Notes on Enrichment

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Abstract

We write down the definition of enriched bicategory. Nothing here is new.

Contents

Enriched category theory is attractive for many reasons. See Kelly [3]. Category theory permits simple definitions of algebraic concepts such as monoids and groups. Enrichment then permits further definitions such as that of algebras over a field. In some respects, these are just fun toy examples, but these sorts of examples supply enriched category theory with remarkable flexibility. We recall a definition of *enriched bicategory*, which has been written down by a number of people sometimes with some variation. It seems this definition first appeared in the thesis of Carmodey [1]. Lack also gave a definition in his thesis over strict monoidal bicategories [4]. Forcey has studied the combinatorics of polytopes associated to enrichment and higher categories in detail. See [2], for example.

Enriched categories are defined over monoidal categories. Why is this? Very simply, the monoidal structure consists, in part, of a functor $\otimes\colon \mathcal{V} \times \mathcal{V} \to \mathcal{V}$, which is used to define composition in the enriched category. In other words, for objects a, b, c of a $\mathcal{V}$-enriched category, there is a composition map in $\mathcal{V}$ given by

$$c_{abc}\colon \hom(a,b) \otimes \hom(b,c) \to \hom(a,c).$$

To draw attention to the idea that enriched bicategories are a weakening of enriched categories, we first remind ourselves of the definition enriched categories. Starting with a monoidal category $\mathcal{V}$, a $\mathcal{V}$-enriched category $\mathcal{C}$ consists of a set of objects $a, b, c, \ldots$, and for each pair of objects a, b, an object $\hom(a,b)$ of $\mathcal{V}$. Further, the structure maps of $\mathcal{C}$, which we will detail below are morphisms in $\mathcal{V}$.

It is useful to note that an enriched category is not necessarily a category, and an enriched bicategory is not necessarily a bicategory. However, there are certain examples of monoidal categories and bicategories for which the resulting enriched structures should be very familiar. We should add a section containing some examples at the end.

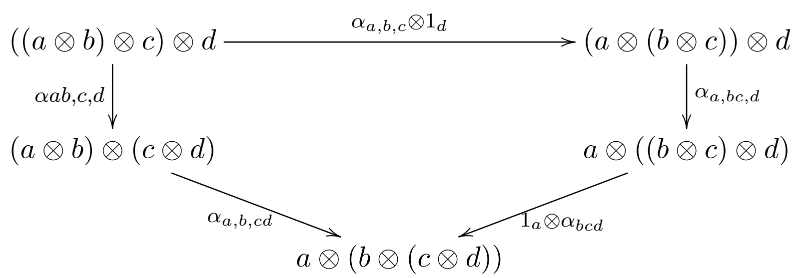
1 Enriched Categories

A monoidal category $\mathcal{V}$ consists of:

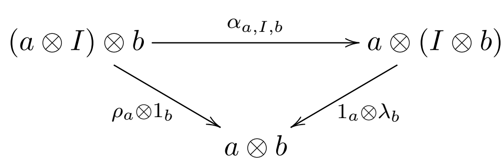
- a functor

$$\otimes\colon V \times V \to V,$$

  called the monoidal product,
- an object I called the monoidal unit, and
- natural isomorphisms α, λ, ρ satisfying, for a, b, c, d in $\mathcal{V}$, the coherence conditions described the commutativity of the following diagrams:



and



Given this data we can define a $\mathcal{V}$-category, also known as a category enriched over $\mathcal{V}$.

Definition 1.1. *A* **$\mathcal{V}$-category** $\mathcal{C}$ *consists of:*

- *a set* $\mathsf{Ob}(\mathcal{C})$ *of objects* $a, b, c, \ldots$*;*
- *for each pair of objects* a, b*, a* **hom-object** $\hom(a, b) \in \mathcal{V}$*, which we will often denote* (a, b)*;*
- *a morphism called* **composition**

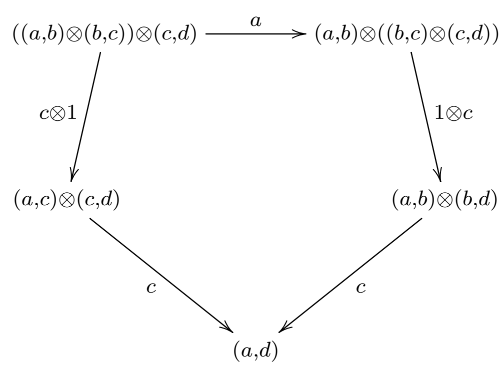
$$c = c_{abc} \colon \hom(a, b) \otimes \hom(b, c) \to \hom(a, c)$$

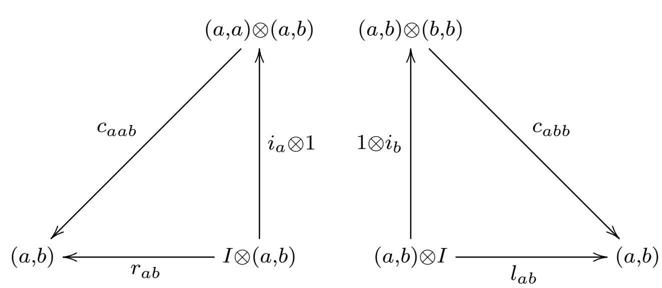
  for each triple of objects $a, b, c \in \mathcal{C}$*;*
- *an* **identity-assigning** *morphism*

$$i_a \colon I \to \hom(a, a)$$

  for each object $a \in \mathcal{C}$*;*

all satisfying the axioms:

- 

  for each quadruple of objects $a, b, c, d \in \mathcal{B}$*;*
- 

  for each pair of objects $a, b \in \mathcal{B}$*.*

2 Enriched Bicategories

Now, we write the definition of a 'category enriched over a monoidal bicategory'. We choose to call these 'enriched bicategories', but alternatively might call them 'weakly enriched categories'.

Definition 2.1. *Let $\mathcal{V}$ be a monoidal bicategory. A* $\mathcal{V}$**-bicategory** $\mathcal{B}$ *consists of:*

- *a set* $\mathsf{Ob}\mathcal{B}$ *of objects* $a, b, c, \ldots$*;*
- *for every pair of objects* a, b*, a* **hom-object** $\hom(a, b) \in \mathcal{V}$*, which we denote* (a, b) *suppressing the tensor product when necessary;*
- *a morphism called* **composition**

$$c = c_{abc} \colon \hom(a, b) \otimes \hom(b, c) \to \hom(a, c)$$

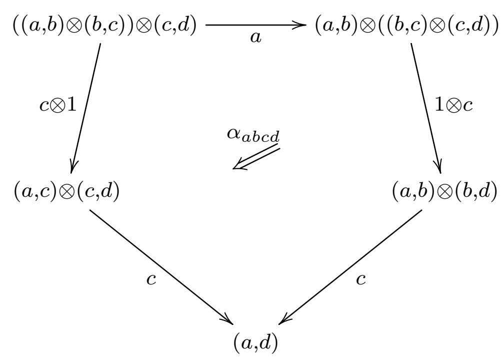
for each triple f objects $a, b, c \in \mathcal{B}$*;*

- *an* **identity-assigning** *morphism*

$$i_a \colon I \to \hom(a, a)$$

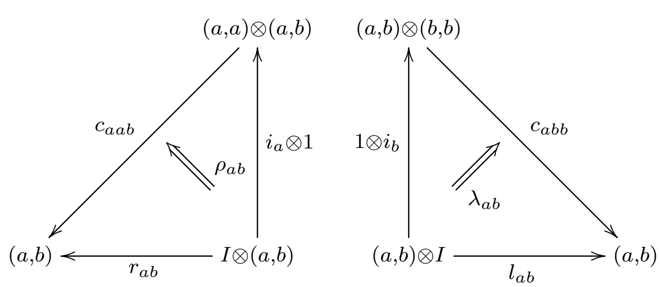
for each object $a \in \mathcal{B}$*;*

- *an invertible* 2*-morphism called the* **associator**



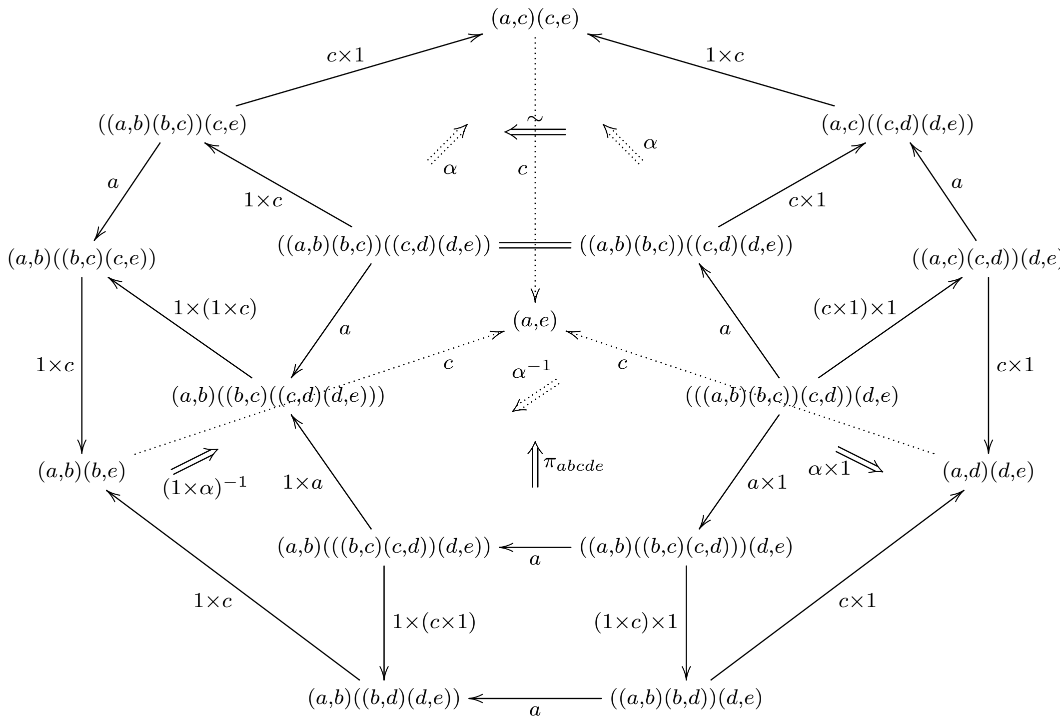
for each quadruple of objects $a, b, c, d \in \mathcal{B}$*;*

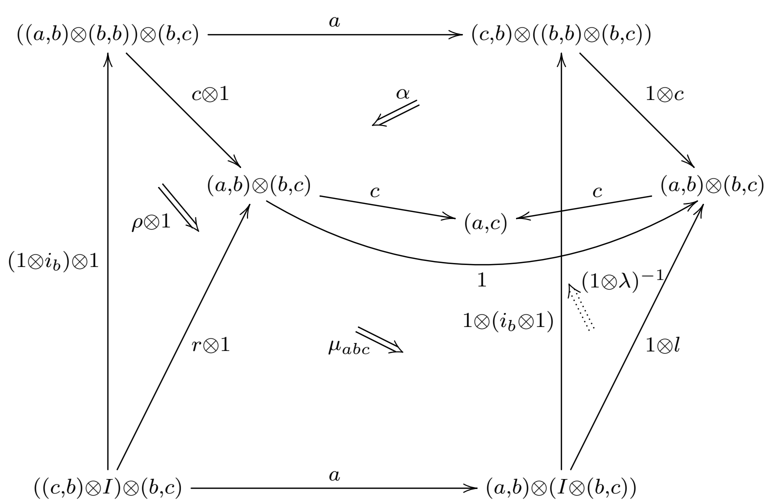
- *and invertible* 2*-morphisms called the* **right unitor** *and* **left unitor**



for every pair of objects $a, b \in \mathcal{B}$*;*

- *satisfying the following axioms*





the arrow marked $\sim$ is a structure cell for $\mathcal{V}$.

2.1 $\mathcal{V}$-Homormophisms

We present a 'weak' notion of $\mathcal{V}$-homomorphism.

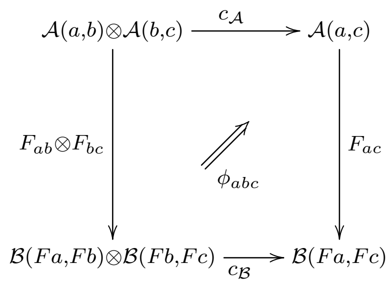
Definition 2.2. *Let $\mathcal{V}$ be a monoidal bicategory and $\mathcal{A}$ and $\mathcal{B}$ be $\mathcal{V}$-bicategories. A* **$\mathcal{V}$-homomorphism**, *or* **pseudo enriched homomorphism**, *$F\colon \mathcal{A} \to \mathcal{B}$ consists of:*

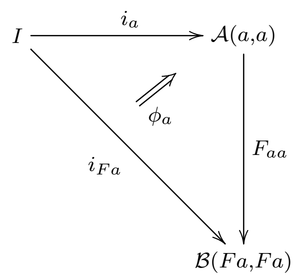
- *a function $F\colon \mathsf{Ob}\mathcal{A} \to \mathsf{Ob}\mathcal{B}$,*

- *for each pair of objects* $a, b \in \mathsf{Ob}\mathcal{A}$, *a* 1*-morphism*

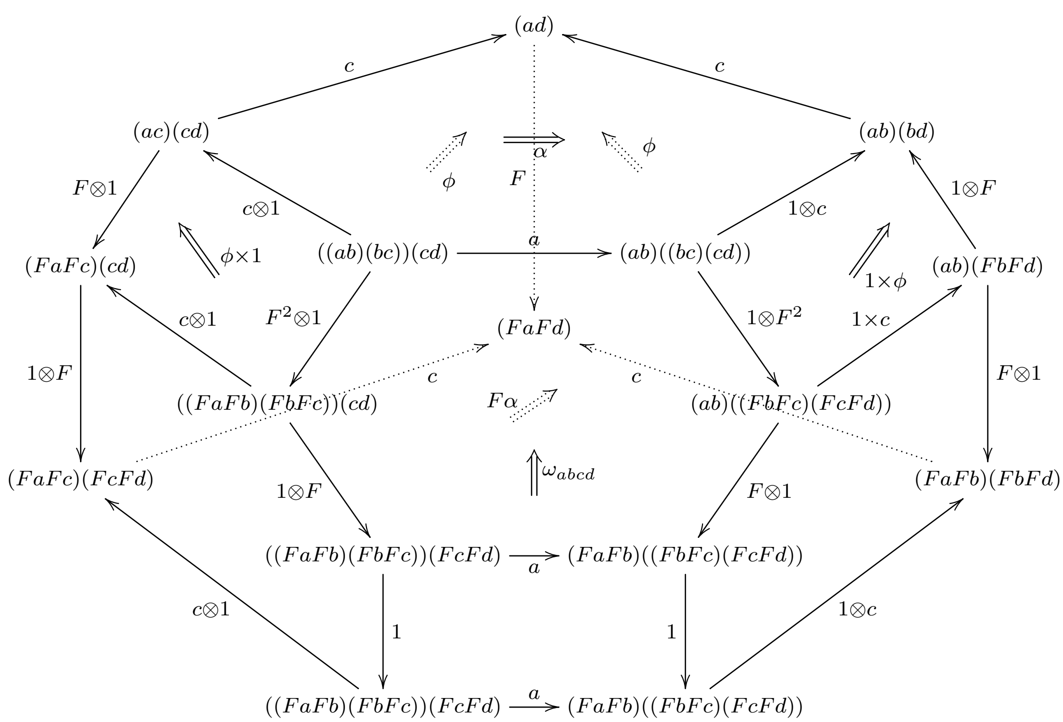
$$F_{ab} \colon \mathcal{A}(a,b) \to \mathcal{B}(Fa, Fb)$$

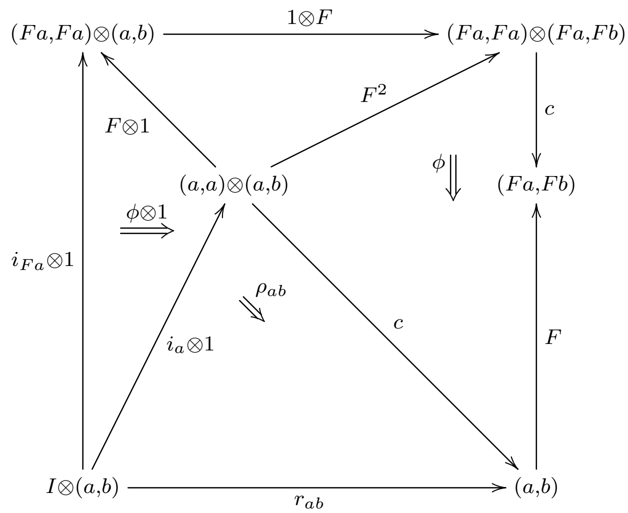
  in $\mathcal{V}$,
- *for each triple of objects* $a, b, c \in \mathsf{Ob}\mathcal{A}$, *a pair of* 2*-morphisms:*





  in $\mathcal{V}$,
- *satisfying the following axioms*





References

[1] S. M. Carmody, Cobordism Categories, PhD thesis, University of Cambridge, 1995.

[2] S. Forcey, Quotients of the Multiplihedron as Categorified Associahedra, Available as arXiv:0803.2694.

[3] G. M. Kelly, *Basic Concepts of Enriched Category Theory*, Cambridge University Press, Cambridge, 1982.

[4] S. G. Lack,The algebra of distributive and extensive categories, PhD thesis, University of Cambridge, 1995.