Ambidexterity §4

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1 Motivations

1.1 Notation. We write **Spc** for the ∞-category of spaces or ∞-groupoids, $\mathbf{Cat}_\infty$ for the ∞-category of ∞-categories, **Sp** for the ∞-category of spectra, and $\mathbf{Sp}_{K(n)}$ for the ∞-category of $K(n)$-local spectra (where $K(n)$ is a Morava K-theory).

Recall that the goal of [1] is to show that for any π-finite space X, prime number p, and Morava K-theory $K(n)$ (at p), the functors

$$\operatorname{colim}_X, \lim_X \colon \operatorname{Fun}(X, \mathbf{Sp}_{K(n)}) \to \mathbf{Sp}_{K(n)}$$

are (canonically) equivalent. If we let $f \colon X \to *$ denote the unique morphism, then:

- The diagonal functor $\mathbf{Sp}_{K(n)} \to \operatorname{Fun}(X, \mathbf{Sp}_{K(n)})$ is identified with the functor $f^\star \colon \operatorname{Fun}(*, \mathbf{Sp}_{K(n)}) \to \operatorname{Fun}(X, \mathbf{Sp}_{K(n)})$ given by precomposition with f.
- The functor colim_X is identified with the left adjoint $f_!$ of $f^\star$, given by left Kan extension along f.
- The functor $\lim_X$ is identified with the right adjoint $f_\star$ of $f^\star$, given by right Kan extension along f.

Rephrasing our problem, we're interested in studying the assignment

$$X \mapsto \begin{tikzcd} \operatorname{Fun}(*, \mathbf{Sp}_{K(n)}) \arrow[r, "f^\star" description] & \operatorname{Fun}(X, \mathbf{Sp}_{K}) \arrow[l, bend right, "f_!"'] \arrow[l, bend left, "f_\star"] \arrow[l, phantom, shift right=3, "\perp"] \arrow[l, phantom, shift left=3, "\perp"] \end{tikzcd} .$$

and determining when there is a natural equivalence $f_! \simeq f_\star$.

(1) To encode all the relevant functoriality in X, we should adopt the relative point of view and determine which morphisms $f \colon X \to Y$ in **Spc** have the property that $f_! \simeq f_\star$.

(2) For setting up the general theory, it is not really relevant that we are working with local systems of $K(n)$-local spectra; we might as well consider any functor that assigns a morphism $f\colon X \to Y$ in **Spc** a chain of three adjunctions $f_! \dashv f^* \dashv f_*$.

(3) The perspective we've taken is to assume that to any morphism $f\colon X \to Y$ in **Spc**, we have three adjoints $f_! \dashv f^* \dashv f_*$, and to try to construct an equivalence $\mathrm{Nm}_f\colon f_! \xrightarrow{\sim} f_*$. We can equivalently *not* assume the existence of the extreme right adjoint f_* (or, alternatively, the extreme left adjoint $f_!$), and try to exhibit $f_!$ as a right adjoint to f^*. That is, we might as well restrict our attention to functors $\mathbf{Spc} \to \mathbf{Cat}_\infty^{ladj}$, where $\mathbf{Cat}_\infty^{ladj}$ is the ∞-category of ∞-categories and *left adjoint* functors (see Notation 2.1).

(4) The fact that we're considering functors to $\mathbf{Cat}_\infty^{ladj}$ with source the ∞-category of spaces isn't particularly relevant for setting up the general theory; we can replace the ∞-category of spaces with an essentially arbitrary index ∞-category $\boldsymbol{X}$.

There is one non-obvious fact particular to our situation that *is* relevant to the general theory (Proposition 1.5). First we recall the colimit formula for left Kan extensions.

1.2 Recollection (comma ∞-categories). The ***comma*** $\boldsymbol{\infty}$***-category*** $X \downarrow_Y Z$ associated to two functors $f\colon X \to Y$ and $g\colon Z \to Y$ is the universal ∞-category fitting into a lax-commutative diagram

$$\begin{tikzcd} X \downarrow_Y Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow"] & X \arrow[d, "f"] \\ Z \arrow[r, "g"'] & Y . \end{tikzcd}$$

Explicitly, $X \downarrow_Y Z$ can be computed as the iterated pullback

$$X \downarrow_Y Z \coloneqq X \times_{\mathrm{Fun}(\{0\},Y)} \mathrm{Fun}(\Delta^1, Y) \times_{\mathrm{Fun}(\{1\},Y)} Z \,. \tag{1.3}$$

Notice that since all morphisms are invertible in an ∞-groupoid, if Y is an ∞-groupoid, then the comma ∞-category $X \downarrow_Y Z$ is simply given by the pullback

$$X \downarrow_Y Z \coloneqq X \times_Y Z \,.$$

(This can also be seen from the formal description (1.3) by noting that an ∞-category Y is an ∞-groupoid if and only if evaluation at 0 or 1 defines an equivalence of ∞-categories $\mathrm{Fun}(\Delta^1, Y) \xrightarrow{\sim} Y$.)

1.4 Recollection (colimit formula for left Kan extensions). Let C be an ∞-category with colimits and let $L_X\colon X \to C$ and $f\colon X \to Y$ be functors. Then for each object $y \in Y$, the value of the left Kan extension $\mathrm{Lan}_f L_X$ of L_X along f at y is given by the colimit

$$(\mathrm{Lan}_f L_X)(y) \simeq \mathrm{colim}\Big(X \downarrow_Y \{y\} \longrightarrow X \xrightarrow{L_X} C \Big) ,$$

where $X \downarrow_Y \{y\}$ is the comma ∞-category.

1.5 Proposition ([1, Proposition 4.3.3]). *Let C be an ∞-category with colimits and*

$$
\begin{tikzcd}
X' \arrow[r, "\bar{f}"] \arrow[d, "\bar{g}"'] \arrow[dr, phantom, very near start, "\lrcorner"] & Y' \arrow[d, "g"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}
\tag{1.6}
$$

a pullback square in **Spc**. *Then the associated square*

$$
\begin{tikzcd}
\mathrm{Fun}(X',C) \arrow[r, "\bar{f}_!"] & \mathrm{Fun}(Y',C) \\
\mathrm{Fun}(X,C) \arrow[r, "f_!"'] \arrow[u, "\bar{g}^*"] & \mathrm{Fun}(Y,C) \arrow[u, "g^*"']
\end{tikzcd}
$$

commutes.

Proof. Note that by the universal property of the left Kan extension, for any local system $L_X \colon X \to C$ we have a natural transformation

$$\theta \colon \mathrm{Lan}_{\bar{f}}(L_X \circ \bar{g}) \to (\mathrm{Lan}_f L_X) \circ g ,$$

i.e., a natural transformation $\bar{f}_! \bar{g}^*(L_X) \to g^* f_!(L_X)$. We want to show that for each $y' \in Y'$, the morphism

$$\theta(y') \colon \mathrm{Lan}_{\bar{f}}(L_X \circ \bar{g})(y') \to (\mathrm{Lan}_f L_X)(g(y'))$$

is an equivalence.

Since C has all colimits, by the pointwise formula for left Kan extensions (Recollection 1.4), for all $y' \in Y'$ we have

$$\mathrm{Lan}_{\bar{f}}(L_X \circ \bar{g})(y') \simeq \operatorname{colim}\left(X' \downarrow_{Y'} \{y'\} \longrightarrow X' \xrightarrow{\bar{g}} X \xrightarrow{L_X} C \right) .$$

Since Y' is an ∞-groupoid,

$$X' \downarrow_{Y'} \{y'\} \simeq X' \times_{Y'} \{y'\}$$

(Recollection 1.2). Since the square (1.6) is a pullback square and Y is an ∞-groupoid, we see that

$$X' \times_{Y'} \{y'\} \simeq X \times_Y \{g(y')\} \simeq X \downarrow_Y \{g(y')\}$$

(Recollection 1.2). Hence

$$
\begin{aligned}
\mathrm{Lan}_{\bar{f}}(L_X \circ \bar{g})(y') &\simeq \operatorname{colim}\left(X \downarrow_Y \{g(y')\} \longrightarrow X \xrightarrow{L_X} C \right) \\
&\simeq \mathrm{Lan}_f(L_X)(g(y')) .
\end{aligned}
$$

Moreover, this equivalence is induced by θ. $\square$

1.7 Remark. For an ∞-category C with colimits, there is a version of Proposition 1.5 involving the $(-)_{\star}$ *right* adjoints that says that given the pullback square (1.6) in **Spc** there's a natural equivalence

$$f^{\star} g_{\star} \xrightarrow{\sim} \bar{g}_{\star} \bar{f}^{\star}$$

of functors $\mathrm{Fun}(Y', C) \to \mathrm{Fun}(X, C)$. The proof is the same as Proposition 1.5; one just uses the limit formula for right Kan extensions.

The identification $g^{\star} f_{!} \simeq \bar{f}_{!} \bar{g}^{\star}$ in Proposition 1.5 is a *Beck–Chevalley condition*, which we examine in the next section.

2 Beck–Chevalley morphisms

2.1 Notation. We write $\mathbf{Cat}_{\infty}^{ladj} \subset \mathbf{Cat}_{\infty}$ for the subcategory with objects any ∞-category and morphisms functors $C \to D$ which are *left adjoints*. We usually write $f_{!} \colon C \to D$ for a left adjoint and $f^{\star} \colon D \to C$ for its corresponding right adjoint. In this case, we write

$$\eta_f \colon \mathrm{id}_C \to f^{\star} f_{!} \quad \text{and} \quad \varepsilon_f \colon f_{!} f^{\star} \to \mathrm{id}_D$$

for the unit and counit of the adjunction $f_{!} \dashv f^{\star}$, respectively.

2.2 Definition. Consider a commutative square σ

$$\begin{tikzcd} C' \arrow[r, "\bar{f}_{!}"] \arrow[d, "\bar{g}_{!}"'] & D' \arrow[d, "g_{!}"] \\ C \arrow[r, "f_{!}"'] & D \end{tikzcd} \tag{2.3}$$

in $\mathbf{Cat}_{\infty}^{ladj}$. The ***Beck–Chevalley morphism*** associated to the square σ is the composite natural transformation

$$\mathrm{BC}(\sigma) \colon \bar{f}_{!} \bar{g}^{\star} \xrightarrow{\bar{f}_{!} \bar{g}^{\star} \eta_f} \bar{f}_{!} \bar{g}^{\star} f^{\star} f_{!} \xrightarrow{\sim} \bar{f}_{!} g^{\star} \bar{f}^{\star} f_{!} \xrightarrow{\varepsilon_{\bar{f}} g^{\star} f_{!}} g^{\star} f_{!} \,,$$

where the middle equivalence comes from the identification of right adjoints

$$\bar{g}^{\star} f^{\star} \xrightarrow{\sim} g^{\star} \bar{f}^{\star} \,.$$

The Beck–Chevalley morphism is depicted diagrammatically as

$$\begin{tikzcd} C' \arrow[r, "\bar{f}_{!}"] \arrow[dr, phantom, "\mathrm{BC}(\sigma) \Searrow"] & D' \\ C \arrow[u, "\bar{g}^{\star}"] \arrow[r, "f_{!}"'] & D \arrow[u, "g^{\star}"'] \end{tikzcd} .$$

We say that the square (2.3) ***satisfies the Beck–Chevalley condition*** if the Beck–Chevalley morphism $\mathrm{BC}(\sigma) \colon \bar{f}_{!} \bar{g}^{\star} \to g^{\star} f_{!}$ is an equivalence.

2.4 Remark. As in Remark 1.7, there is a Beck–Chevalley morphism for the $(-)_\star$ *right* adjoints: it is a natural transformation

$$f^\star g_\star \to \bar{g}_\star \bar{f}^\star \,. \tag{2.5}$$

This is the basechange morphism that one often sees in algebraic geometry, for example, in the smooth and proper basechange theorems for étale cohomology (see [3, Chapter VI Corollary 2.3 & Theorem 4.1]).

An alternative approach to ambidexterity is to dispose of the $(-)_!$ adjoints and instead work with the $(-)_\star$ adjoints and the other Beck–Chevalley morphism (2.5).

2.6 Example. By giving a more careful proof of Proposition 1.5, we can conclude that for any ∞-category C with colimits and pullback square

$$\begin{tikzcd} X' \arrow[r, "\bar{f}"] \arrow[d, "\bar{g}"'] \arrow[dr, phantom, very near start, "\lrcorner"] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

in **Spc**, the induced square

$$\begin{tikzcd} \mathrm{Fun}(X', C) \arrow[r, "\bar{f}_!"] \arrow[d, "\bar{g}_!"'] & \mathrm{Fun}(Y', C) \arrow[d, "g_!"] \\ \mathrm{Fun}(X, C) \arrow[r, "f_!"'] & \mathrm{Fun}(Y, C) \end{tikzcd}$$

satisfies the Beck–Chevalley condition. (See also [2].)

2.7 Observation. Please observe that given commutative squares

$$\begin{tikzcd} C' \arrow[r] \arrow[d] & D' \arrow[r] \arrow[d] & E' \arrow[d] \\ C \arrow[r] & D \arrow[r] & E \,, \end{tikzcd}$$

σ and τ (from left to right) in $\mathbf{Cat}_\infty^{ladj}$, the Beck–Chevalley morphism of the outer rectangle is equivalent to natural transformation given by the *horizontal* composite of the Beck–Chevalley morphisms

$$\begin{tikzcd} C' \arrow[r] & D' \arrow[r] & E' \\ C \arrow[r] \arrow[u] \arrow[ur, phantom, "\mathrm{BC}(\sigma)\ \Searrow"] & D \arrow[r] \arrow[u] \arrow[ur, phantom, "\mathrm{BC}(\tau)\ \Searrow"] & E \,. \arrow[u] \end{tikzcd}$$

Similarly, given commutative squares

$$\begin{tikzcd} C'' \arrow[r] \arrow[d] & D'' \arrow[d] \\ C' \arrow[r] \arrow[d] & D' \arrow[d] \\ C \arrow[r] & D , \end{tikzcd}$$

σ' and σ (from top to bottom) in $\mathbf{Cat}_{\infty}^{ladj}$, the Beck–Chevalley morphism of the outer rectangle is equivalent to natural transformation given by the *vertical* composite of the Beck–Chevalley morphisms

$$\begin{tikzcd} C'' \arrow[r] \arrow[dr, phantom, "\mathrm{BC}(\sigma') \searrow"] & D'' \\ C' \arrow[r] \arrow[u] \arrow[dr, phantom, "\mathrm{BC}(\sigma) \searrow"] & D' \arrow[u] \\ C \arrow[r] \arrow[u] & D \arrow[u] . \end{tikzcd}$$

Given our generalizations and reformulations of our goal from §1, we are interested in considering functors $\boldsymbol{X} \to \mathbf{Cat}_{\infty}^{ladj}$ sending pullback squares to squares satisfying the Beck–Chevalley condition, and exhibiting the left adjoint $f_!$ of f^* as a right adjoint of f_* for a certain class of morphisms $f : X \to Y$ in $\boldsymbol{X}$.

2.8 Definition. Let $\boldsymbol{X}$ be an ∞-category with pullbacks. A functor $C : \boldsymbol{X} \to \mathbf{Cat}_{\infty}^{ladj}$ is a ***Beck–Chevalley functor*** if for every pullback square

$$\begin{tikzcd} X' \arrow[r, "\bar{f}"] \arrow[d, "\bar{g}"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

in $\boldsymbol{X}$, the induced square

$$\begin{tikzcd} C_{X'} \arrow[r, "\bar{f}_!"] \arrow[d, "\bar{g}_!"'] & C_{Y'} \arrow[d, "g_!"] \\ C_X \arrow[r, "f_!"'] & C_Y \end{tikzcd}$$

in $\mathbf{Cat}_{\infty}^{ladj}$ satisfies the Beck–Chevalley condition.

2.9 Example. For any ∞-category C with colimits, the functor

$$\mathrm{Fun}(-, C) : \mathbf{Spc} \to \mathbf{Cat}_{\infty}^{ladj} ,$$

where the functoriality is in the $(-)_!$ adjoints, is a Beck–Chevalley functor.

2.10 Remark. Beck–Chevalley functors $\boldsymbol{X} \to \mathbf{Cat}_\infty^{ladj}$ are classified by functors $q\colon C \to \boldsymbol{X}$ that are both cartesian and cocartesian fibrations and for each pullback square in $\boldsymbol{X}$ the associated square in $\mathbf{Cat}_\infty^{ladj}$ satisfies the Beck–Chevalley condition. Such fibrations are called ***Beck–Chevalley fibrations*** [1, Definition 4.1.3].

3 The definition of ambidexterity & basic properties

In this section we define ambidexterity of morphisms in an ∞-category $\boldsymbol{X}$ with pullbacks with respect to a Beck–Chevalley functor $C\colon \boldsymbol{X} \to \mathbf{Cat}_\infty^{ladj}$. This definition is a rather intricate inductive definition; to warm up we recall how to define truncatedness in an ∞-category with pullbacks as well as the universal property of the counit of an adjunction

3.1 Recollection ([HTT, Lemma 5.5.6.15])**.** Let $\boldsymbol{X}$ be an ∞-category with finite limits (e.g., $\boldsymbol{X} = \mathbf{Spc}$). For each integer $n \geq -2$ we define the class of ***n-truncated*** morphisms of $\boldsymbol{X}$ inductively as follows. A morphism $f\colon X \to Y$ in $\boldsymbol{X}$ is (-2)-***truncated*** if f is an equivalence. Now suppose that n-truncated morphisms in $\boldsymbol{X}$ have been defined for some integer $n \geq -2$. Then we say that a morphism $f\colon X \to Y$ in $\boldsymbol{X}$ is $(n+1)$-***truncated*** if the diagonal morphism $\delta_f\colon X \to X \times_Y X$ is n-truncated.

Note that if $\boldsymbol{X} = \mathbf{Spc}$ is the ∞-category of spaces, then a morphism f is n-truncated in the sense just defined if and only if the fibers of f are n-truncated spaces, so the notion just defined agrees with the classical notion of truncatedness.

3.2 Recollection (universal property of the (co)unit [HTT, Definition 5.2.2.7 & Proposition 5.2.2.8])**.** Given functors between ∞-categories $f\colon C \rightleftarrows D : g$, the functor f is left adjoint to g if and only if there exists a natural transformation $\eta\colon \mathrm{id}_C \to gf$ such that for every pair of objects $c \in C$ and $d \in D$, the composite

$$\mathrm{Map}_D(f(c), d) \longrightarrow \mathrm{Map}_C(gf(c), g(d)) \xrightarrow{\eta(c)^*} \mathrm{Map}_C(c, g(d))$$

is an equivalence in $\mathbf{Spc}$. Dually, f is left adjoint to g if and only if there exists a natural transformation $\varepsilon\colon fg \to \mathrm{id}_D$ such that for every pair of objects $c \in C$ and $d \in D$, the composite

$$\mathrm{Map}_C(c, g(d)) \longrightarrow \mathrm{Map}_D(f(c), fg(d)) \xrightarrow{\varepsilon(d)_*} \mathrm{Map}_C(f(c), d)$$

is an equivalence in $\mathbf{Spc}$.

Our approach to exhibiting a left adjoint $f_! \dashv f^*$ as a right adjoint to f^* is to define a counit transformation that exhibits $f_!$ as right adjoint to f^*.

3.3 Construction (definition of ambidexterity [1, Construction 4.1.8])**.** Let $\boldsymbol{X}$ be an ∞-category with pullbacks and $C\colon \boldsymbol{X} \to \mathbf{Cat}_\infty^{ladj}$ a Beck–Chevalley functor. We define the following data for each integer $n \geq -2$:

(a_n) A class of morphisms in $\boldsymbol{X}$ which we call ***n-ambidextrous morphisms*** (with respect to the Beck–Chevaley functor C).

(b_n) For each n-ambidextrous morphism $f\colon X \to Y$ in $\boldsymbol{X}$, a natural transformation $\mu_f^{(n)}\colon \mathrm{id}_{C_Y} \to f_! f^\star$ (well-defined up to equivalence) that exhibits $f_!$ as a right adjoint to $f^\star$.

A morphism $f\colon X \to Y$ in $\boldsymbol{X}$ is (-2)-***ambidextrous*** if f is an equivalence. In this case, we let $\mu_f^{(-2)} \coloneqq \varepsilon_f^{-1}$ be an inverse to the counit $\varepsilon_f\colon f_! f^\star \xrightarrow{\sim} \mathrm{id}_{C_Y}$ of the adjunction $f_! \dashv f^\star$.

Assume that (a_n) and (b_n) have been defined for some integer $n \geq -2$. Let $f\colon X \to Y$ be a morphism in $\boldsymbol{X}$, and write $\delta_f\colon X \to X \times_Y X$ for the diagonal map, which fits into a commutative diagram

$$
\begin{tikzcd}
X \arrow[dr, "\delta_f"] \arrow[drr, equal, bend left] \arrow[ddr, equal, bend right] & & \\
& X \times_Y X \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"] \arrow[dr, phantom, "\lrcorner" very near start] & X \arrow[d, "f"] \\
& X \arrow[r, "f"'] & Y\,.
\end{tikzcd}
\tag{3.4}
$$

Write σ for the pullback square in the diagram (3.4), so that the Beck–Chevalley morphism $\mathrm{BC}(\sigma)\colon \mathrm{pr}_{1,!}\,\mathrm{pr}_2^\star \to f^\star f_!$ is an equivalence. We say that f is ***weakly*** $(n+1)$-***ambidextrous*** if the diagonal δ_f is n-ambidextrous. If f is weakly $(n+1)$-ambidextrous, we define a natural transformation

$$\nu_f^{(n+1)}\colon f^\star f_! \to \mathrm{id}_{C_X}$$

as the composite natural transformation

$$f^\star f_! \xrightarrow[\mathrm{BC}(\sigma)^{-1}]{\sim} \mathrm{pr}_{1,!}\,\mathrm{pr}_2^\star \xrightarrow{\mathrm{pr}_{1,!}\,\mu_{\delta_f}^{(n)}\,\mathrm{pr}_2^\star} \mathrm{pr}_{1,!}\,\delta_{f,!}\delta_f^\star\,\mathrm{pr}_2^\star \simeq \mathrm{id}_{C_X} \circ \mathrm{id}_{C_X} = \mathrm{id}_{C_X}\ .$$

We say that a morphism $f\colon X \to Y$ is $(n+1)$-***ambidextrous*** if the following condition is satisfied: for every pullback diagram

$$
\begin{tikzcd}
X' \arrow[r, "\bar{f}"] \arrow[d, "\bar{g}"'] \arrow[dr, phantom, "\lrcorner" very near start] & Y' \arrow[d, "g"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}
$$

in $\boldsymbol{X}$, the map $\bar{f}$ is weakly $(n+1)$-ambidextrous and $\nu_{\bar{f}}^{(n+1)}\colon \bar{f}^\star \bar{f}_! \to \mathrm{id}_{C_{X'}}$ is the counit for an adjunction $\bar{f}^\star \dashv \bar{f}_!$.

If f is $(n+1)$-ambidextrous, we let $\mu_f^{(n+1)}\colon \mathrm{id}_{C_Y} \to f_! f^\star$ denote a compatible unit for the adjunction

$$f^\star\colon C_Y \rightleftarrows C_X : f_!$$

determined by $\nu_f^{(n+1)}$.

3.5 Definition ([1, Definition 4.1.1]). Let $\boldsymbol{X}$ be an ∞-category with pullbacks and let $C\colon \boldsymbol{X} \to \mathbf{Cat}_\infty^{ladj}$ be a Beck–Chevalley functor. We say that a morphism $f\colon X \to Y$ in $\boldsymbol{X}$ is (***weakly***) ***ambidextrous*** (with respect to the Beck–Chevalley functor C) if f is (weakly) n-ambidextrous for some integer $n \geq -2$.

The following basic properties are easily deduced from the definitions.

3.6 Proposition ([1, Proposition 4.1.10]). *Let $\boldsymbol{X}$ be an ∞-category with pullbacks, $C\colon \boldsymbol{X} \to \mathbf{Cat}_\infty^{ladj}$ a Beck–Chevalley functor, and $f\colon X \to Y$ a morphism in $\boldsymbol{X}$. Then:*

(3.6.1) *If f is weakly n-ambidextrous for some integer $n \geq -2$, then f is n-truncated.*

(3.6.2) *For each integer $n \geq -2$, the class of n-ambidextrous morphisms are stable under pullback.*

(3.6.3) *For each integer $n \geq -1$, the class of weakly n-ambidextrous morphisms are stable under pullback.*

(3.6.4) *Let $-1 \leq m \leq n$ be integers. If f is weakly m-ambidextrous, then f is weakly n-ambidextrous. Moreover, the natural transformations $\nu_f^{(m)}, \nu_f^{(n)}\colon f^\star f_! \to \mathrm{id}_{C_X}$ agree up to homotopy.*

(3.6.5) *Let $-2 \leq m \leq n$ be integers. If f is m-ambidextrous, then f is n-ambidextrous. Moreover, the natural transformations $\mu_f^{(m)}, \mu_f^{(n)}\colon \mathrm{id}_{C_Y} \to f_! f^\star$ agree up to homotopy.*

(3.6.6) *Let $-1 \leq m \leq n$ be integers. If f is (weakly) n-ambidextrous, then f is (weakly) m-ambidextrous if and only if f is m-truncated.*

3.7 Notation. Let $\boldsymbol{X}$ be an ∞-category with pullbacks, $C\colon \boldsymbol{X} \to \mathbf{Cat}_\infty^{ladj}$ a Beck–Chevalley functor, and $f\colon X \to Y$ a morphism in $\boldsymbol{X}$ that is weakly ambidextrous. Then we simply write ν_f for $\nu_f^{(n)}$ for some integer $n \geq -2$ such that f is weakly n-ambidextrous (so that ν_f is well-defined up to homotopy). If f is ambidextrous, we write μ_f for a compatible unit of ν_f.

3.8 Reformulation (the norm map [1, Remark 4.1.12]). Let $\boldsymbol{X}$ be an ∞-category with pullbacks, $C\colon \boldsymbol{X} \to \mathbf{Cat}_\infty^{ladj}$ a Beck–Chevalley functor, and $f\colon X \to Y$ a morphism in $\boldsymbol{X}$. If $f^\star\colon C_Y \to C_X$ admits a right adjoint, we denote the right adjoint to $f^\star$ by $f_\star\colon C_X \to C_Y$. We then have an equivalence

$$\mathrm{Map}_{\mathrm{Fun}(C_X, C_X)}(f^\star f_!, \mathrm{id}_{C_X}) \simeq \mathrm{Map}_{\mathrm{Fun}(C_X, C_Y)}(f_!, f_\star)\,. \tag{3.9}$$

If f is weakly ambidextrous, we let $\mathrm{Nm}_f\colon f_! \to f_\star$ denote the image of $\nu_f\colon f^\star f_! \to \mathrm{id}_{C_X}$ under the equivalence (3.9). We call Nm_f the ***norm map*** associated to f.

We can reformulate the definition of ambidexterity as follows. A weakly ambidextrous morphism $f\colon X \to Y$ is ***ambidextrous*** if and only if for every pullback square

$$\begin{tikzcd} X' \arrow[r, "\bar{f}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y' \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

in $\boldsymbol{X}$, the following conditions are satisfied:

(3.8.1) The morphism $\bar{f}$ is weakly ambidextrous.

(3.8.2) The functor $\bar{f}^\star\colon C_{Y'} \to C_{X'}$ admits a right adjoint $\bar{f}_\star$.

(3.8.3) The norm map $\mathrm{Nm}_{\bar{f}}\colon \bar{f}_! \to \bar{f}_\star$ is an equivalence.

4 Naturality properties of the norm

The goal of this section is to establish two naturality properties of the construction $f \mapsto \mu_f$ (or, equivalently, $f \mapsto \mathrm{Nm}_f$).

4.1 Proposition ([1, Proposition 4.2.1]). *Let $\boldsymbol{X}$ be an ∞-category with pullbacks, $C\colon \boldsymbol{X} \to \mathbf{Cat}_\infty^{ladj}$ a Beck–Chevalley functor, and let τ be a pullback diagram*

$$\begin{tikzcd} X' \arrow[r, "\bar{f}"] \arrow[d, "\bar{g}"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

in $\boldsymbol{X}$. Then:

(4.1.1) *If f is weakly ambidextrous, then $\bar{f}$ is weakly ambidextrous and the diagram*

$$\begin{tikzcd} \bar{f}^\star \bar{f}_! \bar{g}^\star \arrow[r, "\bar{f}^\star \mathrm{BC}(\tau)"] \arrow[d, "\nu_{\bar{f}} \bar{g}^\star"'] & \bar{f}^\star g^\star f_! \arrow[r, "\sim"] & \bar{g}^\star f^\star f_! \arrow[d, "\bar{g}^\star \nu_f"] \\ \bar{g}^\star \arrow[rr, equal] & & \bar{g}^\star \end{tikzcd}$$

commutes up to homotopy.

(4.1.2) *If f is ambidextrous, then $\bar{f}$ is ambidextrous and the diagram*

$$\begin{tikzcd} g^\star \arrow[rr, equal] \arrow[d, "\mu_{\bar{f}} g^\star"'] & & g^\star \arrow[d, "g^\star \mu_f"] \\ \bar{f}_! \bar{f}^\star g^\star \arrow[r, "\sim"] & \bar{f}_! \bar{g}^\star f^\star \arrow[r, "\mathrm{BC}(\tau) f^\star"'] & g^\star f_! f^\star \end{tikzcd}$$

commutes up to homotopy.

4.2 Reformulation (Proposition 4.1 in terms of norms [1, Remark 4.2.3]). In the situation of Proposition 4.1, assume that f and $\bar{f}$ are weakly ambidextrous and that the functors $f^\star$ and $\bar{f}^\star$ admit right adjoints $f_\star$ and $\bar{f}_\star$. Then we can reformulate assertion (4.1.1) as follows: the morphism

$$\bar{f}_! \bar{g}^\star \xrightarrow[\sim]{\mathrm{BC}(\tau)} g^\star f_! \xrightarrow{g^\star \mathrm{Nm}_f} g^\star f_\star \longrightarrow \bar{f}^\star \bar{g}^\star .$$

us homotopic to $\mathrm{Nm}_{\bar{f}}\, \bar{g}^\star$, where the last morphism is the Beck–Chevalley morphism involving the $(-)_\star$ adjoints (Remark 2.4).

Proof of Proposition 4.1. The statement that f is (weakly) ambidextrous implies that $\bar{f}$ is (weakly) ambidextrous is immediate from the definitions.

Recall that if f is weakly ambidextrous, then f is n-truncated for some integer $n \geq -2$ (3.6.1). We prove both (4.1.1) and (4.1.2) simultaneously by induction on the truncatedness of f, which we denote by n. If $n = -2$, then f is an equivalence, hence $\bar{f}$ is an equivalence, and (4.1.1) and (4.1.2) are obvious.

So assume that $n \geq -1$ and that f (and hence $\bar{f}$) is weakly ambidextrous. Thus we have a pullback square ρ

$$\begin{tikzcd} X' \arrow[r, "\delta_{\bar{f}}"] \arrow[d, "\bar{g}"'] \arrow[dr, phantom, "\lrcorner", very near start] & X' \times_{Y'} X' \arrow[d, "\pi"] \\ X \arrow[r, "\delta_f"'] & X \times_Y X , \end{tikzcd}$$

where the morphism π is induced by g and $\bar{g}$, and δ_f and $\delta_{\bar{f}}$ are $(n-1)$-truncated and ambidextrous by the assumption that f is weakly ambidextrous. Let σ denote the pullback square

$$\begin{tikzcd} X \times_Y X \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y , \end{tikzcd}$$

let $\bar{\sigma}$ denote the pullback square

$$\begin{tikzcd} X' \times_{Y'} X' \arrow[r, "\overline{\mathrm{pr}}_1"] \arrow[d, "\overline{\mathrm{pr}}_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & X' \arrow[d, "\bar{f}"] \\ X' \arrow[r, "\bar{f}"'] & Y' , \end{tikzcd}$$

and let ξ denote the pullback square

$$\begin{tikzcd} X' \times_{Y'} X' \arrow[r, "\overline{\mathrm{pr}}_1"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\lrcorner", very near start] & X' \arrow[d, "\bar{g}"] \\ X \times_Y X \arrow[r, "\mathrm{pr}_1"'] & X . \end{tikzcd}$$

Now consider the diagram

$$
\begin{tikzcd}
\bar{f}^\star \bar{f}_! \bar{g}^\star \arrow[r, "\mathrm{BC}(\bar{\sigma})^{-1}", "\sim"'] \arrow[dd, "\mathrm{BC}(\tau)"', "\wr"] & \overline{\mathrm{pr}}_{1,!}\overline{\mathrm{pr}}_2^\star \bar{g}^\star \arrow[r, "\mu_{\delta_{\bar{f}}}"] \arrow[dd, "\wr"] & \overline{\mathrm{pr}}_{1,!}\delta_{\bar{f},!}\delta_{\bar{f}}^\star \overline{\mathrm{pr}}_2^\star \bar{g}^\star \arrow[r, "\sim"] \arrow[d, "\wr"] & \bar{g}^\star \arrow[dd, equal] \\
 & & \overline{\mathrm{pr}}_{1,!}\delta_{\bar{f},!}\delta_{\bar{f}}^\star \pi^\star \, \mathrm{pr}_2^\star \arrow[d, "\wr"] & \\
\bar{f}^\star g^\star f_! \arrow[dd, "\wr"] & \overline{\mathrm{pr}}_{1,!}\pi^\star \, \mathrm{pr}_2^\star \arrow[ur, "\mu_{\delta_{\bar{f}}}"] \arrow[dr, "\mu_{\delta_f}"'] \arrow[dd, "\mathrm{BC}(\xi)"', "\wr"] & \overline{\mathrm{pr}}_{1,!}\delta_{\bar{f},!}\pi^\star \delta_f^\star \, \mathrm{pr}_2^\star \arrow[r, "\sim"] \arrow[d, "\wr"', "\mathrm{BC}(\rho)"] & \bar{g}^\star \arrow[dd, equal] \\
 & & \overline{\mathrm{pr}}_{1,!}\pi^\star \delta_{f,!}\delta_f^\star \, \mathrm{pr}_2^\star \arrow[d, "\wr"', "\mathrm{BC}(\xi)"] & \\
\bar{g}^\star f^\star f_! \arrow[r, "\sim", "\mathrm{BC}(\sigma)^{-1}"'] & \bar{g}^\star \, \mathrm{pr}_{1,!} \, \mathrm{pr}_2^\star \arrow[r, "\mu_{\delta_f}"'] & \bar{g}^\star \, \mathrm{pr}_{1,!} \, \delta_{f,!}\delta_f^\star \, \mathrm{pr}_2^\star \arrow[r, "\sim"] & \bar{g}^\star \, ,
\end{tikzcd}
\tag{4.3}
$$

where the long composites at the top and bottom of the diagram are the definitions of $\nu_{\bar{f}}\bar{g}^\star$ and $\bar{g}^\star \nu_f$, respectively, and morphisms labeled with '$\sim$' and no other decorations are given by identification of adjoints. Our goal is to show that the outer rectangle of (4.4) commutes up to homotopy; we do this by showing that each sub-diagram commutes up to homotopy.

The diagrams in the middle column of (4.4) commute up to homotopy by the inductive hypothesis (for (4.1.2)). The upper-right diagram in (3.4) commutes because we're just identifying adjoints. The lower-right diagram in (3.4) commutes because Beck–Chevalley morphisms compose horizontally (Observation 2.7) and we have a commutative diagram

$$
\begin{tikzcd}
X' \arrow[r, "\delta_{\bar{f}}"] \arrow[d, "\bar{g}"'] \arrow[dr, phantom, "\lrcorner", very near start] & X' \times_{Y'} X' \arrow[r, "\overline{\mathrm{pr}}_1"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\lrcorner", very near start] & X' \arrow[d, "\bar{g}"] \\
X \arrow[r, "\delta_f"'] & X \times_Y X \arrow[r, "\mathrm{pr}_1"'] & X \, ,
\end{tikzcd}
$$

where the long composites on the top and bottom are the identity on X' and X, respectively. To see that the left third rectangle in (3.4) commutes up to homotopy, it suffices to show that the diagram

$$
\begin{tikzcd}
\bar f^\star \bar f_! \bar g^\star \arrow[d, "\mathrm{BC}(\tau)"', "\wr"] & \overline{\mathrm{pr}}_{1,!}\overline{\mathrm{pr}}_2^\star \bar g^\star \arrow[l, "\mathrm{BC}(\bar\sigma)"', "\sim"] \arrow[d, "\wr"] \\
\bar f^\star g^\star f_! \arrow[d, "\wr"] & \overline{\mathrm{pr}}_{1,!}\pi^\star \mathrm{pr}_2^\star \arrow[d, "\mathrm{BC}(\xi)", "\wr"'] \\
\bar g^\star f^\star f_! & \bar g^\star \mathrm{pr}_{1,!}\mathrm{pr}_2^\star \arrow[l, "\sim"', "\mathrm{BC}(\sigma)"]
\end{tikzcd}
\tag{4.4}
$$

commutes up to homotopy. This again follows from the fact that Beck–Chevalley morphisms compose vertically (Observation 2.7) and the fact that the outer rectangles of the two diagrams

$$
\begin{tikzcd}
X' \times_{Y'} X' \arrow[r, "\overline{\mathrm{pr}}_1"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\lrcorner"] & X' \arrow[d, "\bar g"] \\
X \times_Y X \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"'] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "f"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
X' \times_{Y'} X' \arrow[r, "\overline{\mathrm{pr}}_1"] \arrow[d, "\overline{\mathrm{pr}}_2"'] \arrow[dr, phantom, "\lrcorner"] & X' \arrow[d, "\bar f"] \\
X' \arrow[r, "\bar f"] \arrow[d, "\bar g"'] \arrow[dr, phantom, "\lrcorner"] & Y' \arrow[d, "g"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}
$$

are the same.

Now we prove the iductive step for (4.1.2). Assume that f is ambidextrous so that the natural transformations $\nu_f\colon f^\star f_! \to \mathrm{id}_{C_X}$ and $\nu_{\bar f}\colon \bar f^\star \bar f_! \to \mathrm{id}_{C_{X'}}$ are counits of adjunctions $f^\star \dashv f_!$ and $\bar f^\star \dashv \bar f_!$. Then by the universal property of the unit μ_f, the composite map

$$
\begin{tikzcd}
\mathrm{Map}_{\mathrm{Fun}(C_Y, C_{Y'})}(g^\star, g^\star f_! g^\star) \arrow[r, "-\circ f_!"] \arrow[dr, "\alpha"'] & \mathrm{Map}_{\mathrm{Fun}(C_X, C_{Y'})}(g^\star f_!, g^\star f_! f^\star f_!) \arrow[d, "(g^\star f_! \nu_f)\circ -"] \\
& \mathrm{Map}_{\mathrm{Fun}(C_X, C_{Y'})}(g^\star f_!, g^\star f_!)
\end{tikzcd}
$$

is an equivalence. Moreover, by the triangle identity, the natural transformation $g^\star\mu_f\colon g^\star \to g^\star f_! f^\star$ corresponds to the identity $g^\star f_! \to g^\star f_!$ under the equivalence α. Thus proving that the diagram appearing in (4.1.2) commutes up to homotopy is equivalent to showing that the composite

$$
g^\star f_! \xrightarrow{\mu_{\bar f} g^\star f_!} \bar f_! \bar f^\star g^\star f_! \simeq \bar f_! \bar g^\star f^\star f_! \xrightarrow[\sim]{\mathrm{BC}(\tau) f^\star f_!} g^\star f_! f^\star f_! \xrightarrow{g^\star f_! \nu_f} g^\star f_!
$$

is homotopic to the identity. To prove this, consider the diagram

$$\begin{tikzcd}
g^* f_! \arrow[r, "\mu_{\bar f}"] & \bar f_! \bar f^* g^* f_! \arrow[r, "\sim"] & \bar f_! \bar g^* f^* f_! \arrow[r, "\mathrm{BC}(\tau)", "\sim"'] \arrow[d, "\nu_f"'] & g^* f_! f^* f_! \arrow[r, "\nu_f"] & g^* f_! \\
\bar f_! \bar g^* \arrow[u, "\mathrm{BC}(\tau)", "\wr"'] \arrow[r, "\mu_{\bar f}"'] & \bar f_! \bar f^* \bar f_! \bar g^* \arrow[u, "\mathrm{BC}(\tau)", "\wr"'] \arrow[r, "\nu_{\bar f}"'] & \bar f_! \bar g^* \arrow[urr, "\sim", "\mathrm{BC}(\tau)"'] & &
\end{tikzcd}.$$

The left-hand square and right-hand triangle obviously commute, and the middle square commutes by the inductive step for (4.1.1). To show that the top composite is homotopic to the identity, it suffices to show that the bottom composite is homotopic to the identity: this is true by the triangle identity sicne $\mu_{\bar f}$ and $\nu_{\bar f}$ are a compatible unit and counit. □

4.5 Corollary (adjoint formulation of Proposition 4.1 [1, Corollary 4.2.6]). *Let $\boldsymbol{X}$ be an ∞-category with pullbacks, $C\colon \boldsymbol{X} \to \mathbf{Cat}_\infty^{ladj}$ a Beck–Chevalley functor, and let τ be a pullback diagram*

$$\begin{tikzcd}
X' \arrow[r, "\bar f"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y' \arrow[d] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

in $\boldsymbol{X}$. Then:

(4.5.1) *If f is weakly ambidextrous, then $\bar f$ is weakly ambidextrous and the diagram*

$$\begin{tikzcd}
\bar g_! \bar f^* \bar f_! \arrow[r, "\mathrm{BC}(\tau)\bar f_!"] \arrow[d, "\bar g_! \nu_{\bar f}"'] & f^* g_! \bar f_! \arrow[r, "\sim"] & f^* f_! \bar g_! \arrow[d, "\nu_f \bar g_!"] \\
\bar g_! \arrow[rr, equal] & & \bar g_!
\end{tikzcd}$$

commutes up to homotopy.

(4.5.2) *If f is ambidextrous, then $\bar f$ is ambidextrous and the diagram*

$$\begin{tikzcd}
g_! \arrow[rr, equal] \arrow[d, "g_! \mu_{\bar f}"'] & & g_! \arrow[d, "\mu_f g_!"] \\
g_! \bar f_! \bar f^* \arrow[r, "\sim"] & f_! \bar g_! \bar f^* \arrow[r, "f_! \mathrm{BC}(\tau)"'] & f_! f^* g_!
\end{tikzcd}$$

commutes up to homotopy.

The proof of the following proposition is similar to the proof of Proposition 4.1, though a little more involved.

4.6 Proposition ([1, Proposition 4.2.2]). *Let $\boldsymbol{X}$ be an ∞-category with pullbacks, $C\colon \boldsymbol{X} \to \mathbf{Cat}_\infty^{ladj}$ a Beck–Chevalley functor, and suppose we are given morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ in $\boldsymbol{X}$. Then:*

(4.1.1) *If f and g are weakly ambidextrous, then gf is weakly ambidextrous and ν_{gf} is homotopic to the composition*

$$(gf)^\star(gf)_! \simeq f^\star g^\star g_! g_! \xrightarrow{f^\star \nu_g f_!} f^\star f_! \xrightarrow{\nu_f} \mathrm{id}_{C_X} .$$

commutes up to homotopy.

(4.1.2) *If f and g are ambidextrous, then gf is ambidextrous and μ_{gf} is homotopic to the composition*

$$\mathrm{id}_{C_Z} \xrightarrow{\mu_g} g_! g^\star \xrightarrow{g_! \mu_f g^\star} g_! f_! f^\star g^\star \simeq (gf)_!(gf)^\star .$$

4.7 Reformulation (Proposition 4.6 in terms of norms [1, Remark 4.2.4])**.** In the situation of Proposition 4.6, assume that f and g are weakly ambidextrous and that the functors $f^\star$ and $g^\star$ admit right adjoints $f_\star$ and $g_\star$. Then $(gf)^\star$ is left adjoint to $(gf)_\star \coloneqq g_\star f_\star$ and we can reformulate assertion (4.6.1) as follows: the norm $\mathrm{Nm}_{gf} \colon (gf)_! \to (gf)_\star$ is given by the composite

$$(gf)_! \simeq g_! f_! \xrightarrow{\mathrm{Nm}_g f_!} g_\star f_! \xrightarrow{g_\star \mathrm{Nm}_f} g_\star f_\star \simeq (gf)_\star .$$

5 Ambidexterity for local systems

So far we have considered ambidexterity for arbitrary Beck–Chevalley functors $\boldsymbol{X} \to \mathbf{Cat}_\infty^{ladj}$. We now specify to the case of C-valued local systems, i.e., $\boldsymbol{X} = \mathbf{Spc}$ and we consider the Beck–Chevalley functor $\mathrm{Fun}(-, C) \colon \mathbf{Spc} \to \mathbf{Cat}_\infty^{ladj}$, where C is an ∞-category with colimits (recall Proposition 1.5).

5.1 Definition ([1, Definition 4.3.4])**.** Let C be an ∞-category with colimits.

- A space X is ***weakly** C**-ambidextrous*** if the unique morphism $f \colon X \to *$ is weakly C-ambidextrous (with respect to the Beck–Chevalley functor $\mathrm{Fun}(-, C) \colon \mathbf{Spc} \to \mathbf{Cat}_\infty^{ladj}$).
- A space X is *C-**ambidextrous*** if X is weakly C-ambidextrous and the natural transformation $\nu_f \colon f^\star f_! \to \mathrm{id}_{\mathrm{Fun}(X,C)}$ is the counit of an adunction (so that $f^\star \dashv f_!$).

5.2 Proposition ([1, Proposition 4.3.5])**.** *Let C be an ∞-category with colimits and $f \colon X \to Y$ and $f \colon X \to Y$ a morphism in* **Spc**. *Then:*

(5.2.1) *The morphism f is ambidextrous if and only if f is n-truncated for some integer $n \geq -2$ and each fiber X_y of f is C-ambidextrous.*

(5.2.2) *The morphism f is weakly ambidextrous if and only if f is n-truncated for some integer $n \geq -2$ and each fiber X_y of f is weakly C-ambidextrous.*

5.3 Corollary ([1, Corollary 4.3.6])**.** *Let C be an ∞-category with colimits and $f \colon X \to Y$ a morphism between truncated spaces. If Y is C-ambidextrous and each fiber X_y of f is C-ambidextrous, then X is C-ambidextrous.*

The proof of Proposition 5.2 uses the following fact to reduce to proving the claim in a special case.

5.4 Lemma ([1, Lemma 4.3.8]). *Let C be an ∞-category with colimits and X a space. Then $\mathrm{Fun}(X, C)$ is generated under colimits by objects of the form $x_! c$, where $x\colon * \to X$ is a point of X and $c \in C \simeq \mathrm{Fun}(*, C)$.*

Lemma 5.4 is also used in the proof of the following proposition.

5.5 Proposition ([1, Proposition 4.3.9]). *Let C be an ∞-category with colimits and X a truncated space. Let $f : X \to *$ denote the unique morphism. Then X is C-ambidextrous if and only if the following conditions are satisfied:*

(5.5.1) *X is weakly C-ambidextrous (i.e., $\mathrm{Map}_X(x, x')$ is C-ambidextrous for all $x, x' \in X$).*

(5.5.2) *The pullback functor $f^\star$ admits a right adjoint $f_\star$.*

(5.5.3) *The functor $f_\star$ preserves colimits.*

Proof. If X is C-ambidextrous, then (5.5.1) is obvious and (5.5.2) and (5.5.3) follow from the fact that the left adjoint $f_!$ of $f^\star$ is also right adjoint to $f^\star$.

Now assume that (5.5.1)–(5.5.3) are satisfied. Using (5.5.2), the counit $\nu_f : f^\star f_! \to \mathrm{id}_{\mathrm{Fun}(X,C)}$ corresponds to the norm map $\mathrm{Nm}_X : f_! \to f_\star$ (Reformulation 3.8), and our goal is to show that Nm_f is an equivalence. That is, we want to show that for every local system $L\colon X \to C$, the natural transformation

$$\mathrm{Nm}_f(L)\colon f_! L \to f_\star L$$

is an equivalence in C. By asumption (5.5.3), the collection of objects $L \in \mathrm{Fun}(X, C)$ for which $\mathrm{Nm}_f(L)$ is an equivalence is closed under colimits, so by Lemma 5.4 we are reduced to the case where $L = x_! c$ for $x\colon * \to X$ a point and $c \in C$. By assumption (5.5.1), for every point $x \in X$ the morphsim $x\colon * \to X$ is C-ambidextrous. By Reformulation 4.7, the composite

$$c \simeq (f x)_! c \simeq f_! L \xrightarrow{\mathrm{Nm}_f(L)} f_\star L \xrightarrow{f_\star \mathrm{Nm}_x(L)} f_\star x_\star c \simeq (f x)_\star c \simeq c$$

is homotopic to the identity. Since $x\colon * \to X$ is X-ambidextrous, the norm Nm_x is an equivalence, hence Nm_f is an equivalence as well. □

5.6 Remark. Note that Proposition 5.5 gives a characterization of (weakly) C-ambidextrous morphisms that doesn't explicitly mention the natural transformations ν_f or μ_f.

6 Semiadditivity & ambidexterity of Eilenberg–MacLane spaces

6.1 Definition ([1, Definition 4.4.1]). Let $n \geq -2$ be an integer. A space X is a ***finite n-type*** if X is π-finite and n-truncated.

6.2 Definition ([1, Definition 4.4.2]). Let C be an ∞-category with colimits and $n \geq -2$ be an integer. We day that C is ***n-semiadditive*** if every finite n-type is C-ambidextrous.

6.3 Examples.

(6.3.1) Since a space X is a finite (-2)-type if and only if X is contractible, every ∞-category with colimits is (-2)-semiadditive.

(6.3.2) Since the only (-1)-types are contractible and empty spaces, an ∞-category with colimits C is (-1)-semiadditive if and only if is C-ambidextrous if and only if C is pointed.

(6.3.3) Since a space X is a finite 0-type if and only if X is equivalent to a finite set, an ∞-category with colimits is 0-semiadditive if and only if C is semiadditive in the usual sense, i.e., the finite products in C are also finite coproducts.

(6.3.4) Any *stable* ∞-category with colimits is 0-semiadditive.

6.4 Notation ([1, Notation 4.4.15]). Let C be a 0-semiadditive ∞-category with colimits and $n \geq 0$ be an integer. Write $[n]\colon \mathrm{id}_C \to \mathrm{id}_C$ for the natural transformation determined by the composite

$$c \xrightarrow{\Delta} c^{\times n} \simeq c^{\sqcup n} \xrightarrow{\nabla} c$$

of the diagonal and codiagonal for each object $c \in C$.

6.5 Proposition ([1, Proposition 4.4.16]). *Let C be a 0-semiadditive ∞-category with limits and assume that there exists a prime number p with the following property:*

(6.5.1) *For every integer $n \geq 1$ which is relatively prime to p, the natural transformation $[n]\colon \mathrm{id}_C \to \mathrm{id}_C$ is an equivalence*

Then C is 1-semiadditive if and only if the Eilenberg–MacLane space $K(\mathbf{Z}/p, 1)$ is C-ambidextrous.

Proof. The fact that the 1-semiadditivity of C implies that $K(\mathbf{Z}/p, 1)$ is C-ambidextrous is immediate from the definition of 1-semiadditivity.

For the other direction, suppose that $K(\mathbf{Z}/p, 1)$ is C-ambidextrous. Let X be a finite 1-type; our goal is to show that X is C-ambidextrous. By applying Corollary 5.3 to the map $X \to \pi_0(X)$, we can reduce to the case where X is connected, so that $X \simeq BG$ for some finite group G.

Now we reduce to the case where G is a P-group Let $P \subset G$ be a p-Sylow subgroup, and consider the maps $g\colon BP \to BG$ induced by the inclusion $P \hookrightarrow G$ and $f\colon BG \to *$. Then g is equivalent to a covering space with finite fibers, hence is C-ambidextrous since C is 0-semiadditive. We want to show that $\mathrm{Nm}_f\colon f_! \to f_*$ is an equivalence. Let $L \in \mathrm{Fun}(BG, C)$ and let α denote the composite

$$\alpha\colon L \longrightarrow g_* g^* L \simeq g_! g^* L \longrightarrow L\,,$$

where the middle equivalence comes from the fact that g is ambidextrous. We claim that α is an equivalence. To prove this, it suffices to show that for every point $x \in BG$, the morphism $x^\star\alpha\colon x^\star L \to x^\star L$ is an equivalence. Unwinding the definitions, we see that $x^\star\alpha$ is given by the morphism $[\#(G/P)]\colon x^\star L \to x^\star L$. Since P is a p-Sylow subgroup of G, the number $\#(G/P)$ is relatively prime to p, so that $[\#(G/P)]$ is an equivalence by assumption (6.5.1). Since α is an equivalence, we see that L is a retract of $g_! g^\star L$. Hence to see that $\mathrm{Nm}_f(L)$ is an equivalence, it suffices to prove that $\mathrm{Nm}_f(g_! g^\star L)$ is an equivalence. We may therefore assume that $L = g_! L'$ for some $L' \in \mathrm{Fun}(BP, C)$. Consider the composite

$$\mathrm{Nm}_{fg}(L')\colon (fg)_! L' \xrightarrow{\mathrm{Nm}_f(g_! L')} f_\star g_! L' \xrightarrow[\sim]{f_\star \mathrm{Nm}_g(L')} (fg)_\star L' \ ,$$

where the second morphism is an equivalence since g is C-ambidextrous. By the 2-of-3 property we see that to prove that Nm_f is an equivalence, it suffices to prove that $\mathrm{Nm}_{fg}(L')$ is an equivalence, so we can replace G by P and assume that G is a p-group.

With this reduction, we now proceed by induction on the cardinality of the p-group G. If G is trivial, there is nothing to prove. For the induction step, we can choose a normal subgroup $N \triangleleft G$ of order p. It follows from the inductive hypothesis that $B(G/N)$ is C-ambidextrous. We have a fiber sequence

$$K(\mathbf{Z}/p, 1) \simeq BN \longrightarrow BG \longrightarrow B(G/N) \ ,$$

so an application of Corollary 5.3 and the assumption that $K(\mathbf{Z}/p, 1)$ is C-ambidextrous show that BG is C-ambidextrous, completing the proof. □

6.6 Proposition ([1, Proposition 4.4.17]). *Let C be a 0-semiadditive ∞-category with limits and p be a prime number. If the natural transformation $[p]\colon \mathrm{id}_C \to \mathrm{id}_C$ is an equivalence, then for every finite p-group G, the Eilenberg–MacLane space BG is C-ambidextrous.*

Proof. As in the proof of Proposition 6.5, we can reduce to the case where $G = \mathbf{Z}/p$. Consider the maps $g\colon * \to BG$ given by any point and $f\colon BG \to *$, so that g is equivalent to a covering space with finite fibers (namely the projection $EG \to BG$). We need to show that $\mathrm{Nm}_f\colon f_! \to f_\star$ is an equivalence. Let $L \in \mathrm{Fun}(BG, C)$ and let α denote the composite

$$\alpha\colon L \longrightarrow g_\star g^\star L \simeq g_! g^\star L \longrightarrow L \ ,$$

where the middle equivalence comes from the fact that g is ambidextrous. As in the proof of Proposition 6.5, we see that for each $x \in BG$, the map $x^\star\alpha\colon x^\star L \to x^\star L$ is homotopic to $[p]\colon x^\star L \to x^\star L$, hence an equivalence by assumption. Thus L is a retract of $g_!(g^\star L)$. Thus it suffices to show that Nm_f induces as equivalence $f_! g_\star(g^\star L) \simeq f_\star g_!(g^\star L)$. Set $L' \coloneqq g^\star L$. Since $fg = \mathrm{id}_*$ the composite map

$$\mathrm{Nm}_{fg}\colon L' \simeq (fg)_! L' \xrightarrow{\mathrm{Nm}_f} f_\star g_! L' \xrightarrow{\mathrm{Nm}_g} (fg)_\star L' \simeq L'$$

is an equivalence. By the 2-of-3 property, we are reduced to proving that Nm_g induces an equivalence $f_\star g_! L' \simeq f_\star g_\star L'$. This follows from the assumption that C is 0-semiadditive. □

6.7 Corollary ([1, Corollary 4.4.18]). *Let C be a 0-semiadditive ∞-category with limits. If for each integer $n \geq 1$ the natural transformation $[n]\colon \mathrm{id}_C \to \mathrm{id}_C$ is an equivalence, then C is 1-semiadditive.*

6.8 Proposition ([1, Proposition 4.4.20]). *Let C be a stable ∞-category with limits and colimits and let p be a prime number such that the natural transformation $[p]\colon \mathrm{id}_C \to \mathrm{id}_C$ is an equivalence. Then the Eilenberg–MacLane spaces $K(\mathbf{Z}/p, m)$ are C-ambidextrous for $m \geq 1$.*

6.9 Corollary ([1, Corollary 4.4.21]). *Let C be a stable ∞-category with limits and colimits. Assume that for each object $c \in C$, the endomorphism ring $\mathrm{Ext}^0_C(c, c)$ is a $\mathbf{Q}$-algebra. Then C is n-semiadditive for every integer $n \geq -2$.*

6.10 Example ([1, Example 4.4.22]). Let R be an E_1-ring spectrum with the property that $\pi_0(R)$ is a $\mathbf{Q}$-vector space. Then the ∞-category of left R-module spectra is n-semiadditive for every integer $n \geq -2$.

6.11 Corollary ([1, Corollary 4.4.23]). *Let C be a stable ∞-category with limits and colimits and p a prime number. Assume that for each object $c \in C$, the endomorphism ring $\mathrm{Ext}^0_C(c, c)$ is a $\mathbf{Z}_{(p)}$-module. Then C is n-semiadditive if and only if the Eilenberg–MacLane spaces $K(\mathbf{Z}/p, m)$ are C-ambidextrous for $1 \leq m \leq n$.*

References

HTT J. Lurie, *Higher topos theory*, Annals of Mathematics Studies. Princeton, NJ: Princeton University Press, 2009, vol. 170, pp. xviii+925, ISBN: 978-0-691-14049-0; 0-691-14049-9.

1. M. J. Hopkins and J. Lurie, *Ambidexterity in $K(n)$-local stable homotopy theory*, Preprint available at math.harvard.edu/~lurie/papers/Ambidexterity.pdf, Dec. 2013.
2. *Math overflow question 286886: Pointwise evaluation of the Beck–Chevalley map in ∞-categories*, MO:286886, Nov. 2017.
3. J. S. Milne, *Étale cohomology*, Princeton Mathematical Series. Princeton University Press, Princeton, N.J., 1980, vol. 33, pp. xiii+323, ISBN: 0-691-08238-3.