EQUIVARIANT ALGEBRAIC TOPOLOGY

by

Sören Illman

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CONTENTS

INTRODUCTION

Our main objective is the construction of an equivariant singular homology and cohomology theory, for spaces on which a compact Lie group G acts, with coefficients in an arbitrary given covariant coefficient system and contravariant coefficient system, respectively. See Definition 1.2 and Theorems 2.1 and 2.2 in Chapter III for precise statements. Our construction is such that G besides being an arbitrary compact Lie group also can be a discrete group or an abelian locally compact group.

For actions of discrete groups equivariant homology and cohomology theories of this type exist before. G. Bredon has constructed an "equivariant classical cohomology theory" for CW complexes with a cellular action of a discrete group, see G. Bredon [2] and [3]. Recently Th. Bröcker constructed an equivariant homology and cohomology theory with pre-described coefficients, for spaces with an action of a discrete group, see Th. Bröcker [4].

The key to our construction is the definition of an equivariant simplex, see Definition 1.1 in Chapter II. The construction of equivariant singular theory is then very much analogous to the construction of ordinary singular theory. In our case the proof of the dimension axiom requires some argument.

In Section 9 of Chapter III we describe an alternative construction of an equivariant singular homology and cohomology theory. This construction is technically much easier to handle. We shall use this construction on later occasions.

In Chapter I we define equivariant CW complexes and prove the equivariant versions of the homotopy extension property, the skeletal approximation theorem, and the Whitehead theorem.

In Chapter II we prove that every differentiable manifold on which a compact Lie group acts differentiably is an equivariant CW complex, see Theorem 3.1 and Corollary 4.1 in Chapter II. This result has also been proved by Takao Matumoto, see [20] Proposition 4.4. Matumoto formulates the definition of an equivariant CW complex (he calls them G-CW complexes) in a different way than we do, but his Proposition 4.4 proves the same result as our Theorem 3.1 . In fact his proof is much shorter. The existence of the article [20] was pointed out to me, when I had already completed this work, by A. Wasserman. I also wish to thank G. Bredon for a very illuminating conversation about the proof of the above result given in [20].

In Chapter IV we construct a transfer homomorphism both in equivariant singular homology and cohomology. We also define a "Kronecker index" and a cup-product in equivariant singular cohomology. In the last section we prove that equivariant singular homology and cohomology of a finite dimensional equivariant CW complex is isomorphic to its "cellular equivariant homology and cohomology," respectively. From this it follows that the equivariant singular homology and cohomology groups of a differentiable G-manifold M vanish in degrees above the dimension of the manifold M. If M moreover is compact, the equivariant singular homology and cohomology groups are finitely generated R-modules if the coefficient systems are finitely generated coefficient systems over a noetherian ring R.

We announced the results of Chapters I, II, III (under the assumption that G is a compact Lie group), and Section 5 of Chapter IV in a talk at the Conference on Transformation Groups at the University of Massachusetts, Amherst, June 7-18, 1971. An article on this will appear in the Proceedings of the Amherst conference.

TERMINOLOGY AND NOTATIONS

Let G be a topological group and X a topological space. A left action of G on X is a continuous map $\varphi: G \times X \longrightarrow X$ such that $\varphi(e, x) = x$, e = identity of G, for all $x \in X$, and $\varphi(g, \varphi(g', x)) = \varphi(g'g, x)$ for all $g, g' \in G$, $x \in X$. We denote $\varphi(g, x) = gx$. If $A \subset X$ we write $GA = \{ga \mid g \in G, a \in A\}$. By a G-space X we mean a topological space X together with a left action of G on X. A G-subset of a G-space X is a subset A of X such that $GA = A$. A G-pair (X, A) consists of a G-space X and a G-subset A of X.

Let X be a G-space. The orbit of a point $x \in X$ is the set $Gx = G\{x\}$. The orbit space, which we denote by $G\backslash X$, is the set $G\backslash X = \{Gx \mid x \in X\}$ with the quotient topology from the projection $\pi: X \longrightarrow G\backslash X$, $\pi(x) = Gx$. It is easy to see that π is an open map.

A map $f: X \longrightarrow Y$, X and Y are G-spaces, is called a G-map if $f(gx) = gf(x)$ for all $g \in G$, $x \in X$. Correspondingly we have the notion of a G-homeomorphism, a G-retraction, etc.

Let X be a G-space. The isotropy group of $x \in X$ is the subgroup $G_x = \{g \in G \mid gx = x\}$ of G. If $\{x\}$ is closed in X then G_x is a closed subgroup of G. For any $x \in X$ the map $\alpha_x: G/G_x \longrightarrow Gx \subset X$, defined by $\alpha_x(gG_x) = gx$, is a continuous one-to-one map onto the orbit Gx. Moreover α_x is a G-map, where G acts in the standard way on G/G_x, that is, by $(\bar{g}, gG_x) \longmapsto \bar{g}gG_x$. If G is compact and X is Hausdorff, then $\alpha_x: G/G_x \longrightarrow Gx$ is a G-homeomorphism for each $x \in X$.

Let H be a subgroup of G. We denote by (H) the collection of all subgroups of G which are conjugate in G to H, that is, $(H) = \{gHg^{-1} | g \in G\}$. Such a collection (H) is called a G-orbit type, or simple orbit type when the group G is clear from the context. We say that a point x of a G-space X has orbit type (H) if $G_x \in (H)$. If $x \in X$ has isotropy group G_x then $gx \in X$ has isotropy group gG_xg^{-1}. Thus all points on one orbit have the same orbit type (hence of course the name orbit type). This allows us to speak about the orbit type of a point in the orbit space $G\backslash X$. Denote $X' = G\backslash X$ and let $\pi: X \longrightarrow X'$ be the projection. Thus we say that the point $x' \in X'$ has orbit type (H) if $G_x \in (H)$ where $\pi(x) = x'$. We use the following notations (they are the standard ones), $X_{(H)} = \{x \in X | G_x \in (H)\}$, and $X'_{(H)} = \pi(X_{(H)})$, and also $X^H = \{x \in X | hx = x$ for all $h \in H\}$.

Now assume that G is a compact Lie group, and consider only G-orbit types which are represented by closed subgroups of G. Define $(H) \leq (K)$ to mean that there exists $H' \in (H)$ and $K' \in (K)$ such that $H' \subset K'$. Clearly $(H) \leq (H)$, and if $(H) \leq (K)$ and $(K) \leq (L)$ then $(H) \leq (L)$. Moreover if $(H) \leq (K)$ and $(K) \leq (H)$ then $(H) = (K)$. This follows from the fact that a closed subgroup H of a compact Lie group G is not conjugate (in G) to a proper subgroup of itself. Thus $\leq$ is a partial ordering.

Let X and Y be G-spaces, and $f_0, f_1: X \longrightarrow Y$ two G-maps. A G-homotopy from f_0 to f_1 is a G-map $H: I \times X \longrightarrow Y$, (G acts on $I \times X$ by acting trivially on I, that is, $(g, (t, x)) \longmapsto (t, gx)$, $g \in G$, $(t, x) \in I \times X$) such that $H(0, x) = f_0(x)$ and $H(1, x) = f_1(x)$ for all $x \in X$. A G-map

$f: X \to Y$ is a G-homotopy equivalence if there exists a G-map $h: Y \to X$ such that hf is G-homotopic to id_X and fh is G-homotopic to id_Y. Correspondingly we have the "G-version" of other related concepts. For example the meaning of the expression: $D: I \times X \to X$ is a strong G-deformation retraction from X to A, is now clear.

Let X be a topological space and $\{A_j\}_{j \in J}$ a family of closed subsets of X such that $\bigcup_{j \in J} A_j = X$. We say that X has the topology coherent with $\{A_j\}_{j \in J}$ if the following is true. A subset B of X is closed in X if and only if $B \cap A_j$ is closed in A_j for every $j \in J$. (The expression "weak topology with respect to $\{A_j\}_{j \in J}$" is also used in the literature.)

A Hausdorff space X is said to be compactly generated if X has the topology coherent with the family of all compact subsets of X. It is a well-known fact that every locally compact space is compactly generated.

The following fact will be used on a number of occasions without further reference.

Lemma. Let X be a topological space, and $\{A_j\}_{j \in J}$ a family of closed subsets of X which cover X, and assume that X has the topology coherent with $\{A_j\}_{j \in J}$. Let C be a compact space (also Hausdorff), then $C \times X$ has the topology coherent with $\{C \times A_j\}_{j \in J}$.

Proof. Let $\dot{\bigcup}_{j \in J} A_j$ be the disjoint union of all A_j, $j \in J$, and let $p: \dot{\bigcup}_{j \in J} A_j \to X$ be the natural projection onto X. Then the statement, X has the topology coherent with $\{A_j\}_{j \in J}$, is equivalent to the statement, p is a quotient map. We have $C \times (\dot{\bigcup}_{j \in J} A_j) = \dot{\bigcup}_{j \in J} (C \times A_j)$ as topological spaces. Since C is compact, it follows that $id \times p: C \times (\dot{\bigcup}_{j \in J} A_j) \longrightarrow C \times X$ is a quotient map if p is a quotient map. (This is well known, for example the second part of the proof of Theorem 4.4 in Steenrod [17], proves exactly this.) This completes the proof.

Let us conclude this section by pointing out that the word "map" will always mean "a continuous map," and that the notation $A \subset B$ always includes the possibility $A = B$. These agreements were already used above.

CHAPTER I

EQUIVARIANT CW COMPLEXES

In this chapter G denotes a compact topological group. We define equivariant CW complexes. This definition is obtained from the definition of an ordinary CW complex in a simple way. Instead of adjoining cells E^n by a map from S^{n-1} one adjoins G-spaces of the form $E^n \times G/H$, where $H \subset G$ is some closed subgroup (not fixed), by an equivariant map from $S^{n-1} \times G/H$. The standard elementary properties like the homotopy extension property, the skeletal approximation theorem, and the Whitehead theorem, are also valid in the equivariant case, and the proofs are completely analogous to the proofs in the ordinary case, as for example given in Spanier [16].

1. ADJOINING EQUIVARIANT CELLS

Definition 1.1. Let X be a Hausdorff G-space, and A a closed G-subset of X, and n a non-negative integer. We say that X is obtainable from A by adjoining equivariant n-cells, if the following is true. There exists a collection $\{c_j^n\}_{j\in J}$ of closed G-subsets of X, such that the following conditions are satisfied:

1. $X = A \cup \left(\bigcup_{j\in J} c_j^n\right)$, and X has the topology coherent with $\{A, c_j^n\}_{j\in J}$.

2. Denote $\dot{c}_j^n = c_j^n \cap A$, then

$$(c_j^n - \dot{c}_j^n) \cap (c_i^n - \dot{c}_i^n) = \emptyset, \qquad \text{for } j \neq i.$$

3. For each $j \in J$, there exists a closed subgroup H_j of G, and a G-map

$$f_j: (E^n \times G/H_j,\ S^{n-1} \times G/H_j) \longrightarrow (c_j^n,\ \dot{c}_j^n)$$

such that $f_j(E^n \times G/H_j) = c_j^n$, and f_j maps $E^n \times G/H_j - S^{n-1} \times G/H_j$ homeomorphically onto $c_j^n - \dot{c}_j^n$.

Assume that X is obtainable from A by adjoining equivariant n-cells. We shall show that a collection $\{c_j^n\}_{j \in J}$ of subsets of X, which satisfies conditions 1-3 in the above definition, is uniquely determined by the pair (X, A), and that so is the number n.

Lemma 1.2. Assume that X is obtainable from A by adjoining equivariant n-cells, and let $\{c_j^n\}_{j \in J}$ be a collection of closed G-subsets which satisfies conditions 1-3 in Definition 1.1. Then $c_j^n - \dot{c}_j^n$ is an open G-subset of X, for any $j \in J$, the set $c_j^n - \dot{c}_j^n$ is open and closed in $X - A$, and there does not exist a path in $X - A$ connecting $c_j^n - \dot{c}_j^n$ and $c_i^n - \dot{c}_i^n$, if $j \neq i$.

Proof. Let $j_0 \in J$, and consider the set $B = A \cup \Big(\bigcup_{\substack{j \in J \\ j \neq j_0}} c_j^n \Big)$. We have

$B \cap A = A$, $B \cap c_j^n = c_j^n$ for $j \neq j_0$, and $B \cap c_{j_0}^n = A \cap c_{j_0}^n$. Hence B is closed, by condition 1 in Definition 1.1, and thus $c_{j_0}^n - \dot{c}_{j_0}^n = X - B$ is an open set in X. Since $X - A = \bigcup_{j\in J}(c_j^n - \dot{c}_j^n)$ and the union is disjoint, it follows that $c_{j_0}^n - \dot{c}_{j_0}^n$ is closed in $X - A$. Hence for any path $\omega: I \longrightarrow X - A$, the inverse image of a set $c_j^n - \dot{c}_j^n$ is both open and closed in I, thus $\omega^{-1}(c_j^n - \dot{c}_j^n) = I$ or $\emptyset$.

q.e.d.

Lemma 1.3. Let X, A and $\{c_j^n\}_{j\in J}$ be as in Lemma 1.2. Let c be a closed G-subset of X, and let f be a G-map

$$f: (E^n \times G/H,\ S^{n-1}\times G/H) \longrightarrow (c, \dot{c}),$$

where $\dot{c} = c \cap A$, H some closed subgroup of G, such that $f(E^n \times G/H) = c$, and f maps $E^n \times G/H - S^{n-1}\times G/H$ homeomorphically onto $c - \dot{c}$. Then $c = c_j^n$, for some $j \in J$.

Proof. Let us first show that the set $c - \dot{c}$ intersects at most one of the sets $c_j^n - \dot{c}_j^n$, $j \in J$. Assume the contrary and say that $c - \dot{c}$ intersects the sets $c_j^n - \dot{c}_j^n$ and $c_i^n - \dot{c}_i^n$, where $i \neq j$. Then there exist $(x_0, g_0H) \in \mathring{E}^n \times G/H$, and $(x_1, g_1H) \in \mathring{E}^n \times G/H$, such that $f(x_0, g_0H) \in c_j^n - \dot{c}_j^n$ and $f(x_1, g_1H) \in c_i^n - \dot{c}_i^n$. Define $\omega: I \longrightarrow X - A$ by $\omega(t) = f((1-t)x_0 + tx_1, g_0H)$ $\in c - \dot{c} \subset X - A$. Thus, $\omega(0) = f(x_0, g_0H) \in c_j^n - \dot{c}_j^n$, and $\omega(1) = f(x_1, g_0H) =$ $(g_0 g_1^{-1}) f(x_1, g_1H) \in c_i^n - \dot{c}_i^n$, which is impossible by Lemma 1.1. Hence $c - \dot{c} \subset c_j^n - \dot{c}_j^n$, for some $j \in J$. We now show that $c - \dot{c} = c_j^n - \dot{c}_j^n$. Clearly the set $c - \dot{c}$ is closed in $c_j^n - \dot{c}_j^n$. The G-homeomorphism $f: \mathring{E}^n \times G/H \longrightarrow c - \dot{c}$ induces a homeomorphism $\bar{f}: \mathring{E}^n \longrightarrow G\backslash(c - \dot{c})$, and similarly f_j induces a homeomorphism $\bar{f}_j: \mathring{E}^n \longrightarrow G\backslash(c_j^n - \dot{c}_j^n)$. Let $\pi: c_j^n - \dot{c}_j^n \longrightarrow G\backslash(c_j^n - \dot{c}_j^n)$ be the projection. Since $\pi^{-1}(\pi(c - \dot{c})) = c - \dot{c}$, it follows that $\pi(c-\dot{c}) = G\backslash(c-\dot{c})$ is a closed subset of $G\backslash(c_j^n - \dot{c}_j^n)$. Now consider the composite map

$$\mathring{E}^n \xrightarrow{\overline{f}} G\backslash(c-\dot{c}) \hookrightarrow G\backslash(c_j^n - \dot{c}_j^n) \xrightarrow{(f_j)^{-1}} \mathring{E}^n$$

and denote it by $v\colon \mathring{E}^n \longrightarrow \mathring{E}^n$. Thus v is a closed and injective map, and thus v induces a homeomorphism from $\mathring{E}^n$ onto $v(\mathring{E}^n)$. By the "Invariance of domain theorem" $v(\mathring{E}^n)$ is an open set in $\mathring{E}^n$. Since $v(\mathring{E}^n)$ is both open and closed in $\mathring{E}^n$, we have $v(\mathring{E}^n) = \mathring{E}^n$, because $v(\mathring{E}^n)$ is not empty. Thus it follows that $c - \dot{c} = c_j^n - \dot{c}_j^n$. Finally, it follows from the properties of the map f_j that $\overline{c_j^n - \dot{c}_j^n} = c_j^n$, and similarly from the properties of the map f that $\overline{c - \dot{c}} = c$. Hence $c = c_j^n$.

q.e.d.

Corollary 1.4. Assume that X is obtainable from A by adjoining equivariant n-cells. Then there exists one and only one collection $\{c_j^n\}_{j\in J}$, of closed G-subsets of X, which satisfies conditions 1-3 in Definition 1.1. Moreover, X is not obtainable from A by adjoining equivariant m-cells, if $m \neq n$.

Proof. The uniqueness of the collection $\{c_j^n\}_{j\in J}$ follows from Lemma 1.3. Assume that X is obtainable from A, both by adjoining equivariant n-cells $\{c_j^n\}_{j\in J}$ and by adjoining equivariant m-cells $\{c_\ell^m\}_{\ell\in L}$. Then $\bigcup_{j\in J}(c_j^n - \dot{c}_j^n) = \bigcup_{\ell\in L}(c_\ell^m - \dot{c}_\ell^m)$, and we see in the same way as in the proof of Lemma 1.3 that each set $c_j^n - \dot{c}_j^n$ intersects at most one set $c_\ell^m - \dot{c}_\ell^m$ and vice versa. Thus $c_j^n - \dot{c}_j^n = c_\ell^m - \dot{c}_\ell^m$ for some ℓ and j, and since the orbit space of this G-space is homeomorphic to $\mathring{E}^n$ and $\mathring{E}^m$, we have $n = m$.

q.e.d.

Assume that X is obtainable from A by adjoining equivariant n-cells. We call the G-subsets c_j^n equivariant n-cells of (X,A). We also say that X is obtained from A by adjoining the equivariant n-cells $\{c_j^n\}_{j\in J}$. The open G-subsets $c_j^n - \dot{c}_j^n$ are called equivariant open n-cells of (X,A), and we denote $\overset{o}{c}{}_j^n = c_j^n - \dot{c}_j^n$. A G-map $f_j\colon E^n \times G/H_j \longrightarrow c_j^n$, that satisfies condition 3 in Definition 1.1, is called an equivariant characteristic map for c_j^n, and its restriction $f_j|\colon S^{n-1}\times G/H_j \longrightarrow \dot{c}_j^n \hookrightarrow A$, is called an equivariant attaching map for c_j^n. Notice that a function u from X into a topological space Y is continuous if and only if $u|A$ and $u|c_j^n$, all $j \in J$, are continuous, and that $u|c_j^n\colon c_j^n \longrightarrow Y$ is continuous if and only if $(u|)f_j\colon E^n \times G/H_j \longrightarrow Y$ is continuous, where f_j is some equivariant characteristic map for c_j^n.

Also observe that if $n = 0$, then X is a disjoint union of A and G-spaces of the form G/H_j where each H_j is a closed subgroup of G.

<u>Lemma 1.5</u>. The G-space $0\times E^n \times G/H \cup I \times S^{n-1} \times G/H$ is a strong G-deformation retract of $I \times E^n \times G/H$.

<u>Proof</u>. Let $\bar{D}\colon I\times I\times E^n \longrightarrow I\times E^n$ be a strong deformation retraction of $I\times E^n$ to $0\times E^n \cup I\times S^{n-1}$. Then $D = \bar{D}\times \mathrm{id}\colon I\times I\times E^n\times G/H \longrightarrow I\times E^n \times G/H$ is a strong G-deformation retraction of $I\times E^n\times G/H$ to $0\times E^n\times G/H \cup I\times S^{n-1}\times G/H$
q.e.d.

<u>Lemma 1.6</u>. Assume that X is obtainable from A by adjoining equivariant n-cells. Then $0\times X \cup I\times A$ is a strong G-deformation retract of $I\times X$.

<u>Proof</u>. Let $\{c_j^n\}_{j\in J}$ be the collection of equivariant n-cells for (X,A).

Let $f_j: (E^n \times G/H_j, S^{n-1} \times G/H_j) \longrightarrow (c_j^n, \dot{c}_j^n)$ be equivariant characteristic maps for c_j^n, $j \in J$. Let $D_j': I \times I \times E^n \times G/H_j \longrightarrow I \times E^n \times G/H_j$ be a strong G-deformation retraction of $I \times E^n \times G/H_j$ to $0 \times E^n \times G/H_j \cup I \times S^{n-1} \times G/H_j$. Then we have the commutative diagram

$$\begin{array}{ccc} I \times I \times E^n \times G/H_j & \xrightarrow{D_j'} & I \times E^n \times G/H_j \\ \Big\downarrow {\scriptstyle id \times id \times f_j} & & \Big\downarrow {\scriptstyle id \times f_j} \\ I \times I \times c_j^n & \xrightarrow{D_j} & I \times c_j^n \end{array}$$

where D_j is well-defined by the condition $D_j(t, s, f_j(x, gH_j)) = (id \times f_j) D_j'(t, s, x, gH_j)$. Then D_j is continuous since $id \times id \times f_j$ is a quotient map. Thus D_j is a strong G-deformation retraction of $I \times c_j^n$ to $0 \times c_j^n \cup I \times \dot{c}_j^n$.

Define $D_A: I \times I \times A \longrightarrow I \times A$ by $D_A(t, s, a) = (s, a)$. Consider D_A and D_j, $j \in J$, as G-maps into $I \times X$. Since the G-maps D_A and D_j, $j \in J$ agree on any common set of definition, and since $I \times I \times X$ has the topology coherent with $\{I \times I \times A, I \times I \times c_j^n\}_{j \in J}$, it follows that D_A and D_j, $j \in J$ determine a unique G-map $D: I \times I \times X \longrightarrow I \times X$. Clearly D is a strong G-deformation retraction of $I \times X$ to $0 \times X \cup I \times A$.

q. e. d.

Corollary 1.7. Assume that X is obtainable from A by adjoining equivariant n-cells. Then (X, A) has the G-homotopy extension property with respect to any G-map.

Definition 1.8. Let (Y, B) be an arbitrary G-pair and n a non-negative integer. We say that (Y, B) satisfies condition π_n if the following is true. For any closed subgroup H of G, any G-map

$$f: (E^n \times G/H, S^{n-1} \times G/H) \longrightarrow (Y, B)$$

is G-homotopic relative to $S^{n-1} \times G/H$ to a G-map from $E^n \times G/H$ into B.

For $n = 0$, this simply means that any G-map $f: 0 \times G/H \longrightarrow Y$ can be extended to a G-map $H: I \times G/H \longrightarrow Y$ such that $H(1 \times G/H) \subset B$.

Definition 1.9. Let (Y, B) be an arbitrary G-pair. We say that (Y, B) is equivariantly n-connected if (Y, B) satisfies condition π_k, for $k = 0, \ldots, n$.

Proposition 1.10. Let (Y, B) be an arbitrary G-pair and n a non-negative integer. Then the following three conditions are equivalent:

1. (Y, B) satisfies condition π_n.
2. For any closed subgroup H of G any map $s: (E^n, S^{n-1}) \longrightarrow (Y^H, B^H)$ is homotopic relative to S^{n-1} to a map from E^n into B^H.
3. $n = 0$. For any closed subgroup H of G every path component of Y^H intersects B^H.

   $n \geq 1$. For any closed subgroup H of G we have $\pi_n(Y^H, B^H, b_0) = 0$, for every $b_0 \in B^H$.

Proof. The equivalence of conditions 2 and 3 is a standard fact. We shall show that conditions 1 and 2 are equivalent.

First assume that (Y, B) satisfies condition π_n. Let H be a closed subgroup of G and let $s: (E^n, S^{n-1}) \longrightarrow (Y^H, B^H)$ be an arbitrary map. Consider the G-map $\bar{f}: (E^n \times G, S^{n-1} \times G) \longrightarrow (Y, B)$ defined by

$\bar{f}(x,g) = gs(x)$. Since $H\ s(x) = s(x)$ for all $x \in E^n$, it follows that $\bar{f}$ factors through $E^n \times G/H$ and thus determines a G-map $f\colon (E^n \times G/H,\ S^{n-1} \times G/H) \to (Y,B)$ and $f(x,gH) = gs(x)$. Thus there exists a G-homotopy, relative to $S^{n-1} \times G/H$, $F\colon I \times (E^n \times G/H,\ S^{n-1} \times G/H) \to (Y,B)$ from f to a G-map from $E^n \times G/H$ into B. Since all the points $(t,x,eH) \in I \times E^n \times G/H$ are fixed under H, it follows that we get a map $S\colon I \times (E^n,\ S^{n-1}) \to (Y^H, B^H)$ by defining $S(t,x) = F(t,x,eH)$. Clearly S is a homotopy, relative to S^{n-1}, from s to a map from E^n into B^H.

Now assume that the G-pair (Y,B) satisfies condition 2. Let H be a closed subgroup of G, and let $f\colon (E^n \times G/H,\ S^{n-1} \times G/H) \to (Y,B)$ be an arbitrary G-map. Define $s\colon (E^n,\ S^{n-1}) \to (Y^H, B^H)$ by $s(x) = f(x,eH)$, $x \in E^n$. Thus there exists a homotopy, relative to S^{n-1}, $S\colon I \times (E^n,\ S^{n-1}) \to (Y^H, B^H)$ from s to a map from E^n into B^H. We define a G-map $F\colon I \times (E^n \times G/H,\ S^{n-1} \times G/H \to (Y,B)$ by $F(t,x,gH) = g\,S(t,x)$. Clearly F is a G-homotopy, relative to $S^{n-1} \times G/H$, from f to a G-map from $E^n \times G/H$ into B.

q. e. d.

Corollary 1.11. Let (Y,B) be an arbitrary G-pair. Then the following three conditions are equivalent:

1. (Y,B) is equivariantly n-connected.
2. For each closed subgroup H of G the pair (Y^H, B^H) is n-connected.
3. For each closed subgroup H of G every path component of Y^H intersects B^H, and $\pi_k(Y^H, B^H, b_0) = 0$ for every $b_0 \in B^H$ and $k = 1, \ldots, n$.

q. e. d.

Remark. Let us comment on the case when $B = \emptyset$ in Definitions 1.8 and 1.9, Lemma 1.10, and Corollary 1.11. Consider Definition 1.8. If $B = \emptyset$ and $Y \neq \emptyset$, then (Y, B) does not satisfy condition π_0. In fact the existence of a closed subgroup H of G such that $B^H = \emptyset$ and $Y^H \neq \emptyset$ implies that (Y, B) does not satisfy condition π_0. On the other hand, if $n \geq 1$ and $B = \emptyset$ then (Y, B) satisfies condition π_n since the required condition is satisfied in an empty way.

Thus, the G-pair $(Y, \emptyset)$ is not equivariantly n-connected for any n.

Observe that Proposition 1.10 and Corollary 1.11 are true also in the case when $B = \emptyset$. For example, if $n \geq 1$ it is true that $\pi_n(Y^H, B^H, b_0) = 0$ for every $b_0 \in B^H$ simply because $B^H = \emptyset$.

Lemma 1.12. Let (Z, C) be a pair which is n-connected in the ordinary sense. Let Y be an arbitrary G-space. Then the G-pair $(Z \times Y, C \times Y)$ is equivariantly n-connected.

Proof. Let H be a closed subgroup of G. Then $((Z \times Y)^H, (C \times Y)^H) = (Z \times Y^H, C \times Y^H)$ is n-connected since (Z, C) is. Thus $(Z \times Y, C \times Y)$ is equivariantly n-connected by Corollary 1.11.

q.e.d.

Let M denote another, compact, Hausdorff topological group and let (Y, B) be an arbitrary M-pair. Let $\varphi: G \longrightarrow M$ be a continuous homomorphism. We make Y into a G-space by defining $gy = \varphi(g)y$, for $g \in G$ and $y \in Y$. We say that the M-space Y is made into a G-space through the homomorphism $\varphi: G \longrightarrow M$. In this way (Y, B) becomes a G-pair. If H is a closed subgroup of G, then we have $(Y^H, B^H) = (Y^{\varphi(H)}, B^{\varphi(H)})$.

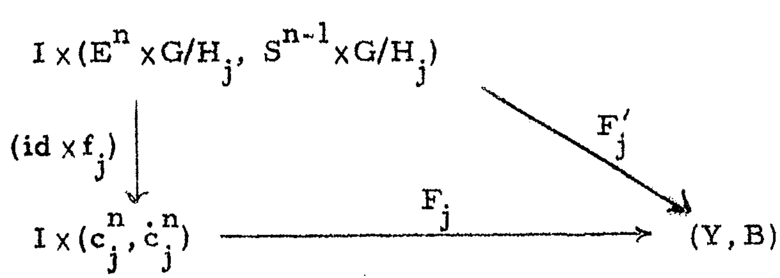
Hence, we have

<u>Corollary 1.13</u>. Assume that the M-pair (Y, B) is equivariantly n-connected. Let $\varphi: G \longrightarrow M$ be any continuous homomorphism and make (Y, B) into a G-pair through φ. Then the G-pair (Y, B) is equivariantly n-connected.

q. e. d.

<u>Lemma 1.14</u>. Assume that X is obtainable from A by adjoining equivariant n-cells. Let (Y, B) be a G-pair which satisfies condition π_n. Then any G-map $f: (X, A) \longrightarrow (Y, B)$ is G-homotopic relative to A, to a G-map from X into B.

<u>Proof.</u> Let $\{c_j^n\}_{j \in J}$ be the equivariant n-cells of (X, A), and let $f_j: (E^n \times G/H_j, S^{n-1} \times G/H_j) \longrightarrow (c_j^n, \dot{c}_j^n)$ be equivariant characteristic maps. Consider the G-maps $f \circ f_j: (E^n \times G/H_j, S^{n-1} \times G/H_j) \longrightarrow (Y, B)$. By assumption there are G-homotopies, relative to $S^{n-1} \times G/H_j$, $F_j': I \times (E^n \times G/H_j, S^{n-1} \times G/H_j) \longrightarrow (Y, B)$, from $f \circ f_j$ to a G-map from $E^n \times G/H_j$ into B. We have the commutative diagram



where F_j is well-defined by $F_j(t, f_j(x, gH)) = F_j'(t, x, gH)$. Thus F_j is a G-homotopy, relative to $\dot{c}_j^n$ into B. Define $F_A: I \times A \longrightarrow Y$ by $F_A(t, a) = f(a)$. Since the G-maps F_A and F_j, $j \in J$ agree on any common set of definition, and since $I \times X$ has the topology coherent with $\{I \times A, I \times c_j^n\}_{j \in J}$ they determine a G-map $F: I \times X \longrightarrow Y$. Clearly F is a G-homotopy from f to a G-map from X into B.

q. e. d.

We shall end this section with one more lemma.

Lemma 1.15. Assume that X is obtainable from A by adjoining equivariant n-cells. If C is a compact subset of X then C is included in the union of A and a finite number of equivariant n-cells of (X, A).

Proof. Assume the contrary and form an infinite set $\{x_i\}$ consisting of one point in C from each equivariant open n-cells which C meets. Thus the set $\{x_i\}$ is closed in X and so is all its subsets. Hence $\{x_i\}$ is an infinite discrete subset of C. This is impossible since C is compact.

q. e. d.

2. EQUIVARIANT CW COMPLEXES

Definition 2.1. An equivariant relative CW complex (X, A) consists of a Hausdorff G-space X, a closed G-subset A of X, and an increasing filtration of X by closed G-subsets $(X, A)^k$, $k = 0, 1, \ldots,$ such that the following conditions are satisfied:

1. $(X, A)^0$ is obtainable from A by adjoining equivariant 0-cells, and for $k \geq 1$, $(X, A)^k$ is obtainable from $(X, A)^{k-1}$ by adjoining equivariant k-cells.

2. $X = \bigcup_{k=0}^{\infty} (X, A)^k$, and X has the topology coherent with $\{(X, A)^k\}_{k \geq 0}$.

The closed G-subset $(X,A)^k$ is called the k-skeleton of the equivariant relative CW complex (X,A). Observe that they are part of the structure, but the G-pair (X,A) can of course have many different filtrations, which make (X,A) into an equivariant relative CW complex.

If $A = \emptyset$ we call X an equivariant CW complex, and denote the k-skeleton by X^k.

Let (X,A) be an equivariant relative CW complex. Then $(X,(X,A)^k)$, any $k \geq 0$, is an equivariant relative CW complex, with skeletons defined as follows:

$$(X,(X,A)^k)^m = \begin{cases} (X,A)^k & \text{for } m \leq k \\ (X,A)^m & \text{for } m > k. \end{cases}$$

Likewise $((X,A)^k, A)$, any $k \geq 0$, is an equivariant relative CW complex with skeletons

$$((X,A)^k, A)^m = \begin{cases} (X,A)^m & \text{for } m \leq k \\ (X,A)^k & \text{for } m > k. \end{cases}$$

Also observe that if (Z,C) is a relative CW complex, in the ordinary sense, with skeletons $(Z,C)^k$, $k = 0,1,\ldots,$ and H is a closed subgroup of G, then $(Z \times G/H,\ C \times G/H)$ is an equivariant relative CW complex with skeletons $(Z,C)^k \times G/H$.

Let (X,A) be an equivariant relative CW complex. If $X = (X,A)^n$, but $X \neq (X,A)^{n-1}$, we say that $\dim(X,A) = n$. If no such integer n exists we say that $\dim(X,A) = \infty$. We agree to have $\dim(A,A) = -1$.

It follows directly from Definition 2.1 and Corollary 1.4 that for an equivariant relative CW complex (X,A) the integer (or ∞) $\dim(X,A)$ is well-defined. We shall prove below that $\dim(X,A)$ depends only on the G-pair (X,A), and does not depend on the filtration which makes (X,A) into an equivariant relative CW complex. This result also follows from the fact that $(G\backslash X, G\backslash A)$ is a relative CW complex in the ordinary sense, and for such a pair $\dim(G\backslash X, G\backslash A)$ is independent of the skeleton filtration. But we shall not use this and instead give a complete proof for the equivariant case.

Proposition 2.2. Let (X,A) be a G-pair which admits the structure of an equivariant relative CW complex. Then $\dim(X,A)$ is well-defined, that is, does not depend on the skeleton filtration.

Proof. Assume that (X,A) is an equivariant relative CW complex with skeletons $(X,A)^0 \subset \ldots \subset (X,A)^n = X$ and such that $(X,A)^{n-1} \neq X$. We shall show that if $A \subset Y^0 \subset \ldots \subset Y^m \subset \ldots$, $\bigcup_{q=0}^{\infty} Y^q = X$ is another filtration of X which gives (X,A) the structure of an equivariant CW complex, then $Y^n = X$ and $Y^{n-1} \neq X$. This proves that $\dim(X,A)$ is well-defined.

We use the notation $A = (X,A)^{-1}$ and $A = Y^{-1}$. First we prove that $X - (X,A)^{n-1} \subset Y^n$. Let c^n be an equivariant n-cell of $(X,(X,A)^{n-1})$, which exists since $(X,A)^{n-1} \neq X$. Since c^n is compact, there exists a finite integer q, such that $(c^n - \dot{c}^n) \subset c^n \subset Y^q$ (see Lemma 2.2 below). Let m be the smallest integer for which $(c^n - \dot{c}^n) \subset Y^m$. Thus there exists an equivariant m-cell e^m of (Y^m, Y^{m-1}) such that $(c^n - \dot{c}^n) \cap (e^m - \dot{e}^m) \neq \emptyset$. Since $c^n - \dot{c}^n$ is an open subset of $(X,A)^n = X$

(by Lemma 1.2), it follows that $(c^n - \dot{c}^n) \cap (e^m - \dot{e}^m)$ is open in $e^m - \dot{e}^m$. On the other hand $(X - Y^m) \cup (e^m - \dot{e}^m)$ is an open subset of X since $e^m - \dot{e}^m$ is open in Y^m, and thus the set $(c^n - \dot{c}^n) \cap (e^m - \dot{e}^m) =$ $(c^n - \dot{c}^n) \cap ((X - Y^m) \cup (e^m - \dot{e}^m))$ is open in $c^n - \dot{c}^n$. It follows that $G\backslash((c^n - \dot{c}^n) \cap (e^m - \dot{e}^m))$ is open in both $G\backslash(c^n - \dot{c}^n)$ and $G\backslash(e^m - \dot{e}^m)$. But since these two spaces are homeomorphic to $\overset{\circ}{E}{}^n$ and $\overset{\circ}{E}{}^m$, respectively, it follows that $m = n$. Thus $X - (X,A)^{n-1} \subset Y^n$.

Now assume by induction that $X - (X,A)^p \subset Y^n$. We shall prove that $X - (X,A)^{p-1} \subset Y^n$. Thus we have to prove that $(X,A)^p - (X,A)^{p-1} \subset Y^n$. Assume the contrary and let c^p be an equivariant p-cell of $((X,A)^p, (X,A)^{p-1})$, such that $c^p - \dot{c}^p \subset Y^m$, where $m > n$, and $c^p - \dot{c}^p \not\subset Y^{m-1}$. Thus there exists an equivariant m-cell e^m of (Y^m, Y^{m-1}) such that $(c^p - \dot{c}^p) \cap (e^m - \dot{e}^m) \neq \emptyset$. Now $(X - (X,A)^p) \cup (c^p - \dot{c}^p)$ is an open subset of X. Since $m > n$, it follows from the induction assumption that $((X - (X,A)^p) \cup (c^p - \dot{c}^p)) \cap (e^m - \dot{e}^m) = (c^p - \dot{c}^p) \cap (e^m - \dot{e}^m)$. Thus $(c^p - \dot{c}^p) \cap (e^m - \dot{e}^m)$ is open in $e^m - \dot{e}^m$. We also have $(c^p - \dot{c}^p) \cap ((X - Y^m) \cup (e^m - \dot{e}^m)) = (c^p - \dot{c}^p) \cap (e^m - \dot{e}^m)$, and thus $(c^p - \dot{c}^p) \cap (e^m - \dot{e}^m)$ is open in $c^p - \dot{c}^p$ since $(X - Y^m) \cup (e^m - \dot{e}^m)$ is an open subset of X. By the same argument as above we get that $m = p$. This is a contradiction since $p < n < m$. Hence $X - (X,A)^{p-1} \subset Y^n$. Thus by induction $X - A \subset Y^n$, and hence $X \subset Y^n$. It follows that $X = Y^n$. If $X = Y^{n-1}$, then by what we just have proved it would follow that $X = (X,A)^{n-1}$, which is a contradiction. Thus we have $X = Y^n$ and $X = Y^{n-1}$.

q.e.d.

Lemma 2.3. Let (X,A) be an equivariant relative CW complex and let C be a compact subset of X. Then there exists an integer m such that $C \subset (X,A)^m$.

Proof. The proof is the same as that of Lemma 1.15. q.e.d.

Definition 2.4. Let (X,A) be an equivariant relative CW complex, Y a closed G-subset of X. We say that $(Y, Y \cap A)$ is a subcomplex of (X,A) if the filtration $Y \cap (X,A)^k$, $k = 0,1,\ldots$ gives $(Y, Y \cap A)$ the structure of an equivariant relative CW complex.

Lemma 2.5. Let $(Y, Y \cap A)$ be a cubsomplex of the equivariant relative CW complex (X,A). Then $(X, Y \cup A)$ is an equivariant relative CW complex, with skeletons $(X, Y \cup A)^k = Y \cup (X,A)^k$.

Proof. First observe that X has the topology coherent with $Y \cup (X,A)^k$, $k = 0,1,\ldots$. It remains to show that $Y \cup (X,A)^k$ is obtainable from $Y \cup (X,A)^{k-1}$ by adjoining equivariant n-cells, for $k = 0,1,\ldots$, where $(X,A)^{-1} = A$. Clearly $Y \cup (X,A)^0$ is obtainable from $Y \cup A$ by adjoining equivariant 0-cells. Now assume that $k \geq 1$, and let e^k be an equivariant k-cell of $(Y \cap (X,A)^k, Y \cap (X,A)^{k-1})$. Then $e^k \cap (X,A)^{k-1} = e^k \cap (Y \cap (X,A)^{k-1}) = \dot{e}^k$, and thus the closed G-subset e^k satisfies the conditions in Lemma 1.3. Hence $e^k = c_j^k$, for some $j \in J$, where $\{c_j^k\}_{j \in J}$ is the collection of all equivariant k-cells of $((X,A)^k, (X,A)^{k-1})$. Thus every equivariant k-cell of $(Y \cap (X,A)^k, Y \cap (X,A)^{k-1})$ is an equivariant k-cell of $((X,A)^k, (X,A)^{k-1})$. It is now easy to see that

$Y \cup (X,A)^k$ is obtained from $Y \cup (X,A)^{k-1}$ by adjoining all the equivariant k-cells of $((X,A)^k, (X,A)^{k-1})$ which are not equivariant k-cells of $(Y \cap (X,A)^k, Y \cap (X,A)^{k-1})$.

q.e.d.

We shall now consider the product of two equivariant relative CW complexes. Let M be another compact, Hausdorff, topological group. Let (X,A) be a G-pair, and (Y,B) an M-pair. Assume that (X,A) is an equivariant relative CW complex with skeletons $(X,A)^k$, $k = 0,1,\ldots,$ and that the M-pair (Y,B) is an equivariant relative CW complex with skeletons $(Y,B)^p$, $p = 0,1,\ldots$. Let $G \times M$ act on $X \times Y$ in the obvious way and consider the $(G \times M)$-pair $(X,A) \times (Y,B) = (X \times Y, X \times B \cup A \times Y)$. Define a filtration of $X \times Y$ by closed $(G \times M)$-subsets $((X,A) \times (Y,B))^n$, $n = 0,1,\ldots$, through the formula

$$((X,A) \times (Y,B))^n = \bigcup_{k+p=n} (X,A)^k \times (Y,B)^p .$$

Here k and p denote arbitrary integers, and $(X,A)^k = A$ for $k < 0$, and $(Y,B)^p = B$ for $p < 0$. Thus all the sets $((X,A) \times (Y,B))^n$, $n = 0,1,\ldots,$ contain $X \times B \cup A \times Y$, and we also get $((X,A) \times (Y,B))^n = X \times B \cup A \times Y$, for $n < 0$.

Consider the $(G \times M)$-pair $((X,A) \times (Y,B))^n, ((X,A) \times (Y,B))^{n-1})$. Let c^k be an equivariant k-cell of (X,A), and e^p an equivariant p-cell of (Y,B), where $k+p=n$. Then $(c^k \times e^p) \cap ((X,A) \times (Y,B))^{n-1} = c^k \times \dot{e}^p \cup \dot{c}^k \times e^p$. Let $f\colon (E^k \times G/H, S^{k-1} \times G/H) \longrightarrow (c^k, \dot{c}^k)$ and $f'\colon (E^p \times M/N, S^{p-1} \times M/N) \longrightarrow (e^p, \dot{e}^p)$ be equivariant characteristic maps for c^k and e^p respectively. Let $h\colon (E^n, S^{n-1}) \longrightarrow (E^k, S^{k-1}) \times (E^p, S^{p-1})$

be a homeomorphism, and let h_1 and h_2 be the factors of h to E^k and E^p, respectively. Define a $(G \times M)$-homeomorphism

$$\bar{h}: (E^n, S^{n-1}) \times (G \times M)/(H \times N) \longrightarrow (E^k, S^{k-1}) \times G/H \times (E^p, S^{p-1}) \times M/N$$

by $\bar{h}(x, (g, m)(H \times N)) = (h_1(x), gH, h_2(x), mN)$. Then the $(G \times M)$-map $(f \times f')\bar{h}: E^n, S^{n-1}) \times (G \times M)/(H \times N) \longrightarrow (c^k \times e^p, c^k \times \dot{e}^p \cup \dot{c}^k \times e^p)$ satisfies condition 3 in Definition 1.1. Also observe that $((X, A) \times (Y, B))^n$ is the union of $((X, A) \times (Y, B))^{n-1}$ and all sets of the form $c^k \times e^p$, where $k+p=n$, and that the sets of the form $c^k \times e^p - (c^k \times \dot{e}^p \cup \dot{c}^k \times e^p)$ are disjoint from each other. We now have:

<u>Proposition 2.6</u>. Let the notation be as above. If both X and Y are locally compact or if X is compact and Y is arbitrary, then $(X, A) \times (Y, B) = (X \times Y, X \times B \cup A \times Y)$ is an equivariant relative CW complex with skeletons as defined above.

<u>Proof</u>. We first prove that under the above assumptions $X \times Y$ has the topology coherent with $\{((X, A) \times (Y, B))^n\}_{n \geq 0}$. Let $F \subset X \times Y$ be a subset such that $F \cap ((X, A) \times (Y, B))^n$ is a closed set for all $n \geq 0$.

If X and Y are locally compact, then $X \times Y$ is locally compact and hence compactly generated. Thus in order to show that F is a closed set, it is enough to show that $F \cap C$ is closed for each compact subset C of $X \times Y$. Let $C \subset X \times Y$ be compact. Then $\pi_X(C) \subset X$ and $\pi_Y(C) \subset Y$ are compact and hence it follows by Lemma 2.3 that $C \subset (X, A)^p \times (Y, B)^r$ for some p and r. Thus $F \cap C = F \cap ((X, A) \times (Y, B))^{p+r} \cap C$ is a closed set. Hence F is closed.

If X is compact, then $X = X^{m_0}$ for some m_0 by Lemma 2.3. Moreover $X \times Y$ has the topology coherent with $\{X \times (Y,B)^n\}_{n \geq 0}$. Since now $X \times (Y,B)^n \subset ((X,A) \times (Y,B))^{m_0+n}$ our claim follows. The fact that under the above assumptions $((X,A) \times (Y,B))^n$ has the topology coherent with the family consisting of $((X,A) \times (Y,B))^{n-1}$ and all n-cells $c^k \times e^p$, $k + p = n$, is proved by arguments which are completely analogous to the ones above.

q. e. d.

Proposition 2.7. Let (X,A) be an equivariant relative CW complex. Then (X,A) has the G-homotopy extension property with respect to any G-map.

Proof. This follows from Corollary 1.7, using induction and the fact that $I \times X$ has the topology coherent with $\{I \times (X,A)^k\}_{k \geq 0}$.

q. e. d.

Proposition 2.8. Let (X,A) be an equivariant relative CW complex, with $\dim(X,A) \leq n$, and let (Y,B) be an equivariantly n-connected G-pair. Then any G-map $f: (X,A) \longrightarrow (Y,B)$ is G-homotopic relative to A, to a G-map from X into B.

Proof. The proof is by induction on n. If $n = 0$, then X is obtainable from A by adjoining equivariant 0-cells, and our assertion follows from Lemma 1.14. Now assume that our assertion is correct for the value $n-1$, where $n \geq 1$. We shall prove that it is correct for the value n. Let $f: (X,A) \longrightarrow (Y,B)$ be a G-map, where $\dim(X,A) \leq n$, and (Y,B) is equivariantly n-connected. Consider the G-map $f|: ((X,A)^{n-1},A) \longrightarrow (Y,B)$.

Since $\dim((X,A)^{n-1},A) \leq n-1$ and since (Y,B) is equivariantly (n-1)-connected, it follows by the induction assumption that there exists a G-homotopy relative to A, $F: I \times ((X,A)^{n-1},A) \longrightarrow (Y,B)$, from $f|$ to a G-map from $(X,A)^{n-1}$ into B. Then by the G-homotopy extension property, there exists a G-homotopy $\overline{F}: I \times X \longrightarrow Y$ such that $\overline{F}: I \times (X,A)^{n-1} = F$ and $\overline{F}(0,x) = f(x)$, for $x \in X$. Define $f_1: (X,(X,A)^{n-1}) \longrightarrow (Y,B)$ by $f_1(x) = \overline{F}(1,x)$. Since (Y,B) satisfies condition π_n, and $X = (X,A)^n$ is obtainable from $(X,A)^{n-1}$ by adjoining equivariant n-cells, it follows by Lemma 1.14 that f_1 is G-homotopic relative to $(X,A)^{n-1}$ to a G-map from X into B. Since f is G-homotopic relative to A to f_1, it follows that f is G-homotopic relative to A to a G-map from X into B.

q.e.d.

<u>Corollary 2.9</u>. Let (X,A) be an equivariant relative CW complex, and let (Y,B) be equivariantly n-connected for all n. Then any G-map $f: (X,A) \longrightarrow (Y,B)$ is G-homotopic relative to A, to a G-map from X into B.

<u>Proof</u>. It follows by induction from Proposition 2.8 and Proposition 2.7 that there exist G-homotopies

$$F_k: I \times (X,A) \longrightarrow (Y,B), \qquad k = 0,1,\ldots$$

such that

1. $F_0(0,x) = f(x)$, for $x \in X$.
2. $F_k(1,x) = F_{k+1}(0,x)$, for $x \in X$.
3. F_k is a G-homotopy relative to $(X,A)^{k-1}$.
4. $F_k(1 \times (X,A)^k) \subset B$.

Now define $f: I\times(X,A) \longrightarrow (Y,B)$ by the formula

$$F(t,x) = F_{k-1}\left(\frac{t-(1-\frac{1}{k})}{\frac{1}{k}-\frac{1}{k+1}}, x\right), \text{ for } 1-\frac{1}{k} \leq t \leq 1-\frac{1}{k+1}, \quad k \geq 1.$$

$$F(1,x) = F_{k+1}(1,x), \quad \text{for } x \in (X,A)^k.$$

It remains to show that F is continuous. Consider $F|I\times(X,A)^m$, where $m \geq 0$. Since the set $[0,1-\frac{1}{m+2}]\times(X,A)^m$ is a finite union of closed subsets on which F is continuous it follows that $F|[0, 1-\frac{1}{m+2}]\times(X,A)^m$ is continuous. Since each G-homotopy F_{m+p}, where $p \geq 1$, is relative to $(X,A)^m$, it follows that $F(t,x) = F_{m+1}(1,x)$, for $1-\frac{1}{m+2} \leq t < 1$, and $x \in (X,A)^m$. Since also $F(1,x) = F_{m+1}$, for $x \in (X,A)^m$, it follows that $F|[1-\frac{1}{m+2}, 1]\times(X,A)^m$ is continuous, and hence $F|I\times(X,A)^m$ is continuous. Thus F is continuous since $I\times X$ has the topology coherent with $\{I\times(X,A)^k\}_{k\geq 0}$. q.e.d.

For any two G-spaces X and Y we denote by $[X;Y]$ the set of G-homotopy classes of G-maps from X into Y.

Proposition 2.10. Let (Y,B) be an equivariantly n-connected G-pair, and denote by $i: B \longrightarrow Y$ the inclusion. The induced function

$$i_{\#}: [X,B] \longrightarrow [X,Y]$$

is surjective for all equivariant CW complexes X with $\dim X \leq n$, and $i_{\#}$ is injective for all equivariant CW complexes with $\dim X \leq n-1$. If (Y,B) is equivariantly n-connected for all n, then $i_{\#}$ is a bijection for all equivariant CW complexes X.

Proof. Assume that (Y, B) is equivariantly n-connected. It follows directly from Proposition 2.8 that $i_{\#}: [X, B] \longrightarrow [X, Y]$ is onto if $\dim X \leq n$. Let $\dim X \leq n - 1$, and assume that the G-maps $f, f': X \longrightarrow B$ are such that there exists a G-homotopy $F: I \times X \longrightarrow Y$ from if to if'. Thus $F: (I \times X, 0 \times X \cup 1 \times X) \longrightarrow (Y, B)$, and since $\dim(I \times X, 0 \times X \cup 1 \times X) \leq n$ it follows by Proposition 2.8 that F is G-homotopic relative to $0 \times X \cup 1 \times X$ to a G-map $\widetilde{F}: I \times X \longrightarrow B$. Thus $\widetilde{F}$ is a G-homotopy in B from f to f'. This shows that $i_{\#}$ is injective if $\dim X \leq n - 1$. If (Y, B) is equivariantly n-connected for all n, we use Corollary 2.9 to show that $i_{\#}$ is a bijection for all equivariant CW complexes X.

q.e.d.

Together with the mapping cylinder construction Proposition 2.10 will give us an equivariant Whitehead theorem. But we shall first continue towards the proof of an equivariant skeletal approximation theorem.

Lemma 2.11. Let Y be a G-space and $f: E^k \times G/H \longrightarrow Y$ a G-map. Let $\{U_j\}_{j \in J}$ be a covering of Y by open G-subsets. Then there exists a triangulation $|K| \cong E^k$ of E^k such that for any simplex $s \in K$ there exists $j \in J$ such that $f(|s| \times G/H) \subset U_j$.

Proof. The sets $(E^k \times \{eH\}) \cap f^{-1}(U_i)$ form an open cover of $E^k \times \{eH\} = E^k$. Thus there exists a triangulation $|K| \cong K^k$ of E^k such that for any simplex $s \in K$ there exists $j \in J$ such that $|s| \subset (E^k \times \{eH\}) \cap f^{-1}(U_j)$. Thus $f(|s|) \subset U_j$ and since U_j is a G-subset of Y we have $f(|s| \times G/H) \subset U_j$.

q.e.d.

Lemma 2.12. Let X be obtainable from A by adjoining equivariant n-cells where $n \geq 1$. Then (X, A) is equivariantly $(n-1)$-connected.

Proof. Let $f: (E^k \times G/H, S^{k-1} \times G/H) \longrightarrow (X, A)$ be a G-map where H is a closed subgroup of G and $0 \leq k \leq n-1$. Let $\{c_j^n\}_{j \in J}$ be the collection of equivariant n-cells of (X, A). Since the set $f(E^k \times G/H)$ is compact, there exists by Lemma 1.15 a finite number of equivariant n-cells, say $c_1^n, \ldots, c_m^n$, such that we have $f(E^k \times G/H) \subset A \cup c_1^n \cup \ldots \cup c_m^n$. For each i, $1 \leq i \leq m$, choose a point $x_i \in c_i^n - \dot{c}_i^n$. Then the sets $\widetilde{A} = A \cup (c_1^n - Gx_1) \cup \ldots \cup (c_m^n - Gx_m)$ and $c_i^n - \dot{c}_i^n$, $1 \leq i \leq m$ form a covering of X by open G-subsets of X. By Lemma 2.11 there exists a triangulation $|K| \cong K^k$ of K^k such that for every simplex $s \in K$, we have either $f(|s| \times G/H) \subset \widetilde{A}$ or $f(|s| \times G/H) \subset c_i^n - \dot{c}_i^n$ for some i, $1 \leq i \leq m$. Let $|L|$ be the subpolyhedron of $|K| \cong E^k$, which is the space of all simplexes $s \in K$ for which we have $f(|s| \times G/H) \subset \widetilde{A}$, and let $|L_i|$ be the subpolyhedron which is the space of all simplexes $s \in K$ for which we have $f(|s| \times G/H) \subset c_i^n - \dot{c}_i^n$, $i = 1, \ldots, m$. We have $S^{k-1} \subset |L|$ and $E^k \cong |K| = |L| \cup |L_1| \cup \ldots \cup |L_m|$, and if $i \neq j$ then $|L_i| \cap |L_j| = \emptyset$. Denote $|\dot{L}_i| = |L_i| \cap |L| = |L_i \cap L|$, $i = 1, \ldots, m$, and observe that $(|L_i|, |\dot{L}_i|)$ is a relative CW-complex, in the ordinary sense, with $\dim(|L_i|, |\dot{L}_i|) \leq k \leq n-1$. Thus $(|L_i| \times G/H, |\dot{L}_i| \times G/H)$ is an equivariant relative CW complex and $\dim(|L_i| \times G/H, |\dot{L}_i| \times G/H) \leq n-1$.

Now observe that the G-pair $(c_i^n - \dot{c}_i^n, (c_i^n - \dot{c}_i^n) - Gx_i)$ is G-homeomorphic with the G-pair $((E^n - S^{n-1}) \times G/H_i, ((E^n - S^{n-1}) - 0) \times G/H_i)$

for some closed subgroup H_i of G, $i = 1,\ldots,m$. Since the pair $(E^n - S^{n-1}, (E^n - S^{n-1}) - 0)$ is (n-1)-connected, in the ordinary sense, it follows by Lemma 1.12 that the G-pair $((E^n - S^{n-1}) \times G/H_i, ((E^n - S^{n-1}) - 0) \times G/H_i)$ is equivariantly (n-1)-connected. Thus it follows from Proposition 2.8 that the G-map $f| : (|L_i| \times G/H, |\dot{L}_i| \times G/H) \longrightarrow (c_i^n - \dot{c}_i^n, (c_i^n - \dot{c}_i^n) - Gx_i)$ is G-homotopic relative to $|\dot{L}_i| \times G/H$ to a G-map from $|L_i| \times G/H$ into $(c_i^n - \dot{c}_i^n) - Gx_i \subset \tilde{A}$. Thus these G-homotopies together with the constant homotopy on $|L| \times G/H$, determine a G-homotopy relative to $|L| \times G/H$, $F : I \times E^k \times G/H \longrightarrow X$ from f to a G-map $f' : E^k \times G/H \longrightarrow \tilde{A}$. Since $S^{k-1} \subset |L|$ it follows that f is G-homotopic relative to $S^{k-1} \times G/H$ to $f' : E^k \times G/H \longrightarrow \tilde{A}$. Clearly A is a strong G-deformation retract of $\tilde{A}$. Thus f' is G-homotopic relative to $S^{k-1} \times G/H$ to a G-map $f'' : E^k \times G/H \longrightarrow A$. Hence f is G-homotopic relative to $S^{k-1} \times G/H$ to a G-map from $E^k \times G/H$ into A.

q. e. d.

Corollary 2.13. Let (X, A) be an equivariant relative CW complex. Then for any $n \geq 0$, $(X, (X,A)^n)$ is equivariantly n-connected.

Proof. We first prove by induction that $((X,A)^m, (X,A)^n)$ is equivariantly n-connected for all $m > n$. Since $(X,A)^{n+1}$ is obtainable from $(X,A)^n$ by adjoining equivariant (n+1)-cells, it follows by Proposition 2.12 above that our claim is true for the value $m = n + 1$. Now let $m > n + 1$ and assume that our claim is true for the value m-1. Since $((X,A)^m, (X,A)^{m-1})$ is equivariantly (m-1)-connected, it is also equivariantly n-connected.

Let $f: (E^k \times G/H, S^{k-1} \times G/H) \longrightarrow ((X,A)^m, (X,A)^n)$ be a G-map where $0 \leq k \leq n$. Thus f is G-homotopic relative to $S^{k-1} \times G/H$ to a G-map $f': E^k \times G/H \longrightarrow (X,A)^{m-1}$. Since $((X,A)^{m-1}, (X,A)^n)$ is equivariantly n-connected by the induction assumption, it follows that f' is G-homotopic relative to $S^{k-1} \times G/H$ to a G-map $f'': E^k \times G/H \longrightarrow (X,A)^n$. Thus f is G-homotopic relative to $S^{k-1} \times G/H$ to a G-map from $E^k \times G/H$ into $(X,A)^n$ which shows that $((X,A)^m, (X,A)^n)$ is equivariantly n-connected. Now let $f: (E^k \times G/H, S^{n-1} \times G/H) \longrightarrow (X,(X,A)^n)$ be any G-map, where $0 \leq k \leq n$. Since the set $f(E^k \times G/H)$ is compact, it follows by Lemma 2.3 that there exists m such that $f(E^k \times G/H) \subset (X,A)^m$. Then by what we already proved, f is G-homotopic relative to $S^{k-1} \times G/H$ to a G-map from $E^k \times G/H$ into $(X,A)^n$.

q.e.d.

We are now able to prove an equivariant skeletal approximation theorem. Let again M denote another compact, Hausdorff, topological group, and assume that the M-pair (Y,B) is an equivariant relative CW complex with skeletons $(Y,B)^k$, $k \geq 0$. Let $\varphi: G \longrightarrow M$ be a continuous homomorphism. Assume that the G-space (X,A) is an equivariant relative CW complex, with skeletons $(X,A)^k$, $k \geq 0$. A map $f: (X,A) \longrightarrow (Y,B)$ is called a φ-map if $f(gx) = \varphi(g)f(x)$, for all $g \in G$, $x \in X$. We say that a φ-map $f: (X,A) \longrightarrow (Y,B)$ is skeletal if $f((X,A)^k) \subset (Y,B)^k$, all $k \geq 0$. Observe that f is a φ-map if and only if f is a G-map into Y when Y is made into a G-space through $\varphi: G \longrightarrow M$. But it is perhaps only as an M-pair that (Y,B) is an equivariant relative CW complex and this is why we have formulated the concept of skeletal map in this generality.

<u>Theorem 2.14.</u> Let the G-pair (X,A) and the M-pair (Y,B) be equivariant relative CW complexes, and let $\varphi: G \to M$ be a continuous homomorphism. Assume that the φ-map $f: (X,A) \to (Y,B)$ is skeletal on the subcomplex $(X', X' \cap A)$ of (X,A). Then there exists a skeletal φ-map $\bar{f}: (X,A) \to (Y,B)$ which is φ-homotopic rel. X' to f.

<u>Proof.</u> By Corollary 2.13 the M-pair $(Y,(Y,B)^k)$ is equivariantly k-connected, $k = 0, 1, \ldots$. Now make Y into a G-space through $\varphi: G \to M$. By Corollary 1.13 the G-pair $(Y,(Y,B)^k)$ is equivariantly k-connected. From now on we shall consider Y as a G-space. Consider the G-map $f: ((X,A)^0, A \cup (X' \cap (X,A)^0)) \to (Y,(Y,B)^0)$. By Proposition 2.8 there exists a G-homotopy relative to $A \cup (X' \cap (X,A)^0)$, $F_0: I \times (X,A)^0 \to Y$ from $f|$ to a G-map from $(X,A)^0$ into $(Y,B)^0$. Extend F_0 to a homotopy relative to $X' \cup A$, $F_0': I \times ((X,A)^0 \cup X') \to Y$. By the G-homotopy extension property there exists a G-homotopy $\bar{F}_0: I \times X \to Y$ from f which extends F_0'. Thus $\bar{F}_0$ is a homotopy relative to X' and to A, and $\bar{F}_0(1 \times (X,A)^0) \subset (Y,B)^0$. Define $f_1: X \to Y$ by $f_1(x) = \bar{F}_0(1,x)$. Then we have $f_1: ((X,A)^1, (X,A)^0 \cup (X' \cap (X,A)^1) \to (Y,(Y,B)^1)$. Now in the same way as above using Proposition 2.8 and the G-homotopy extension property we see that there exists a G-homotopy relative to X' and to $(X,A)^0$, $\bar{F}_1: I \times X \to Y$ from f_1, and such that $\bar{F}_1(1 \times (X,A)^1) \subset (Y,B)^1$. Continuing in this way we see that there exist G-homotopies relative to X', $\bar{F}_k: I \times X \to Y$, $k = 0,1,\ldots,$ such that

1. $\overline{F}_0(0,x) = f(x)$, for $x \in X$.

2. $\overline{F}_k(1,x) = \overline{F}_{k+1}(0,x)$, for $x \in X$.

3. $\overline{F}_k$ is a homotopy relative to $(X,A)^{k-1}$.

4. $\overline{F}_k(1 \times (X,A)^k) \subset (Y,B)^k$.

Now define $F: I \times X \longrightarrow Y$, by the formula

$$F(t,x) = \overline{F}_{k-1}\left(\frac{t - (1 - \frac{1}{k})}{\frac{1}{k} - \frac{1}{k+1}}, x\right), \quad \text{for } 1 - \frac{1}{k} \leq t \leq 1 - \frac{1}{k+1}, \ k \geq 1.$$

$$F(1,x) = \overline{F}_{k+1}(1,x), \qquad \text{for } x \in (X,A)^k.$$

It is easily seen that F is continuous (see the proof of Corollary 2.9). Clearly the G-map $\overline{f}: (X,A) \longrightarrow (Y,B)$ defined by $\overline{f}(x) = F(1,x)$ is a map with the desired properties.

q.e.d.

Corollary 2.15. Let the G-space X and the M-space Y be equivariant CW complexes, and let $\varphi: G \longrightarrow M$ be a continuous homomorphism. Then any φ-map from X to Y is φ-homotopic to a skeletal φ-map. If skeletal φ-maps from X into Y are φ-homotopic, there exists a skeletal φ-homotopy between them.

q.e.d.

3. EQUIVARIANT WHITEHEAD THEOREM

Let X and Y be two arbitrary G-spaces, and $f: X \longrightarrow Y$ be a G-map. We shall consider the mapping cylinder of f. Although there is no difference between the ordinary and the equivariant case, we shall give the details in order to fix the notation and make it clear that the mapping cylinder inherits a continuous G-action.

Let $I \times \dot{\cup}\, Y$ be the disjoint union of $I \times X$ and Y; it is a G-space in the obvious way. Define a relation $\sim$ in $I \times X \,\dot{\cup}\, Y$ by

$$\begin{array}{ll} (1,x) \sim y \ \text{ and } \ y \sim (1,x) & \text{if } f(x) = y,\ x \in X,\ y \in Y. \\ (1,x) \sim (1,x') & \text{if } (x) = f(x'),\ x,\ x' \in X. \\ \alpha \quad \sim \alpha & \text{for } \alpha \in I \times X \,\dot{\cup}\, Y. \end{array}$$

Thus $\sim$ is an equivalence relation in $I \times X \,\dot{\cup}\, Y$. Let Z_f denote the set of equivalence classes, and let $\pi: I \times X \,\dot{\cup}\, Y \longrightarrow Z_f$ be the natural projection. We make Z_f into a topological space by giving it the quotient topology from π. We denote $\pi(t,x) = [t,x]$ and $\pi(y) = [y]$. Let $\tilde{\gamma}: G \times (I \times X \,\dot{\cup}\, Y) \longrightarrow I \times X \,\dot{\cup}\, Y$ be the G-action on $I \times X \,\dot{\cup}\, Y$. If $\alpha \sim \beta$, where $\alpha, \beta \in I \times X \,\dot{\cup}\, Y$, it follows, since f is a G-map, that $\tilde{\gamma}(g,\alpha) \sim \tilde{\gamma}(g,\beta)$ for all $g \in G$. Thus we have the commutative diagram

$$\begin{array}{ccc} G \times (I \times X \,\dot{\cup}\, Y) & \xrightarrow{\ \tilde{\gamma}\ } & I \times X \,\dot{\cup}\, Y \\ {\scriptstyle id \times \pi} \downarrow & & \downarrow {\scriptstyle \pi} \\ G \times Z_f & \xrightarrow{\ \gamma\ } & Z_f \end{array}$$

where γ is well-defined by $\gamma(g,[\alpha]) = [\tilde{\gamma}(g,\alpha)]$.

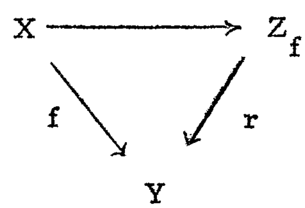
Since G is compact and π is a quotient map, it follows that $id \times \pi$ is a quotient map. Thus γ is continuous and makes Z_f into a G-space.

We have the G-map $i: X \longrightarrow Z_f$, defined by $i(x) = [0,x]$. This is an imbedding of X as a closed G-subset of Z_f. We shall consider X as a closed G-subspace of Z_f through i. We also have the G-map $j: Y \to Z_f$, defined by $j(y) = [y]$. We shall consider Y as a closed G-subset of Z_f, through j. Finally, we have a G-retraction from Z_f onto Y, $r: Z_f \to Y$, defined by $r[t,x] = [1,x] = [f(x)]$, for $(t,x) \in I \times X$,

and $r[y] = [y]$, for $y \in Y$.

Exactly as in the ordinary case we now have

Proposition 3.1. Let the notation be as above. The diagram



commutes. The G-retraction r is a G-homotopy equivalence, with G-homotopy inverse $j: Y \to Z_f$. More precisely, $j \circ r: Z_f \to Z_f$ is G-homotopic relative to Y, to the identity. Moreover (Z_f, X) has the G-homotopy extension property with respect to any G-map.

Proof. That the diagram commutes follows from the definitions. Define $F: I \times Z_f \to Z_f$ by

$$F(\tau,[t,x]) = [\tau t + 1 - \tau, x], \qquad \text{for } (\tau, t, x) \in I \times I \times X.$$

$$F(\tau, [y] = [y] \qquad , \qquad \text{for } y \in Y.$$

Then F is a G-homotopy relative to Y from $j \circ r$ to the identity.

It remains to show that (Z_f, X) has the G-homotopy extension property. Let $h: Z_f \to W$ and $H': I \times X \to W$ be G-maps such that

$H'(0,x) = h([0,x])$ for $x \in X$. Then define $H: I \times Z_f \longrightarrow W$ by

$$H(\tau,[t,x]) = \begin{cases} h\left[\frac{2t-\tau}{2-\tau}, x\right], & 0 \leq \tau \leq 2t \leq 2,\ x \in X \\ H'\left(\frac{\tau-2t}{1-t}, x\right), & 0 \leq 2t \leq \tau \leq 1,\ x \in X \end{cases}$$

$$H(\tau,[y]) = h[y], \qquad y \in Y.$$

Then H is a G-homotopy from h, and H extends H'.

q.e.d

Definition 3.2. Let X and Y be two G-spaces. We say that a G-map $f: X \longrightarrow Y$ is an equivariant n-equivalence if the G-pair (Z_f, X) is equivariantly n-connected.

Proposition 3.3. A G-map $f: X \longrightarrow Y$ is an equivariant n-equivalence if and only if the induced map $f^H: X^H \longrightarrow Y^H$ is an n-equivalence in the ordinary sense for every closed subgroup H of G.

Proof. It follows from Corollary 1.11 that $f: X \longrightarrow Y$ is an equivariant n-equivalence if and only if $((Z_f)^H, X^H)$ is n-connected for each closed subgroup H of G. But it is easy to see that the pair $((Z_f)^H, X^H)$ is homeomorphic with the pair (Z_{f^H}, X^H). This completes the proof since $f^H: X^H \longrightarrow Y^H$ is a n-equivalence if and only if (Z_{f^H}, X^H) is n-connected (either by definition or a standard fact).

q.e.d.

Remark. A map $h: V \longrightarrow W$ between topological spaces is an n-equivalence if $h_*: \pi_k(V,v) \longrightarrow \pi_k(W,h(v))$ is bijective for $0 \leq k < n$ and onto for $k = n$, for every $v \in V$. This is equivalent to (Z_h, V) being n-connected. Observe that if $h: V \longrightarrow W$ is a homotopy equivalence, then

$h_*: \pi_k(V, v) \longrightarrow \pi_k(W, h(v))$ is bijective for all $k \geq 0$, and all $v \in V$, and also observe that this fact needs a proof since a homotopy inverse to h need not map $h(v)$ back to v.

Corollary 3.4. A G-homotopy equivalence is an equivariant n-equivalence for all $n \geq 0$.

Proof. If $f: X \longrightarrow Y$ is a G-homotopy equivalence, then clearly the induced map $f^H: X^H \longrightarrow Y^H$ is a homotopy equivalence and thus an n-equivalence for all $n \geq 0$.

q.e.d.

Theorem 3.5. Let $f: X \longrightarrow Y$ be an equivariant n-equivalence. The induced function

$$f_{\#}: [C; X] \longrightarrow [C; Y]$$

is surjective for all equivariant CW complexes C with $\dim C \leq n$, and $f_{\#}$ is injective for all equivariant CW complexes C with $\dim C \leq n - 1$. If f is an equivariant n-equivalence for all n, then $f_{\#}$ is a bijection for all equivariant CW complexes C.

Proof. It follows from Proposition 3.1 that we have a commutative diagram

$$\begin{array}{ccc} [C; X] & \xrightarrow{\quad i_{\#} \quad} & [C; Z_f] \\ & {\scriptstyle f_{\#}} \searrow \quad \swarrow {\scriptstyle r_{\#}} & \\ & [C; Y] & \end{array}$$

where $r_{\#}$ is a bijection. Our claim now follows by Proposition 2.9.

q.e.d.

Corollary 3.6. Let X and Y be equivariant CW complexes and $f: X \to Y$ an equivariant n-equivalence where $\max(\dim X, \dim Y) \leq n-1$. Then f is a G-homotopy equivalence.

Proof. Since $f_{\#}: [Y;X] \to [Y;Y]$ is onto there exists a G-map $h: Y \to X$ such that fh is G-homotopic to id_Y. Since $f_{\#}: [X;X] \to [X,Y]$ is injective and $f_{\#}[hf] = [fhf] = [id_Y f] = [f\, id_Y] = f_{\#}[id_X]$ it follows that hf is G-homotopic to id_X. Thus $h: Y \to X$ is a G-homotopy inverse to f.

q.e.d.

Corollary 3.7. Let X and Y be equivariant CW complexes. Then a G-map $f: X \to Y$ is a G-homotopy equivalence if and only if for each closed subgroup H of G the induced map $f^H: X^H \to Y^H$ induces a one-to-one correspondence between the path components of X^H and Y^H, and isomorphisms $f_*^H: \pi_k(X^H, x) \to \pi_k(Y^H, f(x))$, for all $k \geq 1$ and every $x \in X^H$.

q.e.d.

CHAPTER II

DIFFERENTIABLE G-MANIFOLDS ARE EQUIVARIANT CW COMPLEXES

In this chapter we prove that any differentiable manifold with a differentiable action of a compact Lie group is an equivariant CW complex. In fact a stronger result is proved. We prove that a differentiable G-manifold has what we call an equivariant triangulation. C. T. Yang has proved that the orbit space of a differentiable G-manifold is triangulable, see C. T. Yang [18]. We prove that the part over a "suitable" simplex in the orbit space is an equivariant simplex of some type. These equivariant simplexes are defined in Definition 1.1. Our proof makes repeated use of the "differentiable slice theorem" and of the "covering homotopy theorem" of Palais in R. Palais [13].

In Section 4 we give a partially new proof of the result by Atiyah-Segal that equivariant K-theory of a compact differentiable G-manifold is finitely generated over R(G), the representation ring of G. The equivariant Whitehead theorem of Chapter I gives a necessary and sufficient condition for a G-map between differentiable G-manifolds to be a G-homotopy equivalence.

1. EQUIVARIANT SIMPLEXES

In this section G denotes a locally compact, Hausdorff, topological group.

Let Δ_n be the standard n-simplex, that is

$$\Delta_n = \{(x_0, \ldots, x_n) \in R^{n+1} \mid \sum_{i=0}^{n} x_i = 1, \quad x_i \geq 0\}.$$

We consider Δ_m, $0 \leq m \leq n$, as a subset of Δ_n through the imbedding of Δ_m into Δ_n which is given by $(x_0, \ldots, x_m) \longmapsto (x_0, \ldots, x_m, \underbrace{0, \ldots, 0}_{n-m})$

<u>Definition 1.1</u>. Let $K_0, \ldots, K_n$ be a sequence of closed subgroups of G such that $K_0 \supset K_1 \supset \cdots \supset K_n$. We define the standard equivariant n-simplex of type $(K_0, \ldots, K_n)$ denoted by

$$(\Delta_n; K_0, \ldots, K_n)$$

to be the G-space constructed in the following way. Consider the G-space $\Delta_n \times G$, and define a relation $\sim$ in $\Delta_n \times G$ as follows:

$$(x, g) \sim (x, g') \iff gK_m = g'K_m \in G/K_m, \text{ where } x \in \Delta_m - \Delta_{m-1}.$$

Thus $\sim$ is an equivalence relation in $\Delta_n \times G$, and we define

$$(\Delta_n; K_0, \ldots, K_n) = (\Delta_n \times G)/\sim.$$

We denote by $p: \Delta_n \times G \longrightarrow (\Delta_n; K_0, \ldots, K_n)$ the natural projection and by $[x, g] \in (\Delta_n; K_0, \ldots, K_n)$ the image of $(x, g) \in \Delta_n \times G$ under this projection. We now have the commutative diagram

$$\begin{array}{ccc} G \times (\Delta_n \times G) & \xrightarrow{\sigma} & \Delta_n \times G \\ \downarrow{\scriptstyle id \times p} & & \downarrow{\scriptstyle p} \\ G \times (\Delta_n; K_0, \ldots, K_n) & \xrightarrow{\bar{\sigma}} & (\Delta_n; K_0, \ldots, K_n) \end{array}$$

where $\sigma(\tilde{g}, (x, g)) = (x, \tilde{g}g)$, and $\bar{\sigma}$ is well-defined by $\bar{\sigma}(\tilde{g}, [x,g]) = [x, \tilde{g}g]$.

We shall show in Lemma 1.3 below that $\mathrm{id} \times p$ is a quotient mapping. Thus $\bar{\sigma}$ is continuous and makes $(\Delta_n; K_0, \ldots, K_n)$ into a G-space.

Lemma 1.2. The space $(\Delta_n; K_0, \ldots, K_n)$ is Hausdorff.

Proof. Let $[x, g] \neq [x', g'] \in (\Delta_n; K_0, \ldots, K_n)$. If $x \neq x'$, we choose disjoint open neighborhoods U and U' of x and x', respectively, in Δ_n. We have $p^{-1}(p(U \times G)) = U \times G$, and similarly for U'. Thus $p(U \times G)$ and $p(U' \times G)$ are disjoint open sets containing $[x, g]$ and $[x', g']$ respectively.

If $x = x'$, let m be such that $x = x' \in \Delta_m - \Delta_{m-1}$, $0 \leq m \leq n$. Since $[x, g] \neq [x', g']$, we have $gK_m \neq g'K_m \in G/K_m$. Since G/K_m is Hausdorff we can choose disjoint open neighborhoods V and V' of gK_m and $g'K_m$ respectively, in G/K_m. Denote $W = \delta^{-1}(V)$ and $W' = \delta^{-1}(V')$, where $\delta: G \longrightarrow G/K_m$ is the natural projection. Then W and W' are disjoint open subsets of G, and moreover we have $WK_m = W$ and $W'K_m = W'$. Thus $WK_i = W$ and $W'K_i = W'$, for $m \leq i \leq n$, since $K_i \subset K_m$ for $m \leq i \leq n$. From this it follows that

$$p^{-1}(p((\Delta_n - \Delta_{m-1}) \times W)) = (\Delta_n - \Delta_{m-1}) \times W$$

and similarly for W'. Hence $p((\Delta_n - \Delta_{m-1}) \times W)$ and $p((\Delta_n - \Delta_{m-1}) \times W')$ are disjoint open sets containing $[x, g]$ and $[x, g']$, respectively.

q.e.d.

Lemma 1.3. The map $\mathrm{id} \times p: G \times (\Delta_n \times G) \longrightarrow G \times (\Delta_n; K_0, \ldots, K_n)$ is a quotient map.

Proof. By a "compactly generated space" we mean a space which is compactly generated and Hausdorff. Since $\Delta_n \times G$ is locally compact and

Hausdorff, it is compactly generated. Since we already showed that $(\Delta_n; K_0, \ldots, K_n)$ is Hausdorff, it follows by 2.6 in Steenrod [17] that $(\Delta_n; K_0, \ldots, K_n)$ is compactly generated. Our claim now follows by Theorems 4.3 and 4.4 in Steenrod [17], since G is locally compact and Hausdorff.

q.e.d.

Remarks. The natural projection $p: \Delta_n \times G \longrightarrow (\Delta_n; K_0, \ldots, K_n)$ is not an open map in general. Let $G = S^1$, then $p: \Delta_1 \times S^1 \longrightarrow (\Delta_1; S^1, \{e\})$ is not an open map.

The space $(\Delta_n; K_0, \ldots, K_n)$ is not locally compact in general. Let $G = R$, the additive group of the real numbers, then $(\Delta_1; R, \{0\})$ is not locally compact.

2. EQUIVARIANT TRIANGULATIONS

In Definition 2.1 and Lemma 2.2 below G denotes a locally compact group. As before a "compactly generated space" means a space which is compactly generated and Hausdorff.

Definition 2.1. Let X be a compactly generated G-space. An equivariant triangulation of X is a triangulation t of the orbit space $G\backslash X$

$$t: |C| \longrightarrow G\backslash X$$

such that for each n-simplex $s \in C$ there exists closed subgroups $K_0 \supset \cdots \supset K_n$ of G and a G-homeomorphism

$$h: (\Delta_n; K_0, \ldots, K_n) \longrightarrow \pi^{-1}(t(|s|))$$

which induces

$$\Delta_n \xrightarrow{\ell} |s| \xrightarrow{t|} t(|s|)$$

on the orbit spaces. Here $\ell\colon \Delta_n \longrightarrow |s|$ denotes some linear homeomorphism.

<u>Lemma 2.2.</u> Let X be a compactly generated G-space and let $t\colon |C| \longrightarrow G\backslash X$ be an equivariant triangulation of X. Then X has the topology coherent with the family $\{\pi^{-1}(t(|C^n|))\}_{n \geq 0}$. Moreover for each n the subspace $\pi^{-1}(t(|C^n|))$ has the topology coherent with the family $\{\pi^{-1}(t(|C^{n-1}|)), \pi^{-1}(t(|s|))\}$ where s runs through all n-simplexes of C.

<u>Proof.</u> Let $B \subset X$ be such that $B \cap \pi^{-1}(t(|C^n|))$ is a closed set for all n. We have to show that B is closed. Since X is compactly generated, it is enough to show that $B \cap F$ is a closed set for each compact subset F of X. Let $F \subset X$ be compact. Since $\pi(F) \subset G\backslash X$ is compact and $G\backslash X$ has the topology coherent with $\{t(|C^n|)\}_{n \geq 0}$ it follows that there exists m such that $\pi(F) \subset t(|C^m|)$. Thus the set $B \cap F = B \cap \pi^{-1}(t(|C^m|)) \cap F$ is closed. This completes the proof of the first assertion in Lamma 2.2. The other claim is proved in an analogous way.

q.e.d.

<u>Proposition 2.3.</u> Let G be a compact group. Let X be a compactly generated G-space which can be equivariantly triangulated. Then X is an equivariant CW complex.

<u>Proof.</u> Let $t\colon |C| \longrightarrow G\backslash X$ be an equivariant triangulation of X. We claim that X is an equivariant CW complex with skeletons $\pi^{-1}(t(|C^n|))$, $n \geq 0$. Since X has the topology coherent with $\{\pi^{-1}(t(|C^n|))\}_{n \geq 0}$ by

Lemma 2.2, it only remains to show that $\pi^{-1}(t(|C^n|))$ is obtainable from $\pi^{-1}(t(|C^{n-1}|))$ by adjoining equivariant n-cells.

We claim that the collection $\{\pi^{-1}(t(|s|))\}$, where s runs through all n-simplexes of C, satisfies conditions 1, 2, and 3 of Definition 1.1 in Chapter I. Condition 1 follows from Lemma 2.2. Observe that $\pi^{-1}(t(|s|)) \cap \pi^{-1}(t(|C^{n-1}|)) = \pi^{-1}(t(|\dot{s}|))$ and that condition 2 is clear. Let

$$h: (\Delta_n; K_0, \ldots, K_n) \longrightarrow \pi^{-1}(t(|s|))$$

be a G-homeomorphism which induces

$$\Delta_n \xrightarrow{\ell} |s| \xrightarrow{t|} t(|s|)$$

on the orbit spaces. Here ℓ denotes a linear homeomorphism. Denote by $(\dot{\Delta}_n; K_0, \ldots, K_n)$ the part of $(\Delta_n; K_0, \ldots, K_n)$ which lies over $\dot{\Delta}_n$. The natural projection $p: \Delta_n \times G \longrightarrow (\Delta_n; K_0, \ldots, K_n)$ factors through $\rho: \Delta_n \times G/K_n \longrightarrow (\Delta_n; K_0, \ldots, K_n)$. We have

$$\rho: (\Delta_n \times G/K_n, \dot{\Delta}_n \times G/K_n) \longrightarrow ((\Delta_n; K_0, \ldots, K_n), (\dot{\Delta}_n; K_0, \ldots, K_n))$$

and ρ restricts to a G-homeomorphism from $\Delta_n \times G/K_n - \dot{\Delta}_n \times G/K_n$ to $(\Delta_n; K_0, \ldots, K_n) - (\dot{\Delta}_n; K_0, \ldots, K_n)$. Let $\alpha: (E^n, S^{n-1}) \longrightarrow (\Delta_n, \dot{\Delta}_n)$ be a homeomorphism. Then the G-map

$$h\rho(\alpha \times \mathrm{id}): (E^n \times G/K_n, S^{n-1} \times G/K_n) \longrightarrow (\pi^{-1}(t(|s|)), \pi^{-1}(t(|\dot{s}|)))$$

shows that condition 3 is satisfied.

q.e.d.

3. EQUIVARIANT TRIANGULATION OF A DIFFERENTIABLE MANIFOLD WITH A DIFFERENTIABLE ACTION OF A COMPACT LIE GROUP

We shall prove that if a compact Lie group G acts differentiably on a differentiable manifold M, then the G-space M can be equivariantly triangulated. By a theorem of C. T. Yang, see [18], the orbit space of such an action can be triangulated. This theorem is of course the basic starting point for the proof of our result. In this chapter G will always denote a compact Lie group. By a differentiable G-manifold M, we mean a differentiable manifold M together with a differentiable action of G on M.

We shall first review some other basic results. Let us begin with the "differentiable slice theorem." Let G be a compact Lie group, and K a closed subgroup of G. Thus K itself is a compact Lie group by a classical result of E. Cartan. Let V be an orthogonal representation space for K, that is V is a finite dimensional real euclidean space on which K acts by orthogonal transformations. By $\mathring{V}(1)$ and $V(1)$, we denote the open and closed disc, respectively, of radius = 1 in V. Then $\mathring{V}(1)$ is a differentiable K-manifold. Consider $\mathring{V}(1) \times G$ and define a right K-action on $\mathring{V}(1) \times G$ by $(v, g, k) \longmapsto (k^{-1}v, gk)$ where $v \in \mathring{V}(1)$, $g \in G$ and $k \in K$. We denote by $\mathring{V}(1) \times_K G = (\mathring{V}(1) \times G)/K$ the orbit space of $\mathring{V}(1) \times G$ under this right K-action and by $\{v, g\} \in \mathring{V}(1) \times_K G$ the image of $(v, g) \in \mathring{V}(1) \times G$ under the natural projection.

Define $p: \mathring{V}(1) \times_K G \longrightarrow G/K$, by $p(\{v, g\}) = gK$. Thus $p: \mathring{V}(1) \times_K G \longrightarrow G/K$ is the fiber bundle with fiber $\mathring{V}(1)$ associated with the principal K-bundle $\rho: G \longrightarrow G/K$. Thus $\mathring{V}(1) \times_K G$ gets the structure of a differentiable manifold. Moreover, we can define a left G-action on $\mathring{V}(1) \times_K G$ by $(\tilde{g}, \{v, g\}) \longmapsto \{v, \tilde{g}g\}$, where $\tilde{g} \in G$ and $\{v, g\} \in \mathring{V}(1) \times_K G$. In this way $\mathring{V}(1) \times_K G$ becomes a differentiable G-manifold. We can identify the differentiable G-manifold G/K with the 0-section in $\mathring{V}(1) \times_K G$ through the imbedding given by $gK \longmapsto \{0, g\} \in \mathring{V}(1) \times_K G$ where $gK \in G/K$. The differentiable K-manifold $\mathring{V}(1)$ can be identified with the K-subset of $\mathring{V}(1) \times_K G$ consisting of all elements of the form $\{v, e\}$ where $v \in \mathring{V}(1)$, through the K-imbedding given by $v \longmapsto \{v, e\}$, $v \in \mathring{V}(1)$ (observe that $\{kv, e\} = \{v, k\} = k\{v, e\}$).

Differentiable slice theorem. Let M be a differentiable G-manifold, and let $x \in M$. Then there exists an orthogonal representation space V for G_x, and a G-diffeomorphism

$$h: \mathring{V}(1) \times_{G_x} G \longrightarrow U,$$

where U is an open G-neighborhood of Gx in M, and we have $h(\{0, e\}) = x$.

This theorem was first proved by J. L. Koszul, see Koszul [9], theorem on page 139. See also Montgomery, Samelson, Yang [10], Lemma 3.1. Let the notation be as above and denote $h(\mathring{V}(1)) = S$. Then $x \in S$, $GS = U$, and the restriction of h to $\mathring{V}(1)$ is a K-diffeomorphism from $\mathring{V}(1)$ onto S. It is the set S that is called a slice at $x \in M$, and U is called a tubular neighborhood of the orbit Gx. For the important generalization of the notion of slice to actions on topological spaces, and a "slice theorem" for such actions see Montgomery, Yang [11], Definition and Theorem 1 on page 108, and Mostow [12], Theorem 3.1. For a good exposition of these questions see Palais [13]. We shall not need a direct use of the general topological slice theorem, but it should be observed that it is used in the proof of the "covering homotopy theorem" by R. Palais, which we shall use.

Let us next consider the following special situation. Assume that X is a completely regular space on which G acts in such a way that the action has only one orbit type, say $= (H')$. Let $\pi: X \longrightarrow G\backslash X$ be the projection onto the orbit space. By a theorem of A. M. Gleason, see Gleason [7], Theorem 3.6, this projection is locally trivial. We shall consider the situation in somewhat more detail. Let the closed subgroup $H \in (H')$ be an arbitrary representative for the orbit type. By $N(H)$ we denote the normalizer of H in G. Then the compact Lie group $N(H)/H$ acts freely on the completely regular space X^H, and the projection onto the orbit space of this action is $\pi|: X^H \longrightarrow G\backslash X$. Thus by Theorem 3.1 (or the already cited Theorem 3.6) in Gleason [7], $\pi|: X^H \longrightarrow G\backslash X$ is a principal (left) $N(H)/H$ bundle, that is, it is locally trivial. Now $N(H)/H$ acts on G/H on the right by $(gH, aH) \longmapsto gaH$, where $gH \in G/H$ and $aH \in N(H)/H$. Define a left $N(H)/H$ action on $G/H \times X^H$ by $(aH, gH, x) \longmapsto (ga^{-1}H, ax)$, where $aH \in N(H)/H$ and $(gH, x) \in G/H \times X^H$, and denote the orbit space under this action by $G/H \times_{N(H)/H} X^H$, and denote by $\{gH, x\}$ the image of (gH, x) under the natural projection.

Define $p: G/H \times_{N(H)/H} X^H \longrightarrow G\backslash X$ by $p(\{gH, x\}) = \pi(x)$. Thus p is the fiber bundle with fiber G/H associated with the principal (left) $N(H)/H$ bundle $\pi| : X^H \longrightarrow G\backslash X$. Moreover, we can define a left G action on $G/H \times_{N(H)/H} X^H$ by $(\tilde{g}, \{gH, x\}) \longmapsto \{\tilde{g}gH, x\}$. Now the mapping

$$\gamma: G/H \times_{N(H)/H} X^H \longrightarrow X$$

defined by $\gamma(\{gH, x\}) = gx$, is a G-homeomorphism (see Borel [1], 1.1 and Lemma 1.2).

Finally, let us consider the "covering homotopy theorem" of Palais. In the following, X and Y denote completely regular G-spaces. A G-map $f: X \longrightarrow Y$ is called isovariant if the induced map $f| : Gx \longrightarrow Gf(x)$ is a bijection for each $x \in X$. In this case $f| : Gx \longrightarrow Gf(x)$ is of course a G-homeomorphism. For any G-map $f: X \longrightarrow Y$ we have $G_x \subset G_{f(x)}$ for all $x \in X$, and f is isovariant if and only if $G_x = G_{f(x)}$ for all $x \in X$. Since a compact Lie group is not conjugate to a proper subgroup of itself, it follows that $f: X \longrightarrow Y$ is isovariant if and only if $f(X_{(H)}) \subset Y_{(H)}$ for every orbit type (H). It will be convenient to extend this terminology as follows. Let $X' = G\backslash X$ and $Y' = G\backslash Y$ be the orbit spaces. A map $s: X' \longrightarrow Y'$ will be called isovariant if $s(X'_{(H)}) \subset Y'_{(H)}$ for every orbit type (H). Thus if an isovariant map $s: X' \longrightarrow Y'$ can be lifted to a G-map $f: X \longrightarrow Y$, then f is isovariant.

It is easy to see that if an isovariant G-map $f: X \longrightarrow Y$, where X is locally compact, induces a homeomorphism between the orbit spaces, then f is a G-homeomorphism (see Proposition 1.1.18 in Palais [13]).

$(\Delta_n; K_0, \ldots, K_n)$ is Hausdorff, and since G now is assumed to be compact it follows that $(\Delta_n; K_0, \ldots, K_n)$ is compact Hausdorff. Thus $(\Delta_n; K_0, \ldots, K_n)$ is normal and hence completely regular by Urysohn's theorem. Since $\Delta_n \times G$ is second countable and $p: \Delta_n \times G \longrightarrow (\Delta_n; K_0, \ldots, K_n)$ is closed it follows by, for example, Theorem 12 in Chapter 3 and Theorem 20 in Chapter 5 in Kelley's book [8], that $(\Delta_n; K_0, \ldots, K_n)$ is second countable. A differentiable manifold M is second countable by definition and it is also completely regular, and thus any subset of M has the same properties.

We now state:

Covering homotopy theorem. (R. Palais) Let X and Y be locally compact second countable G-spaces. Let $f: X \longrightarrow Y$ be an isovariant G-map, and $f': X' \longrightarrow Y'$ the induced map on the orbit spaces. If $S: I \times X' \longrightarrow Y'$ is any isovariant homotopy such that $S(0, \ ) = f'$, then there exists an isovariant G-homotopy $F: I \times X \longrightarrow Y$ which covers S and such that $F(0, \ ) = f$.

This is Theorem 2.4.1 in Palais [13]. Observe that we are still assuming that X and Y are completely regular spaces.

We shall now prove:

Theorem 3.1. Let M be a differentiable G-manifold. Then there exists an equivariant triangulation of M.

Proof. We have the following basic result by C. T. Yang.

Theorem (C. T. Yang). Let M be a differentiable G-manifold. Then

$G\backslash M$ can be triangulated in such a way that all points in the interior of any simplex belong to the same orbit type.

This is the theorem in Yang [18]. The property of the triangulation, that all points in the interior of any simplex belong to the same orbit type, is not stated in [18], but it follows from the proof that the constructed triangulation has this property. We denote $G\backslash M = Y$. We shall say that an imbedding

$$j: \Delta_n \longrightarrow Y$$

is of type $((\overline{K}_0), \ldots, (\overline{K}_n))$ where $(\overline{K}_i)$, $0 \leq i \leq n$, are orbit types, if we have

$$j(\Delta_m - \Delta_{m-1}) \subset Y_{(\overline{K}_m)}$$

for $m = 0, \ldots, n$. It follows immediately that $(\overline{K}_0) \geq (\overline{K}_1) \geq \cdots \geq (\overline{K}_n)$. We also say that a subset $B \subset Y$ is of type $((\overline{K}_0), \ldots, (\overline{K}_n))$ if there exists an imbedding of Δ_n of type $((\overline{K}_0), \ldots, (\overline{K}_n))$ onto B. Thus in this case $(\overline{K}_0), \ldots, (\overline{K}_n)$ represent the orbit types in B, but of course there need not be n distinct orbit types in B; some $(\overline{K}_i)$ may equal $(\overline{K}_{i+1})$. It is easy to show that the n-sequence $((\overline{K}_0), \ldots, (\overline{K}_n))$ is uniquely determined by B when it exists.

Now consider a triangulation of Y in which all points of the interior of any simplex belong to the same orbit type. Take the barycentric subdisivion of this triangulation. Then it follows immediately by induction that each n-simplex in this new triangulation is of type $((\overline{K}_0), \ldots, (\overline{K}_n))$ for some orbit types $(\overline{K}_i)$, $i = 0, \ldots, n$. Hence it follows that in order to complete the proof of Theorem 1.1 we have to prove the following.

Lemma 3.2. Let the notation be as above. Let $j\colon \Delta_n \longrightarrow Y$ be an imbedding of type $((\overline{K}_0),\ldots,(\overline{K}_n))$. Then there exist closed subgroups $K_i \in (\overline{K}_i)$, $i = 0,\ldots,n$, and a G-homeomorphism $\beta\colon (\Delta_n; K_0,\ldots,K_n) \longrightarrow \pi^{-1}(j(\Delta_n))$ such that β induces the map $j\colon \Delta_n \longrightarrow j(\Delta_n)$ on the orbit spaces.

Here $\pi\colon M \longrightarrow Y$ is the projection.

Proof of Lemma. The proof is by induction on n. The case $n = 0$ is clear. Although the case $n = 1$ is included in the induction proof, we shall prove it separately in order to make the rest of the proof more clear. Let

$$j\colon \Delta_1 \longrightarrow Y$$

be an imbedding of type $((\overline{K}_0), (\overline{K}_1))$. Choose any $x_0 \in M$ such that $\pi(x_0) = j((1,0)) = j(\Delta_0)$, where $(1,0) \in \Delta_1$. Denote $G_{x_0} = K_0$. Let

$$h_0\colon \mathring{V}_0(1) \times_{K_0} G \longrightarrow U$$

be a presentation of a tubular neighborhood U of Gx_0 in M, where $h_0(\{0,e\}) = x_0$, and with corresponding K_0-slice $S_0 = h_0(\mathring{V}_0(1))$. The existence of all this is given by the "differentiable slice theorem." Then $\pi(S_0) = \pi(U)$ is an open neighborhood of $j((1,0))$ in Y. Thus there exists $0 < \delta_0 \leq 1$ such that $j((1-\delta,\delta)) \in \pi(S_0)$ for $0 \leq \delta \leq \delta_0$. Choose any $x_1 \in \pi^{-1}(j(1-\delta_0,\delta_0)) \cap S_0$, and denote $G_{x_1} = K_1$. Thus $K_1 \subset K_0$, and we also have $K_0 \in (\overline{K}_0)$ and $K_1 \in (\overline{K}_1)$.

Now let $\{v_1, e\} \in \mathring{V}_0(1) \subset \mathring{V}_0(1) \times_{K_0} G$ be the unique point for which $h(\{v_1, e\}) = x_1$. Define

$$\omega\colon \Delta_1 \longrightarrow S_0 \subset U\ ,$$

by $\omega((1-\tau,\tau)) = h(\{\tau v_1, e\})$, where $(1-\tau,\tau) \in \Delta_1$. Thus $\omega((1,0)) = x_0$ and $\omega((0,1)) = x_1$. Moreover, every point in $\omega(\Delta_1 - \Delta_0)$ has isotropy group K_1, and $\omega(\Delta_0) = x_0$ has isotropy group K_0. Thus the map

$$\alpha\colon (\Delta_1 ; K_0, K_1) \longrightarrow U$$

defined by $\alpha([(1-\tau,\tau), g]) = g\omega((1-\tau,\tau))$ is a well-defined isovariant G-map. We wish to apply the "covering homotopy theorem" of Palais to "move" α to a G-map that covers $j\colon \Delta_1 \longrightarrow Y$. We now proceed to do this. Denote $\Delta_1(\delta) = \{(1-\tau,\tau) \in \Delta_1 \mid 0 \leq \tau \leq \delta\}$. Thus we have $j(\Delta_1(\delta_0)) \subset \pi(S_0)$. Consider the G-space $\pi^{-1}(j(\Delta_1(\delta_0) - \Delta_0)) \subset U$. Since every point in it has orbit type $(\overline{K}_1) = (K_1)$, and since $j(\Delta_1(\delta_0) - \Delta_0)$ is contractible, it follows that we have a commutative diagram

$$\begin{array}{ccc}
\pi^{-1}(j(\Delta_1(\delta_0) - \Delta_0)) & \xleftarrow{\ \gamma\ } & G/K_1 \times j(\Delta_1(\delta_0) - \Delta_0) \\
\pi| \searrow & & \swarrow pr_2 \\
& j(\Delta_1(\delta_0) - \Delta_0) &
\end{array}$$

where γ is a G-homeomorphism. Thus there exists a map

$$s\colon (\Delta_1 - \Delta_0) \longrightarrow \pi^{-1}(j(\Delta_1(\delta_0) - \Delta_0)) \subset U$$

such that $s((0,1)) = x_1$, and $\pi s = j_{\delta_0}|\colon (\Delta_1 - \Delta_0) \longrightarrow Y$, where $j_{\delta_0}\colon \Delta_1 \longrightarrow Y$ is defined by $j_{\delta_0}((1-\tau,\tau)) = j((1-\delta_0\tau, \delta_0\tau))$, and such that all points in $s(\Delta_1 - \Delta_0)$ have isotropy group K_1.

Using the G-homeomorphism $h_0\colon \mathring{V}_0(1) \times_{K_0} G \longrightarrow U$, we define

$$D\colon I \times U \longrightarrow U$$

by $D(t, h_0(\{v, g\})) = h_0(\{tv, g\})$. Thus D is a strong G-deformation retraction of U to Gx_0. At $t = 0$, D gives the retraction of U onto Gx_0, and at $t = 1$, D gives the identity. Observe that if $x \in U - Gx_0$ has isotropy group G_x, then also every point $D(t, x)$, where $0 < t \leq 1$, has isotropy group G_x. Also observe that the map $\omega: \Delta_1 \longrightarrow S_0 \subset U$ we defined before is given by $\omega((1-\tau, \tau)) = D(\tau, x_1)$, where $x_1 \in S_0 \subset U$ is as before. Now we define a homotopy from $\omega| : (\Delta_1 - \Delta_0) \longrightarrow U$ to $s: (\Delta_1 - \Delta_0) \longrightarrow U$ as follows. Define

$$H: I \times (\Delta_1 - \Delta_0) \longrightarrow U$$

by

$$H(t, (1-\tau, \tau)) = \begin{cases} s((1-\tau, \tau)) & , \quad 0 \leq (1-\tau) \leq t \leq 1, \quad 0 < \tau. \\ D\left(\frac{\tau}{1-t}, \; s((t, 1-t))\right), & 0 \leq t \leq (1-\tau) < 1. \end{cases}$$

The space $I \times (\Delta_1 - \Delta_0)$ is the union of the two closed subsets,

$\{(t, (1-\tau, \tau)) \in I \times (\Delta_1 - \Delta_0) \,|\, 0 \leq (1-\tau) \leq t \leq 1, \; 0 < \tau\}$ and

$\{(t, (1-\tau, \tau)) \in I \times (\Delta_1 - \Delta_0) \,|\, 0 \leq t \leq (1-\tau) < 1\}$.

Since it is clear from the definition of H, that H restricted to these closed subsets is continuous and since H is well-defined on the intersection, it follows that H is continuous.

Now, $H(0, (1-\tau, \tau)) = D(\tau, s((0, 1))) = D(\tau, x_1) = \omega((1-\tau, \tau))$,

and $H(1, (1-\tau, \tau)) = s((1-\tau, \tau))$, where $0 < \tau \leq 1$.

Thus H is a homotopy from $\omega|$ to s. Moreover, observe that every point in $H(I \times (\Delta_1 - \Delta_0))$ has isotropy group K_1. Thus

$$T = \pi H: I \times (\Delta_1 - \Delta_0) \longrightarrow \pi(U) \subset Y$$

is a homotopy from $\pi\omega| : (\Delta_1 - \Delta_0) \longrightarrow Y$ to $j_{\delta_0}| : (\Delta_1 - \Delta_0) \longrightarrow Y$, and moreover, $T(I \times (\Delta_1 - \Delta_0)) \subset Y_{(\overline{K}_1)}$. We now claim that T can be extended to a homotopy

$$\overline{T}: I \times \Delta_1 \longrightarrow Y$$

by defining

$$\overline{T}(t, (1-\tau, \tau)) = \begin{cases} j((1,0)), & \tau = 0 \\ T(t, (1-\tau, \tau)), & 0 < \tau \leq 1. \end{cases}$$

We have to show that $\overline{T}$ is continuous. This is more or less obvious from the definition of H, but we shall give a formal proof. We denote $j((1,0)) = j_{\delta_0}((1,0)) = y_0 \epsilon Y$. We have to show that T is continuous at every point of the form $(t, (1,0)) \epsilon I \times \Delta_1$, $0 \leq t \leq 1$. Thus it is clearly enough to show the following. Given an open neighborhood B of y_0 in $\pi(U)$, there exists $\epsilon > 0$, such that

$$\overline{T}(I \times (1-\tau, \tau)) \subset B, \quad \text{for all } 0 \leq \tau < \epsilon.$$

This is equivalent to showing that

$$H(I \times (1-\tau, \tau)) \subset \pi^{-1}(B), \quad \text{for all } 0 < \tau < \epsilon.$$

Since $S_0 \cap \pi^{-1}(B)$ is an open neighborhood of x_0 in X_0, it follows that there exists $t_1 > 0$, $(t_1 \leq 1)$, such that $h_0(\{v, e\}) \subset \pi^{-1}(B)$ if $\|v\| < t_1$. Thus $h_0(\{v, g\}) \subset \pi^{-1}(B)$, if $\|v\| < t_1$, for all $g \epsilon G$. Hence

$$D(t \times U) \subset \pi^{-1}(B), \quad \text{if } 0 \leq t < t_1.$$

Next observe that since $D(I \times Gx_0) \subset \pi^{-1}(B)$, we have $I \times Gx_0 \subset D^{-1}(\pi^{-1}(B)) \subset I \times U$. Since both I and Gx_0 are compact, there exists an open neighborhood W of Gx_0 in U such that $I \times W \subset D^{-1}(\pi^{-1}(B))$. Thus

$$D(I \times W) \subset \pi^{-1}(B).$$

Now choose $\epsilon_0 > 0$ $(\epsilon_0 < 1)$ such that

$$s((1-\tau,\tau)) \in \pi^{-1}(B) \cap W, \quad \text{for } 0 < \tau \leq \varepsilon_0.$$

Denote $1 - \varepsilon_0 = t_0$. Next choose $\varepsilon_1 > 0$, such that $\varepsilon_1/1-t_0 = \varepsilon_1/\varepsilon_0 < t_1$. Thus especially $\varepsilon_1 < \varepsilon_0$. We claim that

$$H(I \times (1-\tau,\tau)) \subset \pi^{-1}(B), \quad \text{for all } 0 < \tau < \varepsilon_1.$$

Consider a point $(t,(1-\tau,\tau))$, where $0 < \tau < \varepsilon_1$. First, if $0 \leq (1-\tau) \leq t \leq 1$, then we have

$$H(t,(1-\tau,\tau)) = s(1-\tau,\tau) \in \pi^{-1}(B) \cap W \subset \pi^{-1}(B),$$

since $0 < \tau < \varepsilon_1 < \varepsilon_0$. Secondly, if $0 \leq t \leq (1-\tau) < 1$ and $t_0 \leq t < 1$, then

$$H(t,(1-\tau,\tau)) = D\left(\frac{\tau}{1-t}, s(t,1-t)\right) \in \pi^{-1}(B)$$

since in this case $(1-t) \leq (1-t_0) = \varepsilon_0$, and hence $s(t,1-t) \in \pi^{-1}(B) \cap W \subset W$, and thus $D\left(\frac{\tau}{1-t}, s(t,1-t)\right) \in D(I \times W) \subset \pi^{-1}(B)$. Thirdly, if $0 \leq t \leq (1-\tau) < 1$, and $0 \leq t \leq t_0$, then

$$H(t,(1-\tau,\tau)) = D\left(\frac{\tau}{1-t}, s(t,1-t)\right) \in \pi^{-1}(B),$$

since now $(1-t) \geq (1-t_0) = \varepsilon_0$, and thus $\frac{\tau}{1-t} \leq \frac{\varepsilon_1}{\varepsilon_0} < t_1$, and hence we have

$$D\left(\frac{\tau}{1-t}, s(t,1-t)\right) \in D\left(\frac{\tau}{1-t} \times U\right) \subset \pi^{-1}(B).$$

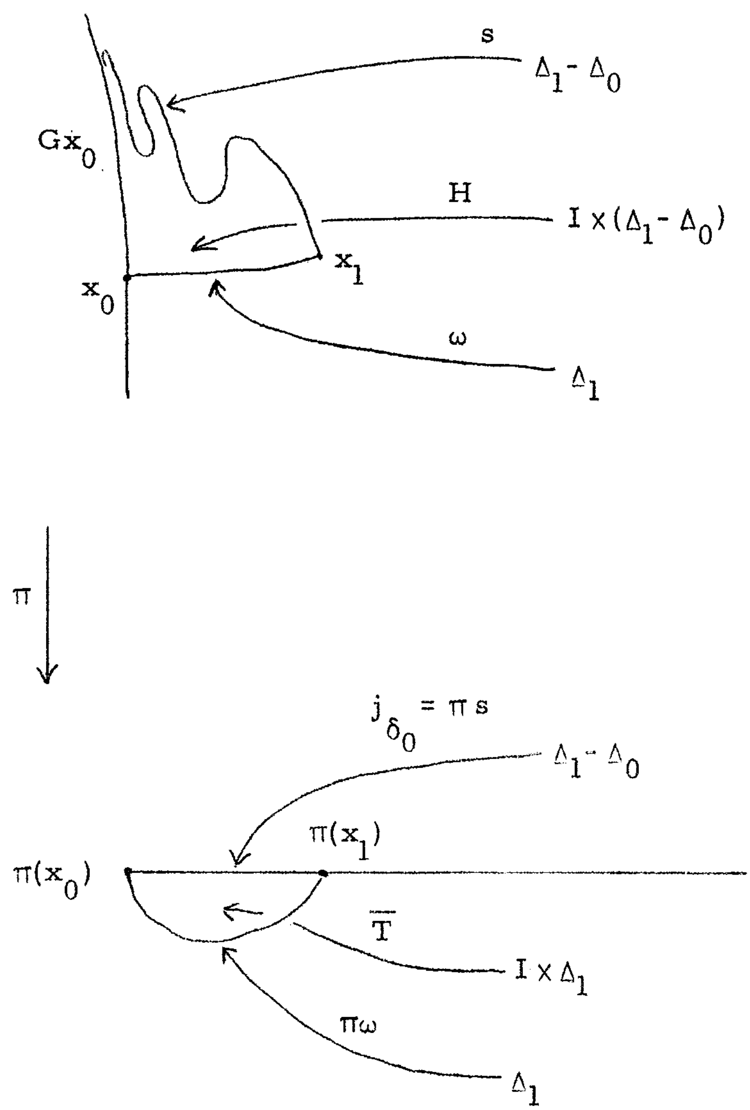
This completes the proof that T is continuous.

Consider Δ_1 as the orbit space of $(\Delta_1; K_0, K_1)$, that is Δ_0 has orbit type $(K_0) = (\overline{K}_0)$, and every point in $\Delta_1 - \Delta_0$ has orbit type $(K_1) = (\overline{K}_1)$. Thus we have constructed an isovariant homotopy

$$\overline{T}: I \times \Delta_1 \longrightarrow Y$$

from $\pi\omega: \Delta_1 \longrightarrow Y$ to $j_{\delta_0}: \Delta_1 \longrightarrow Y$. Recall the isovariant G-map

The figure below illustrates the case n=1.



$\alpha: (\Delta_1; K_0, K_1) \longrightarrow M$ and observe that α induces the map $\pi\omega: \Delta_1 \longrightarrow Y$. Thus by the "covering homotopy theorem" by Palais, the whole homotopy $\overline{T}$ can be lifted; especially there exists an isovariant G-map

$$\beta: (\Delta_1; K_0, K_1) \longrightarrow M$$

such that the induced map on the orbit spaces is $j_{\delta_0}: \Delta_1 \longrightarrow Y$. Since j_{δ_0} is a homeomorphism onto $j_{\delta_0}(\Delta_1) = j(\Delta_1(\delta_0))$, it follows that β gives a G-homeomorphism

$$\beta: (\Delta_1; K_0, K_1) \xrightarrow{\cong} \pi^{-1}(j(\Delta_1(\delta_0)))$$

and β induces $j_{\delta_0}: \Delta_1 \longrightarrow j_{\delta_0}(\Delta_1) = j(\Delta_1(\delta_0))$ on the orbit spaces. To see that there exists a G-homeomorphism

$$\overline{\beta}: (\Delta_1; K_0, K_1) \xrightarrow{\cong} \pi^{-1}(j(\Delta_1))$$

such that $\overline{\beta}$ induces $j: \Delta_1 \longrightarrow j(\Delta_1)$ on the orbit spaces, it only remains to observe that $j_{\delta_0}: \Delta_1 \longrightarrow Y$ is isovariantly homotopic to $j: \Delta_1 \longrightarrow Y$, and then apply the "covering homotopy theorem" of Palais once more. This completes the proof of the case $n = 1$.

We shall now turn to the general case, that is, to the induction argument. We begin with some notations and remarks.

We denote,

$$\Delta_n = \left\{(a_0, \ldots, a_n) \in R^{n+1} \,\middle|\, \sum_{i=0}^{n} a_i = 1,\ a_i \geq 0,\ 0 \leq i \leq n\right\},$$

and for $0 < \delta \leq 1$,

$$\Delta_n(\delta) = \{(a_0, \ldots, a_n) \in \Delta_n \mid 1 - a_0 \leq \delta\}.$$

As before we denote $d^0 = (1, 0, \ldots, 0) \in \Delta_n$, $d^1 = (0, 1, 0, \ldots, 0) \in \Delta_n, \ldots, d^n =$

$(0,\ldots,0,1) \in \Delta_n$, and we also sometimes use the sum notation, that is, we write $(a_0,\ldots,a_n) = \sum_{i=0}^{n} a_i d^i$, where $\sum_{i=0}^{n} a_i = 1$ and $a_i \geq 0$, $0 \leq i \leq n$.

Let $\delta_0, \delta_1, \ldots, \delta_{n-1}$ be positive real numbers such that $1 \geq \delta_0 \geq \delta \geq \cdots \geq \delta_{n-1} > 0$. Consider the points

$$
\begin{aligned}
e^0 &= (1,0,\ldots,0) = d^0 \in \Delta_n \\
e^1 &= (1-\delta_0, \delta_0, 0,\ldots,0) \in \Delta_n \\
e^2 &= (1-\delta_0, \delta_0 - \delta_1, \delta_1, 0,\ldots,0) \in \Delta_n \\
&\vdots \\
e^m &= (1-\delta_0, \delta_0 - \delta_1, \ldots, \delta_{m-2} - \delta_{m-1}, \delta_{m-1}, 0,\ldots,0) \in \Delta_n \\
&\vdots \\
e^n &= (1-\delta_0, \delta_0 - \delta_1, \ldots, \delta_{n-2} - \delta_{n-1}, \delta_{n-1}) .
\end{aligned}
$$

We denote the convex hull of the points $e_0, \ldots, e_n$ by $\Delta_n(\delta_0,\ldots,\delta_{n-1})$. Thus $\Delta_n(\delta_0,\ldots,\delta_n)$ is so to speak "a small n-simplex" inside Δ_n. The reader should think about the δ_i's as "small" numbers. Observe that if $\delta_i = 1$, $i = 0,\ldots,n-1$, then $e^i = d^i$, $i = 0,\ldots,n$. Also observe that if $\delta_0 = \delta_1 = \cdots = \delta_{n-1}$, then $\Delta_n(\delta_0, \delta_1, \ldots, \delta_{n-1}) = \Delta_n(\delta_0)$.

Define

$$i(\delta_0,\ldots,\delta_{n-1}) : \Delta_n \longrightarrow \Delta_n$$

by $i(\delta_0,\ldots,\delta_{n-1})\left(\sum_{i=0}^{n} a_i d^i\right) = \sum_{i=0}^{n} a_i e^i$, where the e^i's are defined by the equations above. Thus $i(\delta_0,\ldots,\delta_{n-1})$ is a homeomorphism from Δ_n onto $\Delta_n(\delta_0,\ldots,\delta_{n-1})$. Observe that we have

$$i(\delta_0,\ldots,\delta_{n-1})(\Delta_m - \Delta_{m-1}) \subset (\Delta_m - \Delta_{m-1}) .$$

Thus $i(\delta_0,\ldots,\delta_{n-1}) : \Delta_n \longrightarrow \Delta_n$ is an isovariant map when Δ_n is regarded as the orbit space of some $(\Delta_n; K_0, \ldots, K_n)$.

Denote

$$e^i(t) = td^i + (1-t)e^i \in \Delta_n, \qquad i = 0, \ldots, n.$$

Thus $e^i(0) = e^i$ and $e^i(1) = d^i$, $\quad i = 0, \ldots, n$.

Now define a homotopy

$$H: I \times \Delta_n \longrightarrow \Delta_n$$

by $H\left(t, \sum_{i=0}^{n} a_i d^i\right) = \sum_{i=0}^{n} a_i e^i(t)$. Thus H is a homotopy from $i(\delta_0, \ldots, \delta_{n-1})$ to the identity. Moreover, we have

$$H(I \times (\Delta_m - \Delta_{m-1})) \subset (\Delta_m - \Delta_{n-1}), \qquad 0 \leq m \leq n,$$

that is H is an isovariant homotopy when Δ_n is regarded as the orbit space of some $(\Delta_n; K_0, \ldots, K_n)$.

Next define

$$b: (\Delta_n - \Delta_0) \longrightarrow \Delta_n$$

by $b((a_0, \ldots, a_n)) = \left(0, \frac{a_1}{1-a_0}, \frac{a_2}{1-a_0}, \ldots, \frac{a_n}{1-a_0}\right)$, that is b "pushes everything to the back face."

Observe that

$$b(\Delta_m - \Delta_{m-1}) \subset (\Delta_m - \Delta_{m-1}), \quad \text{for } 1 \leq m \leq n.$$

Define a homotopy

$$B: I \times (\Delta_n - \Delta_0) \longrightarrow \Delta_n$$

by $B(t, (a_0, \ldots, a_n)) = \left(t a_0, \frac{a_1(1-t a_0)}{(1-a_0)}, \ldots, \frac{a_n(1-t a_0)}{(1-a_0)}\right)$.

Observe that

$$B(I \times (\Delta_m - \Delta_{m-1})) \subset (\Delta_m - \Delta_{m-1}), \quad \text{for } 1 \leq m \leq n.$$

Thus both b and B are isovariant whenever Δ_n is regarded as the orbit space of some $(\Delta_n; K_0, \ldots, K_n)$.

Finally we point out the following. Let $K_0 \supset K_1 \supset \cdots \supset K_k$ be closed subgroups of G, and let V be an orthogonal representation space for K_0, with open unit disc $\mathring{V}(1)$. Assume that the map

$$\sigma_b: \Delta_{k-1} \longrightarrow \mathring{V}(1)$$

is such that all points in $\sigma_b(\Delta_m - \Delta_{m-1})$ have isotropy group K_{m+1}, where $0 \leq m \leq k - 1$. We define

$$\sigma: \Delta_k \longrightarrow \mathring{V}(1)$$

by

$$\sigma((a_0, \ldots, a_k)) = \begin{cases} (1-a_0)\sigma_b\left(\left(\frac{a_1}{1-a_0}, \ldots, \frac{a_k}{1-a_0}\right)\right), & a_0 \neq 1 \\ 0 & , a_0 = 1 \end{cases}$$

Thus σ is continuous. Now observe that since the action by K_0 in V is linear, it follows that each point in $\sigma(\Delta_m - \Delta_{m-1})$ has isotropy group K_m, where $0 \leq m \leq k$.

Let us now resume the proof.

Let

$$j: \Delta_n \longrightarrow Y$$

be an imbedding of type $((\overline{K}_0), \ldots, (\overline{K}_n))$. Let $x_0 \in M$ be any point such that $\pi(x_0) = j(d^0)$. Denote $G_{x_0} = K_0$, and let

$$h_0: \mathring{V}_0(1) \times_{K_0} G \longrightarrow U_0$$

be a presentation, as given by the "differentiable slice theorem," of a tubular neighborhood U of Gx_0, such that $h_0(\{0,e\}) = x_0$, and with corresponding slice $h_0(\mathring{V}_0(1) = S_0$. We have $K_0 \epsilon (\overline{K}_0)$. Since $\pi(S_0) = \pi(U_0)$ is an open neighborhood of $j(d^0)$ in Y there exists $1 \geq \delta_0 > 0$ such that

$$j(\Delta_n(\delta_0)) \subset \pi(S_0).$$

Let $x_1 \epsilon \pi^{-1}(j(1-\delta_0, \delta_0, 0, \ldots, 0)) \cap S_0$, and denote $G_{x_1} = K_1$. Thus $K_1 \epsilon (\overline{K}_1)$. Consider $x_1 \epsilon S_0$ as a point in the differentiable K_0-space S_0 and apply the "differentiable slice theorem" to this situation. Thus we have a K_0-diffeomorphism

$$h_1 : \mathring{V}_1(1) \times_{K_1} K_0 \longrightarrow U_1 \subset S_0$$

where U_1 is a tubular neighborhood of $K_0 x_0$ in S_0, and $h_1(\{0,e\}) = x_1$. Here V_1 denotes an orthogonal representation space for K_1. Denote the corresponding K_1-slice at x_1 by S_1, that is, $S_1 = h_1(\mathring{V}_1(1)$. Let $\pi_{K_0} : S_0 \longrightarrow K_0 \backslash S_0$ be the projection from the K_0-space S_0 to its orbit space. Then $\pi_{K_0}(S_1)$ is open in $K_0 \backslash S_0$. Since $K_0 \backslash S_0$ is homeomorphic with $G \backslash G S_0$, through the mapping $\pi_{K_0}(s_0) \longmapsto \pi(s_0)$ (see Proposition 1.7.6 in Palais [13]) it follows that $\pi(S_1)$ is open in $G \backslash G S_0 = \pi(S_0)$. But since $\pi(S_0)$ is open in Y it follows that $\pi(S_1)$ is open in Y. In fact, it is good to already at this point observe that S_1 is also a differentiable K_1-slice at x_1 in M. We can define

$$\overline{h}_1 : \mathring{V}_1(1) \times_{K_1} G \longrightarrow G S_1 = G U_1$$

by $\overline{h}_1(\{v,g\}) = g h_1\{v,e\}$, and GS_1 is an open tubular neighborhood of Gx_1 in M.

Now $\pi(S_1)$ is an open neighborhood of $\pi(x_1) = j(1-\delta_0, \delta_0, 0, \ldots, 0)$ in Y. Consider the composite map

$$\Delta_{n-1} \xrightarrow{i_b(\delta_0)} \Delta_n \xrightarrow{j} Y$$

where $(i_b(\delta_0))(a_0, \ldots, a_{n-1}) = (1-\delta_0, \delta_0 a_0, \ldots, \delta_0 a_{n-1})$.

Then $(j \circ i_b(\delta_0))(1, 0, \ldots, 0) = \pi(x_1)$. Hence there exists $1 \geq \delta_1' > 0$, such that

$$(j \circ i_b(\delta_0))(\Delta_{n-1})\delta_1')) \subset \pi(S_1).$$

Denote $\delta_1 = \delta_0 \cdot \delta_1'$. Then we have

$$(i_b(\delta_0)(1-\delta_1', \delta_1', 0, \ldots, 0) = (1-\delta_0, \delta_0 - \delta_1, \delta_1, 0, \ldots, 0).$$

Thus $j(1-\delta_0, \delta_0-\delta_1, \delta_1, 0, \ldots, 0) \in \pi(S_1)$.

Let $x_2 \in \pi^{-1}(j(1-\delta_0, \delta_0-\delta_1, \delta_1, \ldots, 0)) \cap S_1$, and denote $G_{x_2} = K_2$. Thus $K_2 \subset K_1$ and $K_2 \in (\overline{K}_2)$. Consider $x_2 \in S_1$ as a point in the differentiable K_1-space S_1, and apply the "differentiable slice theorem" to this situation. Thus let S_2 be a K_2-slice at x_2 in S_1. Now $\pi(S_2)$ is an open neighborhood of $\pi(x_2) = j(1-\delta_0, \delta_0 - \delta_1, \delta_1, 0, \ldots, 0)$ in Y. Consider the composite map

$$\Delta_{n-2} \xrightarrow{i_b(\delta_0, \delta_1)} \Delta_n \xrightarrow{j} Y$$

where $i_b(\delta_0, \delta_1)(a_0, \ldots, a_{n-2}) = (1-\delta_0, \delta_0-\delta_1, \delta_1 a_0, \ldots, \delta_1 a_{n-2})$. Then $(j \circ i_b(\delta_0, \delta_1))(1, 0, \ldots, 0) = \pi(x_2)$. Hence there exists $1 \geq \delta_2' > 0$, such that

$$(j \circ i_b(\delta_0, \delta_1))(\Delta_{n-2}(\delta_2')) \subset \pi(S_2) .$$

Denote $\delta_2 = \delta_1 \delta_2'$. Then we have

$i_b(\delta_0, \delta_1)(1-\delta'_2, \delta'_2, 0, \ldots, 0) = (1-\delta_0, \delta_0-\delta_1, \delta_1-\delta_2, \delta_2, 0, \ldots, 0)$ and thus $j(1-\delta_0, \delta_0-\delta_1, \delta_1-\delta_2, \delta_2, \cdots, 0) \in \pi(S_2)$. Now continue as before, choose any $x_3 \in \pi^{-1}(j(1-\delta_0, \delta_1-\delta_0, \delta_1-\delta_2, \delta_2, \ldots, 0)) \cap S_2$, and so on.

In this way we get points

$$x_0, x_1, \ldots, x_n \in M$$

with corresponding isotropy groups

$$G_{x_i} = K_i, \qquad i = 0, \ldots, n,$$

and differentiable slices

$$S_0 \supset S_1 \supset \cdots \supset S_{n-1},$$

and positive real numbers

$$1 \geq \delta_0 \geq \delta_1 \geq \cdots \geq \delta_{n-1} > 0,$$

such that the following is valid.

We have $x_i \in S_i$, $i = 0, \ldots, n-1$, $x_n \in S_{n-1}$, and S_i is a differentiable K_i-slice at x_i in S_{i-1}, $i = 0, \ldots, n-1$ (here we interpret $S_{-1} = M$). Thus $K_0 \supset K_1 \supset \cdots \supset K_n$.

We also have

$$\pi(x_i) = j(1-\delta_0, \delta_0-\delta_1, \ldots, \delta_{i-2}-\delta_{i-1}, \delta_{i-1}, 0, \ldots, 0) = j(e^i)$$

for $i = 0, \ldots, n$, and hence $K_i \in (\overline{K}_i)$, $i = 0, \ldots, n$. Moreover, we have

$$j(\{e^i, e^{i+1}, \ldots, e^n\}) \subset \pi(S_i), \qquad i = 0, \ldots, n-1,$$

where $\{e^i, \ldots, e^n\}$ denotes the convex hull of the points $e^i, \ldots, e^n \in \Delta_n$.

We now construct a mapping

$$\omega_n : \Delta_n \longrightarrow S_0$$

in the following way. We denote

$$\Delta_k(b) = \{(0, \ldots, 0, a_{n-k}, \ldots, a_n) \in \Delta_n\}.$$

Let $h_i : \overset{\circ}{V}_i(1) \xrightarrow{\cong} S_i$, $i = 0, \ldots, n-1$ be presentations of the slices S_i; thus V_i is an orthogonal representation space for K_i, $i = 0, \ldots, n-1$. Let

$$t_i : \overset{\circ}{V}_i(1) \longrightarrow \overset{\circ}{V}_{i-1}(1), \qquad i = 1, \ldots, n-1,$$

be the mapping defined by $t_i(v) = h_{i-1}^{-1}(h_i(v))$, $v \in \overset{\circ}{V}_i(1)$. Thus t_i corresponds to the inclusion $S_i \hookrightarrow S_{i-1}$; more precisely the following diagram is commutative:

$$\begin{array}{ccccccc}
S_{n-1} & \hookrightarrow & S_{n-2} & \hookrightarrow & \cdots & \hookrightarrow & S_0 \\
h_{n-1}\uparrow\cong & & h_{n-2}\uparrow\cong & & & & h_0\uparrow\cong \\
\overset{\circ}{V}_{n-1} & \xrightarrow{t_{n-1}} & \overset{\circ}{V}_{n-2}(1) & \xrightarrow{t_{n-2}} & \cdots & \xrightarrow{t_1} & \overset{\circ}{V}_0(1)\,.
\end{array}$$

Observe that v and $t_i(v)$ always have the same isotropy groups.

Now let $v_n \in \overset{\circ}{V}_{n-1}(1)$ be the element for which $h_{n-1}(v_n) = x_n \in S_{n-1}$. Define

$$\sigma_1 : \Delta_1(b) \longrightarrow \overset{\circ}{V}_{n-1}(1)$$

by $\sigma_1(a_{n-1}, a_n) = a_n v_n$. Then consider

$$t_{n-1}\sigma_1 : \Delta_1(b) \longrightarrow \overset{\circ}{V}_{n-2}(1),$$

and extend this map to a map

$$\sigma_2 : \Delta_2(b) \longrightarrow \overset{\circ}{V}_{n-2}(1)$$

by defining

$$\sigma_2(a_{n-2}, a_{n-1}, a_n) = \begin{cases} (1-a_{n-2})t_{n-1}\sigma_1\left(\frac{a_{n-1}}{1-a_{n-2}}, \frac{a_n}{1-a_{n-2}}\right), & a_{n-2} \neq 1 \\ 0 & , \quad a_{n-2} = 1 \end{cases}$$

Then consider

$$t_{n-2}\sigma_2 : \Delta_2(b) \longrightarrow \mathring{V}_{n-3}(1)$$

and extend it to a map

$$\sigma_3 : \Delta_3(b) \longrightarrow \mathring{V}_{n-3}(1)$$

by defining

$$\sigma_3(a_{n-3}, \ldots, a_n) = \begin{cases} (1-a_{n-3})t_{n-2}\sigma_2\left(\frac{a_{n-2}}{1-a_{n-3}}, \frac{a_{n-1}}{1-a_{n-3}}, \frac{a_n}{1-a_{n-3}}\right), & a_{n-3} \neq 1 \\ 0 & , \quad a_{n-3} = 1 \end{cases}$$

Continuing in this way we construct the map

$$\sigma_n : \Delta_n \longrightarrow \mathring{V}_0(1) .$$

Now define

$$\omega_n = h_0\sigma_n : \Delta_n \longrightarrow S_0 .$$

We have

$$\omega_n(d^i) = x_i \in M, \qquad i = 0, \ldots, n,$$

and moreover every point in

$$\omega_n(\Delta_m - \Delta_{m-1}), \qquad 0 \leq m \leq n$$

has isotropy group exactly K_m. Thus we get an isovariant G-map

$$\alpha_n : (\Delta_n; K_0, \ldots, K_n) \longrightarrow U \subset M$$

by defining $\alpha_n([a,g]) = g\,\omega_n(a)$.

<u>Induction hypothesis</u>: Construct an isovariant G-map

$$\alpha_n : (\Delta_n ; K_0, \ldots, K_n) \longrightarrow U_0 \subset M$$

in the way it was done above. Then there exists an isovariant G-homotopy

$$F_n : I \times (\Delta_n ; K_0, \ldots, K_n) \longrightarrow U_0 \subset M$$

such that $F_n(0, \ ) = \alpha_n$, and the isovariant G-map

$$\beta_n = F_n(1, \ ) : (\Delta_n ; K_0, \ldots, K_n) \longrightarrow U_0$$

induces the map

$$j \circ i(\delta_0, \ldots, \delta_{n-1}) : \Delta_n \longrightarrow \pi(U_0) = \pi(S_0)$$

on the orbit spaces. Here $i(\delta_0, \ldots, \delta_{n-1}) : \Delta_n \longrightarrow \Delta_n$ is as before.

We already proved this in the case $n = 1$ (in the case $n=0$ there is nothing to prove). Now assume that the "induction hypothesis" has been established for the value $n - 1$ where $n \geq 2$. We shall show that it is valid for the value n.

Thus let all the notation be as before, and consider the imbedding

$$j_b = j \circ i(\delta_0, \ldots, \delta_{n-1}) \circ e_n^0 : \Delta_{n-1} \longrightarrow Y,$$

that is $j_b : \Delta_{n-1} \longrightarrow Y$ is the back face of $j \circ i(\delta_0, \ldots, \delta_{n-1}) : \Delta_n \longrightarrow Y$. Thus j_b is of type $(\overline{K}_1), \ldots, (\overline{K}_n)$.

Now consider the points

$$x_1, \ldots, x_n \in M$$

with corresponding isotropy groups $G_{x_i} = K_i$, $i = 1, \ldots, n$, and the differentiable slices

$$S_1 \supset S_2 \supset \cdots \supset S_{n-1},$$

(the index should be thought of as beginning at 0 instead of 1) and the real numbers

$$1 = \bar{\delta}_0 = \bar{\delta}_1 = \cdots = \bar{\delta}_{n-2} \, .$$

Recall that we already pointed out that S_1 is not only a K_1-slice at x_1 in S_0 but also a K_1-slice at x_1 in M, and thus S_1 "is a correct start." We also have $\pi(x_i) = j_b(d^{i-1})$, $i = 1, \ldots, n$, and $j_b(\{d^{i-1}, \ldots, d^n\}) \subset \pi(S_i)$, $i = 1, \ldots, n$. Now observe that when we construct

$$\omega_{n-1} : \Delta_{n-1} \longrightarrow S_1$$

from the above data in the same way as we constructed $\omega_n : \Delta_n \longrightarrow S_0$ we get exactly the map

$$\omega_{n-1} = \omega_n e_n^0 : \Delta_{n-1} \longrightarrow S_1$$

that is the back face of ω_n. Thus the corresponding α_{n-1} is given by

$$\alpha_{n-1} = \alpha_n e_n^0 : (\Delta_{n-1}; K_1, \ldots, K_n) \longrightarrow U_1 .$$

Thus by induction hypothesis there exists an isovariant G-homotopy

$$F_1 : I \times (\Delta_{n-1}; K_1, \ldots, K_n) \longrightarrow U_1 ,$$

such that

$$F_1(0, \ ) = \alpha_{n-1} : (\Delta_{n-1}; K_1, \ldots, K_n) \longrightarrow U_1$$

and such that

$$\beta_{n-1} = F_1(1, \ ) : (\Delta_{n-1}; K_1, \ldots, K_n) \longrightarrow U_1$$

induces the map

$$j_b : \Delta_{n-1} \longrightarrow j_b(\Delta_{n-1}) = j(\{e^1, \ldots, e^n\}) \subset \pi(U_1)$$

on the orbit spaces. Next using the map β_{n-1} we define an isovariant G-map

$$\gamma: (\Delta_n; K_0, \ldots, K_n) - (\Delta_0; K_0) \longrightarrow U_0$$

by $\gamma\Big([(a_0, \ldots, a_n),]\Big) = \beta_{n-1}\Big([(\frac{a_1}{1-a_0}, \ldots, \frac{a_n}{1-a_0}), g]\Big)$. Then γ induces the map

$$j \circ i(\delta_0, \ldots, \delta_{n-1}) \circ b: (\Delta_n - \Delta_0) \longrightarrow \pi(U_0)$$

on the orbit spaces, where $b: (\Delta_n - \Delta_0) \longrightarrow \Delta_n$ is as before. To shorten the notation we denote $\bar{j} = j \circ i(\delta_0, \ldots, \delta_{n-1}): \Delta_n \longrightarrow \pi(U_0)$. Then

$$\bar{j}B: I \times (\Delta_n - \Delta_0) \longrightarrow \pi(U_0)$$

is an isovariant homotopy from $\bar{j}b$ to $\bar{j}|: (\Delta_n - \Delta_0) \longrightarrow \pi(U_0)$. Thus by the "covering homotopy theorem" of Palais there exists an isovariant homotopy

$$F_2: I \times ((\Delta_n; K_0, \ldots, K_n) - (\Delta_0; K_0)) \longrightarrow U_0$$

such that $F_2(0, \ ) = \gamma$, and such that F_2 induces $\bar{j}B: I \times (\Delta_n - \Delta_0) \longrightarrow \pi(U_0)$ on the orbit spaces.

Now restrict the mapping $F_2(1, \ )$ to $\Delta_n - \Delta_0$ and call it

$$s: \Delta_n - \Delta_0 \longrightarrow U_0.$$

Thus $s(a) = F_2(1, [a, e])$, $a \in \Delta_n - \Delta_0$. We have

$$\pi s = \bar{j}: (\Delta_n - \Delta_0) \longrightarrow \pi(U_0) .$$

Observe that every point in

$$s(\Delta_m - \Delta_{m-1}), \qquad 1 \leq m \leq n$$

has isotropy group K_m.

Restrict F_1 to $I \times \Delta_{n-1}$ and call it

$$h_1 : I \times \Delta_{n-1} \longrightarrow U_1 \subset U_0.$$

That is h_1 is defined by $h_1(t, a') = F_1(t, [a', e])$. Define

$$h_2 : I \times \Delta_{n-1} \longrightarrow U_0$$

by $h_2(t, a') = F_2(t, [e_n^0(a'), e])$.

Observe that $h_1(1, \ ) = h_2(0, \ )$. Moreover, every point in

$$h_1(I \times (\Delta_m - \Delta_{m-1})) \text{ and } h_2(I \times (\Delta_m - \Delta_{m-1}))$$

has isotropy group K_{m+1}, $0 \leq m \leq n - 1$.

Recall the strong G-deformation retraction

$$D : I \times U_0 \longrightarrow U_0$$

of U_0 to Gx_0. We define

$$H_1 : I \times (\Delta_n - \Delta_0) \longrightarrow U_0$$

by

$$H_1(t, (a_0, a_1, \ldots, a_n)) = D\Big((1-a_0), h_1\Big(t, (\frac{a_1}{1-a_0}, \ldots, \frac{a_n}{1-a_0})\Big)\Big)$$

and similarly

$$H_2 : I \times (\Delta_n - \Delta_0) \longrightarrow U_0$$

by

$$H_2(t, (a_0, a_1, \ldots, a_n)) = D\Big((1-a_0), h_2\Big(t, (\frac{a_1}{1-a_0}, \ldots, \frac{a_n}{1-a_0})\Big)\Big).$$

Now first observe that

$$H_1(0, \ ) = \omega_n| : (\Delta_n - \Delta_0) \longrightarrow U_0.$$

Secondly, observe that the map

$$\varphi = H_2(1, \ ) : (\Delta_n - \Delta_0) \longrightarrow U_0$$

has the property that

$$\varphi(0, a_1, \ldots, a_n) = s(0, a_1, \ldots, a_n),$$

that is, restricted to the back face of $\Delta_n - \Delta_0$, φ and s agres.

Now we define a homotopy from φ to s in the following way.

Define

$$H_3 : I \times (\Delta_n - \Delta_0) \longrightarrow U_0$$

by

$$H_3(t_1(a_0, \ldots, a_n)) = \begin{cases} s(a_0, \ldots, a_n) & , \ 0 \leq a_0 \leq t \leq 1, \quad a_0 < 1 \\ D\left(\frac{1-a_0}{1-t}, s\left(t, \frac{a_1(1-t)}{1-a_0}, \ldots, \frac{a_n(1-t)}{1-a_0}\right)\right), & 0 \leq t \leq a_0 < 1 . \end{cases}$$

Thus H_3 is a continuous homotopy from φ to s.

Observe that all three homotopies H_1, H_2, and H_3 have the property that every point in

$$H_i(I \times (\Delta_m - \Delta_{m-1})), \qquad 1 \leq m \leq n, \quad i = 1, 2, 3,$$

has isotropy group K_m. Thus the homotopies

$$T_i = \pi H_i : I \times (\Delta_n - \Delta_0) \longrightarrow \pi(U_0), \qquad i = 1, 2, 3$$

are isovariant homotopies, and together they form an isovariant homotopy from

$$\pi \omega_n | : (\Delta_n - \Delta_0) \longrightarrow \pi(U_0)$$

to the map

$$\bar{j} | : (\Delta_n - \Delta_0) \longrightarrow \pi(U_0).$$

We claim that each T_i, $i = 1, 2, 3$, can be extended to a homotopy

$$\bar{T}_i : I \times \Delta_n \longrightarrow \pi(U_0), \qquad i = 1, 2, 3,$$

by defining

$$\overline{T}_i(t,(a_0,\ldots,a_n)) = \begin{cases} \overline{j}(1,0,\ldots,0) = \pi(x_0), & a_0 = 1 \\ T_i(t,(a_0,\ldots,a_n)), & a_0 < 1 . \end{cases}$$

We shall show below that each $\overline{T}_i$, $i = 1,2,3$ is continuous. Assume that this has been done. Then the $\overline{T}_i$'s, $i = 1,2,3$ form an isovariant homotopy from

$$\pi\omega_n : \Delta_n \longrightarrow \pi(U_0)$$

to the map

$$\overline{j} : \Delta_n \longrightarrow \pi(U_0).$$

Since the isovariant G-map

$$\alpha_n : (\Delta_n; K_0,\ldots,K_n) \longrightarrow U_0$$

induces the map $\pi\omega_n$ on the orbit spaces, it thus follows by the "covering homotopy theorem" of Palais that there exists an isovariant G-homotopy

$$F_n : I \times (\Delta_n; K_0,\ldots,K_n) \longrightarrow U_0$$

from α_n to a map

$$\beta_n : (\Delta_n; K_0,\ldots,K_n) \longrightarrow U_0$$

such that β_n induces $\overline{j} : \Delta_n \longrightarrow \pi(U_0)$ on the orbit spaces. This completes the induction step. Observe that β_n is a G-homeomorphism from $(\Delta_n; K_0,\ldots,K_n)$ onto $\pi^{-1}(\overline{j}(\Delta_n)) = \pi^{-1}(j(\{e^1,\ldots,e^n\}))$. To get a G-homeomorphism

$$\overline{\beta}_n : (\Delta_n; K_0,\ldots,K_n) \longrightarrow \pi^{-1}(j(\Delta_n))$$

it only remains to recall that $\overline{j} = j \circ i(\delta_0,\ldots,\delta_{n-1}) : \Delta_n \longrightarrow \pi(U_0)$ is isovariantly homotopic to $j : \Delta_n \longrightarrow \pi(U_0)$, and to apply the "covering homotopy theorem" by Palais once more.

To show that $T_i: \Delta_n \longrightarrow \pi(U_0)$, $i = 1, 2, 3$ is continuous, it is clearly enough to show that if B is an open neighborhood in $\pi(U_0)$ of $\bar{j}(1, 0, \ldots, 0) = \pi(x_0) = y_0$, then there exists $\epsilon > 0$ such that

$$H_i(I \times (a_0, \ldots, a_n)) \subset \pi^{-1}(B)$$

for all $(a_0, \ldots, a_n) \in \Delta_n - \Delta_0$ with $1 - a_0 < \epsilon$.

Let $t_1 > 0$ $(t_1 \leq 1)$ be such that

$$D(t \times U_0) \subset \pi^{-1}(B) \qquad \text{if } 0 \leq t < t_1$$

and let W be an open neighborhood of Gx_0 in U_0 such that

$$D(I \times W) \subset \pi^{-1}(B).$$

Thus for H_1 and H_2 it is enough to take $\epsilon = t_1$. Consider H_3. Choose $\epsilon_0 > 0$ $(\epsilon_0 < 1)$ such that

$$s(a_0, \ldots, a_n) \subset \pi^{-1}(B) \cap W, \quad \text{if } 1 - a_0 \leq \epsilon_0 .$$

Denote $1 - \epsilon_0 = t_0$. Then choose $\epsilon_1 > 0$ such that $\frac{\epsilon_1}{1-t_0} = \frac{\epsilon_1}{\epsilon_0} < t_1$. Thus especially $\epsilon_1 < \epsilon_0$. We claim that

$$H_3(I \times (a_0, \ldots, a_n)) \subset \pi^{-1}(B), \qquad \text{if } 1 - a_0 < \epsilon_1 .$$

Consider a point $(t, (a_0, \ldots, a_n))$ where $1 - a_0 < \epsilon_1$. First if $0 \leq a_0 \leq t \leq 1$, $a_0 < 1$, then

$$H_3(t, (a_0, \ldots, a_n)) = s(a_0, \ldots, a_n) \in \pi^{-1}(B) \cap W \subset \pi^{-1}(B).$$

Secondly, if $0 \leq t \leq a_0 < 1$ and $t_0 \leq t \leq 1$, then

$$H_3(t, (a_0, \ldots, a_n)) = D\left(\frac{1-a_0}{1-t}, s\left(t, \frac{a_1(1-t)}{1-a_0}, \ldots, \frac{a_n(1-t)}{1-a_0}\right)\right) \in \pi^{-1}(B)$$

since in this case $1 - t \leq 1 - t_0 = \epsilon_0$, and hence

$$s\left(t, \frac{a_1(1-t)}{1-a_0}, \ldots, \frac{a_n(1-t)}{1-a_0}\right) \epsilon \pi^{-1}(B) \cap W \subset W,$$ and thus the conclusion

follows.

Thirdly, if $0 \leq t \leq a_0 < 1$ and $0 \leq t \leq t_0$, then

$$H_3(t, (a_0, \ldots, a_n)) = D\left(\frac{1-a_0}{1-t}, s(\ \ldots\ )\right) \epsilon \pi^{-1}(B)$$

since in this case $1 - t \geq 1 - t_0 = \epsilon_0$, and hence $\frac{1-a_0}{1-t} \leq \frac{\epsilon_1}{\epsilon_0} < t_1$, and thus the conclusion follows. This completes the proof of Lemma 3.2 and hence of Theorem 3.1.

q.e.d.

4. THREE COROLLARIES

The following corollary follows from Theorem 3.1 and Proposition 2.3.

Corollary 4.1. Let G be a compact Lie group and let M be a differentiable G-manifold. Then M is an equivariant CW complex.

q.e.d.

By Corollary 3.7 in Chapter I we thus have:

Corollary 4.2. Let G be a compact Lie group and let M and N be differentiable G-manifolds. Then a G-map $f: M \longrightarrow N$ is a G-homotopy equivalence if and only if for each closed subgroup H of G the induced map $f^H: M^H \longrightarrow N^H$ induces a one-to-one correspondence between the path components of M^H and N^H, and isomorphisms $f_*^H: \pi_k(M^H, x) \longrightarrow \pi_k(N^H, f(x))$, for all $k \geq 1$ and every $x \in M^H$.

q.e.d.

For a semi-free action, that is, an action in which the only isotropy groups are G and $\{e\}$, Corollary 4.2 says that a G-map $f: M \longrightarrow N$ is a G-homotopy equivalence if $f: M \longrightarrow N$ is an ordinary homotopy equivalence, when we forget about the G-action, and the restriction to the fixed point set $f^G: M^G \longrightarrow N^G$ is a homotopy equivalence.

The following result is due to Atiyah-Segal. See Proposition 5.4 in Segal [14]. The statement there is more general than the one we give below. The proof in [14] uses the spectral sequence for equivariant K-theory, and all the details are not given.

Corollary 4.3. (Atiyah-Segal) Let G be a compact Lie group and M a compact differentiable G-manifold. Then $K_G^*(M)$ is a finitely generated R(G)-module.

Proof. Since M is compact, it is a finite equivariant CW complex. Denote it by X and the skeletons by $X = X^m, \dots, X^0$. We have the exact sequence

$$K_G^*(X^n, X^{n-1}) \xrightarrow{j^*} K_G^*(X^n) \xrightarrow{i^*} K_G^*(X^{n-1}) .$$

The module $K_G^*(X^n, X^{n-1}) = \tilde{K}^*(X^n/X^{n-1})$ is a finite direct sum of modules of the form

$$\tilde{K}_G^*(S^n \times G/H / \{b\} \times G/H) = \begin{cases} R(H), & \text{for } * + n = \text{even} \\ 0 \;, & \text{for } * + n = \text{odd} . \end{cases}$$

By a theorem of Atiyah, R(G) is noetherian and R(H) is finitely generated over R(G) (see Proposition 3.2 and Corollary 3.3 in Segal [15]). Assume by induction that $K_G^*(X^{n-1})$ is finitely generated over R(G).

Thus, in the short exact sequence

$$0 \longrightarrow \mathrm{im}(j^*) \hookrightarrow K_G^*(X^n) \xrightarrow{i^*} \mathrm{im}(i^*) \longrightarrow 0$$

both $\mathrm{im}(j^*)$ and $\mathrm{im}(i^*)$ are finitely generated over $R(G)$. Hence $K_G^*(X^n)$ is finitely generated over $R(G)$. Since $X = X^m$, induction completes the proof.

q. e. d.

CHAPTER III

EQUIVARIANT SINGULAR THEORY

In this chapter G denotes a good locally compact group, by which we mean that G is a compact Lie, or G is a discrete group, or G is an abelian locally compact group. We construct an equivariant singular homology and cohomology theory with coefficients in an arbitrary given covariant coefficient system and contravariant coefficient system, respectively, on the category of all G-spaces and G-maps. The construction is very much analogous to the construction of ordinary singular theory. We use the equivariant simplexes, defined in Definition 1.1 in Chapter II, in place of standard simplexes. Ordinary singular theory in its present form is due to S. Eilenberg [5]. We have chosen the exposition in Eilenberg-Steenrod [6] as the ground for our imitation. This applies especially to the proofs of the homotopy and excision axioms. The proof of the dimension axiom requires some argument, and it is here that we have to assume that G is a good locally compact group, see Lemma 7.3.

1. COEFFICIENT SYSTEMS

Recall that in this chapter G denotes a good locally compact group, that is, a compact Lie group, a discrete group, or an abelian locally compact group.

<u>Definition 1.1</u>. A family $\mathcal{F}$ of closed subgroups of G is called an orbit

type family for G, if the following condition is satisfied: if $H \in \mathcal{F}$ and H' is conjugate to H, then $H' \in \mathcal{F}$.

Thus, the family of all closed subgroups, and the family of all finite subgroups of G, are examples of orbit type families for G. A more special example is the following. Let $G = O(n)$ and let $\mathcal{F}$ be the family of all subgroups conjugate to $O(m)$ (standard imbedding) for some m, where $0 \leq m \leq n$.

In the following R will denote an arbitrary ring with unit. By an R-module we mean a unitary left R-module.

Definition 1.2. Let $\mathcal{F}$ be an orbit type family for G. A covariant equivariant coefficient system k for $\mathcal{F}$, over the ring R, is a covariant functor from the category of G-spaces of the form G/H, where $H \in \mathcal{F}$, and G-homotopy classes of G-maps, to the category of R-modules.

A contravariant equivariant coefficient system ℓ is defined by the contravariant version of the above definition.

If $\alpha\colon G/H \longrightarrow G/K$ is a G-map, and $H, K \in \mathcal{F}$ we denote

$$k(\alpha) = \alpha_*\colon k(G/H) \longrightarrow k(G/K),$$

and

$$\ell(\alpha) = \alpha^*\colon \ell(G/K) \longrightarrow \ell(G/H).$$

Let k and k' be covariant equivariant coefficient systems for $\mathcal{F}$. A natural transformation

$$h\colon k \longrightarrow k'$$

will be called a homomorphism of covariant equivariant coefficient systems. If h is a natural equivalence, we call h an isomorphism.

Similarly for contravariant equivariant coefficient systems.

From now on we shall shorten the terminology so that we simply speak about "coefficient systems."

2. EQUIVARIANT SINGULAR HOMOLOGY AND COHOMOLOGY

Theorem 2.1. Let G be a good locally compact group. Let $\mathcal{F}$ be an orbit type family for G and let k be an arbitrary covariant coefficient system for $\mathcal{F}$.

Then there exists an equivariant homology theory $H_*^G(\ ;k)$, defined on the category of all G-pairs and all G-maps, which satisfies all seven equivariant Eilenberg-Steenrod axioms, and which has the given coefficient system k as coefficients.

This means:

For each G-pair (X,A) we have an R-module $H_n^G(X,A;k)$ for every integer n.

Each G-map $f: (X,A) \longrightarrow (Y,B)$ induces a homomorphism

$$f_*: H_n^G(X,A;k) \longrightarrow H_n^G(Y,B;k)$$

for every integer n.

Each G-pair (X,A) determines a boundary homomorphism

$$\partial: H_n^G(X,A;k) \longrightarrow H_{n-1}^G(A;k)$$

for every integer n.

In addition, the following axioms are satisfied.

A.1. If f = identity, then f_* = identity.

A.2. If $f: (X,A) \longrightarrow (Y,B)$ and $f': (Y,B) \longrightarrow (Z,C)$ are G-maps, then

$$(f'f)_* = f'_* f_* .$$

A.3. For any G-map $f: (X,A) \longrightarrow (Y,B)$ we have

$$\partial f_* = (f|A)_* \partial .$$

A.4. (Exactness axiom). Any G-pair (X,A) gives rise to an exact homology sequence

$$\cdots \xleftarrow{i_*} H^G_{n-1}(A;k) \xleftarrow{\partial} H^G_n(X,A;k) \xleftarrow{j_*} H^G_n(X;k) \xleftarrow{i_*} H^G_n(A;k) \xleftarrow{\partial} \cdots$$

A.5. (Homotopy axiom). If $f_0, f_1: (X,A) \longrightarrow (Y,B)$ are G-homotopic, then

$$(f_0)_* = (f_1)_*$$

A.6. (Excision axiom). An inclusion of the form

$$i: (X-U, A-U) \longrightarrow (X,A)$$

where $\overline{U} \subset A^\circ$ (U is a G-subset) induces an isomorphism

$$i_*: H^G_n(X-U, A-U; k) \xrightarrow{\cong} H^G_n(X,A;k)$$

for every integer n.

A.7. (Dimension axiom). If $H \in \mathcal{F}$, then

$$H^G_m(G/H; k) = 0 \qquad \text{for all } m \neq 0.$$

Moreover, for every $H \in \mathcal{F}$ we have an isomorphism

$$\gamma: H^G_0(G/H; k) \xrightarrow{\cong} k(G/H),$$

such that if also $K \in \mathcal{F}$ and $\alpha: G/H \longrightarrow G/K$ is a G-map, then the diagram

$$\begin{array}{ccc} H^G_0(G/H; k) & \xrightarrow{\gamma} & k(G/H) \\ \alpha_* \downarrow & & \downarrow \alpha_* \\ H^G_0(G/K; k) & \xrightarrow{\gamma} & k(G/K) \end{array}$$

commutes.

Moreover, this equivariant homology theory has no "negative homology," that is, for any G-pair (X, A) we have

$$H_m^G(X, A; k) = 0 \qquad \text{if } m < 0.$$

We call this equivariant homology theory for "equivariant singular homology with coefficients in k."

<u>Theorem 2.2.</u> Let G be a good locally compact group. Let $\mathcal{F}$ be an orbit type family for G, and let ℓ be an arbitrary contravariant coefficient system for $\mathcal{F}$.

Then there exists an equivariant cohomology theory $H_G^*(\ ;\ell)$ defined on the category of all G-pairs and all G-maps, which satisfies all seven equivariant Eilenberg-Steenrod axioms and which has the given coefficient system ℓ as coefficients.

This means:

For each G-pair (X, A) we have an R-module $H_G^n(X, A; \ell)$ for every integer n. Each G-map $f: (X, A) \longrightarrow (Y, B)$ induces a homomorphism

$$f^*: H_G^n(Y, B; \ell) \longrightarrow H_G^n(X, A; \ell)$$

for every integer n. Each G-pair (X, A) determines a coboundary homomorphism

$$\delta: H_G^{n-1}(A; \ell) \longrightarrow H_G^n(X, A; \ell)$$

for every integer n. In addition, the following axioms are satisfied.

<u>A.1</u>. If f = identity, then f^* = identity.

<u>A.2</u>. If $f: (X, A) \longrightarrow (Y, B)$ and $f': (Y, B) \longrightarrow (Z, C)$ are G-maps, then

$$(f'f)^* = f^* f'^*.$$

A.3. For any G-map $f: (X,A) \longrightarrow (Y,B)$ we have

$$f^* \delta = \delta(f|A)^*.$$

A.4. (Exactness axiom). Any G-pair (X,A) gives rise to an exact cohomology sequence

$$\cdots \xrightarrow{i^*} H_G^{n-1}(A;\ell) \xrightarrow{\delta} H_G^n(X,A;\ell) \xrightarrow{j^*} H_G^n(X;\ell) \xrightarrow{i^*} H_G^n(A;\ell) \xrightarrow{\delta} \cdots$$

A.5. (Homotopy axiom). If $f_0, f_1 : (X,A) \longrightarrow (Y,B)$ are G-homotopic, then

$$(f_0)^* = (f_1)^* .$$

A.6. (Excision axiom). An inclusion of the form

$$i: (X-U, A-U) \longrightarrow (X,A),$$

where $\overline{U} \subset A^o$, (U is a G-subset) induces an isomorphism

$$i^* : H_G^n(X,A;\ell) \xrightarrow{\cong} H_G^n(X-U, A-U;\ell)$$

for every integer n.

A.7. (Dimension axiom). If $H \in \mathcal{F}$, then

$$H_G^m(G/H;\ell) = 0, \qquad \text{for all } m \neq 0.$$

Moreover, for every $H \in \mathcal{F}$, we have an isomorphism

$$\xi: H_G^0(G/H;\ell) \xrightarrow{\cong} \ell(G/H)$$

such that if also $K \in \mathcal{F}$ and $\alpha: G/H \longrightarrow G/K$ is a G-map, then the diagram

$$\begin{array}{ccc} H_G^0(G/H;\ell) & \xrightarrow{\xi} & \ell(G/H) \\ \alpha^* \uparrow & & \uparrow \alpha^* \\ H_G^0(G/K;\ell) & \xrightarrow{\xi} & \ell(G/K) \end{array}$$

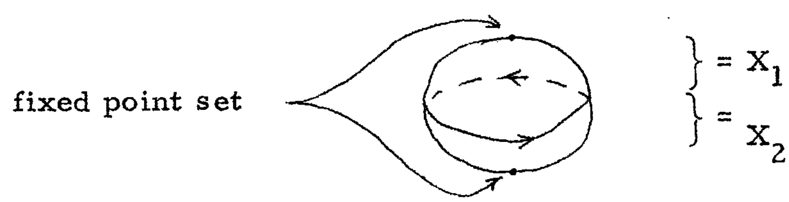
commutes.

Moreover, this equivariant cohomology theory has no "negative cohomology," that is, for any G-pair (X,A) we have

$$H_G^m(X,A;\ell) = 0, \qquad \text{if } m < 0.$$

We call this equivariant cohomology theory for "equivariant singular cohomology with coefficients in ℓ."

<u>Example.</u> As a simple illustration, we determine the equivariant singular homology of the following example. Let $G = S^1$ the circle group and let $X = S^2$ the two-sphere. Assume that S^1 acts on S^2 by the standard "free rotation" leaving the "south and north poles" fixed. The following picture describes the situation.



Here X_1 and X_2 denote the northern and southern hemispheres, respectively, and $X_0 = X_1 \cap X_2$ is the equator.

It is a formal consequence of the axioms that we in this situation have the following exact Mayer-Vietoris sequence

$$0 \longleftarrow H_0^G(X;k) \xleftarrow{j_{1*}+j_{2*}} H_0^G(X_1;k) \oplus H_0^G(X_2;k) \xleftarrow{(i_{1*},-i_{2*})} H_0^G(X_0;k)$$

$$\xleftarrow{\partial} H_1^G(X;k) \longleftarrow 0 .$$

Since both X_1 and X_2 are G-homotopy equivalent to a point and $X_0 \cong G$ as G-spaces, it follows that the above sequence is

$$0 \longleftarrow H_0^G(X;k) \longleftarrow k(G/G) \oplus k(G/G) \xleftarrow{(p_*,-p_*)} k(G) \longleftarrow H_1^G(X;k) \longleftarrow 0$$

where $p_*: k(G) \longrightarrow k(G/G)$ is induced by the G-map $p: G \longrightarrow G/G$. Thus

$$H_0^G(X;k) \cong (k(G/G) \oplus k(G/G)) / \{(p_*(a), -p_*(a)) \mid a \in k(S^1)\}$$

$$H_1^G(X;k) \cong \ker(p_*: k(G) \longrightarrow k(G/G))$$

$$H_m^G(X;k) = 0 \qquad \text{for } m \neq 0,1.$$

Let us consider this result for some specific covariant coefficient systems. Let the orbit type family $\mathcal{F}$ be the family of all closed subgroups of $G = S^1$, and let the ring R be the integers Z.

1. Define a covariant coefficient system k_1 as follows. Let $k_1(G/H) = Z$ if $H \neq G$ and $k(G/G) = Z_2$, and let $p: G/H \longrightarrow G/G$, where $H \neq G$, induce the natural projection $Z \longrightarrow Z_2$ and all other induced homomorphisms on k_1 are the identity on Z. Then

$$H_0^G(X;k_1) \cong Z_2$$

$$H_1^G(X;k_1) \cong Z$$

$$H_m^G(X;k_1) = 0 \qquad \text{for } m \neq 0,1.$$

2. Define k_2 by: $k_2(G/\{e\}) = Z$, and $k_2(G/H) = 0$ for $H \neq \{e\}$. Then

$$H_0^G(X;k_2) = 0$$

$$H_1^G(X;k_2) \cong Z$$

$$H_m^G(X;k_2) = 0 \qquad \text{for } m \neq 0,1.$$

3. Define k_3 by: $k_3(G/H) = 0$ for $H \neq G$, and $k_3(G/G) = Z$. Then

$$H_0^G(X; k_3) = Z \oplus Z$$
$$H_n^G(X; k_3) = 0 \qquad \text{for } n \neq 0.$$

4. Define k_4 by: $k_4(G/H) = Z$ for all closed subgroups H of G and all induced homomorphisms are the identity on Z. Then

$$H_0^G(X; k_4) \cong Z$$
$$H_n^G(X; k_4) = 0 \qquad \text{for } n \neq 0.$$

5. Define k_5 by: $k_5(G/H) = Z$ for all closed subgroups H of G, and every G-map $\alpha : G/H \longrightarrow G/K$, where $H \subset K$ and $H \neq K$ induces the zero homomorphism. Then

$$H_0^G(X; k_5) \cong Z \oplus Z$$
$$H_1^G(X; k_5) \cong Z$$
$$H_m^G(X; k_5) = 0 \qquad \text{for } m \neq 0, 1.$$

3. A LEMMA

Recall the definition of the standard equivariant n-simplex

$$(\Delta_n; k_0, \ldots, K_n)$$

of type $(K_0, \ldots, K_n)$, see Definition 1.1 in Chapter II. We shall use the same notation as in Section 1 of Chapter II. Consider the standard equivariant n-simplexes $(\Delta_n; K_0, \ldots, K_n)$ and $(\Delta_n; K_0', \ldots, K_n')$ and let

$$h: (\Delta_n; K_0, \ldots, K_n) \longrightarrow (\Delta_n; K_0', \ldots, K_n')$$

be a G-map which covers $\text{id}: \Delta_n \longrightarrow \Delta_n$, that is the following diagram commutes.

$$\begin{array}{ccc} (\Delta_n; K_0, \ldots, K_n) & \xrightarrow{h} & (\Delta_n; K'_0, \ldots, K'_n) \\ \pi \downarrow & & \downarrow \pi' \\ \Delta_n & \xrightarrow{id} & \Delta_n \end{array}$$

Thus h induces a G-map

$$h| : \pi^{-1}(x) \longrightarrow (\pi')^{-1}(x)$$

for every $x \in \Delta_n$.

Assume that $x \in \Delta_m - \Delta_{m-1}$. We define a G-map

$$h_x : G/K_m \longrightarrow G/K'_m$$

by requiring that the diagram

$$\begin{array}{ccc} G/K_m & \xrightarrow{h_x} & G/K'_m \\ \gamma_x \downarrow & & \gamma'_x \downarrow \cong \\ \pi^{-1}(x) & \xrightarrow{h|} & (\pi')^{-1}(x) \end{array}$$

commutes. Here γ_x and γ'_x are the G-homeomorphisms defined by

$$\gamma_x(gK_m) = [x, g] \in (\Delta_n; K_0, \ldots, K_n)$$

and $$\gamma'_x(gK'_m) = [x, g] \in (\Delta_n; K'_0, \ldots, K'_n).$$

Definition 3.1. Let $\mathcal{F}$ be an orbit type family for G. We say that the standard equivariant n-simplex $(\Delta_n; K_0, \ldots, K_n)$ belongs to $\mathcal{F}$ if $K_i \in \mathcal{F}$ for $i = 0, \ldots, n$.

Lemma 3.2. Let k be a covariant coefficient system for the orbit type family $\mathcal{F}$. Assume that $(\Delta_n; K_0, \ldots, K_n)$ and $(\Delta_n; K'_0, \ldots, K'_n)$ belong to $\mathcal{F}$, and that the G-map

$$h: (\Delta_n; K_0, \ldots, K_n) \longrightarrow (\Delta_n; K'_0, \ldots, K'_n)$$

covers $\mathrm{id}: \Delta_n \to \Delta_n$. Then h determines for each m, $0 \leq m \leq n$ a unique homomorphism

$$(h_m)_*: k(G/K_m) \longrightarrow k(G/K'_m)$$

and we have $(h_m)_* = (h_x)_*$ for any $x \in \Delta_m - \Delta_{m-1}$. Moreover, for any m, q such that $0 \leq q \leq m \leq n$, the diagram

$$\begin{array}{ccc} k(G/K_m) & \xrightarrow{(h_m)_*} & k(G/K'_m) \\ p_* \downarrow & & \downarrow p'_* \\ k(G/K_q) & \xrightarrow{(h_q)_*} & k(G/K'_q) \end{array}$$

is commutative. Here $p: G/K_m \to G/K_q$ is the natural projection, that is, $p(gK_m) = gK_q$ and correspondingly for p'.

If h is a G-homeomorphism, then $(h_m)_*$ is an isomorphism, and we have

$$(h_m)_*^{-1} = ((h^{-1})_m)_* .$$

<u>Proof</u>. Let $x \in \Delta_m - \Delta_{m-1}$ and $z \in \Delta_q - \Delta_{q-1}$ where $0 \leq q \leq m \leq n$. We shall show that the following diagram of G-spaces and G-maps is G-homotopy commutative.

$$\begin{array}{ccc} G/K_m & \xrightarrow{h_x} & G/K'_m \\ p \downarrow & & \downarrow p' \\ G/K_q & \xrightarrow{h_z} & G/K'_q \end{array}$$

Since $x \in \Delta_m - \Delta_{m-1}$ and $z \in \Delta_q - \Delta_{q-1}$ and $0 \leq q \leq m \leq n$, we have

$(1-t)x + tz \in \Delta_m - \Delta_{m-1}$ for $0 \leq t < 1$. Denote $|x,z| = \{(1-t)x + tz \in \Delta_n \mid 0 \leq t \leq 1\}$.

We have the commutative diagrams

$$\begin{array}{ccc} G/K_m & \xrightarrow[\cong]{\gamma_x} & \pi^{-1}(x) \\ p \downarrow & & \downarrow \rho \\ G/K_q & \xrightarrow[\cong]{\gamma_z} & \pi^{-1}(z) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} G/K'_m & \xrightarrow[\cong]{\gamma'_x} & (\pi')^{-1}(x). \\ p' \downarrow & & \downarrow \rho' \\ G/K'_q & \xrightarrow[\cong]{\gamma'_z} & (\pi')^{-1}(z) \end{array}$$

where $\rho([x,g]) = [x,g] \in (\Delta_n; K_0, \ldots, K_n)$

and $\rho'([x,g]) = [z,g] \in (\Delta_n; K'_0, \ldots, K'_n)$. Now define the G-map

$$F: I \times G/K_m \longrightarrow G/K'_q$$

to be the composite

$$I \times G/K_m \xrightarrow{\gamma} \pi^{-1}(|x,z|) \xrightarrow{h|} (\pi')^{-1}(|x,z|) \xrightarrow{\bar{\rho}'} (\pi')^{-1}(z) \xrightarrow{(\gamma_z)^{-1}} G/K'_q$$

where $\gamma(t, gK_m) = [(1-t)x + tz, g] \in (\Delta_n; K_0, \ldots, K_n)$

and $\bar{\rho}'([(1-t)x + tz, g]) = [z,g] \in (\Delta_n; K'_0, \ldots, K'_n)$.

Now

$$F(0, \ ) = (\gamma'_z)^{-1} \circ \rho' \circ (h|) \circ \gamma_x = p' \circ (\gamma'_x)^{-1} \circ (h|) \circ \gamma_x = p' \circ h_x$$

and $$F(1, \ ) = (\gamma_z)^{-1} \circ (h|) \circ \rho \circ \gamma_x = (\gamma_z)^{-1} \circ (h|) \circ \gamma_z \circ p = h_z \circ p .$$

Thus $F: I \times G/K_m \longrightarrow G/K'_q$ is a G-homotopy from $p' \circ h_x$ to $h_z \circ p$.

This completes the proof of Lemma 3.2. q.e.d.

4. CONSTRUCTION OF EQUIVARIANT SINGULAR HOMOLOGY

In this section we construct the equivariant singular homology groups of a G-pair (X, A) with coefficients in a given covariant coefficient system k.

Definition 4.1. A G-map

$$T: (\Delta_n; K_0, \ldots, K_n) \longrightarrow X$$

is called an equivariant singular n-simplex of type $(K_0, \ldots, K_n)$ in X. We call K_n for the main type of T, and denote

$$t(T) = K_n.$$

The equivariant singular $(n-1)$-simplex of type $(K_0, \ldots, \hat{K}_i, \ldots, K_n)$

$$T^{(i)} = T\,\bar{e}_n^i: (\Delta_{n-1}; K_0, \ldots, \hat{K}_i, \ldots, K_n) \longrightarrow X$$

is called the i:th face of T, $i = 0, \ldots, n$.

Observe that we have

$$t(T^{(i)}) = t(T) = K_n, \qquad \text{for } i = 0, \ldots, n-1$$
$$t(T^{(n)}) = K_{n-1} .$$

Definition 4.2. Let $\mathcal{F}$ be an orbit type family for G. We say that the equivariant singular n-simplex $T: (\Delta_n; K_0, \ldots, K_n) \longrightarrow X$ belongs to $\mathcal{F}$ if $(\Delta_n; K_0, \ldots, K_n)$ belongs to $\mathcal{F}$, that is, if $K_i \in \mathcal{F}$, for $i = 0, \ldots, n$.

From now on we assume that we are given an orbit type family $\mathcal{F}$ for G, and a covariant coefficient system k for $\mathcal{F}$ over some ring R.

Given an equivariant singular n-simplex

$$T: (\Delta_n; K_0, \ldots, K_n) \longrightarrow X$$

which belongs to $\mathcal{F}$, we form

$$Z_T \otimes k(G/t(T)) = Z_T \otimes k(G/K_n).$$

Here Z_T denotes the infinite cyclic group on the generator T, and the tensor product is over the integers. The R-module structure on $k(G/t(T))$ makes $Z_T \otimes k(G/t(T))$ into an R-module such that $i: k(G/t(T)) \longrightarrow Z_T \otimes k(G/t(T))$ defined by $i(a) = T \otimes a$ is an isomorphism of R-modules.

Definition 4.3. By $\hat{C}_n^G(X; k)$ we denote the direct sum

$$\sum_T \oplus (Z_T \otimes k(G/t(T))$$

where the direct sum is over all equivariant singular n-simplexes in X, which belong to $\mathcal{F}$. Thus for $n < 0$ we have $\hat{C}_n^G(X; K) = \{0\}$.

We define the boundary homomorphism

$$\hat{\partial}_n : \hat{C}_n^G(X; k) \longrightarrow \hat{C}_{n-1}^G(X; k)$$

as follows. For $n \leq 0$ we define $\hat{\partial}_n = 0$. Assume $n > 0$, and let T be an equivariant singular n-simplex and $a \in k(G/t(T))$. Then we define

$$\hat{\partial}_n(T \otimes a) = \sum_{i=0}^{n} (-1)^i T^{(i)} \otimes (p_i)_*(a) \in \hat{C}_{n-1}^G(X; k).$$

Here $(p_i)_*: k(G/t(T)) \longrightarrow k(G/t(T^{(i)}))$, $i = 0, \ldots, n$ is the homomorphism induced by the natural projection $p_i: G/t(T) \longrightarrow G/t(T^{(i)})$. Thus we have

$$\hat{\partial}_n(T \otimes a) = \sum_{i=0}^{n-1} (-1)^i T^{(i)} \otimes a + (-1)^n T^{(n)} \otimes (p_n)_*(a).$$

This defines the R-module homomorphism $\hat{\partial}_n$.

<u>Lemma 4.4.</u> $\hat{\partial}_{n-1}\hat{\partial}_n = 0.$

<u>Proof.</u> For $n \leq 1$ this is clear since then $\hat{\partial}_{n-1} = 0$. Thus assume $n \geq 2$. Let T be an equivariant singular n-simplex. First notice that we then have the identity

$$(T^{(i)})^{(j)} = (T^{(j)})^{(i-1)}, \qquad \text{for } 0 \leq j < i \leq n.$$

Denote by $p^i_j: G/t(T^{(i)}) \longrightarrow G/t((T^{(i)})^{(j)})$, the natural projection. Thus

$$p^i_j p_i = p^j_{i-1} p_j, \qquad \text{for } 0 \leq j < i \leq n.$$

Assume that T belongs to $\mathcal{F}$ and let $a \in k(t(T))$. Then

$$(\hat{\partial}_{n-1}\hat{\partial}_n)(T \otimes a) = \sum_{i=0}^{n} (-1)^i \hat{\partial}_{n-1}(T^{(i)} \otimes (p_i)_*(a))$$

$$= \sum_{i=0}^{n}\sum_{j=0}^{n-1} (-1)^i(-1)^j (T^{(i)})^{(j)} \otimes (p^i_j)_*(p_i)_*(a)$$

$$= \sum_{0 \leq j < i \leq n} (-1)^{i+j}(T^{(i)})^{(j)} \otimes (p^i_j p_i)_*(a)$$

$$+ \sum_{0 \leq i \leq j < n} (-1)^{i+j}(T^{(i)})^{j} \otimes (p^i_j p_i)_*(a).$$

The first sum equals

$$\sum_{0 \leq j < i \leq n} (-1)^{i+j}(T^{(j)})^{(i-1)} \otimes (p^j_{i-1} p_j)_*(a).$$

Changing the notation so that $i-1$ becomes j and j becomes i, we see that this sum is the negative of the second sum above, and thus the two sums cancel.

q.e.d.

Thus we get a chain complex $\{\hat{C}^G_n(X; k), \hat{\partial}_n\}$. We shall denote it by $\hat{S}^G(X; k)$. Our main interest is not in $\hat{S}^G(X, k)$, but in a quotient of $\hat{S}^G(X; k)$. We now proceed to define this quotient.

Let for the moment $\mathscr{S}_n \subset \hat{C}_n^G(X; k)$ denote the set of all elements in $\hat{C}_n^G(X; k)$ that have at most one coordinate $\neq 0$. Every element in $\mathscr{S}_n$ has a unique expression of the form

$$T \otimes a .$$

where T is some equivariant singular n-simplex belonging to $\mathscr{F}$ in X, and $a \in k(G/t(T))$. We define a relation $\sim$ in $\mathscr{S}_n$ in the following way. Let $T \otimes a$ and $T' \otimes a'$ be two arbitrary elements in $\mathscr{S}_n$ where

$$T: (\Delta_n; K_0, \ldots, K_n) \longrightarrow X, \quad T': (\Delta_n; K_0', \ldots, K_n') \longrightarrow X$$

are equivariant singular n-simplexes belonging to $\mathscr{F}$ in X and $a \in k(G/K_n)$, $a' \in k(G/K_n')$. We now define

$T \otimes a \sim T' \otimes a' \Longleftrightarrow$ there exists a G-homeomorphism

$h: (\Delta_n; K_0, \ldots, K_n) \longrightarrow (\Delta_n; K_0', \ldots, K_n')$

covering $id: \Delta_n \longrightarrow \Delta_n$ such that

$$T = T'h \quad \text{and} \quad (h_n)_*(a) = a'.$$

Here $(h_n)_*: k(G/K_n) \longrightarrow k(G/K_n')$ is the isomorphism of R-modules determined by h as in Lemma 3.2. It is immediately seen that $\sim$ is an equivalence relation in $\mathscr{S}_n$.

<u>Definition 4.5</u>. Let the notation be as above. We define

$$\overline{C}_n^G(X; k) \subset \hat{C}_n^G(X; k)$$

to be the submodule of $\hat{C}_n^G(X; k)$ consisting of all elements of the form

$$\sum_{i=1}^{s} (T_i \otimes a_i - T_i' \otimes a_i')$$

where $T_i \otimes a_i \sim T_i' \otimes a_i'$ for $i = 1, \ldots, s$.

Definition 4.6. We define the R-module $C_n^G(X; k)$ by

$$C_n^G(X; k) = \hat{C}_n^G(X; k) / \bar{C}_n^G(X; k)$$

Lemma 4.7. The boundary homomorphism

$$\hat{\partial}_n : \hat{C}_n^G(X; k) \longrightarrow \hat{C}_{n-1}^G(X; k)$$

restricts to a homomorphism

$$\bar{\partial}_n : \bar{C}_n^G(X; k) \longrightarrow \bar{C}_{n-1}^G(X; k)$$

and thus also induces a homomorphism

$$\partial_n : C_n^G(X; k) \longrightarrow C_{n-1}^G(X; k).$$

Proof. We must show that if $T \otimes a \sim T' \otimes a'$ then $\hat{\partial}_n(T \otimes a - T' \otimes a') \in \bar{C}_{n-1}^G(X; k)$. For this it is enough to show that we have

$$T^{(i)} \otimes (p_i)_*(a) \sim (T')^{(i)} \otimes (p_i')_*(a').$$

Assume that $T: (\Delta_n; K_0, \ldots, K_n) \longrightarrow X$ and $T': (\Delta_n; K_0', \ldots, K_n') \longrightarrow X$, and hence that $a \in k(G/K_n)$, $a' \in k(G/K_n')$. Since $T \otimes a \sim T' \otimes a'$, there exists a G-homeomorphism $h: (\Delta_n; K_0, \ldots, K_n) \longrightarrow (\Delta_n; K_0', \ldots, K_n')$ which covers $id: \Delta_n \longrightarrow \Delta_n$ such that $T = T'h$ and $(h_n)_*(a) = a'$. Then h induces a G-homeomorphism

$$h^{(i)}: (\Delta_{n-1}; K_0, \ldots, \hat{K}_i, \ldots, K_n) \longrightarrow (\Delta_{n-1}; K_0', \ldots, \hat{K}_i', \ldots, K_n')$$

which covers $id: \Delta_{n-1} \longrightarrow \Delta_{n-1}$ and we have $T^{(i)} = (T')^{(i)} h^{(i)}$, $i = 0, \ldots, n$. The isomorphism

$$((h^{(i)})_{n-1})_*: k(G/t(T^{(i)})) \longrightarrow k(G/t((T')^{(i)}))$$

determined by $h^{(i)}$ equals

$$(h_n)_*: k(G/K_n) \longrightarrow k(G/K'_n) \quad \text{for } i = 0, \ldots, n-1$$

$$(h_{n-1})_*: k(G/K_{n-1}) \longrightarrow k(G/K'_{n-1}) \quad \text{for } i = n$$

where $(h_n)_*$ and $(h_{n-1})_*$ are the isomorphisms determined by h. Thus for $i = 0, \ldots, n-1$ we have

$$((h^{(i)})_{n-1})_*(p_i)_*(a) = (h_n)_*(a) = a' = (p'_i)_*(a').$$

This shows that $T^{(i)} \otimes (p_i)_*(a) \sim (T')^{(i)} \otimes (p'_i)_*(a)$ for $i = 1, \ldots, n-1$. Now consider the case $i = n$. By Lemma 3.2 we have

$$(p'_n)_*(h_n)_* = (h_{n-1})_*(p_n)_*.$$

Thus $((h^{(n)})_{n-1})_*(p_n)_*(a) = (h_{n-1})_*(p_n)_*(a) = (p'_n)_*(h_n)_*(a) = (p'_n)_*(a')$. This shows that

$$T^{(n)} \otimes (p_n)_*(a) \sim (T')^{(h)} \otimes (p'_n)_*(a').$$

This completes the proof. q.e.d.

Since $\hat{\partial}_{n-1}\hat{\partial}_n = 0$ it follows that $\bar{\partial}_{n-1}\bar{\partial}_n = 0$ and $\partial_{n-1}\partial_n = 0$. Thus we get the chain complexes

$$\bar{S}^G(X; k) = \left\{\bar{C}^G_n(X; k), \bar{\partial}_n\right\}$$

and

$$S^G(X; k) = \left\{C^G_n(X; k), \partial_n\right\}.$$

It is the chain complex $S^G(X; k)$ that gives us the equivariant singular homology groups with coefficients in k of X.

Let (X, A) be a G-pair. The inclusion $i: A \hookrightarrow X$ incudes a monomorphism of chain complexes

$$\hat{i}\colon \hat{S}^G(A;\,k) \longrightarrow \hat{S}^G(X;\,k).$$

Moreover, the image $\hat{i}(\hat{C}^G_n(A;\,k))$ is a direct summand in $\hat{C}^G_n(X;\,k)$, for all n. We identify $\hat{C}^G_n(A;\,k)$ with $\hat{i}(\hat{C}^G_n(A;\,k))$, that is, we consider $\hat{S}^G(A;\,k)$ as a subcomplex of $\hat{S}^G(X;\,k)$ through the monomorphism $\hat{i}$. We denote

$$\hat{C}^G_n(X, A;\,k) = \hat{C}^G_n(X;\,k) \big/ \hat{C}^G_n(A;\,k)$$

and the corresponding chain complex by

$$\hat{S}^G(X, A;\,k) = \left\{\hat{C}^G_n(X, A;\,k),\ \hat{\partial}_n\right\}.$$

We have the short exact sequence of chain complexes

$$0 \longrightarrow \hat{S}^G(A;\,k) \xrightarrow{\hat{i}} \hat{S}^G(X;\,k) \xrightarrow{\hat{j}} \hat{S}^G(X, A;\,k) \longrightarrow 0.$$

Clearly $\hat{i}$ restricts to

$$\bar{i}\colon \bar{S}^G(A;\,k) \longrightarrow \bar{S}^G(X;\,k)$$

and hence $\hat{i}$ induces a homomorphism of chain complexes

$$i\colon S^G(A;\,k) \longrightarrow S^G(X;\,k)$$

Lemma 4.8. The homomorphism $i\colon S^G(A;\,k) \longrightarrow S^G(X;\,k)$ induced by $\hat{i}$ is a monomorphism. Moreover, $i(C^G_n(A;\,k))$ is a direct summand in $C^G_n(X;\,k)$ for each n.

Proof. Consider the commutative diagram

$$\begin{array}{ccc}
\bar{S}^G(A;\,k) & \xrightarrow{\bar{i}} & \bar{S}^G(X;\,k) \\
\downarrow & & \downarrow \\
\hat{S}^G(A;\,k) & \xrightarrow{\hat{i}} & \hat{S}^G(X;\,k)
\end{array}$$

Observe that the claim that the induced homomorphism $(i: S^G(A;k) \to S^G(X;k)$ is a monomorphism is equivalent to the claim that we have

$$\hat{S}^G(A;\, k) \cap \bar{S}^G(X;\, k) = \bar{S}^G(A;\, k).$$

We define a homomorphism

$$\hat{\alpha}: \hat{S}^G(X;\, k) \longrightarrow \hat{S}^G(A;\, k)$$

by

$$\hat{\alpha}(T \otimes a) = \begin{cases} T \otimes a & \text{if } \operatorname{Im}(T) \subset A \\ 0 & \text{if } \operatorname{Im}(T) \cap (X\text{-}A) \neq \emptyset \,. \end{cases}$$

Thus $\hat{\alpha}$ is a left inverse to $\hat{i}$. If $T \otimes a \sim T' \otimes a'$, then $\operatorname{Im}(T) = \operatorname{Im}(T')$. Thus it follows that $\hat{\alpha}$ restricts to

$$\bar{\alpha}: \bar{S}^G(X;\, k) \longrightarrow \bar{S}^G(A;\, k)$$

and thus $\hat{\alpha}$ induces

$$\alpha: S^G(X;\, k) \longrightarrow S^G(A;\, k)$$

and α is a left inverse to i. q.e.d.

We denote

$$C_n^G(X, A;\, k) = C_n^G(X;\, k) / C_n^G(A;\, k)$$

and the corresponding chain complex by

$$S^G(X, A;\, k) = \left\{ C_n^G(X, A;\, k),\ \partial_n \right\}.$$

Definition 4.9. We define

$$H_n^G(X, A;\, k)$$

to be the nth homology module of the chain complex $S^G(X, A;\, k)$.

By Lemma 4.8 and by definition we have the short exact sequence of chain complexes

$$0 \longrightarrow S^G(A; k) \longrightarrow S^G(X; k) \longrightarrow S^G(X, A; k) \longrightarrow 0 .$$

This gives us the boundary homomorphism

$$\partial: H_n^G(X, A; k) \longrightarrow H_{n-1}^G(A; k)$$

and the exact homology sequence in the standard way.

More or less as a side remark let us point out the following. Define the chain complex $\overline{S}^G(X, A; k)$ in the obvious way. Then

$$0 \longrightarrow \overline{S}^G(X, A; k) \longrightarrow \hat{S}^G(X, A; k) \longrightarrow S^G(X, A; k) \longrightarrow 0$$

is a short exact sequence of chain complexes. This can either be seen "directly" or by drawing the obvious commutative 3×3 diagram and applying the 3×3 lemma.

We denote the homology groups of $\overline{S}^G(X, A; k)$ and $\hat{S}^G(X, A; k)$ by $\overline{H}_*^G(X, A; k)$ and $\hat{H}_*^G(X, A; k)$ respectively. Thus we get a long exact sequence

$$\cdots \longleftarrow \overline{H}_{n-1}^G(X, A; k) \longleftarrow H_n^G(X, A; k) \longleftarrow \hat{H}_n^G(X, A; k) \longleftarrow \overline{H}_n^G(X, A; k) \longleftarrow \cdots$$

Our main interest is in $H_*^G(\ ; k)$. But in the process of the proof of the fact that $H_*^G(\ ; k)$ satisfies all seven axioms, it will also be shown that both $\hat{H}_*(\ ; k)$ and $\overline{H}_*^G(\ ; k)$ satisfy the first six axioms.

Let (X, A) and (Y, B) be G-pairs and let

$$f: (X, A) \longrightarrow (Y, B)$$

be a G-map. If $T: (\Delta_n; K_0, \ldots, K_n) \longrightarrow X$ is an equivariant singular n-simplex of type $(K_0, \ldots, K_n)$ belonging to $\mathcal{F}$ in X, then

$fT: (\Delta_n; K_0, \ldots, K_n) \longrightarrow Y$ is an equivariant singular n-simplex of type $(K_0, \ldots, K_n)$ belonging to $\mathcal{F}$ in Y. Thus we get a homomorphism

$$\hat{f}_\#: \hat{C}^G_n(X, A; k) \longrightarrow \hat{C}^G_n(Y, B; k)$$

by defining $f_\#(T \otimes a) = (fT) \otimes a$. Since $(fT)^{(i)} = fT^{(i)}$, for $i = 0, \ldots, n$, it follows that we ha^re a homomorphism of chain complexes

$$\hat{f}_\#: \hat{S}^G(X, A; k) \longrightarrow \hat{S}^G(Y, B; k).$$

If $T \otimes a \sim T' \otimes a'$, then $f_\#(T \otimes a) \sim f_\#(T' \otimes a')$ and hence $f_\#$ restricts to

$$\bar{f}_\#: \bar{S}^G(X, A; k) \longrightarrow \bar{S}^G(Y, B; k)$$

and hence $\hat{f}_\#$ induces a chain homomorphism

$$f_\#: S^G(X, A; k) \longrightarrow S^G(Y, B; k).$$

It is now clear that we have proved everything up to the exactness axiom in the statement of Theorem 2.1. The homotopy, excision, and dimension axioms will be proved in the following sections.

5. THE HOMOTOPY AXIOM

We define the standard equivariant n-prism $(n \geq 1)$ of type $(K_0, \ldots, K_{n-1})$ to be the G-space

$$(\pi_n; K_0, \ldots, K_{n-1}) = I \times (\Delta_{n-1}; K_0, \ldots, K_{n-1}).$$

We have the G-maps $(n \geq 2)$

$$\bar{r}^i_n: (\pi_{n-1}; K_0, \ldots, \hat{K}_i, \ldots, K_{n-1}) \longrightarrow (\pi_n; K_0, \ldots, K_{n-1})$$

for $i = 0, \ldots, n-1$, defined by $\bar{r}^i_n(t, x) = (t, \bar{e}^i_{n-1}(x))$. We also have the

G-maps $(n \geq 1)$

$$\bar{\ell}_n: (\Delta_{n-1}; K_0, \dots, K_{n-1}) \longrightarrow (\pi_n; K_0, \dots, K_{n-1})$$
$$\bar{u}_n: (\Delta_{n-1}; K_0, \dots, K_{n-1}) \longrightarrow (\pi_n; K_0, \dots, K_{n-1})$$

defined by $\bar{\ell}_n(x) = (0, x)$ and $\bar{u}_n(x) = (1, x)$.

The following identities follow directly from the above definitions.

$$\bar{r}_n^i \bar{r}_{n-1}^j = \bar{r}_n^j \bar{r}_{n-1}^{i-1} \qquad \text{for } 0 \leq j < i \leq n-1,\ n \geq 3,$$
$$\bar{r}_n^i \bar{\ell}_{n-1} = \bar{\ell}_n \bar{e}_{n-1}^i \qquad \text{for } 0 \leq i \leq n-1, \quad n \geq 2,$$
$$\bar{r}_n^i \bar{u}_{n-1} = \bar{u}_n \bar{e}_{n-1}^i \qquad \text{for } 0 \leq i \leq n-1, \quad n \geq 2.$$

<u>Definition 5.1</u>. A continuous G-map

$$P: (\pi_n; K_0, \dots, K_{n-1}) \longrightarrow X \qquad (n \geq 1)$$

is called an equivariant singular n-prism of type $(K_0, \dots, K_{n-1})$ in X.

We call K_{n-1} for the main type of P, and denote

$$t(P) = K_{n-1}\ .$$

The equivariant singular (n-1)-simplex of type $(K_0, \dots, \hat{K}_i, \dots, K_{n-1})$

$$P^{(i)} = P\bar{r}_n^i : (\pi_{n-1}; K_0, \dots, \hat{K}_i, \dots, K_{n-1}) \longrightarrow X, \quad (n \geq 2)$$

is called the ith face of P, $i = 0, \dots, n-1$. An equivariant singular 1-prism has no faces. The equivariant singular (n-1)-simplexes of type $(K_0, \dots, K_{n-1})$

$$P_\ell = P\bar{\ell}_n : (\Delta_{n-1}; K_0, \dots, K_{n-1}) \longrightarrow X$$
$$P_u = P\bar{u}_n : (\Delta_{n-1}; K_0, \dots, K_{n-1}) \longrightarrow X$$

are called the lower and upper base of P, respectively.

Observe that we have

$$t(P^{(i)}) = t(P) = K_{n-1}, \qquad \text{for } i = 0, \ldots, n-2$$

$$t(P^{(n-1)}) = K_{n-2},$$

and

$$t(P_\ell) = t(P_u) = t(P) = K_{n-1}.$$

We also have the following identities

$$(P^{(i)})^{(j)} = (P^{(j)})^{(i-1)} \qquad \text{for } 0 \leq j < i \leq n-1, \quad n \geq 3$$

$$(P^{(i)})_\ell = (P_\ell)^{(i)} \qquad \text{for } 0 \leq i \leq n-1, \quad n \geq 2$$

$$(P^{(i)})_u = (P_u)^{(i)} \qquad \text{for } 0 \leq i \leq n-1, \quad n \geq 2.$$

Thus we can write $P_\ell^{(i)}$ and $P_u^{(i)}$ without ambiguity.

Given an equivariant singular n-prism

$$P: (\pi_n; K_0, \ldots, K_{n-1}) \longrightarrow X, \qquad (n \geq 1)$$

which belongs to $\mathcal{F}$, that is, $K_i \in \mathcal{F}$, $i = 0, \ldots, n-1$, we consider the R-module

$$Z_P \otimes k(G/t(P)) = Z_P \otimes k(G/K_{n-1})$$

where Z_P denotes the infinite cyclic group on the generator P, and the tensor product is over the integers.

Definition 5.2. By $\hat{C}_n^G P(X; k)$ we denote the direct sum

$$\sum_S \oplus (Z_S \otimes k(G/t(S)))$$

where the direct sum is taken over all equivariant singular n-simplexes and all equivariant singular n-prisms belonging to $\mathcal{F}$ in X. We define the boundary homomorphism

$$\hat{\partial}_n : \hat{C}_n^G P(X; k) \longrightarrow \hat{C}_{n-1}^G P(X; k)$$

as follows. If $n \leq 0$, then $\partial_n = 0$. Assume that $n \geq 1$ and let P be an equivariant singular n-prism and $a \in k(G/t(P))$. Then we define

$$\hat{\partial}_n(P \otimes a) = P_u \otimes a - P_\ell \otimes a - \sum_{i=0}^{n-1} (-1)^i P^{(i)} \otimes (p_i)_*(a).$$

For an equivariant singular n-simplex T we define $\hat{\partial}_n (T \otimes a)$ by the same formula as before. This defines the homomorphism $\hat{\partial}_n$.

<u>Lemma 5.3.</u> $\hat{\partial}_{n-1}\hat{\partial}_n = 0$.

<u>Proof.</u> The standard calculation works.

q. e. d.

Thus we get a chain complex $\left\{\hat{C}^G_n P(X; k), \hat{\partial}_n\right\}$. We shall denote it by $\hat{S}^G P(X; k)$. We now proceed to define the chain complex $S^G P(X; k)$ which is a quotient of $\hat{S}^G P(X; k)$.

We define a relation $\sim$ in the subset of $\hat{C}^G_n (X; k)$ consisting of all elements of the form $T \otimes a$ or $P \otimes a$ as follows. We define $T \otimes a \sim T' \otimes a'$ to mean the same thing as before. Consider the elements $P \otimes a$ and $P' \otimes a'$, where

$$P: (\pi_n; K_0, \ldots, K_{n-1}) \longrightarrow X, \; P': (\pi_n; K'_0, \ldots, K'_{n-1}) \longrightarrow X$$

are equivariant singular n-prisms belonging to $\mathcal{F}$ in X, and $a \in k(G/K_{n-1})$, $a' \in k(G/K'_{n-1})$. We now define

$P \otimes a \sim P' \otimes a' \Longleftrightarrow$ there exists a G-homeomorphism

$h: (\Delta_{n-1}; K_0, \ldots, K_{n-1}) \longrightarrow (\Delta_{n-1}; K'_0, \ldots, K'_{n-1})$

which covers $id: \Delta_{n-1} \longrightarrow \Delta_{n-1}$, such that

$P = P'(id \times h)$, and $(h_{n-1})_*(a) = a'$.

Here $(h_{n-1})_*: k(G/K_{n-1}) \longrightarrow k(G/K'_{n-1})$ is the isomorphism determined by h as in Lemma 3.2.

<u>Definition 5.4.</u> Let the notation be as above. We define

$$\overline{C}_n^G P(X; k) \subset \hat{C}_n^G P(X; k)$$

to be the submodule of $\hat{C}_n^G P(X; k)$ consisting of all elements of the form

$$\sum_{i=1}^{s} (P_i \otimes a_i - P_i' \otimes a_i') + \sum_{i=1}^{t} (T_i \otimes b_i - T_i' \otimes b_i')$$

where $P_i \otimes a_i \sim P_i' \otimes a_i'$ for $i = 1, \ldots, s$ and $T_i \otimes b_i \sim T_i' \otimes b_i'$ for $i = 1, \ldots, t$.

We then define

$$C_n^G P(X; k) = \hat{C}_n^G P(X; k) / \overline{C}_n^G P(X; k) .$$

<u>Lemma 5.5.</u> The boundary homomorphism

$$\hat{\partial}_n: \hat{C}_n^G P(X; k) \longrightarrow \hat{C}_{n-1}^G P(X; k)$$

restricts to a homomorphism

$$\overline{\partial}_n: \overline{C}_n^G P(X; k) \longrightarrow \overline{C}_{n-1}^G P(X; k)$$

and thus induces a homomorphism

$$\partial_n: C_n^G P(X; k) \longrightarrow C_{n-1}^G P(X; k).$$

<u>Proof.</u> The proof is completely analogous to the proof of Lemma 4.7.

q.e.d.

Thus we get the chain complexes

$$\overline{S}^G P(X; k) = \left\{ \overline{C}_n^G P(X; k), \overline{\partial}_n \right\}, \quad \text{and}$$

$$S^G P(X; k) = \left\{ C_n^G P(X; k), \partial_n \right\} .$$

Let (X, A) be a G-pair. Then the inclusion $i: A \hookrightarrow X$ induces an inclusion of chain complexes

$$\hat{i}: \hat{S}^G P(A; k) \hookrightarrow \hat{S}^G P(X; k)$$

which again induces an inclusion

$$i: S^G P(A; k) \hookrightarrow S^G P(X; k).$$

We define $S^G P(X, A; k)$ to be the quotient chain complex of $S^G P(X; k)$ by $S^G P(A; k)$, and analogously for the "bar" and "roof" chain complexes.

Now observe that $\hat{S}^G(X; k)$ is a subcomplex of $\hat{S}^G P(X; k)$ and that this inclusion also induces an inclusion

$$\hat{S}^G(X, A; k) \hookrightarrow \hat{S}^G P(X, A; k) .$$

This inclusion again induces an inclusion

$$S^G(X, A; k) \hookrightarrow S^G P(X, A; k) .$$

We wish to construct a retraction

$$\rho: S^G P(X, A; k) \longrightarrow S^G(X, A; k).$$

In order to do this it is convenient to introduce the "equivariant linear complex" of the G-spaces $(\Delta_n; K_0, \ldots, K_n)$ and $(\pi_n; K_0, \ldots, K_{n-1})$ We proceed to do this now.

We shall consider Δ_n as a subset of $(\Delta_n; K_0, \ldots, K_n)$ through the imbedding given by $x \longmapsto [x, e]$ and similarly $\pi_n = I \times \Delta_{n-1}$ as a subset of $(\pi_n; K_0, \ldots, K_{n-1}) = I \times (\Delta_{n-1}; K_0, \ldots, K_{n-1})$ through the imbedding given by $(t, x) \longmapsto (t, [x, e]) = [(t, x), e]$.

<u>Definition 5.6.</u> Let

$$T: (\Delta_m; L_0, \ldots, L_m) \longrightarrow (\Delta_n; K_0, \ldots, K_n)$$

be an equivariant singular m-simplex of type $(L_0, \ldots, L_m)$ in $(\Delta_n; K_0, \ldots, K_n)$. We say that T is linear if

$$T(\Delta_m) \subset \Delta_n \quad \text{and} \quad T|: \Delta_m \longrightarrow \Delta_n$$

is linear in the ordinary sense. Similarly

$$T: (\Delta_m; L_0, \ldots, L_m) \longrightarrow (\pi_n; K_0, \ldots, K_{n-1})$$

is called linear if

$$T(\Delta_m) \subset \pi_n \quad \text{and} \quad T|: \Delta_m \longrightarrow \pi_n$$

is linear in the ordinary sense.

Thus a linear equivariant singular m-simplex of type $(L_0, \ldots, L_m)$ in $(\Delta_n; K_0, \ldots, K_n)$ is determined by the values

$$T([d^i, e]) = [v^i, e], \quad v^i \in \Delta_n, \quad i = 0, \ldots, m.$$

Similarly a linear equivariant singular m-simplex of type $(L_0, \ldots, L_m)$ in $(\pi_n; K_0, \ldots, K_{n-1})$ is determined by the values

$$T([d^i, e]) = [w^i, e], \quad w^i \in \pi_n, \quad i = 0, \ldots, m.$$

<u>Lemma 5.7.</u> Let $v^i \in \Delta_n$, $i = 0, \ldots, m$. Then the assignment

$$T([d^i, e]) = [v^i, e], \qquad i = 0, \ldots, m$$

determines a linear equivariant singular m-simplex

$$T: (\Delta_m; L_0, \ldots, L_m) \longrightarrow (\Delta_n; K_0, \ldots, K_n)$$

if and only if $L_i \subset$ isotropy group of $[v^i, e]$ in $(\Delta_n; K_0, \ldots, K_n)$, $i = 0, \ldots, m$.

Let $w^i \in \pi_n$, $i = 0, \ldots, m$. Then the assignment

$$T([d^i, e]) = [w^i, e], \quad i = 0, \ldots, m$$

determines a linear equivariant singular m-simplex

$$T: (\Delta_m; L_0, \ldots, L_m) \longrightarrow (\pi_n; K_0, \ldots, K_{n-1})$$

if and only if $L_i \subset$ isotropy group of $[w^i, e]$ in $(\pi_n; K_0, \ldots, K_{n-1})$, $i = 0, \ldots, m$

Proof. Say that $v^q \in \Delta_{i_q} - \Delta_{i_q - 1} \subset \Delta_n$, $0 \leq q \leq m$. Thus the isotropy group of $[v^q, e]$ in $(\Delta_n; K_0, \ldots, K_n)$ is K_{i_q}. Let $i_M = \max\{i_0, \ldots, i_q\}$, where $0 \leq M \leq q$. Then $\sum_{j=0}^{q} a_j v^j \in \Delta_{i_M}$, if $\sum_{j=0}^{q} a_j = 1$ and $a_j \geq 0$, $0 \leq j \leq q$. Thus we have isotropy group of $\left[\sum_{j=0}^{q} a_j v^j, e\right]$ in $(\Delta_n; K_0, \ldots, K_n) \supset K_{i_M} \supset L_M \supset L_q$, for all $q = 0, \ldots, m$. Therefore the map $\Delta_m \longrightarrow (\Delta_n; K_0, \ldots, K_n)$ given by $\sum_{j=0}^{m} a_j d^j \longmapsto \left[\sum_{j=0}^{m} a_j v^j, e\right]$ determines a G-map from $(\Delta_m; L_0, \ldots, L_m)$. Similarly for the "prism case." This proves the "if" part, the "only if" part is clear.

q.e.d.

We denote the linear equivariant singular m-simplex

$$T: (\Delta_m; L_0, \ldots, L_m) \longrightarrow (\Delta_n; K_0, \ldots, K_n)$$

given by $T([d^i, e]) = [v^i, e]$, $i = 0, \ldots, m$, by the symbol

$$(L_0, v^0) \ldots (L_m, v^m).$$

For $w^i \in \pi_n \subset (\pi_n; K_0, \ldots, K_n)$, $i = 0, \ldots, m$ the symbol

$$(L_0, w^0) \ldots (L_m, w^m)$$

has the analogous meaning. Thus Lemma 5.7 gives a necessary and sufficient condition for these symbols to be well-defined. Assume that $(L_0, v^0) \ldots (L_m, v^m)$ belongs to $\mathcal{F}$ and let $a \in k(G/L_m)$. With these notations the boundary homomorphism is given by

$$\hat{\partial}_m((L_0, v^0)\ldots(L_m, v^m) \otimes a) = \sum_{i=0}^{m} (-1)^i (L_0, v^0)\ldots\widehat{(L_i, v^i)}\ldots(L_m, v^m) \otimes (p_i)_*(a).$$

The linear equivariant singular simplexes in $(\Delta_n; K_0, \ldots, K_n)$ and $(\pi_n; K_0, \ldots, K_{n-1})$ generate subcomplexes of $\hat{S}^G((\Delta_n; K_0, \ldots, K_n); k)$ and $\hat{S}^G((\pi_n; K_0, \ldots, K_{n-1}); k)$ respectively. We denote these by

$$\hat{S}^G Q((\Delta_n; K_0, \ldots, K_n); k) = \left\{\hat{C}^G_m Q((\Delta_n; K_0, \ldots, K_n)),\ \hat{\partial}_m\right\} \text{ and}$$

$$\hat{S}^G Q((\pi_n; K_0, \ldots, K_{n-1}); k) = \left\{\hat{C}^G_m Q((\pi_n; K_0, \ldots, K_{n-1})),\ \hat{\partial}_m\right\}.$$

The G-maps

$$\bar{\ell}_n : (\Delta_{n-1}; K_0, \ldots, K_{n-1}) \longrightarrow (\pi_n; K_0, \ldots, K_{n-1})$$
$$\bar{u}_n : (\Delta_{n-1}; K_0, \ldots, K_{n-1}) \longrightarrow (\pi_n; K_0, \ldots, K_{n-1})$$

induce chain homomorphisms

$$(\bar{\ell}_n)_\# : \hat{S}^G Q((\Delta_{n-1}; K_0, \ldots, K_{n-1}); k) \longrightarrow \hat{S}^G Q((\pi_n; K_0, \ldots, K_{n-1}); k)$$

and $(u_n)_\#$.

Thus for example

$$(\bar{\ell}_n)_\#((L_0, v^0)\ldots(L_m, v^m)) = (L_0, \ell_n v^0)\ldots(L_m, \ell_n v^m),$$

where $\ell_n : \Delta_{n-1} \longrightarrow \pi_n$.

We now construct a chain homotopy

$$D^n : \hat{S}^G Q((\Delta_{n-1}; K_0, \ldots, K_{n-1}); k) \longrightarrow \hat{S}^G Q((\pi_n; K_0, \ldots, K_{n-1}); k)$$

from $(\bar{u}_n)_\#$ to $(\bar{\ell}_n)_\#$ in the following way. Define the homomorphism

$$D^n_m : \hat{C}^G_m Q((\Delta_{n-1}; K_0, \ldots, K_{n-1}); k) \longrightarrow \hat{C}^G_{m+1} Q((\pi_n; K_0, \ldots, K_{n-1}); k)$$

by the formula

$$D^n_m((L_0, v^0) \ldots (L_m, v^m) \otimes a)$$

$$= \sum_{i=0}^{m} (-1)^i (L_0, \ell_n v^0) \ldots (L_i, \ell_n v^i)(L_i, u_n v^i) \ldots (L_m, u_n v^m) \otimes a.$$

This determines the homomorphism D^n_m. The formula

$$\hat{\partial}_{m+1} D^n_m + D^n_{m-1} \hat{\partial}_m = (\bar{u}_n)_\# - (\bar{\ell}_n)_\#$$

is established by the same calculation as in the standard case (see Eilenberg-Steenrod [6], pp. 164-165). The presence of the elements a and $(p_i)_*(a)$ does not affect the outcome.

The face maps

$$\bar{e}^i_{n-1} : (\Delta_{n-2}; K_0, \ldots, \hat{K}_i, \ldots, K_{n-1}) \longrightarrow (\Delta_{n-1}; K_0, \ldots, K_{n-1})$$

$$\bar{r}^i_n : (\pi_{n-1}; K_0, \ldots, \hat{K}_i, \ldots, K_{n-1}) \longrightarrow (\pi_n; K_0, \ldots, K_{n-1})$$

induce chain homomorphisms

$$(\bar{e}^i_{n-1})_\# : \hat{S}^G Q((\Delta_{n-2}; K_0, \ldots, \hat{K}_i, \ldots, K_{n-1}); k) \longrightarrow \hat{S}^G Q((\Delta_{n-1}; K_0, \ldots, K_{n-1}); k)$$

and $(\bar{r}^i_n)_\#$ correspondingly.

The formula

$$(\bar{r}^i_n)_\# D^{n-1}_m = D^n_m (\bar{e}^i_{n-1})_\#$$

is immediately verified.

We are now ready for

Lemma 5.8. There exists a chain homomorphism

$$\hat{\rho}: (\hat{S}^G P(X; k), \hat{S}^G P(A; k)) \longrightarrow (\hat{S}^G(X; k), \hat{S}^G(A; k))$$

which is the identity on $\hat{S}^G(X; k)$.

Proof. We define the homomorphism

$$\hat{\rho}_n : \hat{C}^G_n P(X; k) \longrightarrow \hat{C}^G_n (X; k)$$

as follows. If T is an equivariant singular n-simplex belonging to $\mathcal{F}$ and $a \in k(G/t(T))$ we define $\hat{\rho}_n(T \otimes a) = T \otimes a$. Let

$$P: (\pi_n; K_0, \ldots, K_{n-1}) \longrightarrow X$$

be an equivariant singular n-prism belonging to $\mathcal{F}$ and $a \in k(G/K_{n-1})$. We then define

$$\hat{\rho}_n(P \otimes a) = (\hat{P}_{\#} D^n_{n-1})((K_0, d^0) \ldots (K_{n-1}, d^{n-1}) \otimes a).$$

Here $\hat{P}_{\#}: \hat{C}^G_n Q((\pi_n; K_0, \ldots, K_{n-1}); k) \longrightarrow \hat{C}^G_n (X; k)$ is the chain map induced by P, and $(K_0, d^0) \ldots (K_{n-1}, d^{n-1})$ is the identity mapping on $(\Delta_{n-1}; K_0, \ldots, K_{n-1})$ and thus $(K_0, d^0) \ldots (K_{n-1}, d^{n-1}) \otimes a$ belongs to $\hat{C}^G_{n-1} Q((\Delta_{n-1}; K_0, \ldots, K_{n-1}); k)$. This determines the homomorphism $\hat{\rho}_n$.

It only remains to verify that $\hat{\rho}$ is a chain map. This is done by the same calculation as in the standard case (see Eilenberg-Steenrod [6], p. 195). It is a straightforward calculation using the fact that D^n is a chain homotopy from $(\bar{u}_n)_{\#}$ to $(\bar{\ell}_n)$ and the fact that the chain homotopies D^n commute with the "face maps."

q.e.d.

Lemma 5.9. The retraction $\hat{\rho}$ of Lemma 5.8 restricts to a retraction

$$\bar{\rho}: (\bar{S}^G P(X; k), \bar{S}^G P(A; k)) \longrightarrow (\bar{S}^G(X; k), \bar{S}^G(A; k))$$

and thus $\hat{\rho}$ induces a retraction

$$\rho: (S^G P(X;k),\ S^G P(A;k)) \longrightarrow (S^G(X;k),\ S^G(A;k)).$$

<u>Proof</u>. Let $P: (\pi_n; K_0, \ldots, K_{n-1}) \longrightarrow X$, $P': (\pi_n; K'_0, \ldots, K'_{n-1}) \longrightarrow X$ and $a \in k(G/K_{n-1})$, $a' \in k(G/K'_{n-1})$, and assume that $P \otimes a \sim P' \otimes a'$.

Let $h: (\Delta_{n-1}; K_0, \ldots, K_{n-1}) \longrightarrow (\Delta_{n-1}; K'_0, \ldots, K'_{n-1})$ be a G-homeomorphism which covers $id: \Delta_{n-1} \longrightarrow \Delta_{n-1}$ and such that $P = P'(id \times h)$ and $(h_{n-1})_*(a) = a'$. Now recall that we have

$$\hat{\rho}_n(P \otimes a) =$$

$$\sum_{i=0}^{n-1} (-1)^i P((K_0, \ell_n d^0) \ldots (K_i, \ell_n d^i)(K_i, u_n d^i) \ldots (K_{n-1}, u_n d^{n-1})) \otimes a.$$

Thus it only remains to show that

$$P((K_0, \ell_n d^0) \ldots (K_i, \ell_n d^i)(K_i, u_n d^i) \ldots (K_{n-1}, u_n d^{n-1})) \otimes a$$

$$\sim P'((K'_0, \ell_n d^0) \ldots (K'_i, \ell_n d^i)(K'_i, u_n d^i) \ldots (K'_{n-1}, u_n d^{n-1})) \otimes a'$$

for $i = 0, \ldots, n-1$.

The G-map

$$(K_0, \ell_n d^0) \ldots (K_i, \ell_n d^i)(K_i, u_n d^i) \ldots (K_{n-1}, u_n d^{n-1})$$
$$: (\Delta_n; K_0, \ldots, K_i, K_i, \ldots, K_{n-1}) \longrightarrow (\pi_n; K_0, \ldots, K_{n-1})$$

is a G-homeomorphism onto its image, that is, it is a G-imbedding, and the same is true for the "prime" version. Therefore the G-map

$$(id \times h): (\pi_n; K_0, \ldots, K_{n-1}) \longrightarrow (\pi_{n-1}; K'_0, \ldots, K'_{n-1})$$

restricted to these images determines a G-homeomorphism

$$\tilde{h}: (\Delta_n; K_0, \ldots, K_i, K_i, \ldots, K_{n-1}) \longrightarrow (\Delta_n; K'_0, \ldots, K'_i, K'_i, \ldots, K'_{n-1})$$

and $\tilde{h}$ covers $id: \Delta_n \longrightarrow \Delta_n$. Since, moreover, $\tilde{h}_{d^n} = h_{d^{n-1}}: G/K_{n-1} \to G/K'_{n-1}$ it follows that $(\tilde{h}_n)_*(a) = (h_{n-1})_*(a) = a'$. Thus $\tilde{h}$ is a G-homeomorphism with the desired properties.

q.e.d.

Proposition 5.10. Two G-homotopic maps

$$f_0, f_1: (X, A) \longrightarrow (Y, B)$$

induce chain homotopic maps

$$(f_0)_\#, (f_1)_\#: S^G(X, A; k) \longrightarrow S^G(Y, B; k).$$

The same is true for the "bar" and "roof" versions.

Proof. Let $F: I \times (X, A) \longrightarrow (Y, B)$ be a G-homotopy from f_0 to f_1. Given an equivariant singular n-simplex belonging to $\mathcal{F}$ in X

$$T: (\Delta_n; K_0, \ldots, K_n) \longrightarrow X$$

we form the equivariant singular (n+1)-prism

$$F(id \times T): (\pi_{n+1}; K_0, \ldots, K_n) \longrightarrow Y.$$

Thus we get a homomorphism

$$\hat{D}_n: \hat{C}^G_n(X; k) \longrightarrow \hat{C}^G_{n+1} P(Y; k)$$

by defining

$$\hat{D}_n(T \otimes a) = (F(id \times T)) \otimes a$$

where $a \in k(G/t(T)) = k(G/t(F(id \times T)))$. The homomorphism $\hat{D}_n$ takes $\hat{C}^G_n(A; k)$ into $\hat{C}^G_{n+1} P(B; k)$. It is clear from the definition that $\hat{D}_n$ restricts to

$$\bar{D}_n: \bar{C}^G_n(X; k) \longrightarrow \bar{C}^G_{n+1} P(Y; k)$$

and hence $\hat{D}_n$ induces

$$D_n : C_n^G(X; k) \longrightarrow C_{n+1}^G P(Y, k).$$

A straightforward computation (compare with Eilenberg-Steenrod [6], p. 196) now shows that

$$\rho_{n+1} D_n : C_n^G(X; k) \longrightarrow C_{n+1}^G(Y; k)$$

is a chain homotopy from $(f_1)_\#$ to $(f_0)_\#$. Moreover, $\rho_{n+1} D_n$ takes $C_n^G(A; k)$ into $C_{n+1}^G(B; k)$. Similarly for the "bar" and "roof" versions.
q. e. d.

This proves the homotopy axiom in Theorem 2.1 and also the homotopy axiom for the theories $\overline{H}_*^G(\ ; k)$ and $\hat{H}_*^G(\ ; k)$.

6. THE EXCISION AXIOM

We shall again in this section use the notion of a linear equivariant singular simplex in $(\Delta_n; K_0, \ldots, K_n)$. We use the same notations as in Section 5. First we construct "subdivision" maps on $\hat{S}^G Q((\Delta_n; K_0, \ldots, K_n); k)$.

Consider the linear equivariant singular m-simplex

$$\sigma = (L_0, v^0) \ldots (L_m, v^m) : (\Delta_m; L_0, \ldots, L_m) \longrightarrow (\Delta_n; K_0, \ldots, K_n)$$

where $v^i \in \Delta_n \subset (\Delta_n; K_0, \ldots, K_n)$, $i = 0, \ldots, m$. The point

$$p_\sigma = \left[(\frac{1}{m+1} v^0 + \ldots + \frac{1}{m+1} v^m), e \right] \in (\Delta_n; K_0, \ldots, K_n)$$

is called the barycenter point of σ. Since the point

$\left[(\frac{1}{m+1}, \ldots, \frac{1}{m+1}), e \right] \in (\Delta_m; L_0, \ldots, L_m)$ has isotropy group L_m it follows that

$$L_m \subset \text{isotropy group of } p_\sigma \text{ in } (\Delta_n; K_0, \ldots, K_n).$$

We define the barycenter of σ to be the linear equivariant singular 0-simplex b_σ, defined by

$$b_\sigma = (L_m, p_\sigma) : (\Delta_0, L_m) \longrightarrow (\Delta_n; K_0, \ldots, K_n).$$

Now given a linear equivariant singular q-simplex

$$\rho = (\overline{L}_0, \overline{v}^0) \ldots (\overline{L}_q, \overline{v}^q) : (\Delta_q; \overline{L}_0, \ldots, \overline{L}_q) \longrightarrow (\Delta_n; K_0, \ldots, K_n)$$

$$\text{with } L_m \subset \overline{L}_q,$$

we define the linear equivariant singular (q+1)-simplex $\rho \cdot b_\sigma$ by

$$\rho \cdot b_\sigma = (\overline{L}_0, \overline{v}^0) \ldots (\overline{L}_q, \overline{v}^q)(L_m, b_\sigma)$$

$$(\Delta_{q+1}; \overline{L}_0, \ldots, \overline{L}_q, L_m) \longrightarrow (\Delta_0; K_0, \ldots, K_n)$$

Thus we see that the symbol $\rho \cdot b_\sigma$ is well-defined whenever $t(\sigma) \subset t(\rho)$.

Let us now for the moment consider the chain complex where the mth chain group is the free abelian group on all linear equivariant singular m-simplexes in $(\Delta_n; K_0, \ldots, K_n)$ and where the boundary is defined in the natural way. To be precise we are considering the case in which the coefficient system is given by $k(G/H) = Z$ for all H (in $\mathcal{F}$) and all induced maps are the identity on Z. We denote this chain complex by

$$\hat{S}^G Q((\Delta_n; K_0, \ldots, K_n)) = \left\{ \hat{C}^G_m Q((\Delta_n; K_0, \ldots, K_n), \hat{\partial}_m) \right\}.$$

Thus we have

$$\rho \cdot b_\sigma \in \hat{C}^G_{q+1} Q((\Delta_n; K_0, \ldots, K_n)).$$

The operation $\cdot\, b_\sigma$ extends uniquely to a homomorphism from the subcomplex of $\hat{S}^G Q((\Delta_n; K_0, \ldots, K_n))$ generated by all linear equivariant singular simplexes whose main type contain $t(\sigma)$. The formulas

$$\hat{\partial}_1(\rho \cdot b_\sigma) = b_\sigma - \rho \qquad \text{for } q = 0$$

$$\hat{\partial}_{q+1}(\rho \cdot b_\sigma) = (\hat{\partial}_q \rho) \cdot b_\sigma + (-1)^{q+1} \rho \qquad \text{for } q \geq 1$$

follow directly from the definitions. By linearity it thus follows that, if c is a linear combination of linear equivariant singular q-simplexes whose main type contain $t(\sigma)$, we have

$$\hat{\partial}_1(c \cdot b_\sigma) = \mathrm{In}(c)\, b_\sigma - c \qquad \text{for } q = 0$$

$$\hat{\partial}_{q+1}(c \cdot b_\sigma) = (\hat{\partial}_q c) \cdot b_\sigma + (-1)^{q+1} c \qquad \text{for } q \geq 1 .$$

Here $\mathrm{In}: \hat{C}_0 Q((\Delta_n; K_0, \ldots, K_n)) \longrightarrow Z$ is the homomorphism determined by $\mathrm{In}(\sigma) = 1$ for each linear equivariant singular 0-simplex σ.

After the above remarks we are now ready to define homomorphisms

$$Sd_m: \hat{C}^G_m Q((\Delta_n; K_0, \ldots, K_n)) \longrightarrow \hat{C}^G_m Q((\Delta_n; K_0, \ldots, K_n))$$

$$R_m: \hat{C}^G_m Q((\Delta_n; K_0, \ldots, K_n)) \longrightarrow \hat{C}^G_{m+1} Q((\Delta_n; K_0, \ldots, K_n))$$

for all m, inductively as follows. For $m = 0$ we define $Sd_0 =$ identity and $R_0 = 0$. Then inductively we define for any linear equivariant singular m-simplex σ, where $m \geq 1$

$$Sd_m(\sigma) = (-1)^m (Sd_{m-1}(\hat{\partial}_m(\sigma))) \cdot b_\sigma$$

$$R_m(\sigma) = (-1)^{m+1}(\sigma - Sd_m(\sigma) - R_{m-1}(\hat{\partial}_m \sigma)) \cdot b_\sigma .$$

Observe that both Sd_m and R_m preserve main types in the following sense. If σ is a linear equivariant singular m-simplex, then $Sd_m(\sigma)$ is a linear combination of linear equivariant singular m-simplexes which all have the same main type as σ, and analogously for $R_m(\sigma)$.

The homomorphisms Sd_m form a chain map Sd, and the homomorphisms R_m form a chain homotopy from id to Sd. That is, we have the formulas

$$\hat{\partial}_m Sd_m = Sd_{m-1} \hat{\partial}_m$$
$$\hat{\partial}_{m+1} R_m + R_{m-1} \hat{\partial}_m = \mathrm{id} - Sd_m .$$

These formulas are proved by induction on m. For $m \leq 0$ they are correct. Let σ be a linear equivariant singular m-simplex, where $m \geq 1$, and assume that the above formulas hold for values $< m$. We then have

$$\hat{\partial}_m Sd_m(\sigma) = \hat{\partial}_m((-1)^m (Sd_{m-1}(\hat{\partial}_m \sigma)) \cdot b_\sigma)$$

$$= (-1)^m((\hat{\partial}_{m-1} Sd_{m-1}(\hat{\partial}_m \sigma)) \cdot b_\sigma + (-1)^m Sd_{m-1}(\hat{\partial}_m \sigma))$$

$$= (-1)^m (Sd_{m-2}(\hat{\partial}_{m-1} \hat{\partial}_m \sigma)) \cdot b_\sigma + Sd_{m-1}(\hat{\partial}_m \sigma)$$

$$= Sd_{m-1}(\hat{\partial}_m \sigma).$$

If $m = 1$ the term $\hat{\partial}_{m-1} Sd_{m-1}(\hat{\partial}_m \sigma)$ should be read as $\mathrm{In}(Sd_0(\hat{\partial}_1 \sigma)) = \mathrm{In}(\hat{\partial}_1 \sigma) = 0$. Also

$$\hat{\partial}_{m+1} R_m(\sigma) = \hat{\partial}_{m+1}((-1)^{m+1}(\sigma - Sd_m(\sigma) - R_{m-1}(\hat{\partial}_m \sigma)) \cdot b_\sigma)$$

$$= (-1)^{m+1}(\hat{\partial}_m(\sigma - Sd_m(\sigma) - R_{m-1}(\hat{\partial}_m \sigma))) \cdot b_\sigma$$
$$+ (-1)^{m+1}(-1)^{m+1}(\sigma - Sd_m(\sigma) - R_{m-1}(\hat{\partial}_m \sigma))$$

$$= (-1)^{m+1}(\hat{\partial}_m \sigma - Sd_{m-1}(\hat{\partial}_m \sigma) - \hat{\partial}_m \sigma + Sd_{m-1}(\hat{\partial}_m \sigma) - R_{m-2}(\hat{\partial}_{m-1} \hat{\partial}_m \sigma)) \cdot b_\sigma$$
$$+ (\sigma - Sd_m(\sigma) - R_{m-1}(\hat{\partial}_m \sigma)) = \sigma - Sd_m(\sigma) - R_{m-1}(\hat{\partial}_m \sigma).$$

Let us now return to consider $\hat{C}^G_m Q((\Delta_n; K_0, \ldots, K_n); k)$. We shall use the following notation. Let $a \in k(G/L)$ and let $c = \sum \sigma_i$ where each σ_i is a linear equivariant singular m-simplex such that $t(\sigma_i) \supset L$. We define $c \otimes p_*(a)$ to be

$$c \otimes p_*(a) = \sum \sigma_i \otimes (p^i)_*(a)$$

where $(p^i)_*: k(G/L) \longrightarrow k(G/t(\sigma_i))$ is induced by the natural projection $p^i: G/L \longrightarrow G/t(\sigma_i)$. Thus $c \otimes p_*(a) \in \hat{C}^G_m((\Delta_n; K_o, \ldots, K_n); k)$. We now define homomorphisms

$$Sd_m: \hat{C}^G_m Q((\Delta_n; K_0, \ldots, K_n); k) \longrightarrow \hat{C}^G_m Q((\Delta_n; K_0, \ldots, K_n); k)$$

$$R_m: \hat{C}^G_m Q((\Delta_n; K_0, \ldots, K_n); k) \longrightarrow \hat{C}^G_{m+1} Q((\Delta_n; K_0, \ldots, K_n); k)$$

by

$$Sd_m(\sigma \otimes a) = Sd_m(\sigma) \otimes a$$

$$R_m(\sigma \otimes a) = R_m(\sigma) \otimes a$$

where σ is a linear equivariant singular m-simplex and $a \in k(G/t(\sigma))$. (The reader should not be confused by the fact that we use the same symbol to denote two different homomorphisms.) Observe that these definitions are well-defined since both Sd_m and R_m preserve main types. This determines the homomorphisms Sd_m and R_m. We claim that the homomorphisms Sd_m form a chain map Sd and that the homomorphisms R_m form a chain homotopy from id to Sd. This is proved by the calculations

$$\hat{\partial}_m Sd_m(\sigma \otimes a) = \hat{\partial}_m((Sd_m \sigma) \otimes a)$$
$$= (\hat{\partial}_m Sd_m \sigma) \otimes p_*(a) = (Sd_{m-1} \hat{\partial}_m \sigma) \otimes p_*(a)$$

$$= Sd_{m-1}((\hat{\partial}_m \sigma) \otimes p_*(a)) = Sd_{m-1}(\hat{\partial}_m(\sigma \otimes a)),$$

$$\hat{\partial}_{m+1} R_m(\sigma \otimes a) = \hat{\partial}_{m+1}((R_m \sigma) \otimes a)$$

$$= (\hat{\partial}_{m+1} R_m \sigma) \otimes p_*(a)$$

$$= (\sigma - Sd_m \sigma - R_{m-1} \hat{\partial}_m \sigma) \otimes p_*(a)$$

$$= \sigma \otimes p_*(a) - (Sd_m \sigma) \otimes p_*(a) - (R_{m-1} \hat{\partial}_m \sigma) \otimes p_*(a)$$

$$= \sigma \otimes a - Sd_m \sigma) \otimes a - R_{m-1}(\hat{\partial}_m \sigma \otimes p_*(a))$$

$$= \sigma \otimes a - Sd_m(\sigma \otimes a) - R_{m-1}(\hat{\partial}_m(\sigma \otimes a)).$$

The reader should observe that the "new" Sd_m's cannot directly be defined by a recursive formula like $Sd_m(\sigma) = (-1)^m (Sd_{m-1}(\hat{\partial}_m \sigma)) \cdot b_\sigma$ due to the fact that the "part" $\sigma^{(m)}$ in $\hat{\partial}_m \sigma$ may have main type strictly greater than the main type of σ. This is the only reason why we had to proceed in the way we did above.

We now define homomorphisms

$$\widehat{Sd}_n : \hat{C}_n^G(X; k) \longrightarrow \hat{C}_n^G(X; k)$$

$$\hat{R}_n : \hat{C}_n^G(X; k) \longrightarrow \hat{C}_{n+1}^G(X; k)$$

in the following way. Let

$$T : (\Delta_n; K_0, \dots, K_n) \longrightarrow X$$

be an equivariant singular n-simplex belonging to $\mathcal{F}$ and $a \in k(G/K_n)$. We then define

$$\widehat{Sd}_n(T \otimes a) = (\hat{T}_\# \ Sd_n)((K_0, d^0) \dots (K_n, d^n) \otimes a)$$

$$\hat{R}_n(T \otimes a) = (\hat{T}_\# \ R_n)((K_0, d^0) \dots (K_n, d^n) \otimes a) .$$

Here $\hat{T}_\# : \hat{C}_q^G Q((\Delta_n; K_0, \dots, K_n); k) \longrightarrow \hat{C}_q^G Q(X; k)$ $(q = n, n+1)$ is the chain map induced by T, and $(K_0, d^0) \dots (K_n, d^n)$ is the identity mapping on

$(\Delta_n; K_0, \ldots, K_n)$. It is easy to see that the homomorphisms $\widehat{Sd}_n$ form a chain map $\widehat{Sd}$ and that the homomorphisms $\hat{R}_n$ form a chain homotopy $\hat{R}$ from id to $\widehat{Sd}$. The proof of this is a formal computation using the fact that both Sd_m and R_m commute with the maps induced by the face maps $\bar{e}_n^i$ on the linear equivariant chain complexes.

Lemma 6.1. The chain map

$$\widehat{Sd}: \hat{S}^G(X; k) \longrightarrow \hat{S}^G(X; k)$$

restricts to

$$\overline{Sd}: \bar{S}^G(X; k) \longrightarrow \bar{S}^G(X; k)$$

and thus $\hat{Sd}$ induces a chain map

$$Sd: S^G(X; k) \longrightarrow S^G(X; k).$$

The corresponding statement for the chain homotopy $\hat{R}$ is true.

Proof. Let $T: (\Delta_n; K_0, \ldots, K_n) \longrightarrow X$, $T': (\Delta_n; K_0', \ldots, K_n') \longrightarrow X$ and $a \in k(G/K_n)$, $a' \in k(G/K_n')$ and assume that $T \otimes a \sim T' \otimes a'$. Thus there exists a G-homeomorphism $h: (\Delta_n; K_0, \ldots, K_n) \longrightarrow (\Delta_n; K_0', \ldots, K_n')$ which covers id: $\Delta_n \longrightarrow \Delta_n$, such that $T = T'h'$ and $(h_n)_*(a) = a'$.

Let us first consider the chain map $\widehat{Sd}$. We have to show that $\widehat{Sd}_n(T \otimes a) - \widehat{Sd}_n(T' \otimes a') \in \bar{C}_n^G(X; k)$, that is, that

$$\hat{T}_\# Sd_n((K_0, d^0) \ldots (K_n, d^n) \otimes a) - \hat{T}'_\# Sd_n((K_0', d^0) \ldots (K_n', d^n) \otimes a')$$

belongs to $\bar{C}_n^G(X; k)$.

Now $Sd_n((K_0, d^0) \ldots (K_n, d^n) \otimes a) = \sum_{j=1}^{(n+1)!} \sigma_j \otimes a$, where each σ_j is a linear equivariant singular n-simplex in $(\Delta_n; K_0, \ldots, K_n)$ which, moreover, is a G-homeomorphism onto its image. Let σ denote of the σ_j's and let σ' be the corresponding one in the expansion for $Sd_n((K'_0,d^0)\ldots(K'_n,d^n)\otimes a')$. Then σ is of the form

$$\sigma = (K_{i_0},v^0)\ldots(K_{i_{n-1}},v^{n-1})(K_n,p)$$
$$: (\Delta_n;K_{i_0},\ldots,K_{i_{n-1}},K_n) \longrightarrow (\Delta_n;K_0,\ldots,K_n)$$

and thus $\sigma' = (K'_{i_0},v^0)\ldots(K'_{i_{n-1}},v^{n-1})(K'_n,p)$, where p is the barycenter in Δ_n. Since h covers $\mathrm{id}: \Delta_n \longrightarrow \Delta_n$ it follows that h restricts to a G-homeomorphism $h|: \mathrm{Im}(\sigma) \longrightarrow \mathrm{Im}(\sigma')$. Thus, since both σ and σ' are G-homeomorphisms onto their images, it follows that h determines a G-homeomorphism

$$\tilde{h}: (\Delta_n;K_{i_0},\ldots,K_{i_{n-1}},K_n) \longrightarrow (\Delta_n;K'_{i_0},\ldots,K'_{i_{n-1}},K'_n)$$

which covers $\mathrm{id}: \Delta_n \longrightarrow \Delta_n$ and such that $\sigma'\tilde{h} = h\sigma$ and moreover, $\tilde{h}_{d^n} = h_p: G/K_n \longrightarrow G/K'_n$. Hence

$$(T\sigma)\otimes a \sim (T'\sigma')\otimes a'$$

for $T\sigma = T'h\sigma = T'\sigma'\tilde{h}$ and $(\tilde{h}_n)_*(a) = (h_n)_*(a) = a'$. This proves our claim, and the statement in Lemma 6.1 has been proved for $\widehat{Sd}$.

The proof of the corresponding result for $\hat{R}$ requires a little more care. Keeping the same notation as above, we have to show that

$$\hat{T}_\# R_n((K_0,d^0)\ldots(K_n d^n)\otimes a) - \hat{T}'_\# R_n((K'_0,d^0)\ldots(K'_n,d^n)\otimes a')$$

belongs to $\overline{C}^G_{n+1}(X;k)$.

Now $R_n((K_0,d^0)\ldots(K_n,d^n)\otimes a) = \sum_j \tau_j \otimes a$, where each τ_j is a linear equivariant singular (n+1)-simplex in $(\Delta_n;K_0,\ldots,K_n)$. Let τ denote one of the τ_j's. Then τ is of the form

$$\tau = (K_{i_0}, v^0)\ldots(K_{i_n}, v^n)(K_n, p)$$
$$: (\Delta_{n+1}; K_{i_0}, \ldots, K_{i_n}, K_n) \longrightarrow (\Delta_n; K_0, \ldots, K_n)$$

and, moreover, it follows by induction from the definition of R_n that we have $i_0 \leq i_1 \leq \cdots \leq i_n$ and that the point $v^j \in (\Delta_n; K_0, \ldots, K_n)$ has isotropy group K_{i_j}, $j = 0, \ldots, n$. The point $p \in (\Delta_n; K_0, \ldots, K_n)$ has isotropy group K_n. From this it follows that the linear map $\tau| : \Delta_{n+1} \longrightarrow \Delta_n$, which is given by $\sum_{i=0}^{n+1} a_i d^i \longmapsto \sum_{i=0}^{n} a_i v^i + a_{n+1} p$, "preserves isotropy groups." Thus τ restricted to one orbit is a G-homeomorphism. Let

$$\tau' = (K'_{i_0}, v^0)\ldots(K'_{i_n}, v^n)(K'_n, p)$$
$$: (\Delta_{n+1}; K'_{i_0}, \ldots, K'_{i_n}, K'_n) \longrightarrow (\Delta_n; K'_0, \ldots, K'_n)$$

be the linear equivariant singular (n+1)-simplex from the expansion for $R_n((K'_0, d^0)\ldots(K'_n, d^n) \otimes a')$ which corresponds to τ. Consider the diagram

$$\begin{array}{ccc}
(\Delta_{n+1}; K_{i_0}, \ldots, K_{i_n}, K_n) & \xrightarrow{\ \tau\ } & (\Delta_n; K_0, \ldots, K_n) \\
\Big\downarrow \tilde{h} & & \Big\downarrow h \\
(\Delta_{n+1}; K'_{i_0}, \ldots, K'_{i_n}, K'_n) & \xrightarrow{\ \tau'\ } & (\Delta_n; K'_0, \ldots, K'_n)
\end{array}$$

Since both τ and τ' induce G-homeomorphisms on the orbits, it follows that there is a unique $\tilde{h}$ which both makes the above diagram commutative and also covers $id: \Delta_{n+1} \longrightarrow \Delta_{n+1}$. Thus if we denote $h([b, e]) = [b, h_2(b)]$, we have

$$\widetilde{h}\left(\left[\sum_{i=0}^{n+1} a_i d^i, e\right]\right) = \left[\sum_{i=0}^{n+1} a_i d^i, h_2\left(\sum_{i=0}^{n} a_i v^i + a_{n+1} p\right)\right].$$

It is not difficult to see that $\widetilde{h}$ is continuous and hence it follows that $\widetilde{h}$ is a G-homeomorphism. Now $\widetilde{h}_{d^{n+1}} = h_p : G/K_n \longrightarrow G/K'_n$. Hence

$$(T\tau)\otimes a \sim (T'\tau')\otimes a'$$

for $T\tau = T'h\tau = (T'\tau')\widetilde{h}$, and $(\widetilde{h}_{n+1})_*(a) = (h_n)_*(a) = a'$.

This proves our claim and completes the proof of Lemma 6.1.

q. e. d.

Exactly as in the case of ordinary singular theory, the subdivision chain map $Sd: S^G(X;k) \longrightarrow S^G(X;k)$ and the chain homotopy $R: S^G(X;k) \longrightarrow S^G(X;k)$ are the crucial ingredients for the proof of the excision axiom. We proceed to give the remaining details.

Definition 6.2. Let $\mathcal{V}$ be a family of G-subsets of the G-space X. An equivariant singular simplex $T: (\Delta_n; K_0, \ldots, K_n) \longrightarrow X$ is said to be in $\mathcal{V}$ if $T((\Delta_n; K_0, \ldots, K_n))$ is contained in at least one of the sets of $\mathcal{V}$.

Clearly all equivariant singular simplexes in X (belonging to $\mathcal{F}$) which are in $\mathcal{V}$ "generate" a subcomplex $\hat{S}^G(X;k;\mathcal{V})$ of the chain complex $\hat{S}^G(X,A;k;\mathcal{V})$ in the obvious way. We then have the inclusion

$$\hat{\eta}: \hat{S}^G(X,A;k;\mathcal{V}) \longrightarrow \hat{S}^G(X,A;k)$$

which restricts to $\bar{\eta}: \bar{S}^G(X,A;k;\mathcal{V}) \longrightarrow \bar{S}^G(X,A;k)$ and $\hat{\eta}$ induces again an inclusion

$$\eta: S^G(X,A;k;\mathcal{V}) \longrightarrow S^G(X,A;k).$$

Denote $\widehat{Sd}^m = \underbrace{\widehat{Sd} \dots \widehat{Sd}}_{m}$, and $\overline{Sd}^m$, Sd^m similarly ($\widehat{Sd}^0 = id$). Thus $\widehat{Sd}^m$ induces Sd^m. Let us also point out that if B is a G-subset of X then both $Int(B) = B^o$ and $\overline{B}$ are G-subsets of X.

<u>Lemma 6.3.</u> Let $\mathcal{V}$ be a family of G-subsets of X such that $X = \bigcup_{B \in \mathcal{V}} B^o$. Let $T: (\Delta_n; K_0, \dots, K_n) \longrightarrow X$ be an equivariant singular n-simplex in X, and $a \in k(G/K_n)$. Then there exists an integer m such that $\widehat{Sd}^m(T \otimes a) \in \hat{S}^G(X; k; \mathcal{V})$.

<u>Proof.</u> Consider the (ordinary) singular n-simplex $T|: \Delta_n \longrightarrow X$. From the corresponding result in the ordinary case we know that there exist m such that $Sd^m(T|) \in S(X; \mathcal{V})$. Here $Sd: S(X) \longrightarrow S(X)$ is the subdivision chain map on the ordinary singular chain complex of X. But since $\mathcal{V}$ is a family of G-subsets, it now follows from the way our $\widehat{Sd}$ is defined that we have $\widehat{Sd}^m(T \otimes a) \in \hat{S}^G(X; k; \mathcal{V})$.

q. e. d.

For any equivariant singular simplex T in X we denote by $m(T)$ the smallest integer such that $\widehat{Sd}^{m(T)}(T \otimes a) \in \hat{S}^G(X; k; \mathcal{V})$. The element a does not affect this situation at all. Clearly we have $m(T^{(i)}) \leq m(T)$. If $T \otimes a \in \hat{S}^G(X; k; \mathcal{V})$ then $m(T) = 0$, and if $T \otimes a \in \hat{S}^G(A; k)$ then also $\widehat{Sd}^{m(T)}(T \otimes a) \in \hat{S}^G(A; k; \mathcal{V})$. Observe that if $T \otimes a \sim T' \otimes a'$ then $m(T) = m(T')$. The following proposition corresponds to Theorem 8.2 on page 197 in Eilenberg-Steenrod [6]. The proof we give follows the proof they give in the Notes, at the end of Chapter VII, and not the proof they give in the text. Note the remark on page 207 in Eilenberg-Steenrod [6].

<u>Proposition 6.4</u>. Let $\mathcal{V}$ be a family of G-subsets of X such that $X = \bigcup_{B \in \mathcal{V}} B^{o}$. Then, for any G-subset A of X, the inclusion

$$\eta : S^{G}(X, A; k; \mathcal{V}) \longrightarrow S^{G}(X, A; k)$$

is a homotopy equivalence. The inclusions $\bar{\eta}$ and $\hat{\eta}$ are also homotopy equivalences.

<u>Proof</u>. Let T be an equivariant singular n-simplex, belonging to $\mathcal{F}$ in X and let $a \in k(G/t(T))$. Define

$$\hat{\tau}(T \otimes a) = \widehat{Sd}^{m(T)}(T \otimes a) + \sum_{i=0}^{n} (-1)^{i} \sum_{j=m(T^{(i)})}^{m(T)-1} \hat{R}\, \widehat{Sd}^{j}(T^{(i)} \otimes (p_i)_*(a))$$

$$\hat{D}(T \otimes a) = \sum_{j=0}^{m(T)-1} \hat{R}\, \widehat{Sd}^{j}(T \otimes a).$$

Observe that $\hat{\tau}(T \otimes a) \in \hat{C}^{G}_{n}(X; k; \mathcal{V})$ and that $\hat{D}(T \otimes a) \in \hat{C}^{G}_{n+1}(X; k)$. This defines homomorphisms

$$\hat{\tau}_n : \hat{C}^{G}_{n}(X, A; k) \longrightarrow \hat{C}^{G}_{n}(X, A; k; \mathcal{V})$$

$$\hat{D}_n : \hat{C}^{G}_{n}(X, A; k) \longrightarrow \hat{C}^{G}_{n+1}(X, A; k).$$

A formal computation shows that

$$\hat{\partial}_{n+1} \hat{D}_n + \hat{D}_{n-1} \hat{\partial}_n = \mathrm{id} - \hat{\eta}_n \hat{\tau}_n .$$

Using this formula and the fact that $\hat{\eta}$ is an inclusion and a chain map, we see that the $\hat{\tau}_n$'s form a chain map. Since $\hat{\tau}\hat{\eta} = \mathrm{id}$ and $\hat{\eta}\hat{\tau}$ is chain homotopic to the identity, $\hat{\tau}$ is a homotopy inverse to $\hat{\eta}$.

Since the maps $\widehat{Sd}$ and $\hat{R}$ restrict to maps $\overline{Sd}$ and $\bar{R}$ and induce maps Sd and R, and since $m(T) = M(T')$ if $T \otimes a \sim T' \otimes a'$, it follows that both $\hat{\tau}$ and $\hat{D}$ restrict to $\bar{\tau}$ and $\bar{D}$ and induce τ and D. Thus τ is a homotopy inverse to η .

q. e. d.

Proposition 6.5. Let (X, A) be a G-pair, and let U be a G-subset of X such that $\bar{U} \subset A^o$. Then the inclusion

$$i: (X-U, A-U) \longrightarrow (X, A)$$

induces a homotopy equivalence

$$i_\#: S^G(X-U, A-U; k) \longrightarrow S^G(X, A; k).$$

The corresponding $\hat{i}_\#$ and $\bar{i}_\#$ are also homotopy equivalences.

Proof. The family $\mathcal{V}$ consisting of the two G-subsets A and X-U satisfies the condition in Proposition 6.4. Since we have

$$S^G(X; k; \mathcal{V}) = S^G(X-U; k) + S^G(A; k)$$

$$S^G(A; k; \mathcal{V}) = S^G(A; k) \quad \text{and}$$

$$S^G(X-U; k) \cap S^G(A; k) = S^G(A-U; k)$$

it follows (by the Noether isomorphism theorem) that

$$j: S^G(X-U, A-U; k) \longrightarrow S^G(X, A; k; \mathcal{V})$$

is an isomorphism, and thus especially a homotopy equivalence. Since $i_\# = \eta j$ where $\eta: S^G(X, A; k; \mathcal{V}) \longrightarrow S^G(X, A; k)$ is a homotopy equivalence by Proposition 6.4, it follows that $i_\#$ is a homotopy equivalence. Similarly for $\hat{i}_\#$ and $\bar{i}_\#$.

q. e. d.

This proves the excision axiom in Theorem 2.1 and also the excision axiom for the theories $\bar{H}^G_*(\ ; k)$ and $\hat{H}^G_*(\ ; k)$.

7. THE DIMENSION AXIOM

Recall that G denotes a good locally compact, Hausdorff topological group. Moreover, $\mathcal{F}$ denotes an orbit type family for G, and k is an arbitrary covariant coefficient system for $\mathcal{F}$ over the ring R. The most natural case is of course the one where $\mathcal{F}$ is the family of all closed subgroups of G.

We shall determine the R-modules $H_n^G(G/H;k)$ for $H \in \mathcal{F}$. From now on we shall not explicitly mention the orbit type family $\mathcal{F}$. It is implicitly assumed that all the closed subgroups we deal with belong to $\mathcal{F}$, and one only has to observe that all the constructions we use do not take us out from $\mathcal{F}$.

Definition 7.1. We define $\hat{C}_n^G \mathrm{Iso}(G/H;k)$ to be the submodule of $\hat{C}_n^G(G/H;k)$ generated by all elements of the form

$$V \otimes a$$

where the equivariant singular n-simplex V is of the type

$$V: (\Delta_n; K, \dots, K) = \Delta_n \times G/K \longrightarrow G/H$$

and, moreover, V is such that the restriction

$$V|: \{x\} \times G/K \longrightarrow G/H$$

is a G-homeomorphism for all $x \in \Delta_n$. As usual $a \in k(G/K)$.

Clearly the modules $\hat{C}_n^G \mathrm{Iso}(G/H;k)$ form a subcomplex $\hat{S}^G \mathrm{Iso}(G/H;k)$ of $\hat{S}^G(G/H;k)$.

We say that an equivariant singular n-simplex V is of type "Iso" if V is as in Definition 7.1.

Given a G-map

$$T: (\Delta_n; K, \ldots, K) = \Delta_n \times G/K \longrightarrow G/H$$

we define the G-map

$$\overline{T}: (\Delta_n; K, \ldots, K) = \Delta_n \times G/K \longrightarrow \Delta_n \times G/H$$

by $\quad \overline{T}(x, gK) = (x, T(x, gK)).$

Thus $\overline{T}$ covers $\mathrm{id}: \Delta_n \longrightarrow \Delta_n$.

Lemma 7.2. An equivariant singular n-simplex of the form $T: \Delta_n \times G/K \longrightarrow G/H$ is of type "Iso" if and only if $\overline{T}: \Delta_n \times G/K \longrightarrow \Delta_n \times G/H$ is a G-homeomorphism.

Proof. If $\overline{T}$ is a G-homeomorphism, then clearly $T = pr_2 \circ \overline{T}$ is of type "Iso." Here $pr_2: \Delta_n \times G/H \longrightarrow G/H$ denotes the projection onto the second factor.

Assume now that T is of type "Iso." Then $\overline{T}: \Delta_n \times G/K \longrightarrow \Delta_n \times G/H$ is a continuous bijection. It remains to show that $\overline{T}^{-1}$ is continuous. In case G is a compact Lie group or a discrete group, the continuity of $\overline{T}^{-1}$ is clear. Assume now that G is abelian. It follows that $H = K$ and G/H is again a topological group. Denote the restriction of T to Δ_n by $\omega: \Delta_n \longrightarrow G/H$. Thus $\overline{T}$ is given by $\overline{T}(a, gH) = (a, (gH)\,\omega(a))$. Hence the inverse $\overline{T}^{-1}$ is given by $\overline{T}^{-1}(a, gH) = (a, (gH)(\omega(a))^{-1})$ which is a continuous map.

q.e.d.

Observe that it follows directly from Definition 7.1 that if $V \otimes a \in \hat{C}_n^G \mathrm{Iso}(G/H; k)$ and $V \otimes a \sim T' \otimes a'$, then also T' is of type "Iso", that is $T' \otimes a' \in \hat{C}_n^G \mathrm{Iso}(G/H; k)$. The inclusion

$$\hat{\eta}: \hat{C}_n^G \mathrm{Iso}(G/H; k) \longrightarrow \hat{C}_n^G (G/H; k)$$

induces again an inclusion

$$\eta: C_n^G \mathrm{Iso}(G/H; k) \longrightarrow C_n^G \mathrm{Iso}(G/H; k)$$

and, moreover, it follows from the above observation that in fact $C_n^G \mathrm{Iso}(G/H; k)$ is a direct summand in $C_n^G (G/H; k)$. We shall show that η is a homotopy equivalence. But first we determine the chain complex $S^G \mathrm{Iso}(G/H; k) = \{C_n^G (G/H; k), \partial_n\}$ completely. Let

$$V: \Delta_n \times G/K \longrightarrow G/H$$

$$V': \Delta_n \times G/K' \longrightarrow G/H$$

be equivariant singular n-simplexes of type "Iso." It follows from Lemma 7.2 that there exists a unique G-homeomorphism $h: \Delta_n \times G/K \longrightarrow \Delta_n \times G/K'$, namely $h = (\overline{V}')^{-1} \overline{V}$, which makes the following diagram commutative

$$\begin{array}{ccccc} \Delta_n \times G/K & \xrightarrow{\overline{V}} & \Delta_n \times G/H & & \\ h \big\downarrow & & \big\downarrow \mathrm{id} & \searrow^{pr_2} & \\ & & & & G/H \\ & & & \nearrow_{pr_2} & \\ \Delta_n \times G/K' & \xrightarrow{\overline{V}'} & \Delta_n \times G/H & & \end{array}$$

Thus it follows that $h = (\overline{V}')^{-1} \overline{V}: \Delta_n \times G/H \longrightarrow \Delta_n \times G/K'$ is a G-homeomorphism which covers $\mathrm{id}: \Delta_n \longrightarrow \Delta_n$ and such that $V = V'h$; moreover, h is the only G-map which satisfies these two conditions. This applies especially when $V: \Delta_n \times G/K \longrightarrow G/H$ and $V' = \pi_n: \Delta_n \times G/H \longrightarrow G/H$ where π_n is the projection onto the second factor. We then have $V = \pi_n \overline{V}$ and thus for any $a \in k(G/K)$ we get

$$V \otimes a \sim \pi_n \otimes (\overline{V}_n)_*(a)$$

where $(\overline{V}_n)_*: k(G/K) \longrightarrow k(G/H)$ (as described in Lemma 3.2)

In order to be able to be very specific, let us introduce one more chain complex. Define C_n^G spec. $(G/H; k) = Z_{\pi_n} \otimes k(G/H)$, and let S^G spec.$(G/H; k) = \{C_n^G \text{ spec. } (G/H; k), \partial_n\}$ be the corresponding chain complex. Here π_n denotes the equivariant singular n-simplex $\pi_n: \Delta_n \times G/H \longrightarrow G/H$, which is projection onto the second factor. Since $\pi_n^{(i)} = \pi_{n-1}$, $i = 0, \ldots, n$, it follows that we have

$$\partial_n(\pi \otimes a) = \sum_{i=0}^{n} (-1)^i \pi_n^{(i)} \otimes a = \begin{cases} \pi_{n-1} \otimes a & n \text{ even, } n \geq 2 \\ 0 & n \text{ odd} \end{cases} .$$

Hence it follows that

$$H_m(S^G \text{ spec. } (G/H; k)) \cong \begin{cases} k(G/H & m = 0 \\ 0 & m \neq 0 \end{cases} .$$

Every element in C_0^G spec. $(G/H; k) = Z_{\pi_0} \otimes k(G/H)$ is a cycle and the zero element is the only boundary. Thus the homomorphism given by $\pi_0 \otimes a \longmapsto a$, $a \in k(G/H)$ gives the wanted isomorphism in degree zero.

We now define a homomorphism

$$\hat{\alpha}_n: \hat{C}_n^G \text{ Iso}(G/H; k) \longrightarrow C_n^G \text{ spec. } (G/H; k)$$

in the following way. Let $V: \Delta_n \times G/K \longrightarrow G/H$ be of type "Iso" and $a \in k(G/K)$. We then define

$$\hat{\alpha}_n(V \otimes a) = \pi_n \otimes (\overline{V}_n)_*(a)$$

where $\pi_n: \Delta_n \times G/H \longrightarrow G/H$ is the projection onto the second factor and $(\overline{V}_n)_*: k(G/K) \longrightarrow k(G/H)$ is as described in Lemma 3.2. Observe that in this special case we have $(\overline{V}_n)_* = (\overline{V}_x)_*$ for any $x \in \Delta_n$, see Lemma 3.2.

From this it follows that the homomorphisms $\hat{\alpha}_n$ form a chain map

$$\hat{\alpha}: \hat{S}^G \operatorname{Iso}(G/H; k) \longrightarrow S^G \text{ spec. } (G/H; k) .$$

Now assume that $V \otimes a \sim V' \otimes a'$ where both $V: \Delta_n \times G/K \longrightarrow G/H$ and $V': \Delta_n \times G/K \longrightarrow G/H$ are of type "Iso." Then we know from earlier remarks that $(((\bar{V}')^{-1}\bar{V})_n)_*(a) = a'$, and hence $(\bar{V}_n)_*(a) = (\bar{V}'_n)_*(a')$. Thus $\hat{\alpha}_n(V \otimes a) = \hat{\alpha}_n(V' \otimes a)$, and it follows that $\hat{\alpha}$ induces a chain map

$$\alpha: S^G \operatorname{Iso}(G/H; k) \longrightarrow S^G \text{ spec. } (G/H; k).$$

Denote by $\{V \otimes a\} \in C_n^G \operatorname{Iso}(G/H; k)$ the image of $V \otimes a \in \hat{C}_n^G \operatorname{Iso}(G/H; k)$ under the natural projection. Thus $\alpha_n(\{V \otimes a\}) = \pi_n \otimes (\bar{V}_n)_*(a)$. We claim that the homomorphism

$$\beta_n: C_n^G \text{ spec. } (G/H; k) \longrightarrow C_n^G \operatorname{Iso}(G/H; k)$$

defined by $\beta_n(\pi_n \otimes b) = \{\pi_n \otimes b\}$, $b \in k(G/H)$, is an inverse to α_n. Since $\bar{\pi}_n = \mathrm{id}: \Delta_n \times G/H \longrightarrow \Delta_n \times G/H$, we have $\alpha_n \beta_n = \mathrm{id}$. Since $V \otimes a \sim \pi_n \otimes (\bar{V}_n)_*(a)$, we have $\{V \otimes a\} = \{\pi_n \otimes (\bar{V}_m)_*(a)\}$, and hence $\beta_n \alpha_n = \mathrm{id}$. This shows that α is an isomorphism of chain complexes.

Thus we have

$$H_m(S^G \operatorname{Iso}(G/H; k)) \cong \begin{cases} k(G/H) & m = 0 \\ 0 & m \neq 0 \end{cases}$$

The isomorphism in degree zero is explicitly described as follows. Let $\{V_0 \otimes a\} \in C_0^G \operatorname{Iso}(G/H; k) = H_0(S^G \operatorname{Iso}(G/H; k))$, where $V_0: G/K \longrightarrow G/H$ is a G-homeomorphism and $a \in k(G/K)$. Then $\{V_0 \otimes a\} \longmapsto (V_0)_*(a) \in k(G/H)$ gives the wanted isomorphism. Here $(V_0)_*: k(G/K) \longrightarrow k(G/H)$ is the isomorphism induced by V_0.

We have now completely determined the chain complex $S^G \mathrm{Iso}(G/H;k)$. The next step is to construct a homotopy inverse to the inclusion

$$\eta: S^G \mathrm{Iso}(G/H;k) \longrightarrow S^G(G/H;k).$$

We proceed to do this now.

Let
$$T: (\Delta_n; K_0, \ldots, K_n) \longrightarrow G/H$$

be an arbitrary equivariant singular n-simplex in G/H. We define

$$\overline{T}: (\Delta_n; K_0, \ldots, K_n) \longrightarrow \Delta_n \times G/H$$

by $\overline{T}([x,g]) = (x, T([x,g]))$. Thus $\overline{T}$ is a G-map and $\overline{T}$ covers $\mathrm{id}: \Delta_n \to \Delta_n$. We shall study the mapping cylinder of $\overline{T}$. We use the notation $M(\overline{T})$ for this mapping cylinder and $\pi: I \times (\Delta_n; K_0, \ldots, K_n) \dot{\cup} \Delta_n \times G/H \to M(\overline{T})$ denotes the natural projection. This time we choose the notation so that $\pi(0,[x,g]) = \pi(\overline{T}([x,g]))$. We shall denote $\pi(t,[x,g]) = \{t,[x,g]\}$ and $\pi(x,gH) = \{x,gH\}$. We consider $(\Delta_n; K_0, \ldots, K_n)$ as a closed G-subset of $M(\overline{T})$ through the G-imbedding $i: (\Delta_n; K_0, \ldots, K_n) \to M(\overline{T})$, given by $i([x,g]) = \{1,[x,g]\}$, and also $\Delta_n \times G/H$ as a closed G-subset of $M(\overline{T})$ through the G-imbedding $j: \Delta_n \times G/H \to M(\overline{T})$ given by $j(x,gH) = \{x,gH\}$. Since both $(\Delta_n; K_0, \ldots, K_n)$ and $\Delta_n \times G/H$ are Hausdorff, it follows that $M(\overline{T})$ is Hausdorff.

We shall show that $M(\overline{T})$ is G-homeomorphic to what we could call an equivariant skew prism. We first describe what we mean by an equivariant skew prism.

Let $L, K_0, \ldots, K_n$ be closed subgroups of G such that $L \supset K_0 \supset \ldots \supset K_n$. We define the equivariant skew n-prism of type

$(L; K_0, \ldots, K_n)$, denoted by

$$(I \times \Delta_n; L; K_0, \ldots, K_n)$$

to be the G-space constructed in the following way. Consider the G-space $I \times \Delta_n \times G$ and define a relation in $I \times \Delta_n \times G$ as follows.

$$(0, x, g) \sim (0, x, g') \iff gL = g'L \in G/L, \qquad \text{for any } x \in \Delta_n$$

$$(t, x, g) \sim (t, x, g') \iff gK_m = g'K_m \in G/K_m, \qquad \text{for } t \neq 0,\ x \in \Delta_m - \Delta_{m-1}.$$

Thus $\sim$ is an equivalence relation in $I \times \Delta_n \times G$, and we define $(I \times \Delta_n; L; K_0, \ldots, K_n) = I \times \Delta_n \times G/\sim$. By $[t, x, g] \in (I \times \Delta_n; L; K_0, \ldots, K_n)$, we denote the image of $(t, x, g) \in I \times \Delta_n \times G$ under the natural projection. It is easy to see that $(I \times \Delta_n; L; K_0, \ldots, K_n)$ is Hausdorff (see Lemma 1.2 in Chapter II) and thus we see in the same way as in Lemma 1.3 in Chapter II that the natural action by G on $(I \times \Delta_n; L; K_0, \ldots, K_n)$ is continuous.

The main step in the proof of the dimension axiom is the construction $M(\overline{T})$ together with the following lemma. We use Palais' "covering homotopy theorem" in the proof of the lemma (for the case, G is a compact Lie group).

<u>Lemma 7.3</u>. Let G be a good locally compact group. Let

$$T: (\Delta_n; K_0, \ldots, K_n) \longrightarrow G/H$$

be a G-map and define

$$\overline{T}: (\Delta_n; K_0, \ldots, K_n) \longrightarrow \Delta_n \times G/H$$

by $\overline{T}([x, g]) = (x, T([x, g]))$. Then there exists a closed subgroup L of G such that $L \supset K_0 \supset \ldots \supset K_n$ and a G-homeomorphism

$$k: (I \times \Delta_n; L; K_0, \ldots, K_n) \longrightarrow M(\overline{T})$$

which covers $id: I \times \Delta_n \longrightarrow I \times \Delta_n$.

<u>Proof</u>. a) Assume that G is a compact Lie group. Define

$$\alpha : I \times \Delta_n \longrightarrow M(\overline{T})$$

to be the composite map

$$I \times \Delta_n \xrightarrow{\alpha'} I \times (\Delta_n; K_0, \ldots, K_n) \xrightarrow{\pi|} M(\overline{T})$$

where

$$\alpha'(t, (a_0, \ldots, a_n)) = (t, [(1-t+ta_0, ta_1, \ldots, ta_n), e]).$$

Thus α is continuous and we have

$$\alpha(t, (a_0, \ldots, a_n)) = \{t, [(1-t+ta_0, ta_1, \ldots, ta_n), e]\}.$$

Observe that every point in $\alpha((0,1] \times (\Delta_m - \Delta_{m-1}))$ has isotropy group K_m, $0 \leq m \leq n$. The set $\alpha(\{0\} \times \Delta_n)$ consists of one point, namely the point $\{0, [d^0, e]\} = \{d^0, g_0H\}$, where $\overline{T}([d^0, e]) = (d^0, T([d^0, e])) = (d^0, g_0H) \in \Delta_n \times G/H$. This point has isotropy group $g_0Hg_0^{-1} = L$. Observe that $L \supset K_0$. Thus α determines an isovariant G-map

$$\overline{\alpha} : (I \times \Delta_n; L; K_0, \ldots, K_n) \longrightarrow M(\overline{T})$$

where $\overline{\alpha}([t, a, g]) = g\alpha(t, a)$. The G-map $\overline{\alpha}$ induces on the orbit spaces the map $\beta : I \times \Delta_n \longrightarrow I \times \Delta_n$ given by

$$\beta(t, (a_0, \ldots, a_n)) = (t, (1-t+ta_0, ta_1, \ldots, ta_n)).$$

Define a homotopy

$$F : I \times (I \times \Delta_n) \longrightarrow I \times \Delta_n$$

by

$$F(s, t, (a_0, \ldots, a_n)) = (t, (1-((1-s)t+s) + ((1-s)t+s)a_0, ((1-s)t+s)a_1, \ldots, ((1-s)t+s)a_n)$$

Thus F is a homotopy from β to $\mathrm{id}: I \times \Delta_n \longrightarrow I \times \Delta_n$. Observe that

$$F(I \times (0,1] \times (\Delta_m - \Delta_{m-1})) \subset (0,1] \times (\Delta_m - \Delta_{m-1})$$
$$F(I \times \{0\} \times \Delta_n) \subset \{0\} \times \Delta_n.$$

Thus it follows that F is an isovariant homotopy. Since $F(0, \ ) = \beta: I \times \Delta_n \longrightarrow I \times \Delta_n$ can be lifted (to $\bar{\alpha}$) it follows by Palais' "covering homotopy theorem" (see Theorem 2.4.1 on page 51 in Palais [13]; the spaces $(I \times \Delta_n; L; K_0, \dots, K_n)$ and $M(\bar{T})$ are second countable) that F can be lifted to an isovariant G-map. Especially it follows that there is an isovariant G-map

$$k: (I \times \Delta_n; L; K_0, \dots, K_n) \longrightarrow M(\bar{T})$$

which covers $F(1, \ ) = \mathrm{id}: I \times \Delta_n \longrightarrow I \times \Delta_n$. Thus k is a G-homeomorphism. This completes the proof in the case a).

b) Assume that G is a discrete group. Define the map

$$\gamma: I \times \Delta_n \longrightarrow M(\bar{T})$$

by $\gamma(t,a) = \{t, [a,e]\}$. Since G/H is discrete, it follows that we have

$$\{0, [a,e]\} = \{\bar{T}([a,e])\} = \{a, T([a,e])\} = \{a, g_0 H\}$$

for some fixed $g_0 H \in G/H$ for all $a \in \Delta_n$. Denote $L = g_0 H g_0^{-1}$. Thus all the points $\{0,[a,e]\} \in M(\bar{T})$, $a \in \Delta_n$ have isotropy group L and $L \supset K_0$. Hence γ determines a G-map

$$\bar{\gamma}: (I \times \Delta_n; L; K_0, \dots, K_n) \longrightarrow M(\bar{T})$$

which is a continuous bijection and which covers $\mathrm{id}: I \times \Delta_n \longrightarrow I \times \Delta_n$. It is easy to see that inverse function $\bar{\gamma}^{-1}$ is continuous, and thus $\bar{\gamma}$ is a G-homeomorphism.

c) Assume that G is an abelian locally compact group. Define the map

$$\gamma: I \times \Delta_n \longrightarrow M(\overline{T})$$

by $\gamma(t, a) = \{t, [a, e]\}$. Since G is abelian, it follows that all points of the form $\{0, [a, e]\} = \{\overline{T}([a, e])\} \in M(\overline{T})$ have isotropy group H, and $H \supset K_0$. thus γ determines a G-map

$$\overline{\gamma}: (I \times \Delta_n; H; K_0, \ldots, K_n) \longrightarrow M(\overline{T})$$

which is a continuous bijection and which covers $id: I \times \Delta_n \longrightarrow I \times \Delta_n$. It remains to show that $\overline{\gamma}^{-1}$ is continuous. Consider the diagram

$$\begin{array}{ccc} I \times (\Delta_n; K_0, \ldots, K_n) \dot{\cup} \Delta_n \times G/H & \xleftarrow{id \dot{\cup} \sigma} & I \times (\Delta_n; K_0, \ldots, K_n) \dot{\cup} \Delta_n \times G/H \\ \kappa \downarrow & & \downarrow \pi \\ (I \times \Delta_n; H; K_0, \ldots, K_n) & \xrightarrow{\overline{\gamma}} & M(\overline{T}) \end{array}$$

Here $\sigma(a, gH) = (a, (gH)(\omega(a))^{-1})$ where $\omega: \Delta_n \longrightarrow G/H$ is given by $\omega(a) = T([a, e])$. The G-map κ is defined by

$$\kappa(t, [a, g]) = [t, a, g]$$

$$\kappa(a, gH) \quad = [0, a, g]$$

Since $\overline{\gamma}\,\kappa(a, gH) = \{a, T(a, gH)\} = \{a, (gH)\,T[a, e]\} = \{a, (gH)\,\omega(a)\}$, it follows that the above diagram commutes. Thus $\overline{\gamma}^{-1}$ is continuous since p is a quotient map.

q.e.d.

We are now ready to construct a homotopy inverse to the inclusion

$$\eta: S^G\, Iso(G/H; k) \longrightarrow S^G(G/H; k).$$

First we define homomorphisms

$$\hat{\varphi}_n : \hat{C}^G_n(G/H;k) \longrightarrow C^G_n \operatorname{Iso}(G/H;k), \quad \text{and}$$

$$\hat{\Phi}_n : \hat{C}^G_n(G/H;k) \longrightarrow C^G_{n+1}(G/H;k) .$$

Let $\quad T \otimes a \in C^G_n(G/H;k)$, where

$$T : (\Delta_n; K_0, \ldots, K_n) \longrightarrow G/H$$

and $a \in k(G/H)$. Form $M(\overline{T})$ and let

$$k : (I \times \Delta_n; L; K_0, \ldots, K_n) \longrightarrow M(\overline{T})$$

be a G-homeomorphism which covers $\text{id}: I \times \Delta_n \longrightarrow I \times \Delta_n$, the existence of which is given by Lemma 7.3. We have the commutative diagram

$$(*) \qquad \begin{array}{ccccccc}
(\Delta_n; K_0, \ldots, K_n) & \xrightarrow[\cong]{k_1} & (\Delta_n; K_0, \ldots, K_n) & \overset{\overline{T}}{\searrow} & & \overset{T}{\curvearrowright} & \\
{\scriptstyle i_1}\downarrow & & \downarrow{\scriptstyle i} & & & & \\
(I \times \Delta_n; L; K_0, \ldots, K_n) & \xrightarrow[\cong]{k} & M(\overline{T}) & \xrightarrow{r} & \Delta_n \times G/H & \xrightarrow{pr_2} & G/H \\
{\scriptstyle i_0}\uparrow & & \uparrow{\scriptstyle j} & \overset{\text{id}}{\nearrow} & & & \\
\Delta_n \times G/L & \xrightarrow[\cong]{k_0} & \Delta_n \times G/H & & & &
\end{array}$$

The meaning of all the maps in the diagram is clear.

Denote $\tilde{a} = ((k_1)_n)_*^{-1}(a)$, where $((k_1)_n)_* : k(G/K_n) \longrightarrow k(G/K_n)$ is the isomorphism determined by k_1 as described in Lemma 3.2. Denote

$$D = pr_2\, r\, k : (I \times \Delta_n; K_0, \ldots, K_n) \longrightarrow G/H.$$

Using these notations, we now define

$$\hat{\varphi}_n(T \otimes a) = \{(D\, i_0) \otimes p_*(\tilde{a})\}$$

where $p_* : k(G/K_n) \longrightarrow k(G/L)$ is induced by the natural projection $p: G/K_n \longrightarrow G/L$. Observe that

$$D\,i_0 = pr_2\,k_0 \colon \Delta_n \times G/L \longrightarrow G/H$$

is of type "Iso" and thus

$$\{(D\,i_0) \otimes p_*(\tilde{a})\} \in C_n^G\ \mathrm{Iso}(G/H; k).$$

We have to show that this definition of $\hat{\varphi}_n(T \otimes a)$ is independent of the choice of k. Let $k' \colon (I \times \Delta_n; L'; K_0, \ldots, K_n) \longrightarrow M(\overline{T})$ be another G-homeomorphism which covers $id \colon I \times \Delta_n \longrightarrow I \times \Delta_n$. We then have the commutative diagram

$$\begin{array}{ccccc}
\Delta_n \times G/L & \xrightarrow{\ i_0\ } & (I \times \Delta_n; L; K_0, \ldots, K_n) & \overset{k}{\searrow} & \\
{\scriptstyle f|}\downarrow & & \downarrow{\scriptstyle f} & & M(\overline{T}) \\
\Delta_n \times G/L' & \xrightarrow{\ i_0'\ } & (I \times \Delta_n; L'; K_0, \ldots, K_n) & \underset{k'}{\nearrow} &
\end{array}$$

where $f = (k')^{-1}k$ is a G-homeomorphism which covers $id \colon I \times \Delta_n \longrightarrow I \times \Delta_n$, and thus $f| \colon \Delta_n \times G/L \longrightarrow \Delta_n \times G/L'$ is a G-homeomorphism which covers $id \colon \Delta_n \longrightarrow \Delta_n$.

Denote $D' = pr_2\, r\, k'$, $\tilde{a}' = ((k_1')_n)_*^{-1}(a)$, and let $p' \colon G/K_n \longrightarrow G/L'$ be the natural projection. We must show that

$$(D\,i_0) \otimes p_*(\tilde{a}) \sim (D' i_0') \otimes p_*'(\tilde{a}').$$

Since we have $D\,i_0 = D' i_0'(f|)$, it only remains to show that $(f|)_*(p_*(\tilde{a})) = p_*'(\tilde{a}')$.

Consider the diagram

$$\begin{array}{ccc}
G/K_n & \xrightarrow{\ f_{(1,d^n)}\ } & G/K_n \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\
G/L & \xrightarrow{\ (f|)_{d^n}\ } & G/L'
\end{array}$$

where $f_{(1,d^n)}$ denotes f restricted to the orbit over $(1, d^n) \in I \times \Delta_n$, and $(f|)_{d^n}$ has the analogous meaning. Exactly in the same way as in Lemma 3.2 we see that this diagram is G-homotopy commutative. Since $f_{(1,d^n)} = (k_1')^{-1}_{d^n} (k_1)_{d^n}$, we have

$(f|)_* p_*(\widetilde{a}) = p_*'(f_{(1,d^n)})_*(\widetilde{a}) = p_*'((k_1')_n)_*^{-1}(a) = p_*'(\widetilde{a}')$. Thus the definition we have given for $\hat{\varphi}_n(T \otimes a)$ is well-defined. This determines the homomorphism $\hat{\varphi}_n : \hat{C}_n^G(G/H; k) \longrightarrow C_n^G \operatorname{Iso}(G/H; k)$. It now follows immediately that the homomorphisms $\hat{\varphi}_n$ form a chain map

$$\hat{\varphi} : \hat{S}^G(G/H; k) \longrightarrow S^G \operatorname{Iso}(G/H; k).$$

Next we define homomorphisms

$$\hat{\Phi}_n : C_n^G(G/H; k) \longrightarrow C_{n+1}^G(G/H; k)$$

and show that they form a chain homotopy from the natural projection $\hat{p} : \hat{C}_n^G(G/H; k) \longrightarrow C_n^G(G/H; k)$ to $\eta\hat{\varphi}_n : \hat{C}_n^G(G/H; k) \longrightarrow C_n^G(G/H; k)$. Recall the diagram (*). Denote $\ell a = (0, a) \in I \times \Delta_n$ and $ua = (1, a) \in (I \times \Delta_n)$ for $a \in \Delta_n$. We use the following linear equivariant singular (n+1)-simplexes in $(I \times \Delta_n; L; K_0, \ldots, K_n)$. Let

$$(\ell d^0, L) \ldots (\ell d^i, L)(ud^i, K_i) \ldots (ud^n, K_n)$$
$$:(\Delta_{n+1}; \underbrace{L, \ldots, L}_{i+1}, K_i, \ldots, K_n) \longrightarrow (I \times \Delta_n; L; K_0, \ldots, K_n)$$

$0 \leq i \leq n$ be the G-map which is determined by the condition that it restricts to a map $\Delta_{n+1} \longrightarrow I \times \Delta_n$ and that this restriction is the linear map given by

$$\sum_{j=0}^{n+1} a_j d^j \longmapsto \sum_{j=0}^{i} a_j \ell d^j + \sum_{j=i}^{n} a_{j+1} u d^j .$$

Observe that each $(\ell d^0, L)\ldots(\ell d^i, L)(ud^i, K_i)\ldots(ud^n, K_n)$ is a G-homeomorphism onto its image. We now define

$$\hat{\Phi}_n(T \otimes a)$$
$$= \left\{ \sum_{i=0}^{n} (-1)^i D((\ell d^0, L)\ldots(\ell d^i, L)(ud^i, K_i)\ldots(ud^n, K_n)) \otimes \tilde{a} \right\} .$$

We have to show that this definition of $\hat{\Phi}(T \otimes a)$ is independent of the choice of k. Let $k' : (I \times \Delta_n; L'; K_0, \ldots, K_n) \longrightarrow M(\overline{T})$ be another G-homeomorphism which covers $\mathrm{id}: I \times \Delta_n \longrightarrow I \times \Delta_n$, and denote $D' = pr_2 \, r \, k'$ and $\tilde{a}' = ((k_1')_n)_*^{-1}(a)$ as before. An argument completely analogous to the one in "the case $\hat{\varphi}_n(T \otimes a)$" now shows that

$$D(\ell d^0, L)\ldots(\ell d^i, L)(ud^i, K_i)\ldots(ud^n, K_n) \otimes \tilde{a}$$
$$\sim D'(\ell d^0, L')\ldots(\ell d^i, L')(ud^i, K_i)\ldots(ud^n, K_n) \otimes \tilde{a}'$$

for $i = 0, \ldots, n$. Thus $\hat{\Phi}_n(T \otimes a)$ is well-defined. This determines the homomorphism $\hat{\Phi}_n : \hat{C}_n^G(G/H; k) \longrightarrow C_{n+1}^G(G/H; k)$.

We now compute $\partial_{n+1} \hat{\Phi}_n(T \otimes a)$.

We have

$$\partial_{n+1} \hat{\Phi}_n(T \otimes a)$$
$$= \left\{ \sum_{i=0}^{n} (-1)^i \left[\sum_{j=0}^{i=1} (-1)^j D(\ell d^0, L)\ldots\widehat{(\ell d^j, L)}\ldots(\ell d^i, L)(ud^i, K_i)\ldots(ud^n, K_n) \otimes \tilde{a} \right] \right.$$
$$+ (-1)^i D(\ell d^0, L)\ldots(\ell d^{i-1}, L)(ud^i, K_i)\ldots(ud^n, K_n) \otimes \tilde{a}$$
$$+ (-1)^{i+1} D(\ell d^0, L)\ldots(\ell d^i)(ud^{i+1}, K_{i+1})\ldots(ud^n, K_n) \otimes p_*(\tilde{a})$$

$$+ \sum_{j=i+1}^{n} (-1)^{j+1} D(\ell d^0, L)\dots(\ell d^i, L)(ud^i, K_i)\dots\widehat{(ud^j, K_j)}\dots(ud^n, K_n)\otimes p_*(\tilde{a})\Big]\Big\}.$$

The sum over i of the two middle lines equals

$$\{D(ud^0, K_0)\dots(ud^n, K_1)\otimes\tilde{a} - D(\ell d^0, L)\dots(\ell d^n, L)\otimes p_*(\tilde{a})\}$$
$$= \{(D\, i_1)\otimes\tilde{a}\} - \{(D\, i_0)\otimes p_*(\tilde{a})\}.$$

Since $D\, i_1 = T k_1$ (see diagram (*)), it follows directly that $T\otimes a \sim (D\, i_1)\otimes \tilde{a}$, and thus this expression equals $\{T\otimes a\} - \eta\hat{\varphi}_n\{T\otimes a\}$. Since we already showed that the definition of $\hat{\Phi}_n$ is independent of the choice of the G-homeomorphism k, it follows that in forming the expression for $\hat{\Phi}_{n-1}(T^{(i)}\otimes(p_i)_*(a))$ $i = 0, \dots, n$, we can use diagram (*) restricted to the appropriate face. Thus we see that the remaining double sum in the expression for $\partial_{n+1}\hat{\Phi}_n(T\otimes a)$ above equals (change the order of summation) $-\hat{\Phi}_{n-1}\hat{\partial}_n(T\otimes a)$. This shows that the homomorphisms

$$\hat{\Phi}_n : \hat{C}_n^G(G/H; k) \longrightarrow C_{n+1}^G(G/H; k)$$

form a chain homotopy from $\hat{p}$ to $\eta\hat{\varphi}$

It now remains to show that $\hat{\varphi}_n$ and $\hat{\Phi}_n$ induce homomorphisms

$$\varphi_n : C_n^G(G/H; k) \longrightarrow C_n^G \mathrm{Iso}(G/H; k), \quad \text{and}$$

$$\Phi_n : C_n^G(G/H; k) \longrightarrow C_{n+1}^G(G/H; k).$$

Thus assume that $T\otimes a \sim T'\otimes a'$ where

$$T : (\Delta_n; K_0, \dots, K_n) \longrightarrow G/H, \quad T' : (\Delta_n; K_0', \dots, K_n') \longrightarrow G/H$$

and $a \in k(G/K_n)$, $a' \in k(G/K_n')$. Let $h : (\Delta_n; K_0, \dots, K_n) \longrightarrow (\Delta_n; K_0', \dots, K_n')$ be a G-homeomorphism which covers $\mathrm{id} : \Delta_n \longrightarrow \Delta_n$, and such that $T = T'h$ and $(h_n)_*(a) = a'$. Thus $\overline{T} = \overline{T}'h$ and therefore h induces a G-homeomorphism

$$\bar{h}\colon M(\bar{T}) \longrightarrow M(\bar{T}')$$

which covers $id\colon I \times \Delta_n \longrightarrow I \times \Delta_n$, and such that the diagram

$$\begin{array}{ccc} M(\bar{T}) & \xrightarrow{\;r\;} & \\ \bar{h}\downarrow & & \Delta_n \times G/H \\ M(\bar{T}') & \xrightarrow{\;r'\;} & \end{array}$$

commutes. We also get the commutative diagram

$$\begin{array}{ccc} (I \times \Delta_n; L; K_0, \ldots, K_n) & \xrightarrow{\;k\;} & M(\bar{T}) \\ f\downarrow & & \downarrow \bar{h} \\ (I \times \Delta_n; L'; K_0', \ldots, K_n') & \xrightarrow{\;k'\;} & M(\bar{T}') \end{array}$$

where by definition $f = (k')^{-1}\,\bar{h}k$, and thus f is a G-homeomorphism which covers $id\colon I \times \Delta_n \longrightarrow I \times \Delta_n$. Using these two commutative diagrams, it is easily seen that we have

$$\hat{\varphi}_n(T \otimes a) = \hat{\varphi}_n(T' \otimes a') \in C_n^G \operatorname{Iso}(G/H; k) \quad \text{and}$$

$$\hat{\Phi}_n(T \otimes a) = \hat{\Phi}_n(T' \otimes a') \in C_{n+1}^G(G/H; k)\ .$$

We now have

$$\partial_{n+1} \Phi_n + \Phi_{n-1} \partial_n = id - \eta_n \varphi_n .$$

Thus the chain map

$$\eta\varphi\colon S^G(G/H; k) \longrightarrow S^G(G/H; k)$$

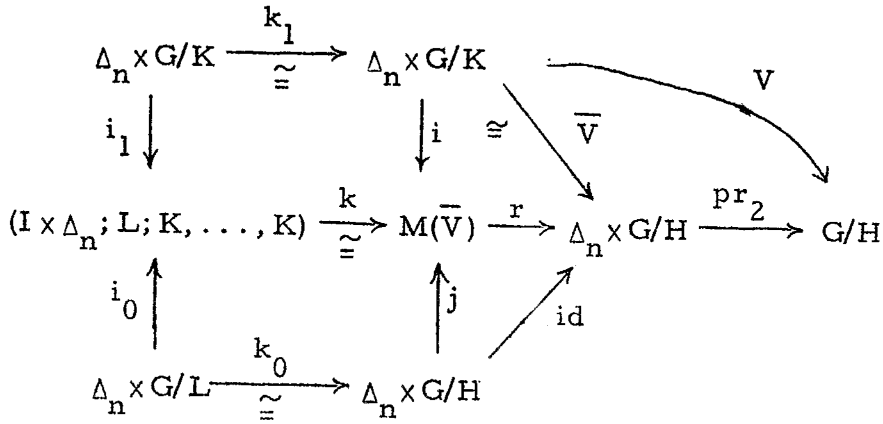
induces the identity on homology.

We claim that

$$\varphi\eta = id\colon S^G \operatorname{Iso}(G/H; k) \longrightarrow S^G \operatorname{Iso}(G/H; k).$$

Let $\{V \otimes a\} \in C_n^G \mathrm{Iso}(G/H; k)$, where $V\colon \Delta_n \times G/K \longrightarrow G/H$ is of type "Iso" and $a \in k(G/K)$. In this case the diagram (*) becomes

(**)



Now $\varphi_n\{V \otimes a\} = \{(pr_2 k_0) \otimes p_*(\tilde{a})\}$, where $\tilde{a} = ((k_1)_n)_*^{-1}(a)$ and $p\colon G/K \longrightarrow G/L$ is the natural projection, $L \supset K$. We have

$$\{V \otimes a\} = \{pr_2 \overline{V} k_1 \otimes \tilde{a}\}.$$

We have

$$pr_2 \overline{V} k_1 = (pr_2 k_0)(k_0)^{-1} \overline{V} k_1$$

and $(k_0)^{-1} \overline{V} k_1\colon \Delta_n \times G/K \longrightarrow \Delta_n \times G/L$ is a G-homeomorphism which covers $\mathrm{id}\colon \Delta_n \longrightarrow \Delta_n$. We claim that

$$((k_0)^{-1} \overline{V} k_1)_*(\tilde{a}) = p_*(\tilde{a}).$$

This follows from the fact that the diagram

$$\begin{array}{ccc}
\{d^n\} \times G/K & \xrightarrow{k_1|} & \{d^n\} \times G/K \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \overline{V}|} \\
\{d^n\} \times G/L & \xrightarrow{k_0|} & \{d^n\} \times G/H
\end{array}$$

is G-homotopy commutative. This again is proved using diagram (**) in exactly the same way as Lemma 3.2 was proved. Thus

$\{V \otimes a\} = \{(pr_2 \overline{V} k_1) \otimes \tilde{a}\} = \{(pr_2 k_0) \otimes p_*(\tilde{a})\} = \varphi_n\{V \otimes a\}$. This shows that $\varphi\eta = \mathrm{id}$. Hence the chain map

$$\varphi\colon S^G(G/H; k) \longrightarrow S^G \, \mathrm{Iso}(G/H; k)$$

induces isomorphism on homology. Thus we have

$$H_m(S^G(G/H; k)) \cong \begin{cases} k(G/H & m = 0 \\ 0 & m \neq 0 \end{cases}.$$

It is easily seen that the explicit isomorphism in degree zero is given as follows.

Let $\{T_0 \otimes a\} \in C_0^G(G/H; k)$, where $T_0\colon G/K \longrightarrow G/H$. Then

$$\{T_0 \otimes a\} \longmapsto (T_0)_*(a) \in G/H$$

where $(T_0)_*\colon k(G/K) \longrightarrow k(G/H)$ is the homomorphism induced by T_0.

Denote this isomorphism by

$$\gamma\colon H_0^G(G/H; k) \longrightarrow k(G/H)$$

From the above explicit expression for γ it follows at once that it commutes with homomorphisms induced from G-maps as claimed in the statement of the "dimension axiom." This completes the proof of the dimension axiom in Theorem 2.1.

This completes the proof of Theorem 2.1.

8. EQUIVARIANT SINGULAR COHOMOLOGY

To construct equivariant singular cohomology we take the "dual" of the chain complex which gave us equivariant singular homology. But in our situation with coefficient systems and "all the rest" this requires some more elaboration than just saying "take Hom (,)."

We proceed as follows. Let R be a ring with unit. By ℓ we denote a contravariant coefficient system for some orbit type family $\mathcal{F}$ over the ring R. From now on we shall not usually mention the orbit type family $\mathcal{F}$. It is implicitly assumed that all the closed subgroups we deal with belong to $\mathcal{F}$. Thus, for example, all equivariant singular n-simplexes are assumed to be equivariant singular n-simplexes which belong to $\mathcal{F}$.

Let X be a G-space. Denote

$$\hat{C}_n^G(X) = \sum_T \oplus Z_T$$

where the direct sum is over all equivariant singular n-simplexes in X. That is, $\hat{C}_n^G(X)$ is the free abelian group on all equivariant singular n-simplexes in X. The boundary homomorphism

$$\hat{\partial}_n : \hat{C}_n^G(X) \longrightarrow \hat{C}_{n-1}^G(X)$$

is defined by

$$\hat{\partial}_n(T) = \sum_{i=0}^{n} (-1)^i T^{(i)} .$$

Then $\hat{\partial}_{n-1}\hat{\partial}_n = 0$, and we thus have the chain complex

$$\hat{S}^G(X) = \{\hat{C}_n^G(X), \hat{\partial}_n\}.$$

That is,

$$\hat{S}^G(X) = \hat{S}^G(X; k)$$

where k is the covariant coefficient system for which $k(G/H) = Z$ for every closed subgroup H of G and all the induced homomorphisms are the identity on Z.

Denote

$$L = \sum_H \oplus \ell(G/H)$$

where the direct sum is over all closed subgroups H in G.

Definition 8.1. We define the R-module $\hat{C}^n_G(X;\ell)$ by

$$\hat{C}^n_G(X;\ell) = \mathrm{Hom}_t(\hat{C}^G_n(X), L).$$

Here $\mathrm{Hom}_t(\hat{C}^G_n(X), L)$ consists of all homomorphisms of abelian groups

$$c: \hat{C}^G_n(X) = \sum_T \oplus Z_T \longrightarrow \sum_H \oplus \ell(G/H) = L$$

which satisfy the condition

$$c(T) \in \ell(G/t(T))$$

for every equivariant singular n-simplex T in X. The R-module structure in L makes $\mathrm{Hom}_Z(\hat{C}^G_n(X), L)$ into an R-module, and $\mathrm{Hom}_t(\hat{C}^G_n(X), L)$ is an R-submodule of that module.

Definition 8.2. Let

$$\hat{\alpha}: \hat{C}^G_n(X) \longrightarrow \hat{C}^G_m(Y)$$

be a homomorphism. We say that $\hat{\alpha}$ is "type increasing" if the following condition is satisfied.

The homomorphism α determines a natural number $q \geq 0$, and $q+1$ integers m_j, $j=0,\ldots,q$ and order preserving functions $\alpha_j: \{0,\ldots,m\} \longrightarrow \{0,\ldots,n\}$ (i.e., $a \leq b \Longrightarrow \alpha_j(a) \leq \alpha_j(b)$) $j=0,\ldots,q$, such that if

$$T: (\Delta_n; K_0,\ldots,K_n) \longrightarrow X$$

then

$$\hat{\alpha}(T) = \sum_{j=0}^{q} m_j S_j, \qquad m_j \in Z$$

where each S_j is an equivariant singular m-simplex in Y of the form

$$S_j : (\Delta_m; K_{\alpha_j(0)}, \ldots, K_{\alpha_j(m)}) \longrightarrow Y.$$

In particular, we have

$$t(T) \subset t(S_j) \qquad j=0,\ldots,q.$$

The boundary homomorphism

$$\hat{\partial}_n : \hat{C}^G_n(X) \longrightarrow \hat{C}^G_{n-1}(X)$$

is "type increasing." In this case we have $q = n$ and

$\alpha_j : \{0,\ldots,n-1\} \longrightarrow \{0,\ldots,n\}$ is given by $\alpha_j(a) = a$ for $0 \leq a < j$,

$\alpha_j(a) = a+1$ for $j \leq a \leq n-1$.

For any G-map $f: X \longrightarrow Y$ the induced homomorphism

$$\hat{f}_\# : \hat{C}^G_n(X) \longrightarrow \hat{C}^G_n(Y)$$

is "type increasing." In this case $q=0$ and $\alpha_0 = \mathrm{id}: \{0,\ldots,n\} \longrightarrow \{0,\ldots,n\}$.

<u>The dual of a "type increasing" homomorphism</u>

Let $\hat{\alpha} : \hat{C}^G_n(X) \longrightarrow \hat{C}^G_m(Y)$ be a "type increasing" homomorphism.

We shall define a homomorphism

$$\hat{\alpha}^\# : \hat{C}^m_G(Y; \ell) \longrightarrow \hat{C}^n_G(X; \ell)$$

which we call the dual of $\hat{\alpha}$.

Let $c \in \hat{C}^m_G(Y; \ell) = \mathrm{Hom}_t(\hat{C}^G_m(Y), L)$. The homomorphism

$$\hat{\alpha}^\#(c) \in \mathrm{Hom}_t(\hat{C}^G_n(X), L) = \hat{C}^n_G(X; \ell)$$

is defined as follows.

Let T be an equivariant singular n-simplex in X and

$$\hat{\alpha}(T) = \sum_{j=0}^{q} m_j S_j, \qquad m_j \in Z$$

where each S_j is an equivariant singular m-simplex in Y. We then define

$$(\hat{\alpha}^{\#}(c))(T) = \sum_{j=0}^{q} m_j (p_j)^* c(S_j) \in \ell(G/t(T))$$

where $(p_j)^*: \ell(G/t(S_j)) \longrightarrow \ell(G/t(T))$ is the homomorphism induced by the natural projection $p_j: G/t(T) \longrightarrow G/t(S_j)$, $j=1, \ldots, q$.

$$\hat{\alpha}^{\#}(c) \in \mathrm{Hom}_t(\hat{C}_n^G(X), L) = \hat{C}_G^n(X; \ell).$$

Since the homomorphisms $(p_j)^*$ above are homomorphisms of R-modules, it follows that $\hat{\alpha}^{\#}: \hat{C}_G^m(Y; \ell) \longrightarrow \hat{C}_G^n(X; \ell)$ is a homomorphism of R-modules.

<u>Lemma 8.3</u>. Let $\hat{\alpha}: \hat{C}_n^G(X) \longrightarrow \hat{C}_m^G(Y)$ and $\hat{\beta}: \hat{C}_m^G(Y) \longrightarrow \hat{C}_p^G(Y')$ be "type increasing" homomorphisms. Then $\hat{\beta}\hat{\alpha}$ is "type increasing" and

$$(\hat{\beta}\hat{\alpha})^{\#} = \hat{\alpha}^{\#}\hat{\beta}^{\#}.$$

Also $\mathrm{id}^{\#} = \mathrm{id}$ and $0^{\#} = 0$.

<u>Proof</u>. The proof is clear.

q. e. d.

The boundary homomorphism $\hat{\partial}_n: \hat{C}_n^G(X) \longrightarrow \hat{C}_{n-1}^G(X)$ is "type increasing." We denote its dual by

$$\hat{\delta}^{n-1} = (\hat{\partial}_n)^{\#}: \hat{C}_G^{n-1}(X; \ell) \longrightarrow \hat{C}_G^n(X; \ell).$$

Then $\hat{\delta}^n \hat{\delta}^{n-1} = (\hat{\partial}_{n+1})^{\#}(\hat{\partial}_n)^{\#} = (\hat{\partial}_n \hat{\partial}_{n+1})^{\#} = 0^{\#} = 0$, and thus we get the cochain complex

$$\hat{S}_G^*(X; \ell) = \{\hat{C}_G^n(X; \ell),\ \hat{\delta}_n\}.$$

Let $f: X \longrightarrow Y$ be a G-map. Then the induced homomorphism $\hat{f}_{\#}: \hat{C}_n^G(X) \longrightarrow \hat{C}_n^G(Y)$ is "type increasing." We denote its dual by

$$\hat{f}^{\#}: \hat{C}_G^n(Y; \ell) \longrightarrow \hat{C}_G^n(X; \ell).$$

These homomorphisms commute with the coboundary and thus form a homomorphism of cochain complexes

$$\hat{f}^{\#}: \hat{S}^{*}_{G}(Y; \ell) \longrightarrow \hat{S}^{*}_{G}(X; \ell).$$

Let (X, A) be a G-pair. We have the inclusion

$$\hat{i}_{\#}: \hat{C}^{G}_{n}(A) \longrightarrow \hat{C}^{G}_{n}(X)$$

and also the homomorphism

$$\hat{\alpha}: \hat{C}^{G}_{n}(X) \longrightarrow \hat{C}^{G}_{n}(A)$$

which is a left inverse to $i_{\#}$ (see the proof of Lemma 4.8). Both $\hat{i}_{\#}$ and $\hat{\alpha}$ are "type increasing." The dual of $\hat{i}_{\#}$ is

$$\hat{i}^{\#}: \hat{C}^{n}_{G}(X; \ell) \longrightarrow \hat{C}^{n}_{G}(A; \ell)$$

and since $\hat{i}^{\#}\hat{\alpha}^{\#} = (\hat{\alpha}\,\hat{i}_{\#})^{\#} = \mathrm{id}^{\#} = \mathrm{id}$, it follows in particular that $\hat{i}^{\#}$ is onto.

Now define $\hat{C}^{n}_{G}(X, A; \ell)$ to be the submodule of $\mathrm{Hom}_{t}(\hat{C}^{G}_{n}(X), L) = \hat{C}^{n}_{G}(X; \ell)$ consisting of all the homomorphisms that vanish on $\hat{C}^{G}_{n}(A)$. Thus we have the short exact sequence

$$0 \longrightarrow \hat{C}^{n}_{G}(X, A; \ell) \longrightarrow \hat{C}^{n}_{G}(X; \ell) \xrightarrow{\hat{i}^{\#}} \hat{C}^{n}_{G}(A; \ell) \longrightarrow 0.$$

This completes the part dealing with the definition and some general properties of the cochain complex $\hat{S}^{*}(X; \ell)$.

In constructing equivariant singular homology we took a quotient of the "roof" chain complex. Here, dually, in constructing equivariant singular cohomology we shall consider an appropriate subcomplex of $\hat{S}^{*}_{G}(X; \ell)$. We now define this one.

Definition 8.4. We define the submodule

$$C^{n}_{G}(X; \ell) \subset \hat{C}^{n}_{G}(X; \ell) = \mathrm{Hom}_{t}(\hat{C}^{G}_{n}(X), L)$$

to be the submodule consisting of all the homomorphisms $c \in \mathrm{Hom}_{t}(\hat{C}^{G}_{n}(X), L)$

which satisfy the following condition. Let $T': (\Delta_n; K'_0, \ldots, K'_n) \longrightarrow X$ be an equivariant singular n-simplex in X, and let $h: (\Delta_n; K_0, \ldots, K_n) \longrightarrow (\Delta_n; K'_0, \ldots, K'_n)$ be a G-homeomorphism which covers $id: \Delta_n \longrightarrow \Delta_n$. Denote $T = T'h: (\Delta_n; K_0, \ldots, K_n) \longrightarrow X$. Then

$$c(T) = (h_n)^* c(T') \in \ell(G/K_n).$$

Here $(h_n)^*: \ell(G/K'_n) \longrightarrow \ell(G/K_n)$ is the isomorphism determined by h as described in Lemma 3.2.

We shall now define a condition on a "type increasing" homomorphism $\hat{\alpha}: \hat{C}_n^G(X) \longrightarrow \hat{C}_m^G(Y)$ which will guarantee that the dual

$$\hat{\alpha}^{\#}: \hat{C}^m_G(Y; \ell) \longrightarrow \hat{C}^n_G(X; \ell)$$

restricts to a homomorphism

$$\alpha^{\#}: C^m_G(Y; \ell) \longrightarrow C^n_G(X; \ell).$$

<u>Definition 8.5.</u> Let

$$\hat{\alpha}: \hat{C}_n^G(X) \longrightarrow \hat{C}_m^G(Y)$$

be a "type increasing" homomorphism. We say that $\hat{\alpha}$ is "isomorphism preserving" if it satisfies the following condition.

Given a G-homeomorphism

$$h: (\Delta_n; K_0, \ldots, K_n) \longrightarrow (\Delta_n; K'_0, \ldots, K'_n)$$

α also determines (besides the natural number $q \geq 0$, the integers m_j and order preserving functions $\alpha_j: \{0, \ldots, m\} \longrightarrow \{0, \ldots, n\}$ $j=0, \ldots, q$) G-homeomorphisms

$$\alpha_j(h): (\Delta_m; K_{\alpha_j(0)}, \ldots, K_{\alpha_j(m)}) \longrightarrow (\Delta_m; K'_{\alpha_j(0)}, \ldots, K'_{\alpha_j(m)})$$

$j=0,\ldots,q$, which cover $\mathrm{id}\colon \Delta_m \to \Delta_m$ and such that the following diagram is G-homotopy commutative,

$$\begin{array}{ccc} G/K_n & \xrightarrow{h_{d^n}} & G/K'_n \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ G/K_{\alpha_j(m)} & \xrightarrow{h_{d^m}} & G/K'_{\alpha_j(m)} \end{array}$$

(notations as in Lemma 3.2), and such that we have the property: If

$$T'\colon (\Delta_n; K'_0,\ldots,K'_n) \to X \quad \text{and} \quad T = T'h,$$

and

$$\hat{\alpha}(T') = \sum_{j=0}^{q} m_j S'_j\,,$$

then we have

$$\hat{\alpha}(T) = \hat{\alpha}(T'h) = \sum_{j=0}^{q} m_j (S'_j \alpha_j(h)).$$

<u>Lemma 8.6</u>. Let $\hat{\alpha}\colon \hat{C}^G_n(X) \to \hat{C}^G_m(Y)$ be a "type increasing" and "isomorphism preserving" homomorphism. Then its dual $\hat{\alpha}^{\#}\colon \hat{C}^m_G(Y;\ell) \to \hat{C}^n_G(X;\ell)$ restricts to a homomorphism

$$\alpha^{\#}\colon C^m_G(Y;\ell) \longrightarrow C^n_G(X;\ell).$$

<u>Proof</u>. Let $c \in C^m_G(Y;\ell)$. We claim that $\hat{\alpha}^{\#}(c) \in C^n_G(X;\ell)$

Let $T'\colon (\Delta_n; K'_0,\ldots,K'_n) \to X$ and let $h\colon (\Delta_n; K_0,\ldots,K_n) \to (\Delta_n; K'_0,\ldots,K'_n)$ be a G-homeomorphism which covers $\mathrm{id}\colon \Delta_n \to \Delta_n$, and denote $T = T'h$. We have to show that

$$(\hat{\alpha}^{\#}(c))(T) = h^*_n(\hat{\alpha}^{\#}(c))(T') \in \ell(G/K_n).$$

Let $\hat{\alpha}(T') = \sum_{j=0}^{q} m_j S'_j$, and hence $\hat{\alpha}(T) = \sum_{j=0}^{q} m_j(S'_j \alpha_j(h))$. We have

$$(\hat{\alpha}^{\#}(c)(T) = \sum_{j=0}^{q} m_j(p_j)^{*} c(S'_j \alpha_j(h))$$

$$= \sum_{j=0}^{q} m_j(p_j)^{*}(\alpha_j(h))^{*}_{m} c(S'_j) = \sum_{j=0}^{q} m_j h^{*}_n(p'_j)^{*} c(S'_j)$$

$$= h^{*}_n\Big(\sum_{j=0}^{q} m_j(p'_j)^{*} c(S'_j)\Big) = h^{*}_n(\hat{\alpha}^{\#}(c)(T')$$

This completes the proof.

q.e.d.

The boundary homomorphism $\hat{\partial}_n: \hat{C}^G_n(X) \longrightarrow \hat{C}^G_{n-1}(X)$ is "type increasing" and "isomorphism preserving."

Let $h: (\Delta_n; K_0, \ldots, K_n) \longrightarrow (\Delta_n; K'_0, \ldots, K'_n)$ be a G-homeomorphism which covers $id: \Delta_n \longrightarrow \Delta_n$. In this case we simply have

$$\alpha_j(h) = h|: (\Delta_n; K_0, \ldots, \hat{K}_j, \ldots, K_n) \longrightarrow (\Delta_n; K'_0, \ldots, \hat{K}'_j, \ldots, K'_n).$$

Thus $\hat{\delta}^n$ restricts to

$$\delta^n: C^n_G(X; \ell) \longrightarrow C^{n+1}_G(X; \ell)$$

and we get the cochain complex

$$S^{*}_G(X; \ell) = \{C^n_G(X; \ell), \ \delta^n\}.$$

For any G-map $f: X \longrightarrow Y$ the induced homomorphism $\hat{f}_{\#}: \hat{C}^G_n(X) \longrightarrow \hat{C}^G_n(Y)$ is "type increasing" and "isomorphism preserving" and thus $\hat{f}^{\#}$ restricts to a homomorphism of cochain complexes

$$f^{\#}: S^{*}_G(Y; \ell) \longrightarrow S^{*}_G(X; \ell).$$

Let (X, A) be a G-pair. Then both $\hat{i}_{\#}: \hat{C}^G_n(A) \longrightarrow \hat{C}^G_n(X)$ and its left inverse $\hat{\alpha}: \hat{C}^G_n(X) \longrightarrow \hat{C}^G_n(A)$ are "type increasing" and "isomorphism preserving." Hence we have the short exact sequence of cochain complexes

$$0 \longrightarrow S^*_G(X,A;\ell) \longrightarrow S^*_G(X;\ell) \xrightarrow{i^\#} S^*_G(A;\ell) \longrightarrow 0$$

where, by definition, $S^*_G(X,A;\ell) = \ker i^\#$.

Definition 8.7. We define

$$H^n_G(X,A;\ell)$$

to be the nth homology module of the cochain complex $S^*_G(X,A;\ell)$.

It is now clear that we have proved everything up to the exactness axiom in the statement of Theorem 2.2.

Proposition 8.8. Two G-homotopic maps

$$f_0, f_1 : (X,A) \longrightarrow (Y,B)$$

induce cochain homotopic maps

$$f_0^\#, f_1^\# : S^*(Y,B;\ell) \longrightarrow S^*(X,A;\ell).$$

Proof. Let $F: I \times (X,A) \longrightarrow (Y,B)$ be a G-homotopy from f_0 to f_1. We shall use the same notations as in Section 5. Recall from the proof of Proposition 5.10 the chain homotopy

$$\hat{\rho}_{n+1}\hat{D}_n : \hat{C}^G_n(X) \longrightarrow \hat{C}^G_{n+1}(Y)$$

from $(\hat{f}_1)_\#$ to $(\hat{f}_0)_\#$. Recall moreover that if $T:(\Delta_n;K_0,\ldots,K_n) \longrightarrow X$, then we have

$(\hat{\rho}_{n+1}\hat{D}_n)(T) =$

$$\sum_{i=0}^{n} (-1)^i (F(\mathrm{id} \times T))((K_0, \ell_n d^0)\ldots(K_i, \ell_n d^i)(K_i, u_n d^i)\ldots(K_n, u_n d^n)).$$

It only remains to show that $\hat{\rho}_{n+1}\hat{D}_n$ is "type increasing" and "isomorphism preserving," our clain then follows using Lemma 8.6. But this is exactly what is shown in the proof of Lemma 5.10, or one could simply say that it is clear from the above expression.

q.e.d.

Proposition 8.9. Let (X,A) be a G-pair and let U be a G-subset of X such that $\overline{U} \subset A^{o}$. Then the inclusion

$$i: (X-U, A-U) \longrightarrow (X,A)$$

induces a homotopy equivalence

$$i^{\#}: S_G^{*}(X,A;\ell) \longrightarrow S_G^{*}(X-U,A-U;\ell).$$

Proof. We shall use the same notations as in Section 6. Let $\mathscr{V}$ be as in Proposition 6.4. Recall the inclusion

$$\hat{\eta}: \hat{C}_n^G(X;\mathscr{V}) \longrightarrow \hat{C}_n^G(X),$$

and that in the proof of Proposition 6.4 we defined homeomorphisms

$$\hat{\tau}: \hat{C}_n^G(X) \longrightarrow \hat{C}_n^G(X;\mathscr{V})$$

$$\hat{D}: \hat{C}_n^G(X) \longrightarrow \hat{C}_{n+1}^G(X)$$

such that $\hat{\tau}\hat{\eta} = \mathrm{id}$ and $\hat{\partial}\hat{D} + \hat{D}\hat{\partial} = \mathrm{id} - \hat{\eta}\hat{\tau}$. We claim that both $\hat{\tau}$ and $\hat{D}$ are "type increasing" and "isomorphism preserving." From the definition of $\hat{\tau}$ and $\hat{D}$ given in the proof of Proposition 6.4, it follows that is is enough to show that the homomorphisms

$$\widehat{Sd}: \hat{C}_n^G(X) \longrightarrow \hat{C}_n^G(X)$$

$$\hat{R}: \hat{C}_n^G(X) \longrightarrow \hat{C}_{n+1}^G(X)$$

are "type increasing" and "isomorphism preserving." But this is exactly what is proved in the proof of Lemma 6.1.

Thus

$$\eta^{\#}: S_G^*(X; \ell) \longrightarrow S_G^*(X; \ell; \mathcal{V})$$

is a homotopy equivalence with homotopy inverse

$$\tau^{\#}: S_G^*(X; \ell; \mathcal{V}) \longrightarrow S_G^*(X; \ell).$$

Now let $\mathcal{V}$ be the family consisting of the G-subsets A and X-U. Recall the proof of Proposition 6.5. The map

$$\hat{i}_{\#}: \hat{S}^G(X\text{-}U,\ A\text{-}U) \longrightarrow \hat{S}^G(X, A)$$

equals the composite

$$\hat{S}^G(X\text{-}U, A\text{-}U) \xrightarrow{\hat{j}} \hat{S}^G(X, A; \mathcal{V}) \xrightarrow{\hat{\eta}} \hat{S}^G(X, A)$$

where $\hat{j}$ is an isomorphism. It follows from what we showed above that $\hat{\eta}$ induces a homotopy equivalence $\eta^{\#}: S_G^*(X, A; \ell) \longrightarrow S_G^*(X, A; \ell; \mathcal{V})$. Since both $\hat{j}$ and its inverse are "type increasing" and "isomorphism preserving" it follows that $\hat{j}$ induces an isomorphism

$$j^{\#}: S_G^*(X, A; \ell; \mathcal{V}) \longrightarrow S_G^*(X\text{-}U,\ A\text{-}U; \ell).$$

Since $i^{\#} = j^{\#}\eta^{\#}$ this completes the proof.

q.e.d.

<u>The dimension axiom</u>

The construction of equivariant singular cohomology and the verification of the first six axioms has consisted of showing that the homomorphisms used in the corresponding homology version are "type increasing" and "isomorphism preserving" and then applying Lemma 8.6. The proof of the dimension axiom does not lend itself directly to this procedure. But still the proof of the dimension axiom for equivariant singular cohomology is completely dual to the proof in homology. We simply have to give direct definitions of the "dual" homomorphisms in each case. We use the contents of Section 5 freely.

We shall determine the homology of the cochain complex $S^*_G(G/H; \ell)$. We denote, as before, $\pi_n = pr_2 : \Delta_n \times G/H \longrightarrow G/H$. Define $C^n_G \text{ spec.}(G/H; \ell) = \text{Hom}\,(Z_{\pi_n}, \ell(G/H))$. This equals $\text{Hom}_t(C^G_n \text{spec.}(G/H), L)$. The mapping $C^n_G \text{ spec.}(G/H; \ell) \longrightarrow \ell(G/H)$ given by $c \longmapsto c(\pi_n)$ is an isomorphism of R-modules.

We have the corresponding cochain complex $S^*_G \text{ spec.}(G/H; \ell)$, and

$$H_m(S^*_G \text{ spec.}(G/H;\ell)) \cong \begin{cases} \ell(G/H) & m = 0 \\ 0 & m \neq 0 \end{cases}.$$

Every element in $C^0_G \text{ spec.}(G/H; \ell)$ is a cocycle and the isomorphism from $H_0(S^*_G \text{ spec.}(G/H; \ell)) = C^0_G \text{ spec.}(G/H; \ell)$ to $\ell(G/H)$ is given by $c \longmapsto c(\pi_0)$.

We define $\hat{C}^n_G \text{ Iso }(G/H; \ell) = \text{Hom}_t(\hat{C}^G_n \text{ Iso }(G/H), L)$ and $C^n_G \text{ Iso }(G/H; \ell) \subset \hat{C}^n_G \text{ Iso }(G/H; \ell)$ to be the submodule consisting of all homomorphisms that also satisfy the condition in Definition 8.4. Now define

$$\hat{\alpha}^{\#} : C^n_G \text{ spec.}(G/H; \ell) \longrightarrow \hat{C}^n_G \text{ Iso }(G/H; \ell)$$

as follows. Let $c \in C^n_G \text{ spec.}(G/H; \ell)$ and $V \in \hat{C}^G_n \text{ Iso }(G/H)$. Then

$$(\hat{\alpha}^{\#}(c))(V) = (\overline{V}_n)^* \; c(\pi_n) \in \ell(G/t(V))$$

where $$(\overline{V}_n)^* : \ell(G/H) \longrightarrow \ell(G/t(V)).$$

It is immediately seen that in fact $\hat{\alpha}^{\#}(c) \in C^n_G \text{ Iso }(G/H; \ell)$. Thus we have the homomorphism

$$\alpha^{\#}: C^n_G \text{ spec.}(G/H; \ell) \longrightarrow C^n_G \text{ Iso }(G/H; \ell).$$

Define

$$\beta^{\#}: C^n_G \text{ Iso }(G/H; \ell) \longrightarrow C^n_G \text{ spec.}(G/H; \ell)$$

by $(\beta^{\#}(c))(\pi_n) = c(\pi_n)$. Both $\alpha^{\#}$ and $\beta^{\#}$ are homomorphisms of cochain complexes.

Let $c \in C^n_G$ spec. $(G/H; \ell)$, then

$$(\beta^{\#}\alpha^{\#}(c))(\pi_n) = (\alpha^{\#}(c))(\pi_n) = (\mathrm{id}_n)^* c(\pi_n) = c(\pi_n).$$

Let $c \in C^n_G$ Iso $(G/H; \ell)$, then

$$(\alpha^{\#}\beta^{\#}(c))(V) = (\overline{V}_n)^*(\beta^{\#}(c))(\pi_n) = (\overline{V}_n)^* c(\pi_n) = c(V).$$

Thus $\beta^{\#}\alpha^{\#} = \mathrm{id}$ and $\alpha^{\#}\beta^{\#} = \mathrm{id}$.

Thus

$$H_m(S^*_G \text{ Iso }(G/H; \ell)) \cong \begin{cases} \ell(G/H & m = 0 \\ 0 & m \neq 0 \end{cases}$$

and every element in C^0_G Iso $(G/H; \ell)$ is a cocycle and the isomorphism from $H_0(S^*_G \text{ Iso }(G/H; \ell)) = C^0_G \text{ Iso }(G/H; \ell)$ to $\ell(G/H)$ is given by $c \longmapsto (\beta^{\#}(c))(\pi_0) = c(\pi_0)$.

We shall now dualize the proof of the fact that the inclusion

$$\eta: S^G \text{ Iso }(G/H; k) \longrightarrow S^G(G/H; k)$$

is a homotopy equivalence.

The inclusion

$$\hat{\eta}: \hat{C}^G_n \text{ Iso }(G/H) \longrightarrow \hat{C}^G_n(G/H)$$

induces

$$\eta^{\#}: C^n_G(G/H; \ell) \longrightarrow C^n_G \text{ Iso } (G/H; \ell).$$

Recall the construction of the homomorphisms

$$\hat{\varphi}: \hat{C}^G_n(G/H) \longrightarrow C^G_n\text{Iso } (G/H)$$

$$\hat{\Phi}: \hat{C}^G_n(G/H) \longrightarrow C^G_{n+1}(G/H).$$

In particular, recall the diagram (*) in Section 5 and the notations there.

Now define

$$\hat{\varphi}^{\#}: C^n_G \text{ Iso } (G/H; \ell) \longrightarrow \hat{C}^n_G(G/H; \ell)$$

as follows. Let $c \in C^n_G$ Iso $(G/H; \ell)$. Define

$$\hat{\varphi}^{\#}(c) \in \hat{C}^n_G(G/H; \ell) = \text{Hom}_t(\hat{C}^G_n(G/H), L)$$

by the following. Let

$$T: (\Delta_n; K_0, \ldots, K_n) \longrightarrow G/H$$

and consider the diagram (*) for some G-homeomorphism k. Define

$$(\hat{\varphi}^{\#}(c))(T) = ((k_1)^*_n)^{-1} p^* c(D i_0) \in \ell (G/K_n).$$

Here $D i_0 = pr_2 k_0: \Delta_n \times G/L \longrightarrow G/H$ is of type "Iso" and $p^*: \ell(G/L) \longrightarrow \ell(G/K_n)$ and $(k_1)^*_n: \ell(G/K_n) \longrightarrow \ell(G/K_n)$. We have to show that $\hat{\varphi}^{\#}$ is well-defined, that is does not depend on the G-homeomorphism k. The proof of this is completely analogous to the proof we gave in the homology case. With the same notations as there, the details are as follows.

We have to show that

$$((k_1)^*_n)^{-1} p^* c(D i_0) = ((k'_1)^*_n)^{-1} (p')^* c(D' i'_0).$$

Since $c \in C^n_G \operatorname{Iso}(G/H; \ell)$ and $D\,i_0 = (D'i'_0)(f|)$, it follows that

$$c(D\,i_0) = (f|)^*_n\, c(D'i'_0).$$

Thus

$$((k_1)^*_n)^{-1}\, p^*\, c(D\,i_0) = ((k_1)^*_n)^{-1}\, p^*(f|)^*_n\, c(D'i'_0)$$

$$= ((k_1)^*_n)^{-1}\, (f_{(1,d^n)})^*\, (p')^*\, c(D'i'_0)$$

$$= ((k_1)^*_n)^{-1}\, ((k'_1)^{-1}_{d^n}\, (k_1)_{d^n})^*\, (p')^*\, c(D'i'_0)$$

$$= ((k'_1)^*_n)^{-1}\, (p')^*\, c(D'i'_0).$$

Now it is easily seen that $\hat{\varphi}^\#$ is a homomorphism of cochain complexes.

Next we define the homomorphism

$$\hat{\Phi}^\#: C^{n+1}_G(G/H; \ell) \longrightarrow \hat{C}^n_G(G/H; \ell)$$

as follows. Let $c \in C^{n+1}_G(G/H; \ell)$. Define

$$\hat{\Phi}^\#(c) \in \hat{C}^n_G(G/H; \ell) = \operatorname{Hom}_t(\hat{C}^G_n(G/H), L)$$

by the following. Let

$$T: (\Delta_n; K_0, \ldots, K_n) \longrightarrow G/H$$

and consider the diagram (*) in Section 5 and the definition of $\hat{\Phi}_n$ given there. We define

$$(\hat{\Phi}^\#(c))(T) =$$

$$= ((k_1)^*_n)^{-1}\, c\Big(\sum_{i=0}^{n} (-1)^i D((\ell d^0, L)\ldots(\ell d^i, L)(ud^i, K_i)\ldots(ud^n, K_n))\Big).$$

It follows immediately that this definition of $\hat{\Phi}^\#(c)$ is independent of the choice of the G-homeomorphism k in diagram (*). From the calculation in Section 5 which showed that $\hat{\Phi}$ is a chain homotopy from $\hat{p}$ to $\eta\hat{\varphi}$, it follows that we now have

$$\hat{\delta}\hat{\Phi}^{\#} + \hat{\Phi}^{\#}\delta = \hat{j} - \hat{\varphi}^{\#}\eta^{\#}$$

where $\hat{j}: C^n_G(G/H; \ell) \longrightarrow \hat{C}^n_G(G/H; \ell)$ is the inclusion.

The next step is to show that in fact

$$\hat{\varphi}^{\#}(c) \in C^n_G(G/H; \ell) \subset \hat{C}^n_G(G/H; \ell) \quad \text{and}$$

$\hat{\Phi}^{\#}(c) \in C^n_G(G/H; \ell) \subset \hat{C}^n_G(G/H; \ell)$. That is we have to show that if $T = T'h$ then $(\hat{\varphi}^{\#}(c))(T) = h^*_n(\hat{\varphi}^{\#}(c))(T')$. This is proved using the same diagrams as in the proof of the corresponding homology statement. Thus we have homomorphisms

$$\varphi^{\#}: C^n_G \text{ Iso } (G/H; \ell) \longrightarrow C^n_G(G/H; \ell)$$

$$\Phi^{\#}: C^{n+1}_G(G/H; \ell) \longrightarrow C^n_G(G/H; \ell)$$

and

$$\delta\Phi^{\#} + \Phi^{\#}\delta = \text{id} - \varphi^{\#}\eta^{\#}.$$

Finally one checks, using diagram (**) in Section 5, that $\eta^{\#}\varphi^{\#} = \text{id}$.

Thus

$$\eta^{\#}: S^{\#}_G(G/H; \ell) \longrightarrow S^{\#}_G \text{ Iso } (G/H; \ell)$$

induces an isomorphism on the homology of these cochain complexes.

Thus

$$H_m(S^*_G(G/H; \ell)) \cong \begin{cases} \ell(G/H) & m = 0 \\ 0 & m \neq 0 \end{cases}.$$

We have the homomorphism $C^0_G(G/H; \ell) \longrightarrow \ell(G/H)$ given by $c \longmapsto \eta^{\#}(c) \longmapsto (\eta^{\#}(c))(\pi_0) = c(\pi_0)$. This homomorphism restricted to the cocycles in $C^0_G(G/H; \ell)$ gives the explicit isomorphism.

Denote this isomorphism by

$$\xi : H^0_G(G/H; \ell) \longrightarrow \ell(G/H).$$

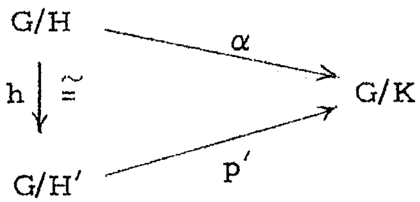
We shall show that ξ commutes with homomorphisms induced from G-maps. In the case of homology the corresponding assertion followed directly from the description of the isomorphism. The cohomology case is not completely equally direct. We need the following lemma.

Lemma 8.10. Let $c \in C^0_G(G/K; \ell)$ be a cocycle. Let $\alpha : G/H \longrightarrow G/K$ be a G-map and $\alpha^* : \ell(G/K) \longrightarrow \ell(G/H)$ the induced homomorphism on the coefficient system.

Now regard α as an equivariant singular 0-simplex in G/K of type H, $\alpha : \Delta_0 \times G/H \longrightarrow G/K$. Also consider the identity $\pi_{0,K} : \Delta_0 \times G/K \longrightarrow G/K$ as an equivariant singular 0-simplex in G/K of type K. We claim that

$$c(\alpha) = \alpha^* c(\pi_{0,K}) \in \ell(G/H).$$

Proof. Denote $\alpha(eH) = g_0 K \in G/H$. Then $H \subset g_0 K g_0^{-1}$ and thus $g_0^{-1} H g_0 \subset K$. Denote $H' = g_0^{-1} H g_0$. Thus $H' \subset K$, and let $p' : G/H' \longrightarrow G/K$ be the natural projection, that is, $p'(eH') = eK$. We have the commutative diagram of G-spaces and G-maps



where h is the G-homeomorphism determined by $h(eH) = g_0 H'$.

Since $c \in C^0_G(G/K; \ell)$, we have

$$c(\alpha) = h^* c(p').$$

Define

$$T_1 : (\Delta_1 ; K, H') \longrightarrow G/K$$

by $T_1([x, e]) = eK \in G/K$. Then

$$\partial(T_1) = p' - \pi_{0,K}.$$

Since $\delta(c) = 0$ it follows that

$$0 = (\delta(c))(T_1) = "c(\partial T_1)" = c(p') - (p')^* c(\pi_{0,K}) \in \ell(G/H').$$

Thus

$$c(\alpha) = h^* c(p') = h^* (p')^* c(\pi_{0,K}) = (p'h)^* c(\pi_{0,K}) = \alpha^* c(\pi_{0,K}).$$

q. e. d.

Let $\alpha : G/H \longrightarrow G/K$ be a G-map.

Let $c \in H^0_G(G/K; \ell)$, that is $c \in C^0_G(G/H; \ell)$ and $\delta c = 0$. Then

$$\alpha^* \xi(c) = \alpha^* c(\pi_{0,K}) \text{ and}$$

$$\xi \alpha^*(c) = (\alpha^*(c))(\pi_{0,H}) = c(\alpha \pi_{0,H}) = c(\alpha).$$

Thus by the above lemma

$$\alpha^* \xi = \xi \alpha^* .$$

This completes the proof of the dimension axiom for equivariant singular cohomology. This completes the proof of Theorem 2.2.

9. AN ALTERNATIVE CONSTRUCTION

In Chapter II we saw that a smooth G-manifold M (G = compact Lie group) is built up by standard n-simplexes $(\Delta_n; K_0, \ldots, K_n)$ as pieces. The construction of equivariant singular homology and cohomology we have described in this chapter is of course inspired by that result. The philosophy behind the construction is thus the following. The standard n-simplexes $(\Delta_n; K_0, \ldots, K_n)$ are the equivariant versions of the standard n-simplexes Δ_n. Hence, imitating the construction of ordinary singular homology and cohomology, we consider all G-maps from standard equivariant simplexes into the G-space X, whose equivariant homology and cohomology groups we want to define. The coefficient system is introduced in order to distinguish between different orbit types. If $h: (\Delta_n; K_0, \ldots, K_n) \to (\Delta_n; K'_0, \ldots, K'_n)$ is a G-homeomorphism which covers $\mathrm{id}: \Delta_n \to \Delta_n$, we consider $(\Delta_n; K_0, \ldots, K_n)$ and $(\Delta_n; K'_0, \ldots, K'_n)$ only as different presentations of the same object. This leads to the definition of the relation $\sim$, and thus to the identification we have used.

But further work with this construction reveals that it is technically quite cumbersome to handle. A good example of this is the proof of Proposition 2.3 in Chapter IV. The situation there is the following: A is a closed subgroup of G of index m, and we wish to prove that (with the appropriate assumptions on the coefficients systems k_A and k_G) the transfer homomorphism $\tau^!: H_n^G(X; k_G) \to H_n^A(X; k_A)$ followed by the homomorphism $i_*: H_n^A(X; k_A) \to H_n^G(X; k_G)$, induced by the inclusion $i: A \to G$, equals multiplication by m on $H_n^G(X; k_G)$. But on the chain level this composite is not multiplication by m. It is only chain homotopic to the homomorphism given by multiplication by m. However, the proof shows that, if we were allowed to "identify" not only by G-homeomorphisms $h: (\Delta_n; K_0, \ldots, K_n) \to (\Delta_n; K_0, \ldots, K_n)$ which cover $\mathrm{id}: \Delta_n \to \Delta_n$, but h could be any G-map which covers $\mathrm{id}: \Delta_n \to \Delta_n$, then this composite would equal multiplication by m already on the chain level. On the other hand, if we "identify" by any G-map $h: (\Delta_n; K_0, \ldots, K_n) \longrightarrow (\Delta_n; L_0, \ldots, L_n)$ which covers $id: \Delta_n \longrightarrow \Delta_n$, it is completely unnecessary to have different orbit types present in the same standard equivariant n-simplex. We may then as well take the G-spaces $(\Delta_n; K, \ldots, K) = \Delta_n \times G/K$ as our equivariant n-simplexes.

Thus another way to construct an equivariant homology theory which satisfies all seven equivariant Eilenberg-Steenrod axioms and has a given covariant coefficient system k as coefficients is the following.

Call the G-space $\Delta_n \times G/K$, K is a closed subgroup of G, for the standard equivariant n-simplex of type K. A G-map

$$T: \Delta_n \times G/K \longrightarrow X$$

is called an equivariant singular n-simplex of type K in X. Denote $t(T) = K$. Define

$$\hat{C}^G_n(X; k) = \sum_T \oplus \, (Z_T \otimes k(G/t(T)))$$

where the direct sum is over all equivariant singular n-simplexes in X. The boundary homomorphism $\hat{\partial}_n: \hat{C}^G_n(X; k) \longrightarrow \hat{C}^G_{n-1}(X; k)$ is defined in the ordinary way.

Define a relation $\sim$ (it is not reflexive) among the elements of the form $T \otimes a$, $a \in k(G/t(T))$, in $\hat{C}^G_n(X; k)$ as follows. Define $T \otimes a \sim T' \otimes b$ if there is a commutative diagram

$$\begin{array}{ccc} \Delta_n \times G/K & \xrightarrow{\;T\;} & \\ \downarrow h & & X \\ \Delta_n \times G/L & \xrightarrow{\;T'\;} & \end{array}$$

where h is a G-map which covers id: $\Delta_n \longrightarrow \Delta_n$ and $h_*(a) = b$. Now define $\overline{C}_n^G(X;k)$ to be the submodule of $\hat{C}_n^G(X;k)$ consisting of all elements of the form $\sum_{i=1}^{m} (T_i \otimes a_i - T' \otimes b_i)$ where $T_i \otimes a_i \sim T_i' \otimes b_i$ or $T_i' \otimes b_i \sim T_i \otimes a_i$, $i = 1, \ldots, m$. Then define

$$C_n^G(X;k) = \hat{C}_n^G(X;k) / \overline{C}_n^G(X;k)$$

and observe that the boundary homomorphism $\hat{\partial}$ induces $\partial_n : C_n^G(X;k) \longrightarrow C_{n-1}^G(X;k)$. Finally define

$$H_n^G(X;k) = \text{nth homology of the chain complex } \{C_n^G(X;k), \partial_n\}.$$

This construction is much easier to handle than our original construction. The proof of the "dimension axiom" becomes a triviality.

We shall use this simplified construction on later occasions, but we stick to our original construction in Chapter IV.

The corresponding remarks also apply for cohomology.

CHAPTER IV

FURTHER PROPERTIES OF EQUIVARIANT SINGULAR HOMOLOGY AND COHOMOLOGY

When not otherwise specified, G denotes an arbitrary good locally compact group as in Chapter III. We also assume in this chapter that the orbit type family $\mathcal{F}$ is the family of all closed subgroups of G. In Section 1, we define the induced homomorphisms on equivariant singular homology and cohomology by a homomorphism on the transformation groups. Section 2 gives the construction of a transfer homomorphism both in equivariant singular homology and cohomology. In Section 3, we define a "Kronecker index," that is, a pairing between equivariant singular homology and cohomology, whenever we are given a pairing of the coefficient systems. We also define a cup-product in equivariant singular cohomology. A more detailed study of these questions is left to another occasion.

We prove in Section 4 that equivariant singular homology and cohomology of a principal G-bundle X is isomorphic to ordinary singular homology and cohomology, respectively, of the orbit space $G\backslash X$, whenever the coefficient systems satisfy the appropriate condition.

In Section 5, we assume that G is a compact Lie group and consider equivariant singular homology and cohomology of finite dimensional equivariant CW complexes. We prove that the spectral sequence, which arises by filtering X by its skeletons, collapses. Another way to express this is to say that, equivariant singular homology and cohomology of a finite dimensional equivariant CW complex is isomorphic to its "cellular equivariant homology and cohomology," respectively. By the result of Chapter II, this applies in particular to differentiable G-manifolds. It thus follows that the equivariant singular homology and cohomology groups of a G-manifold M vanish in degrees above the dimension of the manifold M.

We conclude Section 5 by showing that if the coefficient system is constant, that is, $k(G/H) = A$ or $\ell(G/H) = A$, where A is some R-module, for each closed subgroup H of G, and all induced homomorphisms are the identity on A, then equivariant singular homology and cohomology of a finite dimensional equivariant CW complex X is isomorphic to ordinary singular homology and cohomology, respectively, with coefficient group A of the orbit space $G\backslash X$.

1. FUNCTORIALITY IN THE GROUP

Let M and G be good locally compact groups and $\varphi: M \longrightarrow G$ a continuous homomorphism such that for any closed subgroup $N \subset M$ the subgroup $\varphi(N) \subset G$ is closed. If both M and G are compact or if both both are discrete groups, the above condition is automatically satisfied.

Let X be an M-space, Y a G-space, and $f: X \longrightarrow Y$ a φ-map. Thus $f(mx) = \varphi(m)f(x)$ for all $m \in M$ and $x \in X$. Make Y into an M-space through the homomorphism $\varphi: M \longrightarrow G$. That is, M acts on Y by

$my = \varphi(m)y$. Denote the space Y together with this M-action by Y_M and Y together with the original G-action by Y_G. Then f is the composite of the M-map $f_M: X \longrightarrow Y_M$ and the φ-map $id: Y_M \longrightarrow Y_G$. From now on we shall be considering the φ-map

$$id: Y_M \longrightarrow Y_G$$

and define the homomorphisms it induces on equivariant singular homology and cohomology.

Let

$$\alpha: M/N \longrightarrow M/N'$$

be an arbitrary M-map (N and N′ are closed subgroups of M). Denote $\alpha(eN) = m_0N'$. Thus $\alpha(mN) = mm_0N'$. We have $N \subset m_0 N' m_0^{-1}$, and hence $\varphi(N) \subset \varphi(m_0)\varphi(N')\varphi(m_0)^{-1}$. Therefore we can define a G-map

$$\varphi(\alpha): G/\varphi(N) \longrightarrow G/\varphi(N')$$

by the condition $\varphi(\alpha)(e\varphi(N)) = \varphi(m_0)\varphi(N')$. We have

$$\varphi(\alpha)(g\varphi(N)) = g\varphi(m_0)\varphi(N').$$

Now let k_M be a covariant coefficient system for the group M over the ring R, and k_G a covariant coefficient system for G over the same ring R. Let

$$\Phi: k_M \longrightarrow k_G$$

be a natural transformation with respect to the homomorphism $\varphi: M \longrightarrow G$. By this we mean that for any closed subgroup $N \subset M$ we have a homomorphism of R-modules

$$\Phi: k_M(M/N) \longrightarrow k_G(G/\varphi(N))$$

such that if $\alpha: M/N \longrightarrow M/N'$ is an M-map, the following diagram commutes

$$\begin{array}{ccc} k_M(M/N & \xrightarrow{\Phi} & k_G(G/\varphi(N)) \\ \alpha_* \downarrow & & \downarrow (\varphi(\alpha))_* \\ k_M(M/N') & \xrightarrow{\Phi} & k_G(G/\varphi(N')) \end{array} .$$

Proposition 1.1. Let the homomorphism $\varphi: M \longrightarrow G$ and the natural transformation $\Phi: k_M \longrightarrow k_G$ be as above. Let (Y_G, B_G) be a G-pair and make it into an M-pair (Y_M, B_M) through the homomorphism φ. Then we have induced homomorphisms

$$\varphi_*: H_n^M(Y_M, B_M; k_M) \longrightarrow H_n^G(Y_G, B_G; k_G)$$

with the following properties.

1. φ_* commutes with the boundary homomorphism.
2. If $s: (Y_G, B_G) \longrightarrow (Y'_G, B'_G)$ is a G-map, then clearly $s: (Y_M, B_M) \longrightarrow (Y'_M, B'_M)$ is an M-map, and we have $s_*\varphi_* = \varphi_* s_*$.
3. If both $\varphi = \text{id}$ and $\Phi = \text{id}$ then $\varphi_* = \text{id}$.
4. If also $\varphi': M' \longrightarrow M$ and $\Phi': k_{M'} \longrightarrow k_M$ as above, then the homomorphism $\varphi\varphi': M' \longrightarrow G$ and the natural transformation $\Phi\Phi': k_{M'} \longrightarrow k_G$ induce the homomorphism $(\varphi\varphi')_* = \varphi_*\varphi'_*$.

Proof. We define a chain mapping

$$\hat{\varphi}_\#: \hat{S}^M(Y_M; k_M) \longrightarrow \hat{S}^G(Y_G; k_G)$$

as follows. Let

$$T_M: (\Delta_n; N_0, \ldots, N_n) \longrightarrow Y_M$$

be an M-equivariant singular n-simplex of type $(N_0, \ldots, N_n)$ in Y_M. We define a G-equivariant singular n-simplex of type $(\varphi(N_0), \ldots, \varphi(N_n))$

in Y_G,

$$T_G: (\Delta_n; \varphi(N_0), \ldots, \varphi(N_n)) \longrightarrow Y_G$$

by demanding that

$$T_G([x,e]_G) = T_M([x,e]_M)$$

and extending the definition of T_G to arbitrary elements by the requirement that T_G be a G-map. Observe that if $x \in \Delta_m - \Delta_{m-1}$, then the point $T_M([x,e]_M) \in Y_M$ is fixed under the subgroup N_m. Thus the same point $T_M([x,e]_M) \in Y_G$ is fixed under the subgroup $\varphi(N_m) \subset G$, and hence the above definition of T_G is well-defined. Let $a \in k_M(M/N_n)$. We define

$$\hat{\varphi}_\#(T_M \otimes a) = T_G \otimes \Phi(a),$$

where $\Phi: k_M(M/N_n) \longrightarrow k_G(G/\varphi(N_n))$. This defines the homomorphism $\hat{\varphi}_\#: \hat{C}_n^M(Y_M; k_M) \longrightarrow \hat{C}_n^G(Y_G; k_G)$, and it is immediately seen that $\hat{\varphi}_\#$ commutes with the boundary. Clearly $\hat{\varphi}_\#$ maps $\hat{C}_n^M(B_M; k_M)$ into $\hat{C}_n^G(B_G; k_G)$.

It remains to show that $\hat{\varphi}_\#$ restricts to $\bar{\varphi}_\#: \bar{C}_n^M(Y_M; k_M) \longrightarrow \bar{C}_n^G(Y_G; k_G)$ and hence induces $\varphi_\#: C_n^M(Y_M; k_M) \longrightarrow C_n^G(Y_G; k_G)$. Assume that $T_M \otimes a \sim T'_M \otimes a'$, where $T'_M: (\Delta_n; N'_0, \ldots, N'_n) \longrightarrow Y_M$ and $a' \in k_M(M/N'_n)$. Let $h_M: (\Delta_n; N_0, \ldots, N_n) \longrightarrow (\Delta_n; N'_0, \ldots, N'_n)$ be an M-homeomorphism which covers $\mathrm{id}: \Delta_n \longrightarrow \Delta_n$, such that $T_M = T'_M h_M$ and $((h_M)_n)_*(a) = a'$. Define the maps in the diagram

$$\begin{array}{ccc}
(\Delta_n; N_0, \ldots, N_n) & \xrightarrow{\Delta(\varphi)} & (\Delta_n; \varphi(N_0), \ldots, \varphi(N_n)) \\
h_M \downarrow & & \downarrow h_G \\
(\Delta_n; N'_0, \ldots, N'_n) & \xrightarrow{\Delta(\varphi)} & (\Delta_n; \varphi(N'_0), \ldots, \varphi(N'_n))
\end{array}$$

as follows. The map $\Delta(\varphi)$ is defined by

$$\Delta(\varphi)([x,m]_M) = [x, \varphi(m)]_G$$

and it is immediately seen that $\Delta(\varphi)$ is a well-defined continuous φ-map. Define h_G on the subset $\Delta_n \subset (\Delta_n; \varphi(N_0), \ldots, \varphi(N_n))$ by

$$h_G([x,e]_G) = \Delta(\varphi)h_M([x,e]_M).$$

This defines a continuous map from Δ_n into $(\Delta_n; \varphi(N'_0), \ldots, \varphi(N'_n))$. If $x \in \Delta_m - \Delta_{m-1}$, then the point $\Delta(\varphi)h_M([x,e]_M)$ is fixed under the subgroup $\varphi(N_m)$ and thus the above definition of h_G on the subset Δ_n extends to give a G-map

$$h_G: (\Delta_n; \varphi(N_0), \ldots, \varphi(N_n)) \longrightarrow (\Delta_n; \varphi(N'_0), \ldots, \varphi(N'_n)).$$

Moreover, the above diagram commutes and h_G covers $\mathrm{id}: \Delta_n \longrightarrow \Delta_n$. Using h_M^{-1}, one constructs an inverse to h_G, and thus h_G is a G-homeomorphism.

Observe that $T_G \Delta(\varphi) = \mathrm{id}\, T_M$ and $T'_G \Delta(\varphi) = \mathrm{id}\, T'_M$, where id denotes the φ-map $\mathrm{id}: Y_M \longrightarrow Y_G$. Hence we have

$$\begin{aligned} T'_G h_G([x,e]_G) &= T'_G \Delta(\varphi) h_M([x,e]_M) \\ &= \mathrm{id}\, T'_M h_M([x,e]_M) \\ &= \mathrm{id}\, T_M([x,e]_M) \\ &= T_G \Delta(\varphi)([x,e]_M) \\ &= T_G([x,e]_G). \end{aligned}$$

Since T_G, T'_G and h_G are G-maps, it follows that $T_G = T'_G h_G$.

We claim that $((h_G)_n)_*(\Phi(a)) = \Phi(a')$. Restricting h_M and h_G to the orbit over $d^n \in \Delta_n$ gives the M-map $(h_M)_{d^n}: M/N_n \longrightarrow M/N'_n$ and the

G-map $(h_G)_{dn}: G/\varphi(N_n) \longrightarrow G/\varphi(N'_n)$ respectively. It follows from the definitions that $(h_G)_{dn} = \varphi((h_M)_{dn})$ and hence that

$$((h_G)_n)_* \Phi = \Phi((h_M)_n)_*.$$

Therefore $((h_G)_n)_*(\Phi(a)) = \Phi(a')$, as we claimed. Thus we have showed that

$$T_G \otimes \Phi(a) \sim T'_G \otimes \Phi(a').$$

It now follows that $\hat{\varphi}_\#$ induces a chain mapping

$$\varphi_\#: S^M(Y_M, B_M; k_M) \longrightarrow S^G(Y_G, B_G; k_G).$$

This chain mapping induces the homomorphisms

$$\varphi_*: H_n^M(Y_M, B_M; k_M) \longrightarrow H_n^G(Y_G, B_G; k_G)$$

and it is clear that the properties 1 - 4 are satisfied.

q.e.d.

Let us now consider the cohomology version of Proposition 1.1. Let the homomorphism $\varphi: M \longrightarrow G$ be as before and let ℓ_M and ℓ_G be contravariant coefficient systems for M and G, respectively, over the ring R. Let

$$\Psi: \ell_G \longrightarrow \ell_M$$

be a natural transformation with respect to the homomorphism $\varphi: M \longrightarrow G$. This means that for any closed subgroup $N \subset M$ we have a homomorphism of R-modules

$$\Psi: \ell_G(G/\varphi(N)) \longrightarrow \ell_M(M/N)$$

such that if $\alpha: M/N \longrightarrow M/N'$ is an M-map, then the following diagram commutes

$$\begin{array}{ccc} \ell_G(G/\varphi(N')) & \xrightarrow{\Psi} & \ell_M(M/N') \\ (\varphi(\alpha))^* \downarrow & & \downarrow \alpha^* \\ \ell_G(G/\varphi(N)) & \xrightarrow{\Psi} & \ell_M(M/N) \end{array}$$

Proposition 1.2. Let the homomorphism $\varphi: M \longrightarrow G$ and the natural transformation $\Psi: \ell_G \longrightarrow \ell_M$ be as above. Let (Y_G, B_G) be a G-pair and make it into an M-pair (Y_M, B_M) through the homomorphism φ. Then we have induced homomorphisms

$$\varphi^*: H^n_G(Y_G, B_G; \ell_G) \longrightarrow H^n_M(Y_M, B_M; \ell_M)$$

and the contravariant versions of the properties 1 - 4 in Proposition 1.1 are valid.

Proof. Define a cochain mapping

$$\hat{\varphi}^\#: \hat{S}^*_G(Y_G; \ell_G) \longrightarrow \hat{S}^*_M(Y_M; \ell_M)$$

as follows. Let $c \in \hat{C}^n_G(Y_G; \ell_G)$ and define $\hat{\varphi}^\#(c)$ by the following. Let $T_M: (\Delta_n; N_0, \ldots, N_n) \longrightarrow Y_M$ be an M-equivariant singular n-simplex in Y_M. Define the G-equivariant singular n-simplex $T_G: (\Delta_n; \varphi(N_0), \ldots, \varphi(N_n)) \longrightarrow Y_G$ as in the proof of Proposition 1.1. Then set

$$(\hat{\varphi}^\#(c))(T_M) = \Psi(c(T_G)) \in \ell_M(M/N_n).$$

This defines the homomorphism $\hat{\varphi}^\#$, and it is immediately seen that $\hat{\varphi}^\#$ is a cochain mapping.

It remains to show that $\hat{\varphi}^\#$ restricts to

$$\varphi^\#: S^*_G(Y_G; \ell_G) \longrightarrow S^*_M(Y_M; \ell_M).$$

Assume that $c \in C^n_G(Y_G; \ell_G)$. Let the notation be the same as in the proof of Proposition 1.1, and let $T_M = T'_M h_M$. Recall that $T_G = T'_G h_G$ and $(h_G)_{dn} = \varphi((h_M)_{dn})$. Thus

$$(\hat{\varphi}^{\#}(c))(T_M) = \Psi(c(T_G)) = \Psi((h_G)^*_n c(T'_G))$$
$$= (h_M)^*_n \Psi(c(T'_G)) = (h_M)^*_n (\hat{\varphi}^{\#}(c))(T'_M).$$

Hence $\hat{\varphi}^{\#}(c) \in C^n_M(Y_M; \ell_M)$. This completes the proof. q.e.d.

2. TRANSFER HOMOMORPHISM

In this section A denotes a fixed closed subgroup of G such that the space of right cosets $A\backslash G$ is a finite set. Assume that $A\backslash G$ consists of m elements, that is,

$$A\backslash G = \{Ag_1, \ldots, Ag_m\}.$$

Since A is closed in G it follows that each point in $A\backslash G$ ($A\backslash G$ has the quotient topology from the projection $\pi: G \longrightarrow A\backslash G$) is closed. Hence the finite space $A\backslash G$ has the discrete topology.

We say that a G-map

$$\beta: G/H \longrightarrow G/H',$$

(H and H' are arbitrary closed subgroups of G) is of type "A" if we have

$$\beta(eH) = a_0 H'$$

where $a_0 \in A$. In this case we have $H \subset a_0 H' a_0^{-1}$ and hence $A \cap H \subset a_0(A \cap H') a_0^{-1}$. Thus we can define an A-map

$$\beta^! : A/A \cap H \longrightarrow A/A \cap H'$$

by the condition $\beta^!(e(A \cap H)) = a_0(A \cap H')$. We have $\beta^!(a(A \cap H)) = aa_0(A \cap H')$,

$a \in A$. Moreover, the A-map $\beta^{!}$ depends only on the G-map β of type "A", and not on the specific choice of the element $a_0 \in A$. For if $\beta(eH) = a_1 H'$, where $a_1 \in A$, then $(a_1)^{-1} a_0 \in (A \cap H')$, and hence $a_0(A \cap H') = a_1(A \cap H')$.

Observe that if $H \subset H'$, then the natural projection $p: G/H \longrightarrow G/H'$ is of type "A", and $p^{!} : A/A \cap H \longrightarrow A/A \cap H'$ is the natural projection.

Now let k_A and k_G be covariant coefficient systems for A and G, respectively, over the ring R. Let

$$\Phi^{!} : k_G \longrightarrow k_A$$

be a natural transformation of transfer type with respect to the inclusion $A \hookrightarrow G$. By this we mean that for any closed subgroup $H \subset G$ we have a homomorphism of R-modules

$$\Phi^{!} : k_G(G/H) \longrightarrow k_A(A/A \cap H)$$

such that if $\beta : G/H \longrightarrow G/H'$ is a G-map of type "A", then the following diagram commutes.

$$\begin{array}{ccc} k_G(G/H) & \xrightarrow{\Phi^{!}} & k_A(A/A \cap H) \\ \beta_* \downarrow & & \downarrow (\beta^{!})_* \\ k_G(G/H') & \xrightarrow{\Phi^{!}} & k_A(A/A \cap H') \end{array}$$

Let Y be a G-space. By restricting the G-action on Y to the subgroup A, Y becomes an A-space. We shall construct a transfer homomorphism

$$\tau^{!} : H_n^G(Y; k_G) \longrightarrow H_n^A(Y; k_A)$$

for all n.

We begin by defining for each element $Ag \in A\backslash G$ an induced chain homomorphism

$$(Ag)_{\#}\colon \hat{C}^{G}_{n}(Y;k_G) \longrightarrow C^{A}_{n}(Y;k_A).$$

Let $g \in G$. Given a G-equivariant standard n-simplex $(\Delta_n; K_0, \ldots, K_n)$ we form the A-equivariant standard n-simplex $(\Delta_n; A \cap gK_0g^{-1}, \ldots, A \cap gK_ng^{-1})$ and consider the map

$$(g)\colon (\Delta_n; A\cap gK_0g^{-1}, \ldots, A\cap gK_ng^{-1}) \longrightarrow (\Delta_n; K_0, \ldots, K_n)$$

defined by $(g)([x,a]) = [x, ag]$. It is immediately seen that the map (g) is well-defined and clearly (g) is an A-map when $(\Delta_n; K_0, \ldots, K_n)$ is considered as an A-space. We have the commutative diagram

$$\begin{array}{ccc}
(\Delta_n; A\cap gK_0g^{-1}, \ldots, A\cap gK_ng^{-1}) & \xrightarrow{(g)} & (\Delta_n; K_0, \ldots, K_n) \\
\big\downarrow \eta & \nearrow [g] & \\
(\Delta_n; gK_0g^{-1}, \ldots, gK_ng^{-1}) & &
\end{array}$$

where $\eta([x,a]) = [x,a]$ and $[g]([x,\bar{g}]) = [x, \bar{g}g]$. Both η and $[g]$ are well-defined maps. The map η is an A-map, and $[g]$ is a G-homeomorphism which covers $\mathrm{id}\colon \Delta_n \longrightarrow \Delta_n$.

Now define

$$(g)_{\#}\colon \hat{C}^{G}_{n}(Y;k_G) \longrightarrow \hat{C}^{A}_{n}(Y;k_A)$$

as follows. Let $T\colon (\Delta_n; K_0, \ldots, K_n) \longrightarrow Y$ be a G-equivariant singular n-simplex in Y, and $b \in k_G(G/K_n)$. We define

$$(g)_{\#}(T \otimes b) = T \circ (g) \otimes (\Phi^{!}([g]_n)_*^{-1}(b)).$$

The A-map $T \circ (g)$ is an A-equivariant singular n-simplex in Y of type

$(A \cap gK_0g^{-1}, \ldots, A \cap gK_ng^{-1})$. The isomorphism

$$([g]_n)_*: k_G(G/gK_ng^{-1}) \longrightarrow k_G(G/K_n)$$

is determined by $[g]$ as described in Lemma 3.2 in Chapter III, and

$$\Phi^! : k_G(G/gK_ng^{-1}) \longrightarrow k_A(A/A \cap gK_ng^{-1}).$$

This defines the homomorphism $(g)_\#$, and it is immediately seen that the homomorphisms $(g)_\#$ commute with the boundary.

Next we show that the chain mapping which is the composite of $(g)_\#$ followed by the natural projection from $\hat{C}^A_n(Y; k_A)$ onto $C^A_n(Y; k_A)$ depends only on the element $Ag \in A\backslash G$ and not on the specific choice of $g \in G$.

Let $a \in A$. We claim that $(g)_\#(T \otimes b) \sim (ag)_\#(T \otimes b)$. Consider the commutative diagram

$$(*) \quad \begin{array}{ccc}
(\Delta_n; gK_0g^{-1}, \ldots, gK_ng^{-1}) & \xrightarrow{[g]} & \\
\uparrow \eta & & \searrow \\
(\Delta_n; A \cap gK_0g^{-1}, \ldots, A \cap gK_ng^{-1}) & \xrightarrow{(g)} & (\Delta_n; K_0, \ldots, K_n) \\
\downarrow \{a^{-1}\} & & \downarrow \mathrm{id} \\
(\Delta_n; A \cap (ag)K_0(ag)^{-1}, \ldots, A \cap (ag)K_n(ag)^{-1}) & \xrightarrow{(ag)} & (\Delta_n; K_0, \ldots, K_n) \\
\downarrow \eta & & \nearrow \\
(\Delta_n; (ag)K_0(ag)^{-1}, \ldots, (ag)K_n(ag)^{-1}) & \xrightarrow{[ag]} &
\end{array}$$

where $\{a^{-1}\}$ is defined by $\{a^{-1}\}([x, \bar{a}]) = [x, \bar{a}a^{-1}]$. The map $\{a^{-1}\}$ is a well-defined A-homeomorphism which covers $\mathrm{id}: \Delta_n \longrightarrow \Delta_n$. We have

$T \circ (g) = (T \circ (ag)) \circ \{a^{-1}\}$. We claim that

$$(**) \qquad (\{a^{-1}\}_n)_*(\Phi^{!}([g]_n)_*^{-1}(b)) = \Phi^{!}([ag]_n)_*^{-1}(b).$$

Let $[a^{-1}]_n : G/gK_n g^{-1} \longrightarrow G/(ag)K_n(ag)^{-1}$ be the G-homeomorphism defined by the condition $[a^{-1}]_n(e(gK_n g^{-1})) = a^{-1}((ag)K_n(ag)^{-1})$. Thus $[a^{-1}]_n$ is of type "A" and we have $[a^{-1}]_n^{!} = \{a^{-1}\}_n$, where $\{a^{-1}\}_n$ is the A-map obtained by restricting the A-map $\{a^{-1}\}$ in diagram (*) to the orbit over $d^n \in \Delta_n$. Thus we have

$$\Phi^{!}([a^{-1}]_n)_* = (\{a^{-1}\}_n)_* \Phi^{!} .$$

Now (**) follows by restricting the commutative diagram (*) to the orbits over $d^n \in \Delta_n$. Thus we have showed that $(g)_\#(T \otimes b) \sim (ag)_\#(T \otimes b)$, and hence that

$$(Ag)_\#: \hat{C}_n^G(Y; k_G) \longrightarrow C_n^A(Y; k_A)$$

is well-defined.

We now define

$$\hat{\tau}_\#: \hat{C}_n^G(Y; k_G) \longrightarrow C_n^A(Y; k_A)$$

to be the homomorphism

$$\hat{\tau}_\# = \sum_{i=1}^{m} (Ag_i)_\# .$$

Thus $\hat{\tau}_\#(T \otimes b) = \sum_{i=1}^{m} (g_i)_\#(T \otimes b)$, where the elements $g_1, \ldots, g_m \in G$ form some complete set of representatives for the set of right cosets $A\backslash G$. Clearly the homomorphisms $\hat{\tau}_\#$ commute with the boundary and thus form a chain mapping.

We shall prove that $\hat{\tau}_\#$ induces a homomorphism

$$\tau_\#: C_n^G(Y; k_G) \longrightarrow C_n^A(Y; k_A).$$

The proof of this requires some preliminary considerations.

Let $(\Delta_n; K_0, \ldots, K_n)$ be the standard G-equivariant n-simplex of type $(K_0, \ldots, K_n)$. We also consider $(\Delta_n; K_0, \ldots, K_n)$ as an A-space by restricting the action of G to the subgroup A. We shall consider a special kind of A-imbeddings of standard A-equivariant n-simplexes into $(\Delta_n; K_0, \ldots, K_n)$. We make the following definition.

Let $(\Delta_n; B_0, \ldots, B_n)$ be a standard A-equivariant n-simplex. An A-map

$$\alpha: (\Delta_n; B_0, \ldots, B_n) \longrightarrow (\Delta_n; K_0, \ldots, K_n)$$

is called a special A-imbedding if α is an A-homeomorphism onto its image and α covers $\mathrm{id}: \Delta_n \longrightarrow \Delta_n$.

By restricting α to the orbit over $d^n \in \Delta_n$ we get the A-map

$$\alpha_n: A/B_n \longrightarrow G/K_n.$$

Denote $\alpha_n(eB_n) = gK_n$. Now define $\omega(\alpha)$ to be the double coset

$$\omega(\alpha) = AgK_n \in A\backslash G/K_n.$$

Clearly $\omega(\alpha)$ is well-defined.

We say that two special A-imbeddings

$\alpha: (\Delta_n; B_0, \ldots, B_n) \longrightarrow (\Delta_n; K_0, \ldots, K_n)$ and

$\alpha': (\Delta_n; B'_0, \ldots, B'_n) \longrightarrow (\Delta_n; K_0, \ldots, K_n)$ are isomorphic if there is a commutative diagram

$$\begin{array}{ccc} (\Delta_n; B_0, \ldots, B_n) & \xrightarrow{\ \alpha\ } & \\ j \big\downarrow & & (\Delta_n; K_0, \ldots, K_n) \\ (\Delta_n; B'_0, \ldots, B'_n) & \xrightarrow{\ \alpha'\ } & \end{array}$$

where j is an A-homeomorphism which covers $id: \Delta_n \longrightarrow \Delta_n$. Observe that j is unique if it exists. By restricting the maps in the above diagram to the orbit over $d^n \in \Delta$ we get the maps α_n, α'_n and j_n. Denote $\alpha_n(eB_n) = gK_n$, $\alpha'_n(eB'_0) = g'K_n$ and $j_n(eB_n) = aB'_n$. Thus $ag'K_n = gK_n$, and hence

$$\omega(\alpha) = AgK_n = Ag'K_n = \omega(\alpha').$$

Denote by $A(\Delta_n; K_0, \ldots, K_n)$ the set of isomorphism classes of special A-imbeddings into $(\Delta_n; K_0, \ldots, K_n)$. Thus we have constructed the function (also denoted by ω)

$$\omega: A(\Delta_n; K_0, \ldots, K_n) \longrightarrow A\backslash G/K_n.$$

Lemma 2.1. The function $\omega: A(\Delta_n; K_0, \ldots, K_n) \longrightarrow A\backslash G/K_n$ is a bijection.

Proof. For any element $\beta \in A\backslash G/K_n$ we define a corresponding A-subset, denoted by P_β, of $(\Delta_n; K_0, \ldots, K_n)$ in the following way.

Let $\pi: (\Delta_n; K_0, \ldots, K_n) \longrightarrow \Delta_n$ be the projection onto the orbit space. For any m, $0 \le m \le n$, we have the map

$$\rho_m: \pi^{-1}(\Delta_m - \Delta_{m-1}) \longrightarrow A\backslash G/K_m$$

defined by $\rho_m([x,g]) = AgK_m$. If $y \in \Delta_p - \Delta_{p-1}$ and $x \in \Delta_m - \Delta_{m-1}$, $0 \le p \le m \le n$, then $\rho_p([y,g]) = p\rho_m([x,g])$, where $p: A\backslash G/K_m \longrightarrow A\backslash G/K_p$ is the natural projection, that is, $p(A\bar{g}K_m) = A\bar{g}K_p$. Let $p_m: A\backslash G/K_n \longrightarrow A\backslash G/K_m$ be the natural projection. Let $\beta \in A\backslash G/K_n$ and define

$$P_\beta = \rho_0^{-1}(p_0(\beta)) \cup \rho_1^{-1}(p_1(\beta)) \cup \cdots \cup \rho_n^{-1}(\beta).$$

Since $\rho_m(a[x,g]) = \rho_m([x,ag]) = \rho_m([x,g])$, for $a \in A$, $x \in \Delta_m - \Delta_{m-1}$,

$0 \leq m \leq n$, it follows that P_β is an A-subset of $(\Delta_n; K_0, \ldots, K_n)$. Moreover, if $\rho_m([x,g]) = \rho_m([x,g'])$ then there exists $a \in A$ such that $a[x,g] = [x,g']$. Hence it follows that the A-subset $P_\beta \subset (\Delta_n; K_0, \ldots, K_n)$ consists of exactly one A-orbit over each point $x \in \Delta_n$.

We shall now show that if $\alpha: (\Delta_n; B_0, \ldots, B_n) \longrightarrow (\Delta_n; K_0, \ldots, K_n)$ is a special A-imbedding then $\mathrm{Im}(\alpha) = P_{\omega(\alpha)}$. Since $P_{\omega(\alpha)}$ is an A-subset of $(\Delta_n; K_0, \ldots, K_n)$ with exactly one A-orbit over each point $x \in \Delta_n$, it follows that is enough to show that $\alpha([x,e]) \in P_{\omega(\alpha)}$ for all $x \in \Delta_n$. It follows from the definitions that $\rho_n \alpha([d^n, e]) = \omega(\alpha) \in A\backslash G/K_n$. Now consider the element $[x,e] \in (\Delta_n; B_0, \ldots, B_n)$ and let m be such that $x \in \Delta_m - \Delta_{m-1}$. The existence of the closed interval between d^n and x in $\Delta_n - \Delta_{m-1}$ shows that there is a path in $A\backslash G/K_m$ from $p_m \rho_n \alpha([d^n, e]) = p_m(\omega(\alpha))$ to $\rho_m(\alpha[x,e])$. Since $A\backslash G/K_m$ is a finite discrete set, it follows that $\rho_m(\alpha[x,e]) = p_m(\omega(\alpha))$. Hence $\alpha([x,e]) \in \rho_m^{-1}(p_m(\omega(\alpha))) \subset P_{\omega(\alpha)}$. It follows that a special A-imbedding α into $(\Delta_n; K_0, \ldots, K_n)$ is an A-homeomorphism onto the subset $P_{\omega(\alpha)} \subset (\Delta_n; K_0, \ldots, K_n)$.

From this it follows that the function $\omega: A(\Delta_n; K_0, \ldots, K_n) \longrightarrow A\backslash G/K_n$ is injective. For if $\omega(\alpha) = \omega(\alpha')$ then both α and α' are A-homeomorphisms onto the same set $P_{\omega(\alpha)}$, and hence it follows that α and α' are isomorphic.

It remains to show that ω is onto $A\backslash G/K$. Assume that $\beta \in A\backslash G/K_n$ and let $g \in G$ be such that $\beta = AgK_n$. It is easy to see that the A-map

$$(g): (\Delta_n; A \cap gK_0 g^{-1}, \ldots, A \cap gK_n g^{-1}) \longrightarrow (\Delta_n; K_0, \ldots, K_n),$$

where $(g)([x, a]) = [x, ag]$ is an A-homeomorphism onto its image. Thus (g) is a special A-imbedding. We have $\alpha((g)) = AgK_n = \beta$.

q. e. d.

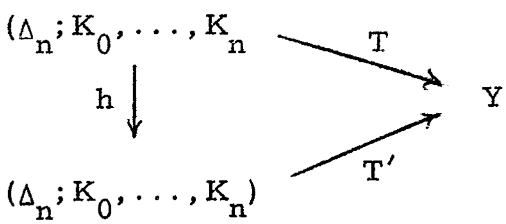
We are now ready to prove that the homomorphism

$$\hat{\tau}_{\#}: \hat{C}_n^G(Y; k_G) \longrightarrow C_n^A(Y; k_A)$$

induces

$$\tau_{\#}: C_n^G(Y; k_G) \longrightarrow C_n^A(Y; k_A).$$

Let T and T' be G-equivariant singular n-simplexes in Y and $b \in k_G(G/t(T))$, $b' \in k_G(G/t(T'))$. Assume that $T \otimes b \sim T' \otimes b'$. Thus we have a commutative diagram



where h is a G-homeomorphism which covers $id: \Delta_n \longrightarrow \Delta_n$ and $(h_n)_*(b) = b'$.

Let $g_1, \ldots, g_m \in G$ be a complete set of representatives for $A\backslash G$, that is, $A\backslash G = \{Ag_1, \ldots, Ag_m\}$. Consider the element

$$\sum_{i=1}^{m} (g_i)_{\#}(T \otimes b) \in \hat{C}_n^A(Y; k_A).$$

The image of this element under the projection onto $C_n^A(Y; k_A)$ is independent of the choice of the representatives g_i, and equals $\hat{\tau}_{\#}(T \otimes b)$. Denote by $h_n: G/K_n \longrightarrow G/K_n'$ the G-homeomorphism we obtain by restricting h to the orbit over $d^n \in \Delta_n$. Choose a fixed element $\bar{g} \in G$ such that $h_n(eK_n) = \bar{g}K_n'$. Thus $h_n(g_iK_n) = g_i\bar{g}K_n'$. Denote $g_i\bar{g} = g_i'$. The elements $g'_1, \ldots, g'_m$ form a complete set of representatives for $A\backslash G$. Hence the element

$$\sum_{i=1}^{m} (g'_i)_\#(T'\otimes b') \in \hat{C}^A_n(Y;k_A)$$

is such that its image under the projection onto $C^A_n(Y;k_A)$ equals $\hat{\tau}_\#(T'\otimes b')$.

We claim that

$$(g_i)_\#(T\otimes b) \sim (g'_i)_\#(T'\otimes b'), \qquad i=1,\ldots,m.$$

Let us denote $g = g_i$ and $g' = g'_i$, and consider the diagram

$$\begin{array}{ccc} (\Delta_n; A\cap gK_o g^{-1}, \ldots, A\cap gK_n g^{-1}) & \xrightarrow{(g)} & (\Delta_n; K_0, \ldots, K_n) \\ j \downarrow & & \downarrow h \\ (\Delta_n; A\cap g'K'_0(g')^{-1}, \ldots, A\cap g'K'_n(g')^{-1}) & \xrightarrow{(g')} & (\Delta_n; K'_0, \ldots, K'_n) \end{array}$$

The composite $h\circ(g)$ is a special A-imbedding into $(\Delta_n; K'_0, \ldots, K'_n)$, and so is (g'). Moreover $\omega(h\circ(g)) = Ag'K'_n = \omega((g'))$. Thus it follows by Lemma 2.1 that $h\circ(g)$ and (g') are isomorphic and hence there exists an A-homeomorphism j which covers $\mathrm{id}: \Delta_n \longrightarrow \Delta_n$ and which makes the above diagram commutative. Recall that

$$(g)_\#(T\otimes b) = T\circ(g)\otimes \Phi^!([g]_n)_*^{-1}(b)$$

$$(g')_\#(T'\otimes b'') = T\circ(g')\otimes \Phi^!([g']_n)_*^{-1}(b').$$

Since $T\circ(g) = (T'\circ(g'))\circ j$ it only remains to show that $(j_n)_*(\Phi^!([g]_n)_*^{-1}(b)) = \Phi^!([g']_n)_*^{-1}(b')$. This last fact is easily verified by arguments completely analogous to the ones after diagram (*). Thus $(g_i)_\#(T\otimes b) \sim (g'_i)_\#(T'\otimes b')$

for $i=1,\ldots,m$, and hence $\hat{\tau}_{\#}(T\otimes b) = \hat{\tau}_{\#}(T'\otimes b')$. We have proved that $\hat{\tau}_{\#}$ induces

$$\tau_{\#}\colon C_n^G(Y;k_G) \longrightarrow C_n^A(Y;k_A)$$

and the homomorphisms $\tau_{\#}$ form a chain mapping. Moreover, it is clear from the way $\tau_{\#}$ is constructed that if B is a G-subset of Y then $\tau_{\#}(C_n^G(B;k_G)) \subset C_n^A(B;k_A)$. Also if $f\colon Y \longrightarrow Y'$ is a G-map, then $f_{\#}\tau_{\#} = \tau_{\#}f_{\#}$. We denote the induced map on homology by $\tau^{!}$ and call it the transfer homomorphism. We have proved

Theorem 2.2. Assume that A is a closed subgroup of G such that $A\backslash G$ is a finite set. Let k_G and k_A be covariant coefficient systems for G and A, respectively, and let $\Phi^{!}\colon k_G \longrightarrow k_A$ be a natural transformation of transfer type. Then for any G-pair (Y,B) we have transfer homomorphisms

$$\tau^{!}\colon H_n^G(Y,B;k_G) \longrightarrow H_n^A(Y,B;k_A)$$

for all n. The homomorphisms $\tau^{!}$ commute with the boundary homomorphism and with homomorphisms induced by G-maps.

q.e.d.

We shall now study the composite of the transfer homomorphism $\tau^{!}$ followed by the homomorphism induced by the inclusion $i\colon A \longrightarrow G$. Let k_G, k_A and $\Phi^{!}\colon k_G \longrightarrow k_A$ be as above, and let $\Phi\colon k_A \longrightarrow k_G$ be a natural transformation with respect to $i\colon A \longrightarrow G$.

Assume that the following condition is satisfied. For each closed subgroup H of G the diagram

$$\begin{array}{ccccc} k_G(G/H) & \xrightarrow{\Phi^!} & k_A(A/A\cap H) & \xrightarrow{\Phi} & k_G(G/A\cap H) \\ & \searrow^{\mathrm{id}} & & & \downarrow p_* \\ & & & & k_G(G/H) \end{array}$$

is commutative. Here $p\colon G/A\cap H \longrightarrow G/H$ is the natural projection.

<u>Proposition 2.3</u>. Assume that A is a closed subgroup of G such that $A\backslash G$ is a finite set of m elements. Let k_G, k_A, $\Phi^!: k_G \longrightarrow k_A$ and $\Phi\colon k_A \longrightarrow k_G$ be as above. Then the composite

$$H_n^G(Y;k_G) \xrightarrow{\tau^!} H_n^A(Y;k_A) \xrightarrow{i_*} H_n^G(Y;k_G)$$

equals multiplication by m.

<u>Proof</u>. Let $g_1,\ldots,g_m \in G$ be such that $A\backslash G = \{Ag_1,\ldots,Ag_m\}$. Let $T\colon (\Delta_n; K_0,\ldots,K_n) \longrightarrow Y$ be a G-equivariant singular n-simplex in Y, and $b \in k_G(G/K_n)$. Then the element

$$(1) \qquad \sum_{i=1}^{m} (T\circ(g_i))_G \otimes \Phi\Phi^!([g_i]_n)_*^{-1}(b) \in \hat{C}_n^G(Y;k_G)$$

is such that its image in $C_n^G(Y;k_G)$ equals $i_\# \hat{\tau}_\#(T\otimes b)$. Recall that $T\circ(g_i)$ is the A-equivariant singular n-simplex in Y

$$(\Delta_n; A\cap g_iK_0g_i^{-1},\ldots,A\cap g_iK_ng_i^{-1}) \xrightarrow{(g_i)} (\Delta_n; K_0,\ldots,K_n) \xrightarrow{T} Y,$$

and that $(T\circ(g_i))_G$ denotes the corresponding G-equivariant singular n-simplex from the standard <u>G-equivariant</u> n-simplex

$$(\Delta_n; A\cap g_iK_0g_i^{-1},\ldots,A\cap g_iK_ng_i^{-1}).$$

The element (1) should be compared with the element

$$(2) \qquad \sum_{i=1}^{m} (T \circ [g_i]) \otimes ([g_i]_n)_*^{-1}(b) \in \hat{C}_n^G(Y; k_G).$$

Here $T \circ [g_i]$ denotes the G-equivariant singular n-simplex in Y

$$(\Delta_n; g_i K_0 g_i^{-1}, \ldots, g_i K_n g_i^{-1}) \xrightarrow{[g_i]} (\Delta_n; K_0, \ldots, K_n) \xrightarrow{T} Y.$$

Since $(T \circ [g_i]) \otimes ([g_i]_n)_*^{-1}(b) \sim T \otimes b$ it follows that the image of the element (2) in $C_n^G(Y; k_G)$ equals $m\{T \otimes b\}$, where $\{T \otimes b\}$ denotes the image of the element $T \otimes b \in \hat{C}_n^G(Y; k_G)$ in $C_n^G(Y; k_G)$. Also observe that $p_* \Phi \Phi^! ([g_i]_n)_*^{-1}(b) = ([g_i]_n)_*^{-1}(b)$, where $p: G/(A \cap g_i K_n g_i^{-1}) \longrightarrow G/g_i K_n g_i^{-1}$ is the natural projection.

Now let

$$(g_i)_\#: \hat{C}_n^G(Y; k_G) \longrightarrow C_n^G(Y; k_G)$$

be the chain mapping defined by

$$(g_i)_\#(T \otimes b) = \{(T \circ (g_i))_G \otimes \Phi \Phi^! ([g_i]_n)_*^{-1}(b)\}.$$

We already know that this chain mapping only depends on the right coset $Ag_i \in A \backslash G$ and not on the specific choice of the representative g_i for the right coset Ag_i.

Define the chain mapping

$$\langle g_i \rangle_\#: \hat{C}_n^G(Y; k_G) \longrightarrow C_n^G(Y; k_G)$$

by

$$\langle g_i \rangle_\#(T \otimes b) = \{(T \circ [g_i]) \otimes ([g_i]_n)_*^{-1}(b)\}.$$

Clearly also $\langle g_i \rangle_\#$ only depends on the right coset $Ag_i \in A \backslash G$.

We shall construct a chain homotopy from $(g_i)_\#$ to $\langle g_i \rangle_\#$. Let

$(I \times \Delta_n; i)$ denote the G-space obtained in the following way. Consider the G-space $I \times (\Delta_n; A \cap g_i K_0 g_i^{-1}, \ldots, A \cap g_i K_n g_i^{-1})$, and at the end $t = 0$ collapse further so that the end $t = 0$ becomes $(\Delta_n; g_i K_0 g_i^{-1}, \ldots, g_i K_n g_i^{-1})$, that is, $(I \times \Delta_n; i)$ is the mapping cylinder of the natural projection

$$\rho: (\Delta_n; A \cap g_i K_0 g_i^{-1}, \ldots, A \cap g_i K_n g_i^{-1}) \longrightarrow (\Delta_n; g_i K_o g_i^{-1}, \ldots, g_i K_n g_i^{-1}).$$

The G-map $(T \circ (g_i))_G$ determines in an obvious way a G-map

$$\overline{T}_i : (I \times \Delta_n; i) \longrightarrow Y$$

such that at $t \neq 0$, $\overline{T}_i$ equals $(T \circ (g_i))_G$ and at $t = 0$, $\overline{T}_i$ equals $T \circ [g_i]$. Using the notion of a linear equivariant singular simplex in $(I \times \Delta_n; i)$ and the same notation as in Sections 5, 6, and 7 of Chapter III, we now define a homomorphism

$$\hat{D}_i : \hat{C}_n^G(Y; k_G) \longrightarrow C_{n+1}^G(Y; k_G)$$

by (we denote in the formula below $g_i = g$)

$$\hat{D}_i(T \otimes b)$$

$$= \Big\{ \sum_{j=0}^{n} (-1)^j \overline{T}_i \Big((\ell d^0, gK_0 g^{-1}) \ldots (\ell d^j, gK_j g^{-1})(ud^j, A \cap gK_j g^{-1}) \\ \ldots (ud^n, A \cap gK_n g^{-1}) \Big) \otimes \Phi\Phi^! ([g]_n)_*^{-1} (b) \Big\}.$$

Here $\ell d^q = (0, d^q)$ and $ud^q = (1, d^q)$ as before. This defines the homomorphism $\hat{D}_i : \hat{C}_n^G(Y; k_G) \longrightarrow C_{n+1}^G(Y; k_G)$. The standard calculation shows that

$$\partial \hat{D}_i + \hat{D}_i \hat{\partial} = (g_i)_\# - \langle g_i \rangle_\#.$$

The homomorphism $\hat{D}_i$ depends only on the right coset Ag_i and not on the specific representative g_i. This is seen by an argument completely analogous to the one we used in showing that $(Ag)_\#: \hat{C}^G_n(Y;k_G) \longrightarrow C^A_n(Y;k_A)$ is well-defined. Using this it follows that the homomorphism

$$\hat{D} = \sum_{i=1}^{m} \hat{D}_i : \hat{C}^G_n(Y;k_G) \longrightarrow C^G_{n+1}(Y;k_G)$$

is such that it induces a homomorphism

$$D : C^G_n(Y;k_G) \longrightarrow C^G_{n+1}(Y;k_G).$$

The argument for this is analogous to the proof that $\hat{\tau}_\#: \hat{C}^G_n(Y;k_G) \longrightarrow C^A_n(Y;k_A)$ induces $\tau_\#: C^G_n(Y;k_G) \longrightarrow C^A_n(Y;k_A)$.

By the remarks at the beginning of this proof it follows that $\hat{D}$ is a chain homotopy from $\Sigma^m_{i=1}(g_i)_\# = i_\# \hat{\tau}_\#$ to the chain map which is the natural projection from $\hat{C}^G_n(Y;k_G)$ onto $C^G_n(Y;k_G)$ followed by multiplication by m. Thus the induced homomorphisms D form a chain map from $i_\# \tau_\#$ to the chain map given by multiplication by m. Hence $i_\# \tau_\#$ induces multiplication by m on the homology.

q.e.d.

The construction of the transfer homomorphism in cohomology is dual to the construction of the transfer homomorphism in homology. We shall give some details.

Let ℓ_A and ℓ_G be contravariant coefficient systems for A and G, respectively, over the ring R. Let

$$\Psi_! : \ell_A \longrightarrow \ell_G$$

be a natural transformation of transfer type with respect to the inclusion $A \hookrightarrow X$. By this we mean that for any closed subgroup $H \subset G$ we have a homomorphism of R-modules

$$\Psi_! : \ell_A(A/A\cap H) \longrightarrow \ell_G(G/H)$$

such that if β: $G/H \longrightarrow G/H'$ is a G-map of type "A", then the following diagram commutes

$$\begin{array}{ccc} \ell_A(A/A\cap H') & \xrightarrow{\Psi_!} & \ell_G(G/H') \\ \Big\downarrow (\beta^!)^* & & \Big\downarrow \beta^* \\ \ell_A(A/A\cap H) & \xrightarrow{\Psi_!} & \ell_G(G/H) \end{array}$$

Let Y be a G-space. By restricting the G-action on Y to the subgroup A, Y becomes an A-space. We shall define a transfer homomorphism

$$\tau_! : H^n_A(Y;\ell_A) \longrightarrow H^n_G(Y;\ell_G)$$

for all n.

We first define for each element $Ag \in A\backslash G$ an induced homomorphism

$$(Ag)^{\#}: C^n_A(Y;\ell_A) \longrightarrow \hat{C}^n_G(Y;\ell_G)$$

Let $g \in G$ and define

$$(g)^{\#}: C^n_A(Y;\ell_A) \longrightarrow \hat{C}^n_G(Y;\ell_G)$$

as follows. Let $c \in C^n_A(Y;\ell_A)$, and define $(g)^{\#}(c)$ by the following. If $T: (\Delta_n; K_0, \ldots, K_n) \longrightarrow Y$ is a G-equivariant singular n-simplex in Y we define the value of $(g)^{\#}(c)$ on T by

$$((g)^{\#}(c))(T) = ([g]_n)_*^{-1} \Psi_! \; c(T \circ (g)).$$

Here $c(T\circ(g)) \in \ell_A(A/A\cap gK_ng^{-1})$ and

$$\Psi_! : \ell_A(A/A\cap gK_ng^{-1}) \longrightarrow \ell_G(G/gK_ng^{-1})$$

and

$$([g]_n)_*^{-1} : \ell_G(G/gK_n g^{-1}) \longrightarrow \ell_G(G/K_n).$$

This defines the homomorphism $(g)^{\#}$ and it is immediately seen that it commutes with the coboundary.

Let $a \in A$. We claim that $(g)^{\#} = (ag)^{\#}$. This is easily seen using the diagram (*). First it follows that we have

$$([a^{-1}]_n)^* \Psi_! = \Psi_! (\{a^{-1}\}_n)^*.$$

Since $c \in C_A^n(Y;\ell_A)$ it thus follows that

$$((g)^{\#}(c))(T) = ([g]_n)_*^{-1} \Psi_! \; c(T \circ (g))$$

$$= ([g]_n)_*^{-1} \Psi_! \; c(T \circ (ag) \circ \{a^{-1}\})$$

$$= ([g]_n)_*^{-1} \Psi_! \; (\{a^{-1}\}_n)^* c(T \circ (ag))$$

$$= ([g]_n)_*^{-1} ([a^{-1}]_n)^* \Psi_! \; c(T \circ (ag))$$

$$= ([ag]_n)_*^{-1} \Psi_! \; c(T \circ (ag)) = ((ag)^{\#}(c))(T).$$

Thus $(ag)^{\#} = (g)^{\#}$ and this gives us the cochain homomorphism

$$(Ag)^{\#} : C_A^n(Y;\ell_A) \longrightarrow \hat{C}_G^n(Y;\ell_G).$$

We now define

$$\hat{\tau}^{\#} : C_A^n(Y;\ell_A) \longrightarrow \hat{C}_G^n(Y;\ell_G)$$

to be the homomorphism

$$\hat{\tau}^{\#} = \sum_{i=1}^{m} (Ag_i)^{\#}.$$

Thus $\hat{\tau}^{\#}(c) = \sum_{i=1}^{m} (g_i)^{\#}(c)$, where the elements $g_1, \ldots, g_m$ form some complete set of representatives for the set of right cosets $A \backslash G$. Clearly $\hat{\tau}^{\#}$ is a cochain homomorphism.

It only remains to show that the image of $\hat{\tau}^{\#}$ lies in $C^n_G(Y; \ell_G)$ and that $\hat{\tau}^{\#}$ therefore induces

$$\tau^{\#}: C^n_A(Y; \ell_A) \longrightarrow C^n_G(Y; \ell_G).$$

This is again proved by the "dual" version of the proof of the corresponding fact for homology. Let the notation be the same as in the discussion after Lemma 2.1. One first shows that

$$((g_i)^{\#}(c))(T) = h_n^*((g_i')^*(c))(T'), \qquad i=1,\dots,m$$

where $T = T'h$. Since $g_1', \dots, g_m'$ also form a complete set of representatives for the set of right cosets $A\backslash G$ it follows that we have

$$(\hat{\tau}^{\#}(c))(T) = \sum_{i=1}^{m} ((g_i)^{\#}(c))(T)$$

$$\sum_{i=1}^{m} h_n^*((g_i')^{\#}(c))(T') = h_n^* \sum_{i=1}^{m} ((g_i')^{\#}(c))(T')$$

$$h_n^*(\hat{\tau}^{\#}(c))(T').$$

Hence $\hat{\tau}^{\#}(c) \in C^n_G(Y; \ell_G)$ as we claimed. Thus we have the cochain homomorphism $\tau^{\#}$. It is clear from the way $\tau^{\#}$ is constructed that if B is a G-subset of Y then $\tau^{\#}(C^n_A(Y, B; \ell_A)) \subset C^n_G(Y, B; \ell_G)$. Also if $f\colon Y \longrightarrow Y'$ is a G-map, then $f^{\#}\tau^{\#} = \tau^{\#}f^{\#}$. We denote the homomorphism $\tau^{\#}$ induces on cohomology by $\tau_!$ and call it the transfer homomorphism. We have proved

Theorem 2.4. Assume that A is a closed subgroup of G such that $A\backslash G$ is a finite set. Let ℓ_A and ℓ_G be contravariant coefficient systems for A and G, respectively, and let $\Psi_! : \ell_A \longrightarrow \ell_G$ be a natural transformation of transfer type. Then for any G-pair (Y, B) we have transfer homomorphisms

$$\tau_! : H^n_A(Y, B; \ell_A) \longrightarrow H^n_G(Y, B; \ell_G)$$

for all n. The homomorphisms $\tau_!$ commute with the coboundary homomorphism and with homomorphisms induced by G-maps.

q. e. d.

Let ℓ_G, ℓ_A and $\Psi_! : \ell_A \longrightarrow \ell_G$ be as above, and let $\Psi\colon \ell_G \longrightarrow \ell_A$ be a natural transformation with respect to $i\colon A \longrightarrow G$. Assume that the following condition is satisfied. For each closed subgroup H of G the diagram

$$\begin{array}{ccccc} \ell_G(G/A\cap H) & \xrightarrow{\Psi} & \ell_A(A/A\cap H) & \xrightarrow{\Psi_!} & \ell_G(G/H) \\ & \searrow_{\mathrm{id}} & & & \downarrow p^* \\ & & & & \ell_G(G/A\cap H) \end{array}$$

is commutative. Here $p\colon G/A\cap H \longrightarrow G/H$ is the natural projection.

The "dual" of the proof of Proposition 2.3 gives us

Proposition 2.5. Assume that A is a closed subgroup of G such that A\G is a finite set of m elements. Let ℓ_G, ℓ_A, $\Psi: \ell_G \longrightarrow \ell_A$ and $\Psi_! : \ell_A \longrightarrow \ell_G$ be as above. Then the composite

$$H^n_G(Y;\ell_G) \xrightarrow{i^*} H^n_A(Y;\ell_A) \xrightarrow{\tau_!} H^n_G(Y;\ell_G)$$

equals multiplication by m.

q.e.d.

3. THE KRONECKER INDEX AND THE CUP-PRODUCT

In this section it is assumed that R is a commutative ring.

Definition 3.1. Let k and ℓ be a covariant and a contravariant coefficient system, respectively, over R. A pairing of k and ℓ consists of the following. For each closed subgroup H of G we have a homomorphism of R-modules

$$\omega: \ell(G/H) \otimes_R k(G/H) \longrightarrow R$$

such that if $\alpha: G/H \longrightarrow G/K$ is a G-map, and $a \in \ell(G/K), b \in k(G/H)$ then

$$\omega(a \otimes_R \alpha_*(b)) = \omega(\alpha^*(a) \otimes_R b).$$

Now let X be a G-space, and $\hat{c} \in \hat{C}^n_G(X;\ell)$ and $\hat{\sigma} \in \hat{C}^G_n(X;k)$. Assume that ω is a pairing of k and ℓ. Define $\langle \hat{c}, \hat{\sigma} \rangle \in R$ by the following. If $\hat{\sigma} = \sum_{i=1}^{m} T_i \otimes a_i$ we set

$$\langle \hat{c}, \hat{\sigma} \rangle = \omega\Big(\sum_{i=1}^{m} \hat{c}(T_i) \otimes_R a_i\Big).$$

It is immediately seen that this gives us a well-defined homomorphism of

R-modules

$$\langle\ ,\ \rangle : \hat{C}^n_G(X;\ell) \otimes_R \hat{C}^G_n(X;k) \longrightarrow R.$$

Let T be an equivariant singular $(n+1)$-simplex in X, and $a \in k(G/\ell(T))$. Then we have

$$\langle \hat{c}, \hat{\partial}(T \otimes a) \rangle = \langle \hat{c}, \sum_{i=0}^{n+1} (-1)^i T^{(i)} \otimes (p_i)_*(a) \rangle$$

$$= \omega\Big(\sum_{i=0}^{n+1} (-1)^i \hat{c}(T^{(i)}) \otimes_R (p_i)_*(a) \Big) = \omega\Big(\sum_{i=0}^{n+1} (-1)^i (p_i)^* \hat{c}(T^{(i)}) \otimes_R a \Big)$$

$$= \omega(\hat{\delta}\hat{c}(T) \otimes_R a) = \langle \hat{\delta}\hat{c},\ T \otimes a \rangle.$$

Thus it follows that we have

$$\langle \hat{c},\ \hat{\partial}\hat{\sigma} \rangle = \langle \hat{\delta}\, \hat{c},\ \hat{\sigma} \rangle .$$

Now assume that $c \in C^n_G(X;\ell)$ and $\sigma \in C^G_n(X;k)$. We claim that the definition

$$\langle c, \sigma \rangle = \langle c,\ \hat{\sigma} \rangle \in R\ ,$$

where $\hat{\sigma} \in \hat{C}^G_n(X;k)$ is any representative for σ gives us a well-defined homomorphism

$$\langle\ ,\ \rangle : C^n_G(X;\ell) \otimes_R C^G_n(X;k) \longrightarrow R.$$

This is seen as follows. Assume that $T \otimes a \sim T' \otimes a'$, and let h be a G-homeomorphism such that $T = T'h$, and $(h_n)_*(a) = a'$. Since $c \in C^n_G(X;\ell)$ it follows that we have

$$\langle c, T \otimes a \rangle = \omega(c(T) \otimes_R a) = \omega((h_n)^* c(T') \otimes_R a)$$

$$= \omega(c(T') \otimes_R (h_n)_*(a)) = \omega(c(T') \otimes_R a')$$

$$= \langle c,\ T' \otimes a' \rangle .$$

This proves our claim.

Since we now have

$$\langle c, \partial\sigma \rangle = \langle \delta c, \sigma \rangle,$$

it follows that $\langle\ ,\ \rangle$ induces a homomorphism

$$\langle\ ,\ \rangle : H^n_G(X; \ell) \otimes_R H^G_n(X; \ell) \longrightarrow R$$

in the obvious way. We call this the Kronecker index.

The Kronecker index gives rise to the homomorphism

$$v: H^n_G(X; \ell) \longrightarrow \mathrm{Hom}_R(H^G_n(X; k), R)$$

defined by $v(\eta)(\xi) = \langle \eta, \xi \rangle$, $\eta \in H^n_G(X; \ell)$ and $\xi \in H^G_n(X; k)$.

We leave a further discussion of the homomorphism v to another occasion.

We shall now define a cup product in equivariant singular cohomology with coefficients in a contravariant ring coefficient system. We say that a contravariant coefficient system ℓ is a ring coefficient system if $\ell(G/H)$ is a ring (with unit) for each closed subgroup H of G and all induced homomorphisms are ring homomorphisms, and, moreover, $\ell(G/G) = R$ and the R-module structure on $\ell(G/H)$ is the same as the one induced by the ring homomorphism $p^*: R = \ell(G/G) \longrightarrow \ell(G/H)$.

Assume that ℓ is a contravariant ring coefficient system. Let $\hat{c} \in \hat{C}^n_G(X; \ell)$ and $\hat{c}' \in \hat{C}^m_G(X; \ell)$. We define the product $\hat{c} \cup \hat{c}' \in \hat{C}^{n+m}_G(X; \ell)$ as follows. Denote $p = n + m$. Let $T: (\Delta_p; K_0, \ldots, K_p) \longrightarrow X$ be an equivariant singular p-simplex in X. Define

$$\alpha_n: (\Delta_n; K_0, \ldots, K_n) \longrightarrow (\Delta_p; K_0, \ldots, K_p)$$

and

$$\beta_m: (\Delta_m; K_n, \ldots, K_p) \longrightarrow (\Delta_p; K_0, \ldots, K_p)$$

by

$$\alpha_n([(x_0, \ldots, x_n), g]) = [(x_0, \ldots, x_n, 0, \ldots, 0), g]$$

and $$\beta_m([(x_0, \ldots, x_m), g]) = [(0, \ldots, 0, x_0, \ldots, x_m), g].$$

Now define the value of $\hat{c} \cup \hat{c}'$ on T by

$$(\hat{c} \cup \hat{c}')(T) = (-1)^{nm}(p^* \hat{c}(T\alpha_n))(\hat{c}'(T\beta_m)) \in \ell(G/K_p)$$

where $p: G/K_p \longrightarrow G/K_n$ is the natural projection.

Thus we have the homomorphism

$$\cup: \hat{C}^n_G(X; \ell) \otimes_R \hat{C}^m_G(X; \ell) \longrightarrow \hat{C}^{n+m}_G(X; \ell).$$

The formula

$$\hat{\delta}(\hat{c} \cup \hat{c}') = \hat{\delta}\, \hat{c} \cup \hat{c}' + (-1)^n \hat{c} \cup \hat{\delta}\, \hat{c}'$$

is established by a standard calculation.

We now claim that if $c \in C^n_G(X; \ell)$ and $c' \in C^m_G(X; \ell)$ then also $c \cup c' \in C^{n+m}_G(X; \ell)$. Let $T': (\Delta_p; K'_0, \ldots, K'_p) \longrightarrow X$ and let $h: (\Delta_p; K_0, \ldots, K_p) \longrightarrow (\Delta_p; K'_0, \ldots, K'_p)$ be a G-homeomorphism which covers $id: \Delta_p \longrightarrow \Delta_p$. We have to show that

$$(c \cup c')(T'h) = (h_p)^*(c \cup c')(T').$$

The G-homeomorphism h determines a G-homeomorphism $h_\alpha: (\Delta_n; K_0, \ldots, K_n) \longrightarrow (\Delta_n; K'_0, \ldots, K'_n)$, which covers $id: \Delta_n \longrightarrow \Delta_n$, such that $h\alpha_n = \alpha'_n h_\alpha$ and also a G-homeomorphism $h_\beta: (\Delta_m; K_n, \ldots, K_p) \longrightarrow (\Delta_m; K_n, \ldots, K_p)$, which covers $id: \Delta_m \longrightarrow \Delta_m$, such that $h\beta_m = \beta'_m h_\beta$. Observe that we have

$$p^*((h_\alpha)_n)^* = (h_p)^* p'^*: \ell(G/K'_n) \longrightarrow \ell(G/K_p)$$

and $$((h_\beta)_m)^* = (h_p)^*: \ell(G/K'_p) \longrightarrow \ell(G/K_p).$$

Thus we have

$$(c \cup c')(T'h) = (-1)^{nm}(p^{*}c(T'h\alpha_n))(c'(T'h\beta_m))$$

$$= (-1)^{nm}(p^{*}c(T'\alpha'_n h_\alpha))(c'(T'\beta'_m h_\beta))$$

$$= (-1)^{nm}(p^{*}(h_\alpha)^{*}_n c(T'\alpha'_n))((h_\beta)^{*}_m c'(T'\beta'_m))$$

$$= (-1)^{nm}((h_p)^{*} p'^{*} c(T'\alpha'_n))((h_p)^{*} c'(T'\beta'_m))$$

$$= (-1)^{nm}(h_p)^{*}((p'^{*}c(T'\alpha'_n))(c'(T'\beta'_m)))$$

$$= (h_p)^{*}(c \cup c')(T').$$

This proves our claim, and hence we have the homomorphism

$$\cup: C^n_G(X;\ell) \otimes_R C^m_G(X;\ell) \longrightarrow C^{n+m}_G(X;\ell).$$

Since we have

$$\delta(c \cup c') = \delta c \cup c' + (-1)^n c \cup \delta c'$$

it follows that the homomorphism $\cup$ induces

$$\cup: H^n_G(X;\ell) \otimes H^m_G(X;\ell) \longrightarrow H^{n+m}_G(X;\ell).$$

4. FREE ACTIONS

In this section we assume that the G-space X is the total space of a principal G-bundle, that is, G acts freely on X and the projection $\pi: X \longrightarrow G\backslash X$ is locally trivial.

We also assume that k is a covariant coefficient system for G with the property that every G-map $\alpha: G \longrightarrow G$ induces the identity $id: k(G) \longrightarrow k(G)$.

Let $T: (\Delta_n; K_0, \ldots, K_n) \longrightarrow X$ be an equivariant singular n-simplex in X. Since G acts freely on X it follows that $\{e\} = K_0 = \ldots = K_n$, and thus $(\Delta_n; K_0, \ldots, K_n) = \Delta_n \times G$.

We shall define a chain map

$$\hat{\gamma}_{\#}: \hat{C}_n^G(X; k) \longrightarrow C_n(G\backslash X; k(G)).$$

Here $C_n(G\backslash X; k(G))$ denotes the nth ordinary singular chain group, with coefficient R-module $k(G)$. Let $T \otimes a \in \hat{C}_n^G(X; k)$ where

$$T: \Delta_n \times G \longrightarrow X$$

and $a \in k(G)$. Define

$$\gamma(T): \Delta_n \longrightarrow G\backslash X$$

by $\gamma(T)(x) = \pi T(x, e)$, where $x \in \Delta_n$ and e is the identity element of G. Now define $\hat{\gamma}_{\#}$ by

$$\hat{\gamma}_{\#}(T \otimes a) = \gamma(T) \otimes a .$$

Clearly the homomorphisms $\hat{\gamma}_{\#}$ commute with the boundary, and thus form a chain map.

If $T \otimes a \sim T' \otimes a'$, then $\gamma(T) = \gamma(T'): \Delta_n \longrightarrow G\backslash X$ and since all induced homomorphisms on $k(G)$ are assumed to be the identity, it follows that $a = a'$. Thus $\hat{\gamma}_{\#}$ induces

$$\gamma_{\#}: C_n^G(X; k) \longrightarrow C_n(G\backslash X; k(G)).$$

We shall show that $\gamma_{\#}$ is an isomorphism. Let

$$S: \Delta_n \longrightarrow G\backslash X.$$

Then the induced principal G-bundle by S over Δ_n is isomorphic to $\Delta_n \times G$. Thus there exists a G-map

$$\beta(S): \Delta_n \times G \longrightarrow X$$

which covers S, and, moreover, $\beta(S)$ is well-defined up to a G-homeomorphism $h: \Delta_n \times G \longrightarrow \Delta_n \times G$ which covers $id: \Delta_n \longrightarrow \Delta_n$. Hence we have a homomorphism

$$\beta_{\#}: C_n(G\backslash X; k(G)) \longrightarrow C_n^G(X; k)$$

defined by $\beta_{\#}(S \otimes a) = \{\beta(S) \otimes a\}$, where $a \in k(G)$ and $\{\beta(S) \otimes a\} \in C_n^G(X; k)$ is the image of $\beta(S) \otimes a \in \hat{C}_n^G(X; k)$ under the natural projection. Clearly $\beta_{\#}$ is a two-sided inverse to $\gamma_{\#}$. We have proved

Theorem 3.1. Assume that G acts freely on X such that the projection $\pi: X \longrightarrow G\backslash X$ is locally trivial. Let k be a covariant coefficient system for G with the property that each G-map $\alpha: G \longrightarrow G$ induces $id: k(G) \longrightarrow k(G)$. Then there exists a natural isomorphism

$$\gamma_*: H_n^G(X; k) \xrightarrow{\cong} H_n(G\backslash X; k(G))$$

for every n.

q.e.d.

We shall now prove the corresponding result for cohomology. Let ℓ be a contravariant coefficient system for G with the property that every G-map $\alpha: G \longrightarrow G$ induces the identity $id: \ell(G) \longrightarrow \ell(G)$.

Denote as before $L = \sum_H \otimes \ell(G/H)$ where the direct sum is over all closed subgroups of G. Since every equivariant singular n-simplex in X is of the form $T: \Delta_n \times G \longrightarrow X$ it follows that

$$\hat{C}_G^n(X; \ell) = \mathrm{Hom}_t(\hat{C}_n^G(X), L) = \mathrm{Hom}(\hat{C}_n^G(X), \ell(G))$$

(see Definition 8.1 in Chapter III).

The homomorphism

$$\hat{\gamma}_{\#}: \hat{C}_n^G(X) \longrightarrow C_n(G\backslash X)$$

gives rise to the dual homomorphism

$$\hat{\gamma}^{\#}: \mathrm{Hom}(C_n(G\backslash X), \ell(G)) \longrightarrow \hat{C}^n_G(X; \ell)$$

where $\hat{\gamma}^{\#}(c') = c'\hat{\gamma}_{\#}$.

If $T': \Delta_n \times G \longrightarrow X$, and $h: \Delta_n \times G \longrightarrow \Delta_n \times G$ is a G-homeomorphism which covers $id: \Delta_n \longrightarrow \Delta_n$, then $\hat{\gamma}_{\#}(T'h) = \hat{\gamma}_{\#}(T')$. Thus $\hat{\gamma}^{\#}$ is a homomorphism into $C^n_G(X; \ell) \subset \hat{C}^n_G(X; \ell)$, and we denote this homomorphism by

$$\gamma^{\#}: \mathrm{Hom}(C_n(G\backslash X), \ell(G)) \longrightarrow C^n_G(X; \ell).$$

We claim that $\gamma^{\#}$ is an isomorphism. Define a homomorphism

$$\beta^{\#}: C^n_G(X; \ell) \longrightarrow \mathrm{Hom}(C_n(G\backslash X), \ell(G))$$

as follows. Let $c \in C^n_G(X; \ell)$, that is, c is a homomorphism $c: \hat{C}^G_n(X) \longrightarrow \ell(G)$ which satisfies the condition $c(T'h) = (h_n)^* c(T') = c(T')$ for every T' and h as above. If $S: \Delta_n \longrightarrow G\backslash X$, we define the value of the homomorphism $\beta^{\#}(c)$ on S by

$$(\beta^{\#}(c))(S) = c(\beta(S)) \in \ell(G),$$

where $\beta(S): \Delta_n \times G \longrightarrow X$ is some G-map which covers $S: \Delta_n \longrightarrow G\backslash X$. This value is independent of the choice of $\beta(S)$. This defines $\beta^{\#}$. Clearly $\beta^{\#}$ is a two-sided inverse to $\gamma^{\#}$. We have proved

<u>Theorem 3.2.</u> Assume that G acts freely on X such that the projection $\pi: X \longrightarrow G\backslash X$ is locally trivial. Let ℓ be a contravariant coefficient system for G with the property that each G-map $\alpha: G \longrightarrow G$ induces $id: \ell(G) \longrightarrow \ell(G)$. Then there exists a natural isomorphism

$$\gamma^*: H^n(G\backslash X; \ell(G)) \longrightarrow H^n_G(X; \ell)$$

for every n.

q.e.d.

Remark 1. If G is connected, then any G-map $\alpha: G \to G$ is G-homotopic to $id: G \to G$ and hence it follows that any G-map $\alpha: G \to G$ induces $id: k(G) \to k(G)$ and $id: \ell(G) \to \ell(G)$ for every covariant coefficient system k and contravariant coefficient system ℓ. Thus Theorems 3.1 and 3.2 apply to principal G-bundles, G connected, for arbitrary coefficient systems.

Remark 2. Let $p: EG \to BG$ be the universal principal G-bundle over the classifying space BG for G. Let M be an R-module. One can define an equivariant homology theory $h_*^G(\ ;M)$ and equivariant cohomology theory $h_G^*(\ ;M)$ as follows. Let X be a G-space, and denote by $X \times_G EG$ the orbit space of the diagonal action by G on $X \times EG$. Then define

$$h_n^G(X; M) = H_n(X \times_G EG; M)$$

and

$$h_G^n(X; M) = H^n(X \times_G EG; M).$$

This theory is due to A. Borel.

Let k and ℓ be as in Theorems 3.1 and 3.2. Then we have

$$h_n^G(X; k(G)) \cong H_n^G(X \times EG; k)$$

and

$$h_G^n(X; \ell(G)) \cong H_G^n(X \times EG; \ell)$$

Let $R = Z$ and $M = Z$. Thus by definition

$$h_n^G(\{*\}; Z) = H_n(BG; Z).$$

That is, the one-point set $\{*\}$ has non-zero homology in positive degrees.

In our theory the homology groups $H_n(BG; Z)$ occur as the equivariant homology groups of the G-space EG whenever the coefficient system is such that $k(G) = Z$ and all induced homomorphisms on $k(G)$ are the identity on Z. Correspondingly for cohomology.

5. EQUIVARIANT SINGULAR HOMOLOGY AND COHOMOLOGY OF EQUIVARIANT CW COMPLEXES

In this section we assume that G is a compact Lie group and that the G-space X is a finite dimensional equivariant CW complex. Let $\emptyset = X^{-1} \subset X^0 \subset \ldots \subset X^m = X$ be the skeleton filtration of X.

Let k be an arbitrary covariant coefficient system for G. We shall show that the chain complex

$$(1) \quad \ldots \leftarrow H^G_{n-1}(X^{n-1}, X^{n-2}; k) \xleftarrow{\partial} H^G_n(X^n, X^{n-1}; k) \xleftarrow{\partial} H^G_{n+1}(X^{n+1}, X^n; k) \leftarrow \ldots$$

has nth homology isomorphic to $H^G_n(X; k)$. First we have

Lemma 5.1. Let H be an arbitrary closed subgroup of G. Then

$$H^G_p(E^n \times G/H,\ S^{n-1} \times G/H; k) \cong \begin{cases} k(G/H) & \text{for } p = n \\ 0 & \text{for } p \neq n. \end{cases}$$

Proof. This follows from the fact that $H^G_*(\ ; k)$ satisfies all seven equivariant Eilenberg-Steenrod axioms and has k as its coefficients, in the same way as the corresponding result for ordinary singular homology follows from the ordinary Eilenberg-Steenrod axioms (see Section 16 of Chapter I in Eilenberg-Steenrod [6]).

q.e.d.

We shall now determine the R-modules

$$H^G_p(X^n, X^{n-1}; k)$$

for all p. We have $X^n = X^{n-1} \cup (\bigcup_{j\in J} c_j^n)$, where $\{c_j^n\}_{j\in J}$ is the collection of all equivariant n-cells in X. Choose an equivariant characteristic map

$$f_j: (E^n \times G/H_j, S^{n-1}\times G/H_j) \longrightarrow (c_j^n, \dot{c}_j^n)$$

for each equivariant n-cell c_j^n, $j \in J$. Denote $f_j(0, eH_j) = x_j \in c_j^n - \dot{c}_j^n$, and let $B = \bigcup_{j\in J} G x_j$. Thus B is the disjoint union of exactly one orbit from the interior of each equivariant n-cell.

Since $S^{n-1}\times G/H$ is a strong G-deformation retract of $(E^n - \{0\}) \times G/H$, it follows that X^{n-1} is a strong G-deformation retract of $X^n - B$. Hence it follows from the exact homology sequence of the pair $(X^n - B, X^{n-1})$ that

$$H_p^G(X^n - B, X^{n-1}; k) = 0, \qquad \text{for all } p.$$

From the exact homology sequence for the triple $(X^n, X^n - B, X^{n-1})$ it thus follows that

$$i_*: H_p^G(X^n, X^{n-1}; k) \xrightarrow{\cong} H_p^G(X^n, X^n - B; k)$$

is an isomorphism for all p.

Let $E^n(\frac{1}{2})$ be the set of all vectors in E^n of length $\leq \frac{1}{2}$, and denote

$$\frac{1}{2} c_j^n = f_j(E^n(\tfrac{1}{2}) \times G/H_j).$$

Let U be the open neighborhood of X^{n-1} in X^n such that $X - U = \bigcup_{j\in J}(\frac{1}{2} c_j^n)$.

By excision it follows that

$$i_*': H_p^G(X^n - U, (X^n - B) - U; k) \xrightarrow{\cong} H_p^G(X^n, X^n - B; k)$$

is an isomorphism for all p.

We now have $X - U = \bigcup_{j\in J}(\frac{1}{2} c_j^n)$ and $(X-B) - U = \bigcup_{j\in J}((\frac{1}{2}c_j^n) - G x_j)$, and the pair $((\frac{1}{2}c_j^n), (\frac{1}{2}c_j^n) - G x_j)$ is G-homeomorphic to $(E^n \times G/H_j, S^{n-1}\times G/H_j)$

Therefore

$$H_p^G(X^n - U, (X^n - B) - U; k) \cong \sum_{j \in J} \oplus H_p^G(E^n \times G/H_j, S^{n-1} \times G/H_j; k)$$

(This additivity property of equivariant singular homology follows easily from the way equivariant singular homology is constructed.) Thus altogether we have

$$(2) \qquad H_p^G(X^n, X^{n-1}; k) \cong \sum_{j \in J} \oplus H_p^G(E^n \times G/H_j, S^{n-1} \times G/H_j; k)$$

and Lemma 5.1 tells us what the right-hand side is. Especially we have

$$(3) \qquad H_p^G(X^n, X^{n-1}; k) = 0 \qquad \text{if } p \neq n.$$

It follows from (3) that the chain complex (1) has nth homology isomorphic to $H_n^G(X; k)$. One way to see this is as follows. Consider the spectral sequence $(E^r_{s,t}, d^r)$ with

$$E^1_{s,t} = H^G_{s+t}(X^s, X^{s-1}; k)$$

and

$$d^1 : E^1_{s,t} \longrightarrow E^1_{s-1,t}$$

equal to the boundary of the triple (X^s, X^{s-1}, X^{s-2}). By (3) we have

$$E^1_{s,t} = 0, \qquad \text{if } t \neq 0.$$

Thus it follows that

$$E^2_{s,t} = E^\infty_{s,t}$$

and

$$E^\infty_{s,t} = H^G_{s+t}(X; k).$$

This proves that the chain complex (1) has nth homology isomorphic to

$H_n^G(X;k)$. Of course, it is not necessary to introduce spectral sequences at all. Some simple arguments using the appropriate exact homology sequences proves the same result. We have proved

<u>Theorem 5.2</u>. Let G be a compact Lie group and assume that the G-space X is a finite dimensional equivariant CW complex. Let k be a covariant coefficient system for G. Then the nth homology of the chain complex

$$(1) \qquad \ldots \leftarrow H_{n-1}^G(X^{n-1}, X^{n-2}; k) \xleftarrow{\partial} H_n^G(X^n, X^{n-1}; k) \leftarrow \ldots$$

is isomorphic to $H_n^G(X;k)$.

q.e.d.

<u>Corollary 5.3</u>. If X is an m-dimensional equivariant CW complex, then

$$H_p^G(X;k) = 0 \qquad \text{for } p > m.$$

q.e.d.

We call a covariant coefficient system k finitely generated if k(G/H) is a finitely generated R-module for every closed subgroup H of G.

If X is a finite equivariant CW complex and k is a finitely generated covariant coefficient system, it follows by (2) that all the modules in the chain complex (1) are finitely generated R-modules. Thus, if we, more-over, assume that the ring R is noetherian, it follows that the homology groups of the chain complex (1) are finitely generated R-modules. Thus, Theorem 5.2 gives us

<u>Corollary 5.4</u>. Let X be a finite equivariant CW complex and k a finitely generated coefficient system over a noetherian ring R. Then the equivariant singular homology modules $H_n^G(X;k)$ are finitely generated R-modules for

all n, and $H_p^G(X;k) = 0$ for $p > \dim X$.

q.e.d.

Let M be a differentiable G-manifold. By Corollary 4.1 in Chapter II, M is an equivariant CW complex and it is clear that M is a finite dimensional equivariant CW complex. To be precise, the dimension of M as an equivariant CW complex is the same as the dimension of the polyhedron $G\backslash M$, and thus in any case not greater than the dimension of the manifold M. If M, moreover, is compact it follows from Lemma 1.15 in Chapter I that M is a finite equivariant CW complex. Thus Corollaries 5.3 and 5.4 apply in the case of smooth actions. We formulate this as a separate theorem.

Theorem 5.5. Let M^n be an n-dimensional differentiable G-manifold, where G is a compact Lie group and k a covariant coefficient system for G. Then

$$H_p^G(M^n;k) = 0 \qquad \text{for } p > n.$$

If M^n is compact and k is a finitely generated coefficient system for G over a noetherian ring R, then every $H_s^G(M^n;k)$, $s=0,1,\ldots$, is a finitely generated R-module.

q.e.d.

The cohomology versions of the above results are proved in a completely analogous way. Let ℓ be a contravariant coefficient system for G. It follows from the fact that $H_G^*(\ ;\ell)$ satisfies all seven equivariant Eilenberg-Steenrod axioms and has ℓ as its coefficients, that we have

$$H_G^p(E^n \times G/H, S^{n-1} \times G/H; \ell) \cong \begin{cases} \ell(G/H) & \text{for } p = n \\ 0 & \text{for } p \neq n. \end{cases}$$

Since equivariant singular cohomology of a disjoint union is the direct product of the equivariant singular cohomology of the "factors" of the disjoint union, it follows that

$$H_G^p(X^n, X^{n-1}; \ell) \cong \prod_{j \in J} H_p^G(E^n \times G/H_j, S^{n-1} \times G/H_j; \ell)$$

where the product is over all equivariant n-cells of X. Thus we have

$$H_G^p(X^n, X^{n-1}; \ell) = 0 \qquad \text{if } p \neq n.$$

This gives us the following results.

Theorem 5.6. Let G be a compact Lie group and assume that the G-space X is a finite dimensional equivariant CW complex. Let ℓ be a contravariant coefficient system for G. Then the nth homology of the cochain complex

$$\cdots \xrightarrow{\delta} H_G^{n-1}(X^{n-1}, X^{n-2}; \ell) \xrightarrow{\delta} H_G^n(X^n, X^{n-1}; \ell) \xrightarrow{\delta} \cdots$$

is isomorphic to $H_G^n(X; \ell)$.

q. e. d.

Theorem 5.7. Let M^n be an n-dimensional differentiable G-manifold, where G is a compact Lie group, and let ℓ be a contravariant coefficient system for G. Then

$$H_G^p(M^n; \ell) = 0 \qquad \text{for } p > n.$$

If M^n is compact and ℓ is a finitely generated coefficient system for G over a noetherian ring R, then every $H_G^s(M^n; \ell)$ s=0, 1, ... , is a finitely generated R-module.

q. e. d.

We conclude this section by showing that if the coefficient system is constant, the equivariant singular homology and cohomology of X is isomorphic to the ordinary singular homology and cohomology, respectively, of the orbit space $G\backslash X$.

Let P be an R-module. The covariant coefficient system k for which $k(G/H) = P$, for every closed subgroup H of G, and all induced homomorphisms are the identity on P is called constant and denoted by P. In the same way the R-module P can also be thought of as a contravariant coefficient system.

We now define homomorphisms

$$\hat{\gamma}_{\#}: \hat{C}^G_n(X;P) \longrightarrow C_n(G\backslash X;P)$$

in the same way as in Section 4. That is, if $T \otimes a \in \hat{C}^G_n(X;P)$, where $T: (\Delta_n; K_0, \ldots, K_n) \longrightarrow X$ and $a \in P$, then we denote by $\gamma(T): \Delta_n \longrightarrow G\backslash X$ the map induced by T on the orbit spaces and define $\hat{\gamma}_{\#}(T \otimes a) = \gamma(T) \otimes a$. The homomorphisms $\hat{\gamma}_{\#}$ commute with boundary homomorphisms and also induce homomorphisms

$$\gamma_{\#}: C^G_n(X;P) \longrightarrow C_n(G\backslash X;P).$$

The corresponding cochain homomorphism

$$\gamma^{\#}: \mathrm{Hom}(C_n(G\backslash X), P) = C^n(G\backslash X;P) \longrightarrow C^n_G(X;P)$$

is again defined as in Section 4. This gives us the homomorphisms

$$\gamma_*: H^G_n(X;P) \longrightarrow H_n(G\backslash X;P)$$

and

$$\gamma^*: H^n(G\backslash X;P) \longrightarrow H^n_G(X;P)$$

for all n. We shall show that γ_* and γ^* are isomorphisms for every n.

If $X = G/H$, where H is some closed subgroup of G, then $H_0^G(G/H; P) \cong P$ and $H_0(\{*\}; P) \cong P$, and γ_* induces the identity on P. If $m \neq 0$ then $H_m^G(G/H; P) = H_m(\{*\}; P) = 0$. For cohomology, the corresponding fact holds. Thus γ_* and γ^* induce isomorphisms on the coefficients, that is, whenever X is of the form G/H. The fact that γ_* and γ^* are isomorphisms, for any finite dimensional equivariant CW complex X, now follows from an equivariant "Dold type uniqueness theorem" given below.

Let $h_*^G = \{h_n^G\}_{n \in Z}$ be a generalized equivariant homology theory defined on the category of all G-pairs and G-maps. By this, we mean that h_*^G satisfies the first six equivariant Eilenberg-Steenrod axioms, and moreover, we require the excision axiom to hold only under the assumption that U is an open set (see A.6 in the statement of Theorem 3.2 in Chapter III).

We say that h_*^G is additive if

$$h_n^G(\dot{\bigcup_{j \in J}} X_j) \xleftarrow{\cong} \sum_{j \in J} \oplus\, h_n^G(X_j)$$

for any disjoint union $\dot{\bigcup_{j \in J}} X_j$ of G-spaces X_j and every $n \in Z$.

A generalized equivariant cohomology theory h_G^* is defined analogously and h_G^* is called additive if

$$h_G^n(\dot{\bigcup_{j \in J}} X_j) \xrightarrow{\cong} \prod_{j \in J} h_G^n(X_j)$$

for every $n \in Z$.

Theorem 5.8. Let h_*^G and $\bar{h}_*^G$ be two additive generalized equivariant homology theories and $\varphi: h_*^G \longrightarrow \bar{h}_*^G$ a natural transformation. If

$$\varphi: h_n^G(G/H) \xrightarrow{\cong} \bar{h}_n^G(G/H)$$

is an isomorphism for every $n \in Z$ and each closed subgroup H of G, then

$$\varphi: h_n^G(X) \xrightarrow{\cong} \bar{h}_n^G(X)$$

is an isomorphism for every finite dimensional equivariant CW complex X, and all $n \in Z$.

Proof. In the same way as in the proof of Theorem 5.2, it follows that we have

$$h_p^G(X^n, X^{n-1}) \cong \sum_{j \in J} \oplus\, h_p^G(E^n \times G/H_j, S^{n-1} \times G/H_j)$$

where the direct sum is over all equivariant n-cells in X. Moreover, we have

$$h_p^G(E^n \times G/H, S^{n-1} \times G/H) \cong h_{p-n}^G(G/H)$$

for every closed subgroup H of G. From this it follows that

$$\varphi: h_p^G(X^n, X^{n-1}) \xrightarrow{\cong} \bar{h}_p^G(X^n, X^{n-1})$$

is an isomorphism for all p and n. Since $X = X^m$ for some m, our claim now follows using induction.

q.e.d.

The corresponding result for a natural transformation $\eta: h_G^* \longrightarrow \bar{h}_G^*$ between additive generalized equivariant cohomology theories is valid and proved in the same way.

Observe that the equivariant singular homology and cohomology theories are additive and that taking ordinary singular homology and cohomology of the orbit spaces gives us an additive equivariant homology and cohomology theory, respectively. Thus we have

Corollary 5.9. For any finite dimensional equivariant CW complex X we have natural isomorphisms

$$\gamma_*: H_n^G(X;P) \xrightarrow{\cong} H_n(G\backslash X;P)$$

and

$$\gamma^*: H^n(G\backslash X;P) \xrightarrow{\cong} H^n_G(X;P)$$

for all n. In particular, this applies when X is a smooth G-manifold.

q.e.d.

Let H be a closed subgroup of G such that $H\backslash G$ consists of m elements. Both for H and G we take the constant coefficient system given by the R-module P. Then Theorems 2.2 and 2.4 and Propositions 2.3 and 2.5 together with Corollary 5.9 give us

Corollary 5.10. Let M be a smooth G-manifold and let H be a closed subgroup of G such that $H\backslash G$ consists of m elements. Then we have transfer homomorphisms

$$\tau^!: H_n(G\backslash M;P) \longrightarrow H_n(H\backslash M;P)$$

and

$$\tau_!: H^n(H\backslash M;P) \longrightarrow H^n(G\backslash M;P)$$

for all n. Moreover, the composite homomorphisms

$$p_*\tau^!: H_n(G\backslash M;P) \longrightarrow H_n(G\backslash M;P)$$

and

$$\tau^! p^*: H^n(G\backslash M;P) \longrightarrow H^n(G\backslash M;P)$$

equal multiplication by m.

q.e.d.

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ABSTRACT

Let G be a compact Lie group, a discrete group, or an abelian locally compact group. By a covariant coefficient system k for G, over the ring R, we mean a covariant functor from the category of G-spaces of the form G/H, where H is a closed subgroup (not fixed) of G, and G-homotopy classes of G-maps, to the category of R-modules. A contravariant coefficient system ℓ is defined analogously.

We construct an equivariant homology theory $H_*^G(\ ;k)$ and an equivariant cohomology theory $H_G^*(\ ;\ell)$, defined on the category of all G-pairs and G-maps, which both satisfy all seven equivariant Eilenberg-Steenrod axioms and which have the given covariant coefficient system k and the contravariant coefficient system ℓ, respectively, as coefficients. By the statement that $H_*^G(\ ;k)$ satisfies the equivariant dimension axiom and has k as coefficients we mean the following. If H is a closed subgroup of G we have

$$H_m^G(G/H;k) = 0 \qquad \text{for } m \neq 0$$

and there exists an isomorphism

$$\gamma: H_0^G(G/H;k) \xrightarrow{\cong} k(G/H)$$

which commutes with homomorphisms induced by G-maps $\alpha: G/H \longrightarrow G/K$. The corresponding explanation applies for $H_G^*(\ ;\ell)$. We call the equivariant homology theory $H_*^G(\ ;k)$ for equivariant singular homology with coefficients in k and the equivariant cohomology theory $H_G^*(\ ;\ell)$ for equivariant singular cohomology with coefficients in ℓ.

We also construct transfer homomorphisms both in equivariant singular homology and cohomology, and define a "Kronecker index" and a cup-product in equivariant singular cohomology.

Assume from now on that G is a compact Lie group. We define equivariant CW complexes and prove the equivariant versions of the homotopy extension property, the skeletal approximation theorem, and the Whitehead theorem. Moreover, we prove that every differentiable G-manifold M is an equivariant CW complex.

Finally, we show that equivariant singular homology and cohomology of a finite dimensional equivariant CW complex is isomorphic to its "cellular equivariant homology and cohomology," respectively. From this it follows that the equivariant singular homology and cohomology groups of a differentiable G-manifold M vanish in degrees above the dimension of the manifold M. If M moreover is compact, the equivariant singular homology and cohomology groups are finitely generated R-modules if the coefficient systems are finitely generated coefficient systems over a noetherian ring R.