Spin Cobordism Determines Real K-Theory

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1 Introduction

It is a classic theorem of Conner and Floyd [CF] that complex bordism determines complex K-homology. That is, the map

$$\mathrm{MU}_*(X) \otimes_{\mathrm{MU}_*} \mathrm{K}_* \to \mathrm{K}_*(X)$$

induced by the Todd genus $\mathrm{MU} \to \mathrm{K}$ is an isomorphism of K_* modules for all spectra X. See [St] for descriptions of all the bordism theories used in this paper. The Conner-Floyd theorem was later generalized by Landweber in his exact functor theorem [Lan]. Conner and Floyd also prove symplectic bordism determines real K-theory:

$$\mathrm{MSp}_*(X) \otimes_{\mathrm{MSp}_*} \mathrm{KO}_* \to \mathrm{KO}_*(X)$$

is an isomorphism of KO_* modules for all X.

The maps $\mathrm{MU} \to \mathrm{K}$ and $\mathrm{MSp} \to \mathrm{KO}$ used by Conner and Floyd were extended to maps $\mathrm{MSpin}^c \to \mathrm{K}$ and $\mathrm{MSpin} \to \mathrm{KO}$ by Atiyah, Bott, and Shapiro in [ABS]. Recall that $\mathrm{Spin}(n)$ is the 1-connected cover of $\mathrm{SO}(n)$ and $\mathrm{Spin}^c(n)$ is another Lie group derived from $\mathrm{Spin}(n)$ and S^1. The relevant property for our purpose is that a bundle is orientable with respect to KO (resp. K)-theory if and only if it is a Spin (resp. Spin^c) bundle. Thus this is the natural place to look for an isomorphism of Conner-Floyd type.

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We should note, however, that Ochanine [Och] has proven that

$$\mathrm{MSU}_*(X) \otimes_{\mathrm{MSU}_*} \mathrm{KO}_* \to \mathrm{KO}_*(X)$$

is not an isomorphism. The problem is that the lefthand side is not a homology theory. There is no known exact functor theorem for any other bordism theory except for those derived from MU.

The purpose of the present paper is to prove

Theorem 1 *The maps*

$$MSpin_*(X) \otimes_{MSpin_*} KO_* \to KO_*(X)$$

and

$$MSpin^c{}_*(X) \otimes_{MSpin^c{}_*} K_* \to K_*(X)$$

induced by the Atiyah- Bott-Shapiro orientations are natural isomorphisms of KO_* (resp. K_*)-modules for all spectra X.

Spanier-Whitehead duality then shows that Theorem 1 is also true in cohomology, if we assume X is finite.

The real case of Theorem 1 fits in with the general philosophy that at the prime 2, one should use covers of the orthogonal group to replace the unitary group: i.e., MSO, MSpin, MO$< 8 >$, etc. instead of MU. At odd primes, KO and K are essentially equivalent, as are MSO and MU. But at the prime 2, KO is a more subtle theory than K, and MU cannot detect this, though Theorem 1 says MSpin can.

There is another cohomology theory exciting much current interest: elliptic cohomology [Lan2]. This theory is v_2-periodic, unlike KO which is v_1-periodic. (See [Rav] for a discussion of v_n-periodic cohomology theories.) However, elliptic cohomology is currently only defined after inverting 2. One then applies the Landweber exact functor theorem to get elliptic cohomology from MU. One might hope, in line with the philosophy above, that one could define elliptic cohomology at 2 by using a cover of the orthogonal group, most likely MO$< 8 >$. This idea is due to Ochanine [Och2]. It has been carried out by Kreck and Stolz in [KS], but using MSpin. The resulting theory is not v_2-periodic at 2 however. Unfortunately, there is no known analog for MO$< 8 >$ of the Anderson-Brown-Peterson splitting (see below) for MSpin which is crucial in our proof of Theorem 1.

Our proof of Theorem 1 is very algebraic in nature. A hint that Theorem 1 might be true is provided by a result of Baum and Douglas [BD]. They give a geometric definition of K (resp. KO)-homology using MSpin^c (resp. Spin) manifolds. One might hope that their work could lead to a more geometric proof of Theorem 1, but we are currently unable to construct such a proof.

Instead, we prove Theorem 1 by proving it after localizing at each prime. Essentially all of the work is at $p = 2$, where we have available the splitting of MSpin and MSpin^c due to Anderson, Brown, and Peterson [ABP]. We recall this splitting in section 2. In section 3, we prove that Theorem 1 is equivalent to showing that $L_1\mathrm{MSpin}$ (resp. $L_1\mathrm{MSpin}^c$) determines KO (resp. K). Here L_1 is K-theory localization. See [Rav] or [B] for a discussion of L_1. Section 4 contains a lemma, which is at the heart of our proof, giving conditions under which a ring spectrum R determines a module spectrum M, so that

$$\mathrm{R}_*(X) \otimes_{\mathrm{R}_*} \mathrm{M}_* \to \mathrm{M}_*(X)$$

is a natural isomorphism for all X. The essential idea is to resolve M as an R-module spectrum. We apply this lemma in section 5, completing the proof in the complex case. In the real case, however, the lemma only tells us that $\mathrm{MSpin} \wedge M(2^n)$ determines $\mathrm{KO} \wedge M(2^n)$, where $M(2^n)$ is the mod 2^n Moore spectrum and $n > 1$. Section 6 is devoted to showing that this is actually enough to prove Theorem 1. Finally, the last section discusses odd primes. In the real case, Theorem 1 is obvious at odd primes. The complex case is a little more difficult, but the classical method of Conner and Floyd works.

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2 The Anderson-Brown-Peterson Splitting

Throughout this section, all spectra will be localized at 2 unless otherwise stated. We recall the work of Anderson, Brown, and Peterson [ABP] on MSpin and MSpin^c. An exposition of their work containing the details of the MSpin^c case can be found in [St]. In addition, an intensive study of the homological aspects of the Anderson-Brown-Peterson splitting has been carried out by Giambalvo and Pengelley [GP].

Let ko (resp. ku) denote the connective cover of the real (resp. complex) K-theory spectrum KO (resp. K). Also, let $ko\langle 2\rangle$ denote the 2-connective cover of ko. Let $H\mathbf{Z}/2\mathbf{Z}$ denote the $\mathbf{Z}/2\mathbf{Z}$-Eilenberg-MacLane spectrum. Anderson, Brown, and Peterson construct an additive splitting of MSpin (resp.

MSpin^c) into a wedge of suspensions of ko, $ko{<}2{>}$, and H$\mathbf{Z}/2\mathbf{Z}$ (resp. ku and H$\mathbf{Z}/2\mathbf{Z}$).

We will describe their results in the real case and later indicate the simplifications that occur in the complex case. Let J be a partition: that is, a possibly empty multiset of positive integers, such as $\{1,1,2\}$. Let $n(J) = \sum_{j \in J} j$, so that $n(\emptyset) = 0$ and $n(\{1,1,2\}) = 4$. Anderson, Brown, and Peterson construct maps π^J : MSpin $\rightarrow$ KO. If $J = \{n\}$ we denote π^J by π^n, and we denote $\pi^\emptyset$, which is the Atiyah-Bott-Shapiro orientation, by π^0.

These maps interact with the multiplication on MSpin and KO by an analogue of the Cartan formula for the Steenrod squares. One way to describe this is as follows. Let P be the set of partitions, and consider the set of formal linear combinations $\mathbf{Z}[P]$. We can make this into a ring by defining a multiplication on the set of partitions by set union, and then into a Hopf algebra by defining $\Delta(\{n\}) = \sum_{k=0}^{n} \{n-k\} \otimes \{k\}$. Suppose $\Delta(J) = \sum J' \otimes J''$. Let

$$\mu : \mathrm{MSpin} \wedge \mathrm{MSpin} \rightarrow \mathrm{MSpin} \quad \text{and} \quad \mu' : \mathrm{KO} \wedge \mathrm{KO} \rightarrow \mathrm{KO}$$

denote the ring spectrum multiplications. Then the Cartan formula says that $\pi^J \mu = \sum \mu'(\pi^{J'} \wedge \pi^{J''})$.

Theorem 2 ([ABP]) *1. Suppose that $1 \notin J$. Then if $n(J)$ is even, π^J lifts to the $4n(J)$-fold connective cover of KO, $\Sigma^{4n(J)} ko$. If $n(J)$ is odd, π^J lifts to the $4n(J)-2$-fold connective cover of KO, $\Sigma^{4n(J)-4} ko{<}2{>}$.*

2. There exist a countable collection $z_k \in H^*(MSpin; \mathbf{Z}/2\mathbf{Z})$ such that

$$\prod_{1 \notin J} \pi^J \times \prod_k z_k : MSpin \;\rightarrow \prod_{1 \notin J, n(J)\, even} \Sigma^{4n(J)} ko$$

$$\times \prod_{1 \notin J, n(J)\, odd} \Sigma^{4n(J)-4} ko{<}2{>} \times \prod_k \Sigma^{deg\ z_k} H\mathbf{Z}/2\mathbf{Z}$$

is a 2-local homotopy equivalence.

Note that, though we have used the product symbol above, the product and the coproduct, i.e. the wedge, are the same in this case. Note as well that we use the same notation for π^J and the lift of π^J to a connective cover of KO. Let us denote the left inverse of π^J arising from the above theorem by ρ^J. Note that ρ^J is only defined if $1 \notin J$, and if $1 \notin I$ we have $\pi^I \rho^J = \delta_{I,J} \cdot 1$. The fact that the π^J with $1 \in J$ do not appear in this splitting, whereas they do appear in the Cartan formula, makes the real case more difficult than the complex case.

There is a similar splitting in the complex case, except partitions containing 1 are included, and every π^J lifts to the $4n(J)$-fold connnective cover of K, $\Sigma^{4n(J)}ku$. The analogue of the Cartan formula remains true.

Note that the above splitting is not a ring spectrum splitting, nor is it even a ko-module spectrum splitting, as a low-dimensional calculation verifies. Mahowald has asked if the above splitting can be modified to make it into a ko-module spectrum splitting [Mah]. If so, the proof of Theorem 1 could be made much simpler. However, Stolz has recently proven that there is in fact no way to do this ([KS]). Perhaps this helps explain why our proof of Theorem 1 is so complicated.

3 K-Theory Localization

In this section we study the K-theory localizations of MSpin and MSpin^c. Let $p \in \pi_8 ko$ be the periodicity element, and let $v = \rho^0 p \in \pi_8\mathrm{MSpin}$. We also use v for the complex analogue in $\pi_2\mathrm{MSpin}^c$. Our main objective in this section is to show that $v^{-1}\mathrm{MSpin}$ and $v^{-1}\mathrm{MSpin}^c$ are K-local, which will imply that it suffices to prove Theorem 1 after applying L_1 . We will stick to the real case, leaving the obvious modifications needed in the complex case to the reader.

To prove this, we will need to understand multiplication by v. The following two lemmas provide some of this understanding.

Lemma 1 *Let J be a partition. Then all $\pi^J(v) = 0$ except for $\pi^0(v) = p$ and possibly $\pi^1(v)$ and $\pi^{1,1}(v)$. These latter two are both even as elements of the appropriate homotopy group. Further, v maps to 0 under the forgetful homomorphism to MO.*

Proof: If $1 \notin J$ and $J \neq \emptyset$, the splitting shows $\pi^J(v) = 0$. Stong shows in [St] that, if $1 \in J$, π^J lifts to the $4n(J)$-connective cover of KO. Thus, for dimensional reasons, the composition

$$S^0 \to ko \xrightarrow{\rho^0} \mathrm{MSpin} \xrightarrow{\pi^J} KO$$

where the first map is the unit, is null for $J \neq \emptyset$. Recall that p denotes the periodicity element, and denote the image of the Hopf map η by α. The image of the unit on positive dimensional homotopy groups is $\{p^n\alpha, p^n\alpha^2 | n \geq 0\}$. Indeed, it is clear that the image must be contained in this set, and suitable elements from the image of J hit these elements. So we have $\pi^J\rho^0(p^n\alpha) = 0$ for all $J \neq \emptyset$. But α is the image of η under the unit, so $\eta \cdot (\pi^J\rho^0 p^n) = 0$. This can only happen if $\pi^J\rho^0 p^n$ is even for all $J \neq \emptyset$.

In particular, $\pi^J v$ is even for all $J \neq \emptyset$. Anderson, Brown, and Peterson [ABP] show that $\mathrm{ch}(\pi^J(x) \otimes \mathbf{C}) = p_J x +$ (higher terms), for $x \in \mathrm{MSpin}^*(X)$. Here $p_J x$ is the Pontrjagin class of x corresponding to J. Thus the $p_J v$ are also even for $J \neq \emptyset$. Since p_2 and $p_{1,1} = p_1^2$ determine oriented cobordism in dimension 8, v goes to an even element in MSO_8, and thus goes to zero in MO_8. QED

Our plan for proving that $v^{-1}\mathrm{MSpin}$ is K-local is based on Bousfield's identification of K-local spectra [B]. Let $A : \Sigma^8 M(2) \to M(2)$ be the Adams map on the mod 2 Moore spectrum. We will show that

$$1 \wedge A : \mathrm{MSpin} \wedge \Sigma^8 M(2) \to \mathrm{MSpin} \wedge M(2)$$

becomes a homotopy equivalence after inverting v. To do this we need the following lemma.

Lemma 2 *Let $I \subset \pi_*(MSpin \wedge M(2))$ consist of those α with $(\pi^J \wedge 1)\alpha = 0$ for all J with $1 \notin J$. Then I is precisely the sub-$MSpin_*$-module of v-torsion elements.*

Proof: First note that if $(\pi^J \wedge 1)\alpha = 0$ for all J with $1 \notin J$, then in fact $(\pi^J \wedge 1)\alpha = 0$ for all J. This follows from the Anderson-Brown-Peterson splitting. Indeed, such an α must factor through a wedge of $\mathrm{H}\mathbf{Z}/2\mathbf{Z}$'s. Since KO $\wedge M(2)$ is K-local, any map from $\mathrm{H}\mathbf{Z}/2\mathbf{Z}$ to KO $\wedge M(2)$ factors through $L_1\mathrm{H}\mathbf{Z}/2\mathbf{Z}$ which is contractible by the results of [AH]. This proves that $(\pi^J \wedge 1)\alpha = 0$ before lifting π^J to a cover of KO. But the map from a cover of KO smashed with $M(2)$ to KO smashed with $M(2)$ is injective on homotopy, so in fact $(\pi^J \wedge 1)\alpha = 0$ after lifting the π^J as well.

Now suppose $\alpha \notin I$. Find a partition J' with $1 \notin J'$ such that the power of 2 dividing $(\pi^{J'} \wedge 1)\alpha$ is minimal (among J with $1 \notin J$.) Then, using the splitting, we can write

$$\alpha = 2^n \sum_{1 \notin J} (\rho^J \wedge 1)(\alpha_J) + \beta$$

where $\alpha_{J'}$ is not divisible by 2 and $\beta \in I$. It is then easy to see using the previous lemma and the Cartan formula that

$$(\pi^{J'} \wedge 1)(v\alpha) \equiv 2^n p \alpha_{J'} \bmod 2^{n+1}.$$

Continuing in this way, we see that $v^i\alpha$ is never 0.

Conversely, suppose $\alpha \in I$. Consider the exact sequence of MSpin_* modules arising from the defining cofibration for $M(2)$:

$$\mathrm{MSpin}_* \xrightarrow{f} \pi_*(\mathrm{MSpin} \wedge M(2)) \xrightarrow{g} \mathrm{MSpin}_{*-1}.$$

These maps commute with the π^J so $\pi^J(g\alpha) = 0$ for all J. Thus $g\alpha$ is in the analog of I for MSpin. Anderson, Brown, and Peterson call this set I_*; i.e. I_* is the subset of MSpin_* consisting of those classes γ with $\pi^J\gamma = 0$ for all J. Now I_* is an ideal mapped monomorphically to MO_* [ABP], and v maps to 0 there, so $g(v\alpha) = vg(\alpha) = 0$. Hence there is a β with $f(\beta) = v\alpha$. Then $f(\pi^J\beta) = (\pi^J \wedge 1)(v\alpha) = 0$ for all J. Since the kernal of f consists of the even elements, we must have $\pi^J\beta$ being even for all J. Thus $\beta = 2\gamma + \delta$ where $\delta \in I_*$. Thus $v^2\alpha = f(v\beta) = f(2v\gamma) = 0$. *QED*

Now we can prove

Theorem 3 *$v^{-1}MSpin$ and $v^{-1}MSpin^c$ are K-local.*

Proof: According to Bousfield [B], it suffices to show that

$$1 \wedge A : \pi_*(v^{-1}\mathrm{MSpin} \wedge M(2)) \to \pi_{*+8}(v^{-1}\mathrm{MSpin} \wedge M(2))$$

is an isomorphism. Since $\pi_*(v^{-1}\mathrm{MSpin} \wedge M(2)) = v^{-1}\pi_*(\mathrm{MSpin} \wedge M(2))$, we must prove the following two facts for $\alpha \in \pi_*(\mathrm{MSpin} \wedge M(2))$.

1. If $(1 \wedge A)\alpha$ is v-torsion, so is α.
2. There is a β and an n such that $(1 \wedge A)\beta - v^n\alpha$ is v-torsion.

Now $1 \wedge A$ respects the Anderson-Brown-Peterson splitting, and on the homotopy of a cover of KO smashed with $M(2)$, $1 \wedge A$ is multiplication by p. Thus we have

$$(1 \wedge A)(\rho^J \wedge 1)x = (\rho^J \wedge 1)(1 \wedge A)x = (\rho^J \wedge 1)(px).$$

Using the splitting, write

$$\alpha \equiv \sum_{1 \notin J} (\rho^J \wedge 1)\alpha_J (\mathrm{mod}\ I).$$

Then

$$(1 \wedge A)\alpha \equiv \sum_{1 \notin J} (\rho^J \wedge 1) p \alpha_J (\text{mod } I).$$

By the preceding lemma, this cannot be v-torsion unless each $p\alpha_J = 0$. But multiplication by p is injective, so each $\alpha_J = 0$ and $\alpha \in I$, so α is v-torsion as well.

We now prove the second fact above. Again, write

$$\alpha \equiv \sum_J (\rho^J \wedge 1)\alpha_J (\text{mod } I).$$

Consider the exact sequence of MSpin_* modules

$$0 \to \text{MSpin}_* \otimes \mathbf{Z}/2\mathbf{Z} \xrightarrow{1\wedge f} \pi_*(\text{MSpin} \wedge M(2)) \xrightarrow{1\wedge g} \text{Tor}(\text{MSpin}_{*-1}) \to 0.$$

This sequence arises from the defining cofiber sequence for $M(2)$, so f and g come from maps of spectra, which we also denote f and g. Note that $(1 \wedge f)(I_*) \subseteq I$ and $(1 \wedge g)(I) \subseteq I_*$. Thus

$$(1 \wedge g)(\alpha) \equiv \sum_J (\rho^J \wedge 1)(1 \wedge g)\alpha_J (\text{mod } I_*).$$

Since we are in a $\mathbf{Z}/2\mathbf{Z}$ vector space, Lemma 1 and the Cartan formula imply that

$$(1 \wedge g)(v\alpha) \equiv \sum_J (\rho^J \wedge 1)(1 \wedge g)p\alpha_J (\text{mod } I_*).$$

This is equivalent (mod I_*) to $(1 \wedge g)(1 \wedge A)\sum_J (\rho^J \wedge 1)\alpha_J$. We saw in the proof of Lemma 2 that v annihilates I_*. Thus

$$(1 \wedge g)(v^2\alpha) = (1 \wedge g)(1 \wedge A)v \sum_J (\rho^J \wedge 1)\alpha_J.$$

If we let $\alpha' = \sum_J (\rho^J \wedge 1)\alpha_J$, we find that there is a β such that

$$(1 \wedge f)\beta = v^2\alpha - (1 \wedge A)v\alpha'.$$

Write

$$\beta \equiv \sum_J (\rho^J \wedge 1)\beta_J (\text{mod } I_*).$$

Again, we are working mod 2, so

$$(1 \wedge f)(v\beta) \equiv \sum_J (\rho^J \wedge 1)(1 \wedge f)p\beta_J \equiv (1 \wedge A)(1 \wedge f)\beta (\text{mod } I_*).$$

Thus $(1 \wedge f)(v\beta) - (1 \wedge A)(1 \wedge f)\beta$ is v-torsion. Hence we get that

$$v^3\alpha - (1 \wedge A)(v^2\alpha' + (1 \wedge f)\beta$$

is v-torsion, completing the proof. *QED*

Note that if the Anderson-Brown-Peterson splitting could be made into a ko-module spectrum splitting (which it can not be [KS]), Theorem 3 would be obvious since then we would have $v^{-1}\mathrm{MSpin} \simeq \bigvee_{1 \notin J} \mathrm{KO}$. It can be shown using [St] that the homotopy groups of $v^{-1}\mathrm{MSpin}$ are what they should be for such an equivalence to hold.

Corollary 1 *For any ring spectrum R,*

$$(MSpin \wedge R)_*(X) \otimes_{(MSpin \wedge R)_*} (KO \wedge R)_* =$$

$$(L_1 MSpin \wedge R)_*(X) \otimes_{(L_1 MSpin \wedge R)_*} (KO \wedge R)_*.$$

Also

$$MSpin^c{}_*(X) \otimes_{MSpin^c{}_*} K_* = L_1 MSpin^c{}_*(X) \otimes_{L_1 MSpin^c{}_*} K_*.$$

Proof: Since p is a unit in $(\mathrm{KO} \wedge \mathrm{R})_*$,

$$(L_1\mathrm{MSpin} \wedge \mathrm{R})_*(X) \otimes_{(L_1\mathrm{MSpin} \wedge \mathrm{R})_*} (\mathrm{KO} \wedge \mathrm{R})_* =$$

$$(v^{-1}(L_1\mathrm{MSpin} \wedge \mathrm{R}))_*(X) \otimes_{(v^{-1}(L_1\mathrm{MSpin} \wedge \mathrm{R}))_*} (\mathrm{KO} \wedge \mathrm{R})_*.$$

Similarly,

$$(\mathrm{MSpin} \wedge \mathrm{R})_*(X) \otimes_{(\mathrm{MSpin} \wedge \mathrm{R})_*} (\mathrm{KO} \wedge \mathrm{R})_* =$$

$$(v^{-1}(\mathrm{MSpin} \wedge \mathrm{R}))_*(X) \otimes_{(v^{-1}(\mathrm{MSpin} \wedge \mathrm{R}))_*} (\mathrm{KO} \wedge \mathrm{R})_*.$$

Applying Theorem 3 and recalling that direct limits commute with smashing and applying L_1, we get

$$v^{-1}(L_1\mathrm{MSpin} \wedge \mathrm{R}) \simeq v^{-1}(L_1\mathrm{MSpin}) \wedge \mathrm{R}) \simeq L_1(v^{-1}\mathrm{MSpin}) \wedge \mathrm{R}$$

$$\simeq v^{-1}(\mathrm{MSpin}) \wedge \mathrm{R} \simeq v^{-1}(\mathrm{MSpin} \wedge \mathrm{R}).QED$$

4 Presentations of Module Spectra

In this section we prove a general lemma that is at the heart of our argument. Suppose R is a ring spectrum, M an R-module spectrum. There is a natural transformation

$$F_M : \mathrm{R}_*(X) \otimes_{\mathrm{R}_*} \mathrm{M}_* \to \mathrm{M}_*(X)$$

obtained as follows. Let $f : S^m \to X \wedge R$ be an element of $\mathrm{R}_*(X)$, and let $g : S^n \to M$ an element of M_*. Then $F_M(f \otimes g)$ is the composite

$$S^{m+n} = S^m \wedge S^n \to X \wedge R \wedge M \to X \wedge M,$$

where the last map is the structure map of M. There is a similar map in cohomology. We want to know under what conditions F_M is an isomorphism. The following lemma partially answers this question. The idea is to find a presentation of M as an R-module spectrum.

Lemma 3 *Suppose M, N, and P are R-module spectra with F_P always surjective, and F_N a natural isomorphism. Suppose there are R-module maps $d_2 : P \to N$ and $d_1 : N \to M$ with $d_1 d_2 \simeq 0$. Suppose in addition there are maps $s_1 : M \to N$ and $s_2 : N \to P$ with $d_1 s_1$ and $s_1 d_1 + d_2 s_2$ homotopy equivalences. Then F_M is an isomorphism.*

Proof: The map d_2 induces maps

$$\mathrm{R}_*(X) \otimes_{\mathrm{R}_*} \mathrm{P}_* \to \mathrm{R}_*(X) \otimes_{\mathrm{R}_*} \mathrm{N}_*$$

and $P_*(X) \to N_*(X)$ which we also denote by d_2. The same is true for d_1. Also s_2 induces a map $N_*(X) \to P_*(X)$, which we also denote s_2, and similarly for s_1. Note that s_2 does not induce a map $\mathrm{R}_*(X) \otimes_{\mathrm{R}_*} \mathrm{N}_* \to \mathrm{R}_*(X) \otimes_{\mathrm{R}_*} \mathrm{P}_*$ as it is not a map of R-module spectra. So we have the following commutative diagram.

$$\begin{array}{ccc}
\mathrm{R}_*(X) \otimes_{\mathrm{R}_*} \mathrm{P}_* & \xrightarrow{F_P} & \mathrm{P}_*(X) \\
{\scriptstyle d_2}\downarrow & & {\scriptstyle d_2}\downarrow \\
\mathrm{R}_*(X) \otimes_{\mathrm{R}_*} \mathrm{N}_* & \xrightarrow{F_N} & \mathrm{N}_*(X) \\
{\scriptstyle d_1}\downarrow & & {\scriptstyle d_1}\downarrow \\
\mathrm{R}_*(X) \otimes_{\mathrm{R}_*} \mathrm{M}_* & \xrightarrow{F_M} & \mathrm{M}_*(X)
\end{array}$$

First we prove surjectivity. Since $d_1 s_1 : M_*(X) \to M_*(X)$ is an isomorphism, $d_1 : N_*(X) \to M_*(X)$ is surjective. Since F_N is an isomorphism, $F_M d_1 = d_1 F_N$ is also surjective. Therefore, F_M is surjective.

Now we prove injectivity. As we pointed out above, $d_1 : N_*(X) \to M_*(X)$ is surjective. In particular, $d_1 : N_* \to M_*$ is surjective. Since the tensor product is right exact, $d_1 : \mathrm{R}_*(X) \otimes_{\mathrm{R}_*} \mathrm{N}_* \to \mathrm{R}_*(X) \otimes_{\mathrm{R}_*} \mathrm{M}_*$ is also surjective.

Now suppose $x \in \mathrm{R}_*(X) \otimes_{\mathrm{R}_*} \mathrm{M}_*$ has $F_M(x) = 0$. By the above remark, there is a $y \in \mathrm{R}_*(X) \otimes_{\mathrm{R}_*} \mathrm{N}_*$ with $d_1(y) = x$. Then $d_1 F_N(y) = F_M d_1(y) = 0$. Since $s_1 d_1 + d_2 s_2 : N_*(X) \to N_*(X)$ is an isomorphism, there is a $z \in N_*(X)$ such that $s_1 d_1(z) + d_2 s_2(z) = F_N(y)$. Then we have

$$0 = d_1 F_N(y) = d_1 s_1 d_1(z) + d_1 d_2 s_2(z) = d_1 s_1 d_1(z)$$

since $d_1 d_2 = 0$. Since $d_1 s_1$ is an isomorphism we have $d_1(z) = 0$. Thus $F_N(y) = d_2 s_2(z)$.

Since F_P is surjective, there is a $w \in \mathrm{R}_*(X) \otimes_{\mathrm{R}_*} \mathrm{P}_*$ with $F_P(w) = s_2(z)$. Then

$$F_N d_2(w) = d_2 F_P(w) = d_2 s_2(z) = F_N(y).$$

Since F_N is an isomorphism, we have $d_2(w) = y$. Thus

$$x = d_1(y) = d_1 d_2(w) = 0$$

and F_M is injective. QED

5 MSpin $\wedge\, M(2^n)$ Determines KO $\wedge\, M(2^n)$

In this section, we apply Lemma 3 to obtain the result in the title of this section. We also complete the proof of the complex case of Theorem 1. All spectra are localized at 2.

We start with the complex case. Take $\mathrm{R} = \mathrm{N} = \mathrm{MSpin}^c$, and let $\mathrm{M} = ku$. Let $d_1 = \pi^0$ be the Atiyah-Bott-Shapiro orientation. Recall that ρ^J denotes the right inverse to π^J coming from the Anderson-Brown-Peterson splitting. Let $s_1 = \rho^0$. Then d_1 is an R-module map, $d_1 s_1$ is a homotopy equivalence, and certainly F_N is an isomorphism.

Now let $\mathrm{P} = \mathrm{MSpin}^c \wedge \bigvee_{J \neq \emptyset} S^{4n(J)}$. Then F_P is an isomorphism. Define s_2 to be 0 on the $\mathrm{H}\mathbf{Z}/2\mathbf{Z}$ summands of R and on the bottom ku summand corresponding to $J = \emptyset$. On the other summands, define s_2 to be the composite

$$\Sigma^{4n(I)} ku \stackrel{\simeq}{\to} ku \wedge S^{4n(I)} \stackrel{s_1 \wedge 1}{\to} \mathrm{MSpin}^c \wedge S^{4n(I)} \hookrightarrow \mathrm{MSpin}^c \wedge \bigvee_{J \neq \emptyset} S^{4n(J)}.$$

Let $\iota^J : S^{4n(J)} \to \Sigma^{4n(J)} ku$ denote the inclusion of the bottom cell. Then define d_2 as the composite

$$\mathrm{MSpin}^c \wedge \bigvee_{J \neq \emptyset} S^{4n(J)} \stackrel{1 \wedge \bigvee \iota^J}{\to} \mathrm{MSpin}^c \wedge \bigvee_{J \neq \emptyset} \Sigma^{4n(J)} ku$$

$$\stackrel{1 \wedge \bigvee \rho^J}{\to} \mathrm{MSpin}^c \wedge \mathrm{MSpin}^c \stackrel{\mu}{\to} \mathrm{MSpin}^c.$$

It is easy to see that d_2 is a module map, and a diagram chase using the Cartan formula shows that, for $J \neq \emptyset$, $\pi^J d_2 s_2 \rho^J$ is a homotopy equivalence, and $\pi^I d_2 s_2 \rho^J$ is 0 for $I \neq J$.

Thus we get that the composite

$$\bigvee_J \Sigma^{4n(J)} ku \stackrel{\bigvee \rho^J}{\to} \mathrm{MSpin}^c \stackrel{d_2 s_2 + s_1 d_1}{\to} \mathrm{MSpin}^c \stackrel{\bigvee \pi^J}{\to} \bigvee_J \Sigma^{4n(J)} ku$$

is a homotopy equivalence. The $\mathrm{H}\mathbf{Z}/2\mathbf{Z}$ summands prevent us from applying the main lemma right away. But $L_1 \mathrm{H}\mathbf{Z}/2\mathbf{Z} = 0$ [AH], so if we apply L_1, each of the maps $\bigvee \rho^J$ and $\bigvee \pi^J$ become homotopy equivalences.

Thus we have proved the following:

Theorem 4

$$L_1 MSpin^c{}_*(X) \otimes_{L_1 MSpin^c{}_*} L_1 ku_* \to L_1 ku_*(X)$$

is a natural isomorphism of $L_1 ku_*$ modules.

Invert the periodicity element on both sides of this isomorphism. Then, since $p^{-1} L_1 ku \simeq L_1 p^{-1} ku \simeq L_1 \mathrm{K} = \mathrm{K}$, we find that

$$L_1 \mathrm{MSpin}^c{}_*(X) \otimes_{L_1 \mathrm{MSpin}^c{}_*} \mathrm{K}_* \to \mathrm{K}_*(X)$$

is a natural isomorphism. Combined with Corollary 1, this completes the proof of the complex case of Theorem 1.

The idea above was to shift a ku summand down to a bottom summand of a shifted copy of MSpin^c and then multiply by the bottom cell to shift it back up. In the real case this works fine for the ko summands but not for the $ko{<}2{>}$summands. Even after applying L_1, there is no bottom cell: $\pi_0 L_1 ko{<}2{>}$ is $\mathbf{Q}/\mathbf{Z}$. However, if we smash with a Moore space $M(2^n)$ the natural map $ko{<}2{>} \to ko$ induces a homotopy equivalence

$$L_1 ko{<}2{>} \wedge M(2^n) \to L_1 ko \wedge M(2^n).$$

So, let N = R = $L_1\mathrm{MSpin} \wedge M(2^n)$ for some fixed $n > 1$. We need $n > 1$ so that $M(2^n)$, and hence R, is a ring spectrum. Let M = $L_1 ko \wedge M(2^n)$. Note that M = KO $\wedge M(2^n)$. Indeed, the map between 2 adjacent connected covers of KO has fiber (a shifted copy of) H$\mathbf{Z}$ or H$\mathbf{Z}/2\mathbf{Z}$. L_1 kills H$\mathbf{Z}/2\mathbf{Z}$ and converts H$\mathbf{Z}$ to H$\mathbf{Q}$[AH]. Smashing with the Moore space then kills H$\mathbf{Q}$. Let $d_1 = L_1\pi^0 \wedge M(2^n)$, which we will just call π^0, and let $s_1 = \rho^0$, using the same convention. Then d_1 is an R-module map, $d_1 s_1$ is a homotopy equivalence, and F_N is an isomorphism.

Let P = R $\wedge(\bigvee_{J'} S^{4n(J')} \vee \bigvee_{J''} S^{4n(J'')-4})$, where J' runs through partitions with $1 \notin J'$ and $n(J')$ even, and J'' runs through partitions with $1 \notin J''$ and $n(J)$ odd. Then F_P is an isomorphism.

Define $\iota^{J'} : S^{4n(J')} \to L_1\Sigma^{4n(J')} ko \wedge M(2^n)$ to be the inclusion of the bottom cell of $\Sigma^{4n(J')} ko$ smashed with the units of $L_1 S^0$ and $M(2^n)$. Similarly, define $\iota^{J''} : S^{4n(J'')-4} \to L_1\Sigma^{4n(J'')-4} ko{<}2{>} \wedge M(2^n)$ as above followed by the inverse of the homotopy equivalence

$$L_1\Sigma^{4n(J'')-4} ko{<}2{>} \wedge M(2^n) \to L_1\Sigma^{4n(J'')-4} ko \wedge M(2^n).$$

We can then define the R-module map d_2 just as we did in the complex case.

We define s_2 in the same way as in the complex case for the ko summands. That is, it is the composite

$$L_1\Sigma^{4n(J')} ko \wedge M(2^n) \stackrel{s_1 \wedge 1}{\to} R \wedge S^{4n(J')} \hookrightarrow P.$$

Of course it is 0 on the bottom ko summand. On the $ko{<}2{>}$ summands we just precede the above composite with the homotopy equivalence

$$L_1\Sigma^{4n(J'')-4} ko{<}2{>} \wedge M(2^n) \to L_1\Sigma^{4n(J'')-4} ko \wedge M(2^n).$$

Now we must check that $d_2 s_2 + s_1 d_1$ induces an isomorphism on homotopy groups. Look at a particular summand, say a $ko{<}2{>}$ summand. Moving the suspension coordinate outside has the effect of dividing by an appropriate power of the periodicity element. Then s_2 includes this in by ρ^0. d_2 then multiplies this by the bottom class of the $ko<2>$ summand. We would like this to simply multiply by the same power of the periodicity element, so that $d_2 s_2$ would just include the $ko<2>$ summand. Unfortunately, the Cartan formula is a nontrivial sum because $\pi^J \rho^0$ may not be 0 on $\pi_*(L_1 ko \wedge M(2^n))$ if $1 \in J$. However, the following lemma shows that it is always even.

Lemma 4 *Suppose $\alpha \in \pi_* L_1 ko \wedge M(2^n)$. Then $\pi^J \rho^0 \alpha$ is divisible by 2 unless $J = \emptyset$.*

Proof: The proof is modeled on the proof of Lemma 1 in section 3. An Adams spectral sequence calculation shows that

$$\pi_*(L_1 ko \wedge M(2^n)) = \mathbf{Z}/2^n\mathbf{Z}[\eta, \beta, q, p, p^{-1}]/L$$

where L is an ideal of relations generated by

$$\eta^3, 2\eta, 2\beta, \eta^2\beta - 2^{n-1}q, \eta q, \beta q, \text{and} q^2 - 4p.$$

The degree of η is 1, that of β is 2, that of q is 4 and that of p is 8. The composition

$$L_1 S^0 \wedge M(2^n) \to L_1 ko \wedge M(2^n) \stackrel{\rho^0 \wedge 1}{\to} \mathrm{L_1MSpin} \wedge M(2^n) \stackrel{\pi^J \wedge 1}{\to} KO \wedge M(2^n)$$

where the first map is the unit, is null for $J \neq \emptyset$. Now the image of the unit map on homotopy contains the $p^n\eta$,$p^n\eta^2$, $p^n\beta$, $p^n\eta\beta$, and $2^{n-1}p^n q$. Therefore all these classes have $\pi^J\rho^0 = 0$. We deduce from this that the other classes must have $\pi^J\rho^0$ even. Indeed, denote the Hopf map by η as well. Then $\eta(\pi^J\rho^0 p^n) = \pi^J\rho^0(p^n\eta) = 0$, so $\pi^J\rho^0 p^n$ must be even. A similar argument using 2 instead of η shows that $\pi^J\rho^0 p^n q$ is even. QED

By the lemma, if we look at the matrix for $d_2 s_2 + s_1 d_1$ in a particular dimension, it will have ones on the diagonal and even numbers everywhere else. This matrix is infinite dimensional because $L_1\mathrm{MSpin} \wedge M(2^n)$ is not locally finite. However, we are working over a bounded 2-torsion group, and in this case a map with such a matrix must be an isomorphism. Indeed, iterating such a map eventually prodeces the identity. Thus we have proved

Theorem 5 *The map*

$$(L_1 MSpin \wedge M(2^n))_*(X) \otimes_{(L_1 MSpin \wedge M(2^n))_*} (KO \wedge M(2^n))_*$$

$$\rightarrow (KO \wedge M(2^n))_*(X)$$

is an isomorphism for any $n > 1$.

The results of section 3 then enable us to deduce

Corollary 2 *The map*

$$(MSpin \wedge M(2^n))_*(X) \otimes_{(MSpin \wedge M(2^n))_*} (KO \wedge M(2^n))_*$$
$$\rightarrow (KO \wedge M(2^n))_*(X)$$

is an isomorphism for any $n > 1$.

6 MSpin and MSpin $\wedge M(2^n)$

In this section we finish the proof of Theorem 1 in the real case. We need the following two lemmas, both of which are presumably well-known. We do not assume our spectra are localized at 2 for these lemmas.

Lemma 5 *The identity map of a finite torsion spectrum T has finite order.*

Proof: Let DT denote the Spanier-Whitehead dual of T. Then self-maps of T are in one to one correspondance with homotopy classes of T $\wedge$ DT. Now T $\wedge$ DT is clearly finite, and a homology calculation shows that it is torsion. Thus all of its homotopy groups are finite torsion. QED

Lemma 6 *If E is a homology theory such that E_* has bounded torsion, then, for all finite X, $E_*(X)$ also has bounded torsion.*

Proof: Induction on cells shows that the torsion-free part of $\pi_*(X)$ is finitely generated. Taking a minimal set of such generators, we obtain a cofiber sequence

$$\Sigma^{-1}T \xrightarrow{f} \bigvee_{k=1}^{l} S^{n_k} \xrightarrow{g} X \xrightarrow{h} T$$

where T is a finite torsion spectrum. By the previous lemma, $\times m : T \rightarrow T$ is null for some m. Consider the exact sequence in E-homology induced by the above cofiber sequence. By hypothesis, there is an n so that if $w \in E_*(\bigvee S^{n_k})$ is torsion, then $nw = 0$. Suppose $x \in E_*(X)$ is torsion, so $Nx = 0$ for some

N. Then $h_*(mx) = 0$, so there is a $y \in E_*(\bigvee S^{n_k})$ such that $g_* y = mx$. We claim that y is torsion. Indeed, $Nx = 0$ so $g_*(Ny) = 0$. Thus $Ny = f_* z$ for some $z \in E_*(\Sigma^{-1} T)$. But then $mz = 0$ so $mNy = 0$ and y is torsion. Thus $ny = 0$ and therefore $nmx = 0$. QED

We will also need the fact that $\mathrm{MSpin}_*(X)$, after localizing at 2, has no nonzero infinitely 2-divisible elements for finite X. This is true since MSpin_* consists of finitely generated abelian groups, so, by induction on cells, so does $\mathrm{MSpin}_*(X)$.

Finally, we need the following lemma.

Lemma 7 *If X is finite, there are no nonzero infinitely 2-divisible elements in $MSpin_*(X) \otimes_{MSpin_*} KO_*$ after localizing at 2.*

Proof: As above, consider the cofiber sequence

$$\bigvee S^{n_k} \xrightarrow{g} X \xrightarrow{h} T \xrightarrow{f} \bigvee S^{n_k+1},$$

where T is a finite torsion spectrum. Localize at 2. Then, for some n, $\times 2^n : T \to T$ is null. Applying MSpin_*, we get the exact sequence

$$\mathrm{MSpin}_*(\bigvee S^{n_k}) \xrightarrow{g} \mathrm{MSpin}_*(X) \xrightarrow{h} \mathrm{Ker}\ f \to 0.$$

Tensoring this with KO_*, we get the exact sequence

$$KO_*(\bigvee S^{n_k}) \xrightarrow{g} \mathrm{MSpin}_*(X) \otimes_{\mathrm{MSpin}_*} \mathrm{KO}_* \xrightarrow{h} \mathrm{Ker}\ f \otimes_{\mathrm{MSpin}_*} KO_* \to 0.$$

Let us rewrite this more economically in a fixed degree as

$$M \xrightarrow{g} N \xrightarrow{h} P \to 0.$$

Here P is all 2^n torsion, and M is the 2-localization of a finitely generated group. Thus $M' = M/\mathrm{Ker}\ g$ is also the 2-localization of a finitely generated abelian group, and we have the short exact sequence

$$0 \to M' \xrightarrow{g} N \xrightarrow{h} P \to 0.$$

Now suppose $x \in N$ is infinitely 2-divisible, so that there is a sequence of elements $x_m \in N$ with $x_0 = x$ and $2x_m = x_{m-1}$. Then $h(x_m)$ is also infinitely 2-divisible, but P is bounded 2-torsion, so $h(x_m) = 0$. Thus $x_m = g(y_m)$ for some $y_m \in M'$. As g is injective, we have $2y_m = y_{m-1}$ so each y_m is infinitely

2-divisible. But M' is the 2-localization of a finitely generated abelian group, so we must have $y_m = 0$, and so $x = 0$. QED

We can now complete the proof of Theorem 1 at the prime 2. It is easy to see that

$$\mathrm{MSpin}_*(X) \otimes_{\mathrm{MSpin}_*} \mathrm{KO}_* \to \mathrm{KO}_*(X)$$

is surjective: indeed, the isomorphism

$$\mathrm{MSp}_*(X) \otimes_{\mathrm{MSp}_*} \mathrm{KO}_* \to \mathrm{KO}_*(X)$$

factors through $\mathrm{MSpin}_*(X) \otimes_{\mathrm{MSpin}_*} \mathrm{KO}_*$, since the Atiyah- Bott-Shapiro orientation is an extension of $\mathrm{MSp} \to \mathrm{KO}$. So it suffices to prove injectivity. Recall that p denotes the periodicity element in $\pi_8 ko$, $v = \rho^0(p)$. Then since $\pi^0 : \mathrm{MSpin}_* \to \mathrm{KO}_*$ is surjective in nonnegative degrees and $\times p$ is an isomorphism on both sides, it suffices to prove injectivity for elements of the form $x \otimes 1$, where $x \in \mathrm{MSpin}_*(X)$. We can also assume that X is (the 2-localization of) a finite spectrum, since both homology functors and tensor products commute with direct limits.

By the isomorphism of the proceeding section, it suffices to prove that the image of $x \otimes 1$ in

$$(\mathrm{MSpin} \wedge M(2^n))_*(X) \otimes_{(\mathrm{MSpin} \wedge M(2^n))_*} (\mathrm{KO} \wedge M(2^n))_*$$

is nonzero for some n. By the preceeding lemmas, we can choose q so large that 2^q does not divide $x \otimes 1 \in \mathrm{MSpin}_*(X) \otimes_{\mathrm{MSpin}_*} \mathrm{KO}_*$ nor $x \in \mathrm{MSpin}_*(X)$, and such that $\times 2^q$ kills the torsion in $\mathrm{MSpin}_*(X)$.

Suppose that $x \otimes 1$ becomes 0 after smashing with the mod 2^{2q} Moore spectrum. Then there is an $i \geq 0$ such that

$$v^i x = \alpha y \in (\mathrm{MSpin} \wedge M(2^{2q}))_*(X)$$

where $\alpha \in (\mathrm{MSpin} \wedge M(2^{2q}))_*$ satisfies $\pi^0 \alpha = 0$, and $y \in (\mathrm{MSpin} \wedge M(2^{2q}))_*(X)$. Let $\lambda : M(2^{2q}) \to M(2^q)$ be the map which is an isomorphism on the bottom cell and $\times 2^q$ on the top cell. Then we can choose ring structures on our Moore spectra such that λ becomes a ring spectrum map [Oka]. Thus, in $(\mathrm{MSpin} \wedge M(2^q))_*(X)$, $v^i x = ((1 \wedge \lambda)\alpha)((1 \wedge \lambda)y)$, and $\pi^0(1 \wedge \lambda)\alpha = 0$.

Now consider the universal coefficient sequence

$$0 \to \mathrm{MSpin}_*(X) \otimes \mathbf{Z}/2^n\mathbf{Z} \to (\mathrm{MSpin} \wedge M(2^n))_*(X)$$

$$\to \mathrm{Tor}(\mathrm{MSpin}_{*-1}(X), \mathbf{Z}/2^n\mathbf{Z}) \to 0.$$

By choice of q, for n a multiple of q the right-hand group is just $\mathrm{Tor}(\mathrm{MSpin}_{*-1}(X))$. Now, λ induces a map between these sequences which is $\times 2^q$, hence 0, on the right-hand groups. Thus,

$$(1 \wedge \lambda)y \in \mathrm{MSpin}_*(X) \otimes \mathbf{Z}/2^q\mathbf{Z}.$$

We can therefore lift $(1 \wedge \lambda)y$ to $y' \in \mathrm{MSpin}_*(X)$. Similarly, we can lift $(1 \wedge \lambda)\alpha$ to $\alpha' \in \mathrm{MSpin}_*$. Then in $\mathrm{MSpin}_*(X)$ we have

$$v^i x = \alpha' y' + 2^q z \in \mathrm{MSpin}_*(X)$$

and $\pi^0 \alpha' = 2^q \beta$. Thus $v^i x \otimes 1 = 2^q z \otimes 1 + y' \otimes 2^q \beta$. Therefore, 2^q divides $v^i x \otimes 1$ and hence $x \otimes 1$. This is a contradiction, and concludes the proof of Theorem 1 at the prime 2.

7 Odd Primes

The purpose of this section is to prove Theorem 1 at odd primes. In this section all spectra are assumed to be localized at an odd prime p. There is nothing to prove in the real case, since at an odd prime $\mathrm{MSpin} \simeq \mathrm{MSp}$.

We follow the method of proof used by Conner and Floyd [CF] to prove

$$\mathrm{MU}_*(X) \otimes_{\mathrm{MU}_*} \mathrm{K}_* \to \mathrm{K}_*(X)$$

is an isomorphism. Their argument requires us to work in cohomology and to consider K^* as a $\mathbf{Z}/2\mathbf{Z}$ graded cohomology theory. After we are done, we can convert back to homology by using Spanier-Whitehead duality and a limit argument.

Their proof has three steps, as follows.

1. Prove that the map

$$G : \mathrm{MU}^*(X) \otimes_{\mathrm{MU}^*} \mathrm{K}^* \to \mathrm{K}^*(X)$$

   is surjective by finding a section $\mathrm{K} \to \mathrm{MU}$, as in Lemma 2.

2. Prove that G is an isomorphism for all X if and only if G is an isomorphism for the universal example $X = \mathrm{MU}$.

3. Prove that G is an isomorphism for $X = \mathrm{MU}$.

In the interest of economical notation, let us denote MSpin^c by E. We want to follow this program for

$$G : \mathrm{E}^*(X) \otimes_{\mathrm{E}^*} \mathrm{K}^* \to \mathrm{K}^*(X).$$

The first two steps are the same as in Conner and Floyd. Since the Atiyah-Bott-Shapiro orientation extends the Todd genus, the composite of their section K $\to$ MU with the natural map MU $\to$ E will be a section for E $\to$ K. Step 2 of their argument goes through verbatim from [CF], except the universal example is now $X = \mathrm{E}$.

For Step 3 we must do a little more work. Recall from [St] that $\pi_*\mathrm{E}$ is evenly graded and torsion-free as a $\mathbf{Z}_{(p)}$ module, as is $\mathrm{H}^*(\mathrm{E}; \mathbf{Z}_{(p)})$. Let $\mathrm{E}(r)$ denote the r-skeleton of E in a minimal CW-decomposition. Then $\mathrm{H}^*(\mathrm{E}(r); \mathbf{Z}_{(p)})$ is also evenly graded and torsion-free. Therefore the Atiyah-Hirzebruch spectral sequence for $\mathrm{E}^*(\mathrm{E}(r))$ collapses. The E^∞ term is $\mathrm{H}^*(\mathrm{E}(r); \mathbf{Z}_{(p)}) \otimes \mathrm{E}^*$. This is a free E^* module, so there is no room for module extensions and we have

$$\mathrm{E}^*(\mathrm{E}(r)) = \mathrm{H}^*(\mathrm{E}(r); \mathbf{Z}_{(p)}) \otimes \mathrm{E}^*.$$

The sequence $\mathrm{H}^*(\mathrm{E}(r); \mathbf{Z}_{(p)}) \otimes \mathrm{E}^*$ satisfies the Mittag-Leffler condition, so there is no $\lim^1$ term and we have

$$\mathrm{E}^*(\mathrm{E}) = \lim \mathrm{E}^*(\mathrm{E}(r)) = \lim \left(\mathrm{H}^*(\mathrm{E}(r); \mathbf{Z}_{(p)}) \otimes \mathrm{E}^*\right)$$

$$= \lim \left(\mathrm{H}^*(\mathrm{E}(r); \mathbf{Z}_{(p)})\right) \otimes \mathrm{E}^* = \mathrm{H}^*(\mathrm{E}; \mathbf{Z}_{(p)}) \otimes \mathrm{E}^*$$

as E^* modules. The third equality above holds because E^* is a free $\mathbf{Z}_{(p)}$ module. A similar argument shows

$$\mathrm{K}^*(\mathrm{E}) = \mathrm{H}^*(\mathrm{E}; \mathbf{Z}_{(p)}) \otimes \mathrm{K}^*$$

as K^* modules. Thus G is an isomorphism for $X = \mathrm{E}$, and the proof of Theorem 1 is complete.

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