NORTHWESTERN UNIVERSITY

STABLE DECOMPOSITIONS
OF CERTAIN LOOP SPACES

A DISSERTATION

SUBMITTED TO THE GRADUATE SCHOOL

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

for the degree

DOCTOR OF PHILOSOPHY

Field of Mathematics

By

Michael J. Hopkins, M.J.
Ph.D.

Evanston, Illinois

August, 1984

ABSTRACT

STABLE DECOMPOSITIONS OF CERTAIN LOOP SPACES

by

MICHAEL J. HOPKINS

This paper studies the stable structure of $\Omega SU(n)$ and $\Omega Sp(n)$. The homology of $\Omega SU(n)$ is a polynomial ring. As a module over the Steenrod algebra it splits as the sum of its homogeneous parts. It is conjectured that this corresponds to a geometric splitting, and a version of this conjecture is proved after inverting one of the generators. In contrast with this, $\Omega Sp(2)$ and $\Omega Sp(3)$ are shown to be stably atomic at the prime 2, in the sense that any stable self-map inducing an isomorphism of $H_2(\ ;\mathbf{Z}/2)$ is a 2-local stable homotopy equivalence.

The spectra arising in the (localized) splitting of $\Omega SU(n)$ are the bordism theories associated to the double loop maps

$$\Omega SU(n-1) \longrightarrow \Omega SU \simeq BU .$$

These bordism theories are fairly interesting in their own right and two of the three chapters of this paper are devoted to studying their properties.

Table of Contents

Introduction

This thesis represents an attempt, only partially successful, to determine the stable structure of $\Omega SU(n)$ and $\Omega Sp(n)$. In trying to understand these spectra one is naturally led to consider Thom spectra $X(n)$ and $Xp(n)$ which arise from certain vector bundles over $\Omega SU(n)$ and $\Omega Sp(n)$. The spectra $X(n)$ are due to Ravenel [19] and in some sense generalize the X_k-construction of Barratt. It turns out that these spectra are fairly interesting in their own right and a large part of the material herein is devoted to developing their properties.

In Chapter 1, the spectra $X(n)$ and $Xp(n)$ are defined and certain elementary properties are exhibited. The spectra $X(n)$ filter MU and it is shown that many of the properties of MU and BP induce analogous properties of the spectra $X(n)$.

In Chapter 2 applications of this and related material are made. These include:

i) A characterization of the 2-primary cyclic BP-module spectra which might admit a commutative multiplication.

ii) Any commutative ring spectrum of characteristic two is a wedge of suspensions of Eilenberg-MacLane spectra.

iii) Elements of order two in a commutative ring spectrum which have nilpotent mod-2 Hurewicz image are nilpotent.

iv) A decomposition of the (p-localized) infinite smash product $\overset{\infty}{\wedge}\Sigma^{-2}\mathbb{C}P^n$ into irreducible spectra, together with applications to the structure of $[\overset{\infty}{\wedge}\Sigma^{-2}\mathbb{C}P^n, \overset{\infty}{\wedge}\Sigma^{-2}\mathbb{C}P^n]$ and to the structure of the infinite loop space $\Omega^{\infty}X(n)$.

v) A study of nilpotence in $\pi_*X(n)$ and a geometric decomposition of $X(2)$ in terms of nilpotent self maps of finite complexes.

Odd primary analogues of applications i), ii) and iii) are also indicated. The other applications are valid at any prime.

Chapter 3 contains the structure theorems pertaining to the suspension spectra of $\Omega SU(n)$ and $\Omega Sp(n)$. The homology of $\Omega SU(n)$ is a polynomial ring. As a module over the Steenrod algebra it splits as a sum of its homogeneous parts. It is conjectured that this corresponds to a geometric splitting, and a version of this conjecture is proved after inverting one of the generators. In contrast with this, $\Omega Sp(2)$ and $\Omega Sp(3)$ are shown to be stably atomic (at the prime 2) in the sense that any stable self map inducing an isomorphism in $H_2(\ ;\mathbb{Z}/2)$ is a stable homotopy equivalence.

It is a pleasure to thank Professor Mark Mahowald for his patience and guidance during our many conversations and during several sailing excursions. Many of the results in this work were conjectured by him and many of the proofs follow his suggestions. It has been a marvelous opportunity to learn about stable homotopy theory from one who knows it so directly. His influence will be apparent throughout this work.

Second, I would like to thank my friend Henry Cejtin for being my first mentor and for kindly waiting for me to finish my thesis before he began writing his. I would also like to thank my friends Jeff Smith and Wolfgang Lellman for countless stimulating conversations and for helping to create a very exciting topological neighborhood here at Northwestern.

Finally, I wish to thank Wagner Associates for their cheerful and speedy job of typing this manuscript.

Chapter 1. The Spectra $X(n)$ and $Xp(n)$

1.1 Elementary Remarks

Given an H-space X and an H-map $f:X \to BO$, the resulting Thom spectrum X^f is a ring spectrum [11], [12]. If X has a higher kind of multiplicative structure (for example if X is an n-fold loop space), and if f is compatible with this higher structure, then X^f will have analogous structure in its multiplication [8].

Let $BSU(n) \to BSU$ and $BSp(n) \to BSU$ be the usual inclusions. Taking double loop spaces and appealing to Bott periodicity yields maps $\Omega SU(n) \to BU$ and $\Omega Sp(n) \to BU$. These maps result in Thom spectra which will be denoted $X(n)$ and $Xp(n)$ respectively. They have a higher multiplicative structure analogous to that of a double loop space. There are clearly maps $X(n) \to X(n+1)$ and one has $\lim_{n\to\infty} X(n) = MU$.

The spectra $X(n)$ were originally constructed by Ravenel [19]. It is quite likely that many of the results of this chapter are known to him. Very few of these appear in the literature or in preprint form, however, so we have taken the liberty to give a fairly detailed account.

We wish to compute the homology of $X(n)$ and $Xp(n)$. Recall [22] that consideration of the subspace of 'reflections' results in an inclusion $S^{2n-1} \times_{S^1} S^1 / S^{2n-1} \times_{S^1} \{1\} \hookrightarrow U(n)$. Translating this back to $SU(n)$ by multiplying by a fixed inclusion $U(1) \hookrightarrow U(n)$ results in a map $\Sigma\mathbb{C}P^{n-1} \to SU(n)$. The following can be extracted from [22].

Proposition 1.1.1. i) $H_*(SU(n);\ \mathbb{Z})$ is the exterior algebra $E[H_*(\Sigma\mathbb{C}P^{n-1};\ \mathbb{Z})]$.

ii) The map $H_*(Sp(n);\ \mathbb{Z}) \to H_*(SU(2n);\ \mathbb{Z})$ is the inclusion of the exterior subalgebra on the generators of dimensions congruent to 3 mod 4.

Let $\mathbb{C}P^{n-1} \to \Omega SU(n)$ be the adjoint of the above map. The following is an easy consequence of the Eilenberg-Moore spectral sequence.

Proposition 1.1.2. i) $H_*(\Omega SU(n);\ \mathbb{Z})$ is isomorphic to the symmetric algebra on $\tilde{H}_*(\mathbb{C}P^{n-1};\ \mathbb{Z})$.

ii) The map $H_*(\Omega Sp(n);\ \mathbb{Z}) \longrightarrow H_*(\Omega SU(2n);\mathbb{Z})$ is the inclusion of a polynomial subalgebra. Modulo decomposables, the generators of $H_*(\Omega Sp(n);\ \mathbb{Z})$ are the generators of $H_*(\Omega SU(2n);\mathbb{Z})$ in dimensions congruent to 2 mod 4.

Combining the above proposition with the Thom isomorphism yields the homology of $X(n)$ and $Xp(n)$. Before describing it, however, we need to make a convention. An appropriate choice of equivalence $\Omega SU \to BU$ makes the composite

$$\mathbb{C}P^{n-1} \longrightarrow \Omega SU(n) \longrightarrow \Omega SU \longrightarrow BU$$

classify the canonical line bundle.* Passing to Thom spectra results in an "orientation" $\Sigma^{-2}\mathbb{C}P^n \to X(n)$. We wish to fix a basis for $H_*(\Sigma^{-2}\mathbb{C}P^n; \mathbb{Z})$. Let $x \in H^2(\mathbb{C}P^{n-1}; \mathbb{Z})$ be a generator and let $\{\beta_0, \ldots, \beta_{n-1}\} \subset H_*(\mathbb{C}P^{n-1}; \mathbb{Z})$ be the basis dual to $\{1, x, x^2, \cdots, x^{n-1}\}$. Finally, let $\{b_0, \ldots, b_{n-1}\} \subset \tilde{H}_*(\Sigma^{-2}\mathbb{C}P^n; \mathbb{Z})$ correspond to $\{\beta_0, \ldots, \beta_{n-1}\}$ under the Thom isomorphism. The abelian group $\tilde{H}_*(\Sigma^{-2}\mathbb{C}P^n; \mathbb{Z})$ is therefore free abelian with basis $\{b_i \mid 0 \leq i \leq n-1\}$ and with $|b_i| = 2i$.

Proposition 1.1.3. i) $H_*(X(n); \mathbb{Z}) =$

$$\mathbb{Z}[b_0, \ldots, b_{n-1}]/(b_0 - 1).$$

ii) The map $H_*(Xp(n); \mathbb{Z}) \to H_*(X(2n); \mathbb{Z})$ is the inclusion of a polynomial subalgebra with generators congruent to b_{2i-1} $(i < n)$ modulo decomposables.

---

* See Appendix B

1.2 A Characterization of X(n)

The analogy of the orientation $\Sigma^{-2}\mathbb{C}P^n \to X(n)$ with $\Sigma^{-2}\mathbb{C}P^\infty \to MU$ can be pursued and yields a characterization of $X(n)$ similar to that of MU via formal group laws. Our treatment parallels that of [1].

Let E be a ring spectrum. We shall say that a map $f:\Sigma^{-2}\mathbb{C}P^n \to E$ <u>carries the unit</u> if the restriction of f to the bottom cell of $\Sigma^{-2}\mathbb{C}P^n$ is the unit of E. Such a map will be called an <u>E-orientation</u> of $\mathbb{C}P^n$ and, when no confusion will arise, just an orientation.

The following is the main result of this section.

<u>Proposition 1.2.1</u>. Let E be a ring spectrum. Any map $\Sigma^{-2}\mathbb{C}P^n \to E$ carrying the unit extends to a unique map of ring spectra $X(n) \to E$.

An E-orientation of $\mathbb{C}P^n$ can be regarded as an element $x \in E^2(\mathbb{C}P^n)$.

<u>Lemma 1.2.2</u>. Let E be a ring spectrum and let $x \in E^2(\mathbb{C}P^n)$ be an orientation. Then

i) $E^*(\mathbb{C}P^n) \approx E^*[x]/(x^{n+1})$.

ii) The external product

$$\overbrace{E^*(\mathbb{C}P^n) \otimes_{E*} \cdots \otimes_{E*} E^*(\mathbb{C}P^n)}^{k} \longrightarrow E^*(\overbrace{\mathbb{C}P^n \times \cdots \times \mathbb{C}P^n}^{k})$$

is an isomorphism.

proof: i) Any representative for x on the E^2-term of the Atiyah-Hirzebruch spectral sequence must survive to E^∞. Since the E^2-term is isomorphic to $E^*[x]/(x^{n+1})$ the spectral sequence collapses. There are no extension problems since we are dealing with free E^*-modules.

ii) This follows from the appropriate pairings of spectral sequences.

Lemma 1.2.3. For $0 \leq i \leq n$ let $\beta_i \in E_{2i}(\mathbb{C}P^n)$ be dual to x^i under the Kronecker pairing. Then

i) $E_*\mathbb{C}P^n$ is a free E_*-module with basis $\{\beta_i\}$.

ii) The external product

$$E_*(\mathbb{C}P^n) \otimes_{E_*} \cdots \otimes_{E_*} E_*(\mathbb{C}P^n) \longrightarrow E_*(\mathbb{C}P^n \times \cdots \times \mathbb{C}P^n)$$

is an isomorphism.

iii) As an E_*-coalgebra, $E_*(\mathbb{C}P^n)$ is determined by the formula

$$\psi(\beta_k) = \sum_{i+j=k} \beta_i \otimes \beta_j \quad .$$

proof: This follows from Lemma 1.2.2 by using the appropriate pairings of spectral sequences.

The elements β_i for $i \leqslant n-1$ can be regarded as elements of $E_*(\mathbb{C}P^{n-1})$. For $0 \leqslant i \leqslant n-1$ let $b_i \in \tilde{E}_{2i}(\Sigma^{-2}\mathbb{C}P^n)$ correspond to β_i under the Thom isomorphism. We will also use the symbol b_i to denote the image of b_i under the map $E_*(\Sigma^{-2}\mathbb{C}P^n) \to E_*(X(n))$. If two spectra, say E and F, are under consideration, superscripts will be appended to distinguish $\{b_i^E\}$ from $\{b_i^F\}$.

Corollary 1.2.4. i) $E_*(X(n)) \approx E_*[b_0, \ldots, b_{n-1}]/(b_0 - 1)$.

ii) $E_*(X(n) \wedge X(n)) \approx E_*(X(n)) \otimes_{E_*} E_*(X(n))$.

iii) $E^*(X(n)) \approx \mathrm{Hom}_{E_*}[E_*(X(n)), E_*]$.

iv) $E^*(X(n) \wedge X(n)) \approx \mathrm{Hom}_{E_*}[E_*(X(n) \wedge X(n)), E_*]$.

v) Under the identification iii), the maps of ring spectra $X(n) \to E$ correspond to $\mathrm{Hom}_{E_*\text{-algebras}}[E_*(X(n)), E_*]$.

proof: Parts i) and ii) follow easily from Lemma 1.2.3 and the Atiyah-Hirzebruch spectral sequence. Parts iii) and iv) also follow from the Atiyah-Hirzebruch spectral sequence modulo convergence questions. These are dealt with below. Part v) follows easily once iv) is interpreted as saying that two maps $X(n) \wedge X(n) \to E$ agree if and only if the induced maps in E-homology agree.

We have used the following lemma.

Lemma 1.2.5. Let X be a spectrum which is the direct limit of a sequence

$$\cdots \longrightarrow X_n \xrightarrow{f_n} X_{n+1} \longrightarrow \cdots$$

of finite dimensional spectra. Suppose further that $f_n^* : H^*(X_{n+1};\pi_* E) \to H^*(X_n;\pi_* E)$ is an epimorphism for all n. If the Atiyah-Hirzebruch spectral sequence for $E^*(X)$ collapses at E^2 then it converges to $E^*(X)$.

proof: We need to show that $E^*(X) \approx \varprojlim_n E^*(X_n)$ -- that is, that $\varprojlim_n^1 E^*(X_n) = 0$. Since the maps $f_n^* : H^*(X_{n+1};\pi_* E) \to H^*(X_n;\pi_* E)$ are epimorphisms, so is the map $H^*(X;\pi_* E)$. It follows that the Atiyah-Hirzebruch spectral sequences for $E^*(X_n)$ collapse at E^2. These spectral sequences converge since the X_n are finite dimensional. This implies that the maps $E^*X_{n+1} \to E^*X_n$ are epimorphisms, since a map of finitely filtered abelian groups which induces an epimorphism of associated graded groups is an epimorphism. It follows that the system $\{E^*(X_n)\}$ is Mittag-Leffler and hence that $\varprojlim^1 E^*(X_n) = 0$. This completes the proof.

The proof of Proposition 1.2.1 now follows easily. By Corollary 1.2.4, the maps of ring spectra $X(n) \to E$ are in one to one correspondence with the (graded) E_*-algebra maps $E_*(X(n)) \cong E_*[b_0, \ldots, b_{n-1}]/(b_0 - 1) \to E_*$. These correspond bijectively to maps $E_*(\Sigma^{-2}\mathbb{C}P^n) \to E_*$ which send b_0 to 1 , using the identification of Lemma 1.2.3 i). By the same lemma these correspond exactly to the orientations $\Sigma^{-2}\mathbb{C}P^n \to E$. This completes the proof.

1.3 Localization of $X(n)$

In this section we will investigate the behavior of $X(n)$ after localizing at a prime. Let $H\mathbb{Z}/p$ denote the mod-p Eilenberg-MacLane spectrum. Recall that for p odd, $H\mathbb{Z}/p_*H\mathbb{Z}/p \approx E[\tau_0,\tau_1,\ldots]\otimes\mathbb{Z}/p[\xi_1,\xi_2,\ldots]$ with $|\tau_i| = 2\cdot p^i - 1$ and $|\xi_i| = 2p^i - 2$. When $p = 2$ one has $H\mathbb{Z}/2_*H\mathbb{Z}/2 \approx \mathbb{Z}/2[\xi_1,\xi_2,\ldots]$ with $|\xi_i| = 2^i - 1$. In order to avoid separating cases we will adopt the odd primary notation and leave to the reader the obvious modifications for the case $p = 2$.

An immediate consequence of the definition of the ξ_i ([14]) is that the unique orientation $\Sigma^{-2}\mathbb{C}P^n \to H\mathbb{Z}/p$ induces a map in homology which sends b_j to ξ_i if $j = p^i - 1$ and to zero otherwise.

<u>Corollary 1.3.1</u>. There is a unique map of ring spectra $X(n) \to H\mathbb{Z}/p$. The induced map in homology has for its image the polynomial subalgebra of A_* generated by $\{\xi_i \mid i \leq \log_p(n)\}$.

Let $P \subset A$ be the sub-algebra of cyclic reduced powers. The vector space dual of P is the polynomial ring $\mathbb{Z}/p[\xi_1,\xi_2,\ldots]$. Let $\Gamma^n(P) \hookrightarrow P$ be the n-fold commutator ideal. It is a Hopf ideal and the quotient P/Γ^{n+1} will be denoted P_n.

Lemma 1.3.2. The vector space dual of P_n is isomorphic to $\mathbb{Z}/p[\xi_1,\ldots,\xi_n]$.

proof: Let $I_n \subset P$ be the annihilator ideal of $\mathbb{Z}/p[\xi_1,\ldots,\xi_n]$. Then $I_{n+1} \subset I_n$ and the Hopf algebra kernel of $P/I_{n+1} \twoheadrightarrow P/I_n$ is abelian. It follows that $I_n \subset \Gamma^{n+1}$. It remains to show that Γ^{n+1} annihilates the elements $1,\xi_1,\ldots,\xi_n$. This follows from dimensional considerations and the following facts:

i) Elements of minimal degree in an ideal of a connected graded Hopf algebra are primitive.

ii) The set of primitives in P is the vector space dual of the set $\{\xi_1,\xi_2,\ldots\}$ with respect to the monomial basis.

iii) The connectivities of the ideals Γ^n form a strictly increasing sequence.

From now on, m and n when they appear together, will be integers in the relation $p^m \leqslant n < p^{m+1}$.

Proposition 1.3.3. The mod-p cohomology $H^*(X(n);\mathbb{Z}/p)$ is free over P_m .

proof: Since $X(n)$ is a commutative ring spectrum $H^*(X(n);\mathbb{Z}/p)$ is a commutative co-algebra. The proposition follows in the usual way from a theorem of Milnor-Moore [15] once one shows that the action of the Steenrod algebra on a generator of $H^0(X(n);\mathbf{Z}/p)$ factors through P_m, and that evaluation on this generator induces a monomorphism $P_m \hookrightarrow H^*(X(n);\mathbf{Z}/p)$. The vector space dual of this assertion is precisely the content of Corollary 1.3.1. This completes the proof.

We now wish to show that Proposition 1.3.3 has a geometric analogue.

<u>Lemma 1.3.5</u>. Any map of ring spectra $\varepsilon: MU \to MU$ determines a family $\{\varepsilon_n : X(n) \to X(n)\}$ of maps of ring spectra with the following properties:

i) There is a commutative diagram

$$\begin{tikzcd} X(n) \arrow[r, hook] \arrow[d, "\varepsilon_n"'] & X(n+1) \arrow[d, "\varepsilon_{n+1}"] \\ X(n) \arrow[r, hook] & X(n+1) \end{tikzcd}$$

ii) $\lim\limits_{n\to\infty} \varepsilon_n = \varepsilon$.

Furthermore, the ε_n are unique with respect to these properties.

<u>proof</u>: The map ε is determined by its restriction to $\Sigma^{-2}\mathbb{C}P^\infty$. Since $X(n) \to MU$ is a $(2n-1)$-equivalence this

determines a unique map $\Sigma^{-2}\mathbb{C}P^n \to X(n)$. The lemma now follows from Proposition 1.2.1.

In order to compare our results with the more standard results from MU-theory it will be convenient to locate another system of polynomial generators in $H_*(X(n);\mathbf{Z})$. Let $\sum_{i\geqslant 0} m_i x^{i+1}$ be the formal power series inverse (under substitution) to $\Sigma b_i x^{i+1}$. It is easy to see that $m_{n-1} \in \mathbf{Z}[b_1,\ldots,b_{n-1}]$ and that $m_{n-1} \equiv -b_{n-1}$ mod decomposables. We can therefore regard $m_i \in H_*(X(n);\mathbf{Z})$ whenever $i < n$. The proof of the following lemma is then an easy calculation.

Lemma 1.3.6. i) $H_*(X(n);\mathbf{Z}) \approx \mathbf{Z}[m_1,\ldots,m_{n-1}]$.

ii) The homology homomorphism induced by the map of ring spectra $X(n) \to H\mathbf{Z}/p$ sends m_j to $\chi(\xi_i)$ if $j = p^i - 1$ and to zero otherwise.

Proposition 1.3.7. After localizing at p there exist maps $\varepsilon_n : X(n) \to X(n)$ of ring spectra satisfying:

i) ε_n is idempotent.

ii) The induced map in homology satisfies

$$\varepsilon_n(m_i) = \begin{cases} m_i & \text{if } i = p^j - 1 \text{ for some } j \\ 0 & \text{otherwise .} \end{cases}$$

iii) The ε_n are compatible in the sense that the following diagram commutes (up to homotopy)

$$\begin{array}{ccc} X(n) & \hookrightarrow & X(n+1) \\ \varepsilon_n \big\downarrow & & \big\downarrow \varepsilon_{n+1} \\ X(n) & \longrightarrow & X(n+1) \end{array}$$

iv) The limit $\lim_{n\to\infty} \varepsilon_n : MU \to MU$ is Quillen's idempotent [18].

<u>proof</u>: This is immediate from the properties of Quillen's idempotent and from Lemma 1.3.5.

Recall our convention that n and m are integers in the relation $p^m \leqslant n < p^{m+1}$. The idempotents ε_n give rise to spectra which will be denoted $X\langle m\rangle$. Part ii) of Proposition 1.3.7 determines the cohomology of $X\langle m\rangle$ and shows that $X\langle m\rangle$ is independent of the choice of n. The $X\langle m\rangle$ fit together to form a sequence whose limit is BP.

<u>Corollary 1.3.8</u>. For $m \geqslant 1$ there exist p-local commutative ring spectra $X\langle m\rangle$ of finite type with $H^*(X\langle m\rangle;\mathbb{Z}/p) \approx P_m$. If E is a ring spectrum admitting an orientation $\Sigma^{-2}\mathbb{C}P^n \to E$, then there is an isomorphism $E_*X\langle m\rangle \approx E_*\otimes \mathbb{Z}_{(p)}[t_1,\dots,t_m]$ with $|t_i| = 2p^i - 2$.

proof: Only the last assertion has not yet been proven. It follows easily from the fact that the Atiyah-Hirzebruch spectral sequence for $E_*X\langle m\rangle$ collapses, since $X\langle m\rangle$ is a retract of $X(n)$ localized at p.

Finally, construct ring spectra M_n out of suitable wedges of spheres, satisfying $H_*(M_n;\mathbb{Z}) \approx \mathbb{Z}[x_i \mid 1 \leq i \leq n-1, i \neq p^j - 1$ for any $j]$ with $|x_i| = 2i$. We will denote $\lim\limits_{n\to\infty} M_n$ by M.

Corollary 1.3.9. After localizing at p, there exists an equivalence of ring spectra $MU \approx BP \wedge M$. Furthermore, any such equivalence restricts to a unique family of (p-local) equivalences $X(n) \to X\langle m\rangle \wedge M_n$.

proof: The first assertion is standard. For the second observe that any map of ring spectra $MU \to BP\wedge M$ is determined by its restriction to $\Sigma^{-2}\mathbb{C}P^\infty$. Any orientation $\Sigma^{-2}\mathbb{C}P^\infty \to BP\wedge M$ determines a unique orientation $\Sigma^{-2}\mathbb{C}P^n \to X\langle m\rangle\wedge M_n$ since $X\langle m\rangle \wedge M_n \to BP\wedge M$ is a $(2n-1)$-equivalence. This completes the proof.

1.4 $X(n)_* X(n)$ and $X\langle m\rangle_* X\langle m\rangle$

Recall that a ring spectrum E is said to be flat if it satisfies one of the following equivalent conditions:

i) $E \wedge E$ splits as a wedge of suspensions of E.

ii) E_*E is a free E_* module.

Corollary 1.2.4 implies that the algebra of co-operations $X(n)_* X(n)$ is isomorphic to $X(n)_*[b_1,\ldots,b_{n-1}]$ with $|b_i| = 2i$. Corollary 1.3.8 implies that $X\langle m\rangle_* X\langle m\rangle \approx X\langle m\rangle_*[t_1,\ldots,t_m]$ with $|t_i| = 2(p^i - 1)$. It follows that $X(n)$ and $X\langle m\rangle$ are both flat. The purpose of this section is to show that the decompositions resulting from part i) above can be made compatible as n and m vary. This enables one to determine much of the structure of the Hopf algebroids $X(n)_*X(n)$ and $X\langle m\rangle_*X\langle m\rangle$ from that of MU_*MU and BP_*BP.

The idempotent ε_n gives rise to a map $X(n) \to X\langle m\rangle$. Composing this with the orientation $\Sigma^{-2}\mathbb{C}P^n \to X(n)$ fixes an orientation $\Sigma^{-2}\mathbb{C}P^n \to X\langle m\rangle$.

Proposition 1.4.1. Let E be a ring spectrum. Then any ring spectra homomorphism $X\langle m\rangle \to E$ is uniquely determined by its restriction to $\Sigma^{-2}\mathbb{C}P^n$.

<u>proof</u>: If there is a map of ring spectra $X\langle m\rangle \to E$ then E is p-local and admits an orientation. Corollary 1.3.9 implies that $E^*(X\langle m\rangle) \to E^*(X(n))$ is a split monomorphism. The proposition then follows from Proposition 1.2.1.

Next we need to discuss the splittings of $MU \wedge MU$ and of $BP \wedge BP$. To construct a splitting one first builds Moore spectra for the rings $\mathbf{Z}[b_1, b_2, \ldots]$ $(|b_i| = 2i)$ and $\mathbf{Z}[t_1, t_2, \ldots]$ $(|t_i| = 2(p^i - 1))$, by imposing appropriate multiplications on wedges of spheres. Smashing these with MU and BP gives ring spectra $MU[b_1, b_2, \ldots]$ and $BP[t_1, t_2, \ldots]$. Next one constructs maps $MU \wedge MU \to MU[b_1, b_2, \ldots]$ and $BP \wedge BP \to BP[t_1, t_2, \ldots]$, and proves them to be equivalences. It is not unreasonable to ask that these be maps of ring spectra. It is also not unreasonable to ask that they preserve the obvious left BP and MU module structures.

<u>Definition</u>. Let E and Ω be ring spectra. An equivalence $s: E \wedge E \to E \wedge \Omega$ is said to be an <u>admissible splitting</u> if it is a map of ring spectra preserving the obvious left E-module structure.

The advantage of admissible splittings is that they are determined by the right unit

$$\eta_R : S^0 \wedge E \longrightarrow E \wedge E \longrightarrow E \wedge \Omega ,$$

which is a map of ring spectra. When E is one of the spectra $X(n)$ or $X\langle m\rangle$, the right unit is in turn determined by its restriction to $\Sigma^{-2}\mathbb{C}P^n$. We will call this restriction the right orientation.

Proposition 1.4.2. Let $s: MU \wedge MU \to MU[t_1, t_2, \ldots]$ be any admissible splitting. The map s determines a family of admissible splittings $s_n: X(n) \wedge X(n) \to X(n)[b_1, \ldots, b_{n-1}]$ which are unique with respect to the following two properties:

i) The s_n are compatible in the sense that

$$\begin{tikzcd}
X(n) \wedge X(n) \arrow[r] \arrow[d, "s_n"'] & X(n+1) \wedge X(n+1) \arrow[d, "s_{n+1}"] \\
X(n)[b_1, \ldots, b_{n-1}] \arrow[r] & X(n+1)[b_1, \ldots, b_n]
\end{tikzcd}$$

commutes.

ii) The limit $\lim\limits_{n\to\infty} s_n$ is s.

<u>Proposition 1.4.3</u>. Let $s: BP \wedge BP \to BP[t_1, t_2, \ldots]$ be any admissible splitting. The map s determines a family of admissible splittings $s_m: X\langle m\rangle \wedge X\langle m\rangle \to X\langle m\rangle[t_1, \ldots, t_m]$ which are unique with respect to the following two properties:

i) The s_n are compatible in the sense that the diagram

$$\begin{tikzcd} X\langle m\rangle \wedge X\langle m\rangle \arrow[r] \arrow[d, "s_m"'] & X\langle m+1\rangle \wedge X\langle m+1\rangle \arrow[d, "s_{m+1}"] \\ X\langle m\rangle[t_1, \ldots, t_m] \arrow[r] & X\langle m+1\rangle[t_1, \ldots, t_{m+1}] \end{tikzcd}$$

commutes.

ii) The limit $\lim\limits_{n\to\infty} s_n$ is s.

<u>proof of Proposition 1.4.2</u>: Any admissible splitting $s: MU \wedge MU \to MU[b_1, b_2, \ldots]$ is uniquely determined by its right orientation $\Sigma^{-2}\mathbb{C}P^\infty \to MU[b_1, b_2, \ldots]$. This in turn determines a unique family of orientations $\Sigma^{-2}\mathbb{C}P^n \to X(n)[b_1, \ldots, b_{n-1}]$, since $X(n)[b_1, \ldots, b_{n-1}] \to MU[b_1, b_2, \ldots]$ is a $(2n-1)$-equivalence. These latter maps induce homomorphisms $X(n) \wedge X(n) \to X(n)[b_1, \ldots, b_{n-1}]$ which are easily calculated to be homology equivalences, hence admissible

splittings. The compatibility and unicity statements follow from the analogous statements about the orientations $\Sigma^{-2}\mathbb{C}P^n \to X(n)[b_1,\ldots,b_{n-1}]$.

<u>proof of Proposition 1.4.3</u>: We wish to mimick the proof of Proposition 1.4.2. The splitting $BP\wedge BP \to BP[t_1,t_2,\ldots]$ is uniquely determined by the right orientation $\eta_R:\Sigma^{-2}\mathbb{C}P^\infty \to BP[t_1,t_2,\ldots]$. This in turn determines unique orientations $\Sigma^{-2}\mathbb{C}P^n \to X\langle m\rangle[t_1,\ldots,t_m]$. We need only show that these extend to maps of ring spectra $X\langle m\rangle \to X\langle m\rangle[t_1,\ldots,t_m]$. By Proposition 1.2.1 they do extend to maps $X(n)_{(p)} \to X\langle m\rangle[t_1,\ldots,t_m]$ since the range is p-local. Composing with the canonical inclusion $X\langle m\rangle \to X(n)_{(p)}$ yields a map $X\langle m\rangle \to X\langle m\rangle[t_1,\ldots,t_m]$. This gives rise to another orientation $\Sigma^{-2}\mathbb{C}P^n \to X\langle m\rangle[t_1,\ldots,t_m]$ and we will be done if we can show that it agrees with the one we started with.

Consider the case $n=\infty$. Here one can factor the maps in question as follows:

$$\begin{tikzcd}
 & & \Sigma^{-2}\mathbb{C}P^\infty \arrow[r] \arrow[d] & BP \arrow[r] & BP[t_1,t_2,\ldots] \\
\Sigma^{-2}\mathbb{C}P^\infty \arrow[r] & BP \arrow[r] & MU_{(p)} \arrow[ur] & &
\end{tikzcd}$$

.

By definition the map $BP \to MU_{(p)} \to BP$ is the identity. The two orientations $\Sigma^{-2}\mathbb{C}P^{\infty} \to BP[t_1, t_2, \ldots]$ therefore agree. This means that the two orientations $\Sigma^{-2}\mathbb{C}P^{n} \to X\langle m\rangle[t_1, \ldots, t_m]$ agree after mapping into $BP[t_1, t_2, \ldots]$. But the map $X\langle m\rangle[t_1, \ldots, t_m] \to BP[t_1, t_2, \ldots]$ is a $(2p^{m+1} - 3)$-equivalence, so they agree as maps $\Sigma^{-2}\mathbb{C}P^{n} \to X\langle m\rangle[t_1, \ldots, t_m]$. This completes the proof.

1.5 The Spectra $Xp(n)$

We record some results which will be needed in chapter 3. Recall that M_n denotes the Moore spectrum for the ring $\mathbb{Z}[x_i \mid i \neq p^j - 1 \text{ for some } j]$ where $|x_i| = 2i$. Let $\overline{M}_n$ be the Moore spectrum for the ring $\mathbb{Z}[x_i \mid i \equiv 0(2),\ i < 2n]$.

Proposition 1.5.1. After localizing at 2, there is an equivalence

$$Xp(n)\wedge\overline{M}_n \longrightarrow X(2n)$$

of ring spectra.

proof: We already have a 2-local equivalence $X\langle 2m\rangle\wedge M_{2m} \to X(2n)$. Let $\overline{M}_n \to M_{2m}$ be the obvious inclusion. Composing with the above equivalence gives a map $\overline{M}_n \to X(2n)$. This can be multiplied by the inclusion $Xp(n) \hookrightarrow X(2n)$ to obtain a map of ring spectra $Xp(n)\wedge\overline{M}_n \to X(2n)$. By construction, it is a homology equivalence. This completes the proof.

Finally, we need to note the existence of one more family of ring spectra. Let $[k]\ \Sigma^{-2}\mathbb{C}P^n \to \Sigma^{-2}\mathbb{C}P^n$ be the map induced by $z \longmapsto z^k : S^1 \longrightarrow S^1$. Observe that $[k]\circ[\ell] = [k\cdot\ell]$ and hence that $\frac{1}{2}([1] - [-1])$ is idempotent and of degree one on $\pi_0(\Sigma^{-2}\mathbb{C}P^n)$. It therefore induces an idempotent $\varepsilon: X(n) \to X(n)$ after inverting 2. Call the resulting ring spectrum $\overline{X(n)}$.

Proposition 1.5.2. After inverting 2, there exist ring spectra $\overline{X(n)} \subset X(n)$ with $H_*(\overline{X(n)}) \approx \mathbb{Z}[\frac{1}{2}][b_2, b_4, \ldots]$. There is an equivalence of ring spectra $X(n) \to \overline{X(n)}[x_1, x_3, \ldots]$.

proof: Only the last assertion needs to be proved. The Atiyah-Hirzebruch spectral sequence for $[\mathbb{C}P^n, \overline{X(n)}[x_1, x_3, \ldots]]$ collapses. This allows one to construct a suitable orientation $\Sigma^{-2}\mathbb{C}P^n \to \overline{X(n)}[x_1, x_3, \ldots]$. If one is interested in compatibility over the various n, one can first construct the splitting of MU and then restrict to the finite cases as in the proof of Corollary 1.3.9. This completes the proof.

Chapter 2 Applications

In this chapter we present the applications of the spectra $X(n)$ and $X\langle m\rangle$ which were enumerated in the introduction. They have been included largely for amusement and serve to illustrate the ease with which certain kinds of information can be obtained about $X(n)$ and $X\langle m\rangle$. Many of these applications arose out of attempts to understand the conjectures of Ravenel [19], and the interested reader may wish to look there for an exposition of our prevailing philosophy.

In the first two sections of this chapter we will work mostly at the prime two. This is largely for technical ease in the statements of results. Most of these results have odd primary analogues which, though more difficult to state, require only trivial modifications of the two primary arguments in their proofs. At the end of each section we will indicate the necessary modifications at odd primes.

2.1 Regular BP Module Spectra

Recall that $BP_* \approx \mathbf{Z}_{(p)}[v_1, v_2, \ldots]$ where $|v_i| = 2p^i - 2$. Whenever v_0 arises it will be taken to mean multiplication by p. Given any sequence $c = (c_0, c_1, \ldots)$ of elements of BP_*, the Baas-Sullivan [2] technique of bordism with singularities yields certain BP module spectra BP_c. In case c is a regular sequence on has $\pi_*(BP_c) = BP_*/(c)$. We will be interested in the special case when c is a subsequence of $(v_0, v_1, \ldots)$.

Definition. A regular BP-module spectrum is a BP module spectrum of the form BP_c with c a subsequence of $(v_0, v_1, \ldots)$. The regular BP module spectra with $c = (v_{n+1}, v_{n+2}, \ldots)$ are denoted $BP\langle n \rangle$.

The following lemma has been proven by Baas and Madsen [3].

Lemma 2.1.1. Let BP_c be a regular BP module spectrum. Then one has $H^*(BP_c; \mathbb{Z}/p) \approx A \otimes_E E[Q_i : v_i \in c]$ where $E = E[Q_0, Q_1, \ldots]$ is the exterior subalgebra of A generated by the primitives of odd degree.

proof: We will prove the dual statement. For an element $\phi \in A_*$ let $\hat{\phi}$ denote $\chi(\phi)$. We will show that any non-trivial map $BP_c \to HZ/p$ induces an isomorphism of $H_*(BP_c;\mathbf{Z}/p)$ with the subalgebra $E[\hat{\tau}_i \mid v_i \in c] \otimes \mathbf{Z}/p[\hat{\xi}_1,\hat{\xi}_2,\ldots]$ of the dual Steenrod algebra.

Suppose first that c consists only of the single element v_i. Let M denote the cofibre of $v_i : S^{2p^i-2} \to BP$. Then the map $BP \to BP_c$ factors through $BP \to M$, and we obtain by composition a map $M \to BP_c \to H\mathbb{Z}/p$. The element v_i has Adams filtration 1. It follows that $H(M;\mathbb{Z}/p)$ is given by a non-trivial A-module extension in $\mathrm{Ext}_A^{1,2p^i-1}(\mathbf{Z}/p, H_*(BP;\mathbb{Z}/p))$. This group is easily calculated to be $\mathbb{Z}/p$ so there is only one possibility for $H_*(M;\mathbf{Z}/p)$ -- namely the sub module of A_* generated by $\mathbf{Z}/p[\hat{\xi}_1,\hat{\xi}_2,\ldots]$ and $\hat{\tau}_i$. The image of $H_*(BP_c;\mathbf{Z}/p) \to A_*$ therefore contains $E[\hat{\tau}_i]$. Since BP_c is a BP-module spectrum it must contain all of $E[\hat{\tau}_i] \otimes \mathbf{Z}/p[\hat{\xi}_1,\hat{\xi}_2,\ldots]$. The cofibre sequence $\Sigma^{2p^i-2}BP \to BP \to BP_c$ shows that H_*BP_c can be no larger than this. It follows that the lemma is true when c consists of one element.

Morava [16] has constructed pairings $BP_c \wedge BP_{c'} \to BP_{(c,c')}$ where (c,c') denotes the concatenation of the sequences c and c'. These pairings induce isomorphisms of $H^0(\ ;\mathbf{Z}/p)$. It follows that the image of $H_*(BP_c;\mathbf{Z}/p)$ in A_* contains the conjectured subalgebra when c has only finitely many elements. The cofibre sequence

$$\Sigma^{|c_n|} BP_{(c_0,\ldots,c_{n-1})} \to BP_{(c_0,\ldots,c_{n-1})} \to BP_{(c_0,\ldots,c_n)}$$

shows by induction that $H_*(BP_c;\mathbf{Z}/p)$ can be no larger than this subalgebra. The lemma is therefore true when c has only finitely many elements. The general case follows by passage to direct limits. This completes the proof.

Since the homology and the homotopy of the regular BP-module spectra are commutative rings one might expect that they are commutative ring spectra. It can be shown ([16], [19], [20]) that the BP_c are ring spectra, but as far as commutativity is concerned, we have

Theorem 2.1.2. The only 2-local regular BP-module spectra which might admit commutative multiplications are the $BP\langle n\rangle$.

Theorem 2.1.2 follows from the fact that commutative ring spectra have a slightly enriched structure. Let E be a commutative ring spectrum and let $u: E\wedge E \to E$ be the multiplication. If $\tau: E\wedge E \to E\wedge E$ denotes the twist, then $u\circ(1-\tau)$ is null-homotopic. The multiplication therefore extends over the cofibre of $(1-\tau)$ which will be denoted by $D_2(E)$. We will call any spectrum E equipped with an extension over $D_2(E)$ of the multiplication a D_2 ring spectrum. A map between D_2 ring spectra E and F will be a map $E \to F$ of ring spectra compatible with the extensions over D_2 .

The homology of $D_2(E)$ is easily computed and results in an extended squaring operation in $H_*(E;\mathbf{Z}/2)$ analogous to the Dyer Lashof operation in the homology of a double loop space. We will call this operation

$$Q_1 : H_n(E) \longrightarrow H_{2n+1}(E) .$$

The spectrum $D_2(E)$ is the subspectrum $S^1 \ltimes_{\Sigma_2} E\wedge E$ of the extended power $E\Sigma_2 \ltimes_{\Sigma_2} E\wedge E$. Ring spectra with extended power operations have been studied in complete generality by May et al. [12]. They have also studied the resulting homology operations.

The following lemma can be found in [23]. Recall that the anti-automorphism $\chi: A_* \to A_*$ is being denoted $\phi \mapsto \hat{\phi}$.

<u>Lemma 2.1.3</u>. The mod-2 Eilenberg-MacLane spectrum $HZ/2$ admits a unique D_2-structure. Furthermore, the operation Q_1 sends $\hat{\xi}_n$ to $\hat{\xi}_{n+1}$.

<u>Lemma 2.1.4</u>. Suppose that E is a (-1)-connected ring spectrum and that the unit generates $\pi_0 E$. Then any non-trivial map $E \to H\mathbf{Z}/2$ is a map of D_2 spectra.

<u>proof</u>: The commutativity of all diagrams follows from the fact that $H^0(D_2E;\mathbf{Z}/2) \approx \mathbf{Z}/2$.

<u>Lemma 2.1.5</u>. Let E satisfy the conditions of Lemma 2.1.4 and let $E \to H\mathbf{Z}/2$ be a non-trivial map. If $\hat{\xi}_n \in \operatorname{im} H_*E$ then so is $\hat{\xi}_{n+1}$.

<u>proof</u>: This is immediate from Lemmas 2.1.3 and 2.1.4.

We can now complete the proof of Theorem 2.1.2. Let BP_c be a regular BP-module spectrum admitting a commutative multiplication -- i.e., a D_2 structure. Suppose $v_i \in c$. Then $\hat{\xi}_i$ is an element of $H_*(BP_c) \subset A_*$ by Lemma 2.1.1. By Lemma 2.1.5 $\hat{\xi}_{i+1}$ is also in H_*BP_c and so by Lemma 2.1.1 again, $v_{i+1} \in C$. It must therefore be the case that $c = (v_{n+1}, v_{n+2}, \ldots)$ for some n -- i.e. $BP_c = BP\langle n \rangle$. This completes the proof.

<u>Remark</u>: Of course Theorem 2.1.2 has an odd primary analogue. Commutativity, however, is not sufficient. In fact, Morava [16] has shown that the odd primary BP_c all admit commutative multiplications. At the prime p one must require that the iterated multiplication extend over $D_p(X) = \tilde{C}(\mathbf{R}^2;p) \ltimes_{\Sigma_p} X^p$ where $\tilde{C}(\mathbf{R}^2;p)$ is the configuration space of p ordered points in $\mathbf{R}^2$, and X^p denotes the p-fold

smash product with the obvious Σ_p action. Under this assumption the proofs of this section go through without trouble.

2.2 Commutative ring spectra

In fact the notion of commutativity at the prime 2 can be quite a strong condition. In this section we will give two more applications of the lemmas of the previous section. These are slight strengthenings of results of Steinberger [23] and Nishida [17].

Theorem 2.2.1. Let E be a commutative ring spectrum. If E_* contains an invertible element of order two then E is weakly equivalent to a wedge of suspensions of Eilenberg-MacLane spectra.

Theorem 2.2.2. Let E be a commutative ring spectrum and let $\alpha \in E_*$ be an element of order two. Then α is nilpotent if and only if the Hurewicz image $H_*\alpha \in H_*(E;\mathbb{Z}/2)$ is nilpotent.

Given a spectrum X and a map $S^0 \to X$ one can form a sequence

$$X \longrightarrow D_2X \longrightarrow D_2(D_2X) \longrightarrow \cdots \longrightarrow D_2^{(n)}(X) \longrightarrow \cdots \quad .$$

Let $D_2^{\infty}(X)$ be the homotopy direct limit of this sequence. The following lemma is the main tool in the proof of Theorems 2.2.1 and 2.2.2.

Lemma 2.2.3. Let $M_{2i} = S^0 \cup_2 e^1$ be the $\mathbb{Z}/2$ Moore spectrum. Then $H^0(D_2^\infty(M_{2i});\mathbf{Z}/2) \approx \mathbf{Z}/2$ and the non-trivial map $D_2^\infty(M_{2i}) \to H\mathbf{Z}/2$ is projection onto a wedge summand.

proof: The calculation $H^0(D_2^\infty M_{2i};\mathbf{Z}/2) \cong \mathbf{Z}/2$ is trivial. We will show that $D_2^\infty M_{2i} \to H\mathbf{Z}/2$ is surjective in homology. The result will then follow from a slight modification of a theorem of Margolis [13] (Lemma 2.2.4 below).

Define a second grading of A_* by $w_t(\hat{\xi}_n) = 2^{n-1}$ and $wt(\xi \cdot \xi') = wt(\xi) + wt(\xi')$. We will show by induction on n that the image of $H_*(D_2^{(n)} M_{2i};\mathbf{Z}/2) \to A_*$ contains all elements of weight $\leqslant 2^n$. This is obviously true for $n = 0$. Assume therefore that we have shown this for $D_2^{(n)} M_{2i}$. The subgroup of A_* of elements of weight $\leqslant 2^{n+1}$ is precisely the vector space generated by the two-fold products of elements of weight $\leqslant n$, and by ξ_{n+2}. The composition

$$D_2^{(n)}(M_{2i}) \wedge D_2^{(n)}(M_{2i}) \longrightarrow D_2^{(n+1)}(M_{2i}) \longrightarrow H\mathbf{Z}/2$$

shows that the image of $H_*(D_2^{(n+1)}(M_{2i});\mathbf{Z}/2)$ contains the two-fold products of elements of weight $\leqslant 2^n$. The factorization

$$D_2\big(D_2^{(n)}(M_{2i})\big) = D_2^{(n+1)}(M_{2i}) \longrightarrow D_2(H\mathbf{Z}/2) \longrightarrow H\mathbf{Z}/2$$

shows that it also contains $Q_1(\hat{\xi}_{n+1}) = \hat{\xi}_{n+2}$. This completes the proof.

We have used the following lemma.

<u>Lemma 2.2.4</u>. Let X be a connected spectrum and let $f: X \to H\mathbb{Z}/p$ be any map. If f induces a surjective map in mod p homology then f is projection onto a wedge summand.

<u>proof</u>: It follows that f induces a monomorphism in cohomology and, since A is an injective A-module, that $H^*(X;\mathbb{Z}/p) \approx A \oplus N$. Suppose that X is $(k-1)$-connected. Then N is $(k-1)$-connected. Guided by the Bockstein spectral sequence, one can map to wedges of suspensions of $H\mathbb{Z}/2^n$ and $H\hat{\mathbb{Z}}_2$, and successively kill the cohomology in N. There results a tower of spectra $\cdots \to X^n \to X^{n-1} \to \cdots \to X^0 = X$ with $H^*(X^n) = A \oplus N_n$ where N_n is $(n+k-1)$-connected. Let F denote the fibre of $\varprojlim_n X^n \to HQ \wedge \varprojlim_n X^n$. The composite

$$F \longrightarrow X \longrightarrow H\mathbb{Z}/p$$

induces an isomorphism in both mod-p and rational cohomology. Since the spectra involved are connected it is an equivalence. This completes the proof.

<u>Remark</u>: The analogous theorem of Margolis ([13], Theorem 2) has a stronger conclusion but makes essential use

of the hypothesis that X be of finite type -- a property <u>not</u> enjoyed by $D_2^\infty(M_{2i})$.

<u>Proof of Theorem 2.2.1</u>. If E_* contains an invertible element of order 2 then $1 \in E_*$ has order 2, and E_* is a $\mathbb{Z}/2$ vector space. The unit $S^0 \to E$ can therefore be extended over M_{2i} . Since E is commutative this in turn extends over $D_2(M_{2i})$ which in turn extends over $D_2(D_2(M_{2i}))$ etc. Iterating this procedure and passing to the limit yields a map $D_2^\infty(M_{2i}) \to E$ extending the unit. By Lemma 2.2.3 there is a map $H\mathbb{Z}/2 \to D_2^\infty(M_{2i})$ inducing an isomorphism of π_0 . The unit $S^0 \to E$ therefore admits an extension $f: H\mathbb{Z}/2 \to E$. Let S be a homogeneous basis for E_* . The map $\underset{s \in S}{\vee} \Sigma^{|s|} H\mathbb{Z}/2 \to E$ whose $s^{\underline{th}}$ component is

$$s \wedge f : S^{|s|} \wedge H\mathbb{Z}/2 \longrightarrow E$$

induces, by construction, an isomorphism in π_* . This completes the proof.

<u>Proof of Theorem 2.2.2</u>. (This is essentially Nishida's argument [17]). Replacing α by some power if necessary we may assume that $H_*(\alpha) = 0$, i.e., that the composite

$$S^{|\alpha|} \longrightarrow E \longrightarrow H\mathbb{Z}/2 \wedge E$$

is null homotopic. We may also assume that $|\alpha|$ is even. The spectrum $H\mathbf{Z}/2$ is a direct limit of finite spectra, so, since homotopy commutes with direct limits, we have a null homotopy of

$$S^{|\alpha|} \longrightarrow E \longrightarrow H\mathbf{Z}/2^{(k)} \wedge E$$

for some finite skeleton $H\mathbf{Z}/2^{(k)}$ of $H\mathbf{Z}/2$. Similarly, the composite $H\mathbf{Z}/2^{(k)} \to H\mathbf{Z}/2 \to D_2^{\infty}(M_{2i})$ must factor through $D_2^{(n)}(M_{2i})$ for some finite n.

Since α has order 2 it extends over $\Sigma^{|\alpha|}M_{2i}$. Its square therefore extends over $D_2(\Sigma^{|\alpha|}M_{2i}) \approx \Sigma^{2|\alpha|}D_2(M_{2i})$ since $|\alpha|$ is even [4]. Repeating this procedure produces an extension of α^{2^n} over $\Sigma^{2^n|\alpha|}D_2^{(n)}M_{2i}$, and hence over $\Sigma^{2^n}H\mathbf{Z}/2^{(k)}$ by the preceding paragraph. The (2^n+1)-st power of α therefore factors as follows:

$$S^{(2^n+1)|\alpha|} \xrightarrow{1\wedge\alpha} S^{2^n} \wedge E \to \Sigma^{2^n} H\mathbf{Z}/2^{(k)} \wedge E \to E \wedge E \to E \ .$$

The composite of the first two maps is null homotopic. This completes the proof.

<u>Remark</u>. As in section 2.1 the odd primary analogues of the results of this section go through virtually unchanged -- provided that one replaces the assumption of commutativity with that of a D_p structure.

2.3 The Infinite Smash Product $\overset{\infty}{\wedge}\Sigma^{-2}\mathbb{C}P^n$.

In this section we return to the spectra $X\langle m\rangle$. We remind the reader that m and n are integers in the relation $p^m \leq n < p^{m+1}$.

Definition. The infinite smash product $\overset{\infty}{\wedge}\Sigma^{-2}\mathbb{C}P^n$ is the direct $\lim\limits_{k\to\infty} \overset{k}{\wedge}\Sigma^{-2}\mathbb{C}P^n$. The map $\overset{k}{\wedge}\Sigma^{-2}\mathbb{C}P^n \to \overset{k+1}{\wedge}\Sigma^{-2}\mathbb{C}P^n$ is obtained by smashing the identity map of $\overset{k}{\wedge}\Sigma^{-2}\mathbb{C}P^n$ with the inclusion of the bottom cell of $\Sigma^{-2}\mathbb{C}P^n$.

The main result of this section is the following structure theorem.

Theorem 2.3.1. After localizing at p there is an equivalence between the infinite smash product $\overset{\infty}{\wedge}\Sigma^{-2}\mathbb{C}P^n$ and a wedge of suspensions of $X\langle m\rangle$.

We will give two applications of Theorem 2.3.1 -- one to the structure of $[\overset{\infty}{\wedge}\Sigma^{-2}\mathbb{C}P^n, \overset{\infty}{\wedge}\Sigma^{-2}\mathbb{C}P^n]$, and one to the structure of the infinite loop space $\Omega^\infty X\langle m\rangle$ (Propositions 2.3.4 and 2.3.6).

Lemma 2.3.2. There is an isomorphism $X\langle m\rangle^*(\overset{\infty}{\wedge}\Sigma^{-2}\mathbb{C}P^n) \approx \varprojlim\limits_{k} X\langle m\rangle^*(\overset{k}{\wedge}\Sigma^{-2}\mathbb{C}P^n)$. In particular the

Atiyah-Hirzebruch spectral sequence for $X\langle m\rangle^*(\overset{\infty}{\Lambda}\Sigma^{-2}\mathbb{C}P^n)$ collapses at E^2 and converges.

proof: This is an application of Lemmas 1.2.2 and 1.2.5.

Lemma 2.3.3. The Atiyah-Hirzebruch spectral sequences for the following groups collapse and converge:

i) $\left[\overset{\ell}{\Lambda}\Sigma^{-2}\mathbb{C}P^k , \overset{\infty}{\Lambda}\Sigma^{-2}\mathbb{C}P^n\right]$ $\qquad k \leqslant n$.

ii) $\left[X(k) , \overset{\infty}{\Lambda}\Sigma^{-2}\mathbb{C}P^n\right]$ $\qquad k \leqslant n$.

iii) $\left[X\langle m'\rangle , \overset{\infty}{\Lambda}\Sigma^{-2}\mathbb{C}P^n_{(p)}\right]$ $\qquad m' \leqslant m$.

proof: Part ii) follows from part i). Indeed, multiplying the orientations $\Sigma^{-2}\mathbb{C}P^k \to X(k)$ results in maps $\overset{\ell}{\Lambda}\Sigma^{-2}\mathbb{C}P^k \to X(k)$ which induce monomorphisms in cohomology with any coefficients through a range tending to infinity with ℓ . Part iii) is immediate from ii) in view of Corollary 1.3.9. It remains to prove part i). The strategy is to produce all of the maps.

Step 1. Let $\alpha \in \pi_j(\overset{\infty}{\Lambda}\Sigma^{-2}\mathbb{C}P^n)$. Then for any q there exists a map $\Sigma^j\overset{q}{\Lambda}\Sigma^{-2}\mathbb{C}P^n \to \overset{\infty}{\Lambda}\Sigma^{-2}\mathbb{C}P^n$ extending α . Indeed, since homotopy commutes with direct limits, there is an element $\bar{\alpha} \in \pi_j(\overset{r}{\Lambda}\Sigma^{-2}\mathbb{C}P^n)$, for some r , which projects to α under the homomorphism

$$\pi_j\left(\overset{r}{\Lambda}\Sigma^{-2}\mathbb{C}P^n\right) \longrightarrow \pi_j\left(\overset{\infty}{\Lambda}\Sigma^{-2}\mathbb{C}P^n\right) .$$

The desired map is then given as the following composite:

$$S^j \wedge \overset{q}{\Lambda}\Sigma^{-2}\mathbb{C}P^k \xrightarrow{\bar{\alpha}\wedge i} \overset{r}{\Lambda}\Sigma^{-2}\mathbb{C}P^n \wedge \overset{q}{\Lambda}\Sigma^{-2}\mathbb{C}P^n \longrightarrow \overset{\infty}{\Lambda}\Sigma^{-2}\mathbb{C}P^n .$$

<u>Step 2</u>. Given $x \in H^i(\overset{\ell}{\Lambda}\Sigma^{-2}\mathbb{C}P^k;\mathbb{Z})$ there exists a q and a map $f:\overset{\ell}{\Lambda}\Sigma^{-2}\mathbb{C}P^k \to \Sigma^i\overset{q}{\Lambda}\Sigma^{-2}\mathbb{C}P^k$ so that the induced map in cohomology sends a fixed generator $\iota \in H^i(\Sigma^i\overset{q}{\Lambda}\Sigma^{-2}\mathbb{C}P^k;\mathbb{Z}) \approx \mathbb{Z}$ to x . In fact, one can use an appropriate diagonal map in view of the ring structure of $H^*(\overset{\ell}{\Pi}\mathbb{C}P^k)$.

<u>Step 3</u>. Let z be an element of $H^i(\overset{\ell}{\Lambda}\Sigma^{-2}\mathbb{C}P^k;\pi_j(\overset{\infty}{\Lambda}\Sigma^{-2}\mathbb{C}P^n))$. Choose a homomorphism $g:\mathbb{Z} \to \pi_j(\overset{\infty}{\Lambda}\Sigma^{-2}\mathbb{C}P^n)$ so that there is an element $x \in H^i(\overset{\ell}{\Lambda}\Sigma^{-2}\mathbb{C}P^k;\mathbb{Z})$ projecting to z under the coefficient homomorphism g . Finally, let $\alpha = g(1) \in \pi_j(\overset{\infty}{\Lambda}\Sigma^{-2}\mathbb{C}P^n)$. By Step 2, there exists an integer q and a map $f:\overset{\ell}{\Lambda}\Sigma^{-2}\mathbb{C}P^k \to \Sigma^i\overset{q}{\Lambda}\Sigma^{-2}\mathbb{C}P^k$ with $f^*(\iota) = x$. By Step 1 there is a map

$$\Sigma^j\overset{q}{\Lambda}\Sigma^{-2}\mathbb{C}P^k \longrightarrow \overset{\infty}{\Lambda}\Sigma^{-2}\mathbb{C}P^n$$

extending α . Consider the following composite in which the first map is the (j-i)-fold suspension of f:

$$\Sigma^{j-i}\overset{\ell}{\wedge}\Sigma^{-2}\mathbb{C}P^k \longrightarrow \Sigma^{j}\overset{m}{\wedge}\Sigma^{-2}\mathbb{C}P^k \longrightarrow \overset{\infty}{\wedge}\Sigma^{-2}\mathbb{C}P^n .$$

The first map represents $x \in H^i(\overset{\ell}{\wedge}\Sigma^{-2}\mathbb{C}P^k;\pi_j(\Sigma^j\overset{m}{\wedge}\Sigma^{-2}\mathbb{C}P^k))$ $= H^i(\overset{\ell}{\wedge}\mathbb{C}P^k;\mathbb{Z})$. By naturality of the Atiyah-Hirzebruch spectral sequence, the composite represents z . This completes the proof.

We are now able to prove Theorem 2.3.1. Let us suppose by induction that we have constructed an equivalence between the p-localization of $\overset{\infty}{\wedge}\Sigma^{-2}\mathbb{C}P^n$ and $F \vee N$, where F is a wedge of suspensions of $X\langle m\rangle$ and N is (k-1)-connected. The inductive step is to construct an equivalence between N and $F' \vee N'$, where F' is a wedge of suspensions of $X\langle m\rangle$ and N' is k-connected. The result will then follow by passing to the limit. Now $\pi_k(N) \approx H_k(N;\mathbb{Z}_{(p)})$ is a (countably generated) free $\mathbb{Z}_{(p)}$-module, since it is a summand of $H_k(\overset{\infty}{\wedge}\Sigma^{-2}\mathbb{C}P^n;\mathbb{Z}_{(p)})$. Choose a basis $\{s_1,s_2,\ldots\}$ for $\pi_k(N)$ and let F' and F'_q denote respectively $\bigvee_{\{s_1,s_2,\ldots\}} \Sigma^k X\langle m\rangle$ and $\bigvee_{\{s_1,\ldots,s_q\}} \Sigma^k X\langle m\rangle$. Let M_i be the free $\mathbb{Z}_{(p)}$-module with generator s_i . Suppose by induction that we can write $N \approx N_q \vee F'_q$ with $\pi_k(N_q) \approx \bigoplus_{i>q} M_i$. We wish to decompose $N_q \approx N_{q+1} \vee \Sigma^k X\langle m\rangle$ with $\pi_k(N_{q+1}) \approx \bigoplus_{i>q+1} M_i$. The desired equivalence then results by passing to the limit.

Since N_q is a wedge summand of $\overset{\infty}{\Lambda}\Sigma^{-2}\mathbb{C}P^n$, the Atiyah-Hirzebruch spectral sequences for $[X\langle m\rangle, N_q]$ and $[N_q, X\langle m\rangle]$ collapse and converge. There result maps $i:\Sigma^k X\langle m\rangle \to N_q$ and $p:N_q \to \Sigma^k X\langle m\rangle$ representing the inclusion and projection of the first factor of $\bigoplus_{i>q} M_i$ in $H^0(X\langle m\rangle;\pi_k(N_q)) \approx \mathrm{Hom}(M_{q+1}, \bigoplus_{i>q} M_i)$ and $H^k(N_q;\pi_0 X\langle m\rangle) \approx \mathrm{Hom}(\bigoplus_{i>q} M_i, M_{q+1})$ respectively. The composite

$$\Sigma^k X\langle m\rangle \longrightarrow N_q \longrightarrow \Sigma^k X\langle m\rangle$$

therefore induces the identity on $H_k(\Sigma^k X\langle m\rangle;\mathbb{Z}_{(p)})$. Since $H^*(X\langle m\rangle;\mathbb{Z}/p)$ is a cyclic A-module the composite is a mod-p homology equivalence. It is therefore a homotopy equivalence since $H^*(X\langle m\rangle;\mathbb{Z}_{(p)})$ is a $\mathbb{Z}_{(p)}$-module of finite type. This completes the proof.

We now present two applications of Theorem 2.3.1. For the rest of this section all spectra will be localized at p. Observe that there is a homomorphism

$$\left[\overset{k}{\Lambda}\Sigma^{-2}\mathbb{C}P^n, \overset{k}{\Lambda}\Sigma^{-2}\mathbb{C}P^n\right] \longrightarrow \left[\overset{k+1}{\Lambda}\Sigma^{-2}\mathbb{C}P^n, \overset{k+1}{\Lambda}\Sigma^{-2}\mathbb{C}P^n\right]$$

given by smashing with the identity map of $\Sigma^{-2}\mathbb{C}P^n$.

<u>Proposition 2.3.4</u>. If $n \geqslant p$, the map

$$\lim_{k\to\infty}\left[\overset{k}{\Lambda}\Sigma^{-2}\mathbb{C}P^n\,,\,\overset{k}{\Lambda}\Sigma^{-2}\mathbb{C}P^n\right] \longrightarrow \left[\overset{\infty}{\Lambda}\Sigma^{-2}\mathbb{C}P^n\,,\,\overset{\infty}{\Lambda}\Sigma^{-2}\mathbb{C}P^n\right]$$

is neither surjective nor injective.

<u>Remark</u>. When $n < p$ the spectra involved are wedges of p-local spheres. In this case the map is injective but not surjective. Details are left to the reader.

<u>proof</u>: Any element of a direct limit must arise at some finite stage. The map

$$\overset{\infty}{\Lambda}\Sigma^{-2}\mathbb{C}P^n \longrightarrow X\langle m\rangle \longrightarrow \overset{\infty}{\Lambda}\Sigma^{-2}\mathbb{C}P^n$$

which induces the identity on π_0 cannot arise at a finite stage. The result would be a map $X\langle m\rangle \to \overset{k}{\Lambda}\Sigma^{-2}\mathbb{C}P^n$ inducing an isomorphism of $H^0(\ ;\mathbf{Z}/p)$, violating the action of the Steenrod algebra. The map in question is therefore not surjective.

Let $\alpha \in \pi_{2p-3}(S^0)$ be the element of Hopf invariant one. Then

$$\alpha\wedge 1 : \Sigma^{2p-3}\overset{\infty}{\Lambda}\Sigma^{-2}\mathbb{C}P^n \longrightarrow \overset{\infty}{\Lambda}\Sigma^{-2}\mathbb{C}P^n$$

is null homotopic by Theorem 2.3.1 since $X\langle m\rangle$ is a ring spectrum, and since

$$S^{2p-3} \xrightarrow[\alpha]{} S^0 \hookrightarrow \Sigma^{-2}\mathbb{C}P^p \longrightarrow X\langle m \rangle$$

is null homotopic. On the other hand, any element which vanishes in a direct limit of abelian groups must vanish at a finite stage. The result now follows from the following lemma.

Lemma 2.3.5. Let F be a finite spectrum. Then $\alpha\wedge 1 : \Sigma^{2p-3}F \to F$ is not null homotopic.

proof: The cofibre of $\alpha\wedge 1$ is $M_\alpha \wedge F$ where $M_\alpha = S^0 \cup_\alpha e^{2p-2}$. We must show that $M_\alpha \wedge F \not\simeq F \vee \Sigma^{2p-2}F$. Let n be the largest integer with the property that the reduced power P^n acts non trivially on $H^*(F;\mathbb{Z}/p)$. Then P^{n+1} acts non-trivially on $H^*(M_\alpha \wedge F;\mathbb{Z}/p)$ but trivially on $H^*(F \vee \Sigma^{2p-2}F;\mathbb{Z}/p)$. This completes the proofs of lemma 2.3.5 and Proposition 2.3.4.

Our second application concerns the structure of the infinite loop space $\Omega^\infty X\langle m \rangle$. The spectra $BP\langle n \rangle$ all have the property that $\Omega^\infty BP\langle n \rangle \simeq \Omega^\infty BP\langle 1 \rangle \times Y$ [24] for some infinite loop space Y. ($\Omega^\infty BP\langle 1 \rangle$ is one of the (p-1) similar factors of $BU_{(p)}$). We will show that this is not the case for $X\langle m \rangle$. In fact we will prove a little more.

Proposition 2.3.6. There does not exist a map $\Sigma^{2p-4}\mathbb{C}P^n \to \Omega^\infty X\langle m-1\rangle$ inducing an isomorphism of π_{2p-2}.

The proof of Proposition 2.3.6 consists of two lemmas.

Lemma 2.3.7. Let E be a ring spectrum and let $\alpha \in \pi_i E$ be in the image of $E^{-i-2}(\mathbb{C}P^n) \to E^{-i-2}(S^2) \approx \pi_i E$. Then every element in the kernel of the $X\langle m\rangle$-Hurewicz homomorphism $E_* \to X\langle m\rangle_* E$ is annihilated by some power of α.

proof: Let $f \in \pi_j E$ be in the kernel of the $X\langle m\rangle$-Hurewicz homomorphism. Since homotopy commutes with direct limits, the composite

$$S^j \xrightarrow{f} E \longrightarrow X\langle m\rangle^{(k)} \wedge E$$

is null homotopic for some finite skeleton $X\langle m\rangle^{(k)}$ of $X\langle m\rangle$. By Theorem 2.3.1 there is a map $X\langle m\rangle \to \overset{\infty}{\wedge}\Sigma^{-2}\mathbb{C}P^n$ inducing an isomorphism of π_0. This map restricts to a map $X\langle m\rangle^{(k)} \to \overset{N}{\wedge}\Sigma^{-2}\mathbb{C}P^n$ for some N. It follows that the composite

$$S^j \xrightarrow{f} E \longrightarrow \overset{N}{\wedge}\Sigma^{-2}\mathbb{C}P^n \wedge E$$

is null homotopic.

The element $\alpha : S^i \to E$ admits an extension to $\Sigma^{i-2}\mathbb{C}P^n$. It follows that α^N can be extended over $\overset{N}{\wedge}\Sigma^{i-2}\mathbb{C}P^n$. The element $\alpha^N \cdot f$ therefore factors as

$$S^{i\cdot N+j} \xrightarrow{1\wedge f} S^{i\cdot N}\wedge E \longrightarrow \overset{N}{\wedge}\Sigma^{i-2}\mathbb{C}P^n \wedge E \longrightarrow E .$$

The composite of the first two maps is null homotopic. This completes the proof.

<u>Lemma 2.3.8</u>. Let $v_1 \in \pi_{2p-2}X\langle m-1\rangle \approx \mathbb{Z}_{(p)}$ be a generator. There exists an element $w \in \pi_* X\langle m-1\rangle$ with the following properties:

i) w is in the kernel of the $X\langle m\rangle$- Hurewicz homomorphism.

ii) None of the elements $v_1^k w$ is zero.

<u>proof</u>: The following line was suggested by Mahowald.

<u>Step 1</u>. Let M_p be the Moore spectrum $S^0 \cup_p e^1$. There exists an element $\bar{\bar{v}} \in \pi_{2p^m-2}(X\langle m-1\rangle \wedge M_p)$ whose image under the map $\pi_*(X\langle m-1\rangle \wedge M_p) \to \pi_*(BP\wedge M_p) \approx \mathbb{Z}/p[v_1, v_2, \ldots]$ is v_m. Indeed, the map of Adams spectral sequences is surjective and collapses in the range $t \geqslant (2p^m - 2)\cdot s$. This is because in this range, the usual change of rings converts it into the map of co-algebra cohomology induced by

$$E[\tau_1,\ldots,\tau_m]\otimes \mathbf{Z}/p[\xi_{n+1}] \longrightarrow E[\tau_1,\ldots,\tau_m] \ .$$

The listed generators are primitive so the map in question is $\mathbb{Z}/p[v_1,\ldots,v_m]\otimes E[x] \to \mathbb{Z}/p[v_1,\ldots,v_m]$. The bidegree of x is $(1, 2p^m - 2)$. Since there are no differentials in the range there can be no differentials in the domain.

Step 2. Consider the exact sequence

$$\pi_*X\langle m-1\rangle \xrightarrow{p} \pi_*X\langle m-1\rangle \longrightarrow \pi_*(X\langle m-1\rangle \wedge M_p) \xrightarrow{\delta} \pi_{*-1}X\langle m-1\rangle .$$

Let w be the element $\delta(\bar{v}) \in \pi_{2p^m-3}(X\langle m-1\rangle)$. Then w is in the kernel of $\pi_*X\langle m-1\rangle \to X\langle m\rangle_* X\langle m-1\rangle$. In fact the range is $(2p^{m+1} - 3)$ equivalent to $BP_*X\langle m-1\rangle$ which is torsion free.

Step 3. None of the elements $v_1^k w \in \pi_*X\langle m\rangle$ is zero. Indeed, suppose one had $v_1^k w = 0$. Then $v_1^k \bar{v}$ would be the mod p reduction of an element of $\pi_*X\langle m-1\rangle$ which would have non-trivial image in π_*BP . This contradicts the (elementary) calculation

$$\pi_*X\langle m-1\rangle \otimes \mathbb{Q} \approx \mathbb{Q}[v_1,\ldots,v_{m-1}] \hookrightarrow \pi_*BP\otimes\mathbb{Q} \approx \mathbb{Q}[v_1,v_2,\ldots] \ .$$

This completes the proof.

2.4 Nilpotence in $\pi_* X\langle m\rangle$

There is a general conjecture that the torsion elements of $\pi_* X\langle m\rangle$ are nilpotent. We will get nowhere near proving this, but in this section we will obtain some preliminary information. Observe that the map $\pi_* X\langle m\rangle \to \pi_* BP$ is a monomorphism modulo torsion, and that the image can be identified with $\mathbb{Z}_{(p)}[v_1, \ldots, v_m]$. The following theorem shows that some classes of torsion elements in $\pi_* X\langle m\rangle$ are nilpotent.

Theorem 2.4.1. Let $v_1 \in \pi_{2p-2}(X\langle m\rangle) \approx \mathbb{Z}_{(p)}$ be a generator and let $\alpha \in \pi_j X\langle m\rangle$ be any element.

i) If $p \cdot \alpha = 0$ then α is nilpotent.

ii) If $v_1 \cdot \alpha = 0$ then α is nilpotent.

proof: In both cases α is not detected in BP homology, hence perforce not in mod-p homology. Part i) therefore follows from Theorem 2.2.2 and its odd primary analogue (and the fact that the $X(n)$ are Thom spectra of double loop maps). For part ii) let Y denote the (2p-1) skeleton of the cofibre of $v_1 : \Sigma^{2p-2} X\langle m\rangle \to X\langle m\rangle$. The spectrum Y has three cells and the image in homology of any non-trivial map $Y \to H\mathbb{Z}/p$ is the vector space with basis $\{1, \hat{\tau}_1, \hat{\xi}_1\}$. The argument in the proof of Lemma

2.2.3 shows that the image in homology of the map $D_p^{\infty}Y \to H\mathbb{Z}/p$ is $E[\hat{\tau}_1, \hat{\tau}_2, \ldots] \otimes \mathbb{Z}/p[\hat{\xi}_1, \hat{\xi}_2, \ldots]$. The map $M_p \wedge D_p^{\infty}(Y) \to H\mathbb{Z}/p$ is therefore surjective in homology and hence by Lemma 2.2.4, projection onto a wedge summand. One can now imitate the proof of Theorem 2.2.2 to show that some power of α is in the kernel of $\pi_* X\langle m \rangle \to \pi_* X\langle m \rangle \wedge M_p$. This means that some power of α is divisible by p. Since α was a p-torsion element it must be nilpotent. This completes the proof.

<u>Remark (Added in proof)</u>. We have recently been able to extend this to an analogous result involving elements annihilated by v_k, $k \leqslant m$. Details will appear elsewhere.

2.5 A Geometric Decomposition of $X(2)$.

In this section we present a decomposition of $X(2)$ localized at 2 in terms of nilpotent self maps of certain finite complexes. Motivation for such a decomposition arises from two sources. For one thing, the squares of the maps in question behave as if they were obtained by smashing the identity map of a finite complex with an element of $\pi_* S^0$. By Nishida's theorem [17] they must be nilpotent. Second, there is a general conjecture (motivated by [19]) which states that for a ring spectrum R, elements in the kernel of $X\langle m\rangle_* R \to X\langle m+1\rangle_* R$ are nilpotent. It is natural therefore to look for a way in which $X\langle m+1\rangle$ is built out of $X\langle m\rangle$ and certain nilpotent maps. The decomposition of this section is the case $m = 0$ for $p = 2$. We will describe at the end of this section how this result can be extended, but as we have no applications in mind, we will only sketch the result.

Before stating our main result, we need to establish some conventions. All spectra will be localized at 2. Recall that the homology $H_*(X(2);\mathbb{Z}_{(2)})$ is a polynomial ring $\mathbb{Z}_{(2)}[b_1]$ with $|b_1| = 2$. The spectrum $X(2)$ therefore admits a CW decomposition with one (2-local) cell in every even dimension. We will use the symbol X_k^n to denote the sub-quotient of $X(n)$ with 2^k cells, starting in dimension $2n\cdot 2^k$. It is the quotient of the

$2((n+1)\cdot 2^k - 1)$ skeleton of $X(2)$ by its $2\cdot(n\cdot 2^k - 1)$ skeleton. When $n = 0$ we will abbreviate this notation to X_k.

Lemma 2.5.1. i) One can find a primitive element $b \in X(2)_2 X(2)$ so that $X(2)_* X(2) \approx X(2)_*[b]$

ii) Let $\theta_k \in X(2)^{2k} X(2)$ be dual to b^k. Then $\theta_k \circ \theta_\ell = (k,\ell)\theta_{k+\ell}$.

iii) The coproduct is given by

$$\theta_n \longrightarrow \sum_{i+j=n} \theta_i \otimes \theta_j$$

proof: Part i) follows from the results of Chapter 1. Parts ii) and iii) are formal consequences of i).

We can regard the operations θ_k as maps $X(2) \to \Sigma^{2k} X(2)$.

Lemma 2.5.2. The map $\theta_{k*} : H_n(X(2);\mathbb{Z}_{(2)}) \to H_{n-2k}(X(2);\mathbb{Z}_{(2)})$ sends b^k to $\binom{n}{k} b^{n-k}$.

proof: This follows formally from Proposition 2.5.1 but the following line is also instructive. We introduce the total operation $\theta_t = \Sigma\theta_k t^k : X(2) \to X(2)[t]$. This is a map of ring spectra and under the equivalence $X(2)\wedge X(2) \approx X(2)[t]$ can be identified with the homomorphism η_R. By definition we have $\theta_t(b) = b+t$. It follows that $\theta_t(b^n) = (b+t)^n = \Sigma\binom{n}{k}b^{n-k}t^k$. This completes the proof.

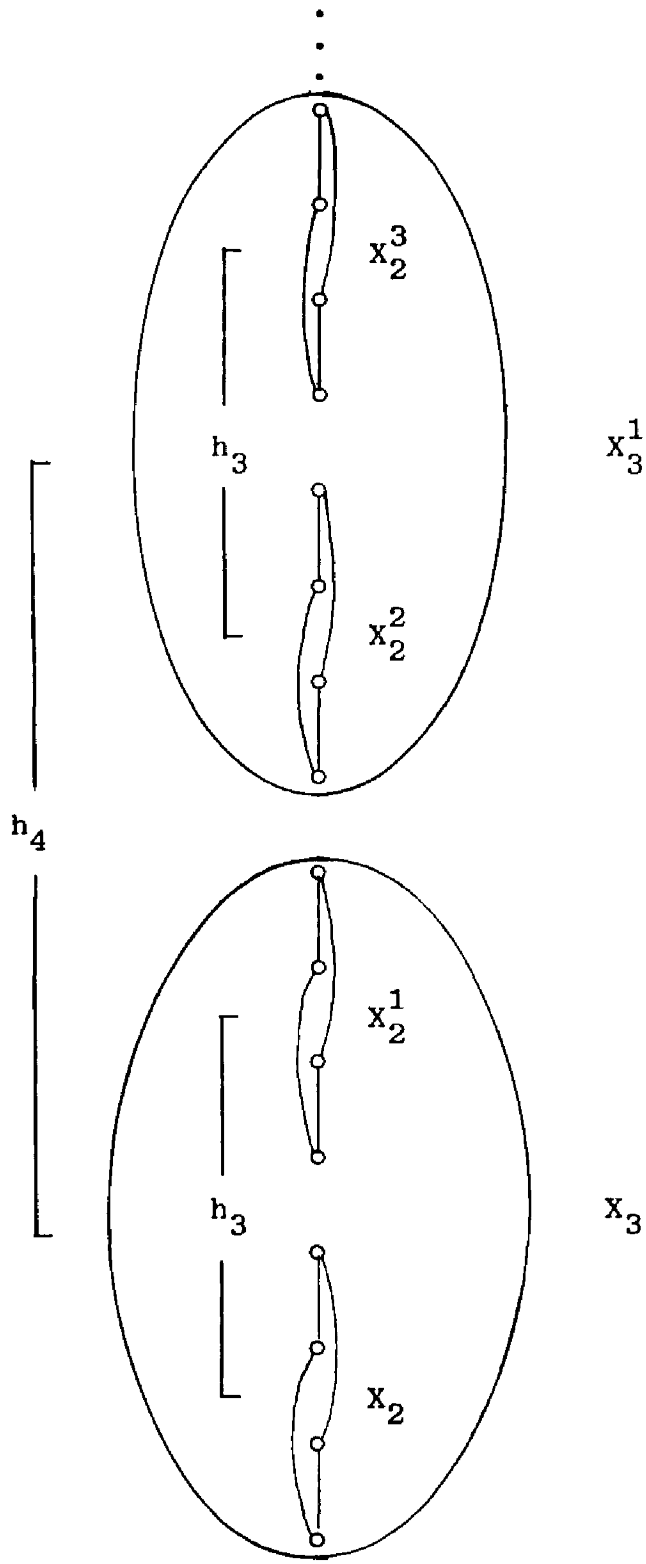
Our main use of the operations θ_k is the following periodicity theorem for the X_k^n. This result is due to Mahowald.

<u>Proposition 2.5.3</u>. There is an equivalence $\bar{\theta}: X_k^n \to \Sigma^{2n\cdot 2^k} X_k$.

<u>proof</u>: Consider the operation $\theta_{n\cdot 2^k} : X(2) \to \Sigma^{2n\cdot 2^k} X(2)$. Passing to sub-quotients yields a map $\bar{\theta}: X_k^n \to \Sigma^{2n\cdot 2^k} X_k$. The effect of $\bar{\theta}$ in homology is determined by Lemma 2.5.2 and is easily seen to be an isomorphism over $\mathbb{Z}_{(2)}$.

By definition there is a cofibration $\Sigma^{-1}X_k^1 \to X_k \to X_{k+1}$. We denote by h_k the composite $\Sigma^{2n\cdot 2^k-1}X_k \xrightarrow{\bar{\theta}^{-1}} \Sigma^{-1}X_k^1 \to X_k$. This is justified since the map is represented in the Adams spectral sequence by the product of the identity map of $H^*(X_k;\mathbb{Z}/2)$ with $h_k \in \mathrm{Ext}_A(\mathbf{Z}/2,\mathbf{Z}/2)$.

The maps h_k give a decomposition of $X(2)$. Indeed, one starts with S^0 and attaches a two cell by the map h_1. Having inductively obtained X_k, one builds X_{k+1} as the cofibre of h_k. Since $X(2)$ is the limit of the X_k, this actually does build $X(2)$. Our main goal is to show that the h_k are nilpotent.



X(2)

<u>Lemma 2.5.4</u>. The map h_k^2 has order 2.

<u>proof</u>: Consider the $2\cdot(3\cdot 2^k - 1)$-skeleton of $X(2)$. By the preceding paragraph it admits a decomposition

$$X_k \cup X_k^1 \cup X_k^2 \approx X_k \cup_f C\Sigma^{2^k-1}X_k \cup_g C\Sigma^{2\cdot 2^k-1}X_k \ .$$

The attaching map f is by definition h_k. We will show that the attaching map g is $2(\text{odd})h_k$. By definition, the map g is the unique map making the following diagram commute (The unspecified maps are the obvious attaching maps):

$$\begin{tikzcd}
\Sigma^{-1}X_k^2 \arrow[r] \arrow[d, "\theta_{2^{k+1}}"] & X_k^1 \arrow[d, "\theta_{2^k}"] \\
\Sigma^{2^{k+1}-1}X_k \arrow[r, "g"'] & X_k
\end{tikzcd} \ .$$

By definition of the maps θ_k there is the following commutative diagram

$$\begin{tikzcd}
\Sigma^{-1}X_k^2 \arrow[r] \arrow[d, "\theta_{2^k}"] & X_k^1 \arrow[d, "\theta_{2^k}"] \\
\Sigma^{-1}X_k^1 \arrow[r] \arrow[d, "\theta_{2^k}"] & X_k \\
X_k \arrow[ur, "h_k"'] &
\end{tikzcd} \ .$$

Since $\theta_{2^k} \circ \theta_{2^k} = (2^k, 2^k)\theta_{2^{k+1}}$ we must have $g = (2^k, 2^k)h_k$. The binomial coefficient $(2^k, 2^k)$ is twice an odd number. This completes the proof.

Lemma 2.5.5. The following composites are null homotopic:

i) $\Sigma^{2^k-1} X_k \xrightarrow{h_k} X_k \longrightarrow X(2) \wedge X_k$

ii) $\Sigma^{2^k-1} X_k \xrightarrow{h_k} X_k \longrightarrow \left(\overset{N}{\Lambda} \Sigma^{-2} \mathbb{C}P^2 \right) \wedge X_k \qquad (N \gg 0)$

iii) $\Sigma^{2^k-1} X_k \xrightarrow{h_k} X_k \longrightarrow \left(\overset{M}{\Lambda} \Sigma^{-1} \mathbb{R}P^2 \right) \wedge X_k \qquad (M \gg 0).$

proof: i) It follows easily from Proposition 1.4.2 that $X(2) \wedge X_k \to X(2) \wedge X_{k+1}$ is the inclusion of a wedge summand. The composite

$$\Sigma^{2^k-1} X_k \longrightarrow X_k \longrightarrow X(2) \wedge X_k \longrightarrow X(2) \wedge X_{k+1}$$

is null homotopic since it factors through the cofibration

$$\Sigma^{2^k-1} X_k \longrightarrow X_k \longrightarrow X_{k+1} \quad .$$

ii) From part i) we know that

$$\Sigma^{2^k-1} X_k \longrightarrow X_k \longrightarrow X(2)^{(n)} \wedge X_k$$

is null homotopic for some finite skeleton $X(2)^{(n)}$ of $X(2)$. For sufficiently large N there is a map $X(2)^{(n)} \to \overset{N}{\wedge}\Sigma^{-2}\mathbb{C}P^2$ extending the inclusion of the bottom cell. The composite in question therefore factors as

$$\Sigma^{2^k-1}X_k \longrightarrow X_k \longrightarrow X(2)^{(n)}\wedge X_k \longrightarrow \overset{N}{\wedge}\Sigma^{-2}\mathbb{C}P^2\wedge X_k .$$

iii) The argument for this case proceeds as in part ii) using the fact that there is a map $\Sigma^{-2}\mathbb{C}P^2 \to \overset{3}{\wedge}\Sigma^{-1}\mathbb{R}P^2$ extending the inclusion of the bottom cell.

<u>Theorem 2.5.6</u>. The map h_k is nilpotent.

<u>proof</u>: By Lemma 2.5.4, the square of h_k extends over the smash product of the domain with $\Sigma^{-1}\mathbb{R}P^2$. The M-fold iterate of h_k can therefore be extended over the smash product of the domain with $\overset{M}{\wedge}\Sigma^{-1}\mathbb{R}P^2$. Using Lemma 2.5.5 and a by now familiar argument, this guarantees that the next iterate of h_k is nullhomotopic. This completes the proof.

<u>Remark</u>. The situation of this section generalizes to the spectra $X(n)$. Instead of filtering by skeleta, one pulls the James filtration of ΩS^{2n-1} back through the fibration $\Omega SU(n) \to \Omega S^{2n-1}$. Passing to Thom spectra

results in a filtration of $X(n)$ whose associated graded is $\vee\Sigma^{2ni}X(n-1)$. The various copies of $X(n-1)$ are related by a divided polynomial subalgebra of $X(n)^*X(n)$. Analogues of the spectra X_k^n can be constructed. Instead of being built out of cells, these are $X(n-1)$-module spectra built out of copies of $X(n-1)$. A periodicity theorem analogous to Proposition 2.5.3 can easily be proven and the generalizations of the maps h_k can be constructed. At odd primes one has $h_k^2=0$. At the prime two we are currently unable to demonstrate nilpotence of the maps h_k but conjecture that this is the case. Again, we have no applications of these ideas and therefore omit the proofs.

Chapter 3. Stable Decompositions of $\Omega SU(n)$ and $\Omega Sp(n)$.

3.1 Introduction and Statement of Results

In this final chapter we will explore the structure of the suspension spectra of $\Omega SU(n)$ and $\Omega Sp(n)$. Recall that $H_*(\Omega SU(n);\mathbb{Z}) \approx \mathbb{Z}[b_1,\ldots,b_{n-1}]$ and let M_n denote the subgroup of homogeneous polynomials of degree n. A result of James [7] asserts the existence of a stable map $\Omega SU(n) \to \mathbb{C}P^{n-1}$ splitting the inclusion $\mathbb{C}P^{n-1} \to \Omega SU(n)$. We conjecture (following Mahowald) that the James splitting can be refined as follows.

Conjecture 1. There exist spectra B_n with $H_*B_n \approx M_n$ and a stable equivalence $\Omega SU(n) \approx \bigvee_{n=1}^{\infty} B_n$.

Remark. This conjecture has been verified by Mahowald when $n = 3$ and by Snaith [21] when $n = \infty$. In fact candidates for the B_n can be constructed as subquotients of the spaces of loops on certain Stiefel Manifolds. The problem, of course, is to produce the maps.

In contrast to Conjecture 1, we have the following conjecture and theorem.

Conjecture 2. After localizing at 2, any stable self map of $\Omega Sp(n)$ $(n > 1)$ inducing an isomorphism of $H_2(\ ;\mathbb{Z}/2)$ is an equivalence.

Theorem 3.1.1. Conjecture 2 is true in case $n = 2$ or 3.

The proof of Theorem 3.1.1 is given in Section 3.5.

Remark. It is shown in Appendix A that the spectra $\Omega Sp(n)$ decompose after inverting 2.

Stably, the space $Sp(2)$ breaks apart. It therefore supplies an example (apparently the first) of a space which stably decomposes, but whose loop space does not.

There is a weak sense in which we can say something about Conjecture 1. For a spectrum X let X_+ denote $X \vee S^0$. The spectra $\Omega SU(n)_+$ and $\Omega Sp(n)_+$ are the bordism theories associated to the (double loop) null homotopic maps. As such they are commutative ring spectra. Let α denote indiscriminately the inclusion of the 2-cell in either $\Omega SU(n)_+$ or $\Omega Sp(n)_+$. We can then form the localized spectra $\alpha^{-1}\Omega SU(n)_+$ and $\alpha^{-1}\Omega Sp(n)_+$.

Theorem 3.1.2. i) There is an equivalence

$$\alpha^{-1}\Omega SU(n)_+ \simeq \bigvee_{i=-\infty}^{\infty} \Sigma^{2i} X(n-1) .$$

ii) After inverting 2, there is an equivalence

$$\alpha^{-1}\Omega Sp(n)_+ \approx \bigvee_{i=-\infty}^{\infty} \Sigma^{2i}\overline{X(2n-1)} \ .$$

iii) After localizing at 2, there is a map

$$\alpha^{-1}\Omega Sp(n)_+ \longrightarrow \bigvee_{i=-\infty}^{\infty} \Sigma^{2i} Xp(n-1)$$

which induces an isomorphism of $Xp(n-1)$ homology.

Since we can completely decompose $X(n)$ and $Xp(n)$ into irreducible spectra after localizing at any prime, Theorem 3.1.2 gives a complete decomposition of $\alpha^{-1}\Omega SU(n)_+$ at any prime and of $\alpha^{-1}\Omega Sp(n)_+$ at odd primes.

An appropriate version of Conjecture 1 would imply part i) of Theorem 3.1.2. This is because inversion of the class α has the effect of assembling the spectra B_j (appropriately desuspended) to form $X(n-1)$.

Added in proof. We have recently been able to show that the map of part iii) in Theorem 3.1.2 is not an equivalence when $n=2$. The fibre is therefore a non-contractible (2-local) spectrum F with the property that $X(2)\wedge F$ is contractible. A theorem of Ravenel's ([19], section 3) asserts the existence of such spectra, and this seems to be a new example of one.

3.2 The Homology of $\Omega Sp(n)$

The proof of Theorem 3.1.1 requires a fairly detailed knowledge of the homology and connective K-theory of $\Omega Sp(n)$. In this section we will determine the algebra $H_*(\Omega Sp(n);\mathbf{Z}/2)$ as a right algebra over the Steenrod algebra. We first need to recall some facts about $Sp(n)$ and $SU(n)$. The assertions of the following lemma can be found in [22].

Lemma 3.2.1. i) There is an embedding

$$S^{4n-1} \times_{Sp(1)} Sp(1)/S^{4n-1} \times_{Sp(1)} \{1\} \longrightarrow Sp(n) ,$$

with $Sp(1)$ acting on itself by conjugation.

ii) The integral homology of $Sp(n)$ is the exterior algebra on the homology of

$$S^{4n-1} \times_{Sp(1)} Sp(1)/S^{4n-1} \times_{Sp(1)} \{1\} .$$

The space $S^{4n-1} \times_{Sp(1)} Sp(1)/S^{4n-1} \times_{Sp(1)} \{1\}$ is easily seen to be the Thom complex of the vector bundle over HP^{n-1} induced by the double cover $Sp(1) \to SO(3)$. For our purposes this space is inconvenient because it is not a suspension. We can remedy this by passing to the bundle induced over $\mathbb{C}P^{2n-2}$. It is easily seen to be the Whitney sum of a trivial line bundle with the bundle induced by the squaring map $SO(2) \to SO(2)$. Its Thom complex is therefore a suspension and carries the homology of $Sp(n)$. We will base our calculations of $H_*(\Omega Sp(n))$ on the adjoint of this map. For ease of notation we will work with the case $n = \infty$. The finite cases are determined by restriction.

Let L denote the canonical line bundle over $\mathbb{C}P^\infty$, and let P be the Thom complex of L^2. Passing to Thom complexes from the squaring map $\mathbb{C}P^\infty \to \mathbb{C}P^\infty$ results in a sequence

$$\mathbb{C}P^\infty \longrightarrow P \longrightarrow \mathbb{C}P^\infty ,$$

in which the first map is the zero section and the composite is the squaring map. The following lemma is an easy consequence of the above sequence, the Thom isomorphism, and the fact that L^2 has zero Stiefel-Whitney classes.

<u>Lemma 3.2.2</u>. i) The zero section $\mathbb{C}P^\infty \to P$ induces a monomorphism of integral cohomology which is of degree two in each positive even dimension. The cohomology ring $H^*(P;\mathbb{Z})$ is therefore generated by elements x_i of degree $2i$, subject to the relation

$$x_i \cdot x_j = 2x_{i+j} .$$

ii) Let $a_n \in H_{2n}(P;\mathbf{Z})$ be dual to x_n. Then the co-algebra structure of $H_*(P;\mathbf{Z})$ is given by

$$a_n \longmapsto a_n \otimes 1 + 2\sum_{i=1}^{n-1} a_{n-i} \otimes a_i + 1 \otimes a_n .$$

iii) As a module over the mod-2 Steenrod algebra, $H^*(P;\mathbb{Z}/2)$ is isomorphic to $H^*(S^2 \vee \Sigma^2\mathbb{C}P^\infty;\mathbb{Z}/2)$. In particular, if a_n denotes as well its own mod 2 reduction, then the right A-module structure of $H_*(P;\mathbf{Z}/2)$ is given by

$$a_n \cdot Sq = \sum \binom{n-k-1}{k} a_{n-k} .$$

The map $\Sigma P \to Sp$ yields by adjunction a map $P \to \Omega Sp$. We will use the symbol a_n to denote indiscriminantly the homology class in $H_{2n}(P;\mathbf{Z})$, its image in $H_*(\Omega Sp;\mathbf{Z})$ under the above map, and the mod 2 reductions of these classes. An easy consequence of the Eilenberg-Moore spectral sequence is that $H_*(\Omega Sp;\mathbb{Z}) \approx \mathbf{Z}[a_1,a_3,\ldots]$. The real work is to express the a_{2i} as polynomials in the a_{2i+1}.

Let a_t denote the formal sum $\sum_{i=0}^{\infty} a_i t^i$, and let a_{ev} and a_{odd} denote respectively $\sum_{i=0}^{\infty} a_{2i} t^{2i}$ and $\sum_{i=0}^{\infty} a_{2i+1} t^{2i+1}$. We make the convention that $a_0 = 1$ whenever it arises. There are the relations $a_{ev} = \frac{1}{2}[a_t + a_{-t}]$ and $a_{odd} = \frac{1}{2}[a_t - a_{-t}]$.

<u>Theorem 3.2.3.</u> i) There is an isomorphism $H_*(\Omega Sp;\mathbf{Z}) \approx \mathbf{Z}[a_1,a_3,\ldots]$.

ii) In $H_*(\Omega Sp;\mathbb{Z})$ the relation

$$a_{ev}^2 - a_{odd}^2 = a_{ev}$$

holds, hence

$$a_{ev} = \frac{1+\sqrt{1+4a_{odd}^2}}{2} .$$

iii) Let $I \subset H_*(\Omega Sp(n);\mathbb{Z})$ be the augmentation ideal. Then

$$a_{ev} \equiv a_{odd}^2 \quad \text{mod} \quad I^4$$

and

$$a_{2^k(2n+1)} \equiv (a_{2n+1})^{2^k} \quad \text{mod} \quad (2) .$$

<u>proof</u>: It suffices to verify the relation in part ii), which is easily seen to be equivalent to

$$*) \qquad a_{ev} = a_t \cdot a_{-t} .$$

Consider the set of formal power series $f(t) = \Sigma f_n t^n$, with $f_n \in H_{2n}(\Omega Sp;\mathbf{Z})$. Such a power series is said to be <u>grouplike</u> if $f(t) \mapsto f(t) \otimes f(t)$ under the coproduct. Observe that this implies $f(0) = 1$. If

$f(t) = 1 + f_n t + f_{n+1} t^{n+1} + \cdots$ is grouplike then f_n is primitive. We will see below that $H_*(\Omega Sp;\mathbf{Z})$ has no primitives in dimensions congruent to zero mod 4. It follows that if $f(t)$ is grouplike and satisfies $f(t) = f(-t)$ then $f(t) = 1$. Now the series $2a_t - 1$ and $2a_{-t} - 1$ are grouplike, being the images of b_t and b_{-t} under the map $H_*(\mathbb{C}P^\infty) \to H_*(P) \to H_*(\Omega Sp;\mathbf{Z})$. The series $f(t) = (2a_t - 1)(2a_{-t} - 1)$ is therefore grouplike and satisfies $f(t) = f(-t)$. This gives

$$(2a_t - 1)(2a_{-t} - 1) = 1$$

which implies *).

It remains to show that $H_*(\Omega Sp;\mathbf{Z})$ has no primitives in dimensions congruent to zero mod 4. Since $H_*(\Omega Sp;\mathbf{Z})$ is torsion free it suffices to verify this for $H_*(\Omega Sp;Q)$. But $H_*(\Omega Sp;Q)$ is a commutative, co-commutative Hopf algebra over Q. For such algebras the map from the primitives to the indecomposables is an isomorphism [15]. The result now follows since the indecomposables all lie in dimensions congruent to 2 mod 4. This completes the proof.

Remark. Taken together, theorems 3.2.2 and 3.2.3 determine the structure of $H_*(\Omega Sp(n);\mathbf{Z}/2)$ as an algebra over the Steenrod algebra. The key to this structure is, of course, the relation *). We will give another proof of this relation in the next section.

3.3 The Image of $H_*(\Omega Sp(n))$ in $H_*(BU)$.

In this section we will determine the images of the generators $a_n \in H_*(\Omega Sp;\mathbb{Z})$ under the monomorphism $H_*(\Omega Sp;\mathbb{Z}) \to H_*(\Omega SU;\mathbb{Z}) = H_*(BU;\mathbb{Z}) \approx \mathbb{Z}[b_1,b_2,\ldots]$. Our first task is to determine the bundle over $\mathbb{C}P^\infty$ classified by the composite:

$$**) \qquad \mathbb{C}P^\infty \longrightarrow P \longrightarrow \Omega Sp \longrightarrow \Omega SU \approx BU .$$

Let L denote the canonical line bundle over $\mathbb{C}P^\infty$ and L^* its dual.

<u>Theorem 3.3.1</u>. i) The vector bundle classified by the above composite is $L - L^*$.

ii) The map in homology induced by $P \to \Omega Sp \to BU$ sends a_t to

$$\frac{1}{2}\left[\frac{b_t}{b_{-t}} + 1\right] = \frac{1}{b_{-t}} \cdot b_{ev} .$$

<u>proof</u>: We calculate the Chern character. Since the map $P \to \Omega Sp$ is first given in the form $\Sigma P \to Sp$, it will be easier to calculate the Chern character of the composite

$$\Sigma^2 \mathbb{C}P^\infty \longrightarrow \Sigma^2 P \longrightarrow BSp \longrightarrow BSU$$

and then deduce our result from Bott periodicity. Let ξ

be the vector bundle classified by the above composite. The Chern classes of ξ are given by

$$c_{2i}(\xi) = -2s^2 \cdot x^{2i-1} \in H^{4i}(\Sigma^2 \mathbb{C}P^\infty; \mathbb{Z})$$

$$c_{2i-1}(\xi) = 0$$

where s^2 denotes the double suspension (see the remark below). The Chern classes determine the Chern character by the formula

$$ch(\xi) = \Sigma \rho_n \frac{t^n}{n!}$$

where $\rho_n = \rho_n(c_1, \ldots, c_n)$ is the Newton polynomial. Now $\rho_n(c_1, \ldots, c_n)$ is congruent to $(-1)^{n-1} n\, c_n$ mod decomposables. Since there are no cup products in $H^*(\Sigma^2 \mathbb{C}P^\infty)$ this determines the Chern character of ξ :

$$ch(\xi) = \sum_{n=1}^{\infty} \frac{(2n)2s^2 x^{2n-1}}{(2n)!}$$

$$= 2 \sum_{n=1}^{\infty} \frac{s^2 x^{2n-1}}{(2n-1)!} .$$

By Bott periodicity, the Chern character of **) is

$$2 \sum_{n=1}^{\infty} \frac{x^{2n-1}}{(2n-1)!} = e^x - e^{-x} .$$

Since $ch: K^*(\mathbb{C}P^\infty) \to H^{ev}(\mathbb{C}P^\infty; Q)$ is a monomorphism, the bundle classified by **) must be $L - L^*$. This proves i).

The homology homomorphisms induced by $L - 1$ and $L^* - 1$ send b_t to b_t and b_{-t} respectively. The homomorphism induced by $L - L^*$ therefore sends b_t to b_t/b_{-t}. Part ii) now follows since $b_t = 2a_t - 1$. This completes the proof.

Remark 1. Some comments are in order concerning the Chern classes of ξ. Lemma 3.2.2 and Theorem 3.2.3 imply that the map in cohomology induced by

$$\Sigma\mathbb{C}P \longrightarrow \Sigma P \longrightarrow Sp \longrightarrow SU$$

is zero in dimensions $4k + 1$, and has index 2 in dimensions $4k - 1$. The Chern classes of ξ are therefore as stated -- up to sign. The signs are determined by the choice of equivalence $\Omega SU \approx BU$. In Appendix B a choice is made which guarantees that the adjoint to the embedding $\Sigma\mathbb{C}P^\infty \to SU$ classifies the canonical line bundle. An elementary calculation of the Chern character shows that the Chern classes of the associated bundle $\Sigma^2\mathbb{C}P^\infty \to BSU$ are given by

$$c_j = (-1)^{j-1} s^2 x^{j-1} \in H^{2j}(\Sigma^2\mathbb{C}P^\infty; \mathbf{Z}) \qquad (j > 1) .$$

This determines the signs used above.

Remark 2. Part ii) of Theorem 3.3.1 enables one to give an alternate proof of Theorem 3.2.3 ii). Indeed, the map $H_*(\Omega Sp;\mathbb{Z}) \to H_*(BU;\mathbb{Z})$ is a monomorphism and the series a_{ev} and $a_t a_{-t}$ are both sent to $\frac{1}{b_t b_{-t}}(b_{ev})^2$.

3.4 $bu_*\Omega Sp(n)$

Since $H_*(\Omega Sp(n);\mathbf{Z})$ is torsion free, the Atiyah-Hirzebruch spectral sequence for $bu_*(\Omega Sp(n))$ collapses. The result is that $bu_*(\Omega Sp(n))$ is isomorphic to the polynomial algebra

$$\mathbf{Z}[v_1][\alpha_1,\alpha_3,\ldots,\alpha_{2n+1}] \ ,$$

where $|\alpha_i| = 2i$.

For any spectrum X there is an isomorphism (the dual of the Chern character)

$$bu_*(X)\otimes Q \approx Q[v_1]\otimes H_*(X;\mathbb{Z}) \ .$$

For our purposes it will be convenient to find generators α_{2i+1} and compute their images under the embedding

$$bu_*(\Omega Sp) \longrightarrow Q[v_1]\otimes\mathbb{Z}[a_1,a_3,\ldots] \ .$$

This can be done as follows. First, a Thom class $U \in bu^2(P)$ must be selected. A basis $\{x_j\}$ for $bu^*(P)$ is then given by

$$x_j = U\cdot(L-1)^j = \Phi_k((L-1)^j)$$

where Φ_k denotes the connective K-theory Thom isomorphism. The set $\{\alpha_k\}$ can then be taken to be the basis of $bu_*(P)$ dual to $\{x_j\}$. The elements α_{2k+1} , $k \leqslant n$, will then give rise to polynomial generators for $bu_*(\Omega Sp(n))$.

We will need to compute the Chern characters of the bundles involved. In this direction, two comments are in order. First, the Chern character can be viewed as the rationalization map

$$[X,bu] \longrightarrow \mathrm{Hom}[H_*(X),\pi_* bu \otimes Q] = H^*(X;Q[v_1]) \ .$$

We will find it useful to use the (somewhat unorthodox) notation

$$ch(\xi) = \Sigma y_i v_1^i$$

where $y_i \in H^*(X;Q)$. The powers of v_1 are usually viewed as placeholders and omitted. We will find it helpful to remember that they are powers of v_1.

Second, it will be more convenient to make our calculations over $\mathbb{C}P^\infty$ and then pass to P via the Thom isomorphism. We will therefore need to know about the relationship between the rational homology and connective K-theory Thom isomorphisms. This is given by a theorem of Riemann-Roch type as described by Dyer [6].

Let ξ be a complex vector bundle over X and let $U \in bu^*(X^\xi)$ be a Thom class. Denote by Φ_H and Φ_K the rational cohomology and connective K-theory Thom isomorphisms respectively, and set $\rho(\xi) = \Phi_H^{-1} \circ ch(U)$.

<u>Proposition 3.4.1.</u> Let x be an element of $bu^*(X)$. With the above notation,

$$\Phi_H^{-1}\circ ch\circ\Phi_K(x) = ch(x)\cdot\rho(\xi) .$$

For our purposes, a convenient Thom class $U \in bu^2(P)$ is given by the composite

$$P \longrightarrow \Omega Sp \longrightarrow \Omega SU = BU .$$

With this choice we have

$$ch(U) = \sum_{k=0}^{\infty} x_{2k+1}\frac{v_1^{2k+1}}{(2k+1)!}$$

and hence

$$\rho(L^2) = \frac{e^{xv_1} - e^{-xv_1}}{2xv_1} = \frac{\sinh(xv_1)}{xv_1} \in H^*(P;Q[v_1]) .$$

Setting $x_j = \Phi_K((L-1)^j)$ yields a basis $\{x_j\}$ for $bu^*(P)$. By Theorem 3.4.1 one has

$$\Phi_H^{-1}\circ ch(x_j) = \frac{\sinh(xv_1)}{xv_1}\left(e^{v_1} - 1\right)^j .$$

We can now define $\{\alpha_j\}$ to be the basis of $bu_*(P)$ dual to $\{x_j\}$. The image of $\{\alpha_j\}$ in $Q[v_1]\otimes H_*(P)$ is given by the basis of $Q[v_1]\otimes H_*(P)$ dual to $\{ch(x_j)\}$. This is in principle calculable, though only practical for small values of j. We tabulate the result for small values of j in the following lemma. This result has been checked by machine calculation.

<u>Lemma 3.4.2</u>. The images of α_j, $j \leq 5$, under the embedding $bu_* P \to Q[v_1] \otimes H_*(P;\mathbb{Z})$ are given by

$$\begin{aligned}
\alpha_1 &\mapsto a_1 \\
\alpha_2 &\mapsto a_2 \\
\alpha_3 &\mapsto -\tfrac{1}{6} v_1^2 a_1 - \tfrac{1}{2} v_1 a_2 + a_3 \\
\alpha_4 &\mapsto \tfrac{1}{6} v_1^3 a_1 + \tfrac{1}{6} v_1^2 a_2 - v_1 a_3 + a_4 \\
\alpha_5 &\mapsto -\tfrac{2}{15} v_1^4 a_1 + \tfrac{3}{4} v_1^2 a_3 - \tfrac{3}{2} v_1 a_4 + a_5 .
\end{aligned}$$

Identifying the α_j with their images in $bu_*(\Omega Sp)$, and using Theorem 3.2.3, results in the following formulae for the map $bu_*(\Omega Sp) \to Q[v_1] \otimes H_*(\Omega Sp;\mathbb{Z})$:

$$\begin{aligned}
\alpha_1 &\mapsto a_1 \\
\alpha_3 &\mapsto -\tfrac{1}{6} v_1^2 a_1 - \tfrac{1}{2} v_1 a_1^2 + a_3 \\
\alpha_5 &\mapsto -\tfrac{2}{15} v_1^4 a_1 + \tfrac{3}{4} v_1^2 a_3 - \tfrac{3}{2} v_1 \left[2a_1 a_3 - a_1^4 \right] + a_5 .
\end{aligned}$$

After localizing at 2 the above formulae can be simplified somewhat. Make the change of variables

$$\bar{\alpha}_3 = \alpha_3 + v_1\alpha_1 + \frac{2}{3}v_1^2\alpha_1$$

$$\bar{\alpha}_5 = (3\alpha_1^2\alpha_3 - \alpha_5) + 3v_1(\alpha_1^4 - \alpha_1\alpha_3) + v_1^2\alpha_3 + \frac{8}{15}v_1^4\alpha_1$$

$$\bar{a}_5 = 3a_1a_3 - a_5 .$$

Lemma 3.4.3. After localizing at 2, there are isomorphisms

$$bu_*(\Omega Sp(3)) \approx \mathbb{Z}_{(2)}[v_1][\alpha_1, \bar{\alpha}_3, \bar{\alpha}_5]$$

$$H_*(\Omega Sp(3);\mathbb{Z}_{(2)}) \approx \mathbb{Z}_{(2)}[a_1, a_3, \bar{a}_5] .$$

The map $bu_*(\Omega Sp(3)) \to H_*(\Omega Sp(3);\mathbb{Z}_{(2)})$ is given by:

$$\alpha_1 \mapsto a_1$$

$$\bar{\alpha}_3 \mapsto a_3 + \frac{1}{2}v_1 a_1^2 + \frac{1}{2}v_1^2 a_1$$

$$\bar{\alpha}_5 \mapsto \bar{a}_5 + \frac{1}{4}v_1^2 a_3 + \frac{1}{2}v_1^4 a_1 .$$

3.5 The Stable Atomicity of $\Omega Sp(2)$ and $\Omega Sp(3)$

In this section we will prove Theorem 3.1.1. We will restrict our attention to $\Omega Sp(3)$, the case of $\Omega Sp(2)$ being similar and in fact easier. Throughout this section all spectra will be localized at 2.

As a first step toward proving Theorem 3.1.1 it is necessary to write $H_*(\Omega Sp(3);\mathbb{Z}/2)$ as a direct sum of irreducible right A-modules. Let M_k denote the right submodule of A_* consisting of polynomials of degree $\leqslant k$ in ξ_1^2 and ξ_2^2.

Lemma 3.5.1. There is an isomorphism of A-modules

$$H_*(\Omega Sp(2);\mathbb{Z}/2) \approx E[a_1] \otimes [\bigoplus_{k\geqslant 0} \Sigma^{4k} M_k]$$

proof: Let the symbols a_1, a_3, and $\bar{a}_5$ denote both the 2-local classes of the previous section, and their mod 2 reductions. Lemma 3.2.2 implies that

$$a_1 Sq = a_1$$

$$a_3 Sq = a_3 + a_1^2$$

$$\bar{a}_5 Sq = \bar{a}_5 + a_3 + a_1^2 .$$

It follows that there is an isomorphism

$$H_*(\Omega Sp(3);\mathbb{Z}/2) \approx E[a_1]\otimes\mathbb{Z}/2[a_1^2,a_3,\bar{a}_5] .$$

Since the Steenrod algebra preserves the degrees of monomials in $\mathbb{Z}/2[a_1^2,a_3,\bar{a}_5]$, it splits as a module into the sum of its homogeneous parts -- the symmetric powers of the vector space with basis $\{a_1^2,a_3,\bar{a}_5\}$. This vector space is isomorphic to the A-module with basis $\{1,\xi_1^2,\xi_2^2\}$. Its k^{th} symmetric power is M_k. This completes the proof.

Our main goal is the following theorem, which easily implies Theorem 3.1.1.

Theorem 3.5.2. Let $f:\Omega Sp(3)\to\Omega Sp(3)$ be any stable self map inducing an isomorphism of $H_2(\ ;\mathbb{Z}/2)$. Then the composites

$$\Sigma^{4k}M_k \to H_*(\Omega Sp(3);\mathbb{Z}/2) \xrightarrow{f_*} H_*(\Omega Sp(3);\mathbb{Z}/2) \to \Sigma^{4k}M_k$$

$$a_1\Sigma^{4k}M_k \to H_*(\Omega Sp(3);\mathbb{Z}/2) \xrightarrow{f_*} H_*(\Omega Sp(3);\mathbb{Z}/2) \to a_1\Sigma^{4k}M_k$$

are non-trivial, hence isomorphisms since the M_k are the duals of cyclic A-modules.

Theorem 3.5.2 will be proved by supplementing the action of the Steenrod algebra with operations from connective K-theory.

Recall that modulo torsion, $H_*(bu;\mathbb{Z}_{(2)}) \approx \mathbb{Z}_{(2)}[t]$ where t has degree 2. One can therefore define a map of ring spectra

$$\theta_t = \sum_{k=0}^{\infty} \theta_k t^k : bu \longrightarrow H\mathbb{Z}_{(2)}[t]$$

by collapsing the torsion from the Hurewicz map $bu \to H\mathbb{Z}_{(2)} \wedge bu$. One can also think of θ_t as a map from $bu_*(X)$ to $H\mathbb{Z}_{(2)}[t]_*(X)$. We fix the generator t by requiring that $\theta_t(v_1) = 2t$. After rationalization the map θ_t is equivalent to the Chern character.

Lemma 3.5.3. The map $\theta_t : bu_*(\Omega Sp(3)) \to H\mathbb{Z}_{(2)}[t]_*(\Omega Sp(3))$ is given by the following formulae:

$$\theta_t(\alpha_1) = a_1$$

$$\theta_t(\bar{\alpha}_3) = a_3 + t a_1^2 + 2t^2 a_1$$

$$\theta_t(\bar{\alpha}_5) = \bar{a}_5 + t^2 a_3 + 8t^4 a_1 \quad .$$

proof: This is immediate from Lemma 3.4.3 and the fact that $\theta_t(v_1) = 2t$.

Give $bu_*(\Omega Sp(3)) \approx \mathbb{Z}_{(2)}[v_1][\alpha_1, \bar{\alpha}_3, \bar{\alpha}_5]$ the monomial basis. In the following lemma, I will generically denote the subgroup generated by all monomials except those under immediate consideration. For example, in the statement $x \equiv a_3 + 2a_1^3 \bmod I$, I is the subgroup generated by all monomials except a_3 and a_1^3. A similar role will be played by J for the ring $H\mathbb{Z}_{(2)}[t]_*(\Omega Sp(3)) \approx \mathbf{Z}_{(2)}[t][a_1, a_3, \bar{a}_5]$.

<u>Lemma 3.5.4</u>. i) Let $x \in bu_*(\Omega Sp(3))$ satisfy $\theta_{2n}(x) \equiv 2^n t^{2n} a_1^n \bmod J + (2^{n+1})$. Then one has

$$x \equiv \alpha_3^n \quad \bmod I + (2) .$$

ii) Let $x \in bu_*(\Omega Sp(3))$ satisfy $\theta_{2n}(X) \equiv 2^n t^{2n} a_1^{n+1} \bmod J + (2)$. Then one has

$$x \equiv \alpha_1 \bar{\alpha}_3^n \quad \bmod I + (2) .$$

<u>proof</u>: We will sketch the proof of i), the proof of ii) being similar. With the notation as in part i), a straightforward calculation shows that

$$\theta_{2n}(I + (2)) \subset J + (2^{n+1}) .$$

Write $x = n_1 a_3^n + m$, with $m \in I + (2)$. Applying θ_{2n} yields

$$n_1 2^n t^{2n} a_1^n \equiv \theta_{2n}(X) \equiv 2^n t^{2n} a_1^n \quad \bmod J + (2^{n+1}) .$$

It follows that $n_1 \equiv 1\ (2)$. This completes the proof.

We can now prove Theorem 3.5.2. Let $f:\Omega Sp(3) \to \Omega Sp(3)$ be any stable map inducing an isomorphism of $H_2(\ ;\mathbb{Z}/2)$. Suppose by induction that we have shown the composites

$$\Sigma^{4k}M_k \to H_*(\Omega Sp(3);\mathbb{Z}/2) \xrightarrow{f_*} H_*(\Omega Sp(3);\mathbb{Z}/2) \to \Sigma^{4k}M_k$$

and

$$a_1\Sigma^{4k}M_k \to H_*(\Omega Sp(3);\mathbb{Z}/2) \xrightarrow{f_*} H_*(\Omega Sp(3);\mathbb{Z}/2) \to a_1\Sigma^{4k}M_k$$

to be isomorphisms for $k < n$. Observe that there is a canonical isomorphism

$$bu_*(\Omega Sp(3))/(2,v_1) \approx H_*(\Omega Sp(3);\mathbb{Z}/2) .$$

There results a collapse $p:bu_*(\Omega Sp(3)) \to H_*(\Omega Sp(3);\mathbb{Z}/2)$.

Consider the element $a_3^n \in \Sigma^{4k}M_n$. We have $a_3^n = p(\bar{\alpha}_3^n)$ and it will suffice to show that $f_*(\bar{\alpha}_3^n) \equiv \bar{\alpha}_3^n \bmod I + (2)$. Applying the operation θ_{2k} yields

$$\theta_{2k}f_*(\bar{\alpha}_3^n) = f_*\theta_{2k}(\bar{\alpha}_3^n) = f_*(2^n a_1^{2n}) .$$

By the inductive hypothesis $f_*(a_1^{2n}) \equiv a_1^{2n} \bmod J + (2)$. It follows that

$$\theta_{2k}f_*(\bar{\alpha}_3^n) \equiv 2^n a_1^{2n} \quad \bmod J + (2^{n+1}) ,$$

and hence $f_*(\bar{\alpha}_3^n) = \bar{\alpha}_3^n \bmod I + (2)$ by Lemma 3.5.4. A similar calculation using $\alpha_1\bar{\alpha}_3^n$ completes the inductive step and therefore the proof of Theorem 3.5.2.

3.6 Proof of Theorem 3.1.2

In this section we will prove Theorem 3.1.2. We first require a simple lemma.

Lemma 3.6.1. Let E be an associative, commutative ring spectrum, and let $\alpha \in \pi_j E$. Then the localized spectrum $\alpha^{-1}E$ is a ring spectrum.

proof: The homotopy commutative diagram

$$\begin{tikzcd} \vdots \arrow[d] & \vdots \arrow[d] \\ \Sigma^n E \wedge \Sigma^n E \arrow[r, "\mu"] \arrow[d, "\alpha\wedge\alpha"] & \Sigma^{2n} E \arrow[d, "\alpha^2"] \\ \Sigma^{n-j} E \wedge \Sigma^{n-j} E \arrow[r] \arrow[d] & \Sigma^{2n-2j} E \arrow[d] \\ \vdots & \vdots \end{tikzcd}$$

can be made to commute. Passage to the limit yields the desired multiplication.

Warning: The ring spectrum $\alpha^{-1}E$ need neither be commutative nor associative. It does however represent a commutative, associative ring valued cohomology theory on the category of spaces.

Recall that there is a map $i:\mathbb{C}P^{n-1} \to \Omega SU(n)_+$ and that $H_*(\Omega SU(n)_+)$ is the symmetric algebra on $H_*(\mathbb{C}P^{n-1})$. Since the Atiyah Hirzebruch spectral sequence for $X(n-1)_*\mathbb{C}P^{n-1}$ collapses, it also collapses for $X(n-1)_*(\Omega SU(n)_+)$, and one finds that $X(n-1)_*(\Omega SU(n)_+)$ is the symmetric algebra (over $X(n-1)_*$) on $X(n-1)_*(\mathbb{C}P^{n-1})$.

Consider the map $\frac{1}{\alpha} \wedge i:\Sigma^{-2}\mathbb{C}P^{n-1} \to \alpha^{-1}\Omega SU(n)_+$. The range is a ring spectrum and the map is an orientation. By Proposition 1.2.1 this extends to a unique map of ring spectra $f:X(n-1) \to \alpha^{-1}\Omega SU(n)_+$. We can therefore form a map of $X(n-1)$-module spectra

$$\bigvee_{i=-\infty}^{\infty} \Sigma^{2i}X(n-1) \longrightarrow \alpha^{-1}\Omega SU(n)_+$$

whose component $\Sigma^{2i}X(n-1) \to \alpha^{-1}\Omega SU(n)_+$ is the map $\alpha^i \wedge f$. This map is easily seen to induce an isomorphism of $X(n-1)$ homology. Since it is a map of $X(n-1)$-module spectra, the induced map of homotopy is a retract of this isomorphism, hence an isomorphism. This completes the proof of part i).

The proof of part ii) is similar. Let P^j denote the $(2j)$-skeleton of P. It is the Thom complex of the restriction of L^2 to $\mathbb{C}P^{2j-2}$. The zero section $\mathbb{C}P^\infty \to P$ induces a map $\mathbb{C}P^{2n-1} \to P^{2n-1}$ of degree 2 on π_2. Since

2 has been inverted, we can form the orientation

$$\frac{1}{2\alpha} \wedge i : \Sigma^{-2}\mathbb{C}P^{2n-1} \longrightarrow \Sigma^{-2}P^{2n-1} \longrightarrow \alpha^{-1}\Omega Sp(n)_+ .$$

This extends to a unique map of ring spectra

$$f : X(2n-1) \longrightarrow \alpha^{-1}\Omega Sp(n)_+ ,$$

and we can use Proposition 1.5.2 to form the composite:

$$\bigvee_{i=-\infty}^{\infty} \Sigma^{2i}\overline{X(2n-1)} \longrightarrow \bigvee_{i=-\infty}^{\infty} \Sigma^{2i}X(2n-1) \xrightarrow{\vee\alpha^{i}\wedge f} \alpha^{-1}\Omega Sp(n)_+ .$$

This is a map of $X(2n-1)$ module spectra. It induces an isomorphism of $X(2n-1)$ homology since it induces an isomorphism of integral homology, and the Atiyah-Hirzebruch spectral sequence for the $X(2n-1)$ homology of the domain, and hence the range, collapses and converges (convergence follows from Lemma 1.2.5). This proves part ii).

For part iii), a map inducing an isomorphism of integral homology is given by the composite:

$$f : \alpha^{-1}\Omega Sp(n)_+ \to \alpha^{-1}\Omega SU(2n)_+ \simeq \bigvee_{i=-\infty}^{\infty} \Sigma^{2i}X(2n-1) \to \bigvee_{i=-\infty}^{\infty} \Sigma^{2i}Xp(n-1) .$$

The last map is provided by Proposition 1.5.1 and Corollary 1.3.9. We need to show that f induces an isomorphism of $Xp(n-1)$ homology. It suffices to show that the Atiyah-Hirzebruch spectral sequence for $Xp(n-1)_*(\alpha^{-1}\Omega Sp(n)_+)$ collapses. For this it is enough to collapse the spectral sequence for $Xp(n-1)_*(P^{2n-1})$. The composite

$$\Sigma^{-2}P^{2n-1} \hookrightarrow \alpha^{-1}\Omega Sp(n)_+ \longrightarrow \bigvee_{i=-\infty}^{\infty} \Sigma^{2i}Xp(n-1) \longrightarrow Xp(n-1)$$

provides an $Xp(n-1)$ orientation of the restriction of L^2 to $\mathbb{C}P^{2n-2}$. The result now follows from the Thom isomorphism since the Atiyah-Hirzebruch spectral sequence for $Xp(n-1)_*\mathbb{C}P^{2n-2}$ collapses. This completes the proof of Theorem 3.1.2.

REFERENCES

1. J. F. Adams: Stable homotopy and generalized homology, University of Chicago Press, Chicago (1974).

2. N. A. Baas: On bordism theory of manifolds with singularities. Math. Scand. 33 (1973), 279-302.

3. N. A. Baas and I. Madsen: On the realization of certain modules over the Steenrod algebra. Math. Scand. 31 (1972), 220-224.

4. F. R. Cohen, M. E. Mahowald, and R. J. Milgram: The stable decomposition for the double loop space of the sphere. A. M. S. Proceedings of Symposia in Pure Math., 32 (1978), 225-228.

5. F. R. Cohen, J. P. May and L. R. Taylor: $K(\mathbb{Z},0)$ and $K(\mathbb{Z}/2,0)$ as Thom Spectra. Illinois J. Math. 25 (1981), 99-106.

6. E. Dyer: Relations between cohomology theories. Algebraic topology - a students guide. London Mathematical Society Lecture Note Series, No. 4. Cambridge University Press, London-New York, 1972, 188-189.

7. I. M. James: The space of bundle maps. Topology 2 (1963), 45-59.

8. G. Lewis: The stable category and generalized Thom spectra, Thesis, University of Chicago 1978.

9. R. Bruner, G. Lewis, J. P. May, J. McClure and M. Steinberger: H_∞ ring spectra and their applications, Springer Lecture Notes in Mathematics, in preparation.

10. M. Mahowald: A new infinite family in ${}_2\pi_*^S$. Topology, 16 (1977), 249-256.

11. ____________: Ring spectra which are Thom complexes. Duke Math. J. 46 (1979), 549-559.

12. J. P. May: H_∞ ring spectra and their applications. A.M.S. Proceedings of Symposia in Pure Math., 32 (1978) 229-243.

13. H. Margolis: Eilenberg-MacLane spectra. Proc Amer. Math. Soc. 43 (1974), 409-415.

14. J. W. Milnor: The Steenrod algebra and its dual, Annals of Math. 67 (1958), 150-171.

15. J. W. Milnor and J. C. Moore: On the structure of Hopf algebras. Annals of Math. 81 (1965), 211-264.

16. J. Morava: A product for the odd-primary bordism of manifolds with singularities. Topology 18 (1979), 177-186.

17. G. Nishida: The nilpotency of elements of the stable homotopy groups of spheres. J. Math. Soc. Japan 25 (1973), 707-732.

18. D. G. Quillen: On the formal group laws of unoriented and complex cobordism theory. Bull. Amer. Math. Soc. 75 (1969), 1293-1298.

19. D. C. Ravenel: Localization with respect to certain periodic homology theories. Amer. J. Math. 106 (1984), 351-414.

20. N. Shimada and N. Yogita: Multiplications in the complex cobordism theory with singularities, Publ. Res. Inst. Math. Sci. 12 (1976), 259-293.

21. V. P. Snaith : Localized stable homotopy of some classifying spaces. Math. Proc. Cam. Phil. Soc. 89 (1981), 325-330.

22. N. E. Steenrod: Cohomology Operations, Lectures by N. E. Steenrod written and revised by D. B. A. Epstein. Annals of Mathematical Studies, No. 50, Princeton: Princeton University press, 1962.

23. M. Steinberger: The homology operations of H_∞ ring spectra. Thesis, Univ. of Chicago, 1977.

24. W. S. Wilson: The Ω-spectrum for Brown-Peterson cohomology II. Amer. J. Math. 97 (1975), 101-123.

Appendix A: $\Omega Sp(n)$ -- away from 2

After inverting the prime 2, the space $\Omega Sp(n)$ stably decomposes. This follows easily from work of James [7], and the fact that the map

$$\mathbb{C}P^{2n-1} \longrightarrow P^{2n-1}$$

is an equivalence away from 2.

For convenience we will follow [22] and refer to the space $S^{4n-1} \times_{Sp(1)} Sp(1)/S^{4n-1} \times_{Sp(1)} \{1\}$ as Q_{n-1}. Away from 2, the idempotent

$$\tfrac{1}{2}([1] - [-1]) : \Sigma\mathbb{C}P^{2n-1} \longrightarrow \Sigma\mathbb{C}P^{2n-1}$$

Shows the composite

$$\Sigma\mathbb{C}P^{2n-1} \longrightarrow \Sigma P^{2n-1} \longrightarrow Q_n$$

to be projection onto a wedge summand. James [7] has produced a stable splitting $Sp(n) \to Q_n$ of the inclusion $Q_n \to Sp(n)$. After inverting 2, one can form the (stable) composite

$$Q_n \longrightarrow \Sigma\mathbb{C}P^{2n-1} \longrightarrow \Sigma P^{2n-1} \longrightarrow \Sigma\Omega Sp(n) \longrightarrow \Omega Sp(n) \longrightarrow Q_n .$$

It is easily seen to be a homotopy equivalence. This shows that there is a stable decomposition

$$\Omega Sp(n) \approx \Sigma^{-1} Q_n \vee Y$$

for some spectrum Y.

Appendix B: The Equivalence $\Omega SU \to BU$.

It would appear that the spectra $X(n)$ depend on the choice of equivalence $\Omega SU \to BU$. It is the purpose of this appendix to show that this is not the case and to fix a choice of equivalence with certain properties.

Proposition B.1. Let $\Omega SU \to BU$ be any H-map which is an equivalence. Then the bordism theory resulting from the composite

$$*) \qquad \Omega SU(n) \longrightarrow \Omega SU \longrightarrow BU$$

is $X(n)$.

proof: Let $X(n)'$ denote the Thom spectrum associated to *). The $X(n)'$ are ring spectra and one has $\lim_{n\to\infty} X(n)' = MU$. The Thom isomorphism implies that $H_*(X(n)';\mathbb{Z}) \simeq \mathbb{Z}[b_1,\ldots,b_{n-1}]$ and hence that $X(n)' \to MU$ is a $(2n-1)$-equivalence. It follows that any orientation $\Sigma^{-2}\mathbb{C}P^\infty \to MU$ restricts to a unique orientation $\Sigma^{-2}\mathbb{C}P^n \to X(n)'$. The orientation $\Sigma^{-2}\mathbb{C}P^n \to X(n)'$ has the further property that the image of $H_*(\Sigma^{-2}\mathbb{C}P^n;\mathbb{Z})$ is a set of generators. Proposition 1.2.1 then produces a map $X(n) \to X(n)'$ inducing an isomorphism of integral homology. This completes the proof.

Remark. A canonical inverse to the map $X(n) \to X(n)'$ can be produced by observing that the proof of Proposition 1.2.1 works as well for $\Sigma^{-2}\mathbb{C}P^n \to X(n)'$.

Since $\Omega SU(n) \to \Omega SU$ is a double loop map, $X(n)$ is a ring spectrum with commutativity in its multiplication analogous to that of a double loop space. It will be convenient to fix a double loop map $\Omega SU \to BU$ which is a homotopy equivalence.

Proposition B.2. Let $\mathbb{C}P^\infty \to \Omega SU$ be the adjoint of $\Sigma\mathbb{C}P^\infty \to SU$. There is a homotopy equivalence $\Omega SU \to BU$ which is a double loop map and which makes the composite

$$\mathbb{C}P^\infty \longrightarrow \Omega SU \longrightarrow BU$$

classify the canonical line bundle.

proof: The map $\Sigma\mathbb{C}P^\infty \to SU$ has an adjoint $i:\Sigma^2\mathbb{C}P^\infty \to BSU$ inducing an epimorphism of complex K theory. It follows that any map $\Sigma^2\mathbb{C}P^\infty \to BSU$ can be extended through i . Let $f:\Sigma^2\mathbb{C}P^\infty \to BSU$ correspond under Bott periodicity to the canonical line bundle. There results a diagram

$$\begin{tikzcd} \Sigma^2\mathbb{C}P^\infty \arrow[r, "f"] \arrow[d, "i"'] & BSU \\ BSU \arrow[ur, dashed] & \end{tikzcd}$$

Taking double loop spaces produces the required equivalence.

VITA

Michael J. Hopkins was born in Alexandria, Virginia on April 18, 1958. He received his B.A. from Northwestern University in June, 1979, and has attended graduate school at Oxford and Northwestern Universities.