Free loop spaces in topology and physics

Kathryn Hess

Institute of Geometry, Algebra and Topology
Ecole Polytechnique Fédérale de Lausanne

Meeting of the Edinburgh Mathematical Society
Glasgow, 14 November 2008

The goal of this lecture

An overview of a few of the many important roles played by free loop spaces in topology and mathematical physics.

Outline

The functional definition

Let X be a topological space.

The space of free loops on X is

$$\mathcal{L}X = \mathrm{Map}(S^1, X).$$

If M is a smooth manifold, then we take into account the smooth structure and set

$$\mathcal{L}M = C^\infty(S^1, M).$$

The pull-back definition

Let X be a topological space. Let $\mathcal{P}X = \mathrm{Map}\big([0,1], X\big)$.

Let $q : \mathcal{P}X \to X \times X$ denote the fibration given by

$$q(\lambda) = \big(\lambda(0), \lambda(1)\big).$$

Then $\mathcal{L}X$ fits into a pull-back square

$$\begin{array}{ccc} \mathcal{L}X & \longrightarrow & \mathcal{P}X \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{\Delta} & X \times X, \end{array}$$

where $e(\lambda) = \lambda(1)$ for all free loops $\lambda : S^1 \to X$.

Note that the fiber of both e and q over a point x_0 is ΩX, the space of loops on X that are based in x_0.

Structure: the circle action

The free loop space $\mathcal{L}X$ admits an action of the circle group S^1, given by rotating the loops.

More precisely, there is an action map

$$\kappa : S^1 \times \mathcal{L}X \to \mathcal{L}X,$$

where

$$\kappa(z, \lambda) : S^1 \to X : z' \mapsto \lambda(z \cdot z').$$

Structure: the power maps

For any natural number r, the free loop space $\mathcal{L}X$ admits an r^{th}-power map

$$\ell_r : \mathcal{L}X \to \mathcal{L}X$$

given by

$$\ell_r(\lambda) : S^1 \to X : z \mapsto \lambda(z^r),$$

i.e., the loop $\ell_r(\lambda)$ goes r times around the same path as λ, moving r times as fast.

A related construction

Let U and V be subspaces of X.

The space of open strings in X starting in U and ending in V is

$$\mathcal{P}_{U,V}X = \Big\{\lambda : [0,1] \to X \mid \lambda(0) \in U, \lambda(1) \in V\Big\},$$

which fits into a pull-back diagram

$$\begin{array}{ccc} \mathcal{P}_{U,V}X & \longrightarrow & \mathcal{P}X \\ {\scriptstyle \bar{q}}\downarrow & & \downarrow{\scriptstyle q} \\ U \times V & \xrightarrow{(pr_U, pr_V)} & X \times X \end{array} .$$

Both the free loop space and the space of open strings are special cases of the homotopy coincidence space of a pair of maps $f : Y \to X$ and $g : Y \to X$.

The enumeration problem

Question

Let M be a closed, compact Riemannian manifold.
How many distinct closed geodesics lie on M?

Betti numbers and geodesics

For any space X and any field $\Bbbk$, let

$$b_n(X;\Bbbk) = \dim_{\Bbbk} H^n(X;\Bbbk).$$

Theorem (Gromoll & Meyer, 1969)

If there is field $\Bbbk$ such that $\{b_n(\mathcal{L}M;\Bbbk)\}_{n\geq 0}$ is unbounded, then M admits infinitely many distinct prime geodesics.

Proof by infinite-dimensional Morse-theoretic methods.

The rational case

Theorem (Sullivan & Vigué, 1975)

If

- *M is simply connected, and*
- *the graded commutative algebra $H^*(M;\mathbb{Q})$ is not monogenic,*

then $\{b_n(\mathcal{L}M;\mathbb{Q})\}_{n\geq 0}$ is unbounded, and therefore M admits infinitely many distinct prime geodesics.

Proof using the Sullivan models of rational homotopy theory.

The case of homogeneous spaces I

Theorem (McCleary & Ziller, 1987)

If M is a simply connected homogeneous space that is not diffeomorphic to a symmetric space of rank 1, then $\{b_n(\mathcal{L}M;\mathbb{F}_2)\}_{n\geq 0}$ is unbounded and therefore M admits infinitely many distinct prime geodesics.

Proof by explicit spectral sequence calculation, given the classification of such M.

The case of homogeneous spaces II

Remark

It is easy to show that if M is diffeomorphic to a symmetric space of rank 1, then $\left\{b_n(\mathcal{L}M;\mathbb{k})\right\}_{n\geq 0}$ is bounded for all $\mathbb{k}$, but

- Hingston proved that a simply connected manifold with the rational homotopy type of a symmetric space of rank 1 generically admits infinitely many closed geodesics, and
- Franks and Bangert showed that S^2 admits infinitely many geodesics, independently of the metric.

A suggestive result for based loop spaces

Theorem (McCleary, 1987)

If X is a simply connected, finite CW-complex such that $H^*(X; \mathbb{F}_p)$ is not monogenic, then $\{b_n(\Omega X; \mathbb{F}_p)\}_{n\geq 0}$ is unbounded.

Proof via an algebraic argument based on the Bockstein spectral sequence.

A conjecture and its consequences

Conjecture

If X is a simply connected, finite CW-complex such that $H^*(X;\mathbb{F}_p)$ is not monogenic, then $\{b_n(\mathcal{L}X;\mathbb{F}_p)\}_{n\geq 0}$ is unbounded.

Corollary

If there is a prime p such that $H^*(M;\mathbb{F}_p)$ is not monogenic, then M admits infinitely many distinct closed geodesics.

Proof strategy

(Joint work with J. Scott.)

Construct "small" algebraic model

$$\begin{array}{ccc} B & \longrightarrow & A \\ \simeq \downarrow & & \downarrow \simeq \\ C^*\mathcal{L}X & \longrightarrow & C^*\Omega X \end{array}$$

of the inclusion of the based loops into the free loops.

By careful analysis of McCleary's argument, show that representatives in A of the classes in $H^*(\Omega X, \mathbb{F}_p)$ giving rise to its unbounded Betti numbers lift to B, giving rise to unbounded Betti numbers for $\mathcal{L}X$.

Hochschild (co)homology of algebras

Let A be a (perhaps differential graded) associative algebra over a field $\Bbbk$.

The Hochschild homology of A is

$$HH_*A = \mathrm{Tor}_*^{A\otimes A^{op}}(A, A)$$

and the Hochschild cohomology of A is

$$HH^*A = \mathrm{Ext}^*_{A\otimes A^{op}}(A, A^\sharp),$$

where $A^\sharp = \mathrm{hom}_{\Bbbk}(A, \Bbbk)$.

If A is a (differential graded) Hopf algebra, then HH^*A is naturally a graded algebra.

HH and free loop spaces

Theorem (Burghelea & Fiedorowicz, Cohen, Goodwillie)

If X is a path-connected space, then there are $\Bbbk$-linear isomorphisms

$$HH_*\big(C_*(\Omega X;\Bbbk)\big) \cong H_*(\mathcal{L}X;\Bbbk)$$

and

$$HH^*\big(C_*(\Omega X;\Bbbk)\big) \cong H^*(\mathcal{L}X;\Bbbk).$$

Theorem (Menichi)

The isomorphism $HH^*\big(C_*(\Omega X;\Bbbk)\big) \cong H^*(\mathcal{L}X;\Bbbk)$ respects multiplicative structure.

Power maps: the commutative algebra case

Theorem (Loday, Vigué)

If A is a commutative (dg) algebra, then HH_*A admits a natural "r^{th}-power map" that is topologically meaningful in the following sense.

If A is the commutative dg algebra of rational piecewise-linear forms on a simplicial complex X, then there is an isomorphism

$$HH_{-*}A \cong H^*(\mathcal{L}X; \mathbb{Q})$$

that commutes with r^{th}-power maps.

Power maps: the cocommutative Hopf algebra case

Theorem (H.-Rognes)

If A is a cocommutative (dg) Hopf algebra, then HH_*A admits a natural "r^{th}-power map" that is topologically meaningful in the following sense.

Let K be a simplicial set that is a double suspension. If A is the cocommutative dg Hopf algebra of normalized chains on GK (the Kan loop group on K), then there is an isomorphism

$$HH_*A \cong H_*(\mathcal{L}|K|)$$

that commutes with r^{th}-power maps.

Cyclic homology of algebras

The cyclic homology of a (differential graded) algebra A, denoted HC_*A, is a graded vector space that fits into a long exact sequence (originally due to Connes)

$$\ldots \to HH_nA \xrightarrow{I} HC_nA \xrightarrow{S} HC_{n-2}A \xrightarrow{B} HH_{n-1}A \to \ldots.$$

HC and free loop spaces

For any G-space Y, where G is a topological group, let Y_{hG} denotes the homotopy orbit space of the G-action.

Theorem (Burghelea & Fiedorowicz, Jones)

For any path-connected space X, there is a $\Bbbk$-linear isomorphism

$$HC_*\big(C_*(\Omega X;\Bbbk)\big) \cong H_*\big((\mathcal{L}X)_{hS^1};\Bbbk\big).$$

Generalizations: ring spectra I

[Bökstedt, Bökstedt-Hsiang-Madsen]

Let R be an S-algebra (ring spectrum), e.g., the Eilenberg-MacLane spectrum $H\mathbb{Z}$ or $S[\Omega X]$, the suspension spectrum of ΩX, for any topological space X.

Topological Hochschild homology

$$THH(R)$$

and topological cyclic homology (mod p)

$$TC(R; p)$$

are important approximations to the algebraic K-theory of R.

Generalizations: ring spectra II

Let X be a topological space, and let $R = S[\Omega X]$.

Then $TC(R;p)$ can be constructed from

$$S[\mathcal{L}X] \quad \text{and} \quad S\big[(\mathcal{L}X)_{hS^1}\big],$$

using the p^{th}-power map $\ell_p : \mathcal{L}X \to \mathcal{L}X$.

Generalizations: (derived) schemes

[Weibel, Weibel-Geller]

Hochschild and cyclic homology can be generalized in a natural way to schemes, so that there is still a Connes-type long exact sequence relating them.

[Toën-Vezzosi]

Hochschild and cyclic homology can then be further generalized to derived schemes and turns out to be expressible in terms of a “free loop space” construction.

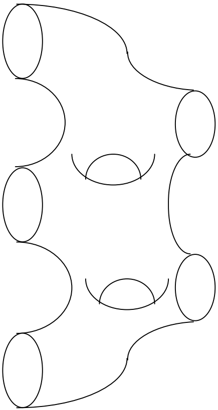
The closed cobordism categories **C** and **HC**

- The objects of **C** and of **HC** are all closed 1-manifolds (disjoint unions of circles), which are in bijective correspondance with $\mathbb{N}$.
- $\mathbf{C}(m,n) = C_*(\mathcal{M}_{m,n})$ and $\mathbf{HC}(m,n) = H_*(\mathcal{M}_{m,n})$, where $\mathcal{M}_{m,n}$ is the moduli space of Riemannian cobordisms from m to n circles.

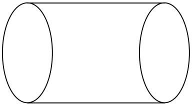
Both **C** and **HC** are monoidal categories, i.e., endowed with a "tensor product," which is given by disjoint union of circles (equivalently, by addition of natural numbers) and disjoint union of cobordisms.

Cobordisms as morphisms

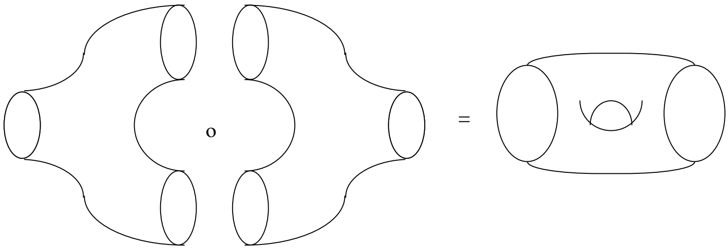
A 3-to-2 cobordism



A 1-to-1 cobordism



Composition of cobordisms



Topological CFT's

Let $\mathbb{k}$ be a field, and let $\mathbf{Ch}_{\mathbb{k}}$ denote the category of chain complexes of $\mathbb{k}$-vector spaces.

A closed TCFT is a linear functor $\Phi : \mathbf{C} \to \mathbf{Ch}_{\mathbb{k}}$ that is monoidal up to chain homotopy.

In particular, for all $n, m \in \mathbb{N}$,

- $\Phi(n)$ is a chain complex;
- there is a natural chain equivalence $\Phi(n) \otimes \Phi(m) \xrightarrow{\simeq} \Phi(n+m)$;
- there are chain maps $\mathbf{C}(m, n) \otimes \Phi(m) \to \Phi(n)$.

Homological CFT's

Let $\mathbf{grVect}_{\Bbbk}$ denote the category of graded $\Bbbk$-vector spaces.

A closed HCFT is a linear functor $\Psi : \mathbf{HC} \to \mathbf{grVect}_{\Bbbk}$ that is strongly monoidal.

In particular, for all $n, m \in \mathbb{N}$,

- $\Psi(n)$ is a graded vector space;
- there is a natural isomorphism $\Psi(n) \otimes \Psi(m) \xrightarrow{\cong} \Psi(n+m)$;
- there are graded linear maps $\mathbf{HC}(m, n) \otimes \Psi(m) \to \Psi(n)$.

If $\Phi : \mathbf{C} \to \mathbf{Ch}_{\Bbbk}$ is a closed TCFT, then $H_*\Phi$ is a closed HCFT

Folklore Theorem

If $\Psi : \mathbf{HC} \to \mathbf{grVect}_{\Bbbk}$ is a closed HCFT, then $\Psi(1)$ is a bicommutative Frobenius algebra, i.e., there exists

- a commutative, unital multiplication map

$$\mu : \Psi(1) \otimes \Psi(1) \to \Psi(1)$$

and

- a cocommutative, counital comultiplication map

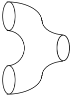
$$\delta : \Psi(1) \to \Psi(1) \otimes \Psi(1)$$

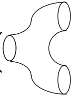
such that

$$(\mu \otimes 1)(1 \otimes \delta) = \delta\mu = (1 \otimes \mu)(\delta \otimes 1) : \Psi(1) \otimes \Psi(1) \to \Psi(1) \otimes \Psi(1).$$

The geometry of μ and δ

Let $\Psi : \mathbf{HC} \to \mathbf{grVect}_{\Bbbk}$ be a closed HCFT. Using the isomorphism $\Psi(1) \otimes \Psi(1) \cong \Psi(1+1)$, we get:

$\mu = \Psi($  $) : \Psi(1) \otimes \Psi(1) \longrightarrow \Psi(1)$

$\delta = \Psi($  $) : \Psi(1) \longrightarrow \Psi(1) \otimes \Psi(1)$

Generalizations

There are open-closed cobordism categories, in which the objects are all compact, 1-dimensional oriented manifolds (disjoint unions of circles and intervals). The notion of open-closed conformal field theories then generalizes in an obvious way that of closed CFT's.

Philosophy

String topology is the study of the (differential and algebraic) topological properties of the spaces of smooth paths and of smooth loops on a manifold, which are themselves infinite-dimensional manifolds.

The development of string topology is strongly driven by analogies with string theory in physics, which is a theory of quantum gravitation, where vibrating "strings" play the role of particles.

As we will see, string topology provides us with a family of HCFT's, one for for each manifold M.

Compact manifolds and intersection products

Let M be a smooth, orientable manifold of dimension n.

Let $\delta_M : H^{n-p}M \xrightarrow{\cong} H_pM$ denote the Poincaré duality isomorphism (the cap product with the fundamental class of M).

The intersection product on H_*M is given by the composite

$$\begin{array}{ccccc} H_pM \otimes H_qM & \xrightarrow{\delta_M^{-1}\otimes\delta_M^{-1}} & H^{n-p}M \otimes H^{n-q}M & \xrightarrow{\cup} & H^{2n-p-q}M \\ & \searrow^{\bullet} & & & \downarrow{\scriptstyle \delta_M} \\ & & & & H_{p+q-n}M \end{array}$$

and endows $\mathbb{H}_*M := H_{*+n}M$ with the structure of a Frobenius algebra.

The Chas-Sullivan product

Theorem (Chas & Sullivan, 1999)

Let M be a smooth, orientable manifold of dimension n. There is a commutative and associative "intersection" product

$$H_p\mathcal{L}M \otimes H_q\mathcal{L}M \to H_{p+q-n}\mathcal{L}M$$

that

- *endows* $\mathbb{H}_*\mathcal{L}M := H_{*+n}\mathcal{L}M$ *with the structure of a Frobenius algebra and*
- *is compatible with the intersection product on* H_*M*, i.e., the following diagram commutes.*

$$\begin{array}{ccc} H_p\mathcal{L}M \otimes H_q\mathcal{L}M & \longrightarrow & H_{p+q-n}\mathcal{L}M \\ \downarrow{\scriptstyle e_*\otimes e_*} & & \downarrow{\scriptstyle e_*} \\ H_pM \otimes H_qM & \longrightarrow & H_{p+q-n}M \end{array}$$

From string topology to HCFT's

Theorem (Godin, Cohen-Jones, Harrelson, Ramirez, Lurie)

For any closed, oriented manifold M, there is an HCFT

$$\Psi_M : \mathbf{HC} \to \mathbf{grVect}_{\Bbbk}$$

such that $\Psi_M(1) = \mathbb{H}_* \mathcal{L} M$.

"Algebraic" string topology and HCFT's

Theorem (Costello, Kontsevich-Soibelman)

If A is an A_∞-symmetric Frobenius algebra (e.g., if A is a bicommutative Frobenius algebra), then there is an HCFT

$$\Psi_A : \mathbf{HC} \to \mathbf{grVect}_{\Bbbk}$$

such that $\Psi_A(1) = HH_*A$.

Positive-energy representations

If G is a connected, compact Lie group, then $\mathcal{L}G$ is the loop group of G.

A projective representation

$$\varphi : \mathcal{L}G \to PU(\mathcal{H}),$$

where $\mathcal{H}$ is an infinite-dimensional Hilbert space, is of positive energy if there is a smooth homomorphism $u : S^1 \to PU(\mathcal{H})$ such that

$$\begin{array}{ccc} S^1 \times \mathcal{L}G & \xrightarrow{u\times\varphi} & U(H) \times PU(H) \\ \downarrow{\scriptstyle \kappa} & & \downarrow{\scriptstyle conj.} \\ \mathcal{L}G & \xrightarrow{\varphi} & PU(H) \end{array}$$

commutes, and

$$H = \bigoplus_{n\geq 0} H_n,$$

where $u(e^{i\theta})(x) = e^{in\theta} \cdot x$ for every $x \in H_n$.

The Verlinde ring

There is a "topological" equivalence relation on the set of projective, positive-energy representations of $\mathcal{L}G$.

Let $R^{\varphi}(G)$ denote the group completion of the monoid of projective, positive-energy representations that are equivalent to a given representation $\varphi : \mathcal{L}G \to PU(\mathcal{H})$.

Verlinde defined a commutative multiplication–the fusion product–on $R^{\varphi}(G)$, giving it the structure of a commutative ring.

In fact, $R^{\varphi}(G) \otimes \mathbb{C}$ is a Frobenius algebra, and there is an HCFT

$$\Psi_{\varphi} : \mathbf{HC} \to \mathbf{grVect}_{\Bbbk}$$

such that $\Psi_{\varphi}(1) = R^{\varphi}(G) \otimes \mathbb{C}$.

The topology behind the algebra

Theorem (Freed-Hopkins-Teleman)

Let G and $\varphi : \mathcal{L}G \to PU(\mathcal{H})$ be as above.

There is a ring isomorphism from $R^{\varphi}(G)$ to a twisted version of the equivariant K-theory of G acting on itself by conjugation.

Free loop spaces and twisted K-theory

Let

$$\mathcal{P}G = \{f \in C^\infty(\mathbb{R}, G) \mid \exists x \in G \text{ s.t. } f(\theta + 2\pi) = x \cdot f(\theta)\ \forall \theta \in \mathbb{R}\}.$$

Consider the principal $\mathcal{L}G$-fibre bundle

$$p : \mathcal{P}G \to G : f \mapsto f(2\pi)f(0)^{-1},$$

where $\mathcal{L}G$ acts freely on $\mathcal{P}G$ by right composition.

Together, φ and p give rise to a twisted Hilbert bundle

$$\mathcal{P}G \underset{\mathcal{L}G}{\times} \mathbb{P}(\mathcal{H}) \to G,$$

where $\mathbb{P}(\mathcal{H}) = \mathcal{H}/S^1$. The twisted equivariant K-theory of G is given in terms of sections of this bundle.