product $\pi_2 \times \pi_2 \to \pi_3$. In categorical language, this becomes the statement that connected, simply-connected homotopy 3-types are modelled by braided monoidal categories in which every morphism is invertible and every object has a tensor pseudoinverse [21]. Since all homotopy 3-types should be modelled by weak 3-groupoids, we can first ask if strict 3-groupoids can model all connected, simply-connected homotopy 3-types. Now the Whitehead product in the nerve of a strict 3-groupoid is the zero map. (See [34] for a full discussion.) Since any reasonable definition of triequivalence should induce a weak equivalence between the corresponding nerves and there are connected, simply-connected homotopy 3-types with non-trivial Whitehead product, we see that strict 3-groupoids do not model all homotopy 3-types.

The correct coherence theorem, proved by Gordon, Power, and Street, is that every tricategory is triequivalent to a **Gray**-category. Here **Gray** denotes a particular monoidal structure on the category of strict 2-categories defined by Gray [18], and a **Gray**-category is then just a category enriched over **Gray**. The reader should take note that we actually use what might be called the strong Gray tensor product, where Gray studied the lax version. Simply put, the problem with strictifying every trigroupoid to a strict 3-groupoid is the existence of a "braiding" in the trigroupoid case (corresponding topologically to the Whitehead product) that is forced to be symmetric in the strict 3-groupoid case. The Gray tensor product of 2-categories builds in an appropriate interchange isomorphism, and the coherence theorem of [17] then states that this interchange isomorphism is the only obstruction to completely strictifying a tricategory.

This coherence theorem is very natural when approached via the example of the tricategory of bicategories, functors, transformations, and modifications, where here all terms refer to the weak version of the notion involved. The coherence theorem for bicategories states that every bicategory is biequivalent to a strict 2-category. Similarly, there is a coherence theorem for functors that produces the result that, when strictifying bicategories, one can also strictify the maps between them to yield strict 2-functors between strict 2-categories. Thus we are able to produce a functor $\mathrm{st} : \mathbf{Bicat} \to \mathbf{2Cat}$ that is left adjoint to the inclusion of 2-categories into bicategories. Now both **Bicat** and **2Cat** form 3-dimensional structures in a natural way. Two questions arise. First, can this functor st be extended to a map of 3-dimensional structures? Secondly, if it can, what properties does this extension have?

Given a bicategory B, there is a canonical comparison functor $\mathrm{st}B \to B$ that is a biequivalence. This leads one to believe that st might be a triequivalence, but this is not the case. The problem arises when trying to understand the composition laws for transformations in **Bicat**. In defining the horizontal composite $\beta * \alpha$ of a pair of transformations, there are two equally good candidates for the component of $\beta * \alpha$ at the object a, and if β is a strict transformation then these two choices agree. But one quickly learns that it is not always possible to replace β by an isomorphic transformation that is strict, so we see that **Bicat** has an unavoidable amount of weakness built into it. This weakness, though, is precisely the fact that interchange for 2-cells is an isomorphism and not an equality. Now **Gray**-categories are the strictest form of 3-dimensional category in which interchange remains weak (i.e., is an isomorphism not an equality), so the example of the tricategory **Bicat** leads one to the study of **Gray**-categories. To answer the questions posed in the previous paragraph we introduce a new tricategory called **Gray** which consists of 2-categories, 2-functors, weak transformations, and modifications. It is now relatively simple to check that the functor st gives a triequivalence $\mathbf{Bicat} \to \mathbf{Gray}'$ (here $\mathbf{Gray}'$ denotes a particular full sub-**Gray**-category of **Gray**), and this statement brings together the many facets of the coherence theory for bicategories in one simple statement. It is worth noting that the tricategory **Gray** is the tricategory obtained from the category of 2-categories by using the closed structure given by the Gray tensor product and its right adjoint.

The definition given by Gordon, Power, and Street has a feature that will be the focus of this work: it is not completely algebraic, and for some applications this is a definite drawback. In the case of tricategories, we mean that some of the data is required to have a certain property but verifying this property makes use of additional data that is not uniquely specified in the definition. This is a by-product of the choice made when categorifying the notion of isomorphism. The data for a bicategory include associativity, left unit, and right unit isomorphisms; these exist as invertible 2-cells in the given bicategory structure. In the definition of tricategory, analogous 2-cells exist but now they are not top-dimensional cells, so we require them to be weakly invertible rather than invertible.

This is where the definition given by Gordon, Power, and Street is not fully algebraic. They choose to require the 2-cells above to be equivalence cells. This is a property of a cell, but leaves some data unspecified: it requires that there exist a pseudoinverse and invertible cells of one dimension higher exhibiting the cells as weakly invertible, but does not require a choice of these cells. This is different from the definition of an isomorphism in a category. Since inverses are unique in a category, requiring that a morphism be invertible and requiring the exhibition of an inverse are logically equivalent conditions. The situation here is genuinely different, as there are many possible pseudoinverses and even then many possible invertible cells exhibiting this pseudoinvertibility.

Giving an algebraic definition of tricategory thus requires changing these equivalence 1-cells to an algebraic condition of weak invertibility. The rest of this work will be concerned with developing the basic coherence theory of a fully algebraic definition of tricategory along these lines. We have taken the notion of adjoint equivalence as our algebraic version of weakly invertible 1-cell in a bicategory. It should be noted that every equivalence 1-cell in a bicategory is part of an adjoint equivalence, but that there is no canonical choice of such extra structure.

The definitions given here are of course similar to those given by Gordon, Power, and Street, but wherever they demand that a transformation be a pseudonatural equivalence, we instead require an adjoint equivalence in the appropriate functor bicategory. This provides canonical pseudoinverses for all of the appropriate structure constraints, as well as the necessary cells of the next dimension up to exhibit this pseudoinvertibility explicitly.

There are many choices for the notion of weak invertibility. An intermediate notion between equivalence and adjoint equivalence might be called specified equivalence. This would require giving a pseudoinverse and the invertible cells exhibiting this pseudoinvertibility, but would not require these cells to satisfy any axioms. The choice of adjoint equivalence has the clear advantage over this intermediate notion that it allows the use of mates. A happy by-product of the theory of mates in a bicategory allows us to refrain from introducing a new set of dual axioms for these additional cells, as they are already implied. This is the phenomenon that is responsible for the fact that the opposite tricategory, defined by reversing the direction of the 1-cells only, satisfies the tricategory axioms.

The coherence theorem for bicategories states that every bicategory is biequivalent to a strict 2-category. The simplest way to prove this theorem is to study the Yoneda embedding for bicategories, a functor

$$B \to \mathbf{Bicat}(B^{\mathrm{op}}, \mathbf{Cat}).$$

The target of this functor is strict since **Cat** is a strict 2-category, and the essential image of this functor is a 2-catgory biequivalent to B.

The proof of the coherence theorem given by Gordon, Power, and Street has two parts. The first is the replacement of an arbitrary tricategory T with a somewhat strict kind of tricategory, called a cubical tricategory. This is done by applying the functor st to all of the data for T and then using the fact that this functor is lax monoidal to get a composition map

$$\mathrm{st}\big(T(b,c)\big) \times \mathrm{st}\big(T(a,b)\big) \to \mathrm{st}\big(T(a,c)\big).$$

The second step in [17] is to construct for any cubical tricategory S a suitably well-behaved embedding of S into a **Gray**-category. The essential image of S inside this new **Gray**-category will then be a smaller **Gray**-category triequivalent to S. Combining these two parts gives the desired theorem. It should be noted that Gordon, Power, and Street do not give an exact 3-dimensional version of this proof. Instead of using the notion of functor tricategory (which remains undefined using their definition), they use the **Gray**-category of pre-representations of a cubical tricategory; one can view this **Gray**-category as the functor tricategory but with some data and axioms omitted.

Our proof follows a strategy that combines both that used to prove coherence for bicategories and that used by Gordon, Power, and Street. We explicitly construct the functor tricategory $\mathbf{Tricat}(S,T)$ in the case when T is a **Gray**-category, and then show that it is again a **Gray**-category. The outline of the proof is as follows. First we show how to replace T with a cubical tricategory as in [17], and then we explicitly construct a Yoneda embedding

$$S \hookrightarrow \mathbf{Tricat}(S^{\mathrm{op}}, \mathbf{Gray})$$

when S is any cubical tricategory. Restricting to the essential image gives the desired triequivalence. This shows the benefit of replacing T with a cubical tricategory, as the general Yoneda embedding would be a functor of the form

$$T \hookrightarrow \mathbf{Tricat}(T^{\mathrm{op}}, \mathbf{Bicat})$$

which would not yield the desired coherence result as **Bicat** is not a **Gray**-category.

This path to the coherence theorem requires defining a multitude of compositions for functors, transformations, modifications, and perturbations. These compositions are given by messy formulas, but inspecting these demonstrates the need for a fully algebraic definition of tricategory as all parts of the definition are necessary for writing down these formulas. We see this as a good indicator of what we have accomplished by making the definition fully algebraic: with all structure in plain sight, it is possible to write down formulas and thus make concrete constructions that required arbitrary choices in the original definition.

The drawback of this approach should also be clear: in trying to write down explicit formulas, one needs to work with very large diagrams. Verifying basic axioms with these diagrams becomes a difficult task. This is solved in the case of bicategories by proving another kind of coherence theorem, one that states that all diagrams of constraints commute. It is, after all, this kind of theorem that allows the explicit construction of the strictification $\mathrm{st}B$ for any bicategory B. Proving an analogue of this theorem, and reaping the attendant benefits, is the focus of the last third of this work.

To prove this theorem for bicategories, we first take a slight detour to prove another kind of coherence theorem (see [22] for the same line of proof but restricted to the case of monoidal categories). Given a set of objects A_0 and for each pair of objects a category $A(a,b)$, we can construct two canonical 2-dimensional structures: the free bicategory on A and the free strict 2-category on A. Each of these has the set A_0 as its set of objects, but the sets of 1- and 2-cells differ. The coherence theorem here states that these two structures are biequivalent by the strict functor induced by the universal property of the free bicategory. The theorem that every diagram of constraints in a bicategory commutes is now a simple corollary of the universal property of the free bicategory and this coherence theorem applied to the case when each of the categories

$A(a, b)$ is discrete. Our first goal, then, will be to mimic this coherence theorem comparing the free weak structure with the free strict structure, except that in our case we compare the free tricategory with the free **Gray**-category.

There is a new difficulty that arises by going up a dimension. This is the fact that there are at least three different choices of underlying graphs for a tricategory, two of which we use here. The same is true for **Gray**-categories, but these two types of graphs are not the same as the two types of graphs that underlie tricategories. This leads to a situation in which we are required to use a variety of universal properties in different categories to produce the desired comparison. The fact that tricategories and functors between them do not form a category enters the picture as well. With these facts in mind, we take care to always state in what category a diagram is to be interpreted.

We then prove that every free tricategory is triequivalent to the free **Gray**-category on the same underlying data via the strict functor given by the universal property. Using this, we are in position to prove a new theorem about diagrams of constraint cells commuting. Note that it is not true that every diagram of constraint 3-cells in a tricategory commutes; the "counterexample" comes from the fact that tricategories with one 0-cell and one 1-cell should be the same (in some sense, see [11] for a treatment of the difficulties in making this statement rigorous for 2-dimensional structures) as braided monoidal categories. If we take B to be a braided monoidal category with braiding γ, then the equation $\gamma^2 = 1$ is the condition that B be symmetric. There are many braided monoidal categories which are not symmetric, giving examples of tricategories for which not every diagram of constraint 3-cells commutes.

The theorem for bicategories that we are emulating has two components, a universal property and a coherence theorem applied to a particular kind of example. Focusing on the particular kind of example involved (an underlying graph in which all the 3-cells are identities, called 2-locally discrete), we prove that in the free tricategory on a 2-locally discrete graph every diagram of constraint 3-cells commutes. This relies on a new result that in the free **Gray**-category on a 2-locally discrete graph, every diagram of 3-cells commutes. The analogous result for free 2-categories on a locally discrete graph is trivial, but the proof in this case is not. Using these results, we exhibit a diagram of constraint 3-cells that does not always commute. Here it is the units in the tricategory that prevent the application of the coherence theorems; see [34] for more discussion of units in higher categories. It should be noted that most of the diagrams encountered in this work are easily shown to commute by this theorem.

Using this theorem, we are able to construct explicitly a **Gray**-category $\mathrm{Gr}T$ and triequivalences $\mathrm{Gr}T \to T$ and $T \to \mathrm{Gr}T$ from any tricategory T. These constructions mimic those given for bicategories, but are by necessity much more complicated.

Finally, we give a parallel treatment of the coherence theory for functors. First we prove that the tricategory freely generated by an underlying graph and the constraint cells for a functor is triequivalent to an appropriate **Gray**-category. Using this triequivalence, we prove that certain diagrams consisting of constraint cells from both a functor and its target must commute. This provides enough information to construct explicitly a strictification $\mathrm{Gr}F$ for any functor F. This completes the project of replacing tricategories and functors between them with **Gray**-categories and **Gray**-functors up to triequivalence.

It should be noted that many of our results, especially in the earlier chapters, are either similar to or the same as those in [17], although with changed definitions. We will record these differences and similarities as they arise.

There are a number of places in this work where we are required to verify axioms involving very large diagrams built from the tricategory constraints. Some of these calculations are not explicitly included because of space issues, but the relevant equations have been checked rigorously.

Now we provide a brief description of each of the chapters and the three appendices.

Chapter 2 consists of a rapid treatment of the coherence theory for bicategories. We include two proofs of coherence for bicategories, one using the Yoneda embedding and the other using the universal property of the free bicategory construction. This chapter is provided both to remind the reader of necessary bicategorical results and to give an idea of the path we will take through the coherence theory for tricagories.

Chapter 3 provides the algebraic definitions of tricategory and the higher cells between them. Our definitions differ from those in [17] in that we require adjoint equivalences where Gordon, Power, and Street require equivalences. We do not require additional axioms even though our definitions require additional data; we explain how the theory of mates makes the addition of extra axioms unnecessary and how this leads to the definition of the opposite tricategory.

Chapter 4 is devoted to proving some important basic results. First we study the composition of functors between tricategories and show why these fail to form a category. We provide some conditions under which an altered composition gives a category structure to tricategories and strict functors. Then we study some operations on transformations that will be necessary later. These first two sections focus on the structure of the putative tetracategory **Tricat**. The third section is concerned with changing known tricategory structures to obtain new ones. Finally we study the appropriate notion of equivalence between tricategories, that is, triequivalence.

Chapter 5 gives the necessary background on Gray's tensor product. We define this in three ways: by giving a generators-and-relations definition, by giving the universal property, and by identifying the right adjoint. We then collect together the relevant properties to describe the closed symmetric monoidal category **Gray**, whose underlying category is the category of strict 2-categories and strict 2-functors between them.

Chapter 6 contains the first constructions of tricategory structures from scratch as well as an important first step in the proof of the coherence theorem. The concept of cubical tricategory is introduced, and strict, cubical tricategories are shown to be **Gray**-categories. This gives **Gray**-categories an interpretation as a semi-strict version of tricategories. Additionally, we show that the closed monoidal category **Gray** inherits a tricategorical structure in this way. We define a full sub-**Gray**-category $\mathbf{Gray}' \subset \mathbf{Gray}$ and show that this structure is triequivalent to the tricategory structure on **Bicat** which we construct directly.

Chapter 7 studies the construction of the tricategory of functors, transformations, modifications, and perturbations between two fixed tricategories. We show that given tricategories S and T and functors $F, G : S \to T$, there is a bicategory with 0-cells the transformations between F and G, 1-cells the modifications between those, and 2-cells the perturbations between those. When the target tricategory is a **Gray**-category, we give a composition functor and the rest of the required data necessary to give a tricategory structure. We additionally prove that this tricategory structure is actually a **Gray**-category.

Chapter 8 contains the proof that every tricategory is triequivalent to a **Gray**-category. This is done by first replacing the tricategory in question with a triequivalent cubical one and then proving a Yoneda Lemma for cubical tricategories. Thus we see how the coherence theorem for tricategories breaks up easily into two steps, the first of which is a direct consequence of coherence for bicategories and the second of which is analogous to the proof of coherence for bicategories.

Chapter 9 contains the construction of free tricategories; this finally brings to bear the full power of the algebraic nature of our definition of tricategory. There are many different options for the underlying data of a tricategory, and we construct free tricategories for the two choices that will be most important for the proof of coherence. We also construct free **Gray**-categories as well, and prove some important results needed in the next chapter. We note that these free constructions are all left adjoints to the obvious forgetful functors.

Chapter 10 contains two new coherence theorems. First, we prove that the free tricategory on a graph is strictly triequivalent to the free **Gray**-category constructed from the same data in a canonical way. Then we go on to prove that certain free **Gray**-categories have very restricted structure. This in turn leads to an easy proof of another coherence theorem stating that certain diagrams of constraints in any tricategory always commute. This theorem allows us to construct, from any tricategory T, a **Gray**-category $\mathrm{Gr}T$ and triequivalences between these two tricategories.

Chapter 11 provides a coherence theorem for functors. We begin by analyzing the free functor on a map of underlying graphs. This leads to a coherence theorem for functors stating that certain diagrams consisting of both constraint cells of a functor and the constraints of its target tricategory always commute. We use this theorem to produce a **Gray**-functor $\mathrm{Gr}F : \mathrm{Gr}S \to \mathrm{Gr}T$ from any functor $F : S \to T$.

Three appendices are included. The first collects a few results concerning adjoint equivalences and biadjoint biequivalences that will be needed throughout the work. We have also included here a brief review of the theory of mates. The second appendix gives unpacked versions of all the data in the definitions in Chapter 3. The third appendix deals with calculational issues that are present in a few places, most notably Chapters 4, 7, and 8.

The idea of making the definition of tricategory fully algebraic has existed informally for some time but the details have never been worked out rigorously. Even though many of the ideas behind the definitions and proofs here are simple, often the calculations are quite involved; the proof of Theorem 10.2.2 and all of the calculations that reference Appendix C are good examples. But these calculations, and the coherence theory that follows, are necessary if tricategories are to be utilized in genuine applications.

Gordon, Power, and Street proved an important coherence theorem for weak 3-categories. We have altered their definition, not because it is incorrect in some way, but because it is not suited for making the kinds of constructions that we desire for future applications. In doing so, we were led to simple proofs of important coherence results that could not be stated using the original definition.

Chapter 2

Coherence for bicategories

In this chapter, we will give a rapid treatment of the coherence theory for bicategories, including a full proof for the coherence theorem for functors. The goal of this chapter is to prepare the reader for the path we will take through the coherence theory for tricategories, as well as to recall some crucial facts that will be used throughout. The overall strategy here is adapted from the one used in [22] for monoidal categories.

We will give two proofs that every bicategory is biequivalent to a strict 2-category, each having its own flavor. The first proof can be dispensed with quickly. The second proof requires some of the tools developed for the first, but also allows us to prove the coherence theorem for functors.

2.1 Bicategorical conventions

In any bicategory B, we shall use the letters a, l, and r to denote the associativity, left unit, and right unit isomorphisms, respectively. Vertical composition of 2-cells will be written as concatenation, and the symbol $*$ will be used to denote horizontal composition. The terms pseudofunctor, weak functor, and homomorphism of bicategories are all used throughout the literature to refer to the same concept. We will always write functor for this notion; any strict or lax functor will be labeled as such. Given a functor F, we will generically denote its constraints by φ since the source and target of this constraint make it clear what kind of constraint cell it is.

We follow the convention of [17] and not of the other references ([5] and [37] for instance) in what is meant by a lax transformation. For our purposes, a lax transformation $\alpha : F \Rightarrow G$ consists of 1-cells $\alpha_a : Fa \to Ga$ and 2-cells

$$\begin{tikzcd} Fa \arrow[r, "Ff"] \arrow[d, "\alpha_a"'] & Fb \arrow[d, "\alpha_b"] \arrow[dl, Rightarrow, "\alpha_f"] \\ Ga \arrow[r, "Gf"'] & Gb \end{tikzcd}$$

subject to two axioms. A transformation is a lax transformation such that the cells α_f are invertible for every $f : a \to b$. A transformation between strict 2-functors is a 2-natural transformation if the cells α_f are identities for all f.

Since we have changed the orientation of the naturality isomorphism in the definition of transformation, it is necessary to alter the definition of modification by changing its axiom. These changes are not substantive, they merely avoid excessive use of the prefix op-.

A numbered prefix, such as in 2-category or 2-functor, will always refer to the strict notion.

Our naming conventions for the corresponding concepts for tricategories will be the same, as we reserve the terms functor, transformation, etc., to mean the weak version. Any strict or lax version of these concepts will always be called such.

2.2 The Yoneda embedding

This section is devoted to proving a coherence theorem by first developing an appropriate Yoneda lemma for bicategories. We will not provide any proofs in this section, we instead refer the reader to [36] or [41].

Proposition 2.2.1. *Let B, C be bicategories. There is a bicategory $\mathbf{Bicat}(B, C)$ whose 0-cells are the functors $F : B \to C$, whose 1-cells are the transformations $\alpha : F \Rightarrow G$, and whose 2-cells are the modifications $\Gamma : \alpha \Rrightarrow \beta$.*

The proof of this proposition requires identifying the constraint cells and then checking the bicategory axioms. These constraint cells are obtained from the constraint cells in the target, giving the following corollary.

Corollary 2.2.2. *If C is a strict 2-category and B is any bicategory, then the functor bicategory $\mathbf{Bicat}(B, C)$ is a strict 2-category.*

Definition 2.2.3. Let B be a bicategory. Then the bicategory B^{op} has the same cells as B, the 1-cell source and target maps are switched, $r^{\mathrm{op}} = l$, $l^{\mathrm{op}} = r$, and $a^{\mathrm{op}}_{fgh} = a^{-1}_{hgf}$.

Now we are in a position to define the Yoneda map $\mathbf{y} : B \to \mathbf{Bicat}(B^{\mathrm{op}}, \mathbf{Cat})$ and state the Yoneda Lemma for bicategories.

Definition 2.2.4. Let B be a bicategory. Then the Yoneda map

$$\mathbf{y} : B \to \mathbf{Bicat}(B^{\mathrm{op}}, \mathbf{Cat})$$

is defined on the underlying 2-globular set as follows. The functor $\mathbf{y}$ acts by sending an object a to the functor which is defined on 0-cells by $b \mapsto B(b, a)$, on 1-cells by the functor which is $g \mapsto gf$ on objects, and on 2-cells by sending α to the transformation with components $1_g * \alpha$. The functor $\mathbf{y}$ acts on the 1-cell $f : a \to a'$ by sending it to the transformation with component at b given by $g \mapsto fg$, and for $h : b \to c$, the 2-cell $\mathbf{y}f_h$ is a^{-1}_{fgh}. The functor $\mathbf{y}$ acts on 2-cells by sending $\alpha : f \Rightarrow f'$ to the modification with component $\alpha * 1_g$.

Definition 2.2.5. Let P be a property of functors between categories. A functor $F : B \to C$ between bicategories is *locally* P if each functor F_{ab} has property P.

Theorem 2.2.6 (Bicategorical Yoneda Lemma)**.** *The Yoneda functor* $\mathbf{y} : B \to \mathbf{Bicat}(B^{op}, \mathbf{Cat})$ *is locally full and faithful.*

Corollary 2.2.7. *Every bicategory is biequivalent to a strict 2-category.*

Proof. Let I be the sub-2-category of $\mathbf{Bicat}(B^{\mathrm{op}}, \mathbf{Cat})$ consisting of those 0-cells which are in the image of $\mathbf{y}$, those 1-cells which are isomorphic to some $\mathbf{y}f$, and all 2-cells between them. It is immediate that this is a 2-category. Then $\mathbf{y} : B \to I$ is locally full and faithful by Theorem 2.2.6, and it is biessentially surjective and locally essentially surjective by definition. ☐

2.3 Coherence for bicategories

This section is devoted to proving a coherence theorem of the form "every free bicategory is biequivalent to a strict free 2-category via a strict functor." Using this, we obtain a biequivalence $\mathrm{st}B \to B$ for every bicategory B, where $\mathrm{st}B$ is a strict 2-category. Other notions of coherence are mentioned.

2.3.1 Graphs and free constructions

Definition 2.3.1. The category $Gr(\mathbf{Cat})$ of category-enriched graphs (which we will also call $\mathbf{Cat}$-graphs) has objects G consisting of a set G_0 of objects and for every pair of objects a, b, a category $G(a, b)$. A map $f : G \to G'$ of $\mathbf{Cat}$-graphs consists of functions $f_0 : G_0 \to G'_0$ and functors $f_{ab} : G(a, b) \to G'(f_0a, f_0b)$.

The free bicategory on a $\mathbf{Cat}$-graph G, denoted $\mathcal{F}G$, has the following underlying 2-globular set. The set of 0-cells of $\mathcal{F}G$ is G_0. The set of 1-cells is inductively defined to include new 1-cells I_a for each $a \in G_0$, 1-cells $f : a \to b$ for each object $f \in G(a, b)$, and 1-cells $f \circ g$ if f, g are both 1-cells of $\mathcal{F}G$. The source and target functions are defined in the obvious fashion.

The set of 2-cells of $\mathcal{F}G$ is defined in three steps. The first is to define a basic 2-cell. These are built inductively from the arrows in all of the $G(a, b)$ and new isomorphism 2-cells a_{fgh}, l_f, r_f by binary horizontal composition. Secondly, we form composable strings of these basic 2-cells. Finally, we quotient out by the equivalence relation generated by naturality of the 2-cells a_{fgh}, l_f, r_f, the middle-four interchange law, the rule that the composition $\alpha \circ \beta$ in $\mathcal{F}G$ agrees with that of G if α, β are arrows in some $G(a,b)$, and the two bicategory axioms. Note that there is an obvious inclusion $i : G \to \mathcal{F}G$ of category-enriched graphs.

Proposition 2.3.2. *1. The data above satisfy the necessary axioms to constitute a bicategory.*
2. Let B be a bicategory. Then given a map $f : G \to B$ of category-enriched graphs, there is a unique strict functor of bicategories $\tilde{f} : \mathcal{F}G \to B$ such that $\tilde{f}i = f$ in $Gr(\mathbf{Cat})$.

Proof. The first statement is obvious by the definition. The second statement follows by defining $\tilde{f}$ using induction and strictness. □

Now we define the free 2-category on a **Cat**-graph G, denoted $\mathcal{F}_sG$. The set of 0-cells is the set G_0. The set of 1-cells is the set of composable strings of length ≥ 0, where the unique string of length zero will be the identity 1-cell. The set of 2-cells from one string $f_n f_{n-1} \cdots f_1$ to another $g_m \cdots g_1$ is empty if $n \neq m$, and otherwise consists of the strings $\alpha_n * \alpha_{n-1} * \cdots * \alpha_1$ where $\alpha_i : f_i \to g_i$ in some $G(a,b)$.

Composition of 1-cells is by concatenation, and composition of 2-cells is given by
$$(\alpha_n * \cdots * \alpha_1) \circ (\beta_n * \cdots * \beta_1) = (\alpha_n \beta_n) * \cdots * (\alpha_1 \beta_1).$$

It is a simple matter to verify the following proposition, where here j denotes the inclusion of G into $\mathcal{F}_sG$.

Proposition 2.3.3. *1. The data above satisfy the necessary axioms to constitute a 2-category.*
2. Let X be a 2-category. Then given a map $f : G \to X$ of category-enriched graphs, there is a unique 2-functor $\tilde{f} : \mathcal{F}_sG \to X$ such that $\tilde{f}j = f$ in $Gr(\mathbf{Cat})$.

Thus the statement of the coherence theorem for bicategories becomes the following.

Theorem 2.3.4 (Coherence for bicategories)**.** *The functor $\Gamma : \mathcal{F}G \to \mathcal{F}_sG$ induced by $j : G \to \mathcal{F}_sG$ is a strict biequivalence.*

2.3.2 Proof of the coherence theorem

Definition 2.3.5. Let G, G' be category-enriched graphs, and let $S, T : G \to G'$ be maps between them. The category-enriched graph $\mathrm{Eq}(S,T)$ is defined to have objects those $a \in G_0$ such that $S_0 a = T_0 a$. The category $\mathrm{Eq}(S,T)(a,b)$ has objects pairs (h, α) where $h : a \to b$ in G and $\alpha : Sh \to Th$ is an isomorphism in $G'(S_0 a, S_0 b)$. The morphisms $\beta : (h,\alpha) \to (h', \alpha')$ are those $\beta : h \to h'$ in G such that
$$\alpha' \circ S(\beta) = T(\beta) \circ \alpha.$$

Note that there is a map $\pi : \mathrm{Eq}(S, T) \to G$ defined by

$$\begin{array}{rcl} \pi(a) & = & a \\ \pi(h, \alpha) & = & h \\ \pi(\beta) & = & \beta. \end{array}$$

Lemma 2.3.6. *Let B, C be bicategories, and $F, G : B \to C$ be functors between them. Then $Eq(F, G)$ supports a bicategory structure such that π can be extended to a strict functor $Eq(F, G) \to B$. Furthermore, there is a transformation*

$$\sigma : F\pi \Rightarrow G\pi$$

whose components are all identity maps.

Proof. For the first claim, we must define composition, identity 1-cells, constraint 2-cells, and check the bicategory axioms. To fix notation, the constraint cells for F will be φ_{fg} and φ_0, while those for G will be ψ_{fg} and ψ_0. Composition of 1-cells is then defined by the formula

$$(g, \beta) \circ (f, \alpha) = (gf, \psi_{fg} \circ (\beta * \alpha) \circ \varphi_{fg}^{-1}).$$

The identity 1-cell for the object a is $(\mathrm{id}_a, \psi_0 \circ \varphi_0^{-1})$. It is simple to check that the associativity and unit constraints from B are 2-cells in $\mathrm{Eq}(F, G)$ with the appropriate sources and targets; from this the bicategory axioms follow immediately.

It is trivial to check that π can be extended to a strict functor.

Finally, we define the transformation $\sigma : F\pi \Rightarrow G\pi$. The component at a is id_a. The component at (f, α) is

$$r^{-1} \circ \alpha \circ l;$$

this is a natural transformation by the definition of morphisms in $\mathrm{Eq}(F, G)$ and the naturality of both l and r. The transformation axioms follow easily. □

Proposition 2.3.7. *Let $F : \mathcal{F}X \to B$ be a functor from a free bicategory into any bicategory. Then there is a strict functor $G : \mathcal{F}X \to B$ and a transformation $\alpha : F \Rightarrow G$ such that $\alpha_a = id_{Fa}$ for every object a.*

Proof. Since $\mathcal{F}X$ is free, there is a unique strict functor $G : \mathcal{F}X \to B$ such that $Fi = Gi$ as maps $X \to B$. We also have a map $\iota : X \to \mathrm{Eq}(F, G)$ which is the identity on objects, sends f to (f, id_{Ff}), and sends β to β. Note that $\pi\iota = i$ and the transformation $\sigma * 1_\iota$ is the identity. This produces, by the universal property of $\mathcal{F}X$, a unique strict functor $\tilde{\iota} : \mathcal{F}X \to \mathrm{Eq}(F, G)$ such that $\tilde{\iota} i = \iota$. This gives the equality $\pi\tilde{\iota} i = i$, and since $\pi\tilde{\iota}$ is strict, it must be the identity functor on $\mathcal{F}X$. Then the transformation $\sigma * 1_{\tilde{\iota}}$ is a transformation from $F\pi\tilde{\iota} = F$ to $G\pi\tilde{\iota} = G$, and it has as its component at a the 1-cell id_{Ga} by the definition of $\sigma * 1_{\tilde{\iota}}$. □

It should be noted that we have used that functors of bicategories compose in a strictly associative and unital fashion in this proof.

Let $f : X \to B$ be a map of category-enriched graphs into a bicategory B. Then we can extend f to a map of category-enriched graphs $\hat{f} : \mathcal{F}_s X \to B$ which is defined as follows. The object function $\hat{f}_0$ agrees with f_0. The identity 1-cell on a gets mapped to the identity 1-cell on $f_0 a$, and $\hat{f}(h) = f(h)$ where $h : a \to b$ is an object of $X(a,b)$. If $h_n \cdots h_1 : a \to b$ in $\mathcal{F}_s X$, then

$$\hat{f}(h_n \cdots h_1) = (\cdots(fh_n \circ fh_{n-1}) \circ fh_{n-2}) \circ \cdots \circ fh_2) \circ fh_1.$$

Similarly, $\hat{f}(\alpha_n \cdots \alpha_1)$ is the 2-cell

$$(\cdots(f\alpha_n * f\alpha_{n-1}) * \cdots * f\alpha_2) * f\alpha_1.$$

Lemma 2.3.8. *Let G be a category-enriched graph, and let $F : \mathcal{F}G \to X$ be a strict functor into a 2-category X. Then there exists a unique strict functor $F_s : \mathcal{F}_s G \to B$ such that $F = F_s \Gamma$.*

Proof. This is an immediate consequence of the universal properties of $\mathcal{F}$, $\mathcal{F}_s$, and the fact that $\Gamma i = j$. □

Lemma 2.3.9. *Let $F, G : B \to C$ be functors between bicategories, and let $\alpha : F \Rightarrow G$ be a transformation between them. Assume that F and G agree on objects, and that $\alpha_a = id_{Fa}$ for all objects a. Then F is locally faithful (locally full) if and only if G is locally faithful (locally full).*

Proof. We need only show that F locally faithful implies G locally faithful since there is a transformation $\alpha^{-1} : G \Rightarrow F$ that has all its components identity maps defined by taking $(\alpha^{-1})_f = l^{-1} \circ r \circ (\alpha_f)^{-1} \circ r^{-1} \circ l$.

Using the naturality of r and the naturality of the 2-cells α_f, we get

$$G\alpha = r \circ \alpha_{f'} \circ (1 * F\alpha) \circ \alpha_f^{-1} \circ r^{-1},$$

where $\alpha : f \Rightarrow f'$. Thus G is locally faithful since the the composite on the right is a locally faithful function of α. The same proof shows local fullness. □

Proof of 2.3.4. It is clear that Γ is surjective on objects, so we need only show that it is locally an equivalence of categories. We have the map $\hat{i} : \mathcal{F}_s G \to \mathcal{F}G$, and it is simple to check that the composite map of category-enriched graphs

$$\mathcal{F}_s G \xrightarrow{\hat{i}} \mathcal{F}G \xrightarrow{\Gamma} \mathcal{F}_s G$$

is the identity, so Γ is locally essentially surjective. From this it also follows that Γ is locally full.

To show that Γ is locally faithful, first note that there is a a locally faithful functor $T : \mathcal{F}G \to X$ into a strict 2-category X by the Yoneda Lemma. There is a strict functor $S : \mathcal{F}G \to X$ and a transformation $\alpha : S \Rightarrow T$ with $\alpha_a = \mathrm{id}_a$ by Proposition 2.3.7. By the universal property of the map Γ, there is a unique strict functor $R : \mathcal{F}_s G \to B$ such that $R\Gamma = S$. Now S is locally faithful since T is, hence Γ must be locally faithful as well. □

2.3.3 Using coherence: strictification

Let B be a bicategory. We use the coherence theorem to construct a strictification $\mathrm{st}B$ of B, along with a biequivalence $e : \mathrm{st}B \to B$.

The 2-category $\mathrm{st}B$ will have the same objects as B. A 1-cell from a to b will be a string of composable 1-cells of B, where there is a unique empty string which will be the identity 1-cell. Before defining 2-cells, we define e on 0- and 1-cells. On 0-cells, e is the identity. On 1-cells, we define

$$e(f_n f_{n-1} \cdots f_1) = (\cdots (f_n f_{n-1}) f_{n-2}) \cdots f_2) f_1;$$

for the empty string $\varnothing : a \to a$, we set $e(\varnothing) = I_a$. The set of 2-cells between the strings $f_n f_{n-1} \cdots f_1$ and $g_m g_{m-1} \cdots g_1$ is defined to be the set of 2-cells between $e(f_n f_{n-1} \cdots f_1)$ and $e(g_m g_{m-1} \cdots g_1)$ in B. It is now obvious how e acts on 2-cells.

The 2-category structure of $\mathrm{st}B$ is defined as follows. Composition of 1-cells is given by concatenation of strings, with the empty string as the identity. It is immediate that this is strictly associative and unital. Vertical composition of 2-cells is as in B, and this is strictly associative and unital since vertical composition of 2-cells in a bicategory is always strict in this way.

Let A be the sub-category-enriched graph of B with all the same objects but with $A(a,b)$ the discrete category with $\mathrm{ob}A(a,b) = \mathrm{ob}B(a,b)$. By coherence, the strict functor $\Gamma : \mathcal{F}A \to \mathcal{F}_s A$ is a biequivalence, and it is easy to see that the 2-category $\mathcal{F}_s A$ is locally discrete. Thus, in $\mathcal{F}A$, the set of 2-cells between any two 1-cells is either empty or a singleton, depending on whether these 1-cells are mapped to the same 1-cell by Γ. (Note that this is one way to prove the "all diagrams of constraint cells commute" form of coherence for bicategories.) In particular, we have a unique coherence isomorphism

$$e(f_n \cdots f_1) e(g_m \cdots g_1) \cong e(f_n \cdots f_1 g_m \cdots g_1).$$

Thus we can now define the horizontal composition $\alpha * \beta$ in $\mathrm{st}B$ as the composite

$$\begin{array}{rcl} e(f_n \cdots f_1 g_m \cdots g_1) & \cong & e(f_n \cdots f_1) e(g_m \cdots g_1) \\ & \xrightarrow{\alpha * \beta} & e(f'_n \cdots f'_1) e(g'_m \cdots g'_1) \\ & \cong & e(f'_n \cdots f'_1 g'_m \cdots g'_1) \end{array}$$

in B, where the unlabeled isomorphisms are induced by the strict map $\mathcal{F}A \to B$. The uniqueness of these isomorphisms ensures that this definition satisfies the middle-four interchange laws as well as being strictly associative and unital.

By definition, e is functorial on vertical composition of 2-cells. The constraint cells for e are induced by the strict map $\mathcal{F}A \to B$ in a similar fashion as above. The uniqueness of these cells immediately forces the functor axioms to hold. Finally, it is trivial to see that e is a biequivalence as it is surjective on objects, locally surjective on 1-cells, and a 2-local isomorphism on 2-cells by definition. Thus we have completed the task of producing, for each bicategory B, a strict 2-category $\mathrm{st}B$ and a biequivalence $e : \mathrm{st}B \to B$.

It will be useful later to note that there exists a biequivalence $f : B \to \mathrm{st}B$ defined as follows. The map f is the identity on objects, includes each 1-cell as the string of length 1, and then is the identity on 2-cells as well. This is functorial on 2-cells, and we can take both constraint cells to be represented by identity 2-cells in B (although they are not identities in $\mathrm{st}B$). The functor axioms are then easy to check. The only thing to check to show that f is a biequivalence is that it is locally essentially surjective, but this is easy as every 1-cell $f_n \cdots f_1$ is clearly isomorphic to a 1-cell of length 1, namely $e(f_n \cdots f_1)$; the empty string is isomorphic to the identity map viewed as a 1-cell of $\mathrm{st}B$, so f is locally essentially surjective. It should be noted that $ef = 1_B$, and fe is biequivalent to $1_{\mathrm{st}B}$ in $\mathbf{Bicat}(\mathrm{st}B, \mathrm{st}B)$ by a transformation whose components on objects can all be taken to be identities and whose components on 1-cells all come from coherence.

Remark 2.3.10. The previous paragraph contains all of the information needed to conclude that every bicategory is *equivalent* to a strict 2-category inside of the 2-category **NHom** studied by Lack and Paoli in [27].

2.4 Coherence for functors

In this section, we prove a coherence result for functors of bicategories. This theorem is analogous to Theorem 2.3.4 in that it states that "free functors are biequivalent to free strict functors." The statement is slightly more delicate, but it produces similar results to those in Section 2.3.3.

2.4.1 Free functors

Let $\varphi : G \to G'$ be a map in $Gr(\mathbf{Cat})$. Our goal is to produce the free functor generated by φ; the source of this functor will be the free bicategory generated by G, but the target is a more complicated object. The idea is that the target will be the free bicategory generated by G' and new 2-cells that will play the role of constraint cells.

We define the bicategory $\mathcal{F}(G', \varphi)$ as follows. The 0-cells of $\mathcal{F}(G', \varphi)$ are the same as the objects of G'. The 1-cells are generated (using binary composites) by new 1-cells $I_a : a \to a$, the 1-cells of G', and new 1-cells $\varphi(r)$ for every 1-cell r in $\mathcal{F}G$. These are subject to the requirement that $\varphi(r) = s$ in $\mathcal{F}(G', \varphi)$ if r is an object $G(a, b)$ and $\varphi(r) = s$ in G', and we extend this over composition.

The 2-cells are defined in a sequence of steps analogous to how we defined the 2-cells of $\mathcal{F}G$. The first step is to form basic 2-cells from the 2-cells of G', 2-cells $\varphi(\alpha)$ with α a 2-cell of $\mathcal{F}G$ (subject to the same kind of condition that we imposed on the 1-cells $\varphi(r)$), and isomorphism constraint cells $a_{fgh}, l_f, r_f, \varphi_a, \varphi_{fg}$ by binary horizontal composition. Then we form strings of vertically composable basic cells, and finally we quotient out by the equivalence relation formed by the necessary naturality conditions along with the axioms for a bicategory and those required of the 2-cells φ_a, φ_{fg} to force φ to extend to a weak functor

$\mathcal{F}G \to \mathcal{F}(G', \varphi)$. The universal property of $\mathcal{F}(G', \varphi)$ is expressed by the following proposition.

Proposition 2.4.1. *Let $\varphi : G \to G'$ be a map of category-enriched graphs. Then there is a commutative square*

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "i"'] & G' \arrow[d, "k"] \\ \mathcal{F}G \arrow[r, "\tilde{\varphi}"'] & \mathcal{F}(G', \varphi) \end{tikzcd}$$

in $Gr(\mathbf{Cat})$ such that for all commutative squares

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "R"'] & G' \arrow[d, "S"] \\ X \arrow[r, "F"'] & Y \end{tikzcd}$$

in $Gr(\mathbf{Cat})$ with $F : X \to Y$ a functor between bicategories, there exists a unique commutative square of functors

$$\begin{tikzcd} \mathcal{F}G \arrow[r, "\tilde{\varphi}"] \arrow[d, "U"'] & \mathcal{F}(G', \varphi) \arrow[d, "V"] \\ X \arrow[r, "F"'] & Y \end{tikzcd}$$

such that

1. *the functors U, V are strict and*
2. *$Ui = R$ and $Vk = S$.*

Proof. There is an obvious inclusion $k : G' \to \mathcal{F}(G', \varphi)$ and the definition of $\mathcal{F}(G', \varphi)$ forces the first square to commute. Now assume we have a commutative square of the form $S\varphi = FR$. The functor U is already determined by the universal property of $\mathcal{F}G$. We define V as follows. On 0-cells, V agrees with S. The action of V on 1-cells is determined inductively by strictness and the relations $Ui = R, Vk = S$; the same holds for 2-cells, with the additional requirement that the constraint cells in $\mathcal{F}(G', \varphi)$ required for $\tilde{\varphi}$ to be a functor are mapped to the constraint cells in Y for the composite functor TU. This demonstrates uniqueness and forces the required diagrams to commute. $\square$

Given any $\varphi : G \to G'$ as above, we can consider the following square.

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "j"'] & G' \arrow[d, "j"] \\ \mathcal{F}_s G \arrow[r, "\mathcal{F}_s \varphi"'] & \mathcal{F}_s G' \end{tikzcd}$$

By our universal property, we thus have the following commutative square.

$$\begin{tikzcd} \mathcal{F}G \arrow[r, "\tilde{\varphi}"] \arrow[d, "\Gamma"'] & \mathcal{F}(G', \varphi) \arrow[d, "\Delta"] \\ \mathcal{F}_s G \arrow[r, "\mathcal{F}_s \varphi"'] & \mathcal{F}_s G' \end{tikzcd}$$

The coherence theorem for functors now takes the following form.

Theorem 2.4.2 (Coherence for functors)**.** *The functor* $\Delta : \mathcal{F}(G', \varphi) \to \mathcal{F}_s G'$ *is a strict biequivalence.*

2.4.2 Proof of the coherence theorem

Lemma 2.4.3. *Assume the following squares of functors commute, where* R, S_i *are strict, for* $i = 1, 2$.

$$\begin{tikzcd} \mathcal{F}G \arrow[r, "\varphi"] \arrow[d, "R"'] & \mathcal{F}(G', \varphi) \arrow[d, "S_i"] \\ X \arrow[r, "F_i"'] & Y \end{tikzcd}$$

Assume that the S_i *have the same object-map, and that the* F_i *have the same object-map. Then for every transformation* $\alpha : F_1 \Rightarrow F_2$ *with* $\alpha_a = id_{F_1 a}$ *for all* a*, there is a unique transformation* $\beta : S_1 \Rightarrow S_2$ *with* $\beta_b = id_{S_1 b}$ *for all* b *and*

$$\alpha * 1_R = \beta * 1_\varphi.$$

Proof. First, we must construct a new bicategory Y^I. It has as 0-cells the identity 1-cells of Y. A 1-cell $\mathrm{id}_a \to \mathrm{id}_b$ is a triple (h_1, h_2, γ) which consists of a pair of 1-cells h_1, h_2 and a 2-cell isomorphism $\gamma : \mathrm{id}_b h_1 \Rightarrow h_2 \mathrm{id}_a$. A 2-cell $(h_1, h_2, \gamma) \Rightarrow (k_1, k_2, \delta)$ consists of a pair of 2-cells $\sigma_i : h_i \Rightarrow k_i$ such that

$$(1_g * \sigma_1) \circ \gamma = \delta \circ (\sigma_2 * 1_f).$$

The identity 1-cell on id_a is the triple $(\mathrm{id}_a, \mathrm{id}_a, 1)$. Composition of 1-cells is given by the formula

$$(h_1, h_2, \gamma) \circ (h_1', h_2', \gamma') = \big(h_1 h_1', h_2 h_2', a \circ (\gamma_2 * 1_{h_1'}) \circ a^{-1} \circ (1_{h_2} * \gamma_1) \circ a\big).$$

The associativity and unit constraints are given by those in Y, and the necessary diagrams are easily checked. Vertical composition of 2-cells is given by vertical composition of 2-cells in Y, as is horizontal composition. It is then easy to verify that 2-cells compose in a strictly associative and unital fashion, and that they satisfy the middle-four interchange law.

Now α induces a functor $F : X \to Y^I$ by the formulas $F(x) = \alpha_x$, $F(f) = (F_1 f, F_2 f, \alpha_f)$, and $F(\sigma) = (F_1 \sigma, F_2 \sigma)$. The constraint cells for F are given by the constraint cells of F_1 and F_2. We must now check that these constraint cells satisfy the necessary equation to be valid 2-cells, but this follows immediately from the transformation axioms. By our universal property of $\mathcal{F}(G', \varphi)$, we obtain the commutative square pictured below.

$$\begin{tikzcd} \mathcal{F}G \arrow[r, "\varphi"] \arrow[d, "R"'] & \mathcal{F}(G', \varphi) \arrow[d, "S"] \\ X \arrow[r, "F"'] & Y^I \end{tikzcd}$$

There are strict functors $\pi_i : Y^I \to Y$ given by $\pi_i(\mathrm{id}_a) = a$, $\pi_i(h_1, h_2, \gamma) = h_i$, $\pi_i(\sigma_1, \sigma_2) = \sigma_i$. It is immediate that $\pi_i F = F_i$, so by the universal property of $\mathcal{F}(G', \varphi)$, we get that $\pi_i S = S_i$ as well. Thus we define β by $\beta_x = \mathrm{id}_{S_1 x}$ and $\beta_f = Sf$. It is now easy to check that this defines a transformation with the desired properties. □

Proof of 2.4.2. We have the inclusion $j : G' \to \mathcal{F}_s G'$ and thus an induced map of category-enriched graphs $\hat{j} : \mathcal{F}_s G' \to \mathcal{F}(G', \varphi)$. It is easy to check that the composite

$$\mathcal{F}_s G' \xrightarrow{\hat{j}} \mathcal{F}(G', \varphi) \xrightarrow{\Delta} \mathcal{F}_s G'$$

is the identity in $Gr(\mathbf{Cat})$, so Δ is locally full and locally essentially surjective. We know that Δ is surjective on objects, so we need only show that it is locally faithful.

By Proposition 2.3.7, there is a strict functor $S : \mathcal{F}G \to \mathcal{F}(G', \varphi)$ and a transformation $\alpha : S \Rightarrow \tilde{\varphi}$ that has components $\alpha_a = \mathrm{id}_{\varphi a}$. Thus the universal property of $\mathcal{F}(G', \varphi)$ gives the following commutative square.

$$\begin{tikzcd} \mathcal{F}G \arrow[r, "\tilde{\varphi}"] \arrow[d, "1"'] & \mathcal{F}(G', \varphi) \arrow[d, "E"] \\ \mathcal{F}G \arrow[r, "S"'] & \mathcal{F}(G', \varphi) \end{tikzcd}$$

We also have the identity square.

$$\begin{tikzcd} \mathcal{F}G \arrow[r, "\tilde{\varphi}"] \arrow[d, "1"'] & \mathcal{F}(G', \varphi) \arrow[d, "1"] \\ \mathcal{F}G \arrow[r, "\tilde{\varphi}"'] & \mathcal{F}(G', \varphi) \end{tikzcd}$$

Using the transformation α, we can apply Lemma 2.4.3; since the identity functor is locally full and faithful, we can use Lemma 2.3.9 to conclude that E is locally full and faithful.

The universal property of $\mathcal{F}(G', \varphi)$ provides the following commutative square.

$$\begin{tikzcd} \mathcal{F}G \arrow[r, "\tilde{\varphi}"] \arrow[d, "1"'] & \mathcal{F}(G', \varphi) \arrow[d, "\Delta_1"] \\ \mathcal{F}G \arrow[r, "\mathcal{F}\varphi"'] & \mathcal{F}G' \end{tikzcd}$$

The universal property also implies that $\Gamma\Delta_1 = \Delta$; since we already know that Γ is locally faithful, we need only show that Δ_1 is locally faithful to complete the proof. There is a unique strict functor $T : \mathcal{F}G' \to \mathcal{F}(G', \varphi)$ which extends the inclusion of G' into $\mathcal{F}(G', \varphi)$. It is a simple calculation to check that $S = T \circ \mathcal{F}\varphi$. Then $T\Delta_1$ is a strict functor $\mathcal{F}(G', \varphi) \to \mathcal{F}(G', \varphi)$ and it is easy to check that it makes the following square commute using the fact that all of the functors are strict.

$$\begin{tikzcd}
\mathcal{F}G \arrow[r, "\tilde{\varphi}"] \arrow[d, "1"'] & \mathcal{F}(G', \varphi) \arrow[d, "T\Delta_1"] \\
\mathcal{F}G \arrow[r, "S"'] & \mathcal{F}(G', \varphi)
\end{tikzcd}$$

Thus $E = T\Delta_1$, and hence Δ_1 is locally faithful since E is. $\square$

2.4.3 Using coherence: strictification

In this section, we use Theorem 2.4.2 to produce for each functor $F : B \to B'$ a strict 2-functor $\mathrm{st}F : \mathrm{st}B \to \mathrm{st}B'$. Thus, up to biequivalence, we can replace functors by strict maps. Since this construction will commute with composition, we can replace diagrams by biequivalent diagrams of strict 2-categories and strict 2-functors between them.

Let $F : X \to Y$ be a functor between bicategories. We define the strict functor $\mathrm{st}F : \mathrm{st}X \to \mathrm{st}Y$ as follows. On 0-cells, $\mathrm{st}F$ agrees with F. On 1-cells, we define

$$\mathrm{st}F(f_n \cdots f_1) = Ff_n \cdots Ff_1,$$

and $\mathrm{st}F(\mathrm{id}_a) = \mathrm{id}_{Fa}$. We will define the action of $\mathrm{st}F$ on 2-cells using the same technique as in Section 2.3.3. Let $\alpha : e(f_n \cdots f_1) \Rightarrow e(g_m \cdots g_1)$ be a 2-cell in $\mathrm{st}X$. Then we define $\mathrm{st}F(\alpha)$ to be the 2-cell

$$e(Ff_n \cdots Ff_1) \cong F\Big(e(f_n \cdots f_1)\Big) \xrightarrow{F\alpha} F\Big(e(g_m \cdots g_1)\Big) \cong e(Fg_m \cdots Fg_1),$$

where the unlabeled isomorphisms are the unique isomorphism 2-cells provided by our coherence theorem by considering the sub-**Cat**-graph of Y with no non-identity 2-cells.

The same proof as in 2.3.3 shows that this is a strict functor; the same techniques also prove that $\mathrm{st}(F \circ G) = \mathrm{st}F \circ \mathrm{st}G$. The commutativity of the square

$$\begin{tikzcd}
X \arrow[r, "F"] \arrow[d, "f"'] & Y \arrow[d, "f"] \\
\mathrm{st}X \arrow[r, "\mathrm{st}F"'] & \mathrm{st}Y
\end{tikzcd}$$

is immediate from the definitions. It is not the case that $Fe = e \circ \mathrm{st}F$, but there is a transformation ω between these with $\omega_a = \mathrm{id}_{Fa}$ for all objects a and ω_f given by the unique coherence 2-cell.

It should also be noted that the functor $\mathrm{st} : \mathbf{Bicat} \to \mathbf{2Cat}$ is a reflection for the inclusion of **2Cat** into **Bicat**.

Chapter 3

The algebraic definition of tricategory

In this chapter, we give the definition of an algebraic tricategory. We shall make note of when this differs from the definition of tricategory given in [17]. Finally, we give the definitions of functor, transformation, modification, and perturbation.

3.1 Basic definition

Notation 3.1.1 (Adjoint equivalences). If B is a bicategory, then we will always write our adjoint equivalences as $(f, f^{\bullet}, \varepsilon, \eta)$. These will be abbreviated as $\boldsymbol{f}$.

Definition 3.1.2 (Algebraic tricategory). A tricategory T consists of the following data subject to the following axioms.
DATA:

- A set ob$T$ of *objects* of T;
- For $(a, b) \in \text{ob}T \times \text{ob}T$, a bicategory $T(a, b)$, called the *hom-bicategory* of T at a and b. The objects of $T(a, b)$ will be referred to as the 1-cells of T with source a and target b, the arrows of $T(a, b)$ will be referred to as 2-cells of T (with their same source and target), and the 2-cells of $T(a, b)$ will be referred to as 3-cells of T (also with their same source and target);
- For objects a, b, c of T, a functor $\otimes : T(b, c) \times T(a, b) \to T(a, c)$ called *composition*;
- For an object a of T, a functor $I_a : 1 \to T(a, a)$, where 1 denotes the unit bicategory;

- For objects a, b, c, d of T, an adjoint equivalence $\boldsymbol{a}$

$$
\begin{tikzcd}
T(c,d) \times T(b,c) \times T(a,b) \arrow[r, "\otimes\times 1"] \arrow[d, "1\times\otimes"'] \arrow[dr, phantom, "\Downarrow \boldsymbol{a}"] & T(b,d) \times T(a,b) \arrow[d, "\otimes"] \\
T(c,d) \times T(a,c) \arrow[r, "\otimes"'] & T(a,d)
\end{tikzcd}
$$

in $\mathbf{Bicat}(T(c,d) \times T(b,c) \times T(a,b), T(a,d))$;

- For objects a, b of T, adjoint equivalences $\boldsymbol{l}$ and $\boldsymbol{r}$

$$
\begin{tikzcd}
& T(b,b) \times T(a,b) \arrow[dr, "\otimes"] & & & T(a,b) \times T(a,a) \arrow[dr, "\otimes"] & \\
T(a,b) \arrow[ur, "I_b\times 1"] \arrow[rr, "1"'] \arrow[rr, phantom, bend left=20, "\Downarrow \boldsymbol{l}"] & & T(a,b) & T(a,b) \arrow[ur, "1\times I_a"] \arrow[rr, "1"'] \arrow[rr, phantom, bend left=20, "\Downarrow \boldsymbol{r}"] & & T(a,b)
\end{tikzcd}
$$

in $\mathbf{Bicat}(T(a,b), T(a,b))$;

- For objects a, b, c, d, e of T, an isomorphism 2-cell π (i.e., an invertible modification)

$$
\begin{tikzcd}
& T^4 \arrow[rr, "\otimes\times1\times1"] \arrow[dl, "1\times1\times\otimes"'] \arrow[dr, "1\times\otimes\times1"] & & T^3 \arrow[dr, "\otimes\times1"] \arrow[dl, phantom, "\Downarrow a\times 1"] & & & T^4 \arrow[rr, "\otimes\times1\times1"] \arrow[dl, "1\times1\times\otimes"'] \arrow[dr, phantom, "="] & & T^3 \arrow[dl, "1\times\otimes"'] \arrow[dr, "\otimes\times1"] \arrow[d, phantom, "\overset{a}{\Leftarrow}"] & \\
T^3 \arrow[dr, "1\times\otimes"'] \arrow[r, phantom, "\overset{1\times a}{\Leftarrow}"] & {} & T^3 \arrow[rr, "\otimes\times1"] \arrow[dl, "1\times\otimes"] \arrow[dr, phantom, "\Downarrow a"] & & T^2 \arrow[dl, "\otimes"] \arrow[r, Rrightarrow, "\pi"] & T^3 \arrow[rr, "\otimes\times1"] \arrow[dr, "1\times\otimes"'] \arrow[rr, phantom, bend right=40, "\Downarrow a"] & & T^2 \arrow[dr, "\otimes"] & {} & T^2 \arrow[dl, "\otimes"] \\
& T^2 \arrow[rr, "\otimes"'] & & T & & & T^2 \arrow[rr, "\otimes"'] & & T &
\end{tikzcd}
$$

in the bicategory $\mathbf{Bicat}(T^4(a,b,c,d,e), T(a,e))$, where $T^4 = T^4(a,b,c,d,e)$ is an abbreviation for $T(d,e) \times T(c,d) \times T(b,c) \times T(a,b)$, for example;

- For objects a, b, c of T, invertible modifications

$$
\begin{tikzcd}
T^2 \arrow[dr, "1\times I\times1"'] \arrow[drr, bend left, "1"] \arrow[ddr, bend right, "1"'] & \arrow[dr, phantom, "\Downarrow r^{\bullet}\times1"] & & T^2 \arrow[ddr, bend left, "1"] \arrow[dd, bend right, "1"'] & \\
\arrow[dr, phantom, "\Leftarrow 1\times l"] & T^3 \arrow[r, "\otimes\times1"'] \arrow[d, "1\times\otimes"] \arrow[dr, phantom, "\Downarrow a"] & T^2 \arrow[d, "\otimes"] \arrow[r, phantom, "\overset{\mu}{\Rrightarrow}"] & \arrow[r, phantom, "\overset{1}{\Swarrow}"] & T^2 \arrow[d, "\otimes"] \\
& T^2 \arrow[r, "\otimes"'] & T & T^2 \arrow[r, "\otimes"'] & T
\end{tikzcd}
$$

$$
\begin{tikzcd}
& T^3 \arrow[dr, "\otimes\times 1"] \arrow[d, phantom, "\Downarrow l\times 1"] & \\
T^2 \arrow[ur, "I\times 1\times 1"] \arrow[rr, "1"] \arrow[d, "\otimes"'] & {} \arrow[d, phantom, "="] & T^2 \arrow[d, "\otimes"] \\
T \arrow[rr, "1"'] & {} & T
\end{tikzcd}
\overset{\lambda}{\Rrightarrow}
\begin{tikzcd}
& T^3 \arrow[dr, "\otimes\times 1"] \arrow[d, "1\times\otimes"] & \\
T^2 \arrow[ur, "I\times 1\times 1"] \arrow[d, "\otimes"'] \arrow[r, phantom, "="] & T^2 \arrow[dr, "\otimes"] \arrow[d, phantom, "\Downarrow l"] \arrow[r, phantom, "\Downarrow a"] & T^2 \arrow[d, "\otimes"] \\
T \arrow[ur, "I\times 1"] \arrow[rr, "1"'] & {} & T
\end{tikzcd}
$$

$$
\begin{tikzcd}
T \arrow[rr, "1"] & {} \arrow[d, phantom, "="] & T \\
T^2 \arrow[u, "\otimes"] \arrow[rr, "1"] \arrow[dr, "1\times 1\times I"'] & {} \arrow[d, phantom, "\Downarrow 1\times r^{\bullet}"] & T^2 \arrow[u, "\otimes"'] \\
& T^3 \arrow[ur, "1\times\otimes"'] &
\end{tikzcd}
\overset{\rho}{\Rrightarrow}
\begin{tikzcd}
T \arrow[rr, "1"] \arrow[dr, "1\times I"'] & {} \arrow[d, phantom, "\Downarrow r^{\bullet}"] & T \\
T^2 \arrow[u, "\otimes"] \arrow[dr, "1\times 1\times I"'] \arrow[r, phantom, "="] & T^2 \arrow[ur, "\otimes"] \arrow[r, phantom, "\Rightarrow a"] & T^2 \arrow[u, "\otimes"'] \\
& T^3 \arrow[u, "\otimes\times 1"] \arrow[ur, "1\times\otimes"'] &
\end{tikzcd}
$$

AXIOMS:

- The following equation of 2-cells holds in the bicategory $T(a_1, a_5)$, where we have used parentheses instead of $\otimes$ for compactness and the unmarked isomorphisms are naturality isomorphisms for a.

$$
\begin{tikzcd}
& & (k(j(hg)))f \arrow[dr, "a"] & & \\
& (k((jh)g))f \arrow[ur, "(1a)1"] \arrow[dr, "a"] \arrow[rr, phantom, "\cong"] \arrow[d, phantom, "\Downarrow\pi"] & & k((j(hg))f) \arrow[dr, "1a"] \arrow[d, phantom, "\Downarrow 1\pi"] & \\
((k(jh))g)f \arrow[ur, "a1"] \arrow[dr, "a"] & {} & k(((jh)g)f) \arrow[ur, "1(a1)"] \arrow[dr, "1a"] & {} & k(j((hg)f)) \arrow[dd, "1(1a)"] \\
& (k(jh))(gf) \arrow[rr, "a"] & & k((jh)(gf)) \arrow[dr, "1a"] \arrow[d, phantom, "\Downarrow\pi"] & \\
(((kj)h)g)f \arrow[uu, "(a1)1"] \arrow[dr, "a"] \arrow[ur, phantom, "\cong"] & & & {} & k(j(h(gf))) \\
& ((kj)h)(gf) \arrow[uu, "a(11)"] \arrow[rr, "a"'] & & (kj)(h(gf)) \arrow[ur, "a"'] &
\end{tikzcd}
$$

$$\Big\Vert$$

$$
\begin{tikzcd}
& & (k(j(hg)))f \arrow[dr, "a"] & & \\
& (k((jh)g))f \arrow[ur, "(1a)1"] \arrow[d, phantom, "\Downarrow\pi 1"] & & k((j(hg))f) \arrow[dr, "1a"] \arrow[d, phantom, "\Downarrow\pi"] & \\
((k(jh))g)f \arrow[ur, "a1"] & {} & ((kj)(hg))f \arrow[uu, "a1"] \arrow[dr, "a"] & {} & k(j((hg)f)) \arrow[dd, "1(1a)"] \\
& & & (kj)((hg)f) \arrow[ur, "a"] \arrow[dd, "(11)a"] \arrow[dr, phantom, "\cong"] & \\
(((kj)h)g)f \arrow[uu, "(a1)1"] \arrow[uurr, "a1"] \arrow[dr, "a"] & {} & & & k(j(h(gf))) \\
& ((kj)h)(gf) \arrow[rr, "a"'] \arrow[u, phantom, "\Downarrow\pi"] & & (kj)(h(gf)) \arrow[ur, "a"'] &
\end{tikzcd}
$$

- The following equation of 2-cells holds in the bicategory $T(a_1, a_4)$, where the unmarked isomorphisms are either naturality isomorphisms for a or unique coherence isomorphisms from the hom-bicategory.

$$
\begin{tikzcd}
 & & (h(Ig))f \arrow[dr, "(1l)1"] \arrow[dd, "a"] & \\
 & ((hI)g)f \arrow[ur, "a1"] \arrow[dd, "a"'] \arrow[dr, phantom, "\Downarrow\pi"] & & (hg)f \arrow[dddd, "a"] \\
 & & h((Ig)f) \arrow[dd, "1a"'] \arrow[dddr, "1(l1)"] \arrow[r, phantom, "\cong"] & \\
(hg)f \arrow[uur, "(r^{\bullet}1)1"] \arrow[ddr, "a"'] \arrow[r, phantom, "\cong"] & (hI)(gf) \arrow[dr, "a"] & \arrow[dr, phantom, "\Downarrow 1\lambda"] & \\
 & & h(I(gf)) \arrow[dr, "1l"'] & \\
 & h(gf) \arrow[uu, "r^{\bullet}(11)"'] \arrow[rr, bend right, "1"'] \arrow[r, phantom, "\Downarrow\mu"] & & h(gf)
\end{tikzcd}
$$

$$=$$

$$
\begin{tikzcd}
 & & (h(Ig))f \arrow[dr, "(1l)1"] & \\
 & ((hI)g)f \arrow[ur, "a1"] \arrow[r, phantom, "\Downarrow\mu 1"] & & (hg)f \arrow[dd, "a"] \\
(hg)f \arrow[ur, "(r^{\bullet}1)1"] \arrow[urrr, "11"'] \arrow[dr, "a"'] & & \cong & \\
 & h(gf) \arrow[rr, "1"'] & & h(gf)
\end{tikzcd}
$$

- The following equation of 2-cells holds in the bicategory $T(a_1, a_4)$.

$$
\begin{tikzcd}
 & h((gI)f) \arrow[dr, "1a"] & & \\
h(gf) \arrow[ur, "1(r^{\bullet}1)"] \arrow[dr, phantom, "\cong"] & & h(g(If)) \arrow[ddr, "1(1l)"] & \\
 & (h(gI))f \arrow[uu, "a"'] \arrow[r, phantom, "\Downarrow\pi"] & & \\
 & \arrow[d, phantom, "\Downarrow\rho 1"] & (hg)(If) \arrow[uu, "a"] \arrow[dd, "(11)l"'] \arrow[r, phantom, "\cong"] & h(gf) \\
 & ((hg)I)f \arrow[uu, "a1"'] \arrow[ur, "a"] & & \\
(hg)f \arrow[uuuu, "a"] \arrow[uuur, "(1r^{\bullet})1"] \arrow[ur, "r^{\bullet}1"'] \arrow[rr, bend right, "1"'] & \Downarrow\mu & (hg)f \arrow[uur, "a"'] &
\end{tikzcd}
$$

$$=$$

$$
\begin{tikzcd}
 & h((gI)f) \arrow[dr, "1a"] & & \\
h(gf) \arrow[ur, "1(r^{\bullet}1)"] \arrow[drrr, "11"] & \Downarrow 1\mu & h(g(If)) \arrow[dr, "1(1l)"] & \\
 & \cong & & h(gf) \\
(hg)f \arrow[uu, "a"] \arrow[rr, "1"'] & & (hg)f \arrow[ur, "a"'] &
\end{tikzcd}
$$

Definition 3.1.3. A tricategory T is *strict* if each of the adjoint equivalences $\boldsymbol{a}, \boldsymbol{l}, \boldsymbol{r}$ is the identity adjoint equivalence and the modifications π, μ, λ, ρ are given by unique coherence isomorphisms.

Remark 3.1.4 Adjoint equivalences. The major difference between the definition given in [17] and the one given here is that all of the equivalences in the [17] definition have been replaced with adjoint equivalences. For example, the associator a in [17] is an equivalence $\otimes \circ \otimes \times 1 \to \otimes \circ 1 \times \otimes$ in the appropriate bicategory; we have replaced this with an adjoint equivalence which includes a distinguished pseudoinverse $a^{\bullet}$ as well as unit and counit isomorphisms that satisfy the triangle identities.

Remark 3.1.5. The definition of r has been changed from that of [17]; our $r^{\bullet}$ here is the r of [17]. This has been arranged so that the unit isomorphisms always have an identity cell in the source and never in the target.

Remark 3.1.6 Suppression of constraints. Note that the diagrams above never have associations given for their sources and targets as they are merely shorthand. By the coherence theorem for bicategories, a pasting diagram of 2-cells in a bicategory has a unique value once a choice of association has been made for the source and target. Unless there is an obvious choice of association, we will always assume that 1-cells in a bicategory have been associated by applying the function e used in the construction of the strictification $\mathrm{st}B$ in the previous chapter.

Additionally, the diagrams do not all type-check in the following sense. Written down in equational form, the axioms would take the form of an equation of 2-cells in some bicategory. This equation would not be well-formed, though, as the sources and targets would not always match up to allow adjacent terms in this equation to be composed. These sources and targets can be made to match up by appropriately inserting constraint 2-cells which arise as either the constraint cells in a bicategory or as the constraint cells of a functor. By the coherence theorem for functors, such a pasting diagram has a unique value regardless of how these constraint cells are inserted. It is in this sense that we interpret the axioms given above.

Remark 3.1.7. It should be noted that λ and ρ seem to have a different status than μ. In particular, the reader will note that the cells are not categorified versions of bicategory axioms, but instead categorified versions of useful *results* about constraint cells in a bicategory. See [22] for a proof of the one-object versions of these bicategorical results and to see how they assist in the proof of coherence for monoidal categories. Thus λ and ρ provide an interesting example of how new data arises in the categorification process.

It should be noted, however, that these cells are determined by the rest of the data for a tricategory and the requirement that the second and third axioms hold. This can be seen for λ by using the second axiom, setting $h = I$, and using unit constraints. These axioms are not redundant, though, and do provide new information, as generating λ and ρ in this fashion does not guarantee that the second and third axioms hold, but only that they hold in the special cases used in this strategy for defining λ and ρ.

Remark 3.1.8. Most of the data for a tricategory can be seen as a direct categorification of the axioms in the definition of bicategory. The datum π is plainly seen to be a categorified version of the Mac Lane pentagon, which is written

$$(1 * a) \circ a \circ (a * 1) = a \circ a$$

in equational form. The final data consists of three parts, two of which have already been discussed. The modification μ is a direct categorification of the single unit axiom for bicategories:

$$r * 1 = 1 * l \circ a.$$

The axioms are less transparent. The first tricategory axiom is called the nonabelian 4-cocycle condition. The picture should be familiar to topologists as K_5 and to category theorists as $\mathcal{O}_5$. These two objects - the fifth associahedron of Stasheff and the fifth oriental of Street - are related, though how has not been rigorously determined. See [43] and [39] for more discussion of these objects.

The other two axioms were introduced by [17], and are normalized versions of the cocycle condition.

3.2 Adjoint equivalences and tricategory axioms

It should be noted that we have only included axioms for the left adjoints of the adjoint equivalences that are the basic data for a tricategory, except in the case of $\boldsymbol{r}$ where we have only used the right adjoint. Thus the major difference between the definition given here and the one in [17] is the addition of specified pseudoinverses and the necessary units and counits, but we require these to satisfy no additional axioms. This is not necessary by the theory of mates in a bicategory; see Appendix B for a quick review of the basic theory and a list of the results necessary for our purposes.

Mates allow us to define the opposite tricategory of T, T^{op}, and by this we see that the relevant axioms for the right adjoints (or left adjoint in the case of $\boldsymbol{r}$) are already satisfied.

Definition 3.2.1. Let T be a tricategory. Then the *opposite tricategory*, denoted T^{op}, is given by the following data. The tricategory T^{op} has the same object set as T, and

$$T^{\mathrm{op}}(a, b) = T(b, a).$$

The composition functor $\otimes^{\mathrm{op}}$ is given by $\otimes \circ \tau$, where τ is the twist isomorphism. We take the same unit homomorphism. The adjoint equivalences $\mathbf{a}^{\mathrm{op}}, \mathbf{l}^{\mathrm{op}}, \mathbf{r}^{\mathrm{op}}$ are the opposite adjoint equivalences of $\mathbf{a}, \mathbf{r}, \mathbf{l}$, in which case we switch the left and right adjoints and take the new unit to be the inverse of the old counit and the new counit to be the inverse of the old unit. We take the isomorphisms $\pi^{\mathrm{op}}, \mu^{\mathrm{op}}$ to be $(\pi^{-1})^{\dagger}, (\mu^{-1})^{\dagger}$, similarly for $\lambda^{\mathrm{op}}, \rho^{\mathrm{op}}$.

As a corollary to the results in the appendix, we have the following.

Corollary 3.2.2. *The data for T^{op} given above satisfy the axioms necessary to be a tricategory.*

The general style of definition will then be as follows. All of the data involving 2-cells in a tricategory (i.e., 1-cells in some hom-bicategory) will be given, when appropriate, as adjoint equivalences. The 3-cells isomorphisms between composites of these will be given in terms of the left adjoints whenever possible. Any required 3-cells isomorphisms between the dual data can then be obtained by taking the relevant mates. The axioms for these 3-cells will be treated similarly. It should be noted that, since we are dealing with adjoint *equivalences*, whenever necessary we can take the opposite adjoint equivalences by switching the left and right adjoints and modifying the unit and counit as required.

3.3 Trihomomorphisms and other higher cells

Definition 3.3.1. Let T and T' be tricategories. A *trihomomorphism* $F : T \to T'$ consists of the following data subject to the following axioms.
DATA:

- A function $\mathrm{ob}T \to \mathrm{ob}T'$;
- For objects a, b of T, a functor $F_{ab} : T(a, b) \to T'(Fa, Fb)$;
- For objects a, b, c of T, an adjoint equivalence $\boldsymbol{\chi} : \otimes' \circ (F \times F) \Rightarrow F \circ \otimes$ with left adjoint shown below;

$$\begin{tikzcd}
T(b,c) \times T(a,b) \arrow[r, "F\times F"] \arrow[d, "\otimes"'] & T'(Fb,Fc) \times T'(Fa,Fb) \arrow[d, "\otimes'"] \arrow[dl, phantom, "\overset{\chi}{\Leftarrow}"] \\
T(a,c) \arrow[r, "F"'] & T'(Fa,Fc)
\end{tikzcd}$$

- For each object a of T, an adjoint equivalence $\boldsymbol{\iota} : I'_{Fa} \Rightarrow F \circ I_a$ with left adjoint shown below;

$$\begin{tikzcd}
\mathbf{1} \arrow[rr, "I'_{Fa}"] \arrow[dr, "I_a"'] & \arrow[d, phantom, "\Downarrow\iota"] & T'(Fa,Fa) \\
& T(a,a) \arrow[ur, "F"'] &
\end{tikzcd}$$

- For objects a, b, c, d of T, an invertible modification as pictured below;

$$\begin{tikzcd}[column sep=small]
& T^3 \arrow[rr, "F\times F\times F"] \arrow[dl, "1\times\otimes"'] \arrow[dr, "\otimes\times 1"] \arrow[drr, phantom, "\Downarrow\chi\times 1"] & & T'^3 \arrow[dr, "\otimes'\times 1"] & & & & T^3 \arrow[rr, "F\times F\times F"] \arrow[dl, "1\times\otimes"'] \arrow[d, phantom, "\Downarrow 1\times\chi"] & & T'^3 \arrow[dl, "1\times\otimes'"'] \arrow[dr, "\otimes'\times 1"] & \\
T^2 \arrow[dr, "\otimes"'] \arrow[rr, phantom, "\overset{a}{\Leftarrow}"] & & T^2 \arrow[rr, "F\times F"] \arrow[dl, "\otimes"] \arrow[d, phantom, "\Downarrow\chi"] & & T'^2 \arrow[dl, "\otimes'"] \arrow[rr, Rrightarrow, "\omega"] & & T^2 \arrow[rr, "F\times F"] \arrow[dr, "\otimes"'] & \arrow[d, phantom, "\Downarrow\chi"] & T'^2 \arrow[dr, "\otimes'"] \arrow[rr, phantom, "\overset{a'}{\Leftarrow}"] & & T'^2 \arrow[dl, "\otimes'"] \\
& T \arrow[rr, "F"'] & & T' & & & & T \arrow[rr, "F"'] & & T' &
\end{tikzcd}$$

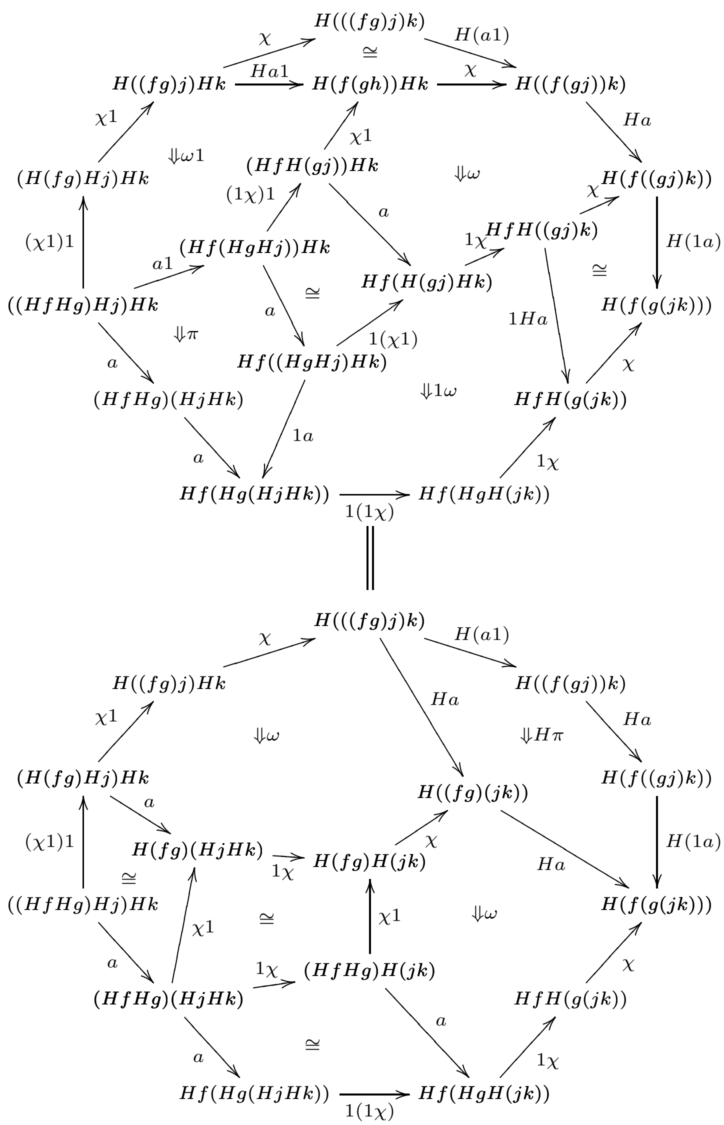
- For objects a, b of T, invertible modifications γ and δ as pictured below;

$$
\begin{tikzcd}
 & & T'^2 \arrow[ddrr, "\otimes'"] & & \\
 & & & & \\
T' \arrow[uurr, "I'\times 1"] \arrow[r, phantom, "\vec{\iota}\times 1"] & & T^2 \arrow[uu, "H\times H"] \arrow[ddrr, "\otimes"] \arrow[r, phantom, "\vec{\chi}"] \arrow[dd, phantom, "\Downarrow l"] & & T' \\
 & & & & \\
T \arrow[uu, "H"] \arrow[uurr, "I\times 1"] \arrow[rrrr, "1"'] & & & & T \arrow[uu, "H"']
\end{tikzcd}
\overset{\gamma}{\Rrightarrow}
\begin{tikzcd}
 & T'^2 \arrow[dr, "\otimes'"] \arrow[d, phantom, "\Downarrow l'"] & \\
T' \arrow[ur, "I'\times 1"] \arrow[rr, "1"'] & {} & T' \\
T \arrow[u, "H"] \arrow[rr, "1"'] \arrow[urr, phantom, "="] & & T \arrow[u, "H"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
T' \arrow[rr, "1"] & & T' \\
T \arrow[u, "H"] \arrow[rr, "1"] \arrow[urr, phantom, "="] & {} \arrow[d, phantom, "\Downarrow r^{\bullet}"] & T \arrow[u, "H"'] \\
 & T^2 \arrow[ur, "\otimes"'] & \\
T \arrow[u, phantom] & &
\end{tikzcd}
\overset{\delta}{\Rrightarrow}
\begin{tikzcd}
T' \arrow[rr, "1"] \arrow[dr, "1\times I'"] & \Downarrow r'^{\bullet} & T' \\
 & T'^2 \arrow[ur, "\otimes'"] & \\
T \arrow[uu, "H"] \arrow[dr, "1\times I"'] \arrow[ur, phantom, "\vec{1\times\iota}"] & & T \arrow[uu, "H"'] \arrow[ul, phantom, "\vec{\chi}"] \\
 & T^2 \arrow[uu, "H\times H"] \arrow[ur, "\otimes"'] &
\end{tikzcd}
$$

AXIOMS:

- For all 1-cells $(x, y, z, w) \in T(d, e) \times T(c, d) \times T(b, c) \times T(a, b)$, the following equation of modifications holds;



- For all 1-cells $(x, y) \in T(b, c) \times T(a, b)$, the following equation of modifications holds.

$$
\begin{tikzcd}
 & & H((fI)g) \arrow[rr, "Ha"] & & H(f(Ig)) \arrow[ddrr, "H(1l)"] & & \\
 & & & \Downarrow\omega & HfH(Ig) \arrow[u, "\chi"'] \arrow[ddddrr, "1Hl"] & \cong & \\
H(fg) \arrow[uurr, "H(r^{\bullet}1)"] & \cong & & Hf(HIHg) \arrow[ur, "1\chi"] & \Downarrow 1\gamma & & H(fg) \\
 & H(fI)Hg \arrow[uuur, "\chi"'] & (HfHI)Hg \arrow[l, "\chi 1"'] \arrow[ur, "a"] & \cong & & & \\
 & \overset{\delta 1}{\Rightarrow} & (HfI)Hg \arrow[u, "(1\iota)1"] \arrow[r, "a"] & Hf(IHg) \arrow[uu, "1(\iota 1)"'] \arrow[drrr, "1l"] & & & \\
HfHg \arrow[uuu, "\chi"] \arrow[uur, "Hr^{\bullet}1"] \arrow[urr, "r^{\bullet}1"'] \arrow[rrrrrr, "1"'] & & & \Downarrow\mu & & & HfHg \arrow[uuu, "\chi"']
\end{tikzcd}
$$

$$\Big\|$$

$$
\begin{tikzcd}
 & H((fI)g) \arrow[r, "Ha"] & H(f(Ig)) \arrow[dr, "H(1l)"] & \\
H(fg) \arrow[ur, "H(r^{\bullet}1)"] \arrow[rrr, "1"'] \arrow[rrr, phantom, shift left=4ex, "\Downarrow H\mu"] & & & H(fg) \\
HfHg \arrow[u, "\chi"] \arrow[rrr, "1"'] \arrow[rrr, phantom, shift left=4ex, "\cong"] & & & HfHg \arrow[u, "\chi"']
\end{tikzcd}
$$

Definition 3.3.2. A *lax funtor* $F : T \to T'$ consists of the same data as a trihomomorphism $F : T \to T'$ with the following changes:

- each F_{ab} is only a lax functor,
- a lax transformation χ in place of $\boldsymbol{\chi}$,
- a lax transformation ι in place of $\boldsymbol{\iota}$, and
- the modifications are no longer required to be invertible.

Definition 3.3.3. 1. A functor F is *locally strict* if each F_{ab} is a strict functor between bicategories.
2. A *strict functor* is a trihomomorphism $F : T \to T'$ such that

- F is locally strict,
- $\boldsymbol{\chi}$ and $\boldsymbol{\iota}$ are the identity adjoint equivalences,
- and the modifications ω, γ, and δ are given by the diagrams below, where all unmarked isomorphisms are unique coherence cells arising either from the functor $\otimes$ or the hom-bicategory.

$$\begin{tikzcd}
(Fh \otimes' Fg) \otimes' Ff \arrow[r, "1"'] \arrow[r, bend left, "1 \otimes' 1"] \arrow[r, phantom, bend left=20, "\cong"] \arrow[d, "a'"'] \arrow[drr, "Fa = a'" description] & F(h \otimes g) \otimes' Ff \arrow[r, "1"] \arrow[dr, phantom, "\cong"] & F((h \otimes g) \otimes f) \arrow[d, "Fa"] \\
Fh \otimes' (Fg \otimes' Ff) \arrow[r, "1"] \arrow[r, bend right, "1 \otimes' 1"'] \arrow[r, phantom, bend right=20, "\cong"] \arrow[u, phantom, shift left=12, "\cong"] & Fh \otimes' F(g \otimes f) \arrow[r, "1"'] & F(h \otimes (g \otimes f))
\end{tikzcd}$$

$$\begin{tikzcd}
I' \otimes' Ff \arrow[r, "1"'] \arrow[r, bend left, "1 \otimes' 1"] \arrow[r, phantom, bend left=20, "\cong"] \arrow[drr, "l'"'] & FI \otimes' Ff \arrow[r, "1"] \arrow[d, phantom, "\cong"] & F(I \otimes f) \arrow[d, "Fl = l'"] \\
 & {} & Ff
\end{tikzcd}$$

$$\begin{tikzcd}
Ff \arrow[r, "(r')^{\bullet}"] \arrow[drr, "(r')^{\bullet} = Fr^{\bullet}"'] & Ff \otimes' I' \arrow[r, "1"'] \arrow[r, bend left, "1 \otimes' 1"] \arrow[r, phantom, bend left=20, "\cong"] \arrow[d, phantom, "\cong"] & Ff \otimes' FI \arrow[d, "1"] \\
 & {} & F(f \otimes I)
\end{tikzcd}$$

Remark 3.3.4. It is clear from the definition above that given a function on objects F_0 and strict functors of hom-bicategories F_{ab}, there is at most one structure of a strict functor with this underlying data.

Remark 3.3.5. This definition differs from the definition of strict functor given in [17] in two ways. First, we require local strictness while the original definition did not. Second, the definition given in [17] requires that the modifications ω, γ, and δ are identities, when this is in fact impossible as their sources do not equal their targets; we have remedied this mistake by requiring these modifications to have unique coherence isomorphisms as their components.

Definition 3.3.6. Let $F, G : T \to T'$ be trihomomorphisms with the same source and target. A *tritransformation* $\theta : F \to G$ consists of a family of 1-cells

$\theta_a : Fa \to Ga$ of T', indexed by the objects of T, adjoint equivalences

$$\begin{tikzcd}
T(a,b) \arrow[r, "F"] \arrow[d, "G"'] & T'(Fa,Fb) \arrow[d, "T'(1,\theta_b)"] \arrow[dl, Rightarrow, "\boldsymbol{\theta}"'] \\
T'(Ga,Gb) \arrow[r, "T'(\theta_a,1)"'] & T'(Fa,Gb)
\end{tikzcd}$$

in $\mathbf{Bicat}(T(a,b), T'(Fa,Gb))$ for all objects a, b of T, and invertible modifications as shown below. We have abbreviated $T(a,b)$ by $[a,b]$, $T(b,c) \times T(a,b)$ by $[b,c;a,b]$, and similarly in T'; no distinction is made between T and T', as lower case letters such as a, b, c, etc., are objects in T while Fa, Gb, etc., are objects in T'.

$$\begin{tikzcd}
 & & {[b,c;a,b]} \arrow[r, "F\times F"] \arrow[dd, "G\times F"] \arrow[ddl, "G\times G"'] \arrow[ddll, bend right, "\otimes"'] \arrow[dr, phantom, "\Downarrow\theta\times 1"] & {[Fb,Fc;Fa,Fb]} \arrow[dd, "T'(1,\theta_c)\times 1"] \\
 & & & \\
 & \arrow[l, phantom, "\overset{\chi}{\Leftarrow}"] & {[Gb,Gc;Fa,Fb]} \arrow[r, "T'(\theta_b,1)\times 1"] \arrow[d, "1\times T'(1,\theta_b)"] \arrow[l, phantom, "\overset{1\times\theta}{\Leftarrow}"] \arrow[dr, phantom, "\Downarrow a"] & {[Fb,Gc;Fa,Fb]} \arrow[d, "\otimes"] \\
{[a,c]} \arrow[drr, "G"'] & {[Gb,Gc;Ga,Gb]} \arrow[r, "1\times T'(\theta_a,1)"] \arrow[dr, "\otimes"] & {[Gb,Gc;Fa,Gb]} \arrow[r, "\otimes"'] \arrow[d, phantom, "\Downarrow a^{\bullet}"] & {[Fa,Gc]} \\
 & & {[Ga,Gc]} \arrow[ur, "T'(\theta_a,1)"'] &
\end{tikzcd}$$

$$\Big\Downarrow \Pi$$

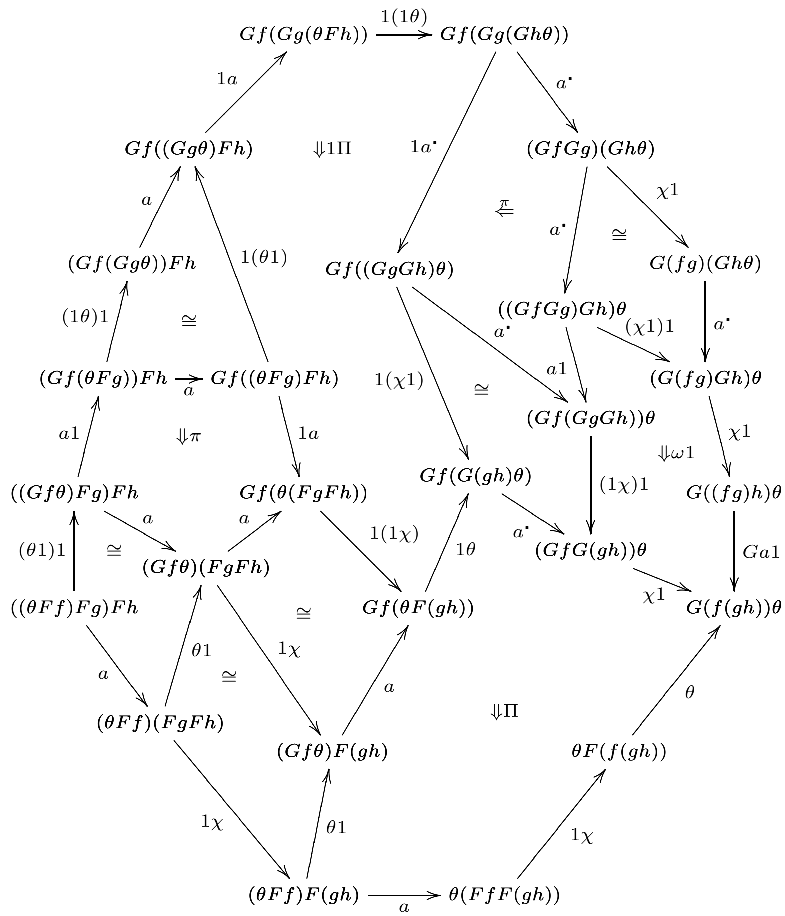
$$\begin{tikzcd}
 & & {[b,c;a,b]} \arrow[r, "F\times F"] \arrow[d, "F\times F"] \arrow[ddll, bend right, "\otimes"'] \arrow[dr, phantom, "\Downarrow a"] & {[Fb,Fc;Fa,Fb]} \arrow[d, "T'(1,\theta_c)\times 1"] \\
 & \arrow[d, phantom, "\overset{\chi}{\Leftarrow}"] & {[Fb,Fc;Fa,Fb]} \arrow[d, "\otimes"] & {[Fb,Gc;Fa,Fb]} \arrow[d, "\otimes"] \\
{[a,c]} \arrow[rr, "F"'] \arrow[drr, "G"'] & & {[Fa,Fc]} \arrow[r, "T'(1,\theta_c)"] \arrow[d, phantom, "\Downarrow\theta"] & {[Fa,Gc]} \\
 & & {[Ga,Gc]} \arrow[ur, "T'(\theta_a,1)"'] &
\end{tikzcd}$$

$$
\begin{tikzcd}
 & 1 \arrow[dl, bend right, "I_a"'] \arrow[d, "I_{Fa}"] \arrow[ddr, bend left, "\theta_a"] \arrow[dl, phantom, "\overset{\iota}{\Leftarrow}"] \arrow[dr, phantom, "\overset{r^{\bullet}}{\Leftarrow}"] & \\
T(a,a) \arrow[r, "F"'] \arrow[d, "G"'] \arrow[dr, phantom, "\Downarrow\theta"] & T'(Fa,Fa) \arrow[dr, "{T'(1,\theta_a)}"] & \\
T'(Ga,Ga) \arrow[rr, "{T'(\theta_a,1)}"'] & & T'(Fa,Ga)
\end{tikzcd}
\overset{M}{\Rrightarrow}
\begin{tikzcd}
 & & 1 \arrow[dll, "I_a"'] \arrow[ddll, bend right, "I_{Ga}"] \arrow[dd, bend left, "\theta_a"] \arrow[d, phantom, "\overset{l^{\bullet}}{\Leftarrow}"] \\
T(a,a) \arrow[d, "G"'] \arrow[r, phantom, "\overset{\iota}{\Leftarrow}"] & & \\
T'(Ga,Ga) \arrow[rr, "{T'(\theta_a,1)}"'] & & T'(Fa,Ga)
\end{tikzcd}
$$

The functor θ_a is the functor whose value at the single object of 1 is the 1-cell θ_a and all of whose constraints are given by unique coherence isomorphisms; these are subject to the following three axioms.
AXIOMS:

$$
\begin{tikzcd}[column sep=tiny]
 & & & Gf(Gg(\theta Fh)) \arrow[rrr, "1(1\theta)"] \arrow[dr, "a^{\bullet}"] \arrow[d, phantom, "\Downarrow\pi"] & & & Gf(Gg(Gh\theta)) \arrow[dr, "a^{\bullet}"] & & \\
 & & Gf((Gg\theta)Fh) \arrow[ur, "1a"] & & (GfGg)(\theta Fh) \arrow[rrr, "1\theta"] \arrow[ddrr, "\chi 1"] \arrow[ur, phantom, "\cong"] & & & (GfGg)(Gh\theta) \arrow[ddr, "\chi 1"] & \\
 & & & ((GfGg)\theta)Fh \arrow[ur, "a"'] \arrow[dd, "(\chi 1)1"'] \arrow[r, phantom, "\cong"] & & & & & \\
 & (Gf(Gg\theta))Fh \arrow[uur, "a"] \arrow[urr, "a^{\bullet}1"] \arrow[dr, phantom, "\Downarrow\Pi 1"] & & & & & G(fg)(\theta Fh) \arrow[rr, "1\theta"'] \arrow[u, phantom, "\cong"] \arrow[dd, phantom, "\Downarrow\Pi"] & & G(fg)(Gh\theta) \arrow[dd, "a^{\bullet}"] \\
 & & & (G(fg)\theta)Fh \arrow[urrr, "a"'] & & & & & \\
 & (Gf(\theta Fg))Fh \arrow[uu, "(1\theta)1"] & & & & & & & (G(fg)Gh)\theta \arrow[dd, "\chi 1"] \\
 & & & (\theta F(fg))Fh \arrow[uu, "\theta 1"] \arrow[drr, "a"] \arrow[dd, phantom, "\cong"] & & & & & \\
((Gf\theta)Fg)Fh \arrow[uur, "a1"] & & (\theta(FfFg))Fh \arrow[ur, "(1\chi)1"] \arrow[drr, "a"'] \arrow[d, phantom, "\Downarrow\pi"] & & & \theta(F(fg)Fh) \arrow[drr, "1\chi"] \arrow[ddr, phantom, "\Downarrow 1\omega"] & & & G((fg)h)\theta \arrow[d, "Ga1"] \\
((\theta Ff)Fg)Fh \arrow[u, "(\theta 1)1"] \arrow[urr, "a1"] \arrow[drr, "a"'] & & & & \theta((FfFg)Fh) \arrow[ur, "1(\chi 1)"'] \arrow[dr, "1a"] & & & \theta F((fg)h) \arrow[ur, "\theta"] \arrow[d, "1Fa"] \arrow[r, phantom, "\cong"] & G(f(gh))\theta \\
 & & (\theta Ff)(FgFh) \arrow[rrr, "a"'] \arrow[dr, "1\chi"'] & & & \theta(Ff(FgFh)) \arrow[dr, "1(1\chi)"] \arrow[dll, phantom, "\cong"] & & \theta F(f(gh)) \arrow[ur, "\theta"'] & \\
 & & & (\theta Ff)F(gh) \arrow[rrr, "a"'] & & & \theta(FfF(gh)) \arrow[ur, "1\chi"'] & &
\end{tikzcd}
$$

$$\Big\|$$



$Gf(Gg(\theta Fh))$
$1(1\theta)$
$Gf(Gg(Gh\theta))$
$1a$
$a^{\bullet}$
$Gf((Gg\theta)Fh)$
$\Downarrow 1\Pi$
$1a^{\bullet}$
$(GfGg)(Gh\theta)$
a
$\chi 1$
$\Leftarrow \pi$
$a^{\bullet}$
$\cong$
$(Gf(Gg\theta))Fh$
$1(\theta 1)$
$Gf((GgGh)\theta)$
$G(fg)(Gh\theta)$
$((GfGg)Gh)\theta$
$(1\theta)1$
$\cong$
$a^{\bullet}$
$(\chi 1)1$
$a^{\bullet}$
$a1$
$(Gf(\theta Fg))Fh$
a
$Gf((\theta Fg)Fh)$
$1(\chi 1)$
$\cong$
$(G(fg)Gh)\theta$
$(Gf(GgGh))\theta$
$a1$
$\Downarrow \pi$
$1a$
$\chi 1$
$\Downarrow \omega 1$
$Gf(G(gh)\theta)$
$(1\chi)1$
$((Gf\theta)Fg)Fh$
$Gf(\theta(FgFh))$
$G((fg)h)\theta$
a
a
$1(1\chi)$
1θ
$a^{\bullet}$
$(\theta 1)1$
$\cong$
$(Gf\theta)(FgFh)$
$(GfG(gh))\theta$
$Ga1$
$\chi 1$
$((\theta Ff)Fg)Fh$
$\cong$
$Gf(\theta F(gh))$
$G(f(gh))\theta$
$\theta 1$
1χ
a
$\cong$
a
θ
$\Downarrow \Pi$
$(\theta Ff)(FgFh)$
$(Gf\theta)F(gh)$
$\theta F(f(gh))$
1χ
$\theta 1$
1χ
$(\theta Ff)F(gh)$
a
$\theta(FfF(gh))$

$$
\begin{tikzcd}
 & & & & GI(\theta Ff) \arrow[dr, "1\theta"] & & & & \\
 & & & (GI\theta)Ff \arrow[ur, "a"] \arrow[rr, phantom, "\Downarrow\Pi"] & & GI(Gf\theta) \arrow[dr, "a^{\bullet}"] & & & \\
 & & (\theta FI)Ff \arrow[ur, "\theta 1"] \arrow[drr, "a"] & & & & (GIGf)\theta \arrow[ddr, "\chi 1"] & & \\
 & & & & \theta(FIFf) \arrow[dr, "1\chi"] & & & & \\
 & (\theta I)Ff \arrow[uur, "(1\iota)1"] \arrow[drr, "a"] \arrow[urr, phantom, "\cong"] \arrow[ddrr, phantom, "\Downarrow\mu"] & & & & \theta F(If) \arrow[rr, "\theta"] \arrow[dd, "1Fl"] \arrow[ddrr, phantom, "\cong"] & & G(If)\theta \arrow[ddr, "Gl1"] & \\
 & & & \theta(IFf) \arrow[uur, "1(\iota 1)"] \arrow[drr, "1l"] \arrow[uurr, phantom, "\Downarrow 1\gamma"] & & & & & \\
\theta Ff \arrow[uur, "r^{\bullet}1"] \arrow[rrrrr, "1"] \arrow[drr, "l^{\bullet}1"] & & & & & \theta Ff \arrow[rrr, "\theta"] & & & Gf\theta \\
 & & (I\theta)Ff \arrow[drr, "a"] \arrow[uurr, phantom, "\Downarrow\lambda"] & & & & I(Gf\theta) \arrow[urr, "l"] & & \\
 & & & & I(\theta Ff) \arrow[uur, "l"] \arrow[urr, "1\theta"] \arrow[uurrrr, phantom, "\cong"] & & & &
\end{tikzcd}
$$

$$\Vert$$

$$
\begin{tikzcd}
 & & & & GI(\theta Ff) \arrow[dr, "1\theta"] & & & & \\
 & & & (GI\theta)Ff \arrow[ur, "a"] & & GI(Gf\theta) \arrow[dr, "a^{\bullet}"] \arrow[ddd, phantom, "\cong"] & & & \\
 & & (\theta FI)Ff \arrow[ur, "\theta 1"] & & & & (GIGf)\theta \arrow[ddr, "\chi 1"] & & \\
 & & & & & & & & \\
 & (\theta I)Ff \arrow[uur, "(1\iota)1"] \arrow[ddr, phantom, "\Downarrow M1"] & & & & & & G(If)\theta \arrow[ddr, "Gl1"] & \\
 & & & & & & (IGf)\theta \arrow[uuu, "(\iota 1)1"] \arrow[drr, "l1"] \arrow[ur, phantom, "\Downarrow\gamma 1"] \arrow[ddr, phantom, "\Downarrow\lambda"] & & \\
\theta Ff \arrow[uur, "r^{\bullet}1"] \arrow[drr, "l^{\bullet}1"] & & & & & & & & Gf\theta \\
 & & (I\theta)Ff \arrow[drr, "a"] \arrow[uuuuuur, "(\iota 1)1"] \arrow[uurr, phantom, "\cong"] & & & & I(Gf\theta) \arrow[urr, "l"] \arrow[uu, "a^{\bullet}"] \arrow[uuuuuul, bend left, "\iota 1"] & & \\
 & & & & I(\theta Ff) \arrow[urr, "1\theta"] \arrow[uuuuuuuu, "\iota 1"] \arrow[uuur, phantom, "\cong"] & & & &
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small]
 & & & & & Gf(\theta FI) \arrow[dr, "1\theta"] & & & \\
 & & Gf(\theta I) \arrow[urrr, "1(1\iota)"] \arrow[rr, phantom, "\cong"] & & & & Gf(GI\theta) \arrow[ddr, "a^{\bullet}"] & & \\
 & & & & (Gf\theta)FI \arrow[uur, "a"] & & & & \\
 & Gf\theta \arrow[uur, "1r^{\bullet}"] \arrow[r, "r^{\bullet}"'] & (Gf\theta)I \arrow[uu, "a"] \arrow[urr, "1\iota"] & & \cong & (\theta Ff)FI \arrow[ul, "\theta 1"] \arrow[dr, "a"] & \Downarrow\Pi & (GfGI)\theta \arrow[ddr, "\chi 1"] & \\
 & \cong & & (\theta Ff)I \arrow[ul, "\theta 1"] \arrow[urr, "1\iota"] \arrow[ddr, "a"] & \cong & & \theta(FfFI) \arrow[dr, "1\chi"] & & \\
\theta Ff \arrow[uur, "\theta"] \arrow[urrr, "r^{\bullet}"'] \arrow[dddrrrrr, bend right, "1"'] & & \Downarrow\rho & & & \Downarrow 1\delta & & \theta F(fI) \arrow[r, "\theta"] & G(fI)\theta \\
 & & & & \theta(FfI) \arrow[uurr, "1(1\iota)"] \arrow[ddr, "1r"] & & & \cong & \\
 & & & & & & & Gf\theta \arrow[uur, "Gr^{\bullet}1"'] & \\
 & & & & & \theta Ff \arrow[uuurr, "1Fr^{\bullet}"] \arrow[urr, "\theta"'] & & &
\end{tikzcd}
$$

$$
\Vert
$$

$$
\begin{tikzcd}[column sep=small]
 & & & & Gf(\theta FI) \arrow[dr, "1\theta"] & & & \\
 & & Gf(\theta I) \arrow[urr, "1(1\iota)"] & \Downarrow 1M & & Gf(GI\theta) \arrow[ddr, "a^{\bullet}"] & & \\
 & & & Gf(I\theta) \arrow[urr, "1(\iota 1)"'] \arrow[ddrr, "a^{\bullet}"] & & \cong & & \\
 & Gf\theta \arrow[uur, "1r^{\bullet}"] \arrow[urr, "1l^{\bullet}"] \arrow[dddrrrrr, "1"] & & \Downarrow\mu & & & (GfGI)\theta \arrow[ddr, "\chi 1"] & \\
 & & & & & (GfI)\theta \arrow[ur, "(1\iota)1"] \arrow[ddr, "r1"] & & \\
\theta Ff \arrow[uur, "\theta"] \arrow[ddrrrr, bend right, "1"'] & & \cong & & & & \Downarrow\delta 1 & G(fI)\theta \\
 & & & & & & Gf\theta \arrow[ur, "Gr^{\bullet}1"'] & \\
 & & & & \theta Ff \arrow[urr, "\theta"'] & & &
\end{tikzcd}
$$

Definition 3.3.7. A *lax transformation* $\theta : F \to G$ between lax functors consists of the same data as a tritransformation between trihomomorphisms with the following changes:

- lax transformations θ in place of the adjoint equivalences θ and
- the modifications are no longer required to be invertible.

Definition 3.3.8. Let θ and ϕ be tritransformations with the same source F and target G. A *trimodification* $m : \theta \Rrightarrow \phi$ consists of a family of 2-cells $m_a : \theta_a \Rrightarrow \phi_a$ in the target tricategory T', indexed by the objects of the source tricategory T, and invertible modifications

$$
\begin{tikzcd}
T(a,b) \arrow[r, "F"] \arrow[d, "G"'] & T'(Fa,Fb) \arrow[d, "T'(1,\theta_b)"] \arrow[dl, phantom, "\Leftarrow\theta"] \\
T'(Ga,Gb) \arrow[r, "T'(\theta_a{,}1)"] \arrow[r, bend right=70, "T'(\phi_a{,}1)"'] \arrow[r, phantom, bend right=35, "\Downarrow(m_a)^*"] & T'(Fa,Gb)
\end{tikzcd}
\overset{m}{\Rrightarrow}
\begin{tikzcd}
T(a,b) \arrow[r, "F"] \arrow[d, "G"'] & T'(Fa,Fb) \arrow[d, "T'(1,\phi_b)"'] \arrow[dl, phantom, "\Leftarrow\phi"] \arrow[d, bend left=70, "T'(1,\theta_b)"] \arrow[d, phantom, bend left=35, "\Leftarrow(m_b)_*"] \\
T'(Ga,Gb) \arrow[r, "T'(\phi_a{,}1)"'] & T'(Fa,Gb)
\end{tikzcd}
$$

such that the following two axioms hold, where we have written tensor as concatenation and all unmarked isomorphisms are naturality isomorphisms.
AXIOMS:

$$
\begin{tikzcd}
(\theta Ff)Fg \arrow[r, "\theta 1"] \arrow[d, "(mF1)F1"'] \arrow[dr, phantom, "\Downarrow m1"] & (Gf\theta)Fg \arrow[r, "a"] \arrow[d, "(G1m)F1"'] \arrow[dr, phantom, "\cong"] & Gf(\theta Fg) \arrow[r, "1\theta"] \arrow[d, "G1(mF1)"'] \arrow[dr, phantom, "\Downarrow^{1m}"] & Gf(Gg\theta) \arrow[r, "a^\bullet"] \arrow[d, "G1(G1m)"'] \arrow[dr, phantom, "\cong"] & (GfGg)\theta \arrow[r, "\chi 1"] \arrow[d, "(G1G1)m"'] \arrow[dr, phantom, "\cong"] & G(fg)\theta \arrow[d, "G1m"] \\
(\phi Ff)Fg \arrow[r, "\phi 1"] \arrow[drr, "a"'] & (Gf\phi)Fg \arrow[r, "a"'] & Gf(\phi Fg) \arrow[r, "1\phi"'] \arrow[d, phantom, "\Downarrow\Pi"] & Gf(Gg\phi) \arrow[r, "a^\bullet"'] & (GfGg)\phi \arrow[r, "\chi 1"'] & G(fg)\phi \\
 & & \phi(FfFg) \arrow[r, "1\chi"'] & \phi F(fg) \arrow[urr, "\phi"'] & &
\end{tikzcd}
$$

$$\Vert$$

$$
\begin{tikzcd}
(\theta Ff)Fg \arrow[r, "\theta 1"] \arrow[d, "(mF1)F1"'] \arrow[drr, "a"] & (Gf\theta)Fg \arrow[r, "a"] & Gf(\theta Fg) \arrow[r, "1\theta"] \arrow[d, phantom, "\Downarrow\Pi"] & Gf(Gg\theta) \arrow[r, "a^\bullet"] & (GfGg)\theta \arrow[r, "\chi 1"] & G(fg)\theta \arrow[d, "G1m"] \\
(\phi Ff)Fg \arrow[r, phantom, "\cong"] \arrow[drr, "a"'] & {} & \theta(FfFg) \arrow[r, "1\chi"'] \arrow[d, "m(F1F1)"] \arrow[dr, phantom, "\cong"] & \theta F(fg) \arrow[urr, "\theta"'] \arrow[d, "mF(11)"'] \arrow[r, phantom, "\Downarrow m"] & {} & G(fg)\phi \\
 & & \phi(FfFg) \arrow[r, "1\chi"'] & \phi F(fg) \arrow[urr, "\phi"'] & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\theta \arrow[rr, "r^\bullet"] \arrow[d, "m"'] \arrow[drr, phantom, "\cong"] & & \theta I \arrow[rr, "1\iota"] \arrow[d, "m1"] \arrow[drr, phantom, "\cong"] & & \theta F I \arrow[rr, "\theta"] \arrow[d, "m1"] \arrow[drr, phantom, "\Downarrow m"] & & GI\theta \arrow[d, "1m"] \\
\phi \arrow[rr, "r^\bullet"] \arrow[drrr, "l^\bullet"'] & & \phi I \arrow[rr, "1\iota"'] & & 1\iota \arrow[rr, "\phi"] \arrow[d, phantom, "\Downarrow M"] & & GI\phi \\
& & & I\phi \arrow[urrr, "\iota 1"'] & & &
\end{tikzcd}
$$

$$\Vert$$

$$
\begin{tikzcd}
\theta \arrow[rr, "r^\bullet"] \arrow[d, "m"'] \arrow[drrr, "l^\bullet"'] & & \theta I \arrow[rr, "1\iota"] \arrow[dr, phantom, "\Downarrow M"] & & \theta F I \arrow[rr, "\theta"] & & GI\theta \arrow[d, "1m"] \\
\phi \arrow[drrr, "l^\bullet"'] \arrow[rrr, phantom, "\cong"] & & & I\theta \arrow[urrr, "\iota 1"'] \arrow[d, "1m"] \arrow[rrr, phantom, "\cong"] & & & GI\phi \\
& & & I\phi \arrow[urrr, "\iota 1"'] & & &
\end{tikzcd}
$$

Note that in equational form, the modifications m above can be written as

$$m : (m_a)^* * 1_G \circ \theta \Rightarrow \phi \circ (m_b)_* * 1_F$$

in the appropriate hom-bicategory. This is how the data is presented in [17].

Definition 3.3.9. A *perturbation* $\sigma : m \Rrightarrow n$ between trimodifications with the same source and target consists of a family of 3-cells $\sigma_a : m_a \Rrightarrow n_a$ in the target tricategory T', indexed by objects of the source tricategory T, such that the following axiom holds.

$$
\begin{tikzcd}
\theta \otimes Ff \arrow[r, "\theta"] \arrow[d, bend right=60, "n\otimes 1"'] \arrow[d, "m\otimes 1"] \arrow[d, phantom, shift right=4ex, "\overset{\sigma\otimes 1}{\Leftarrow}"] & Gf \otimes \theta \arrow[d, "1\otimes m"] \arrow[dl, Rightarrow, "m"'] \\
\phi \otimes Ff \arrow[r, "\phi"'] & Gf \otimes \phi
\end{tikzcd}
=
\begin{tikzcd}
\theta \otimes Ff \arrow[r, "\theta"] \arrow[d, "n\otimes 1"'] & Gf \otimes \theta \arrow[d, "1\otimes n"'] \arrow[d, bend left=60, "1\otimes m"] \arrow[d, phantom, shift left=4ex, "\overset{1\otimes\sigma}{\Leftarrow}"] \arrow[dl, Rightarrow, "n"'] \\
\phi \otimes Ff \arrow[r, "\phi"'] & Gf \otimes \phi
\end{tikzcd}
$$

Remark 3.3.10. This equation is presented in [17] as the equality of modifications shown below.

$$
\begin{tikzcd}
(\theta_b)_* \circ F \arrow[r, "\theta"] \arrow[d, bend right=60, "(n_b)_* * 1"'] \arrow[d, bend left=60, "(m_b)_* * 1"] \arrow[d, phantom, "\overset{\Leftarrow}{(\sigma_b)_* * 1}"] \arrow[dr, phantom, "\overset{\Leftarrow}{m}"] & (\theta_a)^* \circ G \arrow[d, "(m_a)^* * 1"] \\
(\phi_b)_* \circ F \arrow[r, "\phi"'] & (\phi_a)^* \circ G
\end{tikzcd}
=
\begin{tikzcd}
(\theta_b)_* \circ F \arrow[r, "\theta"] \arrow[d, "(n_b)_* * 1"] \arrow[dr, phantom, "\overset{\Leftarrow}{n}"] & (\theta_a)^* \circ G \arrow[d, bend right=60, "(n_a)^* * 1"'] \arrow[d, bend left=60, "(m_a)^* * 1"] \arrow[d, phantom, "\overset{\Leftarrow}{(\sigma_a)^* * 1}"] \\
(\phi_b)_* \circ F \arrow[r, "\phi"'] & (\phi_a)^* \circ G
\end{tikzcd}
$$

Since two modifications are equal if and only if they have the same components, the equation given in the definition and the one here are equivalent axioms.

Notation 3.3.11. We shall drop the prefixes bi- and tri- when the context is understood. Thus both homomorphisms of bicategories and trihomomorphisms of tricategories will be called functors, with weak being understood; a lax map will always be called such.

3.4 Comparing definitions

This section will briefly compared the definitions given here with those given in [17]. The only way in which we have changed the definitions in [17] is by adding additional data, so that the "pseudonatural equivalences" of [17] have been replaced with ajoint equivalences in the functor bicategory. Thus it is obvious that every tricategory in our sense (we shall refer to these as algebraic tricategories) gives rise to a tricategory in the sense of Gordon, Power, and Street by neglect of structure. Given a tricategory in the sense of Gordon, Power, and Street, it is possible to construct an algebraic tricategory by choosing adjoint equivalences. There is no canonical choice, but any two such choices are related by a functor that is the identity on objects and hom-bicategories. It is similarly easy to see that every functor in our sense gives rise to a functor in the sense of Gordon, Power, and Street by neglect of structure, and every functor in the sense of Gordon, Power, and Street can be noncanonically given additional data to produce a functor in our sense.

Every transformation in our sense also gives a transformation in the sense of Gordon, Power, and Street, but the converse is more delicate. Since the definition of transformation involves the cells $a^{\bullet}$, the definition given by Gordon, Power, and Street is ambiguous. Since no one choice of $a^{\bullet}$ is fixed, we must use the fact that any two choices are uniquely isomorphic as adjoints of a to produce the data for a transformation as defined here. Aside from this technical point, we can once again produce from a transformation of Gordon, Power, and Street a transformation as defined here, but noncanonically.

The definitions given here of modification and perturbation are exactly the same as those given in [17].

Let $F : S \to T$ be a functor as defined here, and let $UF : US \to UT$ be the functor, in the sense of Gordon, Power, and Street, obtained by neglecting structure. Using the definitions given at the beginning of the next chapter, it should be clear that $UG \circ UF = U(G \circ F)$ and $U1 = 1$. This does not extend to transformations since the composite of transformations is undefined using the definitions in [17] because of the necessity of using some cells $a^{\bullet}$.

Chapter 4

Basic structure

This chapter will be devoted to studying some aspects of the total algebraic structure consisting of tricategories, functors, transformations, modifications, and perturbations. This chapter will only establish some basic properties that will be used later.

4.1 Structure of functors

This section will give an explicit formula for composing functors between tricategories. The formula here is more interesting than the corresponding one for functors between bicategories in the following way. Functors between bicategories compose strictly so that bicategories with lax, weak, or strict functors form a *category*. This gives rise to the fact that the tricategory **Bicat** has strict composition of 1-cells. We will see that this is not the case with tricategories, and that even making a category out of tricategories and strict functors requires some work.

Let R, S, and T be tricategories, and let $H : R \to S$ and $J : S \to T$ be functors. We now define the composite functor $JH : R \to T$.

- The function on objects $\mathrm{ob}R \to \mathrm{ob}T$ is given by the composite of the object functions for H and J.
- The functors on hom-bicategories $JH_{ab} : R(a,b) \to T(JHa, JHb)$ are given by
$$R(a,b) \xrightarrow{H_{ab}} S(Ha, Hb) \xrightarrow{J_{HaHb}} T(JHa, JHb).$$
- The adjoint equivalence $\boldsymbol{\chi}^{JH}$ is defined as follows. The transformation χ^{JH} is the composite
$$\otimes^T \circ J \times J \circ H \times H \xrightarrow{\chi^J * 1} J \circ \otimes^S \circ H \times H \xrightarrow{1 * \chi^H} J \circ H \circ \otimes^R.$$

Similarly, the transformation $(\chi^{JH})^{\bullet}$ is the composite

$$J \circ H \circ \otimes^R \xrightarrow{1*\chi^{\bullet}} J \circ \otimes^S \circ H \times H \xrightarrow{\chi^{\bullet}*1} \otimes^T \circ J \times J \circ H \times H.$$

The counit of this adjunction is the composite displayed below.

$$\begin{aligned}
&(\chi\chi^{\bullet})_{gf} = \big(J(\chi_{gf}) \circ \chi_{HgHf}\big) \circ \big(\chi^{\bullet}_{Hg,Hf} \circ J(\chi^{\bullet}_{gf})\big) \xrightarrow{\cong} \\
&J(\chi_{gf}) \circ \Big(\big(\chi_{HgHf} \circ \chi^{\bullet}_{HgHf}\big) \circ J(\chi^{\bullet}_{gf})\Big) \xrightarrow{1*(\varepsilon*1)} \\
&J(\chi_{gf}) \circ \Big(1 \circ J(\chi^{\bullet}_{gf})\Big) \xrightarrow{1*l} J(\chi_{gf}) \circ J(\chi^{\bullet}_{gf}) \xrightarrow{\phi_2^J} \\
&J(\chi_{gf}\chi^{\bullet}_{gf}) \xrightarrow{J\varepsilon} J(1_{H(g\otimes f)}) \xrightarrow{\phi_0^J} 1_{JH(g\otimes f)}
\end{aligned}$$

The unit is defined similarly, and a check shows that this gives an adjoint equivalence in the appropriate bicategory.

- If we denote the units by $I^R : \mathbf{1} \to R(a,a)$, etc., then the adjoint equivalence $\boldsymbol{\iota}^{JH}$ is defined as follows. The transformation ι^{JH} is the composite

$$I^T \xrightarrow{\iota^J} J \circ I^S \xrightarrow{1*\iota^H} J \circ H \circ I^R.$$

The transformation $(\iota^{JH})^{\bullet}$ is the composite

$$J \circ H \circ I^R \xrightarrow{1*\iota^{\bullet}} J \circ I^S \xrightarrow{\iota^{\bullet}} I^T.$$

The unit and counit of this adjunction are determined in a manner similar to that used for χ.

- The component at (h,g,f) of the invertible modification ω^{JH} is defined by the pasting diagram below. The unmarked isomorphisms are given by unique coherence cells by the coherence for functors theorem or naturality isomorphisms, and the unlabeled 2-cells are uniquely determined as the source and target of $J\omega^H$.

$$\begin{tikzcd}[column sep=small]
(JHh\otimes'' JHg)\otimes'' JHf \arrow[rrrrrr, "a"] \arrow[dr, "J\chi\otimes 1"] \arrow[dd, "\chi\otimes'' 1"'] & & & & & & JHh\otimes''(JHg\otimes'' JHf) \arrow[dl, "1\otimes''\chi"'] \arrow[dd, "1\otimes''\chi"] \\
\cong & J(Hh\otimes' Hg)\otimes'' JHf \arrow[dl, "J\chi\otimes'' 1"] \arrow[dr, "\chi"] & & \Downarrow\omega^J & & JHh\otimes'' J(Hg\otimes' Hf) \arrow[dl, "\chi"'] \arrow[dr, "1\otimes'' J\chi"] & \cong \\
JH(h\otimes g)\otimes'' JHf \arrow[dd, "\chi"'] \arrow[dr, "\chi"] & \cong & J((Hh\otimes' Hg)\otimes' Hf) \arrow[rr, "Ja"] \arrow[dl, "J(\chi\otimes' 1)"'] \arrow[ddrrrr] \arrow[ddrrrr, bend right] & & J(Hh\otimes'(Hg\otimes' Hf)) \arrow[dr, "J(1\otimes'\chi)"'] & \cong & JHh\otimes'' JH(g\otimes f) \arrow[dl, "\chi"'] \arrow[dd, "\chi"] \\
:= & J(H(h\otimes g)\otimes' Hf) \arrow[dl, "J\chi"] & & \Downarrow J\omega^H \quad \cong & & J(Hh\otimes' H(g\otimes f)) \arrow[dr, "J\chi"] & =: \\
JH((h\otimes g)\otimes f) \arrow[rrrrrr, "JHa"'] & \cong & & & & & JH(h\otimes(g\otimes f))
\end{tikzcd}$$

- The component at f of the invertible modification γ^{JH} is defined by the pasting diagram below, where the unmarked isomorphisms are once again unique coherence cells or naturality isomorphisms; δ^{JH} is defined similarly.

$$\begin{tikzcd}
I''\otimes'' JHf \arrow[dr, "\iota\otimes'' 1"] \arrow[dd, "\iota\otimes'' 1"'] \arrow[ddddrr, bend left=60, "l''"] & & \\
\cong & JI'\otimes'' JHf \arrow[dl, "J\iota\otimes'' 1"] \arrow[dr, "\chi"] & \Uparrow\gamma^J \\
JHI\otimes'' JHf \arrow[dr, "\chi"] \arrow[dd, "\chi"'] & \cong & J(I'\otimes' Hf) \arrow[dl, "J(\iota\otimes' 1)"'] \arrow[dd, "Jl'"] \\
\cong & J(HI\otimes' Hf) \arrow[dl, "\chi"] & \Uparrow J\gamma^H \\
JH(I\otimes f) \arrow[rr, "JHl"'] & & JHf
\end{tikzcd}$$

Note that in the definitions above no associations were given. This is because functors between bicategories compose in a strictly associative manner. Calculation then yields the following result.

Proposition 4.1.1. *The data above satisfies the axioms for a functor between tricategories.*

Proposition 4.1.2. *Tricategories and strict functors do not form a category when equipped with the composition law above.*

Proof. We show that the composite functor $\mathrm{id} \circ \mathrm{id}$ does not have the same underlying data as the functor id, so that composition of functors is not strictly unital.

The identity functor id_T on a tricategory T has each component functor the identity, and χ is the identity transformation $\otimes \Rightarrow \otimes$. For an object (g, f), the component of this transformation is $\mathrm{id} : g \otimes f \to g \otimes f$. The transformation χ for the composite $\mathrm{id}_T \circ \mathrm{id}_T$ has component $\mathrm{id} \circ \mathrm{id} : g \otimes f \to g \otimes f$. In general, this is not equal to the identity map on $g \otimes f$. □

Corollary 4.1.3. *Tricategories and functors do not form a category using the above composition law.*

The difficulty in defining a category of tricategories and strict functors is that the composite of strict functors will no longer be strict. It is possible to rectify this by changing the composition law in the following way. Let $Gr(\mathbf{Bicat})$ be the category of bicategory-enriched graphs. An object G consists of a set G_0 and for each pair $a, b \in G_0$, a bicategory $G(a, b)$. A morphism $f : G \to H$ in $Gr(\mathbf{Bicat})$ consists of a map $f : G_0 \to H_0$ and functors

$$f_{a,b} : G(a, b) \longrightarrow H(fa, fb).$$

Every tricategory has an underlying bicategory-enriched graph, and every functor has an underlying map of these graphs.

Definition 4.1.4. Let S, T be tricategories. A functor $F : S \to T$ is *virtually strict* if there exists a strict functor $\widetilde{F} : S \to T$ such that the underlying graph-map of F is equal to the underlying graph-map of $\widetilde{F}$.

Note that if F is already strict, then $F = \tilde{F}$. Thus for every virtually strict functor F, there is a unique strictification $\widetilde{F}$.

Theorem 4.1.5. *There is a category* $\mathbf{Tricat}_v$ *with objects tricategories, morphisms strict functors, and composition given by*

$$F \circ_v G = \widetilde{F \circ G}.$$

Definition 4.1.6. Let B be a bicategory, and let $\boldsymbol{f}, \boldsymbol{g}$ be adjoint equivalences with f, g parallel 1-cells. Then an isomorphism α from $\boldsymbol{f}$ to $\boldsymbol{g}$ consists of isomorphisms $\alpha : f \Rightarrow g, \alpha^\bullet : f^\bullet \Rightarrow g^\bullet$ in B such that the diagrams below commute.

$$\begin{tikzcd}
 & y \arrow[dr, "f^\bullet"'] \arrow[dr, bend left=60, "g^\bullet"] \arrow[dr, phantom, bend left=30, "\Uparrow\alpha^\bullet"] & \\
x \arrow[ur, "f"'] \arrow[ur, bend left=60, "g"] \arrow[ur, phantom, bend left=30, "\alpha\Uparrow"] \arrow[rr, "1"'] \arrow[rr, phantom, shift left=4, "\Uparrow\eta_f"] & & x
\end{tikzcd}
=
\begin{tikzcd}
 & y \arrow[dr, "g^\bullet"] & \\
x \arrow[ur, "g"] \arrow[rr, "1"'] \arrow[rr, phantom, shift left=4, "\Uparrow\eta_g"] & & x
\end{tikzcd}$$

$$
\begin{tikzcd}
 & x \arrow[dr, "f"'] \arrow[dr, bend left=50, "g"] \arrow[d, phantom, "\varepsilon_f\Downarrow"] & \\
y \arrow[ur, "f^\bullet"] \arrow[ur, bend left=50, "g^\bullet"] \arrow[rr, "1"'] & {} & y
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & x \arrow[dr, "g"] \arrow[d, phantom, "\varepsilon_g\Downarrow"] & \\
y \arrow[ur, "g^\bullet"] \arrow[rr, "1"'] & {} & y
\end{tikzcd}
$$

with $\alpha^{-1}\Downarrow$ between $g^\bullet$ and $f^\bullet$, and $\Downarrow(\alpha^\bullet)^{-1}$ between f and g.

Remark 4.1.7. It is clear that α determines $\alpha^\bullet$, and that in fact $\alpha^\bullet$ is the mate of α^{-1}. The two conditions above follow immediately using the theory of mates.

Lemma 4.1.8. *Let F, G be composable strict functors. Then there is an isomorphism between $\boldsymbol{\chi}^{\boldsymbol{F\circ G}}$ and the identity adjoint equivalence, and there is an isomorphism between $\boldsymbol{\iota}^{\boldsymbol{F\circ G}}$ and the identity adjoint equivalence.*

Proof. These isomorphisms are given by unique coherence cells by the coherence for functors theorem since the component of $\chi^{F\circ G}$ at g, f is $F(1)\circ 1$ and similarly for the other transformations. $\square$

Proof of 4.1.5. Given composable strict functors F, G, we need only show that $F \circ G$ is virtually strict, that $1 \circ_v F = F = F \circ_v 1$, and that $\circ_v$ is associative. Associativity follows from the fact that there is at most one structure of a strict functor on a given map of bicategory-enriched graphs by the definition of strict functor. The unit conditions are trivial since $1 \circ F, F, F \circ 1$ all have the same underlying maps of **Bicat**-graphs.

We define ω^H to be the modification $\omega^{F\circ G}$, composed with the isomorphisms in the lemma as appropriate so as to obtain the correct source and target. By coherence for functors and the local strictness of the functors involved, this is equal to the diagram giving the definition of the modification ω for a strict functor. We similarly define δ, γ, and they are both equal by coherence to modifications of the sort given in the definition of a strict functor. Thus we have given a strict functor H and by definition it has the same underlying map of bicategory-enriched graphs as $F \circ G$, completing the proof. $\square$

Remark 4.1.9. The category $\mathbf{Tricat}_v$ is equivalent to the category of algebras for the free tricategory monad on the category of bicategory-enriched graphs and locally strict functors, although this point of view will not be studied here.

4.2 Structure of transformations

It will be necessary in later sections to understand some of the basic structure of transformations, so we collect in this section the relevant results. Most of the proofs are simple diagram chases, so we omit these details whenever possible.

Proposition 4.2.1. *Let $\alpha : F \to G$ and $\beta : G \to H$ be transformations. Then there is a transformation $\beta\alpha : F \to H$ with $(\beta\alpha)_a = \beta_a \otimes \alpha_a$.*

Sketch of proof. The adjoint equivalence $\beta\alpha$ is given by setting $(\beta\alpha)_f$ equal to the composite

$$\begin{gathered}(\beta_b \otimes \alpha_b) \otimes Ff \xrightarrow{a} \beta_b \otimes (\alpha_b \otimes Ff) \xrightarrow{1\otimes\alpha_f} \beta_b \otimes (Gf \otimes \alpha_a) \xrightarrow{a^\bullet} \\ (\beta_b \otimes Gf) \otimes \alpha_a \xrightarrow{\beta_f \otimes 1} (Hf \otimes \beta_a) \otimes \alpha_a \xrightarrow{a} Hf \otimes (\beta_a \otimes \alpha_a)\end{gathered}$$

and $(\beta\alpha)^\bullet_f$ is the obvious adjoint, with the unit and counit given by the obvious composition of constraint cells with units and counits for all of the adjoint equivalences involved. The definitions of Π and M are given by diagrams similar to those in Theorem 7.2.1 with additional coherence cells inserted where necessary. Checking the necessary axioms requires using the tricategory axioms in the target as well as the axioms for each transformation separately. $\square$

Proposition 4.2.2. *Let $F, F' : R \to S$ and $G, G' : S \to T$ be functors, and let $\alpha : F \to F'$, $\beta : G \to G'$ be transformations. Then there are transformations $\beta * 1_F : GF \to G'F$ and $1_G * \alpha : GF \to GF'$ whose components are given by β_{Fa} and $G\alpha_a$, respectively.*

Proof. We will only prove the statement for $\beta * 1_F$ as the other proof is analogous. The adjoint equivalences $\beta * 1_F$ are defined by the following formulas.

$$\begin{aligned}(\beta * 1_F)_f &= \beta_{Ff} \\ (\beta * 1_F)^\bullet_f &= \beta^\bullet_{Ff} \\ \varepsilon^{\beta * 1_F}_f &= \varepsilon^\beta_{Ff} \\ \eta^{\beta * 1_F}_f &= \eta^\beta_{Ff} \\ (\beta * 1_F)_\theta &= \beta_{F\theta}\end{aligned}$$

These define appropriate transformations and modifications since these cells are just components of β. The component at f, g of the invertible modification Π is given by the diagram below.

$$\begin{tikzcd}
(\beta_{Fc} \otimes GFg) \otimes GFf \arrow[r, "\beta_{Fg} \otimes 1"] \arrow[d, "a"'] & (G'Fg \otimes \beta_{Fb}) \otimes GFf \arrow[r, "a"] & G'Fg \otimes (\beta_{Fb} \otimes GFf) \arrow[r, "1 \otimes \beta_{Ff}"] & G'Fg \otimes (G'Ff \otimes \beta_{Fa}) \arrow[d, "a^\bullet"] \\
\beta_{Fc} \otimes (GFg \otimes GFf) \arrow[dr, "1 \otimes \chi^G"] \arrow[ddr, bend right, "1 \otimes \chi^{GF}"'] \arrow[rrr, phantom, "\Swarrow \Pi_{Fg,Ff}"] & & & (G'Fg \otimes G'Ff) \otimes \beta_{Fa} \arrow[dl, "\chi^{G'} \otimes 1"'] \arrow[ddl, bend left, "1 \otimes \chi^{G'F}"] \\
& \beta_{Fc} \otimes G(Fg \otimes Ff) \arrow[r, "\beta_{Fg \otimes Ff}"] \arrow[d, "1 \otimes G(\chi^F)"] \arrow[dr, phantom, "\Swarrow \beta^{-1}_{\chi^F}"] \arrow[d, phantom, shift right=8ex, "\cong"] & G'(Fg \otimes Ff) \otimes \beta_{Fa} \arrow[d, "G'(\chi^F) \otimes 1"'] \arrow[d, phantom, shift left=8ex, "\cong"] & \\
& \beta_{Fc} \otimes GF(g \otimes f) \arrow[r, "\beta_{F(g \otimes f)}"'] & G'F(g \otimes f) \otimes \beta_{Fa} &
\end{tikzcd}$$

The two isomorphisms are the composites of unit isomorphisms in the hom-bicategory with the functoriality isomorphism for $\otimes$.

For each object a, the single component of the invertible modification M is given by the diagram below.

$$\begin{tikzcd}
\beta_{Fa} \arrow[rr, "r^\bullet"] \arrow[dddrrr, bend right, "l^\bullet"'] & & \beta_{Fa}\otimes I_{GFa} \arrow[rr, "1\otimes\iota^{GF}"] \arrow[dr, "1\otimes\iota^{G}"'] & & \beta_{Fa}\otimes GFI_a \arrow[rr, "\beta_{FI_a}"] \arrow[dr, phantom, "\Downarrow\beta_{\iota^F}"] & & G'FI_a\otimes\beta_{Fa} \\
& & & \beta_{Fa}\otimes GI_{Fa} \arrow[ur, "1\otimes G\iota^F"'] \arrow[dr, "\beta_{I_{Fa}}"'] \arrow[u, phantom, "\cong"] \arrow[ddll, phantom, "\Downarrow M_a"] & & & \\
& & & & G'I_{Fa}\otimes\beta_{Fa} \arrow[uurr, "G'\iota^F\otimes 1"'] \arrow[rr, phantom, "\cong"] & & \\
& & & I_{G'Fa}\otimes\beta_{Fa} \arrow[ur, "\iota^{G'}\otimes 1"] \arrow[uuurrr, bend right, "\iota^{G'F}\otimes 1"'] & & &
\end{tikzcd}$$

The transformation axioms are now easy to check using that β is a transformation and the fact that all the coherence cells used in the definitions above are either those of the hom-bicategory or of the functor $\otimes$. $\square$

It will be necessary later to use associativity and unit transformations. If we were to construct the tetracategory **Tricat** from first principles, these transformations would be a necessary part of that structure.

Proposition 4.2.3. *1. Let $F : Q \to R, G : R \to S, H : S \to T$ be functors. Then there are transformations*

$$\begin{aligned} \alpha &: (H\circ G)\circ F \to H\circ(G\circ F) \\ \alpha^\bullet &: H\circ(G\circ F) \to (H\circ G)\circ F \end{aligned}$$

which have as their components at the object a the identity 1-cell I_{HGFa}.
2. Let id_T denote the identity functor on the tricategory T. Then there are transformations

$$\begin{aligned} \rho &: F\circ id \to F \\ \rho^\bullet &: F \to F\circ id \end{aligned}$$

which have as their components at the object a the identity 1-cell I_{Fa}.

Proof. We will only prove the first claim, as the second follows by analogous arguments. First note that $(H\circ G)\circ F$ and $H\circ(G\circ F)$ have the same underlying map on cells. The components are the identity cells I_{HGFa}, and the transformations α are defined by the formulas below.

$$\begin{aligned} \alpha_f &= r^\bullet \circ l \\ \alpha^\bullet_f &= l^\bullet \circ r \end{aligned}$$

The unit and counit of this adjoint equivalence are given by the obvious composites of constraint cells in the target hom-bicategory and units and counits of the adjoint equivalences $\mathbf{l}, \mathbf{r}$. The naturality isomorphism α_θ is given by the composite of the naturality constraints for the transformations involved. Thus this adjoint equivalence is just the adjoint equivalence $r^\bullet\circ l \dashv l^\bullet\circ r$.

The component at f, g of the invertible modification Π is given by the diagram below.

$$
\begin{tikzcd}
(I\otimes HGFg)\otimes HGFf \arrow[rr, "(r^\bullet l)\otimes 1"] \arrow[dr, "l\otimes 1"] \arrow[ddd, "a"'] \arrow[dddr, phantom, "\Downarrow\lambda"] & & (HGFg\otimes I)\otimes HGFf \arrow[rr, "a"] \arrow[drr, phantom, "\Downarrow\mu"] & & HGFg\otimes(I\otimes HGFf) \arrow[rrr, "1\otimes(r^\bullet l)"] \arrow[dr, "1\otimes l"] & & & HGFf\otimes(HGFf\otimes I) \arrow[d, "a^\bullet"] \\
& HGFg\otimes HGFf \arrow[ur, "r^\bullet\otimes 1"'] \arrow[rrrr, "1"'] \arrow[u, phantom, "\cong"] \arrow[ddrr, phantom, "\cong"] & & & & HGFg\otimes HGFf \arrow[urr, "1\otimes r^\bullet"] \arrow[drr, "r^\bullet"'] \arrow[ddl, "\chi^H"'] \arrow[u, phantom, "\cong"] \arrow[rr, phantom, "\Downarrow\rho"] & & \\
& & & & & & & (HGFg\otimes HGFf)\otimes I \arrow[ddl, "\chi^H\otimes 1"'] \arrow[ddddd, "\chi^{H(GF)}\otimes 1"] \\
I\otimes(HGFg\otimes HGFf) \arrow[uur, "l"'] \arrow[rr, "1\otimes\chi^H"'] \arrow[ddr, "1\otimes\chi^{HG}"'] \arrow[ddddr, bend right, "1\otimes\chi^{(HG)F}"'] \arrow[ddrr, phantom, "\cong"] \arrow[dddd, phantom, "\cong"] & & I\otimes H(GFg\otimes GFf) \arrow[rr, "l"] \arrow[ddl, "1\otimes H\chi^G"] \arrow[dd, phantom, "\cong"] & & H(GFg\otimes GFf) \arrow[drr, "r^\bullet"] \arrow[ddl, "H\chi^G"'] \arrow[ur, phantom, "\cong"] \arrow[dd, phantom, "\cong"] & & & \\
& & & & & & H(GFg\otimes GFf)\otimes I \arrow[ddl, "H\chi^G\otimes 1"'] \arrow[dddr, "H\chi\otimes 1"] \arrow[r, phantom, "\cong"] \arrow[dd, phantom, "\cong"] & \\
& I\otimes HG(Fg\otimes Ff) \arrow[rr, "l"'] \arrow[dd, "1\otimes HG\chi^F"] \arrow[ddrr, phantom, "\cong"] & & HG(Fg\otimes Ff) \arrow[drr, "r^\bullet"'] \arrow[ddr, "HG\chi"] \arrow[ddrr, phantom, "\cong"] & & & & \\
& & & & & HG(Fg\otimes Ff)\otimes I \arrow[drr, "HG\chi^F"'] & & \\
& I\otimes HGF(g\otimes f) \arrow[rrr, "l"'] & & & HGF(g\otimes f) \arrow[rrr, "r^\bullet"'] & & & HGF(g\otimes f)\otimes I
\end{tikzcd}
$$

The unmarked isomorphisms are either given by the composite of a unit constraint from the hom-bicategory with the functoriality constraint of $\otimes$ in the case of the triangular regions, or by a naturality isomorphism in the case of the square regions. Note that we actually require a mate of ρ and not ρ itself in the upper right corner.

For each object a, the single component of the invertible modification M is given by the diagram below.

$$
\begin{tikzcd}
I \arrow[r, "r^{\bullet}"] \arrow[ddr, "1"'] \arrow[dddrr, bend right=60, "l^{\bullet}"'] \arrow[dr, phantom, "\cong"] \arrow[ddr, phantom, shift right=6ex, "C"] & I\otimes I \arrow[r, "1\otimes\iota^{(HG)F}"] \arrow[d, "1\otimes\iota"] \arrow[dd, bend right, "l"'] \arrow[dr, phantom, "\cong"] & I\otimes HGFI \arrow[r, "l"] \arrow[dr, phantom, "\cong"] & HGFI \arrow[r, "r^{\bullet}"] \arrow[dr, phantom, "\cong"] & HGFI\otimes I \\
& I\otimes HI \arrow[r, "1\otimes H\iota"] \arrow[dr, "l"] \arrow[d, phantom, "\cong"] & I\otimes HGI \arrow[u, "1\otimes HG\iota"] \arrow[r, "l"] \arrow[d, phantom, "\cong"] & HGI \arrow[u, "HG\iota"'] \arrow[r, "r^{\bullet}"] \arrow[d, phantom, "\cong"] & HGI\otimes I \arrow[u, "HG\iota\otimes 1"] \arrow[d, phantom, "C"] \\
& I \arrow[r, "\iota"] \arrow[dr, "r^{\bullet}"'] & HI \arrow[r, "r^{\bullet}"'] \arrow[ur, "H\iota"] \arrow[d, phantom, "\cong"] & HI\otimes I \arrow[ur, "H\iota\otimes 1"] & \\
& & I\otimes I \arrow[ur, "\iota\otimes 1"'] \arrow[uuurr, bend right=60, "\iota^{H(GF)}\otimes 1"'] & &
\end{tikzcd}
$$

The two regions marked C have isomorphisms given by composites of unit isomorphisms with the functoriality constraint for $\otimes$, the isomorphisms $lr^{\bullet} \cong 1$ and $r^{\bullet}1 \cong l^{\bullet}$ are mates of the isomorphism $l_I \cong r_I$ in Appendix A (composed with a unit in the latter case), and all the other isomorphisms are naturality isomorphisms.

The three transformation axioms can now be checked by lengthy calculation. See Appendix C for a discussion of the proof of this and other omitted calculations. □

Our final result about transformations concerns the two different compositions available for strict functors, $G \circ F$ and $G \circ_v F$. It will be necessary later to know that there are transformations relating these two functors, so we construct them here.

Proposition 4.2.4. *Let $F : R \to S$ and $G : S \to T$ be strict functors. Then there are transformations*

$$
\begin{aligned}
\phi &: G \circ_v F \to G \circ F \\
\phi^{\bullet} &: G \circ F \to G \circ_v F
\end{aligned}
$$

whose components are identities.

Proof. We only provide the details for ϕ as $\phi^{\bullet}$ is similar. As stated, we have defined the components by $\phi_a = I_{GFa}$, and we define the adjoint equivalences ϕ to be the same ones used in the previous proposition, $r^{\bullet} \circ l \dashv l^{\bullet} \circ r$.

The component at f, g of the invertible modification Π is given by the diagram below.

$$
\begin{tikzcd}
(I\otimes GFg)\otimes GFf \arrow[rr, "(r^\bullet l)\otimes 1"] \arrow[dddd, "a"'] \arrow[ddrr, "l\otimes 1"'] & & (GFg\otimes I)\otimes GFf \arrow[rr, "a"] \arrow[dd, phantom, "\Downarrow\mu"] & & GFg\otimes(I\otimes GFf) \arrow[rr, "1\otimes(r^\bullet l)"] \arrow[dddd, "1\otimes l"'] & & GFg\otimes(GFf\otimes I) \arrow[dr, "a^\bullet"] \arrow[dd, phantom, "\Downarrow\rho"] & \\
 & \cong & & & & \cong & & (GFg\otimes GFf)\otimes I \arrow[ddl, "1\otimes 1"'] \arrow[ddd, bend left, "\chi^{G\circ F}\otimes 1"] \\
 & \arrow[l, phantom, "\Leftarrow\lambda"] & GFg\otimes GFf \arrow[uu, "r^\bullet\otimes 1"'] \arrow[ddrr, "1"] & & & & & \\
 & & \cong & & & \cong & G(Fg\otimes Ff)\otimes I \arrow[dr, "G1\otimes 1"'] & \cong \\
I\otimes(GFg\otimes GFf) \arrow[rr, "1\otimes 1"'] \arrow[uurr, "l"] & & I\otimes GF(g\otimes f) \arrow[rr, "l"'] & & GF(g\otimes f) \arrow[rrr, "r^\bullet"'] \arrow[uuuurr, "1\otimes r^\bullet"] \arrow[uuurrr, "r^\bullet"] & & & GF(g\otimes f)\otimes I
\end{tikzcd}
$$

The cell marked ρ is actually a mate of ρ, and the unmarked isomorphisms are one naturality isomorphism for l and composites of unit isomorphisms in the hom-bicategories, functoriality isomorphisms for $\otimes$, and the isomorphisms $1 \otimes 1 \cong 1$, $G1 \cong 1$. Note that we have used repeatedly that χ^F, χ^G, and $\chi^{G\circ_v F}$ are all identity transformations.

For each object a, the single component of the invertible modification M is given by the diagram below.

$$
\begin{tikzcd}
I \arrow[r, "r^\bullet"] \arrow[rrr, bend right, "1"'] \arrow[ddrr, bend right=40, "l^\bullet"'] & I\otimes I \arrow[r, "1\otimes 1"] \arrow[d, phantom, "\cong"] & I\otimes I \arrow[r, "l"] & I \arrow[r, "r^\bullet"] & I\otimes I \\
 & & \cong & I\otimes I \arrow[ur, "G1\otimes 1"] & \cong \\
 & & I\otimes I \arrow[ur, "1\otimes 1"] \arrow[uurr, bend right, "\iota^{G\circ F}\otimes 1"'] & &
\end{tikzcd}
$$

The lower right isomorphism is the composite of a unit isomorphism and the functoriality isomorphism for $\otimes$; the upper right isomorphism is the composite of the isomorphism $1 \otimes 1 \cong 1$, a unit isomorphism, and a mate of the isomorphism $l_I \cong r_I$; and the middle isomorphism is the composite of the isomorphism $G1 \cong 1$, two copies of $1 \otimes 1 \cong 1$, three unit isomorphisms, and a mate of $l_I \cong r_I$. Note that we have used repeatedly that F, G, and $G \circ_v F$ preserve units strictly and that ι is the identity transformation for these functors.

The three transformation axioms now follow by calculation. $\square$

4.3 Change of structure

This section will give three results, each of which explains how it is possible to obtain new tricategory structures from known ones. The first result shows how to transport a tricategory structure along a map of its underlying data.

This is the first step towards showing that every tricategory is triequivalent to a particular kind of semi-strict 3-category. The theorem given here will be used repeatedly to construct tricategory structures throughout this work. The second and third result of this section show how to perturb a known tricategory structure by altering its composition law. The result is a new tricategory structure on the same cells that is closely related to the original structure.

For the following theorem, we require the notion of a biadjoint biequivalence in a tricategory T. The definition can be found in Appendix A.

Theorem 4.3.1 (Transport of Structure)**.** *Let T be a tricategory, and let S be a set. Let $S(a,b)$ be an $S\times S$-indexed set of bicategories. Given a function $H_0 : S \to obT$ and an $S\times S$-indexed set of biadjoint biequivalences $(H_{ab}, H^{\bullet}_{ab})$,*

$$H_{ab} : S(a,b) \to T(Ha, Hb),$$

there is a unique tricategory structure on S and a unique functor H that agrees with H_0 on objects and H_{ab} on hom-bicategories such that the following conditions hold.

1. *The functor $\otimes : S(b,c)\times S(a,b) \to S(a,c)$ is the composite*

$$S(b,c)\times S(a,b) \xrightarrow{H\times H} T(Hb,Hc)\times T(Ha,Hb) \xrightarrow{\otimes} T(Ha,Hc) \xrightarrow{H^{\bullet}} S(a,c).$$

2. *The transformation χ is*

$$\otimes^T \circ H\times H = id \circ \otimes^T \circ H \times H \xrightarrow{\alpha^{\bullet}*1} HH^{\bullet}\otimes^T (H\times H) = H\circ \otimes^S,$$

and the transformation $\chi^{\bullet}$ is

$$H\circ\otimes^S = HH^{\bullet}\otimes^T (H\times H) \xrightarrow{\alpha*1} id \otimes^T (H\times H) = \otimes^T \circ H\times H.$$

The counit of this adjunction $\chi \dashv \chi^{\bullet}$ is the following composite.

$$\begin{tikzcd}
HH^{\bullet}\otimes^T (H\times H) \arrow[r, "\alpha*1"] & \otimes^T H\times H \arrow[r, "\alpha^{\bullet}*1"] \arrow[d, equal] & HH^{\bullet}\otimes^T (H\times H) \\
HH^{\bullet}\otimes^T (H\times H) \arrow[r, "(\alpha^{\bullet}\alpha)*1"] \arrow[d, phantom, ""] & HH^{\bullet}\otimes^T (H\times H) \arrow[d, phantom, "\Downarrow \Gamma^{-1}*1"] & \\
HH^{\bullet}\otimes^T (H\times H) \arrow[r, "1"] & HH^{\bullet}\otimes^T (H\times H) &
\end{tikzcd}$$

The unit is determined similarly, and a check shows that this gives an adjoint equivalence in the appropriate bicategory.

3. *The functor $\mathbf{1} \to S(a,a)$ is the composite*

$$\mathbf{1} \xrightarrow{I_{Ha}} T(Ha,Ha) \xrightarrow{H^{\bullet}} S(a,a).$$

4. *The transformation ι is*

$$I_{Ha} = id \circ I_{Ha} \xrightarrow{\alpha^{\bullet} * 1} HH^{\bullet} I_{Ha}$$

and $\iota^{\bullet}$ is

$$HH^{\bullet} I_{Ha} \xrightarrow{\alpha * 1} id \circ I_{Ha} = I_{Ha}.$$

The counit of this adjunction $\iota \dashv \iota^{\bullet}$ is given by the composite below.

$$\begin{tikzcd}
I_{Ha} \arrow[r, "\alpha * 1"] & HH^{\bullet} I_{Ha} \arrow[r, "\alpha^{\bullet} * 1"] \arrow[d, phantom, "\|"] & I_{Ha} \\
I_{Ha} \arrow[rr, "(\alpha^{\bullet}\alpha) * 1"] \arrow[d, phantom, ""] & \arrow[d, phantom, "\Downarrow \underline{\Gamma}^{-1} * 1"] & I_{Ha} \\
I_{Ha} \arrow[rr, "1"] & & I_{Ha}
\end{tikzcd}$$

The unit is determined similarly, and a check shows that this is an adjoint equivalence in the appropriate bicategory.

5. *The modifications ω, γ, and δ for the functor H are all identities.*

Proof. We have provided the first four pieces of data directly. The rest of the data for the tricategory S is determined by (5) as follows. The modification π is determined by the first functor axiom and the fact that each H_{ab} is locally faithful, and the modification μ is determined by the second functor axiom. The second and third tricategory axioms then determine λ and ρ, and the first tricategory axiom follows by applying H, using the tricategory axioms in T, and then noting that each H_{ab} is locally faithful. $\square$

Our next result shows how it is possible to change the composition law of a tricategory to a new composition law.

Theorem 4.3.2 (Change of Composition)**.** *Let T be a tricategory with composition $\otimes$. Let $\boxtimes_{abc} : T(b,c) \times T(a,b) \to T(a,c)$ be a family of functors indexed by triples of objects of T, and let $\mathbf{s}_{abc} : \otimes \Rightarrow \boxtimes$ be a similarly indexed family of adjoint equivalences. Then there is a tricategory $T_{\boxtimes}$ with*

- *$obT_{\boxtimes} = obT$,*
- *$T_{\boxtimes}(a,b) = T(a,b)$, and*
- *composition law $\boxtimes_{abc} : T(b,c) \times T(a,b) \to T(a,c)$*

and a functor $S : T \to T_{\boxtimes}$ which is the identity on objects and on hom-bicategories.

Proof. We need to provide the remaining data for $T_\boxtimes$ and show that it satisfies the tricategory axioms. First, we specify that $T_\boxtimes$ has the same unit as T. The transformation $a_\boxtimes$ is given by

$$\boxtimes \circ (\boxtimes \times 1) \xrightarrow{s*(s\times 1)} \otimes \circ (\otimes \times 1) \xrightarrow{a} \otimes \circ (1 \times \otimes) \xrightarrow{s^\bullet *(1\times s^\bullet)} \boxtimes \circ (1 \times \boxtimes),$$

and $a^\bullet_\boxtimes$ is given by

$$\Big(\big(s * (1 \times s)\big) \circ a^\bullet\Big) \circ \big(s^\bullet * (s^\bullet \times 1)\big).$$

The unit and counit of this adjoint equivalence are the obvious composites of units and counits for $\mathbf{a}$ and $\mathbf{s}$.

Similarly, $\mathbf{l}_\boxtimes$ and $\mathbf{r}_\boxtimes$ are defined by the diagrams below, where $\mathbf{s}^\bullet$ is the opposite adjoint equivalence of $\mathbf{s}$.

$$\begin{tikzcd}
T(b,b)\times T(a,b) \arrow[dr, "\otimes"] \arrow[dr, bend right=60, "\boxtimes"'] \arrow[dr, phantom, bend right=25, "\Uparrow \mathbf{s}^\bullet"'] & T(a,b) \arrow[l, "I_b \times 1"'] \arrow[d, "1"] \arrow[d, phantom, shift right=6ex, "\Rightarrow \mathbf{l}"] \\
 & T(a,b)
\end{tikzcd}
\quad
\begin{tikzcd}
T(a,b)\times T(a,a) \arrow[dr, "\otimes"] \arrow[dr, bend right=60, "\boxtimes"'] \arrow[dr, phantom, bend right=25, "\Uparrow \mathbf{s}^\bullet"'] & T(a,b) \arrow[l, "1 \times I_a"'] \arrow[d, "1"] \arrow[d, phantom, shift right=6ex, "\Rightarrow \mathbf{r}"] \\
 & T(a,b)
\end{tikzcd}$$

The modifications $\pi_\boxtimes, \mu_\boxtimes, \lambda_\boxtimes, \rho_\boxtimes$ are all obtained by pasting appropriate identity modifications for the transformations $s\times 1\times 1, s\times 1, s, s^\bullet, 1\times s^\bullet, 1\times 1\times s^\bullet$ to the exterior of π, μ, λ, ρ after applying inverses of units for each of the adjoint equivalences $1 \times \mathbf{s}, \mathbf{s} \times 1, 1 \times \mathbf{s} \times 1, \mathbf{s}$ and unit isomorphisms (from the functor bicategories) where appropriate.

Using this definition and the fact that $\boldsymbol{s}$ is an adjoint equivalence, it is a simple matter to check the three tricategory axioms.

For the final claim, we need to give the constraint data for S. The adjoint equivalence χ is the adjoint equivalence s, and the adjoint equivalence ι is the identity adjoint equivalence. The component at h, g, f of the invertible modification ω is given by the diagram below.

$$\begin{tikzcd}
(h\boxtimes g)\boxtimes f \arrow[r, "s\boxtimes 1"] \arrow[d, "s"'] & (h\otimes g)\boxtimes f \arrow[r, "s"] \arrow[d, phantom, "\cong"] & (h\otimes g)\otimes f \arrow[r, "a"] & h\otimes (g\otimes f) \\
(h\boxtimes g)\otimes f \arrow[d, "s\otimes 1"'] & & \arrow[l, phantom, "\cong"] & h\boxtimes (g\otimes f) \arrow[u, "s"'] \arrow[dl, phantom, "\cong"] \\
(h\otimes g)\otimes f \arrow[r, "a"'] \arrow[uurr, "1"] & h\otimes (g\otimes f) \arrow[r, "s^\bullet"'] \arrow[uurr, "1"'] & h\boxtimes (g\otimes f) \arrow[r, "1\boxtimes s^\bullet"'] & h\boxtimes (g\boxtimes f) \arrow[u, "1\boxtimes s"']
\end{tikzcd}$$

The left isomorphism is the composite of a unit isomorphism for the hom-bicategory with a naturality isomorphism for s, the middle isomorphism is the composite of two unit isomorphisms for the hom-bicategory, and the right isomorphism is the composite of inverses of counits and unit isomorphisms. The component of the invertible modification γ at f is given by composing the isomorphism $1 \boxtimes 1 \cong 1$ with a unit isomorphism in the hom-bicategory; δ is defined similarly.

It is now easy to check the functor axioms using the fact that $\boldsymbol{s}$ is an adjoint equivalence. □

Finally, we introduce a result that allows one to alter the units in a tricategory, in much the same way that the previous result allowed a change in the composition law. We will not prove this, as the details are similar to those in the previous proof.

Theorem 4.3.3 (Change of Units)**.** *Let T be a tricategory with units $I_a : 1 \to T(a,a)$. Let $\tilde{I}_a : 1 \to T(a,a)$ be a collection of functors indexed by the objects of T, and let $\boldsymbol{r}_a$ be a similarly indexed collection of adjoint equivalences between I_a and $\tilde{I}_a$. Then there is a tricategory $T_{\tilde{I}}$ with*

- *$obT_{\tilde{I}} = obT$,*
- *$T_{\tilde{I}}(a,b) = T(a,b)$, and*
- *unit given by $\tilde{I}_a : 1 \to T(a,a)$*

and a functor $R : T \to T_{\tilde{I}}$ that is the identity on objects and hom-bicategories.

4.4 Triequivalences

This section will introduce the notion of triequivalence. It is a direct categorification of the notion of equivalence of categories. We replace the condition of the functor being an isomorphism on hom-sets with being a biequivalence on hom-bicategories, and replace essential surjectivity with the notion of triessential surjectivity. This in turn relies on the notion of an internal biequivalence in a tricategory T.

Definition 4.4.1. 1. A 1-cell $f : a \to b$ in a tricategory T is an *internal biequivalence* if there exists a 1-cell $g : b \to a$ such that $f \otimes g$ is equivalent to id_b in the bicategory $T(b,b)$ and $g \otimes f$ is equivalent to id_a in the bicategory $T(a,a)$.
2. A *specified biequivalence* in a tricategory T consists of

- a pair of 1-cells $f : a \to b$ and $g : b \to a$;
- four 2-cells $\alpha : f \otimes g \Rightarrow \mathrm{id}_b$, $\alpha^\bullet : \mathrm{id}_b \Rightarrow f \otimes g$, $\beta : g \otimes f \Rightarrow \mathrm{id}_a$, and $\beta^\bullet : \mathrm{id}_a \Rightarrow g \otimes f$;
- and two specified equivalences $(\alpha, \alpha^\bullet, \varepsilon_{fg}, \eta_{fg})$ and $(\beta, \beta^\bullet, \varepsilon_{gf}, \eta_{gf})$ in $T(b,b)$ and $T(a,a)$, respectively.

Remark 4.4.2. Note that a 1-cell f is a biequivalence if and only if there exists a specified biequivalence containing f. It is also useful to note that f is a biequivalence if and only if there exists a specified biequivalence containing f such that each of $(\alpha, \alpha^{\bullet}, \varepsilon_{fg}, \eta_{fg})$ and $(\beta, \beta^{\bullet}, \varepsilon_{gf}, \eta_{gf})$ are adjoint equivalences.

The biadjoint biequivalences mentioned in the previous section have underlying specified biequivalences.

Definition 4.4.3. A functor $H : T \to T'$ is *triessentially surjective* if every object of T' is internally biequivalent to an object of the form Ha, $a \in T$.

Definition 4.4.4. A functor $H : T \to T'$ is a *triequivalence* if each H_{ab} is a biequivalence and H is triessentially surjective.

Remark 4.4.5. The functors $S : T \to T_{\boxtimes}, R : T \to T_{\tilde{I}}$ constructed in the previous section are triequivalences.

Theorem 4.4.6. *Every tricategory T is triequivalent to a tricategory T' with the same objects as T and $T'(a, b)$ a strict 2-category for all objects a, b.*

Proof. For each pair of objects $a, b \in T$, we can choose a biadjoint biequivalence $T'(a, b) \to T(a, b)$ with $T'(a, b)$ a strict 2-category using the coherence theorem for bicategories. By Proposition 4.3.1, we extend this to a tricategory T' and a functor $T' \to T$. It is clear that this is a triequivalence. $\square$

Chapter 5

The Gray tensor product

In this chapter, we will give the necessary background needed on the Gray tensor product of 2-categories. The discussion consists of three parts. In the first, we give the "generators and relations" definition of the Gray tensor product. We will rarely need this definition, but it is useful to know. In the second part, we will give the relationship between the Gray tensor product $A \otimes B$ and cubical functors with domain $A \times B$. Finally, we will discuss the right adjoint of the functor $- \otimes B$ which turns out to be very easy to describe.

Throughout this chapter, we shall always deal with strict 2-categories. Functor will, as always, mean weak functor; strict 2-functors will always be called such.

Nothing in this chapter is new, we have merely collected the required results. The main references are Gray's works [18] and [19], although the handwritten notes of Street [40] provide another perspective.

5.1 The Gray tensor product

Our goal in describing the Gray tensor product of 2-categories will be to use the resulting monoidal structure as a category over which to enrich. The resulting objects, categories enriched over **2Cat** with the Gray tensor product, will be a semi-strict form of 3-category used in our coherence theorem. It is possible to define this tensor using only a universal property, but we prefer to define it from the ground up and show later that it satisfies a universal property.

The Gray tensor product of X and Y, denoted $X \otimes Y$, has objects ordered pairs (A, B), where $A \in \mathrm{ob}X$ and $B \in \mathrm{ob}Y$. The morphisms of $X \otimes Y$ are generated by two kinds of morphisms. The first type of generator is an ordered pair of the form $(f, 1) : (A, B) \to (A', B)$ with $f : A \to A'$ a morphism of X; the second type is $(1, g) : (A, B) \to (A, B')$ with $g : B \to B'$ a morphism of Y. The morphisms of $X \otimes Y$ are equivalence classes of composable strings of these two types of generators. The equivalence relation is the smallest one such that the following conditions hold, when they make sense.

- $(f,1)(f',1) \sim (ff',1)$
- $(1,g)(1,g') \sim (1,gg')$
- If w, w' are two equivalent strings, then $wv \sim w'v$ and $uw \sim uw'$.

Note that if $w \sim w'$, then w and w' have the same source and target.

The 2-cells of $X \otimes Y$ are formed in a similar, but slightly more complicated manner. There are three basic types of generating 2-cells, and a 2-cell in the tensor product is an equivalence class of composites, vertical and horizontal, of these basic 2-cells. The first type of 2-cell is one of the form $(\alpha,1) : (f,1) \Rightarrow (f',1)$ where $\alpha : f \Rightarrow f'$ is a 2-cell in X. The second type of 2-cell is one of the form $(1,\beta) : (1,g) \Rightarrow (1,g')$ where $\beta : g \Rightarrow g'$ is a 2-cell in Y. The third kind of 2-cell is an isomorphism $\gamma_{f,g} : (f,1)(1,g) \Rightarrow (1,g)(f,1)$, with inverse $\gamma_{f,g}^{-1} : (1,g)(f,1) \Rightarrow (f,1)(1,g)$, where both f and g are non-identity morphisms in their respective 2-categories. If either f or g is the identity, then $\gamma_{f,g}$ is the identity. We now form equivalence classes of formal composites of such 2-cells in two steps. First, we compose them horizontally with the same conditions we placed on composing 1-cells. Second, we compose them vertically and impose conditions like the ones above and additional ones to force the resulting structure to be a 2-category.

First we deal with horizontal composition. Let w, w' be strings of the three basic types of generating 2-cells in $X \otimes Y$. Then $w \sim w'$ if they are made so by the smallest equivalence relation such that the following conditions hold, when they make sense.

- $(\alpha,1) * (\alpha',1) \sim (\alpha * \alpha',1)$
- $(1,\beta) * (1,\beta') \sim (1,\beta * \beta')$
- If σ, σ' are two equivalent strings, then $\sigma * \tau \sim \sigma' * \tau$ and $\rho * \sigma \sim \rho * \sigma'$.

Note that if $\sigma \sim \sigma'$, then σ and σ' have the same source and target 0-cells. We shall denote these equivalence classes by $[\sigma], [\tau]$, etc.

A 2-cell in $X \otimes Y$ is then an equivalence class of vertically composable strings $[\alpha_1][\alpha_2]\cdots[\alpha_n]$, where the equivalence relation is the smallest one such that the following conditions hold, when they make sense.

- $(\alpha,1)(\alpha',1) \sim (\alpha\alpha',1)$
- $(1,\beta)(1,\beta') \sim (1,\beta\beta')$
- $\Big(\gamma_{f',g} * (1_f,1)\Big)\Big((1_{f'},1) * \gamma_{f,g}\Big) \sim \gamma_{f'f,g}$
- $\Big((1,1_{g'}) * \gamma_{f,g}\Big)\Big(\gamma_{f,g'} * (1,1_g)\Big) \sim \gamma_{f,g'g}$
- $\Big((1,1_{g'}) * (1_{f'},1) * \gamma_{f,g}\Big)\Big(\gamma_{f',g'} * (1_f,1) * (1,1_g)\Big)$
  $\sim \Big(\gamma_{f',g'} * (1_f,1) * (1,1_g)\Big)\Big((1,1_{g'}) * (1_{f'},1) * \gamma_{f,g}\Big)$

- If $\alpha : f \Rightarrow f'$ and $\beta : g \Rightarrow g'$ in X and Y, respectively, then $\Big((1,\beta) * (\alpha,1)\Big)\gamma_{f,g} \sim \gamma_{f',g'}\Big((\alpha,1) * (1,\beta)\Big)$.
- If $[\alpha] \sim [\alpha']$, then $[\alpha][\beta] \sim [\alpha'][\beta]$ and $[\delta][\alpha] \sim [\delta][\alpha']$; the same condition holds for horizontal composition as described below.

It is now easy to write down horizontal and vertical composition of such equivalence classes. For vertical composition, we have concatenation of strings. For horizontal composition, let w be represented by $[\alpha_1][\alpha_2]\cdots[\alpha_n]$. Note that the 0-cell source and target of w can be computed by taking the 0-cell source and target of any of the α_i. Thus if w' is represented by $[\alpha'_1][\alpha'_2]\cdots[\alpha'_m]$ and has the same 0-cell source as w's 0-cell target, we make the following construction. If $m < n$, insert $n - m$ vertical identity 2-cells in any way into w'; we write the resulting string as $[\tilde{\alpha}_1][\tilde{\alpha}_2]\cdots[\tilde{\alpha}_n]$ and define $w * w'$ to be the equivalence class of $([\alpha_1] * [\tilde{\alpha}_1])([\alpha_2] * [\tilde{\alpha}_2])\cdots([\alpha_n] * [\tilde{\alpha}_n])$. If $m \geq n$, we perform a similar construction on w. It is easy to show that this equivalence class is independent of how the identities were inserted.

We omit the details that $X \otimes Y$ forms a 2-category. The only difficult axiom to check is the interchange law, and the conditions required two paragraphs previously force this to hold. We also omit the details that the above tensor product gives a monoidal structure on **2Cat**. This monoidal structure has a symmetry defined on generating objects and 1-cells by switching the order, on generating 2-cells of the form $(\alpha, 1)$ or $(1, \beta)$ by switching the order, and on generating 2-cells of the form $\gamma_{f,g}$ as $\gamma^{-1}_{g,f}$. Additionally, this monoidal structure is closed, with an adjoint hom-functor to be determined later.

5.2 Cubical functors

In this section, we present a different perspective on the Gray tensor product using cubical functors. This is in analogy with the definition of the usual tensor product of R-modules, in which the module $A \otimes_R B$ is the target of a universal bilinear map $A \times B \to A \otimes_R B$. The Gray tensor product $X \otimes Y$ will receive a universal cubical functor $X \times Y \to X \otimes Y$. We first define cubical functors of n variables, describe them in elementary terms, and then prove the above universal property.

5.2.1 Defining cubical functors

Definition 5.2.1. A functor $F : A_1 \times A_2 \times \cdots A_n \to B$ is *cubical* if the following condition holds:
if $(f_1, f_2, \ldots, f_n)(g_1, g_2, \ldots, g_n)$ is a composable pair of morphisms in the 2-category $A_1 \times A_2 \times \cdots A_n$ such that for all $i > j$, either g_i or f_j is an identity map, then the comparison 2-cell

$$\phi : F(f_1, f_2, \ldots, f_n)F(g_1, g_2, \ldots, g_n) \Rightarrow F\Big((f_1, f_2, \ldots, f_n)(g_1, g_2, \ldots, g_n)\Big)$$

is an identity.

First, note that every cubical functor strictly preserves identity 1-cells. This follows from the unit axioms for a functor and the fact that the 2-cell

$$\phi_{fI} : Ff \circ FI \Rightarrow F(f \circ I)$$

is always an identity 2-cell (similarly for ϕ_{If}) since it satisfies the cubical condition. For the case $n = 1$, a cubical functor is trivially a strict 2-functor.

Proposition 5.2.2. *A cubical functor $F : A_1 \times A_2 \to B$ determines, and is uniquely determined by*

1. *For each object $a_1 \in obA_1$, a strict 2-functor F_{a_1};*

2. *For each object $a_2 \in obA_2$, a strict 2-functor F_{a_2};*

3. *For each pair of objects a_1, a_2 in A_1, A_2, respectively, the equation*

$$F_{a_1}(a_2) = F_{a_2}(a_1) := F(a_1, a_2)$$

   holds;

4. *For each pair of 1-cells $f_1 : a_1 \to a_1'$, $f_2 : a_2 \to a_2'$ in A_1, A_2, respectively, a 2-cell isomorphism*

$$\begin{tikzcd} F(a_1, a_2) \arrow[r, "F_{a_1}(f_2)"] \arrow[d, "F_{a_2}(f_1)"'] & F(a_1, a_2') \arrow[d, "F_{a_2'}(f_1)"] \arrow[dl, Rightarrow, "\gamma_{f_1,f_2}"', "\cong"] \\ F(a_1', a_2) \arrow[r, "F_{a_1'}(f_2)"'] & F(a_1', a_2') \end{tikzcd}$$

   which is an identity 2-cell if either f_1 or f_2 is an identity 1-cell;

subject to the following 3 axioms for all diagrams of the form

$$\begin{tikzcd} (a_1, a_2) \arrow[r, bend left, "(f_1,f_2)"] \arrow[r, bend right, "(g_1,g_2)"'] \arrow[r, phantom, "\Downarrow(\alpha_1,\alpha_2)"] & (a_1', a_2') \arrow[r, "(h_1,h_2)"] & (a_1'', a_2'') \end{tikzcd}$$

in $A_1 \times A_2$.

$$
\begin{tikzcd}
F(a_1,a_2) \arrow[r, bend left=50, "F_{a_1}(f_2)"] \arrow[r, "F_{a_1}(g_2)"'] \arrow[r, phantom, shift left=2.5ex, "\Downarrow F_{a_1}\alpha_2"] \arrow[d, bend right=50, "F_{a_2}(g_1)"'] \arrow[d, bend left=50, "F_{a_2}(f_1)"] \arrow[d, phantom, "\Leftarrow F_{a_2}\alpha_1"] \arrow[dr, phantom, "\Downarrow\gamma"] & F(a_1,a_2') \arrow[d, "F_{a_2'}(f_1)"] \\
F(a_1',a_2) \arrow[r, "F_{a_1'}(g_2)"'] & F(a_1',a_2')
\end{tikzcd}
=
\begin{tikzcd}
F(a_1,a_2) \arrow[r, "F_{a_1}(f_2)"] \arrow[d, "F_{a_2}(g_1)"'] \arrow[dr, phantom, "\Downarrow\gamma"] & F(a_1,a_2') \arrow[d, bend right=50, "F_{a_2'}(g_1)"'] \arrow[d, bend left=50, "F_{a_2'}(f_1)"] \arrow[d, phantom, "\Leftarrow F_{a_2'}\alpha_1"] \\
F(a_1',a_2) \arrow[r, "F_{a_1'}(f_2)"] \arrow[r, bend right=50, "F_{a_1'}(g_2)"'] \arrow[r, phantom, shift right=2.5ex, "\Downarrow F_{a_1'}\alpha_2"] & F(a_1',a_2')
\end{tikzcd}
$$

$$
\begin{tikzcd}
F(a_1,a_2) \arrow[r, "F_{a_1}(f_2)"] \arrow[d, "F_{a_2}(f_1)"'] \arrow[dr, phantom, "\Downarrow\gamma"] & F(a_1,a_2') \arrow[d, "F_{a_2'}(f_1)"] \\
F(a_1',a_2) \arrow[r, "F_{a_1'}(f_2)"] \arrow[d, "F_{a_2}(h_1)"'] \arrow[dr, phantom, "\Downarrow\gamma"] & F(a_1',a_2') \arrow[d, "F_{a_2'}(h_1)"] \\
F(a_1'',a_2) \arrow[r, "F_{a_1''}(f_2)"'] & F(a_1'',a_2')
\end{tikzcd}
=
\begin{tikzcd}
F(a_1,a_2) \arrow[r, "F_{a_1}(f_2)"] \arrow[d, "F_{a_2}(h_1 f_1)"'] \arrow[dr, phantom, "\Downarrow\gamma"] & F(a_1,a_2') \arrow[d, "F_{a_2'}(h_1 f_1)"] \\
F(a_1'',a_2) \arrow[r, "F_{a_1''}(f_2)"'] & F(a_1'',a_2')
\end{tikzcd}
$$

$$
\begin{tikzcd}
F(a_1,a_2) \arrow[r, "F_{a_1}(f_2)"] \arrow[d, "F_{a_2}(f_1)"'] \arrow[dr, phantom, "\Downarrow\gamma"] & F(a_1,a_2') \arrow[r, "F_{a_1}(h_2)"] \arrow[d, "F_{a_2'}(f_1)"] \arrow[dr, phantom, "\Downarrow\gamma"] & F(a_1,a_2'') \arrow[d, "F_{a_2''}(f_1)"] \\
F(a_1',a_2) \arrow[r, "F_{a_1'}(f_2)"] & F(a_1',a_2') \arrow[r, "F_{a_1'}(h_2)"] & F(a_1',a_2'')
\end{tikzcd}
$$

$$\Vert$$

$$
\begin{tikzcd}
F(a_1,a_2) \arrow[r, "F_{a_1}(h_2 f_2)"] \arrow[d, "F_{a_2}(f_1)"'] \arrow[dr, phantom, "\Downarrow\gamma"] & F(a_1,a_2'') \arrow[d, "F_{a_2''}(f_1)"] \\
F(a_1',a_2) \arrow[r, "F_{a_1'}(h_2 f_2)"'] & F(a_1',a_2'')
\end{tikzcd}
$$

Proof. It is easy to see that $F_{a_1} : A_2 \to B$ is a 2-functor, as

$$F(1, fg) = F(1, f)F(1, g)$$
$$F(1, 1)F(1, g) = F(1, g) = F(1, g)F(1, 1),$$

where all displayed equalities are actually 2-cell constraints. The same argument shows that F_{a_2} is a strict 2-functor.

Let $f_1 : a_1 \to a_1'$, $f_2 : a_2 \to a_2'$ be a pair of 1-cells in A_1, A_2, respectively. Then the 2-cell γ_{f_1,f_2} is the composite of the constraint 2-cell with the identity 2-cell.

$$\begin{tikzcd}
F(a_1, a_2) \arrow[r, "F(1,f_2)"] \arrow[d, "F(f_1,1)"'] \arrow[dr, "F(f_1,f_2)" description] & F(a_1, a_2') \arrow[d, "F(f_1,1)"] \arrow[dl, phantom, "\Swarrow\cong" near start] \\
F(a_1', a_2) \arrow[r, "F(1,f_2)"'] \arrow[ur, phantom, "\Nearrow=" near start] & F(a_1', a_2')
\end{tikzcd}$$

Coherence for functors gives that each of the three axioms holds.

Given the data above, we construct a cubical functor F. The functor F is already defined on objects, so we define it on 1-cells by

$$F(f_1, f_2) = F(1, f_2) \circ F(f_1, 1)$$

and on 2-cells by

$$F(\alpha_1, \alpha_2) = F(1, \alpha_2) * F(\alpha_1, 1).$$

Here we have written $F(1, -), F(-, 1)$, for $F_{a_1}(-)$, resp. $F_{a_2}(-)$.The constraint cells are given by γ or are identities as necessitated by the definition of cubical functor, and it is simple to check that the axioms above give the axioms for for a weak functor. □

Proposition 5.2.3. *A cubical functor $F : A_1 \times A_2 \times A_3 \to B$ determines, and is uniquely determined by*

1. *For each object $a_1 \in A_1$, a cubical functor of 2 variables $F_{a_1} : A_2 \times A_3 \to B$, and similarly for objects $a_2 \in A_2$, $a_3 \in A_3$;*
2. *For each pair of objects a_1, a_2 in A_1, A_2, respectively, the equation*

$$F_{a_1}(a_2, -) = F_{a_2}(a_1, -)$$

holds, and similarly for pairs a_1, a_3 and a_2, a_3

such that the following axiom holds:
Given a 1-cell $(f_1, f_2, f_3) : (a_1, a_2, a_3) \to (a_1', a_2', a_3')$ in $A_1 \times A_2 \times A_3$, the

equation below holds.

$$
\begin{tikzcd}
& F(a_1,a_2,a_3') \arrow[dr, "F_{a_1}(f_2,1)"] \arrow[dd, phantom, "\Downarrow\gamma"] & \\
F(a_1,a_2,a_3) \arrow[ur, "F_{a_1}(1,f_3)"] \arrow[dr, "F_{a_1}(f_2,1)"] \arrow[dd, "F_{a_3}(f_1,1)"'] \arrow[dddr, phantom, "\Downarrow\gamma"] & & F(a_1,a_2',a_3') \arrow[dd, "F_{a_2}(f_1,1)"] \arrow[dddl, phantom, "\Downarrow\gamma"] \\
& F(a_1,a_2',a_3) \arrow[ur, "F_{a_1}(1,f_3)"] \arrow[dd, "F_{a_3}(f_1,1)"] & \\
F(a_1',a_2,a_3) \arrow[dr, "F_{a_3}(1,f_2)"'] & & F(a_1',a_2',a_3') \\
& F(a_1',a_2',a_3) \arrow[ur, "F_{a_2}(1,f_3)"'] &
\end{tikzcd}
$$

$$\|$$

$$
\begin{tikzcd}
& F(a_1,a_2,a_3') \arrow[dr, "F_{a_1}(f_2,1)"] \arrow[dd, "F_{a_2}(f_1,1)"] & \\
F(a_1,a_2,a_3) \arrow[ur, "F_{a_1}(1,f_3)"] \arrow[dd, "F_{a_3}(f_1,1)"'] \arrow[dr, phantom, "\Downarrow\gamma"] & & F(a_1,a_2',a_3') \arrow[dd, "F_{a_2}(f_1,1)"] \arrow[dl, phantom, "\Downarrow\gamma"] \\
& F(a_1',a_2,a_3') \arrow[dr, "F_{a_1'}(f_2,1)"'] \arrow[dd, phantom, "\Downarrow\gamma"] & \\
F(a_1',a_2,a_3) \arrow[ur, "F_{a_2}(1,f_3)"'] \arrow[dr, "F_{a_3}(1,f_2)"'] & & F(a_1',a_2',a_3') \\
& F(a_1',a_2',a_3) \arrow[ur, "F_{a_2}(1,f_3)"'] &
\end{tikzcd}
$$

Proposition 5.2.4. *A cubical functor $F : A_1 \times A_2 \times \cdots A_n \to B$, $n \geq 3$, determines, and is uniquely determined by*

1. *For each $(a_1, a_2, \ldots, a_n) \in A_1 \times A_2 \times \cdots A_n$ and each $i < j < k$, the restriction to*

$$F(a_1, a_2, \ldots, \hat{a}_i, \ldots, \hat{a}_j, \ldots, \hat{a}_k, \ldots, a_n) : A_i \times A_j \times A_k \to B$$

   is a cubical functor of 3 variables (where $\hat{a}_i$ indicates that object has been omitted and the variable is free), and

2. *These functors are compatible in the sense of Proposition 5.2.3.*

Proposition 5.2.5. *Let $i_1, \ldots, i_k$ be positive integers, and let*

$$F_j : A_{j,1} \times \cdots \times A_{j,i_j} \to B_j$$

be cubical functors. Then for any cubical functor

$$F : B_1 \times \cdots \times B_k \to C,$$

the composite $F \circ (F_1 \times \cdots \times F_k)$ is a cubical functor.

Proof. This is a functor, so we must check that certain constraints are identities. Recall that the constraint 2-cell for a composite $G \circ F$ is given by the formula

$$\phi^{G \circ F} = \phi^G \circ G(\phi^F).$$

Since G preserves identity 2-cells, it is enough to establish that ϕ^F and ϕ^G are appropriate identities.

Let $(f_{m,n}), (f'_{m,n})$ be composable arrows in the product 2-category $\prod A_{p,q}$ such that whenever $(a, b) < (a', b')$, either $f'_{a',b'}$ or $f_{a,b}$ is the identity. For (a, b) to be less than (a', b') in the total order, either $a < a'$ or $b < b'$. In particular, for a fixed a, either $f'_{a,b'}$ or $f_{a,b}$ is the identity. Thus the constraint ϕ_a for F_a is the identity, so the contraint for $F_1 \times \cdots \times F_k$ is the identity.

Now we must show that the constraint for F is the identity. This amounts to proving that if $a > a'$, then either $F_a(f_{a,1}, f_{a,2}, \ldots, f_{a,i_a})$ or $F_{a'}(f'_{a',1}, \ldots, f'_{a',i_{a'}})$ is the identity. If, for a fixed a, all the $f_{a,b}$ are identities, then $F_a(f_{a,1}, \ldots, f_{a,i_a})$ will be an identity as well since F_a preserves 1-cell identities strictly because it is cubical. Now assume that $a' > a$ and that $F_{a'}(f'_{a',1}, \ldots, f'_{a',i_{a'}})$ is not the identity. Then some $f_{a',b}$ is not the identity. Since $(a', b) > (a, c)$ for every c, every $f_{a,c}$ must be the identity by the cubical assumption. This shows that $F_a(f_{a,1}, \ldots, f_{a,i_a})$ is the identity, and we may now conclude that ϕ^F is the identity by the fact that F is cubical. This completes the proof that the composition constraint for F at $(f_{a,b}) \circ (f'_{a,b})$ is the identity, so the composite functor is cubical. □

5.2.2 The universal cubical functor

We are now in a position to prove that the Gray tensor product provides a solution to the problem of finding a universal cubical functor

$$A \times B \to C.$$

Let $\mathbf{Cub}(A_1 \times A_2, B)$ denote the set of cubical functors $A_1 \times A_2 \to B$.

Theorem 5.2.6. *Let A, B, and C be 2-categories. There is a cubical functor*

$$c : A \times B \to A \otimes B,$$

natural in A and B, such that composition with c induces an isomorphism

$$\mathbf{Cub}(A \times B, C) \cong \mathbf{2Cat}(A \otimes B, C).$$

Proof. We define c using Proposition 5.2.2. We define the 2-functor c_a by

$$\begin{aligned} c_a(b) &= (a, b) \\ c_a(f) &= (1_a, f) \\ c_a(\alpha) &= (1_{1_a}, \alpha); \end{aligned}$$

the 2-functor c_b is defined similarly. The 2-cell isomorphism $\gamma_{f,g}$ is the same $\gamma_{f,g}$ that is part of the data for $A \otimes B$. The three axioms for a cubical functor are exactly the axioms for the Gray tensor product, so we have defined a cubical functor $c : A \times B \to A \otimes B$. Naturality in both variables is clear.

To prove that this cubical functor has the claimed universal property, assume that $F : A \times B \to C$ is a cubical functor. We define a strict 2-functor

$\overline{F} : A \otimes B \to C$ by the following formulas.

$$\begin{aligned}
\overline{F}(a,b) &= F(a,b)\\
\overline{F}(f,1) &= F_b(f)\\
\overline{F}(1,g) &= F_a(g)\\
\overline{F}(\alpha,1) &= F_b(\alpha)\\
\overline{F}(1,\beta) &= F_a(\beta)\\
\overline{F}(\gamma^{A\otimes B}_{f,g}) &= \gamma^F_{f,g}
\end{aligned}$$

This defines $\overline{F}$ on objects, generating 1-cells, and generating 2-cells. We extend $\overline{F}$ to the whole of $A \otimes B$ by making it a strict 2-functor, i.e., it preserves all types of compositions and identities. The axioms for cubical functors and the Gray tensor product ensure that this is well-defined. It is clear that $\overline{F}$ is the unique strict 2-functor making the diagram

$$\begin{tikzcd}
A \times B \arrow[r, "F"] \arrow[d, "c"'] & C \\
A \otimes B \arrow[ur, "\overline{F}"'] &
\end{tikzcd}$$

commute, completing the proof. □

5.3 The monoidal category Gray

In this section, we will establish the basic results necessary to introduce the monoidal category **Gray**. We will not prove that this monoidal structure satisfies the necessary coherence laws (see [18] and [19]).

Notation 5.3.1. Let $\mathbf{Bicat}^s(A,B)$ denote the full sub-bicategory of $\mathbf{Bicat}(A,B)$ with objects the strict functors.

Lemma 5.3.2. *For any 2-categories A, B, $\mathbf{Bicat}^s(A,B)$ is a 2-category.*

Proof. The bicategory $\mathbf{Bicat}(X,Y)$ is a 2-category if Y is a 2-category. By definition, $\mathbf{Bicat}^s(A,B)$ is a full sub-2-category of this. □

Proposition 5.3.3. *1. The evaluation map $e : \mathbf{Bicat}^s(A,B) \times A \to B$ is a cubical functor.*
2. The function which sends a 2-functor $F : A_1 \to \mathbf{Bicat}^s(A_2,B)$ to the composite

$$A_1 \times A_2 \xrightarrow{F\times 1} \mathbf{Bicat}^s(A_2,B) \times A_2 \xrightarrow{e} B$$

is a natural isomorphism between 2-functors $A_1 \to \mathbf{Bicat}^s(A_2,B)$ and cubical functors $A_1 \times A_2 \to B$.

Proof. For the first part, the evaluation map e is defined by the following formulas.

$$\begin{gathered} e(F,a) = Fa \\ e(1_F, f) = Ff \\ e(\sigma, 1_a) = \sigma_a \text{ (the component of } \sigma \text{ at } a) \\ e(1_{1_F}, \alpha) = F\alpha \\ e(\Gamma, 1_{1_a}) = \Gamma_a \\ \gamma_{\sigma,f} = \sigma_f \end{gathered}$$

It is easy to check that this is a 2-functor when each variable is held fixed, and satisfies the necessary conditions to give a cubical functor.

For the second claim, first note that the composite displayed is actually a cubical functor by Proposition 5.2.5. Now given a cubical functor $F : A_1 \times A_2 \to B$, we must construct a strict 2-functor $\tilde{F} : A_1 \to \mathbf{Bicat}^s(A_2, B)$. To fix notation, we have objects a_1, a_1' in A_1, morphisms f_1, f_1' in A_1 each with source a_1 and target a_1', and a 2-cell $\alpha_1 : f_1 \Rightarrow f_1'$ in A_1; similarly for A_2 with subscript 2 instead of 1. The strict 2-functor is given by the formulas below.

$$\begin{gathered} \tilde{F}(a_1) = F_{a_1} \\ \tilde{F}(f_1)_{a_2} = F_{a_2}(f_1) \\ \tilde{F}(f_1)_{f_2} = \gamma_{f_1,f_2} \\ \tilde{F}(\alpha)_{a_2} = F_{a_2}(\alpha) \end{gathered}$$

It is easy to check that $\tilde{F}(f_1)$ gives a weak transformation, that $\tilde{F}(\alpha)$ gives a modification, that this assignment is a 2-functor, and that it is inverse to the assignment $F \mapsto e \circ F \times 1$. □

Corollary 5.3.4. *The functor $-\otimes B$ is left adjoint to the functor $\mathbf{Bicat}^s(B, -)$.*

Proposition 5.3.5. *Let A, B, C be 2-categories, and $G : A \times B \to C$ be a functor such that each $G(a,-)$, $G(-,b)$ is a strict 2-functor. Then there is a cubical functor $F : A \times B \to C$ such that*

1. *F agrees with G on objects and*
2. *there is a transformation $\nu : G \Rightarrow F$ which is the identity on objects.*

Proof. Define F to agree with G on objects. For a 1-cell $(f,g) : (a,b) \to (a',b')$, define F to be the composite

$$F(a,b) \xrightarrow{G(f,1)} F(a',b) \xrightarrow{G(1,g)} F(a',b'),$$

where we have already used that $F(a,b) = G(a,b)$. For a 2-cell $(\alpha,\beta) : (f,g) \to (f',g')$, define $F(\alpha,\beta)$ to be the horizontal composite

$$\begin{tikzcd} F(a,b) \arrow[r, bend left, "G(f{,}1)"] \arrow[r, bend right, "G(f'{,}1)"'] \arrow[r, phantom, "\Downarrow G(\alpha{,}1)"] & F(a',b) \arrow[r, bend left, "G(1{,}g)"] \arrow[r, bend right, "G(1{,}g')"'] \arrow[r, phantom, "\Downarrow G(\beta{,}1)"] & F(a',b'). \end{tikzcd}$$

The structure constraint for this functor is given by the following formula, where the indicated isomorphisms are obtained from the structure constraints for the funtor G.

$$\begin{array}{rcl} F(f_2,g_2)\circ F(f_1,g_1) & := & G(1,g_2)G(f_2,1)G(1,g_1)G(f_1,1) \\ & \cong & G(1,g_2)G(f_2,g_1)G(f_1,1) \\ & \cong & G(1,g_2)G(1,g_1)G(f_2,1)G(f_1,1) \\ & = & G(1,g_2g_1)G(f_2f_1,1) \\ & =: & F(f_2f_1,g_2g_1) \end{array}$$

Since this is defined using only the structure constraints of the functor G, coherence immediately implies that this new constraint will also satisfy the axioms necessary for F to be a cubical functor. Thus we have defined a cubical functor $F : A \times B \to C$.

To define the transformation ν, first set $\nu_{(a,b)} : G(a,b) \to F(a,b)$ equal to the identity on $G(a,b)$. We now need a 2-cell isomorphism $\nu_{(f,g)}$ in the square below.

$$\begin{tikzcd} G(a,b) \arrow[r, "\mathrm{id}_{G(a,b)}"] \arrow[d, "G(f,g)"'] & F(a,b) \arrow[d, "F(f,g)"] \arrow[dl, Rightarrow, "\cong"'] \\ G(a',b') \arrow[r, "\mathrm{id}_{G(a',b')}"'] & F(a',b') \end{tikzcd}$$

This amounts to a 2-cell $G(1,g)\circ G(f,1) \Rightarrow G(f,g)$, and for this we take the structure 2-cell for the functor G. Once again, coherence for functors immediately implies that this choice will satisfy the axioms for being a transformation. □

Remark 5.3.6. In [17], the procedure above is called nudging, and the transformation ν nudges G into a cubical functor.

Definition 5.3.7. A functor $F : A_1 \times A_2 \times \cdots A_n \to B$ is *opcubical* if the following condition holds:
if $(f_1, f_2, \ldots, f_n)(g_1, g_2, \ldots, g_n)$ is a composable pair of morphisms in the 2-category $A_1 \times A_2 \times \cdots A_n$ such that for all $i < j$, either g_i or f_j is an identity map, then the comparison 2-cell

$$\phi : F(f_1, f_2, \ldots, f_n)F(g_1, g_2, \ldots, g_n) \Rightarrow F\Big((f_1, f_2, \ldots, f_n)(g_1, g_2, \ldots, g_n)\Big)$$

is an identity.

Remark 5.3.8. Note that the difference between the definitions of cubical and opcubical functors is the switching of $i > j$ for cubical functors to $i < j$ for opcubical functors. It is easy to check that, given a cubical functor $F : A\times B \to C$, we can produce an opcubical functor $F^* : A \times B \to C$ by defining

$$F^*(f,g) = F(f,1)F(1,g)$$

and replacing the necessary structure 2-cells with their inverses. Nudging F^* gives back F, and in this case the transformation ν has components at each object being the identity, and naturality isomorphisms given by the structure constraints for F. Thus we obtain an isomorphism between cubical functors $A \times B \to C$ and opcubical functors $A \times B \to C$.

On the other hand, it is clear that there is an isomorphism between cubical functors $A \times B \to C$ and opcubical functors $B \times A \to C$ by the definition of opcubical functor. Combining these gives an isomorphism between cubical functors $A \times B \to C$ and cubical functors $B \times A \to C$. This procedure is one way of producing a symmetry isomorphism $A \otimes B \cong B \otimes A$.

We have now established the basic results necessary to state the following theorem.

Theorem 5.3.9. *The category* **2Cat** *of 2-categories and 2-functors has the structure of a closed symmetric monoidal category when equipped with*

- *the Gray tensor product,* $A \otimes B$*,*
- *unit object the terminal 2-category,*
- *the internal hom-functor* $\mathbf{Bicat}^s(A, B)$*, and*
- *symmetry given by the construction given in Section 1.*

Remark 5.3.10. Note that this is a different closed symmetric monoidal structure than the one given by cartesian product and the usual hom-2-category having 0-cells 2-functors, 1-cells 2-natural transformations, and 2-cells modifications. We shall refer to the monoidal structure using the Gray tensor product as **Gray**, and the cartesian monoidal structure as **2Cat**.

Chapter 6

Gray-categories and the tricategory Bicat

In this chapter, we will establish an important relationship between categories enriched over the monoidal category **Gray** and certain kinds of semi-strict tricategories. In particular, this section will establish a weak form of the coherence theorem for tricategories. We will then give the construction of the tricategory structure on bicategories and the higher cells between them. The theory developed will then allow a more robust formulation of the coherence theorem in Chapter 2. This theorem we call Coherence for **Bicat**, as it gives an appropriate strictification of the tricategory of bicategories. One should understand that, in this entire chapter, it is the coherence theory for bicategories that is at work. This chapter only repackages the coherence theory by using the language of tricategories and the tools developed so far. Thus we see an interesting interplay between the coherence theory for tricategories and the correct formulation of the coherence results for bicategories.

6.1 Cubical tricategories

This section is devoted to proving a weak form of the coherence theorem for tricategories. The theorem proved here will be used as a stepping stone to the stronger version of coherence. This weak form will introduce many of the concepts necessary to continue, and will be a simple consequence of a few results that are important later.

Definition 6.1.1. A tricategory T is *cubical* if

1. each bicategory $T(a, b)$ is a strict 2-category,
2. each functor $I_a : 1 \to T(a, a)$ is a cubical functor, and
3. each functor $\otimes : T(b, c) \times T(a, b) \to T(a, c)$ is a cubical functor.

Remark 6.1.2. It should be noted that condition 2 above does not appear in the definition of cubical tricategory given in [17].

The main result of this section is the following theorem.

Theorem 6.1.3. *Every tricategory T is triequivalent to a cubical tricategory stT with the same objects.*

To prove this, we need to use the functor $\mathrm{st} : \mathbf{Bicat} \to \mathbf{2Cat}$ which was explored in Chapter 2. Recall that if X is a bicategory, $\mathrm{st}X$ has the same objects, a 1-cell f from x to y is a composable string of arrows

$$x \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} y$$

(where for $n = 0$, we have a unique arrow in $\mathrm{st}X$ from x to x), and a 2-cell $\alpha : f \Rightarrow g$ is a 2-cell in X from $e(f)$ to $e(g)$, where we define $e(f)$ inductively by

- $e(f) = \mathrm{id}_x$ if $n = 0$,
- $e(f) = f_1$ if $n = 1$, and
- $e(f) = e(f') \circ f_1$ if $n > 1$, where f' is the 1-cell given by $f_n f_{n-1} \cdots f_2$.

The "inclusion" $X \to \mathrm{st}X$ sending each object to itself, each 1-cell f to the length 1 composable string, and each 2-cell α to itself is a biequivalence, and there is a distinguished retraction given by sending each object to itself, each 1-cell f of $\mathrm{st}X$ to $e(f)$, and each 2-cell α to itself. It is easy to prove Theorem 6.1.3 after we prove some preliminary results.

Proposition 6.1.4. *Let X, Y be bicategories. Then there exists a cubical functor $\hat{st} : stX \times stY \to st(X \times Y)$ such that*

1. *$\hat{st}$ is the identity on objects and*
2. *there is an adjoint equivalence $\boldsymbol{\zeta}$, with left adjoint pictured below,*

$$\begin{tikzcd} & st(X \times Y) \arrow[dr, "e_{X \times Y}"] \arrow[d, phantom, "\Downarrow\zeta"] & \\ stX \times stY \arrow[ur, "\hat{st}"] \arrow[rr, "e_X \times e_Y"'] & & X \times Y \end{tikzcd}$$

such that all the component maps at each object are the identity map.

Proof. We shall define $\hat{\mathrm{st}}$ using Proposition 5.2.2, so we must define it with each variable held fixed and define a structure 2-cell satisfying certain axioms. By necessity, it is the identity on objects.

Note that identity 1-cells in $\mathrm{st}X$ are the length zero composable strings which we shall write as 1_x; the identity 1-cell in the bicategory X will be written as id_x. Thus we define $\hat{\mathrm{st}}(f,1)$ to be the composable string

$$(f_n, \mathrm{id}_x)(f_{n-1}, \mathrm{id}_x)\cdots(f_1, \mathrm{id}_x),$$

and $\hat{\mathrm{st}}(1,g)$ is defined similarly. Let I^n be the 1-cell in X given by $e(\mathrm{id}_x\mathrm{id}_x\cdots\mathrm{id}_x)$, where the identity appears n times. To define $\hat{\mathrm{st}}$ on 2-cells $(\alpha,1)$, where $\alpha : f \Rightarrow f'$ in $\mathrm{st}X$, we must give a 2-cell in $\mathrm{st}(X\times Y)$

$$\hat{\mathrm{st}}\Big((\alpha,1)\Big) : \hat{\mathrm{st}}\Big((f,1)\Big) \Rightarrow \hat{\mathrm{st}}\Big((f',1)\Big).$$

By definition, this is a 2-cell in $X\times Y$

$$e\Big((f_n,\mathrm{id})(f_{n-1},\mathrm{id})\cdots(f_1,\mathrm{id})\Big) \Rightarrow e\Big((f'_m,\mathrm{id})(f'_{m-1},\mathrm{id})\cdots(f'_1,\mathrm{id})\Big).$$

Since composition in $X\times Y$ is componentwise, this 2-cell now has source $(e(f), I^n)$ and target $(e(f'), I^m)$. Define $\hat{\mathrm{st}}\Big((\alpha,1)\Big)$ to be $(\alpha, \gamma_{n,m})$ where $\gamma_{n,m} : I^n \Rightarrow I^m$ is the isomorphism given by structure constraints, unique by coherence. It is now easy to check that we have defined strict 2-functors $\hat{\mathrm{st}}_x$ and $\hat{\mathrm{st}}_y$ by holding each variable fixed separately.

The next step is to define the structure 2-cell $\gamma_{f,g}$ for $(f,g) : (x,y) \to (x',y')$.

$$\begin{tikzcd}
(x,y) \arrow[r, "\hat{\mathrm{st}}(1,g)"] \arrow[d, "\hat{\mathrm{st}}(f,1)"'] & (x,y') \arrow[d, "\hat{\mathrm{st}}(f,1)"] \arrow[dl, Rightarrow, "\gamma_{f,g}"'] \\
(x',y) \arrow[r, "\hat{\mathrm{st}}(1,g)"'] & (x',y')
\end{tikzcd}$$

By definition, this is a 2-cell in $X\times Y$ with

$$e\Big((f_n,\mathrm{id})\cdots(f_1,\mathrm{id})(\mathrm{id},g_m)\cdots(\mathrm{id},g_1)\Big)$$

as its source and

$$e\Big((\mathrm{id},g_m)\cdots(\mathrm{id},g_1)(f_n,\mathrm{id})\cdots(f_1,\mathrm{id})\Big)$$

as its target. We define $\gamma_{f,g}$ to be the unique isomorphism given by coherence between these 1-cells. There are now 3 axioms to be checked, but these all follow from coherence and various naturality conditions.

For the second statement, we have already defined the components at the objects to be identity maps. Let (f,g) be a 1-cell in $\mathrm{st}X\times\mathrm{st}Y$. The component $\zeta_{(f,g)}$ of this transformation is a 2-cell

$$(\mathrm{id},\mathrm{id})\circ\big(\hat{\mathrm{st}}(1,g)\circ\hat{\mathrm{st}}(f,1)\big) \Rightarrow \big(e(g),e(f)\big)\circ(\mathrm{id},\mathrm{id}).$$

By definition, this 2-cell has source given below.

$$(\mathrm{id}, \mathrm{id}) \circ \Big(e\big((\mathrm{id}, g_m)(\mathrm{id}, g_{m-1}) \cdots (\mathrm{id}, g_1)\big) \circ e\big((f_n, \mathrm{id}) \cdots (f_1, \mathrm{id})\big)\Big)$$

There is a unique coherence isomorphism between the above cell and $\big(e(g), e(f)\big) \circ (\mathrm{id}, \mathrm{id})$ that provides the component of ζ at (f, g). It is now trivial to check that this is a transformation. The right adjoint $\zeta^{\bullet}$ is obtained similarly. The unit and counit of this adjoint equivalence are given by modifications whose components are the coherence isomorphisms l_{id}^{-1} and l_{id}, respectively. The triangle identities follow from coherence.

□

Remark 6.1.5. It is a simple matter to prove that the cubical functor $\hat{\mathrm{st}}$ is natural in both variables. Doing so would complete the bulk of the proof that the functor st : **Bicat** → **Gray** is lax monoidal.

See the end of Section 2.4.3 for mention of the next lemma.

Proposition 6.1.6. *Let $F : B \to C$ be a functor between bicategories. Then there is an adjoint equivalence $\boldsymbol{\omega}$, with left adjoint pictured below,*

$$\begin{tikzcd} stB \arrow[r, "e"] \arrow[d, "stF"'] \arrow[dr, phantom, "\swarrow\omega"] & B \arrow[d, "F"] \\ stC \arrow[r, "e"'] & C \end{tikzcd}$$

such that all the component maps at each object are the identity map.

Proof. We have already stipulated that ω_a and $\omega_a^{\bullet}$ are both identity maps, so we need only provide the components of these transformations at a 1-cell f of st$B$ to complete the data for these transformations. Let f be such a 1-cell, so that f is a composable string $(f_n, \ldots, f_1)$. If we write Ff for the string $(Ff_n, \ldots, Ff_1)$, then ω_f is a 2-cell in C of the form

$$\mathrm{id} \circ F\big(e(f)\big) \Rightarrow e\big(Ff\big) \circ \mathrm{id}.$$

There is a unique coherence isomorphism by the coherence theorem for functors. The component $\omega_f^{\bullet}$ is constructed similarly. Since these are coherence cells, the transformation axioms follow automatically.

The unit and counit are given by l_{id}^{-1} and l_{id}, respectively. □

Proposition 6.1.7. *The biequivalences $e : stX \to X$ and $f : X \to stX$ extend to give a biadjoint biequivalence between X and stX.*

Proof. We need to give adjoint equivalences $\eta : 1 \to fe$, $\varepsilon : ef \to 1$ and two additional modifications, and then check two axioms. The transformation $\eta : 1 \Rightarrow fe$ has component $\eta_a = \mathrm{id}_a$ and naturality constraint given by the unique coherence cell

$$e(\mathrm{id}_b, h) \Rightarrow e(h) \circ \mathrm{id}_a,$$

where $h = (h_n, \dots, h_1)$. The transformation $\eta^{\bullet}$ is defined similarly. The adjoint equivalence ε is the identity adjoint equivalence as $ef : X \to X$ is the identity functor.

The components of the two required modifications have, as their sources and targets, composites of identities. Thus we take their components to be the relevant coherence isomorphisms, and the necessary axioms follow immediately. □

We can now prove the main result of this section.

Proof of 6.1.3. Let T be a tricategory. The cubical tricategory $\mathrm{st}T$ will have the same objects as T with $(\mathrm{st}T)(a,b) = \mathrm{st}\big(T(a,b)\big)$. We apply the Transport of Structure theorem to the identity function on the set of objects of T and the biadjoint biequivalences $e : \mathrm{st}T(a,b) \to T(a,b)$, $f : T(a,b) \to \mathrm{st}T(a,b)$. Combining Propositions 6.1.6 and 6.1.4 gives an adjoint equivalence between

$$\mathrm{st}T(b,c) \times \mathrm{st}T(a,b) \xrightarrow{e\times e} T(b,c) \times T(a,b) \xrightarrow{\otimes} T(a,c)$$

and

$$\mathrm{st}T(b,c) \times \mathrm{st}T(a,b) \xrightarrow{\hat{\mathrm{st}}} \mathrm{st}(T(b,c) \times T(a,b)) \xrightarrow{\mathrm{st}\otimes} \mathrm{st}T(a,c) \xrightarrow{e} T(a,c).$$

Taking the appropriate mate gives an adjoint equivalence between $(\mathrm{st}\otimes) \circ \hat{\mathrm{st}}$ and the composition functor used in the Transport of Structure theorem. Similarly, we can take the unit $1 \to \mathrm{st}(T(a,a))$ to be the unique strict functor whose image on the unique object is I. There is an adjoint equivalence between this functor and the unit given by the Change of Structure theorem. By the Change of Composition and Change of Units theorems, this constructs the tricategory structure on $\mathrm{st}T$ with the desired composition and units, as well as a triequivalence $\mathrm{st}T \to T$. □

Theorem 6.1.8. *There is a triequivalence $T \to stT$ that is the identity on 0-cells and is the inclusion $f : T(a,b) \to st\big(T(a,b)\big)$ on hom-bicategories.*

We will not provide a proof of this theorem as it is completely straightforward. All that remains is to identify the remaining constraint data and check the functor axioms; all of the data is obtained by pasting together units/counits of the biadjoint biequivalence (e,f) and the adjoint equivalences used in the previous proof. The axioms are then simple to check.

6.2 Gray-categories

In this section, we will highlight the relationship between categories enriched over **Gray** and tricategories. Since the final form of the coherence theorem for tricategories will state that every tricategory is triequivalent to a **Gray**-category, we must first explain how **Gray**-categories are tricategories.

Theorem 6.2.1. *1. Every strict 3-category is a* **Gray***-category.*
2. The structure of a **Gray***-category determines, and is uniquely determined by, the structure of a strict, cubical tricategory.*

Proof. First, we note that every strict 2-functor $A \times B \to C$ is also a cubical functor. Thus the composition 2-functor for a strict 3-category X gives rise by universal property to a composition 2-functor $X(b,c) \otimes X(a,b) \to X(a,c)$. The rest of the **Gray**-category structure is simple to check.

For the second statement, it is a simple matter to directly compare data and axioms. Note that the underlying data for a strict, cubical tricategory always satisfies the tricategory axioms, so that the data for a **Gray**-category corresponds to the first four pieces of data for a strict, cubical tricategory, and the axioms for a **Gray**-category correspond to the rest of the data for a strict, cubical tricategory. □

Corollary 6.2.2. *There is a strict, cubical tricategory* **Gray** *with objects strict 2-categories and hom-2-category* $\mathbf{Bicat}^s(A, B)$*.*

Proof. Since **Gray** is a closed monoidal category with internal hom-functor $\mathbf{Bicat}^s$, it is in particular enriched over **Gray**. □

Remark 6.2.3. It should be remarked that **Gray** is not a small tricategory as it does not have a set of objects. The same will obviously be true of **Bicat**, but this should not cause any concern. None of our constructions will ever result in a tricategory-type structure that does not have small hom-bicategories, i.e., hom-bicategories which have sets of 0-, 1-, and 2-cells.

Lemma 6.2.4. *There is a category* $\mathbf{Tricat}_{cub}$ *with objects strict, cubical tricategories and morphisms the strict functors between them; composition is given by the formulas in Section 4.1.*

Proof. This follows immediately from the formulas in Section 4.1 and the results concerning the construction of the category $\mathbf{Tricat}_v$ in Section 4.1. □

Theorem 6.2.5. *The inclusion* $\mathbf{Gray}\text{-}\mathbf{Cat} \hookrightarrow \mathbf{Tricat}_{cub}$ *is an equivalence of categories.*

Proof. First, we show how every **Gray**-enriched functor can be viewed as a strict functor between the corresponding tricategories. Let $F : A \to B$ be a **Gray**-functor between **Gray**-categories. We take the strict functor F to have the same function on objects and the same 2-functor on hom-2-categories. The adjoint equivalences χ and ι can be taken as identity adjoint equivalences since it is clear that $F(x \otimes y) = Fx \otimes Fy$ (for x, y being 1-, 2-, or 3-cells) and $FI_x = I_{Fx}$. The modifications ω, γ, δ can also all be taken to be identity modifications by similar reasoning. Thus we have constructed the inclusion functor above as this assignment obviously preserves composition and identities.

We have already shown that this inclusion is essentially surjective. This functor is an isomorphism on hom-sets since a strict functor between strict, cubical tricategories determines, and is uniquely determined by, a function on objects and strict 2-functors on hom-2-categories that strictly preserve identities and compositions. □

6.3 The tricategory Bicat

This section will establish two key results. The basic result is that the collection of bicategories, functors, transformations, and modifications forms a tricategory. This will be shown directly by calculation. We additionally prove it by transporting the tricategory structure from the tricategory **Gray**. This will also prove that **Bicat** is triequivalent to an easily determined full sub-**Gray**-category of **Gray** which we call **Gray′**.

It should also be noted that there are two natural tricategory structures on such data; this becomes clear when defining the horizontal composite of transformations, as there are two obvious choices and a canonical comparison map between them. This bifurcation will be noted, but it will not be important to the theory developed here.

6.3.1 Constructing Bicat directly

Here we will construct the tricategory structure on **Bicat** directly and without the use of the Transport of Structure theorem. This line of proof is largely calculational. In the next section we will use transport to construct a triequivalent tricategory structure denoted **B**.

The first piece of data we must construct is the hom-bicategory $\mathbf{Bicat}(A, B)$ for bicategories A and B. It has objects the functors $F : A \to B$, 1-cells the transformations $\alpha : F \Rightarrow G$, and 2-cells the modifications $\Gamma : \alpha \Rrightarrow \beta$. We will not construct this bicategory explicitly, but only mention that the structure constraints in it are obtained from the structure constraints of the target B. The unit $I_A : 1 \to \mathbf{Bicat}(A, A)$ is given by a functor whose value on the unique object of 1 is the identity functor $\mathrm{id}_A : A \to A$.

Proposition 6.3.1. *There is a functor*

$$\otimes : \mathbf{Bicat}(B, C) \times \mathbf{Bicat}(A, B) \to \mathbf{Bicat}(A, C)$$

whose function on objects is given by $G \otimes F = G \circ F$.

Proof. We have defined $\otimes$ on objects, now we must define it on hom-categories. Let $\alpha : F \Rightarrow F'$ and $\beta : G \Rightarrow G'$ be transformations. Then we define the transformation $G * \alpha : GF \Rightarrow GF'$ to have its component at a as $G\alpha_a$ and its component at $f : a \to a'$ to be the 2-cell

$$G\alpha_{a'} \circ GFf \xrightarrow{\phi} G(\alpha_{a'} \circ Ff) \xrightarrow{G\alpha_f} G(F'f \circ \alpha_a) \xrightarrow{\phi^{-1}} GF'f \circ G\alpha_a,$$

where ϕ is the structure constraint for G. It is easy to check that this is a transformation with the claimed source and target. We similarly define $\beta * F$. Now define

$$\beta \otimes \alpha := (G' * \alpha) \circ (\beta * F).$$

This transformation has as its component at a the 1-cell

$$G'\alpha_a \circ \beta_{Fa}.$$

Thus given modifications $\Gamma : \alpha \Rrightarrow \alpha'$ and $\Delta : \beta \Rrightarrow \beta'$, we define $\Delta \otimes \Gamma$ to be the modification with component

$$(\Delta \otimes \Gamma)_a = G'\Gamma_a * \Delta_{Fa}.$$

It is a simple matter to check that this does define a modification with source $\beta \otimes \alpha$ and target $\beta' \otimes \alpha'$.

These assignments preserve composition of modifications and preserve identities by the interchange law. Thus we have defined a functor on hom-categories, so the next step is to give structure constraints.

Let $\alpha' : F' \Rightarrow F''$ and $\beta' : G' \Rightarrow G''$ be transformations. We must provide an isomorphism modification between $(\beta' \otimes \alpha') \circ (\beta \otimes \alpha)$ and $(\beta' \circ \beta) \otimes (\alpha' \circ \alpha)$. The first of these transformations has component

$$(G''\alpha'_a \circ \beta'_{F'a}) \circ (G'\alpha_a \circ \beta_{Fa})$$

at a, while the second has component

$$G''(\alpha'_a\alpha_a) \circ (\beta'_{Fa} \circ \beta_{Fa})$$

at a. The structure constraint for composition is the modification $\otimes_2$ with component at a given by the following composite, where coherence 2-cells are unmarked isomorphisms.

$$\begin{gathered}(G''\alpha'_a \circ \beta'_{F'a}) \circ (G'\alpha_a \circ \beta_{Fa}) \cong G''\alpha'_a \circ ((\beta'_{F'a} \circ G'\alpha_a) \circ \beta_{Fa}) \xrightarrow{1*(\beta'_{G'\alpha_a}*1)} \\ G''\alpha'_a \circ ((G''\alpha_a \circ \beta'_{Fa}) \circ \beta_{Fa}) \cong (G''\alpha'_a \circ G''\alpha_a) \circ (\beta'_{Fa} \circ \beta_{Fa}) \xrightarrow{\phi^{-1}*1} \\ G''(\alpha'_a \circ \alpha_a) \circ (\beta'_{Fa} \circ \beta_{Fa})\end{gathered}$$

A lengthy calculation shows that this is a modification; it is clearly invertible. The constraint cell for the identity is constructed similarly.

Finally, we must check the functor axioms. These follow directly from coherence and the transformation axioms. $\square$

Remark 6.3.2. Note that we could have defined $\beta \otimes \alpha$ by the formula

$$(\beta * F') \circ (G * \alpha).$$

This has the effect of giving **Bicat** an opcubical composition instead of the cubical one defined here. The rest of the results of this section can be reformulated in terms of this composition, giving a different tricategory structure on bicategories, functors, transformations, and modifications. We will refer to this tricategory structure as **Bicat***.

Proposition 6.3.3. *There is an adjoint equivalence* $\mathbf{a} : \otimes \circ (\otimes \times 1) \Rightarrow \otimes \circ (1 \times \otimes)$ *with the component of* a *at the object* (H, G, F) *being the identity transformation and the component of* $a^\bullet$ *at* (H, G, F) *being the identity transformation.*

Proof. We need only give each component at a triple (γ, β, α) of transformations, check that this does give the claimed transformation, provide the unit and counit modifications, and check the triangle identities. The modification $a_{\gamma\beta\alpha}$ has component at a given by the following composite.

$$\begin{gathered}\mathrm{id} \circ (H'G'\alpha_a \circ (H'\beta_{Fa} \circ \gamma_{GFa})) \cong (H'G'\alpha_a \circ H'\beta_{Fa}) \circ \gamma_{GFa} \xrightarrow{\phi^H * 1} \\ H'(G'\alpha_a \circ \beta_{Fa}) \circ \gamma_{GFa} \cong (H'(G'\alpha_a \circ \beta_{Fa}) \circ \gamma_{GFa}) \circ \mathrm{id}\end{gathered}$$

This is easily shown to be a modification, and the component $a^{\bullet}_{\gamma\beta\alpha}$ is defined similarly. The transformation axioms follow from coherence for functors as all the 2-cells involved are constraint cells.

Now we must define the unit and counit of this adjoint equivalence. These are modifications $1 \Rrightarrow a^{\bullet}a$ and $aa^{\bullet} \Rrightarrow 1$; both are given by coherence cells, from which the triangle identities follow immediately. □

We state the next two propositions without proof, as they follow from similar arguments as the previous propositions.

Proposition 6.3.4. *There is an adjoint equivalence* $\mathbf{l} : \otimes \circ (I_A \times 1) \Rightarrow 1$ *with the component of* l *at the object* F *being the identity transformation and the component of* $l^{\bullet}$ *at the object* F *being the identity.*

There is an adjoint equivalence $\mathbf{r} : \otimes \circ (1 \times I_A) \Rightarrow 1$ *with the component of* r *at the object* F *being the identity transformation and the component of* $r^{\bullet}$ *at the object* F *being the identity.*

Proposition 6.3.5. *There are invertible modifications* π, μ, λ, ρ *as in the definition of a tricategory with the component of each at any object being the modification given by unique coherence isomorphisms.*

Theorem 6.3.6. *The data provided above gives a tricategory structure on the collections of bicategories, functors, transformations, and modifications.*

Proof. All three axioms follow from the observation that for any of the modifications involved, the components are all given by constraint cells in the target bicategory. Thus coherence implies that all necessary diagrams commute. □

The last two results of this section are presented without proof. They will not be used in the remainder of this work.

Lemma 6.3.7. *There is a biequivalence* $\mathbf{Bicat}(A, B) \to \mathbf{Bicat}^*(A, B)$ *which is the identity on objects.*

Theorem 6.3.8. *There is a triequivalence* $\mathbf{Bicat} \to \mathbf{Bicat}^*$ *which is the identity on 0- and 1-cells.*

6.3.2 The tricategories Bicat and Gray

The main result of this section is that **Bicat** is triequivalent to a full sub-tricategory of the tricategory **Gray** constructed in Corollary 6.2.2. This will be accomplished in two steps. The first is to construct a tricategory **B** by transport and then check that it is triequivalent to a full sub-tricategory of **Gray**. The second step is to compare this with the tricategory **Bicat** constructed in the previous section.

Before proving these theorems, we need to establish the following local result which is a consequence of the coherence theorem for functors and properties of the functor st.

Proposition 6.3.9. *The function sending each functor of bicategories $F : X \to Y$ to the strict 2-functor $stF : stX \to stY$ extends to a biequivalence of bicategories $st_{XY} : \mathbf{Bicat}(X, Y) \to \mathbf{Bicat}^s(stX, stY)$. Moreover, this biequivalence is part of a biadjoint biequivalence in the tricategory* **Bicat**.

Proof. First, we must define st_{XY} on the 1-cells and 2-cells of $\mathbf{Bicat}(X, Y)$. Given a transformation $\alpha : F \Rightarrow G$, define $\mathrm{st}\alpha$ to be the transformation with component $(\mathrm{st}\alpha)_a = \alpha_a$ at a and with naturality constraint $(\mathrm{st}\alpha)_f$ given by the commutativity of the following diagram, where $f = (f_n, \ldots, f_1)$ and the unmarked isomorphisms come from coherence.

$$\begin{tikzcd}
e\Big(\alpha_b, Ff_n, \ldots, Ff_1\Big) \arrow[r, "(\mathrm{st}\alpha)_f"] \arrow[d, "\cong"'] & e\Big(Gf_n, \ldots, Gf_1, \alpha_a\Big) \arrow[d, "\cong"] \\
\alpha_b \circ F\Big(e(f)\Big) \arrow[r, "\alpha_{e(f)}"'] & G\Big(e(f)\Big) \circ \alpha_a
\end{tikzcd}$$

The transformation axioms then follow from the fact that α is a transformation and coherence.

Now given $\Gamma : \alpha \Rrightarrow \beta$, we construct $\mathrm{st}\Gamma$ by giving it the component $(\mathrm{st}\Gamma)_a = \Gamma_a$. Coherence implies that this is a modification.

It is clear that this is a functor on the relevant hom-categories since modifications are composed component-wise. Now we define the structure constraints and prove that they give a functor of bicategories. In each case, the relevant modification has as its component at a the appropriate constraint isomorphism. The modification axioms are satisfied because of coherence.

Proving that st_{XY} is a biequivalence requires proving that it is biessentially surjective and locally an equivalence of categories. To prove the first of these claims, recall that there are biequivalences $f : X \to \mathrm{st}X$ and $e : \mathrm{st}Y \to Y$. Given a 2-functor $F : \mathrm{st}X \to \mathrm{st}Y$, let $\overline{F} : X \to Y$ be the composite eFf. It is easy to show, using the transformation ω from Proposition 6.1.6, that F is equivalent to $\mathrm{st}\overline{F}$.

To show that st_{XY} is locally an equivalence of categories, let $F, G : X \to Y$ be a pair of functors and $\alpha : \mathrm{st}F \Rightarrow \mathrm{st}G$ be a transformation. We define $\overline{\alpha}$ by the following formulas.

$$(\overline{\alpha})_a = \alpha_a$$
$$(\overline{\alpha})_f = \alpha_f$$

Note that in the second formula, we are identifying the 1-cell f with the length one string (f). In the notation of Proposition 6.3.1, $\overline{\alpha} = e * \alpha * f$. There is an isomorphism between α and $\mathrm{st}\overline{\alpha}$ given by a modification all of whose components are constraint isomorphisms. Thus st_{XY} is locally essentially surjective. To see that st_{XY} is locally full and faithful, note that a modification is determined by its components, so $\Gamma \mapsto \mathrm{st}\Gamma$ is injective. On the other hand, any modification $\Delta : \mathrm{st}\alpha \Rrightarrow \mathrm{st}\beta$ gives rise to a modification $\overline{\Delta}$ with the same components by restriction. It is immediate that $\overline{\Delta}$ is a modification and that $\mathrm{st}\overline{\Delta} = \Delta$. Thus st_{XY} is locally full and faithful, therefore a biequivalence.

For the final statement, we merely indicate the rest of the required data; all of the axioms follow easily. The functor $\mathrm{st}^{\bullet}_{XY} : \mathbf{Bicat}^s(\mathrm{st}X, \mathrm{st}Y) \to \mathbf{Bicat}(X, Y)$ is given by the following data. On objects, it maps $F : \mathrm{st}X \to \mathrm{st}Y$ to eFf. On 1-cells, it maps $\alpha : F \Rightarrow G$ to $e * \alpha * f$. On 2-cells, it maps $\Gamma : \alpha \Rrightarrow \beta$ to the modification with the same components; as before, it is immediate that this is a modification since it is defined by "restricting" the original modification. It is clear that this is locally a functor on hom-categories. The constraint modifications all have unique coherence isomorphisms as their components. It is simple to demonstrate that this functor is a biequivalence using arguments similar to those used for st_{XY}.

We must now give adjoint equivalences $\boldsymbol{\varepsilon} : \mathrm{st}_{XY}\mathrm{st}^{\bullet}_{XY} \to 1$ and $\boldsymbol{\eta} : 1 \to \mathrm{st}^{\bullet}_{XY}\mathrm{st}_{XY}$. To give $\boldsymbol{\varepsilon}$, we define ε_F to be the transformation with its components identities and naturality constraints identity 2-cells (since $\mathrm{st}Y$ is a strict 2-category). The modification ε_α for a transformation $\alpha : F \Rightarrow G$ is the identity modification. For $\boldsymbol{\eta}$, we define it to be the transformation given as follows. The component η_F is the identity transformation, and the naturality constraint η_α is the modification with identity maps as its components. The other transformations are defined similarly. Units and counits are given by the obvious coherence cells.

Finally we must give modifications (see Appendix A) and show that they satisfy two axioms. These modifications have, as their sources and targets, composites of constraints and the transformations defined above. Since all of the components of the above transformations are themselves identity transformations, our modifications are given by unique coherence isomorphisms. This defines a modification by coherence, and coherence also implies that the axioms for a biadjoint biequivalence are satisfied.

□

Theorem 6.3.10. *There is a tricategory* **B** *with objects bicategories A and hom-bicategories given by* $\mathbf{Bicat}(A, B)$ *that is triequivalent to the full sub-***Gray***-category of* **Gray** *determined by those 2-categories of the form stB for some bicategory B.*

Proof. We apply the Transport of Structure theorem to the function that sends a bicategory X to st$X$ and to the biadjoint biequivalence given in the previous proposition. By construction, there is a triequivalence $\mathbf{B} \to \mathbf{Gray}$ which agrees with st on objects and st_{XY} on hom-bicategories. □

Remark 6.3.11. It will be necessary for the next proof to know explicitly the tricategory structure on $\mathbf{B}$. The composition in $\mathbf{B}$, denoted $\boxtimes$, is given on objects by the following formula.

$$G \boxtimes F := e \circ \mathrm{st}G \circ \mathrm{st}F \circ f = e \circ \mathrm{st}(GF) \circ f$$

The unit map $1 \to \mathbf{B}(X, X)$ is given by the functor taking the value of id_X on the unique object, the value of the identity on the unique 1-cell, and with all constraints given by unique coherence isomorphisms. The adjoint equivalences $\mathbf{a}, \mathbf{l}, \mathbf{r}$ are all identity adjoint equivalences. The invertible modifications μ, λ, ρ are identities as well. Thus $\mathbf{B}$ is a strict tricategory which has a "cubical" composition law but is not locally a 2-category.

Theorem 6.3.12. *There is a triequivalence* $\mathbf{Bicat} \to \mathbf{B}$ *which is the identity on objects and hom-bicategories.*

Proof. We need only provide the remaining data for a functor and check the appropriate axioms to complete the proof.

The adjoint equivalences χ, ι are identity adjoint equivalences. The invertible modification ω gives, for each triple (H, G, F) of functors, a modification whose component 2-cells have source $\mathrm{id} \circ (H\mathrm{id} \circ \mathrm{id})$ and target $\mathrm{id} \circ (\mathrm{id} \circ F\mathrm{id})$. We take this to be the 2-cell $1 * (\phi_H^{-1} * \phi_F)$. Note that this is the unique coherence cell between these 1-cells.

The invertible modification γ gives, for each functor F, a modification whose component 2-cells have source $\mathrm{id} \circ (\mathrm{id} \circ \mathrm{id})$ and target $\mathrm{id} \circ \mathrm{id}$; we take these components to be the unique coherence cells.

The two functor axioms now follow from the fact that all of the 2-cells involved are unique coherence cells. □

Definition 6.3.13. Let A, B be 2-categories. Then A and B are *strictly biequivalent* if there exist strict 2-functors $F : A \to B$ and $G : B \to A$ such that GF is equivalent to 1_A in $\mathbf{Bicat}(A, A)$ and FG is equivalent to 1_B in $\mathbf{Bicat}(B, B)$.

Remark 6.3.14. Since A, B are strict 2-categories and the functors $FG, GF, 1_A$, and 1_B are strict 2-functors, we could have demanded that GF be equivalent to 1_A in $\mathbf{Gray}(A, A)$, and similarly for FG, for a logically equivalent definition. It is now easy to check that two strict 2-categories are strictly biequivalent if and only if they are internally biequivalent in the tricategory $\mathbf{Gray}$.

Definition 6.3.15. Let $\mathbf{Gray}'$ be the full sub-$\mathbf{Gray}$-category of $\mathbf{Gray}$ determined by all the strict 2-categories which are strictly biequivalent to a 2-category of the form st$B$ for some bicategory B.

Corollary 6.3.16 (Coherence for **Bicat**)**.** *The tricategory* **Bicat** *is triequivalent to the tricategory* **Gray**$'$*.*

Remark 6.3.17. It should be noted that the tricategory **Bicat** is not triequivalent to the tricategory **Gray**, as shown by Lack in [26]. It is easy to see that the inclusion **Gray**$' \hookrightarrow$ **Gray** is not a triequivalence, as the 2-category I with

- a single object x,
- a single idempotent $f : x \to x$, and
- only identity 2-cells

is not strictly biequivalent to any 2-category of the form st$B$. Lack uses a similar example to show that the inclusion **Gray** $\hookrightarrow$ **Bicat** is not a triequivalence, and then proves that any triequivalence **Gray** $\to$ **Bicat** would be forced to be appropriately equivalent to the inclusion. This produces an immediate contradiction, hence **Gray** is not triequivalent to **Bicat**.

Chapter 7

Functor tricategories: Gray-structures

In this chapter, we will begin proving the required results to establish the existence of a tricategory structure on the collection of functors, transformations, modifications, and perturbations between fixed source and target tricategories. We will not complete the full proofs here, but we will establish the complete local structure – for tricategories S, T and functors $F, G : S \to T$ between them, we construct the hom-bicategory $\mathbf{Tricat}(S, T)(F, G)$. We then show that if T is a **Gray**-category, this bicategory is actually a 2-category. Finally, we produce the remaining data for the tricategory $\mathbf{Tricat}(S, T)$ in the particular case that T is a **Gray**-category, and show that the resulting tricategory structure is also a **Gray**-category. The full result that for any pair of tricategories S, T there is a tricategory $\mathbf{Tricat}(S, T)$ whose 0-cells are functors, whose 1-cells are transformations, whose 2-cells are modifications, and whose 3-cells are perturbations will not be provided in this work. There is no substantial obstruction to proving this, however, but doing so is not necessary for our proof of coherence.

7.1 Local structure

The first section will focus on local results that apply when S, T is any pair of tricategories. We will prove that if $F, G : S \to T$ is any pair of functors between tricategories, then there is a bicategory $\mathbf{Tricat}(S, T)(F, G)$ whose objects are transformations $\alpha : F \to G$, whose 1-cells are the modifications between these, and whose 2-cells are the perturbations between these.

Theorem 7.1.1. *Let S, T be tricategories, and $F, G : S \to T$ be functors. Then there is a bicategory $\mathbf{Tricat}(S, T)(F, G)$ with 0-cells the transformations $\alpha : F \to G$, 1-cells the modifications $m : \alpha \Rightarrow \beta$, and 2-cells the perturbations $\sigma : m \Rrightarrow n$.*

Proof. To define such a bicategory, we must give hom-categories, a composition functor, associativity and unit isomorphisms, and then verify two axioms. The hom-category $\mathbf{Tricat}(S,T)(F,G)(\alpha,\beta)$, hereafter abbreviated $[\alpha,\beta]$, is defined to have objects the modifications $m:\alpha\Rightarrow\beta$ and morphisms the perturbations $\sigma: m \Rrightarrow n$. Composition of morphisms is given by defining the component at a of the composite $\tau\circ\sigma$ to be $\tau_a\circ\sigma_a$, where this composition is the vertical composition of 2-cells in the appropriate hom-bicategory. Similarly, the identity arrow $1_m : m \Rrightarrow m$ has as its component at a the identity 2-cell 1_{m_a}, once again taken in the appropriate hom-bicategory. It is immediate that these are perturbations. It is easy to see that this does give the structure of a category, as vertical composition of 2-cells in a bicategory is strictly associative and strictly unital.

The next step in establishing the local bicategory structure is to provide a composition functor
$$\centerdot : [\beta,\gamma]\times[\alpha,\beta]\to[\alpha,\gamma].$$
On objects, we define $n\centerdot m$ to have as its component at a the composite $n_a m_a$, where we now use the composition of 1-cells in the appropriate hom-bicategory. To give a modification, we must also provide an invertible modification (in the bicategorical sense). This consists of, for each $f\in T(a,b)$, a 2-cell

$$1_{Gf}\otimes(n_a m_a)\circ\alpha_f \Rightarrow (n_b m_b)\otimes 1_{Ff}\circ\gamma_f.$$

This 2-cell is given by the pasting diagram below; the unmarked isomorphisms are unique constraint isomorphisms.

$$\begin{tikzcd}
\alpha_b\otimes Ff \arrow[rr, "\alpha_f"] \arrow[dr, "m_b\otimes 1"] \arrow[ddrr, bend right=40, "(n_a m_a)\otimes 1"'] & & Gf\otimes\alpha_a \arrow[rr, "1\otimes(n_a m_a)"] \arrow[dr, "1\otimes m_a"'] \arrow[dl, phantom, "\Swarrow m_f"] & & Gf\otimes\gamma_a \\
\cong & \beta_b\otimes Ff \arrow[rr, "\beta_f"'] \arrow[dr, "n_a\otimes 1"] & \cong & Gf\otimes\beta_a \arrow[ur, "1\otimes n_a"'] \arrow[dl, phantom, "\Searrow n_f"] & \\
& & \gamma_b\otimes Ff \arrow[uurr, bend right=40, "\gamma_f"'] & &
\end{tikzcd}$$

It is immediate that this is invertible, and the modification axiom is trivial to check using that m_f and n_f give modifications.

We now define $\tau\centerdot\sigma$ to have component $\tau_a * \sigma_a$ at a, where this horizontal composite is formed in the appropriate hom-bicategory. Functoriality follows since it is merely the interchange law for the hom-bicategories used in our constructions.

The next step is to define the associativity and unit structure constraints. The associativity constraint is given by the perturbation $A:(p\centerdot n)\centerdot m\Rrightarrow p\centerdot(n\centerdot m)$ having as its component at the object a the 2-cell

$$A_a : (p_a\circ n_a)\circ m_a \Rightarrow p_a\circ(n_a\circ m_a)$$

which is the associativity constraint in the appropriate hom-bicategory. The single axiom for being a perturbation follows immediately as a consequence of coherence. Similar definitions provide the left and right unit constraints, L and R, respectively. There are now two bicategory axioms to check, but these follow directly from the fact that they hold locally by coherence, i.e., in each hom-bicategory separately. □

Corollary 7.1.2. *Let S be a tricategory and let T be a tricategory such that each $T(a,b)$ is a strict 2-category. Then for any pair of weak functors $F, G : S \to T$, the bicategory $\mathbf{Tricat}(S,T)(F,G)$ is a strict 2-category.*

Proof. Since the associativity and unit constraints are given by the constraints in the hom-bicategories of the target, the result is immediate. □

7.2 Global results

For this section, S will be any tricategory and T will be any strict, cubical tricategory, i.e., a **Gray**-category.

Theorem 7.2.1 (Cubical composition)**.** *Under the above hypotheses, there is a cubical composition functor*

$$\otimes : \mathbf{Tricat}(S,T)(G,H) \times \mathbf{Tricat}(S,T)(F,G) \to \mathbf{Tricat}(S,T)(F,H)$$

such that $\beta \otimes \alpha$ is the transformation defined by

- *the component at the object a is given by*

$$(\beta \otimes \alpha)_a = \beta_a \otimes \alpha_a;$$

- *the adjoint equivalence $\beta \otimes \alpha$ is given by*
  1. *$(\beta \otimes \alpha)_f$ is the composite*

$$\begin{array}{c}(\beta_b \otimes \alpha_b) \otimes Ff \xrightarrow{=} \beta_b \otimes (\alpha_b \otimes Ff) \xrightarrow{1 \otimes \alpha_f} \beta_b \otimes (Gf \otimes \alpha_a) \xrightarrow{=} \\ (\beta_b \otimes Gf) \otimes \alpha_a \xrightarrow{\beta_f \otimes 1} (Hf \otimes \beta_a) \otimes \alpha_a \xrightarrow{=} Hf \otimes (\beta_a \otimes \alpha_a),\end{array}$$

  and

  2. *$(\beta \otimes \alpha)^{\bullet}_f$ is the composite*

$$(1 \otimes \alpha^{\bullet}_f) \circ (\beta^{\bullet}_f \otimes 1),$$

  3. *the counit of this adjunction is the obvious composite of counits, and the unit is the obvious composite of units;*

- *the invertible modification* Π *is provided by the pasting diagram below, where we have written tensor as concatenation;*

$$
\begin{tikzcd}
\beta_c\alpha_c FgFf \arrow[r, "1\alpha_g 1"] \arrow[dd, "11\chi"'] & \beta_c Gg\alpha_b Ff \arrow[r, "\beta_g 11"] \arrow[dr, "11\alpha_f"] \arrow[d, phantom, "\swarrow 1\otimes\Pi^{\alpha}"] & Hg\beta_b\alpha_b Ff \arrow[r, "11\alpha_f"] \arrow[d, phantom, "\cong"] & Hg\beta_b Gf\alpha_a \arrow[d, "1\beta_f 1"] \\
& {} & \beta_c GgGf\alpha_a \arrow[ur, "\beta_g 11"'] \arrow[d, "1\chi 1"] \arrow[r, phantom, "\swarrow \Pi^{\beta}\otimes 1"] & HgHf\beta_a\alpha_a \arrow[d, "\chi 11"] \\
\beta_c\alpha_c F(gf) \arrow[rr, "1\alpha_{gf}"'] & & \beta_c G(gf)\alpha_a \arrow[r, "\beta_{gf}1"'] & H(gf)\beta_a\alpha_a
\end{tikzcd}
$$

and

- *the invertible modification* M *is given by the pasting diagram below.*

$$
\begin{tikzcd}
\beta_a\alpha_a \arrow[r, "="] \arrow[dd, "="'] \arrow[dr, "="] & \beta_a\alpha_a I_{Fa} \arrow[r, "11\iota^F"] \arrow[d, phantom, "\swarrow 1\otimes M^{\alpha}"] & \beta_a\alpha_a FI_a \arrow[d, "1\alpha_{I_a}"] \\
& \beta_a I_{Ga}\alpha_a \arrow[r, "1\iota^G 1"'] \arrow[d, phantom, "\swarrow M^{\beta}\otimes 1"] & \beta_a GI_a\alpha_a \arrow[d, "\beta_{I_a}1"] \\
I_{Ha}\beta_a\alpha_a \arrow[rr, "\iota^H 11=\iota^H 1"'] & {} & HI_a\beta_a\alpha_a
\end{tikzcd}
$$

Proof. To give a cubical functor as above, we first need to provide strict 2-functors $\otimes_\alpha$ and $\otimes_\beta$ which each hold one variable constant. First, note that the formulas above do indeed give a transformation $\beta \otimes \alpha : F \Rightarrow H$. Thus we have defined the values of these functors on 0-cells, so we now extend them to 1- and 2-cells. Here we give explicit formulas for $\otimes_\beta$; those for $\otimes_\alpha$ are similar. For a modification $m : \alpha \Rightarrow \alpha'$, we define $\otimes_\beta(m)$ to be the following trimodification. The component at a is

$$\otimes_\beta(m)_a = 1_{\beta_a} \otimes m_a,$$

where the identity 2-cell is taken in the relevant hom-bicategory. For each $f : a \to b$ in S, the modification $\otimes_\beta(m)$ is defined to have component at f given by the following pasting diagram.

$$
\begin{tikzcd}
\beta_b\alpha_b Ff \arrow[r, "1\otimes\alpha_f"] \arrow[d, "1\otimes m_b\otimes 1"'] \arrow[dr, phantom, "\swarrow 1\otimes m_f"] & \beta_b Gf\alpha_a \arrow[r, "\beta_f\otimes 1"] \arrow[d, "1\otimes 1\otimes m_a"] \arrow[dr, phantom, "\cong"] & Hf\beta_a\alpha_a \arrow[d, "1\otimes 1\otimes m_a"] \\
\beta_b\alpha'_b Ff \arrow[r, "1\otimes\alpha'_f"'] & \beta_b Gf\alpha'_a \arrow[r, "\beta_f\otimes 1"'] & Hf\beta_a\alpha'_a
\end{tikzcd}
$$

On 2-cells, we define $\otimes_\beta$ by the formula

$$\otimes_\beta(\sigma)_a = 1_{1_{\beta_a}} \otimes \sigma_a.$$

The perturbation axiom is immediate.

Now we check that $\otimes_\beta$ is a strict 2-functor. First, note that

$$\otimes_\beta(n)_a \circ \otimes_\beta(m)_a = \otimes_\beta(nm)_a$$

since T is a **Gray**-category. Similarly, the modifications $\otimes_\beta(n) \circ \otimes_\beta(m)$ and $\otimes_\beta(nm)$ coincide. If m is the identity modification, it is easy to check that $\otimes_\beta(m)$ is the identity as well. Finally, $\otimes_\beta(\tau)\circ\otimes_\beta(\sigma) = \otimes_\beta(\tau\circ\sigma)$ and $\otimes_\beta(1_m) = 1$ by similar arguments.

Finally, to define a cubical composition functor we must provide a structure 2-cell and check that it satisfies three axioms. This perturbation will have as its component at a the coherence cell

$$(n_a \otimes 1_{\alpha'_a}) \circ (1_{\beta_a} \otimes m_a) \overset{\cong}{\Rrightarrow} (1_{\beta'_a} \otimes m_a) \circ (n_a \otimes 1_{\alpha_a})$$

given by the isomorphism γ arising from the **Gray**-category structure on T. The perturbation axiom is a consequence of the naturality of the isomorphism γ from the **Gray**-category structure on T. It is immediate that this satisfies the necessary axioms to give the comparison cell for a cubical functor, as they are satisfied locally by the **Gray**-category axioms. □

We are now in a position to prove the main theorem of this section.

Theorem 7.2.2 (**Gray**-category structure)**.** *Let S be any tricategory and let T be a strict, cubical tricategory. Then there is a* **Gray***-category* $\mathbf{Tricat}(S,T)$ *with*

- *objects weak functors $F : S \to T$,*
- *hom-2-categories* $\mathbf{Tricat}(S,T)(F,G)$ *as given above, and*
- *composition 2-functor*

$$\mathbf{Tricat}(S,T)(G,H) \otimes \mathbf{Tricat}(S,T)(F,G) \to \mathbf{Tricat}(S,T)(F,H)$$

induced by the cubical functor in Theorem 7.2.1.

Proof. All that remains is to provide a unit map $\mathbf{1} \to \mathbf{Tricat}(S,T)(F,F)$ and to prove that composition is strictly unital and associative. The unit is given by the 2-funtor which sends the unique object to the identity transformation $1_F : F \to F$ given by the following. The component at a is the 1-cell I_{Fa} given by the unit in T. The adjoint equivalence

$$\mathbf{1_F} : (\mathrm{id}_{Fa})_* \circ F \to (\mathrm{id}_{Fa})^* \circ F$$

is taken to be the identity (recall that T has strict units), and the invertible modifications are both the identity. The rest of the unit 2-functor is determined since it is a strict 2-functor. It is immediate that this gives $\otimes$ a strict unit by the proof of the previous theorem.

Finally, we check associativity. From the definition of $\beta \otimes \alpha$, we see that

$$(\gamma \otimes \beta) \otimes \alpha = \gamma \otimes (\beta \otimes \alpha)$$

since the composition in T is strictly associative and unital. An easy computation shows that the same holds for 1- and 2-cells. Since $\otimes$ is strictly associative and unital, $\mathbf{Tricat}(S, T)$ has been given the structure of a **Gray**-category. $\square$

Remark 7.2.3. In [17], there is a strategy outlined for providing a tricategory structure on the 3-globular set whose 0-cells are functors between fixed tricategories, whose 1-cells are transformations, whose 2-cells are modifications, and whose 3-cells are perturbations. It would be a simple matter to use the results above and the Transport of Structure theorem to realize that strategy, but we have refrained from doing so as it is not necessary for our proof of the coherence theorem for tricategories. Additionally, this tricategory structure would *not* be the naive one with the composition functor

$$\otimes : \mathbf{Tricat}(S, T)(G, H) \times \mathbf{Tricat}(S, T)(F, G) \to \mathbf{Tricat}(S, T)(F, H)$$

given by composition of transformations on 0-cells. This is analogous to the fact that the tricategories **Bicat** and **B** in the previous chapter do not coincide, but instead are only triequivalent.

Chapter 8

The Yoneda lemma and coherence

In this chapter, we prove a restricted type of Yoneda Lemma. A full tricategorical Yoneda Lemma would express the existence of a functor

$$T \to \mathbf{Tricat}(T^{\mathrm{op}}, \mathbf{Bicat})$$

having certain properties; in particular, it should be a triequivalence when the target is appropriately restricted. We will not prove this theorem here, as it would require a large quantity of tedious calculations in constructing the functor tricategory in the target. Instead, we will restrict ourselves to the case when T is a cubical tricategory, and then prove a similar result for the functor

$$T \to \mathbf{Tricat}(T^{\mathrm{op}}, \mathbf{Gray}).$$

Since T is cubical, we can replace **Bicat** with **Gray**, and now the functor tricategory in the target is itself a **Gray**-category. Proving that this functor affords a triequivalence between T and its essential image then gives the required coherence result, as we have already shown that any tricategory S can be replaced with a triequivalent cubical tricategory $\mathrm{st}S$.

Our Yoneda-type result will be proved in two steps. First, we establish the existence of the claimed functor. Second, we exhibit the properties necessary for the coherence result.

8.1 The cubical Yoneda Lemma

This section will focus on the case when the target tricategory T is cubical, and that assumption will now be made throughout this section. We proceed with a number of calculational lemmata in order to make the proofs more digestible.

Most of the proofs in this section are unenlightening calculations. Many follow directly from the tricategory axioms, but some are quite involved. We omit these difficult calculations and explain how to prove them in Appendix C.

Lemma 8.1.1. *Let a be an object of T. Then there is a functor*

$$T(-,a) : T^{op} \to \mathbf{Gray}$$

whose value at b is the 2-category $T(b,a)$.

Proof. Recall that the tricategory **Gray** has 0-cells strict 2-categories, 1-cells strict 2-functors, 2-cells transformations, and 3-cells modifications. First, we have that $T(-,a)(b) = T(b,a)$ which is a strict 2-category since T is cubical. If $f : b \to b'$ is a 1-cell in T, then $T(-,a)(f) : T(b',a) \to T(b,a)$ (which we shall now call f^*) is defined as follows.

- On the 0-cells of the hom-2-categories, $f^*(g) = g \otimes f$.
- On the 1-cells $\alpha : g \to h$, $f^*(\alpha) = \alpha \otimes 1_f$.
- On the 2-cells $\Gamma : \alpha \Rightarrow \beta$, $f^*(\Gamma) = \Gamma \otimes 1_{1_f}$.

Since the hom-bicategories for T are strict 2-categories and the composition functor is cubical, we have that

$$\begin{aligned} f^*(\beta \circ \alpha) &= (\beta \circ \alpha) \otimes 1_f \\ &= (\beta \circ \alpha) \otimes (1_f \circ 1_f) \\ &= \beta \otimes 1_f \circ \alpha \otimes 1_f \\ &= f^*(\beta) \circ f^*(\alpha), \end{aligned}$$

so f^* strictly preserves composition. Composition being cubical also forces f^* to strictly preserve units, thus proving that f^* is a strict 2-functor.

For $\alpha : f \to f'$, we define the transformation $T(-,a)(\alpha) : f^* \Rightarrow f'^*$ (now denoted α^*) as follows.

- For $g : b' \to a$, the component α^*_g is $1_g \otimes \alpha : g \otimes f \to g \otimes f'$.
- For a 1-cell $\beta : g \to g'$, we define the 2-cell α^*_β to be the inverse of the structure 2-cell for cubical composition.

$$\begin{tikzcd} & g' \otimes f \arrow[dr, "1\otimes\alpha"] \arrow[dd, phantom, "\Downarrow \gamma^{-1}_{\beta,\alpha}"] & \\ g \otimes f \arrow[ur, "\beta\otimes 1"] \arrow[dr, "1\otimes\alpha"'] & & g' \otimes f' \\ & g \otimes f' \arrow[ur, "\beta\otimes 1"'] & \end{tikzcd}$$

- The transformation axioms follow immediately from the cubical functor axioms.

For $\Gamma : \alpha \Rightarrow \alpha'$, we define the modification $T(-,a)(\Gamma) : \alpha^* \Rrightarrow \alpha'^*$ (now denoted Γ^*) by the following.

- For a 0-cell g in the hom-2-category, the component Γ_g is $1_{1_g} \otimes \Gamma$.

- The modification axiom is a result of the naturality axioms for the cubical composition.

Now that we have defined $T(-,a)$ on cells, we must show that it is a functor when equipped with appropriate constraint data. First, we check that it defines a homomorphism of bicategories on the appropriate hom-bicategories. It is clear that composition of 3-cells is preserved strictly, as are identity 3-cells; therefore we have functors

$$T^{\mathrm{op}}(b,b')(g,g') \to \mathbf{Gray}\Big(T(b,a),T(b',a)\Big)(g^*,g'^*).$$

Now let $\alpha : f \to f'$ and $\alpha' : f' \to f''$ be 1-cells in $T^{\mathrm{op}}(b,b')$. Then $(\alpha' \circ \alpha)^*$ has component at g

$$1_g \otimes (\alpha' \circ \alpha) = 1_g \otimes \alpha' \circ 1_g = \alpha'^*_g \circ \alpha^*_g \otimes \alpha$$

by the same argument as above. By the characterization of cubical functors, it is easy to see that the 2-cells $(\alpha' \circ \alpha)^*_\beta$ and $\alpha'^*_\beta \circ \alpha^*_\beta$ are equal as well. Thus we see that on the hom-bicategories – which are actually strict 2-categories – we have defined strict functors.

Next we construct the adjoint equivalence χ for $T(-,a)$. This consists of a pair of transformations and a pair of invertible modifications satisfying the triangle identities. The transformation χ has component at $h \in T^{\mathrm{op}}(x,y)$ the associator $a_{hfg} : (h \otimes f) \otimes g \to h \otimes (f \otimes g)$, so that the adjoint equivalence χ is just the adjoint equivalence $\mathbf{a}$ (for T) with two of the variables held fixed.

The adjoint equivalence $\boldsymbol{\iota}$ is just the opposite of the adjoint equivalence $\mathbf{r}$ for T. The invertible modification ω is a mate of π (for T), and the invertible modifications γ and δ are mates of ρ and μ, respectively.

The first functor axiom follows from the first tricategory axiom, and the second functor axiom follows from the third tricategory axiom. □

Lemma 8.1.2. *Let $f : a \to a'$ be a 1-cell of T. Then there is a transformation $T(-,f) : T(-,a) \to T(-,a')$ whose component at the object b is a functor which is $g \mapsto f \otimes g$ on objects.*

Proof. The component at an object b will be the strict 2-functor

$$f_* : T(b,a) \to T(b,a')$$

defined by

- $f_*(g) = f \otimes g$,
- $f_*(\alpha) = 1_f \otimes \alpha$, and
- $f_*(\Gamma) = 1_{1_f} \otimes \Gamma$.

This is a 2-functor by the same arguments used to show that f^* is a 2-functor. Next we construct an adjoint equivalence

$$\mathbf{T}(-, \mathbf{f}) : (f_*)_* \circ T(-, a) \to (f_*)^* \circ T(-, a')$$

in the appropriate functor bicategory. First, we must define the transformation $T(-, f)$ to have a component at $g : b \to b'$ (in T^{op}); this component will be a 1-cell in $\mathbf{Gray}(T(a, b), T(a', b'))$, that is, a transformation between strict 2-functors. The source 2-functor is defined on objects by

$$j \mapsto f \otimes (j \otimes g),$$

and the target 2-functor is defined on objects by

$$j \mapsto (f \otimes j) \otimes g.$$

The adjoint equivalence is then the opposite of the adjoint equivalence $\mathbf{a}$ (since $\mathbf{a}$ is the associativity adjoint equivalence for T, this is actually the associativity adjoint equivalence for T^{op}). The invertible modification Π is the mate of π^{-1} with source $a \circ (a^\bullet \otimes 1) \circ a^\bullet$ and target $a^\bullet \circ (1 \otimes a)$. The invertible modification M is the mate of ρ^{-1} with source $a^\bullet \circ (1 \otimes r^\bullet)$ and target $r^\bullet$.

The first transformation axiom follows from the first tricategory axiom, the second is proved using the strategies outlined in Appendix C, and the third is an immediate consequence of the third tricategory axiom. $\square$

Lemma 8.1.3. *Let $\alpha : f \Rightarrow f'$ be a 2-cell in T. Then there is a modification $T(-, \alpha) : T(-, f) \Rightarrow T(-, f')$ whose component at the object b is a transformation whose component at g is*

$$f \otimes g \xrightarrow{\alpha \otimes 1} f' \otimes g.$$

Proof. A transformation has as its data components at objects and naturality isomorphisms for each 1-cell. The naturality isomorphism is the modification which is given componentwise by the isomorphism γ^{-1} given by the cubical composition.

The invertible modification $T(-, \alpha)$ is defined to have its component at j be the naturality isomorphism for $a^\bullet$.

The two modification axioms are consequences of the fact that Π and M given in the previous lemma are modifications.

$\square$

Lemma 8.1.4. *Let $\Gamma : \alpha \Rrightarrow \alpha'$ be a 3-cell in T. Then there is a perturbation $T(-, \Gamma) : T(-, \alpha) \Rrightarrow T(-, \alpha')$ whose component at the object b is the modification whose component at g is*

$$\alpha \otimes 1_g \xRightarrow{\Gamma \otimes 1} \alpha' \otimes 1_g.$$

Proof. The single axiom is trivial using the naturality of the isomorphism γ^{-1} that is the naturality isomorphism for $T(-, \alpha)$. $\square$

Theorem 8.1.5. *Let T be a cubical tricategory. Then there is a functor*

$$y : T \to \mathbf{Tricat}(T^{op}, \mathbf{Gray})$$

that is defined on cells as below.

$$\begin{aligned} a &\mapsto T(-, a) \\ f &\mapsto T(-, f) \\ \alpha &\mapsto T(-, \alpha) \\ \Gamma &\mapsto T(-, \Gamma) \end{aligned}$$

Proof. Now that we have defined y on cells, we must examine its functoriality and provide constraints to give it the structure of a functor between tricategories. For ease of notation, we will write the composition in $\mathbf{Tricat}(T^{\mathrm{op}}, \mathbf{Gray})$ as $\boxtimes$. First, we examine y on hom-bicategories, which in our case are strict 2-categories. It is immediate from the definition given in Lemma 8.1.4 that y strictly preserves identity 3-cells and that

$$y(\Gamma \circ \Delta) = y(\Gamma) \circ y(\Delta).$$

Finally, we need to compare $y(\alpha'\alpha)$ to $y(\alpha')y(\alpha)$, where we are writing the composition of 1-cells in the hom-bicategories as concatenation. Since the composition in T is cubical, the transformations $y(\alpha'\alpha)_b$ and $\Big(y(\alpha')y(\alpha)\Big)_b$ have the same components at g; similarly, these transformations have the same naturality isomorphisms by the cubical composition axioms. We now compare the invertible modifications $y(\alpha'\alpha)_g$ and $\Big(y(\alpha')y(\alpha)\Big)_g$. It follows from the fact that T is locally a 2-category and that its composition is cubical that these two modifications have the same components, hence are in fact equal. It follows similarly that if α is the identity, then so is $y(\alpha)$. Thus y is given the structure of a strict 2-functor on each of the hom-2-categories.

Next, we must define an adjoint equivalence $\boldsymbol{\chi} : \boxtimes \circ y \times y \Rightarrow y \circ \boxtimes$. For an object of the source (g, f), we need a 1-cell

$$y(g) \boxtimes y(f) \to y(g \otimes f).$$

Such a 1-cell is a modification between transformations; the component at an object b of T is the transformation $a^{\centerdot}$. The required invertible modification is the naturality isomorphism for $a^{\centerdot}$. The adjoint $\chi^{\centerdot}$ is defined similarly, and the unit and counit for this adjunction are given by the inverses of the units and counits for the adjoint equivalence $\mathbf{a}$.

Next, we must determine the unit adjoint equivalence $\boldsymbol{\iota}$. The modification ι has source $1_{y(a)}$ and target $y(I_a)$. Thus we define the component at b to be the transformation $l^{\centerdot}$. The required invertible modification is the naturality isomorphism for $l^{\centerdot}$. The rest of the definition is made in analogy with the definition of χ.

The invertible modification ω is the mate of π^{-1} with source $(a \otimes 1) \circ a^{\centerdot} \circ a^{\centerdot}$ and target $a^{\centerdot} \circ (1 \otimes a^{\centerdot})$. The invertible modification γ is the mate of λ with source $(l \otimes 1) \circ a^{\bullet} \circ l^{\bullet}$ and target the identity; the invertible modification δ is defined similarly.

The first functor axiom follows immediately from the first tricategory axiom. The second functor axiom then follows immediately from the second tricategory axiom. ☐

Theorem 8.1.6 (Cubical Yoneda Lemma). *Let T be a cubical tricategory, and $y : T \to \mathbf{Tricat}(T^{op}, \mathbf{Gray})$ be the functor constructed above. Then y is a local biequivalence, i.e., each 2-functor*

$$y_{a,a'} : T(a,a') \to \mathbf{Tricat}(T^{op}, \mathbf{Gray})\Big(T(-,a), T(-,a')\Big)$$

is a biequivalence.

Proof. We must show that this 2-functor is locally an equivalence and is biessentially surjective.

1. The 2-functor $y_{a,a'}$ is locally faithful.

Let $\Delta, \Gamma : \alpha \Rightarrow \beta$ be parallel 2-cells in $T(a,a')$, and assume that $y(\Gamma) = y(\Delta)$. Two perturbations are equal if and only if they have identical components for all objects. Thus we see that $\Gamma \otimes 1_{1_g} = \Delta \otimes 1_{1_g}$ for all $g : b \to a$. In particular, taking $b = a$ and $g = I_a$, we get that $\Delta \otimes 1_{1_I} = \Gamma \otimes 1_{1_I}$. The following diagram commutes by the naturality of r.

$$\begin{tikzcd}
f\otimes I_a \arrow[r, "\alpha\otimes 1"] \arrow[d, "r_f"'] \arrow[dr, phantom, "\Swarrow r_\alpha"] & f'\otimes I_a \arrow[d, "r_{f'}"] & & f\otimes I_a \arrow[r, bend left=50, "\alpha\otimes 1"] \arrow[r, phantom, bend left=25, "\Downarrow\Gamma\otimes 1"] \arrow[r, "\beta\otimes 1"'] \arrow[d, "r_f"'] \arrow[dr, phantom, "\Swarrow r_\beta"] & f'\otimes I_a \arrow[d, "r_{f'}"] \\
f \arrow[r, "\alpha"] \arrow[r, bend right=50, "\beta"'] \arrow[r, phantom, bend right=25, "\Downarrow\Gamma"] & f' & = & f \arrow[r, "\beta"'] & f'
\end{tikzcd}$$

This gives the following equality of 2-cells in the 2-category $T(a,b)$, using the same diagram with Δ instead of Γ.

$$(\Gamma * 1_{r_f}) \circ r_\alpha = (\Delta * 1_{r_f}) \circ r_\alpha$$

But since r_α is invertible and r_f is an equivalence 1-cell, this implies that $\Gamma = \Delta$.

2. The 2-functor $y_{a,a'}$ is locally full.

Let $\alpha, \beta : f \to f'$ be parallel 1-cells in $T(a,a')$, and let $\sigma : y(\alpha) \Rightarrow y(\beta)$ be a perturbation between them. Thus for each object b in T^{op}, we have a 3-cell $\sigma_b : y(\alpha)_b \Rrightarrow y(\beta)_b$ in **Gray**. Such a 3-cell consists of a modification between the transformations $y(\alpha)_b$ and $y(\beta)_b$. The modification σ_b has as its component at the object $g \in T(b,a)$ a 2-cell $(\sigma_b)_g : \alpha \otimes 1_g \Rightarrow \beta \otimes 1_g$. Thus we obtain the 2-cell below, denoted $\overline{\sigma}$, where we have taken appropriate mates of the naturality isomorphisms for r to obtain the unmarked cells.

$$
\begin{tikzcd}
f \arrow[r, "r^\bullet"] \arrow[rrr, bend left=50, "\alpha"] \arrow[rrr, bend right=50, "\beta"'] & f\otimes I \arrow[r, bend left, "\alpha\otimes 1_I"] \arrow[r, bend right, "\beta\otimes 1_I"'] \arrow[r, phantom, "\Downarrow (\sigma_a)_I"] & f'\otimes I \arrow[r, "r"] & f'
\end{tikzcd}
$$

We now claim that $y(\overline{\sigma}) = \sigma$. The perturbation $y(\overline{\sigma})$ has as its component at b the modification with component at g given by $\overline{\sigma} \otimes 1_{1_g}$, and we must show that this is equal to $(\sigma_b)_g$. Now the 3-cell $\overline{\sigma} \otimes 1_{1_g}$ is given by the pasting diagram below.

$$
\begin{tikzcd}
f\otimes g \arrow[r, "r^\bullet\otimes 1"] \arrow[rrr, bend left=50, "\alpha\otimes 1"] \arrow[rrr, bend right=50, "\beta\otimes 1"'] & (f\otimes I)\otimes g \arrow[r, bend left, "(\alpha\otimes 1_I)\otimes 1"] \arrow[r, bend right, "(\beta\otimes 1_I)\otimes 1"'] \arrow[r, phantom, "\Downarrow (\sigma_a)_I\otimes 1"] & (f'\otimes I)\otimes g \arrow[r, "r\otimes 1"] & f'\otimes g
\end{tikzcd}
$$

This is equal to the pasting diagram

$$
\begin{tikzcd}
 & f\otimes g \arrow[r, bend left, "\alpha\otimes 1"] \arrow[r, phantom, "\cong"] \arrow[d, "r^\bullet\otimes 1"] & f'\otimes g \arrow[d, "r^\bullet\otimes 1"'] \arrow[dr, phantom, "\cong"] & \\
f\otimes g \arrow[ur, "1"] \arrow[r, "r^\bullet\otimes 1"] \arrow[dr, "1"'] \arrow[ur, phantom, "\cong"'] \arrow[dr, phantom, "\cong"] & (f\otimes I)\otimes g \arrow[r, bend left, "(\alpha\otimes 1)\otimes 1"] \arrow[r, bend right, "(\beta\otimes 1)\otimes 1"'] \arrow[r, phantom, "\Downarrow (\sigma_a)_I\otimes 1"] & (f'\otimes I)\otimes g \arrow[r, "r\otimes 1"] & f'\otimes g \\
 & f\otimes g \arrow[r, bend right, "\beta\otimes 1"'] \arrow[r, phantom, "\cong"] \arrow[u, "r^\bullet\otimes 1"'] & f'\otimes g \arrow[u, "r^\bullet\otimes 1"] \arrow[ur, "1"'] \arrow[ur, phantom, "\cong"] & \\
 & & & f'\otimes g \arrow[from=1-3, to=2-4, "1"]
\end{tikzcd}
$$

by expanding out the mates involved; note that we have used in an essential way that locally T is a 2-category. The unmarked isomorphisms are either naturality isomorphisms (for $r^\bullet$) tensored with an identity or unit isomorphisms (for the adjoint equivalence $\mathbf{r}$) tensored with an identity.

Since each σ_b is a modification and σ is a perturbation, we have the following equality of 3-cells in T,

$$\begin{tikzcd}[column sep=large]
f\otimes g \arrow[rr, bend left, "\alpha\otimes 1"] \arrow[d, "r^{\bullet}\otimes 1"'] & \arrow[d, phantom, "\cong"] & f'\otimes g \arrow[d, "r^{\bullet}\otimes 1"] \\
(f\otimes I)\otimes g \arrow[rr, bend left, "(\alpha\otimes 1)\otimes 1"] \arrow[rr, bend right, "(\beta\otimes 1)\otimes 1"'] & \Downarrow(\sigma_a)_I\otimes 1 & (f'\otimes I)\otimes g
\end{tikzcd}
=
\begin{tikzcd}[column sep=large]
f\otimes g \arrow[rr, bend left, "\alpha\otimes 1"] \arrow[rr, bend right, "\beta\otimes 1"'] \arrow[d, "r^{\bullet}\otimes 1"'] & \Downarrow(\sigma_b)_g & f'\otimes g \arrow[d, "r^{\bullet}\otimes 1"] \\
(f\otimes I)\otimes g \arrow[rr, bend right, "(\beta\otimes 1)\otimes 1"'] & \cong & (f'\otimes I)\otimes g
\end{tikzcd}$$

where once again the unmarked isomorphisms are naturality isomorphisms for $r^{\bullet}$ tensored with identities. Combining the above pasting diagram with this equality gives that $(y(\overline{\sigma})_b)_g = (\sigma_b)_g$ since the rest of the cells in the resulting diagram are pairs of isomorphisms with their inverses.

3. The 2-functor $y_{a,a'}$ is locally essentially surjective.
To show this, let $\alpha : y(f) \to y(f')$ be a modification. We must show that there is a 2-cell $\overline{\alpha} : f \to f'$ and an invertible perturbation $y(\overline{\alpha}) \cong \alpha$.

The component of α at the object b in T^{op} is a transformation α_b with component

$$(\alpha_b)_g : f \otimes g \to f' \otimes g$$

and naturality isomorphism shown below.

$$\begin{tikzcd}
f\otimes g \arrow[r, "(\alpha_b)_g"] \arrow[d, "1\otimes\beta"'] & f'\otimes g \arrow[d, "1\otimes\beta"] \\
f\otimes g' \arrow[r, "(\alpha_b)_{g'}"'] \arrow[ur, phantom, "\overset{(\alpha_b)_\beta}{\nearrow}"] & f'\otimes g'
\end{tikzcd}$$

In particular, we also have the 2-cell in T shown below.

$$f \xrightarrow{r^{\bullet}} f \otimes I \xrightarrow{(\alpha_a)_I} f' \otimes I \xrightarrow{r} f'$$

We shall denote this 2-cell by $\overline{\alpha}$, and the claim is that $y(\overline{\alpha}) \cong \alpha$ in the functor tricategory. An invertible perturbation exhibiting such an isomorphism would have data consisting of, for every object b in T^{op}, an invertible modification $y(\overline{\alpha})_b \Rrightarrow \alpha_b$. This would consist of, for every $g : b \to a$ in T, an isomorphism between $(\alpha_b)_g$ and $(y(\overline{\alpha})_b)_g$; since the composition in T is cubical, this is an isomorphism between $(\alpha_b)_g$ and

$$f \otimes g \xrightarrow{r^{\bullet}\otimes 1} (f \otimes I) \otimes g \xrightarrow{(\alpha_b)_I \otimes 1} (f' \otimes I) \otimes g \xrightarrow{r\otimes 1} f' \otimes g$$

satisfying the axiom for being a modification. The data for α also gives, for every $j : b \to b'$ in T^{op}, an invertible 3-cell α_j in **Gray**. Such an invertible modification gives an isomorphism

$$(\alpha_j)_g : \big((\alpha_b)_g \otimes 1\big) \circ a^{\bullet} \Rightarrow a^{\bullet} \circ (\alpha_b)_{g\otimes j}.$$

Thus the required perturbation has its component at g given by the following pasting diagram,

$$\begin{tikzcd}
f \otimes g \arrow[r, "r^{\bullet} \otimes 1"] \arrow[ddr, "1"'] \arrow[dr, phantom, "\Swarrow"] & (f \otimes I) \otimes g \arrow[r, "(\alpha_b)_I \otimes 1"] \arrow[d, "a"] \arrow[dr, phantom, "\Swarrow"] & (f' \otimes I) \otimes g \arrow[r, "r \otimes 1"] \arrow[d, "a"] \arrow[dr, phantom, "\Swarrow"] & f' \otimes g \arrow[ddl, "1"] \\
 & f \otimes (I \otimes g) \arrow[r, "(\alpha_b)_{I \otimes g}"'] \arrow[d, "1 \otimes l"'] \arrow[dr, phantom, "\Swarrow"] & f' \otimes (I \otimes g) \arrow[d, "1 \otimes l"] & \\
 & f \otimes g \arrow[r, "(\alpha_b)_g"'] & f' \otimes g &
\end{tikzcd}$$

where the triangular regions are μ and the appropriate mate of μ from left to right, the top square is the mate of $(\alpha_g)_I$, and the bottom square is the naturality isomorphism for α. These 3-cells piece together to give an invertible modification.

The single perturbation axiom then holds since this is a modification.

4. The 2-functor $y_{a,a'}$ is locally biessentially surjective.

Let $f : T(-,a) \to T(-,a')$ be any transformation. Then the component at a of this transformation gives a functor $f_a : T(a,a) \to T(a,a')$. Evaluation at I_a then gives $f_a(I_a) : a \to a'$, which we now write as $\overline{f}$. The claim is that $y(\overline{f})$ is equivalent to f.

We will construct a modification $\alpha : f \Rightarrow y(\overline{f})$ that is an equivalence; for a modification to be an equivalence, it suffices that each component α_x is an equivalence 2-cell in the hom-bicategory of the target. Thus such an equivalence modification requires, for each object b in T, a transformation $f_b \Rightarrow y(\overline{(}f)_b$ that is an equivalence. Such a transformation has its component at $g : b \to a$ an equivalence $f_b(g) \to f_a(I_a) \otimes g$.

The transformation f gives, for every $\beta : b \to b'$ in T, an adjoint equivalence between the functors $\beta^* \circ f_{b'}$ and $f_b \circ \beta^*$. Setting $\beta = g$ and evaluating at I_a, we get an equivalence $f_b(I_a \otimes g) \to f_g(b)$. Composing this with the equivalence $f_b(g) \to f_b(I_a \otimes g)$ given by $f_b(l^{\bullet})$, we produce the desired component of the transformation. The naturality isomorphism and the transformation axioms follow from those of f and $l^{\bullet}$.

The modification α also requires an invertible 3-cell α_h in **Gray** for each 1-cell h of T. This is easily constructed as the composite of Π for the transformation f, coherence isomorphisms from T, and naturality isomorphisms for the transformation f. Coherence and the transformation axioms for f imply that α_h is indeed a modification, and that the modification axioms hold for α. Thus y is locally biessentially surjective. □

Remark 8.1.7. The proof given here is very similar to the proof in [17], especially the first two parts. The third part differs in that we are required to check different axioms to ensure that the same construction produces the appropriate isomorphism. We have not avoided the calculational work present in [17], rather we have used similar calculations to produce the functor tricategory and to show that our Yoneda embedding is a functor.

Our definition of tricategory should allow for a definition of the tricategory of prerepresentations $\mathbf{Prep}(T)$ analogous to the one given in [17], and there should be a forgetful functor

$$\mathbf{Tricat}(T^{\mathrm{op}}, \mathbf{Gray}) \to \mathbf{Prep}(T).$$

Thus the proof given here should be seen as a lift of the proof in [17] to the functor tricategory.

8.2 Coherence for tricategories

Here we finally give the coherence theorem for tricategories. The proof is simple using the results of the last section and Section 6.1.

Corollary 8.2.1 (Coherence for tricategories)**.** *For every tricategory T there is a* **Gray***-category T' and a triequivalence $T \to T'$ which is an isomorphism on objects.*

Proof. We constructed in Section 6.1 a triequivalence $T \to \mathrm{st}T$ that is the identity on objects. By Theorem 8.1.6, the functor

$$y : \mathrm{st}T \to \mathbf{Tricat}(\mathrm{st}T^{\mathrm{op}}, \mathbf{Gray})$$

is locally a local equivalence. Thus we define T' to have objects $y(a)$, a in T, 1-cells all transformations $y(a) \to y(b)$ that are equivalent to a transformation of the form $y(f)$ in the appropriate hom-2-category, 2-cells all modifications between these, and 3-cells all perturbations between these. This is a sub-**Gray**-category of $\mathbf{Tricat}(\mathrm{st}T^{\mathrm{op}}, \mathbf{Gray})$ by construction, and y provides a triequivalence between $\mathrm{st}T$ and T'. The composite

$$T \to \mathrm{st}T \to T'$$

is the desired triequivalence. □

Corollary 8.2.2. *Every tricategory T with one object is triequivalent to a monoid in the monoidal category* **Gray***.*

Proof. A monoid in **Gray** determines, and is determined by (up to the choice of object), a one-object **Gray**-category. □

Chapter 9

Free tricategories

This chapter will develop the basic tools necessary to construct free tricategories and free **Gray**-categories. First we must decide on the underlying data from which a tricategory is to be generated freely. Second, we must construct both the free tricategory and the free **Gray**-category on this data. This requires a bit of care as one must pay careful attention to how the universal property is stated. Finally, we prove some results analogous to those leading up to the proof of the coherence theorem for bicategories.

9.1 Graphs

The first step in producing a free tricategory is to decide from what data we will generate such a tricategory. The natural choice is that of a bicategory-enriched graph, but we wish to construct free **Gray**-categories as well and so we must also work with category-enriched 2-graphs.

Definition 9.1.1. 1. A *category-enriched 2-graph* X consists of a directed graph $X_1 \rightrightarrows X_0$ along with, for each pair of parallel arrows f, g in X_1, a category $X(f,g)$. The category of category-enriched 2-graphs, written $2Gr(\mathbf{Cat})$, has for morphisms $X \to X'$ the pairs (P, F), where P is a map of the underlying directed graphs and F is a collection of functors $F_{f,g} : X(f,g) \to X'(Pf, Pg)$.
2. A *bicategory-enriched graph* Y consists of a set Y_0 along with, for each pair of elements $a, b \in Y_0$, a bicategory $Y(a,b)$. The category of bicategory-enriched graphs has for morphisms $Y \to Y'$ the pairs (Q, G), where $Q : Y_0 \to Y_0'$ is a function and G is a collection of functors $G_{a,b} : Y(a,b) \to Y'(Qa, Qb)$. We shall write this category as $Gr(\mathbf{Bicat})$.

Notation 9.1.2. We shall denote by $Gr(\mathbf{2Cat})$ the subcategory of the category of bicategory-enriched graphs for which each $Y(a,b)$ is a strict 2-category and each $G_{a,b}$ is a strict 2-functor. We also write $Gr(\mathbf{Bicat}_s)$ for the subcategory of $Gr(\mathbf{Bicat})$ consisting of all the objects and only the maps for which each functor $G_{a,b}$ is a strict functor. There are obvious inclusions

$$Gr(\mathbf{2Cat}) \subset Gr(\mathbf{Bicat}_s) \subset Gr(\mathbf{Bicat}).$$

Remark 9.1.3. 1. There is an obvious forgetful functor $Gr(\mathbf{Bicat}) \to 2Gr(\mathbf{Cat})$. This functor, when restricted to $Gr(\mathbf{Bicat}_s)$, has an obvious left adjoint induced by the free bicategory functor.
2. There is also an obvious underlying 3-globular set functor

$$U : 2Gr(\mathbf{Cat}) \to 3\mathbf{GlobSet}$$

which assigns to the category-enriched 2-graph X the 3-globular set UX having

- $UX_0 = X_0$,
- $UX_1 = X_1$,
- $UX_2 = \coprod_{f,g} \mathrm{ob}X(f,g)$,
- $UX_3 = \coprod_{f,g} \mathrm{ar}X(f,g)$,

and having the obvious source and target functions. The functor U has a left adjoint $F_{\mathbf{Cat}}$, which when applied to a 3-globular set G, produces the category-enriched 2-graph $F_{\mathbf{Cat}}G$ having $F_{\mathbf{Cat}}G_0 = G_0$, $F_{\mathbf{Cat}}G_1 = G_1$, and $F_{\mathbf{Cat}}G(f,g) = F(G_3 \rightrightarrows G_2)$, where F is the free category functor from the category of directed graphs to $\mathbf{Cat}$ and this free category functor is applied to the 2- and 3-cells of G whose 1-cell source is f and whose 1-cell target is g. This is the prototype for how we will construct free tricategories on a category-enriched 2-graph.

Recall that the free bicategory on a category-enriched graph, $\mathcal{F}G$ for G a category-enriched graph, has the following universal property: given any bicategory B and a map of $\mathbf{Cat}$-graphs $G \to B$, there is a unique strict functor $\mathcal{F}G \to B$ making the following triangle commute.

$$\begin{tikzcd} G \arrow[r] \arrow[dr] & \mathcal{F}G \arrow[d] \\ & B \end{tikzcd}$$

Let $\mathcal{F}_B : 2Gr(\mathbf{Cat}) \to Gr(\mathbf{Bicat})$ be the functor that is defined by letting $\mathcal{F}_B X$ be the bicategory-enriched graph with $\mathcal{F}_B X_0 = X_0$, and $\mathcal{F}_B X(x,y) = \mathcal{F}(trX_{x,y})$, where $trX_{x,y}$ is the category-enriched graph with

$$(trX_{x,y})_0 = \{f \in X_1 : s(f) = x, t(f) = y)\}$$

and $(trX_{x,y})(f,g) = X(f,g)$; the functor is defined on morphisms in the obvious fashion. Similarly, we can define a functor

$$\mathcal{F}_{2C} : 2Gr(\mathbf{Cat}) \to Gr(\mathbf{2Cat}).$$

The following result is now an obvious consequence of the coherence theorem for bicategories.

Theorem 9.1.4. *Let X be a category-enriched 2-graph. Then the map $\mathcal{F}_B(X) \to \mathcal{F}_{2C}(X)$ which is the identity on objects and is given on hom-bicategories by the universal property of free bicategories is, for every pair of objects x, y, a strict biequivalence $\mathcal{F}_B X(x,y) \to \mathcal{F}_{2C} X(x,y)$.*

This theorem motivates the following definition.

Definition 9.1.5. A map (Q, G) of bicategory-enriched graphs is a *local biequivalence* if each functor $G_{a,b}$ is a biequivalence. We will say that the map (Q, G) is a *locally strict local biequivalence* if it is a local biequivalence and a map in $Gr(\mathbf{Bicat}_s)$.

9.2 Free tricategories

We will now define the free tricategory, $\mathcal{F}A$, generated by a bicategory-enriched graph A. This tricategory will have a universal property with respect to locally strict maps $A \to T$ of **Bicat**-graphs, and using this we will construct the free tricategory on a category-enriched 2-graph.

Let A be a bicategory-enriched graph. Then the free tricategory on A, denoted $\mathcal{F}A$, has object set

$$\mathrm{ob}\mathcal{F}A = A_0.$$

Let $a, b \in A_0$. Then $\mathcal{F}A(a,b)$ is the bicategory whose objects are built inductively from the basic building blocks

1. objects f of $A(c,d)$ and
2. new objects $I_a : a \to a$

by tensoring when source matches target. Thus a generic object of $\mathcal{F}A(a,b)$ will look like

$$((f \otimes g) \otimes I_{a_2}) \otimes (h \otimes j),$$

where $j \in A(a, a_1)$, $h \in A(a_1, a_2)$, $g \in A(a_2, a_3)$, and $f \in A(a_3, b)$; we write these as $\overline{f}$.

The 1-cells of $\mathcal{F}A(a,b)$ are built from the basic building blocks

1. 1-cells $\alpha : f \to g$ in $A(c,d)$,
2. new 1-cells $i_a : I_a \to I_a$,
3. the constraint cells $l_f : I \otimes f \to f$, $l^{\bullet}_f : f \to I \otimes f$,
4. the constraint cells $r_f : f \otimes I \to f$, $r^{\bullet}_f : f \to f \otimes I$, and
5. the constraint cells $a_{fgh} : (f \otimes g) \otimes h \to f \otimes (g \otimes h)$, $a^{\bullet}_{fgh} : f \otimes (g \otimes h) \to (f \otimes g) \otimes h$

by tensoring along object boundaries and composing along 0-cell boundaries (the 0-cells in each $\mathcal{F}A(a,b)$, not the objects of the new tricategory), subject to the equivalence relation generated by setting

$$(\alpha)\circ(\beta) = (\alpha\circ\beta),$$

where the lefthand side is a composite in the free tricategory while the right is a composite in A.

The 2-cells are similarly built up inductively from the 2-cells in each $A(a,b)$ and from constraint 2-cells. These constraint 2-cells are units and counits for the various adjoint equivalences, the constraint isomorphisms forcing the unit and composition to be functors, π, μ, λ, ρ, and new hom-bicategory constraint cells involving the new 1-cells. These 2-cells are subject to the required relations relating composition along 0- and 1-cell boundaries in the new hom-bicategories with those of the old hom-bicategories, for the functoriality and naturality conditions of the functors, transformations, and modifications involved, for the conditions forcing certain pairs of cells to be adjoint equivalences, and for the three axioms for a tricategory.

Proposition 9.2.1. *Let A be a bicategory-enriched graph. Then there is a locally strict map $i : A \to \mathcal{F}A$ which is the identity on objects and sends each cell to the cell of the same name in $\mathcal{F}A$.*

The following theorem follows immediately from the previous proposition and the definition of strict functor.

Theorem 9.2.2. *Let A be a bicategory-enriched graph, and let T be a tricategory. If $F : A \to T$ is a locally strict map of bicategory-enriched graphs, then there is a unique strict functor $\tilde{F} : \mathcal{F}A \to T$ making the following triangle commute in $Gr(\mathbf{Bicat}_s)$.*

$$\begin{tikzcd} A \arrow[r] \arrow[dr] & \mathcal{F}A \arrow[d] \\ & T \end{tikzcd}$$

Definition 9.2.3. Let X be a category-enriched 2-graph. The *free tricategory on X*, also denoted $\mathcal{F}X$, is $\mathcal{F}(\mathcal{F}_B X)$.

The following corollary provides justification for calling $\mathcal{F}(\mathcal{F}_B X)$ the free tricategory on the category-enriched 2-graph X.

Corollary 9.2.4. *Let X be a category-enriched 2-graph. Then for every tricategory T and every map of category-enriched 2-graphs $F : X \to T$, there is a unique strict functor $\tilde{F} : \mathcal{F}X \to T$ such that the following triangle commutes in $2Gr(\mathbf{Cat})$.*

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & \mathcal{F}X \arrow[d] \\ & T \end{tikzcd}$$

Proof. If we apply the universal properties of both $\mathcal{F}_B$ and $\mathcal{F}$, we have the following diagram.

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & \mathcal{F}_B X \arrow[r] \arrow[d] & \mathcal{F}(\mathcal{F}_B X) \arrow[dl] \\ & T & \end{tikzcd}$$

The result follows immediately, since the middle and rightmost downward arrows are unique once the arrow $X \to T$ is chosen. □

Let $F : X \to Y$ be a map of category-enriched 2-graphs. Then the universal property of $\mathcal{F}$ gives a unique strict functor making the diagram below commute in $2Gr(\mathbf{Cat})$.

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "F"'] & \mathcal{F}X \arrow[d, "!"] \\ Y \arrow[r, "i"'] & \mathcal{F}Y \end{tikzcd}$$

We shall call this functor $\mathcal{F}F$.

Remark 9.2.5. The reader should take care when interpreting these universal properties. It is not possible for the free tricategory construction to give a functor $\mathcal{F} : 2Gr(\mathbf{Cat}) \to \mathbf{Tricat}$ as the induced strict functor from a composite $G \circ F$ of maps in $2Gr(\mathbf{Cat})$ is not the composite in **Tricat** of the individual strict functors $\mathcal{F}G$ and $\mathcal{F}F$. This is due only to the fact that **Tricat** does not form a category even when the morphisms are restricted to strict functors, so long as the usual composition law is retained. On the other hand, it is trivial that $\mathcal{F}$ *does* give a functor

$$\mathcal{F} : 2Gr(\mathbf{Cat}) \to \mathbf{Tricat}_v$$

where $\mathbf{Tricat}_v$ is the category of strict functors with composition $\circ_v$ from Chapter 4. Both of the universal properties given in this section can be reinterpreted as adjunctions between some category of enriched graphs and the category $\mathbf{Tricat}_v$.

Before moving on to the construction of free **Gray**-categories, we prove a much-needed result about the free tricategory construction.

Theorem 9.2.6. *Let A, B be* **Bicat***-graphs, and let $f : A \to B$ be a map between them. If f is a locally strict local biequivalence and an isomorphism on objects, then the strict functor $\mathcal{F}f : \mathcal{F}A \to \mathcal{F}B$ is a local biequivalence, hence a triequivalence.*

Proof. We must show that each functor $\mathcal{F}f_{a,b}$ is locally full, locally faithful, locally essentially surjective, and biessentially surjective. First note that $\mathcal{F}f_{a,b}$ sends constraint cells to constraint cells, tensors to tensors, and compositions to compositions since it is both strict and locally strict.

We prove the first two claims by induction over tensor length. If $\alpha : \overline{g} \Rightarrow \overline{h}$ and $\beta : \overline{g} \Rightarrow \overline{h}$ are parallel 2-cells in $\mathcal{F}A(a,b)$ which are represented by 2-cells in $A(a,b)$, then if is clear that

$$\mathcal{F}f(\alpha) = \mathcal{F}f(\beta) \Longrightarrow \alpha = \beta$$

since f is locally faithful; the same holds if α or β is a constraint cells. This suffices, by induction, to show that $\mathcal{F}f_{a,b}$ is locally faithful as it strictly preserves tensors, all compositions, and the equivalence relation imposed by the tricategory axioms.

To show that $\mathcal{F}f_{a,b}$ is locally full, first let $\beta : \mathcal{F}f(\overline{g}) \Rightarrow \mathcal{F}f(\overline{h})$ be a 2-cell which is represented by a 2-cell in $B(fa, fb)$. Then we can find an $\alpha : \overline{g} \Rightarrow \overline{h}$ such that $\mathcal{F}f(\alpha) = \beta$ since f is locally full; the same holds if β is a constraint cell. Since $\mathcal{F}f_{a,b}$ is strict, tensors of 2-cells and constraint cells are also in the image.

To show that $\mathcal{F}f_{a,b}$ is locally essentially surjective, first note that tensors of isomorphism 2-cells are again isomorphism 2-cells; similarly with horizontal compositions of isomorphism 2-cells. Since every 1-cell of $\mathcal{F}B(fa, fb)$ is built from the 1-cells of the $B(c,d)$'s and new 1-cells, it suffices to show that all of these are isomorphic to the images of 1-cells in $\mathcal{F}A(a,b)$. This follows immediately from the strictness of $\mathcal{F}f$, the fact that each $f_{a,b}$ is a biequivalence, and the fact that f is an isomorphism on objects.

Now all that remains is to show that $\mathcal{F}f_{a,b}$ is biessentially surjective. The proof is analogous to the one given in the previous paragraph. $\square$

9.3 Free Gray-categories

In this section, we construct the free **Gray**-category on a **2Cat**-graph Y. This is less messy than constructing the free tricategory as there are fewer "interesting" pieces of new data to generate.

Let Y be a 2-category-enriched graph, so Y consists of a set Y_0 and for each $a, b \in Y_0$, a 2-category $Y(a,b)$. The free **Gray**-category on Y, denoted $\mathcal{F}_G Y$, has object set

$$\mathrm{ob}\mathcal{F}_G Y = Y_0.$$

The 2-category $\mathcal{F}_G Y(a,b)$ is constructed as follows. The objects of $\mathcal{F}_G Y(a,b)$ are composable strings in

1. the objects $f \in Y(c,d)$ and
2. a new object $I_a : a \to a$ for each $a \in Y_0$,

subject to the condition that $sIt = st$ for all strings s and t. Thus a typical object of $\mathcal{F}_G Y(a,b)$ is

$$fgI_{a_2}hj = fghj,$$

where $j \in Y(a, a_1)$, $h \in Y(a_1, a_2)$, $g \in Y(a_2, a_3)$, and $f \in Y(a_3, b)$; we write these as $\overline{f}$, just as we did in the free tricategory.

The set of 1-cells of $\mathcal{F}_G Y(a,b)$ between strings $f_n \cdots f_1$ and $g_m \cdots g_1$ is empty if $n \neq m$. If $n = m$ then it consists of composites of strings of 1-cells $\alpha_n \cdots \alpha_1$ where at most one α_i is a non-identity 1-cell in some $Y(c,d)$. These strings shall be written $1\alpha_i 1$, indicating that α_i is the 1-cell in the ith position. We subject these to the relation that if $\alpha_k : f_k \to f'_k$ and $\alpha'_k : f'_k \to f''_k$, then the composition of strings $(1\alpha'_k 1) \circ (1\alpha_k 1)$ is equal to the string $1(\alpha'_k \alpha_k)1$.

The basic 2-cells between $1\alpha_i 1$ and $1\beta_i 1$ are of the form $1\Gamma_i 1$ with Γ_i a 2-cell $\alpha_i \Rightarrow \beta_i$, and we impose the same condition on vertical composition that we did on composition of 1-cells. Each 2-cell is a composable string built from formal horizontal composites of basic 2-cells and invertible 2-cells of the form

$$\gamma_{\alpha_i,\beta_j} : (1\alpha_{i+m}1)(1\beta_j 1) \overset{\cong}{\Rightarrow} (1\beta_j 1)(1\alpha_{i+m}1),$$

where the 1-cell $1\beta_j 1$ has length m. We impose on these 2-cells the axioms required for the **Gray** tensor product. Composition of 1-cells is given by concatenation, as is vertical composition of 2-cells. Horizontal composites of 2-cells are obtained in the same fashion that we obtained them for the 2-category $X \otimes Y$ in Section 5.1. Once again, we omit the details for showing that $\mathcal{F}_G Y(a,b)$ is a 2-category.

The last thing to define is a composition map

$$\mathcal{F}_G Y(b,c) \otimes \mathcal{F}_G Y(a,b) \to \mathcal{F}_G Y(a,b),$$

where we must use the Gray tensor product on the left. On the 0-cells of these 2-categories, composition is just concatenation. If $\alpha = (1\alpha_{i_k}1)(1\alpha_{i_{k-1}}1)\cdots(1\alpha_{i_1}1)$ and $\beta = (1\beta_{i_l}1)(1\beta_{i_{l-1}}1)\cdots(1\beta_{i_1}1)$ are 1-cells in $\mathcal{F}_G Y(b,c), \mathcal{F}_G Y(a,b)$, respectively, then $\alpha \otimes \beta$ is the 1-cell given by the string

$$(1\beta_{i_l}1)(1\beta_{i_{l-1}}1)\cdots(1\beta_{i_1}1)(1\alpha_{i_k+m}1)(1\alpha_{i_{k-1}+m}1)\cdots(1\alpha_{i_1+m}1),$$

where each $(1\beta_{i_j}1)$ has length m. The tensor product of a pair of 2-cells from these 2-categories is defined by

$$\Gamma_i \otimes \Delta_j = (1\Delta_j 1) * (1\Gamma_{i+m}1)$$

on basic 2-cells and extended in the obvious manner. The following proposition is now routine to check.

Proposition 9.3.1. *Let Y be a 2-category-enriched graph. Then the data given above for $\mathcal{F}_G Y$ satisfy the axioms for being a* **Gray***-category.*

If Y is a category-enriched 2-graph, then we call $\mathcal{F}_G(\mathcal{F}_{2C} Y)$ the free **Gray**-category on Y. This is justified by the following theorem.

Theorem 9.3.2. *1. Let Y be a 2-category enriched graph. Then for every* **Gray***-category T and every map of 2-category-enriched graphs $F : Y \to T$, there is a unique* **Gray***-functor $\tilde{F} : \mathcal{F}_G Y \to T$ such that the following diagram commutes in $Gr(\mathbf{2Cat})$.*

$$\begin{tikzcd} Y \arrow[r] \arrow[dr] & \mathcal{F}_G Y \arrow[d] \\ & T \end{tikzcd}$$

2. Let X be a category-enriched 2-graph. Then for every **Gray***-category T and every map of category-enriched 2-graphs $F : X \to T$, there is a unique* **Gray***-functor $\tilde{F} : \mathfrak{F}_G(\mathfrak{F}_{2C}X) \to T$ such that the following diagram commutes in $2Gr(\mathbf{Cat})$.*

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & \mathfrak{F}_G(\mathfrak{F}_{2C}X) \arrow[d] \\ & T \end{tikzcd}$$

Proof. The second statement follows from the first just as in the proof of Corollary 9.2.4. The first statement follows immediately by noting that a **Gray**-functor strictly preserves all types of units and compositions, and sends the isomorphism γ in $\mathfrak{F}_G X$ to the corresponding isomorphism in T. Since the entire structure of $\mathfrak{F}_G X$ is built from these cells using the **Gray**-category axioms, the functor $\tilde{F}$ is uniquely determined. □

9.4 Preliminary results

This section is devoted to proving the tricategorical versions of the results in Section 2.3.2. The following lemma has a proof that is completely analogous to the proof given there.

Lemma 9.4.1. *Let $F, G : S \to T$ be functors between tricategories, and $\alpha : F \to G$ be a transformation between them. Assume that F, G agree on objects and that $\alpha_a = I_{Fa}$ for every object a. Then F is 2-locally faithful (2-locally full) if and only if G is 2-locally faithful (2-locally full).*

Definition 9.4.2. Let X, Y be bicategory-enriched graphs, and let $F, G : X \to Y$ be maps between them. The category-enriched 2-graph $\mathrm{Eq}(F, G)$ is defined to have objects those $a \in X_0$ such that $F_0 a = G_0 a$. The category-enriched graph $\mathrm{Eq}(F, G)(a, b)$ has objects pairs $(h, \boldsymbol{\alpha})$ with $h : a \to b$ in X and $\boldsymbol{\alpha} : F(h) \to G(h)$ an adjoint equivalence in Y (with our usual conventions about units, counits, and $\alpha \dashv \alpha^{\bullet}$). The category $\mathrm{Eq}(F, G)(a, b)\Big((h, \boldsymbol{\alpha}), (h', \boldsymbol{\alpha'})\Big)$ has objects the pairs (β, Γ) with $\beta : h \to h'$ in X and Γ an invertible 2-cell in $Y(Fa, Ga)$ of the form

$$\Gamma : G(\beta) \circ \alpha \Rightarrow \alpha' \circ F(\beta).$$

The category $\mathrm{Eq}(F, G)(a, b)\Big((h, \boldsymbol{\alpha}), (h', \boldsymbol{\alpha'})\Big)$ has 1-cells with source (β, Γ) and target (β', Γ') those 2-cells $\Delta : \beta \Rightarrow \beta'$ such that

$$(1_{\alpha'} * F\Delta) \circ \Gamma = \Gamma \circ (G\Delta * 1_\alpha).$$

Lemma 9.4.3. *1. The category-enriched 2-graph $Eq(F, G)$ can be equipped with the structure of a bicategory-enriched graph admitting a locally strict map*

$$\pi : Eq(F, G) \to X.$$

2. If X, Y are tricategories and F, G are functors between them, then $Eq(F, G)$ admits the structure of a tricategory such that

1. *π can be given the structure of a strict functor and*

2. *there is a transformation $\sigma : F\pi \to G\pi$ whose components are $\sigma_a = I_{Fa}$ for every object a.*

Proof. For the first claim, we need to define horizontal compositions, 1-cell identities, and the requisite constraint isomorphisms to provide each category-enriched graph $\mathrm{Eq}(F, G)(a, b)$ with the structure of a bicategory. The 1-cell identity for (h, α) is

$$\big(1_h, (1_\alpha * (\phi_0^F)^{-1}) \circ r_\alpha^{-1} \circ l_\alpha \circ (\phi_0^G * 1_\alpha)\big).$$

Composition of 1-cells is given by setting the first component of $(\beta', \Gamma') \circ (\beta, \Gamma)$ equal to $\beta' \circ \beta$ and the second component equal to the pasting diagram below.

$$\begin{tikzcd}
 & Fh \arrow[r, "\alpha"] \arrow[d, "F\beta"'] \arrow[dd, bend right=70, "F(\beta'\beta)"'] & Gh \arrow[d, "G\beta"] \arrow[dd, bend left=70, "G(\beta'\beta)"] \arrow[dl, phantom, "\swarrow\Gamma"] & \\
\Leftarrow_{\phi^F} & Fh' \arrow[r, "\alpha'"] \arrow[d, "F\beta'"'] & Gh' \arrow[d, "G\beta'"] \arrow[dl, phantom, "\swarrow\Gamma'"] & \Leftarrow_{(\phi^G)^{-1}} \\
 & Fh'' \arrow[r, "\alpha''"'] & Gh'' &
\end{tikzcd}$$

Horizontal composition of 2-cells is given by horizontal composition in Y; it is simple to check that this gives a composition functor. The constraint 2-cells are all given by the constraint 2-cells in the hom-bicategories of Y, and coherence implies that these satisfy the two bicategory axioms. We now define π by

$$\begin{aligned}
\pi(h, \alpha) &= h \\
\pi(\beta, \Gamma) &= \beta \\
\pi(\Delta) &= \Delta.
\end{aligned}$$

It is trivial to check that we can equip π with the structure of a map in $Gr(\mathbf{Bicat}_s)$.

For the second claim, we must first give a composition functor

$$\mathrm{Eq}(F, G)(b, c) \times \mathrm{Eq}(F, G)(a, b) \to \mathrm{Eq}(F, G)(a, c).$$

On objects, we define $(h', \boldsymbol{\alpha'}) \otimes (h, \boldsymbol{\alpha})$ to have its first component be $h' \otimes h$ and its second component have left adjoint be given by the following composite.

$$F(h' \otimes h) \xrightarrow{\chi^\bullet} F(h') \otimes F(h) \xrightarrow{\alpha' \otimes \alpha} G(h') \otimes G(h) \xrightarrow{\chi} G(h' \otimes h)$$

The remainder of the adjoint equivalence is then defined in the obvious way. On 1-cells, we define

$$(\delta', \Gamma') \otimes (\delta, \Gamma) : (h', \boldsymbol{\alpha'}) \otimes (h, \boldsymbol{\alpha}) \to (j', \boldsymbol{\beta'}) \otimes (j, \boldsymbol{\beta})$$

to have its first component be $\delta' \otimes \delta$. The second component is defined by the pasting diagram below. (Note that we have used $u = (G\delta' \circ \alpha') \otimes (G\delta \circ \alpha)$ and $v = (\beta' \circ F\delta') \otimes (\beta \circ F\delta)$ for space reasons.)

$$\begin{tikzcd}
F(h'\otimes h) \arrow[r, "\chi^{\bullet}"] \arrow[d, "F(\delta'\otimes\delta)"'] \arrow[dr, phantom, "\cong"] & Fh'\otimes Fh \arrow[r, "\alpha'\otimes\alpha"] \arrow[d, "F\delta'\otimes F\delta"'] \arrow[dr, bend right, "v"'] \arrow[dr, bend left, "u"] \arrow[dr, phantom, "\Swarrow \Gamma'\otimes\Gamma"] & Gh'\otimes Gh \arrow[r, "\chi"] \arrow[d, "G\delta'\otimes G\delta"] \arrow[dr, phantom, "\cong"] & G(h'\otimes h) \arrow[d, "G(\delta'\otimes\delta)"] \\
F(j'\otimes j) \arrow[r, "\chi^{\bullet}"'] & Fj'\otimes Fj \arrow[r, "\beta'\otimes\beta"'] & Gj'\otimes Gj \arrow[r, "\chi"'] & G(j'\otimes j)
\end{tikzcd}$$

The isomorphisms in the square regions are naturality isomorphisms and the isomorphisms in the triangular regions are the functoriality isomorphisms of $\otimes$. It is immediate that this is an invertible 2-cell.

On 2-cells, we define the composition $\Delta' \otimes \Delta$ by the same formula in X. Naturality of the isomorphisms $F\beta' \circ F\beta \Rightarrow F(\beta' \circ \beta)$, $G\beta' \circ G\beta \Rightarrow G(\beta' \circ \beta)$ ensures that this cell satisfies the required axiom. The unit constraint cell is given by the isomorphism $1 \otimes 1 \cong 1$ for the functor $\otimes$, and the constraint cell for composition is given by the isomorphism

$$(j' \otimes j) \circ (h' \otimes h) \cong (j' \circ h') \otimes (j \circ h)$$

obtained from the functor $\otimes$. Coherence for functors implies that the requisite diagrams commute.

The associativity transformation a is defined to have its component at the triple $(h'', \boldsymbol{\alpha''}), (h', \boldsymbol{\alpha'}), (h, \boldsymbol{\alpha})$ be given by the 1-cell with first component $a_{h''h'h}$ and second component the composite below.

$$\begin{tikzcd}
F\big((h''\otimes h')\otimes h\big) \arrow[r, "\chi^{\bullet}"] \arrow[ddd, "Fa"'] & F(h''\otimes h')\otimes Fh \arrow[r, "\big((\chi\circ\alpha''\otimes\alpha')\circ\chi^{\bullet}\big)\otimes\alpha"] \arrow[d, "\chi^{\bullet}\otimes 1"'] \arrow[dr, phantom, "\cong"] & G(h''\otimes h')\otimes Gh \arrow[r, "\chi"] & G\big((h''\otimes h')\otimes h\big) \arrow[ddd, "Ga"] \\
\arrow[r, phantom, "\Downarrow"] & (Fh''\otimes Fh')\otimes Fh \arrow[r, "(\alpha''\otimes\alpha')\otimes\alpha"'] \arrow[d, "a"'] \arrow[dr, phantom, "\Swarrow"] & (Gh''\otimes Gh')\otimes Gh \arrow[u, "\chi\otimes 1"'] \arrow[d, "a"] \arrow[r, phantom, "\Downarrow"] & {} \\
& Fh''\otimes(Fh'\otimes Fh) \arrow[r, "\alpha''\otimes(\alpha'\otimes\alpha)"] \arrow[dr, phantom, "\cong"] & Gh''\otimes(Gh'\otimes Gh) \arrow[d, "1\otimes\chi"] & \\
F\big(h''\otimes(h'\otimes h)\big) \arrow[r, "\chi^{\bullet}"'] & Fh''\otimes F(h'\otimes h) \arrow[r, "\alpha''\otimes\big((\chi\circ\alpha'\otimes\alpha)\circ\chi^{\bullet}\big)"'] \arrow[u, "1\otimes\chi^{\bullet}"] & Gh''\otimes G(h'\otimes h) \arrow[r, "\chi"'] & G\big(h''\otimes(h'\otimes h)\big)
\end{tikzcd}$$

The 2-cells in the diagram are given by the mate of ω^F on the left, ω^G on the right, a naturality isomorphism in the middle square, and unique coherence cells in the top and bottom middle regions. The 2-cell $a^{\bullet}$ is defined similarly, and the unit and counit of this adjoint equivalence are given by the unit and counit for $\mathbf{a}$ in X. The naturality isomorphisms are also given by the naturality isomorphisms for $a, a^{\bullet}$ in X, and it is a simple matter to check that this gives an adjoint equivalence in the appropriate functor-bicategory.

The unit $(I_a, \underline{i}) : a \to a$ is defined by setting $\underline{i}$ equal to the composite below.

$$FI_a \xrightarrow{\iota^{\bullet}} I_{Fa} \xrightarrow{\iota} GI_a$$

The left unit transformation l is defined to have component $l_{(h,\alpha)}$ with first component l_h and second component the composite below.

$$\begin{tikzcd}
F(I_b \otimes h) \arrow[r, "\chi^{\bullet}"] \arrow[dd, "Fl"'] \arrow[dr, phantom, "\Downarrow l^{\bullet}"] & FI_b \otimes Fh \arrow[r, "\underline{i} \otimes \alpha"] \arrow[d, "\iota^{\bullet} \otimes 1"'] \arrow[dr, phantom, "\cong"] & GI_b \otimes Gh \arrow[r, "\chi"] \arrow[dr, phantom, "\Downarrow l"] & G(I_b \otimes h) \arrow[dd, "Gl"] \\
& I_{Fb} \otimes Fh \arrow[r, "1 \otimes \alpha"'] \arrow[d, phantom, "\cong"] & I_{Gb} \otimes Gh \arrow[u, "\iota \otimes 1"'] \arrow[dr, "l"] & \\
Fh \arrow[ur, "l^{\bullet}"] \arrow[rrr, "\alpha"'] & & & Gh
\end{tikzcd}$$

The upper left and upper right 2-cells are the mates of γ^F and γ^G, respectively, and the upper middle 2-cell is a unique coherence cell while the lower middle 2-cell is the mate of the naturality isomorphism for l. The naturality isomorphism for l is given by the naturality isomorphism in X. A similar definition gives $l^{\bullet}$, and the unit and counit of this adjoint equivalence are the same as those for $\mathbf{l}$ in X. The same definitions give the adjoint equivalence $\mathbf{r}$.

The modifications π, μ, λ, and ρ are given by those same modifications in X. A lengthy calculation shows that these are 3-cells in $\mathrm{Eq}(F, G)$. This data obviously satisfies the axioms necessary for $\mathrm{Eq}(F, G)$ to be a tricategory as they are the same axioms that hold in X. Thus we have given $\mathrm{Eq}(F, G)$ the structure of a tricategory. It is immediate that we can choose the adjoint equivalence χ for the functor π to be the identity adjoint equivalence, similarly for ι. The rest of the proof that we can equip π with the structure of a strict functor is trivial.

The transformation σ is constructed as follows. The component at a is $I_{Fa} = I_{Ga}$ as stipulated above. The adjoint equivalences σ are defined by

$$\sigma_{(h,\boldsymbol{\alpha})} = (r^{\bullet}_{Gh} \circ \alpha) \circ l_{Fh}$$
$$\sigma^{\bullet}_{(h,\boldsymbol{\alpha})} = (l^{\bullet}_{Fh} \circ \alpha^{\bullet}) \circ r_{Gh},$$

with the obvious units and counits defined by the units and counits of the adjoint equivalence α as well as those for $\mathbf{l}, \mathbf{r}$. For $(\beta, \Gamma) : (h, \boldsymbol{\alpha}) \to (h', \boldsymbol{\alpha'})$, we have the invertible 3-cell displayed below.

$$\begin{tikzcd}
I \otimes Fh \arrow[r, "l"] \arrow[d, "1 \otimes F\beta"'] \arrow[dr, phantom, "\cong"] & Fh \arrow[r, "\alpha"] \arrow[d, "F\beta"'] \arrow[dr, phantom, "\swarrow \Gamma"] & Gh \arrow[r, "r^{\bullet}"] \arrow[d, "G\beta"] \arrow[dr, phantom, "\cong"] & Gh \otimes I \arrow[d, "G\beta \otimes 1"] \\
I \otimes Fh' \arrow[r, "l"'] & Fh' \arrow[r, "\alpha'"'] & Gh' \arrow[r, "r^{\bullet}"'] & Gh' \otimes I
\end{tikzcd}$$

The outer cells are naturality isomorphisms, and thus naturality follows from the definition of the 3-cells in $\mathrm{Eq}(F,G)$.

The 3-cell $\Pi_{(h',\boldsymbol{\alpha'}),(h,\boldsymbol{\alpha})}$ is given by the pasting diagram below, where we have written $\otimes$ as concatenation on 1-cells.

$$\begin{tikzcd}
(IFh')Fh \arrow[rr, "((r^\bullet\circ\alpha')\circ l)\otimes 1"] \arrow[d, "a"'] \arrow[dddr, bend left, "l\otimes 1"] \arrow[rr, phantom, shift right=8ex, "C"] & & (Gh'I)Fh \arrow[r, "a"] \arrow[dr, phantom, "\Downarrow\mu"] & Gh'(IFh) \arrow[rr, "1\otimes((r^\bullet\circ\alpha)\circ l)"] \arrow[dd, "1\otimes l"] \arrow[ddrr, phantom, "C"] & & Gh'(GhI) \arrow[dd, "a^\bullet"] \\
I(Fh'Fh) \arrow[d, "1\otimes\chi"] \arrow[d, bend right=60, "1\otimes(\chi\circ 1)"'] \arrow[d, phantom, shift right=3ex, "R"] \arrow[ddr, bend left, "l"] \arrow[r, phantom, "\Downarrow\lambda"] & & & & \arrow[r, phantom, "\Downarrow\rho"] & \\
IF(h'h) \arrow[d, "l"] \arrow[dr, phantom, "N"] & & Gh'Fh \arrow[r, "1"] \arrow[uu, "r^\bullet\otimes 1"] \arrow[d, phantom, "C"] & Gh'Fh \arrow[d, "1\otimes\alpha"] & & (Gh'Gh)I \arrow[d, "\chi\otimes 1"'] \arrow[d, bend left=60, "(\chi\circ 1)\otimes 1"] \arrow[d, phantom, shift left=3ex, "R"] \\
F(h'h) \arrow[r, "\chi^\bullet"'] & Fh'Fh \arrow[rr, "\alpha'\otimes\alpha"'] \arrow[ur, "\alpha'\otimes 1"] & & Gh'Gh \arrow[r, "\chi"'] \arrow[uuurr, "1\otimes r^\bullet"] \arrow[urr, bend right, "r^\bullet"'] \arrow[urr, phantom, shift right=3ex, "N"] & G(h'h) \arrow[r, "r^\bullet"'] & G(h'h)I
\end{tikzcd}$$

The regions marked with R are unique isomorphisms involving a right unit isomorphism, regions marked with C are unique isomorphisms involving the functoriality of $\otimes$ as well as unit isomorphisms, regions marked with N are naturality isomorphisms (or their mates), and regions marked with Greek letters are the appropriate 3-cells in Y (or their mates).

The 3-cell M_a is given by the pasting diagram below.

$$\begin{tikzcd}
I \arrow[r, "r^\bullet"] \arrow[ddd, "l^\bullet"'] \arrow[dr, phantom, "\cong"] & I\otimes I \arrow[r, "1\otimes\iota"'] \arrow[r, bend left, "1\otimes(F1\circ\iota)"] \arrow[r, phantom, shift left=2ex, "\cong"] \arrow[dr, bend right, "1"'] & I\otimes FI \arrow[r, "l"] \arrow[d, "1\otimes\iota^\bullet"] \arrow[dl, phantom, "\Swarrow"] & FI \arrow[d, "\iota^\bullet"] \arrow[dl, phantom, "\Swarrow"] \\
& & I\otimes I \arrow[r, "l"] \arrow[ddll, "1"] \arrow[d, phantom, "\cong"] & I \arrow[d, "\iota"] \arrow[ddlll, "r^\bullet"] \\
& & \arrow[r, phantom, "\Longleftarrow"] & GI \arrow[d, "r^\bullet"] \\
I\otimes I \arrow[rrr, "\iota\otimes 1"] \arrow[rrr, bend right, "(G1\circ\iota)\otimes 1"'] \arrow[rrr, phantom, shift right=2ex, "\cong"] & & & GI\otimes I
\end{tikzcd}$$

The isomorphisms in the top and bottom of the diagram are the obvious composites of unit isomorphisms with functoriality isomorphisms; the isomorphisms in the middle of the diagram are obtained from the isomorphism $l_I \cong r_I$ given in Appendix A; the 2-cell in the upper-middle triangular region is the obvious composite of a functoriality isomorphism, the inverse of a unit isomorphism for the adjoint equivalence, and a unit isomorphism for the $\otimes$; and the other 2-cells are naturality isomorphisms.

These 3-cells give modifications between the appropriate transformations since they are composed of modifications or naturality isomorphisms.

The three transformation axioms follow from the strategies outlined in Appendix C and the functor axioms. $\square$

The following lemma is straightforward to prove.

Lemma 9.4.4. *Let $\alpha : F \to G$ be a transformation between functors of tricategories. Let $\beta_a : Fa \to Ga$ be a family of 1-cells in the target indexed by the objects of the source. Let $\mathbf{m_a} : \alpha_a \to \beta_a$ be a family of adjoint equivalences indexed by the objects of the source. Then there is a transformation β with components given by the cells β_a and a modification $m : \alpha \Rightarrow \beta$ with components given by the cells m_a.*

Corollary 9.4.5. *Let $\alpha : F \to G$ and $\beta : G \to H$ be transformations. Assume that F, G, H agree on objects, and assume that there are adjoint equivalences $\mathbf{m_a}$ between α_a and I_{Fa} and between $\mathbf{n_a}$ and I_{Ga}. Then there is a transformation $\gamma : F \to H$ with $\gamma_a = I_{Fa}$ and a modification $nm : \beta\alpha \Rightarrow \gamma$ with each component 2-cell an equivalence in the appropriate bicategory.*

Proof. This follows immediately from the lemma and the definition of the composite transformation $\beta\alpha$. □

Proposition 9.4.6. *Let X be a category-enriched 2-graph, and let $F : \mathcal{F}X \to T$ be a functor from a free tricategory into any tricategory. Then there exists a strict functor $G : \mathcal{F}X \to T$ and a transformation $\alpha : F \to G$ with $\alpha_a = I_{Fa}$ for every object a.*

Proof. Let $i : X \to \mathcal{F}X$ denote the usual inclusion. By definition, there is a strict functor $G : \mathcal{F}X \to T$ with $Fi = Gi$. There is also a map of category-enriched 2-graphs $\iota : X \to \mathrm{Eq}(F, G)$ which is the identity on objects and is defined by $\iota(f) = (f, 1_{Ff})$, $\iota(\alpha) = (\alpha, l_{F\alpha}^{-1} \circ r_{G\alpha})$, and $\iota(\Gamma) = \Gamma$. Note that $\pi\iota = i$ by construction.

By the universal property of $\mathcal{F}X$, there is a unique strict functor $\tilde{\iota} : \mathcal{F}X \to \mathrm{Eq}(F, G)$ such that $\tilde{\iota}i = \iota$. This gives the equality $\pi\tilde{\iota}i = \pi\iota = i$ as maps of category-enriched 2-graphs. Now π and $\tilde{\iota}$ are both strict, so $\pi \circ_v \tilde{\iota}$ is as well, and it has the same underlying map of category-enriched 2-graphs as $\pi\tilde{\iota}$. This implies that $(\pi \circ_v \tilde{\iota})i = i$, so $\pi \circ_v \tilde{\iota}$ is the identity functor on $\mathcal{F}X$.

Consider the following composite of transformations.

$$\begin{gathered} F \xrightarrow{r^\bullet} F \circ (\pi \circ_v \tilde{\iota}) \xrightarrow{1_F * \phi} F \circ (\pi \circ \tilde{\iota}) \xrightarrow{a^\bullet} (F \circ \pi) \circ \tilde{\iota} \xrightarrow{\sigma * 1_{\tilde{\iota}}} \\ (G \circ \pi) \circ \tilde{\iota} \xrightarrow{a} G \circ (\pi \circ \tilde{\iota}) \xrightarrow{1_G * \phi^\bullet} G \circ (\pi \circ_v \tilde{\iota}) \xrightarrow{r} G \end{gathered}$$

Each of these transformations has component at the object a equivalent to an identity 1-cell, so by repeated application of the lemma and its corollary there is a transformation $\alpha : F \to G$ with components $\alpha_a = I_{Fa}$. □

Chapter 10

Coherence via free constructions

In this chapter, we will prove a coherence theorem of the form "every free tricategory is triequivalent to a free **Gray**-category." As in the case of the coherence theory for bicategories, we will then use this result to prove that diagrams of constraint 3-cells of a certain type always commute. This result differs from the analogous theorem for bicategories in that only some diagrams commute for tricategories but all diagrams of constraint 2-cells commute in a bicategory.

With this theorem in hand, we can mimic the proofs in [22] to construct, for each tricategory T, a strictification $\mathrm{Gr}T$ and a triequivalence $\mathrm{Gr}T \to T$. This strictification functor will have a distinguished pseudo-inverse, and both of these triequivalences will be used in later sections to explore the coherence theory for functors between tricategories.

10.1 Coherence for tricategories

Let X be a category-enriched 2-graph. Then the inclusion $X \hookrightarrow \mathcal{F}_G(\mathcal{F}_{2C}X)$ induces a strict functor

$$\Gamma : \mathcal{F}(\mathcal{F}_B X) \to \mathcal{F}_G(\mathcal{F}_{2C}X)$$

by the universal property of the free tricategory. Thus our coherence theorem for tricategories is as follows.

Theorem 10.1.1 (Coherence for tricategories)**.** *Let X be a category enriched 2-graph. Then the strict functor*

$$\Gamma : \mathcal{F}(\mathcal{F}_B X) \to \mathcal{F}_G(\mathcal{F}_{2C}X)$$

is a triequivalence between the free tricategory on X and the free **Gray**-*category on X.*

Before proving this theorem, we need two results just as in the proof of coherence for bicategories. The first is that Γ has a universal property.

Lemma 10.1.2. *Let X be a category-enriched 2-graph, and let $F : \mathcal{F}X \to G$ be a strict functor into a* **Gray***-category G. Then there exists a unique strict functor $F_s : \mathcal{F}_G(\mathcal{F}_{2C}X) \to G$ such that $F = F_s\Gamma$ as maps of the underlying* **Bicat***-graphs.*

The second result we need is a simple construction which allows us to extend maps of **Bicat**-graphs $X \to T$ with T a tricategory to maps of **Bicat**-graphs $\mathcal{F}_G X \to T$.

Lemma 10.1.3. *Let $f : X \to T$ be a map of* **Bicat***-graphs from a 2-category-enriched graph X into a tricategory T. Then it is possible to extend f to a map of bicategory-enriched graphs $\hat{f} : \mathcal{F}_G X \to T$ such that the following diagram commutes in $Gr($***Bicat***$)$.*

$$\begin{tikzcd}
X \arrow[r] \arrow[dr] & \mathcal{F}_G X \arrow[d] \\
& T
\end{tikzcd}$$

Proof. The object function $\hat{f}_0$ is the same as f_0. Now let a, b be objects of X. We define

$$\hat{f}_{a,b} : \mathcal{F}_G X(a,b) \to T(fa, fb)$$

to be the weak functor given by the following data. On the object $\overline{h} = h_n \cdots h_1$, we define

$$\hat{f}(\overline{h}) = (\cdots(fh_n \otimes fh_{n-1}) \otimes fh_{n-2}) \otimes \cdots \otimes fh_2) \otimes fh_1.$$

On the basic 1-cell $1\alpha_i 1$, we define

$$\hat{f}(1\alpha_i 1) = (\cdots(1 \otimes 1) \otimes \cdots \otimes f\alpha_i) \otimes \cdots \otimes 1) \otimes 1.$$

On the 1-cell $\alpha = (1\alpha_{i_n}1, 1\alpha_{i_{n-1}}1, \ldots, 1\alpha_{i_1}1)$, we define

$$\hat{f}(\alpha) = \big(\cdots\big(\hat{f}(1\alpha_{i_n}1) \circ \hat{f}(1\alpha_{i_{n-1}}1)\big) \circ \cdots \circ \hat{f}(1\alpha_{i_2}1)\big) \circ \hat{f}(1\alpha_{i_1}1);$$

we also set $\hat{f}(I_a) = I_{fa}$. On a basic 2-cell $1\Gamma_i 1 : 1\alpha_i 1 \Rightarrow 1\beta_i 1$, we define

$$\hat{f}(1\Gamma_i 1) = (\cdots(1 \otimes 1) \otimes \cdots \otimes f\Gamma_i) \otimes \cdots \otimes 1) \otimes 1.$$

We extend this to strings of basic 2-cells in analogy with how we defined $\hat{f}$ on strings of 1-cells. We define $\hat{f}(\gamma_{\alpha_i,\beta_j})$ to be the canonical isomorphism given by the functoriality constraint in T of the functor $\otimes$. This is extended over composites of 2-cells in the obvious fashion, and clearly gives a map of category-enriched 2-graphs.

Now we need to give structure constraints $\hat{f}(\beta) \circ \hat{f}(\alpha) \cong \hat{f}(\beta\alpha)$ and $\hat{f}(1_h) \cong 1_{\hat{f}h}$. The first of these is given by the associativity constraint in the target bicategory and the second is given by the unit constraint for f. Coherence for functors implies that the two axioms are satisfied, hence we have given a map of bicategory-enriched graphs $\mathcal{F}_G X \to T$. $\square$

Proof of 10.1.1. First, note that Γ is the identity on objects, so we need only check that it is a local biequivalence.

1. The functor Γ is 2-locally full, 2-locally essentially surjective, and locally biessentially surjective.

Let M be any 2-category-enriched graph. Note that we have the inclusion $i : M \to \mathcal{F}M$, thus the induced map $\hat{i} : \mathcal{F}_G M \to \mathcal{F}M$ of bicategory-enriched graphs. We also have the strict functor $K : \mathcal{F}M \to \mathcal{F}_G M$ given by the universal property of the free tricategory. It is then easy to check that

$$\mathcal{F}_G M \xrightarrow{\hat{i}} \mathcal{F}M \xrightarrow{K} \mathcal{F}_G M$$

is the identity in $Gr(\mathbf{Bicat})$ using the fact that K is strict. This gives that for every pair of objects a, b in M, the following composite is the identity in the category **Bicat**.

$$\mathcal{F}_G M(a,b) \xrightarrow{\hat{i}} \mathcal{F}M(a,b) \xrightarrow{K} \mathcal{F}_G M(a,b)$$

Now if f is any object of $\mathcal{F}_G M(a,b)$, then $\hat{i}(f)$ is an object of $\mathcal{F}_G M(a,b)$ that maps to f under K, so K is locally biessentially surjective. If $\alpha : Kf \to Kg$ is any 1-cell in $\mathcal{F}_G M(a,b)$, then there are composites of the constraints $a, a^{\bullet}, l, l^{\bullet}, r, r^{\bullet}$ that give a (nonunique) 1-cell

$$c_f : f \to \hat{i}Kf,$$

since f and $\hat{i}Kf$ differ only in association and by the presence of units from the definitions of $\hat{i}$ and K; the same holds for g. Since K maps all of these constraints to identities, the image of

$$f \xrightarrow{c} \hat{i}Kf \xrightarrow{\hat{i}\alpha} \hat{i}Kg \xrightarrow{c^{\bullet}} g$$

is α, so K is 2-locally essentially surjective. The same argument proves that K is 2-locally full.

Specializing to the case when $M = \mathcal{F}_{2C} X$ for some category-enriched 2-graph X, we get that Γ factors as the composite (in the category of bicategory-enriched graphs) $K \circ \mathcal{F}(\Gamma^l)$, where $\Gamma^l : \mathcal{F}_B X \to \mathcal{F}_{2C} X$ is the locally strict local biequivalence given by coherence for bicategories. By Theorem 9.2.6, $\mathcal{F}(\Gamma^{loc})$ is a triequivalence. Therefore both K and $\mathcal{F}(\Gamma^{loc})$ are 2-locally full, 2-locally essentially surjective, and locally biessentially surjective, so Γ is as well.

2. The functor Γ is 2-locally faithful.

First, we have a 2-locally faithful functor $H : \mathcal{F}X \to G$ into a **Gray**-category G by the coherence theorem for tricategories. Thus we can produce a strict $K : \mathcal{F}X \to G$ and a transformation $\alpha : H \to K$ with $\alpha_a = I_{Ha}$. The universal property of Γ then gives a functor J with $J\Gamma = K$ as maps of the underlying **Bicat**-graphs. We know that K is 2-locally faithful since H is and there is a transformation α with components the identity, so Γ is 2-locally faithful as well. □

10.2 Coherence and diagrams of constraints

An important type of coherence theorem is one stating that a certain large class of diagrams commutes. In this section, we develop one such theorem as it will be necessary for constructing strictifications. In practice, this is perhaps the most useful form of coherence as it allows one to avoid checking diagrams by hand.

Before proving this theorem, we first recall how it is possible to prove that every diagram of constraint 2-cells in a bicategory commutes using the fact that the strict functor $\mathcal{F}X \to \mathcal{F}_s X$ is a biequivalence between the free bicategory on a category-enriched graph and the free 2-category on the same graph. Given a diagram of constraint 2-cells in a bicategory B, there is a locally discrete sub-category-enriched graph D of B for which the diagram in question is the image, under the strict functor $\mathcal{F}D \to B$, of a diagram in $\mathcal{F}D$. Thus proving that the diagram commutes in B reduces to proving that it commutes in $\mathcal{F}D$. Now the diagram in question is mapped to a composite of identities in $\mathcal{F}_s D$, thus commutes there. But since the map $\mathcal{F}D \to \mathcal{F}_s D$ is a biequivalence, it is locally an equivalence of categories and therefore the original diagram commutes in $\mathcal{F}D$ as well.

We follow an analogous strategy using the free tricategory and free **Gray**-category functors. The first step is proving that, in certain free **Gray**-categories, every diagram of 3-cells commutes. A simple definition is required before proving this.

Definition 10.2.1. A category-enriched 2-graph X is *2-locally discrete* if each category $X(f, g)$ is a discrete category.

Theorem 10.2.2. *Let X be a 2-locally discrete category-enriched 2-graph. Then in the free* **Gray***-category on X, every diagram of 3-cells commutes.*

Proof. First note that since X is 2-locally discrete, each 2-category $\mathcal{F}_{2C}X(a, b)$ is locally discrete. Thus every 3-cell in $\mathcal{F}_G(\mathcal{F}_{2C}X)$ is a composite of the isomorphisms

$$\gamma_{\alpha_i,\beta_j} : (1\alpha_{i+m}1)(1\beta_j 1) \Rightarrow (1\beta_j 1)(1\alpha_{i+m}1),$$

where $1\beta_j 1$ has length m. It suffices to prove that any two composites of the isomorphisms $\gamma_{\alpha_i,\beta_j}$ with the same source and target are equal, and since everything is invertible, it suffices to prove that any 3-cell $\alpha \Rrightarrow \alpha$ is the identity. We prove this by induction over the length n of the string $\alpha = (1\alpha_{i_n}1, 1\alpha_{i_{n-1}}1, \ldots, 1\alpha_{i_1}1)$.

When $n = 1$, it is clear that the only 3-cell is the identity. When $n = 2$, the only morphism $(1\alpha_{2,i_2+m}1, 1\alpha_{1,i_1}1) \Rrightarrow (1\alpha_{1,i_1}1, 1\alpha_{2,i_2+m}1)$ is $\gamma_{\alpha_{2,i_2},\alpha_{1,i_1}}$. The same applies with source and target switched, so the only 3-cell with source and target both $(1\alpha_{2,i_2+m}1, 1\alpha_{1,i_1}1)$ is the identity. When $n = 3$, the only new case holds by the axioms for the **Gray** tensor product.

The claim is that every morphism $\Delta : \alpha \Rrightarrow \alpha$ is equal to one of the form $\Delta' * 1$. For simplicity, we write (k, j) for $\gamma_{\alpha_{k,i_k},\alpha_{j,i_j}}$. Thus we can represent each 3-cell Δ as a string $(k_p, j_p), (k_{p-1}, j_{p-1}), \ldots, (k_1, j_1)$ and each source or target 2-cell of this 3-cell as $(\alpha_{\tau(n)}, \alpha_{\tau(n-1)}, \ldots, \alpha_{\tau(1)})$ for some permutation $\tau \in \Sigma_n$. For each i with $1 \leq i \leq p$, we write the target of $(k_i, j_i), \ldots, (j_1, k_1)$ as $(\alpha_{[i](n)}, \ldots, \alpha_{[i](1)})$ where $[i]$ is a permutation in Σ_n. Using this notation, $[i](m) = l$ if in the target of $(k_i, j_i), \ldots, (j_1, k_1)$, $1\alpha_{l,i_l}1$ is in position m.

Fix a string $S = (k_p, j_p), (k_{p-1}, j_{p-1}), \ldots, (k_1, j_1)$ whose composite is Δ. For such a string representing Δ, we associate a positive integer $\mathrm{Ht}\Delta_S$, the height of Δ as represented by the string S. We define the height by

$$\mathrm{Ht}\Delta_S = \max_{1 \leq i \leq p} [i](1).$$

If $\mathrm{Ht}\Delta_S = 1$, then none of the j_l or k_l is 1, and thus we have written Δ as a composite of 3-cells all of which fix $1\alpha_{1,i_1}1$, so $\Delta = \Delta' * 1$.

Assume that $\mathrm{Ht}\Delta_S = H$. Let q be the largest index for which $[q](H) = 1$ and let r be the largest index $r \leq q$ for which $[r](H-1) = 1$. Thus q is the last time $1\alpha_{1,i_1}1$ is as far left as possible, and r is the last time $1\alpha_{1,i_1}1$ is at position $H-1$ before moving to position H for the last time. If we let Δ_a^- denote the composite $(k_a, j_a), \ldots, (k_1, j_1)$ then Δ_{q+1}^- can be written as the composite below.

$$\begin{array}{l}
\alpha_n, \ldots, \alpha_1 \xrightarrow{\Delta_r^-} \alpha_{[r](n)}, \ldots, \alpha_{[r](H)}, \alpha_1, \alpha_{[r](H-2)}, \ldots, \alpha_{[r](1)} \xrightarrow{1*([r](H),1)*1} \\
\alpha_{[r](n)}, \ldots, \alpha_1, \alpha_{[r](H)}, \ldots, \alpha_{[r](1)} \xrightarrow{=} \\
\alpha_{[r+1](n)}, \ldots, \alpha_1, \alpha_{[r+1](H-1)}, \ldots, \alpha_{[r+1](1)} \xrightarrow{!*1*!} \\
\alpha_{[q](n)}, \ldots, \alpha_1, \alpha_{[q](H-1)}, \ldots, \alpha_{[q](1)} \xrightarrow{1*(1,[q](H-1))*1} \\
\alpha_{[q](n)}, \ldots, \alpha_{[q](H)}, \alpha_1, \ldots, \alpha_{[q](1)} \xrightarrow{=} \\
\alpha_{[q+1](n)}, \ldots, \alpha_{[q+1](H)}, \alpha_1, \ldots, \alpha_{[q+1](1)}
\end{array}$$

The equalities are both by the definition of the indices, and the exclamation marks indicate unique isomorphisms given by induction. Using interchange, we can rewrite Δ_r^- to include the left unique isomorphism on the second line; therefore, after Δ_r^- until the end of Δ_{q+1}^-, we can discard all of the string to the left of position H as all of the morphisms are the identity on those 2-cells. Rewriting gives that the remaining composite is equal to the leftmost composite in the diagram below.

$$\begin{tikzcd}
 & \alpha_{[r](H)}\alpha_1\cdots\alpha_{[r](1)} \arrow[dl, "{([r](H),1)*1}"'] \arrow[dr, "1*1*!"] & \\
\alpha_1\alpha_{[r](H)}\cdots\alpha_{[r](1)} \arrow[dd, "1*1*!"'] & & \alpha_{[r](H)}\alpha_1\alpha_{[q](H-1)}\tilde{\alpha}_{[q](H-3)}\cdots\tilde{\alpha}_{[q](1)} \arrow[ddll, "{([r](H),1)*1}"'] \arrow[d, "{1*(1,[q](H-1))*1}"] \\
 & & \alpha_{[r](H)}\alpha_{[q](H-1)}\alpha_1\tilde{\alpha}_{[q](H-3)}\cdots\tilde{\alpha}_{[q](1)} \arrow[d, "{([r](H),[q](H-1))*1}"] \\
\alpha_1\alpha_{[r+1](H-1)}\alpha_{[q](H-1)}\tilde{\alpha}_{[q](H-3)}\cdots\tilde{\alpha}_{[q](1)} \arrow[d, "{1*([r+1],[q])*1}"'] & & \alpha_{[q](H-1)}\alpha_{[r](H)}\alpha_1\tilde{\alpha}_{[q](H-3)}\cdots\tilde{\alpha}_{[q](1)} \arrow[d, "{1*([r](H),1)*1}"] \\
\alpha_1\alpha_{[q](H-1)}\alpha_{[q](H-2)}\cdots\alpha_{[q](1)} \arrow[d, "1*1*!"'] \arrow[rr, "{(1,[q](H-1))*1}"] & & \alpha_{[q](H-1)}\alpha_1\alpha_{[q](H-2)}\cdots\alpha_{[q](1)} \arrow[d, "1*1*!"] \\
\alpha_1\alpha_{[q](H-1)}\cdots\alpha_{[q](1)} \arrow[rr, "{(1,[q](H-1))*1}"'] & & \alpha_{[q](H-1)}\alpha_1\alpha_{[q](H-2)}\cdots\alpha_{[q](1)}
\end{tikzcd}$$

We have once again used the convention that an exclamation mark indicates a unique isomorphism by induction and we have repeatedly used that

$$[r+1](H-1) = [r](H).$$

The tildes used indicate that the cells are in the correct order, but that one is missing; thus the only difference between

$$\alpha_{[r+1](H-1)}\tilde{\alpha}_{[q](H-3)}\cdots\tilde{\alpha}_{[q](1)}$$

and

$$\alpha_{[q](H-2)}\alpha_{[q](H-3)}\cdots\alpha_{[q](1)}$$

is that $\alpha_{[r+1](H-1)}$ (which is $\alpha_{[q](a)}$ for some a) has been removed from the sequence and placed at the beginning.

The top and bottom squares commute by interchange, and the middle hexagonal region is equivalent to the axiom for the **Gray** tensor product relating the two ways of switching the order of three 2-cells from *abc* to *cba* (using the conventions established above). Thus we can replace the leftmost composite in the string S representing Δ with the rightmost. Inspecting the diagram then shows that, for the portion of Δ considered here, α_1 is never in position H; thus we have reduced the number of indices q for which $[q](H) = 1$. Repeating this, we can write Δ as a string S' with $\mathrm{Ht}\Delta_{S'} = H-1$, and therefore as a string T with $\mathrm{Ht}\Delta_T = 1$. This proves that $\Delta = \Delta' * 1$ for some 3-cell Δ'. But by induction, we already know that Δ' is the identity since it has length less than n, so Δ must be the identity as well. $\square$

Corollary 10.2.3. *Let X be a 2-locally discrete category-enriched 2-graph. Then in the free tricategory on X, $\mathcal{F}X$, every diagram of 3-cells commutes.*

We now use this result to show that a certain class of diagrams of constraint 3-cells always commute in any tricategory.

Definition 10.2.4. 1. A *diagram D* of constraint 3-cells in a tricategory T consists of two finite sequences $\{a_n, a_{n-1}, \ldots, a_1\}, \{b_m, b_{m-1}, \ldots, b_1\}$ of 3-cells in T such that

- each cell a_i or b_j is the composite via $\otimes$ or horizontal composition in the hom-bicategory of finitely many cells $a_{i,s}$ or $b_{j,t}$, respectively, each of which is a constraint 3-cell or an identity 3-cell,
- the source of a_i is the target of a_{i-1} for all $1 \leq i \leq n$,
- the source of b_j is the target of b_{j-1} for all $1 \leq j \leq m$, and
- the source of a_1 is the source of b_1, and the target of a_n is the target of b_m.

We call the cells $a_{i,s}$, $b_{j,t}$ the *constituent 3-cells* of D.
2. A diagram D of constraint 3-cells in a tricategory T is called *$\mathcal{F}$-admissible* if there is a 2-locally discrete sub-category-enriched 2-graph E of T and a diagram $\tilde{D}$ of constraint 3-cells in $\mathcal{F}E$ such that D is the image of $\tilde{D}$ under the strict functor $\mathcal{F}E \to T$.

The following is now an immediate corollary of the theorem above and the explanation of why every diagram of constraint of constraint 2-cells in a bicategory commutes.

Corollary 10.2.5. *Let T be a tricategory. Then every $\mathcal{F}$-admissible diagram of constraint 3-cells commutes in T.*

Remark 10.2.6. It is easy to construct examples of diagrams of constraint 3-cells that do not automatically commute. Taking our tricategory to be a **Gray**-category for simplicity, let $\alpha, \beta : I \Rightarrow I$ be two 2-cells each with source and target the unit 1-cell. Using the isomorphism γ and the relevant unit structure, it is possible to produce an automorphism of $\beta \otimes \alpha$ that is not required to be the identity using the results above. But in the free tricategory on any 2-locally discrete category-enriched 2-graph, the unit I is only the source (or target) of constraint cells (and composites of these), so the results above do not apply if α and β are not composites of constraint 2-cells.

10.3 Strictifying tricategories

Here we will construct a strictification $\mathrm{Gr}T$ for a tricategory T. The tricategory $\mathrm{Gr}T$ will be a **Gray**-category and will support a triequivalence $\mathrm{Gr}T \to T$.

Definition 10.3.1. Let $f_1, f_2, \ldots, f_n$ be a sequence of composable 1-cells in a tricategory T. Then a *choice of association* for this sequence consists of numbers $i_0, i_1, \ldots, i_{n-1}, i_n$ and a choice of composition, using binary composites, of the composable sequence $I^{i_0}, f_1, I^{i_1}, f_2, I^{i_2}, \ldots, f_n, I^{i_n}$, where I^m indicates that there are m copies of the unit I in the sequence in that position.

The tricategory $\mathrm{Gr}T$ has the same objects as T. The 2-category $\mathrm{Gr}T(a,b)$ has for 0-cells strings of composable 1-cells of T, written $\{f_i\}$. Note that the identity for an object a is the unique empty string beginning and ending at a. A 1-cell $\overline{\alpha} : \{f_i\} \to \{g_j\}$ consists of composable strings of the following:

1. three numbers k, l_1, l_2 with $k \leq l_1$, $k \leq l_2$ such that
   - if $m < k$, then $f_m = g_m$, and
   - if $n > 0$, then $f_{l_1+n} = g_{l_2+n}$ if either side exists;
2. a pair (σ, τ), where σ is a choice of association for the substring $\{f_i\}_{k \leq i \leq l_1}$ and τ is a choice of association for the substring $\{g_j\}_{k \leq j \leq l_2}$;
3. a 1-cell $\alpha : [f_i]_\sigma \to [g_j]_\tau$ in T, where $[f_i]_\sigma$ indicates that we have associated the substring $\{f_i\}_{k \leq i \leq l_1}$ according to σ.

We additionally include the empty 1-cell, denoted $\varnothing$, which is the identity.

Before defining the 2-cells of $\mathrm{Gr}T(a,b)$, we must define an evaluation function $e : \mathrm{Gr}T_2 \to T_2$ on the underlying 2-globular sets $\mathrm{Gr}T_2$ and T_2. On 0-cells, e is the identity function. On 1-cells,

$$e(\{f_i\}) = (\cdots(f_n \otimes f_{n-1}) \otimes f_{n-2}) \otimes \cdots \otimes f_2) \otimes f_1;$$

we write this particular association as $[f_i]$. We also define the value of e on the empty 1-cell from a to a as I_a, so $[\,] = I_a$. For each association γ of n terms using only binary tensors, we write $[f_i]_\gamma$ for the expression with the f_i's tensored together according to γ. For each pair γ, γ' of non-identical associations of n terms, we choose one 2-cell $[\gamma] \Rightarrow [\gamma']$ in $\mathcal{F}[n]$ (the free tricategory on the 2-locally discrete category-enriched 2-graph with underlying directed graph $[n]$ the linear graph with n arrows and each hom-category empty); here we have written $[\gamma]$ to indicate the string of length n with each of the generating arrows appearing once and associated according to γ. We call this 2-cell $a_{\gamma,\gamma'}$, and it induces a 2-cell $[f_i]_\gamma \Rightarrow [f_i]_{\gamma'}$ in T, also written $a_{\gamma,\gamma'}$. Note that since $[\,] = I_a$, we also include "associators" such as $l^\bullet : f \to f \otimes [\,]$.

We can now define e on the 2-cells of $\mathrm{Gr}T_2$. A 2-cell of $\mathrm{Gr}T_2$ is, as defined above, a string of basic cells which each consist of a choice of substring, a choice of associations for the source and target, and an actual cell between those associations. Let α be such a basic cell from $\{f_i\}$ to $\{g_j\}$. If we treat the associated substring $[f_i]_\sigma$ as a single cell, then there is a 1-cell

$$a : [f_i] \to [f_{i+}, [f_i]_\sigma, f_{i-}]$$

where f_{i-} is the string consisting of those cells with index less than k and f_{i+} is the string consisting of those cells with index greater than l_1. Note that we also have a cell

$$a^{\bullet} : [f_{i+}, [f_i]_\sigma, f_{i-}] \to [f_i]$$

given by reversing the order of composition and replacing instances of a_{hgf} with instances of $a^{\bullet}_{hgf}$. We define $e(\alpha)$ to be the cell

$$[f_i] \xrightarrow{a} [f_{i+}, [f_i]_\sigma, f_{i-}] \xrightarrow{(\cdots(1\otimes 1)\otimes\cdots\otimes\alpha)\otimes\cdots\otimes 1} [f_{i+}, [g_j]_\tau, g_{j-}] \xrightarrow{a^{\bullet}} [g_j],$$

where we follow the convention that for unparenthesized strings of length greater than two, we compose using a leftward bias so that

$$\alpha_n \alpha_{n-1} \cdots \alpha_1$$

means the cell

$$(\cdots(\alpha_n \circ \alpha_{n-1}) \circ \alpha_{n-2}) \circ \cdots \circ \alpha_2) \circ \alpha_1.$$

Any 1-cell $\overline{\alpha}$ in $\mathrm{Gr}T(a,b)$ is a string of such basic cells,

$$\overline{\alpha} = \alpha_n \alpha_{n-1} \cdots \alpha_1.$$

We define $e(\overline{\alpha})$ to be the composite below, parenthesized according to our convention.

$$e(\overline{\alpha}) = e(\overline{\alpha}_n) e(\overline{\alpha}_{n-1}) \cdots e(\overline{\alpha}_1)$$

It is immediate that the difference between $e(\overline{\beta}) \circ e(\overline{\alpha})$ and $e(\overline{\beta}\overline{\alpha})$ is merely one of association, so these two cells differ by a unique isomorphism arising from the associativity isomorphism in the hom-bicategory. We also define the value of e on the empty 1-cell to be $(\cdots(1 \otimes 1) \otimes \cdots 1) \otimes 1$.

A 2-cell $\Gamma : \overline{\alpha} \Rightarrow \overline{\beta}$ in $\mathrm{Gr}T(a,b)$ is a 2-cell $\Gamma : e(\overline{\alpha}) \Rightarrow e(\overline{\beta})$ in T. It is now necessary to equip $\mathrm{Gr}T(a,b)$ with compositions and units, and then show that these choices give $\mathrm{Gr}T(a,b)$ the structure of a 2-category. The 1-cell identities are the empty strings, and the 2-cell identities are obtained as the identity 2-cells in T. The composition of 1-cells is given by concatenation of strings, and it is clearly associative and unital. Vertical composition of 2-cells is inherited from T, and hence is strictly associative and unital. Horizontal composition is also inherited from T, in that we define $\Delta * \Gamma$ to be the 2-cell

$$e(\overline{\beta}\overline{\alpha}) \cong e(\overline{\beta}) e(\overline{\alpha}) \xrightarrow{\Delta * \Gamma} e(\overline{\beta'}) e(\overline{\alpha'}) \cong e(\overline{\beta'\alpha'}),$$

where the unlabelled isomorphisms are the unique cells given by the coherence theorem. It follows by the uniqueness of the isomorphisms that composition satisfies interchange and is strictly associative. Thus $\mathrm{Gr}T(a,b)$ is a strict 2-category.

To provide $\mathrm{Gr}T$ with the structure of a **Gray**-category, we must construct a cubical composition functor

$$\star : \mathrm{Gr}T(b,c) \times \mathrm{Gr}T(a,b) \to \mathrm{Gr}T(a,c)$$

and show that it satisfies appropriate associativity and unit conditions. On 0-cells, we define

$$\{f_i\} \star \{g_j\} = \{f_i, g_j\}$$

by concatenating lists. If $(k, l_1, l_2, \sigma, \tau, \alpha) : \{f_i\} \to \{f'_j\}$ is a basic 1-cell and $\{g_h\}$ is any other 0-cell such that $\{f_i\} \star \{g_h\}$ is defined, then there is a basic 1-cell $\alpha \star \varnothing$ given by

$$(k + H, l_1 + H, l_2 + H, \sigma, \tau, \alpha),$$

where H is the length of $\{g_h\}$. This can be extended to an arbitrary 1-cell $\overline{\alpha} = (\alpha_n, \ldots, \alpha_1)$ by

$$\overline{\alpha} \star \varnothing = (\alpha_n \star \varnothing, \ldots, \alpha_1 \star \varnothing),$$

and we can similarly define $\varnothing \star \overline{\alpha}$ when $\{g_h\} \star \{f_i\}$ is defined. Thus we define $\overline{\beta} \star \overline{\alpha}$ to be the cell given by the string

$$(\varnothing \star \overline{\alpha}) \circ (\overline{\beta} \star \varnothing),$$

where composition means concatenation of strings. We define $\varnothing \star \varnothing = \varnothing$.

To define $\star$ on 2-cells, it suffices to define both $1 \star \Gamma : \varnothing \star \overline{\alpha} \Rightarrow \varnothing \star \overline{\beta}$ and $\Gamma \star 1 : \overline{\alpha} \star \varnothing \Rightarrow \overline{\beta} \star \varnothing$ for a 2-cell $\Gamma : \overline{\alpha} \Rightarrow \overline{\beta}$. We then extend this to a definition of $\Gamma \star \Delta : \overline{\alpha} \star \overline{\alpha'} \Rightarrow \overline{\beta} \star \overline{\beta'}$ by the following formula.

$$\Gamma \star \Delta = (1 \star \Delta) * (\Gamma \star 1)$$

To begin, let $\overline{\alpha}, \overline{\beta}$ be 1-cells $\{f_i\} \to \{g_j\}$, and let $\{h_k\}$ be another 0-cell such that $\{h_k\} \star \{f_i\}$ is defined. If $\overline{\alpha}$ and $\overline{\beta}$ are basic 1-cells, then $e(\varnothing \star \overline{\alpha})$ is the 1-cell displayed below.

$$[h_k, f_i] \xrightarrow{a} [h_k, f_{i+}, [f_i]_\sigma, f_{i-}] \xrightarrow{(\cdots\otimes\alpha)\otimes\cdots\otimes 1} [h_k, f_{i+}, [g_j]_\sigma, f_{i-}] \xrightarrow{a^\bullet} [h_k, g_j]$$

This gives the following pasting diagram of isomorphism 2-cells in $T(a, b)$, where the unlabelled 1-cells are given by our choice of associations and all the 2-cell isomorphisms are the unique isomorphisms given by coherence.

$$\begin{tikzcd}
{[h_k,f_i]} \arrow[r, "a"] \arrow[d, "\tilde{a}"'] & {[h_k,f_{i-},[f_i]_\sigma,f_{i+}]} \arrow[r, "(\cdots\otimes\alpha)\otimes\cdots\otimes 1"] \arrow[d] \arrow[dl, phantom, "\cong"] & {[h_k,f_{i-},[g_j]_\sigma,f_{i+}]} \arrow[r, "a^\bullet"] \arrow[d] \arrow[dl, phantom, "\cong"] & {[h_k,g_j]} \arrow[dl, phantom, "\cong\Downarrow"] \\
{[h_k]\otimes[f_i]} \arrow[r, "(\cdots(1\otimes 1)\cdots\otimes 1)\otimes a"'] \arrow[rrr, bend right, "(\cdots(1\otimes 1)\cdots\otimes 1)\otimes\left(a^\bullet(\cdots\otimes\alpha\otimes\cdots\otimes 1)a\right)"'] & {[h_k]\otimes[f_{i-},[f_i]_\sigma,f_{i+}]} \arrow[r, "(\cdots(1\otimes 1)\cdots\otimes 1)\otimes(\cdots\otimes\alpha\cdots 1)"'] \arrow[r, phantom, shift right=5ex, "\Downarrow\cong"] & {[h_k]\otimes[f_{i-},[g_j]_\sigma,f_{i+}]} \arrow[r, "(\cdots(1\otimes 1)\cdots\otimes 1)\otimes a^\bullet"'] & {[h_k]\otimes[g_j]} \arrow[u, "\tilde{a}^\bullet"']
\end{tikzcd}$$

Thus we have a 2-cell isomorphism $e(\varnothing\star\overline{\alpha}) \Rightarrow \tilde{a}^\bullet(e(\varnothing)\otimes e(\overline{\alpha}))\tilde{a}$, so we can define $1 \star \Gamma$ to be the composite

$$e(\varnothing \star \overline{\alpha}) \Rightarrow \tilde{a}^\bullet(e(\varnothing) \otimes e(\overline{\alpha}))\tilde{a} \overset{1*(1\otimes\Gamma)*1}{\Longrightarrow} \tilde{a}^\bullet(e(\varnothing) \otimes e(\overline{\beta}))\tilde{a} \Rightarrow e(\varnothing \star \overline{\beta}).$$

Now we extend this definition to strings of basic cells. Let

$$\overline{\alpha} = (\alpha_n, \ldots, \alpha_1)$$

be a 1-cell in GrT. Using the above construction, we define a canonical isomorphism $e(\varnothing \star \overline{\alpha}) \cong \tilde{a}^{\bullet}\Big(e(\varnothing) \star e(\overline{\alpha})\Big)\tilde{a}$ below.

$$\begin{array}{rcll}
e(\varnothing \star \overline{\alpha}) & = & e(\varnothing \star \alpha_n, \ldots, \varnothing \star \alpha_1) & \text{definition of } \varnothing \star - \\
 & = & e(\varnothing \star \alpha_n) \cdots e(\varnothing \star \alpha_1) & \text{definition of } e \\
 & \cong & \tilde{a}^{\bullet}\big(e(\varnothing) \otimes e(\alpha_n)\big)\tilde{a} \cdots \tilde{a}^{\bullet}\big(e(\varnothing) \otimes e(\alpha_1)\big)\tilde{a} & \text{by the above} \\
 & \cong & \tilde{a}^{\bullet}\big(e(\varnothing) \otimes e(\alpha_n)\big) \cdots \big(e(\varnothing) \otimes e(\alpha_1)\big)\tilde{a} & \text{counit of } \tilde{a} \dashv \tilde{a}^{\bullet} \\
 & \cong & \tilde{a}^{\bullet}\Big(\big(e(\varnothing) \cdots e(\varnothing)\big) \otimes \big(e(\alpha_n) \cdots e(\alpha_1)\big)\Big)\tilde{a} & \text{unique coherence iso} \\
 & \cong & \tilde{a}^{\bullet}\Big(e(\varnothing) \otimes e(\overline{\alpha})\Big)\tilde{a} & \text{unique coherence iso}
\end{array}$$

We now make the same definition of $1 \star \Gamma$ as above, using our canonical isomorphism and its inverse. This immediately implies that $1 \star \Gamma \circ 1 \star \Delta = 1 \star (\Gamma \circ \Delta)$ and $1 \star 1 = 1$.

Assume that Γ and Δ are 3-cells in GrT such that $\Gamma * \Delta$ is defined. We now show that

$$(1 \star \Gamma) * (1 \star \Delta) = 1 \star (\Gamma * \Delta).$$

Note that we have the diagram below in T, where we have focused on the last step of the canonical isomorphism above with the omission of the associators.

$$\begin{tikzcd}
\Big(e(\varnothing)\cdots e(\varnothing)\Big)\otimes\Big(e(\beta_m)\cdots e(\beta_1)e(\alpha_n)\cdots e(\alpha_1)\Big) \arrow[rr, "\cong"] \arrow[dd, "\cong"'] & & \Big(e(\varnothing)\otimes e(\overline{\beta})\Big)\circ\Big(e(\varnothing)\otimes e(\overline{\alpha})\Big) \arrow[d, "1\otimes\Gamma*1\otimes\Delta"] \\
 & & \Big(e(\varnothing)\otimes e(\overline{\beta'})\Big)\circ\Big(e(\varnothing)\otimes e(\overline{\alpha'})\Big) \arrow[d, "\cong"] \\
e(\varnothing)\otimes e(\overline{\beta}\overline{\alpha}) \arrow[r, "1\otimes(\Gamma*\Delta)"'] & e(\varnothing)\otimes e(\overline{\beta'}\,\overline{\alpha'}) \arrow[r, "\cong"'] & e(\varnothing)\otimes e(\overline{\beta'\alpha'})
\end{tikzcd}$$

All of the isomorphisms above are unique coherence isomorphisms. This diagram commutes by the naturality of the various coherence isomorphisms involved. Writing out the composites that give $(1\star\Gamma)*(1\star\Delta)$ and $1\star(\Gamma*\Delta)$, we see that the two composites that make up this diagram appear, one in $(1 \star \Gamma) * (1 \star \Delta)$ and one in $1 \star (\Gamma * \Delta)$. Since the rest of the definitions of these two cells are identical, we can conclude that they are in fact equal. This concludes the proof that $\varnothing \star -$ is a 2-functor; a similar proof shows the same of $- \star \varnothing$.

The final piece of data for the **Gray**-category structure of GrT is an isomorphism

$$\gamma_{\overline{\beta},\overline{\alpha}} : (\overline{\beta} \star \varnothing) \circ (\varnothing \star \overline{\alpha}) \overset{\cong}{\Longrightarrow} (\varnothing \star \overline{\alpha}) \circ (\overline{\beta} \star \varnothing)$$

satisfying three axioms. This amounts to an isomorphism

$$e(\overline{\beta} \star \varnothing)e(\varnothing \star \overline{\alpha}) \overset{\cong}{\Longrightarrow} e(\varnothing \star \overline{\alpha})e(\overline{\beta} \star \varnothing)$$

in T. Assume first that $\overline{\alpha}$ is a basic 2-cell α, and similarly for $\overline{\beta}$. We then define γ by the following pasting diagram of isomorphisms, where the composite around the top and right is $e(\overline{\beta} \star \varnothing)e(\varnothing \star \overline{\alpha})$ and the composite around the left and bottom is $e(\varnothing \star \overline{\alpha})e(\overline{\beta} \star \varnothing)$.

$$\begin{tikzcd}
{[g_k, f_i]} \arrow[r, "a"] \arrow[dd, "a"'] \arrow[dr, phantom, "\cong"] & {[g_{k+}, [g_k]_\tau, g_{k-}, f_i]} \arrow[r, "(\cdots\otimes\beta)\cdots\otimes 1"] \arrow[d, "a"] \arrow[ddr, phantom, "\cong"] & {[g_{k+}, [g'_{k'}]_{\tau'}, g_{k-}, f_i]} \arrow[r, "a^\bullet"] \arrow[ddd, "a"] \arrow[dr, phantom, "\cong"] & {[g'_{k'}, f_i]} \arrow[dd, "a"] \\
& {[g_{k+}, [g_k]_\tau, g_{k-}, f_{i+}, [f_i]_\sigma, f_{i-}]} \arrow[ddd, "(\cdots\otimes\alpha)\cdots\otimes 1"'] \arrow[ddr, "(\cdots\otimes\beta)\cdots\otimes 1"] \arrow[dr, phantom, "\cong"] & & \\
{[g_k, f_{i+}, [f_i]_\sigma, f_{i-}]} \arrow[ur, "a"'] \arrow[ddd, "(\cdots\otimes\alpha)\cdots\otimes 1"'] \arrow[dr, phantom, "\cong"] & & & {[g'_{k'}, f_{i+}, [f_i]_\sigma, f_{i-}]} \arrow[ddd, "(\cdots\otimes\alpha)\cdots\otimes 1"] \\
& & {[g_{k+}, [g'_{k'}]_{\tau'}, g_{k-}, f_{i+}, [f_i]_\sigma, f_{i-}]} \arrow[ur, "a^\bullet"] \arrow[ddd, "(\cdots\otimes\alpha)\cdots\otimes 1"] \arrow[dr, phantom, "\cong"] & \\
& {[g_{k+}, [g_k]_\tau, g_{k-}, f_{i+}, [f'_{i'}]_{\sigma'}, f_{i-}]} \arrow[ddd, "a^\bullet"] \arrow[ddr, "(\cdots\otimes\beta)\cdots\otimes 1"] \arrow[ddr, phantom, "\cong"'] & & \\
{[g_k, f_{i+}, [f'_{i'}]_{\sigma'}, f_{i-}]} \arrow[ur, "a"'] \arrow[dd, "a^\bullet"'] \arrow[dr, phantom, "\cong"] & & & {[g'_{k'}, f_{i+}, [f'_{i'}]_{\sigma'}, f_{i-}]} \arrow[dd, "a^\bullet"] \\
& & {[g_{k+}, [g'_{k'}]_{\tau'}, g_{k-}, f_{i+}, [f'_{i'}]_{\sigma'}, f_{i-}]} \arrow[ur, "a^\bullet"] \arrow[d, "a^\bullet"] \arrow[dr, phantom, "\cong"] & \\
{[g_k, f'_{i'}]} \arrow[r, "a"'] & {[g_{k+}, [g_k]_\tau, g_{k-}, f'_{i'}]} \arrow[r, "(\cdots\otimes\beta)\cdots\otimes 1"'] & {[g_{k+}, [g'_{k'}]_{\tau'}, g_{k-}, f'_{i'}]} \arrow[r, "a^\bullet"'] & {[g'_{k'}, f'_{i'}]}
\end{tikzcd}$$

All of these isomorphisms are unique coherence isomorphisms by coherence. From this, the naturality axiom for γ follows immediately. The other two axioms follow by the uniqueness of the isomorphisms. Thus we have defined γ when α and β are basic 2-cells, and the second and third axioms for the isomorphism γ serve to define it in general.

Theorem 10.3.2. *Let T be a tricategory. Then the definitions above serve to give* $\mathrm{Gr}T$ *the structure of a* **Gray***-category.*

Proof. Since we have already given the cubical composition functor, all that remains is to show that it is strictly unital and associative. The unit condition is trivial as the unit 1-cell is the empty string. For associativity, first note that concatenation of lists is strictly associative, so $\star$ is strictly associative on 1-cells. For 2-cells, we have the following computation.

$$\begin{aligned}
\overline{\delta} \star (\overline{\beta} \star \overline{\alpha}) &= \overline{\delta} \star (\varnothing \star \overline{\alpha} \circ \overline{\beta} \star \varnothing) \\
&= \varnothing \star (\varnothing \star \overline{\alpha} \circ \overline{\beta} \star \varnothing) \circ \overline{\delta} \star \varnothing \\
&= \varnothing \star \varnothing \star \overline{\alpha} \circ \varnothing \star \overline{\beta} \star \varnothing \circ \overline{\delta} \star \varnothing \star \varnothing \\
&= \varnothing \star \overline{\alpha} \circ (\varnothing \star \overline{\beta} \circ \overline{\delta} \star \varnothing) \star \varnothing \\
&= (\overline{\delta} \star \overline{\beta}) \star \overline{\alpha}
\end{aligned}$$

A similar calculation, using the naturality of the associativity constraint in the hom-bicategory, shows that $\star$ is strictly associative on the 3-cells of $\mathrm{Gr}T$. ☐

Theorem 10.3.3. *The map e of the underlying 2-globular sets $\mathrm{Gr}T_2 \to T_2$ can be extended to a map of category-enriched 2-graphs by setting $e(\Gamma) = \Gamma$. This map can then be given the structure of a functor $\mathrm{Gr}T \to T$. This functor is a triequivalence.*

Proof. The first statement is trivial. For the second, we must construct the remaining data for the functor e. Restricting to the hom-bicategories, we have a map of category-enriched graphs

$$e_{ab} : \mathrm{Gr}T(a,b) \to T(a,b);$$

this is given the structure of a functor of bicategories by using the coherence isomorphisms of the hom-bicategory for structure constraints and the definition of e. Coherence for bicategories then immediately implies that the necessary diagrams commute.

The transformation χ has component at $\{g_j\}, \{f_i\}$ the 2-cell

$$[g_j] \otimes [f_i] \xrightarrow{a} [g_j, f_i]$$

chosen previously. For space purposes, we will write $\underline{\alpha}$ for the cell abbreviated

$$(\cdots \otimes \alpha) \otimes 1) \otimes \cdots \otimes 1$$

above. There is a unique coherence isomorphism

$$e(\underline{\beta}) \otimes e(\underline{\alpha}) \cong a^{\bullet} \otimes a^{\bullet} \circ \underline{\beta} \otimes \underline{\alpha} \circ a \otimes a$$

given by coherence for functors. Upon composition with the inverse of this isomorphism, the naturality isomorphism for χ at the pair of basic 2-cells β, α is given by the pasting diagram below.

$$\begin{tikzcd}
{[g_j]\otimes[f_i]} \arrow[r, "a"] \arrow[dd, "a\otimes a"'] & {[g_j,f_i]} \arrow[r, "a"] & {[g_{j+},[g_j]_\tau,g_{j-},f_i]} \arrow[d, "\underline{\beta}"] \\
& & {[g_{j+},[g'_{j'}]_{\tau'},g_{j-},f_i]} \arrow[dd, "a^{\bullet}"] \\
{[g_{j+},[g_j]_\tau,g_{j-}]\otimes[f_{i+},[f_i]_\sigma,f_{i-}]} \arrow[uurr, "a"] \arrow[dr, "\underline{\beta}\otimes 1"] \arrow[dd, "\underline{\beta}\otimes\underline{\alpha}"'] & & \\
& {[g_{j+},[g'_{j'}]_{\tau'},g_{j-}]\otimes[f_{i+},[f_i]_\sigma,f_{i-}]} \arrow[uur, "a"] \arrow[dl, "1\otimes\underline{\alpha}"] \arrow[ddr, "a"] & {[g'_{j'},f_i]} \arrow[dd, "a"] \\
{[g_{j+},[g'_{j'}]_{\tau'},g_{j-}]\otimes[f_{i+},[f'_{i'}]_{\sigma'},f_{i-}]} \arrow[dd, "a^{\bullet}\otimes a^{\bullet}"'] \arrow[ddrr, "a"] & & \\
& & {[g'_{j'},f_{i+},[f_i]_\sigma,f_{i-}]} \arrow[d, "\underline{\alpha}"] \\
{[g'_{j'}]\otimes[f'_{i'}]} \arrow[r, "a"'] & {[g'_{j'},f'_{i'}]} & {[g'_{j'},f_{i+},[f'_{i'}]_{\sigma'},f_{i-}]} \arrow[l, "a^{\bullet}"]
\arrow[from=1-1, to=3-2, phantom, "\cong"]
\arrow[from=2-2, to=3-2, phantom, "\cong"]
\arrow[from=3-1, to=5-2, phantom, "\cong"]
\arrow[from=4-2, to=4-3, phantom, "\cong"]
\arrow[from=5-2, to=6-2, phantom, "\cong"]
\arrow[from=5-1, to=7-2, phantom, "\cong"]
\end{tikzcd}$$

The isomorphisms are all unique coherence isomorphisms. Uniqueness then gives the definition for when the cells involved are not basic 2-cells. The transformation $\chi^{\bullet}$ is defined in precisely the same fashion, using $a^{\bullet}$ instead of a; the unit and counit of this adjoint equivalence are given by those for a and $a^{\bullet}$.

Both I_{ea} and $e(I_a)$ are the identity 1-cell I_a. Thus we define the adjoint equivalence ι to be the identity adjoint equivalence.

The modifications ω, γ, and δ are all given by unique coherence cells by coherence. From this we also see that the required axioms hold.

Now we must show that e is a triequivalence. First, it is surjective on objects. Given objects a, b in GrT, we must show that each functor Gr$T(a,b) \to T(a,b)$ is a biequivalence. It is surjective on 0-cells since each 0-cell f is the image of the string $\{f\}$. Now let $\{f_i\}$ and $\{g_j\}$ be 0-cells with length I, J, respectively. Any $\alpha : [f_i] \to [g_j]$ is the image of $(0, I, J, [\,], [\,], \alpha)$ (where $[\,]$ refers to our standard association) by definition. Finally, this functor is clearly 2-locally full and faithful by the definition of GrT. $\square$

Remark 10.3.4. In our construction of GrT, it was required that we make arbitrary choices of 2-cell associators $a_{\gamma,\gamma'}$. This construction depended on these choices, as did the construction of the constraint data for the triequivalence e. If we denote the set of these associators by A, then our definitions are actually of a **Gray**-category Gr(T, A) and a triequivalence $e_A :$ Gr$(T, A) \to T$. For a different set of associators A', there is a strict triequivalence $C_{A,A'} :$ Gr$(T, A) \to$ Gr(T, A') which is the identity on 0-, 1-, and 2-cells and is compatible with the evaluation triequivalences e_A and $e_{A'}$ in the sense that there is a transformation $\alpha : e_A \to e_{A'} C_{A,A'}$ whose component at each object is the identity, whose component at each 1-cell f is $r^{\bullet}l$, and whose modifications Π and M are both given by unique coherence isomorphisms. From this point forward, we will assume that a single choice of A has been made and that Gr(T) means Gr(T, A) for this choice of A for all tricategories T.

Now we construct the (essentially obvious) pseudoinverse to e, denoted f as in the case of bicategories.

Theorem 10.3.5. *The map $f : T \to$ GrT of category-enriched 2-graphs given by*

$$\begin{aligned} f(x) &= x \\ f(g) &= \{g\} \\ f(\alpha) &= (0, 1, 1, [\,], [\,], \alpha) \\ f(\Gamma) &= (1 * \Gamma) * 1 \end{aligned}$$

can be given the structure of a functor. This functor is a triequivalence.

Proof. For the first claim, we need to give the rest of the data for f to be a functor and check the required axioms. First, we need to give structure constraints to make f a map of bicategory-enriched graphs. The composition constraint $f(\beta) \circ f(\alpha) \cong f(\beta \circ \alpha)$ is the unique coherence isomorphism in the hom-bicategory; the same is true of the constraint $f(1_g) \cong 1_{f(g)}$. Thus we have a map of bicategory-enriched graphs.

The adjoint equivalence $\boldsymbol{\chi}$ is defined as follows. The component χ_{hg} is given by the cell $(0, 2, 1, [\,], [\,], 1_{h \otimes g})$, and $\chi^{\bullet}_{hg}$ is $(0, 1, 2, [\,], [\,], 1_{h \otimes g})$. The unit and counit are given by the unique coherence isomorphisms in the hom-bicategory. The naturality isomophisms are given by the unique coherence isomorphism from the coherence for functors theorem using the functoriality constraint of $\otimes$ and the constraints in the hom-bicategory. It is now trivial to check the transformation axioms and that this is an adjoint equivalence.

The adjoint equivalence $\boldsymbol{\iota}$ has components defined by $\iota_a = (0, 0, 1, [\,], [\,], 1_{I_a})$ and $\iota_a^{\bullet} = (0, 1, 0, [\,], [\,], 1_{I_a})$. The unit and counit are given by the unique coherence isomorphisms in the hom-bicategory. The invertible 3-cells $\iota : \iota_a \circ i_a \Rightarrow fi_a \circ \iota_a$ and $\iota^{\bullet}$ are also given by unique coherence isomorphisms from the coherence for functors theorem. Once again, it is routine to check the transformations axioms and that this is an adjoint equivalence.

The modifications ω, γ, and δ are all given by unique coherence isomorphisms as above. These clearly give modifications, and the axioms for a functor are now immediate by the coherence theorem for functors.

For the second claim, first note that f is an isomorphism on objects. Thus we need only prove that f is a local biequivalence to show that it is a triequivalence. It is trivial to see that $ef(g) = g$ for g a 1-cell of T, so ef is locally biessentially surjective. By definition, $ef(\alpha) = (1\alpha)1$ which is isomorphic to α in the hom-bicategory of T, so ef is also 2-locally essentially surjective. Also from the definition, it is easy to see that $ef(\Gamma) = (1 * \Gamma) * 1$; using the left and right unit isomorphisms in the hom-bicategory, we see that this function is an isomorphism on 3-cells, hence ef is a local biequivalence. But since e is a local biequivalence, f must be as well. Therefore f is a triequivalence. $\square$

Chapter 11

Coherence for functors

In this chapter, we will establish a coherence result for functors between tricategories. This requires the same attention to detail that the coherence theorem itself demanded, as once again we will need to employ universal properties in the categories $2Gr(\mathbf{Cat})$ and $Gr(\mathbf{Bicat})$ since tricategories and functors between them do not form a category. The coherence theorem proved in the first section will allow us to construct from any functor $F : S \to T$ a strict functor $\mathrm{Gr}F : \mathrm{Gr}S \to \mathrm{Gr}T$, at which point we will have replaced tricategories and functors with triequivalent **Gray**-categories and **Gray**-functors.

11.1 The coherence theorem

Our first goal is to prove analogues of the results in Section 2.3. We begin by producing the free functor generated by a map of bicategory-enriched graphs. The following proposition constructs this functor and provides its universal property.

Proposition 11.1.1. *Let $J : B \to B'$ be a morphism in $Gr(\mathbf{Bicat}_s)$. Then there exists a tricategory $\mathfrak{F}_J B'$, a map $j : B' \to \mathfrak{F}_J B'$ in $Gr(\mathbf{Bicat})$, and a locally strict functor $\tilde{J} : \mathfrak{F}B \to \mathfrak{F}_J B'$ with the following properties.*

1. *The square*

$$\begin{tikzcd} B \arrow[r, "J"] \arrow[d, "i"'] & B' \arrow[d, "j"] \\ \mathfrak{F}B \arrow[r, "\tilde{J}"'] & \mathfrak{F}_J B' \end{tikzcd}$$

   commutes in $Gr(\mathbf{Bicat}_s)$.

2. *Given a square*

$$\begin{tikzcd} B \arrow[r, "J"] \arrow[d, "K"'] & B' \arrow[d, "L"] \\ S \arrow[r, "F"'] & T \end{tikzcd}$$

that is commutative in $Gr(\mathbf{Bicat})$ *with* S, T *tricategories and* F *a locally strict functor between them, there exists a unique square*

$$\begin{tikzcd} \mathcal{F}B \arrow[r, "\tilde{J}"] \arrow[d, "\overline{K}"'] & \mathcal{F}_J B' \arrow[d, "\overline{L}"] \\ S \arrow[r, "F"'] & T \end{tikzcd}$$

such that

- $\overline{L}\tilde{J} = F\overline{K}$ *in* $Gr(\mathbf{Bicat})$,
- $\overline{K}, \overline{L}$ *are strict functors,*
- $\overline{K}i = K$ *and* $\overline{L}j = L$ *as morphisms of the underlying bicategory-enriched graphs, and*
- $\overline{L}$ *maps the adjoint equivalences* $\boldsymbol{\chi}$ *and* $\boldsymbol{\iota}$ *in* $\mathcal{F}_J B'$ *to the adjoint equivalences of the same name in* T.

Proof. The tricategory $\mathcal{F}_J B'$ is constructed as follows. The 0-cells of $\mathcal{F}_J B'$ are the 0-cells of B'. The 1-cells of $\mathcal{F}_J B'$ are generated by new 1-cells I_a, the 1-cells of B', and 1-cells Jf for $f \in \mathcal{F}B_1$, subject to the relation that $Jf = f'$ if f is a 1-cell of B such that $Jf = f'$ in B'. The 2-cells of $\mathcal{F}_J B'$ are built from the basic building blocks

1. 2-cells $\alpha : f \Rightarrow g$ in B',
2. new 2-cells $i_a : I_a \Rightarrow I_a$,
3. the constraint cells $l_f, l^{\bullet}_f, r_f, r^{\bullet}_f, a_{hgf}$, and $a^{\bullet}_{hgf}$,
4. the constraint cells $\chi_{gf}, \chi^{\bullet}_{gf}, \iota_a$, and $\iota^{\bullet}_a$, and
5. 2-cells $J\alpha$ for $\alpha \in \mathcal{F}B_2$

by tensoring along 0-cell boundaries and composing along 1-cell boundaries, subject to the relations $(\beta)\circ(\alpha) = (\beta\circ\alpha)$ (where here the left side is composition in $\mathcal{F}_J B'$ while the right side is composition in B') and $J\alpha = \alpha'$ if $\alpha \in B_2$ and $J\alpha = \alpha'$ in B'.

The 3-cells are built similarly from the 3-cells of B', 3-cells $J\Gamma$ for $\Gamma \in \mathcal{F}B_3$, constraint cells for the tricategory structure, and constraint cells for the functor $\tilde{J}$, all subject to the required relations for both the tricategory structure on $\mathcal{F}_J B'$ and the functor $\tilde{J}$.

The functor $\tilde{J}$ is defined on cells by the formula $\tilde{J}(w) = Jw$, where the cell Jw is one of the defining cells for $\mathcal{F}_J B'$. The constraint cells for $\tilde{J}$ are those given by the definition of $\mathcal{F}_J B'$, and the functor axioms hold by construction. The square in part 1 of the statement of the theorem then commutes automatically.

For the second part of the statement, the strict functor $\overline{K}$ is determined by the universal property of $\mathcal{F}B$. The strict functor $\overline{L}$ is defined as follows. On 0-cells, $\overline{L}$ agrees with L. The rest of the functor $\overline{L}$ is determined by strictness, local strictness, the relations $\overline{K}i = K$, $\overline{L}j = L$, and requiring $\overline{L}$ to map the constraint cells in the definition of $\mathcal{F}_J B'$ to the constraint cells of the functor $\overline{K}$. This gives the definition of $\overline{L}$ and immediately proves uniqueness.

□

Let $J : X \to Y$ be any map in $2Gr(\mathbf{Cat})$. We can apply the construction of the free functor of bicategories between category-enriched graphs locally to produce a locally strict map of bicategory-enriched graphs $J^{loc} : \mathcal{F}_B X \to \mathcal{F}_J^{loc} Y$. This gives the commutative square in $2Gr(\mathbf{Cat})$ displayed below.

$$\begin{tikzcd}
X \arrow[r, "J"] \arrow[d] & Y \arrow[d] \\
\mathcal{F}_B X \arrow[r, "J^{loc}"'] & \mathcal{F}_J^{loc} Y
\end{tikzcd}$$

Applying the universal property locally, we get a unique commutative square of bicategory-enriched graphs

$$\begin{tikzcd}
\mathcal{F}_B X \arrow[r, "J^{loc}"] \arrow[d] & \mathcal{F}_J^{loc} Y \arrow[d] \\
\mathcal{F}_G(\mathcal{F}_{2C} X) \arrow[r, "\mathcal{F}_G(\mathcal{F}_{2C} J)"'] & \mathcal{F}_G(\mathcal{F}_{2C} Y)
\end{tikzcd}$$

which when pasted with the previous square yields the square below.

$$\begin{tikzcd}
X \arrow[r, "J"] \arrow[d] & Y \arrow[d] \\
\mathcal{F}_G(\mathcal{F}_{2C} X) \arrow[r, "\mathcal{F}_G(\mathcal{F}_{2C} J)"'] & \mathcal{F}_G(\mathcal{F}_{2C} Y)
\end{tikzcd}$$

Now applying the universal property of $\mathcal{F}_{J^{loc}}$, we get the following square which commutes in $Gr(\mathbf{Bicat})$.

$$\begin{tikzcd}
\mathcal{F} X \arrow[r, "\widetilde{J^{loc}}"] \arrow[d] & \mathcal{F}_{J^{loc}}(\mathcal{F}_J^{loc} Y) \arrow[d, "\Delta"] \\
\mathcal{F}_G(\mathcal{F}_{2C} X) \arrow[r, "\mathcal{F}_G(\mathcal{F}_{2C} J)"'] & \mathcal{F}_G(\mathcal{F}_{2C} Y)
\end{tikzcd}$$

The left vertical map is the triequivalence from our coherence theorem for tricategories given by the universal property of the map $X \hookrightarrow \mathcal{F}X$. The coherence theorem for functors is now the following statement.

Theorem 11.1.2 (Coherence for functors)**.** *For all maps $J : X \to Y$ of category-enriched 2-graphs, the strict functor $\Delta : \mathcal{F}_{J^{loc}}(\mathcal{F}_J^{loc}Y) \to \mathcal{F}_G(\mathcal{F}_{2C}Y)$ is a triequivalence.*

This proof requires the following lemma. It has a proof similar to that given in Section 2.4.1.

Lemma 11.1.3. *Assume that the following squares of functors satisfy the four conditions of the second part of the previous proposition with R, S_i strict for $i = 1, 2$.*

$$\begin{tikzcd}
\mathcal{F}X \arrow[r, "\widetilde{J^{loc}}"] \arrow[d, "R"'] & \mathcal{F}_{J^{loc}}(\mathcal{F}_J^{loc}Y) \arrow[d, "S_i"] \\
A \arrow[r, "F_i"'] & B
\end{tikzcd}$$

Assume that the S_i have the same object map, and that the F_i have the same object map. Then for every transformation $\alpha : F_1 \to F_2$ with $\alpha_a = I_{F_1 a}$ for every object a, there is a transformation $\beta : S_1 \to S_2$ with $\beta_b = I_{S_1 b}$ for every object b and

$$(\alpha * 1_R)_x = (\beta * 1_{\widetilde{J^{loc}}})_x$$

for all objects x in $\mathcal{F}X$.

Proof of . Since $\Delta^{loc} : \mathcal{F}_J^{loc}Y \to \mathcal{F}_{2C}Y$ is a local biequivalence, there is a map of bicategory-enriched graphs going in the opposite direction which is a local pseudoinverse and is defined by the following formulas.

$$\begin{aligned}
x &\mapsto x \\
f &\mapsto f \\
\alpha_n \cdots \alpha_1 &\mapsto (\cdots(\alpha_n \alpha_{n-1})\alpha_{n-2})\cdots)\alpha_1 \\
\varnothing_f &\mapsto 1_f \\
\Gamma_n \cdots \Gamma_1 &\mapsto (\cdots(\Gamma_n * \Gamma_{n-1}) * \cdots) * \Gamma_1
\end{aligned}$$

The structure constraints are given either by associativity isomorphisms or identities; it is simple to check the required axioms using coherence. If we write r for the composite of this map with the inclusion $\mathcal{F}_J^{loc}Y \to \mathcal{F}_{J^{loc}}(\mathcal{F}_J^{loc}Y)$, then we can produce a map of bicategory-enriched graphs $\hat{r} : \mathcal{F}_G(\mathcal{F}_{2C}Y) \to \mathcal{F}_{J^{loc}}(\mathcal{F}_J^{loc}Y)$ using Lemma 10.1.3. Using the strictness of $\Delta : \mathcal{F}_{J^{loc}}(\mathcal{F}_J^{loc}Y) \to \mathcal{F}_G(\mathcal{F}_{2C}Y)$ and the definition of $\hat{r}$, it is easy to check that $\Delta\hat{r}$ is the identity in the category of bicategory-enriched graphs. Using this fact and the same arguments used in the proof of the coherence theorem for tricategories, we see that Δ is locally biessentially surjective, 2-locally essentially surjective, and 2-locally full.

By Proposition 9.4.6, there is a strict functor

$$S : \mathcal{F}X \to \mathcal{F}_{J^{loc}}(\mathcal{F}_J^{loc}Y)$$

and a transformation $\alpha : S \to \widetilde{J^{loc}}$ with every component an identity. The universal property then gives the following commutative square in $Gr(\mathbf{Bicat}_s)$.

$$\begin{tikzcd}
\mathcal{F}X \arrow[r, "\widetilde{J^{loc}}"] \arrow[d, "1"'] & \mathcal{F}_{J^{loc}}(\mathcal{F}_J^{loc}Y) \arrow[d, "E"] \\
\mathcal{F}X \arrow[r, "S"'] & \mathcal{F}_{J^{loc}}(\mathcal{F}_J^{loc}Y)
\end{tikzcd}$$

The identity square

$$\begin{tikzcd}
\mathcal{F}X \arrow[r, "\widetilde{J^{loc}}"] \arrow[d, "1"'] & \mathcal{F}_{J^{loc}}(\mathcal{F}_J^{loc}Y) \arrow[d, "1"] \\
\mathcal{F}X \arrow[r, "\widetilde{J^{loc}}"'] & \mathcal{F}_{J^{loc}}(\mathcal{F}_J^{loc}Y)
\end{tikzcd}$$

also satisfies the four conditions in the proposition. By the previous lemma and the existence of α, we can conclude that E is 2-locally faithful.

The universal property of $\mathcal{F}_{J^{loc}}(\mathcal{F}_J^{loc}Y)$ also provides the square below.

$$\begin{tikzcd}
\mathcal{F}X \arrow[r, "\widetilde{J^{loc}}"] \arrow[d, "1"'] & \mathcal{F}_{J^{loc}}(\mathcal{F}_J^{loc}Y) \arrow[d, "\Delta_1"] \\
\mathcal{F}X \arrow[r, "\mathcal{F}J"'] & \mathcal{F}Y
\end{tikzcd}$$

The universal property of Γ implies that $\Gamma \circ_v \Delta_1 = \Delta$; since we know that Γ is locally faithful, we need only prove that Δ_1 is as well. By the definition of $\mathcal{F}_{J^{loc}}(\mathcal{F}_J^{loc}Y)$ and the fact that $\mathcal{F}$ is a left adjoint, there is a unique strict functor T such that the diagram below commutes.

$$\begin{tikzcd}
\mathcal{F}_B Y \arrow[r] \arrow[d] & \mathcal{F}_{J^{loc}}(\mathcal{F}_J^{loc}Y) \\
\mathcal{F}Y \arrow[ur, "T"'] &
\end{tikzcd}$$

It is now easy to check that $S = T \circ_v \mathcal{F}J$ using the definition of S given by the construction in the previous chapter. This gives that $T \circ_v \Delta_1$ is a strict functor. But since $S = T \circ_v \mathcal{F}J$, the following square commutes in $Gr(\mathbf{Bicat})$.

$$\begin{tikzcd}
\mathcal{F}X \arrow[r, "\widetilde{J^{loc}}"] \arrow[d, "1"'] & \mathcal{F}_{J^{loc}}(\mathcal{F}_J^{loc}Y) \arrow[d, "T \circ_v \Delta_1"] \\
\mathcal{F}X \arrow[r, "S"'] & \mathcal{F}_{J^{loc}}(\mathcal{F}_J^{loc}Y)
\end{tikzcd}$$

We now need to check that this square satisfies the four properties listed in the second part of Proposition 11.1.1 to conclude that there is a transformation $\alpha : T \circ_v \Delta_1 \to E$ with each component the identity; then $T \circ_v \Delta_1$ will be 2-locally faithful since E is, and thus Δ_1 will be 2-locally faithful as well. The first two properties are immediate. The third and fourth follow by direct calculation using the fact that S and T are strict functors. $\square$

11.2 Coherence and diagrams of constraints

This section consists of results analogous to those in Section 9.2. The goal will be to prove results of the form "all diagrams of constraint 3-cells of a certain form commute." In the next section, we will put these results to work in providing an explicit strictification for functors between tricategories. All of the proofs in this section are simple modifications of the proofs given in Section 9.2, so we either omit them or give shortened versions.

The following important corollary is an immediate consequence of the coherence theorem for functors and Theorem 10.2.2.

Corollary 11.2.1. *Let $J : X \to Y$ be a map of category-enriched 2-graphs, and assume that Y is 2-locally discrete. Then in $\mathcal{F}_{J^{loc}}(\mathcal{F}_J^{loc} Y)$, every diagram of 3-cells commutes.*

Definition 11.2.2. A diagram D in T of constraint cells of T, constraint cells of a functor $J : S \to T$, and the images under J of constraint cells in S is called $(\mathcal{F}, J)$-*admissible* if there is a commutative square of category-enriched 2-graphs

$$\begin{tikzcd} \overline{S} \arrow[r, "K"] \arrow[d] & \overline{T} \arrow[d] \\ S \arrow[r, "J"'] & T \end{tikzcd}$$

with both vertical arrows inclusions and $\overline{T}$ 2-locally discrete and a diagram $\tilde{D}$ in $\mathcal{F}_{K^{loc}}(\mathcal{F}_K^{loc}\overline{T})$ consisting of constraints for the tricategory structure and the functor $\tilde{J}$ such that D is the image of $\tilde{D}$ under the map

$$\mathcal{F}_{K^{loc}}(\mathcal{F}_K^{loc}\overline{T}) \to T$$

induced by the universal property of $\mathcal{F}_{K^{loc}}(\mathcal{F}_K^{loc}\overline{T})$ applied to the square above.

Corollary 11.2.3. *Let $J : S \to T$ be a functor. Then every $(\mathcal{F}, J)$-admissible diagram in T commutes.*

11.3 Strictifying functors

In this section, we will use our coherence theorem to produce, from any functor $F : S \to T$, a strict functor $\mathrm{Gr}F : \mathrm{Gr}S \to \mathrm{Gr}T$.

The definition of GrF on objects is the same as that of the functor F on objects. Since a 1-cell of GrS is either empty or a string $\{f_i\}$, we can also define GrF on 1-cells by the simple formulas below.

$$\mathrm{Gr}F(\varnothing) = \varnothing$$
$$\mathrm{Gr}F(\{f_i\}) = \{Ff_i\}$$

For the definition of GrF on the 1-cells of the hom-2-categories, we note that it is only necessary to define GrF on basic 1-cells and then extend this to strings by strict functoriality. Thus we need only define GrF on the basic 1-cell $(k, l_1, l_2, \sigma, \tau, \alpha)$.

First, choose composites of constraint cells $c_\sigma : [Ff_i]_\sigma \to F([f_i]_\sigma)$ for every association σ just as we did for choosing associators $a_{\gamma,\gamma'}$. These choices also give rise to cells $c^\bullet_\sigma : F([f_i]_\sigma) \to [Ff_i]_\sigma$. Thus we now define Gr$F$ on the basic 1-cells of the hom-2-categories by

$$\mathrm{Gr}F(k, l_1, l_2, \sigma, \tau, \alpha) = (k, l_1, l_2, \sigma, \tau, (c^\bullet_\tau F\alpha)c_\sigma).$$

We additionally define $\mathrm{Gr}F(\varnothing) = \varnothing$.

We will define GrF on 3-cells by using a canonical isomorphism that we construct next. The 2-cell $e\big((k, l_1, l_2, \sigma, \tau, (c^\bullet_\tau F\alpha)c_\sigma)\big)$ is given by the composite

$$a^\bullet \circ \underline{c^\bullet_\tau} \circ \underline{F\alpha} \circ \underline{c_\sigma} \circ a,$$

where we have written $\underline{\delta}$ for the cell $(\cdots(1 \otimes 1) \otimes 1) \cdots \otimes \delta) \otimes \cdots \otimes 1$, as in the last chapter. We thus have the isomorphism given by the pasting diagram below, where each individual isomorphism is unique by our coherence theorem and $(\mathcal{F}, F)$-admissibility.

$$\begin{tikzcd}
{[Ff_i]} \arrow[r, "a"] \arrow[d, "c"'] & {[Ff_{i+},[Ff_i]_\sigma,Ff_{i-}]} \arrow[r, "\underline{c_\sigma}"] \arrow[d, phantom, "\cong"] & {[Ff_{i+},F[f_i]_\sigma,Ff_{i-}]} \arrow[d, "\underline{F\alpha}"] \\
F[f_i] \arrow[d, "Fa"'] & \cong & {[Ff_{i+},F[g_j]_\tau,Ff_{i-}]} \arrow[d, "\underline{c^\bullet_\tau}"] \\
F[f_{i+},[f_i]_\sigma,f_{i-}] \arrow[d, "F\underline{\alpha}"'] \arrow[uurr, "c^\bullet"] & \cong & {[Ff_{i+},[Fg_j]_\tau,Ff_{i-}]} \arrow[d, "a^\bullet"] \\
F[f_{i+},[g_i]_\tau,f_{i-}] \arrow[r, "Fa^\bullet"'] \arrow[uurr, "c^\bullet"] & F[g_j] \arrow[r, "c^\bullet"'] & {[Fg_j]}
\end{tikzcd}$$

Composing this with the composition constraint for F gives a unique isomorphism

$$a^\bullet \circ \underline{c^\bullet_\tau} \circ \underline{F\alpha} \circ \underline{c_\sigma} \circ a \overset{\cong}{\Longrightarrow} c \circ F(a^\bullet \circ \underline{\alpha} \circ a) \circ c^\bullet.$$

It is easy to extend this isomorphism to when α and β are strings of basic 2-cells. Now we define Gr$F(\Gamma)$ to be the composite

$$a^\bullet \underline{c^\bullet_\tau}\,\underline{F\alpha}\,\underline{c_\sigma} a \cong cF(a^\bullet \underline{\alpha} a)c^\bullet \overset{1*F(\Gamma)*1}{\Longrightarrow} cF(a^\bullet \underline{\beta} a)c^\bullet \cong a^\bullet \underline{c^\bullet_\tau}\,\underline{F\beta}\,\underline{c_\sigma} a.$$

Theorem 11.3.1. *Let $F : S \to T$ be a functor between tricategories. Then* GrF *as defined above is a strict functor between* **Gray**-*categories, i.e., a* **Gray**-*functor. Additionally, there are transformations*

$$
\begin{tikzcd}
GrS \arrow[r, "GrF"] \arrow[d, "e"'] & GrT \arrow[d, "e"] \arrow[dl, Rightarrow, "\varphi"'] \\
S \arrow[r, "F"'] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
S \arrow[r, "F"] \arrow[d, "f"'] & T \arrow[d, "f"] \arrow[dl, Rightarrow, "\psi"'] \\
GrS \arrow[r, "GrF"'] & GrT
\end{tikzcd}
$$

with the component at each object the identity.

Proof. For the first claim, we need to prove that GrF strictly preserves all compositions and identities. This holds by definition for the 1-cells of GrS. By definition GrF strictly preserves identity 2-cells and composition along 1-cell boundaries. Thus we need only check that

$$\mathrm{Gr}F(\beta \star \alpha) = \mathrm{Gr}F(\beta) \star \mathrm{Gr}F(\alpha).$$

The definitions of GrF and $\varnothing \star \alpha, \beta \star \varnothing$ make it clear that $\mathrm{Gr}F(\varnothing \star \alpha) = \varnothing \star \mathrm{Gr}F(\alpha)$ and $\mathrm{Gr}F(\beta \star \varnothing) = \mathrm{Gr}F(\beta) \star \varnothing$. Thus we have the following calculation.

$$
\begin{array}{rcl}
\mathrm{Gr}F(\beta \star \alpha) & = & \mathrm{Gr}F(\varnothing \star \alpha \circ \beta \star \varnothing) \\
& = & \mathrm{Gr}F(\varnothing \star \alpha) \circ \mathrm{Gr}F(\beta \star \varnothing) \\
& = & \varnothing \star \mathrm{Gr}F(\alpha) \circ \mathrm{Gr}F(\beta) \star \varnothing \\
& = & \mathrm{Gr}F(\beta) \star \mathrm{Gr}F(\alpha)
\end{array}
$$

For 3-cells, it is obvious that $\mathrm{Gr}F(\Gamma \circ \Delta) = \mathrm{Gr}F(\Gamma) \circ \mathrm{Gr}F(\Delta)$ by interchange in the hom-2-categories and the functoriality of F. Similarly $\mathrm{Gr}F(1) = 1$ since $F(1) = 1$ by functoriality on 3-cells. Using the definition of $\Gamma * \Delta$ in GrT, it is routine to check that $\mathrm{Gr}F(\Gamma * \Delta) = \mathrm{Gr}F(\Gamma) * \mathrm{Gr}F(\Delta)$. To check that $\mathrm{Gr}F(\Gamma \star \Delta) = \mathrm{Gr}F(\Gamma) \star \mathrm{Gr}F(\Delta)$, we only need to verify that this equation holds when either of Γ or Δ is the identity; the definition of $\Gamma \star \Delta$ and the fact that GrF strictly preserves composition along 1-cells boundaries then ensure that the equation holds in general. This is a simple calculation using the definition of $1 \star \Gamma$, $\Gamma \star 1$ resp., and Corollary 11.2.3.

Finally, we must show that $\mathrm{Gr}F(\gamma) = \gamma$. Since γ is defined by a unique coherence isomorphism, we need only show that $\mathrm{Gr}F(\gamma)$ is as well. This follows quickly by the definition of the action of GrF on 3-cells. We have now completed the proof that GrF is a **Gray**-functor between **Gray**-categories.

To define the transformation $\varphi : e \circ \mathrm{Gr}F \to F \circ e$, we first set $\varphi_x = I_{Fx}$. The adjoint equivalence $\boldsymbol{\varphi}_{\{f_i\}}$ is the composite of the adjoint equivalence $\mathbf{r}^{\bullet}\mathbf{l}$ and the adjoint equivalence $\mathbf{c}$ given by 1-cells $c : [Ff_i] \to F[f_i], c^{\bullet} : F[f_i] \to [Ff_i]$ and the obvious unit and counit. The naturality isomorphism $\varphi_{\overline{\theta}}$ is the composite of naturality isomorphisms for c and $r^{\bullet}l$. The modifications Π and M are given by unique coherence isomorphisms using Proposition 11.2.3, and the transformation axioms follow immediately.

To define the transformation $\psi : f \circ F \to \mathrm{Gr}F \circ f$, we first set $\psi_x = \varnothing_{Fx}$. Similarly, $\psi_f = \varnothing_{Ff}$ and $\psi_f^{\bullet} = \varnothing_{Ff}$ with identity unit and counit. The naturality isomorphism ψ_θ is the identity. Once again, Π and M are given by unique coherence isomorphisms and the transformation axioms follow immediately. $\square$

Remark 11.3.2. Note that for ψ, the only nontrivial data are Π and M. That is because $f \circ F = \mathrm{Gr}F \circ f$ as maps of bicategory-enriched graphs.

With the proof of Theorem 11.3.1, we have shown how to replace tricategories and functors between them with **Gray**-categories and **Gray**-functors, up to triequivalence. This furthers the coherence theory begun in [17] and gives a rigorous justification to the use of **Gray**-functors instead of functors of tricategories as appropriate maps (for the purposes of 3-dimensional category theory) between **Gray**-categories. Interesting avenues for future work include finding a strictification for transformations and comparing the resulting 2-cells with **Gray**-transformations, using the theory developed here to study pseudomonads in an arbitrary tricategory, and studying alternate strictifications as different approaches to coherence.

Appendix A

Adjointness in bi- and tricategories

The purpose of this appendix is to collect together all the results needed concerning adjoint equivalences in a bicategory and biadjoint biequivalences in a tricategory. We shall prove the analogue, for an arbitrary bicategory, of the result that in **Cat**, every equivalence can be made into an adjoint equivalence. As a corollary, we also produce the result that an adjoint equivalence in a bicategory is determined by its 1-cells, either η or ε, and the bicategorical triangle identities.

The first section is concerned with the basics. First, we give the definition of an adjoint equivalence in an arbitrary bicategory. The main difference from the usual definitions is the addition of associativity and unit isomorphisms needed for an arbitrary bicategory. Then we present the usual proof in **Cat** of the fact that every equivalence of categories can be made into an adjoint equivalence. Then we give the full proof of that same result in any bicategory. The second section focuses on the theory of mates, an important calculational tool when working with adjunctions in a bicategory. We give the definitions and provide a number of results that will be used without specific mention in this work.

The final section gives the definition of a biadjoint biequivalence in an arbitrary tricategory and relates it to the usual definition of a biadjunction between functors $F : B \to C, G : C \to B$ with B, C bicategories.

A.1 Adjoint equivalences in a bicategory

Here we give an account of the basic theory of adjunctions and adjoint equivalences in an arbitrary bicategory.

A.1.1 Definitions

This section is devoted to providing the necessary definitions. Throughout this note, B will denote a bicategory with associativity isomorphism a and unit isomorphisms l and r.

Definition A.1.1. 1. A specified equivalence (f, g, α, β) in B consists of a pair of 1-cells $f : x \to y$ and $g : y \to x$ and a pair of 2-cells isomorphisms $\alpha : fg \Rightarrow I_y$ and $\beta : I_x \Rightarrow gf$.
2. A 1-cell f in B is called an equivalence if there exist g, α, and β such that (f, g, α, β) is a specified equivalence in B.

To define adjoint equivalence, we first must define what an adjunction is in a bicategory.

Definition A.1.2. Let $f : x \to y$ and $g : y \to x$ be 1-cells in B. An adjunction $f \dashv g$ consists of a 2-cell $\varepsilon : fg \Rightarrow I_y$ and a 2-cell $\eta : I_x \Rightarrow gf$ such that the following two diagrams (the bicategorical triangle identities) commute.

$$\begin{tikzcd}
g \arrow[r, "l_g^{-1}"] \arrow[drrrr, "1_g"'] & I_x g \arrow[r, "\eta * 1_g"] & (gf)g \arrow[r, "a"] & g(fg) \arrow[r, "1_g * \varepsilon"] & gI_y \arrow[d, "r"] \\
 & & & & g
\end{tikzcd}$$

$$\begin{tikzcd}
f \arrow[r, "r_f^{-1}"] \arrow[drrrr, "1_f"'] & fI_x \arrow[r, "1_f * \eta"] & f(gf) \arrow[r, "a^{-1}"] & (fg)f \arrow[r, "\varepsilon * 1_f"] & I_y f \arrow[d, "l"] \\
 & & & & f
\end{tikzcd}$$

We then say that f is left adjoint to g, or that g is right adjoint to f.

Remark A.1.3. In the bicategory **Cat**, the associativity and unit isomorphisms are all identities. In that case, this definition reduces to the usual definition of an adjunction between functors.

Definition A.1.4. An adjoint equivalence $(f, g, \varepsilon, \eta)$ in B consists of a specified equivalence $(f, g, \varepsilon, \eta)$ such that ε and η constitute an adjunction $f \dashv g$.

A.1.2 The bicategory Cat

In this section, we will present the usual proof in **Cat** that every equivalence can be improved to an adjoint equivalence. This proof relies heavily on the fact that 2-cells in **Cat** have an explicit description in terms of families of 1-cells.

Theorem A.1.5. *Let $F : X \to Y$ and $G : Y \to X$ be functors, and let $\alpha : FG \Rightarrow 1_Y$ and $\beta : 1_X \Rightarrow GF$ be natural isomorphisms. Then there is a unique adjoint equivalence $(F, G, \varepsilon, \eta)$ in* **Cat** *such that $\varepsilon = \alpha$.*

Proof. Let $\varepsilon = \alpha$. The second triangle identity states that $\varepsilon F \circ F\eta = 1_F$. By the invertibility of ε, this equation is the same as $F\eta = (\varepsilon F)^{-1}$. The righthand side of this equation is well-defined and F is full and faithful since it is an equivalence of categories, so we define $\eta_x : x \to GFx$ to be the unique arrow such that $F\eta_x = (\varepsilon_{Fx})^{-1}$.

We must now check that η is natural and that the first triangle identity holds. For naturality, we consider the square below.

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "\eta_x"'] & y \arrow[d, "\eta_y"] \\ GFx \arrow[r, "GFf"'] & GFy \end{tikzcd}$$

Applying F to the diagram and using functoriality gives this square.

$$\begin{tikzcd} Fx \arrow[r, "Ff"] \arrow[d, "F\eta_x"'] & Fy \arrow[d, "F\eta_y"] \\ FGFx \arrow[r, "FGFf"'] & FGFy \end{tikzcd}$$

By the definition of η, this square is the naturality square of $(\varepsilon F)^{-1}$ and thus must commute. By the faithfulness of F, the original square commutes as well and so η is natural.

For the first triangle identity, we consider the composite

$$Gy \xrightarrow{\eta_{Gy}} GFGy \xrightarrow{G\varepsilon_y} Gy.$$

Applying F to this yields $FG\varepsilon_y \circ F\eta_{Gy}$, which is by definition $FG\varepsilon_y \circ \varepsilon^{-1}_{FGy}$. Now the following square commutes by the naturality of ε.

$$\begin{tikzcd} FGFGy \arrow[r, "\varepsilon_{FGy}"] \arrow[d, "FG\varepsilon_y"'] & FGy \arrow[d, "\varepsilon_y"] \\ FGy \arrow[r, "\varepsilon_y"'] & y \end{tikzcd}$$

By the invertibility of ε_y, we get that $\varepsilon_{FGy} = FG\varepsilon_y$. Therefore $FG\varepsilon_y \circ \varepsilon^{-1}_{FGy} = I_{FGy}$. Once again by the faithfulness of F, $G\varepsilon_y \circ \eta_{Gy} = I_y$ and thus the first triangle identity is satisfied. □

Remark A.1.6. We could have just as easily constructed an adjoint equivalence with $\eta = \beta$ instead of $\varepsilon = \alpha$. In general it is not possible to require both of these conditions, though.

A.1.3 The proof for bicategories

We will now present a generalization of Theorem A.1.5 that will apply to any bicategory, not just **Cat**. This proof is, by necessity, longer and quite different. The proof for **Cat** used naturality of ε and the full and faithfulness of the functor F. A key step in the general proof is establishing similar results in the general case.

To establish some notation, let g, h be parallel 1-cells in B. Then $B(g, h)$ will denote the set of 2-cells $g \Rightarrow h$.

Lemma A.1.7. *Let $f : y \to z$ be an equivalence 1-cell in B and let $g, h : x \to y$ be a pair of parallel 1-cells in B. Then the function $B(g, h) \to B(fg, fh)$ given by sending a 2-cell α to $1_f * \alpha$ is an isomorphism.*

Proof. Choose any specified equivalence $(f, f^\bullet, \alpha, \beta)$. The following diagram commutes for any 2-cell $\theta : g \Rightarrow h$.

$$\begin{tikzcd}
f^\bullet(fg) \arrow[r, "a^{-1}"] \arrow[d, "1_{f^\bullet}*(1_f*\theta)"'] & (f^\bullet f)g \arrow[r, "\alpha*1_g"] \arrow[d, "1_{f^\bullet f}*\theta"'] & I_z g \arrow[d, "1_I*\theta"] \\
f^\bullet(fh) \arrow[r, "a^{-1}"'] & (f^\bullet f)h \arrow[r, "\alpha*1_h"'] & I_z h
\end{tikzcd}$$

The left square commutes by naturality, and the right square commutes by interchange.

Now let $\phi, \psi : g \Rightarrow h$ be 2-cells in B, and assume that $1_f * \phi = 1_f * \psi$. Then

$$l_h \circ (\alpha * 1_h) \circ a^{-1} \circ \Big(1_{f^\bullet} * (1_f * \phi)\Big) \circ a \circ (\alpha^{-1} * 1_g) \circ l_g^{-1} =$$
$$l_h \circ (\alpha * 1_h) \circ a^{-1} \circ \Big(1_{f^\bullet} * (1_f * \psi)\Big) \circ a \circ (\alpha^{-1} * 1_g) \circ l_g^{-1}.$$

By pasting the naturality square for l at g to the diagram above, we see that the lefthand side of equation A.1.3 is ϕ and the righthand side is ψ. Thus $\phi = \psi$ and the function $1_f * -$ is injective.

We now define an inverse function by the assignment $\theta \mapsto \tilde{\theta}$ where

$$\tilde{\theta} = l_h \circ (\alpha * 1_h) \circ a^{-1} \circ (1_{f^\bullet} * \theta) \circ a \circ (\alpha^{-1} * 1_g) \circ l_g^{-1}.$$

This function is also injective since all the 2-cells involved except the argument are invertible and $1_f * -$ is injective. We will show that $\widetilde{1_f * \theta} = \theta$, and thus that the function $\theta \mapsto \tilde{\theta}$ is surjective. Then $1_f * -$ will be a right inverse for an invertible function, hence invertible itself.

The 2-cell $\widetilde{1_f * \theta}$ is

$$l_h \circ (\alpha * 1_h) \circ a^{-1} \Big(\circ 1_{f^\bullet} * (1_f * \theta)\Big) \circ a \circ (\alpha^{-1} * 1_g) \circ l_g^{-1}.$$

Now $a^{-1} \circ \Big(1_{f^\bullet} * (1_f * \theta)\Big) \circ a = 1_{f^\bullet f} * \theta$ by naturality. By interchange,

$$(\alpha * 1_h) \circ (1_{f^\bullet f} * \theta) \circ (\alpha^{-1} * 1_g) = (\alpha \circ 1_{f^\bullet f} \circ \alpha^{-1}) * (1_h \circ \theta \circ 1_g) = 1_I * \theta.$$

By the naturality of l,

$$l_h \circ (1_I * \theta) \circ l_g^{-1} = \theta,$$

so $\widetilde{1_f * \theta} = \theta$. ◻

Remark A.1.8. If $f : x \to y$ is an equivalence 1-cell and $g, h : y \to z$ are parallel 1-cells in B, then the function $B(g, h) \to B(gf, hf)$ given by $\beta \mapsto \beta * 1_f$ is also an isomorphism. It is now easy to show that the functors $f^* : B(y, z) \to B(x, z)$ and $f_* : B(w, x) \to B(w, y)$ are both equivalences of categories.

Lemma A.1.9. *Let $f : x \to y$ and $g : y \to x$ be 1-cells in B, and let $\varepsilon : fg \Rightarrow I_y$ be an invertible 2-cell. Then*

$$l_{fg} \circ \varepsilon * 1_{fg} = r_{fg} \circ 1_{fg} * \varepsilon$$

as 2-cells $(fg)(fg) \Rightarrow fg$.

Proof. Consider the following diagram.

$$\begin{tikzcd}
(fg)(fg) \arrow[r, "\varepsilon*1"] \arrow[d, "1*\varepsilon"'] \arrow[dr, phantom, "\square"] & I_y(fg) \arrow[r, "1*\varepsilon"] \arrow[d, "l"'] \arrow[dr, phantom, "\circlearrowleft"] & I_yI_y \arrow[d, "l"] \\
(fg)I_y \arrow[r, "r"'] \arrow[d, "\varepsilon*1"'] \arrow[dr, phantom, "\circlearrowleft"] & fg \arrow[r, "\varepsilon"] & I_y \\
I_yI_y \arrow[urr, "r"'] & &
\end{tikzcd}$$

The two regions marked with $\circlearrowleft$ commute by naturality. Now $r_I = l_I$ by [22], so the outside commutes by interchange. Since all the arrows are invertible, the square marked $\square$ commutes as well. ◻

Theorem A.1.10. *Let (f, g, α, β) be a specified equivalence in B. Then there is a unique adjoint equivalence $(f, g, \varepsilon, \eta)$ in B such that $\varepsilon = \alpha$.*

Proof. Set $\varepsilon = \alpha$. The first triangle identity is the equation

$$r_g \circ 1_g * \varepsilon \circ a \circ \eta * 1_g \circ l_g^{-1} = 1_g. \tag{A.1}$$

This can be rearranged, using the invertibility of all the terms involved, to yield

$$\eta * 1_g = a^{-1} \circ 1_g * \varepsilon^{-1} \circ r_g^{-1} \circ l_g. \tag{A.2}$$

Define η using Remark A.1.8. We must now check that η satisfies the second triangle identity. Consider the diagram below, where we write I for an identity

1-cell.

$$
\begin{tikzcd}
fg \arrow[r, "r^{-1}*1"] \arrow[dr, "1*l^{-1}"'] \arrow[dddr, "1*r^{-1}"'] \arrow[ddddr, bend right, "r^{-1}"'] & (fI)g \arrow[r, "(1*\eta)*1"] \arrow[d, "a"] \arrow[dr, phantom, "N"] & (f(gf))g \arrow[r, "a^{-1}*1"] \arrow[d, "a"] & ((fg)f)g \arrow[r, "(\varepsilon*1)*1"] & (If)g \arrow[r, "l*1"] \arrow[dr, phantom, "P"] & fg \\
& f(Ig) \arrow[r, "1*(\eta*1)"] \arrow[d, "1*l"] & f((gf)g) \arrow[dr, phantom, "A"] & & & \\
& fg \arrow[d, "1*r^{-1}"] \arrow[r, phantom, "\triangle"] & {} & \arrow[r, phantom, "N"] & I(fg) \arrow[uu, "a^{-1}"] \arrow[uur, "l"'] \arrow[d, phantom, "\square"] & \\
& f(gI) \arrow[r, "1*(1*\varepsilon^{-1})"] \arrow[d, "a^{-1}"] \arrow[dr, phantom, "N"] & f(g(fg)) \arrow[uu, "1*a^{-1}"] \arrow[r, "a^{-1}"] & (fg)(fg) \arrow[uuu, "a^{-1}"] \arrow[ur, "\varepsilon*1"] \arrow[dr, "1*\varepsilon"] \arrow[d, phantom, "="] & {} & \\
& (fg)I \arrow[urr, "1*\varepsilon^{-1}"] \arrow[rrr, "1"'] & & {} & (fg)I \arrow[uuuur, bend right, "r"'] &
\end{tikzcd}
$$

The regions marked $=$ commute trivially, and the regions marked A are bicategory axioms. The regions marked N commute by naturality. The two regions marked P commute by proposition (JS), or by coherence for bicategories. The region marked $\triangle$ is obtained from the first triangle identity using invertibility and by horizontal composition with 1_f. Finally, the region marked $\square$ commutes by Lemma A.1.9. Thus the diagram commutes.

Examining the exterior of the diagram, we get the equation

$$1_{fg} = l * 1_g \circ (\varepsilon * 1_f) * 1_g \circ a^{-1} * 1_g \circ (1_f * \eta) * 1_g \circ r^{-1} * 1_g. \tag{A.3}$$

Since $1_f * 1_g = 1_{fg}$ and the function $- * 1_g$ is injective by A.1.8, equation A.3 yields that

$$1_f = l \circ \varepsilon * 1_f \circ a^{-1} \circ 1_f * \eta \circ r^{-1}, \tag{A.4}$$

which is the second triangle identity. $\square$

Remark A.1.11. We could just as easily have required $\eta = \beta$ as in Remark A.1.6. It is not possible, in general, to require both $\varepsilon = \alpha$ and $\eta = \beta$.

Corollary A.1.12. *An adjoint equivalence $(f, g, \varepsilon, \eta)$ in a bicategory B is uniquely determined by f, g, the bicategorical triangle identities, and either ε or η.*

A.1.4 Useful results

Here we provide two useful results about adjoint equivalences in an arbitrary bicategory. These results will be used throughout, often without explicit mention.

Proposition A.1.13. *Let B be a bicategory and let*

$$\begin{aligned}\boldsymbol{f} &= (f, f^\bullet, \varepsilon_f, \eta_f)\\ \boldsymbol{g} &= (g, g^\bullet, \varepsilon_g, \eta_g)\end{aligned}$$

be two adjoint equivalences in B such that the target of f is the source of g. Then there is an adjoint equivalence $\boldsymbol{gf} = (gf, f^\bullet g^\bullet, \alpha, \beta)$ with counit α given by the diagram below and unit β uniquely determined.

$$\begin{tikzcd}
 & & x \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow \varepsilon_f"] & & \\
 & y \arrow[ur, "f^\bullet"] \arrow[rr, "I"'] \arrow[d, phantom, "\cong"] & {} & y \arrow[dr, "g"] \arrow[d, phantom, "\Downarrow \varepsilon_g"] & \\
z \arrow[ur, "g^\bullet"] \arrow[urrr, "g^\bullet"'] \arrow[rrrr, "I"'] & {} & & {} & z
\end{tikzcd}$$

Proposition A.1.14. *Let $\boldsymbol{f} = (f, f^\bullet, \varepsilon, \eta)$ be an adjoint equivalence in a bicategory B. Then $\boldsymbol{f}^\bullet = (f^\bullet, f, \eta^{-1}, \varepsilon^{-1})$ is also an adjoint equivalence in B.*

A.2 Mates in a bicategory

In this section, we will quickly review the necessary results from the theory of mates in a bicategory that are used in our definitions. The main reference in the case that the bicategory involved is actually a strict 2-category is [24].

Lemma A.2.1. *Let B be a bicategory, and let $(f, f^\bullet, \varepsilon_f, \eta_f)$ and $(g, g^\bullet, \varepsilon_g, \eta_g)$ be a pair of adjunctions in B. Then there is a bijection between 2-cells $\alpha : tf^\bullet \Rightarrow g^\bullet s$ and 2-cells $\beta : gt \Rightarrow sf$.*

Proof. Define the isomorphism by sending the 2-cell α to the 2-cell α^+ given by the following pasting diagram.

$$\begin{tikzcd}
 & b \arrow[rr, "s"] \arrow[dr, "f^\bullet"] \arrow[rrrr, bend left, "s"] \arrow[d, phantom, "\Uparrow \eta_f"] & & c \arrow[rr, "1"] \arrow[dr, "g^\bullet"'] \arrow[u, phantom, "\Uparrow l"] & & c \\
a \arrow[ur, "f"] \arrow[rr, "1"'] \arrow[rrrr, bend right, "t"'] & {} & a \arrow[rr, "t"'] \arrow[ur, phantom, "\alpha"] \arrow[d, phantom, "\Uparrow r^{-1}"] & & d \arrow[ur, "g"'] \arrow[u, phantom, "\Uparrow \varepsilon_g"] &
\end{tikzcd}$$

The inverse function $\beta \mapsto \beta_+$ should be obvious, and this is an isomorphism by the triangle identities and coherence for bicategories. $\square$

We call α^+ the *mate* of α under the pair of adjunctions $\mathbf{f}, \mathbf{g}$. It should be noted that the mate of an invertible 2-cell is invertible. The rest of this appendix will be devoted to stating a variety of propositions that will be needed in dealing with tricategories; no proofs will be provided, as they generally follow from large diagram chases involving only the triangle identities, coherence for bicategories, and the axioms for functors and transformations.

If $\mathbf{f} = (f, f^\bullet, \varepsilon, \eta)$ is an adjoint equivalence in B, and $(F, \varphi) : B \to C$ is a functor, then

$$(Ff, Ff^\bullet, \varphi_0^{-1} \cdot F\varepsilon \cdot \varphi_2, \varphi_2^{-1} \cdot F\eta \cdot \varphi_0)$$

is an adjoint equivalence in C.

Proposition A.2.2. *Assume that $F, G : B \to C$ are weak functors, and that $\alpha : F \Rightarrow G$ is a transformation between them. If $\mathbf{f}$ is an adjoint equivalence in B, then*

$$\alpha_f^+ = (\alpha_{f^\bullet})^{-1}$$

.

It should be noted that here we are using the opposite adjoint equivalence of the one stated above.

Proposition A.2.3. *Assume that $F, G : B \to C$ are functors, and that $(\alpha, \alpha^\bullet, \varepsilon, \eta)$ is an adjoint equivalence in $\mathbf{Bicat}(B, C)$ with $\alpha : F \Rightarrow G$ and $\alpha^\bullet : G \Rightarrow F$. Then*

$$\alpha^\bullet_f = (\alpha_f^{-1})^+.$$

There is a special case that will be important to us, and that is when there are no additional 1-cells s, t. In that case, we obtain an isomorphism between 2-cells $\alpha : f^\bullet \Rightarrow g^\bullet$ and 2-cells $\hat{\alpha} : g \Rightarrow f$. Here we define the mate $\alpha^\dagger$ by first defining $\underline{\alpha} = r^{-1}\alpha l$, and then

$$\alpha^\dagger = l\underline{\alpha}^+ r^{-1}.$$

Proposition A.2.4. *Let $\mathbf{f}, \mathbf{g}, \mathbf{h}$ be three adjunctions in a bicategory B. If $\alpha : f^\bullet \Rightarrow g^\bullet$ and $\beta : g^\bullet \Rightarrow h^\bullet$ are composable 2-cells, then*

$$(\beta\alpha)^\dagger = (\alpha)^\dagger(\beta)^\dagger,$$

where the mate of $\beta\alpha$ is taken via the pair of composite adjunctions.

An important case is the following. Let B be a bicategory, and let $\mathbf{f_1}, \mathbf{f_2}, \mathbf{g_1}, \mathbf{g_2}$ be four adjoint equivalences such that the left adjoints form a square as below.

$$\begin{tikzcd} \cdot \arrow[r, "g_1"] \arrow[d, "f_1"'] & \cdot \arrow[d, "f_2"] \\ \cdot \arrow[r, "g_2"'] & \cdot \end{tikzcd}$$

If $\alpha : g_2 f_1 \Rightarrow f_2 g_1$ is a 2-cell, then we denote by α^+ the mate of α with respect to the opposite of the adjunctions $\mathbf{f_1}, \mathbf{f_2}$. Similarly, we denote by β^- the mate of $\beta : f_2g_1 \Rightarrow g_2f_1$ under the adjunctions $\mathbf{g_1}, \mathbf{g_2}$. Note the different directions of the 2-cells, and the necessary choices of which adjunction to use; thus α^{+-} makes sense, while α^{-+} does not.

Proposition A.2.5. *Given the situation above, let $\alpha : g_2f_1 \Rightarrow f_2g_1$ be an invertible 2-cell. Then*
1. $(\alpha^{+-})^{-1} = (\alpha^{-1})_{+-}$,
2. $(\alpha_{-+})^{-1} = (\alpha^{-1})^{-+}$, *and*
3. $\alpha^{+-} = \alpha^{\dagger}$.

Corollary A.2.6. *Assume that $F, G : B \to C$ are weak functors, and that $(\alpha, \alpha^{\bullet}, \varepsilon, \eta)$ is an adjoint equivalence in $\mathbf{Bicat}(B, C)$ with $\alpha : F \Rightarrow G$ and $\alpha^{\bullet} : G \Rightarrow F$. If $\mathbf{f}$ is an adjoint equivalence in B, then*

$$\alpha^{\bullet}_{f^{\bullet}} = \alpha_f^{+-} = \alpha_{f^{\bullet}}^{\dagger}.$$

Proof. Combining Proposition A.2.2 and Proposition A.2.3 gives the first equality, and the second is the third part of Proposition A.2.5. □

We next turn to the relationship between mates and the constraint 2-cells $\varphi : fg : FfFg \Rightarrow F(fg)$ of a weak functor $(F, \varphi) : B \to C$.

Proposition A.2.7. *Let $(F, \varphi) : B \to C$ be a weak functor between bicategories. Then the following equation holds for any appropriate pair of adjoint equivalence $\mathbf{f}, \mathbf{g}$ in B.*

$$(\varphi_{fg})^{\dagger} = \varphi^{-1}_{g^{\bullet}f^{\bullet}}.$$

Finally, we end this appendix with a discussion of the relationship between the bicategory constraint cells a_{fgh}, l_f, r_f and their mates.

Proposition A.2.8. *Let B be a bicategory with constraints given above. Then the following equations hold for any appropriate triple of adjoint equivalences $\mathbf{f}, \mathbf{g}, \mathbf{h}$.*

$$\begin{aligned}(r_f)^{\dagger} &= l^{-1}_{f^{\bullet}}\\ (l_f)^{\dagger} &= r^{-1}_{f^{\bullet}}\\ (a_{fgh})^{\dagger} &= a^{-1}_{h^{\bullet}g^{\bullet}f^{\bullet}}.\end{aligned}$$

Using these results, we can now take mates of diagrams of 2-cells inside a bicategory B.

A.3 Biadjoint biequivalences

The final section of this appendix is concerned with categorifying the definition of adjoint equivalence to yield the notion of biadjoint biequivalence. We then use this definition in some of the crucial theorems in Chapter 3.

Before defining biadjoint biequivalence, we need to prove a preliminary lemma. The diagrams used in this definition require the existence of an isomorphism $l_I \cong r_I : II \to I$ that we now construct.

Lemma A.3.1. *Let T be a tricategory. Then the 1-cells l_I and r_I are isomorphic in the bicategory $T(a,a)$.*

Proof. An isomorphism is given by the following composite; isomorphisms coming from the constraint cells in the bicategory $T(a,a)$ are unmarked.

$$\begin{gathered} l \cong l1 \overset{1*\eta_r}{\cong} l(rr^\bullet) \cong (lr)r^\bullet \overset{\text{Nat.}}{\cong} (r \circ l \otimes 1)r^\bullet \overset{(1*\lambda)*1}{\cong} (r(la))r^\bullet \cong \\ (r(l(1 \otimes l^\bullet \circ r \otimes 1)))r^\bullet \cong (r((l \circ 1 \otimes l^\bullet)r \otimes 1))r^\bullet \overset{\text{Nat.}}{\cong} \\ (r((l^\bullet l)r \otimes 1))r^\bullet \overset{(1*(\varepsilon_l*1))*1}{\cong} (r(1 \circ r \otimes 1))r^\bullet \cong r(r \otimes 1 \circ r^\bullet) \overset{\text{Nat.}}{\cong} r(r^\bullet r) \overset{1*\varepsilon_r^{-1}}{\cong} r1 \cong r \end{gathered}$$

□

Remark A.3.2. It should be noted that, once we have proven our coherence theorem, the above isomorphism will be the unique isomorphism constructed from the tricategory coherence cells from r_I to l_I. In the definition below, any isomorphism between l_I and r_I is assumed to be the one constructed in the lemma.

Definition A.3.3. Let T be a tricategory. Then a *biadjoint biequivalence* (f,g) in T consists of

- a pair of 1-cells $f : a \to b$, $g : b \to a$,
- a pair of adjoint equivalences $\alpha = (\alpha, \alpha^\bullet, \Gamma, \underline{\Gamma})$, $\beta = (\beta, \beta^\bullet, \Delta, \underline{\Delta})$ with

$$\begin{array}{rr} \alpha : f \otimes g \to I_b & \alpha^\bullet : I_b \to f \otimes g \\ \Gamma : \alpha\alpha^\bullet \overset{\cong}{\Longrightarrow} 1_{I_b} & \underline{\Gamma} : 1_{I_b} \overset{\cong}{\Longrightarrow} \alpha^\bullet\alpha \\ \beta : g \otimes f \to I_a & \beta^\bullet : I_a \to g \otimes f \\ \Delta : \beta\beta^\bullet \overset{\cong}{\Longrightarrow} 1_{I_a} & \underline{\Delta} : 1_{I_a} \overset{\cong}{\Longrightarrow} \beta^\bullet\beta, \end{array}$$

- and a pair of 3-cell isomorphisms Φ, Ψ

$$\begin{tikzcd} f \arrow[r, "r^\bullet"] \arrow[drrr, "1"'] & f \otimes I \arrow[r, "1 \otimes \beta^\bullet"] & f \otimes (g \otimes f) \arrow[r, "a^\bullet"] \arrow[d, phantom, "\Swarrow \Phi"] & (f \otimes g) \otimes f \arrow[d, "\alpha \otimes 1"] \\ & & & I \otimes f \arrow[d, "l"] \\ & & & f \end{tikzcd}$$

$$\begin{tikzcd} g \arrow[r, "l^\bullet"] \arrow[drrr, "1"'] & I \otimes g \arrow[r, "\beta^\bullet \otimes 1"] & (g \otimes f) \otimes g \arrow[r, "a"] \arrow[d, phantom, "\Swarrow \Psi"] & g \otimes (f \otimes g) \arrow[d, "1 \otimes \alpha"] \\ & & & g \otimes I \arrow[d, "r"] \\ & & & g \end{tikzcd}$$

such that the two pasting diagrams below are identities. Once again we have used the convention that concatenation denotes tensor, and that naturality isomorphisms that are unique constraint isomorphisms from the functor $\otimes$ are unmarked; additionally, some cells are the mates of those indicated.

$$
\begin{tikzcd}
 & & & & & fg \arrow[ddr, "l^{\bullet}"] \arrow[dddddrrr, bend left, "\alpha"] & & & \\
 & & & & (If)g \arrow[ur, "l1"] \arrow[drr, "a"'] \arrow[r, phantom, "\Downarrow\lambda"] \arrow[dddd, phantom, "\cong"] & & & & \\
 & & & ((fg)f)g \arrow[ur, "(\alpha 1)1"] \arrow[dddr, "a"'] \arrow[ll, phantom, "\Downarrow\Phi^{-1}1"] & & & I(fg) \arrow[dddr, "1\alpha"] \arrow[drr, phantom, "\cong"] & & \\
 & & (f(gf))g \arrow[ur, "a^{\bullet}1"] \arrow[dddd, bend left, "a"] & & & & & & \\
 & (fI)g \arrow[ur, "(1\beta^{\bullet})1"] \arrow[dd, "a"] \arrow[dr, phantom, "\cong"] & & & & & & & \\
fg \arrow[ur, "r^{\bullet}1"] \arrow[dr, "1l^{\bullet}"'] \arrow[uuuuurrrrr, bend left, "1"] \arrow[dddddrrrrr, bend right, "1"'] \arrow[r, phantom, "\Downarrow\mu"] & & & & (fg)(fg) \arrow[uuurr, "\alpha 1"'] \arrow[dddrr, "1\alpha"] \arrow[l, phantom, "\Downarrow\pi"] \arrow[rrr, phantom, "\cong"] & & & II \arrow[r, bend left, "l"] \arrow[r, bend right, "r"'] \arrow[r, phantom, "\cong"] & I \\
 & f(Ig) \arrow[dr, "1(\beta^{\bullet}1)"'] & & & & & & & \\
 & & f((gf)g) \arrow[dr, "1a"'] & & & & & & \\
 & & & f(g(fg)) \arrow[uuur, "a^{\bullet}"] \arrow[dr, "1(1\alpha)"'] \arrow[ll, phantom, "\Downarrow 1\Psi"] & & & (fg)I \arrow[uuur, "\alpha 1"'] \arrow[urr, phantom, "\cong"] & & \\
 & & & & f(gI) \arrow[urr, "a^{\bullet}"] \arrow[dr, "1r"'] \arrow[r, phantom, "\Downarrow\rho"] \arrow[uuuu, phantom, "\cong"] & & & & \\
 & & & & & fg \arrow[uur, "r^{\bullet}"'] \arrow[uuuuurrr, bend right, "\alpha"'] & & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& & & gf \arrow[drr, "l^\bullet 1"] \arrow[ddddddrrrrrrr, bend left, "1"] & & & & & & & \\
& & & \Downarrow\lambda & & (Ig)f \arrow[dr, "(\beta^\bullet 1)1"] & & & & & \\
& & & & & & ((gf)g)f \arrow[ddr, "a1"] & \Downarrow \Psi^{-1}1 & & & \\
& & I(gf) \arrow[uuur, "l"] \arrow[uurrr, "a^\bullet"] \arrow[dddrr, "\beta^\bullet 1"] & & & & & & & & \\
& \cong & & & \cong & & & (g(fg))f \arrow[dr, "(1\alpha)1"] \arrow[dddd, bend right, "a"'] & & & \\
& & & & & & & & (gI)f \arrow[drr, "r1"] \arrow[dd, "a"'] & & \\
I \arrow[r, bend left, "l^\bullet"] \arrow[r, bend right, "r^\bullet"'] \arrow[r, phantom, "\cong"] \arrow[uuuuuurrr, bend left, "\beta^\bullet"] \arrow[ddddddrrr, bend right, "\beta^\bullet"'] & II \arrow[uuur, "1\beta^\bullet"] \arrow[dddr, "\beta^\bullet 1"'] & \cong & & (gf)(gf) \arrow[uuuurr, "a^\bullet"'] \arrow[ddddrr, "a"] & & \Downarrow\pi & \cong & & \Downarrow\mu & gf \\
& & & & & & & & g(If) \arrow[urr, "1l"'] & & \\
& \cong & & & \cong & & & g((fg)f) \arrow[ur, "1(\alpha 1)"'] & & & \\
& & (gf)I \arrow[uuurr, "1\beta^\bullet"] \arrow[ddrrr, "a"] \arrow[dddr, "r"'] & & & & & & & & \\
& & & & & & g(f(gf)) \arrow[uur, "1a^\bullet"'] & \Downarrow 1\Phi & & & \\
& & & \Downarrow\rho & & g(fI) \arrow[ur, "1(1\beta^\bullet)"'] & & & & & \\
& & & gf \arrow[urr, "1r^\bullet"'] \arrow[uuuuuurrrrrrr, bend right, "1"'] & & & & & & &
\end{tikzcd}
$$

Remark A.3.4. In the presence of the simplifying assumption that the tricategory T is actually a strict, cubical tricategory (i.e., a **Gray**-category), the axioms above simplify to the condition that the diagrams below are identity diagrams. See [45] for the original definition.

$$\begin{tikzcd}
& fg \arrow[dr, "\alpha"] & \\
fg \arrow[ur, "1"] \arrow[r, "f\beta^{\bullet}g"] \arrow[dr, "1"'] \arrow[ur, phantom, "\Searrow \Phi^{-1}g"] \arrow[dr, phantom, "\Swarrow f\Psi"] & fgfg \arrow[u, "\alpha fg"'] \arrow[d, "fg\alpha"] \arrow[r, phantom, "\cong"] & I \\
& fg \arrow[ur, "\alpha"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
& gf \arrow[d, "\beta^{\bullet}gf"'] \arrow[dr, "1"] & \\
I \arrow[ur, "\beta^{\bullet}"] \arrow[dr, "\beta^{\bullet}"'] \arrow[r, phantom, "\cong"] & gfgf \arrow[r, "g\alpha f"'] \arrow[ur, phantom, "\Swarrow \Psi^{-1}f"] \arrow[dr, phantom, "\Searrow g\Phi"] & gf \\
& gf \arrow[u, "gf\beta^{\bullet}"] \arrow[ur, "1"'] &
\end{tikzcd}$$

We pay special attention to the case when the tricategory in question is **Bicat**; we shall write the 1-cells as functors $F : A \to B, G : B \to A$. In this case the adjoint equivalences $\boldsymbol{a}, \boldsymbol{l}, \boldsymbol{r}$ in **Bicat** are all identity adjoint equivalences. The adjoint equivalences $\boldsymbol{\alpha}, \boldsymbol{\beta}$ show that the functors F, G are biequivalences. The invertible modification Φ amounts to an invertible 2-cell

$$\Phi_a : \alpha_{Fa} \circ F\beta^{\bullet}_a \Rightarrow 1_{Fa}$$

for each object a, similarly for Ψ.

There are functors $B(Fa, -) : B \to \mathbf{Cat}$ and $A(a, G-) : B \to \mathbf{Cat}$, and the definition of a biadjunction given in [37] is an equivalence between these two functors in the functor bicategory. Giving such an equivalence amounts to giving a transformation σ between these two functors with each component 1-cell an equivalence in the target.

We define σ as follows. The component σ_b is the functor $B(Fa, b) \to A(a, Gb)$ that sends f to $Gf \circ \beta^{\bullet}_a$ and $\gamma : f \Rightarrow g$ to $G\gamma * 1$. Note that this functor is an equivalence of categories since G is and $\beta^{\bullet}_a$ is an equivalence 1-cell. For every 1-cell $h : b \to b'$, we have the naturality isomorphism given by the invertible 2-cell $\varphi^G_{hf} * 1_{\beta^{\bullet}_a}$. It is easy to check that this is a transformation using coherence for bicategories.

On the other hand, we can define a transformation $\tau : A(a, G-) \Rightarrow B(Fa, -)$ as follows. The component τ_b is the functor $A(a, Gb) \to B(Fa, b)$ that sends f to $\alpha_b \circ Ff$ and $\gamma : f \Rightarrow g$ to $1 * F\gamma$. The naturality isomorphism is given by

$$(1_{\alpha_{b'}} * \varphi^F_{Gf,h}) \circ (\alpha_f * 1_{F(Gf \circ h)}).$$

The transformation axioms then follow from coherence and the naturality of α. This transformation is an equivalence 1-cell in the functor bicategory since the functors τ_b are all equivalences of categories using the same argument as above. The asymmetry in the definition of σ and τ is due to the fact that we are holding the variable a fixed; holding b fixed would produce a dual asymmetry.

The modifications Φ, Ψ then provide a unit and counit for σ and τ. The two axioms for a biadjoint biequivalence yield the triangle identities. Thus we have provided σ and τ with the structure of an adjoint equivalence. It should now be clear that our definition of biadjoint biequivalence, in the case of the tricategory **Bicat**, is a fully algebraic version of the combination of biequivalence and biadjunction as given by Street in [37].

Appendix B

Unpacked definitions

This appendix will give unpacked versions of the definitions appearing in Chapter 2. Nothing in this appendix is new. We have included it both as a reference and to display all of the constraint cells necessary for the construction of the free tricategory. Only data will be unpacked as the formulas for the axioms are already presented as the equality of pasting diagrams using the cells from the unpacked definitions. An unpacked version of the definition of perturbation is not given, as the original definition is already maximally unpacked.

B.1 Unpacked tricategories

A tricategory T has the data of

- a set ob$T$ of objects,
- for each pair of objects a, b, a bicategory $T(a, b)$,
- for each triple of objects a, b, c, a functor

  $$\otimes : T(b, c) \times T(a, b) \to T(a, c)$$

  which includes isomorphisms

  $$(\beta' \otimes \alpha') \circ (\beta \otimes \alpha) \cong (\beta'\beta) \otimes (\alpha'\alpha),$$

  $$1_g \otimes 1_f \cong 1_{g\otimes f},$$

- for each object a, an object $I_a \in T(a, a)$ and a morphism $i_a : I_a \to I_a$ along with an isomorphism $i_a \cong 1_{I_a}$,
- for each triple of composable 1-cells h, g, f, 2-cells

  $$\begin{aligned} a_{hgf} &: (h \otimes g) \otimes f \to h \otimes (g \otimes f) \\ a^{\bullet}_{hgf} &: h \otimes (g \otimes f) \to (h \otimes g) \otimes f \end{aligned}$$

and invertible 3-cells

$$\begin{gathered}\varepsilon^a_{hgf} : a_{hgf} \circ a^{\bullet}_{hgf} \cong 1_{h\otimes(g\otimes f)} \\ \eta^a_{hgf} : 1_{(h\otimes g)\otimes f} \cong a^{\bullet}_{hgf} a_{hgf},\end{gathered}$$

- for each pair of triples of composable 1-cells, h, g, f and h', g', f', and a triple of 2-cells between them γ, β, α, invertible 3-cells (natural in γ, β, α)

$$\begin{gathered}a_{\gamma,\beta,\alpha} : a_{h'g'f'} \circ (\gamma\otimes\beta)\otimes\alpha \cong \gamma\otimes(\beta\otimes\alpha)\circ a_{hgf} \\ a^{\bullet}_{\gamma,\beta,\alpha} : a^{\bullet}_{h'g'f'} \circ \gamma\otimes(\beta\otimes\alpha) \Rightarrow (\gamma\otimes\beta)\otimes\alpha\circ a^{\bullet}_{hgf},\end{gathered}$$

- for each 1-cell f, 2-cells

$$\begin{gathered}l_f : I_b \otimes f \to f \\ l^{\bullet}_f : f \to I_b \otimes f \\ r_f : f \otimes I_a \to f \\ r^{\bullet}_f : f \to f \otimes I_a\end{gathered}$$

and invertible 3-cells

$$\begin{gathered}\varepsilon^l_f : l_f l^{\bullet}_f \Rightarrow 1_f \\ \eta^l_f : 1_{I_b\otimes f} \Rightarrow l^{\bullet}_f l_f \\ \varepsilon^r_f : r_f r^{\bullet}_f \Rightarrow 1_f \\ \eta^r_f : 1_{f\otimes I_a} \Rightarrow r^{\bullet}_f r_f,\end{gathered}$$

- for each pair of 1-cells f, f' and 2-cell between them α, invertible 3-cells (natural in α)

$$\begin{gathered}l_\alpha : l_{f'} \circ (1\otimes\alpha) \Rightarrow \alpha\circ l_f \\ l^{\bullet}_\alpha : l^{\bullet}_{f'} \circ \alpha \Rightarrow (1\otimes\alpha)\circ l^{\bullet}_f \\ r_\alpha : r_{f'} \circ (\alpha\otimes 1) \Rightarrow \alpha\circ r_f \\ r^{\bullet}_\alpha : r^{\bullet}_{f'} \circ \alpha \Rightarrow (\alpha\otimes 1)\circ r^{\bullet}_f,\end{gathered}$$

- for every quadruple of composable 1-cells j, h, g, f, an invertible 3-cell as displayed below,

$$\begin{tikzcd}
 & (j\otimes(h\otimes g))\otimes f \arrow[r, "a"] & j\otimes((h\otimes g)\otimes f) \arrow[dr, "1\otimes a"] & \\
((j\otimes h)\otimes g)\otimes f \arrow[ur, "a\otimes 1"] \arrow[drr, "a"'] & & \arrow[d, phantom, "\Downarrow \pi_{jhgf}"] & j\otimes(h\otimes(g\otimes f)) \\
 & & (j\otimes h)\otimes(g\otimes f) \arrow[ur, "a"'] &
\end{tikzcd}$$

- and for every pair of composable 1-cells f, g, invertible 3-cells as displayed below.

$$\begin{tikzcd}
 & (g\otimes I)\otimes f \arrow[r, "a"] \arrow[d, phantom, "\Downarrow \mu_{gf}"] & g\otimes(I\otimes f) \arrow[dr, "1\otimes l"] & \\
g\otimes f \arrow[ur, "r^{\bullet}\otimes 1"] \arrow[rrr, "1"'] & & & g\otimes f
\end{tikzcd}$$

$$
\begin{tikzcd}
(I\otimes g)\otimes f \arrow[rr, "l\otimes 1"] \arrow[dr, "a"'] & & g\otimes f \\
& I\otimes(g\otimes f) \arrow[ur, "l"'] \arrow[u, phantom, "\Downarrow\lambda_{gf}"] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
g\otimes f \arrow[rr, "1\otimes r^{\bullet}"] \arrow[dr, "r^{\bullet}"'] & & g\otimes(f\otimes I) \\
& (g\otimes f)\otimes I \arrow[ur, "a"'] \arrow[u, phantom, "\Downarrow\rho_{gf}"] &
\end{tikzcd}
$$

B.2 Unpacked functors

Let S, T be tricategories. A functor $F : S \to T$ has the data of

- a function $\mathrm{ob}F : \mathrm{ob}S \to \mathrm{ob}T$,
- for each pair of objects a, b in S, a functor
$$F_{ab} : S(a,b) \to T(Fa, Fb),$$
- for all pairs of composable 1-cells f, g in S, 2-cells in T
$$\begin{aligned} \chi_{gf} &: Fg \otimes' Ff \to F(g\otimes f) \\ \chi^{\bullet}_{gf} &: F(g\otimes f) \to Fg \otimes' Ff \end{aligned}$$
and invertible 3-cells
$$\begin{aligned} \varepsilon^{\chi}_{gf} &: \chi_{gf} \circ \chi^{\bullet}_{gf} \Rightarrow 1_{F(g\otimes f)} \\ \eta^{\chi}_{gf} &: 1_{Fg\otimes' Ff} \Rightarrow \chi^{\bullet}_{gf} \circ \chi_{gf}, \end{aligned}$$
- for all pairs of pairs of composable 1-cells, f, g and f', g', and all pairs of 2-cells between them β, α, invertible 3-cells (natural in β, α) as displayed below,

$$
\begin{tikzcd}
Fg\otimes' Ff \arrow[r, "\chi"] \arrow[d, "F\beta\otimes' F\alpha"'] & F(g\otimes f) \arrow[d, "F(\beta\otimes\alpha)"] \\
Fg'\otimes' Ff' \arrow[r, "\chi"'] \arrow[ur, phantom, "\Nearrow\chi_{\beta\alpha}"] & F(g'\otimes f')
\end{tikzcd}
$$

$$
\begin{tikzcd}
F(g\otimes f) \arrow[r, "\chi^{\bullet}"] \arrow[d, "F(\beta\otimes\alpha)"'] & Fg\otimes' Ff \arrow[d, "F\beta\otimes' F\alpha"] \\
F(g'\otimes f') \arrow[r, "\chi^{\bullet}"'] \arrow[ur, phantom, "\Nearrow\chi^{\bullet}_{\beta\alpha}"] & Fg'\otimes' Ff'
\end{tikzcd}
$$

- for all objects a in S, 2-cells

$$\begin{aligned} \iota_a &: I_{Fa} \to FI_a \\ \iota_a^{\bullet} &: FI_a \to I_{Fa} \end{aligned}$$

and invertible 3-cells

$$\begin{aligned} \varepsilon_a^{\iota} &: \iota_a \circ \iota_a^{\bullet} \Rightarrow 1_{FI_a} \\ \eta_a^{\iota} &: 1_{I_{Fa}} \Rightarrow \iota_a^{\bullet} \circ \iota_a \\ \iota &: \iota_a \circ i_a \Rightarrow Fi_a \circ \iota_a \\ \iota^{\bullet} &: \iota_a^{\bullet} \circ Fi_a \Rightarrow i_a \circ \iota_a^{\bullet}, \end{aligned}$$

- for every triple of composable 1-cells h, g, f in S, an invertible 3-cell in T as displayed below,

$$\begin{tikzcd}
(Fh \otimes' Fg) \otimes' Ff \arrow[r, "\chi \otimes' 1"] \arrow[d, "a'"'] & F(h \otimes g) \otimes' Ff \arrow[r, "\chi"] \arrow[d, phantom, "\swarrow \omega_{hgf}"] & F((h \otimes g) \otimes f) \arrow[d, "Fa"] \\
Fh \otimes' (Fg \otimes' Ff) \arrow[r, "1 \otimes' \chi"'] & Fh \otimes' F(g \otimes f) \arrow[r, "\chi"'] & F(h \otimes (g \otimes f))
\end{tikzcd}$$

- for every 1-cell f in S, two invertible 3-cells as displayed below.

$$\begin{tikzcd}
I' \otimes' Ff \arrow[r, "\iota \otimes' 1"] \arrow[drr, "l'"'] & FI \otimes' Ff \arrow[r, "\chi"] \arrow[d, phantom, "\swarrow \gamma_f"] & F(I \otimes f) \arrow[d, "Fl"] \\
 & \; & Ff
\end{tikzcd}$$

$$\begin{tikzcd}
Ff \arrow[r, "(r')^{\bullet}"] \arrow[drr, "Fr^{\bullet}"'] & Ff \otimes' I' \arrow[r, "1 \otimes' \iota"] \arrow[d, phantom, "\nearrow \delta_f"] & Ff \otimes' FI \arrow[d, "\chi"] \\
 & \; & F(f \otimes I)
\end{tikzcd}$$

B.3 Unpacked transformations

Let $F, G : S \to T$ be functors. Then the data of a transformation $\alpha : F \to G$ consists of

- for every object a of S, a 1-cell $\alpha_a : Fa \to Ga$ in T,
- for every 1-cell $f : a \to b$ in S, 2-cells

$$\begin{aligned} \alpha_f &: \alpha_b \otimes' Ff \to Gf \otimes' \alpha_a \\ \alpha_f^{\bullet} &: Gf \otimes' \alpha_a \to \alpha_b \otimes' Ff \end{aligned}$$

and invertible 3-cells as displayed below,

$$\begin{aligned}\varepsilon_f^\alpha &: \alpha_f \circ \alpha_f^\bullet \Rightarrow 1_{Gf\otimes'\alpha_a}\\ \eta_f^\alpha &: 1_{\alpha_b\otimes' Ff} \Rightarrow \alpha_f^\bullet \circ \alpha_f\end{aligned}$$

- for every 2-cell $\theta : f \to g$ in S, an invertible 3-cell (natural in θ) in T as displayed below,

$$\begin{tikzcd}
\alpha_b \otimes' Ff \arrow[r, "\alpha_f"] \arrow[d, "1\otimes' F\theta"'] & Gf \otimes' \alpha_a \arrow[d, "G\theta\otimes' 1"] \\
\alpha_b \otimes' Fg \arrow[r, "\alpha_g"'] \arrow[ur, phantom, "\Nearrow \alpha_\theta"] & Gg \otimes' \alpha_a
\end{tikzcd}$$

- for every pair of composable 1-cells g, f in S, an invertible 3-cell in T as displayed below,

$$\begin{tikzcd}
(\alpha_c\otimes' Fg)\otimes' Ff \arrow[r, "\alpha_g\otimes' 1"] \arrow[dd, "a'"'] & (Gg\otimes'\alpha_b)\otimes' Ff \arrow[r, "a'"] & Gg\otimes'(\alpha_b\otimes' Ff) \arrow[r, "1\otimes'\alpha_f"] \arrow[ddl, phantom, "\Swarrow \Pi_{gf}"] & Gg\otimes'(Gf\otimes'\alpha_a) \arrow[d, "(a')^\bullet"] \\
& & & (Gg\otimes' Gf)\otimes'\alpha_a \arrow[d, "\chi^G\otimes' 1"] \\
\alpha_c\otimes'(Fg\otimes' Ff) \arrow[rr, "1\otimes'\chi^F"'] & & \alpha_c\otimes' F(g\otimes f) \arrow[r, "\alpha_{g\otimes f}"'] & G(g\otimes f)\otimes'\alpha_a
\end{tikzcd}$$

- and for every object a in S, an invertible 3-cell as displayed below.

$$\begin{tikzcd}
\alpha_a \arrow[r, "(r')^\bullet"] \arrow[drr, "(l')^\bullet"'] & \alpha_a \otimes' I' \arrow[r, "1\otimes'\iota^F"] & \alpha_a \otimes' FI \arrow[r, "\alpha_I"] \arrow[d, phantom, "\Downarrow M_a"] & GI \otimes' \alpha_a \\
& & I' \otimes' \alpha_a \arrow[ur, "\iota^G\otimes' 1"'] &
\end{tikzcd}$$

B.4 Unpacked modifications

Let $\alpha, \beta : F \to G$ be transformations. Then the data for a modification $m : \alpha \Rrightarrow \beta$ consists of

- for every object a of S, a 2-cell $m_a : \alpha_a \to \beta_a$ of T and
- for every 1-cell $f : a \to b$ in S, an invertible 3-cell in T as displayed below.

$$\begin{tikzcd}
& Gf \otimes' \alpha_a \arrow[dr, "1\otimes' m_a"] \arrow[dd, phantom, "\Downarrow m_f"] & \\
\alpha_b \otimes' Ff \arrow[ur, "\alpha_f"] \arrow[dr, "m_b\otimes' 1"'] & & Gf \otimes' \beta_a \\
& \beta_b \otimes' Ff \arrow[ur, "\beta_f"'] &
\end{tikzcd}$$

Appendix C

Calculations

This appendix will give some calculational tools necessary for many of the proofs given in the body of this work. We will not provide the proofs for these results, as this is unfeasible because of the size of the diagrams involved. We will, however, explain the techniques necessary to reproduce these proofs. The main ingredient in these calculations is the following proposition.

Proposition C.0.1. *Let T be a tricategory. Then the following equations of 3-cells hold.*

$$
\begin{tikzcd}
f(gh) \arrow[rrr, "1(1r^\bullet)"] \arrow[rrd, "1r^\bullet"'] \arrow[rdd, "r^\bullet"'] & & \Downarrow 1\rho & f(g(hI)) \\
& \Downarrow\rho & f((gh)I) \arrow[ru, "1a"'] & \\
(fg)h \arrow[uu, "a"] \arrow[rd, "r^\bullet"'] \arrow[r, phantom, "\cong"] & (f(gh))I \arrow[ru, "a"'] & \Downarrow\pi & (fg)(hI) \arrow[uu, "a"'] \\
& ((fg)h)I \arrow[u, "a1"'] \arrow[rru, "a"'] & &
\end{tikzcd}
=
\begin{tikzcd}
f(gh) \arrow[rr, "1(1r^\bullet)"] & \cong & f(g(hI)) \\
(fg)h \arrow[u, "a"] \arrow[rr, "1r^\bullet"] \arrow[rd, "r^\bullet"'] & \Downarrow\rho & (fg)(hI) \arrow[u, "a"'] \\
& ((fg)h)I \arrow[ru, "a"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
((If)g)h \arrow[rrr, "(l1)1"] \arrow[rd, "a1"'] \arrow[ddd, "a"'] & \Downarrow\lambda 1 & & f(gh) \arrow[ddd, "a"] \\
& (I(fg))h \arrow[rru, "l1"] \arrow[rd, "a"'] & \Downarrow\lambda & \\
& \Downarrow\pi & I((fg)h) \arrow[ruu, "l"'] \arrow[dd, "1a"] \arrow[r, phantom, "\cong"] & {} \\
(If)(gh) \arrow[rrd, "a"'] & & & f(gh) \\
& & I(f(gh)) \arrow[ru, "l"'] &
\end{tikzcd}
=
\begin{tikzcd}
((If)g)h \arrow[rr, "(l1)1"] \arrow[d, "a"'] & \cong & (fg)h \arrow[d, "a"] \\
(If)(gh) \arrow[rr, "l1"] \arrow[rd, "a"'] & \Downarrow\lambda & f(gh) \\
& I(f(gh)) \arrow[ru, "l"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
f \arrow[rr, "r^{\bullet}"] & & fI \\
If \arrow[u, "l"] \arrow[dr, "r^{\bullet}"'] \arrow[r, phantom, "\cong"] & \, & I(fI) \arrow[u, "l"'] \arrow[l, phantom, "\Downarrow \lambda"'] \\
& (If)I \arrow[ur, "a"'] \arrow[uur, bend left, "l1"] &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
f \arrow[rr, "r^{\bullet}"] & & fI \\
If \arrow[u, "l"] \arrow[rr, "1r^{\bullet}"] \arrow[urr, phantom, "\cong"] \arrow[dr, "r^{\bullet}"'] & & I(fI) \arrow[u, "l"'] \\
& (If)I \arrow[ur, "a"'] \arrow[u, phantom, "\Downarrow \rho"] &
\end{tikzcd}
$$

Proof. These same equations are proved in [17] under the hypothesis that T is a cubical tricategory. The calculation done there applies equally well to our situation, so we will not repeat it. We will thus prove that the third equation holds in any tricategory using the fact that it holds in any cubical tricategory as an example. The other two equations can be proved in a similar fashion, though the calculations are longer as the diagrams are more complicated.

We will start with the left side of the equation above in an arbitrary tricategory T. Since the equation holds in any cubical tricategory, it holds in the cubical tricategory st$T$. We will write $\varepsilon : \mathrm{st}T \to T$ for the triequivalence produced by Theorem 6.1.2 defining st$T$.

It can easily be checked that the cell l in st$T$ is given by the cell l in T and maps to l under ε. The same holds for the constraints a and r, as well as their adjoints; it is also easy to check that a similar statement regarding naturality constraints is true. Therefore the left side of the equation in consideration in st$T$ maps under ε to the same diagram in T. The same holds for the right side, and since ε is a triequivalence, the fact that the equation holds in st$T$ implies that it holds in T as well. □

The proofs of the omitted calculations in the body of this work can be recovered in the following manner. Each of these calculations follows from the tricategory axioms and the three equations via the use of various naturality conditions and the equation that is the modification axiom. It is at times necessary to append invertible cells to the diagrams in question to use these equations, though.

As an example, to check the first transformation axiom for α in Proposition 4.2.3, we use the following steps. After appending a naturality isomorphism, apply the third tricategory axiom to the left side. Using naturality and the fact that λ and μ are modifications, we can then use the second equation above on the left side. On the right side of the equation, first use naturality and the fact that ρ and μ are modifications, and then apply the second tricategory axiom and the third equation above. It is now simple to check that the resulting diagrams are equal.

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