Mathematisches Institut, Seminars, (Y. Tschinkel, ed.), p. 1–32
Universität Göttingen, 2004-05

NOTES ON DIFFERENTIABLE STACKS

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Abstract. We discuss differentiable stacks and their cohomology. We try to give all necessary definitions, avoiding technical machinery as far as possible. In the last section we focus on the example of S^1-gerbes and explain the relation to projective (Hilbert-)bundles.

Introduction

These are notes of two lectures given at the Forschungsseminar Bunke-Schick during the Spring term 2004. My task was to explain the notions of stacks and twists. Since this should serve as an introduction to the subject I tried to avoid most of the algebraic language, hoping to make the concept of stacks more understandable. These notes do not claim much originality, all concepts from the theory of algebraic stacks are explained in the book of Laumon and Moret-Bailly [**LMB00**]. I tried to translate the differentiable setting which is used in [**LTX**] and [**FHT**] into this language.

The plan of the text is as follows. We start with the example of the stack classifying $G-$bundles, to motivate the abstract definition of stacks. This definition, given in the first section does not look very geometric, therefore we introduce the notion of charts (sometimes called presentations) in the second section. This allows us to define topological and differentiable stacks. In the

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November 2004.

algebraic setting, this concept was introduced by Deligne and Mumford in their famous article on the irreducibility of the moduli space of curves. Their definition allowed to introduce a lot of geometric notions for stacks and it provided a way of thinking about a differentiable stack as a manifold in which points are allowed to have automorphisms. In the third section we then compare this approach with the groupoid–approach which seems to be better known in topological contexts. The fourth section then defines sheaves, bundles and their cohomology on differentiable stacks. We also provide some easy examples to give an idea of how to do calculations in this setup.

In the last two sections we then give a definition of twists or S^1-gerbes and we show that they are classified by elements in $H^2(\_\_, S^1)$. To compare this with the approach via projective bundles, we then introduce the notion of a local quotient stack, which is used in [**FHT**] to give a definition of twisted K-theory. For S^1-gerbes on a local quotient stack we give a construction of a PU-bundle on the stack which defines the gerbe.

1. Motivation and the first definition of stacks

The simplest example of a stack is the classifying stack of $G-$bundles: Let G be a topological group. In topology one defines a classifying space BG characterized by the property that for any good space (e.g., CW-Complex):

$$\mathrm{Map}(X, BG)/\mathrm{homotopy} = \{\text{Isom. classes of locally trivial } G-\text{bundles on } X\}.$$

This defines BG uniquely up to homotopy. For finite groups G this space has the additional property, that the homotopy classes of homotopies between two classifying maps are identified with isomorphisms between the corresponding $G-$bundles.

Such a definition of BG is not well suited for algebraic categories, because there a good notion of homotopy is not easy to define. Moreover even in analytic categories the spaces BG usually are infinite dimensional and therefore more difficult to handle.

Regarding the first problem, one could ask the naive question: Why don't we look for a space BG for which $\mathrm{Map}(X, BG)$ is the set of isomorphism classes of $G-$bundles on X? Of course such a space cannot exist because locally every bundle is trivial, thus the corresponding map should be locally constant, thus constant on connected components of X. But not every bundle is globally trivial.

On the other hand, this argument is somewhat bizarre, because usually $G-$bundles are defined by local data. The problem only arises because we passed to isomorphism classes of bundles.

Thus the first definition of the stack $\underline{BG}$ will be as the (2-)functor assigning to any space X the *category* of $G-$bundles on X. The axioms for such a functor to be a stack will be modeled on the properties of this particular example. Namely the axioms assure that we can glue bundles given on an open covering. This basic example should be held in mind for the following definition of a stack.

Further, to compare this definition with usual spaces one has to keep in mind the Yoneda lemma: Any space/manifold M is uniquely determined by the functor $\mathrm{Map}(\quad, M)$: Manifolds → Sets. This holds in any category (see Lemma 1.3 below).

Therefore, instead of describing the space, we will first consider the corresponding functor and try to find a geometric description afterwards.

Definition 1.1. A stack $\mathscr{M}$ is a $(2-)$functor

$$\mathscr{M} : \text{Manifolds} \to \text{Groupoids} \subset Cat,$$

i.e.:

– for any manifold X we get a category $\mathscr{M}(X)$ in which all morphisms are isomorphisms, and

– for any morphism $f : Y \to X$ we get a functor

$$f^* : \mathscr{M}(X) \to \mathscr{M}(Y)$$

(id^* has to be the identity),

– for any $Z \xrightarrow{g} Y \xrightarrow{f} X$ a natural transformation $\Phi_{f,g} : g^*f^* \cong (g \circ f)^*$, which is associative whenever we have 3 composable morphisms.

For a *stack* $\mathscr{M}$ we require the 2-functor to have glueing-properties (to make these more readable$^{(1)}$, we write $|_U$ instead of j^*, whenever $U \overset{j}{\hookrightarrow} X$ is an open embedding):

1. We can *glue objects*: Given an open covering U_i of X, objects $P_i \in \mathscr{M}(U_i)$ and isomorphisms $\varphi_{ij} : P_i|_{U_i \cap U_j} \to P_j|_{U_i \cap U_j}$ which satisfy the cocycle condition on threefold intersections $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}|_{U_i \cap U_j \cap U_k}$ there is an object $P \in \mathscr{M}(X)$ together with isomorphisms $\varphi_i : P|_{U_i} \to P_i$ such that $\varphi_{ij} = \varphi_j \circ \varphi_i^{-1}$.

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(1) cf. first remark below

2. We can *glue morphisms*: Given objects $P, P' \in \mathscr{M}(X)$, an open covering U_i of X and isomorphisms $\varphi_i : P|_{U_i} \to P'|_{U_i}$ such that $\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j}$, then there is a unique $\varphi : P \to P'$ such that $\varphi_i = \varphi|_{U_i}$.

Remarks 1.2.

1. Formally the glueing conditions make use of the natural transformations for the inclusions $U_i \cap U_j \hookrightarrow U_i \hookrightarrow X$, this is not visible above, because of our notation $|_{U_i \cap U_j}$. For example write $U_{ijk} = U_i \cap U_j \cap U_k$, denote $j_{ijk,ij} : U_{ijk} \to U_{ij}$, $j_{ij,i} : U_{ij} \to U_i$, $j_{ijk,i} : U_{ijk} \to U_i$ the inclusions. Then we have natural transformations

$$\Phi_{ijk,ij,i} : j^*_{ijk,ij} j^*_{ij,i} \to j^*_{ijk,i}.$$

In the condition to glue objects

$$\varphi_{jk}|_{U_{ijk}} \circ \varphi_{ij}|_{U_{ijk}} = \varphi_{ik}|_{U_{ijk}}$$

we would formally have to replace $\varphi_{ij}|_{U_{ijk}}$ by the composition:

$$j^*_{ijk,i} P_i \xrightarrow{\Phi_{ijk,ij,i}} j^*_{ijk,ij} j^*_{ij,i} P_i \xrightarrow{j^*_{ijk,ij}\varphi_{ij}} j^*_{ijk,ij} j^*_{ij,j} P_j \xrightarrow{\Phi^{-1}_{ijk,ij,j}} j^*_{ijk,j} P_j$$

and similarly for the other maps, but this makes the condition hard to read.

2. Our functor $\underline{BG}$, assigning to any manifold the category of $G-$bundles is a stack.

3. We could replace manifolds by topological spaces in the above definition. This is usually phrased as giving a stack *over manifolds* and a stack *over topological spaces* respectively.

4. Stacks form a $2-$category: Morphisms $F : \mathscr{M} \to \mathscr{N}$ of stacks are given by a collection of functors $F^*_X : \mathscr{N}(X) \to \mathscr{M}(X)$ and, for any $f : X \to Y$, a natural transformation $F_f : f^* F^*_X \xrightarrow{\cong} F^*_Y f^*$. Thus morphisms of stacks form a category, morphisms between morphisms of stacks (i.e., natural transformations $\varphi_X : F_X \to G_X$ satisfying $G_f \circ \varphi_X = \varphi_Y \circ F_f$) are written as $\begin{tikzcd} \mathscr{M} \arrow[r, bend left, ""{name=U, below}] \arrow[r, bend right, ""{name=D, above}] & \mathscr{N} \arrow[Rightarrow, from=U, to=D] \end{tikzcd}$. Note that all 2-morphisms are invertible, since all maps in the categories $\mathscr{M}(X)$ and $\mathscr{N}(X)$ are invertible.

5. The inclusion Sets $\to$ Groupoids (associating to each set the category whose objects are elements of the set and the only morphisms are identities) is a full embedding. By the Yoneda lemma we know that the functor Top $\to$ Functors is a full embedding, thus we get a full embedding Top $\to$ Stacks. This embedding assigns to a space X the stack $\underline{X}$ defined as $\underline{X}(Y) = \mathrm{Map}(Y, X)$, this is a stack, since maps can be glued, pull-back functors are given by the composition of maps.

6. Grothendieck topology of maps with local sections: Bundles, (in fact any stack), satisfy a better glueing condition, namely we do not need that the

$j : U_i \hookrightarrow X$ are injective. Whenever we have a map $\cup U_i \xrightarrow{p} X$ such that p has local sections (i.e., for all points $x \in X$ there is a neighborhood U_x and a section $s : U_x \to \cup U_i$ of p, in particular p is surjective). Then the glueing condition also holds, if we replace $U_i \cap U_j$ by the fibered product $U_i \times_X U_j$. We say that the stack is a stack for the local-section-topology. This point of view will be important to define charts for stacks.

(If we wanted to stay in the category of manifolds instead of topological spaces, we should require the map p to be a submersion, in order to have fibered products.)

The following lemma shows, that with the above definition of $\underline{BG}$ we really get a classifying object for $G-$bundles:

Lemma 1.3 (Yoneda lemma for stacks). *Let $\mathscr{M}$ be a stack (defined for manifolds or topological spaces). For any space X denote by $\underline{X}$ the associated stack (i.e., $\underline{X}(Y) = \mathrm{Map}(Y,X)$). Then there is a canonical equivalence of categories: $\mathscr{M}(X) \cong Mor_{Stacks}(\underline{X}, \mathscr{M})$.*

Proof. Given $P \in \mathscr{M}(X)$ we define a morphism $F_P : \underline{X} \to \mathscr{M}$ by

$$\underline{X}(Y) \ni (Y \xrightarrow{f} X) \mapsto f^*P \in \mathscr{M}(Y).$$

For any isomorphism $\varphi : P \to P'$ in $\mathscr{M}(X)$ we define a natural transformation $A_\varphi : F_P \to F_{P'}$ by $f^*\varphi : f^*P \to f^*P'$. Conversely, given a morphism $F : \underline{X} \to \mathscr{M}$ we get an object $P_F := F(id_X) \in \mathscr{M}(X)$, any automorphism $F \to F$ defines an isomorphism of P_F.

One checks that the composition of these constructions is equivalent to the identity functor. ☐

Remark 1.4. Will often write X instead of $\underline{X}$.

Example 1.5 (Quotient stacks). Let G be a Lie group acting on a manifold X via $act : G \times X \to X$. We define the *quotient stack* $[X/G, act]$ (or simply $[X/G]$) as

$$[X/G, act](Y) := \langle (P \xrightarrow{p} Y, P \xrightarrow{f} X) \,|\, P \to Y \text{ a G-bundle,} f \text{ G-equivariant} \rangle.$$

Morphisms of objects are $G-$equivariant isomorphisms.

Remarks 1.6.

1. For G acting trivially on $X = pt$ the quotient $[pt/G]$ is the stack $\underline{BG}$ classifying G-bundles.

2. If G acts properly and freely, i.e. $X \to X/G$ is a $G-$bundle, then $[X/G] \cong X/G$, because any $f : P \to X$ defines a map on the quotient $P/G = Y \to X/G$ and the canonical morphism $P \to Y \times_{X/G} X$ is then an isomorphism of $G-$bundles.

2. Geometry I: Charts

To translate geometric concepts to the (2-)category of stacks, Deligne and Mumford introduced a notion of charts for stacks.

In our example $\underline{BG}$ the Yoneda-lemma 1.3 shows that the trivial bundle on a point *pt* defines a map $pt \to \underline{BG}$. By the same lemma any $X \xrightarrow{f_P} BG$ is given by a bundle $P \to X$. Therefore, if we take a covering $U_i \to X$ on which the bundle is trivial, then $f_P|_{U_i}$ factors through $pt \to BG$. In particular, this trivial map is in some sense surjective (see Definition 2.3 for a precise definition, we will say that this map has local sections)!

Even more is true: First note that the (2-)category of stacks has fibered products:

Definition 2.1. Given a diagram of morphisms of stacks:

$$\begin{tikzcd} & \mathscr{M} \arrow[d, "F"] \\ \mathscr{M}' \arrow[r, "F'"] & \mathscr{N} \end{tikzcd}$$

we define the fibered product $\mathscr{M} \times_{\mathscr{N}} \mathscr{M}'$ to be the stack given by:

$$\mathscr{M} \times_{\mathscr{N}} \mathscr{M}'(X) := \langle (f, f', \varphi) | f : X \to \mathscr{M}, f' : X \to \mathscr{M}', \varphi : F \circ f \Rightarrow F' \circ f' \rangle.$$

Morphisms $(f, f', \varphi) \to (g, g', \psi)$ are pairs of morphisms

$$(\varphi_{f,g} : f \to g, \varphi_{f',g'} : f' \to g')$$

such that

$$\psi \circ F(\varphi_{f,g}) = F'(\varphi_{f',g'}) \circ \varphi.$$

(We will use brackets $\langle\,\rangle$ as above to denote groupoids instead of sets $\{\,\}$)

Remark 2.2. This defines a stack, because objects of $\mathscr{M}, \mathscr{N}$ glue and morphisms of $\mathscr{N}, \mathscr{M}, \mathscr{M}'$ glue.

We calculate the fibered product in our example above: Given

$$\begin{tikzcd} & X \arrow[d, "f_P"] \\ pt \arrow[r] & BG \end{tikzcd}$$

the fibered product $pt \times_{BG} X$ as the stack given by:

$$\begin{aligned} pt \times_{BG} X(Y) &= \left\langle \begin{tikzcd} Y \arrow[r, "g"] \arrow[d] & X \arrow[d] \arrow[dl, Rightarrow, "\varphi"] \\ pt \arrow[r] & BG \end{tikzcd} \right\rangle \\ &= \langle (g,\varphi) | g : Y \to X \text{ and } \varphi : g^*P \xrightarrow{\cong} G \times Y \rangle \\ &\cong \{(g,s) | g : Y \to X \text{ and } s : Y \to g^*P \text{ a section}\} \\ &\cong \{\tilde{g} : Y \to P\} = P(Y) \end{aligned}$$

The first $\cong$ notes that to give a trivialization of g^*P is the same as to give a section of g^*P, in particular the category defined above is equivalent to a set. The second $\cong$ assigns to $\tilde{g}$ the composition of $\tilde{g}$ with the projection $P \to X$ and the section induced by $\tilde{g}$.

By the last description, we get an equivalence $pt \times_{BG} X \cong \underline{P}$, i.e., $pt \to BG$ is the universal bundle over BG.

Definition 2.3. A stack $\mathscr{M}$ is called a *topological stack* (resp. differentiable stack) if there is a space (resp. manifold) X and a morphism $p : X \to \mathscr{M}$ such that:

1. For all $Y \to \mathscr{M}$ the stack $X \times_{\mathscr{M}} Y$ is a space (resp. manifold).
2. p has local sections (resp. is a submersion), i.e., for all $Y \to \mathscr{M}$ the projection $Y \times_{\mathscr{M}} X \to Y$ has local sections (resp. is a submersion).

The map $X \to \mathscr{M}$ is then called a covering or an *atlas* of $\mathscr{M}$ (in the local-section-topology).

The first property is very important, it therefore gets an extra name:

Definition 2.4. A morphism of stacks $F : \mathscr{M} \to \mathscr{N}$ is called *representable* if for any $Y \to \mathscr{N}$ the fibered product $\mathscr{M} \times_{\mathscr{N}} Y$ is a stack which is equivalent to a topological space.

This definition is the requirement that the fibres of a morphism should be topological spaces and not just stacks. We will see later, that for topological stacks this condition is equivalent to the condition that the morphism F is injective on automorphism groups of objects. The easiest example of such a map is the map $pt \to BG$ we have seen above. The easiest example of a map which is *not* representable is the map $BG \to pt$ forgetting everything (take $Y = pt$).

Example 2.5 (Quotient-stacks). The example of quotients by group actions $[X/G]$ are topological stacks (resp. differentiable, if X, G are smooth). An atlas is given by the quotient map $X \to [X/G]$, defined by the trivial $G-$bundle $G \times X \to X$, the action map $G \times X \xrightarrow{act} X$ is $G-$equivariant.

Just as in the case of $G-$bundles one shows that for any $Y \to [X/G]$ given by a $G-$bundle $P \to Y$ there is a canonical isomorphism $Y \times_{[X/G]} X \cong P$ (the argument is given a second time in Lemma 3.1 below).

Some easy properties of representable morphisms are:

Lemma 2.6.

1. *(Composition) If* $F : \mathscr{K} \to \mathscr{M}$ *and* $G : \mathscr{M} \to \mathscr{N}$ *are representable, then* $F \circ G$ *is representable.*

2. *(Pull-back) If* $F : \mathscr{M} \to \mathscr{N}$ *is representable, and* $G : \mathscr{M}' \to \mathscr{N}$ *is arbitrary then the projection* $\mathscr{M}' \times_{\mathscr{N}} \mathscr{M} \to \mathscr{M}'$ *is representable.*

3. *(Locality) A morphism* $F : \mathscr{M} \to \mathscr{N}$ *of topological stacks is representable if and only if for one atlas* $Y \to \mathscr{N}$ *the product* $\mathscr{M} \times_{\mathscr{N}} Y \to \mathscr{M}$ *is again an atlas.*

4. *If* $\mathscr{M}$ *is a topological stack, then for any two morphisms* $f_i : Y_i \to \mathscr{M}$ *the fibered product* $Y_1 \times_{\mathscr{M}} Y_2$ *is again a topological space.*

Proof. For the first claim note that $Y \times_{\mathscr{N}} \mathscr{K} \cong (Y \times_{\mathscr{N}} \mathscr{M}) \times_{\mathscr{M}} \mathscr{K}$, the latter is a space by assumption.

The second is similar: $Y \times_{\mathscr{M}'} (\mathscr{M}' \times_{\mathscr{N}} \mathscr{M}) \cong Y \times_{\mathscr{N}} \mathscr{M}$.

If $\mathscr{M} \to \mathscr{N}$ is representable, then $Y \times_{\mathscr{N}} \mathscr{M}$ is a topological space and for any $T \to \mathscr{M}$ we have $T \times_{\mathscr{M}} (Y \times_{\mathscr{N}} \mathscr{M}) = T \times_{\mathscr{N}} Y \to Y$ has local sections.

On the other hand, if $Y \times_{\mathscr{N}} \mathscr{M} \to \mathscr{M}$ is an atlas, then for all $T \to \mathscr{N}$ which factor through $T \to Y \to \mathscr{N}$ the pull back $T \times_{\mathscr{N}} \mathscr{M}$ is again a space. For an arbitrary $T \to \mathscr{N}$ the projection $Y \times_{\mathscr{N}} T \to T$ has local sections by assumption. This shows, that the fibered product $T \times_{\mathscr{N}} \mathscr{M}$ is a stack which is equivalent to a functor, and that there is a covering U_i of T, such that $U_i \times_{\mathscr{N}} \mathscr{M}$ is a space. Now functoriality of fibered products assures, that these spaces can be glued, thus $T \times_{\mathscr{N}} \mathscr{M}$ is a space.

For the last statement, note that $Y_1 \times_{\mathscr{M}} Y_2 \cong (Y_1 \times Y_2) \times_{\mathscr{M} \times \mathscr{M}} \mathscr{M}$ where the map $\Delta : \mathscr{M} \to \mathscr{M} \times \mathscr{M}$ is the diagonal map. Thus the assumption may be rephrased as “the diagonal $\Delta : \mathscr{M} \to \mathscr{M} \times \mathscr{M}$ is representable” and then the claim follows from (3). $\square$

Remark 2.7. In the last statement of the lemma, there is a natural map $Y_1 \times_{\mathscr{M}} Y_2 \to Y_1 \times Y_2$, but in general this is not an embedding, thus the diagonal $\mathscr{M} \to \mathscr{M} \times \mathscr{M}$ is *not* an embedding in general.

One should also note that the fibered product $Y_1 \times_{\mathscr{M}} Y_2$ represents the functor of maps $T \to Y_1 \times Y_2$ together with an isomorphism of the two pull backs of the objects $p_i^*(Y_i \to \mathscr{M})$, therefore it is sometimes denoted

$$\mathrm{Isom}(Y_1 \xrightarrow{f_1} \mathscr{M}, Y_2 \xrightarrow{f_2} \mathscr{M})$$

or simply $Isom(f_1, f_2)$. In particular one sees that the automorphisms of a map $f : Y \to \mathscr{M}$ are given by sections of the map $Aut(f) := (Y \times_{\mathscr{M}} Y) \times_{Y \times Y} Y \to Y$, because a map from a space T to $Aut(f)$ is the same as a map $s : T \to Y$ together with an isomorphism $\varphi : f \circ s \Rightarrow f \circ s$.

Any property of maps which can be checked on submersions can now be defined for representable morphisms of differentiable stacks, simply requiring that the property holds for one atlas:

Definition 2.8. A representable morphism $\mathscr{M} \to \mathscr{N}$ is an *open embedding, (resp. closed embedding, submersion, proper, ...)* if for one (equivalently any) atlas $Y \to \mathscr{N}$ the map $\mathscr{M} \times_{\mathscr{N}} Y \to \mathscr{Y}$ is an open embedding (resp. closed embedding, submersion, proper, ...).

Note that if $\mathscr{M}$ and $\mathscr{N}$ are spaces then every map is representable and we get the usual notion of open embedding, etc.
In particular, this definition gives us a notion of *open and closed substacks.*

Example 2.9. For quotient-stacks $[X/G]$ open and closed substacks are given by open and closed $G-$equivariant subspaces $Y \hookrightarrow X$, which define embeddings $[Y/G] \hookrightarrow [X/G]$.

Properties that can be checked on coverings of the source of a map (i.e., to have local sections, or in the differentiable category to be smooth or submersive) can even be defined for any morphism of stacks:

Definition 2.10. An arbitrary morphism $\mathscr{M} \to \mathscr{N}$ of differentiable (resp. topological) stacks is *smooth* (or *a submersion*) (resp. *has local sections)*, if for one (equivalently any) atlas $X \to \mathscr{M}$ the composition $X \to \mathscr{N}$ is smooth (or a submersion) (resp. has local sections), i.e., for one (equivalently any) atlas

$Y \to \mathscr{N}$ the fibered product $X \times_{\mathscr{N}} Y \to Y$ is smooth (or a submersion) (resp. has local sections).

The equivalence of the condition to be satisfied for one or for any atlas is proved as in Lemma 2.6.

Note that we can *glue morphisms of stacks*, i.e., given an atlas $X \to \mathscr{M}$ and a morphism $\mathscr{M} \to \mathscr{N}$ of topological or differentiable stacks we get an induced morphism $X \to \mathscr{N}$ together with an isomorphism of the two induced morphisms $X \times_{\mathscr{M}} X \rightrightarrows \mathscr{N}$, which satisfies the cocycle condition on $X \times_{\mathscr{M}} X \times_{\mathscr{M}} X$.

Conversely, given $f : X \to \mathscr{N}$ together with an isomorphism $p_1 \circ f \Rightarrow p_2 \circ f$ of the two induced maps $X \times_{\mathscr{M}} X \to \mathscr{N}$, which satisfies $p_{23}^*\varphi \circ p_{12}^*\varphi = p_{13}^*\varphi$ on $X^{\times^3_{\mathscr{M}}}$ we get a morphism $\mathscr{M} \to \mathscr{N}$ as follows: For any $T \to \mathscr{M}$ we get a map with local sections $X \times_{\mathscr{M}} T \to T$ and a map $X \times_{\mathscr{M}} T \to \mathscr{N}$ together with a glueing data on $X \times_{\mathscr{M}} X \times_{\mathscr{M}} T = (X \times_{\mathscr{M}} T) \times_T (X \times_{\mathscr{M}} T)$, and by the glueing condition for stacks this canonically defines an element in $\mathscr{N}(T)$.

In particular, a morphism $\mathscr{M} \to \underline{BG}$ is the same as a G-bundle on an atlas X together with a glueing datum on $X \times_{\mathscr{M}} X$ satisfying the cocycle condition on $X \times_{\mathscr{M}} X \times_{\mathscr{M}} X$. If $\mathscr{M} = [X/H]$ is a quotient stack then $X \times_{\mathscr{M}} X \cong H \times X$, thus this is the same as an $H-$equivariant bundle on X.

More generally, for any class of objects which satisfy descent, i.e., which can be defined locally by glueing data, we can define the corresponding objects over stacks to be given as a glueing-data on one atlas. For example *vector bundles, Hilbert-bundles, smooth fibrations.*

Definition 2.11. A *$G-$bundle over a stack $\mathscr{M}$* is given by a $G-$bundle $\mathscr{P}_X$ over an atlas $X \to \mathscr{P}$ together with an isomorphism of the two pull-backs of $p_1^*\mathscr{P}_X \to p_2^*\mathscr{P}_X$ on $X \times_{\mathscr{M}} X$ satisfying the cocycle condition on $X \times_{\mathscr{M}} X \times_{\mathscr{M}} X$.

The same definition applies to *vector bundles, Hilbert bundles, locally trivial fibrations with fiber F.*

Remark 2.12. Note that for any $f : T \to \mathscr{M}$ (in particular for any atlas) this datum defines a $G-$bundle $P_{T,f} \to T$, because by definition $X \times_{\mathscr{M}} T \to T$ has local sections, and we can pull-back the glueing datum to

$$(X \times_{\mathscr{M}} X \times_{\mathscr{M}} T) \cong (X \times_{\mathscr{M}} T) \times_T (X \times_{\mathscr{M}} T).$$

Therefore this automatically defines a differentiable/topological stack $\mathscr{P} \xrightarrow{p} \mathscr{M}$ (and p is representable) via:

$$\mathscr{P}(T) = \langle (f : T \to \mathscr{M}, s : T \to P_{T,f} \text{ a section}) \rangle.$$

An atlas of this stack is given by $(\mathscr{P}_X, s : \mathscr{P}_X \overset{\text{diag}}{\longrightarrow} \mathscr{P}_X \times_X P_X)$. The multiplication map glues, therefore this stack also carries a natural morphism $G \times \mathscr{P} \to \mathscr{P}$.

Remark 2.13. This shows that universal bundles on stacks classifying $G-$bundles or other geometric objects exist automatically. Further, since we can glue morphisms of stacks the classifying stack will also classify $G-$bundles on stacks.

Remark 2.14. Note further, that given a $G-$bundle $\mathscr{P}$ on a stack $\mathscr{M}$ and a map $f : T \to \mathscr{M}$ the glueing datum for the two pull backs of $P_{T,f}$ to $T \times_{\mathscr{M}} T$ defines an action of $Aut(s)$ on $f^*\mathscr{P} = P_{T,f}$.

The notion of a $G-$bundle could be defined directly in the language of stacks. These definitions tend to get clumsy, because one has to take care of automorphisms:

Let G be a Lie group, a locally trivial G-bundle over an analytic stack $\mathscr{M}$ is a stack $\mathscr{P}$ together with a representable morphism $\mathscr{P} \overset{p}{\longrightarrow} \mathscr{M}$, an action $G \times \mathscr{P} \overset{act}{\longrightarrow} \mathscr{P}$ together with an isomorphism $\varphi : p \circ act \overset{\cong}{\longrightarrow} p$, such that act is simply transitive on the fibers of p, an isomorphism φ_2 making the diagram

$$\begin{tikzcd}
G \times G \times \mathscr{P} \arrow[r, "id_G{,}act"] \arrow[d, "m{,}id_P"] & \mathscr{P} \arrow[d, "act"] \\
G \times \mathscr{P} \arrow[r, "act"] & \mathscr{P}
\end{tikzcd}$$

commute, such that in the induced isomorphisms in the associativity diagram coincide. Further, there has to be a two morphism making the diagram

$$\begin{tikzcd}
\mathscr{P} \arrow[r, "e{,}id"] \arrow[dr, "id"'] & G \times \mathscr{P} \arrow[d, "act"] \\
& \mathscr{P}
\end{tikzcd}$$

commute, compatible with multiplication. Finally to make a bundle locally trivial there should exist an atlas $X \to \mathscr{M}$ such that the induced bundle $\mathscr{P} \times_{\mathscr{M}} X \to X$ is trivial.

Claim. The two notions of G-bundles coincide. (We will never use this.)

Example 2.15. Once more, note that $pt \to [pt/G]$ is a $G-$bundle over pt. The action map $G \times pt \to pt$ is trivial. And we note that $pt \to [pt/G]$ corresponds to the trivial bundle, thus a trivialization of this bundle induces canonical isomorphisms φ.

3. Topological stacks as topological groupoids

We can generalize the example of quotients by group actions as follows: Given an atlas $X \to \mathscr{M}$, the two projections $X \times_{\mathscr{M}} X \rightrightarrows X$ define the source an target morphisms of a groupoid, the diagonal is the identity, interchanging the factors the inverse and the composition is given by the projection to the first and third factor of

$$X \times_{\mathscr{M}} X \times_{\mathscr{M}} X \cong (X \times_{\mathscr{M}} X) \times_X (X \times_{\mathscr{M}} X) \to X \times_{\mathscr{M}} X.$$

We will denote this groupoid by $X_\bullet$.

Conversely, any groupoid $\Gamma_1 \rightrightarrows \Gamma_0$ defines a topological stack:

$$[\Gamma_0/\Gamma_1](Y) := \langle (P \xrightarrow{p} Y, P \xrightarrow{f} \Gamma_0) \text{ a locally trivial } \Gamma - \text{bundle } \rangle$$

Recall that a locally trivial Γ-bundle is a diagram

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "p"] & \Gamma_0 \\ Y & \end{tikzcd}$$

together with an action $\Gamma_1 \times_{\Gamma_0} P \to \Gamma_0$ which is equivariant with respect to composition of morphisms in Γ_1, such that there is a covering $U \to Y$ and maps $f_i : U \to \Gamma_0$ such that $P|_U \cong f_i^* \Gamma_\bullet$. Note that such a trivialization is the same as a section $U \to P$ (obtained from the identity section of Γ).

Since we can glue Γ-bundles this is a stack. As in the case of quotients we have:

Lemma 3.1. *The trivial* Γ*-bundle* $\Gamma_1 \to \Gamma_0$ *induces a map* $\Gamma_0 \xrightarrow{\pi} [\Gamma_0/\Gamma_1]$ *which is an atlas for* $[\Gamma_0/\Gamma_1]$*, the map* π *is the universal* Γ*-bundle over* $[\Gamma_0/\Gamma_1]$*. The groupoids* Γ *and* $\Gamma_{0,\bullet}$ *are canonically isomorphic.*

Proof. We only need to show, that for any $Y \xrightarrow{f_P} [\Gamma_0/\Gamma_1]$ given by a bundle P, there is a canonical isomorphism $P \xrightarrow{\cong} \Gamma_0 \times_{[\Gamma_0/\Gamma_1]} Y$. This is seen as before:

$$\begin{aligned} (\Gamma_0 \times_{[\Gamma_0/\Gamma_1]} Y)(T) &\cong \langle (T \xrightarrow{f} Y, T \xrightarrow{g} \Gamma_0, \varphi : f_P \circ f \to \pi \circ g \rangle \\ &\cong \langle (f, g, \varphi : f^* P \cong g^* \Gamma_1) \rangle \\ &\cong \{ (f, \tilde{f} : T \to P) | pr_Y \circ \tilde{f} = f \} \\ &\cong \{ \tilde{f} : T \to P \} = P(T). \end{aligned}$$

□

Next one wants to know, whether two groupoids $\Gamma_\bullet, \Gamma'_\bullet$ define isomorphic stacks. From the point of view of atlases this is easy: Given two atlases

$$X \xrightarrow{p} \mathscr{M}, \;\; X' \xrightarrow{p'} \mathscr{M}$$

we get another atlas refining both, namely

$$X \times_{\mathscr{M}} \times X' \to X \to \mathscr{M}$$

is again an atlas (since both maps are representable and have local sections, the same is true for the composition).

Furthermore $X \times_{\mathscr{M}} X' \to X$ is a locally trivial $X'_\bullet$ bundle. This shows:

Lemma 3.2. *Two groupoids* $\Gamma_\bullet, \Gamma'_\bullet$ *define isomorphic stacks if and only if there is a groupoid* $\Gamma''_\bullet$ *which is a left* $\Gamma_\bullet$ *bundle over* Γ'_0 *and a right* $\Gamma'_\bullet$ *bundle over* Γ_0 *such that both actions commute.*

Example 3.3. If we have a subgroup $H \subset G$ acting on a space X, then $[X/H] \cong [X \times^H G/G]$, since the maps $X \leftarrow X \times G \to X \times^H G$ define a G-bundle over X and an H-bundle over $X \times^H G$.

Similarly, if $H \subset G$ is a normal subgroup, acting freely on X, such that $X \to X/H$ is a principal $H-$bundle, then $[X/G] \cong [(X/H)/(G/H)]$, because $G \times^H X$ is a G/H-bundle over X and a G-bundle over X/H.

Finally we can identify morphisms of stacks in terms of groupoids, if the morphism is a submersion, then in [**LTX**] these are called *generalized homomorphisms.*

Given a morphism $\mathscr{M} \xrightarrow{f} \mathscr{N}$ of topological stacks, and atlases $X \to \mathscr{M}, Y \to \mathscr{N}$ we can form the fibered product $X \times_{\mathscr{N}} Y \to X$. Since $\mathscr{N} \to Y$ is a locally trivial $Y_\bullet$ bundle, this is a (right) $Y_\bullet$ bundle as well. Furthermore, since the map $X \to \mathscr{N}$ factors through $\mathscr{M}$ we also get a $X_\bullet$ (left) action on $X \times_{\mathscr{N}} Y$. Note that (by definition) the map $X \times_{\mathscr{N}} Y \to Y$ is a submersion if and only if $\mathscr{M} \to \mathscr{N}$ is a submersion.

Conversely, suppose we are given $X \leftarrow P \to Y$, together with commuting actions of $X_\bullet$ and $Y_\bullet$ on P, such that P is a locally trivial $Y_\bullet$ bundle over X. Then the $X_\bullet$ action on P is a descent datum for the $Y_\bullet$-bundle, which defines a $Y_\bullet$ bundle over $\mathscr{M}$, thus a morphism $\mathscr{M} \to \mathscr{N}$.

Of course, the simplest case of this is the most useful, namely a morphism of groupoids $X_\bullet \to Y_\bullet$ induces a morphism of the associated quotient stacks, (P as above is then obtained by pulling back $Y_\bullet$ to $X = X_0$).

4. Geometry II: Sheaves, cohomology, tangent spaces, dimension, normal bundles

Given a representable submersion $\mathscr{M} \to \mathscr{N}$ we define the *dimension of the fibers* $rel.dim(\mathscr{M}/\mathscr{N})$ as the dimension of the fibers of $\mathscr{M} \times_{\mathscr{N}} Y \to Y$ for any $Y \to \mathscr{N}$. This is well defined, because the relative dimension does not change under pull-backs.

Given an analytic stack $\mathscr{M}$ define its *dimension* by choosing an atlas $X \to \mathscr{M}$ and defining $\dim \mathscr{M} := dim(X) - rel.dim(X/\mathscr{M})$. This is independent of the atlas (check this for a submersion $X' \to X \to \mathscr{M}$).

Definition 4.1. A *sheaf* $\mathscr{F}$ *on a stack* is a collection of sheaves $\mathscr{F}_{X \to \mathscr{M}}$ for any $X \to \mathscr{M}$, together with, for any triangle

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "h"'] & {} \arrow[d, phantom, "\Rightarrow"] & Y \arrow[dl, "g"] \\
& \mathscr{M} &
\end{tikzcd}$$

with an isomorphism $\varphi : g \circ f \to h$, a morphism of sheaves $\Phi_{f,\varphi} : f^* \mathscr{F}_{Y \to \mathscr{M}} \to \mathscr{F}_{X \to \mathscr{M}}$, compatible for $X \to Y \to Z$ (we often write Φ_f instead of $\Phi_{f,\varphi}$). Such that Φ_f is an isomorphism, whenever f is an open covering.

The sheaf $\mathscr{F}$ is called *cartesian* if all $\Phi_{f,\varphi}$ are isomorphisms.

Remarks 4.2.

1. Instead of giving sheaves $\mathscr{F}_{X \to \mathscr{M}}$ for all $X \to \mathscr{M}$, we could as well only give the global sections $\mathscr{F}_{X \to \mathscr{M}}(X)$, together with restriction maps for $U \to X$. Thus a reader not afraid of sites, will prefer to say that $\mathscr{F}$ is a sheaf on the big site of spaces over $\mathscr{M}$ (with the standard open topology).

2. A cartesian sheaf $\mathscr{F}$ is the same as a sheaf $\mathscr{F}_{X \to \mathscr{M}} =: \mathscr{F}_X$ on some atlas $X \to \mathscr{M}$ together with a descent datum, i.e., an isomorphism $\Phi : pr_1^* \mathscr{F}_X \to pr_2^* \mathscr{F}_X$ on $X \times_{\mathscr{M}} X$ which satisfies the cocycle condition on $X^{\times^3_{\mathscr{M}}}$:

Given such a sheaf this defines a sheaf on every $T \to \mathscr{M}$, because we get an induced descent datum on $X \times_{\mathscr{M}} T \to T$, this defines a sheaf on T. Of course, this is compatible with morphisms, since for $S \xrightarrow{f} T \to \mathscr{M}$ the pull back commutes with descent.

Conversely, given a cartesian sheaf $\mathscr{F}$ and an atlas $X \to \mathscr{M}$ we get an isomorphism $\Phi := \Phi_{pr_2}^{-1} \circ \Phi_{pr_1} : pr_1^* \mathscr{F}_X \to pr_2^* \mathscr{F}_X$ on $X \times_{\mathscr{M}} X$. This satisfies the cocycle condition, since on $X^{\times^3_{\mathscr{M}}}$ we have $pr_{12}^*(\Phi_{pr_1}) = \Phi_{pr_{12}}^{-1} \circ \Phi_{pr_1}$ and therefore $pr_{12}^* \Phi = \Phi_{pr_2}^{-1} \circ \Phi_{pr_1}$.

3. One might prefer to think only of cartesian sheaves on a stack, unfortunately this category does usually not contain enough injectives. But the subcategory of cartesian sheaves is a thick subcategory of all sheaves, i.e. a full category closed under kernels, quotients and extensions.

We can define *global sections of a sheaf on* $\mathscr{M}$. For cartesian sheaves we can simply choose an atlas $X \to \mathscr{M}$ and define

$$\Gamma(\mathscr{M},\mathscr{F}) := Ker(\ \Gamma(X,\mathscr{F}) \rightrightarrows \Gamma(X \times_{\mathscr{M}} X)\ ). \tag{1}$$

Lemma 4.3. *For a cartesian sheaf* $\mathscr{F}$ *on* $\mathscr{M}$ *the group* $\Gamma(\mathscr{M},\mathscr{F})$ *does not depend on the choice of the atlas.*

Proof. First note that the lemma holds if X is replaced by an open covering $X' = \cup U_i \to X \to \mathscr{M}$, because $\mathscr{F}_{X\to\mathscr{M}}$ is a sheaf.

Secondly we only need to check the lemma for refinements, i.e. an atlas $X' \to \mathscr{M}$ which factors $X' \xrightarrow{f} X \to \mathscr{M}$ such that f has local sections. But then by assumption any global section defined via X' induces one on X. $\square$

Similarly to the above construction, one can – as for $G-$bundles – give a simplicial description of cartesian sheaves on a stack as follows: Choose an atlas $X \to \mathscr{M}$. Then a sheaf on $\mathscr{M}$ defines a sheaf on the simplicial space $X_\bullet$, i.e. a sheaf $\mathscr{F}_n$ on all X_n, together with isomorphisms for all simplicial maps $f : [m] \to [n]$ from $f^*\mathscr{F}_n \to \mathscr{F}_m$.

Again we call a sheaf on a simplicial space cartesian, is all f^* are isomorphisms.

Conversely for any map $T \to \mathscr{M}$ a cartesian sheaf on $X_\bullet$ defines a sheaf on the covering $X \times_{\mathscr{M}} T \to T$, via the formula 1. This formula only defines global sections, but we can do the same for any open subset $U \subset T$.

Remark 4.4. Note that the functor $Shv(\mathscr{M}) \to Shv(X_\bullet)$ defined above is *exact.*

Example 4.5. A cartesian sheaf on a quotient stack $[X/G]$ is the same as a $G-$equivariant sheaf on X.

The category of sheaves of abelian groups on a stack $\mathscr{M}$ has enough injectives, so we want to define the cohomology of $H^*(\mathscr{M},\mathscr{F})$ as the derived functor of the global section functor. By the last example, for quotients $[X/G]$ this will be the same as equivariant cohomology on X.

As noted before, to define cohomological functors we have to consider arbitrary sheaves on $\mathscr{M}$ resp. on $X_\bullet$. We define global sections as:

$$\Gamma(\mathscr{M},\mathscr{F}) := \lim_{\leftarrow} \Gamma(X,\mathscr{F}_{X\to\mathscr{M}})$$

Where the limit is taken over all atlases $X \to \mathscr{M}$, the transition functions for a commutative triangle

$$\begin{tikzcd} X' \arrow[rr, "f"] \arrow[dr, "h"'] & \Rightarrow & X \arrow[dl, "g"] \\ & \mathscr{M} & \end{tikzcd}$$

are given by the restriction maps $\Phi_{f,\varphi}$.

Lemma 4.6. *For a cartesian sheaf* $\mathscr{F}$ *on a stack* $\mathscr{M}$ *the two notions of global sections coincide.*

Proof. For any atlas $X \to \mathscr{M}$ the maps $X \times_{\mathscr{M}} X \to \mathscr{M}$ are atlases as well. Thus we get a map

$$\varprojlim \Gamma(X', \mathscr{F}_{X' \to \mathscr{M}}) \to Ker(\ \Gamma(X, \mathscr{F}) \rightrightarrows \Gamma(X \times_{\mathscr{M}} X)\ ).$$

Conversely we have seen in lemma 4.3 that we can define a map in the other direction as well. And it is not difficult to check that these are mutually inverse. ☐

One tool to compute the cohomology of a sheaf on $\mathscr{M}$ is the spectral sequence given by the simplicial description above:

Proposition 4.7. *Let* $\mathscr{F}$ *be a cartesian sheaf of abelian groups on a stack* $\mathscr{M}$*. Let* $X \to \mathscr{M}$ *be an atlas and* $\mathscr{F}_\bullet$ *the induced sheaf on the simplicial space* $X_\bullet$ *then there is an* E_1 *spectral sequence:*

$$E_1^{p,q} = H^q(X_p, \mathscr{F}_p) \Rightarrow H^{p+q}(\mathscr{M}, \mathscr{F}).$$

The spectral sequence is functorial with respect to morphisms

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \arrow[dl, Rightarrow] \\ \mathscr{M} \arrow[r] & \mathscr{N} \end{tikzcd}$$

, for atlases X, Y *of* $\mathscr{M}$ *and* $\mathscr{N}$*.*

Proof. (e.g., [**Del74**],[**Fri82**]) For a cartesian sheaf $\mathscr{F}$ on $\mathscr{M}$ we denote by $\mathscr{F}_\bullet$ the induced sheaf on $X_\bullet$. We first show that $H^*(\mathscr{M}, \mathscr{F})$ is the same as the cohomology of the simplicial space $X_\bullet$ with values in $\mathscr{F}_\bullet$.

Recall that global sections of a sheaf $\mathscr{F}_\bullet$ on $X_\bullet$ are defined as

$$\Gamma(X_\bullet, \mathscr{F}_\bullet) := Ker(\ \Gamma(X, \mathscr{F}) \rightrightarrows \Gamma(X \times_{\mathscr{M}} X), \mathscr{F}\ ).$$

Thus for any cartesian sheaf $\mathscr{F}$ on $\mathscr{M}$ we have $H^0(X_\bullet, \mathscr{F}_\bullet) = H^0(\mathscr{M}, \mathscr{F})$.

We can factor the the cohomology functor on $X_\bullet$ as follows: First $\mathbf{R}\pi_{\bullet,*}$ from the derived category of sheaves on $X_\bullet$ to the derived category of simplicial sheaves on $\mathscr{M}$, then the exact functor *tot* taking the total complex of a simplicial complex and finally take the cohomology over $\mathscr{M}$.

Now for any $U \to \mathscr{M}$ we can calculate $(\mathbf{R}\pi_{\bullet,*}\mathscr{F}_\bullet)|_U$ as the direct image of the simplicial space $X_\bullet \times_{\mathscr{M}} U \xrightarrow{\pi_U} U$ over U. But for any sheaf $\mathscr{F}_U$ on U we know that $tot(\mathbf{R}\pi_{U,\bullet,*}\pi_U^*\mathscr{F}_U) \cong \mathscr{F}$, because π_U has local sections: Indeed, since the claim is local on U we may assume that π_U has a section $s : U \to X \times_{\mathscr{M}} U$. But if we denote $X_U := X \times_{\mathscr{M}} U$ then $X_n \times_{\mathscr{M}} U = X_U \times_U \cdots \times_U X_U$ and therefore the section s induces sections $X_n \to X_{n+1}$ which induce a homotopy on $tot(\mathbf{R}\pi_{U,\bullet,*}\pi^*\mathscr{F})$ proving that this complex is isomorphic to $\mathscr{F}$.

Thus we have shown that $H^*(\mathscr{M}, \mathscr{F}) = H^*(X_\bullet, \mathscr{F}_\bullet)$.

The spectral sequence is defined via the same construction, factoring H^* into $\mathbf{R}\Gamma_\bullet(K) := (\mathbf{R}\Gamma(K_n))_n$, which takes values in the derived category of simplicial complexes and the (exact) functor taking the associated total complex *tot*.

The spectral sequence is the spectral sequence of the double complex corresponding to the simplicial complex. □

This spectral sequence gives one way to transport the properties of the cohomology of manifolds to stacks:

Proposition 4.8.

1. *(Künneth Isomorphism) There is a natural isomorphism*

$$H^*(\mathscr{M} \times \mathscr{N}, \mathbb{Q}) \cong H^*(\mathscr{M}, \mathbb{Q}) \otimes H^*(\mathscr{N}, \mathbb{Q}).$$

2. *(Gysin sequence) For smooth embeddings $\mathscr{Z} \hookrightarrow \mathscr{M}$ of codimension c there is a long exact sequence:*

$$\to H^{k-c}(\mathscr{Z}, \mathbb{Q}) \to H^k(\mathscr{M}, \mathbb{Q}) \to H^k(\mathscr{M} - \mathscr{Z}, \mathbb{Q}) \to$$

In particular, the restriction $H^k(\mathscr{M}, \mathbb{Q}) \to H^k(\mathscr{M} - \mathscr{Z}, \mathbb{Q})$ is an isomorphism for $k < c - 1$.

This helps to do some well known cohomology computations in the language of stacks:

Example 4.9. Let G be a group acting trivially on a space X. To give a $G-$equivariant morphism from a $G-$bundle on a space T to X is the same as to give a map $T \to X$, thus $[X/G] \cong X \times [pt/G]$. And thus

$$H^*([X/G], \mathbb{Q}) \cong H^*(X, \mathbb{Q}) \otimes H^*([pt/G], \mathbb{Q}).$$

Let $T \cong (S^1)^n$ be a torus. Then $\underline{BT} \cong (\underline{BS^1})^n$, because any T-bundle is canonically the product of S^1-bundles, once an isomorphism $T \cong (S^1)^n$ is chosen. Thus $H^*(BT, \mathbb{Q}) \cong H^*(BS^1, \mathbb{Q})^{\otimes n}$.

Finally we want to calculate $H^*([pt/S^1], \mathbb{Q}) \cong \mathbb{Q}[c_1]$ a polynomial ring with one generator of degree 2. One way to do this is as follows: By the spectral sequence 4.7 we see that the morphisms $[\mathbb{C}/\mathbb{C}^*] \to [pt/\mathbb{C}^*] \leftarrow [pt/S^1]$ induce isomorphisms in cohomology, where the action of $\mathbb{C}^*$ on $\mathbb{C}$ is the standard action. This is because $H^*(\mathbb{C} \times (\mathbb{C}^*)^n, \mathbb{Q}) \cong H^*((S^1)^n, \mathbb{Q})$. The same is true for $[\mathbb{C}^N/\mathbb{C}^*] \to [pt/\mathbb{C}]$. But here we can use the Gysin sequence: The inclusion $0 \to \mathbb{C}^N$ induces a closed embedding $[pt/\mathbb{C}^*] \to [\mathbb{C}^N/\mathbb{C}^*]$ of codimension N. The open complement $[\mathbb{C}^N - 0/\mathbb{C}^*] \cong \mathbb{CP}^{N-1}$, because the $\mathbb{C}^*$ action is free outside the origin. This proves the claim.

For the definition of $f_!$ maps in $K-$theory we need to define normal bundles, at least for nice representable morphisms:

Lemma/Definition 4.10. *Let $f : \mathscr{M} \to \mathscr{N}$ be a representable morphism of differentiable stacks satisfying one of the following conditions:*

1. *f is a smooth submersion.*
2. *f is a smooth embedding.*

Let $Y \xrightarrow{p} \mathscr{N}$ be any smooth atlas of $\mathscr{N}$. Then normal bundle $T_{\mathscr{M}\times_{\mathscr{N}}Y\to Y}$ descends to a vector bundle $T_{\mathscr{M}\to\mathscr{N}}$ on $\mathscr{M}$. This does not depend on the choice of Y and is called the normal bundle *to f.*

*Proof***.** We only need to note that formation of the normal bundle commutes with pull-back. Therefore the two pull backs of the normal-bundle of $\mathscr{M}\times_{\mathscr{N}} \to Y$ to $(\mathscr{M}\times_{\mathscr{N}}Y)\times_{\mathscr{M}}(\mathscr{M}\times_{\mathscr{N}}Y)$ are both canonically isomorphic to the normal bundle to $(\mathscr{M}\times_{\mathscr{N}}(Y\times_{\mathscr{N}}Y) \to (Y\times_{\mathscr{N}}Y)$. Therefore the bundle descends to a bundle on $\mathscr{M}$. ☐

Since for manifolds formation of the normal bundle commutes with pull-backs, the same holds for stacks:

Corollary 4.11. *If $\mathscr{M} \to \mathscr{N}$ is a morphism as in the above lemma and $g : \mathscr{N}' \to \mathscr{N}$ is an arbitrary morphism, then $T_{\mathscr{M}\times_{\mathscr{N}}\mathscr{N}'\to\mathscr{N}'} \cong g^*T_{\mathscr{M}\to\mathscr{N}}$.*

Similarly one gets short exact sequences for the normal bundle of a composition, because the corresponding sequences for an atlas descend.

Tangent spaces to differentiable stacks will only be stack-versions of vector bundles. Nevertheless define:

Lemma/Definition 4.12 (Tangent stacks). *Let $\mathscr{M}$ be a differentiable stack and $X \to \mathscr{M}$ be a smooth atlas. Then we can take the tangent spaces to the groupoid $X_\bullet$:*

$$T(X\times_{\mathscr{M}}X\times_{\mathscr{M}}X) \mathrel{\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}} T(X_\times\mathscr{M}X) \rightrightarrows TX$$

by functoriality this is again a groupoid, the quotient $[TX/T(X\times_{\mathscr{M}}X)]$ is independent of the choice of X and is called $T\mathscr{M}$, the tangent stack *to $\mathscr{M}$.*

The fibers of the projection $T\mathscr{M} \to \mathscr{M}$ *are isomorphic to* $[V/W]$*, where* V, W *are finite dimensional vector spaces, and* W *acts on* V *by some linear map* $W \to V$*, which is not injective in general.*

5. S^1-Gerbes or twists

Informally a gerbe(2) over some space X is a stack $\mathscr{X} \to X$ which has the same points as X, i.e. the points of X are isomorphism classes of objects in $\mathscr{X}(pt)$. An S^1-gerbe is a gerbe such that the automorphism groups of all points $pt \to \mathscr{X}$ are isomorphic to S^1 in a continuous way.

The easiest example of such an object is $[pt/S^1] \to pt$. More generally these objects occur naturally in many moduli-problems, e.g. every $U(n)$-bundle with flat connection on a compact Riemann surface has an automorphism group S^1, in good situations the stack of such objects is a S^1-gerbe over the coarse moduli space. This gerbe gives the obstruction to the existence of a Poincaré bundle on the coarse moduli-space. Finally these objects seem to appear naturally in K-theoretic constructions, since the choices of $Spin^c$-structures on an oriented bundle form a S^1-gerbe (locally there is only one such choice, but the trivial $Spin^c$-bundle has more automorphisms).

Definition 5.1. Let X be a space. A stack $\mathscr{X} \xrightarrow{\pi} X$ is called a *gerbe* over X if

1. π has local sections, i.e., there is an open covering $\cup U_i = X$ and sections $s_i : U_i \to \mathscr{X}$ of $\pi|_{U_i}$.

2. Locally over X all objects of $\mathscr{X}$ are isomorphic, i.e., for any two objects $t_1, t_2 \in \mathscr{X}(T)$ there is a covering $\cup U_i = T$ such that $t_1|_{U_i} \cong t_2|_{U_i}$.

A gerbe $\mathscr{X} \to X$ is called a *(continuous)* S^1*-gerbe* if for any $T \to X$, together with a section $s : T \to \mathscr{X}$ there is an isomorphism $Aut(s) := (T \times_{\mathscr{X}} T) \times_{T\times T} T \cong S^1 \times T$ as family of groups over T, which is compatible with composition of morphisms $T' \xrightarrow{s'} T \xrightarrow{s} \mathscr{X}$.

Remarks 5.2.

1. As one might expect, the condition that the automorphism group of any object is S^1 implies that for any section $s : T \to \mathscr{X}$ the map $T \times_{\mathscr{X}} T \to T \times_X T$ is an S^1-bundle. Since the fibres of this map are given by two points together with a morphism between the images in $\mathscr{X}$ the fibres are S^1-torsors. To see that the map is indeed a locally trivial bundle one can replace T by $T \times_{\mathscr{X}} T$

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(2) Gerbe is the french word for sheaf, to avoid another wrong translation (cf. faisceaux, champ etc.) there seems to be an agreement to keep the french word - or at least its spelling.

in the above definition to get an isomorphism $(T \times_{\mathscr{X}} T) \times_{(T \times_X T)} (T \times_{\mathscr{X}} T) \cong S^1 \times (T \times_{\mathscr{X}} T)$ (one only has to write down, the functor represented by the left hand side).

2. An S^1-gerbe over a topological/differentiable space X is always a topological/differentiable stack, an atlas is given by the sections s_i. By the previous remark we know that $U_i \times_{\mathscr{X}} U_i$ is a space and the two projections are S^1-bundles, in particular smooth.

This also shows that we might replace the condition that the automorphism groups are isomorphic to S^1 for all objects, by the same condition for sections of one atlas of X. Representability of arbitrary fibered products $T_1 \times_{\mathscr{X}} T_2$ then follows, since locally over X we can glue S^1-bundles. (This definition will be explained more carefully below.)

3. As in the case of bundles, one there is also a notion of discrete S^1-gerbe, simply by choosing the discrete topology for S^1 in the above definition.

4. Any S^1-gerbe on a contractible space is trivial, i.e. isomorphic to

$$X \times [pt/S^1] \to X.$$

Perhaps this is obvious. If not, one might reason as follows: Choose a covering U_i of X with sections $s_i : U_i \to \mathsf{X}$, such that all $U_i, U_i \cap U_j$ are contractible. Then $U_i \times_{\mathscr{X}} U_j \to U_i \cap U_j$ is a locally trivial S^1 bundle, thus trivial. Therefore the obstruction to glue the sections s_i gives an element in $H^2(X, S^1) = 0$ (the classification of gerbes will show that this H^2 classifies S^1-gerbes).

5. A gerbe with a section is called *neutral.* Gerbes which are isomorphic to $X \times [pt/G] \to X$ for some group G are called *trivial* gerbes over X.

We will need a generalization of the above, to include gerbes over topological stacks $\mathscr{M}$ instead of spaces X. Again we only have to replace coverings by representable morphisms with local sections:

Definition 5.3. Let $\mathscr{M}$ be a topological stack. A stack $\mathscr{M}^\tau \xrightarrow{\pi} \mathscr{M}$ is called a *gerbe* over $\mathscr{M}$ if

1. π has local sections, i.e. there is an atlas $X \to \mathscr{M}$ and a section $s : X \to \mathscr{M}^\tau$ of $\pi|_X$.

2. Locally over $\mathscr{M}$ all objects of $\mathscr{M}^\tau$ are isomorphic, i.e. for any two objects $t_1, t_2 \in \mathscr{M}(T)$ and lifts $s_1, s_2 \in \mathscr{M}^\tau(T)$ with $\pi(s_i) \cong t_i$, there is a covering $\cup U_i = T$ such that $s_1|_{U_i} \cong s_2|_{U_i}$.

A gerbe $\mathscr{M}^\tau \to \mathscr{M}$ is called a *(continuous)* S^1*-gerbe* if there is an atlas $X \xrightarrow{p} \mathscr{M}$ of $\mathscr{M}$, a section $(s : X \to \mathscr{M}^\tau, \varphi : \pi \circ s \Rightarrow p)$ such that there is an isomorphism $\Phi : Aut(s/p) := (X \times_{\mathscr{M}^\tau} X) \times_{X \times_{\mathscr{M}} X} X \cong S^1 \times X$ as family of groups over X, such that on $X \times_{\mathscr{M}} X$ the diagram

$$\begin{tikzcd}
Aut(s \circ pr_1/p \circ pr_1) \arrow[rr, "\cong"] \arrow[dr, "pr_1^*\Phi"'] & & Aut(s \circ pr_2/p \circ pr_2) \arrow[dl, "pr_2^*\Phi"] \, , \\
& X \times_{\mathscr{M}} X \times S^1 &
\end{tikzcd}$$

where the horizontal map is the isomorphism given by the universal property of the fibered product, commutes (i.e. the automorphism groups of objects of $\tilde{\mathscr{M}}$ are central extensions of those of $\mathscr{M}$ by S^1).

Example 5.4.

1. The easiest example of a S^1-gerbe on a quotient stack $[X/G]$ is given by a central extension $S^1 \to \tilde{G} \xrightarrow{pr} G$, then $\tilde{G}$ also acts on X and pr induces a map $[X/\tilde{G}] \xrightarrow{\pi} [X/G]$, which defines a gerbe over $[X/G]$: The atlas $X \to [X/G]$ lifts to $[X/\tilde{G}]$, this shows (1). And (2) follows, because locally any map $T \to G$ can be lifted to $\tilde{G}$.

Finally the map $S^1 \to \tilde{G}$ induces a morphism $[X/S^1] \to [X/\tilde{G}]$ which induces an isomorphism $X \times [pt/S^1] \cong [X/S^1] \xrightarrow{\cong} X \times_{[X/G]} [X/\tilde{G}]$. This shows the last condition of the definition.

2. This generalizes to groupoids: An extensions of a groupoid $\Gamma_1 \rightrightarrows \Gamma_0$ by S^1 is a groupoid $\tilde{\Gamma}_1 \rightrightarrows \Gamma_0$ with a morphism:

$$\begin{tikzcd}
\tilde{\Gamma}_1 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "p"] & \Gamma_0 \arrow[d, "id"] \\
\Gamma_1 \arrow[r, shift left] \arrow[r, shift right] & \Gamma_0
\end{tikzcd}$$

such that p is an S^1-bundle and the S^1-action commutes with the source and target morphisms.

As before this defines a S^1-gerbe $[\Gamma_0/\tilde{\Gamma}_1] \to [\Gamma_0/\Gamma_1]$.

Remarks 5.5.

1. As before a S^1-gerbe is always a differentiable stack, the section $s : X \to \mathscr{M}^\tau$ of the particular atlas $X \to \mathscr{M}$ is an atlas for $\mathscr{M}^\tau$:

The map s is representable, because by base-change (Lemma 2.6) $X \times_{\mathscr{M}} \mathscr{M}^\tau \to \mathscr{M}^\tau$ is representable and the canonical map $X \to X \times_{\mathscr{M}} \mathscr{M}^\tau$ induced by s is surjective by definition and representable since $X \times_{X \times_{\mathscr{M}} \mathscr{M}^\tau} X \cong Aut(s/p) \cong S^1 \times X$.

Thus the free action of $Aut(s/p)$ induces a structure of an S^1 bundle on $X \times_{\mathscr{M}^\tau} X \to X \times_{\mathscr{M}} X$. As in remark 5.2(1) one can prove that this map is a locally trivial S^1-bundle.

Furthermore, the last condition of the definition ensures, that this defines an S^1–extension of groupoids. Thus every S^1 gerbe can be constructed as in the example given above.

2. Since we just saw that for any $T \xrightarrow{s} \mathscr{M}^\tau \xrightarrow{\pi} \mathscr{M}$ the group $Aut(s/s \circ \pi)$ is representable, locally canonically isomorphic to S^1 we get a canonical isomorphism $Aut(s/s \circ \pi) \cong S^1 \times T$. Thus again we could have used this as a definition of S^1-gerbes.

3. Thus we can pull-back gerbes: For any $\mathscr{N} \to \mathscr{M}$ and any S^1-gerbe $\mathscr{M}^\tau \to \mathscr{M}$ the stack $\mathscr{N}^\tau := \mathscr{M}^\tau \times_{\mathscr{M}} \mathscr{N}$ is a S^1-gerbe over $\mathscr{N}$, since for any $T \to \mathscr{N}^\tau$ we have $T \times_{\mathscr{N}} \mathscr{N}^\tau = T \times_{\mathscr{N}} (\mathscr{N} \times_{\mathscr{M}} \mathscr{M}^\tau)T = T \times_{\mathscr{M}} \mathscr{M}^\tau$.

4. A morphism of S^1-gerbes is a morphism of the corresponding stacks overthe base stack, which induces the identity on the central S^1 automorphisms of the objects.

As before we call a gerbe *neutral* if it has a section. To state this in a different way recall that for any bundle $\mathscr{P}$ on $\tilde{\mathscr{M}}$ and any $s : T \to \tilde{\mathscr{M}}$ we get an action of $Aut(s)$ on $s^*\mathscr{P}$. In particular for a line bundle $\mathscr{L}$ the pull back carries an S^1-action. Thus S^1 acts on every fibre by a character $\chi = ()^n : S^1 \to S^1$, where n is some integer, constant on connected components of T resp. $\tilde{\mathscr{M}}$. A line bundle on $\tilde{\mathscr{M}}$ is called of *weight* n if n is constant on all connected components.

Lemma 5.6. *For a S^1-gerbe $\pi : \tilde{\mathscr{M}} \to \mathscr{M}$ the following are equivalent:*

1. *$\tilde{\mathscr{M}} \to \mathscr{M}$ has a section s.*
2. *$\tilde{\mathscr{M}} \cong [pt/S^1] \times \mathscr{M}$ as stacks over $\mathscr{M}$.*
3. *There is a unitary line bundle of weight 1 on $\tilde{\mathscr{M}}$.*

Proof: Of course 2. ⇒ 1.. Furthermore, the universal bundle $pt \to [pt/S^1]$ is of weight 1, thus 2. ⇒ 3..

Given a unitary line bundle of weight 1 we get a morphism $\tilde{\mathscr{M}} \to \mathscr{M} \times BS^1$. This map induces an isomorphism on automorphism groups of objects, because the kernel of the map $Aut_{\tilde{\mathscr{M}}} \to Aut_{\mathscr{M}}$ is S^1 and this kernel is mapped isomorphically to the automorphisms of S^1 bundles, since we started from a bundle of weight 1. The map is also locally essentially surjective on objects, because locally every object of $\mathscr{M}$ can be lifted to an object of $\tilde{\mathscr{M}}$ and locally every S^1-bundle is trivial. And finally the map is a gerbe, since locally all objects in the fibre are isomorphic. This implies that the map is an isomorphism.

This also shows, that the total space of the S^1-bundle is isomorphic to $\mathscr{M}$, thus any line bundle of weight 1 induces a section.

Finally, given a section $\mathscr{M} \to \tilde{\mathscr{M}}$ we get an isomorphism $S^1 \times \mathscr{M} \cong \mathscr{M} \times_{\tilde{\mathscr{M}}} \mathscr{M} \times_{\mathscr{M} \times_{\mathscr{M}} \mathscr{M}} \mathscr{M} = \mathscr{M} \times_{\tilde{\mathscr{M}}} \mathscr{M}$. The compatibility condition shows, that this makes $\mathscr{M}$ into an S^1-bundle over $\mathscr{M}$. ☐

Remark 5.7. The descriptions 2. and 3. of the lemma show that line bundles on $\mathscr{M}$ act on trivializations of a S^1-gerbe. In description 2. this is because a morphism to $[pt/S^1]$ is the same as a unitary line bundle on $\mathscr{M}$ and in description 3. one sees, that two line bundles of weight 1 differ by a line bundle on $\mathscr{M}$.

There is a description of isomorphism classes of gerbes in terms of cocycles, see for example [**Bre94**] and [**Cra**]. We write $\mathscr{S}^1$ for the sheaf of continuous sections of the trivial bundle $S^1 \times \mathscr{M} \to \mathscr{M}$:

Proposition 5.8.

1. *Let $\mathscr{M}$ be a topological stack. Then there is a natural bijection*

$$\{\textit{Isom. classes of } S^1\textit{-gerbes over } \mathscr{M}\} \cong H^2(\mathscr{M}, \mathscr{S}^1).$$

The same holds if S^1 is replaced by any abelian, topological group.

2. *If $\mathscr{M}$ is a differentiable stack such that the diagonal $\Delta : \mathscr{M} \to \mathscr{M} \times \mathscr{M}$ is proper, then the boundary map of the exponential sequence induces an isomorphism $H^2(\mathscr{M}, \mathscr{S}^1) \xrightarrow{\cong} H^3(\mathscr{M}, \mathbb{Z})$.*

Indication of the proof: The two parts of the theorem are of very different nature, they are only put in one statement, because the cocycles in (2), called Dixmier-Douady classes, are often used to characterize gerbes.

For the first part we will first describe how to associate a cohomology class to a gerbe $\mathscr{M}^\tau$.

Choose an atlas $X \to \mathscr{M}$ which is the disjoint union of contractible spaces, e. g., take any atlas Y and then chose a covering of Y by contractible spaces. We use the spectral sequence $H^p(X^{\times^{q+1}_{\mathscr{M}}}, \mathscr{S}^1) \Rightarrow H^{p+q}(\mathscr{M}, \mathscr{S}^1)$ to calculate $H^2(\mathscr{M}, \mathscr{S}^1)$. By the choice of X this is:

$$\begin{tikzcd}
H^2(X, \mathscr{S}^1) = 0 & \cdot & \cdot & \\
H^1(X, \mathscr{S}^1) = 0 & H^1(X \times_{\mathscr{M}} X, \mathscr{S}^1) \arrow[r, "d^1"] \arrow[drr, dashed, "d^2"] & H^1(X^{\times^3_{\mathscr{M}}}, \mathscr{S}^1) & \cdot \\
H^0(X, \mathscr{S}^1) \arrow[r] & H^0(X \times_{\mathscr{M}} X, \mathscr{S}^1) \arrow[r, "d^1"] & H^0(X^{\times^3_{\mathscr{M}}}, \mathscr{S}^1) \arrow[r, "d^1"] & H^0(X^{\times^4_{\mathscr{M}}}, \mathscr{S}^1)
\end{tikzcd}$$

Where the differentials d^1 are given by the alternating sum over the pull-backs (since the spectral sequence is constructed from a simplicial object by taking alternating sums of the simplicial maps).

As explained before the choice of a trivialization of the pull-back $X \xrightarrow{s} X^\tau = X \times_{\mathscr{M}} \mathscr{M}^\tau$ of $\mathscr{M}^\tau$ to X induces a map $\tilde{p} : X \to \mathscr{M}^\tau$ and an S^1-bundle

$$P := X \times_{\mathscr{M}^\tau} X = Isom(\tilde{p} \circ p_1, \tilde{p} \circ p_2) \to Isom(p \circ p_1, p \circ p_2) = X \times_{\mathscr{M}} X$$

thus a class in $H^1(X \times_{\mathscr{M}} X, \mathscr{S}^1)$.

This actually lies in the kernel of d^1, because on $X^{\times^{3}_{\mathscr{M}}}$ the composition induces an isomorphism

$$\Phi_{123} : Isom(\tilde{p} \circ p_1, \tilde{p} \circ p_2) \otimes Isom(\tilde{p} \circ p_2, \tilde{p} \circ p_3) \xrightarrow{\cong} Isom(\tilde{p} \circ p_1, \tilde{p} \circ p_3).$$

We will see below, that the associativity of the composition exactly means that this also lies in the kernel of d^2. Furthermore we may view Φ_{123} as a section of the bundle $p_{12}^* P \otimes p_{23}^* P \otimes (p_{13}^* P)^{-1}$. This shows that the choices of Φ_{123}, which define an associative composition form a torsor for $ker(H^0(X^{\times^{3}_{\mathscr{M}}}, \mathscr{S}^1) \to H^0(X^{\times^{4}_{\mathscr{M}}}, \mathscr{S}^1))$. Two such choices define isomorphic gerbes, whenever we change Φ_{123} by an automorphism of P, i.e., an element of $H^0(X^{\times^{2}_{\mathscr{M}}}, \mathscr{S}^1)$.

To see that this construction defines an element in $H^2(\mathscr{M}, \mathscr{S}^1)$ we have to check that the we found an element in the correct extension of the E_2^{11} by the $E_2^{0,2}$ term and that the differential d^2 corresponds to associativity. Accepting this for a moment, we see that the process can be reversed:

Cohomology classes as above can be used to glue a groupoid over $X \times_{\mathscr{M}} X \to X$. The boundary maps in the spectral sequence assure the associativity of the composition. (One should note that in the construction Φ_{123} also defines isomorphisms $P^{-1} \cong tw^* P$ where $tw = ()^{-1} : X \times_{\mathscr{M}} X \to X \times_{\mathscr{M}} X$ is the inverse map of the groupoid $X^{\times^{\bullet}_{\mathscr{M}}}$, and trivialization of the restriction of P to the diagonal $P|_{\Delta(X)}$.)

To analyze the differentials of the spectral sequence we have to recall its construction: We have to chose acyclic resolutions of S^1 on $X^{\times^{i}_{\mathscr{M}}}$. Thus we choose a covering X^2_α of $X \times_{\mathscr{M}} X$ such that all the intersections $X^2_{\alpha_1} \cap \cdots \cap X^2_{\alpha_3}$ are acyclic (this condition could be avoided if we would allow for another index). Then we chose a covering X^3_β of $X^{\times^{3}_{\mathscr{M}}}$ which has the same property, such that all projections $pr_{ij} : X^{\times^{3}_{\mathscr{M}}} \to X^{\times^{2}_{\mathscr{M}}}$ map X^3_β to some $X^2_{pr_{ij}(\beta)}$. We do the same for $X^{\times^{4}_{\mathscr{M}}}$ and get a covering X^4_γ. Taking global sections of S^1 over these spaces we get a double complex from which the spectral sequence is induced, the total complex calculates $H^*(\mathscr{M}, \mathscr{S}^1)$. Thus writing $X^2_{\alpha\alpha'}$ for the intersection

$X^2_\alpha \cap X^2_{\alpha'}$ we calculate $H^2(\mathscr{M}, \mathscr{S}^1)$ as the cohomology of:

$$\bigoplus_\alpha H^0(X^2_\alpha) \xrightarrow{d_1} \bigoplus_{\alpha,\alpha'} H^0(X^2_{\alpha\alpha'}, \mathscr{S}^1) \oplus \bigoplus_\beta H^0(X^3_\beta, \mathscr{S}^1)$$

$$\xrightarrow{d_2} \bigoplus_{\alpha\alpha'\alpha''} H^0(X^2_{\alpha\alpha'\alpha''}, \mathscr{S}^1) \oplus \bigoplus_{\beta,\beta'} H^0(X^3_{\beta\beta'}, \mathscr{S}^1) \oplus \bigoplus_\gamma H^{(}X^4_\gamma, \mathscr{S}^1)$$

And the differentials are the sum of the simplicial and the covering differentials. Thus the components of d_2 are:

$$d_2(s_{\alpha\alpha'}, s_\beta)_{\alpha,\alpha',\alpha''} = s_{\alpha\alpha'} s^{-1}_{\alpha\alpha''} s_{\alpha'\alpha''}$$

$$d_2(s_{\alpha\alpha'}, s_\beta)_{\beta,\beta'} = pr^*_{12} s_{pr_{12}(\beta)pr_{12}(\beta')} pr^*_{13} s^{-1}_{pr_{13}(\beta)pr_{13}(\beta')} pr^*_{23} s_{pr_{23}(\beta)pr_{23}(\beta')} - s_\beta + s_{\beta'}$$

$$d_2(s_{\alpha\alpha'}, s_\beta)_\gamma = pr^*_{123} s_{pr_{123}(\gamma)} pr^*_{124} s^{-1}_{pr_{124}(\gamma)} pr^*_{134} s_{pr_{134}(\gamma)} pr^*_{234} s^{-1}_{pr_{234}(\gamma)}$$

More precisely, the indices on the right hand side depend on the projections. If the first component is zero $s_{\alpha\alpha'}$ defines an S^1-bundle P on $X \times_{\mathscr{M}} X$. The vanishing of the second summand assures that s_β defines a section of $pr^*_{12}P \otimes pr_{13} * P^{-1} \otimes pr^*_{23}P$. And finally the third summand guarantees associativity as claimed.

$\square_{(1)}$

The second part of the proposition depends on the existence of a Haar-measure on compact groupoids (i.e. groupoids defining stacks with proper diagonal $\mathscr{M} \to \mathscr{M} \times \mathscr{M}$, in particular all automorphism groups of objects are proper over the parameter space).

Using this Crainic [**Cra**] shows that a generalization of the Poincaré lemma holds for such stacks, i.e. the sheaves of continuous **R**-valued functions are acyclic. Therefore by the exponential sequence $H^2(\mathscr{M}, \mathscr{S}^1) \cong H^3(\mathscr{M}, \mathbb{Z})$.

$\square$

Remark 5.9. As one might expect from the proof above, the group structure of $H^2(X, \mathscr{S}^1)$ can also be implemented as an operation on stacks: Given S^1-gerbes $\mathscr{M}^\tau$, $\mathscr{M}^{\tau'}$ on $\mathscr{M}$ one can take the fibred product $\mathscr{M}^\tau \times_{\mathscr{M}} \mathscr{M}^{\tau'}$, which is an $S^1 \times S^1$ and forget the anti-diagonal S^1-automorphisms. To avoid technical arguments we can simply choose an atlas $X \to \mathscr{M}$ on which both gerbes are trivial. Then we have already seen that $X \times_{\mathscr{M}^\tau \times_{\mathscr{M}} \mathscr{M}^{\tau'}} X \to X \times_{\mathscr{M}} X$ is an $S^1 \times S^1$-bundle and the multiplication $S^1 \times S^1 \to S^1$ defines an associated S^1-bundle $X_1 \to X \times_{\mathscr{M}} X$ and it is not difficult to check, that this defines a groupoid $X_1 \rightrightarrows X$.

In the special case of quotient stacks and gerbes given by two group extensions this is simply the Yoneda product of extensions.

Another description of gerbes is via projective bundles. Given any (possibly finite dimensional) Hilbert space H. One gets an exact sequence of groups:

$$1 \to S^1 \to \mathsf{U}(H) \to \mathsf{PU}(H) \to 1$$

By the first example of gerbes this defines an S^1–gerbe $B\mathsf{U} \to B\mathsf{PU}$. In particular for any PU bundle P on a space X we can pull back this gerbe to X via the classifying morphism $X \to B\mathsf{PU}$. The category of sections $X \times_{B\mathsf{PU}} B\mathsf{U}(T)$ is the category of U bundles on T together with an isomorphism of the associated PU bundle and the pull back of P to T.

This shows that the gerbe obtained in this way corresponds to the image of P under the boundary map $\delta : H^1(X, \mathsf{PU}) \to H^2(X, \mathscr{S}^1)$. In particular if H is n-dimensional we may factorize this map via the sequence:

$$0 \to \mathbb{Z}/n\mathbb{Z} \to \mathsf{SU}(n) \to \mathsf{PU}(n) \to 0,$$

i.e., the classes obtained in this way are n-torsion.

For the purpose of this Seminar it will be sufficient to note that the gerbes that arise naturally in $K-$theory are always obtained by PU bundles, this will be explained in the next section.

If X is a manifold (and not a stack), then the fact that $B\mathsf{PU}$ is a $K(\mathbb{Z}, 3)$-space (if H is an infinite dimensional Hilbert space) shows, that δ is an isomorphism, thus any S^1 gerbe arises in this way.

This is less clear for differentiable stacks, and Proposition 2.38 in [**LTX**] gives the result. Unfortunately, since I am not an analyst, their proof is to short for me. In section 6 we will prove that all S^1-gerbes arise from projective bundles, if the stack is a local quotient stack, a notion also defined in that section.

In K-theory one can define Thom-isomorphisms for Spin^c-bundles and one can do the same for bundles on stacks (although one has to be a bit careful with the definition the Thom-space of a bundle). As remarked before the choices of $Spin^c$-structure define a S^1 gerbe, simply pulling back the universal gerbe $\underline{B\mathsf{Spin}^c} \to \underline{B\mathsf{SU}(n)}$. Thus every bundle P on a a space X defines a gerbe $X^\tau \to X$ such that the pull back of P to X^τ has a canonical Spin^c-structure. (We get a stack and not a space, because the sequence of groups is $S^1 \to Spin^c \to \mathsf{SO}$ in contrast to orientation problems where the cokernel imposes the obstruction).

If the bundle is not orientable one first has to chose some $\mathbb{Z}/2$ covering on which one chooses an orientation. And then one takes the above gerbe on the orientation covering.

Again one has to be careful defining a group structure on these objects, since if we have two bundles which admit $Spin^c$-structures on the orientation cover, their tensor product does not necessarily admit a $Spin^c$ structure on the sum of the orientation coverings.

The obstruction comes from the universal example on $B\mathbb{Z}/2 \times B\mathbb{Z}/2$ and this gives a geometric description of the cup product of two torsion-classes:

Lemma 5.10. *Given finite abelian groups* A, B, C *and a bilinear form* $<,>: A \times B \to C$*, then:*

1. $<,>$ *defines an abelian extension* $0 \to C \to G \to A \times B \to 0$ *by the cocycle* $\sigma(a,b,a',b') =< a, -b' > + < a', -b >$.

2. *Given an* A*-bundle* P_A *and a* B*-bundle* P_B *on a space* X *corresponding to classes* $c(P_A) \in H^1(X,A), c(P_B) \in H^1(X,B)$*. Define a* C *gerbe on* X*, given by the pull back of the gerbe* $BG \to BA \times BB$ *defined in (1), via the classifying map* $X \to BA \times BB$*. The Dixmier Douady class of this gerbe is the cup product* $c(P_A) \cup c(P_B)$.

Proof. Since the cup product commutes with pull-backs, we only may assume $X = BA \times BB$ and take P_A, P_B the universal bundles.

In this case the standard atlas $pt \to BA \times BB$ is acyclic, as well as all fibered products $pt \times_{B(A\times B)} \cdots \times_{B(A\times B)} pt$.

Thus the spectral sequence we used to calculate the Dixmier-Douady classes is a complex. The class of the universal C-gerbe therefore is given by the cocycle $s(a,b,a',b') =< a, -b' > + < a', -b >$. And the same cocycle represents the cup-product. □

6. Local quotient stacks

Freed, Hopkins and Teleman define $K-$functors only for local quotient stacks, so we need to introduce this concept and we show that for these stacks any gerbe arises from a projective Hilbert bundle, and the latter is almost uniquely determined by the gerbe. References for this section are [**FHT**],[**LTX**] and the preprint of Atiyah and Segal [**AS**].

Definition 6.1. A differentiable stack $\mathscr{M}$ is called a *local quotient stack* if there is a covering $\mathscr{U}_i$ of $\mathscr{M}$ by open substacks, such that each $\mathscr{U}_i \cong [U_i/G_i]$, where G_i is a compact Lie group acting on a manifold U_i.

Quite a lot of stacks have this property, a very general result was recently given in [**Zun**]. Of course if a stack $\mathscr{M}$ is a local quotient stack, then the diagonal $\mathscr{M} \to \mathscr{M} \times \mathscr{M}$ is proper. We say that $\mathscr{M}$ has *proper isotropy.*

By the standard slice theorems (e.g. [**DK00**] Chapter 2) to be a local quotient stack is a local property as follows (note that we assumed the Lie groups to be compact):

Lemma 6.2. *(To be local quotient stack is a local property) Let $\mathscr{M}$ be a local quotient stack, $X \to \mathscr{M}$ an atlas. Given a point $x \in X$ and $x \in U \subset X$ open there is an open substack $\mathscr{U} \subset \mathscr{M}$ together with a presentation $\mathscr{U} \cong [Y/G]$ where G is a compact Lie group acting on a contractible manifold Y, and a commuting diagram:*

$$\begin{tikzcd} U \arrow[r] & \mathscr{M} \\ Y \arrow[u, "f"] \arrow[r] & {[Y/G]} \arrow[u] \end{tikzcd}$$

and $x \in Im(f)$.

Corollary 6.3. *Any S^1-gerbe on a local quotient stack is again a local quotient stack.*

Proof of Lemma 6.2. Shrinking $\mathscr{M}$ we may assume that $\mathscr{M} \cong [X'/G']$ is a global quotient stack. Further we may assume that $X = X'$, because the projections of the fibered product $X' \leftarrow X \times_{\mathscr{M}} X' \to X$ are submersions, thus we may choose a preimage $\tilde{x}$ of x in the fibered product and a local section $X' \supset U \to X' \times_{\mathscr{M}} X$ passing through $\tilde{x}$.

But now we can find a contractible slice of the group action, which gibes us a local presentation as $\mathscr{U} = [D/Stab_G(x)]$, where D is a ball and the action of the stabilizer of x comes from the linear action on the tangent space at x. □

Proof of Corollary 6.3. We may assume $\mathscr{M} = [X/G]$ is a global quotient. Since gerbes on contractible spaces are trivial, we may apply the last lemma to get a covering of $\mathscr{M}$ by open substacks of the form $[Y/H]$ such that the given gerbe is trivial on Y. Since Y is contractible, the gerbe is induced form a S^1-extension of H. □

To end the section on local quotient stacks, we want to show that for these stacks any S^1-gerbe is defined by a projective bundle, which can be chosen in an almost canonical way (up to non canonical isomorphism). To this end we first need the concept of a universal Hilbert bundle, as defined in [**FHT**].

Definition 6.4 (Freed, Hopkins, Teleman [FHT])
A Hilbert bundle H on a differentiable stack $\mathscr{M}$ is called *universal* if any other Hilbert bundle H' is a direct summand of H. A universal Hilbert bundle is called *local* if its restriction to any open substack is universal.

Lemma 6.5 ([FHT] C.3). *A universal bundle H on a stack $\mathscr{M}$ has the* absorption property*: For any Hilbert bundle H' on $\mathscr{M}$ there is an isomorphism $H \oplus H' \cong H$.*

The basic proposition is:

Proposition 6.6 ([FHT] C.4). *Let $\mathscr{M}$ be a local quotient stack. Then there exists a universal Hilbert H bundle on $\mathscr{M}$. This bundle is local, and its group of unitary automorphisms is weakly contractible.*

We sketch the argument of [**FHT**]: On manifolds all Hilbert bundles are trivial, because the infinite unitary group U is contractible. Now let $\mathscr{M}$ be a global quotient stack $[X/G]$ (G a compact Lie group). Let $\pi : X \to [X/G]$ be the universal G bundle on $[X/G]$. Then for any Hilbert bundle $\mathscr{H}$ on $[X/G]$ the bundle $\pi^*\mathscr{H}$ is trivial, and there is a canonical injection $\mathscr{H} \to \pi_*\pi^*\mathscr{H}$, where π_* means the bundle of fiber wise L^2 sections. Thus π_* of the trivial Hilbert bundle on X is a universal bundle which is local.

Now the global automorphisms of this Hilbert bundle are G-equivariant maps from $X \to \mathsf{U}(H \otimes L^2(G))$, and the space of these maps is contractible ([**AS**] Proposition A3.1). Thus for a local quotient stack one can glue the local bundles and the result is unique up to isomorphism. Thus it gives a universal bundle on $\mathscr{M}$.

6.1. S^1-gerbes on local quotient stacks.

To see that any S^1 gerbe arises from a projective bundle one is tempted to use the cohomology sequence coming from the short exact sequence $1 \to S^1 \to \mathsf{U} \to \mathsf{PU} \to 1$. Unfortunately there is no nice definition of H^2 for non-abelian groups, therefore we need some preparations, to get canonical elements in $H^1(\mathscr{M}, \mathsf{PU})$.

First we need an absorption property for projective bundles, which I learned from [**AS**].

Lemma 6.7. *Let $\mathscr{M}$ be any topological stack.*

1. *The tensor product induces a map*

$$\otimes : H^1(\mathscr{M}, \mathsf{U}) \times H^1(\mathscr{M}, \mathsf{PU}) \to H^1(\mathscr{M}, \mathsf{PU}).$$

2. *The tensor product does not change the induced gerbe, i.e., denote by* ∂ *the boundary map* $\partial : H^1(\mathscr{M}, \mathsf{PU}) \to H^2(\mathscr{M}, \mathscr{S}^1)$*, then for any Hilbert bundle* H *and any projective bundle* P *on* $\mathscr{M}$ *we have* $\partial(H \otimes P) = \partial(P)$.

Proof. Any isomorphism $H \otimes H \cong H$ induces a group homomorphism $U \times \mathsf{PU} \to \mathsf{PU}$. This is well defined up to inner automorphisms of PU.

For the second part we only have to note that the choice of a Hilbert U structure on P also induces one on $H \otimes P$, and this is compatible with the S^1 action on U, thus the gerbes coming from the obstruction to such a lift are isomorphic. ☐

Definition 6.8. (Atiyah-Segal(3) [**AS**]) A projective Hilbert bundle P (i.e. a PU-Bundle) on a differentiable stack $\mathscr{M}$ has the *absorption property* if for any Hilbert bundle H on $\mathscr{M}$ there is an isomorphism $H \otimes P \cong P$.

We denote the set of isomorphism classes of projective bundles having the absorption property by $H^1(\mathscr{M}, \mathsf{PU})_{abs}$.

Remark 6.9. If H_{univ} is a universal Hilbert bundle on a stack $\mathscr{M}$ and P is any projective bundle, then $H_{\mathrm{univ}} \otimes P$ has the absorption property.

Lemma 6.10. *Let* $\mathscr{M}$ *be any differentiable stack. Then the map*

$$H^1(\mathscr{M}, \mathsf{PU})_{abs} \to H^2(\mathscr{M}, \mathscr{S}^1)$$

is injective.

Proof. Let P be a projective bundle, having the absorption property and let $\pi : \tilde{\mathscr{M}} \to \mathscr{M}$ be the S^1-gerbe of Hilbert bundle structures on P. Then $\pi^* P \cong \mathbb{P}(H)$ for some Hilbert bundle H on $\tilde{\mathscr{M}}$.

Aside on weights: Because S^1 is canonically contained in the automorphism group of any object of $\tilde{\mathscr{M}}$, it acts on the sections of any Hilbert bundle $\mathscr{H}$ on $\mathscr{M}$. Thus the canonical decomposition of the sheaf of sections of $\mathscr{H}$ induces a decomposition o $\mathscr{H} = \oplus_{i \in \mathbb{Z}} \mathscr{H}_i$, according to the characters of S^1, called weights. Bundles of weight 0 – i.e. bundles for which $\mathscr{H} = \mathscr{H}_0$ – are pull-backs of Hilbert bundles on $\mathscr{M}$. Bundles of weight 1 – i.e. $\mathscr{H} = \mathscr{H}_1$ – are exactly the bundles, which induce projective bundles on $\mathscr{M}$ whose associated gerbe is $\tilde{\mathscr{M}}$.

Thus in our situation $\mathscr{H}$ is a bundle of weight 1 and we want to show, that it has the absorption property for Hilbert bundles of weight 1 on $\tilde{\mathscr{M}}$. Let $\mathscr{H}'$ be an irreducible Hilbert bundle of weight one on $\tilde{\mathscr{M}}$. Then $\mathscr{H} \otimes \mathscr{H}'^{,*}$ has

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(3) In their article [**AS**] this property is called *regular*, we keep the terminology of [**FHT**]

weight 0, thus $\mathscr{H} \otimes \mathscr{H}'^* \cong \pi^*(\mathscr{H}_{\mathscr{M}})$. Since P has the absorption property, we know that $\mathscr{H} \cong \mathscr{H} \otimes \pi^*(\mathscr{H}''^*)\otimes$. Thus $\mathscr{H} \otimes \mathscr{H}\prime*$ has a non vanishing section (even countably many linear independent sections), which proves the absorption property.

By uniqueness of universal bundles this shows that $\mathscr{H}$ is determined by the gerbe. □

Remark 6.11. If there is a universal Hilbert bundle on $\mathscr{M}$, which is local, then the restriction to open substacks preserves the absorption property. And conversely it is then enough to check this property locally.

Proposition 6.12. *Every S^1-gerbe on a local quotient stack $\mathscr{M}$ comes from a projective bundle. Moreover, the natural map*

$$H^1(\mathscr{M}, \mathsf{PU})_{abs} \to H^2(\mathscr{M}, \mathscr{S}^1)$$

is an isomorphism.

Proof. Let $\tilde{\mathscr{M}}$ be an S^1-gerbe on $\mathscr{M}$. By Lemma 6.3 this is again a local quotient stack and therefore it has a universal Hilbert bundle $\tilde{\mathscr{H}}$. As in the previous lemma, we denote the direct summand of weight 1 of $\tilde{\mathscr{H}}$ by $\tilde{\mathscr{H}}_1$. This bundle is non-trivial, since it is locally the gerbe is defined by a group extension, thus locally the bundle is non trivial. Thus $\tilde{\mathscr{H}}_1$ defines a projective bundle on $\mathscr{M}$, which gives the gerbe. □

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