nLab

Introduction to Stable homotopy theory -- 1-2

We give an introduction to the stable homotopy category and to its key computational tool, the Adams spectral sequence. To that end we introduce the modern tools, such as model categories and highly structured ring spectra. In the accompanying seminar we consider applications to cobordism theory and complex oriented cohomology such as to converge in the end to a glimpse of the modern picture of chromatic homotopy theory.

Lecture notes.

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Stable homotopy theory – Structured spectra

The key result of part 1.1 was (thm.) the construction of a stable homotopy theory of spectra, embodied by a stable model structure on topological sequential spectra $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}$ (thm.) with its corresponding stable homotopy category $\mathrm{Ho}(\mathrm{Spectra})$, which stabilizes the canonical looping/suspension adjunction on pointed topological spaces in that it fits into a diagram of (Quillen-)adjunctions of the form

$$\begin{array}{ccc} (\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}} & \underoverset{\underset{\Omega}{\longrightarrow}}{\overset{\Sigma}{\longleftarrow}}{\bot} & (\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}} \\ {}^{\mathllap{\Sigma^\infty}}\downarrow \dashv \uparrow^{\mathrlap{\Omega^\infty}} && {}^{\mathllap{\Sigma^\infty}}\downarrow \dashv \uparrow^{\mathrlap{\Omega^\infty}} \\ \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}} & \underoverset{\underset{\Omega}{\longrightarrow}}{\overset{\Sigma}{\longleftarrow}}{\simeq_Q} & \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}} \end{array} \;\;\; \overset{\gamma}{\longrightarrow} \;\;\; \begin{array}{ccc} \mathrm{Ho}(\mathrm{Top}^{*/}) & \underoverset{\underset{\Omega}{\longrightarrow}}{\overset{\Sigma}{\longleftarrow}}{\bot} & \mathrm{Ho}(\mathrm{Top}^{*/}) \\ {}^{\mathllap{\Sigma^\infty}}\downarrow \dashv \uparrow^{\mathrlap{\Omega^\infty}} && {}^{\mathllap{\Sigma^\infty}}\downarrow \dashv \uparrow^{\mathrlap{\Omega^\infty}} \\ \mathrm{Ho}(\mathrm{Spectra}) & \underoverset{\underset{\Omega}{\longrightarrow}}{\overset{\Sigma}{\longleftarrow}}{\simeq} & \mathrm{Ho}(\mathrm{Spectra}) \end{array}\,.$$

But fitting into such a diagram does not yet uniquely characterize the stable homotopy category. For instance the trivial category on a single object would also form such a diagram. On the other hand, there is more canonical structure on the category of pointed topological spaces which is not yet reflected here.

Namely the smash product

$$\wedge \;:\; \mathrm{Ho}(\mathrm{Top}^{*/}) \longrightarrow \mathrm{Ho}(\mathrm{Top}^{*/})$$

of pointed topological spaces gives it the structure of a monoidal category (def. 1.1 below), and so it is natural to ask that the above stabilization diagram reflects and respects that extra structure. This means that there should be a smash product of spectra

$$\wedge \;:\; \mathrm{Ho}(\mathrm{Spectra}) \longrightarrow \mathrm{Ho}(\mathrm{Spectra})$$

such that $(\Sigma^\infty \dashv \Omega^\infty)$ is compatible, in that

$$\Sigma^\infty(X \wedge Y) \simeq (\Sigma^\infty X) \wedge (\Sigma^\infty Y)$$

(a "strong monoidal functor", def. 1.47 below).

We had already seen in part 1.1 that $\mathrm{Ho}(\mathrm{Spectra})$ is an additive category, where wedge sum of spectra is a direct sum operation $\oplus$. We discuss here that the smash product of spectra is the corresponding operation analogous to a tensor product of abelian groups.

abelian groups	**spectra**
$\oplus$ direct sum	$\vee$ wedge sum
$\otimes$ tensor product	$\wedge$ smash product

This further strenghtens the statement that spectra are the analog in homotopy theory of abelian groups. In particular, with respect to the smash product of spectra, the sphere spectrum becomes a ring spectrum that is the coresponding analog of the ring of integers.

With the analog of the tensor product in hand, we may consider doing algebra – the theory of rings and their modules – internal to spectra. This "higher algebra" accordingly is the theory of *ring spectra* and *module spectra*.

algebra	**homological algebra**	**higher algebra**
abelian group	chain complex	spectrum
ring	dg-ring	ring spectrum
module	dg-module	module spectrum

Where a ring is equivalently a monoid with respect to the tensor product of abelian groups, we are after a corresponding tensor product of spectra. This is to be the smash product of spectra, induced by the smash product on pointed topological spaces.

In particular the sphere spectrum becomes a ring spectrum with respect to this smash product and plays the role analogous to the ring of integers in abelian groups

abelian groups	**spectra**
$\mathbb{Z}$ integers	$\mathbb{S}$ sphere spectrum

Using this structure there is finally a full characterization of stable homotopy theory, we state (without proof) this *Schwede-Shipley uniqueness* as theorem 5.13 below.

There is a key point to be dealt with here: the smash product of spectra has to exhibit a certain *graded commutativity*. Informally, there are two ways to see this:

First, we have seen above that under the Dold-Kan correspondence chain complexes yield examples of spectra. But the tensor product of chain complexes is graded commutative.

Second, more fundamentally, we see in the discussion of the Brown representability theorem (here) that every (sequential) spectrum A induces a generalized homology theory given by the formula $X \mapsto \pi_\bullet(E \wedge X)$ (where the smash product is just the degreewise smash of pointed objects). By the suspension isomorphism this is such that for $X = S^n$ the n-sphere, then $\pi_{\bullet \geq 0}(E \wedge S^n) \simeq \pi_{\bullet \geq 0}(E_n)$. This means that instead of thinking of a sequential spectrum (def.) as indexed on the natural numbers equipped with addition $(\mathbb{N}, +)$, it may be more natural to think of sequential spectra as indexed on the n-spheres equipped with their smash product of pointed spaces $(\{S^n\}_n, \wedge)$.

Proposition 0.1. *There are homeomorphisms between n-spheres and their smash products*

$$\phi_{n_1,n_2} : S^{n_1} \wedge S^{n_2} \overset{\simeq}{\longrightarrow} S^{n_1+n_2}$$

such that in Ho(Top) there are commuting diagrams like so:

$$\begin{array}{ccc} (S^{n_1} \wedge S^{n_2}) \wedge S^{n_3} & \overset{\simeq}{\longrightarrow} & S^{n_1} \wedge (S^{n_2} \wedge S^{n_3}) \\ {}^{\mathllap{\phi_{n_1,n_2} \wedge \mathrm{id}}}\downarrow && \downarrow^{\mathrlap{\mathrm{id} \wedge \phi_{n_2,n_3}}} \\ S^{n_1+n_2} \wedge S^{n_3} && S^{n_1} \wedge S^{n_2+} \\ & {}_{\mathllap{\phi_{n_1+n_2,n_3}}}\searrow \quad \swarrow_{\mathrlap{\phi_{n_1,n_2+n_3}}} \\ & S^{n_1+n_2+n_3} \end{array}$$

and

$$\begin{array}{ccc} S^{n_1} \wedge S^{n_2} & \overset{b_{n_1,n_2}}{\longrightarrow} & S^{n_2} \wedge S^{n_1} \\ {}^{\mathllap{\phi_{n_1,n_2}}}\downarrow && \downarrow^{\mathrlap{\phi_{n_2,n_1}}} \\ S^{n_1+n_2} & \overset{(-1)^{n_1 n_2}}{\longrightarrow} & S^{n_1+n_2} \end{array}$$

where here $(-1)^n : S^n \to S^n$ denotes the homotopy class of a continuous function of degree $(-1)^n \in \mathbb{Z} \simeq [S^n, S^n]$.

Proof. With the n-sphere S^n realized as the one-point compactification of the Cartesian space $\mathbb{R}^n$, then ϕ_{n_1,n_2} is given by the identity on coordinates and the braiding homeomorphism

$$b_{n_1,n_2} : S^{n_1} \wedge S^{n_2} \overset{\sigma}{\longrightarrow} S^{n_2} \wedge S^{n_1}$$

is given by permuting the coordinates:

$$(x_1, \cdots, x_{n_1}, y_1, \cdots, y_{n_2}) \mapsto (y_1, \cdots, y_{n_2}, x_1, \cdots, x_{n_1}) \,.$$

This has degree $(-1)^{n_1 n_2}$. ▮

This phenomenon suggests that as we "categorify" the natural numbers to the n-spheres, hence the integers to the sphere spectrum, and as we think of the nth component space of a sequential spectrum as being the value assigned to the n-sphere

$$E_n \simeq E(S^n)$$

then there should be a possibly non-trivial action of the symmetric group Σ_n on E_n, due to the fact that there is such an action of S^n which is non-trivial according to prop. 0.1.

We discuss two ways of making this precise below in *Symmetric and orthogonal spectra*, and we discuss how these are unified by a concept of module objects over a monoid object representing the sphere spectrum below in *S-modules*.

The general abstract theory for handling this is *monoidal and enriched category theory*. We first develop the relevant basics in *Categorical algebra*.

1. Categorical algebra

When defining a commutative ring as an abelian group A equipped with an associative, commutative and untial bilinear pairing

$$A \otimes_{\mathbb{Z}} A \overset{(-)\cdot(-)}{\longrightarrow} A$$

one evidently makes crucial use of the tensor product of abelian groups $\otimes_{\mathbb{Z}}$. That tensor product itself gives the category Ab of all abelian groups a structure similar to that of a ring, namely it equips it with a pairing

$$\mathrm{Ab} \times \mathrm{Ab} \overset{(-)\otimes_{\mathbb{Z}}(-)}{\longrightarrow} \mathrm{Ab}$$

that is a functor out of the product category of Ab with itself, satisfying category-theoretic analogs of the properties of associativity, commutativity and unitality.

One says that a ring A is a *commutative monoid* in the category Ab of abelian groups, and that this concept makes sense since Ab itself is a *symmetric monoidal category*.

Now in stable homotopy theory, as we have seen above, the category Ab is improved to the stable homotopy category $\mathrm{Ho}(\mathrm{Spectra})$ (def. \ref{TheStableHomotopyCategory}), or rather to any stable model structure on spectra presenting it. Hence in order to correspondingly refine commutative monoids in Ab (namely commutative rings) to commutative monoids in Ho(Spectra) (namely commutative ring spectra), there needs to be a suitable symmetric monoidal category structure on the category of spectra. Its analog of the tensor product of abelian groups is to be called the *symmetric monoidal smash product of spectra*. The problem is how to construct it.

The theory for handling such a problem is *categorical algebra*. Here we discuss the minimum of categorical algebra that will allow us to elegantly construct the symmetric monoidal smash product of spectra.

Monoidal topological categories

We want to lift the concepts of *ring* and *module* from *abelian groups* to *spectra*. This requires a general idea of what it means to generalize these concepts at all. The abstract theory of such generalizations is that of *monoid in a monoidal category*.

We recall the basic definitions of monoidal categories and of monoids and modules internal to monoidal categories. We list archetypical examples at the end of this section, starting with example 1.9 below. These examples are all fairly immediate. The point of the present discussion is to construct the non-trivial example of Day convolution monoidal stuctures below.

Definition 1.1. A **(pointed) topologically enriched monoidal category** is a (pointed) topologically enriched category $\mathcal{C}$ (def.) equipped with

1. a (pointed) topologically enriched functor (def.)

$$\otimes \;:\; \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$$

   out of the (pointed) topologival product category of $\mathcal{C}$ with itself (def. 1.26), called the **tensor product**,

2. an object

$$1 \in \mathcal{C}$$

   called the **unit object** or **tensor unit**,

3. a natural isomorphism (def.)

$$a \;:\; ((-)\otimes(-))\otimes(-) \overset{\simeq}{\longrightarrow} (-)\otimes((-)\otimes(-))$$

   called the **associator**,

4. a natural isomorphism

$$\ell \;:\; (1 \otimes (-)) \overset{\simeq}{\longrightarrow} (-)$$

   called the **left unitor**, and a natural isomorphism

$$r \;:\; (-)\otimes 1 \overset{\simeq}{\longrightarrow} (-)$$

   called the **right unitor**,

such that the following two kinds of diagrams commute, for all objects involved:

1. **triangle identity**:

$$\begin{array}{ccc} (x \otimes 1) \otimes y & \overset{a_{x,1,y}}{\longrightarrow} & x \otimes (1 \otimes y) \\ {}_{\rho_x \otimes 1_y}\searrow && \swarrow_{1_x \otimes \lambda_y} \\ & x \otimes y & \end{array}$$

2. the **pentagon identity**:

$$\begin{array}{ccccc} & & (w \otimes x) \otimes (y \otimes z) & & \\ & {}^{\alpha_{w \otimes x, y, z}} \nearrow & & \searrow^{\alpha_{w,x,y \otimes z}} & \\ ((w \otimes x) \otimes y) \otimes z & & & & (w \otimes (x \otimes (y \otimes z))) \\ {}^{\alpha_{w,x,y} \otimes \mathrm{id}_z} \downarrow & & & & \uparrow^{\mathrm{id}_w \otimes \alpha_{x,y,z}} \\ (w \otimes (x \otimes y)) \otimes z & & \underset{\alpha_{w, x \otimes y, z}}{\longrightarrow} & & w \otimes ((x \otimes y) \otimes z) \end{array}$$

Lemma 1.2. ***(Kelly 64)***

Let $(\mathcal{C}, \otimes, 1)$ be a monoidal category, def. 1.1. Then the left and right unitors ℓ and r satisfy the following conditions:

1. $\ell_1 = r_1 : 1 \otimes 1 \overset{\simeq}{\longrightarrow} 1$;
2. *for all objects $x, y \in \mathcal{C}$ the following diagrams commutes:*

$$\begin{array}{ccc} (1 \otimes x) \otimes y & & \\ {}^{\alpha_{1,x,y}} \downarrow & \searrow^{\ell_x \otimes \mathrm{id}_y} & \\ 1 \otimes (x \otimes y) & \underset{\ell_{x \otimes y}}{\longrightarrow} & x \otimes y \end{array} ;$$

and

$$\begin{array}{ccc} x \otimes (y \otimes 1) & & \\ {}^{\alpha^{-1}_{1,x,y}} \downarrow & \searrow^{\mathrm{id}_x \otimes r_y} & \\ (x \otimes y) \otimes 1 & \underset{r_{x \otimes y}}{\longrightarrow} & x \otimes y \end{array} ;$$

For **proof** see at *monoidal category* this lemma and this lemma.

Remark 1.3. Just as for an associative algebra it is sufficient to demand $1a = a$ and $a1 = a$ and $(ab)c = a(bc)$ in order to have that expressions of arbitrary length may be re-bracketed at will, so there is a *coherence theorem for monoidal categories* which states that all ways of freely composing the unitors and associators in a monoidal category (def. 1.1) to go from one expression to another will coincide. Accordingly, much as one may drop the notation for the bracketing in an associative algebra altogether, so one may, with due care, reason about monoidal categories without always making all unitors and associators explicit.

(Here the qualifier "freely" means informally that we must not use any non-formal identification between objects, and formally it means that the diagram in question must be in the image of a strong monoidal functor from a *free* monoidal category. For example if in a particular monoidal category it so happens that the object $X \otimes (Y \otimes Z)$ is actually *equal* to $(X \otimes Y) \otimes Z$, then the various ways of going from one expression to another using only associators *and* this equality no longer need to coincide.)

Definition 1.4. A **(pointed) topological braided monoidal category**, is a (pointed) topological monoidal category $\mathcal{C}$ (def. 1.1) equipped with a natural isomorphism

$$\tau_{x,y} : x \otimes y \to y \otimes x$$

called the **braiding**, such that the following two kinds of diagrams commute for all objects involved ("hexagon identities"):

$$\begin{array}{ccccc} (x \otimes y) \otimes z & \overset{a_{x,y,z}}{\longrightarrow} & x \otimes (y \otimes z) & \overset{\tau_{x, y \otimes z}}{\longrightarrow} & (y \otimes z) \otimes x \\ \downarrow^{\tau_{x,y} \otimes \mathrm{Id}} & & & & \downarrow^{a_{y,z,x}} \\ (y \otimes x) \otimes z & \overset{a_{y,x,z}}{\longrightarrow} & y \otimes (x \otimes z) & \overset{\mathrm{Id} \otimes \tau_{x,z}}{\longrightarrow} & y \otimes (z \otimes x) \end{array}$$

and

$$\begin{array}{ccccc} x \otimes (y \otimes z) & \overset{a^{-1}_{x,y,z}}{\longrightarrow} & (x \otimes y) \otimes z & \overset{\tau_{x \otimes y, z}}{\longrightarrow} & z \otimes (x \otimes y) \\ \downarrow^{\mathrm{Id} \otimes \tau_{y,z}} & & & & \downarrow^{a^{-1}_{z,x,y}} \\ x \otimes (z \otimes y) & \overset{a^{-1}_{x,z,y}}{\longrightarrow} & (x \otimes z) \otimes y & \overset{\tau_{x,z} \otimes \mathrm{Id}}{\longrightarrow} & (z \otimes x) \otimes y \end{array} ,$$

where $a_{x,y,z} : (x \otimes y) \otimes z \to x \otimes (y \otimes z)$ denotes the components of the associator of $\mathcal{C}^{\otimes}$.

Definition 1.5. A **(pointed) topological symmetric monoidal category** is a (pointed) topological braided monoidal category (def. 1.4) for which the braiding

$$\tau_{x,y} : x \otimes y \to y \otimes x$$

satisfies the condition:

$$\tau_{y,x} \circ \tau_{x,y} = 1_{x \otimes y}$$

for all objects x, y

Remark 1.6. In analogy to the coherence theorem for monoidal categories (remark 1.3) there is a coherence theorem for symmetric monoidal categories (def. 1.5), saying that every diagram built freely (see remark 1.6) from associators, unitors and braidings such that both sides of the diagram correspond to the same permutation of objects, coincide.

Definition 1.7. Given a (pointed) topological symmetric monoidal category $\mathcal{C}$ with tensor product $\otimes$ (def. 1.5) it is called a **closed monoidal category** if for each $Y \in \mathcal{C}$ the functor $Y \otimes (-) \simeq (-) \otimes Y$ has a right adjoint, denoted $\hom(Y, -)$

$$\mathcal{C} \underoverset{\underset{\hom(Y,-)}{\longrightarrow}}{\overset{(-)\otimes Y}{\longleftarrow}}{\bot} \mathcal{C},$$

hence if there are natural bijections

$$\mathrm{Hom}_{\mathcal{C}}(X \otimes Y, Z) \;\simeq\; \mathrm{Hom}_{\mathcal{C}}\, C(X, \hom(Y,Z))$$

for all objects $X, Z \in \mathcal{C}$.

Since for the case that $X = 1$ is the tensor unit of $\mathcal{C}$ this means that

$$\mathrm{Hom}_{\mathcal{C}}(1, \hom(Y,Z)) \simeq \mathrm{Hom}_{\mathcal{C}}(Y,Z)\,,$$

the object $\hom(Y,Z) \in \mathcal{C}$ is an enhancement of the ordinary hom-set $\mathrm{Hom}_{\mathcal{C}}(Y,Z)$ to an object in $\mathcal{C}$. Accordingly, it is also called the **internal hom** between Y and Z.

In a closed monoidal category, the adjunction isomorphism between tensor product and internal hom even holds internally:

Proposition 1.8. *In a symmetric closed monoidal category (def. 1.7) there are natural isomorphisms*

$$\hom(X \otimes Y, Z) \;\simeq\; \hom(X, \hom(Y,Z))$$

whose image under $\mathrm{Hom}_{\mathcal{C}}(1,-)$ are the defining natural bijections of def. 1.7.

Proof. Let $A \in \mathcal{C}$ be any object. By applying the defining natural bijections twice, there are composite natural bijections

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(A, \hom(X \otimes Y, Z)) & \simeq \mathrm{Hom}_{\mathcal{C}}(A \otimes (X \otimes Y), Z) \\ & \simeq \mathrm{Hom}_{\mathcal{C}}((A \otimes X) \otimes Y, Z) \\ & \simeq \mathrm{Hom}_{\mathcal{C}}(A \otimes X, \hom(Y,Z)) \\ & \simeq \mathrm{Hom}_{\mathcal{C}}(A, \hom(X, \hom(Y,Z))) \end{aligned} \,.$$

Since this holds for all A, the Yoneda lemma (the fully faithfulness of the Yoneda embedding) says that there is an isomorphism $\hom(X \otimes Y, Z) \simeq \hom(X, \hom(Y,Z))$. Moreover, by taking $A = 1$ in the above and using the left unitor isomorphisms $A \otimes (X \otimes Y) \simeq X \otimes Y$ and $A \otimes X \simeq X$ we get a commuting diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(1, \hom(X \otimes Y,)) & \overset{\simeq}{\longrightarrow} & \mathrm{Hom}_{\mathcal{C}}(1, \hom(X, \hom(Y,Z))) \\ {}^{\simeq}\downarrow && \downarrow^{\simeq} \\ \mathrm{Hom}_{\mathcal{C}}(X \otimes Y, Z) & \overset{\simeq}{\longrightarrow} & \mathrm{Hom}_{\mathcal{C}}(X, \hom(Y,Z)) \end{array} \,.$$

▮

Example 1.9. The category Set of sets and functions between them, regarded as enriched in discrete topological spaces, becomes a symmetric monoidal category according to def. 1.5 with tensor product the Cartesian product $\times$ of sets. The associator, unitor and braiding isomorphism are the evident (almost unnoticable but nevertheless nontrivial) canonical identifications.

Similarly the category $\mathrm{Top}_{\mathrm{cg}}$ of compactly generated topological spaces (def.) becomes a symmetric monoidal category with tensor product the corresponding Cartesian products, hence the operation of forming k-ified (cor.) product topological spaces (exmpl.). The underlying functions of the associator, unitor and braiding isomorphisms are just those of the underlying sets, as above.

Symmetric monoidal categories, such as these, for which the tensor product is the Cartesian product are called *Cartesian monoidal categories*.

Both examples are closed monoidal categories (def. 1.7), with internal hom the mapping spaces (prop.).

Example 1.10. The category $\mathrm{Top}_{\mathrm{cg}}^{*/}$ of pointed compactly generated topological spaces with tensor product the smash product $\wedge$ (def.)

$$X \wedge Y \coloneqq \frac{X \times Y}{X \vee Y}$$

is a symmetric monoidal category (def. 1.5) with unit object the pointed 0-sphere S^0.

The components of the associator, the unitors and the braiding are those of Top as in example 1.9, descended to the quotient topological spaces which appear in the definition of the smash product. This works for pointed compactly generated spaces (but not for general pointed topological spaces) by this prop..

The category $\mathrm{Top}_{\mathrm{cg}}^{*/}$ is also a closed monoidal category (def. 1.7), with internal hom the pointed mapping space $\mathrm{Maps}(-,-)_*$ (exmpl.)

Example 1.11. The category Ab of abelian groups, regarded as enriched in discrete topological spaces, becomes a symmetric monoidal category with tensor product the actual tensor product of abelian groups $\otimes_{\mathbb{Z}}$ and with tensor unit the additive group $\mathbb{Z}$ of integers. Again the associator, unitor and braiding isomorphism are the evident ones coming from the underlying sets, as in example 1.9.

This is a closed monoidal category with internal hom $\mathrm{hom}(A,B)$ being the set of homomorphisms $\mathrm{Hom}_{\mathrm{Ab}}(A,B)$ equipped with the pointwise group structure for $\phi_1, \phi_2 \in \mathrm{Hom}_{\mathrm{Ab}}(A,B)$ then $(\phi_1 + \phi_2)(a) \coloneqq \phi_1(a) + \phi_2(b) \ \in B$.

This is the archetypical case that motivates the notation "$\otimes$" for the pairing operation in a monoidal category:

Example 1.12. The category category of chain complexes $\mathrm{Ch}_\bullet$, equipped with the tensor product of chain complexes is a symmetric monoidal category (def. 1.5).

In this case the braiding has a genuinely non-trivial aspect to it, beyond just the swapping of coordinates as in examples 1.9, 1.10 and def. 1.11, namely for $X, Y \in \mathrm{Ch}_\bullet$ then

$$(X \otimes Y)_n = \underset{n_1 + n_2 = n}{\otimes} X_{n_1} \otimes_{\mathbb{Z}} X_{n_2}$$

and in these components the braiding isomorphism is that of Ab, but with a minus sign thrown in whener two odd-graded components are commuted.

This is a first shadow of the graded-commutativity that also exhibited by spectra.

(e.g. Hovey 99, prop. 4.2.13)

Algebras and modules

Definition 1.13. Given a (pointed) topological monoidal category $(\mathcal{C}, \otimes, 1)$, then a **monoid internal to** $(\mathcal{C}, \otimes, 1)$ is

1. an object $A \in \mathcal{C}$;
2. a morphism $e : 1 \longrightarrow A$ (called the *unit*)
3. a morphism $\mu : A \otimes A \longrightarrow A$ (called the *product*);

such that

1. (associativity) the following diagram commutes

$$\begin{array}{ccccc} (A \otimes A) \otimes A & \underset{\simeq}{\overset{a_{A,A,A}}{\longrightarrow}} & A \otimes (A \otimes A) & \overset{A \otimes \mu}{\longrightarrow} & A \otimes A \\ {}^{\mu \otimes A}\downarrow & & & & \downarrow^{\mu} \\ A \otimes A & \longrightarrow & & \overset{\mu}{\longrightarrow} & A \end{array} ,$$

where a is the associator isomorphism of $\mathcal{C}$;

2. (unitality) the following diagram commutes:

$$\begin{array}{ccccc} 1 \otimes A & \overset{e \otimes \mathrm{id}}{\longrightarrow} & A \otimes A & \overset{\mathrm{id} \otimes e}{\longleftarrow} & A \otimes 1 \\ & {}_{\ell}\searrow & \downarrow^{\mu} & \swarrow_{r} & \\ && A && \end{array} ,$$

where ℓ and r are the left and right unitor isomorphisms of $\mathcal{C}$.

Moreover, if $(\mathcal{C}, \otimes, 1)$ has the structure of a symmetric monoidal category (def. 1.5) $(\mathcal{C}, \otimes, 1, B)$ with symmetric braiding τ, then a monoid (A, μ, e) as above is called a **commutative monoid in** $(\mathcal{C}, \otimes, 1, B)$ if in addition

- (commutativity) the following diagram commutes

$$\begin{array}{ccc} A \otimes A & \underset{\simeq}{\overset{\tau_{A,A}}{\longrightarrow}} & A \otimes A \\ & {}_{\mu}\searrow \quad \swarrow_{\mu} & \\ & A & \end{array} .$$

A homomorphism of monoids $(A_1, \mu_1, e_1) \longrightarrow (A_2, \mu_2, f_2)$ is a morphism

$$f : A_1 \longrightarrow A_2$$

in $\mathcal{C}$, such that the following two diagrams commute

$$\begin{array}{ccc} A_1 \otimes A_1 & \overset{f \otimes f}{\longrightarrow} & A_2 \otimes A_2 \\ {}^{\mu_1}\downarrow && \downarrow^{\mu_2} \\ A_1 & \underset{f}{\longrightarrow} & A_2 \end{array}$$

and

$$\begin{array}{ccc} 1_c & \overset{e_1}{\longrightarrow} & A_1 \\ & {}_{e_2}\searrow & \downarrow^{f} \\ && A_2 \end{array} .$$

Write $\mathrm{Mon}(\mathcal{C}, \otimes, 1)$ for the **category of monoids** in $\mathcal{C}$, and $\mathrm{CMon}(\mathcal{C}, \otimes, 1)$ for its subcategory of commutative monoids.

Example 1.14. Given a (pointed) topological monoidal category $(\mathcal{C}, \otimes, 1)$, then the tensor unit 1 is a monoid in $\mathcal{C}$ (def. 1.13) with product given by either the left or right unitor

$$\ell_1 = r_1 : 1 \otimes 1 \overset{\simeq}{\longrightarrow} 1 .$$

By lemma 1.2, these two morphisms coincide and define an associative product with unit the identity $\mathrm{id} : 1 \to 1$.

If $(\mathcal{C}, \otimes, 1)$ is a symmetric monoidal category (def. 1.5), then this monoid is a commutative monoid.

Example 1.15. Given a symmetric monoidal category $(\mathcal{C}, \otimes, 1)$ (def. 1.5), and given two commutative monoids (E_i, μ_i, e_i) $i \in \{1,2\}$ (def. 1.13), then the tensor product $E_1 \otimes E_2$ becomes itself a commutative monoid with unit morphism

$$e : 1 \overset{\simeq}{\longrightarrow} 1 \otimes 1 \overset{e_1 \otimes e_2}{\longrightarrow} E_1 \otimes E_2$$

(where the first isomorphism is, $\ell_1^{-1} = r_1^{-1}$ (lemma 1.2)) and with product morphism given by

$$E_1 \otimes E_2 \otimes E_1 \otimes E_2 \overset{\mathrm{id} \otimes \tau_{E_2, E_1} \otimes \mathrm{id}}{\longrightarrow} E_1 \otimes E_1 \otimes E_2 \otimes E_2 \overset{\mu_1 \otimes \mu_2}{\longrightarrow} E_1 \otimes E_2$$

(where we are notationally suppressing the associators and where τ denotes the braiding of $\mathcal{C}$).

That this definition indeed satisfies associativity and commutativity follows from the corresponding properties of (E_i, μ_i, e_i), and from the hexagon identities for the braiding (def. 1.4) and from symmetry of the braiding.

Similarly one checks that for $E_1 = E_2 = E$ then the unit maps

$$\begin{array}{c} E \simeq E \otimes 1 \overset{\mathrm{id} \otimes e}{\longrightarrow} E \otimes E \\ E \simeq 1 \otimes E \overset{e \otimes 1}{\longrightarrow} E \otimes E \end{array}$$

and the product map

$$\mu : E \otimes E \longrightarrow E$$

and the braiding

$$\tau_{E,E} : E \otimes E \longrightarrow E \otimes E$$

are monoid homomorphisms, with $E \otimes E$ equipped with the above monoid structure.

Definition 1.16. Given a (pointed) topological monoidal category $(\mathcal{C}, \otimes, 1)$ (def. 1.1), and given (A, μ, e) a monoid in $(\mathcal{C}, \otimes, 1)$ (def. 1.13), then a **left module object** in $(\mathcal{C}, \otimes, 1)$ over (A, μ, e) is

1. an object $N \in \mathcal{C}$;
2. a morphism $\rho : A \otimes N \longrightarrow N$ (called the *action*);

such that

1. (unitality) the following diagram commutes:

$$\begin{tikzcd} 1 \otimes N \arrow[r, "e \otimes \mathrm{id}"] \arrow[dr, "\ell"'] & A \otimes N \arrow[d, "\rho"] \\ & N \end{tikzcd} ,$$

   where ℓ is the left unitor isomorphism of $\mathcal{C}$.

2. (action property) the following diagram commutes

$$\begin{tikzcd} (A \otimes A) \otimes N \arrow[r, "a_{A,A,N}", "\simeq"'] \arrow[d, "\mu \otimes N"'] & A \otimes (A \otimes N) \arrow[r, "A \otimes \rho"] & A \otimes N \arrow[d, "\rho"] \\ A \otimes N \arrow[rr, "\rho"] & & N \end{tikzcd} ,$$

A homomorphism of left A-module objects

$$(N_1, \rho_1) \longrightarrow (N_2, \rho_2)$$

is a morphism

$$f : N_1 \longrightarrow N_2$$

in $\mathcal{C}$, such that the following diagram commutes:

$$\begin{tikzcd} A \otimes N_1 \arrow[r, "A \otimes f"] \arrow[d, "\rho_1"'] & A \otimes N_2 \arrow[d, "\rho_2"] \\ N_1 \arrow[r, "f"'] & N_2 \end{tikzcd} .$$

For the resulting **category of modules** of left A-modules in $\mathcal{C}$ with A-module homomorphisms between them, we write

$$A\,\mathrm{Mod}(\mathcal{C}) \,.$$

This is naturally a (pointed) topologically enriched category itself.

Example 1.17. Given a monoidal category $(\mathcal{C}, \otimes, 1)$ (def. 1.1) with the tensor unit 1 regarded as a monoid in a monoidal category via example 1.14, then the left unitor

$$\ell_C : 1 \otimes C \longrightarrow C$$

makes every object $C \in \mathcal{C}$ into a left module, according to def. 1.16, over C. The action property holds due to lemma 1.2. This gives an equivalence of categories

$$\mathcal{C} \simeq 1\mathrm{Mod}(\mathcal{C})$$

of $\mathcal{C}$ with the category of modules over its tensor unit.

Example 1.18. The archetypical case in which all these abstract concepts reduce to the basic familiar ones is the symmetric monoidal category Ab of abelian groups from example 1.11.

1. A monoid in $(\mathrm{Ab}, \otimes_{\mathbb{Z}}, \mathbb{Z})$ (def. 1.13) is equivalently a ring.
2. A commutative monoid in in $(\mathrm{Ab}, \otimes_{\mathbb{Z}}, \mathbb{Z})$ (def. 1.13) is equivalently a commutative ring R.

3. An R-module object in $(\mathrm{Ab}, \otimes_{\mathbb{Z}}, \mathbb{Z})$ (def. 1.16) is equivalently an R-module;

4. The tensor product of R-module objects (def. 1.21) is the standard tensor product of modules.

5. The category of module objects $R\,\mathrm{Mod}(\mathrm{Ab})$ (def. 1.21) is the standard category of modules $R\,\mathrm{Mod}$.

Example 1.19. Closely related to the example 1.18, but closer to the structure we will see below for spectra, are monoids in the category of chain complexes $(\mathrm{Ch}_\bullet, \otimes, \mathbb{Z})$ from example 1.12. These monoids are equivalently differential graded algebras.

Proposition 1.20. *In the situation of def. 1.16, the monoid (A, μ, e) canonically becomes a left module over itself by setting $\rho \coloneqq \mu$. More generally, for $C \in \mathcal{C}$ any object, then $A \otimes C$ naturally becomes a left A-module by setting:*

$$\rho : A \otimes (A \otimes C) \underset{\simeq}{\overset{a^{-1}_{A,A,C}}{\longrightarrow}} (A \otimes A) \otimes C \overset{\mu \otimes \mathrm{id}}{\longrightarrow} A \otimes C .$$

*The A-modules of this form are called **free modules**.*

The free functor F constructing free A-modules is left adjoint to the forgetful functor U which sends a module (N, ρ) to the underlying object $U(N, \rho) \coloneqq N$.

$$A\,\mathrm{Mod}(\mathcal{C}) \underset{U}{\overset{F}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathcal{C} .$$

Proof. A homomorphism out of a free A-module is a morphism in $\mathcal{C}$ of the form

$$f : A \otimes C \longrightarrow N$$

fitting into the diagram (where we are notationally suppressing the associator)

$$\begin{array}{ccc} A \otimes A \otimes C & \overset{A \otimes f}{\longrightarrow} & A \otimes N \\ {}^{\mu \otimes \mathrm{id}}\downarrow && \downarrow^{\rho} \\ A \otimes C & \underset{f}{\longrightarrow} & N \end{array} .$$

Consider the composite

$$\tilde f : C \underset{\simeq}{\overset{\ell_C}{\longrightarrow}} 1 \otimes C \overset{e \otimes \mathrm{id}}{\longrightarrow} A \otimes C \overset{f}{\longrightarrow} N ,$$

i.e. the restriction of f to the unit "in" A. By definition, this fits into a commuting square of the form (where we are now notationally suppressing the associator and the unitor)

$$\begin{array}{ccc} A \otimes C & \overset{\mathrm{id} \otimes \tilde f}{\longrightarrow} & A \otimes N \\ {}^{\mathrm{id} \otimes e \otimes \mathrm{id}}\downarrow && \downarrow^{=} \\ A \otimes A \otimes C & \underset{\mathrm{id} \otimes f}{\longrightarrow} & A \otimes N \end{array} .$$

Pasting this square onto the top of the previous one yields

$$\begin{array}{ccc} A \otimes C & \overset{\mathrm{id} \otimes \tilde f}{\longrightarrow} & A \otimes N \\ {}^{\mathrm{id} \otimes e \otimes \mathrm{id}}\downarrow && \downarrow^{=} \\ A \otimes A \otimes C & \overset{A \otimes f}{\longrightarrow} & A \otimes N \\ {}^{\mu \otimes \mathrm{id}}\downarrow && \downarrow^{\rho} \\ A \otimes C & \underset{f}{\longrightarrow} & N \end{array} ,$$

where now the left vertical composite is the identity, by the unit law in A. This shows that f is uniquely determined by $\tilde f$ via the relation

$$f = \rho \circ (\mathrm{id}_A \otimes \tilde f) .$$

This natural bijection between f and $\tilde f$ establishes the adjunction. ▮

Definition 1.21. Given a (pointed) topological closed symmetric monoidal category $(\mathcal{C}, \otimes, 1)$ (def. 1.5, def. 1.7), given (A, μ, e) a commutative monoid in $(\mathcal{C}, \otimes, 1)$ (def. 1.13), and given (N_1, ρ_1) and (N_2, ρ_2) two left A-module objects (def.1.13), then

1. the **tensor product of modules** $N_1 \otimes_A N_2$ is, if it exists, the coequalizer

$$N_1 \otimes A \otimes N_2 \underset{\rho_1 \circ (\tau_{N_1,A} \otimes N_2)}{\overset{N_1 \otimes \rho_2}{\rightrightarrows}} N_1 \otimes N_1 \overset{\mathrm{coeq}}{\longrightarrow} N_1 \otimes_A N_2$$

and if $A \otimes (-)$ preserves these coequalizers, then this is equipped with the left A-action induced from the left A-action on N_1

2. the **function module** $\hom_A(N_1, N_2)$ is, if it exists, the equalizer

$$\hom_A(N_1, N_2) \overset{\mathrm{equ}}{\longrightarrow} \hom(N_1, N_2) \underset{\hom(A \otimes N_1, \rho_2) \circ (A \otimes (-))}{\overset{\hom(\rho_1, N_2)}{\rightrightarrows}} \hom(A \otimes N_1, N_2) \,.$$

equipped with the left A-action that is induced by the left A-action on N_2 via

$$\frac{A \otimes \hom(X, N_2) \longrightarrow \hom(X, N_2)}{A \otimes \hom(X, N_2) \otimes X \overset{\mathrm{id} \otimes \mathrm{ev}}{\longrightarrow} A \otimes N_2 \overset{\rho_2}{\longrightarrow} N_2} \,.$$

(e.g. Hovey-Shipley-Smith 00, lemma 2.2.2 and lemma 2.2.8)

Proposition 1.22. *Given a (pointed) topological closed symmetric monoidal category $(\mathcal{C}, \otimes, 1)$ (def. 1.5, def. 1.7), and given (A, μ, e) a commutative monoid in $(\mathcal{C}, \otimes, 1)$ (def. 1.13). If all coequalizers exist in $\mathcal{C}$, then the tensor product of modules $\otimes_A$ from def. 1.21 makes the category of modules $A\,\mathrm{Mod}(\mathcal{C})$ into a symmetric monoidal category, $(A\,\mathrm{Mod}, \otimes_A, A)$ with tensor unit the object A itself, regarded as an A-module via prop. 1.20.*

If moreover all equalizers exist, then this is a closed monoidal category (def. 1.7) with internal hom given by the function modules $\hom_A$ of def. 1.21.

(e.g. Hovey-Shipley-Smith 00, lemma 2.2.2, lemma 2.2.8)

Proof sketch. The associators and braiding for $\otimes_A$ are induced directly from those of $\otimes$ and the universal property of coequalizers. That A is the tensor unit for $\otimes_A$ follows with the same kind of argument that we give in the proof of example 1.23 below. ▮

Example 1.23. For (A, μ, e) a monoid (def. 1.13) in a symmetric monoidal category $(\mathcal{C}, \otimes, 1)$ (def. 1.1), the tensor product of modules (def. 1.21) of two free modules (def. 1.20) $A \otimes C_1$ and $A \otimes C_2$ always exists and is the free module over the tensor product in $\mathcal{C}$ of the two generators:

$$(A \otimes C_1) \otimes_A (A \otimes C_2) \simeq A \otimes (C_1 \otimes C_2) \,.$$

Hence if $\mathcal{C}$ has all coequalizers, so that the category of modules is a monoidal category $(A\,\mathrm{Mod}, \otimes_A, A)$ (prop. 1.22) then the free module functor (def. 1.20) is a strong monoidal functor (def. 1.47)

$$F \,:\, (\mathcal{C}, \otimes, 1) \longrightarrow (A\,\mathrm{Mod}, \otimes_A, A) \,.$$

Proof. It is sufficient to show that the diagram

$$A \otimes A \otimes A \underset{\mathrm{id} \otimes \mu}{\overset{\mu \otimes \mathrm{id}}{\rightrightarrows}} A \otimes A \overset{\mu}{\longrightarrow} A$$

is a coequalizer diagram (we are notationally suppressing the associators), hence that $A \otimes_A A \simeq A$, hence that the claim holds for $C_1 = 1$ and $C_2 = 1$.

To that end, we check the universal property of the coequalizer:

First observe that μ indeed coequalizes $\mathrm{id} \otimes \mu$ with $\mu \otimes \mathrm{id}$, since this is just the associativity clause in def. 1.13. So for $f : A \otimes A \longrightarrow Q$ any other morphism with this property, we need to show that there is a unique morphism $\phi : A \longrightarrow Q$ which makes this diagram commute:

$$\begin{array}{ccc} A \otimes A & \overset{\mu}{\longrightarrow} & A \\ {}^{\mathllap{f}}\downarrow & \swarrow_{\mathrlap{\phi}} & \\ Q & & \end{array} \,.$$

We claim that

$$\phi \,:\, A \underset{\simeq}{\overset{r^{-1}}{\longrightarrow}} A \otimes 1 \overset{\mathrm{id} \otimes e}{\longrightarrow} A \otimes A \overset{f}{\longrightarrow} Q \,,$$

where the first morphism is the inverse of the right unitor of $\mathcal{C}$.

First to see that this does make the required triangle commute, consider the following pasting composite of commuting diagrams

$$
\begin{array}{ccc}
A \otimes A & \overset{\mu}{\longrightarrow} & A \\
{}^{\mathrm{id}\otimes r^{-1}}_{\simeq}\downarrow && \downarrow^{r^{-1}}_{\simeq} \\
A \otimes A \otimes 1 & \overset{\mu \otimes \mathrm{id}}{\longrightarrow} & A \otimes 1 \\
{}^{\mathrm{id}\otimes e}\downarrow && \downarrow^{\mathrm{id}\otimes e} \\
A \otimes A \otimes A & \overset{\mu \otimes \mathrm{id}}{\longrightarrow} & A \otimes A \\
{}^{\mathrm{id}\otimes \mu}\downarrow && \downarrow^{f} \\
A \otimes A & \underset{f}{\longrightarrow} & Q
\end{array}
$$

Here the the top square is the naturality of the right unitor, the middle square commutes by the functoriality of the tensor product $\otimes\ : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ and the definition of the product category (def. 1.26), while the commutativity of the bottom square is the assumption that f coequalizes $\mathrm{id} \otimes \mu$ with $\mu \otimes \mathrm{id}$.

Here the right vertical composite is ϕ, while, by unitality of (A, μ, e), the left vertical composite is the identity on A, Hence the diagram says that $\phi \circ \mu = f$, which we needed to show.

It remains to see that ϕ is the unique morphism with this property for given f. For that let $q : A \to Q$ be any other morphism with $q \circ \mu = f$. Then consider the commuting diagram

$$
\begin{array}{ccc}
A \otimes 1 & \overset{\simeq}{\longleftarrow} & A \\
{}^{\mathrm{id}\otimes e}\downarrow & \searrow^{\simeq} & \downarrow^{=} \\
A \otimes A & \overset{\mu}{\longrightarrow} & A \\
{}^{f}\downarrow & \swarrow_{q} & \\
Q & &
\end{array}
$$

where the top left triangle is the unitality condition and the two isomorphisms are the right unitor and its inverse. The commutativity of this diagram says that $q = \phi$. ▮

Definition 1.24. Given a monoidal category of modules $(A\,\mathrm{Mod}, \otimes_A, A)$ as in prop. 1.22, then a monoid (E, μ, e) in $(A\,\mathrm{Mod}, \otimes_A, A)$ (def. 1.13) is called an A-**algebra**.

Propposition 1.25. *Given a monoidal category of modules $(A\,\mathrm{Mod}, \otimes_A, A)$ in a monoidal category $(\mathcal{C}, \otimes, 1)$ as in prop. 1.22, and an A-algebra (E, μ, e) (def. 1.24), then there is an equivalence of categories*

$$
A\,\mathrm{Alg}_{\mathrm{comm}}(\mathcal{C}) \coloneqq \mathrm{CMon}(A\,\mathrm{Mod}) \simeq \mathrm{CMon}(\mathcal{C})^{A/}
$$

between the category of commutative monoids in $A\,\mathrm{Mod}$ and the coslice category of commutative monoids in $\mathcal{C}$ under A, hence between commutative A-algebras in $\mathcal{C}$ and commutative monoids E in $\mathcal{C}$ that are equipped with a homomorphism of monoids $A \longrightarrow E$.

(e.g. EKMM 97, VII lemma 1.3)

Proof. In one direction, consider a A-algebra E with unit $e_E : A \longrightarrow E$ and product $\mu_{E/A} : E \otimes_A E \longrightarrow E$. There is the underlying product μ_E

$$
\begin{array}{ccccc}
E \otimes A \otimes E & \underoverset{\longrightarrow}{\longrightarrow}{} & E \otimes E & \overset{\mathrm{coeq}}{\longrightarrow} & E \otimes_A E \\
&& {}^{\mu_E}\searrow && \downarrow^{\mu_{E/A}} \\
&&&& E
\end{array}
$$

By considering a diagram of such coequalizer diagrams with middle vertical morphism $e_E \circ e_A$, one find that this is a unit for μ_E and that $(E, \mu_E, e_E \circ e_A)$ is a commutative monoid in $(\mathcal{C}, \otimes, 1)$.

Then consider the two conditions on the unit $e_E : A \longrightarrow E$. First of all this is an A-module homomorphism, which means that

$$
(\star) \qquad
\begin{array}{ccc}
A \otimes A & \overset{\mathrm{id}\otimes e_E}{\longrightarrow} & A \otimes E \\
{}^{\mu_A}\downarrow && \downarrow^{\rho} \\
A & \underset{e_E}{\longrightarrow} & E
\end{array}
$$

commutes. Moreover it satisfies the unit property

$$\begin{array}{ccc} A \otimes_A E & \overset{e_A \otimes \mathrm{id}}{\longrightarrow} & E \otimes_A E \\ & {}_{\simeq}\searrow & \downarrow^{\mu_{E/A}} \\ && E \end{array}$$

By forgetting the tensor product over A, the latter gives

$$\begin{array}{ccc} A \otimes E & \overset{e \otimes \mathrm{id}}{\longrightarrow} & E \otimes E \\ \downarrow && \downarrow \\ A \otimes_A E & \overset{e_E \otimes \mathrm{id}}{\longrightarrow} & E \otimes_A E \\ {}^{\simeq}\downarrow && \downarrow^{\mu_{E/A}} \\ E & = & E \end{array} \;\;\simeq\;\; \begin{array}{ccc} A \otimes E & \overset{e_E \otimes \mathrm{id}}{\longrightarrow} & E \otimes E \\ {}^{\rho}\downarrow && \downarrow^{\mu_E} \\ E & \underset{\mathrm{id}}{\longrightarrow} & E \end{array} \,,$$

where the top vertical morphisms on the left the canonical coequalizers, which identifies the vertical composites on the right as shown. Hence this may be pasted to the square $(\star)$ above, to yield a commuting square

$$\begin{array}{ccccc} A \otimes A & \overset{\mathrm{id} \otimes e_E}{\longrightarrow} & A \otimes E & \overset{e_E \otimes \mathrm{id}}{\longrightarrow} & E \otimes E \\ {}^{\mu_A}\downarrow && {}^{\rho}\downarrow && \downarrow^{\mu_E} \\ A & \underset{e_E}{\longrightarrow} & E & \underset{\mathrm{id}}{\longrightarrow} & E \end{array} \;\;=\;\; \begin{array}{ccc} A \otimes A & \overset{e_E \otimes e_E}{\longrightarrow} & E \otimes E \\ {}^{\mu_A}\downarrow && \downarrow^{\mu_E} \\ A & \underset{e_E}{\longrightarrow} & E \end{array} \,.$$

This shows that the unit e_A is a homomorphism of monoids $(A, \mu_A, e_A) \longrightarrow (E, \mu_E, e_E \circ e_A)$.

Now for the converse direction, assume that (A, μ_A, e_A) and (E, μ_E, e'_E) are two commutative monoids in $(\mathcal{C}, \otimes, 1)$ with $e_E : A \to E$ a monoid homomorphism. Then E inherits a left A-module structure by

$$\rho : A \otimes E \overset{e_A \otimes \mathrm{id}}{\longrightarrow} E \otimes E \overset{\mu_E}{\longrightarrow} E \,.$$

By commutativity and associativity it follows that μ_E coequalizes the two induced morphisms $E \otimes A \otimes E \underset{\longrightarrow}{\overset{\longrightarrow}{}} E \otimes E$. Hence the universal property of the coequalizer gives a factorization through some $\mu_{E/A} : E \otimes_A E \longrightarrow E$. This shows that $(E, \mu_{E/A}, e_E)$ is a commutative A-algebra.

Finally one checks that these two constructions are inverses to each other, up to isomorphism. ▮

Topological ends and coends

For working with pointed topologically enriched functors, a certain shape of limits/colimits is particularly relevant: these are called (pointed topological enriched) *ends* and *coends*. We here introduce these and then derive some of their basic properties, such as notably the expression for topological left Kan extension in terms of coends (prop. 1.38 below). Further below it is via left Kan extension along the ordinary smash product of pointed topological spaces ("Day convolution") that the symmetric monoidal smash product of spectra is induced.

Definition 1.26. Let $\mathcal{C}, \mathcal{D}$ be pointed topologically enriched categories (def.), i.e. enriched categories over $(\mathrm{Top}_{\mathrm{cg}}^{*/}, \wedge, S^0)$ from example 1.10.

1. The **pointed topologically enriched opposite category** $\mathcal{C}^{\mathrm{op}}$ is the topologically enriched category with the same objects as $\mathcal{C}$, with hom-spaces

$$\mathcal{C}^{\mathrm{op}}(X,Y) \coloneqq \mathcal{C}(Y,X)$$

and with composition given by braiding followed by the composition in $\mathcal{C}$:

$$\mathcal{C}^{\mathrm{op}}(X,Y) \wedge \mathcal{C}^{\mathrm{op}}(Y,Z) = \mathcal{C}(Y,X) \wedge \mathcal{C}(Z,Y) \underset{\simeq}{\overset{\tau}{\longrightarrow}} \mathcal{C}(Z,Y) \wedge \mathcal{C}(Y,X) \overset{\circ_{Z,Y,X}}{\longrightarrow} \mathcal{C}(Z,X) = \mathcal{C}^{\mathrm{op}}(X,Z) \,.$$

2. the **pointed topological product category** $\mathcal{C} \times \mathcal{D}$ is the topologically enriched category whose objects are pairs of objects (c,d) with $c \in \mathcal{C}$ and $d \in \mathcal{D}$, whose hom-spaces are the smash product of the separate hom-spaces

$$(\mathcal{C} \times \mathcal{D})((c_1, d_1), (c_2, d_2)) \coloneqq \mathcal{C}(c_1, c_2) \wedge \mathcal{D}(d_1, d_2)$$

and whose composition operation is the braiding followed by the smash product of the separate composition operations:

$$
\begin{array}{ccc}
(\mathcal{C}\times\mathcal{D})((c_1,d_1),\ (c_2,d_2)) \wedge (\mathcal{C}\times\mathcal{D})((c_2,d_2),\ (c_3,d_3)) && \\
{}^{=}\downarrow && \\
(\mathcal{C}(c_1,c_2)\wedge\mathcal{D}(d_1,d_2)) \wedge (\mathcal{C}(c_2,c_3)\wedge\mathcal{D}(d_2,d_3)) && \\
\downarrow^{\tau}_{\simeq} && \\
(\mathcal{C}(c_1,c_2)\wedge\mathcal{C}(c_2,c_3)) \wedge (\mathcal{D}(d_1,d_2)\wedge\mathcal{D}(d_2,d_3)) & \xrightarrow{(\circ_{c_1,c_2,c_3})\wedge(\circ_{d_1,d_2,d_3})} & \mathcal{C}(c_1,c_3)\wedge\mathcal{D}(d_1,d_3) \\
&& \downarrow^{=} \\
&& (\mathcal{C}\times\mathcal{D})((c_1,d_1),\ (c_3,d_3))
\end{array}
.
$$

Example 1.27. A pointed topologically enriched functor (def.) into $\mathrm{Top}_{\mathrm{cg}}^{*/}$ (exmpl.) out of a pointed topological product category as in def. 1.26

$$
F : \mathcal{C}\times\mathcal{D} \longrightarrow \mathrm{Top}_{\mathrm{cg}}^{*/}
$$

(a “pointed topological bifunctor”) has component maps of the form

$$
F_{(c_1,d_1),(c_2,d_2)} : \mathcal{C}(c_1,c_2)\wedge\mathcal{D}(d_1,d_2) \longrightarrow \mathrm{Maps}(F_0((c_1,d_1)), F_0((c_2,d_2)))_* \,.
$$

By functoriality and under passing to adjuncts (cor.) this is equivalent to two commuting *actions*

$$
\rho_{c_1,c_2}(d) : \mathcal{C}(c_1,c_2)\wedge F_0((c_1,d)) \longrightarrow F_0((c_2,d))
$$

and

$$
\rho_{d_1,d_2}(c) : \mathcal{D}(d_1,d_2)\wedge F_0((c,d_1)) \longrightarrow F_0((c,d_2)) \,.
$$

In the special case of a functor out of the product category of some $\mathcal{C}$ with its opposite category (def. 1.26)

$$
F : \mathcal{C}^{\mathrm{op}}\times\mathcal{C} \longrightarrow \mathrm{Top}_{\mathrm{cg}}^{*/}
$$

then this takes the form of a “pullback action” in the first variable

$$
\rho_{c_2,c_1}(d) : \mathcal{C}(c_1,c_2)\wedge F_0((c_2,d)) \longrightarrow F_0((c_1,d))
$$

and a “pushforward action” in the second variable

$$
\rho_{d_1,d_2}(c) : \mathcal{C}(d_1,d_2)\wedge F_0((c,d_1)) \longrightarrow F_0((c,d_2)) \,.
$$

Definition 1.28. Let $\mathcal{C}$ be a small pointed topologically enriched category (def.), i.e. an enriched category over $(\mathrm{Top}_{\mathrm{cg}}^{*/}, \wedge, S^0)$ from example 1.10. Let

$$
F : \mathcal{C}^{\mathrm{op}}\times\mathcal{C} \longrightarrow \mathrm{Top}_{\mathrm{cg}}^{*/}
$$

be a pointed topologically enriched functor (def.) out of the pointed topological product category of $\mathcal{C}$ with its opposite category, according to def. 1.26.

1. The **coend** of F, denoted $\int^{c\in\mathcal{C}} F(c,c)$, is the coequalizer in $\mathrm{Top}_{\mathrm{cg}}^{*}$ (prop., exmpl., prop., cor.) of the two actions encoded in F via example 1.27:

$$
\coprod_{c,d\in\mathcal{C}} \mathcal{C}(c,d)\wedge F(d,c) \underoverset{\underset{c,d}{\sqcup}\rho_{(d,c)}(c)}{\underset{c,d}{\sqcup}\rho_{(c,d)}(d)}{\rightrightarrows} \coprod_{c\in\mathcal{C}} F(c,c) \overset{\mathrm{coeq}}{\longrightarrow} \int^{c\in\mathcal{C}} F(c,c) \,.
$$

2. The **end** of F, denoted $\int_{c\in\mathcal{C}} F(c,c)$, is the **equalizer** in $\mathrm{Top}_{\mathrm{cg}}^{*/}$ (prop., exmpl., prop., cor.) of the adjuncts of the two actions encoded in F via example 1.27:

$$
\int_{c\in\mathcal{C}} F(c,c) \overset{\mathrm{equ}}{\longrightarrow} \prod_{c\in\mathcal{C}} F(c,c) \underoverset{\underset{c,d}{\sqcup}\tilde\rho_{(c,d)}(c)}{\underset{c,d}{\sqcup}\tilde\rho_{d,c}(d)}{\rightrightarrows} \prod_{c\in\mathcal{C}} \mathrm{Maps}(\mathcal{C}(c,d),\ F(c,d))_* \,.
$$

Example 1.29. Let G be a topological group. Write $\mathbf{B}(G_+)$ for the pointed topologically enriched category that has a single object $*$, whose single hom-space is G_+ (G with a basepoint freely adjoined (def.))

$$
\mathbf{B}(G_+)(*,*) \coloneqq G_+
$$

and whose composition operation is the product operation $(-)\cdot(-)$ in G under adjoining basepoints (exmpl.)

$$G_+ \wedge G_+ \simeq (G \times G)_+ \overset{((-)\cdot(-))_+}{\longrightarrow} G_+ \;.$$

Then a topologically enriched functor

$$(X, \rho_l) \;:\; \mathbf{B}(G_+) \longrightarrow \mathrm{Top}_{\mathrm{cg}}^{*/}$$

is a pointed topological space $X \coloneqq F(*)$ equipped with a continuous function

$$\rho_l \;:\; G_+ \wedge X \longrightarrow X$$

satisfying the action property. Hence this is equivalently a continuous and basepoint-preserving left action (non-linear representation) of G on X.

The opposite category (def. 1.26) $(\mathbf{B}(G_+))^{\mathrm{op}}$ comes from the opposite group

$$(\mathbf{B}(G_+))^{\mathrm{op}} = \mathbf{B}(G_+^{\mathrm{op}}) \;.$$

(The canonical continuous isomorphism $G \simeq G^{\mathrm{op}}$ induces a canonical euqivalence of topologically enriched categories $(\mathbf{B}(G_+))^{\mathrm{op}} \simeq \mathbf{B}(G_+)$.)

So a topologically enriched functor

$$(Y, \rho_r) \;:\; (\mathbf{B}(G_+))^{\mathrm{op}} \longrightarrow \mathrm{Top}^*_{\mathrm{cg}}$$

is equivalently a basepoint preserving continuous *right* action of G.

Therefore the coend of two such functors (def. 1.28) coequalizes the relation

$$(xg,\; y) \sim (x,\; gy)$$

(where juxtaposition denotes left/right action) and hence is equivalently the canonical smash product of a right G-action with a left G-action, hence the quotient of the plain smash product by the diagonal action of the group G:

$$\int^{* \in \mathbf{B}(G_+)} (Y, \rho_r)(*) \wedge (X, \rho_l)(*) \;\simeq\; Y \wedge_G X \;.$$

Example 1.30. Let $\mathcal{C}$ be a small pointed topologically enriched category (def.). For $F, G : \mathcal{C} \longrightarrow \mathrm{Top}_{\mathrm{cg}}^{*/}$ two pointed topologically enriched functors, then the end (def. 1.28) of $\mathrm{Maps}(F(-), G(-))_*$ is a topological space whose underlying pointed set is the pointed set of natural transformations $F \to G$ (def.):

$$U\Big(\int_{c \in \mathcal{C}} \mathrm{Maps}(F(c), G(c))_* \Big) \;\simeq\; \mathrm{Hom}_{[\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}]}(F, G) \;.$$

Proof. The underlying pointed set functor $U : \mathrm{Top}_{\mathrm{cg}}^{*/} \to \mathrm{Set}^{*/}$ preserves all limits (prop., prop., prop.). Therefore there is an equalizer diagram in $\mathrm{Set}^{*/}$ of the form

$$U\Big(\int_{c \in \mathcal{C}} \mathrm{Maps}(F(c), G(c))_* \Big) \overset{\mathrm{equ}}{\longrightarrow} \prod_{c \in \mathcal{C}} \mathrm{Hom}_{\mathrm{Top}_{\mathrm{cg}}^{*/}}(F(c), G(c)) \underset{\underset{c,d}{\sqcup} U(\tilde\rho_{(c,d)}(c))}{\overset{\underset{c,d}{\sqcup} U(\tilde\rho_{d,c}(d))}{\rightrightarrows}} \prod_{c,d \in \mathcal{C}} \mathrm{Hom}_{\mathrm{Top}_{\mathrm{cg}}^{*/}}(\mathcal{C}(c,d), \mathrm{Maps}(F(c), G(d))_*) \;.$$

Here the object in the middle is just the set of collections of component morphisms $\left\{ F(c) \overset{\eta_c}{\to} G(c) \right\}_{c \in \mathcal{C}}$. The two parallel maps in the equalizer diagram take such a collection to the functions which send any $c \overset{f}{\to} d$ to the result of precomposing

$$\begin{array}{ccc} F(c) && \\ {}^{\mathllap{f(f)}}\downarrow && \\ F(d) & \underset{\eta_d}{\longrightarrow} & G(d) \end{array}$$

and of postcomposing

$$\begin{array}{ccc} F(c) & \overset{\eta_c}{\longrightarrow} & G(c) \\ && \downarrow^{\mathrlap{G(f)}} \\ && G(d) \end{array}$$

each component in such a collection, respectively. These two functions being equal, hence the collection

$\{\eta_c\}_{c\in\mathcal{C}}$ being in the equalizer, means precisley that for all c, d and all $f : c \to d$ the square

$$\begin{array}{ccc} F(c) & \overset{\eta_c}{\longrightarrow} & G(c) \\ {}^{\mathllap{F(f)}}\downarrow && \downarrow^{\mathrlap{G(f)}} \\ F(d) & \underset{\eta_d}{\longrightarrow} & G(g) \end{array}$$

is a commuting square. This is precisley the condition that the collection $\{\eta_c\}_{c\in\mathcal{C}}$ be a natural transformation. ▮

Conversely, example 1.30 says that ends over bifunctors of the form $\mathrm{Maps}(F(-), G(-))_*$ constitute hom-spaces between pointed topologically enriched functors:

Definition 1.31. Let $\mathcal{C}$ be a small pointed topologically enriched category (def.). Define the structure of a pointed topologically enriched category on the category $[\mathcal{C}, \mathrm{Top}^{*/}_{\mathrm{cg}}]$ of pointed topologically enriched functors to $\mathrm{Top}^{*/}_{\mathrm{cg}}$ (exmpl.) by taking the hom-spaces to be given by the ends (def. 1.28) of example 1.30:

$$[\mathcal{C}, \mathrm{Top}^{*/}_{\mathrm{cg}}](F, G) \;\coloneqq\; \int_{c\in\mathcal{C}} \mathrm{Maps}(F(c), G(c))_* \,.$$

The composition operation on these is defined to be the one induced by the composite maps

$$\left(\int_{c\in\mathcal{C}} \mathrm{Maps}(F(c), G(c))_*\right) \wedge \left(\int_{c\in\mathcal{C}} \mathrm{Maps}(G(c), H(c))_*\right) \longrightarrow \prod_{c\in\mathcal{C}} \mathrm{Maps}(F(c), G(c))_* \wedge \mathrm{Maps}(G(c), H(c))_* \xrightarrow{(\circ_{F(c),G(c),H(c)})_{c\in\mathcal{C}}} \prod_{c\in\mathcal{C}} \mathrm{Maps}(F(c), H(c$$

where the first, morphism is degreewise given by projection out of the limits that defined the ends. This composite evidently equalizes the two relevant adjunct actions (as in the proof of example 1.30) and hence defines a map into the end

$$\left(\int_{c\in\mathcal{C}} \mathrm{Maps}(F(c), G(c))_*\right) \wedge \left(\int_{c\in\mathcal{C}} \mathrm{Maps}(G(c), H(c))_*\right) \longrightarrow \int_{c\in\mathcal{C}} \mathrm{Maps}(F(c), H(c))_* \,.$$

The resulting pointed topologically enriched category $[\mathcal{C}, \mathrm{Top}^{*/}_{\mathrm{cg}}]$ is also called the $\mathrm{Top}^{*/}_{\mathrm{cg}}$-**enriched functor category** over $\mathcal{C}$ with coefficients in $\mathrm{Top}^{*/}_{\mathrm{cg}}$.

This yields an equivalent formulation in terms of ends of the pointed topologically enriched Yoneda lemma (prop.):

Proposition 1.32. ***(topologically enriched Yoneda lemma)***

Let $\mathcal{C}$ be a small pointed topologically enriched categories (def.). For $F : \mathcal{C} \to \mathrm{Top}^{*/}_{\mathrm{cg}}$ a pointed topologically enriched functor (def.) and for $c \in \mathcal{C}$ an object, there is a natural isomorphism

$$[\mathcal{C}, \mathrm{Top}^{*/}_{\mathrm{cg}}](\mathcal{C}(c, -),\ F) \;\simeq\; F(c)$$

between the hom-space of the pointed topological functor category, according to def. 1.31, from the functor represented by c to F, and the value of F on c.

In terms of the ends (def. 1.28) defining these hom-spaces, this means that

$$\int_{d\in\mathcal{C}} \mathrm{Maps}(\mathcal{C}(c, d), F(d))_* \;\simeq\; F(c) \,.$$

*In this form the statement is also known as **Yoneda reduction**.*

The **proof** of prop. 1.32 is formally dual to the proof of the next prop. 1.33.

Now that natural transformations are expressed in terms of ends (example 1.30), as is the Yoneda lemma (prop. 1.32), it is natural to consider the dual statement involving coends:

Proposition 1.33. ***(co-Yoneda lemma)***

Let $\mathcal{C}$ be a small pointed topologically enriched category (def.). For $F : \mathcal{C} \to \mathrm{Top}^{*/}_{\mathrm{cg}}$ a pointed topologically enriched functor (def.) and for $c \in \mathcal{C}$ an object, there is a natural isomorphism

$$F(-) \simeq \int^{c\in\mathcal{C}} \mathcal{C}(c,-) \wedge F(c)\,.$$

Moreover, the morphism that hence exhibits $F(c)$ as the coequalizer of the two morphisms in def. 1.28 is componentwise the canonical action

$$\mathcal{C}(c,d) \wedge F(c) \longrightarrow F(d)$$

which is adjunct to the component map $\mathcal{C}(d,c) \to \mathrm{Maps}(F(c),F(d))_*$ of the topologically enriched functor F.

(e.g. MMSS 00, lemma 1.6)

Proof. The coequalizer of pointed topological spaces that we need to consider has underlying it a coequalizer of underlying pointed sets (prop., prop., prop.). That in turn is the colimit over the diagram of underlying sets with the basepointe adjoined to the diagram (prop.). For a coequalizer diagram adding that extra point to the diagram clearly does not change the colimit, and so we need to consider the plain coequalizer of sets.

That is just the set of equivalence classes of pairs

$$(c \to c_0,\ x) \quad \in \mathcal{C}(c,c_0) \wedge F(c)\,,$$

where two such pairs

$$(c \overset{f}{\to} c_0,\ x \in F(c))\,, \quad (d \overset{g}{\to} c_0,\ y \in F(d))$$

are regarded as equivalent if there exists

$$c \overset{\phi}{\to} d$$

such that

$$f = g \circ \phi\,, \quad \text{and} \quad y = \phi(x)\,.$$

(Because then the two pairs are the two images of the pair (g,x) under the two morphisms being coequalized.)

But now considering the case that $d = c_0$ and $g = \mathrm{id}_{c_0}$, so that $f = \phi$ shows that any pair

$$(c \overset{\phi}{\to} c_0,\ x \in F(c))$$

is identified, in the coequalizer, with the pair

$$(\mathrm{id}_{c_0},\ \phi(x) \in F(c_0))\,,$$

hence with $\phi(x) \in F(c_0)$.

This shows the claim at the level of the underlying sets. To conclude it is now sufficient (prop.) to show that the topology on $F(c_0) \in \mathrm{Top}_{\mathrm{cg}}^{*/}$ is the final topology (def.) of the system of component morphisms

$$\mathcal{C}(d,c) \wedge F(c) \longrightarrow \int^{c} \mathcal{C}(c,c_0) \wedge F(c)$$

which we just found. But that system includes

$$\mathcal{C}(c,c) \wedge F(c) \longrightarrow F(c)$$

which is a retraction

$$\mathrm{id} : F(c) \longrightarrow \mathcal{C}(c,c) \wedge F(c) \longrightarrow F(c)$$

and so if all the preimages of a given subset of the coequalizer under these component maps is open, it must have already been open in $F(c)$. ▮

Remark 1.34. The statement of the co-Yoneda lemma in prop. 1.33 is a kind of categorification of the following statement in analysis (whence the notation with the integral signs):

For X a topological space, $f : X \to \mathbb{R}$ a continuous function and $\delta(-,x_0)$ denoting the Dirac distribution, then

$$\int_{x\in X} \delta(x,x_0) f(x) = f(x_0)\,.$$

It is this analogy that gives the name to the following statement:

Proposition 1.35. ***(Fubini theorem for (co)-ends)***

For F a pointed topologically enriched bifunctor on a small pointed topological product category $\mathcal{C}_1 \times \mathcal{C}_2$ (def. 1.26), i.e.

$$F : (\mathcal{C}_1 \times \mathcal{C}_2)^{\mathrm{op}} \times (\mathcal{C}_1 \times \mathcal{C}_2) \longrightarrow \mathrm{Top}_{\mathrm{cg}}^{*/}$$

then its end and coend (def. 1.28) is equivalently formed consecutively over each variable, in either order:

$$\int^{(c_1,c_2)} F((c_1,c_2),(c_1,c_2)) \simeq \int^{c_1}\int^{c_2} F((c_1,c_2),(c_1,c_2)) \simeq \int^{c_2}\int^{c_1} F((c_1,c_2),(c_1,c_2))$$

and

$$\int_{(c_1,c_2)} F((c_1,c_2),(c_1,c_2)) \simeq \int_{c_1}\int_{c_2} F((c_1,c_2),(c_1,c_2)) \simeq \int_{c_2}\int_{c_1} F((c_1,c_2),(c_1,c_2)) \,.$$

Proof. Because limits commute with limits, and colimits commute with colimits. ▮

Remark 1.36. Since the pointed compactly generated mapping space functor (exmpl.)

$$\mathrm{Maps}(-,-)_* : \left(\mathrm{Top}_{\mathrm{cg}}^{*/}\right)^{\mathrm{op}} \times \mathrm{Top}_{\mathrm{cg}}^{*/} \longrightarrow \mathrm{Top}_{\mathrm{cg}}^{*/}$$

takes colimits in the first argument and limits in the second argument to limits (cor.), it in particular takes coends in the first argument and ends in the second argument, to ends (def. 1.28):

$$\mathrm{Maps}(X, \int_c F(c,c))_* \simeq \int_c \mathrm{Maps}(X, F(c,c)_*)$$

and

$$\mathrm{Maps}(\int^c F(c,c),\, Y)_* \simeq \int_c \mathrm{Maps}(F(c,c),\, Y)_* \,.$$

With this coend calculus in hand, there is an elegant proof of the defining universal property of the smash tensoring of topologically enriched functors $[\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*}]$ (def.)

Proposition 1.37. *For $\mathcal{C}$ a pointed topologically enriched category, there are natural isomorphisms*

$$[\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}](X \wedge K, Y) \;\simeq\; \mathrm{Maps}(K,\, [\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}](X,Y))_*$$

and

$$[\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}](X,\, \mathrm{Maps}(K,Y)_*) \;\simeq\; \mathrm{Maps}(K,\, [\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}](X,Y))$$

for all $X, Y \in [\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}]$ and all $K \in \mathrm{Top}_{\mathrm{cg}}^{*/}$.

In particular there is the combined natural isomorphism

$$[\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}](X \wedge K, Y) \;\simeq\; [\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}](X, \mathrm{Maps}(K,Y)_*)$$

exhibiting a pair of adjoint functors

$$[\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}] \underset{\mathrm{Maps}(K,-)_*}{\overset{(-)\wedge K}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} [\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*}] \,.$$

Proof. Via the end-expression for $[\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}](-,-)$ from def. 1.31 and the fact (remark 1.36) that the pointed mapping space construction $\mathrm{Maps}(-,-)_*$ preserves ends in the second variable, this reduces to the fact that $\mathrm{Maps}(-,-)_*$ is the internal hom in the closed monoidal category $\mathrm{Top}_{\mathrm{cg}}^{*/}$ (example 1.10) and hence satisfies the internal tensor/hom-adjunction isomorphism (prop. 1.8):

$$
\begin{aligned}
[\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}](X \wedge K, Y) & = \int_c \mathrm{Maps}((X \wedge K)(c), Y(c))_* \\
& \simeq \int_c \mathrm{Maps}(X(c) \wedge K, Y(x))_* \\
& \simeq \int_c \mathrm{Maps}(K, \mathrm{Maps}(X(c), Y(c))_*)_* \\
& \simeq \mathrm{Maps}(K, \int_c \mathrm{Maps}(X(c), Y(c)))_* \\
& = \mathrm{Maps}(K, [\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}](X, Y))_*
\end{aligned}
$$

and

$$
\begin{aligned}
[\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}](X, \mathrm{Maps}(K, Y)_*) & = \int_c \mathrm{Maps}(X(c), (\mathrm{Maps}(K, Y)_*)(c))_* \\
& \simeq \int_c \mathrm{Maps}(X(c), \mathrm{Maps}(K, Y(c))_*)_* \\
& \simeq \int_c \mathrm{Maps}(X(c) \wedge K, Y(c))_* \\
& \simeq \int_c \mathrm{Maps}(K, \mathrm{Maps}(X(c), Y(c))_*)_* \\
& \simeq \mathrm{Maps}(K, \int_c \mathrm{Maps}(X(c), Y(c))_*)_* \\
& \simeq \mathrm{Maps}(K, [\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}](X, Y))_* \;.
\end{aligned}
$$

▮

Proposition 1.38. ***(left Kan extension via coends)***

Let $\mathcal{C}, \mathcal{D}$ be small pointed topologically enriched categories (def.) and let

$$p : \mathcal{C} \longrightarrow \mathcal{D}$$

be a pointed topologically enriched functor (def.). Then precomposition with p constitutes a functor

$$p^* : [\mathcal{D}, \mathrm{Top}_{\mathrm{cg}}^{*/}] \longrightarrow [\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}]$$

*$G \mapsto G \circ p$. This functor has a left adjoint Lan_p, called **left Kan extension** along p*

$$[\mathcal{D}, \mathrm{Top}_{\mathrm{cg}}^{*/}] \underset{p^*}{\overset{\mathrm{Lan}_p}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} [\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}]$$

which is given objectwise by a coend (def. 1.28):

$$(\mathrm{Lan}_p F) : d \;\mapsto\; \int^{c \in \mathcal{C}} \mathcal{D}(p(c), d) \wedge F(c) \;.$$

Proof. Use the expression of natural transformations in terms of ends (example 1.30 and def. 1.31), then use the respect of $\mathrm{Maps}(-,-)_*$ for ends/coends (remark 1.36), use the smash/mapping space adjunction (cor.), use the Fubini theorem (prop. 1.35) and finally use Yoneda reduction (prop. 1.32) to obtain a sequence of natural isomorphisms as follows:

$$
\begin{aligned}
[\mathcal{D}, \mathrm{Top}_{\mathrm{cg}}^{*/}](\mathrm{Lan}_p F,\, G) & = \int_{d \in \mathcal{D}} \mathrm{Maps}((\mathrm{Lan}_p F)(d),\, G(d))_* \\
& = \int_{d \in \mathcal{D}} \mathrm{Maps}\Big(\int^{c \in \mathcal{C}} \mathcal{D}(p(c), d) \wedge F(c),\; G(d)\Big)_* \\
& \simeq \int_{d \in \mathcal{D}} \int_{c \in \mathcal{C}} \mathrm{Maps}(\mathcal{D}(p(c), d) \wedge F(c)\,,\; G(d))_* \\
& \simeq \int_{c \in \mathcal{C}} \int_{d \in \mathcal{D}} \mathrm{Maps}(F(c), \mathrm{Maps}(\mathcal{D}(p(c), d),\, G(d))_*)_* \cdot \\
& \simeq \int_{c \in \mathcal{C}} \mathrm{Maps}(F(c),\, \int_{d \in \mathcal{D}} \mathrm{Maps}(\mathcal{D}(p(c), d),\, G(d))_*)_* \\
& \simeq \int_{c \in \mathcal{C}} \mathrm{Maps}(F(c), G(p(c)))_* \\
& = [\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}](F, p^* G)
\end{aligned}
$$

▮

Topological Day convolution

Given two functions $f_1, f_2 : G \longrightarrow \mathbb{C}$ on a group (or just a monoid) G, then their convolution product is, whenever well defined, given by the sum

$$f_1 \star f_2 : g \mapsto \sum_{g_1 \cdot g_2 = g} f_1(g_1) \cdot f_2(g_2) .$$

The operation of *Day convolution* is the categorification of this situation where functions are replaced by functors and monoids by monoidal categories. Further below we find the symmetric monoidal smash product of spectra as the Day convolution of topologically enriched functors over the monoidal category of finite pointed CW-complexes, or over sufficiently rich subcategories thereof.

Definition 1.39. Let $(\mathcal{C}, \otimes, 1)$ be a small pointed topological monoidal category (def. 1.1).

Then the **Day convolution tensor product** on the pointed topological enriched functor category $[\mathcal{C}, \mathrm{Top}^{*/}_{\mathrm{cg}}]$ (def. 1.31) is the functor

$$\otimes_{\mathrm{Day}} : [\mathcal{C}, \mathrm{Top}^{*/}_{\mathrm{cg}}] \times [\mathcal{C}, \mathrm{Top}^{*/}_{\mathrm{cg}}] \longrightarrow [\mathcal{C}, \mathrm{Top}^{*/}_{\mathrm{cg}}]$$

out of the pointed topological product category (def. 1.26) given by the following coend (def. 1.28)

$$X \otimes_{\mathrm{Day}} Y : c \mapsto \int^{(c_1, c_2) \in \mathcal{C} \times \mathcal{C}} \mathcal{C}(c_1 \otimes c_2, c) \wedge X(c_1) \wedge Y(c_2) .$$

Example 1.40. Let Seq denote the category with objects the natural numbers, and only the zero morphisms and identity morphisms on these objects (we consider this in a braoder context below in def. 2.4):

$$\mathrm{Seq}(n_1, n_2) \coloneqq \begin{cases} S^0 & \text{if } n_1 = n_2 \\ * & \text{otherwise} \end{cases} .$$

Regard this as a pointed topologically enriched category in the unique way. The operation of addition of natural numbers $\otimes = +$ makes this a monoidal category.

An object $X_\bullet \in [\mathrm{Seq}, \mathrm{Top}^{*/}_{\mathrm{cg}}]$ is an $\mathbb{N}$-sequence of pointed topological spaces. Given two such, then their Day convolution according to def. 1.39 is

$$\begin{aligned} (X \otimes_{\mathrm{Day}} Y)_n &= \int^{(n_1, n_2)} \mathrm{Seq}(n_1 + n_2, n) \wedge X_{n_1} \wedge X_{n_2} \\ &= \coprod_{\substack{n_1 + n_2 \\ = n}} (X_{n_1} \wedge X_{n_2}) \end{aligned} .$$

We observe now that Day convolution is equivalently a left Kan extension (def. 1.38). This will be key for understanding monoids and modules with respect to Day convolution.

Definition 1.41. Let $\mathcal{C}$ be a small pointed topologically enriched category (def.). Its **external tensor product** is the pointed topologically enriched functor

$$\overline{\wedge} : [\mathcal{C}, \mathrm{Top}^{*/}_{\mathrm{cg}}] \times [\mathcal{C}, \mathrm{Top}^{*/}_{\mathrm{cg}}] \longrightarrow [\mathcal{C} \times \mathcal{C}, \mathrm{Top}^{*/}_{\mathrm{cg}}]$$

from pairs of topologically enriched functors over mmathcal $\mathcal{C}$ to topologically enriched functors over the product category $\mathcal{C} \times \mathcal{C}$ (def. 1.26) given by

$$X \overline{\wedge} Y \coloneqq \wedge \circ (X, Y) ,$$

i.e.

$$(X \overline{\wedge} Y)(c_1, c_2) = X(c_1) \wedge X(c_2) .$$

Proposition 1.42. *For $(\mathcal{C}, \otimes\, 1)$ a pointed topologically enriched monoidal category (def. 1.1) the Day convolution product (def. 1.39) of two functors is equivalently the left Kan extension (def. 1.38) of their external tensor product (def. 1.41) along the tensor product $\otimes : \mathcal{C} \times \mathcal{C}$: there is a natural isomorphism*

$$X \otimes_{\mathrm{Day}} Y \simeq \mathrm{Lan}_{\otimes} (X \overline{\wedge} Y) .$$

Hence the adjunction unit is a natural transformation of the form

$$\begin{array}{ccc} \mathcal{C} \times \mathcal{C} & \overset{X \overline{\wedge} Y}{\longrightarrow} & \mathrm{Top}^{*/}_{\mathrm{cg}} \\ {}^{\otimes}\searrow & \Downarrow & \nearrow_{X \otimes_{\mathrm{Day}} Y} \\ & \mathcal{C} & \end{array} .$$

This perspective is highlighted in (MMSS 00, p. 60).

Proof. By prop. 1.38 we may compute the left Kan extension as the following coend:

$$
\begin{aligned}
\mathrm{Lan}_{\otimes_{\mathcal{C}}}(X \,\overline{\wedge}\, Y)(c) & \simeq \int^{(c_1,c_2)} \mathcal{C}(c_1 \otimes_{\mathcal{C}} c_2, c) \wedge (X \,\overline{\wedge}\, Y)(c_1,c_2) \\
& = \int^{(c_1,c_2)} \mathcal{C}(c_1 \otimes c_2, c) \wedge X(c_1) \wedge X(c_2)
\end{aligned}
.
$$

▮

Proposition 1.42 implies the following fact, which is the key for the identification of "functors with smash product" below and then for the description of ring spectra further below.

Corollary 1.43. *The operation of Day convolution $\otimes_{\mathrm{Day}}$ (def. 1.39) is universally characterized by the property that there are natural isomorphisms*

$$
[\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}](X \otimes_{\mathrm{Day}} Y, Z) \simeq [\mathcal{C} \times \mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}](X \,\overline{\wedge}\, Y,\; Z \circ \otimes\,)\,,
$$

where $\overline{\wedge}$ is the external product of def. 1.41, hence that natural transformations of functors on $\mathcal{C}$ of the form

$$
(X \otimes_{\mathrm{Day}} Y)(c) \longrightarrow Z(c)
$$

are in natural bijection with natural transformations of functors on the product category mmathcal $\mathcal{C} \times \mathcal{C}$ *(def. 1.26) of the form*

$$
X(c_1) \wedge Y(c_2) \longrightarrow Z(c_1 \otimes c_2)\;.
$$

Write

$$
y \,:\, \mathcal{C}^{\mathrm{op}} \longrightarrow [\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}]
$$

for the $\mathrm{Top}_{\mathrm{cg}}^{*/}$-Yoneda embedding, so that for $c \in \mathcal{C}$ any object, $y(c)$ is the corepresented functor $y(c) : d \mapsto \mathcal{C}(c,d)$.

Proposition 1.44. *For $(\mathcal{C}, \otimes, 1)$ a small pointed topological monoidal category (def. 1.1), the Day convolution tensor product $\otimes_{\mathrm{Day}}$ of def. 1.39 makes the pointed topologically enriched functor category*

$$
([\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}],\; \otimes_{\mathrm{Day}}, y(1))
$$

into a pointed topological monoidal category (def. 1.1) with tensor unit $y(1)$ co-represented by the tensor unit 1 of $\mathcal{C}$.

Moreover, if $(\mathcal{C}, \otimes, 1)$ is equipped with a (symmetric) braiding $\tau^{\mathcal{C}}$ (def. 1.4), then so is $([\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}],\; \otimes_{\mathrm{Day}}, y(1))$.

Proof. Regarding associativity, observe that

$$
\begin{aligned}
(X \otimes_{\mathrm{Day}} (Y \otimes_{\mathrm{Day}} Z))(c) & \simeq \int^{(c_1,c_2)} \mathcal{C}(c_1 \otimes c_2,\, c) \wedge X(c_1) \wedge \int^{(d_1,d_2)} \mathcal{C}(d_1 \otimes d_2, c_2)(Y(d_1) \wedge Z(d_2)) \\
& \simeq \int^{c_1,d_1,d_2} \underbrace{\int^{c_2} \mathcal{C}(c_1 \otimes c_2, c) \wedge \mathcal{C}(d_1 \otimes d_2, c_2)}_{\simeq \mathcal{C}(c_1 \otimes (d_1 \otimes_{\mathcal{C}} d_2), c)} \wedge (X(c_1) \wedge (Y(d_1) \wedge Z(d_2))) \\
& \simeq \int^{c_1,d_1,d_2} \mathcal{C}(c_1 \otimes (d_1 \otimes d_2), c) \wedge (X(c_1) \wedge (Y(d_1) \wedge Z(d_2))) \\
& \simeq \int^{c_1,c_2,c_3} \mathcal{C}(c_1 \otimes (c_2 \otimes c_3), c) \wedge (X(c_1) \wedge (Y(c_2) \wedge Z(c_3)))
\end{aligned}
,
$$

where we used the Fubini theorem for coends (prop. 1.35) and then twice the co-Yoneda lemma (prop. 1.33). Similarly

$$
\begin{aligned}
((X \otimes_{\mathrm{Day}} Y) \otimes_{\mathrm{Day}} Z)(c) & \simeq \int^{(c_1,c_2)} \mathcal{C}(c_1 \otimes c_2, c) \wedge \int^{(d_1,d_2)} \mathcal{C}(d_1 \otimes d_2, c_1) \wedge (X(d_1) \wedge Y(d_2)) \wedge Y(c_2) \\
& \simeq \int^{c_2,d_1,d_2} \underbrace{\int^{c_1} \mathcal{C}(c_1 \otimes c_2, c) \wedge \mathcal{C}(d_1 \otimes d_2, c_1)}_{\simeq \mathcal{C}((d_1 \otimes d_2) \otimes c_2)} \wedge ((X(d_1) \wedge Y(d_2)) \wedge Z(c_2)) \\
& \simeq \int^{c_2,d_1,d_2} \mathcal{C}((d_1 \otimes d_2) \otimes c_2) \wedge ((X(d_1) \wedge Y(d_2)) \wedge Z(c_2)) \\
& \simeq \int^{c_1,c_2,c_3} \mathcal{C}((c_1 \otimes c_2) \otimes c_3) \wedge ((X(c_1) \wedge Y(c_2)) \wedge Z(c_3))
\end{aligned}
.
$$

So we obtain an associator by combining, in the integrand, the associator $\alpha^{\mathcal{C}}$ of $(\mathcal{C}, \otimes, 1)$ and $\tau^{\mathrm{Top}_{\mathrm{cg}}^{*/}}$ of $(\mathrm{Top}_{\mathrm{cg}}^{*/}, \wedge, S^0)$ (example 1.10):

$$
\begin{array}{ccc}
((X \otimes_{\mathrm{Day}} Y) \otimes_{\mathrm{Day}} Z)(c) & \simeq & \int^{c_1, c_2, c_3} \mathcal{C}((c_1 \otimes c_2) \otimes c_3) \wedge ((X(c_1) \wedge Y(c_2)) \wedge Z(c_3)) \\
{}^{\alpha^{\mathrm{Day}}_{X,Y,Z}(c)}\downarrow && \downarrow^{\int^{c_1,c_2,c_3} \mathcal{C}(\alpha^{\mathcal{C}}_{c_1,c_2,c_3}, c) \wedge \alpha^{\mathrm{Top}^{*/}_{\mathrm{cg}}}_{X(c_1),X(c_2),X(c_3)}} \\
(X \otimes_{\mathrm{Day}} (Y \otimes_{\mathrm{Day}} Z))(c) & \simeq & \int^{c_1, c_2, c_3} \mathcal{C}(c_1 \otimes (c_2 \otimes c_3), c) \wedge (X(c_1) \wedge (Y(c_2) \wedge Z(c_3)))
\end{array}
$$

It is clear that this satisfies the pentagon identity, since $\tau^{\mathcal{C}}$ and $\tau^{\mathrm{Top}^{*/}_{\mathrm{cg}}}$ do.

To see that $y(1)$ is the tensor unit for $\otimes_{\mathrm{Day}}$, use the Fubini theorem for coends (prop. 1.35) and then twice the co-Yoneda lemma (prop. 1.33) to get for any $X \in [\mathcal{C}, \mathrm{Top}^{*/}_{\mathrm{cg}}]$ that

$$
\begin{aligned}
X \otimes_{\mathrm{Day}} y(1) &= \int^{c_1, c_2 \in \mathcal{C}} \mathcal{C}(c_1 \otimes_{\mathcal{D}} c_2, -) \wedge X(c_1) \wedge \mathcal{C}(1, c_2) \\
&\simeq \int^{c_1 \in \mathcal{C}} \int^{c_2 \in \mathcal{C}} \mathcal{C}(c_1 \otimes_{\mathcal{C}} c_2, -) \wedge \mathcal{C}(1, c_2) \wedge X(c_1) \\
&\simeq \int^{c_1 \in \mathcal{C}} \mathcal{C}(c_1 \otimes_{\mathcal{C}} 1, -) \wedge X(c_1) \\
&\simeq \int^{c_1 \in \mathcal{C}} \mathcal{C}(c_1, -) \wedge X(c_1) \\
&\simeq X(-) \\
&\simeq X
\end{aligned}
.
$$

Hence the right unitor of Day convolution comes from the unitor of $\mathcal{C}$ under the integral sign:

$$
\begin{array}{ccc}
(X \otimes_{\mathrm{Day}} y(1))(c) & \simeq & \int^{c_1} \mathcal{C}(c_1 \otimes 1, c) \wedge X(c_1) \\
{}^{r^{\mathrm{Day}}_X(c)}\downarrow && \downarrow^{\int^{c_1} \mathcal{C}(r^{\mathcal{C}}_{c_1}, c) \wedge X(c_1)} \\
X(c) & \simeq & \int^{c_1} \mathcal{C}(c_1, c) \wedge X(c_1)
\end{array}
$$

Analogously for the left unitor. Hence the triangle identity for $\otimes_{\mathrm{Day}}$ follows from the triangle identity in $\mathcal{C}$ under the integral sign.

Similarly, if $\mathcal{C}$ has a braiding $\tau^{\mathcal{C}}$, it induces a braiding τ^{Day} under the integral sign:

$$
\begin{array}{ccc}
(X \otimes_{\mathrm{Day}} Y)(c) & = & \int^{c_1, c_2} \mathcal{C}(c_1 \otimes c_2, c) \wedge X(c_1) \wedge Y(c_2) \\
{}^{\tau^{\mathrm{Day}}_{X,Y}(c)}\downarrow && \downarrow^{\int^{c_1,c_2} \mathcal{C}(\tau^{\mathcal{C}}_{c_1,c_2}, c) \wedge \tau^{\mathrm{Top}^{*/}}_{X(c(1)),X(c_2)}} \\
(Y \otimes_{\mathrm{Day}} X)(c) & = & \int^{c_1, c_2} \mathcal{C}(c_2 \otimes c_1, c) \wedge Y(c_2) \wedge X(c_1)
\end{array}
$$

and the hexagon identity for τ^{Day} follows from that for $\tau^{\mathcal{C}}$ and $\tau^{\mathrm{Top}^{*/}_{\mathrm{cg}}}$ ▮

Moreover:

Proposition 1.45. *For* $(\mathcal{C}, \otimes, 1)$ *a small pointed topological symmetric monoidal category (def. 1.5), the monoidal category with Day convolution* $([\mathcal{C}, \mathrm{Top}^{*/}_{\mathrm{cg}}], \otimes_{\mathrm{Day}}, y(1))$ *from def. 1.44 is a closed monoidal category (def. 1.7). Its internal hom* $[-,-]_{\mathrm{Day}}$ *is given by the end (def. 1.28)*

$$
[X, Y]_{\mathrm{Day}}(c) \simeq \int_{c_1, c_2} \mathrm{Maps}(\mathcal{C}(c \otimes c_1, c_2),\ \mathrm{Maps}(X(c_1), Y(c_2))_*)_* .
$$

Proof. Using the Fubini theorem (def. 1.35) and the co-Yoneda lemma (def. 1.33) and in view of definition 1.31 of the enriched functor category, there is the following sequence of natural isomorphisms:

$$
\begin{aligned}
[\mathcal{C}, V](X, [Y, Z]_{\mathrm{Day}}) &\simeq \int_c \mathrm{Maps}\big(X(c), \int_{c_1, c_2} \mathrm{Maps}(\mathcal{C}(c \otimes c_1, c_2), \mathrm{Maps}(Y(c_1), Z(c_2))_*)_*\big)_* \\
&\simeq \int_c \int_{c_1, c_2} \mathrm{Maps}(\mathcal{C}(c \otimes c_1, c_2) \wedge X(c) \wedge Y(c_1),\ Z(c_2))_* \\
&\simeq \int_{c_2} \mathrm{Maps}\Big(\int^{c, c_1} \mathcal{C}(c \otimes c_1, c_2) \wedge X(c) \wedge Y(c_1),\ Z(c_2)\Big)_* \\
&\simeq \int_{c_2} \mathrm{Maps}\Big((X \otimes_{\mathrm{Day}} Y)(c_2), Z(c_2)\Big)_* \\
&\simeq [\mathcal{C}, V](X \otimes_{\mathrm{Day}} Y, Z)
\end{aligned}
.
$$

▮

Proposition 1.46. *In the situation of def. 1.44, the Yoneda embedding $c \mapsto \mathcal{C}(c,-)$ constitutes a strong monoidal functor (def. 1.47)*

$$(\mathcal{C}, \otimes, 1) \hookrightarrow ([\mathcal{C}, V], \otimes_{\mathrm{Day}}, y(1)) .$$

Proof. That the tensor unit is respected is part of prop. 1.44. To see that the tensor product is respected, apply the co-Yoneda lemma (prop. 1.33) twice to get the following natural isomorphism

$$\begin{aligned} (y(c_1) \otimes_{\mathrm{Day}} y(c_2))(c) & \simeq \int^{d_1, d_2} \mathcal{C}(d_1 \otimes d_2, c) \wedge \mathcal{C}(c_1, d_1) \wedge \mathcal{C}(c_2, d_2) \\ & \simeq \mathcal{C}(c_1 \otimes c_2, c) \\ & = y(c_1 \otimes c_2)(c) \end{aligned} .$$

∎

Functors with smash product

Since the symmetric monoidal smash product of spectra discussed below is an instance of Day convolution (def. 1.39), and since ring spectra are going to be the monoids (def. 1.13) with respect to this tensor product, we are interested in characterizing the monoids with respect to Day convolution. These turn out to have a particularly transparent expression as what is called *functors with smash product*, namely lax monoidal functors from the base monoidal category to $\mathrm{Top}_{\mathrm{cg}}^{*/}$. Their components are pairing maps of the form

$$R_{n_1} \wedge R_{n_2} \longrightarrow R_{n_1 + n_2}$$

satisfying suitable conditions. This is the form in which the structure of ring spectra usually appears in examples. It is directly analogous to how a dg-algebra, which is equivalently a monoid with respect to the tensor product of chain complexes (example 1.19), is given in components .

Here we introduce the concepts of monoidal functors and of functors with smash product and prove that they are equivalently the monoids with respect to Day convolution.

Definition 1.47. Let $(\mathcal{C}, \otimes_{\mathcal{C}}, 1_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, 1_{\mathcal{D}})$ be two (pointed) topologically enriched monoidal categories (def. 1.1). A topologically enriched **lax monoidal functor** between them is

1. a topologically enriched functor

$$F : \mathcal{C} \longrightarrow \mathcal{D} ,$$

2. a morphism

$$\epsilon : 1_{\mathcal{D}} \longrightarrow F(1_{\mathcal{C}})$$

3. a natural transformation

$$\mu_{x,y} : F(x) \otimes_{\mathcal{D}} F(y) \longrightarrow F(x \otimes_{\mathcal{C}} y)$$

for all $x, y \in \mathcal{C}$

satisfying the following conditions:

1. **(associativity)** For all objects $x, y, z \in \mathcal{C}$ the following diagram commutes

$$\begin{array}{ccc} (F(x) \otimes_{\mathcal{D}} F(y)) \otimes_{\mathcal{D}} F(z) & \underset{\simeq}{\overset{a^{\mathcal{D}}_{F(x),F(y),F(z)}}{\longrightarrow}} & F(x) \otimes_{\mathcal{D}} (F(y) \otimes_{\mathcal{D}} F(z)) \\ {}^{\mu_{x,y} \otimes \mathrm{id}}\downarrow && \downarrow^{\mathrm{id} \otimes \mu_{y,z}} \\ F(x \otimes_{\mathcal{C}} y) \otimes_{\mathcal{D}} F(z) && F(x) \otimes_{\mathcal{D}} (F(x \otimes_{\mathcal{C}} y)) \\ {}^{\mu_{x \otimes_{\mathcal{C}} y, z}}\downarrow && \downarrow^{\mu_{x, y \otimes_{\mathcal{C}} z}} \\ F((x \otimes_{\mathcal{C}} y) \otimes_{\mathcal{C}} z) & \underset{F(a^{\mathcal{C}}_{x,y,z})}{\longrightarrow} & F(x \otimes_{\mathcal{C}} (y \otimes_{\mathcal{C}} z)) \end{array} ,$$

where $a^{\mathcal{C}}$ and $a^{\mathcal{D}}$ denote the associators of the monoidal categories;

2. **(unitality)** For all $x \in \mathcal{C}$ the following diagrams commutes

$$\begin{array}{ccc} 1_{\mathcal{D}} \otimes_{\mathcal{D}} F(x) & \overset{\epsilon \otimes \mathrm{id}}{\longrightarrow} & F(1_{\mathcal{C}}) \otimes_{\mathcal{D}} F(x) \\ {}^{\ell^{\mathcal{D}}_{F(x)}}\downarrow && \downarrow^{\mu_{1_{\mathcal{C}}, x}} \\ F(x) & \overset{F(\ell^{\mathcal{C}}_x)}{\longleftarrow} & F(1 \otimes_{\mathcal{C}} x) \end{array}$$

and

$$
\begin{array}{ccc}
F(x) \otimes_{\mathcal{D}} 1_{\mathcal{D}} & \overset{\mathrm{id} \otimes \epsilon}{\longrightarrow} & F(x) \otimes_{\mathcal{D}} F(1_{\mathcal{C}}) \\
{}^{r^{\mathcal{D}}_{F(x)}}\downarrow && \downarrow^{\mu_{x, 1_{\mathcal{C}}}} \\
F(x) & \overset{F(r^{\mathcal{C}}_x)}{\longleftarrow} & F(x \otimes_{\mathcal{C}} 1)
\end{array} \,,
$$

where $\ell^{\mathcal{C}}$, $\ell^{\mathcal{D}}$, $r^{\mathcal{C}}$, $r^{\mathcal{D}}$ denote the left and right unitors of the two monoidal categories, respectively.

If ϵ and alll $\mu_{x,y}$ are isomorphisms, then F is called a **strong monoidal functor**.

If moreover $(\mathcal{C}, \otimes_{\mathcal{C}}, 1_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, 1_{\mathcal{D}})$ are equipped with the structure of braided monoidal categories (def. 1.4) with braidings $\tau^{\mathcal{C}}$ and $\tau^{\mathcal{D}}$, respectively, then the lax monoidal functor F is called a **braided monoidal functor** if in addition the following diagram commutes for all objects $x, y \in \mathcal{C}$

$$
\begin{array}{ccc}
F(x) \otimes_{\mathcal{C}} F(y) & \overset{\tau^{\mathcal{D}}_{F(x), F(y)}}{\longrightarrow} & F(y) \otimes_{\mathcal{D}} F(x) \\
{}^{\mu_{x,y}}\downarrow && \downarrow^{\mu_{y,x}} \\
F(x \otimes_{\mathcal{C}} y) & \underset{F(\tau^{\mathcal{C}}_{x,y})}{\longrightarrow} & F(y \otimes_{\mathcal{C}} x)
\end{array} \,.
$$

A homomorphism $f : (F_1, \mu_1, \epsilon_1) \longrightarrow (F_2, \mu_2, \epsilon_2)$ between two (braided) lax monoidal functors is a **monoidal natural transformation**, in that it is a natural transformation $f_x : F_1(x) \longrightarrow F_2(x)$ of the underlying functors

compatible with the product and the unit in that the following diagrams commute for all objects $x, y \in \mathcal{C}$:

$$
\begin{array}{ccc}
F_1(x) \otimes_{\mathcal{D}} F_1(y) & \overset{f(x) \otimes_{\mathcal{D}} f(y)}{\longrightarrow} & F_2(x) \otimes_{\mathcal{D}} F_2(y) \\
{}^{(\mu_1)_{x,y}}\downarrow && \downarrow^{(\mu_2)_{x,y}} \\
F_1(x \otimes_{\mathcal{C}} y) & \underset{f(x \otimes_{\mathcal{C}} y)}{\longrightarrow} & F_2(x \otimes_{\mathcal{C}} y)
\end{array}
$$

and

$$
\begin{array}{ccccc}
& & 1_{\mathcal{D}} & & \\
& {}^{\epsilon_1}\swarrow & & \searrow^{\epsilon_2} & \\
F_1(1_{\mathcal{C}}) & & \underset{f(1_{\mathcal{C}})}{\longrightarrow} & & F_2(1_{\mathcal{C}})
\end{array} \,.
$$

We write $\mathrm{MonFun}(\mathcal{C}, \mathcal{D})$ for the resulting category of lax monoidal functors between monoidal categories $\mathcal{C}$ and $\mathcal{D}$, similarly $\mathrm{BraidMonFun}(\mathcal{C}, \mathcal{D})$ for the category of braided monoidal functors between braided monoidal categories, and $\mathrm{SymMonFun}(\mathcal{C}, \mathcal{D})$ for the category of braided monoidal functors between symmetric monoidal categories.

Remark 1.48. In the literature the term “monoidal functor” often refers by default to what in def. 1.47 is called a *strong monoidal functor*. But for the purpose of the discussion of functors with smash product below, it is crucial to admit the generality of lax monoidal functors.

If $(\mathcal{C}, \otimes_{\mathcal{C}}, 1_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, 1_{\mathcal{D}})$ are symmetric monoidal categories (def. 1.5) then a braided monoidal functor (def. 1.47) between them is often called a **symmetric monoidal functor**.

Proposition 1.49. *For $\mathcal{C} \overset{F}{\longrightarrow} \mathcal{D} \overset{G}{\longrightarrow} \mathcal{E}$ two composable lax monoidal functors (def. 1.47) between monoidal categories, then their composite $F \circ G$ becomes a lax monoidal functor with structure morphisms*

$$
\epsilon^{G \circ F} : 1_{\mathcal{E}} \overset{\epsilon^G}{\longrightarrow} G(1_{\mathcal{D}}) \overset{G(\epsilon^F)}{\longrightarrow} G(F(1_{\mathcal{C}}))
$$

and

$$
\mu^{G \circ F}_{c_1, c_2} : G(F(c_1)) \otimes_{\mathcal{E}} G(F(c_2)) \overset{\mu^G_{F(c_1), F(c_2)}}{\longrightarrow} G(F(c_1) \otimes_{\mathcal{D}} F(c_2)) \overset{G(\mu^F_{c_1, c_2})}{\longrightarrow} G(F(c_1 \otimes_{\mathcal{C}} c_2)) \,.
$$

Proposition 1.50. *Let $(\mathcal{C}, \otimes_{\mathcal{C}}, 1_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, 1_{\mathcal{D}})$ be two monoidal categories (def. 1.1) and let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a lax monoidal functor (def. 1.47) between them.*

Then for (A, μ_A, e_A) a monoid in $\mathcal{C}$ (def. 1.13), its image $F(A) \in \mathcal{D}$ becomes a monoid $(F(A), \mu_{F(A)}, e_{F(A)})$ by setting

$$
\mu_{F(A)} : F(A) \otimes_{\mathcal{C}} F(A) \longrightarrow F(A \otimes_{\mathcal{C}} A) \overset{F(\mu_A)}{\longrightarrow} F(A)
$$

(where the first morphism is the structure morphism of F) and setting

$$e_{F(A)} \;:\; 1_{\mathcal{D}} \longrightarrow F(1_{\mathcal{C}}) \overset{F(e_A)}{\longrightarrow} F(A)$$

(where again the first morphism is the corresponding structure morphism of F).

This construction extends to a functor

$$\mathrm{Mon}(F) \;:\; \mathrm{Mon}(\mathcal{C}, \otimes_{\mathcal{C}}, 1_{\mathcal{C}}) \longrightarrow \mathrm{Mon}(\mathcal{D}, \otimes_{\mathcal{D}}, 1_{\mathcal{D}})$$

from the category of monoids of $\mathcal{C}$ (def. 1.13) to that of $\mathcal{D}$.

Moreover, if $\mathcal{C}$ and $\mathcal{D}$ are symmetric monoidal categories (def. 1.5) and F is a braided monoidal functor (def. 1.47) and A is a commutative monoid (def. 1.13) then so is $F(A)$, and this construction extends to a functor

$$\mathrm{CMon}(F) \;:\; \mathrm{CMon}(\mathcal{C}, \otimes_{\mathcal{C}}, 1_{\mathcal{C}}) \longrightarrow \mathrm{CMon}(\mathcal{D}, \otimes_{\mathcal{D}}, 1_{\mathcal{D}}) \,.$$

Proof. This follows immediately from combining the associativity and unitality (and symmetry) constraints of F with those of A. ▮

Definition 1.51. Let $(\mathcal{C}, \otimes_{\mathcal{C}}, 1_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, 1_{\mathcal{D}})$ be two (pointed) topologically enriched monoidal categories (def. 1.1), and let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a topologically enriched lax monoidal functor between them, with product operation μ.

Then a left **module over the lax monoidal functor** is

1. a topologically enriched functor

$$G \;:\; \mathcal{C} \longrightarrow \mathcal{D} \,;$$

2. a natural transformation

$$\rho_{x,y} \;:\; F(x) \otimes_{\mathcal{D}} G(y) \longrightarrow G(x \otimes_{\mathcal{C}} y)$$

such that

- **(action property)** For all objects $x,y,z \in \mathcal{C}$ the following diagram commutes

$$\begin{array}{ccc} (F(x) \otimes_{\mathcal{D}} F(y)) \otimes_{\mathcal{D}} G(z) & \underoverset{\simeq}{a^{\mathcal{D}}_{F(x),F(y),F(z)}}{\longrightarrow} & F(x) \otimes_{\mathcal{D}} (F(y) \otimes_{\mathcal{D}} G(z)) \\ {}^{\mu_{x,y} \otimes \mathrm{id}}\downarrow && \downarrow^{\mathrm{id} \otimes \rho_{y,z}} \\ F(x \otimes_{\mathcal{C}} y) \otimes_{\mathcal{D}} G(z) && F(x) \otimes_{\mathcal{D}} (G(x \otimes_{\mathcal{C}} y)) \\ {}^{\rho_{x \otimes_{\mathcal{C}} y, z}}\downarrow && \downarrow^{\rho_{x, y \otimes_{\mathcal{C}} z}} \\ G((x \otimes_{\mathcal{C}} y) \otimes_{\mathcal{C}} z) & \underset{F(a^{\mathcal{C}}_{x,y,z})}{\longrightarrow} & G(x \otimes_{\mathcal{C}} (y \otimes_{\mathcal{C}} z)) \end{array} \,,$$

A homomorphism $f : (G_1, \rho_1) \longrightarrow (G_2, \rho_2)$ between two modules over a monoidal functor (F, μ, ϵ) is

- a natural transformation $f_x \;:\; G_1(x) \longrightarrow G_2(x)$ of the underlying functors

compatible with the action in that the following diagram commutes for all objects $x,y \in \mathcal{C}$:

$$\begin{array}{ccc} F(x) \otimes_{\mathcal{D}} G_1(y) & \overset{\mathrm{id} \otimes_{\mathcal{D}} f(y)}{\longrightarrow} & F(x) \otimes_{\mathcal{D}} G_2(y) \\ {}^{(\rho_1)_{x,y}}\downarrow && \downarrow^{(\mathrm{rhi}_2)_{x,y}} \\ G_1(x \otimes_{\mathcal{C}} y) & \underset{f(x \otimes_{\mathcal{C}} y)}{\longrightarrow} & G_2(x \otimes_{\mathcal{C}} y) \end{array}$$

We write $F\,\mathrm{Mod}$ for the resulting category of modules over the monoidal functor F.

Now we may finally state the main proposition on *functors with smash product*:

Proposition 1.52. *Let $(\mathcal{C}, \otimes, 1)$ be a pointed topologically enriched (symmetric) monoidal category (def. 1.1). Regard $(\mathrm{Top}_{\mathrm{cg}}^{*/}, \wedge, S^0)$ as a topological symmetric monoidal category as in example 1.10.*

Then (commutative) monoids in (def. 1.13) the Day convolution monoidal category $([\mathcal{C}, \mathrm{Top}^{*/}_{\mathrm{cg}}], \otimes_{\mathrm{Day}}, y(1_{\mathcal{C}}))$ of prop. 1.44 are equivalent to (braided) lax monoidal functors (def. 1.47) of the form

$$(\mathcal{C}, \otimes, 1) \longrightarrow (\mathrm{Top}^*_{\mathrm{cg}}, \wedge, S^0) \,,$$

*called **functors with smash products** on $\mathcal{C}$, i.e. there are equivalences of categories of the form*

$$\begin{aligned} \mathrm{Mon}([\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}], \otimes_{\mathrm{Day}}, y(1_{\mathcal{C}})) &\simeq \mathrm{MonFunc}(\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}) \\ \mathrm{CMon}([\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}], \otimes_{\mathrm{Day}}, y(1_{\mathcal{C}})) &\simeq \mathrm{SymMonFunc}(\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}) \end{aligned} .$$

Moreover, module objects over these monoid objects are equivalent to the corresponding modules over monoidal functors (def. 1.51).

This is stated in some form in (Day 70, example 3.2.2). It is highlighted again in (MMSS 00, prop. 22.1).

Proof. By definition 1.47, a lax monoidal functor $F : \mathcal{C} \to \mathrm{Top}_{\mathrm{cg}}^{*/}$ is a topologically enriched functor equipped with a morphism of pointed topological spaces of the form

$$S^0 \longrightarrow F(1_{\mathcal{C}})$$

and equipped with a natural system of maps of pointed topological spaces of the form

$$F(c_1) \wedge F(c_2) \longrightarrow F(c_1 \otimes_{\mathcal{C}} c_2)$$

for all $c_1, c_2 \in \mathcal{C}$.

Under the Yoneda lemma (prop. 1.32) the first of these is equivalently a morphism in $[\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}]$ of the form

$$y(S^0) \longrightarrow F .$$

Moreover, under the natural isomorphism of corollary 1.43 the second of these is equivalently a morphism in $[\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}]$ of the form

$$F \otimes_{\mathrm{Day}} F \longrightarrow F .$$

Translating the conditions of def. 1.47 satisfied by a lax monoidal functor through these identifications gives precisely the conditions of def. 1.13 on a (commutative) monoid in object F under $\otimes_{\mathrm{Day}}$.

Similarly for module objects and modules over monoidal functors. ▮

Proposition 1.53. *Let $f : \mathcal{C} \longrightarrow \mathcal{D}$ be a lax monoidal functor (def. 1.47) between pointed topologically enriched monoidal categories (def. 1.1). Then the induced functor*

$$f^* : [\mathcal{D}, \mathrm{Top}_{\mathrm{cg}}^{*/}] \longrightarrow [\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*}]$$

given by $(f^* X)(c) \coloneqq X(f(c))$ preserves monoids under Day convolution

$$f^* : \mathrm{Mon}([\mathcal{D}, \mathrm{Top}_{\mathrm{cg}}^{*/}], \otimes_{\mathrm{Day}}, y(1_{\mathcal{D}})) \longrightarrow \mathrm{Mon}([\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*}], \otimes_{\mathrm{Day}}, y(1_{\mathcal{C}})$$

Moreover, if $\mathcal{C}$ and $\mathcal{D}$ are symmetric monoidal categories (def. 1.5) and f is a braided monoidal functor (def. 1.47), then f^* also preserves commutative monoids

$$f^* : \mathrm{CMon}([\mathcal{D}, \mathrm{Top}_{\mathrm{cg}}^{*/}], \otimes_{\mathrm{Day}}, y(1_{\mathcal{D}})) \longrightarrow \mathrm{CMon}([\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*}], \otimes_{\mathrm{Day}}, y(1_{\mathcal{C}}) .$$

Similarly, for

$$A \in \mathrm{Mon}([\mathcal{D}, \mathrm{Top}_{\mathrm{cg}}^{*/}], \otimes_{\mathrm{Day}}, y(1_{\mathcal{D}}))$$

any fixed monoid, then f^* sends A-modules to $f^*(A)$-modules

$$f^* : A\,\mathrm{Mod}(\mathcal{D}) \longrightarrow (f^* A)\mathrm{Mod}(\mathcal{C}) .$$

Proof. This is an immediate corollary of prop. 1.52, since the composite of two (braided) lax monoidal functors is itself canonically a (braided) lax monoidal functor by prop. 1.49. ▮

2. $\mathbb{S}$-Modules

We give a unified discussion of the categories of

1. sequential spectra
2. symmetric spectra
3. orthogonal spectra
4. pre-excisive functors

(all in topological spaces) as categories of modules with respect to Day convolution monoidal structures on Top-enriched functor categories over restrictions to faithful sub-sites of the canonical representative of the sphere spectrum as a pre-excisive functor on $\mathrm{Top}^{*/}_{\mathrm{fin}}$.

This approach is due to (Mandell-May-Schwede-Shipley 00) following (Hovey-Shipley-Smith 00).

Pre-Excisive functors

We consider an almost tautological construction of a pointed topologically enriched category equipped with a closed symmetric monoidal product: the category of *pre-excisive functors*. Then we show that this tautological category restricts, in a certain sense, to the category of sequential spectra. However, under this restriction the symmetric monoidal product breaks, witnessing the lack of a functorial smash product of spectra on sequential spectra. However from inspection of this failure we see that there are categories of structured spectra "in between" those of all pre-excisive functors and plain sequential spectra, notably the categories of *orthogonal spectra* and of *symmetric spectra*. These intermediate categories retain the concrete tractable nature of sequential spectra, but are rich enough to also retain the symmetric monoidal product inherited from pre-excisive functors: this is the *symmetric monoidal smash product of spectra* that we are after.

Literature (MMSS 00, Part I and Part III)

Definition 2.1. Write

$$\iota_{\mathrm{fin}} \,:\, \mathrm{Top}^{*/}_{\mathrm{cg,fin}} \hookrightarrow \mathrm{Top}^{*/}_{\mathrm{cg}}$$

for the full subcategory of pointed compactly generated topological spaces (def.) on those that admit the structure of a finite CW-complex (a CW-complex (def.) with a finite number of cells).

We say that the pointed topological enriched functor category (def. 1.31)

$$\mathrm{Exc}(\mathrm{Top}_{\mathrm{cg}}) \coloneqq [\mathrm{Top}^{*/}_{\mathrm{cg,fin}}, \mathrm{Top}^{*/}_{\mathrm{cg}}]$$

is the category of **pre-excisive functors**. (We had previewed this in Part P, this example).

Write

$$\mathbb{S}_{\mathrm{exc}} \coloneqq y(S^0) \coloneqq \mathrm{Top}^{*/}_{\mathrm{cg,fin}}(S^0, -)$$

for the functor co-represented by 0-sphere. This is equivalently the inclusion ι_{fin} itself:

$$\mathbb{S}_{\mathrm{exc}} = \iota_{\mathrm{fin}} \,:\, K \mapsto K \,.$$

We call this the standard incarnation of the **sphere spectrum** as a pre-excisive functor.

By prop. 1.44 the smash product of pointed compactly generated topological spaces induces the structure of a closed (def. 1.7) symmetric monoidal category (def. 1.5)

$$\left(\mathrm{Exc}(\mathrm{Top}_{\mathrm{cg}}),\ \wedge \coloneqq \otimes_{\mathrm{Day}}\,,\ \mathbb{S}_{\mathrm{exc}}\right)$$

with

1. tensor unit the sphere spectrum $\mathbb{S}_{\mathrm{exc}}$;
2. tensor product the Day convolution product $\otimes_{\mathrm{Day}}$ from def. 1.39,

   called the **symmetric monoidal smash product of spectra** for the model of pre-excisive functors;
3. internal hom the dual operation $[-,-]_{\mathrm{Day}}$ from prop. 1.45,

   called the **mapping spectrum** construction for pre-excisive functors.

Remark 2.2. By example 1.14 the sphere spectrum incarnated as a pre-excisive functor $\mathbb{S}_{\mathrm{exc}}$ (according to def. 2.1) is canonically a commutative monoid in the category of pre-excisive functors (def. 1.13).

Moreover, by example 1.17, every object of $\mathrm{Exc}(\mathrm{Top}_{\mathrm{cg}})$ (def. 2.1) is canonically a module object over $\mathbb{S}_{\mathrm{exc}}$. We may therefore tautologically identify the category of pre-excisive functors with the module category over the sphere spectrum:

$$\mathrm{Exc}(\mathrm{Top}_{\mathrm{cg}}) \simeq \mathbb{S}_{\mathrm{exc}}\,\mathrm{Mod}\,.$$

Lemma 2.3. *Identified as a functor with smash product under prop. 1.52, the pre-excisive sphere spectrum* $\mathbb{S}_{\mathrm{exc}}$ *from def. 2.1 is given by the identity natural transformation*

$$\mu_{(K_1,K_2)} \,:\, \mathbb{S}_{\mathrm{exc}}(K_1) \wedge \mathbb{S}_{\mathrm{exc}}(K_2) = K_1 \wedge K_2 \overset{=}{\longrightarrow} K_1 \wedge K_2 = \mathbb{S}_{\mathrm{exc}}(K_1 \wedge K_2) \,.$$

Proof. We claim that this is in fact the unique structure of a monoidal functor that may be imposed on the canonical inclusion $\iota : \mathrm{Top}^{*/}_{\mathrm{cg,fin}} \hookrightarrow \mathrm{Top}^{*/}_{\mathrm{cg}}$, hence it must be the one in question. To see the uniqueness, observe that naturality of the matural transformation μ in particular says that there are commuting squares of the form

$$\begin{array}{ccc} S^0 \wedge S^0 & \overset{=}{\longrightarrow} & S^0 \wedge S^0 \\ {}^{\mathllap{x_1, x_2}}\downarrow && \downarrow^{\mathrlap{x_1,x_2}} \\ K_1 \wedge K_2 & \underset{\mu_{K_1,K_2}}{\longrightarrow} & K_1 \wedge K_2 \end{array} \,,$$

where the vertical morphisms pick any two points in K_1 and K_2, respectively, and where the top morphism is necessarily the canonical identification since there is only one single isomorphism $S^0 \to S^0$, namely the identity. This shows that the bottom horizontal morphism has to be the identity on all points, hence has to be the identity. ∎

We now consider restricting the domain of the pre-excisive functors of def. 2.1.

Definition 2.4. Define the following pointed topologically enriched (def.) symmetric monoidal categories (def. 1.5):

1. Seq is the category whose objects are the natural numbers and which has only identity morphisms and zero morphisms on these objects, hence the hom-spaces are

   $$\mathrm{Seq}(n_1, n_2) \;\coloneqq\; \left\{ \begin{array}{ll} S^0 & \text{for}\; n_1 = n_2 \\ \ast & \text{otherwise} \end{array} \right.$$

   The tensor product is the addition of natural numbers, $\otimes = +$, and the tensor unit is 0. The braiding is, necessarily, the identity.

2. Sym is the standard skeleton of the core of FinSet with zero morphisms adjoined: its objects are the finite sets $\overline{n} \coloneqq \{1, \cdots, n\}$ for $n \in \mathbb{N}$ (hence $\overline{0}$ is the empty set), all non-zero morphisms are automorphisms and the automorphism group of $\{1, \cdots, n\}$ is the symmetric group $\Sigma(n)$ on n elements, hence the hom-spaces are the following discrete topological spaces:

   $$\mathrm{Sym}(n_1, n_2) \;\coloneqq\; \left\{ \begin{array}{ll} (\Sigma(n_1))_+ & \text{for}\; n_1 = n_2 \\ \ast & \text{otherwise} \end{array} \right.$$

   The tensor product is the disjoint union of sets, tensor unit is the empty set. The braiding

   $$\tau^{\mathrm{Sym}}_{n_1,n_2} \,:\, \overline{n_1} \cup \overline{n_2} \longrightarrow \overline{n_2} \cup \overline{n_1}$$

   is given by the canonical permutation in $\Sigma(n_1 + n_2)$ that shuffles the first n_1 elements past the remaining n_2 elements.

   (MMSS 00, example 4.2)

3. Orth has as objects the finite dimenional real linear inner product spaces $(\mathbb{R}^n, \langle -,-\rangle)$ and as non-zero morphisms the linear isometric isomorphisms between these; hence the automorphism group of the object $(\mathbb{R}^n, \langle -,-\rangle)$ is the orthogonal group $O(n)$; the monoidal product is direct sum of linear spaces, the tensor unit is the 0-vector space; again we turn this into a $\mathrm{Top}^{*/}_{\mathrm{cg}}$-enriched category by adjoining a basepoint to the hom-spaces;

   $$\mathrm{Orth}(V_1, V_2) \;\coloneqq\; \left\{ \begin{array}{ll} O(V_1)_+ & \text{for}\; \dim(V_1) = \dim(V_2) \\ \ast & \text{otherwise} \end{array} \right.$$

   The tensor product is the direct sum of linear inner product spaces, tensor unit is the 0-vector space. The braiding

   $$\tau^{\mathrm{Orth}}_{V_1,V_2} \,:\, V_1 \oplus V_2 \longrightarrow V_2 \oplus V_1$$

   is the canonical orthogonal transformation that switches the summands.

   (MMSS 00, example 4.4)

Notice that in the notation of example 1.29

1. the full subcategory of Orth on V is $\mathbf{B}(O(V)_+)$;

2. the full subcategory of Sym on $\{1, \cdots, n\}$ is $\mathbf{B}(\Sigma(n)_+)$;

3. the full subcategory of Seq on n is $\mathbf{B}(1_+)$.

Moreover, after discarding the zero morphisms, then these categories are the disjoint union of categories of the form $\mathbf{B}\,O(n)$, $\mathbf{B}\,\Sigma(n)$ and $\mathbf{B}1 = *$, respectively.

There is a sequence of canonical faithful pointed topological subcategory inclusions

$$\begin{array}{ccccccc} \mathrm{Seq} & \overset{\mathrm{seq}}{\hookrightarrow} & \mathrm{Sym} & \overset{\mathrm{sym}}{\hookrightarrow} & \mathrm{Orth} & \overset{\mathrm{orth}}{\hookrightarrow} & \mathrm{Top}^{*/}_{\mathrm{cg,fin}} \\ n & \mapsto & \{1,\cdots,n\} & \mapsto & \mathbb{R}^n & \mapsto & S^n \end{array},$$

into the pointed topological category of pointed compactly generated topological spaces of finite CW-type (def. 2.1).

Here S^V denotes the one-point compactification of V. On morphisms $\mathrm{sym}\colon (\Sigma_n)_+ \hookrightarrow (O(n))_+$ is the canonical inclusion of permutation matrices into orthogonal matrices and $\mathrm{orth}\colon O(V)_+ \hookrightarrow \mathrm{Aut}(S^V)$ is on $O(V)$ the topological subspace inclusions of the pointed homeomorphisms $S^V \to S^V$ that are induced under forming one-point compactification from linear isometries of V ("representation spheres").

Below we will often use these identifications to write just "n" for any of these objects, leaving implicit the identifications $n \mapsto \{1,\cdots,n\} \mapsto S^n$.

Consider the pointed topological diagram categries (def. 1.31, exmpl.) over these categories:

- $[\mathrm{Seq}, \mathrm{Top}^{*/}_{\mathrm{cg}}]$ is called the category of **sequences** of pointed topological spaces (e.g. HSS 00, def. 2.3.1);
- $[\mathrm{Sym}, \mathrm{Top}^{*/}_{\mathrm{cg}}]$ is called the category of **symmetric sequences** (e.g. HSS 00, def. 2.1.1);
- $[\mathrm{Orth}, \mathrm{Top}^{*/}_{\mathrm{cg}}]$ is called the category of **orthogonal sequences**.

Consider the sequence of restrictions of topological diagram categories, according to prop. 1.53 along the above inclusions:

$$\mathrm{Exc}(\mathrm{Top}_{\mathrm{cg}}) \overset{\mathrm{orth}^*}{\longrightarrow} [\mathrm{Orth}, \mathrm{Top}^{*/}_{\mathrm{cg}}] \overset{\mathrm{sym}^*}{\longrightarrow} [\mathrm{Sym}, \mathrm{Top}^{*/}_{\mathrm{cg}}] \overset{\mathrm{seq}^*}{\longrightarrow} [\mathrm{Seq}, \mathrm{Top}^{*/}_{\mathrm{cg}}] \,.$$

Write

$$\mathbb{S}_{\mathrm{orth}} \coloneqq \mathrm{orth}^* \mathbb{S}_{\mathrm{exc}}\,, \qquad \mathbb{S}_{\mathrm{sym}} \coloneqq \mathrm{sym}^* \mathbb{S}_{\mathrm{orth}}\,, \qquad \mathbb{S}_{\mathrm{seq}} \coloneqq \mathrm{seq}^* \mathbb{S}_{\mathrm{sym}}$$

for the restriction of the excisive functor incarnation of the sphere spectrum (from def. 2.1) along these inclusions.

Proposition 2.5. *The functors seq, sym and orth in def. 2.4 become strong monoidal functors (def. 1.47) when equipped with the canonical isomorphisms*

$$\mathrm{seq}(n_1) \cup \mathrm{seq}(n_2) = \{1,\cdots,n_1\} \cup \{1,\cdots,n_2\} \simeq \{1,\cdots,n_1+n_2\} = \mathrm{seq}(n_1+n_2)$$

and

$$\mathrm{sym}(\{1,\cdots,n_1\}) \oplus \mathrm{sym}(\{1,\cdots,n_2\}) = \mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2} \simeq \mathbb{R}^{n_1+n_2} = \mathrm{sym}(\{1,\cdots,n_1\} \cup \{1,\cdots,n_2\})$$

and

$$\mathrm{orth}(V_1) \wedge \mathrm{orth}(V_2) = S^{V_1} \wedge S^{V_2} \simeq S^{V_1 \oplus V_2} = \mathrm{orth}(V_1 \oplus V_2)\,.$$

Moreover, orth and sym are braided monoidal functors (def. 1.47) (hence symmetric monoidal functors, remark 1.48). But seq is not braided monoidal.

Proof. The first statement is clear from inspection.

For the second statement it is sufficient to observe that all the nontrivial braiding of n-spheres in $\mathrm{Top}^{*/}_{\mathrm{cg}}$ is given by the maps induced from exchanging coordinates in the realization of n-spheres as one-point compactifications of Cartesian spaces $S^n \simeq (\mathbb{R}^n)^*$. This corresponds precisely to the action of the symmetric group inside the orthogonal group acting via the canonical action of the orthogonal group on $\mathbb{R}^n$. This shows that sym and orth are braided, for they include precisely these objects (the n-spheres) with these braidings on them. Finally it is clear that seq is not braided, because the braiding on Seq is trivial, while that on Sym is not, so seq necessrily fails to preserve precisely these non-trivial isomorphisms. ▮

Remark 2.6. Since the standard excisive incarnation $\mathbb{S}_{\mathrm{exc}}$ of the sphere spectrum (def. 2.1) is the tensor unit with repect to the Day convolution product on pre-excisive functors, and since it is therefore canonically a commutative monoid, by example 1.14, prop. 1.53 says that the restricted sphere spectra

$\mathbb{S}_{\mathrm{orth}}$, $\mathbb{S}_{\mathrm{sym}}$ and $\mathbb{S}_{\mathrm{seq}}$ are still monoids, and that under restriction every pre-excisive functor, regarded as a $\mathbb{S}_{\mathrm{exc}}$-module via remark 2.2, canonically becomes a module under the restricted sphere spectrum:

$$\begin{aligned} \mathrm{orth}^\ast &\colon \mathrm{Exc}(\mathrm{Top}_{\mathrm{cg}}) \simeq \mathbb{S}_{\mathrm{exc}}\,\mathrm{Mod} \longrightarrow \mathbb{S}_{\mathrm{orth}}\,\mathrm{Mod} \\ \mathrm{sym}^\ast &\colon \mathrm{Exc}(\mathrm{Top}_{\mathrm{cg}}) \simeq \mathbb{S}_{\mathrm{exc}}\,\mathrm{Mod} \longrightarrow \mathbb{S}_{\mathrm{sym}}\,\mathrm{Mod}\;. \\ \mathrm{seq}^\ast &\colon \mathrm{Exc}(\mathrm{Top}_{\mathrm{cg}}) \simeq \mathbb{S}_{\mathrm{exc}}\,\mathrm{Mod} \longrightarrow \mathbb{S}_{\mathrm{seq}}\,\mathrm{Mod} \end{aligned}$$

Since all three functors orth, sym and seq are strong monoidal functors by prop. 2.5, all three restricted sphere spectra $\mathbb{S}_{\mathrm{orth}}$, $\mathbb{S}_{\mathrm{sym}}$ and $\mathbb{S}_{\mathrm{seq}}$ canonically are monoids, by prop. 1.53. Moreover, according to prop. 2.5, orth and sym are braided monoidal functors, while functor seq is not braided, therefore prop. 1.53 furthermore gives that $\mathbb{S}_{\mathrm{orth}}$ and $\mathbb{S}_{\mathrm{sym}}$ are commutative monoids, while $\mathbb{S}_{\mathrm{seq}}$ is not commutative:

sphere spectrum	$\mathbb{S}_{\mathrm{exc}}$	$\mathbb{S}_{\mathrm{orth}}$	$\mathbb{S}_{\mathrm{sym}}$	$\mathbb{S}_{\mathrm{seq}}$
monoid	yes	yes	yes	yes
commutative monoid	yes	yes	yes	no
tensor unit	yes	no	no	no

Explicitly:

Lemma 2.7. *The monoids $\mathbb{S}_{\mathrm{dia}}$ from def. 2.4 are, when identified as functors with smash product via prop. 1.52 given by assigning*

$$\begin{aligned} \mathbb{S}_{\mathrm{seq}} &: n \mapsto S^n \\ \mathbb{S}_{\mathrm{sym}} &: \overline{n} \mapsto S^n \\ \mathbb{S}_{\mathrm{orth}} &: V \mapsto S^V\,, \end{aligned}$$

respectively, with product given by the canonical isomorphisms

$$S^{V_1} \wedge S^{V_2} \longrightarrow S^{V_1 \oplus V_2}\,.$$

Proof. By construction these functors with smash products are the composites, according to prop. 1.49, of the monoidal functors seq, sym, orth, respectively, with the lax monoidal functor corresponding to $\mathbb{S}_{\mathrm{exc}}$. The former have as structure maps the canonical identifications by definition, and the latter has as structure map the canonical identifications by lemmma 2.3. ▮

Proposition 2.8. *There is an equivalence of categories*

$$(-)^{\mathrm{seq}} \;:\; \mathbb{S}_{\mathrm{seq}}\,\mathrm{Mod} \longrightarrow \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$$

which identifies the category of modules (def. 1.16) over the monoid $\mathbb{S}_{\mathrm{seq}}$ (remark 2.6) in the Day convolution monoidal structure (prop. 1.44) over the topological functor category $[\mathrm{Seq}, \mathrm{Top}^{*/}_{\mathrm{cg}}]$ from def. 2.4 with the category of sequential spectra (def.)

Under this equivalence, an $\mathbb{S}_{\mathrm{seq}}$-module X is taken to the sequential pre-spectrum X^{seq} whose component spaces are the values of the pre-excisive functor X on the standard n-sphere $S^n = (S^1)^{\wedge n}$

$$(X^{\mathrm{seq}})_n \coloneqq X(\mathrm{seq}(n)) = X(S^n)$$

and whose structure maps are the images of the action morphisms

$$\mathbb{S}_{\mathrm{seq}} \otimes_{\mathrm{Day}} X \longrightarrow X$$

under the isomorphism of corollary 1.43

$$\mathbb{S}_{\mathrm{seq}}(n_1) \wedge X(n_1) \longrightarrow X_{n_1 + n_2}$$

evaluated at $n_1 = 1$

$$\begin{array}{ccc} \mathbb{S}_{\mathrm{seq}}(1) \wedge X(n) & \longrightarrow & X_{n+1} \\ {}^{\simeq}\downarrow && \downarrow^{\simeq} \\ S^1 \wedge X_n & \longrightarrow & X_{n+1} \end{array}\,.$$

(Hovey-Shipley-Smith 00, prop. 2.3.4)

Proof. After unwinding the definitions, the only point to observe is that due to the action property,

$$\begin{array}{ccc} \mathbb{S}_{\mathrm{seq}} \otimes_{\mathrm{Day}} \mathbb{S}_{\mathrm{seq}} \otimes_{\mathrm{Day}} X & \overset{\mathrm{id} \otimes_{\mathrm{Day}} \rho}{\longrightarrow} & \mathbb{S}_{\mathrm{seq}} \otimes_{\mathrm{Day}} X \\ {}^{\mu \otimes_{\mathrm{Day}} \mathrm{id}}\downarrow && \downarrow^{\rho} \\ \mathbb{S}_{\mathrm{seq}} \otimes_{\mathrm{Day}} X & \underset{\rho}{\longrightarrow} & X \end{array}$$

any $\mathbb{S}_{\mathrm{seq}}$-action

$$\rho : \mathbb{S}_{\mathrm{seq}} \otimes_{\mathrm{Day}} X \longrightarrow X$$

is indeed uniquely fixed by the components of the form

$$\mathbb{S}_{\mathrm{seq}}(1) \wedge X(n) \longrightarrow X(n) \;.$$

This is because under corollary 1.43 the action property is identified with the componentwise property

$$\begin{array}{ccc} S^{n_1} \wedge S^{n_2} \wedge X_{n_3} & \overset{\mathrm{id} \wedge \rho_{n_2,n_3}}{\longrightarrow} & S^{n_1} \wedge X_{n_2+n_3} \\ {}^{\simeq}\downarrow && \downarrow^{\rho_{n_1,n_2+n_3}} \\ S^{n_1+n_2} \wedge X_{n_3} & \underset{\rho_{n_1+n_2,n_3}}{\longrightarrow} & X_{n_1+n_2+n_3} \end{array}$$

where the left vertical morphism is an isomorphism by the nature of $\mathbb{S}_{\mathrm{seq}}$. Hence this fixes the components $\rho_{n',n}$ to be the n'-fold composition of the structure maps $\sigma_n \coloneqq \rho(1,n)$. ▮

However, since, by remark 2.8, $\mathbb{S}_{\mathrm{seq}}$ is not commutative, there is no tensor product induced on $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$ under the identification in prop. 2.8. But since $\mathbb{S}_{\mathrm{orth}}$ and $\mathbb{S}_{\mathrm{sym}}$ are commutative monoids by remark 2.8, it makes sense to consider the following definition.

Definition 2.9. In the terminology of remark 2.6 we say that

$$\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}}) \coloneqq \mathbb{S}_{\mathrm{orth}}\,\mathrm{Mod}$$

is the **category of orthogonal spectra**; and that

$$\mathrm{SymSpec}(\mathrm{Top}_{\mathrm{cg}}) \coloneqq \mathbb{S}_{\mathrm{sym}}\,\mathrm{Mod}$$

is the **category of symmetric spectra**.

By remark 2.6 and by prop. 1.22 these categories canonically carry a symmetric monoidal tensor product $\otimes_{\mathbb{S}_{\mathrm{orth}}}$ and $\otimes_{\mathbb{S}_{\mathrm{seq}}}$, respectively. This we call the **symmetric monoidal smash product of spectra**. We usually just write for short

$$\wedge \coloneqq \otimes_{\mathbb{S}_{\mathrm{orth}}} \;:\; \mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}}) \times \mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}}) \longrightarrow \mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})$$

and

$$\wedge \coloneqq \otimes_{\mathbb{S}_{\mathrm{sym}}} \;:\; \mathrm{SymSpec}(\mathrm{Top}_{\mathrm{cg}}) \times \mathrm{SymSpec}(\mathrm{Top}_{\mathrm{cg}}) \longrightarrow \mathrm{SymSpec}(\mathrm{Top}_{\mathrm{cg}})$$

In the next section we work out what these symmetric monoidal categories of orthogonal and of symmetric spectra look like more explicitly.

Symmetric and orthogonal spectra

We now define symmetric spectra and orthogonal spectra and their symmetric monoidal smash product. We proceed by giving the explicit definitions and then checking that these are equivalent to the abstract definition 2.9 from above.

Literature. (Hovey-Shipley-Smith 00, section 1, section 2, Schwede 12, chapter I)

Definition 2.10. A topological **symmetric spectrum** X is

1. a sequence $\{X_n \in \mathrm{Top}_{\mathrm{cg}}^{*/} \mid n \in \mathbb{N}\}$ of pointed compactly generated topological spaces;
2. a basepoint preserving continuous right action of the symmetric group $\Sigma(n)$ on X_n;
3. a sequence of morphisms $\sigma_n : S^1 \wedge X_n \longrightarrow X_{n+1}$

such that

- for all $n,k \in \mathbb{N}$ the composite

$$S^k \wedge X_n \simeq S^{k-1} \wedge S^1 \wedge X_n \overset{\mathrm{id}\wedge \sigma_n}{\longrightarrow} S^{k-1} \wedge X_{n+1} \simeq S^{k-2} \wedge S^1 \wedge X_{n+2} \overset{\mathrm{id}\wedge\sigma_{n+1}}{\longrightarrow} \cdots \overset{\sigma_{n+k-1}}{\longrightarrow} X_{n+k}$$

intertwines the $\Sigma(n) \times \Sigma(k)$-action.

A homomorphism of symmetric spectra $f : X \longrightarrow Y$ is

- a sequence of maps $f_n : X_n \longrightarrow Y_n$

such that

1. each f_n intetwines the $\Sigma(n)$-action;
2. the following diagrams commute

$$\begin{array}{ccc} S^1 \wedge X_n & \overset{f_n \wedge \mathrm{id}}{\longrightarrow} & S^1 \wedge Y_n \\ \downarrow^{\sigma_n^X} && \downarrow^{\sigma_n^Y} \\ X_{n+1} & \overset{f_{n+1}}{\longrightarrow} & Y_{n+1} \end{array} \,.$$

We write $\mathrm{SymSpec}(\mathrm{Top}_{\mathrm{cg}})$ for the resulting category of symmetric spectra.

(Hovey-Shipley-Smith 00, def. 1.2.2, Schwede 12, I, def. 1.1)

The definition of orthogonal spectra has the same structure, just with the symmetric groups replaced by the orthogonal groups.

Definition 2.11. A topological **orthogonal spectrum** X is

1. a sequence $\{X_n \in \mathrm{Top}_{\mathrm{cg}}^{*/} \mid n \in \mathbb{N}\}$ of pointed compactly generated topological spaces;
2. a basepoint preserving continuous right action of the orthogonal group $O(n)$ on X_n;
3. a sequence of morphisms $\sigma_n : S^1 \wedge X_n \longrightarrow X_{n+1}$

such that

- for all $n, k \in \mathbb{N}$ the composite

$$S^k \wedge X_n \simeq S^{k-1} \wedge S^1 \wedge X_n \overset{\mathrm{id} \wedge \sigma_n}{\longrightarrow} S^{k-1} \wedge X_{n+1} \simeq S^{k-2} \wedge S^1 \wedge X_{n+2} \overset{\mathrm{id} \wedge \sigma_{n+1}}{\longrightarrow} \cdots \overset{\sigma_{n+k-1}}{\longrightarrow} X_{n+k}$$

intertwines the $O(n) \times \mathrm{Ok}()$-action.

A homomorphism of orthogonal spectra $f : X \longrightarrow Y$ is

- a sequence of maps $f_n : X_n \longrightarrow Y_n$

such that

1. each f_n intetwines the $O(n)$-action;
2. the following diagrams commute

$$\begin{array}{ccc} S^1 \wedge X_n & \overset{f_n \wedge \mathrm{id}}{\longrightarrow} & S^1 \wedge Y_n \\ \downarrow^{\sigma_n^X} && \downarrow^{\sigma_n^Y} \\ X_{n+1} & \overset{f_{n+1}}{\longrightarrow} & Y_{n+1} \end{array} \,.$$

We write $\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})$ for the resulting category of orthogonal spectra.

(e.g. Schwede 12, I, def. 7.2)

Proposition 2.12. *Definitions 2.10 and 2.11 are indeed equivalent to def. 2.9:*

orthogonal spectra are euqivalently the module objects over the incarnation $\mathbb{S}_{\mathrm{orth}}$ of the sphere spectrum

$$\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}}) \simeq \mathbb{S}_{\mathrm{orth}}\, \mathrm{Mod}$$

and symmetric spectra sre equivalently the module objects over the incarnation $\mathbb{S}_{\mathrm{sym}}$ of the sphere spectrum

$$\mathrm{SymSpec}(\mathrm{Top}_{\mathrm{cg}}) \simeq \mathbb{S}_{\mathrm{sym}}\, \mathrm{Mod} \,.$$

(Hovey-Shipley-Smith 00, prop. 2.2.1)

Proof. We discuss this for symmetric spectra. The proof for orthogonal spectra is of the same form.

First of all, by example 1.29 an object in $[\mathrm{Sym}, \mathrm{Top}_{\mathrm{cg}}^{*/}]$ is equivalently a "symmetric sequence", namely a sequence of pointed topological spaces X_k, for $k \in \mathbb{N}$, equipped with an action of $\Sigma(k)$ (def. 2.4).

By corollary 1.43 and lemma 2.7, the structure morphism of an $\mathbb{S}_{\mathrm{sym}}$-module object on X

$$\mathbb{S}_{\mathrm{sym}} \otimes_{\mathrm{Day}} X \longrightarrow X$$

is equivalently (as a functor with smash products) a natural transformation

$$S^{n_1} \wedge X_{n_2} \longrightarrow X_{n_1+n_2}$$

over $\mathrm{Sym} \times \mathrm{Sym}$. This means equivalently that there is such a morphism for all $n_1, n_2 \in \mathbb{N}$ and that it is $\Sigma(n_1) \times \Sigma(n_2)$-equivariant.

Hence it only remains to see that these natural transformations are uniquely fixed once the one for $n_1 = 1$ is given. To that end, observe that lemma 2.7 says that in the following commuting squares (exhibiting the action property on the level of functors with smash product, where we are notationally suppressing the associators) the left vertical morphisms are isomorphisms:

$$\begin{array}{ccc} S^{n_1} \wedge S^{n_2} \wedge X_{n_3} & \longrightarrow & S^{n_1} \wedge X_{n_2+n_3} \\ {}^{\simeq}\downarrow && \downarrow \\ S^{n_1+n_2} \wedge X_{n_3} & \longrightarrow & X_{n_1+n_2+n_3} \end{array} \,.$$

This says exactly that the action of $S^{n_1+n_2}$ has to be the composite of the actions of S^{n_2} followed by that of S^{n_1}. Hence the statement follows by induction.

Finally, the definition of homomorphisms on both sides of the equivalence are just so as to preserve precisely this structure, hence they conincide under this identification. ▮

Definition 2.13. Given $X, Y \in \mathrm{SymSpec}(\mathrm{Top}_{\mathrm{cg}})$ two symmetric spectra, def. 2.10, then their **smash product of spectra** is the symmetric spectrum

$$X \wedge Y \ \in \mathrm{SymSpec}(\mathrm{Top}_{\mathrm{cg}})$$

with component spaces the coequalizer

$$\underset{p+1+q=n}{\bigvee} \Sigma(p+1+q)_+ \underset{\Sigma_p \times \Sigma_1 \times \Sigma_q}{\wedge} X_p \wedge S^1 \wedge Y_q \underoverset{r}{\ell}{\rightrightarrows} \underset{p+q=n}{\bigvee} \Sigma(p+q)_+ \underset{\Sigma_p \times \Sigma_q}{\wedge} X_p \wedge Y_q \overset{\mathrm{coeq}}{\longrightarrow} (X \wedge Y)(n)$$

where ℓ has components given by the structure maps

$$X_p \wedge S^1 \wedge Y_q \overset{\mathrm{id} \wedge \sigma_q}{\longrightarrow} X_p \wedge Y_q$$

while r has components given by the structure maps conjugated by the braiding in $\mathrm{Top}_{\mathrm{cg}}^{*/}$ and the permutation action $\chi_{p,1}$ (that shuffles the element on the right to the left)

$$X_p \wedge S^1 \wedge X_q \overset{\tau^{\mathrm{Top}_{\mathrm{cg}}^{*/}}_{X_p, S^1} \wedge \mathrm{id}}{\longrightarrow} S^1 \wedge X_p \wedge X_q \overset{\sigma_p \wedge \mathrm{id}}{\longrightarrow} X_{p+1} \wedge X_q \overset{\chi_{p,1} \wedge \mathrm{id}}{\longrightarrow} X_{1+p} \wedge X_q \,.$$

Finally The structure maps of $X \wedge Y$ are those induced under the coequalizer by

$$S^1 \wedge (X_p \wedge Y_q \wedge) \simeq (S^1 \wedge X_p) \wedge Y_q \overset{\sigma_p^X \wedge \mathrm{id}}{\longrightarrow} X_{p+1} \wedge Y_q \,.$$

Analogously for orthogonal spectra.

(Schwede 12, p. 82)

Proposition 2.14. *Under the identification of prop. 2.12, the explicit smash product of spectra in def. 2.13 is equivalent to the abstractly defined tensor product in def. 2.9:*

in the case of symmetric spectra:

$$\wedge \simeq \otimes_{\mathbb{S}_{\mathrm{sym}}}$$

in the case of orthogonal spectra:

$$\wedge \simeq \otimes_{\mathbb{S}_{\mathrm{orth}}} \,.$$

(Schwede 12, E.1.16)

Proof. By def. 1.21 the abstractly defined tensor product of two $\mathbb{S}_{\mathrm{sym}}$-modules X and Y is the coequalizer

$$X \otimes_{\mathrm{Day}} \mathbb{S}_{\mathrm{sym}} \otimes_{\mathrm{Day}} Y \underset{\rho_1 \circ (\tau^{\mathrm{Day}}_{X,\mathbb{S}_{\mathrm{sym}}} \otimes \mathrm{id})}{\overset{X \otimes \rho_2}{\rightrightarrows}} X \otimes Y \overset{\mathrm{coeq}}{\longrightarrow} X \otimes_{\mathbb{S}_{\mathrm{sym}}} Y \,.$$

The Day convolution product appearing here is over the category Sym from def. 2.4. By example 1.29 and unwinding the definitions, this is for any two symmetric spectra A and B given degreewise by the wedge sum of component spaces summing to that total degree, smashed with the symmetric group with basepoint adjoined and then quotiented by the diagonal action of the symmetric group acting on the degrees separately:

$$\begin{aligned} (A \otimes_{\mathrm{Day}} B)(n) &= \int^{n_1,n_2} \underbrace{\Sigma(n_1+n_2,n)}_{= \left\{ \begin{array}{ll} \Sigma(n_1+n_2,n)_+ & \text{if } n_1+n_2=n \\ * & \text{otherwise} \end{array} \right._+} \wedge A_{n_1} \wedge B_{n_1} \\ &\simeq \bigvee_{n_1+n_2=n} \Sigma(n_1+n_2)_+ \underset{O(n_1)\times O(n_2)}{\wedge} \left(A_{n_1} \wedge B_{n_2}\right) \end{aligned} \,.$$

This establishes the form of the coequalizer diagram. It remains to see that under this identification the two abstractly defined morphisms are the ones given in def. 2.13.

To see this, we apply the adjunction isomorphism between the Day convolution product and the external tensor product (cor. 1.43) twice, to find the following sequence of equivalent incarnations of morphisms:

$(X \otimes_{\mathrm{Day}} (\mathbb{S}_{\mathrm{orth}} \otimes_{\mathrm{Day}} Y))(n)$	$\longrightarrow$	$(X \otimes_{\mathrm{Day}} Y)(n)$	$\longrightarrow$	Z_n
$X_{n_1} \wedge (\mathbb{S}_{\mathrm{sym}} \otimes_{\mathrm{Day}} Y)(n'_2)$	$\longrightarrow$	$X_{n_1} \wedge Y(n'_2)$	$\longrightarrow$	$Z_{n_1+n'_2}$
$(\mathbb{S}_{\mathrm{sym}} \otimes_{\mathrm{Day}} Y)(n'_2)$	$\longrightarrow$	$Y(n'_2)$	$\longrightarrow$	$\mathrm{Maps}(X_{n_1}, Z_{n_1+n'_2})$
$S^{n_2} \wedge Y_{n_3}$	$\longrightarrow$	$Y_{n_2+n_3}$	$\longrightarrow$	$\mathrm{Maps}(X_{n_1}, Z_{n_1+n_2+n_3})$
$X_{n_1} \wedge S^{n_2} \wedge Y_{n_3}$	$\longrightarrow$	$X_{n_1} \wedge Y_{n_2+n_3}$	$\longrightarrow$	$Z_{n_1+n_2+n_3}$

This establishes the form of the morphism ℓ. By the same reasoning as in the proof of prop. 2.12, we may restrict the coequalizer to $n_2 = 1$ without changing it.

The form of the morphism r is obtained by the analogous sequence of identifications of morphisms, now with the parenthesis to the left. That it involves $\tau^{\mathrm{Top}^{*/}_{\mathrm{cg}}}$ and the permutation action τ^{sym} as shown above follows from the formula for the braiding of the Day convolution tensor product from the proof of prop. 1.44:

$$\tau^{\mathrm{Day}}_{A,B}(n) = \int^{n_1,n_2} \mathrm{Sym}(\tau^{\mathrm{Sym}}_{n_1,n_2}, n) \wedge \tau^{\mathrm{Top}^{*/}_{\mathrm{cg}}}_{A_{n_1},B_{n_2}}$$

by translating it to the components of the precomposition

$$X \otimes_{\mathrm{Day}} \mathbb{S}_{\mathrm{sym}} \overset{\tau^{\mathrm{Day}}_{X,\mathbb{S}_{\mathrm{sym}}}}{\longrightarrow} \mathbb{S}_{\mathrm{sym}} \otimes_{\mathrm{Day}} X \longrightarrow X$$

via the formula from the proof of prop. 1.38 for the left Kan extension $A \otimes_{\mathrm{Day}} B \simeq \mathrm{Lan}_{\otimes} A \overline{\wedge} B$ (prop. 1.42):

$$\begin{aligned} [\mathrm{Sym}, \mathrm{Top}^{*/}_{\mathrm{cg}}](\tau^{\mathrm{Day}}_{X,\mathbb{S}_{\mathrm{sym}}}, X) &\simeq \int_n \mathrm{Maps}\Big(\int^{n_1,n_2} \mathrm{Sym}(\tau^{\mathrm{sym}}_{n_1,n_2}, n) \wedge \tau^{\mathrm{Top}^{*/}_{\mathrm{cg}}}_{X_{n_1},S^{n_2}}, X(n) \Big)_* \\ &\simeq \int_{n_1,n_2} \mathrm{Maps}\big(\tau^{\mathrm{Top}^{*/}_{\mathrm{cg}}}_{X_{n_1},S^{n_2}}, X(\tau^{\mathrm{sym}}_{n_1,n_2}) \big)_* \end{aligned} \,.$$

This last expression is the function on morphisms which precomposes components under the coend with the braiding $\tau^{\mathrm{Top}^{*/}_{\mathrm{cg}}}_{X_{n_1},S^{n_2}}$ in topological spaces and postcomposes them with the image of the functor X of the braiding in Sym. But the braiding in Sym is, by def. 2.4, given by the respective shuffle permutations $\tau^{\mathrm{sym}}_{n_1,n_2} = \chi_{n_1,n_2}$, and by prop. 2.12 the image of these under X is via the given $\Sigma_{n_1+n_2}$-action on $X_{n_1+n_2}$.

Finally to see that the structure map is as claimed: By prop. 2.12 the structure morphisms are the degree-1 components of the $\mathbb{S}_{\mathrm{sym}}$-action, and by prop. 1.21 the $\mathbb{S}_{\mathrm{sym}}$-action on a tensor product of $\mathbb{S}_{\mathrm{sym}}$-modules is induced via the action on the left tensor factor. ▮

Definition 2.15. A **commutative symmetric ring spectrum** E is

1. a sequence of component spaces $E_n \in \mathrm{Top}^{*/}_{\mathrm{cg}}$ for $n \in \mathbb{N}$;

2. a basepoint preserving continuous left action of the symmetric group $\Sigma(n)$ on E_n;

3. for all $n_1, n_2 \in \mathbb{N}$ a *multiplication map*

$$\mu_{n_1,n_2} : E_{n_1} \wedge E_{n_2} \longrightarrow E_{n_1+n_2}$$

(a morphism in $\mathrm{Top}_{\mathrm{cg}}^{*/}$)

4. two *unit maps*

$$\begin{aligned} \iota_0 &: S^0 \longrightarrow E_0 \\ \iota_1 &: S^1 \longrightarrow E_1 \end{aligned}$$

such that

1. (equivariance) μ_{n_1,n_2} intertwines the $\Sigma(n_1) \times \Sigma(n_2)$-action;
2. (associativity) for all $n_1, n_2, n_3 \in \mathbb{N}$ the following diagram commutes (where we are notationally suppressing the associators of $(\mathrm{Top}_{\mathrm{cg}}^{*/}, \wedge, S^0)$)

$$\begin{tikzcd} E_{n_1} \wedge E_{n_2} \wedge E_{n_3} \arrow[r, "\mathrm{id} \wedge \mu_{n_2,n_3}"] \arrow[d, "\mu_{n_1,n_2} \wedge \mathrm{id}"'] & E_{n_1} \wedge E_{n_2+n_3} \arrow[d, "\mu_{n_1,n_2+n_3}"] \\ E_{n_1+n_2} \wedge E_{n_3} \arrow[r, "\mu_{n_1+n_2,n_3}"'] & E_{n_1+n_2+n_3} \end{tikzcd}$$

3. (unitality) for all $n \in \mathbb{N}$ the following diagram commutes

$$\begin{tikzcd} S^0 \wedge E_n \arrow[r, "\iota_0 \wedge \mathrm{id}"] \arrow[dr, "\ell^{\mathrm{Top}_{\mathrm{cg}}^{*/}}_{E_n}"'] & E_0 \wedge E_n \arrow[d, "\mu_{0,n}"] \\ & E_n \end{tikzcd}$$

and

$$\begin{tikzcd} E_n \wedge S^0 \arrow[r, "\mathrm{id} \wedge \iota_0"] \arrow[dr, "r^{\mathrm{Top}_{\mathrm{cg}}^{*/}}_{E_n}"'] & E_n \wedge E_0 \arrow[d, "\mu_{n,0}"] \\ & E_n \end{tikzcd},$$

where the diagonal morphisms ℓ and r are the left and right unitors in $(\mathrm{Top}_{\mathrm{cg}}^{*/}, \wedge, S^0)$, respectively.

4. (commutativity) for all $n_1, n_2 \in \mathbb{N}$ the following diagram commutes

$$\begin{tikzcd} E_{n_1} \wedge E_{n_2} \arrow[r, "\tau^{\mathrm{Top}_{\mathrm{cg}}^{*/}}_{E_{n_1},E_{n_2}}"] \arrow[d, "\mu_{n_1,n_2}"'] & E_{n_2} \wedge E_{n_1} \arrow[d, "\mu_{n_2,n_1}"] \\ E_{n_1+n_2} \arrow[r, "\chi_{n_1,n_2}"'] & E_{n_2+n_1} \end{tikzcd}$$

where the top morphism τ is the braiding in $(\mathrm{Top}_{\mathrm{cg}}^{*/}, \wedge, S^0)$ (def. 1.10) and where $\chi_{n_1,n_2} \in \Sigma(n_1+n_2)$ denotes the permutation action which shuffles the first n_1 elements past the last n_2 elements.

A homomorphism of symmetric commutative ring spectra $f : E \longrightarrow E'$ is a sequence $f_n : E_n \longrightarrow E'_n$ of $\Sigma(n)$-equivariant pointed continuous functions such that the following diagrams commute for all $n_1, n_2 \in \mathbb{N}$

$$\begin{tikzcd} E_{n_1} \wedge E_{n_2} \arrow[r, "f_{n_1} \wedge f_{n_2}"] \arrow[d, "\mu_{n_1,n_2}"'] & E'_{n_1} \wedge E'_{n_2} \arrow[d, "\mu_{n_2,n_1}"] \\ E_{n_1+n_2} \arrow[r, "\chi_{n_1,n_2}"'] & E_{n_2+n_1} \end{tikzcd}$$

and $f_0 \circ \iota_0 = \iota_0$ and $f_1 \circ \iota_1 = \iota_1$.

Write

$$\mathrm{CRing}(\mathrm{SymSpec}(\mathrm{Top}_{\mathrm{cg}}))$$

for the resulting category of symmetric commutative ring spectra.

We regard a symmetric ring spectrum in particular as a symmetric spectrum (def. 2.10) by taking the structure maps to be

$$\sigma_n \;:\; S^1 \wedge E_n \overset{\iota_1 \wedge \mathrm{id}}{\longrightarrow} E_1 \wedge E_n \overset{\mu_{1,n}}{\longrightarrow} E_{n+1} \,.$$

This defines a forgetful functor

$$\mathrm{CRing}(\mathrm{SymSpec}(\mathrm{Top}_{\mathrm{cg}})) \longrightarrow \mathrm{SymSpec}(\mathrm{Top}_{\mathrm{cg}})$$

There is an analogous definition of **orthogonal ring spectrum** and we write

$$\mathrm{CRing}(\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}}))$$

for the category that these form.

(e.g. Schwede 12, def. 1.3)

We discuss **examples** below in a dedicated section *Examples*.

Proposition 2.16. *The symmetric (orthogonal) commutative ring spectra in def. 2.15 are equivalently the commutative monoids in (def. 1.13) the symmetric monoidal category $\mathbb{S}_{\mathrm{sym}}\,\mathrm{Mod}$ ($\mathbb{S}_{\mathrm{orth}}\,\mathrm{Mod}$) of def. 2.9 with respect to the symmetric monoidal smash product of spectra $\wedge = \otimes_{\mathbb{S}_{\mathrm{sym}}}$ ($\wedge = \otimes_{\mathbb{S}_{\mathrm{orth}}}$). Hence there are equivalences of categories*

$$\mathrm{CRing}(\mathrm{SymSpec}(\mathrm{Top}_{\mathrm{cg}})) \;\simeq\; \mathrm{CMon}(\mathbb{S}_{\mathrm{sym}}\,\mathrm{Mod}, \otimes_{\mathbb{S}_{\mathrm{sym}}}, \mathbb{S}_{\mathrm{sym}})$$

and

$$\mathrm{CRing}(\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})) \;\simeq\; \mathrm{CMon}(\mathbb{S}_{\mathrm{orth}}\,\mathrm{Mod}, \otimes_{\mathbb{S}_{\mathrm{orth}}}, \mathbb{S}_{\mathrm{orth}}) \,.$$

Moreover, under these identifications the canonical forgetful functor

$$\mathrm{CMon}(\mathbb{S}_{\mathrm{sym}}\,\mathrm{Mod}, \otimes_{\mathbb{S}_{\mathrm{sym}}}, \mathbb{S}_{\mathrm{sym}}) \longrightarrow \mathrm{SymSpec}(\mathrm{Top}_{\mathrm{cg}})$$

and

$$\mathrm{CMon}(\mathbb{S}_{\mathrm{orth}}\,\mathrm{Mod}, \otimes_{\mathbb{S}_{\mathrm{orth}}}, \mathbb{S}_{\mathrm{orth}}) \longrightarrow \mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})$$

coincides with the forgetful functor defined in def. 2.15.

Proof. We discuss this for symmetric spectra. The proof for orthogonal spectra is directly analogous.

By prop. 1.25 and def. 2.9, the commutative monoids in $\mathbb{S}_{\mathrm{sym}}\,\mathrm{Mod}$ are equivalently commtutative monoids E in $([\mathrm{Sym}, \mathrm{Top}_{\mathrm{cg}}^{*/}], \otimes_{\mathrm{Day}}, y(0))$ equipped with a homomorphism of monoids $\mathbb{S}_{\mathrm{sym}} \longrightarrow E$. In turn, by prop. 1.52 this are equivalently braided lax monoidal functors (which we denote by the same symbols, for convenience) of the form

$$E \;:\; (\mathrm{Sym}, +, 0) \longrightarrow (\mathrm{Top}_{\mathrm{cg}}^{*/}, \wedge, S^0)$$

equipped with a monoidal natural transformation (def. 1.47)

$$\iota \;:\; \mathbb{S}_{\mathrm{sym}} \longrightarrow E \,.$$

The structure morphism of such a lax monoidal functor E has as components precisely the morphisms $\mu_{n_1,n_2} \colon E_{n_1} \wedge E_{n_2} \to E_{n_1+n_2}$. In terms of these, the associativity and braiding condition on the lax monoidal functor are manifestly the above associativity and commutativity conditions.

Moreover, by the proof of prop. 1.25 the $\mathbb{S}_{\mathrm{sym}}$-module structure on an an $\mathbb{S}_{\mathrm{sym}}$-algebra E has action given by

$$\mathbb{S}_{\mathrm{sym}} \wedge E \overset{e \wedge \mathrm{id}}{\longrightarrow} E \wedge E \overset{\mu}{\longrightarrow} E \,,$$

which shows, via the identification in prop 2.12, that the forgetful functors to underlying symmetric spectra coincide as claimed.

Hence it only remains to match the nature of the units. The above unit morphism ι has components

$$\iota_n \;:\; S^n \longrightarrow E_n$$

for all $n \in \mathbb{N}$, and the unitality condition for ι_0 and ι_1 is manifestly as in the statement above.

We claim that the other components are uniquely fixed by these:

By lemma 2.7, the product structure in $\mathbb{S}_{\mathrm{sym}}$ is by isomorphisms $S^{n_1} \wedge S^{n_2} \simeq S^{n_1+n_2}$, so that the commuting square for the coherence condition of this monoidal natural transformation

$$
\begin{array}{ccc}
S^{n_1} \wedge S^{n_2} & \overset{\iota_{n_1} \wedge \iota_{n_2}}{\longrightarrow} & E_{n_1} \wedge E_{n_2} \\
{}^{\simeq}\downarrow && \downarrow^{\mu_{n_1,n_2}} \\
S^{n_1+n_2} & \underset{\iota_{n_1+n_2}}{\longrightarrow} & E_{n_1+n_2}
\end{array}
$$

says that $\iota_{n_1+n_2} = \mu_{n_1,n_2} \circ (\iota_{n_1} \wedge \iota_{n_2})$. This means that $\iota_{n \geq 2}$ is uniquely fixed once ι_0 and ι_1 are given.

Finally it is clear that homomorphisms on both sides of the equivalence precisely respect all this structure under both sides of the equivalence. ▮

Similarly:

Definition 2.17. Given a symmetric (orthogonal) commutative ring spectrum E (def. 2.15), then a left symmetric (orthogonal) **module spectrum** N over E is

1. a sequence of component spaces $N_n \in \mathrm{Top}_{\mathrm{cg}}^{*/}$ for $n \in \mathbb{N}$;

2. a basepoint preserving continuous left action of the symmetric group $\Sigma(n)$ on N_n;

3. for all $n_1, n_2 \in \mathbb{N}$ an *action map*

$$
\rho_{n_1,n_2} : E_{n_1} \wedge N_{n_2} \longrightarrow N_{n_1+n_2}
$$

(a morphism in $\mathrm{Top}_{\mathrm{cg}}^{*/}$)

such that

1. (equivariance) ρ_{n_1,n_2} intertwines the $\Sigma(n_1) \times \Sigma(n_2)$-action;

2. (action property) for all $n_1, n_2, n_3 \in \mathbb{N}$ the following diagram commutes (where we are notationally suppressing the associators of $(\mathrm{Top}_{\mathrm{cg}}^{*/}, \wedge, S^0)$)

$$
\begin{array}{ccc}
E_{n_1} \wedge E_{n_2} \wedge N_{n_3} & \overset{\mathrm{id} \wedge \rho_{n_2,n_3}}{\longrightarrow} & E_{n_1} \wedge N_{n_2+n_3} \\
{}^{\mu_{n_1,n_2} \wedge \mathrm{id}}\downarrow && \downarrow^{\rho_{n_1,n_2+n_3}} \\
E_{n_1+n_2} \wedge N_{n_3} & \underset{\rho_{n_1+n_2,n_3}}{\longrightarrow} & N_{n_1+n_2+n_3}
\end{array}
$$

3. (unitality) for all $n \in \mathbb{N}$ the following diagram commutes

$$
\begin{array}{ccc}
S^0 \wedge N_n & \overset{\iota_0 \wedge \mathrm{id}}{\longrightarrow} & E_0 \wedge N_n \\
& {}_{\ell^{\mathrm{Top}_{\mathrm{cg}}^{*/}}_{N_n}}\searrow & \downarrow^{\mu_{0,n}} \\
&& N_n
\end{array}
.
$$

A homomorphism of left E-module spectra $f : N \longrightarrow N'$ is a sequence of pointed continuous functions $f_n : N_n \longrightarrow N'_n$ such that for all $n_1, n_2 \in \mathbb{N}$ the following diagrams commute:

$$
\begin{array}{ccc}
E_{n_1} \wedge N_{n_2} & \overset{\mathrm{id} \wedge f_{n_2}}{\longrightarrow} & E_{n_1} \wedge N'_{n_2} \\
{}^{\rho_{n_1,n_2}}\downarrow && \downarrow^{\rho_{n_1,n_2}} \\
N_{n_1+n_2} & \underset{f_{n_1+n_2}}{\longrightarrow} & N'_{n_1+n_2}
\end{array}
.
$$

We write

$$
E\,\mathrm{Mod}(\mathrm{SymSpec}(\mathrm{Top}_{\mathrm{cg}})) \quad , \quad E\,\mathrm{Mod}(\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}}))
$$

for the resulting category of symmetric (orthogonal) E-module spectra.

(e.g. Schwede 12, I, def. 1.5)

Proposition 2.18. *Under the identification, from prop. 2.16, of commutative ring spectra with commutative monoids with respect to the symmetric monoidal smash product of spectra, the E-module spectra of def. 2.17 are equivalently the left module objects (def. 1.16) over the respective monoids, i.e. there are equivalences of categories*

$$
E\,\mathrm{Mod}(\mathrm{SymSpec}(\mathrm{Top}_{\mathrm{cg}})) \simeq E\,\mathrm{Mod}([\mathrm{Sym}, \mathrm{Top}_{\mathrm{cg}}^{*/}], \otimes_{\mathrm{Day}}, y(0))
$$

and

$$E\,\mathrm{Mod}(\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})) \;\simeq\; E\,\mathrm{Mod}([\mathrm{Orth}, \mathrm{Top}_{\mathrm{cg}}^{*/}], \otimes_{\mathrm{Day}}, y(0))\,,$$

where on the right we have the categories of modules from def. 1.16.

Proof. The proof is directly analogous to that of prop. 2.16. Now prop. 1.25 and prop. 1.52 give that the module objects in question are equivalently modules over a monoidal functor (def. 1.51) and writing these out in components yields precisely the above structures and properties. ▮

As diagram spectra

In *Introduction to Stable homotopy theory -- 1-1* we obtained the strict/level model structure on topological sequential spectra by identifying the category $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$ of sequential spectra with a category of topologically enriched functors with values in $\mathrm{Top}_{\mathrm{cg}}^{*/}$ (prop.) and then invoking the general existence of the projective model structure on functors (thm.).

Here we discuss the analogous construction for the more general structured spectra from above.

Proposition 2.19. *Let $(\mathcal{C}, \otimes_{\mathcal{C}}, 1_{\mathcal{C}})$ be a topologically enriched monoidal category (def. 1.1), and let $A \in \mathrm{Mon}([\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}], \otimes_{\mathrm{Day}}, y(1_{\mathcal{C}}))$ be a monoid in (def. 1.13) the pointed topological Day convolution monoidal category over $\mathcal{C}$ from prop. 1.44.*

Then the category of left A-modules (def. 1.16) is a pointed topologically enriched functor category category (exmpl.)

$$A\,\mathrm{Mod} \;\simeq\; [A\,\mathrm{Free}_{\mathcal{C}}\mathrm{Mod}^{\mathrm{op}},\; \mathrm{Top}_{\mathrm{cg}}^{*/}]\,,$$

over the category of free modules over A (prop. 1.20) on objects in $\mathcal{C}$ (under the Yoneda embedding $y \colon \mathcal{C}^{\mathrm{op}} \to [\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}]$). Hence the objects of $A\,\mathrm{Free}_{\mathcal{C}}\mathrm{Mod}$ are identified with those of $\mathcal{C}$, and its hom-spaces are

$$A\,\mathrm{Free}_{\mathcal{C}}\mathrm{Mod}(c_1, c_2) \;=\; A\,\mathrm{Mod}(A \otimes_{\mathrm{Day}} y(c_1),\; A \otimes_{\mathrm{Day}} y(c_2))\,.$$

(MMSS 00, theorem 2.2)

Proof. Use the identification from prop. 1.52 of A with a lax monoidal functor and of any A-module object N as a functor with the structure of a module over a monoidal functor, given by natural transformations

$$A(c_1) \otimes N(c_2) \overset{\rho_{c_1,c_2}}{\longrightarrow} N(c_1 \otimes c_2)\,.$$

Notice that these transformations have just the same structure as those of the enriched functoriality of N (def.) of the form

$$\mathcal{C}(c_1, c_2) \otimes N(c_1) \longrightarrow N(c_2)\,.$$

Hence we may unify these two kinds of transformations into a single kind of the form

$$\mathcal{C}(c_3 \otimes c_1, c_2) \otimes A(c_3) \otimes N(c_1) \overset{\mathrm{id} \otimes \rho_{c_3,c_1}}{\longrightarrow} \mathcal{C}(c_3 \otimes c_1, c_2) \otimes N(c_3 \otimes c_2) \longrightarrow N(c_2)$$

and subject to certain identifications.

Now observe that the hom-objects of $A\,\mathrm{Free}_{\mathcal{C}}\mathrm{Mod}$ have just this structure:

$$\begin{aligned} A\,\mathrm{Free}_{\mathcal{C}}\mathrm{Mod}(c_2, c_1) &= A\,\mathrm{Mod}(A \otimes_{\mathrm{Day}} y(c_2), A \otimes_{\mathrm{Day}} y(c_1)) \\ &\simeq [\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}](y(c_2), A \otimes_{\mathrm{Day}} y(c_1)) \\ &\simeq (A \otimes_{\mathrm{Day}} y(c_1))(c_2) \\ &\simeq \int^{c_3, c_4} \mathcal{C}(c_3 \otimes c_4, c_2) \wedge A(c_3) \wedge \mathcal{C}(c_1, c_4) \\ &\simeq \int^{c_3} \mathcal{C}(c_3 \otimes c_1, c_2) \wedge A(c_3) \end{aligned}\,.$$

Here we used first the free-forgetful adjunction of prop. 1.20, then the enriched Yoneda lemma (prop. 1.32), then the coend-expression for Day convolution (def. 1.39) and finally the co-Yoneda lemma (prop. 1.33).

Then define a topologically enriched category $\mathcal{D}$ to have objects and hom-spaces those of $A\,\mathrm{Free}_{\mathcal{C}}\mathrm{Mod}^{\mathrm{op}}$ as above, and whose composition operation is defined as follows:

$$
\begin{aligned}
\mathcal{D}(c_2, c3) \wedge \mathcal{D}(c_1, c_2) & \simeq \left(\int^{c_5} \mathcal{C}(c_5 \otimes_{\mathcal{C}} c_2, c_3) \wedge A(c_5)\right) \wedge \left(\int^{c_4} \mathcal{C}(c_4 \otimes_{\mathcal{C}} c_1, c_2) \wedge A(c_4)\right) \\
& \simeq \int^{c_4, c_5} \mathcal{C}(c_5 \otimes_{\mathcal{C}} c_2, c_3) \wedge \mathcal{C}(c_4 \otimes_{\mathcal{C}} c_1, c_2) \wedge A(c_5) \wedge A(c_4) \\
& \longrightarrow \int^{c_4, c_5} \mathcal{C}(c_5 \otimes_{\mathcal{C}} c_2, c_3) \wedge \mathcal{C}(c_5 \otimes_{\mathcal{C}} c_4 \otimes_{\mathcal{C}} c_1, c_5 \otimes_{\mathcal{C}} c_2) \wedge A(c_5 \otimes_{\mathcal{C}} c_4)' \\
& \longrightarrow \int^{c_4, c_5} \mathcal{C}(c_5 \otimes_{\mathcal{C}} c_4 \otimes_{\mathcal{C}} c_1, c_5 \otimes_{\mathcal{C}} c_2) \wedge A(c_5 \otimes_{\mathcal{C}} c_4) \\
& \longrightarrow \int^{c_4} \mathcal{C}(c_4 \otimes_{\mathcal{C}} c_1, c_3) \otimes_V A(c_4)
\end{aligned}
$$

where

1. the equivalence is braiding in the integrand (and the Fubini theorem, prop. 1.35);
2. the first morphism is, in the integrand, the smash product of
   1. forming the tensor product of hom-objects of $\mathcal{C}$ with the identity morphism on c_5;
   2. the monoidal functor incarnation $A(c_5) \wedge A(c_4) \longrightarrow A(c_5 \otimes_{\mathcal{C}} c_4)$ of the monoid structure on A;
3. the second morphism is, in the integrand, given by composition in $\mathcal{C}$;
4. the last morphism is the morphism induced on coends by regarding extranaturality in c_4 and c_5 separately as a special case of extranaturality in $c_6 \coloneqq c_4 \otimes c_5$ (and then renaming).

With this it is fairly straightforward to see that

$$A\,\mathrm{Mod} \simeq [\mathcal{D}, \mathrm{Top}_{\mathrm{cg}}^{*/}]\,,$$

because, by the above definition of composition, functoriality over $\mathcal{D}$ manifestly encodes the A-action property together with the functoriality over $\mathcal{C}$.

This way we are reduced to showing that actually $\mathcal{D} \simeq A\,\mathrm{Free}_{\mathcal{C}}\mathrm{Mod}^{\mathrm{op}}$.

But by construction, the image of the objects of $\mathcal{D}$ under the Yoneda embedding are precisely the free A-modules over objects of $\mathcal{C}$:

$$\mathcal{D}(c, -) \simeq A\,\mathrm{Free}_{\mathcal{C}}\mathrm{Mod}(-, c) \simeq (A \otimes_{\mathrm{Day}} y(c))(-)\ .$$

Since the Yoneda embedding is fully faithful, this shows that indeed

$$\mathcal{D}^{\mathrm{op}} \simeq A\,\mathrm{Free}_{\mathcal{C}}\mathrm{Mod} \hookrightarrow A\,\mathrm{Mod}\ .$$

▮

Example 2.20. For the sequential case $\mathrm{Dia} = \mathrm{Seq}$ in def. 2.4, then the opposite category of free modules on objects in Seq over $\mathbb{S}_{\mathrm{seq}}$ (def.) is identified as the category $\mathrm{StdSpheres}$ (def.):

$$\mathbb{S}_{\mathrm{seq}}\,\mathrm{Free}_{\mathrm{seq}}\mathrm{Mod}^{\mathrm{op}} \;\simeq\; \mathrm{StdSpheres}$$

Accordingly, in this case prop. 2.19 reduces to the identification (prop.) of sequential spectra as topological diagrams over $\mathrm{StdSpheres}$:

$$[\mathbb{S}_{\mathrm{seq}}\,\mathrm{Free}_{\mathrm{seq}}\mathrm{Mod}^{\mathrm{op}}, \mathrm{Top}_{\mathrm{cg}}^{*/}] \simeq [\mathrm{StdSpheres}, \mathrm{Top}_{\mathrm{cg}}^{*/}] \simeq \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})\ .$$

Proof. There is one object $y(n)$ for each $n \in \mathbb{N}$. Moreover, from the expression in the proof of prop. 2.19 we compute the hom-spaces between these to be

$$
\begin{aligned}
\mathbb{S}_{\mathrm{seq}}\,\mathrm{Free}_{\mathrm{seq}}\mathrm{Mod}(\mathbb{S}_{\mathrm{seq}} \otimes_{\mathrm{Day}} y_{k_2}, \mathbb{S}_{\mathrm{seq}} \otimes_{\mathrm{Day}} y_{k_1}) & \simeq \int^n \mathrm{Seq}(n + k_1, k_2) \wedge \mathbb{S}_{\mathrm{seq}}(n) \\
& \simeq \begin{cases} S^{k_2 - k_1} & \text{if } k_2 \geq k_1 \\ * & \text{otherwise} \end{cases}
\end{aligned}
.
$$

These are the objects and hom-spaces of the category $\mathrm{StdSpheres}$. It is straightforward to check that the definition of composition agrees, too. ▮

Stable weak homotopy equivalences

We consider the evident version of stable weak homotopy equivalences for structured spectra and prove a few technical lemmas about them that are needed in the proof of the stable model structure below

Definition 2.21. For $\mathrm{Dia} \in \{\mathrm{Top}_{\mathrm{cg,fin}}^{*/}, \mathrm{Orth}, \mathrm{Sym}, \mathrm{Seq}\}$ one of the shapes of structured spectra from def. 2.4, let

$\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}$ be the corresponding category of structured spectra (def. 2.1, prop. 2.8, def. 2.9).

1. The stable homotopy groups of an object $X \in \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}$ are those of the underlying sequential spectrum (def.):

$$\pi_\bullet(X) \coloneqq \pi_\bullet(\mathrm{seq}^* X)\ .$$

2. An object $X \in \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}$ is a **structured Omega-spectrum** if the underlying sequential spectrum $\mathrm{seq}^* X$ (def. 2.4) is a sequential Omega spectrum (def.)

3. A morphism f in $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}$ is a **stable weak homotopy equivalence** (or: **$\pi_\bullet$-isomorphism**) if the underlying morphism of sequential spectra $\mathrm{seq}^*(f)$ is a stable weak homotopy equivalence of sequential spectra (def.);

4. a morphism f is a **stable cofibration** if it is a cofibration in the strict model structure $\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}}$ from prop. 3.1.

(MMSS 00, def. 8.3 with the notation from p. 21, Mandell-May 02, III, def. 3.1, def. 3.2)

Lemma 2.22. *Given a morphism $f : X \longrightarrow Y$ in $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}$, then there are long exact sequences of stable homotopy groups (def. 2.21) of the form*

$$\cdots \longrightarrow \pi_{\bullet+1}(Y) \longrightarrow \pi_\bullet(\mathrm{Path}_*(f)) \longrightarrow \pi_\bullet(X) \overset{f_*}{\longrightarrow} \pi_\bullet(Y) \longrightarrow \pi_{\bullet-1}(\mathrm{Path}_*(f)) \longrightarrow \cdots$$

and

$$\cdots \longrightarrow \pi_{\bullet+1}(Y) \longrightarrow \pi_{\bullet+1}(\mathrm{Cone}(f)) \longrightarrow \pi_\bullet(X) \overset{f_*}{\longrightarrow} \pi_\bullet(Y) \longrightarrow \pi_\bullet(\mathrm{Cone}(f)) \longrightarrow \cdots,$$

where $\mathrm{Cone}(f)$ denotes the mapping cone and $\mathrm{Path}_*(f)$ the mapping cocone of f (def.) formed with respect to the standard cylinder spectrum $X \wedge (I_+)$ hence formed degreewise with respect to the standard reduced cylinder of pointed topological spaces.

(MMSS 00, theorem 7.4 (vi))

Proof. Since limits and colimits in the diagram category $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}$ are computed objectwise, the functor seq^* that restricts $\mathbb{S}_{\mathrm{dia}}$-modules to their underlying sequential spectra preserves both limits and colimits, hence it is sufficient to consider the statement for sequential spectra.

For the first case, there is degreewise the long exact sequence of homotopy groups to the left of pointed topological spaces (exmpl.)

$$\cdots \to \pi_2(Y) \longrightarrow \pi_1(\mathrm{Path}_*(f)) \longrightarrow \pi_1(X) \overset{f_*}{\longrightarrow} \pi_1(Y) \longrightarrow \pi_0(\mathrm{Path}_*(f)) \longrightarrow \pi_0(X_n) \overset{f_*}{\longrightarrow} \pi_0(Y_n)\ .$$

Observe that the sequential colimit that defines the stable homotopy groups (def.) preserves exact sequences of abelian groups, because generally filtered colimits in Ab are exact functors (prop.). This implies that by taking the colimit over n in the above sequences, we obtain a long exact sequence of stable homotopy groups as shown.

Now use that in sequential spectra the canonical morphism morphism $\mathrm{Path}_*(f) \longrightarrow \Omega\,\mathrm{Cone}(f)$ is a stable weak homotopy equivalence and is compatible with the map f (prop.) so that there is a commuting diagram of the form

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & \pi_{\bullet+1}(Y) & \longrightarrow & \pi_\bullet(\mathrm{Path}_*(f)) & \longrightarrow & \pi_\bullet(X) & \overset{f_*}{\longrightarrow} & \pi_\bullet(Y) & \longrightarrow & \pi_{\bullet-1}(\mathrm{Path}_*(f)) & \longrightarrow & \cdots \\ && \downarrow^{=} && \downarrow^{\simeq} && \downarrow^{=} && \downarrow^{=} && \downarrow^{\simeq} \\ \cdots & \longrightarrow & \pi_{\bullet+1}(Y) & \longrightarrow & \pi_{\bullet+1}(\mathrm{Cone}(f)) & \longrightarrow & \pi_\bullet(X) & \overset{f_*}{\longrightarrow} & \pi_\bullet(Y) & \longrightarrow & \pi_\bullet(\mathrm{Cone}(f)) & \longrightarrow & \cdots \end{array}\ .$$

Since the top sequence is exact, and since all vertical morphisms are isomorphisms, it follows that also the bottom sequence is exact. ▮

Lemma 2.23. *For $K \in \mathrm{Top}^{*/}_{\mathrm{cg,fin}}$ a CW-complex then the operation of smash tensoring $(-)\wedge K$ preserves stable weak homotopy equivalences in $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}$.*

Proof. Since limits and colimits in the diagram category $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}$ are computed objectwise, the functor seq^* that restricts $\mathbb{S}_{\mathrm{dia}}$-modules to their underlying sequential spectra preserves both limits and colimits, and it also preserves smash tensoring. Hence it is sufficient to consider the statement for sequential spectra.

Fist consider the case of a finite cell complex K.

Write

$$* = K_0 \hookrightarrow \cdots \hookrightarrow K_i \hookrightarrow K_{i+1} \hookrightarrow \cdots \hookrightarrow K$$

for the stages of the cell complex K, so that for each i there is a pushout diagram in $\mathrm{Top}_{\mathrm{cg}}$ of the form

$$\begin{array}{ccccc} S^{n_i - 1} & \longrightarrow & K_i & \longrightarrow & * \\ \downarrow & (\mathrm{po}) & \downarrow & (\mathrm{po}) & \downarrow \\ D^{n_i - 1} & \longrightarrow & K_{i+1} & \longrightarrow & S^{n_i} \end{array} .$$

Equivalently these are pushoutdiagrams in $\mathrm{Top}_{\mathrm{cg}}^{*/}$ of the form

$$\begin{array}{ccccc} S_+^{n_i - 1} & \longrightarrow & K_i & \longrightarrow & * \\ \downarrow & (\mathrm{po}) & \downarrow & (\mathrm{po}) & \downarrow \\ D_+^{n_i - 1} & \longrightarrow & K_{i+1} & \longrightarrow & S^{n_i} \end{array} .$$

Notice that it is indeed S^{n_i} that appears in the top right, not $S_+^{n_i}$.

Now forming the smash tensoring of any morphism $f : X \longrightarrow Y$ in $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(\mathrm{Top}_{\mathrm{cg}})$ by the morphisms in the pushout on the right yields a commuting diagram in $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}$ of the form

$$\begin{array}{ccccc} X \wedge K_i & \longrightarrow & X \wedge K_{i+1} & \longrightarrow & X \wedge S^{n_i} \\ \downarrow && \downarrow && \downarrow \\ Y \wedge K_i & \longrightarrow & Y \wedge K_{i+1} & \longrightarrow & Y \wedge S^{n_i} \end{array} .$$

Here the horizontal morphisms on the left are degreewise cofibrations in $\mathrm{Top}_{\mathrm{cg}}^{*/}$, hence the morphism on the right is degreewise their homotopy cofiber. This way lemma 2.22 implies that there are commuting diagrams

$$\begin{array}{ccccccccc} \pi_{\bullet+1}(X \wedge S^{n_i}) & \longrightarrow & \pi_\bullet(X \wedge K_i) & \longrightarrow & \pi_\bullet(X \wedge K_{i+1}) & \longrightarrow & \pi_\bullet(X \wedge S^{n_i}) & \longrightarrow & \pi_{\bullet-1}(X \wedge K_i) \\ \downarrow && \downarrow && \downarrow^{f \wedge K_{i+1}} && \downarrow && \downarrow \\ \pi_{\bullet+1}(Y \wedge S^{n_i}) & \longrightarrow & \pi_\bullet(Y \wedge K_i) & \longrightarrow & \pi_\bullet(Y \wedge K_{i+1}) & \longrightarrow & \pi_\bullet(Y \wedge S^{n_i}) & \longrightarrow & \pi_{\bullet-1}(X \wedge K_i) \end{array} ,$$

where the top and bottom are long exact sequences of stable homotopy groups.

Now proceed by induction. For $i = 0$ then clearly smash tensoring with $K_0 = *$ preserves stable weak homotopy equivalences. So assume that smash tensoring with K_i does, too. Observe that $(-) \wedge S^n$ preserves stable weak homotopy equivalences, since $\Sigma X[1] \to X$ is a stable weak homotopy equivalence (lemma). Hence in the above the two vertical morphisms on the left and the two on the right are isomorphism. Now the five lemma implies that also $f \wedge K_{i+1}$ is an isomorphism.

Finally, the statement for a non-finite cell complex follows with these arguments and then using that spheres are compact and hence maps out of them into a transfinite composition factor through some finite stage (prop.). ▮

Lemma 2.24. *The pushout in $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}$ of a stable weak homotopy equivalence along a morphism that is degreewise a cofibration in $(\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}$ is again a stable weak homotopy equivalence.*

Proof. Given a pushout square

$$\begin{array}{ccc} X & \overset{g}{\longrightarrow} & Z \\ {}^{f}\downarrow & (\mathrm{po}) & \downarrow \\ Y & \longrightarrow & Y \underset{X}{\sqcup} Z \end{array}$$

observe that the pasting law implies an isomorphism between the horizontal cofibers

$$\begin{array}{ccccc} X & \overset{g}{\longrightarrow} & Z & \longrightarrow & \mathrm{cofib}(g) \\ {}^{f}\downarrow & (\mathrm{po}) & \downarrow && \downarrow^{\simeq} \\ Y & \longrightarrow & Y \underset{X}{\sqcup} Z & \longrightarrow & \mathrm{cofib}(g) \end{array} .$$

Moreover, since cofibrations in $(\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}$ are preserves by pushout, and since pushout of spectra are computed degreewise, both the top and the bottom horizontal sequences here are degreewise homotopy cofiber sequence in $(\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}$. Hence lemma 2.22 applies and gives a commuting diagram

$$\begin{array}{ccccccccc} \pi_{\bullet+1}(\mathrm{cofib}(g)) & \longrightarrow & \pi_\bullet(X) & \longrightarrow & \pi_\bullet(Z) & \longrightarrow & \pi_\bullet(\mathrm{cofib}(g)) & \longrightarrow & \pi_{\bullet-1}(X) \\ \downarrow^{\simeq} && {}^{\pi_\bullet(f)}\downarrow_{\simeq} && \downarrow && \downarrow^{\simeq} && \downarrow^{\simeq} \\ \pi_{\bullet+1}(\mathrm{cofib}(g)) & \longrightarrow & \pi_\bullet(Y) & \longrightarrow & \pi_\bullet(Y \underset{X}{\sqcup} Z) & \longrightarrow & \pi_\bullet(\mathrm{cofib}(g)) & \longrightarrow & \pi_{\bullet-1}(Y) \end{array} ,$$

where the top and the bottom row are both long exact sequences of stable homotopy groups. Hence the claim now follows by the five lemma. ▮

Free spectra and Suspension spectra

The concept of *free spectrum* is a generalization of that of *suspension spectrum*. In fact the stable homotopy types of free spectra are precisely those of iterated loop space objects of suspension spectra. But for the development of the theory what matters is free spectra before passing to stable homotopy types, for as such they play the role of the basic cells for the stable model structures on spectra analogous to the role of the n-spheres in the classical model structure on topological spaces (def. 3.2 below).

Moreover, while free sequential spectra are just re-indexed suspension spectra, free symmetric spectra and free orthogonal spectra in addition come with suitably freely generated actions of the symmetric group and the orthogonal group. It turns out that this is not entirely trivial; it leads to a subtle issue (lemma 2.33 below) where the adjuncts of certain canonical inclusions of free spectra are stable weak homotopy equivalences for sequential and orthogonal spectra, but *not* for symmetric spectra.

Definition 2.25. For $\mathrm{Dia} \in \{\mathrm{Top}^{*/}_{\mathrm{fin}}, \mathrm{Orth}, \mathrm{Sym}, \mathrm{Seq}\}$ any one of the four diagram shapes of def. 2.4, and for each $n \in \mathbb{N}$, the functor

$$(-)_n \;:\; \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod} \overset{\mathrm{seq}^*}{\longrightarrow} \mathbb{S}_{\mathrm{seq}}\,\mathrm{Mod} \simeq \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}}) \overset{(-)_n}{\longrightarrow} \mathrm{Top}^{*/}_{\mathrm{cg}}$$

that sends a structured spectrum to the nth component space of its underlying sequential spectrum has, by prop. 1.38, a left adjoint

$$F^{\mathrm{dia}}_n \;:\; \mathrm{Top}^{*/} \longrightarrow \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}\;.$$

This is called the **free structured spectrum**-functor.

For the special case $n = 0$ it is also called the **structured suspension spectrum** functor and denoted

$$\Sigma^\infty_{\mathrm{dia}} K \;\coloneqq\; F^{\mathrm{dia}}_0 K$$

(Hovey-Shipley-Smith 00, def. 2.2.5, MMSS 00, section 8)

Lemma 2.26. *Let $\mathrm{Dia} \in \{\mathrm{Top}^{*/}_{\mathrm{fin}}, \mathrm{Orth}, \mathrm{Sym}, \mathrm{Seq}\}$ be any one of the four diagram shapes of def. 2.4. Then*

1. *the free spectrum on $K \in \mathrm{Top}^{*/}_{\mathrm{cg}}$ (def. 2.25) is equivalently the smash tensoring with K (def.) of the free module (def. 1.20) over $\mathbb{S}_{\mathrm{dia}}$ (remark 2.6) on the representable $y(n) \in [\mathrm{Dia}, \mathrm{Top}^{*/}_{\mathrm{cg}}]$*

$$\begin{aligned} F^{\mathrm{dia}}_n K &\simeq (\mathbb{S}_{\mathrm{dia}} \otimes_{\mathrm{Day}} y(n)) \wedge K \\ &\simeq \mathbb{S}_{\mathrm{dia}} \otimes_{\mathrm{Day}} (y(n) \wedge K) \end{aligned};$$

2. *on $n' \in \mathrm{Dia}^{\mathrm{op}} \overset{y}{\hookrightarrow} [\mathrm{Dia}, \mathrm{Top}^{*/}_{\mathrm{cg}}]$ its value is given by the following coend expression (def. 1.28)*

$$(F^{\mathrm{dia}}_n K)(n') \;\simeq\; \int^{n_1 \in \mathrm{Dia}} \mathrm{Dia}(n_1 \otimes n, n') \wedge S^{n_1} \wedge K\;.$$

In particular the structured sphere spectrum is the free spectrum in degree 0 on the 0-sphere:

$$\mathbb{S}_{\mathrm{dia}} \simeq F^{\mathrm{dia}}_0 S^0$$

and generally for $K \in \mathrm{Top}^{*/}_{\mathrm{cg}}$ then

$$F^{\mathrm{dia}}_0 K \simeq \mathbb{S}_{\mathrm{dia}} \wedge K$$

is the smash tensoring of the strutured sphere spectrum with K.

(Hovey-Shipley-Smith 00, below def. 2.2.5, MMSS00, p. 7 with theorem 2.2)

Proof. Under the equivalence of categories

$$\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod} \simeq [\mathbb{S}_{\mathrm{dia}}\,\mathrm{Free}_{\mathrm{dia}}\mathrm{Mod}^{\mathrm{op}}, \mathrm{Top}^{*/}_{\mathrm{cg}}]$$

from prop. 2.19, the expression for $F^{\mathrm{dia}}_n K$ is equivalently the smash tensoring with K of the functor that n represents over $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Free}_{\mathrm{dia}}\mathrm{Mod}$:

$$\begin{aligned} F^{\mathrm{dia}}_n K &\simeq y_{\mathbb{S}_{\mathrm{dia}}\,\mathrm{Free}_{\mathrm{Dia}}\mathrm{Mod}}(n) \wedge K \\ &\simeq \mathbb{S}_{\mathrm{dia}}\,\mathrm{Free}_{\mathrm{dia}}\mathrm{Mod}(-, \mathbb{S}_{\mathrm{dia}} \wedge y_{\mathrm{Dia}}(n)) \wedge K \end{aligned}$$

(by fully faithfulness of the Yoneda embedding).

This way the first statement is a special case of the following general fact: For $\mathcal{C}$ a pointed topologically enriched category, and for $c \in \mathcal{C}$ any object, then there is an adjunction

$$[\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}] \underoverset{\underset{(-)_c}{\longrightarrow}}{\overset{y(c)\wedge(-)}{\longleftarrow}}{\bot} \mathrm{Top}_{\mathrm{cg}}^{*/}$$

(saying that evaluation at c is right adjoint to smash tensoring the functor represented by c) witnessed by the following composite natural isomorphism:

$$[\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}](y(c) \wedge K, F) \;\simeq\; \mathrm{Maps}(K, [\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}](y(c), F))_* \;\simeq\; \mathrm{Maps}(K, F(c))_* \;=\; \mathrm{Top}_{\mathrm{cg}}^{*/}(K, F(c)) \,.$$

The first is the characteristic isomorphism of tensoring from prop. 1.37, while the second is the enriched Yoneda lemma of prop. 1.32.

From this, the second statement follows by the proof of prop. 2.19.

For the last statement it is sufficient to observe that $y(0)$ is the tensor unit under Day convolution by prop. 1.44 (since 0 is the tensor unit in Dia), so that

$$\begin{aligned} F_0^{\mathrm{dia}} S^0 & = \mathbb{S}_{\mathrm{dia}} \otimes_{\mathrm{Day}} (y(0) \wedge S^0) \\ & \simeq \mathbb{S}_{\mathrm{dia}} \otimes y(S^0) \\ & \simeq \mathbb{S}_{\mathrm{dia}} \end{aligned} \,.$$

▮

Proposition 2.27. *Explicitly, the free spectra according to def. 2.25, look as follows:*

For sequential spectra:

$$(F_n^{\mathrm{Seq}} K)_q \simeq \left\{ \begin{matrix} S^{q-n} \wedge K & \text{if}\; q \geq n \\ \ast & \text{otherwise} \end{matrix} \right\}$$

for symmetric spectra:

$$(F_n^{\mathrm{Sym}} K)_q \simeq \left\{ \begin{matrix} \Sigma(q)_+ \wedge_{\Sigma(q-n)} S^{q-n} \wedge K & \text{if}\; q \geq n \\ \ast & \text{otherwise} \end{matrix} \right.$$

for orthogonal spectra:

$$(F_n^{\mathrm{Orth}} K)_q \simeq \left\{ \begin{matrix} O(q)_+ \wedge_{O(q-n)} \wedge S^{q-n} \wedge K & \text{if}\; q \geq n \\ \ast & \text{otherwise} \end{matrix} \right. ,$$

where "$\wedge_G$" is as in example 1.29.

(e.g. Schwede 12, example 3.20)

Proof. With the formula in item 2 of lemma 2.26 we have for the case of orthogonal spectra

$$\begin{aligned} (F_n^{\mathrm{Orth}} K)(\mathbb{R}^q) & \simeq \int^{n_1 \in \mathrm{Orth}} \underset{= \left\{ \begin{matrix} O(q)_+ & \text{if}\, n_1 + n = q \\ \ast & \text{otherwise} \end{matrix} \right.}{\underbrace{\mathrm{Orth}(n_1 + n, q)}} \wedge S^{n_1} \wedge K \\ & \simeq \left\{ \begin{matrix} \int^{n_1 = \ast \in \mathbf{B}(O(q-n))} O(q)_+ \underset{O(q-n)}{\wedge} S^{q-n} \wedge K & \text{if}\; q \geq n \\ \ast & \text{otherwise} \end{matrix} \right. \end{aligned} ,$$

where in the second line we used that the coend collapses to $n_1 = q - n$ ranging in the full subcategory

$$\mathbf{B}(O(q-n)_+) \hookrightarrow \mathrm{Orth}$$

on the object $\mathbb{R}^{q-n}$ and then we applied example 1.29. The case of symmetric spectra is verbatim the same, with the symmetric group replacing the orthogonal group, and the case of sequential spectra is again verbatim the same, with the orthogonal group replaced by the trivial group. ▮

Lemma 2.28. *For $\mathrm{Dia} \in \{\mathrm{Orth}, \mathrm{Sym}, \mathrm{Seq}\}$ the diagram shape for orthogonal spectra, symmetric spectra or sequential spectra, then the free structured spectra*

$$F_n^{\mathrm{dia}} S^0 \in \mathbb{S}_{\mathrm{dia}}\, \mathrm{Mod}$$

from def. 2.25 have component spaces that admit the structure of CW-complexes.

Proof. We consider the case of orthogonal spectra. The case of symmetric spectra works verbatim the same, and the case of sequential spectra is tivial.

By prop. 2.27 we have to show that for all $q \geq n \in \mathbb{N}$ the topological spaces of the form

$$O(q)_+ \wedge_{O(q-n)} S^{q-n} \quad \in \mathrm{Top}_{\mathrm{cg}}^{*/}$$

admit the structure of CW-complexes.

To that end, use the homeomorphism

$$S^{q-n} \simeq D^{q-n} / \partial D^{q-n}$$

which is a diffeomorphism away from the basepoint and hence such that the action of the orthogonal group $O(q-n)$ induces a smooth action on D^{q-n} (which happens to be constant on ∂D^{q-n}).

Also observe that we may think of the above quotient by the group action

$$(x, gy) \simeq (xg, y)$$

as being the quotient by the diagonal action

$$O(q-n) \times (O(q)_+ \wedge S^{q-n}) \longrightarrow (O(q)_+ \wedge S^{q-n})$$

given by

$$(g, (x,y)) \mapsto (xg^{-1}, gy) \; .$$

Using this we may rewrite the space in question as

$$\begin{aligned} (O(q)_+ \wedge_{O(q-n)} S^{q-n}) & \simeq (O(q)_+ \wedge S^{q-n}) / O(q-n) \\ & \simeq \frac{O(q) \times D^{q-n}}{O(q) \times \partial D^{q-n}} / O(q-n) \\ & \simeq \frac{O(q) \times D^{q-n}}{\partial (O(q) \times D^{q-n})} / O(q-n) \\ & \simeq \frac{(O(q) \times D^{q-n})/O(q-n)}{(\partial(O(q) \times D^{q-n}))/O(q-n)} \end{aligned} \; .$$

Here $O(q) \times D^{q-n}$ has the structure of a smooth manifold with boundary and equipped with a smooth action of the compact Lie group $O(q-n)$. Under these conditions (Illman 83, corollary 7.2) states that $O(q) \times D^{q-n}$ admits the structure of a G-CW complex for $G = O(q-n)$, and moreover (Illman 83, line above theorem 7.1) states that this may be chosen such that the boundary is a G-CW subcomplex.

Now the quotient of a G-CW complex by G is a CW complex, and so the last expression above exhibits the quotient of a CW-complex by a subcomplex, hence exhibits CW-complex structure. ▮

Proposition 2.29. *Let $\mathrm{Dia} \in \{\mathrm{Top}_{\mathrm{cg,fin}}^{*/}, \mathrm{Orth}, \mathrm{Sym}\}$ be the diagram shape of either pre-excisive functors, orthogonal spectra or symmetric spectra. Then under the symmetric monoidal smash product of spectra (def. 2.1, def. 2.1, def.2.9) the free structured spectra of def. 2.25 behave as follows*

$$F_{n_1}^{\mathrm{dia}}(K_1) \otimes_{\mathbb{S}_{\mathrm{dia}}} F_{n_2}^{\mathrm{dia}}(K_2) \;\simeq\; F_{n_1+n_2}(K_1 \wedge K_2) \; .$$

In particular for structured suspension spectra $\Sigma_{\mathrm{dia}}^\infty \coloneqq F_0^{\mathrm{dia}}$ (def. 2.25) this gives isomorphisms

$$\Sigma_{\mathrm{dia}}^\infty(K_1) \otimes_{\mathbb{S}_{\mathrm{dia}}} \Sigma_{\mathrm{dia}}^\infty(K_2) \;\simeq\; \Sigma_{\mathrm{dia}}^\infty(K_1 \wedge K_2) \; .$$

Hence the structured suspension spectrum functor $\Sigma_{\mathrm{dia}}^\infty$ is a strong monoidal functor (def. 1.47) and in fact a braided monoidal functor (def. \ref{braided monoidal functor}) from pointed topological spaces equipped with the smash product of pointed objects, to structured spectra equipped with the symmetric monoidal smash product of spectra

$$\Sigma_{\mathrm{dia}}^\infty \;:\; (\mathrm{Top}_{\mathrm{cg}}^{*/}, \wedge, S^0) \longrightarrow (\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}, \otimes_{\mathbb{S}_{\mathrm{dia}}}, \mathbb{S}_{\mathrm{dia}}) \; .$$

More generally, for $X \in \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}$ then

$$X \otimes_{\mathbb{S}_{\mathrm{dia}}} (\Sigma_{\mathrm{dia}}^\infty K) \simeq X \wedge K \, ,$$

where on the right we have the smash tensoring of X with $K \in \mathrm{Top}_{\mathrm{cg}}^{*/}$.

(MMSS 00, lemma 1.8 with theorem 2.2, Mandell-May 02, prop. 2.2.6)

Proof. By lemma 2.26 the free spectra are free modules over the structured sphere spectrum $\mathbb{S}_{\mathrm{dia}}$ of the form $F_n^{\mathrm{dia}}(K) \simeq \mathbb{S}_{\mathrm{dia}} \otimes_{\mathrm{Day}} (y(n) \wedge K)$. By example 1.23 the tensor product of such free modules is given by

$$\left(\mathbb{S}_{\mathrm{dia}} \otimes_{\mathrm{Day}} (y(n_1) \wedge K_1)\right) \otimes_{\mathrm{Day}} \left(\mathbb{S}_{\mathrm{dia}} \otimes_{\mathrm{Day}} (y(n_2) \wedge K_2)\right) \;\simeq\; \mathbb{S}_{\mathrm{dia}} \otimes_{\mathrm{Day}} (y(n_1) \wedge K) \otimes_{\mathrm{Day}} (y(n_2) \wedge K)\,.$$

Using the co-Yoneda lemma (prop. 1.33) the expression on the right is

$$\begin{aligned} \left((y(n_1) \wedge K_1) \otimes_{\mathrm{Day}} (y(n_2) \wedge K_2)\right)(c) &= \int^{c_{1,2}} \mathrm{Dia}(c_1 + c_2, c) \wedge y(n_1)(c_1) \wedge K_1 \wedge y(n_2)(c_2) \wedge K_2 \\ &\simeq \int^{c_1, c_2} \mathrm{Dia}(c_1 + c_2, c) \wedge \mathrm{Dia}(n_1, c_1) \wedge \mathrm{Dia}(n_2, c_2) \wedge K_1 \wedge K_2\,. \\ &\simeq \mathrm{Dia}(n_1 + n_2, c) \wedge K_1 \wedge K_2 \\ &\simeq (y(n_1 + n_2) \wedge (K_1 \wedge K_2))(c) \end{aligned}$$

For the last statement we may use that $\Sigma^\infty_{\mathrm{dia}} K \simeq \mathbb{S}_{\mathrm{dia}} \wedge K$, by lemma 2.26), and that $\mathbb{S}_{\mathrm{dia}}$ is the tensor unit for $\otimes_{\mathbb{S}_{\mathrm{dia}}}$ by prop. 1.22.

To see that $\Sigma^\infty_{\mathrm{dia}}$ is braided, write $\Sigma^\infty_{\mathrm{dia}} K \simeq \mathbb{S} \wedge K$. We need to see that

$$\begin{matrix} (\mathbb{S} \wedge K_1) \otimes_{\mathbb{S}} (\mathbb{S} \wedge K_2) & \longrightarrow & (\mathbb{S} \wedge K_2) \otimes_{\mathbb{S}} (\mathbb{S} \wedge K_1) \\ \downarrow && \downarrow \\ \mathbb{S} \wedge (K_1 \wedge K_2) & \longrightarrow & \mathbb{S} \wedge (K_2 \wedge K_1) \end{matrix}$$

commutes. Chasing the smash factors through this diagram and using symmetry (def. 1.5) and the hexagon identities (def. 1.4) shows that indeed it does. ▮

One use of free spectra is that they serve to co-represent adjuncts of structure morphisms of spectra. To this end, first consider the following general existence statement.

Lemma 2.30. *For each $n \in \mathbb{N}$ there exists a morphism*

$$\lambda_n \;:\; F^{\mathrm{dia}}_{n+1} S^1 \longrightarrow F^{\mathrm{dia}}_n S^0$$

between free spectra (def. 2.25) such that for every structured spectrum $X \in \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}$ precomposition $\lambda_n^\ast$ forms a commuting diagram of the form

$$\begin{matrix} \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(F^{\mathrm{dia}}_n S^0, X) & \simeq & \mathrm{Top}^{*/}(S^0, X_n) & \simeq & X_n \\ \downarrow^{\lambda_n^\ast} &&&& \downarrow^{\tilde \sigma_n^X} \\ \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(F^{\mathrm{dia}}_{n+1} S^1, X) & \simeq & \mathrm{Top}^{*/}(S^1, X_{n+1}) & \simeq & \Omega X_{n+1} \end{matrix}\,,$$

where the horizontal equivalences are the adjunction isomorphisms and the canonical identification, and where the right morphism is the $(\Sigma \dashv \Omega)$-adjunct of the structure map σ_n of the sequential spectrum $\mathrm{seq}^\ast X$ underlying X (def. 2.4).

Proof. Since all prescribed morphisms in the diagram are natural transformations, this is in fact a diagram of copresheaves on $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}$

$$\begin{matrix} \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(F^{\mathrm{dia}}_n S^0, -) & \simeq & \mathrm{Top}^{*/}(S^0, (-)_n) & \simeq & (-)_n \\ \downarrow &&&& \downarrow^{\tilde \sigma_n^{(-)}} \\ \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(F^{\mathrm{dia}}_{n+1} S^1, -) & \simeq & \mathrm{Top}^{*/}(S^1, (-)_{n+1}) & \simeq & \Omega (-)_{n+1} \end{matrix}\,.$$

With this the statement follows by the Yoneda lemma. ▮

Now we say explicitly what these maps are:

Definition 2.31. For $n \in \mathbb{N}$, write

$$\lambda_n \;:\; F_{n+1} S^1 \longrightarrow F_n S^0$$

for the adjunct under the (free structured spectrum $\dashv$ n-component)-adjunction in def. 2.25 of the composite morphism

$$S^1 \xrightarrow{=} (F^{\mathrm{Seq}}_n(S^0))_{n+1} \overset{(f_n^{\mathrm{Seq}})_{n+1}}{\hookrightarrow} (F_n^{\mathrm{dia}} S^0)_{n+1}\,,$$

where the first morphism is via prop. 2.27 and the second comes from the adjunction units according to def. 2.25.

(MMSS 00, def. 8.4, Schwede 12, example 4.26)

Lemma 2.32. *The morphisms of def. 2.31 are those whose existence is asserted by prop. 2.30.*

(MMSS 00, lemma 8.5, following Hovey-Shipley-Smith 00, remark 2.2.12)

Proof. Consider the case $\mathrm{Dia} = \mathrm{Seq}$ and $n = 0$. All other cases work analogously.

By lemma 2.27, in this case the morphism λ_0 has components like so:

$$\begin{array}{ccc} \vdots && \vdots \\ S^3 & \overset{\mathrm{id}}{\longrightarrow} & S^3 \\ S^2 & \overset{\mathrm{id}}{\longrightarrow} & S^2 \\ S^1 & \overset{\mathrm{id}}{\longrightarrow} & S^1 \\ \ast & \overset{0}{\longrightarrow} & S^0 \\ \\ \underbrace{} && \underbrace{} \\ F_1 S^1 & \overset{\lambda_0}{\longrightarrow} & F_0 S^0 \end{array} \,.$$

Now for X any sequential spectrum, then a morphism $f \colon F_0 S^0 \to X$ is uniquely determined by its 0th components $f_0 \colon S^0 \to X_0$ (that's of course the very free property of $F_0 S^0$); as the compatibility with the structure maps forces the first component, in particular, to be $\sigma_0^X \circ \Sigma f$:

$$\begin{array}{ccc} \Sigma S^0 & \overset{\Sigma f}{\longrightarrow} & \Sigma X_0 \\ \downarrow^{\simeq} && \downarrow^{\sigma_0^X} \\ S^1 & \underset{}{\overset{\sigma_0^X \circ \Sigma f}{\longrightarrow}} & X_1 \end{array}$$

But that first component is just the component that similarly determines the precompositon of f with λ_0, hence $\lambda_0^\ast f$ is fully fixed as being the map $\sigma_0^X \circ \Sigma f$. Therefore $\lambda_0^\ast$ is the function

$$\lambda_0^\ast \;\colon\; X_0 = \mathrm{Maps}(S^0, X_0) \overset{f \mapsto \sigma_0^X \circ \Sigma f}{\longrightarrow} \mathrm{Maps}(S^1, X_1) = \Omega X_1 \,.$$

It remains to see that this is the $(\Sigma \dashv \Omega)$-adjunct of σ_0^X. By the general formula for adjuncts, this is

$$\tilde \sigma_0^X \;\colon\; X_0 \overset{\eta}{\longrightarrow} \Omega \Sigma X_0 \overset{\Omega \sigma_0^X}{\longrightarrow} \Omega X_1 \,.$$

To compare to the above, we check what this does on points: $S^0 \overset{f_0}{\longrightarrow} X_0$ is sent to the composite

$$S^0 \overset{f_0}{\longrightarrow} X_0 \overset{\eta}{\longrightarrow} \Omega \Sigma X_0 \overset{\Omega \sigma_0^X}{\longrightarrow} \Omega X_1 \,.$$

To identify this as a map $S^1 \to X_1$ we use the adjunction isomorphism once more to throw all the Ω-s on the right back to Σ-s the left, to finally find that this is indeed

$$\sigma_0^X \circ \Sigma f \;\colon\; S^1 = \Sigma S^0 \overset{\Sigma f}{\longrightarrow} \Sigma X_0 \overset{\sigma_0^X}{\longrightarrow} X_1 \,.$$

▮

Lemma 2.33. *The maps $\lambda_n \;\colon\; F_{n+1} S^1 \longrightarrow F_n S^0$ in def. 2.31 are*

1. *stable weak homotopy equivalences for sequential spectra, orthogonal spectra and pre-excisive functors, i.e. for $\mathrm{Dia} \in \{\mathrm{Top}^{\ast/}, \mathrm{Orth}, \mathrm{Seq}\}$;*
2. ***not** stable weak homotopy equivalences for the case of symmetric spectra $\mathrm{Dia} = \mathrm{Sym}$.*

(Hovey-Shipley-Smith 00, example 3.1.10, MMSS 00, lemma 8.6, Schwede 12, example 4.26)

Proof. This follows by inspection of the explicit form of the maps, via prop. 2.27. We discuss each case separately:

sequential case

Here the components of the morphism eventually stabilize to isomorphisms

$$
\begin{array}{cccc}
 & \vdots & & \vdots \\
(\lambda_n)_{n+3} & S^3 & \overset{\mathrm{id}}{\longrightarrow} & S^3 \\
(\lambda_n)_{n+2} & S^2 & \overset{\mathrm{id}}{\longrightarrow} & S^2 \\
(\lambda_n)_{n+1} & S^1 & \overset{\mathrm{id}}{\longrightarrow} & S^1 \\
(\lambda_n)_n : & * & \overset{0}{\longrightarrow} & S^0 \\
 & * & \longrightarrow & * \\
 & \vdots & & \vdots \\
 & * & \longrightarrow & * \\
 & \underbrace{} & & \underbrace{} \\
\lambda_n : & F_{n+1}S^1 & \longrightarrow & F_n S^0
\end{array}
$$

and this immediately gives that λ_n is an isomorphism on stable homotopy groups.

orthogonal case

Here for $q \geq n+1$ the q-component of λ_n is the quotient map

$$
(\lambda_n)_q \;:\; O(q)_+ \wedge_{O(q-n-1)} S^{q-n} \simeq O(q)_+ \wedge_{O(q-n-1)} S^1 \wedge S^{q-n-1} \longrightarrow O(q)_+ \wedge_{O(q-n)} S^{q-n} \,.
$$

By the suspension isomorphism for stable homotopy groups, λ_n is a stable weak homotopy equivalence precisely if any of its suspensions is. Hence consider instead $\Sigma^n \lambda_n \coloneqq S^n \wedge \lambda_n$, whose q-component is

$$
(\Sigma^n \lambda_n)_q \;:\; O(q)_+ \wedge_{O(q-n-1)} S^q \longrightarrow O(q)_+ \wedge_{O(q-n)} S^q \,.
$$

Now due to the fact that $O(q-k)$-action on S^q lifts to an $O(q)$-action, the quotients of the diagonal action of $O(q-k)$ equivalently become quotients of just the left action. Formally this is due to the existence of the commuting diagram

$$
\begin{array}{ccccc}
O(q)_+ \wedge S^q & \overset{\mathrm{id}}{\longrightarrow} & O(q)_+ \wedge S^q & \overset{\mathrm{id}}{\longrightarrow} & O(q)_+ \wedge S^q \\
\downarrow & & \downarrow & & \downarrow^{p_2} \\
Q(q)_+ \wedge_{Q(q-k)} S^q & \longrightarrow & Q(q)_+ \wedge_{Q(q)} S^q & \overset{\simeq}{\longrightarrow} & S^q
\end{array}
$$

which says that the image of any $(g,s) \in O(q)_+ \wedge S^q$ in the quotient $Q(q)_+ \wedge_{Q(q-k)} S^q$ is labeled by $([g],s)$.

It follows that $(\Sigma^n \lambda_n)_q$ is the smash product of a projection map of coset spaces with the identity on the sphere:

$$
(\Sigma^n \lambda_n)_q \simeq \mathrm{proj}_+ \wedge \mathrm{id}_{S^q} \;:\; O(q)/O(q-n-1)_+ \wedge S^q \longrightarrow O(q)/O(q-n)_+ \wedge S^q \,.
$$

Now finally observe that this projection function

$$
\mathrm{proj} \;:\; O(q)/O(q-n-1) \longrightarrow O(q)/O(q-n)
$$

is $(q-n-1)$-connected (see here). Hence its smash product with S^q is $(2q-n-1)$-connected.

The key here is the fast growth of the connectivity with q. This implies that for each s there exists q such that $\pi_{s+q}((\Sigma^n \lambda_n)_q)$ becomes an isomorphism. Hence $\Sigma^n \lambda_n$ is a stable weak homotopy equivalence and therefore so is λ_n.

symmetric case

Here the morphism λ_n has the same form as in the orthogonal case above, except that all occurences of orthogonal groups are replaced by just their sub-symmetric groups.

Accordingly, the analysis then proceeds entirely analogously, with the key difference that the projection

$$
\Sigma(q)/\Sigma(q-n-1) \longrightarrow \Sigma(q)/\Sigma(q-n)
$$

does *not* become highly connected as q increases, due to the discrete topological space underlying the symmetric group. Accordingly the conclusion now is the opposite: λ_n is not a stable weak homotopy equivalence in this case. ▮

Another use of free spectra is that their pushout products may be explicitly analyzed, and checking the pushout-product axiom for general cofibrations may be reduced to checking it on morphisms between free spectra.

Lemma 2.34. *The symmetric monoidal smash product of spectra of the free spectrum constructions (def.*

2.25) on the generating cofibrations $\{S^{n-1} \overset{i_n}{\hookrightarrow} D^n\}_{n \in \mathbb{B}}$ of the classical model structure on topological spaces is given by addition of indices

$$(F_k i_{n_1}) \Box_{\mathbb{S}_{\mathrm{dia}}} (F_\ell i_{n_2}) \simeq F_{k+\ell}(i_{n_1+n_2}) \,.$$

Proof. By lemma 2.29 the commuting diagram defining the pushout product of free spectra

$$\begin{array}{ccccc} & & F_k S_+^{n_1-1} \wedge_{\mathbb{S}_{\mathrm{dia}}} F_\ell S_+^{n_2-1} & & \\ & \swarrow & & \searrow & \\ F_k D_+^{n_1} \wedge_{\mathbb{S}_{\mathrm{dia}}} F_\ell S_+^{n_2-1} & & & & F_k S_+^{n_1-1} \wedge_{\mathbb{S}_{\mathrm{dia}}} F_\ell D_+^{n_2-1} \\ & \searrow & & \swarrow & \\ & & F_k D_+^{n_1-1} \wedge_{\mathbb{S}_{\mathrm{dia}}} F_k D_+^{n_2-1} & & \end{array}$$

is equivalent to this diagram:

$$\begin{array}{ccccc} & & F_{k+\ell}((S^{n_1-1} \times S^{n_2-1})_+) & & \\ & \swarrow & & \searrow & \\ F_{k+\ell}((D^{n_1} \times S^{n_2-1})_+) & & & & F_{k+\ell}((S^{n_1-1} \times D^{n_2})_+) \\ & \searrow & & \swarrow & \\ & & F_{k+\ell}((D^{n_1} \times D^{n_2})_+) & & \end{array} \,.$$

Since the free spectrum construction is a left adjoint, it preserves pushouts, and so

$$(F_k i_{n_1}) \Box_{\mathbb{S}_{\mathrm{dia}}} (F_\ell i_{n_2}) \simeq F_{k+\ell}(i_{n_1} \Box i_{n_2}) \simeq F_{k+\ell}(i_{n_1+n_2}) \,,$$

where in the second step we used this lemma. ▮

3. The strict model structure on structured spectra

Theorem 3.1. *The four categories of*

1. *pre-excisive functors* $\mathrm{Exc}(\mathrm{Top}_{\mathrm{cg}})$;
2. *orthogonal spectra* $\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}}) = \mathbb{S}_{\mathrm{orth}}\,\mathrm{Mod}$;
3. *symmetric spectra* $\mathrm{SymSpec}(\mathrm{Top}_{\mathrm{cg}}) = \mathbb{S}_{\mathrm{sym}}\,\mathrm{Mod}$;
4. *sequential spectra* $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}}) = \mathbb{S}_{\mathrm{seq}}\,\mathrm{Mod}$

*(from def. 2.1, prop. 2.8, def. 2.9) each admit a model category structure (def.) whose weak equivalences and fibrations are those morphisms which induce on all component spaces weak equivalences or fibrations, respectively, in the classical model structure on pointed topological spaces $(\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}$. (thm., prop.). These are called the **strict model structures** (or **level model structures**) on structured spectra.*

Moreover, under the equivalences of categories of prop. 2.8 and prop. 2.12, the restriction functors in def. 2.4 constitute right adjoints of Quillen adjunctions (def.) between these model structures:

$$\begin{array}{ccccccc} \mathrm{Exc}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}} & & \mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}} & & \mathrm{SymSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}} & & \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}} \\ \downarrow^{\simeq} & & \downarrow^{\simeq} & & \downarrow^{\simeq} & & \downarrow^{\simeq} \\ \mathbb{S}\,\mathrm{Mod}_{\mathrm{strict}} & \underoverset{\underset{\mathrm{orth}^*}{\longrightarrow}}{\overset{\mathrm{orth}_!}{\longleftarrow}}{\bot} & \mathbb{S}_{\mathrm{Orth}}\,\mathrm{Mod}_{\mathrm{strict}} & \underoverset{\underset{\mathrm{sym}^*}{\longrightarrow}}{\overset{\mathrm{sym}_!}{\longleftarrow}}{\bot} & \mathbb{S}_{\mathrm{Sym}}\,\mathrm{Mod}_{\mathrm{strict}} & \underoverset{\underset{\mathrm{seq}^*}{\longrightarrow}}{\overset{\mathrm{seq}_!}{\longleftarrow}}{\bot} & \mathbb{S}_{\mathrm{Seq}}\,\mathrm{Mod}_{\mathrm{strict}} \end{array} \,.$$

(MMSS 00, theorem 6.5)

Proof. By prop. 2.19 all four categories are equivalently categories of pointed topologically enriched functors

$$\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod} \simeq [\mathbb{S}_{\mathrm{dia}}\,\mathrm{Free}_{\mathrm{dia}}\mathrm{Mod}, \mathrm{Top}_{\mathrm{cg}}^{*/}]$$

and hence the existence of the model structures with componentwise weak equivalences and fibrations is a special case of the general existence of the projective model structure on enriched functors (thm.).

The three restriction functors dia^* each have a left adjoint $\mathrm{dia}_!$ by topological left Kan extension (prop. 1.38).

Moreover, the three right adjoint restriction functors are along inclusions of objects, hence evidently preserve componentwise weak equivalences and fibrations. Hence these are Quillen adjunctions. ▮

Definition 3.2. Recall the sets

$$I_{\mathrm{Top}^{*/}} \coloneqq \{S^{n-1}_+ \overset{(i_n)_+}{\hookrightarrow} D^n_+\}_{n \in \mathbb{N}}$$
$$J_{\mathrm{Top}^{*/}} \coloneqq \{D^n_+ \overset{(j_n)_+}{\hookrightarrow} (D^n \times I)_+\}_{n \in \mathbb{N}}$$

of generating cofibrations and generating acyclic cofibrations, respectively, of the classical model structure on pointed topological spaces (def.)

Write

$$I^{\mathrm{strict}}_{\mathrm{dia}} \;\coloneqq\; \left\{ F^{\mathrm{dia}}_c((i_n)_+) \right\}_{c \in \mathrm{Dia}, n \in \mathbb{N}}$$

for the set of images under forming free spectra, def. 2.25, on the morphisms in $I_{\mathrm{Top}^{*/}}$ from above. Similarly, write

$$J^{\mathrm{strict}}_{\mathrm{dia}} \;\coloneqq\; \left\{ F^{\mathrm{dia}}_c((j_n)_+) \right\},$$

for the set of images under forming free spectra of the morphisms in $J_{\mathrm{Top}^{*/}_{\mathrm{cg}}}$.

Proposition 3.3. *The sets $I^{\mathrm{strict}}_{\mathrm{dia}}$ and $J^{\mathrm{strict}}_{\mathrm{dia}}$ from def. 3.2 are, respectively sets of generating cofibrations and generating acyclic cofibrations that exhibit the strict model structure $\mathbb{S}_{\mathrm{Dia}}\,\mathrm{Mod}_{\mathrm{strict}}$ from theorem 3.1 as a cofibrantly generated model category (def.).*

(MMSS 00, theorem 6.5)

Proof. By theorem 3.1 the strict model structure is equivalently the projective pointed model structure on topologically enriched functors

$$\mathbb{S}_{\mathrm{Dia}}\,\mathrm{Mod}_{\mathrm{strict}} \simeq [\mathbb{S}_{\mathrm{Dia}}\,\mathrm{Free}_{\mathrm{Dia}}\mathrm{Mod}^{\mathrm{op}}, \mathrm{Top}^{*/}]_{\mathrm{proj}}$$

of the opposite of the category of free spectra on objects in $\mathcal{C} \hookrightarrow [\mathcal{C}, \mathrm{Top}^{*/}_{\mathrm{cg}}]$.

By the general discussion in *Part P -- Classical homotopy theory* (this theorem) the projective model structure on functors is cofibrantly generated by the smash tensoring of the representable functors with the elements in $I_{\mathrm{Top}^{*/}_{\mathrm{cg}}}$ and $J_{\mathrm{Top}^{*/}_{\mathrm{cg}}}$. By the proof of lemma 2.26, these are precisely the morphisms of free spectra in $I^{\mathrm{strict}}_{\mathrm{dia}}$ and $J^{\mathrm{strict}}_{\mathrm{dia}}$, respectively. ▮

Topological enrichment

By the general properties of the projective model structure on topologically enriched functors, theorem 3.1 implies that the strict model category of structured spectra inherits the structure of an enriched model category, enriched over the classical model structure on pointed topological spaces. This proceeds verbatim as for sequential spectra (in *part 1.1 – Topological enrichement*), but for ease of reference we here make it explicit again.

Definition 3.4. Let $\mathrm{Dia} \in \{\mathrm{Top}^{*/}_{\mathrm{cg,fin}}, \mathrm{Orth}, \mathrm{Sym}, \mathrm{Seq}\}$ one of the shapes for structured spectra from def. 2.4.

Let $f : X \to Y$ be a morphism in $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}$ (as in prop. 3.1) and let $i : A \to B$ a morphism in $\mathrm{Top}^{*/}_{\mathrm{cg}}$.

Their **pushout product with respect to smash tensoring** is the universal morphism

$$f \,\Box\, i \coloneqq ((\mathrm{id}, i), (f, \mathrm{id}))$$

in

$$\begin{array}{ccccc} & & X \wedge A & & \\ & {}^{(f,\mathrm{id})}\swarrow & & \searrow^{(\mathrm{id},i)} & \\ Y \wedge A & & (\mathrm{po}) & & X \wedge B \\ & \searrow & & \swarrow & \\ & & (Y \wedge A) \underset{X \wedge A}{\sqcup} (X \wedge B) & & \\ & & \downarrow^{((\mathrm{id},i),(f,\mathrm{id}))} & & \\ & & Y \wedge B & & \end{array},$$

where

$$(-)\wedge(-) : \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod} \times \mathrm{Top}^{*/}_{\mathrm{cg}} \simeq [\mathbb{S}_{\mathrm{dia}}\,\mathrm{Fre}_{\mathrm{dia}}\mathrm{Mod}^{\mathrm{op}},\, \mathrm{Top}^{*/}_{\mathrm{cg}}] \times \mathrm{Top}^{*/}_{\mathrm{cg}} \longrightarrow [\mathbb{S}_{\mathrm{dia}}\,\mathrm{Fre}_{\mathrm{dia}}\mathrm{Mod}^{\mathrm{op}},\, \mathrm{Top}^{*/}_{\mathrm{cg}}] \simeq \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}$$

denotes the smash tensoring of pointed topologically enriched functors with pointed topological spaces

(def.)

Dually, their **pullback powering** is the universal morphism

$$f^{\square i} \coloneqq (\mathrm{Maps}(B,f)_*, \mathrm{Maps}(i,X)_*)$$

in

$$\begin{array}{ccccc} & & \mathrm{Maps}(B,X)_* & & \\ & & \downarrow^{(\mathrm{Maps}(B,f)_*, \mathrm{Maps}(i,X)_*)} & & \\ & & \mathrm{Maps}(B,Y)_* \underset{\mathrm{Maps}(A,Y)_*}{\times} \mathrm{Maps}(A,X)_* & & \\ & \swarrow & & \searrow & \\ \mathrm{Maps}(B,Y)_* & & (\mathrm{pb}) & & \mathrm{Maps}(A,X)_* \\ & {}_{\mathrm{Maps}(i,Y)_*}\searrow & & \swarrow_{\mathrm{Maps}(A,p)_*} & \\ & & \mathrm{Maps}(A,Y)_* & & \end{array} ,$$

where

$$\mathrm{Maps}(-,-)_* \;:\; (\mathrm{Top}^*_{\mathrm{cg}})^{\mathrm{op}} \times \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod} \simeq (\mathrm{Top}^{*/}_{\mathrm{cg}})^{\mathrm{op}} \times [\mathbb{S}_{\mathrm{dia}}\,\mathrm{Free}_{\mathrm{Dia}}\mathrm{Mod}^{\mathrm{op}}, \mathrm{Top}^{*/}_{\mathrm{cg}}] \longrightarrow [\mathbb{S}_{\mathrm{dia}}\,\mathrm{Free}_{\mathrm{Dia}}\mathrm{Mod}^{\mathrm{op}}, \mathrm{Top}^{*/}_{\mathrm{cg}}] \simeq \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}$$

denotes the smash powering (def.).

Finally, for $f : X \to Y$ and $i : A \to B$ both morphisms in $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}$, then their pullback powering is the universal morphism

$$f^{\square i} \coloneqq (\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(B,f), \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(i,X))$$

in

$$\begin{array}{ccccc} & & \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(B,X) & & \\ & & \downarrow^{(\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(B,f), \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(i,X))} & & \\ & & \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(B,Y) \underset{\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(A,Y)}{\times} \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(A,X) & & \\ & \swarrow & & \searrow & \\ \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(B,Y) & & (\mathrm{pb}) & & \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(A,X) \\ & {}_{\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(i,Y)}\searrow & & \swarrow_{\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(A,p)} & \\ & & \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(A,Y) & & \end{array} ,$$

where now $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(-,-)$ is the hom-space functor of $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod} \simeq [\mathbb{S}_{\mathrm{dia}}\,\mathrm{Free}_{\mathrm{Dia}}\mathrm{Mod}^{\mathrm{op}}, \mathrm{Top}^{*/}_{\mathrm{cg}}]$ from def. 1.31.

Proposition 3.5. *The operations of forming pushout products and pullback powering with respect to smash tensoring in def. 3.4 is compatible with the strict model structure $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}_{\mathrm{strict}}$ on structured spectra from theorem 3.1 and with the classical model structure on pointed topological spaces $(\mathrm{Top}^{*/}_{\mathrm{cg}})_{\mathrm{Quillen}}$ (thm., prop.) in that pushout product takes two cofibrations to a cofibration, and to an acyclic cofibration if at least one of the inputs is acyclic, and pullback powering takes a fibration and a cofibration to a fibration, and to an acylic one if at least one of the inputs is acyclic:*

$$\begin{aligned} \mathrm{Cof}_{\mathrm{strict}} \square \mathrm{Cof}_{\mathrm{cl}} &\subset \mathrm{Cof}_{\mathrm{strict}} \\ \mathrm{Cof}_{\mathrm{strict}} \square (\mathrm{Cof}_{\mathrm{cl}} \cap W_{\mathrm{cl}}) &\subset \mathrm{Cof}_{\mathrm{strict}} \cap W_{\mathrm{strict}} \\ (\mathrm{Cof}_{\mathrm{strict}} \cap W_{\mathrm{strict}}) \square \mathrm{Cof}_{\mathrm{cl}} &\subset \mathrm{Cof}_{\mathrm{strict}} \cap W_{\mathrm{strict}} \end{aligned} .$$

Dually, the pullback powering (def. 3.4) satisfies

$$\begin{aligned} \mathrm{Fib}_{\mathrm{strict}}^{\square \mathrm{Cof}_{\mathrm{cl}}} &\subset \mathrm{Fib}_{\mathrm{strict}} \\ \mathrm{Fib}_{\mathrm{strict}}^{\square(\mathrm{Cof}_{\mathrm{cl}} \cap W_{\mathrm{cl}})} &\subset \mathrm{Fib}_{\mathrm{strict}} \cap W_{\mathrm{strict}} \\ (\mathrm{Fib}_{\mathrm{strict}} \cap W_{\mathrm{strict}})^{\square \mathrm{Cof}_{\mathrm{cl}}} &\subset \mathrm{Fib}_{\mathrm{strict}} \cap W_{\mathrm{strict}} \end{aligned} .$$

Proof. The statement concering the pullback powering follows directly from the analogous statement for topological spaces (prop.) by the fact that, via theorem 3.1, the fibrations and weak equivalences in $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}_{\mathrm{strict}}$ are degree-wise those in $(\mathrm{Top}^{*/}_{\mathrm{cg}})_{\mathrm{Quillen}}$, and since smash tensoring and powering is defined degreewise. From this the statement about the pushout product follows dually by Joyal-Tierney calculus (prop.). ▮

Remark 3.6. In the language of model category-theory, prop. 3.5 says that $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}_{\mathrm{strict}}$ is an *enriched model category*, the enrichment being over $(\mathrm{Top}^{*/}_{\mathrm{cg}})_{\mathrm{Quillen}}$. This is often referred to simply as a "topological model category".

We record some immediate consequences of prop. 3.5 that will be useful.

Proposition 3.7. *Let $K \in \mathrm{Top}^*_{\mathrm{cg}}$ be a retract of a cell complex (def.), then the smash-tensoring/powering adjunction from prop. 1.37 is a Quillen adjunction (def.) for the strict model structure from theorem 3.1*

$$\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}} \underoverset{\underset{\mathrm{Maps}(K,-)_*}{\longrightarrow}}{\overset{(-)\wedge K}{\longleftarrow}}{\bot} \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}}\,.$$

Proof. By assumption, K is a cofibrant object in the classical model structure on pointed topological spaces (thm., prop.), hence $* \to K$ is a cofibration in $(\mathrm{Top}^{*/}_{\mathrm{cg}})_{\mathrm{Quillen}}$. Observe then that the the pushout product of any morphism f with $* \to K$ is equivalently the smash tensoring of f with K:

$$f \,\Box\, (* \to K) \simeq f \wedge K\,.$$

This way prop. 3.5 implies that $(-)\wedge K$ preserves cofibrations and acyclic cofibrations, hence is a left Quillen functor. ▮

Lemma 3.8. *Let $X \in \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}_{\mathrm{strict}}$ be a structured spectrum, regarded in the strict model structure of theorem 3.1.*

1. *The smash powering of X with the standard topological interval I_+ (exmpl.) is a good path space object (def.)*

$$\Delta_X \,:\, X \overset{\in W_{\mathrm{strict}}}{\longrightarrow} X^{I_+} \overset{\in \mathrm{Fib}_{\mathrm{strict}}}{\longrightarrow} X \times X\,.$$

2. *If X is cofibrant, then its smash tensoring with the standard topological interval I_+ (exmpl.) is a good cylinder object (def.)*

$$\nabla_X \,:\, X \vee X \overset{\in \mathrm{Cof}_{\mathrm{strict}}}{\longrightarrow} X \wedge (I_+) \overset{\in W_{\mathrm{strict}}}{\longrightarrow} X\,.$$

Proof. It is clear that we have weak equivalences as shown ($I \to *$ is even a homotopy equivalence), what requires proof is that the path object is indeed good in that $X^{(I_+)} \to X \times X$ is a fibration, and the cylinder object is indeed good in that $X \vee X \to X \wedge (I_+)$ is indeed a cofibration.

For the first statement, notice that the pullback powering (def. 3.4) of $* \sqcup * \overset{(i_0,i_1)}{\longrightarrow} I$ into the terminal morphism $X \to *$ is the same as the powering $X^{(i_0,i_1)}$:

$$((X \to *)^{\Box (i_0,i_1)}) \;\simeq\; X^{(i_0,i_1)},\,.$$

But since every object in $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}_{\mathrm{strict}}$ is fibrant, so that $X \to *$ is a fibration, and since (i_0,i_1) is a relative cell complex inclusion and hence a cofibration in $(\mathrm{Top}^{*/}_{\mathrm{cg}})_{\mathrm{Quilln}}$, prop. 3.5 says that $X^{(i_0,i_1)} : X^{I_+} \to X \times X$ is a fibration.

Dually, observe that

$$(* \to X) \,\Box\, (i_0,i_1) \;\simeq\; X \wedge (i_0,i_1)\,.$$

Hence if X is assumed to be cofibrant, so that $* \to X$ is a cofibration, then prop. 3.5 implies that $X \wedge (i_0,i_1) : X \wedge X \to X \wedge (I_+)$ is a cofibration. ▮

Proposition 3.9. *For $X \in \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}$ a structured spectrum, $f \in \mathrm{Mor}(\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod})$ any morphism of structured spectra, and for $g \in \mathrm{Mor}(\mathrm{Top}^{*/}_{\mathrm{cpt}})$ a morphism of pointed topological spaces, then the hom-spaces of def. 1.31 (via prop. 2.19) interact with the pushout-product and pullback-powering from def. 3.4 in that there is a natural isomorphism*

$$\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(f \,\Box\, g, X) \simeq (\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(f,X))^{\Box g}\,.$$

Proof. Since the pointed compactly generated mapping space functor (exmpl.)

$$\mathrm{Maps}(-,-)_* \,:\, \left(\mathrm{Top}^{*/}_{\mathrm{cg}}\right)^{\mathrm{op}} \times \mathrm{Top}^{*/}_{\mathrm{cg}} \longrightarrow \mathrm{Top}^{*/}_{\mathrm{cg}}$$

takes colimits in the first argument to limits (cor.) and ends in the second argument to ends (remark 1.36), and since limits and colimits in $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}$ are computed objectswise (this prop. via prop. 2.19) this follows with the end-formula for the mapping space (def. 1.31):

$$
\begin{aligned}
\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(f \,\square\, g, X) &= \int_c \mathrm{Maps}((f \,\square\, g)(c), X(c))_* \\
&\simeq \int_c \mathrm{Maps}(f(c) \,\square\, g, X(c))_* \\
&\simeq \int_c \mathrm{Maps}(f(c), X(c))_*^{\square g} \\
&\simeq \Big(\int_c \mathrm{Maps}(f(c), X(c))_*\Big)^{\square g} \\
&\simeq (\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(f, X))^{\square g}
\end{aligned}
\,.
$$

▮

Proposition 3.10. *For $X, Y \in \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(\mathrm{Top}_{\mathrm{cg}})$ two structured spectra with X cofibrant in the strict model structure of def. 3.1, then there is a natural bijection*

$$
\pi_0 \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(X, Y) \simeq [X, Y]_{\mathrm{strict}}
$$

between the connected components of the hom-space (def. 1.31 via prop. 2.19) and the hom-set in the homotopy category (def.) of the strict model structure from theorem 3.1.

Proof. By prop. 1.37 the path components of the hom-space are the left homotopy classes of morphisms of structured spectra with respect to the standard cylinder spectrum $X \wedge (I_+)$:

$$
\frac{I_+ \longrightarrow \mathrm{SeqSpec}(X, Y)}{X \wedge (I_+) \longrightarrow Y}\,.
$$

Moreover, by lemma 3.8 the degreewise standard reduced cylinder $X \wedge (I_+)$ of structured spectra is a good cylinder object on X in $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}_{\mathrm{strict}}$. Hence hom-sets in the strict homotopy category out of a cofibrant into a fibrant object are given by standard left homotopy classes of morphisms

$$
[X, Y]_{\mathrm{strict}} \simeq \mathrm{Hom}_{\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}}(X, Y)_{/\sim}
$$

(this lemma). Since X is cofibrant by assumption and since every object is fibrant in $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}_{\mathrm{strict}}$, this is the case. Hence the notion of left homotopy here is that seen by the standard interval, and so the claim follows. ▮

Monoidal model structure

We now combine the concepts of model category (def.) and monoidal category (def. 1.1).

Given a category $\mathcal{C}$ that is equipped both with the structure of a monoidal category and of a model category, then one may ask whether these two structures are compatible, in that the left derived functor (def.) of the tensor product exists to equip also the homotopy category with the structure of a monoidal category. If so, then one may furthermore ask if the localization functor $\gamma : \mathcal{C} \longrightarrow \mathrm{Ho}(\mathcal{C})$ is a monoidal functor (def. 1.47).

The axioms on a *monoidal model category* (def. 3.11 below) are such as to ensure that this is the case.

A key consequence is that, via prop. 1.50, for a monoidal model category the localization functor γ carries monoids to monoids. Applied to the stable model category of spectra established below, this gives that structured ring spectra indeed represent ring spectra in the homotopy category. (In fact much more is true, but requires further proof: there is also a model structure on monoids in the model structure of spectra, and with respect to that the structured ring spectra represent A-infinity rings/E-infinity rings.)

Definition 3.11. A (symmetric) **monoidal model category** is a model category $\mathcal{C}$ (def.) equipped with the structure of a closed (def. 1.7) symmetric (def. 1.5) monoidal category $(\mathcal{C}, \otimes, I)$ (def. 1.1) such that the following two compatibility conditions are satisfied

1. **(pushout-product axiom)** For every pair of cofibrations $f : X \to Y$ and $f' : X' \to Y'$, their pushout-product, hence the induced morphism out of the cofibered coproduct over ways of forming the tensor product of these objects

   $$
   f \,\square_{\otimes}\, g \;\coloneqq\; (X \otimes Y') \underset{X \otimes X'}{\sqcup} (Y \otimes X') \longrightarrow Y \otimes Y'\,,
   $$

   is itself a cofibration, which, furthermore, is acyclic if at least one of f or f' is.

   (Equivalently this says that the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a *left* *Quillen bifunctor*.)

2. **(unit axiom)** For every cofibrant object X and every cofibrant resolution $\emptyset \overset{\in \mathrm{Cof}}{\longrightarrow} Q1 \underset{p_1}{\overset{\in W}{\longrightarrow}} 1$ of the tensor unit 1, the resulting morphism

   $$
   Q1 \otimes X \overset{p_1 \otimes X}{\longrightarrow} 1 \otimes X \underset{\ell}{\overset{\in \mathrm{Iso} \subset W}{\longrightarrow}} X
   $$

is a weak equivalence.

(Hovey 99, def. 4.2.6 Schwede-Shipley 00, def. 3.1, remark 3.2)

Observe some immediate consequences of these axioms:

Remark 3.12. Since a monoidal model category (def. 3.11) is assumed to be closed monoidal (def. 1.7), for every object X the tensor product $X \otimes (-) \simeq (-) \otimes X$ is a left adjoint and hence preserves all colimits. In particular it preserves the initial object $\emptyset$ (which is the colimit over the empty diagram).

If follows that the tensor-pushout-product axiom in def. 3.11 implies that for X a cofibrant object, then the functor $X \otimes (-)$ preserves cofibrations and acyclic cofibrations, since

$$f \Box_{\otimes} (\emptyset \to X) \simeq f \otimes X \,.$$

This implies that if the tensor unit 1 happens to be cofibrant, then the unit axiom in def. 3.11 is already implied by the pushout-product axiom. This is because then we have a lift in

$$\begin{array}{ccc} \emptyset & \longrightarrow & Q1 \\ {}^{\mathllap{\in Cof}}\downarrow & \nearrow & \downarrow^{\mathrlap{p_1}}_{\mathrlap{\in W}} \\ 1 & = & 1 \end{array}$$

This lift is a weak equivalence by two-out-of-three (def.). Since it is hence a weak equivalence between cofibrant objects, it is preserved by the left Quillen functor $(-) \otimes X$ (for any cofibrant X) by Ken Brown's lemma (prop.). Hence now $p_1 \otimes X$ is a weak equivalence by two-out-of-three.

Since for all the categories of spectra that we are interested in here the tensor unit is always cofibrant (it is always a version of the sphere spectrum, being the image under the left Quillen functor $\Sigma^\infty_{\mathrm{dia}}$ of the cofibrant pointed space S^0, prop. 3.18), we may ignore the unit axiom.

Proposition 3.13. *Let $(\mathcal{C}, \otimes, I)$ be a monoidal model category (def. 3.11) with cofibrant tensor unit 1.*

Then the left derived functor $\otimes^L$ (def.) of the tensor product $\otimes$ exsists and makes the homotopy category (def.) into a monoidal category $(\mathrm{Ho}(\mathcal{C}), \otimes^L, \gamma(1))$ (def. 1.1) such that the localization functor $\gamma : \mathcal{C}_c \to \mathrm{Ho}(\mathcal{C})$ (thm.) on the category of cofibrant objects (def.) carries the structure of a strong monoidal functor (def. 1.47)

$$\gamma \,:\, (\mathcal{C}, \otimes, 1) \longrightarrow (\mathrm{Ho}(\mathcal{C}), \otimes^L, \gamma(1)) \,.$$

The first statement is also for instance in (Hovey 99, theorem 4.3.2).

Proof. For the left derived functor (def.) of the tensor product

$$\otimes \; \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$$

to exist, it is sufficient that its restriction to the subcategory

$$(\mathcal{C} \times \mathcal{C})_c \simeq \mathcal{C}_c \times \mathcal{C}_c$$

of cofibrant objects preserves acyclic cofibrations (by Ken Brown's lemma, here).

Every morphism (f, g) in the product category $\mathcal{C}_c \times \mathcal{C}_c$ (def. 1.26) may be written as a composite of a pairing with an identity morphisms

$$(f, g) \,:\, (c_1, d_1) \overset{(\mathrm{id}_{c_1}, g)}{\longrightarrow} (c_1, d_2) \overset{(f, \mathrm{id}_{c_2})}{\longrightarrow} (c_2, d_2) \,.$$

Now since the pushout product (with respect to tensor product) with the initial morphism $(* \to c_1)$ is equivalently the tensor product

$$(* \to c_1) \Box_{\otimes} g \;\simeq\; \mathrm{id}_{c_1} \otimes g$$

and

$$f \Box_{\otimes} (* \to c_2) \;\simeq\; f \otimes \mathrm{id}_{c_2}$$

the pushout-product axiom (def. 3.11) implies that on the subcategory of cofibrant objects the functor $\otimes$ preserves acyclic cofibrations. (This is why one speaks of a *Quillen bifunctor*, see also Hovey 99, prop. 4.3.1).

Hence $\otimes^L$ exists.

By the same decomposition and using the universal property of the localization of a category (def.) one finds that for $\mathcal{C}$ and $\mathcal{D}$ any two categories with weak equivalences (def.) then the localization of their product category is the product category of their localizations:

$$(\mathcal{C} \times \mathcal{D})[(W_{\mathcal{C}} \times W_{\mathcal{D}})^{-1}] \simeq (\mathcal{C}[W_{\mathcal{C}}^{-1}]) \times (\mathcal{D}[W_{\mathcal{D}}^{-1}]) \,.$$

With this, the universal property as a localization (def.) of the homotopy category of a model category (thm.) induces associators α^L and unitors ℓ^L, r^L on $(\mathrm{Ho}(\mathcal{C}, \otimes^L))$:

First write

$$\mu : \gamma(-) \otimes^L \gamma(-) \overset{\simeq}{\longrightarrow} \gamma((-) \otimes (-))$$

for (the inverse of) the corresponding natural isomorphism in the localization diagram

$$\begin{array}{ccc} \mathcal{C} \times \mathcal{C} & \overset{\otimes}{\longrightarrow} & \mathcal{C} \\ {}^{\mathllap{\gamma \times \gamma}}\downarrow & \swArrow_{\mu^{-1}} & \downarrow^{\mathrlap{\gamma}} \\ \mathrm{Ho}(\mathcal{C}) \times \mathrm{Ho}(\mathcal{C}) & \underset{\otimes^L}{\longrightarrow} & \mathrm{Ho}(\mathcal{C}) \end{array} \,.$$

Then consider the associators:

The essential uniqueness of derived functors shows that the left derived functor of $(-) \otimes ((-) \otimes (-))$ and of $((-) \otimes (-)) \otimes (-)$ is the composite of two applications of $\otimes^L$, due to the factorization

$$\begin{array}{ccc} \mathcal{C}_c \times \mathcal{C}_c \times \mathcal{C}_c & \overset{(-)\otimes((-)\otimes(-))}{\longrightarrow} & \mathcal{C}_c \\ {}^{\mathllap{\gamma\times\gamma\times\gamma}}\downarrow & \swArrow & \downarrow^{\mathrlap{\gamma}} \\ \mathrm{Ho}(\mathcal{C}) \times \mathrm{Ho}(\mathcal{C}) \times \mathrm{Ho}(\mathcal{C}) & \underset{\mathbb{L}((-)\otimes((-)\otimes(-)))}{\longrightarrow} & \mathrm{Ho}(\mathcal{C}) \end{array}$$

$$\simeq$$

$$\begin{array}{ccccc} \mathcal{C}_c \times \mathcal{C}_c \times \mathcal{C}_c & \overset{\mathrm{id}\times\otimes}{\longrightarrow} & \mathcal{C}_c \times \mathcal{C}_c & \overset{\otimes}{\longrightarrow} & \mathcal{C}_c \\ {}^{\mathllap{\gamma\times\gamma\times\gamma}}\downarrow & \swArrow_{\mathrm{id}\times\mu^{-1}} & {}^{\mathllap{\gamma\times\gamma}}\downarrow & \swArrow_{\mu^{-1}} & \downarrow^{\mathrlap{\gamma}} \\ \mathrm{Ho}(\mathcal{C}) \times \mathrm{Ho}(\mathcal{C}) \times \mathrm{Ho}(\mathcal{C}) & \underset{\mathrm{id}\times\otimes^L}{\longrightarrow} & \mathrm{Ho}(\mathcal{C}) \times \mathrm{Ho}(\mathcal{C}) & \underset{\otimes^L}{\longrightarrow} & \mathrm{Ho}(\mathcal{C}) \end{array}$$

and similarly for the case with the parenthesis to the left.

So let

$$\begin{array}{ccc} \mathcal{C}_c \times \mathcal{C}_c \times \mathcal{C}_c & \overset{((-)\otimes(-))\otimes(-)}{\longrightarrow} & \mathcal{C} \\ {}^{\mathllap{\gamma\times\gamma\times\gamma}}\downarrow & \swArrow_{\mu^{-1}\cdot(\mu^{-1}\times \mathrm{id})} & \downarrow^{\mathrlap{\gamma}} \\ \mathrm{Ho}(\mathcal{C}) \times \mathrm{Ho}(\mathcal{C}) \times \mathrm{Ho}(\mathcal{C}) & \overset{((-)\otimes^L(-))\otimes^L(-)}{\longrightarrow} & \mathrm{Ho}(\mathcal{C}) \end{array} \,, \quad \begin{array}{ccc} \mathcal{C}_c \times \mathcal{C}_c \times \mathcal{C}_c & \overset{(-)\otimes((-)\otimes(-))}{\longrightarrow} & \mathcal{C} \\ {}^{\mathllap{\gamma\times\gamma\times\gamma}}\downarrow & \swArrow_{\mu^{-1}\cdot(\mathrm{id}\times\mu^{-1})} & \downarrow^{\mathrlap{\gamma}} \\ \mathrm{Ho}(\mathcal{C}) \times \mathrm{Ho}(\mathcal{C}) \times \mathrm{Ho}(\mathcal{C}) & \overset{(-)\otimes^L((-)\otimes^L(-))}{\longrightarrow} & \mathrm{Ho}(\mathcal{C}) \end{array}$$

be the natural isomorphism exhibiting the derived functors of the two possible tensor products of three objects, as shown at the top. By pasting the second with the associator natural isomorphism of $\mathcal{C}$ we obtain another such factorization for the first, as shown on the left below,

$$(\star) \quad \begin{array}{ccc} \mathcal{C}_c \times \mathcal{C}_c \times \mathcal{C}_c & \overset{((-)\otimes(-))\otimes(-)}{\longrightarrow} & \mathcal{C} \\ {}^{\mathllap{=}}\downarrow & \swArrow_{\alpha} & \downarrow^{\mathrlap{=}} \\ \mathcal{C}_c \times \mathcal{C}_c \times \mathcal{C}_c & \overset{(-)\otimes((-)\otimes(-))}{\longrightarrow} & \mathcal{C} \\ {}^{\mathllap{\gamma\times\gamma\times\gamma}}\downarrow & \swArrow_{\mu^{-1}\cdot(\mathrm{id}\times\mu^{-1})} & \downarrow^{\mathrlap{\gamma}} \\ \mathrm{Ho}(\mathcal{C}) \times \mathrm{Ho}(\mathcal{C}) \times \mathrm{Ho}(\mathcal{C}) & \overset{(-)\otimes^L((-)\otimes^L(-))}{\longrightarrow} & \mathrm{Ho}(\mathcal{C}) \end{array} \quad = \quad \begin{array}{ccc} \mathcal{C}_c \times \mathcal{C}_c \times \mathcal{C}_c & \overset{((-)\otimes(-))\otimes(-)}{\longrightarrow} & \mathcal{C} \\ {}^{\mathllap{\gamma\times\gamma\times\gamma}}\downarrow & \swArrow_{\mu^{-1}\cdot(\mathrm{id}\times\mu)} & \downarrow^{\mathrlap{\gamma}} \\ \mathrm{Ho}(\mathcal{C}) \times \mathrm{Ho}(\mathcal{C}) \times \mathrm{Ho}(\mathcal{C}) & \overset{((-)\otimes^L(-))\otimes^L(-)}{\longrightarrow} & \mathrm{Ho}(\mathcal{C}) \\ {}^{\mathllap{=}}\downarrow & \swArrow_{\alpha^L} & \downarrow^{\mathrlap{=}} \\ \mathrm{Ho}(\mathcal{C}) \times \mathrm{Ho}(\mathcal{C}) \times \mathrm{Ho}(\mathcal{C}) & \underset{(-)\otimes^L((-)\otimes^L(-))}{\longrightarrow} & \mathrm{Ho}(\mathcal{C}) \end{array}$$

and hence by the universal property of the factorization through the derived functor, there exists a unique natural isomorphism α^L such as to make this composite of natural isomorphisms equal to the one shown on the right. Hence the pentagon identity satisfied by α implies a pentagon identity for α^L, and so α^L is an associator for $\otimes^L$.

Moreover, this equation of natural isomorphisms says that on components the following diagram commutes

$$\begin{array}{ccc} (\gamma(X) \otimes^L \gamma(Y)) \otimes^L \gamma(Z) & \overset{\alpha^L_{\gamma(X),\gamma(Y),\gamma(Z)}}{\longrightarrow} & \gamma(X) \otimes^L (\gamma(Y) \otimes^L \gamma(Z)) \\ {}^{\mathllap{\mu^{-1}\cdot(\mu^{-1}\times\mathrm{id})}}\uparrow && \uparrow^{\mathrlap{\mu^{-1}\cdot(\mathrm{id}\times\mu^{-1})}} \\ \gamma((X \otimes Y) \otimes Z) & \underset{\gamma(\alpha)}{\longrightarrow} & \gamma(X \otimes (Y \otimes Z)) \end{array} \,.$$

This is just the coherence law for the the compatibility of the monoidal functor μ with the associators.

Similarly consider now the unitors.

The essential uniqueness of the derived functors gives that the left derived functor of $1 \otimes (-)$ is $\gamma(1) \otimes^L (-)$

$$
\begin{tikzcd}
\mathcal{C}_c \arrow[r, "1\otimes(-)"] \arrow[d, "\gamma"'] & \mathcal{C}_c \arrow[d, "\gamma"] \\
\mathrm{Ho}(\mathcal{C}) \arrow[r, "\mathbb{L}(1\otimes(-))"'] & \mathrm{Ho}(\mathcal{C})
\end{tikzcd}
\simeq
\begin{tikzcd}
\mathcal{C}_c \arrow[r, "(1,\mathrm{id})"] \arrow[d, "\gamma"'] & \mathcal{C}_c \times \mathcal{C}_c \arrow[r, "\otimes"] \arrow[d, "\gamma\times\gamma"'] \arrow[dr, phantom, "\swArrow_{\mu^{-1}}"] & \mathcal{C} \arrow[d, "\gamma"] \\
\mathrm{Ho}(\mathcal{C}) \arrow[r, "(\gamma(1),\mathrm{id})"'] & \mathrm{Ho}(\mathcal{C}) \times \mathrm{Ho}(\mathcal{C}) \arrow[r, "\otimes^L"'] & \mathrm{Ho}(\mathcal{C})
\end{tikzcd}
.
$$

Hence the left unitor ℓ of $\mathcal{C}$ induces a derived unitor ℓ^L by the following factorization

$$
\begin{tikzcd}
\mathcal{C}_c \arrow[r, "1\otimes(-)"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\swArrow_{\ell}"] & \mathcal{C}_c \arrow[d, "\gamma"] \\
\mathcal{C}_c \arrow[r, "\mathrm{id}"] \arrow[d, "\gamma"'] & \mathcal{C}_c \arrow[d, "\gamma"] \\
\mathrm{Ho}(\mathcal{C}) \arrow[r, "\mathrm{id}"'] & \mathrm{Ho}(\mathcal{C})
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{C}_c \arrow[r, "1\otimes(-)"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\swArrow_{\mu^{-1}_{1,(-)}}"] & \mathcal{C}_c \arrow[d, "\gamma"] \\
\mathrm{Ho}(\mathcal{C}) \arrow[r, "\gamma(1)\otimes^L(-)"] \arrow[d, "="'] \arrow[dr, phantom, "\swArrow_{\ell^L}"] & \mathrm{Ho}(\mathcal{C}) \arrow[d, "="] \\
\mathrm{Ho}(\mathcal{C}) \arrow[r, "\mathrm{id}"'] & \mathrm{Ho}(\mathcal{C})
\end{tikzcd}
.
$$

Moreover, in components this equation of natural isomorphism expresses the coherence law stating the compatibility of the monoidal functor μ with the unitors.

Similarly for the right unitors. ▮

The restriction to cofibrant objects in prop. 3.13 serves the purpose of giving explicit expressions for the associators and unitors of the derived tensor product $\otimes^L$ and hence to establish the monoidal category structure $(\mathrm{Ho}(\mathcal{C}), \otimes^L, \gamma(1))$ on the homotopy category of a monoidal model category. With that in hand, it is natural to ask how the localization functor on all of $\mathcal{C}$ interacts with the monoidal structure:

Proposition 3.14. *For $(\mathcal{C}, \otimes, 1)$ a monoidal model category (def. 3.11) then the localization functor to its monoidal homotopy category (prop. 3.13) is a lax monoidal functor*

$$
\gamma : (\mathcal{C}, \otimes, 1) \longrightarrow (\mathrm{Ho}(\mathcal{C}), \otimes^L, \gamma(1)) \,.
$$

The explicit **proof** of prop. 3.14 is tedious. An abstract proof using tools from homotopical 2-category theory is here.

Definition 3.15. Given monoidal model categories $(\mathcal{C}, \otimes_{\mathcal{C}}, 1_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, 1_{\mathcal{D}})$ (def. 3.11) with cofibrant tensor units $1_{\mathcal{C}}$ and $1_{\mathcal{D}}$, then a **strong monoidal Quillen adjunction** between them is a Quillen adjunction

$$
\mathcal{C} \underoverset{\underset{R}{\longrightarrow}}{\overset{L}{\longleftarrow}}{\bot} \mathcal{D}
$$

such that L (hence equivalently R) has the structure of a strong monoidal functor.

Proposition 3.16. *Given a strong monoidal Quillen adjunction (def. 3.15)*

$$
\mathcal{C} \underoverset{\underset{R}{\longrightarrow}}{\overset{L}{\longleftarrow}}{\bot} \mathcal{D}
$$

between monoidal model categories $(\mathcal{C}, \otimes_{\mathcal{C}}, 1_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, 1_{\mathcal{D}})$ with cofibrant tensor units $1_{\mathcal{C}}$ and $1_{\mathcal{D}}$, then the left derived functor of L canonically becomes a strong monoidal functor between homotopy categories

$$
\mathbb{L}L : (\mathrm{Ho}(\mathcal{C}), \otimes_{\mathcal{C}}, \gamma(1)_{\mathcal{C}}) \longrightarrow (\mathrm{Ho}(\mathcal{D}), \otimes_{\mathcal{D}}, \gamma(1)_{\mathcal{D}}) \,.
$$

Proof. As in the proof of prop. 3.13, consider the following pasting composite of commuting diagams:

$$
\begin{tikzcd}
\mathcal{D}_c \times \mathcal{D}_c \arrow[r, "\otimes_{\mathcal{D}}"] \arrow[d, "="'] & \mathcal{D}_c \arrow[r, "L"] \arrow[dr, phantom, "\swArrow_{\simeq}"] & \mathcal{C}_c \arrow[d, "="] \\
\mathcal{D}_c \times \mathcal{D}_c \arrow[r, "L\times L"] \arrow[d, "\gamma_{\mathcal{D}}\times\gamma_{\mathcal{D}}"'] & \mathcal{C}_c \times \mathcal{C}_c \arrow[r, "\otimes_{\mathcal{C}}"] \arrow[d, "\gamma_{\mathcal{C}}\times\gamma_{\mathcal{C}}"] & \mathcal{C}_c \arrow[d, "\gamma_{\mathcal{C}}"] \\
\mathrm{Ho}(\mathcal{D}) \times \mathrm{Ho}(\mathcal{D}) \arrow[r, "\mathbb{L}L\times\mathbb{L}L"'] & \mathrm{Ho}(\mathcal{C}) \times \mathrm{Ho}(\mathcal{C}) \arrow[r, "\otimes^L_{\mathcal{C}}"'] & \mathrm{Ho}(\mathcal{C})
\end{tikzcd}
\simeq
\begin{tikzcd}
\mathcal{D}_c \times \mathcal{D}_c \arrow[r, "\otimes_{\mathcal{D}}"] \arrow[d, "\gamma_{\mathcal{D}}\times\gamma_{\mathcal{D}}"'] & \mathcal{D}_c \arrow[r, "L"] \arrow[d, "\gamma_{\mathcal{D}}"] & \mathcal{C}_c \arrow[d, "\gamma_{\mathcal{C}}"] \\
\mathrm{Ho}(\mathcal{D}) \times \mathrm{Ho}(\mathcal{D}) \arrow[r, "\otimes^L_{\mathcal{D}}"] \arrow[d, "="'] & \mathrm{Ho}(\mathcal{D}) \arrow[r, "\mathbb{L}L"] \arrow[d, phantom, "\swArrow_{\simeq}"] & \mathrm{Ho}(\mathcal{C}) \arrow[d, "="] \\
\mathrm{Ho}(\mathcal{D}) \times \mathrm{Ho}(\mathcal{D}) \arrow[r, "\mathbb{L}L\times\mathbb{L}L"'] & \mathrm{Ho}(\mathcal{C}) \times \mathrm{Ho}(\mathcal{C}) \arrow[r, "\otimes^L_{\mathcal{C}}"'] & \mathrm{Ho}(\mathcal{C})
\end{tikzcd}
.
$$

On the top left we have the natural transformation that exhibits L as a strong monoidal functor. By universality of localization and derived functors (def.) this induces the unique factorization through the natural transformation on the bottom right. This exhibits strong monoidal structure on the left derived functor $\mathbb{L}L$. ▮

With some general monoidal homotopy theory established, we now discuss that structured spectra indeed constitute an example. The version of the following theorem for the stable model structure of actual interest is theorem 4.14 further below.

Theorem 3.17.

1. *The classical model structure on pointed topological spaces equipped with the smash product is a monoidal model category*

$$((\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}, \wedge, S^0) .$$

2. *Let $\mathrm{Dia} \in \{\mathrm{Top}^{*/}_{\mathrm{cg,fin}}, \mathrm{Orth}, \mathrm{Sym}\}$. The strict model structures on structured spectra modeled on Dia from theorem 3.1 equipped with the symmetric monoidal smash product of spectra (def. 2.1, def. 2.9) is a monoidal model category (def. 3.11)*

$$\left(\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}_{\mathrm{strict}},\ \wedge = \otimes_{\mathbb{S}_{\mathrm{dia}}}, \mathbb{S}_{\mathrm{dia}}\right) .$$

(MMSS 00, theorem 12.1 (iii) with prop. 12.3)

Proof. By cofibrant generation of both model structures (this theorem and prop. 3.3) it is sufficient to check the pushout-product axiom on generating (acylic) cofibrations (this is as in the proof of this proposition).

Those of $\mathrm{Top}^{*/}_{\mathrm{cg}}$ are as recalled in def. 4.4. These satisfy (exmpl.) the relations

$$i_{k_1} \Box i_{k_2} = i_{k_1+k_2}$$

and

$$i_{k_1} \Box j_{k_2} = j_{k_1+k_2} .$$

This shows that

$$I_{\mathrm{Top}^{*/}} \Box_{\otimes_{\mathbb{S}_{\mathrm{dia}}}} I_{\mathrm{Top}^{*/}} \subset I_{\mathrm{Top}^{*/}}$$

and

$$I_{\mathrm{Top}^{*/}} \Box_{\otimes_{\mathbb{S}_{\mathrm{dia}}}} J_{\mathrm{Top}^{*/}} \subset J_{\mathrm{Top}^{*/}}$$

which implies the pushout-product axiom for $\mathrm{Top}^{*/}_{\mathrm{cg}}$. (However the monoid axiom (def.\ref{MonoidAxiom}) is problematic.)

Now by def. 3.2 the generating (acyclic) cofibrations of $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}_{\mathrm{strict}}$ are of the form $F_n^{\mathrm{dia}}(i_k)_+$ and $F_n^{\mathrm{dia}}(j_k)_+$, respectively. By prop. 2.29 these satisfy

$$F_{n_1}(i_{k_1})_+ \;\Box_\wedge\; F_{n_2}(i_{k_2})_+ \;\simeq\; F_{n_1+n_2}(i_{k_1} \Box_\wedge i_{k_2})_+$$

and

$$F_{n_1}(i_{k_1})_+ \;\Box\; F_{n_2}(j_{k_2})_+ \;\simeq\; F_{n_1+n_2}(i_{k_1} \Box j_{k_2})_+ .$$

Hence with the previous set of relations this shows that

$$I_{\mathrm{dia}}^{\mathrm{strict}} \Box_{\otimes_{\mathbb{S}_{\mathrm{dia}}}} I_{\mathrm{dia}}^{\mathrm{strict}} \subset I_{\mathrm{dia}}^{\mathrm{strict}}$$

and

$$I_{\mathrm{dia}}^{\mathrm{strict}} \Box_{\otimes_{\mathbb{S}_{\mathrm{dia}}}} J_{\mathrm{dia}}^{\mathrm{strict}} \subset J_{\mathrm{dia}}^{\mathrm{strict}}$$

and so the pushout-product axiom follows also for $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}_{\mathrm{strict}}$.

It is clear that in both cases the tensor unit is cofibrant: for $\mathrm{Top}^{*/}_{\mathrm{cg}}$ the tensor unit is the 0-sphere, which clearly is a CW-complex and hence cofibrant. For $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}$ the tensor unit is the standard sphere spectrum, which, by prop. 2.26 is the free structured spectrum (def. 2.25) on the 0-sphere

$$\mathbb{S}_{\mathrm{dia}} \simeq F_0^{\mathrm{dia}}(S^0) .$$

Now the free structured spectrum functor is a left Quillen functor (prop. 3.18) and hence $\mathbb{S}_{\mathrm{dia}}$ is cofibrant. ▮

Suspension and looping

For the strict model structure on topological sequential spectra, forming suspension spectra consitutes a Quillen adjunction $(\Sigma^\infty \dashv \Omega^\infty)$ with the classical model structure on pointed topological spaces (prop.) which is the precursor of the stabilization adjunction involving the stable model structure (thm.). Here we briefly discuss the lift of this strict adjunction to structured spectra.

Proposition 3.18. *Let $\mathrm{Dia} \in \{\mathrm{Top}^{*/}_{\mathrm{cg,fin}}, \mathrm{Orth}, \mathrm{Sym}, \mathrm{Seq}\}$ be one of the shapes of structured spectra from def. 2.4.*

For every $n \in \mathbb{N}$, the functors $\mathrm{Ev}^{\mathrm{dia}}_n$ of extracting the nth component space of a structured spectrum, and the functors F^{dia}_n of forming the free structured spectrum in degree n (def. 2.25) constitute a Quillen adjunction (def.) between the strict model structure on structured spectra from theorem 3.1 and the classical model structure on pointed topological spaces (thm., prop.):

$$\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}_{\mathrm{strict}} \underoverset{\underset{\mathrm{Ev}^{\mathrm{dia}}_n}{\longrightarrow}}{\overset{F^{\mathrm{dia}}_n}{\longleftarrow}}{\bot} (\mathrm{Top}^{*/}_{\mathrm{cg}})_{\mathrm{Quillen}} \,.$$

For $n = 0$ and writing $\Sigma^\infty_{\mathrm{dia}} \coloneqq F^{\mathrm{dia}}_0$ and $\Omega^\infty_{\mathrm{dia}} \coloneqq \mathrm{Ev}^{\mathrm{dia}}_0$, $\Sigma^\infty_{\mathrm{dia}}$ this yields a strong monoidal Quillen adjunction (def. 3.15)

$$\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}_{\mathrm{strict}} \underoverset{\underset{\Omega^\infty_{\mathrm{dia}}}{\longrightarrow}}{\overset{\Sigma^\infty_{\mathrm{dia}}}{\longleftarrow}}{\bot} (\mathrm{Top}^{*/}_{\mathrm{cg}})_{\mathrm{Quillen}} \,.$$

Moreover, these Quillen adjunctions factor as

$$(\Sigma^\infty_{\mathrm{dia}} \dashv \Omega^\infty_{\mathrm{dia}}) \;:\; \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}} \underoverset{\underset{\mathrm{seq}^*}{\longrightarrow}}{\overset{\mathrm{seq}_!}{\longleftarrow}}{\bot} \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}} \underoverset{\underset{\Omega^\infty}{\longrightarrow}}{\overset{\Sigma^\infty}{\longleftarrow}}{\bot} (\mathrm{Top}^{*/}_{\mathrm{cg}})$$

where the Quillen adjunction $(\mathrm{seq}_! \dashv \mathrm{seq}^*)$ is that from theorem 3.1 and where $(\Sigma^\infty \dashv \Omega^\infty)$ is the suspension spectrum adjunction for sequential spectra (prop.).

Proof. By the very definition of the projective model structure on functors (thm.) it is immediate that $\mathrm{Ev}^{\mathrm{dia}}_n$ preserves fibrations and weak equivalences, hence it is a right Quillen functor. F^{dia}_n is its left adjoint by definition.

That $\Sigma^\infty_{\mathrm{dia}}$ is a strong monoidal functor is part of the statement of prop. 2.29.

Moreover, it is clear from the definitions that

$$\Omega^\infty_{\mathrm{dia}} \simeq \Omega^\infty \circ \mathrm{seq}^*,$$

hence the last statement follows by uniqueness of adjoints. ▮

Remark 3.19. In summary, we have established the following situation. There is a commuting diagram of Quillen adjunctions of the form

$$\begin{array}{ccc} (\mathrm{Top}^{*/}_{\mathrm{cg}})_{\mathrm{Quillen}} & \underoverset{\underset{\Omega}{\longrightarrow}}{\overset{\Sigma}{\longleftarrow}}{\bot} & (\mathrm{Top}^{*/}_{\mathrm{cg}})_{\mathrm{Quillen}} \\ {}^{\mathllap{\Sigma^\infty}}\downarrow \dashv \uparrow^{\mathrlap{\Omega^\infty}} && {}^{\mathllap{\Sigma^\infty}}\downarrow \dashv \uparrow^{\mathrlap{\Omega^\infty}} \\ \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}} & \underoverset{\underset{\Omega}{\longrightarrow}}{\overset{\Sigma}{\longleftarrow}}{\bot} & \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}} \\ {}^{\mathllap{\mathrm{dia}_!}}\downarrow \dashv \uparrow^{\mathrlap{\mathrm{dia}^*}} && {}^{\mathllap{\mathrm{dia}_!}}\downarrow \dashv \uparrow^{\mathrlap{\mathrm{dia}^*}} \\ \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}_{\mathrm{strict}} && \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}_{\mathrm{strict}} \end{array} \,.$$

The top square stabilizes to the actual stable homotopy theory (thm.). On the other hand, the top square does not reflect the symmetric monoidal smash product of spectra (by remark 2.6). But the total vertical composite $\Sigma^\infty_{\mathrm{dia}} = \mathrm{dia}_! \Sigma^\infty$ does, in that it is a strong monoidal Quillen adjunction (def. 3.15) by prop. 3.18.

Hence to obtain a stable model category which is also a monoidal model category with respect to the symmetric monoidal smash product of spectra, it is now sufficient to find such a monoidal model structure on $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}$ such that $(\mathrm{seq}_! \dashv \mathrm{seq}^*)$ becomes a Quillen equivalence (def.)

This we now turn to in the section *The stable model structure on structured spectra*.

4. The stable model structure on structured spectra

Theorem 4.1. *The category $\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})$ of orthogonal spectra carries a model category structure (def.) where*

- *the weak equivalences W_{stable} are the stable weak homotopy equivalences (def. 2.21);*

- *the cofibrations $\mathrm{Cof}_{\mathrm{stable}}$ are the cofibrations of the strict model stucture of prop. 3.1;*
- *the fibrant objects are precisely the Omega-spectra (def. 2.21).*

Moreover, this is a cofibrantly generated model category (def.) with generating (acyclic) cofibrations the sets I^{stable} (J^{stable}) from def. 3.2.

(Mandell-May 02, theorem 4.2)

We give the **proof** below, after

Proof of the model structure

The generating cofibrations and acylic cofibrations are going to be the those induced via tensoring of representables from the classical model structure on topological spaces (giving the strict model structure), together with an additional set of morphisms to the generating acylic cofibrations that will force fibrant objects to be Omega-spectra. To that end we need the following little preliminary.

Definition 4.2. For $n \in \mathbb{N}$ let

$$\lambda_n : F_{n+1} S^1 \overset{k_n}{\longrightarrow} \mathrm{Cyl}(\lambda_n) \longrightarrow F_n S^0$$

be the factorization as in the factorization lemma of the morphism λ_n of lemma 2.30 through its mapping cylinder (prop.) formed with respect to the standard cylinder spectrum $(F_{n+1}S^1) \wedge (I_+)$:

Notice that:

Lemma 4.3. *The factorization in def. 4.2 is through a cofibration followed followed by a left homotopy equivalence in $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}}$*

Proof. Since the cell S^1 is cofibrant in $(\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}$, and since $F_{n+1}(-)$ is a left Quillen functor by prop. 3.18, the free spectrum $F_{n+1}S^1$ is cofibrant in $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}}$. Therefore lemma 3.8 says that its standard cylinder spectrum is a good cylinder object and then the factorization lemma (lemma) says that k_n is a cofibration. Moreover, the morphism out of the standard mapping cylinder is a homotopy equivalence, with homotopies induced under tensoring from the standard homotopy contracting the standard cylinder. ▮

With this we may state the classes of morphisms that are going to be shown to be the classes of generating (acyclic) cofibrations for the stable model structures:

Definition 4.4. Recall the sets of generating (acyclic) cofibrations of the strict model structre def. 3.2. Set

$$I^{\mathrm{stable}}_{\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(\mathrm{Top}_{\mathrm{cg}})} \coloneqq I^{\mathrm{strict}}_{\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(\mathrm{Top}_{\mathrm{cg}})}$$

and

$$J^{\mathrm{stable}}_{\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(\mathrm{Top}_{\mathrm{cg}})} \coloneqq J^{\mathrm{strict}}_{\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(\mathrm{Top}_{\mathrm{cg}})} \sqcup \{k_n \,\Box\, i_+\}_{\substack{n \in \mathbb{N} \\ i \in I}}$$

for the disjoint union of the strict acyclic generating cofibration with the pushout products under smash tensoring of the resolved maps k_n from def. 4.2 with the elements in I.

(MMSS 00, def.6.2, def. 9.3)

Lemma 4.5. *Let $\mathrm{Dia} \in \{\mathrm{Top}^{*/}_{\mathrm{cg,fin}}, \mathrm{Orth}, \mathrm{Seq}\}$ (but not Sym). Then every element in $J^{\mathrm{stable}}_{\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(\mathrm{Top}_{\mathrm{cg}})}$ (def. 4.4) is both:*

1. *a cofibration with respect to the strict model structure (prop. 3.1);*
2. *a stable weak homotopy equivalence (def. 2.21).*

Proof. First regarding strict cofibrations:

By the Yoneda lemma, the elements in J have right lifting property against the strict fibrations, hence in particular they are strict cofibrations. Moreover, by Joyal-Tierney calculus (prop.), $k_n \,\Box\, i_+$ has left lifting against any acyclic strict fibration f precisely if k_n has left lifting against $f^{\Box i}$. By prop. 3.5 the latter is still a strict acyclic fibration. Since k_n by construction is a strict cofibration, the lifting follows and hence also $k_n \,\Box\, i_+$ is a strict cofibration.

Now regarding stable weak homotopy equivalences:

The morphisms in J^{strict} by design are strict weak equivalences, hence they are in particular stable weak homotopy equivalences. The morphisms k_n are stable weak homotopy equivalences by lemma 2.33 and by two-out-of-three.

To see that also the pushout products $k_n \Box (i_n)_+$ are stable weak homotopy equivalences. (e.g. Mandell-May 02, p.46):

First $k_n \wedge (S^{n-1})_+$ is still a stable weak homotopy equivalence, by lemma. 2.23.

Moreover, observe that $\mathrm{dom}(k_n) \wedge i_+$ is degreewise a relative cell complex inclusion, hence degreewise a cofibration in the classical model structure on pointed topological spaces. This follows from lemma 2.28, which says that $\mathrm{dom}(k_n) \wedge i_+$ is degreewise the smash product of a CW complex with i_+, and from the fact that smashing with CW-complexes is a left Quillen functor $(\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}} \longrightarrow (\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}$ (prop.) and hence preserves cofibrations.

Altogether this implies by lemma 2.24 that the pushout of the stable weak homotopy equivalence $k_n \wedge (S^{n-1})_+$ along the degreewise cofibration $\mathrm{dom}(k_n) \wedge i_+$ is still a stable weak homtopy equivalence, and so the pushout product $k_n \Box i_+$ is, too, by two-out-of-three. ▮

The point of the class K in def. 3.2 is to make the following true:

Lemma 4.6. *A morphism $f : X \to Y$ in $\mathbb{S}_{\mathrm{dia}}\mathrm{Mod}$ is a J^{stable}-injective morphism (for K from def. 4.4) precisely if*

1. *it is a fibration in the strict model structure (hence degreewise a fibration);*
2. *for all $n \in \mathbb{N}$ the commuting squares of structure map compatibility on the underlying sequential spectra*

$$\begin{array}{ccc} X_n & \overset{\tilde\sigma}{\longrightarrow} & \Omega X_{n+1} \\ \downarrow && \downarrow \\ Y_n & \underset{\tilde\sigma}{\longrightarrow} & \Omega Y_{n+1} \end{array}$$

are homotopy pullbacks (def.).

(MMSS 00, prop. 9.5)

Proof. By prop 3.3, lifting against J^{strict} alone characterizes strict fibrations, hence degreewise fibrations. Lifting against the remaining pushout product morphism $k_n \Box i_+$ is, by Joyal-Tierney calculus, equivalent to left lifting i_+ against the dual pullback product of $f^{\Box k_n}$, which means that $f^{\Box k_n}$ is a weak homotopy equivalence. But by construction of k_n and by lemma 2.30, $f^{\Box k_n}$ is the comparison morphism into the homotopy pullback under consideration. ▮

Corollary 4.7. *The J^{stable}-injective objects are precisely the Omega-spectra (def. 2.21).*

Lemma 4.8. *A morphism in $\mathbb{S}_{\mathrm{dia}}\mathrm{Mod}$ which is both*

1. *a stable weak homotopy equivalence (def. 2.21);*
2. *a J^{stable}-injective morphisms*

is an acyclic fibration in the strict model structure of prop. 3.1, hence is degreewise a weak homotopy equivalence and Serre fibration of topological spaces;

(MMSS 00, corollary 9.8)

Proof. Let $f : X \to B$ be both a stable weak homotopy equivalence as well as a J^{stable}-injective morphism. Since J^{stable} contains, by prop. 3.3, the generating acyclic cofibrations for the strict model structure of prop. 3.1, f is in particular a strict fibration, hence a degreewise fibration. Therefore the fiber F of f is its homotopy fiber in the strict model structure.

Hence by lemma 2.22 there is an exact sequence of stable homotopy groups of the form

$$\pi_{\bullet+1}(X) \overset{\pi_{\bullet+1}(f)}{\longrightarrow} \pi_{\bullet+1}(Y) \longrightarrow \pi_\bullet(F) \longrightarrow \pi_\bullet(X) \overset{\pi_\bullet(f)}{\longrightarrow} \pi_\bullet(Y) \,.$$

By exactness and by the assumption that $\pi_\bullet(f)$ is an isomorphism, this implies that $\pi_\bullet(F) \simeq 0$, hence that $F \to *$ is a stable weak homotopy equivalence.

Observe also that F, being the pullback of a J^{stable}-injective morphisms (by the standard closure properties) is a J^{stable}-injective object, so that by corollary 4.7 F is an Omega-spectrum. Since stable weak homotopy equivalences between Omega-spectra are already degreewise weak homotopy equivalences, together this says that $F \to *$ is a weak equivalence in the strict model structure, hence degreewise a weak homotopy equivalence. From this the long exact sequence of homotopy groups implies that $\pi_{\bullet \geq 1}(f_n)$ is a weak homotopy equivalence for all n and for each homotopy group in positive degree.

To deduce the remaining case that also $\pi_0(f_0)$ is an isomorphism, observe that, by assumption of

J^{stable}-injectivity, lemma 4.6 gives that f_n is a homotopy pullback (in topological spaces) of $\Omega(f_{n+1})$. But, by the above, $\Omega(f_{n+1})$ is a weak homotopy equivalence, since $\pi_\bullet(\Omega(-)) = \pi_{\bullet+1}(-)$. Therefore f_n is the homotopy pullback of a weak homotopy equivalence and hence itself a weak homotopy equivalence. ▮

Lemma 4.9. *The retracts of J^{stable}-relative cell complexes are precisely the morphisms which are*

1. *stable weak homotopy equivalences (def. 2.21),*
2. *as well as cofibrations with respect to the strict model structure of prop. 3.1.*

(MMSS 00, prop. 9.9 (i))

Proof. Since all elements of J^{stable} are stable weak homotopy equivalences as well as strict cofibrations by lemma 4.5, it follows that every retract of a relative K-cell complex has the same property.

In the other direction, if f is a stable weak homotopy equivalence and a strict cofibration, by the small object argument it factors $f \colon \overset{i}{\to} \overset{p}{\to}$ as a relative J^{stable}-cell complex i followed by a J^{stable}-injective morphism p. By the previous statement i is a stable weak homotopy equivalence, and so by assumption and by two-out-of-three so is p. Therefore lemma 4.8 implies that p is a strict acyclic fibration. But then the assumption that f is a strict cofibration means that it has the left lifting property against p, and so the retract argument implies that f is a retract of the relative K-cell complex i. ▮

Corollary 4.10. *The J^{stable}-injective morphisms are precisely those which are injective with respect to the cofibrations of the strict model structure that are also stable weak homotopy equivalences.*

(MMSS 00, prop. 9.9 (ii))

Lemma 4.11. *A morphism in $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}$ (for $\mathrm{Diq} \neq \mathrm{Sym}$) is both*

1. *a stable weak homotopy equivalence (def. \ref{StableEquivalencesForDiagramSpectra})*
2. *injective with respect to the cofibrations of the strict model structure that are also stable weak homotopy equivalences;*

precisely if it is an acylic fibration in the strict model structure of theorem 3.1.

(MMSS 00, prop. 9.9 (iii))

Proof. Every acyclic fibration in the strict model structure is injective with respect to strict cofibrations by the strict model structure; and it is a clearly a stable weak homotopy equivalence.

Conversely, a morphism injective with respect to strict cofibrations that are stable weak homotopy equivalences is a J^{stable}-injective morphism by corollary 4.10, and hence if it is also a stable equivalence then by lemma 4.8 it is a strict acylic fibration. ▮

Proof. (of theorem 4.1)

The non-trivial points to check are the two weak factorization systems.

That $(\mathrm{cof}_{\mathrm{stable}} \cap \mathrm{weq}_{\mathrm{stable}}\,,\, \mathrm{fib}_{\mathrm{stable}})$ is a weak factorization system follows from lemma 4.9 and the small object argument.

By lemma 4.11 the stable acyclic fibrations are equivalently the strict acyclic fibrations and hence the weak factorization system $(\mathrm{cof}_{\mathrm{stable}}\,,\, \mathrm{fib}_{\mathrm{stable}} \cap \mathrm{we}_{\mathrm{stable}})$ is identified with that of the strict model structure $(\mathrm{cof}_{\mathrm{strict}}\,,\, \mathrm{fib}_{\mathrm{strict}} \cap \mathrm{we}_{\mathrm{strict}})$. ▮

Stability of the homotopy theory

We show now that the model structure on orthogonal spectra $\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}$ from theorem 4.1 is Quillen equivalent (def.) to the stable model structure on topological sequential spectra $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}$ (thm.), hence that they model the same stable homotopy theory.

Theorem 4.12. *The free-forgetful adjunction $(\mathrm{seq}_! \dashv \mathrm{seq}^*)$ of def. 2.4 and theorem 3.1 is a Quillen equivalence (def.) between the stable model structure on topological sequential spectra (thm.) and the stable model structure on orthogonal spectra from theorem 4.1.*

$$\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}} \underset{\mathrm{seq}^*}{\overset{\mathrm{seq}_!}{\underset{\longrightarrow}{\overset{\longleftarrow}{\simeq_{\mathrm{Quillen}}}}}} \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}$$

(MMSS 00, theorem 10.4)

Proof. Since the forgetful functor seq^* “creates weak equivalences”, in that a morphism of orthogonal spectra is a weak equivalence precisely if the underlying morphism of sequential spectra is (by def. 2.21) it is sufficient to show (by this prop.) that for every cofibrant sequential spectrum X, the adjunction unit

$$X \longrightarrow \mathrm{seq}^* \mathrm{seq}_! X$$

is a stable weak homotopy equivalence.

By cofibrant generation of the stable model structure on topological sequential spectra $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}$ (thm.) every cofibrant sequential spectrum is a retract of an $I_{\mathrm{seq}}^{\mathrm{stable}}$-relative cell complex (def., def.), where

$$I_{\mathrm{seq}}^{\mathrm{stable}} = \left\{ F_{n_1} S_+^{n_2-1} \overset{F_{n_1}(i_{n_2})_+}{\longrightarrow} F_{n_1} D_+^{n_2} \right\} .$$

Since $\mathrm{seq}_!$ and seq^* both preserve colimits (seq^* because it evaluates at objects and colimits in the diagram category $\mathrm{OrthSpec}$ are computed objectwise, and $\mathrm{seq}_!$ because it is a left adjoint) we have for $X \simeq \underset{\longrightarrow i}{\lim} X_i$ a relative $I_{\mathrm{seq}}^{\mathrm{stable}}$-decompositon of X, that $\eta_X : X \to \mathrm{seq}^* \mathrm{seq}_! X$ is equivalently

$$\underset{\longrightarrow i}{\lim} \eta_{X_i} \;:\; \underset{\longrightarrow i}{\lim} X_i \longrightarrow \underset{\longrightarrow i}{\lim} \mathrm{seq}_! \mathrm{seq}^* X_i \;.$$

Now observe that the colimits involved in a relative $I_{\mathrm{seq}}^{\mathrm{stable}}$-complex (the coproducts, pushouts, transfinite compositions) are all homotopy colimits (def.): First, all objects involved are cofibrant. Now for the transfinite composition all the morphisms involved are cofibrations, so that their colimit is a homotopy colimit by this example, while for the pushout one of the morphisms out of the “top” objects is a cofibration, so that this is a homotopy pushout by (def.).

It follows that if all η_{X_i} are weak equivalences, then so is $\eta = \underset{\longrightarrow i}{\lim} \eta_{X_i}$.

Unwinding this, one finds that it is sufficient to show that

$$\eta_{F_{n_1} S^{n_2}} \;:\; F_{n_1} S_+^{n_2} \longrightarrow \mathrm{seq}^* \mathrm{seq}_! F_{n_1} S^{n_2}$$

is a stable weak homotopy equivalence for all $n_1, n_2 \in \mathbb{N}$.

Consider this for $n_2 \geq n_2$. Then there are canonical morphisms

$$F_{n_1} S^{n_2} \longrightarrow F_0 S^{n_2 - n_1}$$

whose components in degree $q \geq n_1$ are the identity. These are the composites of the maps $\lambda_k \wedge S^{k + n_2 - n_1}$ for $k < n_1$ with λ_n from def. \reg{CorepresentationOfAdjunctsOfStructureMaps}. By prop. 2.33 also $\mathrm{seq}^* \mathrm{seq}_! \lambda_n$ are weak homotopy equivalences. Hence we have commuting diagrams of the form

$$\begin{array}{ccc} F_{n_1}^{\mathrm{seq}} S^{n_2} & \longrightarrow & F_0 S^{n_2 - n_1} \\ {}^{\eta}\downarrow && \downarrow^{\simeq} \\ \mathrm{seq}^* F_{n_1}^{\mathrm{orth}} S^{n_2 - n_1} & \longrightarrow & \mathrm{seq}^* F_0^{\mathrm{orth}} S^{n_2 - n_1} \end{array} \,,$$

where the horizontal maps are stable weak homotopy equivalences by the previous argument and the right vertical morphism is an isomorphism by the formula in prop. 2.27.Hence the left vertical morphism is a stable weak homotopy equivalence by two-out-of-three.

If $n_2 < n_1$ then one reduces this to the above case by smashing with $S^{n_1 - n_2}$. ▮

Remark 4.13. Theorem 4.12 means that the homotopy categories of $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}$ and $\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}$ are equivalent (prop.) via

$$\mathrm{Ho}(\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}) \underset{\mathbb{R}\,\mathrm{seq}^*}{\overset{\mathbb{L}\,\mathrm{seq}_!}{\underset{\longrightarrow}{\overset{\longleftarrow}{\simeq}}}} \mathrm{Ho}(\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}) \;.$$

Since $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}$ is a stable model category (thm.) in that the derived suspension looping adjunction is an equivalence of categories, and and since this is a condition only on the homotopy categories, and since $\mathbb{R}\,\mathrm{seq}^{\mathrm{ast}}$ manifestly preserves the construction of loop space objects, this implies that we have a commuting square of adjoint equivalences of homotopy categories

$$\begin{array}{ccc} \mathrm{Ho}(\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}) & \underset{\Omega}{\overset{\Sigma}{\underset{\longrightarrow}{\overset{\longleftarrow}{\simeq}}}} & \mathrm{Ho}(\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}) \\ {}^{\mathbb{L}\,\mathrm{seq}_!}\downarrow \simeq \uparrow^{\mathbb{R}\,\mathrm{seq}^*} && {}^{\mathbb{L}\,\mathrm{seq}_!}\downarrow \simeq \uparrow^{\mathbb{R}\,\mathrm{seq}^*} \\ \mathrm{Ho}(\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}) & \underset{\Omega}{\overset{\Sigma}{\underset{\longrightarrow}{\overset{\longleftarrow}{\simeq}}}} & \mathrm{Ho}(\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}) \end{array}$$

and so in particular also $\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}$ is a stable model category.

Due to the vertical equivalences here we will usually not distinguish between these homotopy categories and just speak of the *stable homotopy category* (def.)

$$\mathrm{Ho}(\mathrm{Spectra}) \coloneqq \mathrm{Ho}(\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}) \simeq \mathrm{Ho}(\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}) \,.$$

Monoidal model structure

We now discuss that the monoidal model category structure of the strict model structure on orthogonal spectra $\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}}$ (theorem 3.17) remains intact as we pass to the stable model structure $\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}$ of theorem 4.1.

Theorem 4.14. *The stable model structure $\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}$ of theorem 4.1 equipped with the symmetric monoidal smash product of spectra (def. 2.9) is a monoidal model category (def. 3.11) with cofibrant tensor unit*

$$(\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}}),\ \wedge = \otimes_{\mathbb{S}_{\mathrm{orth}}},\ \mathbb{S}_{\mathrm{orth}}) \,.$$

(MMSS 00, prop. 12.6)

Proof. Since $\mathrm{Cof}_{\mathrm{stable}} = \mathrm{Cof}_{\mathrm{strict}}$, the fact that the pushout product of two stable cofibrations is again a stable cofibration is part of theorem 3.17.

It remains to show that if at least one of them is a stable weak homotopy equivalence (def. 2.21), then so is the pushout-product.

Since $\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})$ is a cofibrantly generated model category by theorem 4.1 and since it has internal homs (mapping spectra) with respect to $\otimes_{\mathbb{S}_{\mathrm{dia}}}$ (prop. 1.45), it suffices (as in the proof of this prop.) to check this on generating (acylic) cofibrations, i.e. to check that

$$I^{\mathrm{stable}} \square_{\otimes} J^{\mathrm{stable}} \subset W_{\mathrm{stable}} \cap \mathrm{Cof}_{\mathrm{stable}} \,.$$

Now $I^{\mathrm{stable}} = I^{\mathrm{strict}}$ and $J^{\mathrm{stable}} = J^{\mathrm{strict}} \sqcup \{k_n \square i_+\}$ so that the special case

$$\begin{aligned} I^{\mathrm{stable}} \square_{\otimes} J^{\mathrm{strict}} &= I^{\mathrm{strict}} \square_{\otimes} J^{\mathrm{strict}} \\ &\subset W_{\mathrm{strict}} \cap \mathrm{Cof}_{\mathrm{strict}} \\ &\subset W_{\mathrm{stable}} \cap \mathrm{Cof}_{\mathrm{stable}} \end{aligned}$$

follows again from the monoidal stucture on the strict model category of theorem 3.17.

It hence remains to see that

$$I^{\mathrm{strict}} \square_{\otimes} (k_{n_1} \square (i_{n_2})_+) \subset W_{\mathrm{stable}} \cap \mathrm{Cof}_{\mathrm{stable}}$$

for all $n_1, n_2 \in \mathbb{N}$.

By lemma 4.5 $k_n \square i_+$ is in $\mathrm{Cof}_{\mathrm{strict}}$ and hence

$$I^{\mathrm{strict}} \square_{\otimes} (k_{n_1} \square (i_{n_2})_+) \subset \mathrm{Cof}_{\mathrm{strict}}$$

follows, once more, from the monoidalness of the strict model structure.

Hence it only remains to show that

$$I^{\mathrm{strict}} \square_{\otimes} (k_{n_1} \square (i_{n_2})_+) \subset W_{\mathrm{stable}} \,.$$

This we now prove by inspection:

By two-out-of-three applied to the definition of the pushout product, it is sufficient to show that for every $F_{n_3}(i_{n_4})_+$ in I^{strict}, the right vertical morphism in the pushout diagram

$$\begin{array}{ccc} & \xrightarrow{F_{n_3}(i_{n_4})_+ \otimes \mathrm{dom}(k_{n_1} \square (i_{n_2})_+)} & \\ {}^{\mathrm{dom}(F_{n_3}(i_{n_4}) \otimes (k_{n_1} \square (i_{n_2})_+)}\downarrow & (\mathrm{po}) & \downarrow \\ & \longrightarrow & \end{array}$$

is a stable weak homotopy equivalence. Since seq^* preserves pushouts, we may equivalently check this on the underlying sequential spectra.

Consider first the top horizontal morphism in this square.

We may rewrite it as

$$
\begin{aligned}
F_{n_3}(i_{n_4})_+ \otimes (\mathrm{dom}(k_{n_1}) \,\Box\, (i_{n_2})_+) & \simeq F_{n_3}(i_{n_4})_+ \otimes \left(F_{n_1} S^0 \wedge S^{n_2-1}_+ \underset{F_{n_1+1} S^1 \wedge S_+^{n_2 - 1}}{\sqcup} F_{n_1+1} S^1 \wedge D^{n_2}_+ \right) \\
& \simeq F_{n_3}(i_{n_4})_+ \otimes F_{n_1} S^0 \wedge S^{n_2-1}_+ \underset{F_{n_3}(i_{n_4})_+ \otimes F_{n_1+1} S^1 \wedge S_+^{n_2 - 1}}{\sqcup} F_{n_3}(i_{n_4})_+ \otimes F_{n_1+1} S^1 \wedge D^{n_2}_+, \\
& \simeq F_{n_1+n_3}(i_{n_4})_+ \wedge S^{n_2-1}_+ \underset{F_{n_1+n_3+1}(i_{n_4})_+ \wedge S_+^{n_2-1}}{\sqcup} F_{n_1+n_3+1}(i_{n_4})_+ \wedge S^1 \wedge D^{n_2}_+
\end{aligned}
$$

where we used that $X \otimes (-)$ is a left adjoint and hence preserves colimits, and we used prop. 2.29 to evaluate the smash product of free spectra.

Now by lemma 2.28 the morphism

$$
F_{n_1+n_3+1} S^{n_4-1}_+ \wedge S^1 \wedge S^{n_2-1}_+ \longrightarrow F_{n_1+n_3+1} S^{n_4-1}_+ \wedge S^1 \wedge D^{n_2}_+
$$

is degreewise the smash product of a CW-complex with a relative cell complex inclusion, hence is itself degreewise a relative cell complex inclusion, and therefore its pushout

$$
F_{n+1+n_3} S^{n_4-1}_+ \otimes F_{n_1} S^0 \wedge S^{n_2-1}_+ \longrightarrow F_{n_3}(S^{n_4-1})_+ \otimes \mathrm{dom}(k_{n_1} \,\Box\, (i_{n_2})_+)
$$

is degreewise a retract of a relative cell complex inclusion. But since it is the identity on the smash factor $S^{n_4-1}_+$ in the argument of the free spectra as above, the morphism is degreewise the smash tensoring with S^{n_4-1} of a retract of a relative cell complex inclusion. Since the domain is degreewise a CW-complex by lemma 2.28, $F_{n_3}(S^{n_4-1})_+ \otimes \mathrm{dom}(k_{n_1} \,\Box\, (i_{n_2})_+)$ is degreewise the smash tensoring with $S^{n_4-1}_+$ of a retract of a cell complex.

The same argument applies to the domain of $F_{n_3}(i_4)_+ \otimes (\mathrm{dom}(k_n) \,\Box\, (i_2)_+)$, and so in conclusion this morphism is degreewise the smash product of a cofibration with a cofibrant object in $(\mathrm{Top}^{*/}_{\mathrm{cg}})_{\mathrm{Quillen}}$, and hence is itself degreewise a cofibration.

Now consider the vertical morphism in the above square

The same argument that we just used shows that this is the smash tensoring of the stable weak homotopy equivalence $k_{n_1} \,\Box\, (i_{n_2})_+$ with a CW-complex. Hence by lemma 2.23 the left vertical morphism is a stable weak homotopy equivalence.

In conclusion, the right vertical morphism is the pushout of a stable weak homotopy equivalence along a degreewise cofibration of pointed topological spaces. Hence lemma 2.24 implies that it is itself a stable weak homotopy equivalence. ▮

Corollary 4.15. *The strong monoidal Quillen adjunction (def. 3.15) $(\Sigma^\infty_{\mathrm{orth}} \dashv \Omega^\infty_{\mathrm{orth}})$ on the strict model structure (prop. 3.18) descends to a strong monoidal Quillen adjunction on the stable monoidal model category from theorem 4.14:*

$$
\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}} \underoverset{\underset{\Omega^\infty_{\mathrm{orth}}}{\longrightarrow}}{\overset{\Sigma^\infty_{\mathrm{orth}}}{\longleftarrow}}{\bot} (\mathrm{Top}^{*/}_{\mathrm{cg}}, \wedge, S^0)_{\mathrm{Quillen}} \,.
$$

Proof. The stable model structure $\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}$ is a left Bousfield localization of the strict model structure (def.) in that it has the same cofibrations and a larger class of acyclic cofibrations. Hence $\Sigma^\infty_{\mathrm{orth}}$ is still a left Quillen functor also to the stable model structure. ▮

5. The monoidal stable homotopy category

We discuss now the consequences for the stable homotopy category (def.) of the fact that by theorem 4.12 and theorem 4.14 it is equivalently the homotopy category of a stable monoidal model category. This makes the stable homotopy category become a *tensor triangulated category* (def. 5.3) below. The abstract structure encoded by this governs much of stable homotopy theory (Hovey-Palmieri-Strickland 97). In particular it is this structure that gives rise to the E-Adams spectral sequences which we discuss in Part 2.

Corollary 5.1. *The stable homotopy category* $\mathrm{Ho}(\mathrm{Spectra})$ *(remark 4.13) inherits the structure of a symmetric monoidal category*

$$
(\mathrm{Ho}(\mathrm{Spectra}), \wedge^L, \mathbb{S} \coloneqq \gamma(\mathbb{S}_{\mathrm{orth}}))
$$

with tensor product the left derived functor $\wedge^L$ *of the symmetric monoidal smash product of spectra (def. 2.9, def. 2.13, prop. 2.14) and with tensor unit the sphere spectrum* $\mathbb{S}$ *(the image in* $\mathrm{Ho}(\mathrm{Spectra})$ *of any of the structured sphere spectra from def. 2.4).*

Moreover, the localization functor (def.) is a lax monoidal functor

$$\gamma \colon (\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}}), \wedge, \mathbb{S}_{\mathrm{orth}}) \longrightarrow (\mathrm{Ho}(\mathrm{Spectra}), \wedge^L, \gamma(\mathbb{S})) \;.$$

Proof. In view of theorem 4.14 this is a special case of prop. 3.13. ▮

Remark 5.2. Let $A, X \in \mathrm{Ho}(\mathrm{Spectra})$ be two spectra in the stable homotopy category, then the stable homotopy groups (def.) of their derived symmetric monoidal smash product of spectra (corollary 5.1) is also called the *generalized homology* of X with coefficients in A and denoted

$$A_\bullet(X) \coloneqq \pi_\bullet(A \wedge X) \;.$$

This is conceptually dual to the concept of generalized (Eilenberg-Steenrod) cohomology (example)

$$A^\bullet(X) \coloneqq [X, A]_\bullet \;.$$

Notice that (def., lemma)

$$\begin{aligned} A_\bullet(X) &= \pi_\bullet(A \wedge X) \\ &\simeq [\mathbb{S}, A \wedge X]_\bullet \end{aligned} \;.$$

In the special case that $X = \Sigma^\infty K$ is a suspension spectrum, then

$$A_\bullet(X) \simeq \pi_\bullet(A \wedge K)$$

(by prop. 2.29) and this is called the *generalized A-homology* of the topological space $K \in \mathrm{Top}_{\mathrm{cg}}^{*/}$.

Since the sphere spectrum $\mathbb{S}$ is the tensor unit for the derived smash product of spectra (corollary 5.1) we have

$$E_\bullet(\mathbb{S}) \simeq \pi_\bullet(E) \;.$$

For that reason often one also writes for short

$$E_\bullet \coloneqq \pi_\bullet(E) \;.$$

Notice that similarly the E-generalized cohomology (exmpl.) of the sphere spectrum is

$$\begin{aligned} E^\bullet &\coloneqq E^\bullet(\mathbb{S}) \\ &= [\mathbb{S}, E]_{-\bullet} \\ &\simeq \pi_{-\bullet}(E) \\ &\simeq E_{-\bullet} \end{aligned} \;.$$

(Beware that, as usual, here we are not displaying a tilde-symbol to indicate reduced cohomology).

Tensor triangulated structure

We discuss that the derived smash product of spectra from corollary 5.1 on the stable homotopy category interacts well with its structure of a triangulated category (def.).

Definition 5.3. A **tensor triangulated category** is a category Ho equipped with

1. the structure of a symmetric monoidal category $(\mathrm{Ho}, \otimes, 1, \tau)$ (def. 1.5);

2. the structure of a triangulated category $(\mathrm{Ho}, \Sigma, \mathrm{CofSeq})$ (def.);

3. for all objects $X, Y \in \mathrm{Ho}$ natural isomorphisms

$$e_{X,Y} \colon (\Sigma X) \otimes Y \overset{\simeq}{\longrightarrow} \Sigma(X \otimes Y)$$

such that

1. (tensor product is additive) for all $V \in \mathrm{Ho}$ the functors $V \otimes (-) \simeq (-) \otimes V$ preserve finite direct sums (are additive functors);

2. (tensor product is exact) for each object $V \in \mathrm{Ho}$ the functors $V \otimes (-) \simeq (-) \otimes V$ preserves distinguished triangles in that for

$$X \overset{f}{\longrightarrow} X \overset{g}{\longrightarrow} Y \overset{h}{\longrightarrow} \Sigma X$$

in CofSeq, then also

$$V \otimes X \overset{\mathrm{id}_V \otimes f}{\longrightarrow} V \otimes X \overset{\mathrm{id}_V \otimes g}{\longrightarrow} V \otimes Y \overset{\mathrm{id}_V \otimes h}{\longrightarrow} V \otimes (\Sigma X) \simeq \Sigma(V \otimes X)$$

is in CofSeq, where the equivalence at the end is $e_{X,V} \circ \tau_{V,\Sigma Y}$.

Jointly this says that for all objects V the equivalences e give $V \otimes (-)$ the structure of a *triangulated functor*.

(Balmer 05, def. 1.1)

In addition we ask that

1. (coherence) for all $X, Y, Z \in \mathrm{Ho}$ the following diagram commutes

$$\begin{array}{ccccc} (\Sigma(X) \otimes Y) \otimes Z & \overset{e_{X,Y} \otimes \mathrm{id}}{\longrightarrow} & (\Sigma(X \otimes Y)) \otimes Z & \overset{e_{X \otimes Y, Z}}{\longrightarrow} & \Sigma((X \otimes Y) \otimes Z) \\ {}^{\alpha_{\Sigma X, Y, Z}}\downarrow && && \downarrow^{\Sigma \alpha_{X,Y,Z}} \\ \Sigma(X) \otimes (Y \otimes Z) && \underset{e_{X, Y \otimes Z}}{\longrightarrow} && \Sigma(X \otimes (Y \otimes Z)) \end{array} ,$$

where α is the associator of $(\mathrm{Ho}, \otimes, 1)$.

2. (graded commutativity) for all $n_1, n_2 \in \mathbb{Z}$ the following diagram commutes

$$\begin{array}{ccc} (\Sigma^{n_1} 1) \otimes (\Sigma^{n_2} 1) & \overset{\simeq}{\longrightarrow} & \Sigma^{n_1 + n_2} 1 \\ {}^{\tau_{\Sigma^{n_1} 1, \Sigma^{n_2} 1}}\downarrow && \downarrow^{(-1)^{n_1 \cdot n_2}} \\ (\Sigma^{n_2} 1) \otimes (\Sigma^{n_1} 1) & \underset{\simeq}{\longrightarrow} & \Sigma^{n_1 + n_2} 1 \end{array} ,$$

where the horizontal isomorphisms are composites of the $e_{\cdot,\cdot}$ and the braidings.

(Hovey-Palmieri-Strickland 97, def. A.2.1)

Proposition 5.4. *The stable homotopy category* $\mathrm{Ho}(\mathrm{Spectra})$ *(def.) equipped with*

1. *its triangulated category structure* $(\mathrm{Ho}(\mathrm{Spectra}), \Sigma, \mathrm{CofSeq})$ *for distinguished triangles the homotopy cofiber sequences (prop.);*

2. *the derived symmetric monoidal smash product of spectra* $(\mathrm{Ho}(\mathrm{Spectra}), \wedge^L, \mathbb{S})$ *(corollary 5.1)*

is a tensor triangulated category in the sense of def. 5.3.

(e.g. Hovey-Palmieri-Strickland 97, 9.4)

We break up the **proof** into lemma 5.5, lemma 5.6, lemma 5.7 and lemma 5.9.

Lemma 5.5. *For* $V \in \mathrm{Ho}(\mathrm{Spectra})$ *any spectrum in the stable homotopy category (remark 4.13), then the derived symmetric monoidal smash product of spectra (corollary 5.1)*

$$V \wedge^L (-) \;:\; \mathrm{Ho}(\mathrm{Spectra}) \longrightarrow \mathrm{Ho}(\mathrm{Spectra})$$

preserves direct sums, in that for all $X, Y \in \mathrm{Ho}(\mathrm{Spectra})$ *then*

$$V \wedge^L (X \oplus Y) \simeq (V \wedge^L X) \oplus (V \wedge^L Y) \,.$$

Proof. The direct sum in $\mathrm{Ho}(\mathrm{Spectra})$ is represented by the wedge sum in $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$ (prop., prop.). Since wedge sum of sequential spectra is the coproduct in $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$ (exmpl.) and since the forgetful functor $\mathrm{seq}^\ast : \mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}}) \longrightarrow \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$ preserves colimits (since by prop. 2.19 it acts by precomposition on functor categories, and since for these colimits are computed objectwise), it follows that also wedge sum of orthogonal spectra represents the direct sum operation in the stable homotopy category.

Now assume without restriction that V, X and Y are cofibrant orthogonal spectra representing the objects of the same name in the stable homotopy catgeory. Since wedge sum is coproduct, it follows that also the wedge sum $X \vee Y$ is cofibrant.

Since $V \wedge^L (-)$ is a left Quillen functor by theorem 4.14, it follows that the derived tensor product $V \wedge^L (X \oplus Y)$ is represented by the plain symmetric monoidal smash product of spectra $V \wedge (X \vee Y)$. By def. 2.9 (or more explicitly by prop. 2.14) this is the coequalizer

$$V \otimes_{\mathrm{Day}} \mathbb{S}_{\mathrm{orth}} \otimes_{\mathrm{Day}} (X \vee Y) \underset{\longrightarrow}{\overset{\longrightarrow}{}} V \otimes_{\mathrm{Day}} (X \vee Y) \overset{\mathrm{coeq}}{\longrightarrow} V \otimes_{\mathbb{S}_{\mathrm{orth}}} (X \vee Y) \,.$$

Inserting the definition of Day convolution (def. 1.39), the middle term here is

$$\begin{aligned} \int^{c_1,c_2} \mathrm{Orth}(c_1 \otimes_{\mathrm{Orth}} c_2, -) \wedge V(c_1) \wedge (X \vee Y)(c_2) & \simeq \int^{c_1,c_2} \mathrm{Orth}(c_1 \otimes_{\mathrm{Orth}} c_2, -) \wedge V(c_1) \wedge (X(c_2) \vee Y(c_2)) \\ & \int^{c_1,c_2} \mathrm{Orth}(c_1 \otimes_{\mathrm{Orth}} c_2, -) \wedge V(c_1) \wedge X(c_2) \;\vee\; \int^{c_1,c_2} \mathrm{Orth}(c_1 \otimes_{\mathrm{Orth}} c_2, -) \wedge V(c_1) \wedge Y(c_2)' \\ & \simeq V \otimes_{\mathrm{Day}} X \;\vee\; V \otimes_{\mathrm{Day}} Y \end{aligned}$$

where in the second but last step we used that the smash product in $\mathrm{Top}_{\mathrm{cg}}^{*/}$ distributes over wedge sum and that coends commute with wedge sums (both being colimits).

The analogous analysis applies to the left term in the coequalizer diagram. Hence the whole diagram splits as the wedge sum of the respective diagrams for $V \wedge X$ and $V \wedge Y$. ▮

Lemma 5.6. *For $X \in \mathrm{Ho}(\mathrm{Spectra})$ any spectrum in the stable homotopy category (remark 4.13), then the derived symmetric monoidal smash product of spectra (corollary 5.1)*

$$X \wedge^L (-) \,:\, \mathrm{Ho}(\mathrm{Spectra}) \longrightarrow \mathrm{Ho}(\mathrm{Spectra})$$

preserves homotopy cofiber sequences.

Proof. We may choose a cofibrant representative of X in $\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}$, which we denote by the same symbol. Then the functor

$$X \wedge (-) \,:\, \mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}} \longrightarrow \mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}\,\mathrm{stable}}$$

is a left Quillen functor in that it preserves cofibrations and acyclic cofibrations by theorem 4.14 and it is a left adjoint by prop. 1.22. Hence its left derived functor is equivalently its restriction to cofibrant objects followed by the localization functor.

But now every homotopy cofiber (def.) is represented by the ordinary cofiber of a cofibration. The left Quillen functor preserves both the cofibration as well as its cofiber. ▮

Lemma 5.7. *The canonical suspension functor on the stable homotopy category*

$$\Sigma \,:\, \mathrm{Ho}(\mathrm{Spectra}) \longrightarrow \mathrm{Ho}(\mathrm{Spectra})$$

commutes with forming the derived symmetric monoidal smash product of spectra $\wedge^L$ from corollary 5.1 in that for $X, Y \in \mathrm{Ho}(\mathrm{Spectra})$ any two spectra, then there are isomorphisms

$$\Sigma(X \wedge^L Y) \simeq (\Sigma X) \wedge^L Y \simeq X \wedge^L (\Sigma Y) \,.$$

Proof. By theorem 4.14 the symmetric monoidal smash product of spectra is a left Quillen functor, and by prop. 3.7 and lemma 3.8 the canonical suspension operation is the left derived functor of the left Quillen functor $(-) \wedge S^1$ of smash tensoring with S^1. Therefore all three expressions are represented by application of the underived functors on cofibrant representatives in $\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})$ (the fibrant replacement that is part of the derived functor construction is preserved by left Quillen functors).

So for X and Y cofibrant orthogonal spectra (which we denote by the same symbol as the objects in the homotopy category which they represent), by def. 2.9 (or more explicitly by prop. 2.14), the object $\Sigma(X \wedge^L Y) \in \mathrm{Ho}(\mathrm{Spectra})$ is represented by the coequalizer

$$(X \otimes_{\mathrm{Day}} \mathbb{S}_{\mathrm{orth}} \otimes Y) \wedge S^1 \rightrightarrows (X \otimes_{\mathrm{Day}} Y) \wedge S^1 \overset{\mathrm{coeq}}{\longrightarrow} (X \otimes_{\mathbb{S}_{\mathrm{orth}}} Y) \wedge S^1 \,,$$

where the two morphisms bing coequalized are the images of those of def. 2.9 under smash tensoring with S^1. Now it is sufficient to observe that for any $K \in \mathrm{Top}_{\mathrm{cg}}^{*/}$ we have canonical isomorphisms

$$(X \otimes_{\mathrm{Day}} Y) \wedge K \simeq (X \otimes_{\mathrm{Day}} (Y \wedge K)) \simeq ((X \wedge K) \otimes_{\mathrm{Day}} Y)$$

and similarly for the triple Day tensor product.

This follows directly from the definition of the Day convolution product (def. 1.39)

$$((X \otimes_{\mathrm{Day}} Y) \wedge K)(V) = \int^{V_1, V_2} \mathrm{Orth}(V_1 \oplus V_2, V) \wedge X(V_1) \wedge Y(V_2) \wedge K$$

and the symmetry of the smash product on $\mathrm{Top}_{\mathrm{cg}}^{*/}$ (example 1.10). ▮

Example 5.8. For $A \in \mathrm{Ho}(\mathrm{Spectra})$ a spectrum, then the A-generalized homology (according to remark 5.2) of a suspension of the spectrum is the stable homotopy groups of A in shifted degree:

$$A_\bullet(\Sigma^n \mathbb{S}) \simeq \pi_{\bullet - n}(A) \,.$$

Proof. We compute

$$
\begin{aligned}
A_\bullet(\Sigma^n \mathbb{S}) &\coloneqq \pi_\bullet(A \wedge \Sigma^n \mathbb{S}) \\
&\simeq \pi_\bullet(\Sigma^n (A \wedge \mathbb{S})) \\
&\simeq \pi_\bullet(\Sigma^n A) \\
&\simeq [\mathbb{S}, \Sigma^n A] \\
&= [\mathbb{S}, A]_{-n} \\
&\simeq \pi_{\bullet - n}(A)
\end{aligned}
\,.
$$

Here we use

- first the definition (remark 5.2);
- then the fact that suspension commutes with smash product (lemma 5.7, part of the tensor triangulated structure of prop. 5.4);
- then the fact that the sphere spectrum is the tensor unit of the smash product of spectra (cor. 5.1);
- then the isomorphism of stable homotopy groups with graded homs out of the spjere spectrum (lemma).

▮

Lemma 5.9. *For $n_1, n_2 \in \mathbb{Z}$ then the following diagram commutes in $\mathrm{Ho}(\mathrm{Spectra})$:*

$$
\begin{array}{ccc}
(\Sigma^{n_1}\mathbb{S}) \wedge^L (\Sigma^{n_2}\mathbb{S}) & \overset{\simeq}{\longrightarrow} & \Sigma^{n_1+n_2}\mathbb{S} \\
{}^{\mathllap{\tau_{\Sigma^{n_1}\mathbb{S}, \Sigma^{n_2}\mathbb{S}}}}\downarrow && \downarrow^{\mathrlap{(-1)}} \\
(\Sigma^{n_2}\mathbb{S}) \wedge^L (\Sigma^{n_1}\mathbb{S}) & \underset{\simeq}{\longrightarrow} & \Sigma^{n_1+n_2}\mathbb{S}
\end{array}
\,.
$$

Proof. It is sufficient to prove this for $n_1, n_2 \in \mathbb{N} \hookrightarrow \mathbb{Z}$. From this the general statement follows by looping and using lemma 5.7.

So assume $n_1, n_2 \geq 0$.

Observe that the sphere spectrum $\mathbb{S} = \gamma(\mathbb{S}_{\mathrm{orth}}) \in \mathrm{Ho}(\mathrm{Spectra})$ is represented by the orthogonal sphere spectrum $\mathbb{S}_{\mathrm{orth}} = \Sigma^\infty_{\mathrm{orth}} S^0$ (def. 2.25) and since $\Sigma^\infty_{\mathrm{orth}}$ is a left Quillen functor (prop. 3.18) and $S^0 \in (\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}$ is cofibrant, this is a cofibrant orthogonal spectrum. Hence, as in the proof of lemma 5.7, $\Sigma^{n_1}\mathbb{S}$ is represented by

$$
\mathbb{S} \wedge S^{n_1} \simeq \Sigma^\infty_{\mathrm{orth}} S^{n_1} \,.
$$

Since $\Sigma^\infty_{\mathrm{orth}}$ is a symmetric monoidal functor by prop. 2.29, it makes the following diagram commute

$$
\begin{array}{ccc}
(\mathbb{S} \wedge S^{n_1}) \otimes_{\mathbb{S}_{\mathrm{orth}}} (\mathbb{S} \wedge S^{n_2}) & \overset{\tau^{\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})}_{\mathbb{S}\wedge S^{n_1}, \mathbb{S} \wedge S^{n_2}}}{\longrightarrow} & (\mathbb{S} \wedge S^{n_2}) \otimes_{\mathbb{S}_{\mathrm{orth}}} (\mathbb{S} \wedge S^{n_1}) \\
\downarrow && \downarrow \\
\mathbb{S} \wedge (S^{n_1} \wedge S^{n_2}) & \underset{\mathbb{S}(\tau^{\mathrm{Top}_{\mathrm{cg}}^{*/}}_{S^{n_1}, S^{n_2}})}{\longrightarrow} & \mathbb{S} \wedge (S^{n_2} \wedge S^{n_1})
\end{array}
\,.
$$

Now the homotopy class of $\tau^{\mathrm{Top}_{\mathrm{cg}}^{*/}}_{S^{n_1}, S^{n_2}}$ in

$$
[S^{n_1+n_2}, S^{n_2+n_1}]_* \simeq \pi_{n_1+n_2}(S^{n_1+n_2}) \simeq \mathbb{Z}
$$

is

$$
[\tau^{\mathrm{Top}_{\mathrm{cg}}^{*/}}_{S^{n_1}, S^{n_2}}] = \left\{ \begin{array}{ll} 1 & \text{if } n_1 \cdot n_2 \text{ even} \\ -1 & \text{if } n_1 \cdot n_2 \text{ odd} \end{array} \right. \,.
$$

This translates to $\mathbb{S} \wedge \tau^{\mathrm{Top}_{\mathrm{cg}}^{*/}}_{S^{n_1}, S^{n_2}}$ under the identification (lemma)

$$
[\mathbb{S}, X]_\bullet \simeq \pi_\bullet(X)
$$

and using the adjunction $(-) \wedge (S^{n_1+n_2}) \dashv \mathrm{Maps}(S^{n_1+n_2}, -)_*$ from prop. 1.37:

$$
[\mathbb{S} \wedge (S^{n_1+n_2}), \mathbb{S} \wedge (S^{n_1+n_2})] \simeq [\mathbb{S}, \mathbb{S} \wedge \mathrm{Maps}(S^{n_1+n_2}, S^{n_1+n_2})] \,.
$$

▮

Homotopy ring spectra

We discuss commutative monoids in the tensor triangulated stable homotopy category (prop. 5.4).

In this section the only tensor product that plays a role is the derived smash product of spectra from corollary 5.1. Therefore to ease notation, in this section (and in all of Part 2) we write for short

$$\wedge \coloneqq \wedge^L \;.$$

Definition 5.10. A commutative monoid (E, μ, e) (def. 1.13) in the monoidal stable homotopy category $(\mathrm{Ho}(\mathrm{Spectra}), \wedge, \mathbb{S})$ of corollary 5.1 is called a **homotopy commutative ring spectrum**.

A module object (def. 1.16) over E is accordingly called a **homotopy module spectrum**.

Proposition 5.11. *For (E, μ, e) a homotopy commutative ring spectrum (def. 5.10), its stable homotopy groups (def.)*

$$\pi_\bullet(E)$$

canonically inherit the structure of a $\mathbb{Z}$-graded-commutative ring.

Moreover, for $X \in \mathrm{Ho}(\mathrm{Spectra})$ any spectrum, then the generalized homology (remark 5.2)

$$E_\bullet(X) \coloneqq \pi_\bullet(E \wedge X)$$

(i.e. the stable homotopy groups of the free module over E on X (prop. 1.20)) canonically inherits the structure of a left graded $\pi_\bullet(E)$-module, and similarly

$$X_\bullet(E) \coloneqq \pi_\bullet(X \wedge E)$$

canonically inherits the structure of a right graded $\pi_\bullet(E)$-module.

Proof. Under the identification (lemma)

$$\begin{aligned} \pi_\bullet(E) &\simeq [\mathbb{S}, E]_\bullet \\ &\simeq [\mathbb{S}, \Sigma^{-\bullet} E] \\ &\simeq [\Sigma^\bullet \mathbb{S}, E] \end{aligned}$$

let

$$\alpha_i \;:\; \Sigma^{n_i} \mathbb{S} \longrightarrow E$$

for $i \in \{1, 2\}$ be two elements of $\pi_\bullet(E)$.

Observe that there is a canonical identification

$$\Sigma^{n_1 + n_2} \mathbb{S} \simeq \Sigma^{n_1} \mathbb{S} \wedge \Sigma^{n_2} \mathbb{S}$$

since $\mathbb{S} \simeq \mathbb{S} \wedge \mathbb{S}$ is the tensor unit (cor. 5.1, lemma 1.2) using lemma 5.7 (part of the tensor triangulated structure from prop. 5.4). With this we may form the composite

$$\alpha_1 \cdot \alpha_2 \;:\; \Sigma^{n_1 + n_2} \mathbb{S} \overset{\simeq}{\longrightarrow} \Sigma^{n_1} \mathbb{S} \wedge \Sigma^{n_2} \mathbb{S} \overset{\alpha_1 \wedge \alpha_2}{\longrightarrow} E \wedge E \overset{\mu}{\longrightarrow} E \;.$$

That this pairing is associative and unital follows directly from the associativity and unitality of μ and the coherence of the isomorphism on the left (prop. 5.4). Evidently the pairing is graded. That it is bilinear follows since addition of morphisms in the stable homotopy category is given by forming their direct sum (prop.) and since $\wedge$ distributes over direct sum (lemma 5.5, part of the tensor triangulated structure of prop. 5.4)).

It only remains to show graded-commutivity of the pairing. This is exhibited by the following commuting diagram:

$$\begin{array}{ccc} \Sigma^{n_1 + n_2} \mathbb{S} & \overset{(-1)^{n_1 \cdot n_2}}{\longrightarrow} & \Sigma^{n_1 + n_2} \mathbb{S} \\ {}^{\simeq}\downarrow && \downarrow^{\simeq} \\ \Sigma^{n_1} \mathbb{S} \wedge \Sigma^{n_2} \mathbb{S} & \overset{\tau_{\Sigma^{n_1}\mathbb{S}, \Sigma^{n_2}\mathbb{S}}}{\longrightarrow} & \Sigma^{n_2} \mathbb{S} \wedge \Sigma^{n_1} \mathbb{S} \\ {}^{\alpha_1 \wedge \alpha_2}\downarrow && \downarrow^{\alpha_2 \wedge \alpha_1} \\ E \wedge E & \overset{\tau_{E,E}}{\longrightarrow} & E \wedge E \\ & {}_{\mu}\searrow \quad \swarrow_{\mu} & \\ & E & \end{array} \;.$$

Here the top square is that of lemma 5.9 (part of the tensor triangulated structure of prop. 5.4)), the middle square is the naturality square of the braiding (def. 1.4, cor. 5.1), and the bottom triangle commutes by definition of (E, μ, e) being a commutative monoid (def. 1.13).

Similarly given

$$\alpha : \Sigma^{n_1}\mathbb{S} \longrightarrow E$$

as before and

$$\nu : \Sigma^{n_2}\mathbb{S} \longrightarrow E \wedge X ,$$

then an action is defined by the composite

$$\alpha \cdot \nu : \Sigma^{n_1+n_2}\mathbb{S} \overset{\simeq}{\longrightarrow} \Sigma^{n_1}\mathbb{S} \wedge \Sigma^{n_2}\mathbb{S} \overset{\alpha\wedge\nu}{\longrightarrow} E \wedge E \wedge X \overset{\mu\wedge \mathrm{id}}{\longrightarrow} E \wedge X .$$

This is clearly a graded pairing, and the action property and unitality follow directly from the associativity and unitality, respectively, of (E, μ, e).

Analogously for the right action on $X_\bullet(E)$. ▮

Example 5.12. **(ring structure on the stable homotopy groups of spheres)**

The sphere spectrum $\mathbb{S} = \gamma(\mathbb{S}_{\mathrm{orth}})$ is a homotopy commutative ring spectrum (def. 5.10).

On the one hand this is because it is the tensor unit for the derived smash product of spectra (by cor. 5.1), and by example 1.14 every such is canonically a (commutative) monoid. On the other hand we have the explicit representation by the orthogonal ring spectrum (def. 2.15) $\mathbb{S}_{\mathrm{orth}}$, according to lemma 2.7, and the localization functor γ is a symmetric lax monoidal functor (prop. 3.14, and in fact a strong monoidal functor on cofibrant objects such as $\mathbb{S}_{\mathrm{orth}}$ according to prop. 3.13) and hence preserves commutative monoids (prop. 1.50).

The stable homotopy groups of the sphere spectrum are of course the stable homotopy groups of spheres (exmpl.)

$$\pi_\bullet^s \coloneqq \pi_\bullet(\mathbb{S}) \simeq \underset{\longrightarrow_k}{\lim}\, \pi_{\bullet+k}(S^k) .$$

Now prop. 5.11 gives the stable homotopy groups of spheres the structure of a graded commutative ring. By the proof of prop. 5.11, the product operation in that ring sends elements $\alpha_i : \Sigma^{n_i}\mathbb{S} \longrightarrow \mathbb{S}$ to

$$\Sigma^{n_1+n_2}\mathbb{S} \overset{\simeq}{\longrightarrow} \Sigma^{n_1}\mathbb{S} \wedge \Sigma^{n_2}\mathbb{S} \overset{\alpha_1\wedge\alpha_2}{\longrightarrow} \mathbb{S} \wedge \mathbb{S} \underset{\simeq}{\overset{\mu^{\mathbb{S}}}{\longrightarrow}} \mathbb{S} ,$$

where now not only the first morphism, but also the last morphism is an isomorphism (the isomorphism from lemma 1.2). Hence up to isomorphism, the ring structure on the stable homotopy groups of spheres *is* the derived smash product of spectra.

This implies that for $X, Y \in \mathrm{Ho}(\mathrm{Spectra})$ any two spectra, then the graded abelian group $[X, Y]_\bullet$ (def.) of morphisms from X to Y in the stable homotopy category canonically becomes a module over the ring $\pi_\bullet^s$

$$\pi_\bullet^s \otimes [X, Y]_\bullet \longrightarrow [X, Y]_\bullet$$

by

$$(\Sigma^{n_1}\mathbb{S} \overset{\alpha}{\to} \mathbb{S}), (\Sigma^{n_2}X \overset{f}{\to} Y) \;\mapsto\; \left(\Sigma^{n_1+n_2}X \overset{\simeq}{\to} \Sigma^{n_1}\mathbb{S} \wedge \Sigma^{n_2}X \overset{\alpha\wedge f}{\longrightarrow} \mathbb{S} \wedge Y \overset{\simeq}{\to} Y \right) .$$

In particular for every spectrum $X \in \mathrm{Ho}(\mathrm{Spectra})$, its stable homotopy groups $\pi_\bullet(X) \simeq [\mathbb{S}, X]_\bullet$ (lemma) canonically form a module over $\pi_\bullet^s$. If $X = E$ happens to carry the structure of a homotopy commutative ring spectrum, then this module structure coincides the one induced from the unit

$$\pi_\bullet(e) : \pi_\bullet^s = \pi_\bullet(\mathbb{S}) \longrightarrow \pi_\bullet(E)$$

under prop. 5.11.

(It is straightforward to unwind all this categorical algebra to concrete component expressions by proceeding as in the proof of this lemma).)

This finally allows to uniquely characterize the stable homotopy theory that we have been discussing:

Theorem 5.13. ***(Schwede-Shipley uniqueness theorem)***

The homotopy category $\mathrm{Ho}(\mathcal{C})$ (def.) of every stable homotopy category $\mathcal{C}$ (def.) canonically has graded hom-groups with the structure of modules over $\pi_\bullet^s = \pi_\bullet(\mathbb{S})$ (example 5.12). In terms of this, the following are equivalent:

1. *There is a zig-zag of Quillen equivalences (def.) between $\mathcal{C}$ and the stable model structure on topological sequential spectra (thm.) (equivalently (thm. 4.12) the stable model structure on orthogonal spectra)*

$$\mathcal{C} \underset{\longrightarrow}{\overset{\longleftarrow}{\simeq_{Qu}}} \underset{\longleftarrow}{\overset{\longrightarrow}{\simeq_{Qu}}} \cdots \underset{\longleftarrow}{\overset{\longrightarrow}{\simeq_{Qu}}} \mathrm{OrthSpec}(\mathrm{Top}_{cg})_{\mathrm{stable}} \underset{\longrightarrow}{\overset{\longleftarrow}{\simeq_{Qu}}} \mathrm{SeqSpec}(\mathrm{Top}_{cg})_{\mathrm{stable}}$$

2. *there is an equivalence of categories between the homotopy category $\mathrm{Ho}(\mathcal{C})$ and the stable homotopy category $\mathrm{Ho}(\mathrm{Spectra})$ (def.)*

$$\mathrm{Ho}(\mathcal{C}) \simeq \mathrm{Ho}(\mathrm{Spectra})$$

which is $\pi_\bullet^s$-linear on all hom-groups.

(Schwede-Shipley 02, Uniqueness theorem)

6. Examples

For reference, we consider some basic examples of orthogonal ring spectra (def. 2.15) E. By prop. 2.16 and corollary 5.1 each of these examples in particular represents a homotopy commutative ring spectrum (def. 5.10) in the tensor triangulated stable homotopy category (prop. 5.4).

We make use of these examples of homotopy commutative ring spectra E in Part 2 in the computation of E-Adams spectral sequences.

For constructing representations as orthogonal ring spectra of spectra that are already known as sequential spectra (def.) two principles are usefully kept in mind:

1. by prop. 2.16 it is sufficient to give an equivariant multiplicative pairing $E_{n_1} \wedge E_{n_2} \to E_{n_1+n_2}$ and equivariant unit maps $S^0 \to E_0$, $S^1 \to E_1$, from these the structure maps $S^{n_1} \wedge E_{n_2} \to E_{n_1+n_2}$ are already uniquely induced;

2. the choice of $O(n)$-action on E_n is governed mainly by the demand that the unit map $S^n \to E_n$ has to be equivariant, with respect to the $O(n)$-action on S^n induced by regarding S^n as the one-point compactification of the defining $O(n)$-representation on $\mathbb{R}^n$ ("representation sphere").

Sphere spectrum

We already described the orthogonal sphere spectrum $\mathbb{S}$ as an orthogonal ring spectrum in lemma 2.7. The component spaces are the spheres S^n with their $O(n)$-action as representation spheres, and the multiplication maps are the canonical identifications

$$S^{n_1} \wedge S^{n_2} \longrightarrow S^{n_1+n_2} \,.$$

More generally, by prop. 2.29 the orthogonal suspension spectrum functor is a strong monoidal functor, and so by prop. 2.16 the suspension spectrum of a monoid in $\mathrm{Top}_{cg}^{*/}$ (for instance G_+ for G a topological group) canonically carries the structure of an orthogonal ring spectrum.

The orthogonal sphere spectrum is the special case of this with $\mathbb{S}_{\mathrm{orth}} \simeq \Sigma^\infty_{\mathrm{orth}} S^0$ for S^0 the tensor unit in $\mathrm{Top}_{cg}^{*/}$ (example 1.10) and hence a monoid by example 1.14.

Eilenberg-MacLane spectra

We discuss the model of Eilenberg-MacLane spectra as symmetric spectra and orthogonal spectra. To that end, notice the following model for Eilenberg-MacLane spaces.

Definition 6.1. For A an abelian group and $n \in \mathbb{N}$, the **reduced A-linearization** $A[S^n]_*$ of the n-sphere S^n is the topological space, whose underlying set is the quotient of the tensor product with A of the free abelian group on the underlying set of S^n,

$$A \otimes_{\mathbb{Z}} [S^n] = A[S^n] \longrightarrow A[S^n]_*$$

by the relation that identifies every formal linear combination of the basepoint of S^n with 0. The topology is the induced quotient topology

$$\underset{k \in \mathbb{N}}{\sqcup} A^k \times (S^n)^k \longrightarrow A[S^n]_*$$

(of the disjoint union of product topological spaces, where A is equipped with the discrete topology).

(Aguilar-Gitler-Prieto 02, def. 6.4.20)

Proposition 6.2. *For A a countable abelian group, then the reduced A-linearization $A[S^n]_*$ (def. 6.1) is an Eilenberg-MacLane space, in that its homotopy groups are*

$$\pi_q(A[S^n]_*) \simeq \begin{cases} A & \text{if } q = n \\ * & \text{otherwise} \end{cases}$$

(in particular for $n \geq 1$ then there is a unique connected component and hence we need not specify a basepoint for the homotopy group).

(Aguilar-Gitler-Prieto 02, corollary 6.4.23)

Definition 6.3. For A a countable abelian group, then the **orthogonal Eilenberg-MacLane spectrum** HA is the orthogonal spectrum (def. 2.11) with

- component spaces

  $$(HA)_n \coloneqq A[S^n]_*$$

  being the reduced A-linearization (def. 6.1) of the representation sphere S^n;

- $O(n)$-action on $A[S^n]_*$ induced from the canonical $O(n)$-action on S^n (representation sphere);

- structure maps

  $$\sigma_n : S^1(HA)_n \longrightarrow (HA)_{n+1}$$

  hence

  $$S^1 \wedge A[S^n] \longrightarrow A[S^{n+1}]$$

  given by

  $$\left(x, \left(\sum_i a_i y_i\right), \right) \mapsto \sum_i a_i (x, y_i) \;.$$

The incarnation of HA as a symmetric spectrum is the same, with the group action of $O(n)$ replaced by the subgroup action of the symmetric group $\Sigma(n) \hookrightarrow O(n)$.

If R is a commutative ring, then the Eilenberg-MacLane spectrum HR becomes a commutative orthogonal ring spectrum or symmetric ring spectrum (def. 2.15) by

1. taking the multiplication

   $$(HR)_{n_1} \wedge (HR)_{n_2} = R[S^{n_1}]_* \wedge R[S^{n_2}]_* \longrightarrow R[S^{n_1+n_2}] = (HR)_{n_1+n_2}$$

   to be given by

   $$\left(\left(\sum_i a_i x_i\right), \left(\sum_j b_j y_j\right) \right) \mapsto \sum_{i,j} (a_i \cdot b_j)(x_i, y_j)$$

2. taking the unit maps

   $$S^n \longrightarrow A[S^n]_* = (HR)_n$$

   to be given by the canonical inclusion of generators

   $$x \mapsto 1x \;.$$

(Schwede 12, example I.1.14)

Proposition 6.4. *The stable homotopy groups (def. 2.21) of an Eilenberg-MacLane spectrum HA (def. 6.3) are*

$$\pi_q(HA) \simeq \begin{cases} A & \text{if } q = 0 \\ 0 & \text{otherwise} \end{cases}$$

Thom spectra

We discuss the realization of *Thom spectra* as orthogonal ring spectra. For background on Thom spectra realized as sequential spectra see Part S the section *Thom spectra*.

Definition 6.5. As an orthogonal ring spectrum (def. 2.15), the **universal Thom spectrum** MO has

* component spaces

$$(MO)_n \coloneqq EO(n)_+ \underset{O(n)}{\wedge} S^n$$

the Thom spaces (def.) of the universal vector bundle (def.) of rank n;

* left $O(n)$-action induced by the remaining canonical left action of $EO(n)$;

* canonical multiplication maps (def.)

$$(EO(n_1)_+ \underset{O(n_1)}{\wedge} S^{n_1}) \wedge (EO(n_2)_+ \underset{O(n_2)}{\wedge} S^{n_2} \longrightarrow EO(n_1+n_2)_+ \underset{O(n_1+n_2)}{\wedge} S^{n_1+n_2}$$

* unit maps

$$S^n \simeq O(n)_+ \wedge_{O(n)} S^n \longrightarrow EO(n)_+ \wedge_{O(n)} S^n$$

induced by the fiber inclusion $O(V) \hookrightarrow EO(V)$.

(Schwede 12, I, example 1.16)

For the universal complex Thom spectrum MU the construction is a priori directly analogous, but with the real Cartesian space $\mathbb{R}^n$ replace by the complex vector space $\mathbb{C}^n$ thoughout. This makes the n-sphere $S^n = S^{(\mathbb{R}^n)}$ be replaced by the $2n$-sphere $S^{2n} \simeq S^{\mathbb{C}^n}$ throughout. Hence the construction requires a second step in which the resulting S^2-spectrum (def.) is turned into an actual orthogonal spectrum. This proceeds differently than for sequential spectra (lemma) due to the need to have compatible orthogonal group-action on all spaces.

Definition 6.6. The **universal complex Thom spectrum** MU is represented as an orthogonal ring spectrum (def. 2.15) as follows

First consider the component spaces

$$\overline{MU}_n \coloneqq EU(n)_+ \wedge_{U(n)} S^{(\mathbb{C}^n)}$$

given by the Thom spaces (def.) of the complex universal vector bundle (def.) of rank n, and equipped with the $O(n)$-action which is induced via the canonical inclusions

$$O(n) \hookrightarrow U(n) \hookrightarrow EU(n) \,.$$

Regard these as equipped with the canonical pairing maps (def.)

$$\bar{\mu}_{n_1,n_2} : \overline{MU}_{n_1} \wedge \overline{MU}_{n_2} \longrightarrow \overline{MU}_{n_1+n_2} \,.$$

These are $U(n)$-equivariant, hence in particular $O(n)$-equivariant.

Then take the actual components spaces to be loop spaces of these:

$$MU_n \coloneqq \mathrm{Maps}(S^n, \overline{MU}_n)$$

and regard these as equipped with the conjugation action by $O(n)$ induced by the above action on $\overline{MU}_n$ and the canonical action on $S^n \simeq S^{(\mathbb{R}^n)}$.

Define the actual pairing maps

$$\mu_{n_1,n_2} : MU_{n_1} \wedge MU_{n_2} \longrightarrow MU_{n_1+n_2}$$

via

$$\begin{aligned} \mathrm{Maps}(S^{n_1}, \overline{MU}_{n_1}) \wedge \mathrm{Maps}(S^{n_2}, \overline{MU}_{n_2}) &\longrightarrow \mathrm{Maps}(S^{n_1+n_2}, \overline{MU}_{n_1+n_2}) \\ (\alpha_1, \alpha_2) &\mapsto \bar{\mu}_{n_1,n_2} \circ (\alpha_1 \wedge \alpha_2) \end{aligned} \,.$$

Finally in order to define the unit maps, consider the isomorphism

$$S^{2n} \simeq S^{\mathbb{C}^n} \simeq S^{\mathbb{R}^n \oplus i\mathbb{R}^n} \simeq S^n \wedge S^n$$

and then take the unit maps

$$S^n \longrightarrow (MU)_n = \mathrm{Maps}(S^n, \overline{MU}_n)$$

to be the adjuncts of the canonical embeddings

$$S^n \wedge S^n \simeq S^{\mathbb{C}^n} \simeq U(n)_+ \wedge_{U(n)} S^{\mathbb{C}^n} \longrightarrow EU(n)_+ \wedge_{U(n)} S^{\mathbb{C}^n} \,.$$

(Schwede 12, I, example 1.18)

7. Conclusion

We summarize the results about stable homotopy theory obtained above.

First of all we have established a commuting diagram of Quillen adjunctions and Quillen equivalences of the form

$$\begin{array}{ccc} (\mathrm{Top}^{*/}_{\mathrm{cg}})_{\mathrm{Quillen}} & \underoverset{\underset{\Omega}{\longrightarrow}}{\overset{\Sigma}{\longleftarrow}}{\bot} & (\mathrm{Top}^{*/}_{\mathrm{cg}})_{\mathrm{Quillen}} \\ {}^{\mathllap{\Sigma^\infty}}\downarrow \dashv \uparrow^{\mathrlap{\Omega^\infty}} && {}^{\mathllap{\Sigma^\infty}}\downarrow \dashv \uparrow^{\mathrlap{\Omega^\infty}} \\ \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}} & \underoverset{\underset{\Omega}{\longrightarrow}}{\overset{\Sigma}{\longleftarrow}}{\bot} & \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}} \\ {}^{\mathllap{\mathrm{id}}}\downarrow \dashv \uparrow^{\mathrlap{\mathrm{id}}} && {}^{\mathllap{\mathrm{id}}}\downarrow \dashv \uparrow^{\mathrlap{\mathrm{id}}} \\ \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}} & \underoverset{\underset{\Omega}{\longrightarrow}}{\overset{\Sigma}{\longleftarrow}}{\simeq_Q} & \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}} \\ {}^{\mathllap{\mathrm{seq}_!}}\downarrow \simeq_Q \uparrow^{\mathrlap{\mathrm{seq}^*}} && {}^{\mathllap{\mathrm{seq}_!}}\downarrow \simeq_Q \uparrow^{\mathrlap{\mathrm{seq}^*}} \\ \mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}} && \mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}} \end{array}$$

where

* $(\mathrm{Top}^{*/}_{\mathrm{cg}})_{\mathrm{Quillen}}$ is the classical model structure on pointed topological spaces (thm., thm.);
* $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}$ is the stable model structure on topological sequential spectra (thm.);
* $\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}$ is the stable model structure on orthogonal spectra from theorem 4.1.

Here the top part of the diagram is from remark 3.19, while the vertical Quillen equivalence $(\mathrm{seq}_! \dashv \mathrm{seq}^*)$ is from theorem 4.1.

Moreover, the top and bottom model categories are monoidal model categories (def. 3.11): $\mathrm{Top}^{*/}_{\mathrm{cg}}$ with respect to the smash product of pointed topological spaces (theorem 3.17) and $\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}}$ as well as $\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}$ with respect to the symmetric monoidal smash product of spectra (theorem 3.17 and theorem 4.14); and the compsite vertical adjunction

$$\begin{array}{c} (\mathrm{Top}^{*/}_{\mathrm{cg}}, \wedge, S^0) \\ {}^{\mathllap{\Sigma^\infty_{\mathrm{orth}}}}\downarrow \dashv \uparrow^{\mathrlap{\Omega^\infty_{\mathrm{orth}}}} \\ (\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}}), \wedge, \mathbb{S}_{\mathrm{orth}}) \end{array}$$

is a strong monoidal Quillen adjunction (def. 3.15, corollary 4.15), and so also the induced adjunction of derived functors

$$\begin{array}{c} (\mathrm{Ho}(\mathrm{Top}^{*/}), \wedge^L, S^0) \\ {}^{\mathllap{\Sigma^\infty}}\downarrow \dashv \uparrow^{\mathrlap{\Omega^\infty}} \\ (\mathrm{Ho}(\mathrm{Spectra}), \wedge^L, \mathbb{S}) \end{array}$$

is a strong monoidal adjunction (by prop. 3.16) from the the derived smash product of pointed topological spaces to the derived symmetric smash product of spectra.

Under passage to homotopy categories this yields a commuting diagram of derived adjoint functors

$$\begin{array}{ccc} \mathrm{Ho}(\mathrm{Top}^{*/}) & \underoverset{\underset{\Omega}{\longrightarrow}}{\overset{\Sigma}{\longleftarrow}}{\bot} & \mathrm{Ho}(\mathrm{Top}^{*/}) \\ {}^{\mathllap{\Sigma^\infty}}\downarrow \dashv \uparrow^{\mathrlap{\Omega^\infty}} && {}^{\mathllap{\Sigma^\infty}}\downarrow \dashv \uparrow^{\mathrlap{\Omega^\infty}} \\ \mathrm{Ho}(\mathrm{Spectra}) & \underoverset{\underset{\Omega}{\longrightarrow}}{\overset{\Sigma}{\longleftarrow}}{\simeq} & \mathrm{Ho}(\mathrm{Spectra}) \end{array}$$

between the (Serre-Quillen-)classical homotopy category $\mathrm{Ho}(\mathrm{Top}^{*/})$ and the stable homotopy category $\mathrm{Ho}(\mathrm{Spectra})$ (remark 4.13). The latter is an additive category (def.) with direct sum the wedge sum of spectra

$\oplus = \vee$ (lemma, lemma) and in fact a triangulated category (def.) with distinguished triangles the homotopy cofiber sequences of spectra (prop.).

While this is the situation already for sequential spectra (thm.), in addition we have now that both the classical homotopy category as well as the stable homotopy category are symmetric monoidal categories with respect to derived smash product of pointed topological spaces and the derived symmetric monoidal smash product of spectra, respectively (corollary 5.1).

Moreover, the derived smash product of spectra is compatible with the additive category structure (direct sums) and the triangulated category structure (homotopy cofiber sequences), this being a *tensor triangulated category* (prop. 5.4).

abelian groups	**spectra**
integers $\mathbb{Z}$	sphere spectrum $\mathbb{S}$
$\mathrm{Ab} \simeq \mathbb{Z}\mathrm{Mod}$	$\mathrm{Spectra} \simeq \mathbb{S}\mathrm{Mod}$
direct sum $\oplus$	wedge sum $\vee$
tensor product $\otimes_{\mathbb{Z}}$	smash product of spectra $\wedge_{\mathbb{S}}$
kernels/cokernels	homotopy fibers/homotopy cofibers

The commutative monoids with respect to this smash product of spectra are precisely the commutative orthogonal ring spectra (def. 2.15, prop. 2.16) and the module objects over these are precisely the orthogonal module spectra (def. 2.17, prop. 2.18).

algebra	**homological algebra**	**higher algebra**
abelian group	chain complex	spectrum
ring	dg-ring	ring spectrum
module	dg-module	module spectrum

The localization functors γ (def.) from the monoidal model categories to their homotopy categories are lax monoidal functors (cor. 5.1)

$$\begin{array}{ccc} (\mathrm{Top}_{\mathrm{cg}}^{*/}, \wedge, S^0) & \longrightarrow & (\mathrm{Ho}(\mathrm{Top}^{*/}), \wedge^L, \gamma(S^0)) \\ (\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}}), \wedge, \mathbb{S}_{\mathrm{orth}}) & \longrightarrow & (\mathrm{Ho}(\mathrm{Spectra}), \wedge^L, \gamma(\mathbb{S})) \end{array} \,.$$

This implies that for $E \in \mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})$ a commutativeorthogonal ring spectrum, then its image $\gamma(E)$ in the stable homotopy category is a homotopy commutative ring spectrum (def. 5.10) and similarly for module spectra (prop. 1.50).

monoidal stable model category	**-localization→**	**tensor triangulated category**
stable model structure on orthogonal spectra $\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}$		stable homotopy category $\mathrm{Ho}(\mathrm{Spectra})$
symmetric monoidal smash product of spectra		derived smash product of spectra
commutative orthogonal ring spectrum (E-infinity ring)		homotopy commutative ring spectrum

Finally, the graded hom-groups $[X,Y]_\bullet$ (def.) in the tensor triangulated stable homotopy category are canonically graded modules over the graded commutative ring of stable homotopy groups of spheres (exmpl. 5.12)

$$[X,Y]_\bullet \in \pi_\bullet(\mathbb{S})\mathrm{Mod} \,.$$

Hence the next question is how to actually compute any of these. This is the topic of *Part 2 -- The Adams spectral sequence*.

8. References

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