nLab

Introduction to Stable Homotopy Theory

This entry is a detailed introduction to the stable homotopy category and to its key computational tool, the Adams spectral sequence. To that end we introduce the modern tools, such as model categories and highly structured ring spectra. In the accompanying seminar we consider applications to cobordism theory and complex oriented cohomology such as to converge in the end to a glimpse of the modern picture of chromatic homotopy theory.

Lecture notes. (web version requires Firefox browser – free download)

Prelude -- Classical homotopy theory (pdf 111 pages)

Part 1 -- Stable homotopy theory

* *Part 1.1 -- Sequential Spectra* (pdf, 79 pages)

* *Part 1.2 -- Structured Spectra* (pdf, 75 pages)

Interlude -- Spectral sequences (pdf, 15 pages)

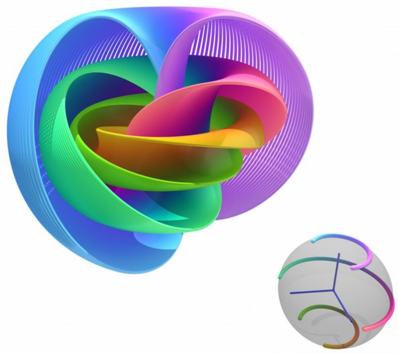
Part 2 -- Adams spectral sequences (pdf, 53 pages)

Examples and Applications -- Cobordism and Complex Oriented Cohomology (pdf, 76 pages)

Background -- Introduction to Homological algebra (pdf, 83 pages)

Background -- Introduction to Topology (pdf, 122 pages)

Contents



> *My initial inclination was to call this book The Music of the Spheres, but I was dissuaded from doing so by my diligent publisher, who is ever mindful of the sensibilities of librarians. (Ravenel 86, preface)*

1. Survey

We are concerned with the theory of *spectra* in the sense of algebraic topology: the proper generalization of abelian groups to homotopy theory.

1) Stable homotopy theory

A group in homotopy theory is equivalently a loop space under concatenation of loops ("∞-group"). A double loop space is a group with some commutativity structure ("Eckmann-Hilton argument"), a triple loop space has more commutativity structure, and so forth. A *spectrum* is where this progression of looping and delooping *stabilizes* (an "∞-abelian group"). Therefore one speaks of *stable homotopy theory*:

$$\text{Spaces} \underset{\text{(linearization)}}{\overset{\text{stabilization}}{\longmapsto}} \text{Spectra}\,.$$

Most of linear algebra and algebraic geometry passes along as abelian groups are generalized to spectra and turns into something remarkably rich, called *brave new algebra*, *higher algebra* and *spectral geometry*. In particular the analog of the theory of (commutative) rings and their modules exist, given by (commutative) ring spectra (E-∞ rings, A-∞ rings) and module spectra (∞-modules).

2) Adams spectral sequences

Since spectra are considerably richer than abelian groups, stable homotopy is much concerned with "fracturing" stable homotopy types into more tractable components:

To that end, notice that from the point of view of arithmetic geometry, an abelian group A is equivalently a quasicoherent sheaf over Spec(Z).

$$\mathrm{AbelianGroups} \simeq \mathrm{QCoh}(\mathrm{Spec}(\mathbb{Z}))\;.$$

This point of view generalizes to homotopy theory and turns out to be very fruitful there. The analog of the integers $\mathbb{Z}$ is the sphere spectrum $\mathbb{S}$, and this is naturally the initial commutative ring spectrum ("E-∞ ring"), just as $\mathbb{Z}$ is the initial commutative ring. The formal dual Spec(S) of $\mathbb{S}$ is hence the terminal space in E-∞ arithmetic geometry ("spectral geometry") and spectra are equivalently the quasicoherent ∞-stacks over $\mathrm{Spec}(\mathbb{S})$

$$\mathrm{Spectra} \simeq \mathrm{QCoh}(\mathrm{Spec}(\mathbb{S}))\;.$$

Therefore the study of spectra "fractures" into the various localizations and formal completions of $\mathrm{Spec}(\mathbb{S})$. Since this is like the white light of $\mathrm{Spec}(\mathbb{S})$ decomposing into various wavelengths, one speaks of *chromatic homotopy theory*.

In particular, an E-∞ ring E is dually a morphism of E_∞-algebraic spaces $\mathrm{Spec}(E) \longrightarrow \mathrm{Spec}(\mathbb{S})$ and under good conditions the 1-image of this map is the formal dual of the localization $L_E \mathbb{S}$ at E:

$$\mathrm{Spec}(E) \overset{\mathrm{epi}_1}{\longrightarrow} \mathrm{Spec}(L_E\mathbb{S}) \overset{\mathrm{mono}_1}{\longrightarrow} \mathrm{Spec}(\mathbb{S})\;.$$

This means that $\mathrm{Spec}(E) \longrightarrow \mathrm{Spec}(L_E\mathbb{S})$ is a cover and that hence E-local spectra are equivalently quasicoherent ∞-stacks on $\mathrm{Spec}(E)$ equipped with descent data: dually they are ∞-modules over E equipped with comodule structure over the Hopf algebroid (Sweedler coring) $E \otimes_{\mathbb{S}} E$.

The computation of homotopy groups of spectra that make use of their decomposition this way into E-∞-modules equipped with descent data is the E-*Adams spectral sequence*, a central tool of the theory.

S) Complex oriented cohomology

For this reason special importance is carried by those E-∞ rings such that $\mathrm{Spec}(E) \to \mathrm{Spec}(\mathbb{S})$ is already a covering, in a suitable sense, for these the E-∞-modules equipped with descent data give an equivalent, but in general more tractable, incarnation of the stable homotopy theory of spectra.

Curiously, this way a good bit of differential topology – cobordism theory – arises within stable homotopy theory: the archetypical $\mathrm{Spec}(E)$ which covers $\mathrm{Spec}(\mathbb{S})$ in a suitable sense is $E =$ MU, the Thom spectrum representing complex cobordism cohomology.

An commutative ring spectrum E over MU, hence a $\mathrm{Spec}(E) \to \mathrm{Spec}(\mathrm{MU})$ is now a multiplicative "complex oriented cohomology theory".

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2. Prelude) Classical homotopy theory

This section is at: *Introduction to Stable homotopy theory -- P*

3. Part 1) Stable homotopy theory

This section is at *Introduction to Stable homotopy theory -- 1*

4. Interlude) Spectral sequences

This section is at *Introduction to Stable homotopy theory -- I*

5. Part 2) Adams spectral sequences

This section is at *Introduction to Stable homotopy theory -- 2*

6. Seminar) Complex oriented cohomology

This section is at *Introduction to Stable homotopy theory -- S*

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7. References

Basic reading

For **Prelude) Classical homotopy theory** a concise and self-contained re-write of the proof (Quillen 67) of the classical model structure on topological spaces is in

- Philip Hirschhorn, *The Quillen model category of topological spaces* (arXiv:1508.01942).

For general model category theory a decent concise account is in

- William Dwyer, J. Spalinski, *Homotopy theories and model categories* (pdf) in Ioan Mackenzie James (ed.), *Handbook of Algebraic Topology* 1995

For the restriction to the convenient category of compactly generated topological spaces good sources are

- Gaunce Lewis, *Compactly generated spaces* (pdf), appendix A of *The Stable Category and Generalized Thom Spectra* PhD thesis Chicago, 1978

- Neil Strickland, *The category of CGWH spaces*, 2009 (pdf)

For section **1) Stable homotopy theory** we follow the modern picture of the stable homotopy category for which an enjoyable survey may be found in

- Cary Malkiewich, *The stable homotopy category*, 2014 (pdf).

The classical account in (Adams 74, part III sections 2, 4-7) is still a good read, but ignore the "Adams category"-construction of the stable homotopy category in sections III.2 and III.3. What we actually do follows

- Michael Mandell, Peter May, Stefan Schwede, Brooke Shipley, *Model categories of diagram spectra*, Proceedings of the London Mathematical Society, 82 (2001), 441-512 (pdf)

For the discussion of ring spectra we pass to symmetric spectra and orthogonal spectra. A compendium on the former is in

- Stefan Schwede, *Symmetric spectra*, 2012 (pdf)

For **Interlude: Spectral sequences** a discussion streamlined for our purposes is in (Rognes 12, section 2).

In **2) Adams spectral sequence** for the general theory we follow

- Frank Adams, *Stable homotopy and generalized homology*, Chicago Lectures in mathematics, 1974
- Aldridge Bousfield, sections 5 and 6 of *The localization of spectra with respect to homology*, Topology 18 (1979), no. 4, 257–281. (pdf)

For the special case of the classical Adams spectral sequence we follow (Kochman 96, chapter V).

For the **Seminar on Complex oriented cohomology** an excellent textbook to hold on to is

- Stanley Kochman, *Bordism, Stable Homotopy and Adams Spectral Sequences*, AMS 1996

Specifically for **S.1) Generalized cohomology** a neat account is in:

- Marcelo Aguilar, Samuel Gitler, Carlos Prieto, section 12 of *Algebraic topology from a homotopical viewpoint*, Springer (2002) (toc pdf)

For **S.2) Cobordism theory** an efficient collection of the highlights is in

- Cary Malkiewich, *Unoriented cobordism and MO*, 2011 (pdf)

except that it omits proof of the Leray-Hirsch theorem/Serre spectral sequence and that of the Thom isomorphism, but see the references there and see (Kochman 96, Aguilar-Gitler-Prieto 02, section 11.7) for details.

For **S.3) Complex oriented cohomology** besides (Kochman 96, chapter 4) have a look at Adams 74, part II and

- Jacob Lurie, lectures 1-10 of *Chromatic Homotopy Theory*, 2010

(These overlap, pick the one that seems more inviting on first reading.)

Further reading

The two originals

- Daniel Quillen, *Axiomatic homotopy theory* in *Homotopical algebra*, Lecture Notes in Mathematics, No. 43 43, Berlin (1967)
- Kenneth Brown, *Abstract Homotopy Theory and Generalized Sheaf Cohomology*, Transactions of the American Mathematical Society, Vol. 186 (1973), 419-458 (JSTOR)

are still an excellent source. For further reading on homotopy theory and stable homotopy theory a useful collection is

- Ioan Mackenzie James, *Handbook of Algebraic Topology* 1995

The modern chromatic picture originates around

- Mike Hopkins, *Complex oriented cohomology theories and the language of stacks*, 1999

a useful survey is in

- Dylan Wilson section 1.2 of *Spectral Sequences from Sequences of Spectra: Towards the Spectrum of the Category of Spectra* lecture at *2013 Pre-Talbot Seminar*, March 2013 (pdf)

a wealth of details is in

- Doug Ravenel, *Complex cobordism and stable homotopy groups of spheres*, 1987/2003 (pdf)

and new foundations have been laid in

- Jacob Lurie, *Higher Algebra*

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Introduction to Homotopy Theory

This pages gives a detailed introduction to classical *homotopy theory*, starting with the concept of homotopy in topological spaces and motivating from this the "abstract homotopy theory" in general model categories.

For background on basic topology see at *Introduction to Topology*.

For application to homological algebra see at *Introduction to Homological algebra*.

For application to stable homotopy theory see at *Introduction to Stable homotopy theory*.

Contents

Context

Homotopy theory

While the field of algebraic topology clearly originates in topology, it is not actually interested in topological spaces regarded up to topological isomorphism, namely homeomorphism ("point-set topology"), but only in topological spaces regarded up to weak homotopy equivalence – hence it is interested only in the "weak homotopy types" of topological spaces. This is so notably because ordinary cohomology groups are invariants of the (weak) homotopy type of topological spaces but do not detect their homeomorphism class.

The category of topological spaces obtained by forcing weak homotopy equivalences to become isomorphisms is the "classical homotopy category" Ho(Top). This homotopy category however has forgotten a little too much information: homotopy theory really wants the weak homotopy equivalences not to become plain isomorphisms, but to become actual homotopy equivalences. The structure that reflects this is called a *model category* structure (short for "category of models for homotopy types"). For classical homotopy theory this is accordingly called the *classical model structure on topological spaces*. This we review here.

1. Topological homotopy theory

This section recalls relevant concepts from actual topology ("point-set topology") and highlights facts that motivate the axiomatics of model categories below. We prove two technical lemmas (lemma 1.40 and lemma 1.52) that serve to establish the abstract homotopy theory of topological spaces further below.

Literature (Hirschhorn 15)

Throughout, let *Top* denote the category whose objects are topological spaces and whose morphisms are continuous functions between them. Its isomorphisms are the homeomorphisms.

(Further below we restrict attention to the full subcategory of compactly generated topological spaces.)

Universal constructions

To begin with, we recall some basics on universal constructions in Top: limits and colimits of diagrams of topological spaces; exponential objects.

Generally, recall:

Definition 1.1. A **diagram** in a category $\mathcal{C}$ is a small category I and a functor

$$\begin{aligned} X_\bullet \; : \; I \longrightarrow \mathcal{C} \\ (i \overset{\phi}{\longrightarrow} j) \;\mapsto (X_i \overset{X(\phi)}{\longrightarrow} X_j) \;. \end{aligned}$$

A **cone** over this diagram is an object Q equipped with morphisms $p_i : Q \longrightarrow X_i$ for all $i \in I$, such that all these triangles commute:

$$\begin{tikzcd} & Q \arrow[dl, "p_i"'] \arrow[dr, "p_j"] & \\ X_i \arrow[rr, "X(\phi)"'] & & X_j \end{tikzcd} \;.$$

Dually, a **co-cone** under the diagram is Q equipped with morphisms $q_i : X_i \longrightarrow Q$ such that all these triangles commute

$$\begin{tikzcd} X_i \arrow[rr, "X(\phi)"] \arrow[dr, "q_i"'] & & X_j \arrow[dl, "q_j"] \\ & Q & \end{tikzcd} \;.$$

A **limit** over the diagram is a universal cone, denoted $\underset{\longleftarrow i \in I}{\lim} X_i$, that is: a cone such that every other cone uniquely factors through it $Q \longrightarrow \underset{\longleftarrow}{\lim}_{i \in I} X_i$, making all the resulting triangles commute.

Dually, a **colimit** over the diagram is a universal co-cone, denoted $\underset{\longrightarrow}{\lim}_{i \in I} X_i$.

We now discuss limits and colimits in $\mathcal{C} =$ Top. The key for understanding these is the fact that there are initial and final topologies:

Definition 1.2. Let $\{X_i = (S_i, \tau_i) \in \mathrm{Top}\}_{i \in I}$ be a set of topological spaces, and let $S \in \mathrm{Set}$ be a bare set. Then

1. For $\{S \overset{f_i}{\to} S_i\}_{i \in I}$ a set of functions out of S, the **initial topology** $\tau_{\mathrm{initial}}(\{f_i\}_{i \in I})$ is the topology on S with the minimum collection of open subsets such that all $f_i : (S, \tau_{\mathrm{initial}}(\{f_i\}_{i \in I})) \to X_i$ are continuous.

2. For $\{S_i \overset{f_i}{\to} S\}_{i \in I}$ a set of functions into S, the **final topology** $\tau_{\mathrm{final}}(\{f_i\}_{i \in I})$ is the topology on S with the maximum collection of open subsets such that all $f_i : X_i \to (S, \tau_{\mathrm{final}}(\{f_i\}_{i \in I}))$ are continuous.

Example 1.3. For X a single topological space, and $\iota_S : S \hookrightarrow U(X)$ a subset of its underlying set, then the initial topology $\tau_{\mathrm{intial}}(\iota_S)$, def. 1.2, is the subspace topology, making

$$\iota_S \;:\; (S, \tau_{\mathrm{initial}}(\iota_S)) \hookrightarrow X$$

a topological subspace inclusion.

Example 1.4. Conversely, for $p_S : U(X) \longrightarrow S$ an epimorphism, then the final topology $\tau_{\mathrm{final}}(p_S)$ on S is the *quotient topology*.

Proposition 1.5. *Let I be a small category and let $X_\bullet : I \longrightarrow \mathrm{Top}$ be an I-diagram in Top (a functor from I to Top), with components denoted $X_i = (S_i, \tau_i)$, where $S_i \in \mathrm{Set}$ and τ_i a topology on S_i. Then:*

1. *The limit of $X_\bullet$ exists and is given by the topological space whose underlying set is the limit in Set of the underlying sets in the diagram, and whose topology is the initial topology, def. 1.2, for the functions p_i which are the limiting cone components:*

$$\begin{array}{ccccc} && \underset{\longleftarrow}{\lim}_{i \in I} S_i \\ & {}^{\mathllap{p_i}}\swarrow && \searrow^{\mathrlap{p_j}} \\ S_i && \longrightarrow && S_j \end{array} \,.$$

Hence

$$\underset{\longleftarrow}{\lim}_{i \in I} X_i \simeq \left(\underset{\longleftarrow}{\lim}_{i \in I} S_i ,\; \tau_{\mathrm{initial}}(\{p_i\}_{i \in I}) \right)$$

2. *The colimit of $X_\bullet$ exists and is the topological space whose underlying set is the colimit in Set of the underlying diagram of sets, and whose topology is the final topology, def. 1.2 for the component maps ι_i of the colimiting cocone*

$$\begin{array}{ccccc} S_i && \longrightarrow && S_j \\ & {}_{\mathllap{\iota_i}}\searrow && \swarrow_{\mathrlap{\iota_j}} \\ && \underset{\longrightarrow}{\lim}_{i \in I} S_i \end{array} \,.$$

Hence

$$\underset{\longrightarrow}{\lim}_{i \in I} X_i \simeq \left(\underset{\longrightarrow}{\lim}_{i \in I} S_i,\ \tau_{\mathrm{final}}(\{\iota_i\}_{i \in I}) \right)$$

(e.g. Bourbaki 71, section I.4)

Proof. The required universal property of $\left(\underset{\longleftarrow}{\lim}_{i \in I} S_i,\ \tau_{\mathrm{initial}}(\{p_i\}_{i \in I}) \right)$ (def. 1.1) is immediate: for

$$\begin{array}{ccccc} & & (S,\tau) & & \\ & {}^{f_i}\swarrow & & \searrow^{f_j} & \\ X_i & & \longrightarrow & & X_j \end{array}$$

any cone over the diagram, then by construction there is a unique function of underlying sets $S \longrightarrow \underset{\longleftarrow}{\lim}_{i \in I} S_i$ making the required diagrams commute, and so all that is required is that this unique function is always continuous. But this is precisely what the initial topology ensures.

The case of the colimit is formally dual. ▮

Example 1.6. The limit over the empty diagram in Top is the point $*$ with its unique topology.

Example 1.7. For $\{X_i\}_{i \in I}$ a set of topological spaces, their coproduct $\underset{i \in I}{\sqcup} X_i \in \mathrm{Top}$ is their *disjoint union*.

In particular:

Example 1.8. For $S \in \mathrm{Set}$, the S-indexed coproduct of the point, $\coprod_{s \in S} *$ is the set S itself equipped with the final topology, hence is the discrete topological space on S.

Example 1.9. For $\{X_i\}_{i \in I}$ a set of topological spaces, their product $\prod_{i \in I} X_i \in \mathrm{Top}$ is the Cartesian product of the underlying sets equipped with the *product topology*, also called the *Tychonoff product*.

In the case that S is a finite set, such as for binary product spaces $X \times Y$, then a sub-basis for the product topology is given by the Cartesian products of the open subsets of (a basis for) each factor space.

Example 1.10. The equalizer of two continuous functions $f, g : X \rightrightarrows Y$ in Top is the equalizer of the underlying functions of sets

$$\mathrm{eq}(f,g) \hookrightarrow S_X \underset{g}{\overset{f}{\rightrightarrows}} S_Y$$

(hence the largets subset of S_X on which both functions coincide) and equipped with the subspace topology, example 1.3.

Example 1.11. The coequalizer of two continuous functions $f, g : X \rightrightarrows Y$ in Top is the coequalizer of the underlying functions of sets

$$S_X \underset{g}{\overset{f}{\rightrightarrows}} S_Y \longrightarrow \mathrm{coeq}(f,g)$$

(hence the quotient set by the equivalence relation generated by $f(x) \sim g(x)$ for all $x \in X$) and equipped with the quotient topology, example 1.4.

Example 1.12. For

$$\begin{array}{ccc} A & \overset{g}{\longrightarrow} & Y \\ {}^{\mathllap{f}}\downarrow & & \\ X & & \end{array}$$

two continuous functions out of the same domain, then the colimit under this diagram is also called the *pushout*, denoted

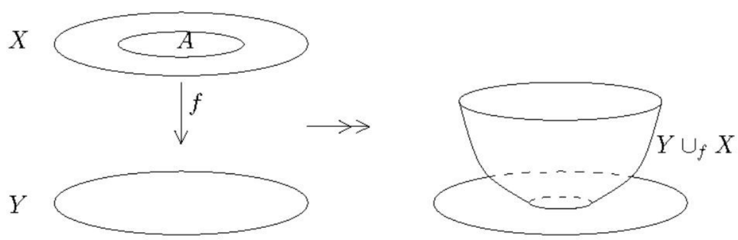
$$\begin{array}{ccc} A & \overset{g}{\longrightarrow} & Y \\ {}^{\mathllap{f}}\downarrow & & \downarrow^{\mathrlap{g_\ast f}} \\ X & \longrightarrow & X \sqcup_A Y \,. \end{array}$$

(Here $g_* f$ is also called the pushout of f, or the *cobase change* of f along g.)

This is equivalently the coequalizer of the two morphisms from A to the coproduct of X with Y (example 1.7):

$$A \rightrightarrows X \sqcup Y \longrightarrow X \sqcup_A Y \,.$$

If g is an inclusion, one also writes $X \cup_f Y$ and calls this the *attaching space*.



By example 1.11 the pushout/attaching space is the quotient topological space

$$X \sqcup_A Y \simeq (X \sqcup Y)/\sim$$

of the disjoint union of X and Y subject to the equivalence relation which identifies a point in X with a point in Y if they have the same pre-image in A.

(graphics from Aguilar-Gitler-Prieto 02)

Notice that the defining universal property of this colimit means that completing the span

$$\begin{array}{ccc} A & \longrightarrow & Y \\ \downarrow & & \\ X & & \end{array}$$

to a commuting square

$$\begin{array}{ccc} A & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & Z \end{array}$$

is equivalent to finding a morphism

$$X \underset{A}{\sqcup} Y \longrightarrow Z \,.$$

Example 1.13. For $A \hookrightarrow X$ a topological subspace inclusion, example 1.3, then the pushout

$$\begin{array}{ccc} A & \hookrightarrow & X \\ \downarrow & (po) & \downarrow \\ \ast & \longrightarrow & X/A \end{array}$$

is the quotient space or *cofiber*, denoted X/A.

Example 1.14. An important special case of example 1.12:

For $n \in \mathbb{N}$ write

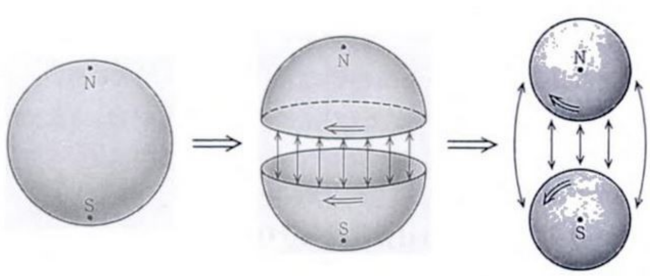
- $D^n \coloneqq \{\vec{x} \in \mathbb{R}^n \vert\, \vert \vec{x} \vert \leq 1\} \hookrightarrow \mathbb{R}^n$ for the standard topological n-disk (equipped with its subspace topology as a subset of Cartesian space);
- $S^{n-1} = \partial D^n \coloneqq \{\vec{x} \in \mathbb{R}^n \vert\, \vert \vec{x} \vert = 1\} \hookrightarrow \mathbb{R}^n$ for its boundary, the standard topological n-sphere.

Notice that $S^{-1} = \emptyset$ and that $S^0 = * \sqcup *$.

Let

$$i_n : S^{n-1} \longrightarrow D^n$$

be the canonical inclusion of the standard (n-1)-sphere as the boundary of the standard n-disk (both regarded as topological spaces with their subspace topology as subspaces of the Cartesian space $\mathbb{R}^n$).



Then the colimit in Top under the diagram

$$D^n \overset{i_n}{\longleftarrow} S^{n-1} \overset{i_n}{\longrightarrow} D^n,$$

i.e. the pushout of i_n along itself, is the n-sphere S^n:

$$\begin{array}{ccc} S^{n-1} & \overset{i_n}{\longrightarrow} & D^n \\ {}^{\mathllap{i_n}}\downarrow & (po) & \downarrow \\ D^n & \longrightarrow & S^n \end{array} .$$

(graphics from Ueno-Shiga-Morita 95)

Another kind of colimit that will play a role for certain technical constructions is transfinite composition. First recall

Definition 1.15. A *partial order* is a set S equipped with a relation $\leq$ such that for all elements $a,b,c \in S$

1) (reflexivity) $a \leq a$;

2) (transitivity) if $a \leq b$ and $b \leq c$ then $a \leq c$;

3) (antisymmetry) if $a \leq b$ and $\mathrm{b} \leq a$ then $a = b$.

This we may and will equivalently think of as a category with objects the elements of S and a unique morphism $a \to b$ precisely if $a \leq b$. In particular an order-preserving function between partially ordered sets is equivalently a functor between their corresponding categories.

A *bottom element* $\bot$ in a partial order is one such that $\bot \leq a$ for all a. A *top element* $\top$ is one for wich $a \leq \top$.

A partial order is a *total order* if in addition

4) (totality) either $a \leq b$ or $b \leq a$.

A total order is a *well order* if in addition

5) (well-foundedness) every non-empty subset has a least element.

An *ordinal* is the equivalence class of a well-order.

The *successor* of an ordinal is the class of the well-order with a top element freely adjoined.

A *limit ordinal* is one that is not a successor.

Example 1.16. The finite ordinals are labeled by $n \in \mathbb{N}$, corresponding to the well-orders $\{0 \leq 1 \leq 2 \cdots \leq n-1\}$. Here $(n+1)$ is the successor of n. The first non-empty limit ordinal is $\omega = [(\mathbb{N}, \leq)]$.

Definition 1.17. Let $\mathcal{C}$ be a category, and let $I \subset \mathrm{Mor}(\mathcal{C})$ be a class of its morphisms.

For α an ordinal (regarded as a category), an α-indexed *transfinite sequence* of elements in I is a diagram

$$X_\bullet : \alpha \longrightarrow \mathcal{C}$$

such that

1. $X_\bullet$ takes all successor morphisms $\beta \overset{\leq}{\to} \beta + 1$ in α to elements in I

$$X_{\beta,\beta+1} \in I$$

2. $X_\bullet$ is *continuous* in that for every nonzero limit ordinal $\beta < \alpha$, $X_\bullet$ restricted to the full-subdiagram $\{\gamma \mid \gamma \leq \beta\}$ is a colimiting cocone in $\mathcal{C}$ for $X_\bullet$ restricted to $\{\gamma \mid \gamma < \beta\}$.

The corresponding **transfinite composition** is the induced morphism

$$X_0 \longrightarrow X_\alpha \coloneqq \underset{\longrightarrow}{\lim} X_\bullet$$

into the colimit of the diagram, schematically:

$$\begin{array}{ccccccc} X_0 & \overset{X_{0,1}}{\longrightarrow} & X_1 & \overset{X_{1,2}}{\longrightarrow} & X_2 & \to & \cdots \\ & \searrow & \downarrow & \swarrow & \cdots & & \\ & & X_\alpha & & & & \end{array} .$$

We now turn to the discussion of mapping spaces/exponential objects.

Definition 1.18. For X a topological space and Y a locally compact topological space (in that for every point, every neighbourhood contains a compact neighbourhood), the **mapping space**

$$X^Y \in \mathrm{Top}$$

is the topological space

* whose underlying set is the set $\mathrm{Hom}_{\mathrm{Top}}(Y, X)$ of continuous functions $Y \to X$,
* whose open subsets are unions of finitary intersections of the following subbase elements of standard open subsets:

  the standard open subset $U^K \subset \mathrm{Hom}_{\mathrm{Top}}(Y, X)$ for

  * $K \hookrightarrow Y$ a compact topological space subset
  * $U \hookrightarrow X$ an open subset

  is the subset of all those continuous functions f that fit into a commuting diagram of the form

$$\begin{array}{ccc} K & \hookrightarrow & Y \\ \downarrow & & \downarrow^{f} \\ U & \hookrightarrow & X \end{array}.$$

Accordingly this is called the *compact-open topology* on the set of functions.

The construction extends to a functor

$$(-)^{(-)} : \mathrm{Top}_{\mathrm{lc}}^{\mathrm{op}} \times \mathrm{Top} \longrightarrow \mathrm{Top}\,.$$

Proposition 1.19. *For X a topological space and Y a locally compact topological space (in that for each point, each open neighbourhood contains a compact neighbourhood), the **topological mapping space** X^Y from def. 1.18 is an exponential object, i.e. the functor $(-)^Y$ is right adjoint to the product functor $Y \times (-)$: there is a natural bijection*

$$\mathrm{Hom}_{\mathrm{Top}}(Z \times Y, X) \simeq \mathrm{Hom}_{\mathrm{Top}}(Z, X^Y)$$

between continuous functions out of any product topological space of Y with any $Z \in \mathrm{Top}$ and continuous functions from Z into the mapping space.

A proof is spelled out here (or see e.g. Aguilar-Gitler-Prieto 02, prop. 1.3.1).

Remark 1.20. In the context of prop. 1.19 it is often assumed that Y is also a Hausdorff topological space. But this is not necessary. What assuming Hausdorffness only achieves is that all alternative definitions of "locally compact" become equivalent to the one that is needed for the proposition: for every point, every open neighbourhood contains a compact neighbourhood.

Remark 1.21. Proposition 1.19 fails in general if Y is not locally compact. Therefore the plain category Top of all topological spaces is not a Cartesian closed category.

This is no problem for the construction of the homotopy theory of topological spaces as such, but it becomes a technical nuisance for various constructions that one would like to perform within that homotopy theory. For instance on general pointed topological spaces the smash product is in general not associative.

On the other hand, without changing any of the following discussion one may just pass to a more convenient category of topological spaces such as notably the full subcategory of compactly generated topological spaces $\mathrm{Top}_{\mathrm{cg}} \hookrightarrow \mathrm{Top}$ (def. 3.35) which is Cartesian closed. This we turn to below.

Homotopy

The fundamental concept of homotopy theory is clearly that of *homotopy*. In the context of topological spaces this is about contiunous deformations of continuous functions parameterized by the standard interval:

Definition 1.22. Write

$$I \coloneqq [0,1] \hookrightarrow \mathbb{R}$$

for the standard topological interval, a compact connected topological subspace of the real line.

Equipped with the canonical inclusion of its two endpoints

$$* \sqcup * \xrightarrow{(\delta_0, \delta_1)} I \overset{\exists !}{\longrightarrow} *$$

this is the standard interval object in Top.

For $X \in \mathrm{Top}$, the product topological space $X \times I$, example 1.9, is called the standard *cylinder object* over X. The endpoint inclusions of the interval make it factor the codiagonal on X

$$\nabla_X \;:\; X \sqcup X \xrightarrow{((\mathrm{id},\delta_0),(\mathrm{id},\delta_1))} X \times I \longrightarrow X \;.$$

Definition 1.23. For $f, g : X \longrightarrow Y$ two continuous functions between topological spaces X, Y, then a **left homotopy**

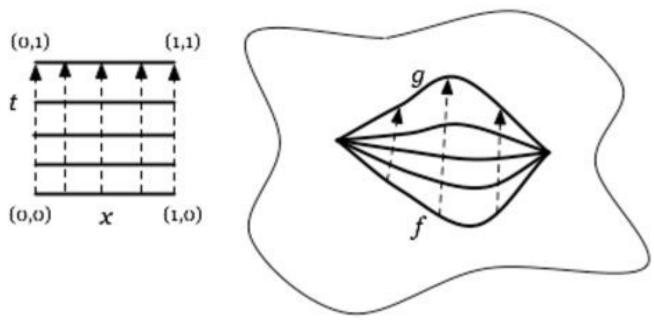
$$\eta : f \Rightarrow_L g$$

is a continuous function

$$\eta \;:\; X \times I \longrightarrow Y$$

out of the standard cylinder object over X, def. 1.22, such that this fits into a commuting diagram of the form

$$\begin{tikzcd}
X \arrow[d, "(\mathrm{id},\delta_0)"'] \arrow[dr, "f"] & \\
X \times I \arrow[r, "\eta"] & Y \,. \\
X \arrow[u, "(\mathrm{id},\delta_1)"] \arrow[ur, "g"'] &
\end{tikzcd}$$



(graphics grabbed from J. Tauber here)

Example 1.24. Let X be a topological space
and let $x, y \in X$ be two of its points, regarded as functions $x, y : * \longrightarrow X$ from the point to X. Then a left homotopy, def. 1.23, between these two functions is a commuting diagram of the form

$$\begin{tikzcd}
* \arrow[d, "\delta_0"'] \arrow[dr, "x"] & \\
I \arrow[r, "\eta"] & Y \,. \\
* \arrow[u, "\delta_1"] \arrow[ur, "y"'] &
\end{tikzcd}$$

This is simply a continuous path in X whose endpoints are x and y.

For instance:

Example 1.25. Let

$$\mathrm{const}_0 \;:\; I \longrightarrow * \xrightarrow{\delta_0} I$$

be the continuous function from the standard interval $I = [0,1]$ to itself that is constant on the value 0. Then there is a left homotopy, def. 1.23, from the identity function

$$\eta \;:\; \mathrm{id}_I \Rightarrow \mathrm{const}_0$$

given by

$$\eta(x,t) \coloneqq x(1-t) \;.$$

A key application of the concept of left homotopy is to the definition of homotopy groups:

Definition 1.26. For X a topological space, then its set $\pi_0(X)$ of *connected components*, also called the **0-th homotopy set**, is the set of left homotopy-equivalence classes (def. 1.23) of points $x : * \to X$, hence the set of path-connected components of X (example 1.24). By composition this extends to a functor

$$\pi_0 : \mathrm{Top} \longrightarrow \mathrm{Set} \,.$$

For $n \in \mathbb{N}$, $n \geq 1$ and for $x : * \to X$ any point, then the nth **homotopy group** $\pi_n(X,x)$ of X at x is the group

- whose underlying set is the set of left homotopy-equivalence classes of maps $I^n \longrightarrow X$ that take the boundary of I^n to x and where the left homotopies η are constrained to be constant on the boundary;
- whose group product operation takes $[\alpha : I^n \to X]$ and $[\beta : I^n \to X]$ to $[\alpha \cdot \beta]$ with

$$\alpha \cdot \beta \;:\; I^n \overset{\simeq}{\longrightarrow} I^n \underset{I^{n-1}}{\sqcup} I^n \overset{(\alpha,\beta)}{\longrightarrow} X \,,$$

where the first map is a homeomorphism from the unit n-cube to the n-cube with one side twice the unit length (e.g. $(x_1, x_2, x_3, \cdots) \mapsto (2x_1, x_2, x_3, \cdots)$).

By composition, this construction extends to a functor

$$\pi_{\bullet \geq 1} \;:\; \mathrm{Top}^{*/} \longrightarrow \mathrm{Grp}^{\mathbb{N}_{\geq 1}}$$

from pointed topological spaces to graded groups.

Notice that often one writes the value of this functor on a morphism f as $f_* = \pi_\bullet(f)$.

Remark 1.27. At this point we don't go further into the abstract reason why def. 1.26 yields group structure above degree 0, which is that positive dimension spheres are H-cogroup objects. But this is important, for instance in the proof of the Brown representability theorem. See the section *Brown representability theorem* in Part S.

Definition 1.28. A continuous function $f : X \longrightarrow Y$ is called a **homotopy equivalence** if there exists a continuous function the other way around, $g : Y \longrightarrow X$, and left homotopies, def. 1.23, from the two composites to the identity:

$$\eta_1 \;:\; f \circ g \Rightarrow_L \mathrm{id}_Y$$

and

$$\eta_2 \;:\; g \circ f \Rightarrow_L \mathrm{id}_X \,.$$

If here η_2 is constant along I, f is said to exhibit X as a **deformation retract** of Y.

Example 1.29. For X a topological space and $X \times I$ its standard cylinder object of def. 1.22, then the projection $p : X \times I \longrightarrow X$ and the inclusion $(\mathrm{id}, \delta_0) : X \longrightarrow X \times I$ are homotopy equivalences, def. 1.28, and in fact are homotopy inverses to each other:

The composition

$$X \overset{(\mathrm{id},\delta_0)}{\longrightarrow} X \times I \overset{p}{\longrightarrow} X$$

is immediately the identity on X (i.e. homotopic to the identity by a trivial homotopy), while the composite

$$X \times I \overset{p}{\longrightarrow} X \overset{(\mathrm{id},\delta_0)}{\longrightarrow} X \times I$$

is homotopic to the identity on $X \times I$ by a homotopy that is pointwise in X that of example 1.25.

Definition 1.30. A continuous function $f : X \longrightarrow Y$ is called a **weak homotopy equivalence** if its image under all the homotopy group functors of def. 1.26 is an isomorphism, hence if

$$\pi_0(f) \, : \, \pi_0(X) \overset{\simeq}{\longrightarrow} \pi_0(X)$$

and for all $x \in X$ and all $n \geq 1$

$$\pi_n(f) \, : \, \pi_n(X,x) \overset{\simeq}{\longrightarrow} \pi_n(Y, f(y)) \, .$$

Proposition 1.31. *Every homotopy equivalence, def. 1.28, is a weak homotopy equivalence, def. 1.30.*

In particular a deformation retraction, def. 1.28, is a weak homotopy equivalence.

Proof. First observe that for all $X \in$ Top the inclusion maps

$$X \overset{(\mathrm{id}, \delta_0)}{\longrightarrow} X \times I$$

into the standard cylinder object, def. 1.22, are weak homotopy equivalences: by postcomposition with the contracting homotopy of the interval from example 1.25 all homotopy groups of $X \times I$ have representatives that factor through this inclusion.

Then given a general homotopy equivalence, apply the homotopy groups functor to the corresponding homotopy diagrams (where for the moment we notationally suppress the choice of basepoint for readability) to get two commuting diagrams

$$\begin{tikzcd} \pi_\bullet(X) \arrow[d, "\pi_\bullet(\mathrm{id},\delta_0)"'] \arrow[dr, "\pi_\bullet(f)\circ\pi_\bullet(g)"] & \\ \pi_\bullet(X \times I) \arrow[r, "\pi_\bullet(\eta)"] & \pi_\bullet(Y) \\ \pi_\bullet(X) \arrow[u, "\pi_\bullet(\mathrm{id},\delta_1)"] \arrow[ur, "\pi_\bullet(\mathrm{id})"'] & \end{tikzcd} \quad , \quad \begin{tikzcd} \pi_\bullet(Y) \arrow[d, "\pi_\bullet(\mathrm{id},\delta_0)"'] \arrow[dr, "\pi_\bullet(g)\circ\pi_\bullet(f)"] & \\ \pi_\bullet(Y \times I) \arrow[r, "\pi_\bullet(\eta)"] & \pi_\bullet(X) \\ \pi_\bullet(Y) \arrow[u, "\pi_\bullet(\mathrm{id},\delta_1)"] \arrow[ur, "\pi_\bullet(\mathrm{id})"'] & \end{tikzcd} \, .$$

By the previous observation, the vertical morphisms here are isomorphisms, and hence these diagrams exhibit $\pi_\bullet(f)$ as the inverse of $\pi_\bullet(g)$, hence both as isomorphisms. ▮

Remark 1.32. The converse of prop. 1.31 is not true generally: not every weak homotopy equivalence between topological spaces is a homotopy equivalence. (For an example with full details spelled out see for instance Fritsch, Piccinini: "Cellular Structures in Topology", p. 289-290).

However, as we will discuss below, it turns out that

1. every weak homotopy equivalence between CW-complexes is a homotopy equivalence (Whitehead's theorem, cor. 3.8);

2. every topological space is connected by a weak homotopy equivalence to a CW-complex (CW approximation, remark 3.12).

Example 1.33. For $X \in \mathrm{Top}$, the projection $X \times I \longrightarrow X$ from the cylinder object of X, def. 1.22, is a weak homotopy equivalence, def. 1.30. This means that the factorization

$$\nabla_X \, : \, X \sqcup X \hookrightarrow X \times I \overset{\simeq}{\longrightarrow} X$$

of the codiagonal ∇_X in def. 1.22, which in general is far from being a monomorphism, may be thought of as factoring it through a monomorphism after replacing X, up to weak homotopy equivalence, by $X \times I$.

In fact, further below (prop. 1.25) we see that $X \sqcup X \to X \times I$ has better properties than the generic monomorphism has, in particular better homotopy invariant properties: it has the left lifting property against all Serre fibrations $E \overset{p}{\longrightarrow} B$ that are also weak homotopy equivalences.

Of course the concept of left homotopy in def. 1.23 is accompanied by a concept of *right homotopy*. This we turn to now.

Definition 1.34. For X a topological space, its **standard topological path space object** is the topological mapping space X^I, prop. 1.19, out of the standard interval I of def. 1.22.

Example 1.35. The endpoint inclusion into the standard interval, def. 1.22, makes the path space X^I of def. 1.34 factor the diagonal on X through the inclusion of constant paths and the endpoint evaluation of paths:

$$\Delta_X \,:\, X \overset{X^{I \to *}}{\longrightarrow} X^I \overset{X^{* \sqcup * \to I}}{\longrightarrow} X \times X \,.$$

This is the formal dual to example 1.22. As in that example, below we will see (prop. 3.14) that this factorization has good properties, in that

1. $X^{I \to *}$ is a weak homotopy equivalence;

2. $X^{* \sqcup * \to I}$ is a Serre fibration.

So while in general the diagonal Δ_X is far from being an epimorphism or even just a Serre fibration, the factorization through the path space object may be thought of as replacing X, up to weak homotopy equivalence, by its path space, such as to turn its diagonal into a Serre fibration after all.

Definition 1.36. For $f,g : X \longrightarrow Y$ two continuous functions between topological spaces X,Y, then a **right homotopy** $f \Rightarrow_R g$ is a continuous function

$$\eta \,:\, X \longrightarrow Y^I$$

into the path space object of X, def. 1.34, such that this fits into a commuting diagram of the form

$$\begin{array}{ccc} && Y \\ & {}^{\mathllap{f}}\nearrow & \uparrow^{\mathrlap{X^{\delta_0}}} \\ X &\overset{\eta}{\longrightarrow}& Y^I \,. \\ & {}_{\mathllap{g}}\searrow & \downarrow^{\mathrlap{Y^{\delta_1}}} \\ && Y \end{array}$$

Cell complexes

We consider topological spaces that are built consecutively by attaching basic cells.

Definition 1.37. Write

$$I_{\text{Top}} \coloneqq \left\{ S^{n-1} \overset{\iota_n}{\hookrightarrow} D^n \right\}_{n \in \mathbb{N}} \subset \text{Mor}(\text{Top})$$

for the set of canonical boundary inclusion maps of the standard n-disks, example 1.14. This going to be called the set of standard **topological generating cofibrations**.

Definition 1.38. For $X \in \text{Top}$ and for $n \in \mathbb{N}$, an n**-cell attachment** to X is the pushout ("attaching space", example 1.12) of a generating cofibration, def. 1.37

$$
\begin{array}{ccc}
S^{n-1} & \overset{\phi}{\longrightarrow} & X \\
{}^{\mathllap{\iota_n}}\downarrow & (\text{po}) & \downarrow \\
D^n & \longrightarrow & X \underset{S^{n-1}}{\sqcup} D^n & = X \cup_\phi D^n
\end{array}
$$

along some continuous function ϕ.

A continuous function $f : X \longrightarrow Y$ is called a **topological relative cell complex** if it is exhibited by a (possibly infinite) sequence of cell attachments to X, in that it is a transfinite composition (def. 1.17) of pushouts (example 1.12)

$$
\begin{array}{ccc}
\coprod_i S^{n_i - 1} & \longrightarrow & X_k \\
{}^{\mathllap{\coprod_i \iota_{n_i}}}\downarrow & (\text{po}) & \downarrow \\
\coprod_i D^{n_i} & \longrightarrow & X_{k+1}
\end{array}
$$

of coproducts (example 1.7) of generating cofibrations (def. 1.37).

A topological space X is a **cell complex** if $\emptyset \longrightarrow X$ is a relative cell complex.

A relative cell complex is called a **finite relative cell complex** if it is obtained from a finite number of cell attachments.

A (relative) cell complex is called a (relative) **CW-complex** if the above transfinite composition is countable

$$
\begin{array}{ccccccc}
X = X_0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & \cdots \\
& {}_{\mathllap{f}}\searrow & \downarrow & \swarrow & & \cdots \\
& & Y = \underset{\longrightarrow}{\lim} X_{\bullet} & & & &
\end{array}
$$

and if X_k is obtained from X_{k-1} by attaching cells precisely only of dimension k.

Remark 1.39. Strictly speaking a relative cell complex, def. 1.38, is a function $f : X \to Y$, *together* with its cell structure, hence together with the information of the pushout diagrams and the transfinite composition of the pushout maps that exhibit it.

In many applications, however, all that matters is that there is *some* (relative) cell decomosition, and then one tends to speak loosely and mean by a (relative) cell complex only a (relative) topological space that admits some cell decomposition.

The following lemma 1.40, together with lemma 1.52 below are the only two statements of the entire development here that involve the concrete particular nature of topological spaces ("point-set topology"), everything beyond that is general abstract homotopy theory.

Lemma 1.40. *Assuming the axiom of choice and the law of excluded middle, every compact subspace of a topological cell complex, def. 1.38, intersects the interior of a finite number of cells.*

(e.g. Hirschhorn 15, section 3.1)

Proof. So let Y be a topological cell complex and $C \hookrightarrow Y$ a compact subspace. Define a subset

$$
P \subset Y
$$

by *choosing* one point in the interior of the intersection with C of each cell of Y that intersects C.

It is now sufficient to show that P has no accumulation point. Because, by the compactness of X, every non-finite subset of C does have an accumulation point, and hence the lack of such shows that P is a finite set and hence that C intersects the interior of finitely many cells of Y.

To that end, let $c \in C$ be any point. If c is a 0-cell in Y, write $U_c \coloneqq \{c\}$. Otherwise write e_c for the unique cell of Y that contains c in its interior. By construction, there is exactly one point of P in the interior of e_c. Hence there is an open neighbourhood $c \in U_c \subset e_c$ containing no further points of P beyond possibly c itself, if c happens to be that single point of P in e_c.

It is now sufficient to show that U_c may be enlarged to an open subset $\tilde U_c$ of Y containing no point of P, except for possibly c itself, for that means that c is not an accumulation point of P.

To that end, let α_c be the ordinal that labels the stage Y_{α_c} of the transfinite composition in the cell complex-presentation of Y at which the cell e_c above appears. Let γ be the ordinal of the full cell complex. Then define the set

$$T \coloneqq \left\{ (\beta, U) \mid \alpha_c \leq \beta \leq \gamma \;,\; U \underset{\text{open}}{\subset} Y_\beta \;,\; U \cap Y_\alpha = U_c \;,\; U \cap P \in \{\emptyset, \{c\}\} \right\},$$

and regard this as a partially ordered set by declaring a partial ordering via

$$(\beta_1, U_1) < (\beta_2, U_2) \quad \Leftrightarrow \quad \beta_1 < \beta_2 \;,\; U_2 \cap Y_{\beta_1} = U_1 \;.$$

This is set up such that every element (β, U) of T with β the maximum value $\beta = \gamma$ is an extension $\tilde U_c$ that we are after.

Observe then that for $(\beta_s, U_s)_{s \in S}$ a chain in $(T, <)$ (a subset on which the relation $<$ restricts to a total order), it has an upper bound in T given by the union $(\cup_s \beta_s, \cup_s U_s)$. Therefore Zorn's lemma applies, saying that $(T, <)$ contains a maximal element $(\beta_{\max}, U_{\max})$.

Hence it is now sufficient to show that $\beta_{\max} = \gamma$. We argue this by showing that assuming $\beta_{\max} < \gamma$ leads to a contradiction.

So assume $\beta_{\max} < \gamma$. Then to construct an element of T that is larger than $(\beta_{\max}, U_{\max})$, consider for each cell d at stage $Y_{\beta_{\max}+1}$ its attaching map $h_d : S^{n-1} \to Y_{\beta_{\max}}$ and the corresponding preimage open set $h_d^{-1}(U_{\max}) \subset S^{n-1}$. Enlarging all these preimages to open subsets of D^n (such that their image back in $X_{\beta_{\max}+1}$ does not contain c), then $(\beta_{\max}, U_{\max}) < (\beta_{\max} + 1, \cup_d U_d)$. This is a contradiction. Hence $\beta_{\max} = \gamma$, and we are done. ∎

It is immediate and useful to generalize the concept of topological cell complexes as follows.

Definition 1.41. For $\mathcal{C}$ any category and for $K \subset \mathrm{Mor}(\mathcal{C})$ any sub-class of its morphisms, a **relative K-cell complexes** is a morphism in $\mathcal{C}$ which is a transfinite composition (def. 1.17) of pushouts of coproducts of morphsims in K.

Definition 1.42. Write

$$J_{\mathrm{Top}} \coloneqq \left\{ D^n \overset{(\mathrm{id}, \delta_0)}{\hookrightarrow} D^n \times I \right\}_{n \in \mathbb{N}} \subset \mathrm{Mor}(\mathrm{Top})$$

for the set of inclusions of the topological n-disks, def. 1.37, into their cylinder objects, def. 1.22, along (for definiteness) the left endpoint inclusion.

These inclusions are similar to the standard topological generating cofibrations I_{Top} of def. 1.37, but in contrast to these they are "acyclic" (meaning: trivial on homotopy classes of maps from "cycles" given by n-spheres) in that they are weak homotopy equivalences (by prop. 1.31).

Accordingly, J_{Top} is to be called the set of standard **topological generating acyclic cofibrations**.

Lemma 1.43. *For X a CW-complex (def. 1.38), then its inclusion $X \overset{(\mathrm{id},\delta_0)}{\longrightarrow} X \times I$ into its standard cylinder (def. 1.22) is a J_{Top}-relative cell complex (def. 1.41, def. 1.42).*

Proof. First erect a cylinder over all 0-cells

$$\begin{array}{ccc} \coprod_{x \in X_0} D^0 & \longrightarrow & X \\ \downarrow & (\mathrm{po}) & \downarrow \\ \coprod_{x \in X_0} D^1 & \longrightarrow & Y_1 \end{array} .$$

Assume then that the cylinder over all n-cells of X has been erected using attachment from J_{Top}. Then the union of any $(n+1)$-cell σ of X with the cylinder over its boundary is homeomorphic to D^{n+1} and is like the cylinder over the cell "with end and interior removed". Hence via attaching along $D^{n+1} \to D^{n+1} \times I$ the cylinder over σ is erected. ▮

Lemma 1.44. *The maps $D^n \hookrightarrow D^n \times I$ in def. 1.42 are finite relative cell complexes, def. 1.38. In other words, the elements of J_{Top} are I_{Top}-relative cell complexes.*

Proof. There is a homeomorphism

$$\begin{array}{ccc} D^n & = & D^n \\ {}^{(\mathrm{id},\delta_0)}\downarrow && \downarrow \\ D^n \times I & \simeq & D^{n+1} \end{array}$$

such that the map on the right is the inclusion of one hemisphere into the boundary n-sphere of D^{n+1}. This inclusion is the result of attaching two cells:

$$\begin{array}{ccc} S^{n-1} & \overset{\iota_n}{\longrightarrow} & D^n \\ {}^{\iota_n}\downarrow & (\mathrm{po}) & \downarrow \\ D^n & \longrightarrow & S^n \\ && \downarrow^{=} \\ S^n & \overset{\mathrm{id}}{\longrightarrow} & S^n \\ {}^{\iota_{n+1}}\downarrow & (\mathrm{po}) & \downarrow \\ D^{n+1} & \underset{\mathrm{id}}{\longrightarrow} & D^{n+1} \end{array} .$$

here the top pushout is the one from example 1.14. ▮

Lemma 1.45. *Every J_{Top}-relative cell complex (def. 1.42, def. 1.41) is a weak homotopy equivalence, def. 1.30.*

Proof. Let $X \longrightarrow \hat X = \underset{\longleftarrow}{\lim}_{\beta \leq \alpha} X_\beta$ be a J_{Top}-relative cell complex.

First observe that with the elements $D^n \hookrightarrow D^n \times I$ of J_{Top} being homotopy equivalences for all $n \in \mathbb{N}$ (by example 1.29), each of the stages $X_\beta \longrightarrow X_{\beta+1}$ in the relative cell complex is also a homotopy equivalence. We make this fully explicit:

By definition, such a stage is a pushout of the form

$$\begin{array}{ccc} \underset{i \in I}{\sqcup} D^{n_i} & \longrightarrow & X_\beta \\ {}^{\underset{i \in I}{\sqcup}(\mathrm{id},\delta_0)}\downarrow & (\mathrm{po}) & \downarrow \\ \underset{i \in I}{\sqcup} D^{n_i} \times I & \longrightarrow & X_{\beta+1} \end{array} .$$

Then the fact that the projections $p_{n_i} : D^{n_i} \times I \to D^{n_i}$ are strict left inverses to the inclusions (id, δ_0) gives a commuting square of the form

$$\begin{array}{ccc} \underset{i \in I}{\sqcup} D^{n_i} & \longrightarrow & X_\beta \\ {}^{\mathllap{\underset{i \in I}{\sqcup} (\mathrm{id}, \delta_0)}}\downarrow && \downarrow^{\mathrlap{\mathrm{id}}} \\ \underset{i \in I}{\sqcup} D^{n_i} \times I && \\ {}^{\mathllap{\underset{i \in I}{\sqcup} p_{n_i}}}\downarrow && \downarrow \\ \underset{i \in I}{\sqcup} D^{n_i} & \longrightarrow & X_\beta \end{array}$$

and so the universal property of the colimit (pushout) $X_{\beta+1}$ gives a factorization of the identity morphism on the right through $X_{\beta+1}$

$$\begin{array}{ccc} \underset{i \in I}{\sqcup} D^{n_i} & \longrightarrow & X_\beta \\ {}^{\mathllap{\underset{i \in I}{\sqcup} (\mathrm{id}, \delta_0)}}\downarrow && \downarrow \\ \underset{i \in I}{\sqcup} D^{n_i} \times I & \longrightarrow & X_{\beta+1} \\ {}^{\mathllap{\underset{i \in I}{\sqcup} p_{n_i}}}\downarrow && \downarrow \\ \underset{i \in I}{\sqcup} D^{n_i} & \longrightarrow & X_\beta \end{array}$$

which exhibits $X_{\beta+1} \to X_\beta$ as a strict left inverse to $X_\beta \to X_{\beta+1}$. Hence it is now sufficient to show that this is also a homotopy right inverse.

To that end, let

$$\eta_{n_i} : D^{n_i} \times I \longrightarrow D^{n_i} \times I$$

be the left homotopy that exhibits p_{n_i} as a homotopy right inverse to p_{n_i} by example 1.29. For each $t \in [0,1]$ consider the commuting square

$$\begin{array}{ccc} \underset{i \in I}{\sqcup} D^{n_i} & \longrightarrow & X_\beta \\ \downarrow && \downarrow \\ \underset{i \in I}{\sqcup} D^{n_i} \times I && X_{\beta+1} \\ {}^{\mathllap{\eta_{n_i}(-,t)}}\downarrow && \downarrow^{\mathrlap{\mathrm{id}}} \\ \underset{i \in I}{\sqcup} D^{n_i} \times I & \longrightarrow & X_{\beta+1} \end{array}.$$

Regarded as a cocone under the span in the top left, the universal property of the colimit (pushout) $X_{\beta+1}$ gives a continuous function

$$\eta(-,t) \;:\; X_{\beta+1} \longrightarrow X_{\beta+1}$$

for each $t \in [0,1]$. For $t = 0$ this construction reduces to the provious one in that $\eta(-,0) : X_{\beta+1} \to X_\beta \to X_{\beta+1}$ is the composite which we need to homotope to the identity; while $\eta(-,1)$ is the identity. Since $\eta(-,t)$ is clearly also continuous in t it constitutes a continuous function

$$\eta \;:\; X_{\beta+1} \times I \longrightarrow X_{\beta+1}$$

which exhibits the required left homotopy.

So far this shows that each stage $X_\beta \to X_{\beta+1}$ in the transfinite composition defining $\hat{X}$ is a homotopy equivalence, hence, by prop. 1.31, a weak homotopy equivalence.

This means that all morphisms in the following diagram (notationally suppressing basepoints and showing only the finite stages)

$$\begin{tikzcd} \pi_n(X) \arrow[r, "\simeq"] \arrow[dr, "\simeq"'] & \pi_n(X_1) \arrow[r, "\simeq"] \arrow[d, "\simeq"] & \pi_n(X_2) \arrow[r, "\simeq"] \arrow[dl, "\simeq"] & \pi_n(X_3) \arrow[r, "\simeq"] & \cdots \\ & \underset{\longleftarrow}{\lim}_\alpha \pi_n(X_\alpha) & \cdots & & \end{tikzcd}$$

are isomorphisms.

Moreover, lemma 1.40 gives that every representative and every null homotopy of elements in $\pi_n(\hat{X})$ already exists at some finite stage X_k. This means that also the universally induced morphism

$$\underset{\longleftarrow}{\lim}_\alpha \pi_n(X_\alpha) \overset{\simeq}{\longrightarrow} \pi_n(\hat{X})$$

is an isomorphism. Hence the composite $\pi_n(X) \overset{\simeq}{\longrightarrow} \pi_n(\hat{X})$ is an isomorphism. ∎

Fibrations

Given a relative C-cell complex $\iota : X \to Y$, def. 1.41, it is typically interesting to study the extension problem along f, i.e. to ask which topological spaces E are such that every continuous function $f : X \longrightarrow E$ has an extension $\tilde{f}$ along ι

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\iota"'] & E \\ Y \arrow[ur, "\exists \tilde{f}"'] & \end{tikzcd}.$$

If such extensions exists, it means that E is sufficiently "spread out" with respect to the maps in C. More generally one considers this extension problem fiberwise, i.e. with both E and Y (hence also X) equipped with a map to some base space B:

Definition 1.46. Given a category $\mathcal{C}$ and a sub-class $C \subset \mathrm{Mor}(\mathcal{C})$ of its morphisms, then a morphism $p : E \longrightarrow B$ in $\mathcal{C}$ is said to have the right lifting property against the morphisms in C if every commuting diagram in $\mathcal{C}$ of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d, "c"'] & E \arrow[d, "p"] \\ Y \arrow[r] & B \end{tikzcd},$$

with $c \in C$, has a lift h, in that it may be completed to a commuting diagram of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d, "c"'] & E \arrow[d, "p"] \\ Y \arrow[r] \arrow[ur, "h"] & B \end{tikzcd}.$$

We will also say that f is a C-**injective morphism** if it satisfies the right lifting property against C.

Definition 1.47. A continuous function $p : E \longrightarrow B$ is called a **Serre fibration** if it is a J_{Top}-injective morphism; i.e. if it has the right lifting property, def. 1.46, against all topological generating acylic cofibrations, def. 1.42; hence if for every commuting diagram of continuous functions of the form

$$\begin{array}{ccc} D^n & \longrightarrow & E \\ {}^{(\mathrm{id},\delta_0)}\downarrow && \downarrow^p, \\ D^n \times I & \longrightarrow & B \end{array}$$

has a lift h, in that it may be completed to a commuting diagram of the form

$$\begin{array}{ccc} D^n & \longrightarrow & E \\ {}^{(\mathrm{id},\delta_0)}\downarrow & {}^{h}\nearrow & \downarrow^p. \\ D^n \times I & \longrightarrow & B \end{array}$$

Remark 1.48. Def. 1.47 says, in view of the definition of left homotopy, that a Serre fibration p is a map with the property that given a left homotopy, def. 1.23, between two functions into its codomain, and given a lift of one the two functions through p, then also the homotopy between the two lifts. Therefore the condition on a Serre fibration is also called the *homotopy lifting property* for maps whose domain is an n-disk.

More generally one may ask functions p to have such homotopy lifting property for functions with arbitrary domain. These are called *Hurewicz fibrations*.

Remark 1.49. The precise shape of D^n and $D^n \times I$ in def. 1.47 turns out not to actually matter much for the nature of Serre fibrations. We will eventually find below (prop. 3.5) that what actually matters here is only that the inclusions $D^n \hookrightarrow D^n \times I$ are relative cell complexes (lemma 1.44) and weak homotopy equivalences (prop. 1.31) and that all of these may be generated from them in a suitable way.

But for simple special cases this is readily seen directly, too. Notably we could replace the n-disks in def. 1.47 with any homeomorphic topological space. A choice important in the comparison to the classical model structure on simplicial sets (below) is to instead take the topological n-simplices Δ^n. Hence a Serre fibration is equivalently characterized as having lifts in all diagrams of the form

$$\begin{array}{ccc} \Delta^n & \longrightarrow & E \\ {}^{(\mathrm{id},\delta_0)}\downarrow && \downarrow^p. \\ \Delta^n \times I & \longrightarrow & B \end{array}$$

Other deformations of the n-disks are useful in computations, too. For instance there is a homeomorphism from the n-disk to its "cylinder with interior and end removed", formally:

$$\begin{array}{ccc} (D^n \times \{0\}) \cup (\partial D^n \times I) & \simeq & D^n \\ \downarrow && \downarrow \\ D^n \times I & \simeq & D^n \times I \end{array}$$

and hence f is a Serre fibration equivalently also if it admits lifts in all diagrams of the form

$$\begin{array}{ccc} (D^n \times \{0\}) \cup (\partial D^n \times I) & \longrightarrow & E \\ {}^{(\mathrm{id},\delta_0)}\downarrow && \downarrow^p. \\ \Delta^n \times I & \longrightarrow & B \end{array}$$

The following is a general fact about closure of morphisms defined by lifting properties which we prove in generality below as prop. 2.10.

Proposition 1.50. *A Serre fibration, def. 1.47 has the right lifting property against all retracts (see remark 2.12) of J_{Top}-relative cell complexes (def. 1.42, def. 1.38).*

The following statement is foreshadowing the long exact sequences of homotopy groups (below) induced by any fiber sequence, the full version of which we come to below (example 4.37) after having developed more of the abstract homotopy theory.

Proposition 1.51. *Let $f : X \longrightarrow Y$ be a Serre fibration, def. 1.47, let $y : * \to Y$ be any point and write*

$$F_y \overset{\iota}{\hookrightarrow} X \overset{f}{\longrightarrow} Y$$

for the fiber inclusion over that point. Then for every choice $x : * \to X$ of lift of the point y through f, the induced sequence of homotopy groups

$$\pi_\bullet(F_y, x) \overset{\iota_*}{\longrightarrow} \pi_\bullet(X, x) \overset{f_*}{\longrightarrow} \pi_\bullet(Y)$$

is exact, in that the kernel of f_* is canonically identified with the image of ι_*:

$$\ker(f_*) \simeq \mathrm{im}(\iota_*) \; .$$

Proof. It is clear that the image of ι_* is in the kernel of f_* (every sphere in $F_y \hookrightarrow X$ becomes constant on y, hence contractible, when sent forward to Y).

For the converse, let $[\alpha] \in \pi_\bullet(X, x)$ be represented by some $\alpha : S^{n-1} \to X$. Assume that $[\alpha]$ is in the kernel of f_*. This means equivalently that α fits into a commuting diagram of the form

$$\begin{array}{ccc} S^{n-1} & \overset{\alpha}{\longrightarrow} & X \\ \downarrow && \downarrow^{f}, \\ D^n & \overset{\kappa}{\longrightarrow} & Y \end{array}$$

where κ is the contracting homotopy witnessing that $f_*[\alpha] = 0$.

Now since x is a lift of y, there exists a left homotopy

$$\eta \; : \; \kappa \Rightarrow \mathrm{const}_y$$

as follows:

$$\begin{array}{ccccc} && S^{n-1} & \overset{\alpha}{\longrightarrow} & X \\ && {}^{\iota_n}\downarrow && \downarrow^{f} \\ && D^n & \overset{\kappa}{\longrightarrow} & Y \\ && \downarrow^{(\mathrm{id},\delta_1)} && \downarrow^{\mathrm{id}} \\ D^n & \overset{(\mathrm{id},\delta_0)}{\longrightarrow} & D^n \times I & \overset{\eta}{\longrightarrow} & Y \\ \downarrow &&&& \downarrow \\ * && \overset{y}{\longrightarrow} && Y \end{array}$$

(for instance: regard D^n as embedded in $\mathbb{R}^n$ such that $0 \in \mathbb{R}^n$ is identified with the basepoint on the boundary of D^n and set $\eta(\vec{v}, t) \coloneqq \kappa(t\vec{v})$).

The pasting of the top two squares that have appeared this way is equivalent to the following commuting square

$$\begin{array}{ccccc} S^{n-1} & \longrightarrow & & \overset{\alpha}{\longrightarrow} & X \\ {}^{(\mathrm{id},\delta_1)}\downarrow &&&& \downarrow^{f}. \\ S^{n-1} \times I & \overset{(\iota_n,\mathrm{id})}{\longrightarrow} & D^n \times I & \overset{\eta}{\longrightarrow} & Y \end{array}$$

Because f is a Serre fibration and by lemma 1.43 and prop. 1.50, this has a lift

$$\tilde{\eta} : S^{n-1} \times I \longrightarrow X .$$

Notice that $\tilde{\eta}$ is a basepoint preserving left homotopy from $\alpha = \tilde{\eta}|_1$ to some $\alpha' \coloneqq \tilde{\eta}|_0$. Being homotopic, they represent the same element of $\pi_{n-1}(X, x)$:

$$[\alpha'] = [\alpha] .$$

But the new representative α' has the special property that its image in Y is not just trivializable, but trivialized: combining $\tilde{\eta}$ with the previous diagram shows that it sits in the following commuting diagram

$$\begin{array}{ccccc} \alpha' : & S^{n-1} & \overset{(\mathrm{id}, \delta_0)}{\longrightarrow} & S^{n-1} \times I & \overset{\tilde{\eta}}{\longrightarrow} & X \\ & \downarrow^{\iota_n} && \downarrow^{(\iota_n, \mathrm{id})} && \downarrow^f \\ & D^n & \overset{(\mathrm{id}, \delta_0)}{\longrightarrow} & D^n \times I & \overset{\eta}{\longrightarrow} & Y \\ & \downarrow &&&& \downarrow \\ & * && \overset{y}{\longrightarrow} && Y \end{array} .$$

The commutativity of the outer square says that $f_* \alpha'$ is constant, hence that α' is entirely contained in the fiber F_y. Said more abstractly, the universal property of fibers gives that α' factors through $F_y \overset{\iota}{\hookrightarrow} X$, hence that $[\alpha'] = [\alpha]$ is in the image of ι_*. ▮

The following lemma 1.52, together with lemma 1.40 above, are the only two statements of the entire development here that crucially involve the concrete particular nature of topological spaces ("point-set topology"), everything beyond that is general abstract homotopy theory.

Lemma 1.52. *The continuous functions with the right lifting property, def. 1.46 against the set $I_{\mathrm{Top}} = \{S^{n-1} \hookrightarrow D^n\}$ of topological generating cofibrations, def. 1.37, are precisely those which are both weak homotopy equivalences, def. 1.30 as well as Serre fibrations, def. 1.47.*

Proof. We break this up into three sub-statements:

A) I_{Top}-injective morphisms are in particular weak homotopy equivalences

Let $p : \hat{X} \to X$ have the right lifting property against I_{Top}

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & \hat{X} \\ {}^{\iota_n}\downarrow & {}^{\exists}\nearrow & \downarrow^{p} \\ D^n & \longrightarrow & X \end{array}$$

We check that the lifts in these diagrams exhibit $\pi_\bullet(f)$ as being an isomorphism on all homotopy groups, def. 1.26:

For $n = 0$ the existence of these lifts says that every point of X is in the image of p, hence that $\pi_0(\hat{X}) \to \pi_0(X)$ is surjective. Let then $S^0 = * \coprod * \longrightarrow \hat{X}$ be a map that hits two connected components, then the existence of the lift says that if they have the same image in $\pi_0(X)$ then they were already the same connected component in $\hat{X}$. Hence $\pi_0(\hat{X}) \to \pi_0(X)$ is also injective and hence is a bijection.

Similarly, for $n \geq 1$, if $S^n \to \hat{X}$ represents an element in $\pi_n(\hat{X})$ that becomes trivial in $\pi_n(X)$, then the existence of the lift says that it already represented the trivial element itself. Hence $\pi_n(\hat{X}) \to \pi_n(X)$ has trivial kernel and so is injective.

Finally, to see that $\pi_n(\hat X) \to \pi_n(X)$ is also surjective, hence bijective, observe that every elements in $\pi_n(X)$ is equivalently represented by a commuting diagram of the form

$$\begin{array}{ccccc} S^{n-1} & \longrightarrow & * & \longrightarrow & \hat X \\ \downarrow && \downarrow && \downarrow \\ D^n & \longrightarrow & X & = & X \end{array}$$

and so here the lift gives a representative of a preimage in $\pi_n(\hat X)$.

B) I_{Top}-injective morphisms are in particular Serre fibrations

By an immediate closure property of lifting problems (we spell this out in generality as prop. 2.10, cor. 2.11 below) an I_{Top}-injective morphism has the right lifting property against all relative cell complexes, and hence, by lemma 1.44, it is also a J_{Top}-injective morphism, hence a Serre fibration.

C) Acyclic Serre fibrations are in particular I_{Top}-injective morphisms

(Hirschhorn 15, section 6).

Let $f : X \to Y$ be a Serre fibration that induces isomorphisms on homotopy groups. In degree 0 this means that f is an isomorphism on connected components, and this means that there is a lift in every commuting square of the form

$$\begin{array}{ccc} S^{-1} = \emptyset & \longrightarrow & X \\ \downarrow && \downarrow^{f} \\ D^0 = * & \longrightarrow & Y \end{array}$$

(this is $\pi_0(f)$ being surjective) and in every commuting square of the form

$$\begin{array}{ccc} S^0 & \longrightarrow & X \\ {}^{\iota_0}\downarrow && \downarrow^{f} \\ D^1 = * & \longrightarrow & Y \end{array}$$

(this is $\pi_0(f)$ being injective). Hence we are reduced to showing that for $n \geq 2$ every diagram of the form

$$\begin{array}{ccc} S^{n-1} & \stackrel{\alpha}{\longrightarrow} & X \\ {}^{\iota_n}\downarrow && \downarrow^{f} \\ D^n & \stackrel{\kappa}{\longrightarrow} & Y \end{array}$$

has a lift.

To that end, pick a basepoint on S^{n-1} and write x and y for its images in X and Y, respectively

Then the diagram above expresses that $f_*[\alpha] = 0 \in \pi_{n-1}(Y,y)$ and hence by assumption on f it follows that $[\alpha] = 0 \in \pi_{n-1}(X,x)$, which in turn mean that there is κ' making the upper triangle of our lifting problem commute:

$$\begin{array}{ccc} S^{n-1} & \stackrel{\alpha}{\longrightarrow} & X \\ {}^{\iota_n}\downarrow & \nearrow_{\kappa'} & \\ D^n && \end{array} \,.$$

It is now sufficient to show that any such κ' may be deformed to a ρ' which keeps making this upper triangle commute but also makes the remaining lower triangle commute.

To that end, notice that by the commutativity of the original square, we already have at least this commuting square:

$$\begin{array}{ccc} S^{n-1} & \overset{\iota_n}{\longrightarrow} & D^n \\ {}^{\iota_n}\downarrow && \downarrow^{f\circ\kappa'} \\ D^n & \underset{\kappa}{\longrightarrow} & Y \end{array}.$$

This induces the universal map $(\kappa, f\circ\kappa')$ from the pushout of its cospan in the top left, which is the n-sphere (see this example):

$$\begin{array}{ccccc} S^{n-1} & \overset{\iota_n}{\longrightarrow} & D^n \\ {}^{\iota_n}\downarrow & (\text{po}) & \downarrow^{f\circ\kappa'} \\ D^n & \underset{\kappa}{\longrightarrow} & S^n \\ && & \searrow^{(\kappa, f\circ\kappa')} \\ && && Y \end{array}.$$

This universal morphism represents an element of the nth homotopy group:

$$[(\kappa, f\circ\kappa')] \in \pi_n(Y,y) \;.$$

By assumption that f is a weak homotopy equivalence, there is a $[\rho]\in\pi_n(X,x)$ with

$$f_*[\rho] = [(\kappa, f\circ\kappa')]$$

hence on representatives there is a lift up to homotopy

$$\begin{array}{ccc} && X \\ & {}^{\rho}\nearrow_{\Downarrow} & \downarrow^{f} \\ S^n & \underset{(\kappa, f\circ\kappa')}{\longrightarrow} & Y \end{array}.$$

Morever, we may always find ρ of the form (ρ', κ') for some $\rho' : D^n \to X$. ("Paste κ' to the reverse of ρ.")

Consider then the map

$$S^n \overset{(f\circ\rho', \kappa)}{\longrightarrow} Y$$

and observe that this represents the trivial class:

$$\begin{aligned} [(f\circ\rho', \kappa)] &= [(f\circ\rho', f\circ\kappa')] + [(f\circ\kappa', \kappa)] \\ &= f_* \underbrace{[(\rho',\kappa')]}_{=[\rho]} + [(f\circ\kappa', \kappa)] \\ &= [(\kappa, f\circ\kappa')] + [(f\circ\kappa', \kappa)] \\ &= 0 \end{aligned}.$$

This means equivalently that there is a homotopy

$$\phi : f\circ\rho' \Rightarrow \kappa$$

fixing the boundary of the n-disk.

Hence if we denote homotopy by double arrows, then we have now achieved the following situation

$$
\begin{array}{ccc}
S^{n-1} & \overset{\alpha}{\longrightarrow} & X \\
{}^{\iota_n}\downarrow & {}^{\rho'} \nearrow_{\Downarrow \phi} & \downarrow^{f} \\
D^n & \longrightarrow & Y
\end{array}
$$

and it now suffices to show that ϕ may be lifted to a homotopy of just ρ', fixing the boundary, for then the resulting homotopic ρ'' is the desired lift.

To that end, notice that the condition that $\phi : D^n \times I \to Y$ fixes the boundary of the n-disk means equivalently that it extends to a morphism

$$
S^{n-1} \underset{S^{n-1}\times I}{\sqcup} D^n \times I \overset{(f\circ\alpha, \phi)}{\longrightarrow} Y
$$

out of the pushout that identifies in the cylinder over D^n all points lying over the boundary. Hence we are reduced to finding a lift in

$$
\begin{array}{ccc}
D^n & \overset{\rho'}{\longrightarrow} & X \\
\downarrow & & \downarrow^{f} \\
S^{n-1} \underset{S^{n-1}\times I}{\sqcup} D^n \times I & \overset{(f\circ\alpha,\phi)}{\longrightarrow} & Y
\end{array}
.
$$

But inspection of the left map reveals that it is homeomorphic again to $D^n \to D^n \times I$, and hence the lift does indeed exist. ▮

2. Abstract homotopy theory

In the above we discussed three classes of continuous functions between topological spaces

1. weak homotopy equivalences;
2. relative cell complexes;
3. Serre fibrations

and we saw first aspects of their interplay via lifting properties.

A fundamental insight due to (Quillen 67) is that in fact *all* constructions in homotopy theory are elegantly expressible via just the abstract interplay of these classes of morphisms. This was distilled in (Quillen 67) into a small set of axioms called a **model category structure** (because it serves to make all objects be *models* for homotopy types.)

This *abstract homotopy theory* is the royal road for handling any flavor of homotopy theory, in particular the stable homotopy theory that we are after in *Part 1*. Here we discuss the basic constructions and facts in abstract homotopy theory, then below we conclude section P1) by showing that the above system of classes of maps of topological spaces is indeed an example.

Literature (Dwyer-Spalinski 95)

Definition 2.1. A **category with weak equivalences** is

1. a category $\mathcal{C}$;
2. a sub-class $W \subset \mathrm{Mor}(\mathcal{C})$ of its morphisms;

such that

1. W contains all the isomorphisms of $\mathcal{C}$;

2. W is closed under **two-out-of-three**: in every commuting diagram in $\mathcal{C}$ of the form

$$\begin{array}{ccccc} && Y && \\ & \nearrow && \searrow & \\ X && \longrightarrow && Z \end{array}$$

   if two of the three morphisms are in W, then so is the third.

Remark 2.2. It turns out that a category with weak equivalences, def. 2.1, already determines a homotopy theory: the one given given by universally forcing weak equivalences to become actual homotopy equivalences. This may be made precise and is called the *simplicial localization* of a category with weak equivalences (Dwyer-Kan 80a, Dwyer-Kan 80b, Dwyer-Kan 80c). However, without further auxiliary structure, these simplicial localizations are in general intractable. The further axioms of a model category serve the sole purpose of making the universal homotopy theory induced by a category with weak equivalences be tractable:

Definition 2.3. A **model category** is

1. a category $\mathcal{C}$ with all limits and colimits (def. 1.1);
2. three sub-classes $W, \mathrm{Fib}, \mathrm{Cof} \subset \mathrm{Mor}(\mathcal{C})$ of its morphisms;

such that

1. the class W makes $\mathcal{C}$ into a **category with weak equivalences**, def. 2.1;
2. The pairs $(W \cap \mathrm{Cof}\,,\ \mathrm{Fib})$ and $(\mathrm{Cap}\,,\ W \cap \mathrm{Fib})$ are both weak factorization systems, def. 2.5.

One says:

- elements in W are *weak equivalences*,
- elements in Cof are *cofibrations*,
- elements in Fib are *fibrations*,
- elements in $W \cap \mathrm{Cof}$ are *acyclic cofibrations*,
- elements in $W \cap \mathrm{Fib}$ are *acyclic fibrations*.

The form of def. 2.3 is due to (Joyal, def. E.1.2). It implies various other conditions that (Quillen 67) demands explicitly, see prop. 2.10 and prop. 2.14 below.

We now dicuss the concept of weak factorization systems appearing in def. 2.3.

Factorization systems

Definition 2.4. Let $\mathcal{C}$ be any category. Given a diagram in $\mathcal{C}$ of the form

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {}^{p}\downarrow && \\ B && \end{array}$$

then an *extension* of the morphism f along the morphism p is a completion to a commuting diagram of the form

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"'] & Y \\ B \arrow[ur, "\tilde{f}"'] & \end{tikzcd}.$$

Dually, given a diagram of the form

$$\begin{array}{ccc} & & A \\ & & \downarrow^{p} \\ X & \overset{f}{\longrightarrow} & Y \end{array}$$

then a **lift** of f through p is a completion to a commuting diagram of the form

$$\begin{tikzcd} & A \arrow[d, "p"] \\ X \arrow[r, "f"] \arrow[ur, "\tilde{f}"] & Y \end{tikzcd}.$$

Combining these cases: given a commuting square

$$\begin{array}{ccc} X_1 & \overset{f_1}{\longrightarrow} & Y_1 \\ {}^{p_l}\downarrow & & \downarrow^{p_r} \\ X_2 & \overset{f_1}{\longrightarrow} & Y_2 \end{array}$$

then a **lifting** in the diagram is a completion to a commuting diagram of the form

$$\begin{array}{ccc} X_1 & \overset{f_1}{\longrightarrow} & Y_1 \\ {}^{p_l}\downarrow & \nearrow & \downarrow^{p_r} \\ X_2 & \overset{f_1}{\longrightarrow} & Y_2 \end{array}.$$

Given a sub-class of morphisms $K \subset \mathrm{Mor}(\mathcal{C})$, then

- a morphism p_r as above is said to have the **right lifting property against** K or to be a K**-injective morphism** if in all square diagrams with p_r on the right and any $p_l \in K$ on the left a lift exists.

dually:

- a morphism p_l is said to have the **left lifting property against** K or to be a K**-projective morphism** if in all square diagrams with p_l on the left and any $p_r \in K$ on the left a lift exists.

Definition 2.5. A **weak factorization system** (WFS) on a category $\mathcal{C}$ is a pair $(\mathrm{Proj}, \mathrm{Inj})$ of classes of morphisms of $\mathcal{C}$ such that

1. Every morphism $f : X \to Y$ of $\mathcal{C}$ may be factored as the composition of a morphism in Proj followed by one in Inj

$$f : X \overset{\in \mathrm{Proj}}{\longrightarrow} Z \overset{\in \mathrm{Inj}}{\longrightarrow} Y .$$

2. The classes are closed under having the lifting property, def. 2.4, against each other:
    1. Proj is precisely the class of morphisms having the left lifting property against every morphisms in Inj;

2. Inj is precisely the class of morphisms having the right lifting property against every morphisms in Proj.

Definition 2.6. For $\mathcal{C}$ a category, a **functorial factorization** of the morphisms in $\mathcal{C}$ is a functor

$$\mathrm{fact} : \mathcal{C}^{\Delta[1]} \longrightarrow \mathcal{C}^{\Delta[2]}$$

which is a section of the composition functor $d_1 : \mathcal{C}^{\Delta[2]} \to \mathcal{C}^{\Delta[1]}$.

Remark 2.7. In def. 2.6 we are using the following standard notation, see at *simplex category* and at *nerve of a category*:

Write $[1] = \{0 \to 1\}$ and $[2] = \{0 \to 1 \to 2\}$ for the ordinal numbers, regarded as posets and hence as categories. The arrow category $\mathrm{Arr}(\mathcal{C})$ is equivalently the functor category $\mathcal{C}^{\Delta[1]} \coloneqq \mathrm{Funct}(\Delta[1], \mathcal{C})$, while $\mathcal{C}^{\Delta[2]} \coloneqq \mathrm{Funct}(\Delta[2], \mathcal{C})$ has as objects pairs of composable morphisms in $\mathcal{C}$. There are three injective functors $\delta_i : [1] \to [2]$, where δ_i omits the index i in its image. By precomposition, this induces functors $d_i : \mathcal{C}^{\Delta[2]} \longrightarrow \mathcal{C}^{\Delta[1]}$. Here

- d_1 sends a pair of composable morphisms to their composition;
- d_2 sends a pair of composable morphisms to the first morphisms;
- d_0 sends a pair of composable morphisms to the second morphisms.

Definition 2.8. A weak factorization system, def. 2.5, is called a **functorial weak factorization system** if the factorization of morphisms may be chosen to be a functorial factorization fact, def. 2.6, i.e. such that $d_2 \circ \mathrm{fact}$ lands in Proj and $d_0 \circ \mathrm{fact}$ in Inj.

Remark 2.9. Not all weak factorization systems are functorial, def. 2.8, although most (including those produced by the small object argument (prop. 2.17 below), with due care) are.

Proposition 2.10. *Let $\mathcal{C}$ be a category and let $K \subset \mathrm{Mor}(\mathcal{C})$ be a class of morphisms. Write $K\,\mathrm{Proj}$ and $K\,\mathrm{Inj}$, respectively, for the sub-classes of K-projective morphisms and of K-injective morphisms, def. 2.4. Then:*

1. *Both classes contain the class of isomorphism of $\mathcal{C}$.*
2. *Both classes are closed under composition in $\mathcal{C}$.*

   $K\,\mathrm{Proj}$ is also closed under transfinite composition.
3. *Both classes are closed under forming retracts in the arrow category $\mathcal{C}^{\Delta[1]}$ (see remark 2.12).*
4. *$K\,\mathrm{Proj}$ is closed under forming pushouts of morphisms in $\mathcal{C}$ ("cobase change").*

   $K\,\mathrm{Inj}$ is closed under forming pullback of morphisms in $\mathcal{C}$ ("base change").
5. *$K\,\mathrm{Proj}$ is closed under forming coproducts in $\mathcal{C}^{\Delta[1]}$.*

   $K\,\mathrm{Inj}$ is closed under forming products in $\mathcal{C}^{\Delta[1]}$.

Proof. We go through each item in turn.

containing isomorphisms

Given a commuting square

$$
\begin{array}{ccc}
A & \stackrel{f}{\to} & X \\
{}^{\mathllap{i}}_{\mathllap{\in \mathrm{Iso}}}\downarrow && \downarrow^{\mathrlap{p}} \\
B & \underset{g}{\longrightarrow} & Y
\end{array}
$$

with the left morphism an isomorphism, then a lift is given by using the inverse of this isomorphism ${}^{f \circ i^{-1}}\nearrow$. Hence in particular there is a lift when $p \in K$ and so $i \in K\,\mathrm{Proj}$. The other case is formally dual.

closure under composition

Given a commuting square of the form

$$
\begin{array}{ccc}
A & \longrightarrow & X \\
\downarrow && \downarrow^{\mathrlap{p_1}}_{\mathrlap{\in K\,\mathrm{Inj}}} \\
{}^{\mathllap{i}}_{\mathllap{\in K}}\downarrow && \downarrow^{\mathrlap{p_2}}_{\mathrlap{\in K\,\mathrm{Inj}}} \\
B & \longrightarrow & Y
\end{array}
$$

consider its pasting decomposition as

$$
\begin{array}{ccc}
A & \longrightarrow & X \\
\downarrow & \searrow & \downarrow^{\mathrlap{p_1}}_{\mathrlap{\in K\,\mathrm{Inj}}} \\
{}^{\mathllap{i}}_{\mathllap{\in K}}\downarrow && \downarrow^{\mathrlap{p_2}}_{\mathrlap{\in K\,\mathrm{Inj}}} \\
B & \longrightarrow & Y
\end{array}
$$

Now the bottom commuting square has a lift, by assumption. This yields another pasting decomposition

$$
\begin{array}{ccc}
A & \longrightarrow & X \\
{}^{\mathllap{i}}_{\mathllap{\in K}}\downarrow && \downarrow^{\mathrlap{p_1}}_{\mathrlap{\in K\,\mathrm{Inj}}} \\
\downarrow & \nearrow & \downarrow^{\mathrlap{p_2}}_{\mathrlap{\in K\,\mathrm{Inj}}} \\
B & \longrightarrow & Y
\end{array}
$$

and now the top commuting square has a lift by assumption. This is now equivalently a lift in the total diagram, showing that $p_1 \circ p_1$ has the right lifting property against K and is hence in $K\,\mathrm{Inj}$. The case of composing two morphisms in $K\,\mathrm{Proj}$ is formally dual. From this the closure of $K\,\mathrm{Proj}$ under transfinite composition follows since the latter is given by colimits of sequential composition and successive lifts against the underlying sequence as above constitutes a cocone, whence the extension of the lift to the colimit follows by its universal property.

closure under retracts

Let j be the retract of an $i \in K\,\mathrm{Proj}$, i.e. let there be a commuting diagram of the form.

$$
\begin{array}{ccccccc}
\mathrm{id}_A \colon & A &\longrightarrow& C &\longrightarrow& A \\
& \downarrow^{\mathrlap{j}} && \downarrow^{\mathrlap{i}}_{\mathrlap{\in K\,\mathrm{Proj}}} && \downarrow^{\mathrlap{j}} \\
\mathrm{id}_B \colon & B &\longrightarrow& D &\longrightarrow& B
\end{array}
.
$$

Then for

$$\begin{array}{ccc} A & \longrightarrow & X \\ {}^{\mathllap{j}}\downarrow && \downarrow^{\mathrlap{f}}_{\mathrlap{\in K}} \\ B & \longrightarrow & Y \end{array}$$

a commuting square, it is equivalent to its pasting composite with that retract diagram

$$\begin{array}{ccccccc} A & \longrightarrow & C & \longrightarrow & A & \longrightarrow & X \\ \downarrow^{\mathrlap{j}} && \downarrow^{\mathrlap{i}}_{\mathrlap{\in K\,\mathrm{Proj}}} && \downarrow^{\mathrlap{j}} && \downarrow^{\mathrlap{f}}_{\mathrlap{\in K}} \\ B & \longrightarrow & D & \longrightarrow & B & \longrightarrow & Y \end{array}$$

Here the pasting composite of the two squares on the right has a lift, by assumption:

$$\begin{array}{ccccccc} A & \longrightarrow & C & \longrightarrow & A & \longrightarrow & X \\ \downarrow^{\mathrlap{j}} && \downarrow^{\mathrlap{i}} && \nearrow && \downarrow^{\mathrlap{f}}_{\mathrlap{\in K}} \\ B & \longrightarrow & D & \longrightarrow & B & \longrightarrow & Y \end{array}$$

By composition, this is also a lift in the total outer rectangle, hence in the original square. Hence j has the left lifting property against all $p \in K$ and hence is in $K\,\mathrm{Proj}$. The other case is formally dual.

closure under pushout and pullback

Let $p \in K\,\mathrm{Inj}$ and and let

$$\begin{array}{ccc} Z \times_f X & \longrightarrow & X \\ {}^{\mathllap{f^* p}}\downarrow && \downarrow^{\mathrlap{p}} \\ Z & \overset{f}{\longrightarrow} & Y \end{array}$$

be a pullback diagram in $\mathcal{C}$. We need to show that $f^* p$ has the right lifting property with respect to all $i \in K$. So let

$$\begin{array}{ccc} A & \longrightarrow & Z \times_f X \\ {}^{\mathllap{i}}_{\mathllap{\in K}}\downarrow && \downarrow^{\mathrlap{f^* p}} \\ B & \overset{g}{\longrightarrow} & Z \end{array}$$

be a commuting square. We need to construct a diagonal lift of that square. To that end, first consider the pasting composite with the pullback square from above to obtain the commuting diagram

$$\begin{array}{ccccc} A & \longrightarrow & Z \times_f X & \longrightarrow & X \\ {}^{\mathllap{i}}\downarrow && \downarrow^{\mathrlap{f^* p}} && \downarrow^{\mathrlap{p}} \\ B & \overset{g}{\longrightarrow} & Z & \overset{f}{\longrightarrow} & Y \end{array} \,.$$

By the right lifting property of p, there is a diagonal lift of the total outer diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow^{\mathrlap{i}} & {}^{\widehat{(f g)}}\nearrow & \downarrow^{\mathrlap{p}} \\ B & \overset{f g}{\longrightarrow} & Y \end{array} \,.$$

By the universal property of the pullback this gives rise to the lift $\hat g$ in

$$
\begin{array}{ccccc}
 & & Z \times_f X & \longrightarrow & X \\
 & {}^{\hat g}\nearrow & \downarrow^{f^\ast p} & & \downarrow^{p} \\
B & \overset{g}{\longrightarrow} & Z & \overset{f}{\longrightarrow} & Y
\end{array}.
$$

In order for $\hat g$ to qualify as the intended lift of the total diagram, it remains to show that

$$
\begin{array}{ccc}
A & \longrightarrow & Z \times_f X \\
\downarrow^{i} & {}^{\hat g}\nearrow & \\
B & &
\end{array}
$$

commutes. To do so we notice that we obtain two cones with tip A:

* one is given by the morphisms

  1. $A \to Z \times_f X \to X$
  2. $A \overset{i}{\to} B \overset{g}{\to} Z$

  with universal morphism into the pullback being

  * $A \to Z \times_f X$

* the other by

  1. $A \overset{i}{\to} B \overset{\hat g}{\to} Z \times_f X \to X$
  2. $A \overset{i}{\to} B \overset{g}{\to} Z$.

  with universal morphism into the pullback being

  * $A \overset{i}{\to} B \overset{\hat g}{\to} Z \times_f X$.

The commutativity of the diagrams that we have established so far shows that the first and second morphisms here equal each other, respectively. By the fact that the universal morphism into a pullback diagram is *unique* this implies the required identity of morphisms.

The other case is formally dual.

closure under (co-)products

Let $\{(A_s \overset{i_s}{\to} B_s) \in K\,\mathrm{Proj}\}_{s \in S}$ be a set of elements of $K\,\mathrm{Proj}$. Since colimits in the presheaf category $\mathcal{C}^{\Delta[1]}$ are computed componentwise, their coproduct in this arrow category is the universal morphism out of the coproduct of objects $\coprod_{s \in S} A_s$ induced via its universal property by the set of morphisms i_s:

$$
\underset{s \in S}{\sqcup} A_s \overset{(i_s)_{s \in S}}{\longrightarrow} \underset{s \in S}{\sqcup} B_s \,.
$$

Now let

$$
\begin{array}{ccc}
\underset{s \in S}{\sqcup} A_s & \longrightarrow & X \\
{}^{(i_s)_{s \in S}}\downarrow & & \downarrow^{f}_{\in K} \\
\underset{s \in S}{\sqcup} B_s & \longrightarrow & Y
\end{array}
$$

be a commuting square. This is in particular a cocone under the coproduct of objects, hence by the universal property of the coproduct, this is equivalent to a set of commuting diagrams

$$\left\{ \begin{array}{ccc} A_s & \longrightarrow & X \\ {}^{i_s}_{\in K\,\mathrm{Proj}}\downarrow && \downarrow^f_{\in K} \\ B_s & \longrightarrow & Y \end{array} \right\}_{s \in S} \,.$$

By assumption, each of these has a lift ℓ_s. The collection of these lifts

$$\left\{ \begin{array}{ccc} A_s & \longrightarrow & X \\ {}^{i_s}_{\in \mathrm{Proj}}\downarrow & {}^{\ell_s}\nearrow & \downarrow^f_{\in K} \\ B_s & \longrightarrow & Y \end{array} \right\}_{s \in S}$$

is now itself a compatible cocone, and so once more by the universal property of the coproduct, this is equivalent to a lift $(\ell_s)_{s\in S}$ in the original square

$$\begin{array}{ccc} \underset{s\in S}{\sqcup} A_s & \longrightarrow & X \\ {}^{(i_s)_{s\in S}}\downarrow & {}^{(\ell_s)_{s\in S}}\nearrow & \downarrow^f_{\in K} \\ \underset{s\in S}{\sqcup} B_s & \longrightarrow & Y \end{array}$$

This shows that the coproduct of the i_s has the left lifting property against all $f \in K$ and is hence in $K\,\mathrm{Proj}$. The other case is formally dual. ▮

An immediate consequence of prop. 2.10 is this:

Corollary 2.11. *Let $\mathcal{C}$ be a category with all small colimits, and let $K \subset \mathrm{Mor}(\mathcal{C})$ be a sub-class of its morphisms. Then every K-injective morphism, def. 2.4, has the right lifting property, def. 2.4, against all K-relative cell complexes, def. 1.41 and their retracts, remark 2.12.*

Remark 2.12. By a retract of a morphism $X \stackrel{f}{\longrightarrow} Y$ in some category $\mathcal{C}$ we mean a retract of f as an object in the arrow category $\mathcal{C}^{\Delta[1]}$, hence a morphism $A \stackrel{g}{\longrightarrow} B$ such that in $\mathcal{C}^{\Delta[1]}$ there is a factorization of the identity on g through f

$$\mathrm{id}_g \;\colon\; g \longrightarrow f \longrightarrow g \,.$$

This means equivalently that in $\mathcal{C}$ there is a commuting diagram of the form

$$\begin{array}{ccccc} \mathrm{id}_A \colon & A & \longrightarrow & X & \longrightarrow & A \\ & \downarrow^g && \downarrow^f && \downarrow^g \\ \mathrm{id}_B \colon & B & \longrightarrow & Y & \longrightarrow & B \end{array} \,.$$

Lemma 2.13. *In every category $\mathcal{C}$ the class of isomorphisms is preserved under retracts in the sense of remark 2.12.*

Proof. For

$$\begin{array}{ccccc} \mathrm{id}_A \colon & A & \longrightarrow & X & \longrightarrow & A \\ & \downarrow^g && \downarrow^f && \downarrow^g \\ \mathrm{id}_B \colon & B & \longrightarrow & Y & \longrightarrow & B \end{array} \,.$$

a retract diagram and $X \stackrel{f}{\to} Y$ an isomorphism, the inverse to $A \stackrel{g}{\to} B$ is given by the composite

$$\begin{array}{ccc} && X & \longrightarrow & A \\ && \uparrow^{f^{-1}} \\ B & \longrightarrow & Y \end{array} \,.$$

▮

More generally:

Proposition 2.14. *Given a model category in the sense of def. 2.3, then its class of weak equivalences is closed under forming retracts (in the arrow category, see remark 2.12).*

(Joyal, prop. E.1.3)

Proof. Let

$$\begin{array}{ccccccc} \mathrm{id} \colon & A & \longrightarrow & X & \longrightarrow & A \\ & {}^{\mathllap{f}}\downarrow && \downarrow^{\mathrlap{w}} && \downarrow^{\mathrlap{f}} \\ \mathrm{id} \colon & B & \longrightarrow & Y & \longrightarrow & B \end{array}$$

be a commuting diagram in the given model category, with $w \in W$ a weak equivalence. We need to show that then also $f \in W$.

First consider the case that $f \in \mathrm{Fib}$.

In this case, factor w as a cofibration followed by an acyclic fibration. Since $w \in W$ and by two-out-of-three (def. 2.1) this is even a factorization through an acyclic cofibration followed by an acyclic fibration. Hence we obtain a commuting diagram of the following form:

$$\begin{array}{ccccccc} \mathrm{id} \colon & A & \longrightarrow & X & \longrightarrow & A \\ & {}^{\mathllap{\mathrm{id}}}\downarrow && \downarrow^{\mathrlap{\in W \cap \mathrm{Cof}}} && \downarrow^{\mathrlap{\mathrm{id}}} \\ \mathrm{id} \colon & A' & \overset{s}{\longrightarrow} & X' & \overset{t}{\longrightarrow} & A' \,, \\ & {}^{\mathllap{f}}_{\mathllap{\in \mathrm{Fib}}}\downarrow && \downarrow^{\mathrlap{\in W \cap \mathrm{Fib}}} && \downarrow^{\mathrlap{f}}_{\mathrlap{\in \mathrm{Fib}}} \\ \mathrm{id} \colon & B & \longrightarrow & Y & \longrightarrow & B \end{array}$$

where s is uniquely defined and where t is any lift of the top middle vertical acyclic cofibration against f. This now exhibits f as a retract of an acyclic fibration. These are closed under retract by prop. 2.10.

Now consider the general case. Factor f as an acyclic cofibration followed by a fibration and form the pushout in the top left square of the following diagram

$$\begin{array}{ccccccc} \mathrm{id} \colon & A & \longrightarrow & X & \longrightarrow & A \\ & {}^{\mathllap{\in W \cap \mathrm{Cof}}}\downarrow & (po) & \downarrow^{\mathrlap{\in W \cap \mathrm{Cof}}} && \downarrow^{\mathrlap{\in W \cap \mathrm{Cof}}} \\ \mathrm{id} \colon & A' & \longrightarrow & X' & \longrightarrow & A' \,, \\ & {}^{\mathllap{\in \mathrm{Fib}}}\downarrow && \downarrow^{\mathrlap{\in W}} && \downarrow^{\mathrlap{\in \mathrm{Fib}}} \\ \mathrm{id} \colon & B & \longrightarrow & Y & \longrightarrow & B \end{array}$$

where the other three squares are induced by the universal property of the pushout, as is the identification of the middle horizontal composite as the identity on A'. Since acyclic cofibrations are closed under forming pushouts by prop. 2.10, the top middle vertical morphism is now an acyclic fibration, and hence by assumption and by two-out-of-three so is the middle bottom vertical morphism.

Thus the previous case now gives that the bottom left vertical morphism is a weak equivalence, and hence the total left vertical composite is. ▮

Lemma 2.15. ***(retract argument)***

Consider a composite morphism

$$f : X \xrightarrow{i} A \xrightarrow{p} Y \,.$$

1. *If f has the left lifting property against p, then f is a retract of i.*

2. *If f has the right lifting property against i, then f is a retract of p.*

Proof. We discuss the first statement, the second is formally dual.

Write the factorization of f as a commuting square of the form

$$\begin{array}{ccc} X & \xrightarrow{i} & A \\ {}^{\mathllap{f}}\downarrow && \downarrow^{\mathrlap{p}} \\ Y & = & Y \end{array} .$$

By the assumed lifting property of f against p there exists a diagonal filler g making a commuting diagram of the form

$$\begin{array}{ccc} X & \xrightarrow{i} & A \\ {}^{\mathllap{f}}\downarrow & {}^{g}\nearrow & \downarrow^{\mathrlap{p}} \\ Y & = & Y \end{array} .$$

By rearranging this diagram a little, it is equivalent to

$$\begin{array}{cccccc} & X & = & X \\ & {}^{\mathllap{f}}\downarrow && {}^{\mathllap{i}}\downarrow \\ \mathrm{id}_Y \colon & Y & \underset{g}{\longrightarrow} & A & \underset{p}{\longrightarrow} & Y \end{array} \,.$$

Completing this to the right, this yields a diagram exhibiting the required retract according to remark 2.12:

$$\begin{array}{cccccc} \mathrm{id}_X \colon & X & = & X & = & X \\ & {}^{\mathllap{f}}\downarrow && {}^{\mathllap{i}}\downarrow && {}^{\mathllap{f}}\downarrow \\ \mathrm{id}_Y \colon & Y & \underset{g}{\longrightarrow} & A & \underset{p}{\longrightarrow} & Y \end{array} \,.$$

∎

Small object argument

Given a set $C \subset \mathrm{Mor}(\mathcal{C})$ of morphisms in some category $\mathcal{C}$, a natural question is how to factor any given morphism $f : X \longrightarrow Y$ through a relative C-cell complex, def. 1.41, followed by a C-injective morphism, def. 1.46

$$f \;\colon\; X \overset{\in C\,\mathrm{cell}}{\longrightarrow} \hat X \overset{\in C\,\mathrm{inj}}{\longrightarrow} Y \,.$$

A first approximation to such a factorization turns out to be given simply by forming $\hat X = X_1$ by attaching **all** possible C-cells to X. Namely let

$$(C/f) \coloneqq \left\{ \array{ \mathrm{dom}(c) &\longrightarrow& X \\ {}^{\mathllap{c \in C}}\downarrow && \downarrow^{\mathrlap{f}} \\ \mathrm{cod}(c) &\longrightarrow& Y } \right\}$$

be the set of **all** ways to find a C-cell attachment in f, and consider the pushout $\hat X$ of the coproduct of morphisms in C over all these:

$$\begin{array}{ccc} \coprod_{c \in (C/f)} \mathrm{dom}(c) & \longrightarrow & X \\ {}^{\coprod_{c \in (C/f)} c}\downarrow && (\mathrm{po}) & \downarrow \\ \coprod_{c \in (C/f)} \mathrm{cod}(c) & \longrightarrow & X_1 \end{array} \,.$$

This gets already close to producing the intended factorization:

First of all the resulting map $X \to X_1$ is a C-relative cell complex, by construction.

Second, by the fact that the coproduct is over all commuting squres to f, the morphism f itself makes a commuting diagram

$$\begin{array}{ccc} \coprod_{c \in (C/f)} \mathrm{dom}(c) & \longrightarrow & X \\ {}^{\coprod_{c \in (C/f)} c}\downarrow && \downarrow^{f} \\ \coprod_{c \in (C/f)} \mathrm{cod}(c) & \longrightarrow & Y \end{array}$$

and hence the universal property of the colimit means that f is indeed factored through that C-cell complex X_1; we may suggestively arrange that factorizing diagram like so:

$$\begin{array}{ccc} \coprod_{c \in (C/f)} \mathrm{dom}(c) & \longrightarrow & X \\ {}^{\mathrm{id}}\downarrow && \downarrow \\ \coprod_{c \in (C/f)} \mathrm{dom}(c) && X_1 \\ {}^{\coprod_{c \in (C/f)} c}\downarrow & \nearrow & \downarrow \\ \coprod_{c \in (C/f)} \mathrm{cod}(c) & \longrightarrow & Y \end{array} \,.$$

This shows that, finally, the colimiting co-cone map – the one that now appears diagonally – **almost** exhibits the desired right lifting of $X_1 \to Y$ against the $c \in C$. The failure of that to hold on the nose is only the fact that a horizontal map in the middle of the above diagram is missing: the diagonal map obtained above lifts not all commuting diagrams of $c \in C$ into f, but only those where the top morphism $\mathrm{dom}(c) \to X_1$ factors through $X \to X_1$.

The idea of the small object argument now is to fix this only remaining problem by iterating the construction: next factor $X_1 \to Y$ in the same way into

$$X_1 \longrightarrow X_2 \longrightarrow Y$$

and so forth. Since relative C-cell complexes are closed under composition, at stage n the resulting $X \longrightarrow X_n$ is still a C-cell complex, getting bigger and bigger. But accordingly, the failure of the accompanying $X_n \longrightarrow Y$ to be a C-injective morphism becomes smaller and smaller, for it now lifts against all diagrams where $\mathrm{dom}(c) \longrightarrow X_n$ factors through $X_{n-1} \longrightarrow X_n$, which intuitively is less and less of a condition as the X_{n-1} grow larger and larger.

The concept of *small object* is just what makes this intuition precise and finishes the small object argument. For the present purpose we just need the following simple version:

Definition 2.16. For $\mathcal{C}$ a category and $C \subset \mathrm{Mor}(\mathcal{C})$ a sub-set of its morphisms, say that these have *small domains* if there is an ordinal α (def. 1.15) such that for every $c \in C$ and for every C-relative cell complex given by a transfinite composition (def. 1.17)

$$f : X \to X_1 \to X_2 \to \cdots \to X_\beta \to \cdots \longrightarrow \hat{X}$$

every morphism $\mathrm{dom}(c) \longrightarrow \hat{X}$ factors through a stage $X_\beta \to \hat{X}$ of order $\beta < \alpha$:

$$\begin{array}{ccc} & & X_\beta \\ & \nearrow & \downarrow \\ \mathrm{dom}(c) & \longrightarrow & \hat X \end{array} .$$

The above discussion proves the following:

Proposition 2.17. ***(small object argument)***

Let $\mathcal{C}$ be a locally small category with all small colimits. If a set $C \subset \mathrm{Mor}(\mathcal{C})$ of morphisms has all small domains in the sense of def. 2.16, then every morphism $f : X \longrightarrow$ in $\mathcal{C}$ factors through a C-relative cell complex, def. 1.41, followed by a C-injective morphism, def. 1.46

$$f : X \overset{\in C\,\mathrm{cell}}{\longrightarrow} \hat X \overset{\in C\,\mathrm{inj}}{\longrightarrow} Y .$$

(Quillen 67, II.3 lemma)

Homotopy

We discuss how the concept of homotopy is abstractly realized in model categories, def. 2.3.

Definition 2.18. Let $\mathcal{C}$ be a model category, def. 2.3, and $X \in \mathcal{C}$ an object.

- A **path space object** $\mathrm{Path}(X)$ for X is a factorization of the diagonal $\Delta_X : X \to X \times X$ as

$$\Delta_X : X \underset{\in W}{\overset{i}{\longrightarrow}} \mathrm{Path}(X) \underset{\in \mathrm{Fib}}{\overset{(p_0, p_1)}{\longrightarrow}} X \times X .$$

where $X \to \mathrm{Path}(X)$ is a weak equivalence and $\mathrm{Path}(X) \to X \times X$ is a fibration.

- A **cylinder object** $\mathrm{Cyl}(X)$ for X is a factorization of the codiagonal (or "fold map") $\nabla_X : X \sqcup X \to X$ as

$$\nabla_X : X \sqcup X \underset{\in \mathrm{Cof}}{\overset{(i_0, i_1)}{\longrightarrow}} \mathrm{Cyl}(X) \underset{\in W}{\overset{p}{\longrightarrow}} X .$$

where $\mathrm{Cyl}(X) \to X$ is a weak equivalence. and $X \sqcup X \to \mathrm{Cyl}(X)$ is a cofibration.

Remark 2.19. For every object $X \in \mathcal{C}$ in a model category, a cylinder object and a path space object according to def. 2.18 exist: the factorization axioms guarantee that there exists

1. a factorization of the codiagonal as

$$\nabla_X : X \sqcup X \overset{\in \mathrm{Cof}}{\longrightarrow} \mathrm{Cyl}(X) \overset{\in W \cap \mathrm{Fib}}{\longrightarrow} X$$

2. a factorization of the diagonal as

$$\Delta_X : X \overset{\in W \cap \mathrm{Cof}}{\longrightarrow} \mathrm{Path}(X) \overset{\in \mathrm{Fib}}{\longrightarrow} X \times X .$$

The cylinder and path space objects obtained this way are actually better than required by def. 2.18: in addition to $\mathrm{Cyl}(X) \to X$ being just a weak equivalence, for these this is actually an acyclic fibration, and dually in addition to $X \to \mathrm{Path}(X)$ being a weak equivalence, for these it is actually an acyclic cofibrations.

Some authors call cylinder/path-space objects with this extra property "very good" cylinder/path-space objects, respectively.

One may also consider dropping a condition in def. 2.18: what mainly matters is the weak equivalence, hence some authors take cylinder/path-space objects to be defined as in def. 2.18 but without the condition that $X \sqcup X \to \mathrm{Cyl}(X)$ is a cofibration and without the condition that $\mathrm{Path}(X) \to X$ is a fibration. Such authors would then refer to the concept in def. 2.18 as "good" cylinder/path-space objects.

The terminology in def. 2.18 follows the original (Quillen 67, I.1 def. 4). With the induced concept of left/right homotopy below in def. 2.22, this admits a quick derivation of the key facts in the following, as we spell out below.

Lemma 2.20. *Let $\mathcal{C}$ be a model category. If $X \in \mathcal{C}$ is cofibrant, then for every cylinder object $\mathrm{Cyl}(X)$ of X, def. 2.18, not only is $(i_0, i_1) : X \sqcup X \to X$ a cofibration, but each*

$$i_0, i_1 : X \longrightarrow \mathrm{Cyl}(X)$$

is an acyclic cofibration separately.

Dually, if $X \in \mathcal{C}$ is fibrant, then for every path space object $\mathrm{Path}(X)$ of X, def. 2.18, not only is $(p_0, p_1) : \mathrm{Path}(X) \to X \times X$ a cofibration, but each

$$p_0, p_1 : \mathrm{Path}(X) \longrightarrow X$$

is an acyclic fibration separately.

Proof. We discuss the case of the path space object. The other case is formally dual.

First, that the component maps are weak equivalences follows generally: by definition they have a right inverse $\mathrm{Path}(X) \to X$ and so this follows by two-out-of-three (def. 2.1).

But if X is fibrant, then also the two projection maps out of the product $X \times X \to X$ are fibrations, because they are both pullbacks of the fibration $X \to *$

$$\begin{array}{ccc} X \times X & \longrightarrow & X \\ \downarrow & (\mathrm{pb}) & \downarrow \\ X & \longrightarrow & * \end{array} .$$

hence $p_i : \mathrm{Path}(X) \to X \times X \to X$ is the composite of two fibrations, and hence itself a fibration, by prop. 2.10. ▮

Path space objects are very non-unique as objects up to isomorphism:

Example 2.21. If $X \in \mathcal{C}$ is a fibrant object in a model category, def. 2.3, and for $\mathrm{Path}_1(X)$ and $\mathrm{Path}_2(X)$ two path space objects for X, def. 2.18, then the fiber product $\mathrm{Path}_1(X) \times_X \mathrm{Path}_2(X)$ is another path space object for X: the pullback square

$$\begin{array}{ccc} X & \overset{\Delta_X}{\longrightarrow} & X \times X \\ \downarrow & & \downarrow \\ \mathrm{Path}_1(X) \underset{X}{\times} \mathrm{Path}_2(X) & \longrightarrow & \mathrm{Path}_1(X) \times \mathrm{Path}_2(X) \\ {}^{\in \mathrm{Fib}}\downarrow & (\mathrm{pb}) & \downarrow^{\in \mathrm{Fib}} \\ X \times X \times X & \overset{(\mathrm{id}, \Delta_X, \mathrm{id})}{\longrightarrow} & X \times X \times X \times X \\ \downarrow^{(\mathrm{pr}_1, \mathrm{pr}_3)}_{\in \mathrm{Fib}} & & \downarrow^{(p_1, p_4)} \\ X \times X & = & X \times X \end{array}$$

gives that the induced projection is again a fibration. Moreover, using lemma 2.20 and two-out-of-three (def. 2.1) gives that $X \to \mathrm{Path}_1(X) \times_X \mathrm{Path}_2(X)$ is a weak equivalence.

For the case of the canonical topological path space objects of def 1.34, with $\mathrm{Path}_1(X) = \mathrm{Path}_2(X) = X^I = X^{[0,1]}$ then this new path space object is $X^{I \vee I} = X^{[0,2]}$, the mapping space out of the standard interval of length 2 instead of length 1.

Definition 2.22. Let $f, g : X \longrightarrow Y$ be two parallel morphisms in a model category.

- A **left homotopy** $\eta : f \Rightarrow_L g$ is a morphism $\eta : \mathrm{Cyl}(X) \longrightarrow Y$ from a cylinder object of X, def. 2.18, such that it makes this diagram commute:

$$\begin{array}{ccccc} X & \longrightarrow & \mathrm{Cyl}(X) & \longleftarrow & X \\ & {}_{f}\searrow & \downarrow^{\eta} & \swarrow_{g} \\ && Y \end{array} \,.$$

- A **right homotopy** $\eta : f \Rightarrow_R g$ is a morphism $\eta : X \to \mathrm{Path}(Y)$ to some path space object of X, def. 2.18, such that this diagram commutes:

$$\begin{array}{ccccc} && X \\ & {}^{f}\swarrow & \downarrow^{\eta} & \searrow^{g} \\ Y & \longleftarrow & \mathrm{Path}(Y) & \longrightarrow & Y \end{array} \,.$$

Lemma 2.23. *Let $f, g : X \to Y$ be two parallel morphisms in a model category.*

1. *Let X be cofibrant. If there is a left homotopy $f \Rightarrow_L g$ then there is also a right homotopy $f \Rightarrow_R g$ (def. 2.22) with respect to any chosen path space object.*

2. *Let X be fibrant. If there is a right homotopy $f \Rightarrow_R g$ then there is also a left homotopy $f \Rightarrow_L g$ with respect to any chosen cylinder object.*

In particular if X is cofibrant and Y is fibrant, then by going back and forth it follows that every left homotopy is exhibited by every cylinder object, and every right homotopy is exhibited by every path space object.

Proof. We discuss the first case, the second is formally dual. Let $\eta : \mathrm{Cyl}(X) \longrightarrow Y$ be the given left homotopy. Lemma 2.20 implies that we have a lift h in the following commuting diagram

$$\begin{array}{ccc} X & \overset{i \circ f}{\longrightarrow} & \mathrm{Path}(Y) \\ {}^{i_0}_{\in W \cap \mathrm{Cof}}\downarrow & {}^{h}\nearrow & \downarrow^{p_0, p_1}_{\in \mathrm{Fib}} \\ \mathrm{Cyl}(X) & \underset{(f \circ p, \eta)}{\longrightarrow} & Y \times Y \end{array} \,,$$

where on the right we have the chosen path space object. Now the composite $\tilde{\eta} \coloneqq h \circ i_1$ is a right homotopy as required:

$$\begin{array}{ccccc} &&&& \mathrm{Path}(Y) \\ &&& {}^{h}\nearrow & \downarrow^{p_0, p_1}_{\in \mathrm{Fib}} \\ X & \overset{i_1}{\longrightarrow} & \mathrm{Cyl}(X) & \underset{(f \circ p, \eta)}{\longrightarrow} & Y \times Y \end{array} \,.$$

∎

Proposition 2.24. *For X a cofibrant object in a model category and Y a fibrant object, then the relations of left homotopy $f \Rightarrow_L g$ and of right homotopy $f \Rightarrow_R g$ (def. 2.22) on the hom set $\mathrm{Hom}(X,Y)$ coincide and are both equivalence relations.*

Proof. That both relations coincide under the (co-)fibrancy assumption follows directly from lemma 2.23.

The symmetry and reflexivity of the relation is obvious.

That right homotopy (hence also left homotopy) with domain X is a transitive relation follows from using example 2.21 to compose path space objects. ■

The homotopy category

We discuss the construction that takes a model category, def. 2.3, and then universally forces all its weak equivalences into actual isomorphisms.

Definition 2.25. Let $\mathcal{C}$ be a model category, def. 2.3. Write $\mathrm{Ho}(\mathcal{C})$ for the category whose

- objects are those objects of $\mathcal{C}$ which are both fibrant and cofibrant;
- morphisms are the homotopy classes of morphisms of $\mathcal{C}$, hence the equivalence classes of morphism under the equivalence relation of prop. 2.24;

and whose composition operation is given on representatives by composition in $\mathcal{C}$.

This is, up to equivalence of categories, the **homotopy category of the model category** $\mathcal{C}$.

Proposition 2.26. *Def. 2.25 is well defined, in that composition of morphisms between fibrant-cofibrant objects in $\mathcal{C}$ indeed passes to homotopy classes.*

Proof. Fix any morphism $X \overset{f}{\to} Y$ between fibrant-cofibrant objects. Then for precomposition

$$(-) \circ [f] \,:\, \mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(Y,Z) \to \mathrm{Hom}_{\mathrm{Ho}(\mathcal{C}(X,Z))}$$

to be well defined, we need that with $(g \sim h) \,:\, Y \to Z$ also $(fg \sim fh) \,:\, X \to Z$. But by prop 2.24 we may take the homotopy $\sim$ to be exhibited by a right homotopy $\eta : Y \to \mathrm{Path}(Z)$, for which case the statement is evident from this diagram:

$$\begin{array}{ccccc} && && Z \\ && & {}^{g}\nearrow & \uparrow^{p_1} \\ X & \overset{f}{\longrightarrow} & Y & \overset{\eta}{\longrightarrow} & \mathrm{Path}(Z) \\ && & {}_{h}\searrow & \downarrow_{p_0} \\ && && Z \end{array} \,.$$

For postcomposition we may choose to exhibit homotopy by left homotopy and argue dually. ■

We now spell out that def. 2.25 indeed satisfies the universal property that defines the localization of a category with weak equivalences at its weak equivalences.

Lemma 2.27. ***(Whitehead theorem in model categories)***

Let $\mathcal{C}$ be a model category. A weak equivalence between two objects which are both fibrant and cofibrant is a homotopy equivalence.

Proof. By the factorization axioms in the model category $\mathcal{C}$ and by two-out-of-three (def. 2.1), every weak equivalence $f : X \longrightarrow Y$ factors through an object Z as an acyclic cofibration followed by an acyclic fibration. In particular it follows that with X and Y both fibrant and cofibrant, so is Z, and hence it is sufficient to prove that acyclic (co-)fibrations between such objects are homotopy equivalences.

So let $f : X \longrightarrow Y$ be an acyclic fibration between fibrant-cofibrant objects, the case of acyclic cofibrations is formally dual. Then in fact it has a genuine right inverse given by a lift f^{-1} in the diagram

$$\begin{array}{ccc} \emptyset & \to & X \\ {}^{\in \mathrm{cof}}\downarrow & {}^{f^{-1}}\nearrow & \downarrow^{f}_{\in \mathrm{Fib} \cap W} \\ X & = & X \end{array}$$

To see that f^{-1} is also a left inverse up to left homotopy, let $\mathrm{Cyl}(X)$ be any cylinder object on X (def. 2.18), hence a factorization of the codiagonal on X as a cofibration followed by a an acyclic fibration

$$X \sqcup X \overset{\iota_X}{\longrightarrow} \mathrm{Cyl}(X) \overset{p}{\longrightarrow} X$$

and consider the commuting square

$$\begin{array}{ccc} X \sqcup X & \overset{(f^{-1}\circ f, \mathrm{id})}{\longrightarrow} & X \\ {}^{\iota_X}_{\in \mathrm{Cof}}\downarrow && \downarrow^{f}_{\in W \cap \mathrm{Fib}} \\ \mathrm{Cyl}(X) & \underset{f \circ p}{\longrightarrow} & Y \end{array}$$

which commutes due to f^{-1} being a genuine right inverse of f. By construction, this commuting square now admits a lift η, and that constitutes a left homotopy $\eta : f^{-1} \circ f \Rightarrow_L \mathrm{id}$. ▮

Definition 2.28. Given a model category $\mathcal{C}$, consider a *choice* for each object $X \in \mathcal{C}$ of

1. a factorization $\emptyset \underset{\in \mathrm{Cof}}{\overset{i_X}{\longrightarrow}} QX \underset{\in W \cap \mathrm{Fib}}{\overset{p_X}{\longrightarrow}} X$ of the initial morphism, such that when X is already cofibrant then $p_X = \mathrm{id}_X$;

2. a factorization $X \underset{\in W \cap \mathrm{Cof}}{\overset{j_X}{\longrightarrow}} PX \underset{\in \mathrm{Fib}}{\overset{q_X}{\longrightarrow}} *$ of the terminal morphism, such that when X is already fibrant then $j_X = \mathrm{id}_X$.

Write then

$$\gamma_{P,Q} : \mathcal{C} \longrightarrow \mathrm{Ho}(\mathcal{C})$$

for the functor to the homotopy category, def. 2.25, which sends an object X to the object PQX and sends a morphism $f : X \longrightarrow Y$ to the homotopy class of the result of first lifting in

$$\begin{array}{ccc} \emptyset & \longrightarrow & QY \\ {}^{i_X}\downarrow & {}^{Qf}\nearrow & \downarrow^{p_Y} \\ QX & \underset{f \circ p_X}{\longrightarrow} & Y \end{array}$$

and then lifting (here: extending) in

$$\begin{array}{ccc} QX & \overset{j_{QY} \circ Qf}{\longrightarrow} & PQY \\ {}^{j_{QX}}\downarrow & {}^{PQf}\nearrow & \downarrow^{q_{QY}} \\ PQX & \longrightarrow & * \end{array}$$

Lemma 2.29. *The construction in def. 2.28 is indeed well defined.*

Proof. First of all, the object PQX is indeed both fibrant and cofibrant (as well as related by a zig-zag of weak equivalences to X):

$$
\begin{tikzcd}
\emptyset \arrow[d, "\in \mathrm{Cof}"'] \arrow[dr, "\in \mathrm{Cof}"] & & \\
QX \arrow[r, "\in W \cap \mathrm{Cof}"'] \arrow[d, "\in W"'] & PQX \arrow[r, "\in \mathrm{Fib}"'] & * \\
X & &
\end{tikzcd}
.
$$

Now to see that the image on morphisms is well defined. First observe that any two choices $(Qf)_i$ of the first lift in the definition are left homotopic to each other, exhibited by lifting in

$$
\begin{tikzcd}
QX \sqcup QX \arrow[r, "((Qf)_1, (Qf)_2)"] \arrow[d, "\in \mathrm{Cof}"'] & QY \arrow[d, "p_Y", "\in W \cap \mathrm{Fib}"'] \\
\mathrm{Cyl}(QX) \arrow[r, "f \circ p_X \circ \sigma_{QX}"'] & Y
\end{tikzcd}
$$

Hence also the composites $j_{QY} \circ (Q_f)_i$ are left homotopic to each other, and since their domain is cofibrant, then by lemma 2.23 they are also right homotopic by a right homotopy κ. This implies finally, by lifting in

$$
\begin{tikzcd}
QX \arrow[r, "\kappa"] \arrow[d, "\in W \cap \mathrm{Cof}"'] & \mathrm{Path}(PQY) \arrow[d, "\in \mathrm{Fib}"] \\
PQX \arrow[r, "(R(Qf)_1, P(Qf)_2)"'] & PQY \times PQY
\end{tikzcd}
$$

that also $P(Qf)_1$ and $P(Qf)_2$ are right homotopic, hence that indeed PQf represents a well-defined homotopy class.

Finally to see that the assignment is indeed functorial, observe that the commutativity of the lifting diagrams for Qf and PQf imply that also the following diagram commutes

$$
\begin{tikzcd}
X \arrow[d, "f"'] & QX \arrow[l, "p_X"'] \arrow[r, "j_{QX}"] \arrow[d, "Qf"] & PQX \arrow[d, "PQf"] \\
Y & QY \arrow[l, "p_y"] \arrow[r, "j_{QY}"'] & PQY
\end{tikzcd}
.
$$

Now from the pasting composite

$$
\begin{tikzcd}
X \arrow[d, "f"'] & QX \arrow[l, "p_X"'] \arrow[r, "j_{QX}"] \arrow[d, "Qf"] & PQX \arrow[d, "PQf"] \\
Y \arrow[d, "g"'] & QY \arrow[l, "p_Y"] \arrow[r, "j_{QY}"'] \arrow[d, "Qg"] & PQY \arrow[d, "PQg"] \\
Z & QZ \arrow[l, "p_Z"] \arrow[r, "j_{QZ}"'] & PQZ
\end{tikzcd}
$$

one sees that $(PQg) \circ (PQf)$ is a lift of $g \circ f$ and hence the same argument as above gives that it is homotopic to the chosen $PQ(g \circ f)$. ▮

For the following, recall the concept of natural isomorphism between functors: for $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ two functors, then a *natural transformation* $\eta : F \Rightarrow G$ is for each object $c \in \mathrm{Obj}(\mathcal{C})$ a morphism $\eta_c : F(c) \longrightarrow G(c)$ in $\mathcal{D}$, such that for each morphism $f : c_1 \to c_2$ in $\mathcal{C}$ the following is a commuting square:

$$\begin{array}{ccc} F(c_1) & \overset{\eta_{c_1}}{\longrightarrow} & G(c_1) \\ {}^{\mathllap{F(f)}}\downarrow && \downarrow^{\mathrlap{G(f)}} \\ F(c_2) & \underset{\eta_{c_2}}{\longrightarrow} & G(c_2) \end{array}.$$

Such η is called a *natural isomorphism* if its η_c are isomorphisms for all objects c.

Definition 2.30. For $\mathcal{C}$ a category with weak equivalences, its **localization at the weak equivalences** is, if it exists,

1. a category denoted $\mathcal{C}[W^{-1}]$
2. a functor

$$\gamma : \mathcal{C} \longrightarrow \mathcal{C}[W^{-1}]$$

such that

1. γ sends weak equivalences to isomorphisms;
2. γ is universal with this property, in that:

   for $F : \mathcal{C} \longrightarrow D$ any functor out of $\mathcal{C}$ into any category D, such that F takes weak equivalences to isomorphisms, it factors through γ up to a natural isomorphism ρ

$$\begin{array}{ccccc} \mathcal{C} && \overset{F}{\longrightarrow} && D \\ & {}_{\mathllap{\gamma}}\searrow & \Downarrow^{\rho} & \nearrow_{\mathrlap{\tilde F}} \\ && \mathrm{Ho}(\mathcal{C}) \end{array}$$

   and this factorization is unique up to unique isomorphism, in that for $(\tilde F_1, \rho_1)$ and $(\tilde F_2, \rho_2)$ two such factorizations, then there is a unique natural isomorphism $\kappa : \tilde F_1 \Rightarrow \tilde F_2$ making the evident diagram of natural isomorphisms commute.

Theorem 2.31. *For $\mathcal{C}$ a model category, the functor $\gamma_{P,Q}$ in def. 2.28 (for any choice of P and Q) exhibits $\mathrm{Ho}(\mathcal{C})$ as indeed being the localization of the underlying category with weak equivalences at its weak equivalences, in the sense of def. 2.30:*

$$\begin{array}{ccc} \mathcal{C} & = & \mathcal{C} \\ {}^{\mathllap{\gamma_{P,Q}}}\downarrow && \downarrow^{\mathrlap{\gamma}} \\ \mathrm{Ho}(\mathcal{C}) & \simeq & \mathcal{C}[W^{-1}] \end{array}.$$

(Quillen 67, I.1 theorem 1)

Proof. First, to see that that $\gamma_{P,Q}$ indeed takes weak equivalences to isomorphisms: By two-out-of-three (def. 2.1) applied to the commuting diagrams shown in the proof of lemma 2.29, the morphism PQf is a weak equivalence if f is:

$$\begin{array}{ccccc} X & \underset{\simeq}{\overset{p_X}{\longleftarrow}} & QX & \underset{\simeq}{\overset{j_{QX}}{\longrightarrow}} & PQX \\ {}^{\mathllap{f}}\downarrow && \downarrow^{\mathrlap{Qf}} && \downarrow^{\mathrlap{PQf}} \\ Y & \underset{p_y}{\overset{\simeq}{\longleftarrow}} & QY & \underset{j_{QY}}{\overset{\simeq}{\longrightarrow}} & PQY \end{array}$$

With this the "Whitehead theorem for model categories", lemma 2.27, implies that PQf represents an isomorphism in $\mathrm{Ho}(\mathcal{C})$.

Now let $F : \mathcal{C} \longrightarrow D$ be any functor that sends weak equivalences to isomorphisms. We need to show that it factors as

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "\gamma"'] & \arrow[d, phantom, "\Downarrow^{\rho}"] & D \\ & \mathrm{Ho}(\mathcal{C}) \arrow[ur, "\tilde F"'] & \end{tikzcd}$$

uniquely up to unique natural isomorphism. Now by construction of P and Q in def. 2.28, $\gamma_{P,Q}$ is the identity on the full subcategory of fibrant-cofibrant objects. It follows that if $\tilde F$ exists at all, it must satisfy for all $X \overset{f}{\to} Y$ with X and Y both fibrant and cofibrant that

$$\tilde F([f]) \simeq F(f) \,,$$

(hence in particular $\tilde F(\gamma_{P,Q}(f)) = F(P Q f)$).

But by def. 2.25 that already fixes $\tilde F$ on all of $\mathrm{Ho}(\mathcal{C})$, up to unique natural isomorphism. Hence it only remains to check that with this definition of $\tilde F$ there exists any natural isomorphism ρ filling the diagram above.

To that end, apply F to the above commuting diagram to obtain

$$\begin{array}{ccccc} F(X) & \underset{\text{iso}}{\overset{F(p_X)}{\longleftarrow}} & F(Q X) & \underset{\text{iso}}{\overset{F(j_{Q X})}{\longrightarrow}} & F(P Q X) \\ {}^{\mathllap{F(f)}}\downarrow && \downarrow^{\mathrlap{F(Q f)}} && \downarrow^{\mathrlap{F(P Q f)}} \\ F(Y) & \underset{F(p_Y)}{\overset{\text{iso}}{\longleftarrow}} & F(Q Y) & \underset{F(j_{Q Y})}{\overset{\text{iso}}{\longrightarrow}} & F(P Q Y) \end{array}$$

Here now all horizontal morphisms are isomorphisms, by assumption on F. It follows that defining $\rho_X \coloneqq F(j_{Q X}) \circ F(p_X)^{-1}$ makes the required natural isomorphism:

$$\begin{array}{cccccccc} \rho_X : & F(X) & \underset{\text{iso}}{\overset{F(p_X)^{-1}}{\longrightarrow}} & F(Q X) & \underset{\text{iso}}{\overset{F(j_{Q X})}{\longrightarrow}} & F(P Q X) & = & \tilde F(\gamma_{P,Q}(X)) \\ & {}^{\mathllap{F(f)}}\downarrow &&&& \downarrow^{\mathrlap{F(P Q f)}} && \downarrow^{\mathrlap{\tilde F(\gamma_{P,Q}(f))}} \\ \rho_Y : & F(Y) & \underset{F(p_Y)^{-1}}{\overset{\text{iso}}{\longrightarrow}} & F(Q Y) & \underset{F(j_{Q Y})}{\overset{\text{iso}}{\longrightarrow}} & F(P Q Y) & = & \tilde F(\gamma_{P,Q}(X)) \end{array} \,.$$

∎

Remark 2.32. Due to theorem 2.31 we may suppress the choices of cofibrant Q and fibrant replacement P in def. 2.28 and just speak of the localization functor

$$\gamma \;:\; \mathcal{C} \longrightarrow \mathrm{Ho}(\mathcal{C})$$

up to natural isomorphism.

In general, the localization $\mathcal{C}[W^{-1}]$ of a category with weak equivalences $(\mathcal{C}, W)$ (def. 2.30) may invert *more* morphisms than just those in W. However, if the category admits the structure of a model category $(\mathcal{C}, W, \mathrm{Cof}, \mathrm{Fib})$, then its localiztion precisely only inverts the weak equivalences.

Proposition 2.33. *Let $\mathcal{C}$ be a model category (def. 2.3) and let $\gamma \;:\; \mathcal{C} \longrightarrow \mathrm{Ho}(\mathcal{C})$ be its localization functor (def. 2.28, theorem 2.31). Then a morphism f in $\mathcal{C}$ is a weak equivalence precisely if $\gamma(f)$ is an isomorphism in $\mathrm{Ho}(\mathcal{C})$.*

(e.g. Goerss-Jardine 96, II, prop 1.14)

While the construction of the homotopy category in def. 2.25 combines the restriction to good (fibrant/cofibrant) objects with the passage to homotopy classes of morphisms, it is often useful to consider intermediate stages:

Definition 2.34. Given a model category $\mathcal{C}$, write

$$\begin{array}{ccc} & \mathcal{C}_{fc} & \\ & \swarrow \quad \searrow & \\ \mathcal{C}_c & & \mathcal{C}_f \\ & \searrow \quad \swarrow & \\ & \mathcal{C} & \end{array}$$

for the system of full subcategory inclusions of:

1. the category of fibrant objects $\mathcal{C}_f$
2. the category of cofibrant objects $\mathcal{C}_c$,
3. the category of fibrant-cofibrant objects $\mathcal{C}_{\mathrm{fc}}$,

all regarded a categories with weak equivalences (def. 2.1), via the weak equivalences inherited from $\mathcal{C}$, which we write $(\mathcal{C}_f, W_f)$, $(\mathcal{C}_c, W_c)$ and $(\mathcal{C}_{fc}, W_{fc})$.

Remark 2.35. Of course the subcategories in def. 2.34 inherit more structure than just that of categories with weak equivalences from $\mathcal{C}$. $\mathcal{C}_f$ and $\mathcal{C}_c$ each inherit "half" of the factorization axioms. One says that $\mathcal{C}_f$ has the structure of a "fibration category" called a "Brown-category of fibrant objects", while $\mathcal{C}_c$ has the structure of a "cofibration category".

We discuss properties of these categories of (co-)fibrant objects below in *Homotopy fiber sequences*.

The proof of theorem 2.31 immediately implies the following:

Corollary 2.36. *For $\mathcal{C}$ a model category, the restriction of the localization functor $\gamma : \mathcal{C} \longrightarrow \mathrm{Ho}(\mathcal{C})$ from def. 2.28 (using remark 2.32) to any of the sub-categories with weak equivalences of def. 2.34*

$$\begin{array}{ccc} & \mathcal{C}_{fc} & \\ & \swarrow \quad \searrow & \\ \mathcal{C}_c & & \mathcal{C}_f \\ & \searrow \quad \swarrow & \\ & \mathcal{C} & \\ & \downarrow^{\gamma} & \\ & \mathrm{Ho}(\mathcal{C}) & \end{array}$$

exhibits $\mathrm{Ho}(\mathcal{C})$ equivalently as the localization also of these subcategories with weak equivalences, at their weak equivalences. In particular there are equivalences of categories

$$\mathrm{Ho}(\mathcal{C}) \simeq \mathcal{C}[W^{-1}] \simeq \mathcal{C}_f[W_f^{-1}] \simeq \mathcal{C}_c[W_c^{-1}] \simeq \mathcal{C}_{fc}[W_{fc}^{-1}] \,.$$

The following says that for computing the hom-sets in the homotopy category, even a mixed variant of the above will do; it is sufficient that the domain is cofibrant and the codomain is fibrant:

Lemma 2.37. *For $X, Y \in \mathcal{C}$ with X cofibrant and Y fibrant, and for P, Q fibrant/cofibrant replacement functors as in def. 2.28, then the morphism*

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(PX, QY) = \mathrm{Hom}_{\mathcal{C}}(PX, QY)/_\sim \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(j_X, p_Y)} \mathrm{Hom}_{\mathcal{C}}(X, Y)/_\sim$$

(on homotopy classes of morphisms, well defined by prop. 2.24) is a natural bijection.

(Quillen 67, I.1 lemma 7)

Proof. We may factor the morphism in question as the composite

$$\mathrm{Hom}_{\mathcal{C}}(PX, QY)/_\sim \overset{\mathrm{Hom}_{\mathcal{C}}(\mathrm{id}_{PX}, p_Y)/_\sim}{\longrightarrow} \mathrm{Hom}_{\mathcal{C}}(PX, Y)/_\sim \overset{\mathrm{Hom}_{\mathcal{C}}(j_X, \mathrm{id}_Y)/_\sim}{\longrightarrow} \mathrm{Hom}_{\mathcal{C}}(X, Y)/_\sim \,.$$

This shows that it is sufficient to see that for X cofibrant and Y fibrant, then

$$\mathrm{Hom}_{\mathcal{C}}(\mathrm{id}_X, p_Y)/_\sim \;:\; \mathrm{Hom}_{\mathcal{C}}(X, QY)/_\sim \to \mathrm{Hom}_{\mathcal{C}}(X, Y)/_\sim$$

is an isomorphism, and dually that

$$\mathrm{Hom}_{\mathcal{C}}(j_X, \mathrm{id}_Y)/_\sim \;:\; \mathrm{Hom}_{\mathcal{C}}(PX, Y)/_\sim \to \mathrm{Hom}_{\mathcal{C}}(X, Y)/_\sim$$

is an isomorphism. We discuss this for the former; the second is formally dual:

First, that $\mathrm{Hom}_{\mathcal{C}}(\mathrm{id}_X, p_Y)$ is surjective is the lifting property in

$$\begin{array}{ccc} \emptyset & \longrightarrow & QY \\ {}^{\mathllap{\in \mathrm{Cof}}}\downarrow && \downarrow^{\mathrlap{p_Y}}_{\mathrlap{\in W \cap \mathrm{Fib}}} \\ X & \overset{f}{\longrightarrow} & Y \end{array}$$

which says that any morphism $f : X \to Y$ comes from a morphism $\hat f : X \to QY$ under postcomposition with $QY \overset{p_Y}{\to} Y$.

Second, that $\mathrm{Hom}_{\mathcal{C}}(\mathrm{id}_X, p_Y)$ is injective is the lifting property in

$$\begin{array}{ccc} X \sqcup X & \overset{(f,g)}{\longrightarrow} & QY \\ {}^{\mathllap{\in \mathrm{Cof}}}\downarrow && \downarrow^{\mathrlap{p_Y}}_{\mathrlap{\in W \cap \mathrm{Fib}}} \\ \mathrm{Cyl}(X) & \underset{\eta}{\longrightarrow} & Y \end{array}$$

which says that if two morphisms $f, g : X \to QY$ become homotopic after postcomposition with $p_Y : QX \to Y$, then they were already homotopic before. ▮

We record the following fact which will be used in part 1.1 (here):

Lemma 2.38. *Let $\mathcal{C}$ be a model category (def. 2.3). Then every commuting square in its homotopy category $\mathrm{Ho}(\mathcal{C})$ (def. 2.25) is, up to isomorphism of squares, in the image of the localization functor $\mathcal{C} \longrightarrow \mathrm{Ho}(\mathcal{C})$ of a commuting square in $\mathcal{C}$ (i.e.: not just commuting up to homotopy).*

Proof. Let

$$\begin{array}{ccc} A & \overset{f}{\longrightarrow} & B \\ {}^{\mathllap{a}}\downarrow && \downarrow^{\mathrlap{b}} \\ A' & \underset{f'}{\longrightarrow} & B' \end{array} \quad \in \mathrm{Ho}(\mathcal{C})$$

be a commuting square in the homotopy category. Writing the same symbols for fibrant-cofibrant objects in $\mathcal{C}$ and for morphisms in $\mathcal{C}$ representing these, then this means that in $\mathcal{C}$ there is a left homotopy of the form

$$\begin{array}{ccc} A & \overset{f}{\longrightarrow} & B \\ {}^{i_1}\downarrow && \downarrow^{b} \\ \mathrm{Cyl}(A) & \underset{\eta}{\longrightarrow} & B' \\ {}^{i_0}\uparrow && \uparrow^{f'} \\ A & \underset{a}{\longrightarrow} & A' \end{array} \,.$$

Consider the factorization of the top square here through the mapping cylinder of f

$$\begin{array}{ccc} A & \overset{f}{\longrightarrow} & B \\ {}^{i_1}\downarrow & (\mathrm{po}) & \downarrow^{\in W} \\ \mathrm{Cyl}(A) & \longrightarrow & \mathrm{Cyl}(f) \\ {}^{i_0}\uparrow & {}_{\eta}\searrow & \downarrow \\ A && B' \\ & {}_{a}\searrow & \uparrow_{f'} \\ && A' \end{array}$$

This exhibits the composite $A \overset{i_0}{\to} \mathrm{Cyl}(A) \to \mathrm{Cyl}(f)$ as an alternative representative of f in $\mathrm{Ho}(\mathcal{C})$, and $\mathrm{Cyl}(f) \to B'$ as an alternative representative for b, and the commuting square

$$\begin{array}{ccc} A & \longrightarrow & \mathrm{Cyl}(f) \\ {}^{a}\downarrow && \downarrow \\ A' & \underset{f'}{\longrightarrow} & B' \end{array}$$

as an alternative representative of the given commuting square in $\mathrm{Ho}(\mathcal{C})$. ▮

Derived functors

Definition 2.39. For $\mathcal{C}$ and $\mathcal{D}$ two categories with weak equivalences, def. 2.1, then a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is called a **homotopical functor** if it sends weak equivalences to weak equivalences.

Definition 2.40. Given a homotopical functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ (def. 2.39) between categories with weak equivalences whose homotopy categories $\mathrm{Ho}(\mathcal{C})$ and $\mathrm{Ho}(\mathcal{D})$ exist (def. 2.30), then its ("total") *derived functor* is the functor $\mathrm{Ho}(F)$ between these homotopy categories which is induced uniquely, up to unique isomorphism, by their universal property (def. 2.30):

$$\begin{array}{ccc} \mathcal{C} & \overset{F}{\longrightarrow} & \mathcal{D} \\ {}^{\gamma_{\mathcal{C}}}\downarrow & \swarrow_{\simeq} & \downarrow^{\gamma_{\mathcal{D}}} \\ \mathrm{Ho}(\mathcal{C}) & \underset{\exists\, \mathrm{Ho}(F)}{\longrightarrow} & \mathrm{Ho}(\mathcal{D}) \end{array} \,.$$

Remark 2.41. While many functors of interest between model categories are not homotopical in the sense of def. 2.39, many become homotopical after restriction to the full subcategories $\mathcal{C}_f$ of fibrant objects or $\mathcal{C}_c$ of cofibrant objects, def. 2.34. By corollary 2.36 this is just as good for the purpose of homotopy theory.

Therefore one considers the following generalization of def. 2.40:

Definition 2.42. Consider a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ out of a model category $\mathcal{C}$ (def. 2.3) into a category with weak equivalences $\mathcal{D}$ (def. 2.1).

1. If the restriction of F to the full subcategory $\mathcal{C}_f$ of fibrant object becomes a homotopical functor (def. 2.39), then the derived functor of that restriction, according to def. 2.40, is

called the *right derived functor* of F and denoted by $\mathbb{R}F$:

$$\begin{array}{ccccc} & \mathcal{C}_f & \hookrightarrow & \mathcal{C} & \overset{F}{\longrightarrow} & \mathcal{D} \\ & {}^{\gamma_{\mathcal{C}_f}}\downarrow & & \swArrow_{\simeq} & & \downarrow^{\gamma_{\mathcal{D}}} \\ \mathbb{R}F: & \mathcal{C}_f[W^{-1}] & \simeq & \mathrm{Ho}(\mathcal{C}) & \underset{\mathrm{Ho}(F)}{\longrightarrow} & \mathrm{Ho}(\mathcal{D}) \end{array},$$

where we use corollary 2.36.

2. If the restriction of F to the full subcategory $\mathcal{C}_c$ of cofibrant object becomes a homotopical functor (def. 2.39), then the derived functor of that restriction, according to def. 2.40, is called the *left derived functor* of F and denoted by $\mathbb{L}F$:

$$\begin{array}{ccccc} & \mathcal{C}_c & \hookrightarrow & \mathcal{C} & \overset{F}{\longrightarrow} & \mathcal{D} \\ & {}^{\gamma_{\mathcal{C}_f}}\downarrow & & \swArrow_{\simeq} & & \downarrow^{\gamma_{\mathcal{D}}} \\ \mathbb{L}F: & \mathcal{C}_c[W^{-1}] & \simeq & \mathrm{Ho}(\mathcal{C}) & \underset{\mathrm{Ho}(F)}{\longrightarrow} & \mathrm{Ho}(\mathcal{D}) \end{array},$$

where again we use corollary 2.36.

The key fact that makes def. 2.42 practically relevant is the following:

Proposition 2.43. ***(Ken Brown's lemma)***

Let $\mathcal{C}$ be a model category with full subcategories $\mathcal{C}_f, \mathcal{C}_c$ of fibrant objects and of cofibrant objects respectively (def. 2.34). Let $\mathcal{D}$ be a category with weak equivalences.

1. *A functor out of the category of fibrant objects*

$$F : \mathcal{C}_f \longrightarrow \mathcal{D}$$

is a homotopical functor, def. 2.39, already if it sends acylic fibrations to weak equivalences.

2. *A functor out of the category of cofibrant objects*

$$F : \mathcal{C}_c \longrightarrow \mathcal{D}$$

is a homotopical functor, def. 2.39, already if it sends acylic cofibrations to weak equivalences.

The following proof refers to the factorization lemma, whose full statement and proof we postpone to further below (lemma 4.9).

Proof. We discuss the case of a functor on a category of fibrant objects $\mathcal{C}_f$, def. 2.34. The other case is formally dual.

Let $f : X \longrightarrow Y$ be a weak equivalence in $\mathcal{C}_f$. Choose a path space object $\mathrm{Path}(X)$ (def. 2.18) and consider the diagram

$$\begin{array}{ccc} \mathrm{Path}(f) & \underset{\in W \cap \mathrm{Fib}}{\longrightarrow} & X \\ {}^{p_1^* f}_{\in W}\downarrow & (\mathrm{pb}) & \downarrow^{f}_{\in W} \\ \mathrm{Path}(Y) & \underset{\in W \cap \mathrm{Fib}}{\overset{p_1}{\longrightarrow}} & Y \\ {}^{p_0}_{\in W \cap \mathrm{Fib}}\downarrow & & \\ Y & & \end{array},$$

where the square is a pullback and $\mathrm{Path}(f)$ on the top left is our notation for the universal cone object. (Below we discuss this in more detail, it is the *mapping cocone* of f, def. 4.1).

Here:

1. p_i are both acyclic fibrations, by lemma 2.20;

2. $\mathrm{Path}(f) \to X$ is an acyclic fibration because it is the pullback of p_1.

3. $p_1^* f$ is a weak equivalence, because the factorization lemma 4.9 states that the composite vertical morphism factors f through a weak equivalence, hence if f is a weak equivalence, then $p_1^* f$ is by two-out-of-three (def. 2.1).

Now apply the functor F to this diagram and use the assumption that it sends acyclic fibrations to weak equivalences to obtain

$$\begin{array}{ccc} F(\mathrm{Path}(f)) & \underset{\in W}{\longrightarrow} & F(X) \\ {}^{F(p_1^* f)}\downarrow && \downarrow^{F(f)} \\ F(\mathrm{Path}(Y)) & \underset{\in W}{\overset{F(p_1)}{\longrightarrow}} & F(Y) \\ {}^{F(p_0)}_{\in W}\downarrow \\ Y \end{array} \,.$$

But the factorization lemma 4.9, in addition says that the vertical composite $p_0 \circ p_1^* f$ is a fibration, hence an acyclic fibration by the above. Therefore also $F(p_0 \circ p_1^* f)$ is a weak equivalence. Now the claim that also $F(f)$ is a weak equivalence follows with applying two-out-of-three (def. 2.1) twice. ▮

Corollary 2.44. *Let $\mathcal{C}, \mathcal{D}$ be model categories and consider $F : \mathcal{C} \longrightarrow \mathcal{D}$ a functor. Then:*

1. *If F preserves cofibrant objects and acyclic cofibrations between these, then its left derived functor (def. 2.42) $\mathbb{L}F$ exists, fitting into a diagram*

$$\begin{array}{ccc} \mathcal{C}_c & \overset{F}{\longrightarrow} & \mathcal{D}_c \\ {}^{\gamma_{\mathcal{C}}}\downarrow & \swArrow_{\simeq} & \downarrow^{\gamma_{\mathcal{D}}} \\ \mathrm{Ho}(\mathcal{C}) & \overset{\mathbb{L}F}{\longrightarrow} & \mathrm{Ho}(\mathcal{D}) \end{array}$$

2. *If F preserves fibrant objects and acyclic fibrants between these, then its right derived functor (def. 2.42) $\mathbb{R}F$ exists, fitting into a diagram*

$$\begin{array}{ccc} \mathcal{C}_f & \overset{F}{\longrightarrow} & \mathcal{D}_f \\ {}^{\gamma_{\mathcal{C}}}\downarrow & \swArrow_{\simeq} & \downarrow^{\gamma_{\mathcal{D}}} \\ \mathrm{Ho}(\mathcal{C}) & \underset{\mathbb{R}F}{\longrightarrow} & \mathrm{Ho}(\mathcal{D}) \end{array} \,.$$

Proposition 2.45. Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor between two model categories (def. 2.3).

1. If F preserves fibrant objects and weak equivalences between fibrant objects, then the total right derived functor $\mathbb{R}F \coloneqq \mathbb{R}(\gamma_{\mathcal{D}} \circ F)$ (def. 2.42) in

$$\begin{array}{ccc} \mathcal{C}_f & \overset{F}{\longrightarrow} & \mathcal{D} \\ {}^{\gamma_{\mathcal{C}_f}}\downarrow & \swArrow_{\simeq} & \downarrow^{\gamma_{\mathcal{D}}} \\ \mathrm{Ho}(\mathcal{C}) & \underset{\mathbb{R}F}{\longrightarrow} & \mathrm{Ho}(\mathcal{D}) \end{array}$$

is given, up to isomorphism, on any object $X \in \mathcal{C} \overset{\gamma_{\mathcal{C}}}{\longrightarrow} \mathrm{Ho}(\mathcal{C})$ by appying F to a fibrant replacement $P X$ of X and then forming a cofibrant replacement $Q(F(P X))$ of the result:

$$\mathbb{R}F(X) \simeq Q(F(P X)) \,.$$

1. If F preserves cofibrant objects and weak equivalences between cofibrant objects, then the total left derived functor $\mathbb{L}F \coloneqq \mathbb{L}(\gamma_{\mathcal{D}} \circ F)$ (def. 2.42) in

$$\begin{tikzcd} \mathcal{C}_c \arrow[r, "F"] \arrow[d, "\gamma_{\mathcal{C}_c}"'] \arrow[dr, phantom, "\swArrow_{\simeq}"] & \mathcal{D} \arrow[d, "\gamma_{\mathcal{D}}"] \\ \mathrm{Ho}(\mathcal{C}) \arrow[r, "\mathbb{L}F"'] & \mathrm{Ho}(\mathcal{D}) \end{tikzcd}$$

is given, up to isomorphism, on any object $X \in \mathcal{C} \overset{\gamma_{\mathcal{C}}}{\longrightarrow} \mathrm{Ho}(\mathcal{C})$ by appying F to a cofibrant replacement $Q X$ of X and then forming a fibrant replacement $P(F(Q X))$ of the result:

$$\mathbb{L}F(X) \simeq P(F(Q X)) \,.$$

Proof. We discuss the first case, the second is formally dual. By the proof of theorem 2.31 we have

$$\begin{aligned} \mathbb{R}F(X) & \simeq \gamma_{\mathcal{D}}(F(\gamma_{\mathcal{C}})) \\ & \simeq \gamma_{\mathcal{D}} F(Q(P(X))) \end{aligned} \,.$$

But since F is a homotopical functor on fibrant objects, the cofibrant replacement morphism $F(Q(P(X))) \to F(P(X))$ is a weak equivalence in $\mathcal{D}$, hence becomes an isomorphism under $\gamma_{\mathcal{D}}$. Therefore

$$\mathbb{R}F(X) \simeq \gamma_{\mathcal{D}}(F(P(X))) \,.$$

Now since F is assumed to preserve fibrant objects, $F(P(X))$ is fibrant in $\mathcal{D}$, and hence $\gamma_{\mathcal{D}}$ acts on it (only) by cofibrant replacement. ▮

Quillen adjunctions

In practice it turns out to be useful to arrange for the assumptions in corollary 2.44 to be satisfied by pairs of adjoint functors. Recall that this is a pair of functors L and R going back and forth between two categories

$$\mathcal{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D}$$

such that there is a natural bijection between hom-sets with L on the left and those with R on the right:

$$\phi_{d,c} \;:\; \mathrm{Hom}_{\mathcal{C}}(L(d), c) \underset{\simeq}{\longrightarrow} \mathrm{Hom}_{\mathcal{D}}(d, R(c))$$

for all objects $d \in \mathcal{D}$ and $c \in \mathcal{C}$. This being natural means that $\phi \colon \mathrm{Hom}_{\mathcal{D}}(L(-), -) \Rightarrow \mathrm{Hom}_{\mathcal{C}}(-, R(-))$ is a natural transformation, hence that for all morphisms $g \colon d_2 \to d_1$ and $f \colon c_1 \to c_2$ the following is a commuting square:

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(L(d_1), c_1) \arrow[r, "\phi_{d_1,c_1}", "\simeq"'] \arrow[d, "L(f)\circ(-)\circ g"'] & \mathrm{Hom}_{\mathcal{D}}(d_1, R(c_1)) \arrow[d, "g\circ(-)\circ R(g)"] \\ \mathrm{Hom}_{\mathcal{C}}(L(d_2), c_2) \arrow[r, "\simeq", "\phi_{d_2,c_2}"'] & \mathrm{Hom}_{\mathcal{D}}(d_2, R(c_2)) \end{tikzcd}$$

We write $(L \dashv R)$ to indicate an adjunction and call L the *left adjoint* and R the *right adjoint* of the adjoint pair.

The archetypical example of a pair of adjoint functors is that consisting of forming Cartesian products $Y \times (-)$ and forming mapping spaces $(-)^Y$, as in the category of compactly generated topological spaces of def. 3.35.

If $f : L(d) \to c$ is any morphism, then the image $\phi_{d,c}(f) : d \to R(c)$ is called its *adjunct*, and conversely. The fact that adjuncts are in bijection is also expressed by the notation

$$\frac{L(c) \overset{f}{\longrightarrow} d}{c \overset{\tilde{f}}{\longrightarrow} R(d)} \,.$$

For an object $d \in \mathcal{D}$, the adjunct of the identity on Ld is called the *adjunction unit* $\eta_d \,:\, d \longrightarrow RLd$.

For an object $c \in \mathcal{C}$, the adjunct of the identity on Rc is called the *adjunction counit* $\epsilon_c \,:\, LRc \longrightarrow c$.

Adjunction units and counits turn out to encode the adjuncts of all other morphisms by the formulas

- $\widetilde{(Ld \overset{f}{\to} c)} = (d \overset{\eta}{\to} RLd \overset{Rf}{\to} Rc)$
- $\widetilde{(d \overset{g}{\to} Rc)} = (Ld \overset{Lg}{\to} LRc \overset{\epsilon}{\to} c)$.

Definition 2.46. Let $\mathcal{C}, \mathcal{D}$ be model categories. A pair of adjoint functors between them

$$(L \dashv R) \,:\, \mathcal{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D}$$

is called a **Quillen adjunction** (and L,R are called left/right **Quillen functors**, respectively) if the following equivalent conditions are satisfied

1. L preserves cofibrations and R preserves fibrations;
2. L preserves acyclic cofibrations and R preserves acyclic fibrations;
3. L preserves cofibrations and acylic cofibrations;
4. R preserves fibrations and acyclic fibrations.

Proposition 2.47. *The conditions in def. 2.46 are indeed all equivalent.*

(Quillen 67, I.4, theorem 3)

Proof. First observe that

- (i) *A left adjoint L between model categories preserves acyclic cofibrations precisely if its right adjoint R preserves fibrations.*
- (ii) *A left adjoint L between model categories preserves cofibrations precisely if its right adjoint R preserves acyclic fibrations.*

We discuss statement (i), statement (ii) is formally dual. So let $f : A \to B$ be an acyclic cofibration in $\mathcal{D}$ and $g : X \to Y$ a fibration in $\mathcal{C}$. Then for every commuting diagram as on the left of the following, its $(L \dashv R)$-adjunct is a commuting diagram as on the right here:

$$\begin{array}{ccc} A & \longrightarrow & R(X) \\ {}^{f}\downarrow && \downarrow^{R(g)} \\ B & \longrightarrow & R(Y) \end{array} \;\;\; , \;\;\; \begin{array}{ccc} L(A) & \longrightarrow & X \\ {}^{L(f)}\downarrow && \downarrow^{g} \\ L(B) & \longrightarrow & Y \end{array} .$$

If L preserves acyclic cofibrations, then the diagram on the right has a lift, and so the $(L \dashv R)$-adjunct of that lift is a lift of the left diagram. This shows that $R(g)$ has the right lifting property against all acylic cofibrations and hence is a fibration. Conversely, if R preserves fibrations, the same argument run from right to left gives that L preserves acyclic fibrations.

Now by repeatedly applying (i) and (ii), all four conditions in question are seen to be equivalent. ▮

Lemma 2.48. *Let $\mathcal{C} \underset{R}{\overset{L}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathcal{D}$ be a Quillen adjunction, def. 2.46.*

1. *For $X \in \mathcal{C}$ a fibrant object and $\mathrm{Path}(X)$ a path space object (def. 2.18), then $R(\mathrm{Path}(X))$ is a path space object for $R(X)$.*
2. *For $X \in \mathcal{C}$ a cofibrant object and $\mathrm{Cyl}(X)$ a cylinder object (def. 2.18), then $L(\mathrm{Cyl}(X))$ is a path space object for $L(X)$.*

Proof. Consider the second case, the first is formally dual.

First Observe that $L(Y \sqcup Y) \simeq LY \sqcup LY$ because L is left adjoint and hence preserves colimits, hence in particular coproducts.

Hence

$$L(\mathrm{X} \sqcup X \overset{\in \mathrm{Cof}}{\longrightarrow} \mathrm{Cyl}(X)) = (L(X) \sqcup L(X) \overset{\in \mathrm{Cof}}{\longrightarrow} L(\mathrm{Cyl}(X)))$$

is a cofibration.

Second, with Y cofibrant then also $Y \sqcup \mathrm{Cyl}(Y)$ is a cofibrantion, since $Y \to Y \sqcup Y$ is a cofibration (lemma 2.20). Therefore by Ken Brown's lemma (prop. 2.43) L preserves the weak equivalence $\mathrm{Cyl}(Y) \overset{\in W}{\longrightarrow} Y$. ▮

Proposition 2.49. *For $\mathcal{C} \underset{R}{\overset{L}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathcal{D}$ a Quillen adjunction, def. 2.46, then also the corresponding left and right derived functors, def. 2.42, via cor. 2.44, form a pair of adjoint functors*

$$\mathrm{Ho}(\mathcal{C}) \underset{\mathbb{R}R}{\overset{\mathbb{L}L}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathrm{Ho}(\mathcal{D}) .$$

(Quillen 67, I.4 theorem 3)

Proof. By def. 2.42 and lemma 2.37 it is sufficient to see that for $X, Y \in \mathcal{C}$ with X cofibrant and Y fibrant, then there is a natural bijection

$$\mathrm{Hom}_{\mathcal{C}}(LX, Y)/_{\sim} \simeq \mathrm{Hom}_{\mathcal{C}}(X, RY)/_{\sim} .$$

Since by the adjunction isomorphism for $(L \dashv R)$ such a natural bijection exists before passing to homotopy classes $(-)/_{\sim}$, it is sufficient to see that this respects homotopy classes. To that end, use from lemma 2.48 that with $\mathrm{Cyl}(Y)$ a cylinder object for Y, def. 2.18, then $L(\mathrm{Cyl}(Y))$ is a cylinder object for $L(Y)$. This implies that left homotopies

$$(f \Rightarrow_L g) : LX \longrightarrow Y$$

given by

$$\eta : \mathrm{Cyl}(LX) = L\,\mathrm{Cyl}(X) \longrightarrow Y$$

are in bijection to left homotopies

$$(\tilde{f} \Rightarrow_L \tilde{g}) : X \longrightarrow RY$$

given by

$$\tilde{\eta} : \mathrm{Cyl}(X) \longrightarrow RX \,.$$

∎

Definition 2.50. For $\mathcal{C}, \mathcal{D}$ two model categories, a Quillen adjunction (def.2.46)

$$(L \dashv R) : \mathcal{C} \underoverset{\underset{R}{\longrightarrow}}{\overset{L}{\longleftarrow}}{\bot} \mathcal{D}$$

is called a **Quillen equivalence**, to be denoted

$$\mathcal{C} \underoverset{\underset{R}{\longrightarrow}}{\overset{L}{\longleftarrow}}{\simeq_Q} \mathcal{D} \,,$$

if the following equivalent conditions hold.

1. The right derived functor of R (via prop. 2.47, corollary 2.44) is an equivalence of categories

$$\mathbb{R}R : \mathrm{Ho}(\mathcal{C}) \overset{\simeq}{\longrightarrow} \mathrm{Ho}(\mathcal{D}) \,.$$

2. The left derived functor of L (via prop. 2.47, corollary 2.44) is an equivalence of categories

$$\mathbb{L}L : \mathrm{Ho}(\mathcal{D}) \overset{\simeq}{\longrightarrow} \mathrm{Ho}(\mathcal{C}) \,.$$

3. For every cofibrant object $d \in \mathcal{D}$, the “derived adjunction unit”, hence the composite

$$d \overset{\eta}{\longrightarrow} R(L(d)) \overset{R(j_{L(d)})}{\longrightarrow} R(P(L(d)))$$

(of the adjunction unit with any fibrant replacement P as in def. 2.28) is a weak equivalence;

and for every fibrant object $c \in \mathcal{C}$, the “derived adjunction counit”, hence the composite

$$L(Q(R(c))) \overset{L(p_{R(c)})}{\longrightarrow} L(R(c)) \overset{\epsilon}{\longrightarrow} c$$

(of the adjunction counit with any cofibrant replacement as in def. 2.28) is a weak equivalence in D.

4. For every cofibrant object $d \in \mathcal{D}$ and every fibrant object $c \in \mathcal{C}$, a morphism $d \longrightarrow R(c)$ is a weak equivalence precisely if its adjunct morphism $L(c) \to d$ is:

$$\frac{d \overset{\in W_{\mathcal{D}}}{\longrightarrow} R(c)}{L(d) \overset{\in W_{\mathcal{C}}}{\longrightarrow} c} \,.$$

Poposition 2.51. *The conditions in def. 2.50 are indeed all equivalent.*

(Quillen 67, I.4, theorem 3)

Proof. That $1) \Leftrightarrow 2)$ follows from prop. 2.49 (if in an adjoint pair one is an equivalence, then so is the other).

To see the equivalence $1), 2) \Leftrightarrow 3)$, notice (prop.) that a pair of adjoint functors is an equivalence of categories precisely if both the adjunction unit and the adjunction counit are natural isomorphisms. Hence it is sufficient to show that the morphisms called "derived adjunction (co-)units" above indeed represent the adjunction (co-)unit of $(\mathbb{L}L \dashv \mathbb{R}R)$ in the homotopy category. We show this now for the adjunction unit, the case of the adjunction counit is formally dual.

To that end, first observe that for $d \in \mathcal{D}_c$, then the defining commuting square for the left derived functor from def. 2.42

$$\begin{array}{ccc} \mathcal{D}_c & \overset{L}{\longrightarrow} & \mathcal{C} \\ {}^{\mathllap{\gamma_P}}\downarrow & \swArrow_{\simeq} & \downarrow^{\mathrlap{\gamma_{P,Q}}} \\ Ho(\mathcal{D}) & \underset{\mathbb{L}L}{\longrightarrow} & Ho(\mathcal{C}) \end{array}$$

(using fibrant and fibrant/cofibrant replacement functors γ_P, $\gamma_{P,Q}$ from def. 2.28 with their universal property from theorem 2.31, corollary 2.36) gives that

$$(\mathbb{L}L)d \simeq PLPd \simeq PLd \quad \in Ho(\mathcal{C}) \,,$$

where the second isomorphism holds because the left Quillen functor L sends the acyclic cofibration $j_d : d \to Pd$ to a weak equivalence.

The adjunction unit of $(\mathbb{L}L \dashv \mathbb{R}R)$ on $Pd \in Ho(\mathcal{C})$ is the image of the identity under

$$Hom_{Ho(\mathcal{C})}((\mathbb{L}L)Pd, (\mathbb{L}L)Pd) \overset{\simeq}{\to} Hom_{Ho(\mathcal{C})}(Pd, (\mathbb{R}R)(\mathbb{L}L)Pd) \,.$$

By the above and the proof of prop. 2.49, that adjunction isomorphism is equivalently that of $(L \dashv R)$ under the isomorphism

$$Hom_{Ho(\mathcal{C})}(PLd, PLd) \overset{Hom(j_{Ld}, id)}{\longrightarrow} Hom_{\mathcal{C}}(Ld, PLd)/_\sim$$

of lemma 2.37. Hence the derived adjunction unit is the $(L \dashv R)$-adjunct of

$$Ld \overset{j_{Ld}}{\longrightarrow} PLd \overset{id}{\to} PLd \,,$$

which indeed (by the formula for adjuncts) is

$$X \overset{\eta}{\longrightarrow} RLd \overset{R(j_{Ld})}{\longrightarrow} RPLd \,.$$

To see that $4) \Rightarrow 3)$:

Consider the weak equivalence $LX \overset{j_{LX}}{\longrightarrow} PLX$. Its $(L \dashv R)$-adjunct is

$$X \overset{\eta}{\longrightarrow} RLX \overset{R j_{LX}}{\longrightarrow} RPLX$$

by assumption 4) this is again a weak equivalence, which is the requirement for the derived unit in 3). Dually for derived counit.

To see $3) \Rightarrow 4)$:

Consider any $f : Ld \to c$ a weak equivalence for cofibrant d, firbant c. Its adjunct $\tilde{f}$ sits in a commuting diagram

$$\begin{array}{ccccc} \tilde f : & d & \overset{\eta}{\longrightarrow} & RLd & \overset{Rf}{\longrightarrow} & Rc \\ & {}^{=}\downarrow && \downarrow^{Rj_{Ld}} && \downarrow^{Rj_c} \\ & d & \underset{\in W}{\longrightarrow} & RPLd & \overset{RPf}{\longrightarrow} & RPc \end{array},$$

where Pf is any lift constructed as in def. 2.28.

This exhibits the bottom left morphism as the derived adjunction unit, hence a weak equivalence by assumption. But since f was a weak equivalence, so is Pf (by two-out-of-three). Thereby also RPf and Rj_Y, are weak equivalences by Ken Brown's lemma 2.43 and the assumed fibrancy of c. Therefore by two-out-of-three (def. 2.1) also the adjunct $\tilde f$ is a weak equivalence. ▮

In certain situations the conditions on a Quillen equivalence simplify. For instance:

Proposition 2.52. *If in a Quillen adjunction $\mathcal{C} \underoverset{\underset{R}{\longrightarrow}}{\overset{L}{\longleftarrow}}{\bot} \mathcal{D}$ (def. 2.46) the right adjoint R "creates weak equivalences" (in that a morphism f in $\mathcal{C}$ is a weak equivalence precisely if $U(f)$ is) then $(L \dashv R)$ is a Quillen equivalence (def. 2.50) precisely already if for all cofibrant objects $d \in \mathcal{D}$ the plain adjunction unit*

$$d \overset{\eta}{\longrightarrow} R(L(d))$$

is a weak equivalence.

Proof. By prop. 2.51, generally, $(L \dashv R)$ is a Quillen equivalence precisely if

1. for every cofibrant object $d \in \mathcal{D}$, the "derived adjunction unit"

$$d \overset{\eta}{\longrightarrow} R(L(d)) \overset{R(j_{L(d)})}{\longrightarrow} R(P(L(d)))$$

   is a weak equivalence;

2. for every fibrant object $c \in \mathcal{C}$, the "derived adjunction counit"

$$L(Q(R(c))) \overset{L(p_{R(c)})}{\longrightarrow} L(R(c)) \overset{\epsilon}{\longrightarrow} c$$

   is a weak equivalence.

Consider the first condition: Since R preserves the weak equivalence $j_{L(d)}$, then by two-out-of-three (def. 2.1) the composite in the first item is a weak equivalence precisely if η is.

Hence it is now sufficient to show that in this case the second condition above is automatic.

Since R also reflects weak equivalences, the composite in item two is a weak equivalence precisely if its image

$$R(L(Q(R(c)))) \overset{R(L(p_{R(c)}))}{\longrightarrow} R(L(R(c))) \overset{R(\epsilon)}{\longrightarrow} R(c)$$

under R is.

Moreover, assuming, by the above, that $\eta_{Q(R(c))}$ on the cofibrant object $Q(R(c))$ is a weak equivalence, then by two-out-of-three this composite is a weak equivalence precisely if the further composite with η is

$$Q(R(c)) \overset{\eta_{Q(R(c))}}{\longrightarrow} R(L(Q(R(c)))) \overset{R(L(p_{R(c)}))}{\longrightarrow} R(L(R(c))) \overset{R(\epsilon)}{\longrightarrow} R(c)\,.$$

By the formula for adjuncts, this composite is the $(L \dashv R)$-adjunct of the original composite, which is just $p_{R(c)}$

$$\frac{L(Q(R(c))) \xrightarrow{L(p_{R(c)})} L(R(c)) \xrightarrow{\epsilon} c}{Q(R(C)) \xrightarrow{p_{R(c)}} R(c)} \,.$$

But $p_{R(c)}$ is a weak equivalence by definition of cofibrant replacement. ▮

3. The model structure on topological spaces

We now discuss how the category Top of topological spaces satisfies the axioms of abstract homotopy theory (model category) theory, def. 2.3.

Definition 3.1. Say that a continuous function, hence a morphism in Top, is

- a **classical weak equivalence** if it is a weak homotopy equivalence, def. 1.30;
- a **classical fibration** if it is a Serre fibration, def. 1.47;
- a **classical cofibration** if it is a retract (rem. 2.12) of a relative cell complex, def. 1.38.

and hence

- a **acyclic classical cofibration** if it is a classical cofibration as well as a classical weak equivalence;
- a **acyclic classical fibration** if it is a classical fibration as well as a classical weak equivalence.

Write

$$W_{\mathrm{cl}},\ \mathrm{Fib}_{\mathrm{cl}},\ \mathrm{Cof}_{\mathrm{cl}} \subset \mathrm{Mor}(\mathrm{Top})$$

for the classes of these morphisms, respectively.

We first prove now that the classes of morphisms in def. 3.1 satisfy the conditions for a model category structure, def. 2.3 (after some lemmas, this is theorem 3.7 below). Then we discuss the resulting classical homotopy category (below) and then a few variant model structures whose proof follows immediately along the line of the proof of $\mathrm{Top}_{\mathrm{Quillen}}$:

- The model structure on pointed topological spaces $\mathrm{Top}^{*/}_{\mathrm{Quillen}}$;
- The model structure on compactly generated topological spaces $(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{Quillen}}$ and $(\mathrm{Top}^{*/}_{\mathrm{cg}})_{\mathrm{Quillen}}$;
- The model structure on topologically enriched functors $[\mathcal{C}, (\mathrm{Top}_{\mathrm{cg}})_{\mathrm{Quillen}}]_{\mathrm{proj}}$ and $[\mathcal{C}, (\mathrm{Top}^{*}_{\mathrm{cg}})_{\mathrm{Quillen}}]_{\mathrm{proj}}$.

Proposition 3.2. *The classical weak equivalences, def. 3.1, satisfy two-out-of-three (def. 2.1).*

Proof. Since isomorphisms (of homotopy groups) satisfy 2-out-of-3, this property is directly inherited via the very definition of weak homotopy equivalence, def. 1.30. ▮

Lemma 3.3. *Every morphism $f : X \longrightarrow Y$ in Top factors as a classical cofibration followed by an acyclic classical fibration, def. 3.1:*

$$f\,:\,X \overset{\in \mathrm{Cof}_{\mathrm{cl}}}{\longrightarrow} \hat X \overset{\in W_{\mathrm{cl}} \cap \mathrm{Fib}_{\mathrm{cl}}}{\longrightarrow} Y\,.$$

Proof. By lemma 1.40 the set $I_{\mathrm{Top}} = \{S^{n-1} \hookrightarrow D^n\}$ of topological generating cofibrations, def. 1.37, has small domains, in the sense of def. 2.16 (the n-spheres are compact). Hence by the small object argument, prop. 2.17, f factors as an I_{Top}-relative cell complex, def. 1.41, hence just a plain relative cell complex, def. 1.38, followed by an I_{Top}-injective morphisms, def. 1.46:

$$f\,:\,X \overset{\in \mathrm{Cof}_{\mathrm{cl}}}{\longrightarrow} \hat X \overset{\in I_{\mathrm{Top}}\,\mathrm{Inj}}{\longrightarrow} Y\,.$$

By lemma 1.52 the map $\hat X \to Y$ is both a weak homotopy equivalence as well as a Serre fibration. ∎

Lemma 3.4. *Every morphism $f : X \longrightarrow Y$ in Top factors as an acyclic classical cofibration followed by a fibration, def. 3.1:*

$$f\,:\,X \overset{\in W_{\mathrm{cl}} \cap \mathrm{Cof}_{\mathrm{cl}}}{\longrightarrow} \hat X \overset{\in \mathrm{Fib}_{\mathrm{cl}}}{\longrightarrow} Y\,.$$

Proof. By lemma 1.40 the set $J_{\mathrm{Top}} = \{D^n \hookrightarrow D^n \times I\}$ of topological generating acyclic cofibrations, def. 1.42, has small domains, in the sense of def. 2.16 (the n-disks are compact). Hence by the small object argument, prop. 2.17, f factors as an J_{Top}-relative cell complex, def. 1.41, followed by a J_{top}-injective morphisms, def. 1.46:

$$f\,:\,X \overset{\in J_{\mathrm{Top}}\,\mathrm{Cell}}{\longrightarrow} \hat X \overset{\in J_{\mathrm{Top}}\,\mathrm{Inj}}{\longrightarrow} Y\,.$$

By definition this makes $\hat X \to Y$ a Serre fibration, hence a fibration.

By lemma 1.44 a relative J_{Top}-cell complex is in particular a relative I_{Top}-cell complex. Hence $X \to \hat X$ is a classical cofibration. By lemma 1.45 it is also a weak homotopy equivalence, hence a clasical weak equivalence. ∎

Lemma 3.5. *Every commuting square in Top with the left morphism a classical cofibration and the right morphism a fibration, def. 3.1*

$$\begin{array}{ccc} & \longrightarrow & \\ {}^{g \in}_{\mathrm{Cof}_{\mathrm{cl}}}\downarrow & & \downarrow^{f \in}_{\mathrm{Fib}_{\mathrm{cl}}} \\ & \longrightarrow & \end{array}$$

admits a lift as soon as one of the two is also a classical weak equivalence.

Proof. **A)** If the fibration f is also a weak equivalence, then lemma 1.52 says that it has the right lifting property against the generating cofibrations I_{Top}, and cor. 2.11 implies the claim.

B) If the cofibration g on the left is also a weak equivalence, consider any factorization into a relative J_{Top}-cell complex, def. 1.42, def. 1.41, followed by a fibration,

$$g\,:\, \overset{\in J_{\mathrm{Top}}\,\mathrm{Cell}}{\longrightarrow} \overset{\in \mathrm{Fib}_{\mathrm{cl}}}{\longrightarrow}\,,$$

as in the proof of lemma 3.4. By lemma 1.45 the morphism $\overset{\in J_{\mathrm{Top}}\,\mathrm{Cell}}{\longrightarrow}$ is a weak homotopy equivalence, and so by two-out-of-three (prop. 3.2) the factorizing fibration is actually an acyclic fibration. By case A), this acyclic fibration has the right lifting property against the cofibration g itself, and so the retract argument, lemma 2.15 gives that g is a retract of a relative J_{Top}-cell complex. With this, finally cor. 2.11 implies that f has the right lifting property against g. ∎

Finally:

Proposition 3.6. *The systems $(\mathrm{Cof}_{\mathrm{cl}}, W_{\mathrm{cl}} \cap \mathrm{Fib}_{\mathrm{cl}})$ and $(W_{\mathrm{cl}} \cap \mathrm{Cof}_{\mathrm{cl}}, \mathrm{Fib}_{\mathrm{cl}})$ from def. 3.1 are weak factorization systems.*

Proof. Since we have already seen the factorization property (lemma 3.3, lemma 3.4) and the lifting properties (lemma 3.5), it only remains to see that the given left/right classes exhaust the class of morphisms with the given lifting property.

For the classical fibrations this is by definition, for the the classical acyclic fibrations this is by lemma 1.52.

The remaining statement for $\mathrm{Cof}_{\mathrm{cl}}$ and $W_{\mathrm{cl}} \cap \mathrm{Cof}_{\mathrm{cl}}$ follows from a general argument (here) for cofibrantly generated model categories (def. 3.9), which we spell out:

So let $f : X \longrightarrow Y$ be in $(I_{\mathrm{Top}}\,\mathrm{Inj})\mathrm{Proj}$, we need to show that then f is a retract (remark 2.12) of a relative cell complex. To that end, apply the small object argument as in lemma 3.3 to factor f as

$$f \;:\; X \overset{I_{\mathrm{Top}}\,\mathrm{Cell}}{\longrightarrow} \hat{Y} \overset{\in I_{\mathrm{Top}}\,\mathrm{Inj}}{\longrightarrow} Y \,.$$

It follows that f has the left lifting property against $\hat{Y} \to Y$, and hence by the retract argument (lemma 2.15) it is a retract of $X \overset{I\,\mathrm{Cell}}{\longrightarrow} \hat{Y}$. This proves the claim for $\mathrm{Cof}_{\mathrm{cl}}$.

The analogous argument for $W_{\mathrm{cl}} \cap \mathrm{Cof}_{\mathrm{cl}}$, using the small object argument for J_{Top}, shows that every $f \in (J_{\mathrm{Top}}\,\mathrm{Inj})\mathrm{Proj}$ is a retract of a J_{Top}-cell complex. By lemma 1.44 and lemma 1.45 a J_{Top}-cell complex is both an I_{Top}-cell complex and a weak homotopy equivalence. Retracts of the former are cofibrations by definition, and retracts of the latter are still weak homotopy equivalences by lemma 2.13. Hence such f is an acyclic cofibration. ▮

In conclusion, prop. 3.2 and prop. 3.6 say that:

Theorem 3.7. *The classes of morphisms in $\mathrm{Mor}(\mathrm{Top})$ of def. 3.1,*

- $W_{\mathrm{cl}} =$ *weak homotopy equivalences*,
- $\mathrm{Fib}_{\mathrm{cl}} =$ *Serre fibrations*
- $\mathrm{Cof}_{\mathrm{cl}} =$ *retracts of relative cell complexes*

*define a model category structure (def. 2.3) $\mathrm{Top}_{\mathrm{Quillen}}$, the **classical model structure on topological spaces** or **Serre-Quillen model structure** .*

In particular

1. every object in $\mathrm{Top}_{\mathrm{Quillen}}$ is fibrant;

2. the cofibrant objects in $\mathrm{Top}_{\mathrm{Quillen}}$ are the retracts of cell complexes.

Hence in particular the following classical statement is an immediate corollary:

Corollary 3.8. ***(Whitehead theorem)***

Every weak homotopy equivalence (def. 1.30) between topological spaces that are homeomorphic to a retract of a cell complex, in particular to a CW-complex (def. 1.38), is a homotopy equivalence (def. 1.28).

Proof. This is the "Whitehead theorem in model categories", lemma 2.27, specialized to $\mathrm{Top}_{\mathrm{Quillen}}$ via theorem 3.7. ▮

In proving theorem 3.7 we have in fact shown a bit more that stated. Looking back, all the structure of $\mathrm{Top}_{\mathrm{Quillen}}$ is entirely induced by the set I_{Top} (def. 1.37) of generating cofibrations and the set J_{Top} (def. 1.42) of generating acyclic cofibrations (whence the terminology). This phenomenon will keep recurring and will keep being useful as we construct further model categories, such as the classical model structure on pointed topological spaces (def. 3.31), the projective model structure on topological functors (thm. 3.76), and finally various model structures on spectra which we turn to in the section on stable homotopy theory.

Therefore we make this situation explicit:

Definition 3.9. A model category $\mathcal{C}$ (def. 2.3) is called **cofibrantly generated** if there exists two subsets

$$I, J \subset \mathrm{Mor}(\mathcal{C})$$

of its class of morphisms, such that

1. I and J have small domains according to def. 2.16,

2. the (acyclic) cofibrations of $\mathcal{C}$ are precisely the retracts, of I-relative cell complexes (J-relative cell complexes), def. 1.41.

Proposition 3.10. *For $\mathcal{C}$ a cofibrantly generated model category, def. 3.9, with generating (acylic) cofibrations I (J), then its classes $W, \mathrm{Fib}, \mathrm{Cof}$ of weak equivalences, fibrations and cofibrations are equivalently expressed as injective or projective morphisms (def. 2.4) this way:*

1. $\mathrm{Cof} = (I\,\mathrm{Inj})\mathrm{Proj}$

2. $W \cap \mathrm{Fib} = I\,\mathrm{Inj}$;

3. $W \cap \mathrm{Cof} = (J\,\mathrm{Inj})\mathrm{Proj}$;

4. $\mathrm{Fib} = J\,\mathrm{Inj}$;

Proof. It is clear from the definition that $I \subset (I\,\mathrm{Inj})\mathrm{Proj}$, so that the closure property of prop. 2.10 gives an inclusion

$$\mathrm{Cof} \subset (I\,\mathrm{Inj})\mathrm{Proj} \;.$$

For the converse inclusion, let $f \in (I\,\mathrm{Inj})\mathrm{Proj}$. By the small object argument, prop. 2.17, there is a factorization $f \colon \overset{\in I\,\mathrm{Cell}}{\longrightarrow} \overset{I\,\mathrm{Inj}}{\longrightarrow}$. Hence by assumption and by the retract argument lemma 2.15, f is a retract of an I-relative cell complex, hence is in Cof.

This proves the first statement. Together with the closure properties of prop. 2.10, this implies the second claim.

The proof of the third and fourth item is directly analogous, just with J replaced for I. ▮

The classical homotopy category

With the classical model structure on topological spaces in hand, we now have good control over the classical homotopy category:

Definition 3.11. The **Serre-Quillen classical homotopy category** is the homotopy category, def. 2.25, of the classical model structure on topological spaces $\mathrm{Top}_{\mathrm{Quillen}}$ from theorem 3.7: we write

$$\mathrm{Ho}(\mathrm{Top}) \coloneqq \mathrm{Ho}(\mathrm{Top}_{\mathrm{Quillen}}) \;.$$

Remark 3.12. From just theorem 3.7, the definition 2.25 (def. 3.11) gives that

$$\mathrm{Ho}(\mathrm{Top}_{\mathrm{Quillen}}) \simeq (\mathrm{Top}_{\mathrm{Retract(Cell)}})/_\sim$$

is the category whose objects are retracts of cell complexes (def. 1.38) and whose morphisms are homotopy classes of continuous functions. But in fact more is true:

Theorem 3.7 in itself implies that every topological space is weakly equivalent to a retract of a cell complex, def. 1.38. But by the existence of CW approximations, this cell complex may even be taken to be a CW complex.

(Better yet, there is Quillen equivalence to the classical model structure on simplicial sets which implies a *functorial* CW approximation $|\mathrm{Sing}\, X| \overset{\in W_{\mathrm{cl}}}{\longrightarrow} X$ given by forming the geometric realization of the singular simplicial complex of X.)

Hence the Serre-Quillen classical homotopy category is also equivalently the category of just the CW-complexes whith homotopy classes of continuous functions between them

$$\begin{aligned} \mathrm{Ho}(\mathrm{Top}_{\mathrm{Quillen}}) & \simeq (\mathrm{Top}_{\mathrm{Retract(Cell)}})/_\sim \\ & \simeq (\mathrm{Top}_{\mathrm{CW}})/_\sim \end{aligned} .$$

It follows that the universal property of the homotopy category (theorem 2.31)

$$\mathrm{Ho}(\mathrm{Top}_{\mathrm{Quillen}}) \simeq \mathrm{Top}[W_{\mathrm{cl}}^{-1}]$$

implies that there is a bijection, up to natural isomorphism, between

1. functors out of $\mathrm{Top}_{\mathrm{CW}}$ which agree on homotopy-equivalent maps;

2. functors out of all of Top which send weak homotopy equivalences to isomorphisms.

This statement in particular serves to show that two different axiomatizations of generalized (Eilenberg-Steenrod) cohomology theories are equivalent to each other. See at *Introduction to Stable homotopy theory -- S* the section *generalized cohomology functors* (this prop.)

Beware that, by remark 1.32, what is **not** equivalent to $\mathrm{Ho}(\mathrm{Top}_{\mathrm{Quillen}})$ is the category

$$\mathrm{hTop} \coloneqq \mathrm{Top}/_\sim$$

obtained from *all* topological spaces with morphisms the homotopy classes of continuous functions. This category is "too large", the correct homotopy category is just the genuine full subcategory

$$\mathrm{Ho}(\mathrm{Top}_{\mathrm{Quillen}}) \simeq (\mathrm{Top}_{\mathrm{Retract(Cell)}})/_\sim \simeq \mathrm{Top}/_\sim \; = \; \hookrightarrow \mathrm{hTop} \,.$$

Beware also the ambiguity of terminology: "classical homotopy category" some literature refers to hTop instead of $\mathrm{Ho}(\mathrm{Top}_{\mathrm{Quillen}})$. However, here we never have any use for hTop and will not mention it again.

Proposition 3.13. *Let X be a CW-complex, def. 1.38. Then the standard topological cylinder of def. 1.22*

$$X \sqcup X \overset{(i_0, i_1)}{\longrightarrow} X \times I \longrightarrow X$$

(obtained by forming the product with the standard topological intervall $I = [0,1]$) is indeed a cylinder object in the abstract sense of def. 2.18.

Proof. We describe the proof informally. It is immediate how to turn this into a formal proof, but the notation becomes tedious. (One place where it is spelled out completely is Ottina 14, prop.

2.9.)

So let $X_0 \to X_1 \to X_2 \to \cdots \to X$ be a presentation of X as a CW-complex. Proceed by induction on the cell dimension.

First observe that the cylinder $X_0 \times I$ over X_0 is a cell complex: First X_0 itself is a disjoint union of points. Adding a second copy for every point (i.e. attaching along $S^{-1} \to D^0$) yields $X_0 \sqcup X_0$, then attaching an inteval between any two corresponding points (along $S^0 \to D^1$) yields $X_0 \times I$.

So assume that for $n \in \mathbb{N}$ it has been shown that $X_n \times I$ has the structure of a CW-complex of dimension $(n+1)$. Then for each cell of X_{n+1}, attach it *twice* to $X_n \times I$, once at $X_n \times \{0\}$, and once at $X_n \times \{1\}$.

The result is X_{n+1} with a *hollow cylinder* erected over each of its $(n+1)$-cells. Now fill these hollow cylinders (along $S^{n+1} \to D^{n+1}$) to obtain $X_{n+1} \times I$.

This completes the induction, hence the proof of the CW-structure on $X \times I$.

The construction also manifestly exhibits the inclusion $X \sqcup X \overset{(i_0, i_1)}{\longrightarrow}$ as a relative cell complex.

Finally, it is clear (prop. 1.31) that $X \times I \to X$ is a weak homotopy equivalence. ▮

Conversely:

Proposition 3.14. *Let X be any topological space. Then the standard topological path space object (def. 1.34)*

$$X \longrightarrow X^I \overset{(X^{\delta_0}, X^{\delta_1})}{\longrightarrow} X \times X$$

(obtained by forming the mapping space, def. 1.18, with the standard topological intervall $I = [0,1]$) is indeed a path space object in the abstract sense of def. 2.18.

Proof. To see that $\mathrm{const}\colon X \to X^I$ is a weak homotopy equivalence it is sufficient, by prop. 1.31, to exhibit a homotopy equivalence. Let the homotopy inverse be $X^{\delta_0}\colon X^I \to X$. Then the composite

$$X \overset{\mathrm{const}}{\longrightarrow} X^I \overset{X^{\delta_0}}{\longrightarrow} X$$

is already equal to the identity. The other we round, the rescaling of paths provides the required homotopy

$$I \times X^I \overset{(t, \gamma) \mapsto \gamma(t \cdot (-))}{\longrightarrow} X^I \,.$$

To see that $X^I \to X \times X$ is a fibration, we need to show that every commuting square of the form

$$\begin{array}{ccc} D^n & \longrightarrow & X^I \\ {}^{\mathllap{i_0}}\downarrow && \downarrow \\ D^n \times I & \longrightarrow & X \times X \end{array}$$

has a lift.

Now first use the adjunction $(I \times (-)) \dashv (-)^I$ from prop. 1.19 to rewrite this equivalently as the following commuting square:

$$\begin{array}{ccc} D^n \sqcup D^n & \overset{(i_0, i_0)}{\longrightarrow} & (D^n \times I) \sqcup (D^n \times I) \\ {}^{\mathllap{(i_0, i_1)}}\downarrow && \downarrow \\ D^n \times I & \longrightarrow & X \end{array} \,.$$

This square is equivalently (example 1.12) a morphism out of the pushout

$$D^n \times I \underset{D^n \sqcup D^n}{\sqcup} ((D^n \times I) \sqcup (D^n \times I)) \longrightarrow X \,.$$

By the same reasoning, a lift in the original diagram is now equivalently a lifting in

$$\begin{array}{ccc} D^n \times I \underset{D^n \sqcup D^n}{\sqcup} ((D^n \times I) \sqcup (D^n \times I)) & \longrightarrow & X \\ \downarrow && \downarrow \\ (D^n \times I) \times I & \longrightarrow & * \end{array} \,.$$

Inspection of the component maps shows that the left vertical morphism here is the inclusion into the square times D^n of three of its faces times D^n. This is homeomorphic to the inclusion $D^{n+1} \to D^{n+1} \times I$ (as in remark 1.49). Therefore a lift in this square exsists, and hence a lift in the original square exists. ▮

Model structure on pointed spaces

A *pointed object* (X,x) is of course an object X equipped with a point $x \colon * \to X$, and a morphism of pointed objects $(X,x) \longrightarrow (Y,y)$ is a morphism $X \longrightarrow Y$ that takes x to y. Trivial as this is in itself, it is good to record some basic facts, which we do here.

Passing to pointed objects is also the first step in linearizing classical homotopy theory to stable homotopy theory. In particular, every category of pointed objects has a zero object, hence has zero morphisms. And crucially, if the original category had Cartesian products, then its pointed objects canonically inherit a non-cartesian tensor product: the smash product. These ingredients will be key below in the section on stable homotopy theory.

Definition 3.15. Let $\mathcal{C}$ be a category and let $X \in \mathcal{C}$ be an object.

The *slice category* $\mathcal{C}_{/X}$ is the category whose

* objects are morphisms $\begin{array}{c} A \\ \downarrow \\ X \end{array}$ in $\mathcal{C}$;
* morphisms are commuting triangles $\begin{array}{ccccc} A && \longrightarrow && B \\ & \searrow && \swarrow & \\ && X && \end{array}$ in $\mathcal{C}$.

Dually, the *coslice category* $\mathcal{C}^{X/}$ is the category whose

* objects are morphisms $\begin{array}{c} X \\ \downarrow \\ A \end{array}$ in $\mathcal{C}$;
* morphisms are commuting triangles $\begin{array}{ccccc} && X && \\ & \swarrow && \searrow & \\ A && \longrightarrow && B \end{array}$ in $\mathcal{C}$.

There are the canonical forgetful functors

$$U : \mathcal{C}_{/X}, \mathcal{C}^{X/} \longrightarrow \mathcal{C}$$

given by forgetting the morphisms to/from X.

We here focus on this class of examples:

Definition 3.16. For $\mathcal{C}$ a category with terminal object $*$, the coslice category (def. 3.15) $\mathcal{C}^{*/}$ is the corresponding *category of pointed objects*: its

* objects are morphisms in $\mathcal{C}$ of the form $* \xrightarrow{x} X$ (hence an object X equipped with a choice of point; i.e. a *pointed object*);
* morphisms are commuting triangles of the form

$$\begin{tikzcd} & * \arrow[dl, "x"'] \arrow[dr, "y"] & \\ X \arrow[rr, "f"] & & Y \end{tikzcd}$$

(hence morphisms in $\mathcal{C}$ which preserve the chosen points).

Remark 3.17. In a category of pointed objects $\mathcal{C}^{*/}$, def. 3.16, the terminal object coincides with the initial object, both are given by $* \in \mathcal{C}$ itself, pointed in the unique way.

In this situation one says that $*$ is a *zero object* and that $\mathcal{C}^{*/}$ is a *pointed category*.

It follows that also all hom-sets $\mathrm{Hom}_{\mathcal{C}^{*/}}(X,Y)$ of $\mathcal{C}^{*/}$ are canonically pointed sets, pointed by the *zero morphism*

$$0 : X \xrightarrow{\exists !} 0 \xrightarrow{\exists !} Y \,.$$

Definition 3.18. Let $\mathcal{C}$ be a category with terminal object and finite colimits. Then the forgetful functor $U : \mathcal{C}^{*/} \to \mathcal{C}$ from its category of pointed objects, def. 3.16, has a left adjoint

$$\mathcal{C}^{*/} \underoverset{\underset{U}{\longrightarrow}}{\overset{(-)_+}{\longleftarrow}}{\bot} \mathcal{C}$$

given by forming the disjoint union (coproduct) with a base point ("adjoining a base point").

Proposition 3.19. *Let $\mathcal{C}$ be a category with all limits and colimits. Then also the category of pointed objects $\mathcal{C}^{*/}$, def. 3.16, has all limits and colimits.*

Moreover:

1. *the limits are the limits of the underlying diagrams in $\mathcal{C}$, with the base point of the limit induced by its universal property in $\mathcal{C}$;*
2. *the colimits are the limits in $\mathcal{C}$ of the diagrams with the basepoint adjoined.*

Proof. It is immediate to check the relevant universal property. For details see at *slice category – limits and colimits*. ▮

Example 3.20. Given two pointed objects (X,x) and (Y,y), then:

1. their product in $\mathcal{C}^{*/}$ is simply $(X \times Y, (x,y))$;
2. their coproduct in $\mathcal{C}^{*/}$ has to be computed using the second clause in prop. 3.19: since the point $*$ has to be adjoined to the diagram, it is given not by the coproduct in $\mathcal{C}$, but by the pushout in $\mathcal{C}$ of the form:

$$\begin{tikzcd} * \arrow[r, "x"] \arrow[d, "y"'] \arrow[dr, phantom, "(\mathrm{po})"] & X \arrow[d] \\ Y \arrow[r] & X \vee Y \end{tikzcd} \,.$$

This is called the *wedge sum* operation on pointed objects.

Generally for a set $\{X_i\}_{i \in I}$ in $\mathrm{Top}^{*/}$

1. their product is formed in Top as in example 1.9, with the new basepoint canonically induced;

2. their coproduct is formed by the colimit in Top over the diagram with a basepoint adjoined, and is called the wedge sum $\vee_{i \in I} X_i$.

Example 3.21. For X a CW-complex, def. 1.38 then for every $n \in \mathbb{N}$ the quotient (example 1.13) of its n-skeleton by its $(n-1)$-skeleton is the wedge sum, def. 3.20, of n-spheres, one for each n-cell of X:

$$X^n / X^{n-1} \simeq \underset{i \in I_n}{\vee} S^n \,.$$

Definition 3.22. For $\mathcal{C}^{*/}$ a category of pointed objects with finite limits and finite colimits, the *smash product* is the functor

$$(-) \wedge (-) : \mathcal{C}^{*/} \times \mathcal{C}^{*/} \longrightarrow \mathcal{C}^{*/}$$

given by

$$X \wedge Y \coloneqq * \underset{X \sqcup Y}{\sqcup} (X \times Y) \,,$$

hence by the pushout in $\mathcal{C}$

$$\begin{array}{ccc} X \sqcup Y & \overset{(\mathrm{id}_X, y), (x, \mathrm{id}_Y)}{\longrightarrow} & X \times Y \\ \downarrow && \downarrow \\ * & \longrightarrow & X \wedge Y \end{array} \,.$$

In terms of the wedge sum from def. 3.20, this may be written concisely as

$$X \wedge Y = \frac{X \times Y}{X \vee Y} \,.$$

Remark 3.23. For a general category $\mathcal{C}$ in def. 3.22, the smash product need not be associative, namely it fails to be associative if the functor $(-) \times Z$ does not preserve the quotients involved in the definition.

In particular this may happen for $\mathcal{C} =$ Top.

A sufficient condition for $(-) \times Z$ to preserve quotients is that it is a left adjoint functor. This is the case in the smaller subcategory of compactly generated topological spaces, we come to this in prop. 3.44 below.

These two operations are going to be ubiquitous in stable homotopy theory:

symbol	name	category theory
$X \vee Y$	wedge sum	coproduct in $\mathcal{C}^{*/}$
$X \wedge Y$	smash product	tensor product in $\mathcal{C}^{*/}$

Example 3.24. For $X, Y \in \mathrm{Top}$, with $X_+, Y_+ \in \mathrm{Top}^{*/}$, def. 3.18, then

- $X_+ \vee Y_+ \simeq (X \sqcup Y)_+$;

- $X_+ \wedge Y_+ \simeq (X \times Y)_+$.

Proof. By example 3.20, $X_+ \vee Y_+$ is given by the colimit in Top over the diagram

$$\begin{array}{ccccccc} & & & * & & & \\ & & \swarrow & & \searrow & & \\ X & * & & & & * & Y \end{array} \,.$$

This is clearly $X \sqcup * \sqcup Y$. Then, by definition 3.22

$$\begin{aligned} X_+ \wedge Y_+ &\simeq \frac{(X \sqcup *) \times (X \sqcup *)}{(X \sqcup *) \vee (Y \sqcup *)} \\ &\simeq \frac{X \times Y \sqcup X \sqcup Y \sqcup *}{X \sqcup Y \sqcup *} \\ &\simeq X \times Y \sqcup * \,. \end{aligned}$$

▮

Example 3.25. Let $\mathcal{C}^{*/} = \mathrm{Top}^{*/}$ be pointed topological spaces. Then

$$I_+ \in \mathrm{Top}^{*/}$$

denotes the standard interval object $I = [0,1]$ from def. 1.22, with a djoint basepoint adjoined, def. 3.18. Now for X any pointed topological space, then

$$X \wedge (I_+) = (X \times I) / (\{x_0\} \times I)$$

is the **reduced cylinder** over X: the result of forming the ordinary cyclinder over X as in def. 1.22, and then identifying the interval over the basepoint of X with the point.

(Generally, any construction in $\mathcal{C}$ properly adapted to pointed objects $\mathcal{C}^{*/}$ is called the "reduced" version of the unpointed construction. Notably so for "reduced suspension" which we come to below.)

Just like the ordinary cylinder $X \times I$ receives a canonical injection from the coproduct $X \sqcup X$ formed in Top, so the reduced cyclinder receives a canonical injection from the coproduct $X \sqcup X$ formed in $\mathrm{Top}^{*/}$, which is the wedge sum from example 3.20:

$$X \vee X \longrightarrow X \wedge (I_+) \,.$$

Example 3.26. For $(X,x), (Y,y)$ pointed topological spaces with Y a locally compact topological space, then the **pointed mapping space** is the topological subspace of the mapping space of def. 1.18

$$\mathrm{Maps}((Y,y),(X,x))_* \hookrightarrow (X^Y, \mathrm{const}_x)$$

on those maps which preserve the basepoints, and pointed by the map constant on the basepoint of X.

In particular, the **standard topological pointed path space object** on some pointed X (the pointed variant of def. 1.34) is the pointed mapping space $\mathrm{Maps}(I_+, X)_*$.

The pointed consequence of prop. 1.19 then gives that there is a natural bijection

$$\mathrm{Hom}_{\mathrm{Top}^{*/}}((Z,z) \wedge (Y,y), (X,x)) \simeq \mathrm{Hom}_{\mathrm{Top}^{*/}}((Z,z), \mathrm{Maps}((Y,y),(X,x))_*)$$

between basepoint-preserving continuous functions out of a smash product, def. 3.22, with pointed continuous functions of one variable into the pointed mapping space.

Example 3.27. Given a morphism $f: X \longrightarrow Y$ in a category of pointed objects $\mathcal{C}^{*/}$, def. 3.16, with finite limits and colimits,

1. its *fiber* or *kernel* is the pullback of the point inclusion

$$\begin{array}{ccc} \mathrm{fib}(f) & \longrightarrow & X \\ \downarrow & (pb) & \downarrow^{f} \\ * & \longrightarrow & Y \end{array}$$

2. its *cofiber* or *cokernel* is the pushout of the point projection

$$\begin{array}{ccc} X & \overset{f}{\longrightarrow} & Y \\ \downarrow & (po) & \downarrow \\ * & \longrightarrow & \mathrm{cofib}(f) \end{array} \,.$$

Remark 3.28. In the situation of example 3.27, both the pullback as well as the pushout are equivalently computed in $\mathcal{C}$. For the pullback this is the first clause of prop. 3.19. The second clause says that for computing the pushout in $\mathcal{C}$, first the point is to be adjoined to the diagram, and then the colimit over the larger diagram

$$\begin{array}{ccc} * & & \\ & \searrow & \\ & & X & \overset{f}{\longrightarrow} & Y \\ & & \downarrow & & \\ & & * & & \end{array}$$

be computed. But one readily checks that in this special case this does not affect the result. (The technical jargon is that the inclusion of the smaller diagram into the larger one in this case happens to be a final functor.)

Proposition 3.29. *Let $\mathcal{C}$ be a model category and let $X \in \mathcal{C}$ be an object. Then both the slice category $\mathcal{C}_{/X}$ as well as the coslice category $\mathcal{C}^{X/}$, def. 3.15, carry model structures themselves – the* ***model structure on a (co-)slice category****, where a morphism is a weak equivalence, fibration or cofibration iff its image under the forgetful functor U is so in $\mathcal{C}$.*

In particular the category $\mathcal{C}^{*/}$ of pointed objects, def. 3.16, in a model category $\mathcal{C}$ becomes itself a model category this way.

The corresponding homotopy category of a model category, def. 2.25, we call the ***pointed homotopy category*** *$\mathrm{Ho}(\mathcal{C}^{*/})$.*

Proof. This is immediate:

By prop. 3.19 the (co-)slice category has all limits and colimits. By definition of the weak equivalences in the (co-)slice, they satisfy two-out-of-three, def. 2.1, because the do in $\mathcal{C}$.

Similarly, the factorization and lifting is all induced by $\mathcal{C}$: Consider the coslice category $\mathcal{C}^{X/}$, the case of the slice category is formally dual; then if

$$\begin{array}{ccc} & X & \\ \swarrow & & \searrow \\ A & \underset{f}{\longrightarrow} & B \end{array}$$

commutes in $\mathcal{C}$, and a factorization of f exists in $\mathcal{C}$, it uniquely makes this diagram commute

$$\begin{array}{ccc} & X & \\ \swarrow & \downarrow & \searrow \\ A & \longrightarrow & C & \longrightarrow & B \end{array} \,.$$

Similarly, if

$$\begin{array}{ccc} A & \longrightarrow & C \\ \downarrow && \downarrow \\ B & \longrightarrow & D \end{array}$$

is a commuting diagram in $\mathcal{C}^{X/}$, hence a commuting diagram in $\mathcal{C}$ as shown, with all objects equipped with compatible morphisms from X, then inspection shows that any lift in the diagram necessarily respects the maps from X, too. ▮

Example 3.30. For $\mathcal{C}$ any model category, with $\mathcal{C}^{*/}$ its pointed model structure according to prop. 3.29, then the corresponding homotopy category (def. 2.25) is, by remark 3.17, canonically enriched in pointed sets, in that its hom-functor is of the form

$$[-,-]_* : \mathrm{Ho}(\mathcal{C}^{*/})^{\mathrm{op}} \times \mathrm{Ho}(\mathcal{C}^{*/}) \longrightarrow \mathrm{Set}^{*/} \,.$$

Definition 3.31. Write $\mathrm{Top}^{*/}_{\mathrm{Quillen}}$ for the **classical model structure on pointed topological spaces**, obtained from the classical model structure on topological spaces $\mathrm{Top}_{\mathrm{Quillen}}$ (theorem 3.7) via the induced coslice model structure of prop. 3.29.

Its homotopy category, def. 2.25,

$$\mathrm{Ho}(\mathrm{Top}^{*/}) \coloneqq \mathrm{Ho}(\mathrm{Top}^{*/}_{\mathrm{Quillen}})$$

we call the **classical pointed homotopy category**.

Remark 3.32. The fibrant objects in the pointed model structure $\mathcal{C}^{*/}$, prop. 3.29, are those that are fibrant as objects of $\mathcal{C}$. But the cofibrant objects in $\mathcal{C}^{*}$ are now those for which the basepoint inclusion is a cofibration in X.

For $\mathcal{C}^{*/} = \mathrm{Top}^{*/}_{\mathrm{Quillen}}$ from def. 3.31, then the corresponding cofibrant pointed topological spaces are tyically referred to as spaces **with non-degenerate basepoints** or . Notice that the point itself is cofibrant in $\mathrm{Top}_{\mathrm{Quillen}}$, so that cofibrant pointed topological spaces are in particular cofibrant topological spaces.

While the existence of the model structure on $\mathrm{Top}^{*/}$ is immediate, via prop. 3.29, for the discussion of topologically enriched functors (below) it is useful to record that this, too, is a cofibrantly generated model category (def. 3.9), as follows:

Definition 3.33. Write

$$I_{\mathrm{Top}^{*/}} = \left\{ S^{n-1}_+ \overset{(\iota_n)_+}{\longrightarrow} D^n_+ \right\} \quad \subset \mathrm{Mor}(\mathrm{Top}^{*/})$$

and

$$J_{\mathrm{Top}^{*/}} = \left\{ D^n_+ \overset{(\mathrm{id},\delta_0)_+}{\longrightarrow} (D^n \times I)_+ \right\} \quad \subset \mathrm{Mor}(\mathrm{Top}^{*/}) \,,$$

respectively, for the sets of morphisms obtained from the classical generating cofibrations, def. 1.37, and the classical generating acyclic cofibrations, def. 1.42, under adjoining of basepoints (def. 3.18).

Theorem 3.34. *The sets $I_{\mathrm{Top}^{*/}}$ and $J_{\mathrm{Top}^{*/}}$ in def. 3.33 exhibit the classical model structure on pointed topological spaces $\mathrm{Top}^{*/}_{\mathrm{Quillen}}$ of def. 3.31 as a cofibrantly generated model category, def. 3.9.*

(This is also a special case of a general statement about cofibrant generation of coslice model structures, see this proposition.)

Proof. Due to the fact that in $J_{\mathrm{Top}^{*/}}$ a basepoint is freely adjoined, lemma 1.52 goes through verbatim for the pointed case, with J_{Top} replaced by $J_{\mathrm{Top}^{*/}}$, as do the other two lemmas above that depend on point-set topology, lemma 1.40 and lemma 1.45. With this, the rest of the proof follows by the same general abstract reasoning as above in the proof of theorem 3.7. ▮

Model structure on compactly generated spaces

The category Top has the technical inconvenience that mapping spaces X^Y (def. 1.18) satisfying the exponential property (prop. 1.19) exist in general only for Y a locally compact topological space, but fail to exist more generally. In other words: Top is not cartesian closed. But cartesian closure is necessary for some purposes of homotopy theory, for instance it ensures that

1. the smash product (def. 3.22) on pointed topological spaces is associative (prop. 3.44 below);
2. there is a concept of topologically enriched functors with values in topological spaces, to which we turn below;
3. geometric realization of simplicial sets preserves products.

The first two of these are crucial for the development of stable homotopy theory in the next section, the third is a great convenience in computations.

Now, since the homotopy theory of topological spaces only cares about the CW approximation to any topological space (remark 3.12), it is plausible to ask for a full subcategory of Top which still contains all CW-complexes, still has all limits and colimits, still supports a model category structure constructed in the same way as above, but which in addition is cartesian closed, and preferably such that the model structure interacts well with the cartesian closure.

Such a full subcategory exists, the category of compactly generated topological spaces. This we briefly describe now.

Literature (Strickland 09)

Definition 3.35. Let X be a topological space.

A subset $A \subset X$ is called **compactly closed** (or k**-closed**) if for every continuous function $f : K \longrightarrow X$ out of a compact Hausdorff space K, then the preimage $f^{-1}(A)$ is a closed subset of K.

The space X is called **compactly generated** if its closed subsets exhaust (hence coincide with) the k-closed subsets.

Write

$$\mathrm{Top}_{\mathrm{cg}} \hookrightarrow \mathrm{Top}$$

for the full subcategory of Top on the compactly generated topological spaces.

Definition 3.36. Write

$$\mathrm{Top} \overset{k}{\longrightarrow} \mathrm{Top}_{\mathrm{cg}} \hookrightarrow \mathrm{Top}$$

for the functor which sends any topological space $X = (S, \tau)$ to the topological space $(S, k\tau)$ with the same underlying set S, but with open subsets $k\tau$ the collection of all k-open subsets with respect to τ.

Lemma 3.37. *Let* $X \in \mathrm{Top}_{\mathrm{cg}} \hookrightarrow \mathrm{Top}$ *and let* $Y \in \mathrm{Top}$. *Then continuous functions*

$$X \longrightarrow Y$$

are also continuous when regarded as functions

$$X \longrightarrow k(Y)$$

with k from def. 3.36.

Proof. We need to show that for $A \subset X$ a k-closed subset, then the preimage $f^{-1}(A) \subset X$ is closed subset.

Let $\phi : K \longrightarrow X$ be any continuous function out of a compact Hausdorff space K. Since A is k-closed by assumption, we have that $(f \circ \phi)^{-1}(A) = \phi^{-1}(f^{-1}(A)) \subset K$ is closed in K. This means that $f^{-1}(A)$ is k-closed in X. But by the assumption that X is compactly generated, it follows that $f^{-1}(A)$ is already closed. ▮

Corollary 3.38. *For $X \in \mathrm{Top}_{\mathrm{cg}}$ there is a natural bijection*

$$\mathrm{Hom}_{\mathrm{Top}}(X, Y) \simeq \mathrm{Hom}_{\mathrm{Top}_{\mathrm{cg}}}(X, k(Y)) \;.$$

This means equivalently that the functor k (def. 3.36) together with the inclusion from def. 3.35 forms an pair of adjoint functors

$$\mathrm{Top}_{\mathrm{cg}} \underoverset{\underset{k}{\longleftarrow}}{\hookrightarrow}{\bot} \mathrm{Top} \;.$$

This in turn means equivalently that $\mathrm{Top}_{\mathrm{cg}} \hookrightarrow \mathrm{Top}$ is a coreflective subcategory with coreflector k. In particular k is idemotent in that there are natural homeomorphisms

$$k(k(X)) \simeq k(X) \;.$$

Hence colimits in $\mathrm{Top}_{\mathrm{cg}}$ exists and are computed as in Top. Also limits in $\mathrm{Top}_{\mathrm{cg}}$ exists, these are obtained by computing the limit in Top and then applying the functor k to the result.

The following is a slight variant of def. 1.18, appropriate for the context of $\mathrm{Top}_{\mathrm{cg}}$.

Definition 3.39. For $X, Y \in \mathrm{Top}_{\mathrm{cg}}$ (def. 3.35) the **compactly generated mapping space** $X^Y \in \mathrm{Top}_{\mathrm{cg}}$ is the compactly generated topological space whose underlying set is the set $C(Y, X)$ of continuous functions $f : Y \to X$, and for which a subbase for its topology has elements $U^{\phi(K)}$, for $U \subset X$ any open subset and $\phi : K \to Y$ a continuous function out of a compact Hausdorff space K given by

$$U^{\phi(\kappa)} \coloneqq \{ f \in C(Y, X) \,|\, f(\phi(K)) \subset U \} \;.$$

Remark 3.40. If Y is (compactly generated and) a Hausdorff space, then the topology on the compactly generated mapping space X^Y in def. 3.39 agrees with the compact-open topology of def. 1.18. Beware that it is common to say “compact-open topology” also for the topology of the compactly generated mapping space when Y is not Hausdorff. In that case, however, the two definitions in general disagree.

Proposition 3.41. *The category $\mathrm{Top}_{\mathrm{cg}}$ of def. 3.35 is cartesian closed:*

for every $X \in \mathrm{Top}_{\mathrm{cg}}$ then the operation $X \times (-) \times (-) \times X$ of forming the Cartesian product in $\mathrm{Top}_{\mathrm{cg}}$ (which by cor. 3.38 is k applied to the usual product topological space) together with the operation $(-)^X$ of forming the compactly generated mapping space (def. 3.39) forms a pair of adjoint functors

$$\mathrm{Top}_{\mathrm{cg}} \underoverset{\underset{(-)^X}{\longrightarrow}}{\overset{X \times (-)}{\longleftarrow}}{\bot} \mathrm{Top}_{\mathrm{cg}} \,.$$

For proof see for instance (Strickland 09, prop. 2.12).

Corollary 3.42. *For $X,Y \in \mathrm{Top}_{\mathrm{cg}}^{*/}$, the operation of forming the pointed mapping space (example 3.26) inside the compactly generated mapping space of def. 3.39*

$$\mathrm{Maps}(Y,X)_* \coloneqq \mathrm{fib}\left(X^Y \overset{\mathrm{ev}_y}{\longrightarrow} X \,, x \right)$$

is left adjoint to the smash product operation on pointed compactly generated topological spaces.

$$\mathrm{Top}_{\mathrm{cg}}^{*/} \underoverset{\underset{\mathrm{Maps}(Y,-)_*}{\longrightarrow}}{\overset{Y \wedge (-)}{\longleftarrow}}{\bot} \mathrm{Top}_{\mathrm{cg}}^{*/} \,.$$

Corollary 3.43. *For I a small category and $X_\bullet \colon I \to \mathrm{Top}_{\mathrm{cg}}^{*/}$ a diagram, then the compactly generated mapping space construction from def. 3.39 preserves limits in its covariant argument and sends colimits in its contravariant argument to limits:*

$$\mathrm{Maps}(X, \underset{\longleftarrow_i}{\lim} Y_i)_* \;\simeq\; \underset{\longleftarrow_i}{\lim} \mathrm{Maps}(X, Y_i)_*$$

and

$$\mathrm{Maps}(\underset{\longrightarrow_i}{\lim} X_i,\, Y)_* \simeq \underset{\longleftarrow_i}{\lim} \mathrm{Maps}(X_i, Y)_* \,.$$

Proof. The first statement is an immediate implication of $\mathrm{Maps}(X,-)_*$ being a right adjoint, according to cor. 3.42.

For the second statement, we use that by def. 3.35 a compactly generated topological space is uniquely determined if one knows all continuous functions out of compact Hausdorff spaces into it. Hence it is sufficient to show that there is a natural isomorphism

$$\mathrm{Hom}_{\mathrm{Top}_{\mathrm{cg}}^{*/}}\left(K,\, \mathrm{Maps}(\underset{\longrightarrow_i}{\lim} X_i,\, Y)_*\right) \simeq \mathrm{Hom}_{\mathrm{Top}_{\mathrm{cg}}^{*/}}\left(K,\, \underset{\longleftarrow_i}{\lim} \mathrm{Maps}(X_i, Y)_*\right)$$

for K any compact Hausdorff space.

With this, the statement follows by cor. 3.42 and using that ordinary hom-sets take colimits in the first argument and limits in the second argument to limits:

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Top}_{\mathrm{cg}}^{*/}}\left(K,\, \mathrm{Maps}(\underset{\longrightarrow_i}{\lim} X_i,\, Y)_*\right) & \simeq \mathrm{Hom}_{\mathrm{Top}_{\mathrm{cg}}^{*/}}\left(K \wedge \underset{\longrightarrow_i}{\lim} X_i,\, Y\right) \\ & \simeq \mathrm{Hom}_{\mathrm{Top}_{\mathrm{cg}}^{*/}}\left(\underset{\longrightarrow_i}{\lim} (K \wedge X_i),\, Y\right) \\ & \simeq \underset{\longleftarrow_i}{\lim} \left(\mathrm{Hom}_{\mathrm{Top}_{\mathrm{cg}}^{*/}}(K \wedge X_i,\, Y) \right) \\ & \simeq \underset{\longleftarrow_i}{\lim} \mathrm{Hom}_{\mathrm{Top}_{\mathrm{cg}}^{*/}}(K,\, \mathrm{Maps}(X_i, Y)_*) \\ & \simeq \mathrm{Hom}_{\mathrm{Top}_{\mathrm{cg}}^{*/}}\left(K,\, \underset{\longleftarrow_i}{\lim} \mathrm{Maps}(X_i, Y)_*\right) \end{aligned} \,.$$

∎

Moreover, compact generation fixes the associativity of the smash product (remark 3.23):

Proposition 3.44. *On pointed (def. 3.16) compactly generated topological spaces (def. 3.35) the smash product (def. 3.22)*

$$(-) \wedge (-) : \mathrm{Top}_{\mathrm{cg}}^{*/} \times \mathrm{Top}_{\mathrm{cg}}^{*/} \longrightarrow \mathrm{Top}_{\mathrm{cg}}^{*/}$$

is associative and the 0-sphere is a tensor unit for it.

Proof. Since $(-) \times X$ is a left adjoint by prop. 3.41, it presevers colimits and in particular quotient space projections. Therefore with $X, Y, Z \in \mathrm{Top}_{\mathrm{cg}}^{*/}$ then

$$\begin{aligned} (X \wedge Y) \wedge Z &= \frac{\frac{X \times Y}{X \times \{y\} \sqcup \{x\} \times Y} \times Z}{(X \wedge Y) \times \{z\} \sqcup \{[x] = [y]\} \times Z} \\ &\simeq \frac{\frac{X \times Y \times Z}{X \times \{y\} \times Z \sqcup \{x\} \times Y \times Z}}{X \times Y \times \{z\}} \\ &\simeq \frac{X \times Y \times Z}{X \vee Y \vee Z} \end{aligned} .$$

The analogous reasoning applies to yield also $X \wedge (Y \wedge Z) \simeq \frac{X \times Y \times Z}{X \vee Y \vee Z}$.

The second statement follows directly with prop. 3.41. ■

Remark 3.45. Corollary 3.42 together with prop. 3.44 says that under the smash product the category of pointed compactly generated topological spaces is a closed symmetric monoidal category with tensor unit the 0-sphere.

$$(\mathrm{Top}_{\mathrm{cg}}^{*/}, \wedge, S^0), .$$

Notice that by prop. 3.41 also unpointed compactly generated spaces under Cartesian product form a closed symmetric monoidal category, hence a cartesian closed category

$$(\mathrm{Top}_{\mathrm{cg}}, \times, *) .$$

The fact that $\mathrm{Top}_{\mathrm{cg}}^{*/}$ is still closed symmetric monoidal but no longer Cartesian exhibits $\mathrm{Top}_{\mathrm{cg}}^{*/}$ as being "more linear" than $\mathrm{Top}_{\mathrm{cg}}$. The "full linearization" of $\mathrm{Top}_{\mathrm{cg}}$ is the closed symmteric monoidal category of structured spectra under smash product of spectra which we discuss in section 1.

Due to the idempotency $k \circ k \simeq k$ (cor. 3.38) it is useful to know plenty of conditions under which a given topological space is already compactly generated, for then applying k to it does not change it and one may continue working as in Top.

Example 3.46. Every CW-complex is compactly generated.

Proof. Since a CW-complex is a Hausdorff space, by prop. 3.53 and prop. 3.54 its k-closed subsets are precisely those whose intersection with every compact subspace is closed.

Since a CW-complex X is a colimit in Top over attachments of standard n-disks D^{n_i} (its cells), by the characterization of colimits in Top (prop.) a subset of X is open or closed precisely if its restriction to each cell is open or closed, respectively. Since the n-disks are compact, this implies one direction: if a subset A of X intersected with all compact subsets is closed, then A is closed.

For the converse direction, since a CW-complex is a Hausdorff space and since compact subspaces of Hausdorff spaces are closed, the intersection of a closed subset with a compact subset is closed. ■

For completeness we record further classes of examples:

Example 3.47. The category $\mathrm{Top}_{\mathrm{cg}}$ of compactly generated topological spaces includes

1. all locally compact topological spaces,

2. all first-countable topological spaces,

hence in particular

1. all metrizable topological spaces,
2. all discrete topological spaces,
3. all codiscrete topological spaces.

(Lewis 78, p. 148)

Recall that by corollary 3.38, all colimits of compactly generated spaces are again compactly generated.

Example 3.48. The product topological space of a CW-complex with a compact CW-complex, and more generally with a locally compact CW-complex, is compactly generated.

(Hatcher "Topology of cell complexes", theorem A.6)

More generally:

Proposition 3.49. For X a compactly generated space and Y a locally compact Hausdorff space, then the product topological space $X \times Y$ is compactly generated.

e.g. (Strickland 09, prop. 26)

Finally we check that the concept of homotopy and homotopy groups does not change under passing to compactly generated spaces:

Proposition 3.50. *For every topological space X, the canonical function $k(X) \longrightarrow X$ (the adjunction unit) is a weak homotopy equivalence.*

Proof. By example 3.46, example 3.48 and lemma 3.37, continuous functions $S^n \to k(X)$ and their left homotopies $S^n \times I \to k(X)$ are in bijection with functions $S^n \to X$ and their homotopies $S^n \times I \to X$. ▮

Theorem 3.51. *The restriction of the model category structure on $\mathrm{Top}_{\mathrm{Quillen}}$ from theorem 3.7 along the inclusion $\mathrm{Top}_{\mathrm{cg}} \hookrightarrow \mathrm{Top}$ of def. 3.35 is still a model category structure, which is cofibrantly generated by the same sets I_{Top} (def. 1.37) and J_{Top} (def. 1.42) The coreflection of cor. 3.38 is a Quillen equivalence (def. 2.50)*

$$(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{Quillen}} \underset{k}{\overset{\hookrightarrow}{\underset{\longleftarrow}{\perp}}} \mathrm{Top}_{\mathrm{Quillen}} \,.$$

Proof. By example 3.46, the sets I_{Top} and J_{Top} are indeed in $\mathrm{Mor}(\mathrm{Top}_{\mathrm{cg}})$. By example 3.48 all arguments above about left homotopies between maps out of these basic cells go through verbatim in $\mathrm{Top}_{\mathrm{cg}}$. Hence the three technical lemmas above depending on actual point-set topology, topology, lemma 1.40, lemma 1.45 and lemma 1.52, go through verbatim as before. Accordingly, since the remainder of the proof of theorem 3.7 of $\mathrm{Top}_{\mathrm{Quillen}}$ follows by general abstract arguments from these, it also still goes through verbatim for $(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{Quillen}}$ (repeatedly use the small object argument and the retract argument to establish the two weak factorization systems).

Hence the (acyclic) cofibrations in $(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{Quillen}}$ are identified with those in $\mathrm{Top}_{\mathrm{Quillen}}$, and so the inclusion is a part of a Quillen adjunction (def. 2.46). To see that this is a Quillen equivalence (def. 2.50), it is sufficient to check that for X a compactly generated space then a continuous function $f : X \longrightarrow Y$ is a weak homotopy equivalence (def. 1.30) precisely if the adjunct $\tilde f : X \to k(Y)$ is a weak homotopy equivalence. But, by lemma 3.37, $\tilde f$ is the same function as f, just considered with different codomain. Hence the result follows with prop. 3.50. ▮

Compactly generated weakly Hausdorff topological spaces

While the inclusion $\mathrm{Top}_{\mathrm{cg}} \hookrightarrow \mathrm{Top}$ of def. 3.35 does satisfy the requirement that it gives a cartesian closed category with all limits and colimits and containing all CW-complexes, one may ask for yet smaller subcategories that still share all these properties but potentially exhibit further convenient properties still.

A popular choice introduced in (McCord 69) is to add the further restriction to topopological spaces which are not only compactly generated but also weakly Hausdorff. This was motivated from (Steenrod 67) where compactly generated Hausdorff spaces were used by the observation ((McCord 69, section 2)) that Hausdorffness is not preserved my many colimit operations, notably not by forming quotient spaces.

On the other hand, in above we wouldn't have imposed Hausdorffness in the first place. More intrinsic advantages of $\mathrm{Top}_{\mathrm{cgwH}}$ over $\mathrm{Top}_{\mathrm{cg}}$ are the following:

- every pushout of a morphism in $\mathrm{Top}_{\mathrm{cgwH}} \hookrightarrow \mathrm{Top}$ along a closed subspace inclusion in Top is again in $\mathrm{Top}_{\mathrm{cgwH}}$
- in $\mathrm{Top}_{\mathrm{cgwH}}$ quotient spaces are not only preserved by cartesian products (as is the case for all compactly generated spaces due to $X \times (-)$ being a left adjoint, according to cor. 3.38) but by all pullbacks
- in $\mathrm{Top}_{\mathrm{cgwH}}$ the regular monomorphisms are the closed subspace inclusions

We will not need this here or in the following sections, but we briefly mention it for completenes:

Definition 3.52. A topological space X is called **weakly Hausdorff** if for every continuous function

$$f : K \longrightarrow X$$

out of a compact Hausdorff space K, its image $f(K) \subset X$ is a closed subset of X.

Proposition 3.53. *Every Hausdorff space is a weakly Hausdorff space, def. 3.52.*

Proof. Since compact subspaces of Hausdorff spaces are closed. ▮

Proposition 3.54. *For X a weakly Hausdorff topological space, def. 3.52, then a subset $A \subset X$ is k-closed, def. 3.35, precisely if for every subset $K \subset X$ that is compact Hausdorff with respect to the subspace topology, then the intersection $K \cap A$ is a closed subset of X.*

e.g. (Strickland 09, lemma 1.4 (c))

Topological enrichment

So far the classical model structure on topological spaces which we established in theorem 3.7, as well as the projective model structures on topologically enriched functors induced from it in theorem 3.76, concern the hom-sets, but not the hom-spaces (def. 3.65), i.e. the model structure so far has not been related to the topology on hom-spaces. The following statements say that in fact the model structure and the enrichment by topology on the hom-spaces are compatible in a suitable sense: we have an "enriched model category". This implies in particular that the product/hom-adjunctions are Quillen adjunctions, which is crucial for a decent discusson of the derived functors of the suspension/looping adjunction below.

Definition 3.55. Let $i_1 : X_1 \to Y_1$ and $i_2 : X_2 \to Y_2$ be morphisms in $\mathrm{Top}_{\mathrm{cg}}$, def. 3.35. Their **pushout product**

$$i_1 \Box i_2 \coloneqq ((\mathrm{id}, i_2), (i_1, \mathrm{id}))$$

is the universal morphism in the following diagram

$$\begin{array}{ccccc} && X_1 \times X_2 \\ & {}^{(i_1,\mathrm{id})}\swarrow && \searrow^{(\mathrm{id},i_2)} \\ Y_1 \times X_2 && (\mathrm{po}) && X_1 \times Y_2 \\ & \searrow && \swarrow \\ && (Y_1 \times X_2) \underset{X_1 \times X_2}{\sqcup} (X_1 \times Y_2) \\ && \downarrow^{((\mathrm{id},i_2),(i_1,\mathrm{id}))} \\ && Y_1 \times Y_2 \end{array}$$

Example 3.56. If $i_1 : X_1 \hookrightarrow Y_1$ and $i_2 : X_2 \hookrightarrow Y_2$ are inclusions, then their pushout product $i_1 \Box i_2$ from def. 3.55 is the inclusion

$$(X_1 \times Y_2 \ \cup \ Y_1 \times X_2) \hookrightarrow Y_1 \times Y_2 \,.$$

For instance

$$(\{0\} \hookrightarrow I) \Box (\{0\} \hookrightarrow I)$$

is the inclusion of two adjacent edges of a square into the square.

Example 3.57. The pushout product with an initial morphism is just the ordinary Cartesian product functor

$$(\emptyset \to X) \Box (-) \simeq X \times (-) \,,$$

i.e.

$$(\emptyset \to X) \Box (A \overset{f}{\to} B) \simeq (X \times A \overset{X \times f}{\longrightarrow} X \times B) \,.$$

Proof. The product topological space with the empty space is the empty space, hence the map $\emptyset \times A \overset{(\mathrm{id},f)}{\longrightarrow} \emptyset \times B$ is an isomorphism, and so the pushout in the pushout product is $X \times A$. From this one reads off the universal map in question to be $X \times f$:

$$\begin{array}{ccccc} && \emptyset \times A \\ & \swarrow && \searrow^{\simeq} \\ X \times A && (\mathrm{po}) && \emptyset \times B \\ & {}_{\simeq}\searrow && \swarrow \\ && X \times A \\ && \downarrow^{((\mathrm{id},f), \exists !)} \\ && X \times B \end{array} \,.$$

▮

Example 3.58. With

$$I_{\mathrm{Top}} : \{ S^{n-1} \overset{i_n}{\hookrightarrow} D^n \} \quad \text{and} \quad J_{\mathrm{Top}} : \{ D^n \overset{j_n}{\hookrightarrow} D^n \times I \}$$

the generating cofibrations (def. 1.37) and generating acyclic cofibrations (def. 1.42) of $(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{Quillen}}$ (theorem 3.51), then their pushout-products (def. 3.55) are

$$
\begin{aligned}
i_{n_1} \Box i_{n_2} &\simeq i_{n_1+n_2} \\
i_{n_1} \Box j_{n_2} &\simeq j_{n_1+n_2}
\end{aligned}
\,.
$$

Proof. To see this, it is profitable to model n-disks and n-spheres, up to homeomorphism, as n-cubes $D^n \simeq [0,1]^n \subset \mathbb{R}^n$ and their boundaries $S^{n-1} \simeq \partial [0,1]^n$. For the idea of the proof, consider the situation in low dimensions, where one readily sees pictorially that

$$
i_1 \Box i_1 : (\;=\; \cup \;||\;) \hookrightarrow \Box
$$

and

$$
i_1 \Box j_0 : (\;=\; \cup \;|\;) \hookrightarrow \Box \,.
$$

Generally, D^n may be represented as the space of n-tuples of elements in $[0,1]$, and S^n as the suspace of tuples for which at least one of the coordinates is equal to 0 or to 1.

Accordingly, $S^{n_1} \times D^{n_2} \hookrightarrow D^{n_1+n_2}$ is the subspace of (n_1+n_2)-tuples, such that at least one of the first n_1 coordinates is equal to 0 or 1, while $D^{n_1} \times S^{n_2} \hookrightarrow D^{n_1+n_2}$ is the subspace of (n_1+n_2)-tuples such that east least one of the last n_2 coordinates is equal to 0 or to 1. Therefore

$$
S^{n_1} \times D^{n_2} \cup D^{n_1} \times S^{n_2} \simeq S^{n_1+n_2} \,.
$$

And of course it is clear that $D^{n_1} \times D^{n_2} \simeq D^{n_1+n_2}$. This shows the first case.

For the second, use that $S^{n_1} \times D^{n_2} \times I$ is contractible to $S^{n_1} \times D^{n_2}$ in $D^{n_1} \times D^{n_2} \times I$, and that $S^{n_1} \times D^{n_2}$ is a subspace of $D^{n_1} \times D^{n_2}$. ▮

Definition 3.59. Let $i : A \to B$ and $p : X \to Y$ be two morphisms in $\mathrm{Top}_{\mathrm{cg}}$, def. 3.35. Their **pullback powering** is

$$
p^{\Box i} \coloneqq (p^B, X^i)
$$

being the universal morphism in

$$
\begin{array}{ccccc}
 & & X^B & & \\
 & & \downarrow^{(p^B, X^i)} & & \\
 & & Y^B \underset{Y^A}{\times} X^A & & \\
 & \swarrow & & \searrow & \\
Y^B & & (\mathrm{pb}) & & X^A \\
 & {}_{Y^i}\searrow & & \swarrow_{p^A} & \\
 & & Y^A & &
\end{array}
$$

Proposition 3.60. *Let i_1, i_2, p be three morphisms in $\mathrm{Top}_{\mathrm{cg}}$, def. 3.35. Then for their pushout-products (def. 3.55) and pullback-powerings (def. 3.59) the following lifting properties are equivalent ("Joyal-Tierney calculus"):*

$$
\begin{array}{cccc}
 & i_1 \Box i_2 & \text{has LLP against} & p \\
\Leftrightarrow & i_1 & \text{has LLP against} & p^{\Box i_2} \\
\Leftrightarrow & i_2 & \text{has LLP against} & p^{\Box i_1}
\end{array}
\,.
$$

Proof. We claim that by the cartesian closure of $\mathrm{Top}_{\mathrm{cg}}$, and carefully collecting terms, one finds a natural bijection between commuting squares and their lifts as follows:

$$
\begin{array}{ccc}
Q & \stackrel{f}{\longrightarrow} & X^B \\
{}^{\mathllap{i_1}}\downarrow && \downarrow^{\mathrlap{p^{\Box i_2}}} \\
P & \underset{(g_1,g_2)}{\longrightarrow} & Y^B \underset{Y^A}{\times} X^A
\end{array}
\qquad \leftrightarrow \qquad
\begin{array}{ccc}
Q \times B \underset{Q\times A}{\sqcup} P \times A & \stackrel{(\tilde f, \tilde g_2)}{\longrightarrow} & X \\
{}^{\mathllap{i_1 \Box i_2}}\downarrow && \downarrow^{\mathrlap{p}} \\
P \times B & \underset{\tilde g_1}{\longrightarrow} & Y
\end{array}
$$

where the tilde denotes product/hom-adjuncts, for instance

$$
\frac{P \stackrel{g_1}{\longrightarrow} Y^B}{P \times B \stackrel{\tilde g_1}{\longrightarrow} Y}
$$

etc.

To see this in more detail, observe that both squares above each represent two squares from the two components into the fiber product and out of the pushout, respectively, as well as one more square exhibiting the compatibility condition on these components:

$$
\begin{aligned}
&
\begin{array}{ccc}
Q & \stackrel{f}{\longrightarrow} & X^B \\
{}^{\mathllap{i_1}}\downarrow && \downarrow^{\mathrlap{p^{\Box i_2}}} \\
P & \underset{(g_1,g_2)}{\longrightarrow} & Y^B \underset{Y^A}{\times} X^A
\end{array}
\\
\simeq &
\left\{
\begin{array}{ccc}
Q & \stackrel{f}{\longrightarrow} & X^B \\
{}^{\mathllap{i_1}}\downarrow && \downarrow^{\mathrlap{p^B}} \\
P & \underset{g_1}{\longrightarrow} & Y^B
\end{array}
\,,\,
\begin{array}{ccc}
Q & \stackrel{f}{\longrightarrow} & X^B \\
{}^{\mathllap{i_1}}\downarrow && \downarrow^{\mathrlap{X^{i_2}}} \\
P & \underset{g_1}{\longrightarrow} & X^A
\end{array}
\,,\,
\begin{array}{ccc}
P & \stackrel{g_2}{\longrightarrow} & X^A \\
{}^{\mathllap{g_1}}\downarrow && \downarrow^{\mathrlap{p^A}} \\
Y^B & \underset{Y^{i_2}}{\longrightarrow} & Y^A
\end{array}
\right\}
\\
\leftrightarrow &
\left\{
\begin{array}{ccc}
Q \times B & \stackrel{\tilde f}{\longrightarrow} & X \\
{}^{\mathllap{(i_1, \mathrm{id})}}\downarrow && \downarrow^{\mathrlap{p}} \\
P \times B & \underset{\tilde g_2}{\longrightarrow} & Y
\end{array}
\,,\,
\begin{array}{ccc}
Q \times A & \stackrel{(\mathrm{id}, i_2)}{\longrightarrow} & Q \times B \\
{}^{\mathllap{(i_1, \mathrm{id})}}\downarrow && \downarrow^{\mathrlap{\tilde f}} \\
P \times A & \underset{\tilde g_2}{\longrightarrow} & X
\end{array}
\,,\,
\begin{array}{ccc}
P \times A & \stackrel{\tilde g_2}{\longrightarrow} & X \\
{}^{\mathllap{(\mathrm{id}, i_2)}}\downarrow && \downarrow^{\mathrlap{p}} \\
P \times B & \underset{\tilde g_1}{\longrightarrow} & Y
\end{array}
\right\}
\\
\simeq &
\begin{array}{ccc}
Q \times B \underset{Q\times A}{\sqcup} P \times A & \stackrel{(\tilde f, \tilde g_2)}{\longrightarrow} & X \\
{}^{\mathllap{i_1 \Box i_2}}\downarrow && \downarrow^{\mathrlap{p}} \\
P \times B & \underset{\tilde g_1}{\longrightarrow} & Y
\end{array}
\end{aligned}
$$

▮

Proposition 3.61. *The pushout-product in* $\mathrm{Top}_{\mathrm{cg}}$ *(def. 3.35) of two classical cofibrations is a classical cofibration:*

$$
\mathrm{Cof}_{\mathrm{cl}} \Box \mathrm{Cof}_{\mathrm{cl}} \subset \mathrm{Cof}_{\mathrm{cl}} \,.
$$

If one of them is acyclic, then so is the pushout-product:

$$
\mathrm{Cof}_{\mathrm{cl}} \Box (W_{\mathrm{cl}} \cap \mathrm{Cof}_{\mathrm{cl}}) \subset W_{\mathrm{cl}} \cap \mathrm{Cof}_{\mathrm{cl}} \,.
$$

Proof. Regarding the first point:

By example 3.58 we have

$$
I_{\mathrm{Top}} \Box I_{\mathrm{Top}} \subset I_{\mathrm{Top}}
$$

Hence

$$
\begin{array}{cccc}
 & I_{\mathrm{Top}} \,\Box\, I_{\mathrm{Top}} & \text{has LLP against} & W_{\mathrm{cl}} \cap \mathrm{Fib}_{\mathrm{cl}} \\
\Leftrightarrow & I_{\mathrm{Top}} & \text{has LLP against} & (W_{\mathrm{cl}} \cap \mathrm{Fib}_{\mathrm{cl}})^{\Box I_{\mathrm{Top}}} \\
\Rightarrow & \mathrm{Cof}_{\mathrm{cl}} & \text{has LLP against} & (W_{\mathrm{cl}} \cap \mathrm{Fib}_{\mathrm{cl}})^{\Box I_{\mathrm{Top}}} \\
\Leftrightarrow & I_{\mathrm{Top}} \,\Box\, \mathrm{Cof}_{\mathrm{cl}} & \text{has LLP against} & W_{\mathrm{cl}} \cap \mathrm{Fib}_{\mathrm{cl}} \\
\Leftrightarrow & I_{\mathrm{Top}} & \text{has LLP against} & (W_{\mathrm{cl}} \cap \mathrm{Fib}_{\mathrm{cl}})^{\mathrm{Cof}_{\mathrm{cl}}} \\
\Rightarrow & \mathrm{Cof}_{\mathrm{cl}} & \text{has LLP against} & (W_{\mathrm{cl}} \cap \mathrm{Fib}_{\mathrm{cl}})^{\mathrm{Cof}_{\mathrm{cl}}} \\
\Leftrightarrow & \mathrm{Cof}_{\mathrm{cl}} \,\Box\, \mathrm{Cof}_{\mathrm{cl}} & \text{has LLP against} & W_{\mathrm{cl}} \cap \mathrm{Fib}_{\mathrm{cl}}
\end{array}
\,,
$$

where all logical equivalences used are those of prop. 3.60 and where all implications appearing are by the closure property of lifting problems, prop. 2.10.

Regarding the second point: By example 3.58 we moreover have

$$I_{\mathrm{Top}} \,\Box\, J_{\mathrm{Top}} \subset J_{\mathrm{Top}}$$

and the conclusion follows by the same kind of reasoning. ∎

Remark 3.62. In model category theory the property in proposition 3.61 is referred to as saying that the model category $(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{Quillen}}$ from theorem \ref{ModelStructureOnTopcg}

1. is a monoidal model category with respect to the Cartesian product on $\mathrm{Top}_{\mathrm{cg}}$;

2. is an enriched model category, over itself.

A key point of what this entails is the following:

Proposition 3.63. *For $X \in (\mathrm{Top}_{\mathrm{cg}})_{\mathrm{Quillen}}$ cofibrant (a retract of a cell complex) then the product-hom-adjunction for Y (prop. 3.41) is a Quillen adjunction*

$$(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{Quillen}} \underset{(-)^X}{\overset{X \times (-)}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} (\mathrm{Top}_{\mathrm{cg}})_{\mathrm{Quillen}} \,.$$

Proof. By example 3.57 we have that the left adjoint functor is equivalently the pushout product functor with the initial morphism of X:

$$X \times (-) \simeq (\emptyset \to X) \,\Box\, (-) \,.$$

By assumption $(\emptyset \to X)$ is a cofibration, and hence prop. 3.61 says that this is a left Quillen functor. ∎

The statement and proof of prop. 3.63 has a direct analogue in pointed topological spaces

Proposition 3.64. *For $X \in (\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}$ cofibrant with respect to the classical model structure on pointed compactly generated topological spaces (theorem 3.51, prop. 3.29) (hence a retract of a cell complex with non-degenerate basepoint, remark 3.32) then the pointed product-hom-adjunction from corollary 3.42 is a Quillen adjunction (def. 2.46):*

$$(\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}} \underset{\mathrm{Maps}(X,-)_*}{\overset{X \wedge (-)}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} (\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}} \,.$$

Proof. Let now $\Box_\wedge$ denote the **smash pushout product** and $(-)^{\Box(-)}$ the **smash pullback powering** defined as in def. 3.55 and def. 3.59, but with Cartesian product replaced by smash product (def. 3.22) and compactly generated mapping space replaced by pointed mapping spaces (def. 3.26).

By theorem 3.34 $(\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}$ is cofibrantly generated by $I_{\mathrm{Top}^{*/}} = (I_{\mathrm{Top}})_+$ and $J_{\mathrm{Top}^{*/}} = (J_{\mathrm{Top}})_+$.

Example 3.24 gives that for $i_n \in I_{\mathrm{Top}}$ and $j_n \in J_{\mathrm{Top}}$ then

$$(i_{n_1})_+ \Box_\wedge (i_{n_2})_+ \simeq (i_{n_1+n_2})_+$$

and

$$(i_{n_1})_+ \wedge_\wedge (i_{n_2})_+ \simeq (i_{n_1+n_2})_+ \,.$$

Hence the pointed analog of prop. 3.61 holds and therefore so does the pointed analog of the conclusion in prop. 3.63. ▮

Model structure on topological functors

With classical topological homotopy theory in hand (theorem 3.7, theorem 3.51), it is straightforward now to generalize this to a homotopy theory of *topological diagrams*. This is going to be the basis for the stable homotopy theory of spectra, because spectra may be identified with certain topological diagrams (prop.).

Technically, "topological diagram" here means "Top-enriched functor". We now discuss what this means and then observe that as an immediate corollary of theorem 3.7 we obtain a model category structure on topological diagrams.

As a by-product, we obtain the model category theory of homotopy colimits in topological spaces, which will be useful.

In the following we say *Top-enriched category* and *Top-enriched functor* etc. for what often is referred to as "topological category" and "topological functor" etc. As discussed there, these latter terms are ambiguous.

Literature (Riehl, chapter 3) for basics of enriched category theory; (Piacenza 91) for the projective model structure on topological functors.

Definition 3.65. A **topologically enriched category** $\mathcal{C}$ is a $\mathrm{Top}_{\mathrm{cg}}$-enriched category, hence:

1. a class $\mathrm{Obj}(\mathcal{C})$, called the **class of objects**;
2. for each $a, b \in \mathrm{Obj}(\mathcal{C})$ a compactly generated topological space (def. 3.35)

   $$\mathcal{C}(a,b) \in \mathrm{Top}_{\mathrm{cg}}\,,$$

   called the **space of morphisms** or the **hom-space** between a and b;
3. for each $a, b, c \in \mathrm{Obj}(\mathcal{C})$ a continuous function

   $$\circ_{a,b,c} \;:\; \mathcal{C}(a,b) \times \mathcal{C}(b,c) \longrightarrow \mathcal{C}(a,c)$$

   out of the cartesian product (by cor. 3.38: the image under k of the product topological space), called the *composition* operation;
4. for each $a \in \mathrm{Obj}(\mathcal{C})$ a point $\mathrm{Id}_a \in \mathcal{C}(a,a)$, called the *identity* morphism on a

such that the composition is associative and unital.

Similarly a **pointed topologically enriched category** is such a structure with $\mathrm{Top}_{\mathrm{cg}}$ replaced by $\mathrm{Top}^{*/}_{\mathrm{cg}}$ (def. 3.16) and with the Cartesian product replaced by the smash product (def. 3.22) of pointed topological spaces.

Remark 3.66. Given a (pointed) topologically enriched category as in def. 3.65, then forgetting the topology on the hom-spaces (along the forgetful functor $U : \mathrm{Top}_{\mathrm{cg}} \to \mathrm{Set}$) yields an ordinary locally small category with

$$\mathrm{Hom}_{\mathcal{C}}(a,b) = U(\mathcal{C}(a,b)) \;.$$

It is in this sense that $\mathcal{C}$ is a category with extra structure, and hence "enriched".

The archetypical example is $\mathrm{Top}_{\mathrm{cg}}$ itself:

Example 3.67. The category $\mathrm{Top}_{\mathrm{cg}}$ (def. 3.35) canonically obtains the structure of a topologically enriched category, def. 3.65, with hom-spaces given by the compactly generated mapping spaces (def. 3.39)

$$\mathrm{Top}_{\mathrm{cg}}(X,Y) \coloneqq Y^X$$

and with composition

$$Y^X \times Z^Y \longrightarrow Z^X$$

given by the adjunct under the (product⊣ mapping-space)-adjunction from prop. 3.41 of the evaluation morphisms

$$X \times Y^X \times Z^Y \overset{(\mathrm{ev},\mathrm{id})}{\longrightarrow} Y \times Z^Y \overset{\mathrm{ev}}{\longrightarrow} Z \;.$$

Similarly, pointed compactly generated topological spaces $\mathrm{Top}_k^{*/}$ form a pointed topologically enriched category, using the pointed mapping spaces from example 3.26:

$$\mathrm{Top}_{\mathrm{cg}}^{*/}(X,Y) \coloneqq \mathrm{Maps}(X,Y)_* \;.$$

Definition 3.68. A topologically enriched functor between two topologically enriched categories

$$F : \mathcal{C} \longrightarrow \mathcal{D}$$

is a $\mathrm{Top}_{\mathrm{cg}}$-enriched functor, hence:

1. a function

   $$F_0 : \mathrm{Obj}(\mathcal{C}) \longrightarrow \mathrm{Obj}(\mathcal{D})$$

   of objects;

2. for each $a, b \in \mathrm{Obj}(\mathcal{C})$ a continuous function

   $$F_{a,b} : \mathcal{C}(a,b) \longrightarrow \mathcal{D}(F_0(a), F_0(b))$$

   of hom-spaces,

such that this preserves composition and identity morphisms in the evident sense.

A homomorphism of topologically enriched functors

$$\eta : F \Rightarrow G$$

is a $\mathrm{Top}_{\mathrm{cg}}$-enriched natural transformation: for each $c \in \mathrm{Obj}(\mathcal{C})$ a choice of morphism $\eta_c \in \mathcal{D}(F(c), G(c))$ such that for each pair of objects $c, d \in \mathcal{C}$ the two continuous functions

$$\eta_d \circ F(-) : \mathcal{C}(c,d) \longrightarrow \mathcal{D}(F(c), G(d))$$

and

$$G(-) \circ \eta_c : \mathcal{C}(c,d) \longrightarrow \mathcal{D}(F(c), G(d))$$

agree.

We write $[\mathcal{C},\mathcal{D}]$ for the resulting category of topologically enriched functors.

Remark 3.69. The condition on an enriched natural transformation in def. 3.68 is just that on an ordinary natural transformation on the underlying unenriched functors, saying that for every morphisms $f : c \to d$ there is a commuting square

$$f \quad \mapsto \quad \begin{array}{ccc} \mathcal{C}(c,c) \times X & \overset{\eta_c}{\longrightarrow} & F(c) \\ {}^{\mathllap{\mathcal{C}(c,f)}}\downarrow && \downarrow^{\mathrlap{F(f)}} \\ \mathcal{C}(c,d) \times X & \underset{\eta_d}{\longrightarrow} & F(d) \end{array}$$

Example 3.70. For $\mathcal{C}$ any topologically enriched category, def. 3.65 then a topologically enriched functor (def. 3.68)

$$F \;:\; \mathcal{C} \longrightarrow \mathrm{Top}_{\mathrm{cg}}$$

to the archetical topologically enriched category from example 3.67 may be thought of as a topologically enriched copresheaf, at least if $\mathcal{C}$ is small (in that its class of objects is a proper set).

Hence the category of topologically enriched functors

$$[\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}]$$

according to def. 3.68 may be thought of as the (co-)presheaf category over $\mathcal{C}$ in the realm of topological enriched categories.

A functor $F \in [\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}]$ is equivalently

1. a compactly generated topological space $F_a \in \mathrm{Top}_{\mathrm{cg}}$ for each object $a \in \mathrm{Obj}(\mathcal{C})$;

2. a continuous function

$$F_a \times \mathcal{C}(a,b) \longrightarrow F_b$$

   for all pairs of objects $a, b \in \mathrm{Obj}(\mathcal{C})$

such that composition is respected, in the evident sense.

For every object $c \in \mathcal{C}$, there is a topologically enriched representable functor, denoted $y(c)$ or $\mathcal{C}(c,-)$ which sends objects to

$$y(c)(d) = \mathcal{C}(c,d) \in \mathrm{Top}_{\mathrm{cg}}$$

and whose action on morphisms is, under the above identification, just the composition operation in $\mathcal{C}$.

Proposition 3.71. *For $\mathcal{C}$ any small topologically enriched category, def. 3.65 then the enriched functor category $[\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}]$ from example 3.70 has all limits and colimits, and they are computed objectwise:*

if

$$F_\bullet \;:\; I \longrightarrow [\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}]$$

is a diagram of functors and $c \in \mathcal{C}$ is any object, then

$$(\underset{\longleftarrow i}{\lim} F_i)(c) \simeq \underset{\longleftarrow i}{\lim} (F_i(c)) \quad \in \mathrm{Top}_{\mathrm{cg}}$$

and

$$(\underset{\longrightarrow i}{\lim} F_i)(c) \simeq \underset{\longrightarrow i}{\lim} (F_i(c)) \quad \in \mathrm{Top}_{\mathrm{cg}} \,.$$

Proof. First consider the underlying diagram of functors F_i° where the topology on the hom-spaces of $\mathcal{C}$ and of $\mathrm{Top}_{\mathrm{cg}}$ has been forgotten. Then one finds

$$(\underset{\longleftarrow i}{\lim} F_i^\circ)(c) \simeq \underset{\longleftarrow i}{\lim} (F_i^\circ(c)) \quad \in \mathrm{Set}$$

and

$$(\underset{\longrightarrow i}{\lim} F_i^\circ)(c) \simeq \underset{\longrightarrow i}{\lim} (F_i^\circ(c)) \quad \in \mathrm{Set}$$

by the universal property of limits and colimits. (Given a morphism of diagrams then a unique compatible morphism between their limits or colimits, respectively, is induced as the universal factorization of the morphism of diagrams regarded as a cone or cocone, respectvely, over the codomain or domain diagram, respectively).

Hence it only remains to see that equipped with topology, these limits and colimits in Set become limits and colimits in $\mathrm{Top}_{\mathrm{cg}}$. That is just the statement of prop. 1.5 with corollary 3.38. ▮

Definition 3.72. Let $\mathcal{C}$ be a topologically enriched category, def. 3.65, with $[\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}]$ its category of topologically enriched copresheaves from example 3.70.

1. Define a functor

   $$(-)\cdot(-) \,:\, [\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}] \times \mathrm{Top}_{\mathrm{cg}} \longrightarrow [\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}]$$

   by forming objectwise cartesian products (hence k of product topological spaces)

   $$F \cdot X \,:\, c \mapsto F(c) \times X \,.$$

   This is called the **tensoring** of $[\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}]$ over $\mathrm{Top}_{\mathrm{cg}}$.

2. Define a functor

   $$(-)^{(-)} \,:\, (\mathrm{Top}_{\mathrm{cg}})^{\mathrm{op}} \times [\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}] \longrightarrow [\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}]$$

   by forming objectwise compactly generated mapping spaces (def. 3.39)

   $$F^X \,:\, c \mapsto F(c)^X \,.$$

   This is called the **powering** of $[\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}]$ over $\mathrm{Top}_{\mathrm{cg}}$.

Analogously, for $\mathcal{C}$ a pointed topologically enriched category, def. 3.65, with $[\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}]$ its category of pointed topologically enriched copresheaves from example 3.70, then:

1. Define a functor

   $$(-)\wedge(-) \,:\, [\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}] \times \mathrm{Top}_{\mathrm{cg}}^{*/} \longrightarrow [\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}]$$

   by forming objectwise smash products (def. 3.22)

   $$F \wedge X \,:\, c \mapsto F(c) \wedge X \,.$$

   This is called the **smash tensoring** of $[\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}]$ over $\mathrm{Top}_{\mathrm{cg}}^{*/}$.

2. Define a functor

$$\mathrm{Maps}(-,-)_* \;:\; \mathrm{Top}_{\mathrm{cg}}^{*/} \times [\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}] \longrightarrow [\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}]$$

by forming objectwise pointed mapping spaces (example 3.26)

$$F^X \;:\; c \mapsto \mathrm{Maps}(X, F(c))_* \;.$$

This is called the **pointed powering** of $[\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}]$ over $\mathrm{Top}_{\mathrm{cg}}$.

There is a full blown $\mathrm{Top}_{\mathrm{cg}}$-enriched Yoneda lemma. The following records a slightly simplified version which is all that is needed here:

Proposition 3.73. ***(topologically enriched Yoneda-lemma)***

Let $\mathcal{C}$ be a topologically enriched category, def. 3.65, write $[\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}]$ for its category of topologically enriched (co-)presheaves, and for $c \in \mathrm{Obj}(\mathcal{C})$ write $y(c) = \mathcal{C}(c,-) \in [\mathcal{C}, \mathrm{Top}_k]$ for the topologically enriched functor that it represents, all according to example 3.70. Recall the tensoring operation $(F, X) \mapsto F \cdot X$ from def. 3.72.

For $c \in \mathrm{Obj}(\mathcal{C})$, $X \in \mathrm{Top}_{\mathrm{cg}}$ and $F \in [\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}]$, there is a natural bijection between

1. *morphisms $y(c) \cdot X \longrightarrow F$ in $[\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}]$;*

2. *morphisms $X \longrightarrow F(c)$ in $\mathrm{Top}_{\mathrm{cg}}$.*

In short:

$$\frac{y(c) \cdot X \longrightarrow F}{X \longrightarrow F(c)}$$

Proof. Given a morphism $\eta : y(c) \cdot X \longrightarrow F$ consider its component

$$\eta_c \;:\; \mathcal{C}(c,c) \times X \longrightarrow F(c)$$

and restrict that to the identity morphism $\mathrm{id}_c \in \mathcal{C}(c,c)$ in the first argument

$$\eta_c(\mathrm{id}_c, -) \;:\; X \longrightarrow F(c) \;.$$

We claim that just this $\eta_c(\mathrm{id}_c, -)$ already uniquely determines all components

$$\eta_d \;:\; \mathcal{C}(c,d) \times X \longrightarrow F(d)$$

of η, for all $d \in \mathrm{Obj}(\mathcal{C})$: By definition of the transformation η (def. 3.68), the two functions

$$F(-) \circ \eta_c \;:\; \mathcal{C}(c,d) \longrightarrow F(d)^{\mathcal{C}(c,c) \times X}$$

and

$$\eta_d \circ \mathcal{C}(c,-) \times X \;:\; \mathcal{C}(c,d) \longrightarrow F(d)^{\mathcal{C}(c,c) \times X}$$

agree. This means (remark 3.69) that they may be thought of jointly as a function with values in commuting squares in $\mathrm{Top}_{\mathrm{cg}}$ of this form:

$$f \quad \mapsto \quad \begin{array}{ccc} \mathcal{C}(c,c) \times X & \overset{\eta_c}{\longrightarrow} & F(c) \\ {}^{\mathcal{C}(c,f)}\downarrow && \downarrow^{F(f)} \\ \mathcal{C}(c,d) \times X & \underset{\eta_d}{\longrightarrow} & F(d) \end{array}$$

For any $f \in \mathcal{C}(c,d)$, consider the restriction of

$$\eta_d \circ \mathcal{C}(c,f) \in F(d)^{\mathcal{C}(c,c)\times X}$$

to $\mathrm{id}_c \in \mathcal{C}(c,c)$, hence restricting the above commuting squares to

$$f \quad \mapsto \quad \begin{array}{ccc} \{\mathrm{id}_c\}\times X & \overset{\eta_c}{\longrightarrow} & F(c) \\ {}^{\mathcal{C}(c,f)}\downarrow & & \downarrow^{F(f)} \\ \{f\}\times X & \underset{\eta_d}{\longrightarrow} & F(d) \end{array}$$

This shows that η_d is fixed to be the function

$$\eta_d(f,x) = F(f)\circ \eta_c(\mathrm{id}_c, x)$$

and this is a continuous function since all the operations it is built from are continuous.

Conversely, given a continuous function $\alpha : X \longrightarrow F(c)$, define for each d the function

$$\eta_d : (f,x) \mapsto F(f)\circ\alpha\;.$$

Running the above analysis backwards shows that this determines a transformation $\eta : y(c)\times X \to F$. ▮

Definition 3.74. For $\mathcal{C}$ a small topologically enriched category, def. 3.65, write

$$I_{\mathrm{Top}}^{\mathcal{C}} \coloneqq \left\{ y(c)\cdot (S^{n-1} \overset{\iota_n}{\longrightarrow} D^n) \right\}_{\substack{n\in\mathbb{N},\\ c\in \mathrm{Obj}(\mathcal{C})}}$$

and

$$J_{\mathrm{Top}}^{\mathcal{C}} \coloneqq \left\{ y(c)\cdot (D^{n} \overset{(\mathrm{id},\delta_0)}{\longrightarrow} D^n\times I) \right\}_{\substack{n\in\mathbb{N},\\ c\in \mathrm{Obj}(\mathcal{C})}}$$

for the sets of morphisms given by tensoring (def. 3.72) the representable functors (example 3.70) with the generating cofibrations (def.1.37) and acyclic generating cofibrations (def. 1.42), respectively, of $(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{Quillen}}$ (theorem 3.51).

These are going to be called the **generating cofibrations** and **acyclic generating cofibrations** for the *projective model structure on topologically enriched functors* over $\mathcal{C}$.

Analgously, for $\mathcal{C}$ a pointed topologically enriched category, write

$$I_{\mathrm{Top}^{*/}}^{\mathcal{C}} \coloneqq \left\{ y(c)\wedge (S^{n-1}_+ \overset{(\iota_n)_+}{\longrightarrow} D^n_+) \right\}_{\substack{n\in\mathbb{N},\\ c\in \mathrm{Obj}(\mathcal{C})}}$$

and

$$J_{\mathrm{Top}^{*/}}^{\mathcal{C}} \coloneqq \left\{ y(c)\wedge (D^{n}_+ \overset{(\mathrm{id},\delta_0)_+}{\longrightarrow} (D^n\times I)_+) \right\}_{\substack{n\in\mathbb{N},\\ c\in \mathrm{Obj}(\mathcal{C})}}$$

for the analogous construction applied to the pointed generating (acyclic) cofibrations of def. 3.33.

Definition 3.75. Given a small (pointed) topologically enriched category $\mathcal{C}$, def. 3.65, say that a morphism in the category of (pointed) topologically enriched copresheaves $[\mathcal{C},\mathrm{Top}_{\mathrm{cg}}]$ ($[\mathcal{C},\mathrm{Top}_{\mathrm{cg}}^{*/}]$), example 3.70, hence a natural transformation between topologically enriched functors, $\eta : F \to G$ is

- a **projective weak equivalence**, if for all $c\in\mathrm{Obj}(\mathcal{C})$ the component $\eta_c : F(c)\to G(c)$ is a weak homotopy equivalence (def. 1.30);

- a **projective fibration** if for all $c \in \mathrm{Obj}(\mathcal{C})$ the component $\eta_c : F(c) \to G(c)$ is a Serre fibration (def. 1.47);
- a **projective cofibration** if it is a retract (rmk. 2.12) of an $I_{\mathrm{Top}}^{\mathcal{C}}$-relative cell complex (def. 1.41, def. 3.74).

Write

$$[\mathcal{C}, (\mathrm{Top}_{\mathrm{cg}})_{\mathrm{Quillen}}]_{\mathrm{proj}}$$

and

$$[\mathcal{C}, (\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}]_{\mathrm{proj}}$$

for the categories of topologically enriched functors equipped with these classes of morphisms.

Theorem 3.76. *The classes of morphisms in def. 3.75 constitute a model category structure on $[\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}]$ and $[\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}]$, called the **projective model structure on enriched functors***

$$[\mathcal{C}, (\mathrm{Top}_{\mathrm{cg}})_{\mathrm{Quillen}}]_{\mathrm{proj}}$$

and

$$[\mathcal{C}, (\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}]_{\mathrm{proj}}$$

These are cofibrantly generated model category, def. 3.9, with set of generating (acyclic) cofibrations the sets $I_{\mathrm{Top}}^{\mathcal{C}}$, $J_{\mathrm{Top}}^{\mathcal{C}}$ and $I_{\mathrm{Top}^{*/}}^{\mathcal{C}}$, $J_{\mathrm{Top}^{*/}}^{\mathcal{C}}$ from def. 3.74, respectively.

(Piacenza 91, theorem 5.4)

Proof. By prop. 3.71 the category has all limits and colimits, hence it remains to check the model structure

But via the enriched Yoneda lemma (prop. 3.73) it follows that proving the model structure reduces objectwise to the proof of theorem 3.7, theorem 3.51. In particular, the technical lemmas 1.40, 1.45 and 1.52 generalize immediately to the present situation, with the evident small change of wording:

For instance, the fact that a morphism of topologically enriched functors $\eta : F \to G$ that has the right lifting property against the elements of $I_{\mathrm{Top}}^{\mathcal{C}}$ is a projective weak equivalence, follows by noticing that for fixed $\eta : F \to G$ the enriched Yoneda lemma prop. 3.73 gives a natural bijection of commuting diagrams (and their fillers) of the form

$$\left(\begin{array}{ccc} y(c) \cdot S^{n-1} & \longrightarrow & F \\ {}^{(\mathrm{id} \cdot \iota_n)}\downarrow && \downarrow^{\eta} \\ y(c) \cdot D^n & \longrightarrow & G \end{array} \right) \leftrightarrow \left(\begin{array}{ccc} S^{n-1} & \longrightarrow & F(c) \\ \downarrow && \downarrow^{\eta_c} \\ D^n & \longrightarrow & G(c) \end{array} \right) ,$$

and hence the statement follows with part A) of the proof of lemma 1.52.

With these three lemmas in hand, the remaining formal part of the proof goes through verbatim as above: repeatedly use the small object argument (prop. 2.17) and the retract argument (prop. 2.15) to establish the two weak factorization systems. (While again the structure of a category with weak equivalences is evident.) ▮

Example 3.77. Given examples 3.67 and 3.70, the next evident example of a pointed topologically enriched category besides $\mathrm{Top}_{\mathrm{cg}}^{*/}$ itself is the functor category

$$[\mathrm{Top}_{\mathrm{cg}}^{*/}, \mathrm{Top}_{\mathrm{cg}}^{*/}]\,.$$

The only technical problem with this is that $\mathrm{Top}_{\mathrm{cg}}^{*/}$ is not a small category (it has a proper class of objects), which means that the existence of all limits and colimits via prop. 3.71 may (and does) fail.

But so we just restrict to a small topologically enriched subcategory. A good choice is the full subcategory

$$\mathrm{Top}_{\mathrm{cg,fin}}^{*/} \hookrightarrow \mathrm{Top}_{\mathrm{cg}}^{*/}$$

of topological spaces homoemorphic to finite CW-complexes. The resulting projective model category (via theorem 3.76)

$$[\mathrm{Top}_{\mathrm{cg,fin}}^{*/}\,,\, (\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}]_{\mathrm{proj}}$$

is also also known as the **strict model structure for excisive functors**. (This terminology is the special case for $n=1$ of the terminology "n-excisive functors" as used in "Goodwillie calculus", a homotopy-theoretic analog of differential calculus.) After enlarging its class of weak equivalences while keeping the cofibrations fixed, this will become Quillen equivalent to a model structure for spectra. This we discuss in part 1.2, in the section on pre-excisive functors.

One consequence of theorem 3.76 is the model category theoretic incarnation of the theory of *homotopy colimits*.

Observe that ordinary limits and colimits (def. 1.1) are equivalently characterized in terms of adjoint functors:

Let $\mathcal{C}$ be any category and let I be a small category. Write $[I,\mathcal{C}]$ for the corresponding functor category. We may think of its objects as I-shaped diagrams in $\mathcal{C}$, and of its morphisms as homomorphisms of these diagrams. There is a canonical functor

$$\mathrm{const}_I \,:\, \mathcal{C} \longrightarrow [I,\mathcal{C}]$$

which sends each object of $\mathcal{C}$ to the diagram that is constant on this object. Inspection of the definition of the universal properties of limits and colimits on one hand, and of left adjoint and right adjoint functors on the other hand, shows that

1. precisely when $\mathcal{C}$ has all colimits of shape I, then the functor const_I has a left adjoint functor, which is the operation of forming these colimits:

$$[I,\mathcal{C}] \underoverset{\underset{\mathrm{const}_I}{\longleftarrow}}{\overset{\underset{\longrightarrow}{\lim}_I}{\longrightarrow}}{\bot} \mathcal{C}$$

2. precisely when $\mathcal{C}$ has all limits of shape I, then the functor const_I has a right adjoint functor, which is the operation of forming these limits.

$$[I,\mathcal{C}] \underoverset{\underset{\underset{\longleftarrow}{\lim}_I}{\longrightarrow}}{\overset{\mathrm{const}_I}{\longleftarrow}}{\bot} \mathcal{C}$$

Proposition 3.78. *Let I be a small topologically enriched category (def. 3.65). Then the* $(\underset{\longrightarrow}{\lim}_I \dashv \mathrm{const}_I)$*-adjunction*

$$[I,(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{Quillen}}]_{\mathrm{proj}} \underoverset{\underset{\mathrm{const}_I}{\longleftarrow}}{\overset{\underset{\longrightarrow}{\lim}_I}{\longrightarrow}}{\bot} (\mathrm{Top}_{\mathrm{cg}})_{\mathrm{Quillen}}$$

is a Quillen adjunction (def. 2.46) between the projective model structure on topological functors on I, from theorem 3.76, and the classical model structure on topological spaces from theorem 3.51.

Similarly, if I is enriched in pointed topological spaces, then for the classical model structure on pointed topological spaces (prop. 3.29, theorem 3.34) the adjunction

$$[I, (\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}]_{\mathrm{proj}} \underoverset{\underset{\mathrm{const}}{\longleftarrow}}{\overset{\underset{\longrightarrow}{\lim}}{\longrightarrow}}{\bot} (\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}$$

is a Quillen adjunction.

Proof. Since the fibrations and weak equivalences in the projective model structure (def. 3.75) on the functor category are objectwise those of $(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{Quillen}}$ and of $(\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}$, respectively, it is immediate that the functor const_I preserves these. In particular it preserves fibrations and acyclic fibrations and so the claim follows (prop. 2.47). ▮

Definition 3.79. In the situation of prop. 3.78 we say that the left derived functor (def. 2.42) of the colimit functor is the **homotopy colimit**

$$\mathrm{hocolim}_I \coloneqq \mathbb{L}\underset{\longrightarrow_I}{\lim} \;:\; \mathrm{Ho}([I, \mathrm{Top}]) \longrightarrow \mathrm{Ho}(\mathrm{Top})$$

and

$$\mathrm{hocolim}_I \coloneqq \mathbb{L}\underset{\longrightarrow_I}{\lim} \;:\; \mathrm{Ho}([I, \mathrm{Top}^{*/}]) \longrightarrow \mathrm{Ho}(\mathrm{Top}^{*/}) \,.$$

Remark 3.80. Since every object in $(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{Quillen}}$ and in $(\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}$ is fibrant, the homotopy colimit of any diagram $X_\bullet$, according to def. 3.79, is (up to weak homotopy equivalence) the result of forming the ordinary colimit of any projectively cofibrant replacement $\hat{X}_\bullet \overset{\in W_{\mathrm{proj}}}{\longrightarrow} X_\bullet$.

Example 3.81. Write $\mathbb{N}^{\leq}$ for the poset (def. 1.15) of natural numbers, hence for the small category (with at most one morphism from any given object to any other given object) that looks like

$$\mathbb{N}^{\leq} = \{0 \to 1 \to 2 \to 3 \to \cdots\} \,.$$

Regard this as a topologically enriched category with the, necessarily, discrete topology on its hom-sets.

Then a topologically enriched functor

$$X_\bullet \;:\; \mathbb{N}^{\leq} \longrightarrow \mathrm{Top}_{\mathrm{cg}}$$

is just a plain functor and is equivalently a sequence of continuous functions (morphisms in $\mathrm{Top}_{\mathrm{cg}}$) of the form (also called a *cotower*)

$$X_0 \overset{f_0}{\longrightarrow} X_1 \overset{f_1}{\longrightarrow} X_2 \overset{f_2}{\longrightarrow} X_3 \longrightarrow \cdots \,.$$

It is immediate to check that those sequences $X_\bullet$ which are cofibrant in the projective model structure (theorem 3.76) are precisely those for which

1. all component morphisms f_i are cofibrations in $(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{Quillen}}$ or $(\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}$, respectively, hence retracts (remark 2.12) of relative cell complex inclusions (def. 1.38);

2. the object X_0, and hence all other objects, are cofibrant, hence are retracts of cell complexes (def. 1.38).

By example 3.81 it is immediate that the operation of forming colimits sends projective (acyclic) cofibrations between sequences of topological spaces to (acyclic) cofibrations in the classical model structure on pointed topological spaces. On those projectively cofibrant sequences where every map is not just a retract of a relative cell complex inclusion, but a plain relative cell complex inclusion, more is true:

Proposition 3.82. *In the projective model structures on cotowers in topological spaces, $[\mathbb{N}^{\leq}, (\mathrm{Top}_{\mathrm{cg}})_{\mathrm{Quillen}}]_{\mathrm{proj}}$ and $[\mathbb{N}^{\leq}, (\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}]_{\mathrm{proj}}$ from def. 3.81, the following holds:*

1. *The colimit functor preserves fibrations between sequences of relative cell complex inclusions;*

2. *Let I be a finite category, let $D_\bullet(-) : I \to [\mathbb{N}^{\leq}, \mathrm{Top}_{\mathrm{cg}}]$ be a finite diagram of sequences of relative cell complexes. Then there is a weak homotopy equivalence*

$$\underset{\longrightarrow_n}{\lim} \left(\underset{\longleftarrow_i}{\lim} D_n(i) \right) \overset{\in W_{\mathrm{cl}}}{\longrightarrow} \underset{\longleftarrow_i}{\lim} \left(\underset{\longrightarrow_n}{\lim} D_n(i) \right)$$

from the colimit over the limit sequnce to the limit of the colimits of sequences.

Proof. Regarding the first statement:

Use that both $(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{Quillen}}$ and $(\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}$ are cofibrantly generated model categories (theorem 3.34) whose generating acyclic cofibrations have compact topological spaces as domains and codomains. The colimit over a sequence of relative cell complexes (being a transfinite composition) yields another relative cell complex, and hence lemma 1.40 says that every morphism out of the domain or codomain of a generating acyclic cofibration into this colimit factors through a finite stage inclusion. Since a projective fibration is a degreewise fibration, we have the lifting property at that finite stage, and hence also the lifting property against the morphisms of colimits.

Regarding the second statement:

This is a model category theoretic version of a standard fact of plain category theory, which says that in the category Set of sets, filtered colimits commute with finite limits in that there is an isomorphism of sets of the form which we have to prove is a weak homotopy equivalence of topological spaces. But now using that weak homotopy equivalences are detected by forming homotopy groups (def. 1.26), hence hom-sets out of n-spheres, and since n-spheres are compact topological spaces, lemma 1.40 says that homming out of n-spheres commutes over the colimits in question. Moreover, generally homming out of anything commutes over limits, in particular finite limits (every hom functor is left exact functor in the second variable). Therefore we find isomorphisms of the form

$$\mathrm{Hom}\left(S^q, \underset{\longrightarrow_n}{\lim} \left(\underset{\longleftarrow_i}{\lim} D_n(i) \right)\right) \simeq \underset{\longrightarrow_n}{\lim} \left(\underset{\longleftarrow_i}{\lim} \mathrm{Hom}(S^q, D_n(i)) \right) \overset{\sim}{\longrightarrow} \underset{\longleftarrow_i}{\lim} \left(\underset{\longrightarrow_n}{\lim} \mathrm{Hom}(S^q D_n(i)) \right) \simeq \mathrm{Hom}\left(S^q, \underset{\longleftarrow_i}{\lim} \left(\underset{\longrightarrow_n}{\lim} D_n(i) \right)\right)$$

and similarly for the left homotopies $\mathrm{Hom}(S^q \times I, -)$ (and similarly for the pointed case). This implies the claimed isomorphism on homotopy groups. ▮

4. Homotopy fiber sequences

A key aspect of homotopy theory is that the universal constructions of category theory, such as limits and colimits, receive a refinement whereby their universal properties hold not just up to isomorphism but up to (weak) homotopy equivalence. One speaks of *homotopy limits* and *homotopy colimits*.

We consider this here just for the special case of homotopy fibers and homotopy cofibers, leading to the phenomenon of homotopy fiber sequences and their induced long exact sequences of homotopy groups which control much of the theory to follow.

Mapping cones

In the context of homotopy theory, a pullback diagram, such as in the definition of the fiber in example 3.27

$$
\begin{array}{ccc}
\mathrm{fib}(f) & \longrightarrow & X \\
\downarrow & & \downarrow^{f} \\
* & \longrightarrow & Y
\end{array}
$$

ought to commute only up to a (left/right) homotopy (def. 2.22) between the outer composite morphisms. Moreover, it should satisfy its universal property up to such homotopies.

Instead of going through the full theory of what this means, we observe that this is plausibly modeled by the following construction, and then we check (below) that this indeed has the relevant abstract homotopy theoretic properties.

Definition 4.1. Let $\mathcal{C}$ be a model category, def. 2.3 with $\mathcal{C}^{*/}$ its model structure on pointed objects, prop. 3.29. For $f : X \longrightarrow Y$ a morphism between cofibrant objects (hence a morphism in $(\mathcal{C}^{*/})_c \hookrightarrow \mathcal{C}^{*/}$, def. 2.34), its **reduced mapping cone** is the object

$$
\mathrm{Cone}(f) \coloneqq * \underset{X}{\sqcup} \mathrm{Cyl}(X) \underset{X}{\sqcup} Y
$$

in the colimiting diagram

$$
\begin{array}{ccccc}
 & & X & \overset{f}{\longrightarrow} & Y \\
 & & \downarrow^{i_1} & & \downarrow^{i} \\
X & \overset{i_0}{\longrightarrow} & \mathrm{Cyl}(X) & & \\
\downarrow & & & \searrow^{\eta} & \downarrow \\
* & \longrightarrow & & \longrightarrow & \mathrm{Cone}(f)
\end{array}
,
$$

where $\mathrm{Cyl}(X)$ is a cylinder object for X, def. 2.18.

Dually, for $f : X \longrightarrow Y$ a morphism between fibrant objects (hence a morphism in $(\mathcal{C}^{*})_f \hookrightarrow \mathcal{C}^{*/}$, def. 2.34), its **mapping cocone** is the object

$$
\mathrm{Path}_*(f) \coloneqq * \underset{Y}{\times} \mathrm{Path}(Y) \underset{Y}{\times} Y
$$

in the following limit diagram

$$
\begin{array}{ccccc}
\mathrm{Path}_*(f) & \longrightarrow & & \longrightarrow & X \\
\downarrow & \searrow^{\eta} & & & \downarrow^{f} \\
 & & \mathrm{Path}(Y) & \underset{p_1}{\longrightarrow} & Y \\
\downarrow & & \downarrow^{p_0} & & \\
* & \longrightarrow & Y & &
\end{array}
,
$$

where $\mathrm{Path}(Y)$ is a path space object for Y, def. 2.18.

Remark 4.2. When we write homotopies (def. 2.22) as double arrows between morphisms, then the limit diagram in def. 4.1 looks just like the square in the definition of fibers in example 3.27, except that it is filled by the right homotopy given by the component map denoted η:

$$\begin{tikzcd} \mathrm{Path}_*(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow_\eta"] & X \arrow[d, "f"] \\ * \arrow[r] & Y \end{tikzcd}.$$

Dually, the colimiting diagram for the mapping cone turns to look just like the square for the cofiber, except that it is filled with a left homotopy

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\swarrow_\eta"] & Y \arrow[d] \\ * \arrow[r] & \mathrm{Cone}(f) \end{tikzcd}$$

Proposition 4.3. *The colimit appearing in the definition of the reduced mapping cone in def. 4.1 is equivalent to three consecutive pushouts:*

$$\begin{tikzcd} & & X \arrow[r, "f"] \arrow[d, "i_1"] \arrow[dr, phantom, "(\mathrm{po})"] & Y \arrow[d, "i"] \\ & X \arrow[r, "i_0"] \arrow[d] \arrow[dr, phantom, "(\mathrm{po})"] & \mathrm{Cyl}(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "(\mathrm{po})"] & \mathrm{Cyl}(f) \arrow[d] \\ & * \arrow[r] & \mathrm{Cone}(X) \arrow[r] & \mathrm{Cone}(f) \end{tikzcd} .$$

The two intermediate objects appearing here are called

- *the plain **reduced cone*** $\mathrm{Cone}(X) \coloneqq * \underset{X}{\sqcup} \mathrm{Cyl}(X)$*;*
- *the **reduced mapping cylinder*** $\mathrm{Cyl}(f) \coloneqq \mathrm{Cyl}(X) \underset{X}{\sqcup} Y$*.*

Dually, the limit appearing in the definition of the mapping cocone in def. 4.1 is equivalent to three consecutive pullbacks:

$$\begin{tikzcd} \mathrm{Path}_*(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "(\mathrm{pb})"] & \mathrm{Path}(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "(\mathrm{pb})"] & X \arrow[d, "f"] \\ \mathrm{Path}_*(Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "(\mathrm{pb})"] & \mathrm{Path}(Y) \arrow[r, "p_1"'] \arrow[d, "p_0"] & Y \\ * \arrow[r] & Y & \end{tikzcd} .$$

The two intermediate objects appearing here are called

- *the **based path space object*** $\mathrm{Path}_*(Y) \coloneqq * \prod_Y \mathrm{Path}(Y)$*;*
- *the **mapping path space** or **mapping co-cylinder*** $\mathrm{Path}(f) \coloneqq X \underset{Y}{\times} \mathrm{Path}(X)$*.*

Definition 4.4. Let $X \in \mathcal{C}^{*/}$ be any pointed object.

1. The mapping cone, def. 4.3, of $X \to *$ is called the **reduced suspension** of X, denoted

$$\Sigma X = \mathrm{Cone}(X \to *) \; .$$

Via prop. 4.3 this is equivalently the coproduct of two copies of the cone on X over their base:

$$\begin{array}{ccccc} & & X & \longrightarrow & * \\ & & \downarrow^{i_1} & (\mathrm{po}) & \downarrow \\ X & \overset{i_0}{\longrightarrow} & \mathrm{Cyl}(X) & \longrightarrow & \mathrm{Cone}(X) \\ \downarrow & (\mathrm{po}) & \downarrow & (\mathrm{po}) & \downarrow \\ * & \longrightarrow & \mathrm{Cone}(X) & \longrightarrow & \Sigma X \end{array} .$$

This is also equivalently the cofiber, example 3.27 of (i_0, i_1), hence (example 3.20) of the wedge sum inclusion:

$$X \vee X = X \sqcup X \overset{(i_0,i_1)}{\longrightarrow} \mathrm{Cyl}(X) \overset{\mathrm{cofib}(i_0,i_1)}{\longrightarrow} \Sigma X \,.$$

2. The mapping cocone, def. 4.3, of $* \to X$ is called the **loop space object** of X, denoted

$$\Omega X = \mathrm{Path}_*(* \to X) \,.$$

Via prop. 4.3 this is equivalently

$$\begin{array}{ccccc} \Omega X & \longrightarrow & \mathrm{Path}_*(X) & \longrightarrow & * \\ \downarrow & (\mathrm{pb}) & \downarrow & (\mathrm{pb}) & \downarrow \\ \mathrm{Path}_*(X) & \longrightarrow & \mathrm{Path}(X) & \underset{p_1}{\longrightarrow} & X \\ \downarrow & (\mathrm{pb}) & \downarrow^{p_0} & & \\ * & \longrightarrow & X & & \end{array} .$$

This is also equivalently the fiber, example 3.27 of (p_0, p_1):

$$\Omega X \overset{\mathrm{fib}(p_0,p_1)}{\longrightarrow} \mathrm{Path}(X) \overset{(p_0,p_1)}{\longrightarrow} X \times X \,.$$

Proposition 4.5. *In pointed topological spaces* $\mathrm{Top}^{*/}$,

- *the reduced suspension objects (def. 4.4) induced from the standard reduced cylinder* $(-) \wedge (I_+)$ *of example 3.25 are isomorphic to the smash product (def. 3.22) with the 1-sphere, for later purposes we choose to smash* ***on the left*** *and write*

$$\mathrm{cofib}(X \vee X \to X \wedge (I_+)) \simeq S^1 \wedge X \,,$$

Dually:

- *the loop space objects (def. 4.4) induced from the standard pointed path space object* $\mathrm{Maps}(I_+, -)_*$ *are isomorphic to the pointed mapping space (example 3.26) with the 1-sphere*

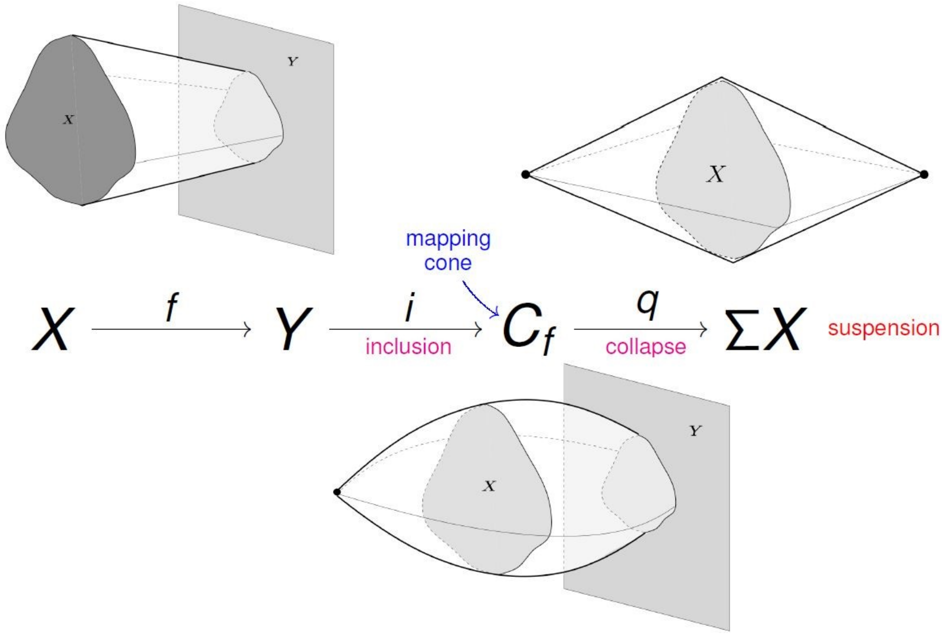
$$\mathrm{fib}(\mathrm{Maps}(I_+, X)_* \to X \times X) \simeq \mathrm{Maps}(S^1, X)_* \,.$$

Proof. By immediate inspection: For instance the fiber of $\mathrm{Maps}(I_+, X)_* \longrightarrow X \times X$ is clearly the subspace of the unpointed mapping space X^I on elements that take the endpoints of I to the basepoint of X. ▮

Example 4.6. For $\mathcal{C} =$ Top with $\mathrm{Cyl}(X) = X \times I$ the standard cyclinder object, def. 1.22, then by example 1.12, the mapping cone, def. 4.1, of a continuous function $f : X \longrightarrow Y$ is obtained by

1. forming the cylinder over X;
2. attaching to one end of that cylinder the space Y as specified by the map f.
3. shrinking the other end of the cylinder to the point.

Accordingly the suspension of a topological space is the result of shrinking both ends of the cylinder on the object two the point. This is homeomoprhic to attaching two copies of the cone on the space at the base of the cone.



(graphics taken from Muro 10)

Below in example 4.19 we find the homotopy-theoretic interpretation of this standard topological mapping cone as a model for the *homotopy cofiber*.

Remark 4.7. The *formula* for the mapping cone in prop. 4.3 (as opposed to that of the mapping co-cone) does not require the presence of the basepoint: for $f : X \longrightarrow Y$ a morphism in $\mathcal{C}$ (as opposed to in $\mathcal{C}^{*/}$) we may still define

$$\mathrm{Cone}'(f) \coloneqq Y \underset{X}{\sqcup} \mathrm{Cone}'(X) \,,$$

where the prime denotes the *unreduced cone*, formed from a cylinder object in $\mathcal{C}$.

Proposition 4.8. *For $f : X \longrightarrow Y$ a morphism in Top, then its unreduced mapping cone, remark 4.7, with respect to the standard cylinder object $X \times I$ def. 1.22, is isomorphic to the reduced mapping cone, def. 4.1, of the morphism $f_+ : X_+ \to Y_+$ (with a basepoint adjoined, def. 3.18) with respect to the standard reduced cylinder (example 3.25):*

$$\mathrm{Cone}'(f) \simeq \mathrm{Cone}(f_+) \,.$$

Proof. By prop. 3.19 and example 3.24, $\mathrm{Cone}(f_+)$ is given by the colimit in Top over the following diagram:

$$\begin{array}{ccccc} * & \longrightarrow & X \sqcup * & \overset{(f,\mathrm{id})}{\longrightarrow} & Y \sqcup * \\ \downarrow && \downarrow && \downarrow \\ X \sqcup * & \longrightarrow & (X \times I) \sqcup * && \\ \downarrow &&&& \downarrow \\ * & \longrightarrow && \longrightarrow & \mathrm{Cone}(f_+) \end{array} \,.$$

We may factor the vertical maps to give

$$\begin{array}{ccccc} * & \longrightarrow & X \sqcup * & \overset{(f,\mathrm{id})}{\longrightarrow} & Y \sqcup * \\ \downarrow && \downarrow && \downarrow \\ X \sqcup * & \longrightarrow & (X \times I) \sqcup * && \\ \downarrow &&&& \downarrow \\ * \sqcup * & \longrightarrow && \longrightarrow & \mathrm{Cone}'(f)_+ \\ \downarrow &&&& \downarrow \\ * & \longrightarrow && \longrightarrow & \mathrm{Cone}'(f) \end{array} \,.$$

This way the top part of the diagram (using the pasting law to compute the colimit in two stages) is manifestly a cocone under the result of applying $(-)_+$ to the diagram for the unreduced cone. Since $(-)_+$ is itself given by a colimit, it preserves colimits, and hence gives the partial colimit $\mathrm{Cone}'(f)_+$ as shown. The remaining pushout then contracts the remaining copy of the point away. ▮

Example 4.6 makes it clear that every cycle $S^n \to Y$ in Y that happens to be in the image of X can be *continuously* translated in the cylinder-direction, keeping it constant in Y, to the other end of the cylinder, where it shrinks away to the point. This means that every homotopy group of Y, def. 1.26, in the image of f vanishes in the mapping cone. Hence in the mapping cone **the image of X under f in Y is removed up to homotopy**. This makes it intuitively clear how $\mathrm{Cone}(f)$ is a homotopy-version of the cokernel of f. We now discuss this formally.

Lemma 4.9. ***(factorization lemma)***

Let $\mathcal{C}_c$ be a category of cofibrant objects, def. 2.34. Then for every morphism $f : X \longrightarrow Y$ the mapping cylinder-construction in def. 4.3 provides a cofibration resolution of f, in that

1. *the composite morphism $X \overset{i_0}{\longrightarrow} \mathrm{Cyl}(X) \overset{(i_1)_* f}{\longrightarrow} \mathrm{Cyl}(f)$ is a cofibration;*
2. *f factors through this morphism by a weak equivalence left inverse to an acyclic cofibration*

$$f \,:\, X \underset{\in \mathrm{Cof}}{\overset{(i_1)_* f \circ i_0}{\longrightarrow}} \mathrm{Cyl}(f) \underset{\in W}{\longrightarrow} Y \,,$$

Dually:

Let $\mathcal{C}_f$ be a category of fibrant objects, def. 2.34. Then for every morphism $f : X \longrightarrow Y$ the mapping cocylinder-construction in def. 4.3 provides a fibration resolution of f, in that

1. *the composite morphism $\mathrm{Path}(f) \overset{p_1^* f}{\longrightarrow} \mathrm{Path}(Y) \overset{p_0}{\longrightarrow} Y$ is a fibration;*
2. *f factors through this morphism by a weak equivalence right inverse to an acyclic fibration:*

$$f \,:\, X \underset{\in W}{\longrightarrow} \mathrm{Path}(f) \underset{\in \mathrm{Fib}}{\overset{p_0 \circ p_1^* f}{\longrightarrow}} Y \,,$$

Proof. We discuss the second case. The first case is formally dual.

So consider the mapping cocylinder-construction from prop. 4.3

$$\begin{array}{ccc} \mathrm{Path}(f) & \overset{\in W \cap \mathrm{Fib}}{\longrightarrow} & X \\ {}^{\mathllap{p_1^* f}}\downarrow & (\mathrm{pb}) & \downarrow^{\mathrlap{f}} \\ \mathrm{Path}(Y) & \underset{\in W \cap \mathrm{Fib}}{\overset{p_1}{\longrightarrow}} & Y \\ {}^{\mathllap{\in W \cap \mathrm{Fib}}}\downarrow^{\mathrlap{p_0}} \\ Y \end{array} \,.$$

To see that the vertical composite is indeed a fibration, notice that, by the pasting law, the above pullback diagram may be decomposed as a pasting of two pullback diagram as follows

$$\begin{array}{ccccc} \mathrm{Path}(f) & \underset{\in \mathrm{Fib}}{\overset{(f,\mathrm{id})^*(p_1,p_0)}{\longrightarrow}} & X \times Y & \overset{\mathrm{pr}_1}{\longrightarrow} & X \\ \downarrow && \downarrow^{(f,\mathrm{Id})} && \downarrow^f \\ \mathrm{Path}(Y) & \overset{(p_1,p_0) \in \mathrm{Fib}}{\longrightarrow} & Y \times Y & \overset{\mathrm{pr}_1}{\longrightarrow} & Y \,. \\ {}^{p_0}\downarrow & & \swarrow_{\mathrm{pr}_2 \atop \in \mathrm{Fib}} \\ Y \end{array}$$

Both squares are pullback squares. Since pullbacks of fibrations are fibrations by prop. 2.10, the morphism $\mathrm{Path}(f) \to X \times Y$ is a fibration. Similarly, since X is fibrant, also the projection map $X \times Y \to Y$ is a fibration (being the pullback of $X \to *$ along $Y \to *$).

Since the vertical composite is thereby exhibited as the composite of two fibrations

$$\mathrm{Path}(f) \overset{(f,\mathrm{id})^*(p_1,p_0)}{\longrightarrow} X \times Y \overset{\mathrm{pr}_2 \circ (f,\mathrm{Id}) = \mathrm{pr}_2}{\longrightarrow} Y \,,$$

it is itself a fibration.

Then to see that there is a weak equivalence as claimed:

The universal property of the pullback $\mathrm{Path}(f)$ induces a right inverse of $\mathrm{Path}(f) \to X$ fitting into this diagram

$$\begin{array}{ccccccc} \mathrm{id}_X : & X & \underset{\in W}{\overset{\exists}{\longrightarrow}} & \mathrm{Path}(f) & \overset{\in W \cap \mathrm{Fib}}{\longrightarrow} & X \\ & {}^{f}\downarrow && \downarrow && \downarrow^f \\ \mathrm{id}_Y : & Y & \underset{\in W}{\overset{i}{\longrightarrow}} & \mathrm{Path}(Y) & \overset{p_1}{\to} & Y \,, \\ && {}_{\mathrm{Id}}\searrow & \downarrow^{p_0} \\ &&& Y \end{array}$$

which is a weak equivalence, as indicated, by two-out-of-three (def. 2.1).

This establishes the claim. ▮

Categories of fibrant objects

Below we discuss the homotopy-theoretic properties of the mapping cone- and mapping cocone-constructions from above. Before we do so, we here establish a collection of general facts that hold in categories of fibrant objects and dually in categories of cofibrant objects, def. 2.34.

Literature (Brown 73, section 4).

Lemma 4.10. *Let $f : X \longrightarrow Y$ be a morphism in a category of fibrant objects, def. 2.34. Then given any choice of path space objects $\mathrm{Path}(X)$ and $\mathrm{Path}(Y)$, def. 2.18, there is a replacement of $\mathrm{Path}(X)$ by a path space object $\widetilde{\mathrm{Path}(X)}$ along an acylic fibration, such that $\widetilde{\mathrm{Path}(X)}$ has a morphism ϕ to $\mathrm{Path}(Y)$ which is compatible with the structure maps, in that the following diagram commutes*

$$
\begin{array}{ccccc}
 && X & \overset{f}{\longrightarrow} & Y \\
 & \swarrow & \downarrow && \downarrow \\
 \mathrm{Path}(X) & \underset{\in W \cap \mathrm{Fib}}{\longleftarrow} & \widetilde{\mathrm{Path}(X)} & \overset{\phi}{\longrightarrow} & \mathrm{Path}(Y) \\
 & {}_{(p_0^X, p_1^X)}\searrow & \downarrow^{(p_0^Y, p_1^Y)} && \downarrow^{(\tilde p_0^X, \tilde p_1^X)} \\
 && X \times X & \overset{(f,f)}{\longrightarrow} & Y \times Y
\end{array}
$$

(Brown 73, section 2, lemma 2)

Proof. Consider the commuting square

$$
\begin{array}{ccccc}
 X & \overset{f}{\longrightarrow} & Y & \longrightarrow & \mathrm{Path}(Y) \\
 \downarrow &&&& \downarrow^{(p_0^Y, p_1^Y)} \\
 \mathrm{Path}(X) & \overset{(p_0^X, p_1^X)}{\longrightarrow} & X \times X & \overset{(f,f)}{\longrightarrow} & Y \times Y
\end{array}
.
$$

Then consider its factorization through the pullback of the right morphism along the bottom morphism,

$$
\begin{array}{ccccc}
 X & \longrightarrow & (f \circ p_0^X, f \circ p_1^X)^* \mathrm{Path}(Y) & \longrightarrow & \mathrm{Path}(Y) \\
 & {}_{\in W}\searrow & \downarrow^{\in W \cap \mathrm{Fib}} && \downarrow^{(p_0^Y, p_1^Y)}_{\in \mathrm{Fib}} \\
 && \mathrm{Path}(X) & \overset{(f \circ p_0^X, f \circ p_1^X)}{\longrightarrow} & Y \times Y
\end{array}
.
$$

Finally use the factorization lemma 4.9 to factor the morphism $X \to (f \circ p_0^X, f \circ p_1^X)^* \mathrm{Path}(Y)$ through a weak equivalence followed by a fibration, the object this factors through serves as the desired path space resolution

$$
\begin{array}{ccccc}
 X & \overset{\in W}{\longrightarrow} & \widetilde{\mathrm{Path}(X)} & \longrightarrow & \mathrm{Path}(Y) \\
 & {}_{\in W}\searrow & \downarrow^{\in W \cap \mathrm{Fib}} && \downarrow^{(p_0^Y, p_1^Y)} \\
 && \mathrm{Path}(X) & \overset{(f \circ p_0^X, f \circ p_1^X)}{\longrightarrow} & Y \times Y
\end{array}
.
$$

∎

Lemma 4.11. *In a category of fibrant objects $\mathcal{C}_f$, def. 2.34, let*

$$
\begin{array}{ccccc}
 A_1 && \overset{f}{\longrightarrow} && A_2 \\
 & {}_{\in \mathrm{Fib}}\searrow && \swarrow_{\in \mathrm{Fib}} & \\
 && B &&
\end{array}
$$

be a morphism over some object B in $\mathcal{C}_f$ and let $u \colon B' \to B$ be any morphism in $\mathcal{C}_f$. Let

$$
\begin{array}{ccccc}
 u^* A_1 && \overset{u^* f}{\longrightarrow} && u^* A_2 \\
 & {}_{\in \mathrm{Fib}}\searrow && \swarrow_{\in \mathrm{Fib}} & \\
 && B' &&
\end{array}
$$

be the corresponding morphism pulled back along u.

Then

- *if f is a fibration then also $u^* f$ is a fibration;*

* *if f is a weak equivalence then also $u^* f$ is a weak equivalence.*

(Brown 73, section 4, lemma 1)

Proof. For $f \in \mathrm{Fib}$ the statement follows from the pasting law which says that if in

$$\begin{array}{ccc} B' \times_B A_1 & \longrightarrow & A_1 \\ \downarrow^{u^* f \in \mathrm{Fib}} && \downarrow^{f \in \mathrm{Fib}} \\ B' \times_B A_2 & \longrightarrow & A_2 \\ \downarrow^{\in \mathrm{Fib}} && \downarrow^{\in \mathrm{Fib}} \\ B' & \stackrel{u}{\longrightarrow} & B \end{array}$$

the bottom and the total square are pullback squares, then so is the top square. The same reasoning applies for $f \in W \cap \mathrm{Fib}$.

Now to see the case that $f \in W$:

Consider the full subcategory $(\mathcal{C}_{/B})_f$ of the slice category $\mathcal{C}_{/B}$ (def. 3.15) on its fibrant objects, i.e. the full subcategory of the slice category on the fibrations

$$\begin{array}{c} X \\ \downarrow^{p}_{\in \mathrm{Fib}} \\ B \end{array}$$

into B. By factorizing for every such fibration the diagonal morphisms into the fiber product $X \underset{B}{\times} X$ through a weak equivalence followed by a fibration, we obtain path space objects $\mathrm{Path}_B(X)$ relative to B:

$$(\Delta_X)/B \;:\; \begin{array}{ccccc} X & \stackrel{\in W}{\longrightarrow} & \mathrm{Path}_B(X) & \stackrel{\in \mathrm{Fib}}{\longrightarrow} & X \underset{B}{\times} X \\ & {}_{\in \mathrm{Fib}}\searrow & \downarrow & \swarrow_{\in \mathrm{Fib}} \\ && B \end{array} \,.$$

With these, the factorization lemma (lemma 4.9) applies in $(\mathcal{C}_{/B})_f$.

(Notice that for this we do need the restriction of $\mathcal{C}_{/B}$ to the fibrations, because this ensures that the projections $p_i \colon X_1 \times_B X_2 \to X_i$ are still fibrations, which is used in the proof of the factorization lemma (here).)

So now given any

$$\begin{array}{ccc} X && \underset{\in W}{\stackrel{f}{\longrightarrow}} && Y \\ & {}_{\in \mathrm{Fib}}\searrow && \swarrow_{\in \mathrm{Fib}} \\ && B \end{array}$$

apply the factorization lemma in $(\mathcal{C}_{/B})_f$ to factor it as

$$\begin{array}{ccccc} X & \stackrel{i \in W}{\longrightarrow} & \mathrm{Path}_B(f) & \stackrel{\in W \cap \mathrm{Fib}}{\longrightarrow} & Y \\ & {}_{\in \mathrm{Fib}}\searrow & \downarrow & \swarrow_{\in \mathrm{Fib}} \\ && B \end{array} \,.$$

By the previous discussion it is sufficient now to show that the base change of i to B' is still a weak equivalence. But by the factorization lemma in $(\mathcal{C}_{/B})_f$, the morphism i is right inverse to another acyclic fibration over B:

$$\begin{array}{ccccc} \mathrm{id}_X : & X & \overset{i \in W}{\longrightarrow} & \mathrm{Path}_B(f) & \overset{\in W \cap \mathrm{Fib}}{\longrightarrow} & X \\ & & {}_{\in \mathrm{Fib}}\searrow & \downarrow & \swarrow_{\in \mathrm{Fib}} \\ & & & B \end{array} .$$

(Notice that if we had applied the factorization lemma of Δ_X in $\mathcal{C}_f$ instead of $(\Delta_X)/B$ in $(\mathcal{C}_{/B})$ then the corresponding triangle on the right here would not commute.)

Now we may reason as before: the base change of the top morphism here is exhibited by the following pasting composite of pullbacks:

$$\begin{array}{ccc} B' \underset{B}{\times} X & \longrightarrow & X \\ \downarrow & (\mathrm{pb}) & \downarrow \\ B' \underset{B}{\times} \mathrm{Path}_B(f) & \longrightarrow & \mathrm{Path}_B(f) \\ \downarrow & (\mathrm{pb}) & \downarrow^{\in W \cap \mathrm{Fib}} \\ B' \underset{B}{\times} X & \longrightarrow & X \\ \downarrow & (\mathrm{pb}) & \downarrow \\ B' & \longrightarrow & B \end{array}$$

The acyclic fibration $\mathrm{Path}_B(f)$ is preserved by this pullback, as is the identity $\mathrm{id}_X : X \to \mathrm{Path}_B(X) \to X$. Hence the weak equivalence $X \to \mathrm{Path}_B(X)$ is preserved by two-out-of-three (def. 2.1).

Lemma 4.12. *In a category of fibrant objects, def. 2.34, the pullback of a weak equivalence along a fibration is again a weak equivalence.*

(Brown 73, section 4, lemma 2)

Proof. Let $u : B' \to B$ be a weak equivalence and $p : E \to B$ be a fibration. We want to show that the left vertical morphism in the pullback

$$\begin{array}{ccc} E \times_B B' & \longrightarrow & B' \\ \downarrow^{\Rightarrow \in W} & & \downarrow^{\in W} \\ E & \overset{\in \mathrm{Fib}}{\longrightarrow} & B \end{array}$$

is a weak equivalence.

First of all, using the factorization lemma 4.9 we may factor $B' \to B$ as

$$B' \overset{\in W}{\longrightarrow} \mathrm{Path}(u) \overset{\in W \cap F}{\longrightarrow} B$$

with the first morphism a weak equivalence that is a right inverse to an acyclic fibration and the right one an acyclic fibration.

Then the pullback diagram in question may be decomposed into two consecutive pullback diagrams

$$\begin{array}{ccc} E \times_B B' & \to & B' \\ \downarrow & & \downarrow \\ Q & \overset{\in \mathrm{Fib}}{\longrightarrow} & \mathrm{Path}(u) \\ \downarrow^{\in W \cap \mathrm{Fib}} & & \downarrow^{\in W \cap \mathrm{Fib}} \\ E & \overset{\in \mathrm{Fib}}{\longrightarrow} & B \end{array} ,$$

where the morphisms are indicated as fibrations and acyclic fibrations using the stability of these under arbitrary pullback.

This means that the proof reduces to proving that weak equivalences $u : B' \overset{\in W}{\longrightarrow} B$ that are right inverse to some acyclic fibration $v : B \overset{\in W \cap F}{\longrightarrow} B'$ map to a weak equivalence under pullback along a fibration.

Given such u with right inverse v, consider the pullback diagram

$$
\begin{array}{ccccc}
 & & E & & \\
 & {}^{(p,\mathrm{id})}_{\in W} & \downarrow & \searrow^{\mathrm{id}} & \\
E_1 \coloneqq & & B \times_{B'} E & \overset{\in W \cap \mathrm{Fib}}{\longrightarrow} & E \\
 & & \downarrow^{\in \mathrm{Fib}} & & \downarrow^{p \in \mathrm{Fib}} \\
 & & & (\mathrm{pb}) & B \\
 & & \downarrow & & \downarrow^{v \in W \cap \mathrm{Fib}} \\
 & & B & \overset{v \in \mathrm{Fib} \cap W}{\longrightarrow} & B'
\end{array}
$$

Notice that the indicated universal morphism $p \times \mathrm{Id} : E \overset{\in W}{\longrightarrow} E_1$ into the pullback is a weak equivalence by two-out-of-three (def. 2.1).

The previous lemma 4.11 says that weak equivalences between fibrations over B are themselves preserved by base extension along $u : B' \to B$. In total this yields the following diagram

$$
\begin{array}{ccccc}
u^\ast E = B' \times_B E & \longrightarrow & E & & \\
{}^{u^\ast(p \times \mathrm{Id})}_{\in W} \downarrow & & {}^{p \times \mathrm{Id}}_{\in W} \downarrow & \searrow^{\mathrm{id}} & \\
u^\ast E_1 & \longrightarrow & E_1 & \overset{\in W \cap \mathrm{Fib}}{\longrightarrow} & E \\
\downarrow^{\in \mathrm{Fib}} & & \downarrow^{\in \mathrm{Fib}} & & \downarrow^{p \in \mathrm{Fib}} \\
 & & & & B \\
\downarrow & & \downarrow & & \downarrow^{v \in W \cap \mathrm{Fib}} \\
B' & \overset{u}{\longrightarrow} & B & \overset{v \in W \cap \mathrm{Fib}}{\longrightarrow} & B'
\end{array}
$$

so that with $p \times \mathrm{Id} : E \to E_1$ a weak equivalence also $u^\ast(p \times \mathrm{Id})$ is a weak equivalence, as indicated.

Notice that $u^\ast E = B' \times_B E \to E$ is the morphism that we want to show is a weak equivalence. By two-out-of-three (def. 2.1) for that it is now sufficient to show that $u^\ast E_1 \to E_1$ is a weak equivalence.

That finally follows now since, by assumption, the total bottom horizontal morphism is the identity. Hence so is the top horizontal morphism. Therefore $u^\ast E_1 \to E_1$ is right inverse to a weak equivalence, hence is a weak equivalence. ▮

Lemma 4.13. *Let $(\mathcal{C}^{*/})_f$ be a category of fibrant objects, def. 2.34 in a model structure on pointed objects (prop. 3.29). Given any commuting diagram in $\mathcal{C}$ of the form*

$$
\begin{array}{ccccc}
X'_1 & \underset{t}{\overset{\in W}{\longrightarrow}} & X_1 & \underset{g}{\overset{f}{\rightrightarrows}} & X_2 \\
 & & \downarrow^{p_1}_{\in \mathrm{Fib}} & & \downarrow^{p_2}_{\in \mathrm{Fib}} \\
 & & B & \overset{u}{\longrightarrow} & C
\end{array}
$$

(meaning: both squares commute and t equalizes f with g) then the localization functor $\gamma:(\mathcal{C}^{*/})_f \to \mathrm{Ho}(\mathcal{C}^{*/})$ (def. 2.28, cor 2.36) takes the morphisms $\mathrm{fib}(p_1) \rightrightarrows \mathrm{fib}(p_2)$ induced by f and g on fibers (example 3.27) to the same morphism, in the homotopy category.

(Brown 73, section 4, lemma 4)

Proof. First consider the pullback of p_2 along u: this forms the same kind of diagram but with the bottom morphism an identity. Hence it is sufficient to consider this special case.

Consider the full subcategory $(\mathcal{C}^{*/}_{/B})_f$ of the slice category $\mathcal{C}^{*/}_{/B}$ (def. 3.15) on its fibrant objects, i.e. the full subcategory of the slice category on the fibrations

$$\begin{array}{c} X \\ \downarrow^{p}_{\in \mathrm{Fib}} \\ B \end{array}$$

into B. By factorizing for every such fibration the diagonal morphisms into the fiber product $X \underset{B}{\times} X$ through a weak equivalence followed by a fibration, we obtain path space objects $\mathrm{Path}_B(X)$ relative to B:

$$\begin{array}{ccccc} (\Delta_X)/B: & X & \overset{\in W}{\longrightarrow} & \mathrm{Path}_B(X) & \overset{\in \mathrm{Fib}}{\longrightarrow} & X \underset{B}{\times} X \\ & & {}_{\in \mathrm{Fib}}\searrow & \downarrow & \swarrow_{\in \mathrm{Fib}} \\ & & & B \end{array} .$$

With these, the factorization lemma (lemma 4.9) applies in $(\mathcal{C}^{*/}_{/B})_f$.

Let then $X \overset{s}{\to} \mathrm{Path}_B(X_2) \xrightarrow{(p_0,p_1)} X_2 \times_B X_2$ be a path space object for X_2 in the slice over B and consider the following commuting square

$$\begin{array}{ccc} X'_1 & \overset{s f t}{\longrightarrow} & \mathrm{Path}_B(X_2) \\ {}^{t}_{\in W}\downarrow && \downarrow^{(p_0,p_1)}_{\in \mathrm{Fib}} \\ X_1 & \overset{(f,g)}{\longrightarrow} & X_2 \underset{B}{\times} X_2 \end{array} .$$

By factoring this through the pullback $(f,g)^*(p_0,p_1)$ and then applying the factorization lemma 4.9 and then two-out-of-three (def. 2.1) to the factoring morphisms, this may be replaced by a commuting square of the same form, where however the left morphism is an acyclic fibration

$$\begin{array}{ccc} X''_1 & \longrightarrow & \mathrm{Path}_B(X_2) \\ {}^{t}_{\in W \cap \mathrm{Fib}}\downarrow && \downarrow^{(p_0,p_1)}_{\in \mathrm{Fib}} \\ X_1 & \overset{(f,g)}{\longrightarrow} & X_2 \underset{B}{\times} X_2 \end{array} .$$

This makes also the morphism $X''_1 \to B$ be a fibration, so that the whole diagram may now be regarded as a diagram in the category of fibrant objects $(\mathcal{C}_{/B})_f$ of the slice category over B.

As such, the top horizontal morphism now exhibits a right homotopy which under localization $\gamma_B : (\mathcal{C}_{/B})_f \longrightarrow \mathrm{Ho}(\mathcal{C}_{/B})$ (def. 2.28) of the slice model structure (prop. 3.29) we have

$$\gamma_B(f) = \gamma_B(g) .$$

The result then follows by observing that we have a commuting square of functors

$$
\begin{array}{ccc}
(\mathcal{C}^{*/}_{/B})_f & \overset{\text{fib}}{\longrightarrow} & \mathcal{C}^{*/} \\
\downarrow^{\gamma_B} & \swArrow & \downarrow^{\gamma} \\
\text{Ho}(\mathcal{C}^{*/}_{/B}) & \longrightarrow & \text{Ho}(\mathcal{C}^{*/})
\end{array}
,
$$

because, by lemma 4.11, the top and right composite sends weak equivalences to isomorphisms, and hence the bottom filler exists by theorem 2.31. This implies the claim. ▮

Homotopy fibers

We now discuss the homotopy-theoretic properties of the mapping cone- and mapping cocone-constructions from above.

Literature (Brown 73, section 4).

Remark 4.14. The factorization lemma 4.9 with prop. 4.3 says that the mapping cocone of a morphism f, def. 4.1, is equivalently the plain fiber, example 3.27, of a fibrant resolution $\tilde{f}$ of f:

$$
\begin{array}{ccc}
\text{Path}_*(f) & \longrightarrow & \text{Path}(f) \\
\downarrow & (\text{pb}) & \downarrow^{\tilde{f}} \\
* & \longrightarrow & Y
\end{array}
.
$$

The following prop. 4.15 says that, up to equivalence, this situation is independent of the specific fibration resolution $\tilde{f}$ provided by the factorization lemma (hence by the prescription for the mapping cocone), but only depends on it being *some* fibration resolution.

Proposition 4.15. *In the category of fibrant objects $(\mathcal{C}^{*/})_f$, def. 2.34, of a model structure on pointed objects (prop. 3.29) consider a morphism of fiber-diagrams, hence a commuting diagram of the form*

$$
\begin{array}{ccccc}
\text{fib}(p_1) & \longrightarrow & X_1 & \underset{\in \text{Fib}}{\overset{p_1}{\longrightarrow}} & Y_1 \\
\downarrow^{h} & & \downarrow^{g} & & \downarrow^{f} \\
\text{fib}(p_2) & \longrightarrow & X_2 & \underset{\in \text{Fib}}{\overset{p_2}{\longrightarrow}} & Y_2
\end{array}
.
$$

If f and g weak equivalences, then so is h.

Proof. Factor the diagram in question through the pullback of p_2 along f

$$
\begin{array}{ccccc}
\text{fib}(p_1) & \longrightarrow & X_1 & & \\
\downarrow^{h} & & {}^{\in W}\downarrow & \searrow^{p_1} & \\
\text{fib}(f^* p_2) & \longrightarrow & f^* X_2 & \underset{\in \text{Fib}}{\overset{f^* p_2}{\longrightarrow}} & Y_1 \\
\downarrow^{\simeq} & & \downarrow^{\in W} & & \downarrow^{f}_{\in W} \\
\text{fib}(p_2) & \longrightarrow & X_2 & \underset{\in \text{Fib}}{\overset{p_2}{\longrightarrow}} & Y_2
\end{array}
$$

and observe that

1. $\text{fib}(f^* p_2) = \text{pt}^* f^* p_2 = \text{pt}^* p_2 = \text{fib}(p_2)$;

2. $f^* X_2 \to X_2$ is a weak equivalence by lemma 4.12;

3. $X_1 \to f^* X_2$ is a weak equivalence by assumption and by two-out-of-three (def. 2.1);

Moreover, this diagram exhibits $h : \mathrm{fib}(p_1) \to \mathrm{fib}(f^* p_2) = \mathrm{fib}(p_2)$ as the base change, along $* \to Y_1$, of $X_1 \to f^* X_2$. Therefore the claim now follows with lemma 4.11. ▮

Hence we say:

Definition 4.16. Let $\mathcal{C}$ be a model category and $\mathcal{C}^{*/}$ its model category of pointed objects, prop. 3.29. For $f : X \longrightarrow Y$ any morphism in its category of fibrant objects $(\mathcal{C}^{*/})_f$, def. 2.34, then its **homotopy fiber**

$$\mathrm{hofib}(f) \longrightarrow X$$

is the morphism in the homotopy category $\mathrm{Ho}(\mathcal{C}^{*/})$, def. 2.25, which is represented by the fiber, example 3.27, of any fibration resolution $\tilde{f}$ of f (hence any fibration $\tilde{f}$ such that f factors through a weak equivalence followed by $\tilde{f}$).

Dually:

For $f : X \longrightarrow Y$ any morphism in its category of cofibrant objects $(\mathcal{C}^{*/})_c$, def. 2.34, then its **homotopy cofiber**

$$Y \longrightarrow \mathrm{hocofib}(f)$$

is the morphism in the homotopy category $\mathrm{Ho}(\mathcal{C})$, def. 2.25, which is represented by the cofiber, example 3.27, of any cofibration resolution of f (hence any cofibration $\tilde{f}$ such that f factors as $\tilde{f}$ followed by a weak equivalence).

Proposition 4.17. *The homotopy fiber in def. 4.16 is indeed well defined, in that for f_1 and f_2 two fibration replacements of any morphisms f in $\mathcal{C}_f$, then their fibers are isomorphic in $\mathrm{Ho}(\mathcal{C}^{*/})$.*

Proof. It is sufficient to exhibit an isomorphism in $\mathrm{Ho}(\mathcal{C}^{*/})$ from the fiber of the fibration replacement given by the factorization lemma 4.9 (for any choice of path space object) to the fiber of any other fibration resolution.

Hence given a morphism $f : Y \longrightarrow X$ and a factorization

$$f : X \underset{\in W}{\longrightarrow} \hat{X} \underset{f_1}{\overset{\in \mathrm{Fib}}{\longrightarrow}} Y$$

consider, for any choice $\mathrm{Path}(Y)$ of path space object (def. 2.18), the diagram

$$\begin{array}{ccc}
\mathrm{Path}(f) & \overset{\in W \cap \mathrm{Fib}}{\longrightarrow} & X \\
{}^{\mathllap{\in W}}\downarrow & (\mathrm{pb}) & \downarrow^{\mathrlap{\in W}} \\
\mathrm{Path}(f_1) & \overset{\in W \cap \mathrm{Fib}}{\longrightarrow} & \hat{X} \\
{}^{\mathllap{\in \mathrm{Fib}}}\downarrow & (\mathrm{pb}) & \downarrow^{\mathrlap{f_1 \atop \in \mathrm{Fib}}} \\
\mathrm{Path}(Y) & \underset{\in W \cap \mathrm{Fib}}{\overset{p_1}{\longrightarrow}} & Y \\
{}^{\mathllap{p_0 \atop \in W \cap \mathrm{Fib}}}\downarrow & & \\
Y & &
\end{array}$$

as in the proof of lemma 4.9. Now by repeatedly using prop. 4.15:

1. the bottom square gives a weak equivalence from the fiber of $\mathrm{Path}(f_1) \to \mathrm{Path}(Y)$ to the fiber of f_1;

2. The square

$$\begin{array}{ccc} \mathrm{Path}(f_1) & \overset{\mathrm{id}}{\longrightarrow} & \mathrm{Path}(f_1) \\ \downarrow && \downarrow \\ \mathrm{Path}(Y) & \underset{p_0}{\longrightarrow} & Y \end{array}$$

gives a weak equivalence from the fiber of $\mathrm{Path}(f_1) \to \mathrm{Path}(Y)$ to the fiber of $\mathrm{Path}(f_1) \to Y$.

3. Similarly the total vertical composite gives a weak equivalence via

$$\begin{array}{ccc} \mathrm{Path}(f) & \overset{\in W}{\longrightarrow} & \mathrm{Path}(f_1) \\ \downarrow && \downarrow \\ Y & \underset{\mathrm{id}}{\longrightarrow} & Y \end{array}$$

from the fiber of $\mathrm{Path}(f) \to Y$ to the fiber of $\mathrm{Path}(f_1) \to Y$.

Together this is a zig-zag of weak equivalences of the form

$$\mathrm{fib}(f_1) \overset{\in W}{\longleftarrow} \mathrm{fib}(\mathrm{Path}(f_1) \to \mathrm{Path}(Y)) \overset{\in W}{\longrightarrow} \mathrm{fib}(\mathrm{Path}(f_1) \to Y) \overset{\in W}{\longleftarrow} \mathrm{fib}(\mathrm{Path}(f) \to Y)$$

between the fiber of $\mathrm{Path}(f) \to Y$ and the fiber of f_1. This gives an isomorphism in the homotopy category. ▮

Example 4.18. **(fibers of Serre fibrations)**

In showing that Serre fibrations are abstract fibrations in the sense of model category theory, theorem 3.7 implies that the fiber F (example 3.27) of a Serre fibration, def. 1.47

$$\begin{array}{ccc} F & \longrightarrow & X \\ && \downarrow^{p} \\ && B \end{array}$$

over any point is actually a homotopy fiber in the sense of def. 4.16. With prop. 4.15 this implies that the weak homotopy type of the fiber only depends on the Serre fibration up to weak homotopy equivalence in that if $p' : X' \to B'$ is another Serre fibration fitting into a commuting diagram of the form

$$\begin{array}{ccc} X & \overset{\in W_{\mathrm{cl}}}{\longrightarrow} & X' \\ \downarrow^{p} && \downarrow^{p'} \\ B & \overset{\in W_{\mathrm{cl}}}{\longrightarrow} & B' \end{array}$$

then $F \overset{\in W_{\mathrm{cl}}}{\longrightarrow} F'$.

In particular this gives that the weak homotopy type of the fiber of a Serre fibration $p : X \to B$ does not change as the basepoint is moved in the same connected component. For let $\gamma : I \longrightarrow B$ be a path between two points

$$b_{0,1} \;:\; * \underset{\in W_{\mathrm{cl}}}{\overset{i_{0,1}}{\longrightarrow}} I \overset{\gamma}{\longrightarrow} B \,.$$

Then since all objects in $(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{Quillen}}$ are fibrant, and since the endpoint inclusions $i_{0,1}$ are weak equivalences, lemma 4.12 gives the zig-zag of top horizontal weak equivalences in the following diagram:

$$
\begin{array}{ccccccc}
F_{b_0} = & b_0^* p & \overset{\in W_{cl}}{\longrightarrow} & \gamma^* p & \overset{\in W_{cl}}{\longleftarrow} & b_1^* p & = F_{b_1} \\
& \downarrow & (pb) & \downarrow^{\gamma^* f}_{\underset{Fib}{\in}} & (pb) & \downarrow \\
& * & \underset{i_0}{\overset{\in W_{cl}}{\longrightarrow}} & I & \underset{i_1}{\overset{\in W_{cl}}{\longleftarrow}} & *
\end{array}
$$

and hence an isomorphism $F_{b_0} \simeq F_{b_1}$ in the classical homotopy category (def. 3.11).

The same kind of argument applied to maps from the square I^2 gives that if $\gamma_1, \gamma_2 : I \to B$ are two homotopic paths with coinciding endpoints, then the isomorphisms between fibers over endpoints which they induce are equal. (But in general the isomorphism between the fibers does depend on the choice of homotopy class of paths connecting the basepoints!)

The same kind of argument also shows that if B has the structure of a cell complex (def. 1.38) then the restriction of the Serre fibration to one cell D^n may be identified in the homotopy category with $D^n \times F$, and may be canonically identified so if the fundamental group of X is trivial. This is used when deriving the Serre-Atiyah-Hirzebruch spectral sequence for p (prop.).

Example 4.19. For every continuous function $f : X \longrightarrow Y$ between CW-complexes, def. 1.38, then the standard topological mapping cone is the attaching space (example 1.12)

$$
Y \cup_f \mathrm{Cone}(X) \quad \in \mathrm{Top}
$$

of Y with the standard cone $\mathrm{Cone}(X)$ given by collapsing one end of the standard topological cyclinder $X \times I$ (def. 1.22) as shown in example 4.6.

Equipped with the canonical continuous function

$$
Y \longrightarrow Y \cup_f \mathrm{Cone}(X)
$$

this represents the homotopy cofiber, def. 4.16, of f with respect to the classical model structure on topological spaces $\mathcal{C} = \mathrm{Top}_{\mathrm{Quillen}}$ from theorem 3.7.

Proof. By prop. 3.13, for X a CW-complex then the standard topological cylinder object $X \times I$ is indeed a cyclinder object in $\mathrm{Top}_{\mathrm{Quillen}}$. Therefore by prop. 4.3 and the factorization lemma 4.9, the mapping cone construction indeed produces first a cofibrant replacement of f and then the ordinary cofiber of that, hence a model for the homotopy cofiber. ▮

Example 4.20. The homotopy fiber of the inclusion of classifying spaces $BO(n) \hookrightarrow BO(n+1)$ is the n-sphere S^n. See this prop. at *Classifying spaces and G-structure*.

Example 4.21. Suppose a morphism $f : X \longrightarrow Y$ already happens to be a fibration between fibrant objects. The factorization lemma 4.9 replaces it by a fibration out of the mapping cocylinder $\mathrm{Path}(f)$, but such that the comparison morphism is a weak equivalence:

$$
\begin{array}{ccccc}
\mathrm{fib}(f) & \longrightarrow & X & \underset{\in Fib}{\overset{f}{\longrightarrow}} & Y \\
\downarrow^{\in W} & & \downarrow^{\in W} & & \downarrow^{id} \\
\mathrm{fib}(\tilde f) & \longrightarrow & \mathrm{Path}(f) & \underset{\in Fib}{\overset{\tilde f}{\longrightarrow}} & Y
\end{array}
.
$$

Hence by prop. 4.15 in this case the ordinary fiber of f is weakly equivalent to the mapping cocone, def. 4.1.

We may now state the abstract version of the statement of prop. 1.51:

Proposition 4.22. *Let $\mathcal{C}$ be a model category. For $f : X \to Y$ any morphism of pointed objects, and for A a pointed object, def. 3.16, then the sequence*

$$[A, \mathrm{hofib}(f)]_* \overset{i_*}{\longrightarrow} [A, X]_* \overset{f_*}{\longrightarrow} [A, Y]_*$$

is exact as a sequence of pointed sets.

(Where the sequence here is the image of the homotopy fiber sequence of def. 4.16 under the hom-functor $[A, -]_* : \mathrm{Ho}(\mathcal{C}^{*/}) \longrightarrow \mathrm{Set}^{*/}$ from example 3.30.)

Proof. Let A, X and Y denote fibrant-cofibrant objects in $\mathcal{C}^{*/}$ representing the given objects of the same name in $\mathrm{Ho}(\mathcal{C}^{*/})$. Moreover, let f be a fibration in $\mathcal{C}^{*/}$ representing the given morphism of the same name in $\mathrm{Ho}(\mathcal{C}^{*/})$.

Then by def. 4.16 and prop. 4.17 there is a representative $\mathrm{hofib}(f) \in \mathcal{C}$ of the homotopy fiber which fits into a pullback diagram of the form

$$\begin{array}{ccc} \mathrm{hofib}(f) & \overset{i}{\longrightarrow} & X \\ \downarrow && \downarrow^{f} \\ * & \longrightarrow & Y \end{array}$$

With this the hom-sets in question are represented by genuine morphisms in $\mathcal{C}^{*/}$, modulo homotopy. From this it follows immediately that $\mathrm{im}(i_*)$ includes into $\ker(f_*)$. Hence it remains to show the converse: that every element in $\ker(f_*)$ indeed comes from $\mathrm{im}(i_*)$.

But an element in $\ker(f_*)$ is represented by a morphism $\alpha : A \to X$ such that there is a left homotopy as in the following diagram

$$\begin{array}{ccccc} && A & \overset{\alpha}{\longrightarrow} & X \\ && {}^{i_0}\downarrow & {}^{\tilde \eta}\nearrow & \downarrow^{f} \\ A & \overset{i_1}{\longrightarrow} & \mathrm{Cyl}(A) & \overset{\eta}{\longrightarrow} & Y \\ \downarrow &&&& \downarrow^{=} \\ * && \longrightarrow && Y \end{array} \,.$$

Now by lemma 2.20 the square here has a lift $\tilde \eta$, as shown. This means that $i_1 \circ \tilde \eta$ is left homotopic to α. But by the universal property of the fiber, $i_1 \circ \tilde \eta$ factors through $i : \mathrm{hofib}(f) \to X$. ▮

With prop. 4.15 it also follows notably that the loop space construction becomes well-defined on the homotopy category:

Remark 4.23. Given an object $X \in \mathcal{C}^{*/}_f$, and picking any path space object $\mathrm{Path}(X)$, def. 2.18 with induced loop space object ΩX, def. 4.4, write $\mathrm{Path}_2(X) = \mathrm{Path}(X) \underset{X}{\times} \mathrm{Path}(X)$ for the path space object given by the fiber product of $\mathrm{Path}(X)$ with itself, via example 2.21. From the pullback diagram there, the fiber inclusion $\Omega X \to \mathrm{Path}(X)$ induces a morphism

$$\Omega X \times \Omega X \longrightarrow (\Omega X)_2 \,.$$

In the case where $\mathcal{C}^{*/} = \mathrm{Top}^{*/}$ and Ω is induced, via def. 4.4, from the standard path space object (def. 1.34), i.e. in the case that

$$\Omega X = \mathrm{fib}(\mathrm{Maps}(I_+, X)_* \longrightarrow X \times X) \,,$$

then this is the operation of concatenating two loops parameterized by $I = [0,1]$ to a single loop parameterized by $[0,2]$.

Proposition 4.24. *Let $\mathcal{C}$ be a model category, def. 2.3. Then the construction of forming loop space objects $X \mapsto \Omega X$, def. 4.4 (which on $\mathcal{C}^{*/}_f$ depends on a choice of path space objects, def.*

2.18) becomes unique up to isomorphism in the homotopy category (def. 2.25) of the model structure on pointed objects (prop. 3.29) and extends to a functor:

$$\Omega : \mathrm{Ho}(\mathcal{C}^{*/}) \longrightarrow \mathrm{Ho}(\mathcal{C}^{*/}) \,.$$

Dually, the reduced suspension operation, def. 4.4, which on $\mathcal{C}^{*/}$ depends on a choice of cylinder object, becomes a functor on the homotopy category

$$\Sigma : \mathrm{Ho}(\mathcal{C}^{*/}) \longrightarrow \mathrm{Ho}(\mathcal{C}^{*/}) \,.$$

Moreover, the pairing operation induced on the objects in the image of this functor via remark 4.23 (concatenation of loops) gives the objects in the image of Ω group object structure, and makes this functor lift as

$$\Omega : \mathrm{Ho}(\mathcal{C}^{*/}) \longrightarrow \mathrm{Grp}(\mathrm{Ho}(\mathcal{C}^{*/})) \,.$$

(Brown 73, section 4, theorem 3)

Proof. Given an object $X \in \mathcal{C}^{*/}$ and given two choices of path space objects $\mathrm{Path}(X)$ and $\widetilde{\mathrm{Path}(X)}$, we need to produce an isomorphism in $\mathrm{Ho}(\mathcal{C}^{*/})$ between ΩX and $\tilde{\Omega} X$.

To that end, first lemma 4.10 implies that any two choices of path space objects are connected via a third path space by a span of morphisms compatible with the structure maps. By two-out-of-three (def. 2.1) every morphism of path space objects compatible with the inclusion of the base object is a weak equivalence. With this, lemma 4.11 implies that these morphisms induce weak equivalences on the corresponding loop space objects. This shows that all choices of loop space objects become isomorphic in the homotopy category.

Moreover, all the isomorphisms produced this way are actually equal: this follows from lemma 4.13 applied to

$$\begin{array}{ccccc} X & \overset{s}{\longrightarrow} & \mathrm{Path}(X) & \rightrightarrows & \widetilde{\mathrm{Path}(X)} \\ && \downarrow && \downarrow \\ && X \times X & \overset{\mathrm{id}}{\longrightarrow} & X \times X \end{array} \,.$$

This way we obtain a functor

$$\Omega : \mathcal{C}^{*/}_f \longrightarrow \mathrm{Ho}(\mathcal{C}^{*/}) \,.$$

By prop. 4.15 (and using that Cartesian product preserves weak equivalences) this functor sends weak equivalences to isomorphisms. Therefore the functor on homotopy categories now follows with theorem 2.31.

It is immediate to see that the operation of loop concatenation from remark 4.23 gives the objects $\Omega X \in \mathrm{Ho}(\mathcal{C}^{*/})$ the structure of monoids. It is now sufficient to see that these are in fact groups:

We claim that the inverse-assigning operation is given by the left map in the following pasting composite

$$\begin{array}{ccccc} \Omega' X & \longrightarrow & \mathrm{Path}'(X) & \longrightarrow & X \times X \\ \downarrow^{\simeq} && \downarrow^{\simeq} & (\mathrm{pb}) & \downarrow^{\mathrm{swap}} \\ \Omega X & \longrightarrow & \mathrm{Path}(X) & \underset{(p_0,p_1)}{\longrightarrow} & X \times X \end{array} ,$$

(where $\mathrm{Path}'(X)$, thus defined, is the path space object obtained from $\mathrm{Path}(X)$ by “reversing the notion of source and target of a path”).

To see that this is indeed an inverse, it is sufficient to see that the two morphisms

$$\Omega X \rightrightarrows (\Omega X)_2$$

induced from

$$\mathrm{Path}(X) \underset{(s\circ p_0, s\circ p_0)}{\overset{\Delta}{\rightrightarrows}} \mathrm{Path}(X) \times_X \mathrm{Path}'(X)$$

coincide in the homotopy category. This follows with lemma 4.13 applied to the following commuting diagram:

$$\begin{array}{ccccc} X & \overset{i}{\longrightarrow} & \mathrm{Path}(X) & \underset{(s\circ p_0, s\circ p_0)}{\overset{\Delta}{\rightrightarrows}} & \mathrm{Path}(X) \times_X \mathrm{Path}'(X) \\ & & {}^{(p_0,p_1)}\downarrow & & \downarrow \\ & & X \times X & \overset{\Delta\circ \mathrm{pr}_1}{\longrightarrow} & X \times X \end{array} \,.$$

∎

Homotopy pullbacks

The concept of homotopy fibers of def. 4.16 is a special case of the more general concept of homotopy pullbacks.

Definition 4.25. A model category $\mathcal{C}$ (def. 2.3) is called a **right proper model category** if pullback along fibrations preserves weak equivalences.

Example 4.26. By lemma 4.12, a model category $\mathcal{C}$ (def. 2.3) in which all objects are fibrant is a right proper model category (def. 4.25).

Definition 4.27. Let $\mathcal{C}$ be a right proper model category (def. 4.25). Then a commuting square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow^{g} \\ C & \underset{f}{\longrightarrow} & D \end{array}$$

in $\mathcal{C}_f$ is called a **homotopy pullback** (of f along g and equivalently of g along f) if the following equivalent conditions hold:

1. for some factorization of the form

$$g : B \overset{\in W}{\longrightarrow} \hat B \overset{\in \mathrm{Fib}}{\longrightarrow} D$$

the universally induced morphism from A into the pullback of $\hat B$ along f is a weak equivalence:

$$\begin{array}{ccc} A & \longrightarrow & B \\ {}^{\in W}\downarrow & & \downarrow^{\in W} \\ C \underset{D}{\times} \hat B & \longrightarrow & \hat B \\ \downarrow & (pb) & \downarrow^{\in \mathrm{Fib}} \\ C & \longrightarrow & D \end{array} \,.$$

2. for some factorization of the form

$$f : C \overset{\in W}{\longrightarrow} \hat C \overset{\in \mathrm{Fib}}{\longrightarrow} D$$

the universally induced morphism from A into the pullback of $\hat{D}$ along g is a weak equivalence:

$$A \overset{\in W}{\longrightarrow} \hat{C} \underset{D}{\times} B \,.$$

3. the above two conditions hold for every such factorization.

(e.g. Goerss-Jardine 96, II (8.14))

Proposition 4.28. *The conditions in def. 4.27 are indeed equivalent.*

Proof. First assume that the first condition holds, in that

$$\begin{array}{ccc} A & \longrightarrow & B \\ {}^{\mathllap{\in W}}\downarrow && \downarrow^{\mathrlap{\in W}} \\ C \underset{D}{\times} \hat{B} & \longrightarrow & \hat{B} \\ \downarrow & (pb) & \downarrow^{\mathrlap{\in Fib}} \\ C & \longrightarrow & D \end{array} \,.$$

Then let

$$f : C \overset{\in W}{\longrightarrow} \hat{C} \overset{\in Fib}{\longrightarrow} D$$

be any factorization of f and consider the pasting diagram (using the pasting law for pullbacks)

$$\begin{array}{ccccc} A & \longrightarrow & \hat{C} \underset{D}{\times} B & \longrightarrow & B \\ {}^{\mathllap{\in W}}\downarrow && \downarrow^{\mathrlap{\in W}} & (pb) & \downarrow^{\mathrlap{\in W}} \\ C \underset{D}{\times} \hat{B} & \overset{\in W}{\longrightarrow} & \hat{C} \underset{D}{\times} \hat{D} & \overset{\in Fib}{\longrightarrow} & \hat{B} \\ \downarrow & (pb) & \downarrow^{\mathrlap{\in Fib}} & (pb) & \downarrow^{\mathrlap{\in Fib}} \\ C & \underset{\in W}{\longrightarrow} & \hat{C} & \underset{\in Fib}{\longrightarrow} & D \end{array} \,,$$

where the inner morphisms are fibrations and weak equivalences, as shown, by the pullback stability of fibrations (prop. 2.10) and then since pullback along fibrations preserves weak equivalences by assumption of right properness (def. 4.25). Hence it follows by two-out-of-three (def. 2.1) that also the comparison morphism $A \to \hat{C} \underset{D}{\times} B$ is a weak equivalence.

In conclusion, if the homotopy pullback condition is satisfied for one factorization of g, then it is satisfied for all factorizations of f. Since the argument is symmetric in f and g, this proves the claim. ▮

Remark 4.29. In particular, an ordinary pullback square of fibrant objects, one of whose edges is a fibration, is a homotopy pullback square according to def. 4.27.

Proposition 4.30. *Let $\mathcal{C}$ be a right proper model category (def. 4.25). Given a diagram in $\mathcal{C}$ of the form*

$$\begin{array}{ccccc} A & \longrightarrow & B & \overset{\in Fib}{\longleftarrow} & C \\ \downarrow^{\mathrlap{\in W}} && \downarrow^{\mathrlap{\in W}} && \downarrow^{\mathrlap{\in W}} \\ D & \longrightarrow & E & \underset{\in Fib}{\longleftarrow} & F \end{array}$$

then the induced morphism on pullbacks is a weak equivalence

$$A \underset{B}{\times} C \overset{\in W}{\longrightarrow} D \underset{E}{\times} F \,.$$

Proof. (The reader should draw the 3-dimensional cube diagram which we describe in words now.)

First consider the universal morphism $C \to E \underset{F}{\times} C$ and observe that it is a weak equivalence by right properness (def. 4.25) and two-out-of-three (def. 2.1).

Then consider the universal morphism $A \underset{B}{\times} C \to A \underset{B}{\times} (E \underset{F}{\times} C)$ and observe that this is also a weak equivalence, since $A \underset{B}{\times} C$ is the limiting cone of a homotopy pullback square by remark 4.29, and since the morphism is the comparison morphism to the pullback of the factorization constructed in the first step.

Now by using the pasting law, then the commutativity of the “left” face of the cube, then the pasting law again, one finds that $A \underset{B}{\times} (E \underset{F}{\times} C) \simeq A \underset{D}{\times} (D \underset{E}{F} \times)$. Again by right properness this implies that $A \underset{B}{\times} (E \underset{F}{\times} C) \to D \underset{E}{\times} F$ is a weak equivalence.

With this the claim follows by two-out-of-three. ▮

Homotopy pullbacks satisfy the usual abstract properties of pullbacks:

Proposition 4.31. *Let $\mathcal{C}$ be a right proper model category (def. 4.25). If in a commuting square in $\mathcal{C}$ one edge is a weak equivalence, then the square is a homotopy pullback square precisely if the opposite edge is a weak equivalence, too.*

Proof. Consider a commuting square of the form

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow && \downarrow \\ C & \underset{\in W}{\longrightarrow} & D \end{array} \,.$$

To detect whether this is a homotopy pullback, by def. 4.27 and prop. 4.28, we are to choose any factorization of the right vertical morphism to obtain the pasting composite

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow && \downarrow^{\mathrlap{\in W}} \\ C \underset{D}{\times} \hat B & \overset{\in W}{\longrightarrow} & \hat B \\ \downarrow &(pb)& \downarrow^{\mathrlap{\in Fib}} \\ C & \underset{\in W}{\longrightarrow} & D \end{array} \,.$$

Here the morphism in the middle is a weak equivalence by right properness (def. 4.25). Hence it follows by two-out-of-three that the top left comparison morphism is a weak equivalence (and so the original square is a homotopy pullback) precisely if the top morphism is a weak equivalence. ▮

Proposition 4.32. *Let $\mathcal{C}$ be a right proper model category (def. 4.25).*

1. *(pasting law) If in a commuting diagram*

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C \\ \downarrow && \downarrow && \downarrow \\ D & \longrightarrow & E & \longrightarrow & F \end{array}$$

the square on the right is a homotoy pullback (def. 4.27) then the left square is, too, *precisely if the total rectangle is;*

2. *in the presence of functorial factorization (def. 2.6) through weak equivalences followed by fibrations:*

   every retract of a homotopy pullback square (in the category $\mathcal{C}_f^{\square}$ of commuting squares in $\mathcal{C}_f$) is itself a homotopy pullback square.

Proof. For the first statement: choose a factorization of $C \overset{\in W}{\longrightarrow} \hat F \overset{\in \mathrm{Fib}}{\longrightarrow} F$, pull it back to a factorization $B \to \hat B \overset{\in \mathrm{Fib}}{\longrightarrow} E$ and assume that $B \to \hat B$ is a weak equivalence, i.e. that the right square is a homotopy pullback. Now use the ordinary pasting law to conclude.

For the second statement: functorially choose a factorization of the two right vertical morphisms of the squares and factor the squares through the pullbacks of the corresponding fibrations along the bottom morphisms, respectively. Now the statement that the squares are homotopy pullbacks is equivalent to their top left vertical morphisms being weak equivalences. Factor these top left morphisms functorially as cofibrations followed by acyclic fibrations. Then the statement that the squares are homotopy pullbacks is equivalent to those top left cofibrations being acyclic. Now the claim follows using that the retract of an acyclic cofibration is an acyclic cofibration (prop. 2.10). ▮

Long sequences

The ordinary fiber, example 3.27, of a morphism has the property that taking it *twice* is always trivial:

$$* \simeq \mathrm{fib}(\mathrm{fib}(f)) \longrightarrow \mathrm{fib}(f) \longrightarrow X \overset{f}{\longrightarrow} Y \,.$$

This is crucially different for the homotopy fiber, def. 4.16. Here we discuss how this comes about and what the consequences are.

Proposition 4.33. *Let $\mathcal{C}_f$ be a category of fibrant objects of a model category, def. 2.34 and let $f : X \longrightarrow Y$ be a morphism in its category of pointed objects, def. 3.16. Then the homotopy fiber of its homotopy fiber, def. 4.16, is isomorphic, in $\mathrm{Ho}(\mathcal{C}^{*/})$, to the loop space object ΩY of Y (def. 4.4, prop. 4.24):*

$$\mathrm{hofib}(\mathrm{hofib}(X \overset{f}{\to} Y)) \simeq \Omega Y \,.$$

Proof. Assume without restriction that $f : X \longrightarrow Y$ is already a fibration between fibrant objects in $\mathcal{C}$ (otherwise replace and rename). Then its homotopy fiber is its ordinary fiber, sitting in a pullback square

$$\begin{array}{ccccc} \mathrm{hofib}(f) \simeq & F & \overset{i}{\longrightarrow} & X \\ & \downarrow && \downarrow^{f} \\ & * & \longrightarrow & Y \end{array} .$$

In order to compute $\mathrm{hofib}(\mathrm{hofib}(f))$, i.e. $\mathrm{hofib}(i)$, we need to replace the fiber inclusion i by a fibration. Using the factorization lemma 4.9 for this purpose yields, after a choice of path space object $\mathrm{Path}(X)$ (def. 2.18), a replacement of the form

$$\begin{array}{ccc} F & \overset{\in W}{\longrightarrow} & F \times_X \mathrm{Path}(X) \\ & {}_{i}\searrow & \downarrow^{\tilde i}_{\in \mathrm{Fib}} \\ && X \end{array} .$$

Hence $\mathrm{hofib}(i)$ is the ordinary fiber of this map:

$$\mathrm{hofib}(\mathrm{hofib}(f)) \simeq F \times_X \mathrm{Path}(X) \times_X * \qquad \in \mathrm{Ho}(\mathcal{C}^{*/}) \,.$$

Notice that

$$F \times_X \mathrm{Path}(X) \;\simeq\; * \times_Y \mathrm{Path}(X)$$

because of the pasting law:

$$\begin{array}{ccc} F \times_X \mathrm{Path}(X) & \longrightarrow & \mathrm{Path}(X) \\ \downarrow & (\mathrm{pb}) & \downarrow \\ F & \overset{i}{\longrightarrow} & X \\ \downarrow & (\mathrm{pb}) & \downarrow^{f} \\ * & \longrightarrow & Y \end{array} \,.$$

Hence

$$\mathrm{hofib}(\mathrm{hofib}(f)) \;\simeq\; * \times_Y \mathrm{Path}(X) \times_X * \,.$$

Now we claim that there is a choice of path space objects $\mathrm{Path}(X)$ and $\mathrm{Path}(Y)$ such that this model for the homotopy fiber (as an object in $\mathcal{C}^{*/}$) sits in a pullback diagram of the following form:

$$\begin{array}{ccc} * \times_Y \mathrm{Path}(X) \times_X * & \longrightarrow & \mathrm{Path}(X) \\ \downarrow & & \downarrow \in W \cap F \\ \Omega Y & \longrightarrow & \mathrm{Path}(Y) \times_Y X \\ \downarrow & (\mathrm{pb}) & \downarrow \\ * & \longrightarrow & Y \times X \end{array} \,.$$

By the pasting law and the pullback stability of acyclic fibrations, this will prove the claim.

To see that the bottom square here is indeed a pullback, check the universal property: A morphism out of any A into $* \underset{Y \times X}{\times} \mathrm{Path}(Y) \times_Y X$ is a morphism $a : A \to \mathrm{Path}(Y)$ and a morphism $b : A \to X$ such that $p_0(a) = *$, $p_1(a) = f(b)$ and $b = *$. Hence it is equivalently just a morphism $a : A \to \mathrm{Path}(Y)$ such that $p_0(a) = *$ and $p_1(a) = *$. This is the defining universal property of $\Omega Y \coloneqq * \underset{Y}{\times} \mathrm{Path}(Y) \underset{Y}{\times} *$.

Now to construct the right vertical morphism in the top square (Quillen 67, page 3.1): Let $\mathrm{Path}(Y)$ be any path space object for Y and let $\mathrm{Path}(X)$ be given by a factorization

$$(\mathrm{id}_X,\; i \circ f,\; \mathrm{id}_X) \,:\, X \overset{\in W}{\longrightarrow} \mathrm{Path}(X) \overset{\in \mathrm{Fib}}{\longrightarrow} X \times_Y \mathrm{Path}(Y) \times_Y X$$

and regarded as a path space object of X by further comoposing with

$$(\mathrm{pr}_1, \mathrm{pr}_3) : X \times_Y \mathrm{Path}(Y) \times_Y X \overset{\in \mathrm{Fib}}{\longrightarrow} X \times X \,.$$

We need to show that $\mathrm{Path}(X) \to \mathrm{Path}(Y) \times_Y X$ is an acyclic fibration.

It is a fibration because $X \times_Y \mathrm{Path}(Y) \times_Y X \to \mathrm{Path}(Y) \times_Y X$ is a fibration, this being the pullback of the fibration $X \overset{f}{\to} Y$.

To see that it is also a weak equivalence, first observe that $\mathrm{Path}(Y) \times_Y X \overset{\in W \cap \mathrm{Fib}}{\longrightarrow} X$, this being the pullback of the acyclic fibration of lemma 2.20. Hence we have a factorization of the identity as

$$\mathrm{id}_X \,:\, X \underset{\in W}{\overset{i}{\longrightarrow}} \mathrm{Path}(X) \longrightarrow \mathrm{Path}(Y) \times_Y X \underset{\in W \cap \mathrm{Fib}}{\longrightarrow} X$$

and so finally the claim follows by two-out-of-three (def. 2.1). ▮

Remark 4.34. There is a conceptual way to understand prop. 4.33 as follows: If we draw double arrows to indicate homotopies, then a homotopy fiber (def. 4.16) is depicted by the following filled square:

$$\begin{array}{ccc} \mathrm{hofib}(f) & \longrightarrow & * \\ \downarrow & \swArrow & \downarrow \\ X & \underset{f}{\longrightarrow} & Y \end{array}$$

just like the ordinary fiber (example 3.27) is given by a plain square

$$\begin{array}{ccc} \mathrm{fib}(f) & \longrightarrow & * \\ \downarrow & & \downarrow \\ X & \underset{f}{\longrightarrow} & Y \end{array} \,.$$

One may show that just like the fiber is the *universal* solution to making such a commuting square (a pullback limit cone def. 1.1), so the homotopy fiber is the universal solution up to homotopy to make such a commuting square up to homotopy – a homotopy pullback homotopy limit cone.

Now just like ordinary pullbacks satisfy the pasting law saying that attaching two pullback squares gives a pullback rectangle, the analogue is true for homotopy pullbacks. This implies that if we take the homotopy fiber of a homotopy fiber, thereby producing this double homotopy pullback square

$$\begin{array}{ccccc} \mathrm{hofib}(g) & \longrightarrow & \mathrm{hofib}(f) & \longrightarrow & * \\ \downarrow & \swArrow & \downarrow^{g} & \swArrow & \downarrow \\ * & \longrightarrow & X & \underset{f}{\longrightarrow} & Y \end{array}$$

then the total outer rectangle here is itself a homotopy pullback. But the outer rectangle exhibits the homotopy fiber of the point inclusion, which, via def. 4.4 and lemma 4.9, is the loop space object:

$$\begin{array}{ccc} \Omega Y & \longrightarrow & * \\ \downarrow & \swArrow & \downarrow \\ * & \longrightarrow & Y \end{array} \,.$$

Proposition 4.35. *Let $\mathcal{C}$ be a model category and let $f : X \to Y$ be morphism in the pointed homotopy category $\mathrm{Ho}(\mathcal{C}^{*/})$ (prop. 3.29). Then:*

1. *There is a long sequence to the left in $\mathcal{C}^{*/}$ of the form*

$$\cdots \longrightarrow \Omega X \overset{\overline{\Omega} f}{\longrightarrow} \Omega Y \longrightarrow \mathrm{hofib}(f) \longrightarrow X \overset{f}{\longrightarrow} Y \,,$$

where each morphism is the homotopy fiber (def. 4.16) of the following one: the ***homotopy fiber sequence*** *of f. Here $\overline{\Omega} f$ denotes Ωf followed by forming inverses with respect to the group structure on $\Omega(-)$ from prop. 4.24.*

Moreover, for $A \in \mathcal{C}^{*/}$ any object, then there is a long exact sequence

$$\cdots \to [A, \Omega^2 Y]_* \longrightarrow [A, \Omega \,\mathrm{hofib}(f)]_* \longrightarrow [A, \Omega X]_* \longrightarrow [A, \Omega Y] \longrightarrow [A, \mathrm{hofib}(f)]_* \longrightarrow [A, X]_* \longrightarrow [A, Y]_*$$

of pointed sets, where $[-,-]_*$ denotes the pointed set valued hom-functor of example 3.30.

1. *Dually, there is a long sequence to the right in $\mathcal{C}^{*/}$ of the form*

$$X \overset{f}{\longrightarrow} Y \longrightarrow \mathrm{hocofib}(f) \longrightarrow \Sigma X \overset{\overline{\Sigma} f}{\longrightarrow} \Sigma Y \to \cdots ,$$

where each morphism is the homotopy cofiber (def. 4.16) of the previous one: the ***homotopy cofiber sequence*** *of f. Moreover, for $A \in \mathcal{C}^{*/}$ any object, then there is a long exact sequence*

$$\cdots \to [\Sigma^2 X, A]_* \longrightarrow [\Sigma\, \mathrm{hocofib}(f), A]_* \longrightarrow [\Sigma Y, A]_* \longrightarrow [\Sigma X, A] \longrightarrow [\mathrm{hocofib}(f), A]_* \longrightarrow [Y, A]_* \longrightarrow [X, A]_*$$

of pointed sets, where $[-,-]_*$ denotes the pointed set valued hom-functor of example 3.30.

(Quillen 67, I.3, prop. 4)

Proof. That there are long sequences of this form is the result of combining prop. 4.33 and prop. 4.22.

It only remains to see that it is indeed the morphisms $\overline{\Omega} f$ that appear, as indicated.

In order to see this, it is convenient to adopt the following notation: for $f : X \to Y$ a morphism, then we denote the collection of generalized element of its homotopy fiber as

$$\mathrm{hofib}(f) = \left\{ (x, f(x) \overset{\gamma_1}{\rightsquigarrow} *) \right\}$$

indicating that these elements are pairs consisting of an element x of X and a "path" (an element of the given path space object) from $f(x)$ to the basepoint.

This way the canonical map $\mathrm{hofib}(f) \to X$ is $(x, f(x) \rightsquigarrow *) \mapsto x$. Hence in this notation the homotopy fiber of the homotopy fiber reads

$$\mathrm{hofib}(\mathrm{hofib}(f)) = \left\{ ((x, f(x) \overset{\gamma_1}{\rightsquigarrow} *), x \overset{\gamma_2}{\rightsquigarrow} *) \right\} .$$

This identifies with ΩY by forming the loops

$$\gamma_1 \cdot f(\overline{\gamma_2}) \,,$$

where the overline denotes reversal and the dot denotes concatenation.

Then consider the next homotopy fiber

$$\mathrm{hofib}(\mathrm{hofib}(\mathrm{hofib}(f))) = \left\{ \left(((x, f(x) \overset{\gamma_1}{\rightsquigarrow} *), x \overset{\gamma_2}{\rightsquigarrow} *), \left(\begin{array}{ccc} x & \overset{\gamma_3}{\rightsquigarrow} & * \\ f(x) & \overset{f(\gamma_3)}{\rightsquigarrow} & * \\ {}^{\gamma_1}\searrow & \Rightarrow & \swarrow \\ & * & \end{array} \right) \right) \right\} ,$$

where on the right we have a path in $\mathrm{hofib}(f)$ from $(x, f(x) \overset{\gamma_1}{\rightsquigarrow} *)$ to the basepoint element. This is a path γ_3 together with a path-of-paths which connects f_1 to $f(\gamma_3)$.

By the above convention this is identified with the loop in X which is

$$\gamma_2 \cdot (\overline{\gamma}_3) \,.$$

But the map to $\mathrm{hofib}(\mathrm{hofib}(f))$ sends this data to $((x, f(x) \overset{\gamma_1}{\rightsquigarrow} *), x \overset{\gamma_2}{\rightsquigarrow} *)$, hence to the loop

$$\begin{aligned} \gamma_1 \cdot f(\overline{\gamma_2}) &\simeq f(\gamma_3) \cdot f(\overline{\gamma_2}) \\ &= f(\gamma_3 \cdot \overline{\gamma_2}) \\ &= f(\overline{\gamma_2 \cdot \overline{\gamma}_3}) \\ &= \overline{f(\gamma_2 \cdot \overline{\gamma}_3)} \end{aligned} ,$$

hence to the reveral of the image under f of the loop in X. ▮

Remark 4.36. In (Quillen 67, I.3, prop. 3, prop. 4) more is shown than stated in prop. 4.35: there the connecting homomorphism $\Omega Y \to \mathrm{hofib}(f)$ is not just shown to exist, but is described in detail via an action of ΩY on $\mathrm{hofib}(f)$ in $\mathrm{Ho}(\mathcal{C})$. This takes a good bit more work. For our purposes here, however, it is sufficient to know that such a morphism exists at all, hence that $\Omega Y \simeq \mathrm{hofib}(\mathrm{hofib}(f))$.

Example 4.37. Let $\mathcal{C} = (\mathrm{Top}_{\mathrm{cg}})_{\mathrm{Quillen}}$ be the classical model structure on topological spaces (compactly generated) from theorem 3.7, theorem 3.51. Then using the standard pointed topological path space objects $\mathrm{Maps}(I_+, X)$ from def. 1.34 and example 3.26 as the abstract path space objects in def. 2.18, via prop. 3.14, this gives that

$$[*, \Omega^n X] \simeq \pi_n(X)$$

is the nth homotopy group, def. 1.26, of X at its basepoint.

Hence using $A = *$ in the first item of prop. 4.35, the long exact sequence this gives is of the form

$$\cdots \to \pi_3(X) \overset{f_*}{\longrightarrow} \pi_3(Y) \longrightarrow \pi_2(\mathrm{hofib}(f)) \longrightarrow \pi_2(X) \overset{-f_*}{\longrightarrow} \pi_2(Y) \longrightarrow \pi_1(\mathrm{hofib}(f)) \longrightarrow \pi_1(X) \overset{f_*}{\longrightarrow} \pi_1(Y) \longrightarrow * \,.$$

This is called the **long exact sequence of homotopy groups** induced by f.

Remark 4.38. As we pass to stable homotopy theory (in Part 1)), the long exact sequences in example 4.37 become long not just to the left, but also to the right. Given then a tower of fibrations, there is an induced sequence of such long exact sequences of homotopy groups, which organizes into an *exact couple*. For more on this see at *Interlude -- Spectral sequences* (this remark).

Example 4.39. Let again $\mathcal{C} = (\mathrm{Top}_{\mathrm{cg}})_{\mathrm{Quillen}}$ be the classical model structure on topological spaces (compactly generated) from theorem 3.7, theorem 3.51, as in example 4.37. For $E \in \mathrm{Top}_{\mathrm{cg}}^{*/}$ any pointed topological space and $i : A \hookrightarrow X$ an inclusion of pointed topological spaces, the exactness of the sequence in the second item of prop. 4.35

$$\cdots \to [\mathrm{hocofib}(i), E] \longrightarrow [X, E]_* \longrightarrow [A, E]_* \to \cdots$$

gives that the functor

$$[-, E]_* \; : \; (\mathrm{Top}_{\mathrm{CW}}^{*/})^{\mathrm{op}} \longrightarrow \mathrm{Set}^{*/}$$

behaves like one degree in an additive reduced cohomology theory (def.). The Brown representability theorem (thm.) implies that all additive reduced cohomology theories are degreewise representable this way (prop.).

5. The suspension/looping adjunction

We conclude this discussion of classical homotopy theory with the key statement that leads over to stable homotopy theory in *Introduction to Stable homotopy theory -- 1*: the suspension and looping adjunction on the classical pointed homotopy category.

Proposition 5.1. *The canonical loop space functor Ω and reduced suspension functor Σ from* *prop. 4.24 on the classical pointed homotopy category from def. 3.31 are adjoint functors, with Σ left adjoint and Ω right adjoint:*

$$(\Sigma \dashv \Omega) \;:\; Ho(Top^{*/}) \underoverset{\underset{\Omega}{\longrightarrow}}{\overset{\Sigma}{\longleftarrow}}{} Ho(Top^{*/}) \;.$$

Moreover, this is equivalently the adjoint pair of derived functors, according to prop. 2.49, of the Quillen adjunction

$$(Top^{*/}_{cg})_{Quillen} \underoverset{\underset{Maps(S^1,-)_*}{\longrightarrow}}{\overset{S^1\wedge(-)}{\longleftarrow}}{\bot} (Top^{*/}_{cg})_{Quillen}$$

of cor. 3.42.

Proof. By prop. 4.24 we may represent Σ and Ω by any choice of cylinder objects and path space objects (def. 2.18).

The standard topological path space $(-)^I$ is generally a path space object by prop. 3.14. With prop. 4.5 this shows that

$$\Omega \simeq \mathbb{R}\, Maps(S^1, -)_* \;.$$

Moreover, by the existence of CW-approximations (remark 3.12) we may represent each object in the homotopy category by a CW-complex. On such, the standard topological cylinder $(-) \times I$ is a cylinder object by prop. 3.13. With prop. 4.5 this shows that

$$\Sigma \simeq \mathbb{L}(S^1 \wedge (-)) \;.$$

▮

Final remark 5.2. What is called *stable homotopy theory* is the result of universally forcing the $(\Sigma \dashv \Omega)$-adjunction of prop. 5.1 to become an equivalence of categories.

This is the topic of the next section at *Introduction to Stable homotopy theory -- 1*.

6. References

A concise and yet self-contained re-write of the proof (Quillen 67) of the classical model structure on topological spaces is provided in

- Philip Hirschhorn, *The Quillen model category of topological spaces* (arXiv:1508.01942).

For general model category theory a decent review is in

- William Dwyer, J. Spalinski, *Homotopy theories and model categories* (pdf) in Ioan Mackenzie James (ed.), *Handbook of Algebraic Topology* 1995

The equivalent definition of model categories that we use here is due to

- André Joyal, appendix E of *The theory of quasi-categories and its applications* (pdf)

The two originals are still a good source to turn to:

- Daniel Quillen, *Axiomatic homotopy theory* in *Homotopical algebra*, Lecture Notes in Mathematics, No. 43 43, Berlin (1967)

- Kenneth Brown, *Abstract Homotopy Theory and Generalized Sheaf Cohomology*, Transactions of the American Mathematical Society, Vol. 186 (1973), 419-458 (JSTOR)

For the restriction to the convenient category of compactly generated topological spaces good sources are

- Gaunce Lewis, *Compactly generated spaces* (pdf), appendix A of *The Stable Category and Generalized Thom Spectra* PhD thesis Chicago, 1978

- Neil Strickland, *The category of CGWH spaces*, 2009 (pdf)

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nLab

Introduction to Stable homotopy theory -- 1-1

We give an introduction to the stable homotopy category and to its key computational tool, the Adams spectral sequence. To that end we introduce the modern tools, such as model categories and highly structured ring spectra. In the accompanying seminar we consider applications to cobordism theory and complex oriented cohomology such as to converge in the end to a glimpse of the modern picture of chromatic homotopy theory._

Lecture notes.

Main page: *Introduction to Stable homotopy theory*.

Previous section: *Prelude -- Classical homotopy theory*

This section_ *Part 1 -- Stable homotopy theory*

This subsection: **Part 1.1 – Stable homotopy theory – Sequential spectra**

Next subsection: *Part 1.2 -- Stable homotopy theory -- Structured Spectra*

Next section: *Part 2 -- Adams spectral sequences*

Stable homotopy theory – Sequential spectra

The *Prelude on Classical homotopy theory* ended with the following phenomenon:

Definition 0.1. The reduced suspension/looping operation on pointed (def.) compactly generated topological spaces (def.) is the smash-tensor/hom-adjunction (cor.) for the standard 1-sphere smash product from the left:

$$(\Sigma \dashv \Omega) \;:\; \mathrm{Top}_{\mathrm{cg}}^{*/} \underoverset{\underset{\mathrm{Maps}(S^1,-)_*}{\longrightarrow}}{\overset{S^1 \wedge (-)}{\longleftarrow}}{\bot} \mathrm{Top}_{\mathrm{cg}}^{*/} \,.$$

Proposition 0.2. *With respect to the classical model structure on pointed compactly generated topological spaces $(\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}$ (thm., prop.)*

1. *the adjunction in def. 0.1 is a Quillen adjunction (def.)*

$$(\Sigma \dashv \Omega) \;:\; (\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}} \underoverset{\underset{\mathrm{Maps}(S^1,-)_*}{\longrightarrow}}{\overset{S^1 \wedge (-)}{\longleftarrow}}{\bot} (\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}} \,,$$

2. *its induced adjoint pair of derived functors on the classical pointed homotopy category (by this prop.) is the canonical suspension/looping adjunction (according to this prop.)*

$$(\Sigma \dashv \Omega) \;:\; \mathrm{Ho}(\mathrm{Top}^{*/}) \underoverset{\underset{\Omega}{\longrightarrow}}{\overset{\Sigma}{\longleftarrow}}{\bot} \mathrm{Ho}(\mathrm{Top}^{*/}) \,.$$

See (this prop.).

The *stable homotopy category* $\mathrm{Ho}(\mathrm{Spectra})$ is to be the result of stabilizing the adjunction in prop. 0.2, in the sense of forcing it to become an equivalence of categories in a compatible way, i.e. such as to fit into a diagram of categories of the form

$$\begin{array}{ccc} \mathrm{Ho}(\mathrm{Top}^{*/}) & \underoverset{\underset{\Omega}{\longrightarrow}}{\overset{\Sigma}{\longleftarrow}}{\bot} & \mathrm{Ho}(\mathrm{Top}^{*/}) \\ {}^{\mathllap{\Sigma^\infty}}\downarrow \dashv \uparrow^{\mathrlap{\Omega^\infty}} && {}^{\mathllap{\Sigma^\infty}}\downarrow \dashv \uparrow^{\mathrlap{\Omega^\infty}} \\ \mathrm{Ho}(\mathrm{Spectra}) & \underoverset{\underset{\Omega}{\longrightarrow}}{\overset{\Sigma}{\longleftarrow}}{\simeq} & \mathrm{Ho}(\mathrm{Spectra}) \end{array} \,.$$

Moreover, for stable homotopy theory proper we are to refine this situation from homotopy categories to model categories and ask it to be the diagram of derived functors (according to this prop.) of a diagram of Quillen adjunctions (def.)

$$\begin{array}{ccc} (\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}} & \underoverset{\underset{\Omega}{\longrightarrow}}{\overset{\Sigma}{\longleftarrow}}{\bot} & (\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}} \\ {}^{\mathllap{\Sigma^\infty}}\downarrow \dashv \uparrow^{\mathrlap{\Omega^\infty}} && {}^{\mathllap{\Sigma^\infty}}\downarrow \dashv \uparrow^{\mathrlap{\Omega^\infty}} \\ \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}} & \underoverset{\underset{\Omega}{\longrightarrow}}{\overset{\Sigma}{\longleftarrow}}{\simeq_Q} & \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}} \end{array} \,,$$

This we establish in theorem 3.25 below.

The notation Σ^∞ and Ω^∞ is meant to be suggestive of the intuition behind how this stabilization will work: The universal way of making a topological space X become stable under suspension is to pass to its infinite suspension in a suitable sense. That suitable sense is going to be called the *suspension spectrum* of X (def. 1.3 below). Conversely, if an object does not change up to equivalence, by forming its loop spaces, it must give an infinite loop space.

In contrast to the classical homotopy category, the stable homotopy category is a triangulated category (a shadow of the fact that the (∞,1)-category of spectra is a stable (∞,1)-category). As such it may be thought of as a refinement of the derived category of chain complexes (of abelian groups): every chain complex gives rise to a spectrum and every chain map to a map between these spectra (the stable Dold-Kan correspondence), but there are many more spectra and maps between them than arise from chain complexes and chain maps.

There is a variety of different models for the stable homotopy theory of spectra, some of which fits into this hierarchy:

1. sequential spectra with their model structure on sequential spectra

2. symmetric spectra with their model structure on symmetric spectra

3. orthogonal spectra with their model structure on orthogonal spectra

4. excisive functors with their model structure for excisive functors

As one moves down this list, the objects modelling the spectra become richer. This means on the one hand that their abstract properties become better as one moves down the list, on the other hand it means that it is more immediate to construct and manipulate examples as one stays further up in the list.

We start with plain sequential spectra as a transparent means to construct the stable homotopy category. In order to discuss ring spectra it is convenient to first pass to the richer model of highly structured spectra, this we do in *Part II*

The most lighweight model for spectra are *sequential spectra*. They support most of stable homotopy theory in a straightforward way, and have the advantage that examples tend to be immediate (for instance the proof of the Brown representability theorem spits out sequential spectra).

The key disadvantage of sequential spectra is that they do not support a functorial smash product of spectra before passing to the stable homotopy category, much less a symmetric smash product of spectra. This is the structure needed for a decent discussion of the higher algebra of ring spectra. To accomodate this, further below we enhance sequential spectra to the more highly structured models given by symmetric spectra and orthogonal spectra. But all these models are connected by a free-forgetful adjunction and for working with either it is useful to have the means to pass back and forth between them.

1. Sequential pre-spectra

The following def. 1.1 is the traditional component-wise definition of sequential spectra. It was first stated in (Lima 58) and became widely appreciated with (Boardman 65).

> *It is generally supposed that G. W. Whitehead also had something to do with it, but the latter takes a modest attitude about that. (Adams 74, p. 131)*

Below in prop. 1.23 we discuss an equivalent definition of sequential spectra as "topological diagram spectra" (Mandell-May-Schwede-Shipley 00), namely as topologically enriched functors (defn.) on a topologically enriched category of n-spheres, which is useful for establishing the stable model category structure (below) and for establishing the symmetric monoidal smash product of spectra (in 1.2).

Throughout, our ambient category of topological spaces is $\mathrm{Top}_{\mathrm{cg}}$, the category of compactly generated topological space (defn.).

Definition 1.1. A **sequential prespectrum in topological spaces**, or just **sequential spectrum** for short (or even just **spectrum), is**

1. an $\mathbb{N}$-graded pointed compactly generated topological space

$$X_\bullet = (X_n \in \mathrm{Top}_{\mathrm{cg}}^{*/})_{n \in \mathbb{N}}$$

(the **component spaces**);

2. pointed continuous functions

$$\sigma_n : S^1 \wedge X_n \to X_{n+1}$$

for all $n \in \mathbb{N}$ (the **structure maps**) from the smash product (defn.) of one component space with the standard 1-sphere to the next component space.

A homomorphism $f : X \to Y$ of sequential spectra is a sequence $f_\bullet : X_\bullet \to Y_\bullet$ of base point-preserving continuous functions between component spaces, such that these respect the structure maps in that all diagrams of the form

$$\begin{array}{ccc} S^1 \wedge X_n & \overset{S^1 \wedge f_n}{\longrightarrow} & S^1 \wedge Y_n \\ \downarrow^{\sigma_n^X} && \downarrow^{\sigma_n^Y} \\ X_{n+1} & \underset{f_{n+1}}{\longrightarrow} & Y_{n+1} \end{array}$$

commute.

Write $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$ for this category of topological sequential spectra.

Due to the classical adjunction

$$\mathrm{Top}_{\mathrm{cg}}^{*/} \underoverset{\underset{\mathrm{Maps}(S^1,-)_*}{\longrightarrow}}{\overset{S^1 \wedge (-)}{\longleftarrow}}{\bot} \mathrm{Top}_{\mathrm{cg}}^{*/}$$

from classical homotopy theory (this prop.), the definition of sequential spectra in def. 1.1 is equivalent to the following definition

Definition 1.2. A **sequential prespectrum in topological spaces**, or just **sequential spectrum** for short (or even just **spectrum), is**

1. an $\mathbb{N}$-graded pointed compactly generated topological space

$$X_\bullet = (X_n \in \mathrm{Top}_{\mathrm{cg}}^{*/})_{n \in \mathbb{N}}$$

(the **component spaces**);

2. pointed continuous functions

$$\tilde{\sigma}_n : X_n \to \mathrm{Maps}(S^1, X_{n+1})_*$$

for all $n \in \mathbb{N}$ (the **adjunct structure maps**) from one component space to the pointed mapping space (def., exmpl.) out of S^1 into the next component space.

A homomorphism $f : X \to Y$ of sequential spectra is a sequence $\widetilde{f_\bullet} : X_\bullet \to Y_\bullet$ of base point-preserving continuous function, such that all diagrams of the form

$$\begin{array}{ccc} X_n & \overset{f_n}{\longrightarrow} & Y_n \\ {}^{\mathllap{\tilde \sigma_n^X}}\downarrow && \downarrow^{\mathrlap{\tilde \sigma_n^Y}} \\ \mathrm{Maps}(S^1, X_{n+1})_* & \underset{\mathrm{Maps}(S^1, f_{n+1})_*}{\longrightarrow} & \mathrm{Maps}(S^1, Y_{n+1})_* \end{array}$$

commute.

Example 1.3. For $X \in \mathrm{Top}^{*/\mathrm{cg}}$ a pointed topological space, its **suspension spectrum** $\Sigma^\infty X$ is the sequential spectrum , def. 1.1, with

- $(\Sigma^\infty X)_n \coloneqq S^n \wedge X$ (smash product of X with the n-sphere);

- $\sigma_n : S^1 \wedge S^n \wedge X \overset{\simeq}{\longrightarrow} S^{n+1} X$ (the canonical homeomorphism).

This construction extends to a functor

$$\Sigma^\infty \;:\; \mathrm{Top}^{*/}_{\mathrm{cg}} \longrightarrow \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}}) \,.$$

Example 1.4. The suspension spectrum (example 1.3) of the point is the **standard sequential sphere spectrum**

$$\mathbb{S}_{\mathrm{seq}} \coloneqq \Sigma^\infty S^0 \,.$$

Its nth component space is the standard n-sphere

$$(\mathbb{S}_{\mathrm{seq}})_n = S^n \,.$$

Example 1.5. A fundamental example of a spectrum that is not just a suspension spectrum is the universal real Thom spectrum, denoted MO. For details on this see *Part S – Thom spectra*.

There are are also the universal complex Thom spectrum denoted MU, and the universal symplectic Thom spectrum denoted MSp. Their standard construction first yields an example of a “sequential S^2-spectrum”; which we introduce below in def. 3.17; and then there is an adjunction (prop. 3.19) that canonically turns this into an ordinary sequential spectrum.

Definition 1.6. Let $X \in \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$ be a sequential spectrum (def. 1.1) and $K \in \mathrm{Top}^{*/}_{\mathrm{cg}}$ a pointed compactly generated topological space. Then

1. $X \wedge K$ (the **smash tensoring** of X with K) is the sequential spectrum given by

   - $(X \wedge K)_n \coloneqq X_n \wedge K$ (smash product on component spaces (defn.))

   - $\sigma_n^{X \wedge K} \coloneqq \sigma_n^X \wedge \mathrm{id}_K$.

2. $\mathrm{Maps}(K, X)_*$ (the **powering** of K into X) is the sequential spectrum with

   - $(\mathrm{Maps}(K, X)_*)_n \;\coloneqq\; \mathrm{Maps}(K, X_n)_*$ (compactly generated pointed mapping space (def., def.))

$$\circ\ \sigma_n^{\mathrm{Maps}(K,X)_*} : S^1 \wedge \mathrm{Maps}(K, X_n) \xrightarrow{(\mathrm{const},\mathrm{id})} \mathrm{Maps}(K, S^1 \wedge X_n)_* \xrightarrow{\mathrm{Maps}(K,\sigma_n)_*} \mathrm{Maps}(K, X_{n+1})_*,$$

where $(\mathrm{const}, \mathrm{id}) : [s, \phi] \mapsto [\mathrm{const}_s, \phi]$.

These operations canonically extend to functors

$$(-) \wedge (-) : \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}}) \times \mathrm{Top}_{\mathrm{cg}}^{*/} \longrightarrow \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$$

and

$$\mathrm{Maps}(-,-)_* : (\mathrm{Top}_{\mathrm{cg}}^{*/})^{\mathrm{op}} \times \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}}) \longrightarrow \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}}) \;.$$

Example 1.7. The tensoring (def. 1.6) of the standard sphere spectrum $\mathbb{S}_{\mathrm{std}}$ (def. 1.4) with a space $X \in \mathrm{Top}_{\mathrm{cg}}$ is isomorphic to the suspension spectrum of X (def. 1.3):

$$\mathbb{S}_{\mathrm{std}} \wedge X \simeq \Sigma^\infty X \;.$$

Proposition 1.8. *For any $K \in \mathrm{Top}_{\mathrm{cg}}^{*/}$ the functors of smash tensoring and powering with K, from def. 1.6, constitute a pair of adjoint functors*

$$\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}}) \underoverset{\underset{\mathrm{Maps}(K,-)_*}{\longrightarrow}}{\overset{(-)\wedge K}{\longleftarrow}}{\bot} \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}}) \;.$$

Proof. For $X, Y \in \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$ and $K \in \mathrm{Top}_{\mathrm{cg}}^{*/}$, let

$$X \wedge K \overset{f}{\longrightarrow} Y$$

be a morphism, with component maps fitting into commuting squares of the form

$$\begin{array}{ccc} S^1 \wedge X_n \wedge K & \overset{S^1 \wedge f_n}{\longrightarrow} & S^1 \wedge Y_n \\ {}^{\mathllap{\sigma_n^X \wedge K}}\downarrow && \downarrow^{\mathrlap{\sigma_n^Y}} \\ X_{n+1} \wedge K & \overset{f_{n+1}}{\longrightarrow} & Y_{n+1} \end{array} \;.$$

Applying degreewise the adjunction

$$\mathrm{Top}_{\mathrm{cg}}^{*/} \underoverset{\underset{\mathrm{Maps}(K,-)_*}{\longrightarrow}}{\overset{(-)\wedge K}{\longleftarrow}}{\bot} \mathrm{Top}_{\mathrm{cg}}^{*/}$$

from classical homotopy theory (this prop.) gives that these squares are in natural bijection with squares of the form

$$\begin{array}{ccc} S^1 \wedge X_n & \overset{\widetilde{S^1 \wedge f_n}}{\longrightarrow} & \mathrm{Maps}(K, S^1 \wedge Y_n)_* \\ {}^{\mathllap{\sigma_n^X}}\downarrow && \downarrow^{\mathrlap{\mathrm{Maps}(K,\sigma_n^Y)_*}} \\ X_{n+1} & \overset{\widetilde{f_{n+1}}}{\longrightarrow} & \mathrm{Maps}(K, Y_{n+1})_* \end{array}$$

But since the map $S^1 \wedge f_n$ is the smash product of two maps, only one of which involves the smash factor of K, one sees that here the top map factors through the map $(\mathrm{const}, \mathrm{id})$ from def. 1.6.

Hence the commuting square above factors as

$$
\begin{array}{ccc}
S^1 \wedge X_n & \overset{S^1 \wedge \widetilde{f_n}}{\longrightarrow} & S^1 \wedge \mathrm{Maps}(K, Y_n)_* \\
{}^{\mathllap{\sigma_n^X}}\downarrow && \downarrow^{\mathrlap{\sigma_n^{\mathrm{Maps}(K,Y)_*}}} \\
X_{n+1} & \overset{\widetilde{f_{n+1}}}{\longrightarrow} & \mathrm{Maps}(K, Y_{n+1})_*
\end{array}
$$

This gives the structure maps for a homomorphism

$$
\tilde f \,:\, X \longrightarrow \mathrm{Maps}(K,Y)_* \,.
$$

Running this argument backwards shows that the map $f \mapsto \tilde f$ given thereby is a bijection. ▮

Remark 1.9. For the adjunction of prop. 1.8 it is crucial that the smash tensoring in def. 1.6 is from the *right*, at least as long as the structure maps in def. 1.1 are defined as they are, with the circle smash factor on the left. We could change both jointly: take the structure maps to be from smash products with the circle on the right, and take smash tensoring to be from the left. But having both on the right or both on the left does not work.

Proposition 1.10. *The functor Σ^∞ that forms suspension spectra (def. 1.3) has a right adjoint functor Ω^∞*

$$
(\Sigma^\infty \dashv \Omega^\infty) \,:\, \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}}) \underoverset{\underset{\Omega^\infty}{\longrightarrow}}{\overset{\Sigma^\infty}{\longleftarrow}}{\bot} \mathrm{Top}_{\mathrm{cg}}^{*/} \,,
$$

given by picking the 0-component space:

$$
\Omega^\infty(X) = X_0 \,.
$$

Proof. By def. 1.1 the components f_n of a homomorphism of sequential spectra of the form

$$
\Sigma^\infty X \overset{f}{\longrightarrow} Y
$$

have to make these diagrams commute

$$
\begin{array}{ccc}
S^1 \wedge S^n X & \overset{S^1 \wedge f_n}{\longrightarrow} & S^1 \wedge Y_n \\
{}^{\mathllap{\simeq}}\downarrow && \downarrow^{\mathrlap{\sigma_n^Y}} \\
S^{n+1} \wedge X & \overset{f_{n+1}}{\longrightarrow} & Y_{n+1}
\end{array}
$$

for all $n \in \mathbb{N}$. Since here the left vertical map is an isomorphism by def. 1.3, this uniquely fixes f_{n+1} in terms of f_n. Hence the only freedom in specifying f is in the choice of the component $f_0 : X \longrightarrow Y_0$, which is equivalently a morphism

$$
X \overset{\tilde f}{\longrightarrow} \Omega^\infty Y \,.
$$

▮

Stable homotopy groups

In analogy to how homotopy groups are the fundamental invariants in classial homotopy theory, the fundamental invariants of stable homtopy theory are *stable homtopy groups*:

Definition 1.11. The **stable homotopy groups** of a sequential prespectrum X, def. 1.1, is the $\mathbb{Z}$-graded abelian group given by the colimit of homotopy groups of the component spaces (def.)

$$\pi_\bullet(X) \coloneqq \underset{\longrightarrow_k}{\lim} \pi_{\bullet+k}(X_k)\,,$$

where the colimit is over the sequential diagram whose component morphisms are given in terms of the structure maps of def. 1.1 by

$$\pi_{q+k}(X_k) \overset{\simeq}{\to} [S^{q+k}, X_k]_* \xrightarrow{(S^1\wedge(-))_{S^{q+k},X_k}} [S^{q+k+1}, S^1 \wedge X_k]_* \xrightarrow{[S^{q+k+1},\sigma_k]} [S^{q+k+1}, X_{k+1}]_* \overset{\simeq}{\to} \pi_{q+k+1}(X_{k+1})$$

and equivalently are given in terms of the adjunct structure maps of def. 1.2 by

$$\pi_{q+k}(X_k) \overset{\simeq}{\longrightarrow} [S^{q+k}, X_k]_* \xrightarrow{[S^{q+k},\tilde\sigma_k]} [S^{q+k}, \mathrm{Maps}(S^1, X_{k+1})_*]_* \simeq [S^1 \wedge S^{q+k}, X_{k+1}]_* \simeq \pi_{q+k+1}(X_{k+1})\,.$$

The colimit starts at

$$k = \begin{cases} 0 & \text{if}\ q \geq 0 \\ |q| & \text{if}\ q < 0 \end{cases}$$

This canonically extends to a functor

$$\pi_\bullet \;\colon\; \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}}) \longrightarrow \mathrm{Ab}^{\mathbb{Z}}\,.$$

Proposition 1.12. *The two component morphisms given in def. 1.11 indeed agree.*

Proof. Consider the following instance of the defining naturality square of the $(S^1 \wedge (-)) \dashv \mathrm{Maps}(S^1,-)_*$-adjunction of prop. 0.2:

$$\begin{array}{ccc} [S^1\wedge X_k, S^1 \wedge X_k]_* &\overset{\simeq}{\to}& [X_k, \mathrm{Maps}(S^1, S^1\wedge X_k)_*]_* \\ {}^{[S^1\wedge \alpha, \sigma_k]}\downarrow && \downarrow^{[\alpha, \mathrm{Maps}(S^1,\sigma_k)_*]_*} \\ [S^1 \wedge S^{q+k}, X_{k+1}]_* &\underset{\simeq}{\to}& [S^{q+k}, \mathrm{Maps}(S^1, X_{k+1})_*]_* \end{array}$$

Then consider the identity element in the top left hom-set. Its image under the left vertical map is the first of the two given component morphisms. Its image under going around the other way is the second of the two component morphisms. By the commutativity of the diagram, these two images agree. ▮

Example 1.13. Given $X \in \mathrm{Top}_{\mathrm{cg}}^{*/}$, then the stable homotopy groups (def. 1.11) of its suspension spectrum (example 1.3) are given by

$$\begin{aligned} \pi_q^S(X) &\coloneqq \pi_q(\Sigma^\infty X) \\ &= \underset{\longrightarrow_k}{\lim} \pi_{q+k}(S^k \wedge X) \\ &\simeq \underset{\longrightarrow_k}{\lim} \pi_q(\Omega^k(\Sigma^k X)) \end{aligned}\,.$$

Specifically for $X = S^0$ the 0-sphere, with suspension spectrum the standard sphere spectrum (def. 1.4), its stable homotopy groups are the stable homotopy groups of spheres:

$$\begin{aligned} \pi_q^S(S^0) &\coloneqq \pi_q(\mathbb{S}) \\ &= \underset{\longrightarrow_k}{\lim} \pi_{q+k}(S^k) \end{aligned}\,.$$

Recall the Freudenthal suspension theorem, which states that if X is an n-connected pointed CW-complex then the comparison map

$$\pi_q(X) \longrightarrow \pi_{q+1}(\Sigma X)$$

is an isomorphism for $q \leq 2n$. This implies first of all that every $\Sigma^k X$ is $(k-1)$-connected

$$\begin{aligned} \pi_0(\Sigma X) &\simeq * \\ \pi_1(\Sigma^2 X) &\simeq \pi_0(\Sigma X) \simeq * \\ \pi_2(\Sigma^3 X) &\simeq \pi_1(\Sigma^2 X) \simeq \pi_0(\Sigma X) \simeq * \\ &\cdots \end{aligned}$$

and then that the qth stable homotopy group of X is attained at stage $k = q+2$ in the colimit:

$$\pi_q^S(X) \simeq \pi_{q+(q+2)}(\Sigma^{q+2} X) \;.$$

Historically, this fact was one of the motivations for finding a stable homotopy category (def. 4.1 below).

Definition 1.14. A morphism $f : X \longrightarrow Y$ of sequential spectra, def. 1.1, is called a **stable weak homotopy equivalence**, if its image under the stable homotopy group-functor of def. 1.11 is an isomorphism

$$\pi_\bullet(f) \;:\; \pi_\bullet(X) \overset{\simeq}{\longrightarrow} \pi_\bullet(Y) \;.$$

Omega-spectra

In order to motivate Omega-spectra consider the following shadow of the structure they will carry:

Example 1.15. A $\mathbb{Z}$-graded abelian group is equivalently a sequence $\{A_n\}_{n\mathbb{Z}}$ of $\mathbb{N}$-graded abelian groups A_n, together with isomorphisms

$$A_n \simeq A_{n+1}[1] \,,$$

(where $[1]$ denotes the operation of shifting all entries in a graded abelian group down in degree by -1). Because this means that the sequence of $\mathbb{N}$-graded abelian groups is of the following form

$$\begin{matrix} \vdots & \vdots & & \\ a_3 & a_2 & a_1 & \cdots \\ a_2 & a_1 & a_0 & \cdots \\ a_1 & a_0 & a_{-1} & \cdots \\ \underbrace{a_0}_{A_0} & \underbrace{a_{-1}}_{A_1} & \underbrace{a_{-2}}_{A_2} & \cdots \end{matrix} \,.$$

This allows to recover the $\mathbb{Z}$-graded abelian group $\{a_n\}_{n\in\mathbb{Z}}$ from an $\mathbb{N}$-sequence of $\mathbb{N}$-graded abelian groups.

Then consider the case that the $\mathbb{N}$-graded abelian groups here are homotopy groups of some topological space. Then shifting the degree of the component groups corresponds to forming loop spaces, because for any topological space X then

$$\pi_\bullet(\Omega X) \simeq \pi_{\bullet+1}(X) \;.$$

(This may be seen concretely in point-set topology or abstractly by looking at the long exact sequence of homotopy groups for the fiber sequence $\Omega X \to \mathrm{Path}_*(X) \to X$.)

We find this kind of behaviour for the stable homotopy groups of Omega-spectra below in example 1.18.

Definition 1.16. An **Omega-spectrum** is a sequential spectrum X of topological spaces, def. 1.1, such that the (smash product ⊣ pointed mapping space)-adjuncts $\tilde\sigma_n$ of the structure maps $\sigma_n : \Sigma X_n \to X_{n+1}$ of X are weak homotopy equivalences (def.), hence classical weak equivalences (def.):

$$\tilde\sigma_n \;:\; X_n \overset{\in W_{\mathrm{cl}}}{\longrightarrow} \mathrm{Maps}(S^1, X_{n+1})_*$$

for all $n \in \mathbb{N}$.

Equivalently: an Omega-spectrum is a sequential spectrum in the incarnation of def. 1.2 such that all adjunct structure maps are weak homotopy equivalences.

Example 1.17. The Brown representability theorem (thm.) implies (prop.) that every generalized (Eilenberg-Steenrod) cohomology theory (def.) is represented by an Omega-spectrum (def. 1.16).

Applied to ordinary cohomology with coefficients some abelian group A, this yields the **Eilenberg-MacLane spectra** $H A$ (exmpl.). These are the Omega-spectra whose nth component space is an Eilenberg-MacLane space

$$(H A)_n \simeq K(A,n) \,.$$

A genuinely generalized (i.e. non-ordinary, hence "extra-ordinary") cohomology theory is topological K-theory $K^\bullet(-)$. Applying the Brown representability theorem to topological K-theory yields the K-theory spectrum denoted KU.

Omega-spectra are singled out among all sequential pre-spectra as having good behaviour under forming stable homotopy groups.

Example 1.18. If a sequential spectrum X is an Omega-spectrum, def. 1.16, then its colimiting stable homotopy groups reduce to the actual homotopy groups of the component spaces, in that:

$$X \text{ Omega-spectrum} \quad\Rightarrow\quad \pi_k(X) \simeq \begin{cases} \pi_k X_0 & \text{if } k \geq 0 \\ \pi_0 X_{\vert k \vert} & \text{if } k < 0 \end{cases} \,.$$

(Hence the stable homotopy groups of an Omega-spectrum realize the general pattern discussed in example 1.15.)

Proof. For an Omega-spectrum, the adjunct structure maps $\tilde\sigma_X$ are weak homotopy equivalences, by definition, hence are classical weak equivalences. Hence $[S^1, \tilde\sigma_n]_*$ is an isomorphism (prop.). Therefore, by prop. 1.12, the sequential colimit in def. 1.11 is entirely over isomorphisms and hence is given already by the first object of the sequence. ∎

We now show that every sequential pre-spectrum may be completed to an Omega-spectrum, up to stable weak homotopy equivalence:

Definition 1.19. For $X \in \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$, define a spectrum $Q X \in \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$ and a morphism

$$\eta_X \;:\; X \longrightarrow Q X$$

(to be called the **spectrification** of X) as follows.

First introduce for the given components X_k and adjunct structure maps $\tilde{\sigma}_k$ of X (from def. 1.2) the notation

$$Z_{0,k} \coloneqq X_k\,, \qquad \tilde{\sigma}_{0,k} \coloneqq \tilde{\sigma}_k\;.$$

Now assume, by induction, that sets of objects $\{Z_{i,k}\}_{k\in\mathbb{N}}$ and maps $\{Z_{i,k} \overset{\tilde{\sigma}_{i,k}}{\longrightarrow} \Omega Z_{i,k+1}\}_{k\in\mathbb{N}}$ have been constructed for some $i \in \mathbb{N}$.

Then construct $Z_{i+1,k} \in \mathrm{Top}_{\mathrm{cg}}$ by factorizing $\tilde{\sigma}_{i,k}$, with respect to the model structure $(\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}$ (thm.) as a classical cofibration followed by a classical weak equivalence. More specifically, apply the small object argument (prop.) with respect to the set of generating cofibrations I_{Top} (def.) to produce functorial factorizations (def.) into a relative cell complex followed by a weak homotopy equivalence (just as in the proof of this lemma):

$$\tilde{\sigma}_{i,k} \;:\; Z_{i,k} \underset{\in I_{\mathrm{Top}}\,\mathrm{Cell}}{\overset{\iota_{i,k}}{\longrightarrow}} Z_{i+1,k} \underset{\in W_{\mathrm{cl}}}{\overset{\phi_{i,k}}{\longrightarrow}} \Omega Z_{i,k+1}\;.$$

Then define $\tilde{\sigma}_{i+1,k}$ as the composite

$$\tilde{\sigma}_{i+1,k} \;:\; Z_{i+1,k} \overset{\phi_{i,k}}{\longrightarrow} \Omega Z_{i,k+1} \overset{\Omega(\iota_{i,k+1})}{\longrightarrow} \Omega Z_{i+1,k+1}\;.$$

This produces for each $i \in \mathbb{N}$ a commuting diagram of the form

$$\begin{array}{ccccccc}
X_k = Z_{0,k} & \underset{\in I_{\mathrm{Top}}\,\mathrm{Cell}}{\overset{\iota_{0,k}}{\longrightarrow}} & Z_{1,k} & \underset{\in I_{\mathrm{Top}}\,\mathrm{Cell}}{\overset{\iota_{1,k}}{\longrightarrow}} & Z_{2,k} & \underset{\in I_{\mathrm{Top}}\,\mathrm{Cell}}{\overset{\iota_{2,k}}{\longrightarrow}} & \cdots \\
{}^{\tilde{\sigma}_k = \tilde{\sigma}_{0,k}}\downarrow && {}^{\tilde{\sigma}_{1,k}}\downarrow && {}^{\tilde{\sigma}_{2,k}}\downarrow && \cdots\,. \\
\Omega X_{k+1} = \Omega Z_{0,k+1} & \overset{\Omega(\iota_{0,k+1})}{\longrightarrow} & \Omega Z_{1,k+1} & \overset{\Omega(\iota_{1,k+1})}{\longrightarrow} & \Omega Z_{2,k+1} & \overset{\Omega(\iota_{2,k+1})}{\longrightarrow} & \cdots
\end{array}$$

That this indeed commutes is the identity

$$\begin{aligned}
\tilde{\sigma}_{i+1,k} \circ \iota_{i,k} &= (\Omega(\iota_{i,k+1}) \circ \phi_{i,k}) \circ \iota_{i,k} \\
&= \Omega(\iota_{i,k+1}) \circ (\phi_{i,k} \circ \iota_{i,k})\;. \\
&= \Omega(\iota_{i,k+1}) \circ \tilde{\sigma}_{i,k}
\end{aligned}$$

Now let QX be the spectrum with component spaces the colimit

$$(QX)_k \coloneqq \underset{\longrightarrow_i}{\lim}\, Z_{i,k}$$

and with adjunct structure maps (via def. 1.2) given by the map induced under colimits by the above diagrams

$$\tilde{\sigma}_k^{QX} \;\coloneqq\; \underset{\longrightarrow}{\lim}\, \tilde{\sigma}_{i,k} \;:\; QX \longrightarrow \Omega(QX)\;.$$

Notice that this is indeed well-defined: since each component map $X_{i,k} \to X_{i+1,k}$ is a relative cell complex and since the 1-sphere S^1 is compact, it follows (lemma) that

$$
\begin{aligned}
\underset{\longrightarrow_i}{\lim} \Omega Z_{i,k} &= \underset{\longrightarrow_i}{\lim} \mathrm{Maps}(S^1, Z_{i,k})_* \\
&\simeq \mathrm{Maps}(S^1, \underset{\longrightarrow_i}{\lim} Z_{i,k})_* \\
&= \Omega \underset{\longrightarrow_i}{\lim} Z_{i,k} \\
&\simeq (\Omega Q X)
\end{aligned}
.
$$

Finally, let

$$
\eta_X : X \to Q X
$$

be degreewise the inclusion of the first component ($i = 0$) into the colimit. By construction, this is a homomorphism of sequential spectra (according to def. 1.2).

Proposition 1.20. *Let $X \in \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$ be a sequential prespectrum with $j_X : X \to QX$ from def. 1.19. Then:*

1. *QX is an Omega-spectrum (def. 1.16);*

2. *$\eta_X : X \to QX$ is a stable weak homotopy equivalence (def. 1.14):*

3. *η_X is a level weak equivalence (is in W_{strict}, def. 2.1) precisely if X is an Omega-spectrum;*

4. *a morphism $f : X \to Y$ is a stable weak homotopy equivalence (def. 1.14), precisely if $Qf : QX \to QY$ is a level weak equivalence (is in W_{strict}, def. 2.1).*

(Schwede 97, lemma 2.1.3 and remark before section 2.2)

Proof. Since the colimit defining QX is a transfinite composition of relative cell complexes, each component map $X_k \to (QX)_k$ is itself a relative cell complex. Since n-spheres are compact topological spaces, it follows (lemma) that each element of a homotopy group in $\pi_\bullet((QX)_k)$ is in the image of a finite stage $\pi_\bullet(Z_{i,k})$ for some $i \in \mathbb{N}$. From this, all statements follow by inspection at finite stages.

Regarding first statement:

Since each $\tilde\sigma_{i,k}$ by construction is a weak homotopy equivalence followed by an inclusion of stages in the colimit, as any element of $\pi_q((QX)_k)$ is sent along $\tilde\sigma_k^{QX}$ it passes through one such $\pi_q(\tilde\sigma_{i,k})$ at some stage i, hence also through all the following, and is hence identically preserved in the colimit.

Regarding the second statement:

By the previous statement and by example 1.18, the map $\pi_\bullet(\eta_X) : \pi_\bullet(X) \to \pi_\bullet(QX)$ is given in degree $q \geq 0$ by

$$
\underbrace{\underset{\longrightarrow_{k \in \mathbb{N}}}{\lim} \pi_{q+k}(X_k)}_{\simeq \underset{\longrightarrow_k}{\lim} \pi_q(\Omega^k X_k)} \longrightarrow \pi_q((QX)_0)
$$

and similarly in degree $q < 0$. Now using the compactness of the spheres and the definition of Q we compute on the right:

$$
\begin{aligned}
\pi_q((QX)_0) &= \pi_q(\underset{\longrightarrow_k}{\lim} Z_{k,0}) \\
&\simeq \underset{\longrightarrow_k}{\lim}\, \pi_q(Z_{k,0}) \\
&\simeq \underset{\longrightarrow_k}{\lim}\, \pi_q(\Omega^k X_k)
\end{aligned}
\,,
$$

where the last isomorphism is π_q applied to the composite of the weak homotopy equivalences

$$
Z_{k,0} \underset{\in W_{\mathrm{cl}}}{\overset{\phi_{k-1,0}}{\longrightarrow}} \Omega Z_{k-1,1} \to \cdots \to \Omega^k Z_{0,k} = \Omega^k X_k \,.
$$

Regarding the third statement:

In one direction:

If X is an Omega-spectrum in that all its adjunct structure maps $\tilde\sigma_k$ are weak homotopy equivalences, then by two-out-of-three also the maps $\iota_{i,k}$ in def. 1.19 are weak homotopy equivalences. Hence $(j_X)_k : X_k \to (QX)_k$ is the map into a sequential colimit over acyclic relative cell complexes, and again by the compactness of the spheres, this means that it is itself a weak homotopy equivalence.

In the other direction:

If η_X is degreewise a weak homotopy equivalence, then by applying two-out-of-three (def.) to the compatibility squares for the adjunct structure morphisms (def. 1.2), using that $\tilde\sigma_n^{QX}$ is a weak homotopy equivalence by the first point above

$$
\begin{array}{ccc}
X_n & \underset{\in W_{\mathrm{cl}}}{\overset{(j_X)_n}{\longrightarrow}} & (QX)_n \\
{}^{\tilde\sigma_n^X}\downarrow && \downarrow^{\tilde\sigma_n^{QX}}_{\in W_{\mathrm{cl}}} \\
\mathrm{Maps}(S^1, X_{n+1}) & \underset{\mathrm{Maps}(S^1,(j_X)_{n+1})}{\overset{\in W_{\mathrm{cl}}}{\longrightarrow}} & \mathrm{Maps}(S^1, (QX)_{n+1})
\end{array}
$$

implies that also $\tilde\sigma_n^X \in W_{\mathrm{cl}}$, hence that X is an Omega-spectrum.

The fourth statement follows with similar reasoning. ▮

Remark 1.21. In the case that X is a CW-spectrum (def. 2.7) then the sequence of resolutions in the definition of spectrification in def. 1.19 is not necessary, and one may simply consider

$$
(Q_{\mathrm{CW}} X)_n \coloneqq \underset{\longrightarrow_k}{\lim}\, \Omega^k X_{n+k} \,.
$$

See for instance (Lewis-May-Steinberger 86, p. 3) and (Weibel 94, 10.9.6 and topology exercise 10.9.2).

As topological diagrams

In order to conveniently understand the stable model category structure on spectra, we now consider an equivalent reformulation of the component-wise definition of sequential spectra, def. 1.1, as topologically enriched functors (defn.).

Definition 1.22. Write

$$
\iota : \mathrm{StdSpheres} \longrightarrow \mathrm{Top}_{\mathrm{cg}}^{*/}
$$

for the non-full topologically enriched subcategory (def.) of that of pointed compactly generated topological spaces (def.) where:

* objects are the standard n-spheres S^n, for $n \in \mathbb{N}$, identified as the smash product powers $S^n \coloneqq (S^1)^{\wedge^n}$ of the standard circle;
* hom-spaces are

$$\mathrm{StdSpheres}(S^n, S^{k+n}) \coloneqq \begin{cases} * & \text{for} \quad k < 0 \\ S^k & \text{otherwise} \end{cases}$$

* composition is induced from composition in $\mathrm{Top}_{\mathrm{cg}}^{*/}$ by regarding the hom-space S^k above as its image in $\mathrm{Maps}(S^n, S^{k+n})_*$ under the adjunct

$$S^k \to \mathrm{Maps}(S^n, S^{k+n})_*$$

of the canonical isomorphism

$$S^k \wedge S^n \xrightarrow{\simeq} S^{k+n} \,.$$

This induces the category

$$[\mathrm{StdSpheres}, \mathrm{Top}_{\mathrm{cg}}^{*/}]$$

of topologically enriched functors on $\mathrm{StdSpheres}$ with values in $\mathrm{Top}_{\mathrm{cg}}^{*/}$ (exmpl.).

Proposition 1.23. *There is an equivalence of categories*

$$(-)^{\mathrm{seq}} : [\mathrm{StdSpheres},\ \mathrm{Top}_{\mathrm{cg}}^{*/}] \xrightarrow{\simeq} \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$$

from the category of topologically enriched functors on the category of standard spheres of def. 1.22 to the category of topological sequential spectra, def. 1.1, which is given on objects by sending $X \in [\mathrm{StdSpheres}, \mathrm{Top}_{\mathrm{cg}}^{*/}]$ to the sequential prespectrum X^{seq} with components

$$X_n^{\mathrm{seq}} \coloneqq X(S^n)$$

and with structure maps

$$\frac{S^1 \wedge X_n^{\mathrm{seq}} \overset{\sigma_n}{\longrightarrow} X_n^{\mathrm{seq}}}{S^1 \longrightarrow \mathrm{Maps}(X_n^{\mathrm{seq}}, X_{n+1}^{\mathrm{seq}})_*}$$

being the adjunct of the component map of X on spheres of consecutive dimension.

Proof. First observe that from its components on consecutive spheres the functor X is already uniquely determined. Indeed, by definition the hom-space between non-consecutive spheres $\mathrm{StdSpheres}(S^n, S^{n+k})$ is the smash product of the hom-spaces between the consecutive spheres, for instance:

$$\begin{array}{ccc} S^1 \wedge S^1 & = & \mathrm{StdSpheres}(S^n, S^{n+1}) \wedge \mathrm{StdSpheres}(S^{n+1}, S^{n+2}) \\ {}^{\simeq}\downarrow && {}^{\simeq}\downarrow^{\circ} \\ S^2 & = & \mathrm{StdSpheres}(S^n, S^{n+2}) \end{array} \,,$$

and so functoriality completely fixes the former by the latter.

This means that we actually have a bijection between classes of objects.

Now observe that a natural transformation $f : X \to Y$ between two functors on $\mathrm{StdSpheres}$ is equivalently a collection of component maps $f_n : X_n \to Y_n$, such that for each $s \in S^1$ then the following squares commute

$$
\begin{array}{ccc}
X(S^n) & \overset{f_n}{\to} & Y^{S^n} \\
{}^{X_{S^n,S^{n+1}}(s)}\downarrow && \downarrow^{Y_{S^n,S^{n+1}}(s)} \\
X(S^{n+1}) & \underset{f_{n+1}}{\longrightarrow} & Y(S^{n+1})
\end{array}
$$

By the smash/hom adjunction, the square equivalently factors as

$$
\begin{array}{ccc}
X(S^n) & \overset{f_n}{\to} & Y^{S^n} \\
{}^{(s,\mathrm{id})}\downarrow && \downarrow^{(s,\mathrm{id})} \\
S^1 \wedge X(S^n) & \underset{\mathrm{id} \times f_n}{\longrightarrow} & S^1 \wedge Y(S^n) \\
{}^{\sigma_n^X}\downarrow && \downarrow^{\sigma_n^Y} \\
X(S^{n+1}) & \mathrm{underset}\, f_{n+1} \longrightarrow & Y(S^{n+1})
\end{array}
\,.
$$

Here the top square commutes in any case, and so the total rectangle commutes precisely if the lower square commutes, hence if under our identification the components $\{f_n\}$ constitute a homomorphism of sequential spectra.

Hence we have an isomorphism on all hom-sets, and hence an equivalence of categories. ▮

Further below we use prop. 1.23 to naturally induce a model structure on the category of topological sequential spectra.

Remark 1.24. Under the equivalence of prop. 1.23, the general concept of tensoring of topologically enriched functors over topological spaces (according to this def.) restricts to the concept of tensoring of sequential spectral over topological spaces according to def. 1.6.

Proposition 1.25. *The category $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cq}})$ of sequential spectra (def. 1.1) has all limits and colimits, and they are computed objectwise:*

Given

$$
X_\bullet \,:\, I \longrightarrow \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})
$$

a diagram of sequential spectra, then:

1. *its colimiting spectrum has component spaces the colimit of the component spaces formed in $\mathrm{Top}_{\mathrm{cg}}$ (via this prop. and this corollary):*

$$
(\underset{\longrightarrow}{\lim}_i X(i))_n \simeq \underset{\longrightarrow}{\lim}_i X(i)_n \,,
$$

2. *its limiting spectrum has component spaces the limit of the component spaces formed in $\mathrm{Top}_{\mathrm{cg}}$ (via this prop. and this corollary):*

$$
(\underset{\longleftarrow}{\lim}_i X(i))_n \simeq \underset{\longleftarrow}{\lim}_i X(i)_n \,;
$$

moreover:

1. *the colimiting spectrum has structure maps in the sense of def. 1.1 given by*

$$S^1 \wedge (\underset{\longrightarrow_i}{\lim} X(i)_n) \simeq \underset{\longrightarrow_i}{\lim} (S^1 \wedge X(i)_n) \overset{\underset{\longrightarrow_i}{\lim} \sigma_n^{X(i)}}{\longrightarrow} \underset{\longrightarrow_i}{\lim} X(i)_{n+1}$$

where the first isomorphism exhibits that $S^1 \wedge (-)$ preserves all colimits, since it is a left adjoint by prop. 0.2;

2. *the limiting spectrum has adjunct structure maps in the sense of def. 1.2 given by*

$$\underset{\longleftarrow_i}{\lim} X(i)_n \overset{\underset{\longleftarrow_i}{\lim} \tilde\sigma_n^{X(i)}}{\longrightarrow} \underset{\longleftarrow_i}{\lim} \mathrm{Maps}(S^1, X(i)_n)_* \simeq \mathrm{Maps}(S^1, \underset{\longleftarrow_i}{\lim} X(i)_n)_*$$

where the last isomorphism exhibits that $\mathrm{Maps}(S^1, -)_*$ preserves all limits, since it is a right adjoint by prop. 0.2.

Proof. That the limits and colimits exist and are computed objectwise follows via prop. 1.23 from the general statement for categories of topological functors (prop.). But it is also immediate to directly check the universal property. ▮

Example 1.26. The initial object and the terminal object in $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$ agree and are both given by the spectrum constant on the point, which is also the suspension spectrum $\Sigma^\infty *$ (def. 1.3) of the point). We will denote this spectrum $*$ or 0 (since it is hence a zero object):

$$\begin{aligned} *_n &= * \\ S^1 \wedge *_n &\simeq * \overset{\simeq}{\to} * \,. \end{aligned}$$

Example 1.27. The coproduct of spectra $X, Y \in \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$, called the **wedge sum of spectra**

$$X \vee Y \coloneqq X \sqcup Y$$

is componentwise the wedge sum of pointed topological spaces (exmpl.)

$$(X \vee Y)_n = X_n \vee Y_n$$

with structure maps

$$\sigma_n^{X \vee Y} : S^1 \wedge (X \vee Y) \simeq S^1 \wedge X \vee S^1 \wedge Y \overset{(\sigma_n^X, \sigma_n^Y)}{\longrightarrow} X_{n+1} \vee Y_{n+1} \,.$$

Example 1.28. For $X \in \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$ a sequential spectrum, def. 1.1, its **standard cylinder spectrum** is its smash tensoring $X \wedge (I_+)$, according to def. 1.6, with the standard interval (def.) with a basepoint freely adjoined (def.). The component spaces of the cylinder spectrum are the standard reduced cylinders (def.) of the component spaces of X:

$$(X \wedge (I_+))_n = X_n \wedge I_+ \,.$$

By the functoriality of the smash tensoring, the factoring

$$\nabla_{S^0} : S^0 \vee S^0 \longrightarrow I_+ \longrightarrow S^0$$

of the codiagonal on the 0-sphere through the standard interval with a base point adjoined, gives a factoring of the codiagonal of X through its standard cylinder spectrum

$$\nabla_X : X \vee X \overset{X \wedge (S^0 \vee S^0 \to I_+)}{\longrightarrow} X \wedge (I_+) \overset{X \wedge (I_+ \to S^0)}{\longrightarrow} X$$

(where we are using that wedge sum is the coproduct in pointed topological spaces (exmpl.).)

Suspension and looping

We discuss models for the operation of reduced suspension and forming loop space objects of sequential spectra.

Definition 1.29. For X a sequential spectrum, then

1. the **standard suspension** of X is the smash product-tensoring $X \wedge S^1$ according to def. 1.6;

2. the **standard looping** of X is the smash powering $\mathrm{Maps}(S^1, X)_*$ according to def. 1.6.

Proposition 1.30. *For $X \in \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$, the standard suspension $X \wedge S^1$ of def. 1.29 is equivalently the cofiber (formed via prop. 1.25) of the canonical inclusion of boundaries into the standard cylinder spectrum $X \wedge (I_+)$ of example 1.28:*

$$X \wedge S^1 \simeq \mathrm{cofib}(X \vee X \to X \wedge (I_+)) \,.$$

Proof. This is immediate from the componentwise construction of the smash tensoring and the componentwise computation of colimits of spectra via prop. 1.25. ■

This means that once we know that $X \vee X \to X \wedge (I_+)$ is suitably a cofibration (to which we turn below) then the standard suspension is a homotopy-correct model for the suspension operation. However, some properties of suspension are hard to prove directly with the standard suspension model. For such there are two other models for suspension and looping of spectra. These three models are not isomorphic to each other in $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$, but (this is lemma 3.22 below) they will become isomorphic in the stable homotopy category (def. 4.1).

Definition 1.31. For X a sequential spectrum (def. 1.1) and $k \in \mathbb{Z}$, the k-fold **shifted spectrum** of X is the sequential spectrum denoted $X[k]$ given by

- $(X[k])_n \coloneqq \begin{cases} X_{n+k} & \text{for } n+k \geq 0 \\ * & \text{otherwise} \end{cases}$;

- $\sigma_n^{X[k]} \coloneqq \begin{cases} \sigma^X_{n+k} & \text{for } n+k \geq 0 \\ 0 & \text{otherwise} \end{cases}$.

Definition 1.32. For X a sequential spectrum, def. 1.1, then

1. the **alternative suspension** of X is the sequential spectrum ΣX with
   1. $(\Sigma X)_n \coloneqq S^1 \wedge X_n$ (smash product on the left (defn.))
   2. $\sigma_n^{\Sigma X} \coloneqq S^1 \wedge (\sigma_n^X)$.

   in the sense of def. 1.1;

2. the **alternative looping** of X is the sequential spectrum ΩX with
   1. $(\Omega X)_n \coloneqq \mathrm{Maps}(S^1, X_n)_*$;

2. $\tilde{\sigma}_n^{\Omega X} \coloneqq \mathrm{Maps}(S^1, \tilde{\sigma}_n^X)_*$

in the sense of def. 1.2.

Remark 1.33. In various references the "alternative suspension" from def. 1.32 is called the "fake suspension" (e.g. Goerss-Jardine 96, p. 499, Jardine 15, section 10.4).

Remark 1.34. There is no direct natural isomorphism between the standard suspension (def. 1.29) and the alternative suspension (def. 1.32). This is due to the non-trivial graded commutativity (braiding) of smash products of spheres. (We discuss braiding of the smash product more in detail in Part 1.2, this example).

Namely a natural isomorphism $\phi\colon \Sigma X \longrightarrow X \wedge S^1$ (or alternatively the other way around) would have to make the following diagrams commute:

$$\begin{array}{ccc} S^1 \wedge S^1 \wedge X_n & \overset{\mathrm{id}_{S^1} \wedge \phi_n}{\longrightarrow} & S^1 \wedge X_n \wedge S^1 \\ {}^{S^1 \wedge \sigma_n}\downarrow && \downarrow^{\sigma_n \wedge S^1} \\ S^1 \wedge X_{n+1} & \underset{\phi_{n+1}}{\longrightarrow} & X_{n+1} \wedge S^1 \end{array} \quad \text{(nc)}$$

and naturally so in X.

The only evident option is to have ϕ be the braiding homomorphisms of the smash product

$$\phi_n = \tau_{S^1, X_n} \;:\; S^1 \wedge X_n \overset{\simeq}{\to} X_n \wedge S^1 \,.$$

It may superficially look like this makes the above diagram commute, but it does not. To make this explicit, consider labeling the two copies of the circle appearing here as S^1_a and S^1_b. Then the diagram we are dealing with looks like this:

$$\begin{array}{ccc} S^1_a \wedge S^1_b \wedge X_n & \longrightarrow & S^1_a \wedge X_n \wedge S^1_b \\ {}^{S^1_a \wedge \sigma_n}\downarrow & \text{(nc)} & \downarrow^{\sigma_n \wedge S^1_b} \\ S^1_a \wedge X_{n+1} & \longrightarrow & X_{n+1} \wedge S^1_b \end{array}$$

If we had $S^1_a \wedge \sigma_n$ on the left and $\sigma_n \wedge S^1_a$ on the right, then the naturality of the braiding would indeed give a commuting diagram. But since this is not the case, the only way to achieve this would be by exchanging in the top left

$$S^1_a \wedge S^1_b \longrightarrow S^1_b \wedge S^1_a \,.$$

However, this map is non-trivial. It represents -1 in $[S^2, S^2]_* = \pi_2(S^2) = \mathbb{Z}$. Hence inserting this map in the top of the previous diagram still does not make it commute.

But this technical problem points to its own solutions: if we were to restrict to the homotopy category of spectra which had structure maps only of the form $S^2 \wedge X_n \to X_{n+2}$, then the braiding required to make the two models of suspension comparable would be

$$S^2_a \wedge S^1_b \longrightarrow S^1_b \wedge S^2_a$$

and this map is indeed trivial, up to homotopy. This we make precise as lemma 3.22 below.

More generally, the kind of issue encountered here is taken care of by the concept of *symmetric spectra*, to which we turn in *Part 1.2*.

Remark 1.35. The looping and suspension operations in def. 1.29 and def. 1.32 commute with shifting, def. 1.31. Therefore in expressions like $\Sigma(X[1])$ etc. we may omit the parenthesis.

Proposition 1.36. *The constructions from def. 1.29, def. 1.31 and def. 1.32 form pairs of adjoint functors* $\mathrm{SeqSpec} \to \mathrm{SeqSpec}$ *like so:*

1. $(-)[-1] \dashv (-)[1]$;
2. $(-) \wedge S^1 \dashv \mathrm{Maps}(S^1, -)_*$;
3. $\Sigma \dashv \Omega$.

Proof. Regarding the first statement:

A morphism of the form $f : X[-1] \longrightarrow Y$ has components of the form

$$\begin{array}{ccc} \vdots && \vdots \\ X_2 & \overset{f_2}{\longrightarrow} & Y_3 \\ X_1 & \overset{f_2}{\longrightarrow} & Y_2 \\ X_0 & \overset{f_1}{\longrightarrow} & Y_1 \\ * & \overset{f_0 = 0}{\longrightarrow} & Y_0 \end{array}$$

and the compatibility condition with the structure maps in lowest degree is automatically satisfied

$$\begin{array}{ccc} * & \overset{(S^1 \wedge f_0) = 0}{\longrightarrow} & S^1 \wedge Y_0 \\ {}^{\mathllap{\sigma_0^{X[-1]} = 0}}\downarrow && \downarrow^{\mathrlap{\sigma_0^Y}} \\ X_0 & \overset{f_1}{\longrightarrow} & Y_1 \end{array} \,.$$

Therefore this is equivalent to components

$$\begin{array}{ccc} \vdots && \vdots \\ X_2 & \overset{f_2}{\longrightarrow} & Y_3 \\ X_1 & \overset{f_2}{\longrightarrow} & Y_2 \\ X_0 & \overset{f_1}{\longrightarrow} & Y_1 \end{array}$$

hence to a morphism $X \longrightarrow Y[1]$.

The second statement is a special case of prop. 1.8.

Regarding the third statement:

This follows by applying the (smash product⊣pointed mapping space)-adjunction isomorphism twice, like so:

Morphisms $f : \Sigma X \to Y$ in the sense of def. 1.1 are in components given by commuting diagrams of this form:

$$
\begin{array}{ccc}
S^1 \wedge S^1 \wedge X_n & \overset{S^1 \wedge f_n}{\longrightarrow} & S^1 \wedge Y_n \\
{}^{S^1 \wedge \sigma_n^X}\downarrow && \downarrow^{\sigma_n^Y} \\
S^1 \wedge X_{n+1} & \underset{f_{n+1}}{\longrightarrow} & Y_{n+1}
\end{array}
\,.
$$

Applying the adjunction isomorphism diagonally gives a natural bijection to diagrams of this form:

$$
\begin{array}{ccc}
S^1 \wedge X_n & \overset{f_n}{\longrightarrow} & Y_n \\
{}^{\sigma_n^X}\downarrow && \downarrow^{\tilde \sigma_n^Y} \\
X_{n+1} & \underset{\widetilde{f_{n+1}}}{\longrightarrow} & \mathrm{Maps}(S^1, Y_{n+1})_*
\end{array}
\,.
$$

(To see this in full detail, for instance for the adjunct of the left and bottom morphism: chase the identity $\mathrm{id}_{S^1 \wedge X_{n+1}}$ in both ways

$$
\begin{array}{ccc}
\mathrm{Hom}(S^1 \wedge X_{n+1}, S^1 \wedge X_{n+1}) & \overset{\simeq}{\longrightarrow} & \mathrm{Hom}(X_{n+1}, \mathrm{Maps}(S^1, S^1 \wedge X_{n+1})_*) \\
{}^{\mathrm{Hom}(S^1 \wedge \sigma_n^X, f_{n+1})}\downarrow && \downarrow^{\mathrm{Hom}(\sigma_n^X, \mathrm{Maps}(S^1, f_{n+1})_*)} \\
\mathrm{Hom}(S^1 \wedge S^1 \wedge X_n, Y_{n+1}) & \overset{\simeq}{\longrightarrow} & \mathrm{Hom}(S^1 \wedge X_n, \mathrm{Maps}(S^1, Y_{n+1})_*)
\end{array}
$$

through the adjunction naturality square. The other cases follow analogously.)

Then applying the adjunction isomorphism diagonally once more gives a further bijection to commuting diagrams of this form:

$$
\begin{array}{ccc}
X_n & \overset{\widetilde{f_n}}{\longrightarrow} & \mathrm{Maps}(S^1, Y_n)_* \\
{}^{\tilde \sigma_n}\downarrow && \downarrow^{\mathrm{Maps}(S^1, \tilde \sigma_n^Y)_*} \\
\mathrm{Maps}(S^1, X_{n+1})_* & \underset{\mathrm{Maps}(S^1, \widetilde{f_{n+1}})_*}{\longrightarrow} & \mathrm{Maps}\big(S^1, \mathrm{Maps}(S^1, Y_{n+1})_*\big)_*
\end{array}
\,.
$$

This, finally, equivalently exhibits homomorphisms of the form

$$
X \longrightarrow \Omega Y
$$

in the sense of def. 1.2. ▮

Proposition 1.37. *The following diagram of adjoint pairs of functors commutes:*

$$
\begin{array}{ccc}
\mathrm{Top}_{\mathrm{cg}}^{*/} & \underoverset{\underset{\Omega}{\longrightarrow}}{\overset{\Sigma}{\longleftarrow}}{\bot} & \mathrm{Top}_{\mathrm{cg}}^{*/} \\
{}^{\Sigma^\infty}\downarrow \dashv \uparrow^{\Omega^\infty} && {}^{\Sigma^\infty}\downarrow \dashv \uparrow^{\Omega^\infty} \\
\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}}) & \underoverset{\underset{\Omega}{\longrightarrow}}{\overset{\Sigma}{\longleftarrow}}{\bot} & \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})
\end{array}
\,,
$$

Here the top horizontal adjunction is from prop. 0.2, the vertical adjunction is from prop. 1.8 and the bottom adjunction is from prop. 1.36.

Proof. It is sufficient to check

$$
\Sigma^\infty \circ \Sigma \simeq \Sigma \circ \Sigma^\infty \,.
$$

From this the statement

$$\Omega^\infty \circ \Omega \simeq \Omega \circ \Omega^\infty$$

follows by uniqueness of adjoints.

So let $X \in \mathrm{Top}_{\mathrm{cg}}^{*/}$. Then

- $(\Sigma \Sigma^\infty X)_n = S^1 \wedge S^n \wedge X$,
- $\sigma_n^{(\Sigma \Sigma^\infty X)} : S^1 \wedge S^1 \wedge S^n \wedge X \xrightarrow{S^1 \wedge \mathrm{id}} S^1 \wedge S^{1+n} \wedge X$,

while

- $(\Sigma^\infty \Sigma X)_n = S^n \wedge S^1 \wedge X$,
- $\sigma_n^{(\Sigma^\infty \Sigma X)} : S^1 \wedge S^n \wedge S^1 \wedge X \xrightarrow{\mathrm{id} \wedge S^1 \wedge X} S^{1+n} \wedge S^1 \wedge X$,

where we write "id" for the canonical isomorphism. Clearly there is a natural isomorphism given by the canonical identifications

$$S^1 \wedge S^n \wedge X \overset{\simeq}{\longrightarrow} (S^1)^{\wedge^{n+1}} \wedge X \overset{\simeq}{\longrightarrow} S^n \wedge S^1 \wedge X \,.$$

(As long as we are not smash-permuting the S^1 factor with the S^n factor – and here we are not – then the fact that they get mixed under this isomorphism is irrelevant. The point where this does become relevant is the content of remark 1.34 below.) ▮

2. The strict model structure on sequential spectra

The model category structure on sequential spectra which presents stable homotopy theory is the "stable model structure" discussed below. Its fibrant-cofibrant objects are (in particular) Omega-spectra, hence are the proper spectrum objects among the pre-spectrum objects.

But for technical purposes it is useful to also be able to speak of a model structure on pre-spectra, which sees their homotopy theory as sequences of simplicial sets equipped with suspension maps, but not their stable structure. This is called the "strict model structure" for sequential spectra. Its main point is that the stable model structure of interest arises from it via left Bousfield localization.

Definition 2.1. Say that a homomorphism $f_\bullet : X_\bullet \to Y_\bullet$ in the category $\mathrm{SeqSpec}(\mathrm{Top})$, def. 1.1 is

- a **strict weak equivalence** if each component $f_n : X_n \to Y_n$ is a weak equivalence in the classical model structure on topological spaces (hence a weak homotopy equivalence);
- a **strict fibration** if each component $f_n : X_n \to Y_n$ is a fibration in the classical model structure on topological spaces (hence a Serre fibration);
- a **strict cofibration** if the maps $f_0 : X_0 \to Y_0$ as well as for all $n \in \mathbb{N}$ the maps

  $$(f_{n+1}, \sigma_n^Y) \,:\, X_{n+1} \underset{S^1 \wedge X_n}{\sqcup} S^1 \wedge Y_n \longrightarrow Y_{n+1}$$

  are cofibrations in the classical model structure on topological spaces (hence retracts of relative cell complexes);

We write W_{strict}, $\mathrm{Fib}_{\mathrm{strict}}$ and $\mathrm{Cof}_{\mathrm{strict}}$ for these classes of morphisms, respectively.

Recall the sets

$$I_{\mathrm{Top}^{*/}} \coloneqq \{ S^{n-1}_+ \overset{(\iota_n)_+}{\longrightarrow} D^n_+ \}_{n \in \mathbb{N}}$$
$$J_{\mathrm{Top}^{*/}} \coloneqq \{ D^n \overset{(j_n)_+}{\longrightarrow} D^n \times I \}_{n \in \mathbb{N}}$$

of standard generating (acyclic) cofibrations (def.) of the classical model structure on pointed topological spaces (thm.).

Definition 2.2. Write

$$I_{\mathrm{seq}}^{\mathrm{strict}} \coloneqq \{ y(S^n) \cdot i_+ \}_{\substack{S^n \in \mathrm{StdSpheres}, \\ i_+ \in I_{\mathrm{Top}^{*/}}}} \;\; \in [\mathrm{StdSpheres}, \mathrm{Top}^{*/}] \simeq \mathrm{SeqSpec}(\mathrm{Top})$$

and

$$J_{\mathrm{seq}}^{\mathrm{strict}} \coloneqq \left\{ y(S^n) \cdot j_+ \right\}_{\substack{S^n \in \mathrm{StdSpheres} \\ j_+ \in J_{\mathrm{Top}^{*/}}}} \;\; \in [\mathrm{StdSpheres}, \mathrm{Top}^{*/}] \simeq \mathrm{SeqSpec}(\mathrm{Top}) \,,$$

for the set of morphisms arising as the tensoring (remark 1.24) of a representable (exmpl.) with a generating acyclic cofibration of the classical model structure on pointed topological spaces (def.).

Theorem 2.3. *The classes of morphisms in def. 2.1 give the structure of a model category (def.) to be denoted $\mathrm{SeqSpec}(\mathrm{Top})_{\mathrm{strict}}$ and called the* ***strict model structure on topological sequential spectra*** *(or:* ***level model structure****).*

Moreover, this is a cofibrantly generated model category with generating (acyclic) cofibrations the set $I_{\mathrm{seq}}^{\mathrm{strict}}$ (resp. $J_{\mathrm{seq}}^{\mathrm{strict}}$) from def. 2.2.

Proof. Prop. 1.23 says that the category of sequential spectra is equivalently an enriched functor category

$$\mathrm{SeqSpec}(\mathrm{Top}) \simeq [\mathrm{StdSpheres},\ \mathrm{Top}_{\mathrm{cg}}^{*/}] \,.$$

Accordingly, this carries the projective model structure on functors (thm.). This immediately gives the statement for the fibrations and the weak equivalences.

It only remains to check that the cofibrations are as claimed. To that end, consider a commuting square of sequential spectra

$$\begin{array}{ccc} X & \overset{h}{\longrightarrow} & A \\ \downarrow^{f} && \downarrow \\ Y & \longrightarrow & B \end{array} \,.$$

By definition, this is equivalently an $\mathbb{N}$-collection of commuting diagrams in $\mathrm{Top}_{\mathrm{cg}}$ of the form

$$\begin{array}{ccc} X_n & \overset{h_n}{\longrightarrow} & A_n \\ \downarrow^{f_n} && \downarrow \\ Y_n & \longrightarrow & B_n \end{array}$$

such that all structure maps are respected.

$$
\begin{array}{ccccccc}
S^1\wedge X_n & \overset{\sigma_n^X}{\longrightarrow} & X_{n+1} && S^1\wedge X_n & \overset{\sigma_n^X}{\longrightarrow} & X_{n+1} \\
\downarrow^{S^1\wedge f_n} && \downarrow^{f_{n+1}} && & \searrow^{S^1\wedge h_n} && \searrow^{h_{n+1}} \\
S^1\wedge Y_n & \overset{\sigma_n^Y}{\longrightarrow} & Y_{n+1} & = & & S^1\wedge A_n & \overset{\sigma_n^A}{\longrightarrow} & A_{n+1} \\
& \searrow && \searrow && \downarrow && \downarrow \\
& S^1\wedge B_n & \overset{\sigma_n^B}{\longrightarrow} & B_{n+1} && S^1\wedge B_n & \overset{\sigma_n^B}{\longrightarrow} & B_{n+1}
\end{array}
\,.
$$

Hence a lifting in the original diagram is a lifting in each degree n, such that the lifting in degree $n+1$ makes these diagrams of structure maps commute.

Since components are parameterized over $\mathbb{N}$, this condition has solutions by induction:

First of all there must be an ordinary lifting in degree 0. Since the strict fibrations are degreewise classical fibrations, this gives the condition that for $f_\bullet$ to be a strict cofibration, then f_0 is to be a classical cofibration.

Then assume that a lifting l_n in degree n has been found

$$
\begin{array}{ccc}
X_n & \overset{h_n}{\longrightarrow} & A_n \\
\downarrow^{f_n} & \nearrow_{l_n} & \downarrow \\
Y_n & \longrightarrow & B_n
\end{array}
\,.
$$

Now the lifting l_{n+1} in the next degree has to also make the following diagram commute

$$
\begin{array}{ccccc}
S^1\wedge X_n & \overset{\sigma_n^X}{\longrightarrow} & X_{n+1} && \\
\downarrow^{S^1\wedge f_n} && \downarrow^{f_{n+1}} & \searrow^{h_{n+1}} & \\
S^1\wedge Y_n & \overset{\sigma_n^Y}{\longrightarrow} & Y_{n+1} && \\
& \searrow^{S^1\wedge l_n} && \searrow^{l_{n+1}} & \downarrow \\
&& S^1\wedge A_n & \overset{\sigma_n^A}{\longrightarrow} & A_{n+1}
\end{array}
\,.
$$

This is a cocone under the commuting square for the structure maps, and therefore the outer diagram is equivalently a morphism out of the domain of the pushout product $f_n \Box \sigma_n^X$ (def.), while the compatible lift l_{n+1} is equivalently a lift against this pushout product:

$$
\begin{array}{ccc}
S^1\wedge Y_n \underset{S^1\wedge X_n}{\sqcup} X_{n+1} & \overset{(\sigma_n^A\circ S^1\wedge l_n,\, h_{n+1})}{\longrightarrow} & A_{n+1} \\
{}^{f_n \Box \sigma_n^X}\downarrow & {}^{l_{n+1}}\nearrow & \downarrow \\
Y_{n+1} & \longrightarrow & B_{n+1}
\end{array}
\,.
$$

This shows that $f_\bullet$ is a strict cofibration precisely if, in addition to f_0 being a classical cofibration, all these pushout products are classical cofibrations. ▮

Suspension and looping

Proposition 2.4. *The $(\Sigma^\infty \dashv \Omega^\infty)$-adjunction from prop. 1.10 is a Quillen adjunction (def.) between the classical model structure on pointed topological spaces (thm., prop.) and the strict model structure on topological sequential spectra of theorem 2.3:*

$$(\Sigma^\infty \dashv \Omega^\infty) \;:\; \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}} \underoverset{\underset{\Omega^\infty}{\longrightarrow}}{\overset{\Sigma^\infty}{\longleftarrow}}{\bot} (\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}} \,.$$

Proof. It is clear that Ω^∞ preserves fibrations and acyclic cofibrations. This is sufficient to deduce a Quillen adjunction.

Just for the record, we spell out a direct argument that also Σ^∞ preserves cofibrations and acyclic cofibrations:

Let $f \colon X \longrightarrow Y$ be a morphism in $\mathrm{Top}_{\mathrm{cg}}^{*/}$ and

$$\Sigma^\infty f \colon \Sigma^\infty X \longrightarrow \Sigma^\infty Y$$

its image.

Since the structure maps in a suspension spectrum, example 1.3, are all isomorphisms, we have for all $n \in \mathbb{N}$ an isomorphism

$$(\Sigma^\infty X)_{n+1} \underset{S^1 \wedge (\Sigma^\infty X)_n}{\coprod} S^1 \wedge (\Sigma^\infty Y)_n \simeq S^1 \wedge (\Sigma^\infty Y)_n \,.$$

Therefore $\Sigma^\infty f$ is a strict cofibration, according to def. 2.1, precisely if $(\Sigma^\infty f)_0 = f$ is a classical cofibration and all the structure maps of $\Sigma^\infty Y$ are classical cofibrations. But the latter are even isomorphisms, so that this is no extra condition (prop.). Hence Σ^∞ sends classical cofibrations of spaces to strict cofibrations of sequential spectra.

Furthermore, since $S^n \wedge (-) \colon (\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}} \to (\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}$ is a left Quillen functor for all $n \in \mathbb{N}$ by prop. 0.2 it sends classical acyclic cofibrations to classical acyclic cofibrations. Hence Σ^∞, which is degreewise given by $S^n \wedge (-)$, sends classical acyclic cofibrations to degreewise acyclic cofibrations, hence in particular to degreewise weak equivalences, hence to weak equivalences in the strict model structure on sequential spectra.

This shows that Σ^∞ is a left Quillen functor. ▮

Proposition 2.5. *The $(\Sigma \dashv \Omega)$-adjunction from prop. 1.36 is a Quillen adjunction (def.) with respect to the strict model structure on sequential spectra of theorem 2.3.*

$$\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}} \underoverset{\underset{\Omega}{\longrightarrow}}{\overset{\Sigma}{\longleftarrow}}{\bot} \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}} \,.$$

Proof. Since the (acyclic) fibrations of $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}}$ are by definition those morphisms that are degreewise (acylic) fibrations in $(\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}$, the statement follows immediately from the fact that the right adjoint Ω is degreewise given by $\mathrm{Maps}(S^1,-)_* \colon (\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}} \to (\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}$, which is a right Quillen functor by prop. 0.2. ▮

In summary, prop. 1.37, prop. 2.4 and prop. 2.5 say that

Corollary 2.6. *The commuting square of adjunctions in prop. 1.37 is a square of Quillen adjunctions with respect to the classical model structure on pointed compactly generated topological spaces (thm., prop.) and the strict model structure on topological sequential spectra of theorem 2.3:*

$$
\begin{array}{ccc}
(\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}} & \underoverset{\underset{\Omega}{\longrightarrow}}{\overset{\Sigma}{\longleftarrow}}{\bot} & (\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}} \\
{}^{\mathllap{\Sigma^\infty}}\downarrow \dashv \uparrow^{\mathrlap{\Omega^\infty}} && {}^{\mathllap{\Sigma^\infty}}\downarrow \dashv \uparrow^{\mathrlap{\Omega^\infty}} \\
\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}} & \underoverset{\underset{\Omega}{\longrightarrow}}{\overset{\Sigma}{\longleftarrow}}{\bot} & \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}}
\end{array}
\,,
$$

Further below we pass to the stable model structure in order to make the bottom adjunction in this diagram become a Quillen equivalence. This stable model structure will have more weak equivalences than the strict model structure, but will have the same cofibrations. Therefore we first consider now cofibrancy conditions already in the strict model structure.

CW-spectra

Definition 2.7. A sequential spectrum X (def. 1.1) is called a **cell spectrum** if

1. all component spaces X_n are cell complexes (def.);

2. all structure maps $\sigma_n : S^1 \wedge X_n \longrightarrow X_{n+1}$ are relative cell complex inclusions.

A **CW-spectrum** is a cell spectrum such that all component spaces X_n are CW-complexes (def.).

Example 2.8. The suspension spectrum $\Sigma^\infty X$ (example 1.3) for $X \in \mathrm{Top}_{\mathrm{cg}}^{*/}$ a CW-complex is a CW-spectrum (def. 2.7).

Remark 2.9. Since, by definition 2.7, a p-cell of a cell spectrum that appears at stage q shows up as its k-fold suspension at stage $q+k$, its attachment to some spectrum X is reflected by a pushout of spectra of the form

$$
\begin{array}{ccccc}
\Sigma^\infty S^{-1}_+[-q] & \longrightarrow & X & \longrightarrow & * \\
{}^{\mathllap{\Sigma^\infty (i_p)_+[-q]}}\downarrow & (\mathrm{po}) & \downarrow & (\mathrm{po}) & \downarrow \\
\Sigma^\infty D^p_+[-q] & \longrightarrow & \hat X & \longrightarrow & \Sigma^\infty S^p[-q]
\end{array}
\,,
$$

where the left vertical morphism is the image under the $-q$th shift spectrum functor (def. 1.31) of the image under the suspension spectrum functor (example 1.3) of the basic cell inclusion $(i_p)_+$ of pointed topological spaces (def.). This is a cofibration by prop. 2.4, and so also the middle vertical morphism is a cofibration, by theorem 2.3. Using the pasting law for pushouts, we find that the cofiber of the middle vertical morphisms (hence its homotopy cofiber (def.) in the strict model structure) is $\Sigma^\infty S^p[-q]$ (not $\Sigma^\infty S^p_+[-q]$ (!)). This is a shift of a trunction of the sphere spectrum.

After having set up the stable model category structure in theorem 3.11 below, we find that this means that cell attachments to CW-spectra in the stable model structure are by cofibers of integer shifts of the sphere spectrum $\mathbb{S}$ (def. 1.4), in that in the stable homotopy category (def. 4.1) the above situation is reflected as a homotopy cofiber sequence of the form

$$
\Sigma^{p-q-1}\mathbb{S} \longrightarrow X \longrightarrow \hat X \longrightarrow \Sigma^{p-q}\mathbb{S} \,.
$$

Lemma 2.10. *Let κ be an regular cardinal and let X be a κ-cell spectrum, hence a cell spectrum (def. 2.7) obtained from at most κ stable cell attachments as in remark 2.9. Then X is κ-small (def.) with respect to morphisms of spectra that are degreewise relative* *cell complex inclusions.*

Proof. By remark 2.9 the attachment of stable cells is by free spectra (def. 3.26) on compact topological spaces. By prop. 3.28 maps out of them are equivalently maps of component spaces in the lowest nontrivial degree. Since compact topological spaces are small with respect to relative cell complex inclusions (lemma), all these cells are small.

Now notice that κ-filtered colimits of sets commute with κ-small limtis of sets (prop.). By assumption X is a κ-small transfinite composition of pushouts of κ-small coproducts, all three of which are κ-small colimits; and let Y be the codomain of a κ-small relative cell complex inclusion, hence itself a κ-small colimit.

Now if $A = \underset{\longrightarrow}{\lim}_n \sigma_n$ is a κ-small colimit of κ-small objects σ_n, and $Y = \underset{\longrightarrow}{\lim}_i Y_i$ is a κ-small colimit, then

$$\begin{aligned} \mathrm{Hom}(A, \underset{\longrightarrow}{\lim}_i Y_i) & \simeq \mathrm{Hom}(\underset{\longrightarrow}{\lim}_\sigma c_\sigma, \underset{\longrightarrow}{\lim}_i Y_i) \\ & \simeq \underset{\longleftarrow}{\lim}_\sigma \mathrm{Hom}(c_\sigma, \underset{\longrightarrow}{\lim}_i Y_i) \\ & \simeq \underset{\longleftarrow}{\lim}_\sigma \underset{\longrightarrow}{\lim}_i \mathrm{Hom}(c_\sigma, Y_i) \\ & \simeq \underset{\longrightarrow}{\lim}_i \underset{\longleftarrow}{\lim}_\sigma \mathrm{Hom}(c_\sigma, Y_i) \\ & \simeq \underset{\longrightarrow}{\lim}_i \mathrm{Hom}(\underset{\longrightarrow}{\lim}_\sigma c_\sigma, Y_i) \\ & \simeq \underset{\longrightarrow}{\lim}_i \mathrm{Hom}(A, Y_i) \end{aligned}.$$

Hence the claim follows. ▮

Proposition 2.11. *The class of CW-spectra is closed under various operations, including*

- *finite wedge sum (def. 1.27)*
- *…*

Proposition 2.12. *A sequential spectrum $X \in \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$ is cofibrant in the strict model structure $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}}$ of theorem 2.3 precisely if*

1. *X_0 is cofibrant;*
2. *each structure map $\sigma_n : S^1 \wedge X_n \to X_{n+1}$ is a cofibration*

in the classical model structure $(\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}$ on pointed compactly generated topological spaces (thm., prop.).

In particular cell spectra and specifically CW-spectra (def. 2.7) are cofibrant.

Proof. The initial object in $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}}$ is the spectrum $*$ that is constant on the point (example 1.26). A morphism $* \to X$ is a cofibration according to def. 2.1 if

1. the morphism $* \to X_0$ is a classical cofibration, hence if the object X_0 is a classical cofibrant object, hence a retract of a cell complex;
2. the morphisms

$$*_{n+1} \underset{S^1 \wedge *_n}{\sqcup} S^1 \wedge X_n \longrightarrow X_{n+1}$$

are classical cofibrations. But since $S^1 \wedge * \simeq * \overset{\simeq}{\to} *$ is an isomorphism in this case the pushout reduces to just its second summand, and so this is now equivalent to

$$S^1 \wedge X_n \longrightarrow X_{n+1}$$

being classical cofibrations; hence retracts of relative cell complexes.

∎

Proposition 2.13. *For $X \in \mathrm{SeqSpec}(\mathrm{Top})_{\mathrm{stable}}$ a CW-spectrum, def. 2.7, then its standard cylinder spectrum $X \wedge (I_+)$ of def. 1.28 satisfies the conditions on an abstract cylinder object (def.) in that the inclusion*

$$X \vee X \longrightarrow X \wedge (I_+)$$

(of the wedge sum of X with itself, example 1.27) is a cofibration in $\mathrm{SeqSpec}(\mathrm{Top})_{\mathrm{stable}}$.

Proof. According to def. 2.1 we need to check that for all n the morphism

$$(X \vee X)_{n+1} \underset{S^1 \wedge (X \vee X)_n}{\sqcup} S^1 \wedge (X \wedge (I_+))_n \longrightarrow (X \wedge (I_+))_{n+1}$$

is a retract of a relative cell complex. After distributing indices and smash products over wedge sums, this is equivalently

$$(X_{n+1} \vee X_{n+1}) \underset{(S^1 \wedge X_n) \vee (S^1 \wedge X_n))}{\sqcup} S^1 \wedge X_n \wedge (I_+) \longrightarrow X_{n+1} \wedge (I_+)\;.$$

Now by the assumption that X is a CW-spectrum, each X_n is a CW-complex, and this implies that $X_n \wedge (I_+)$ is a relative cell complex in $\mathrm{Top}^{*/}$. With this, inspection shows that also the above morphism is a relative cell complex. ∎

We now turn to discussion of CW-approximation of sequential spectra. First recall the relative version of CW-approximation for topological spaces.

For the following, recall that a continuous function $f : X \to Y$ between topological spaces is called an **n-connected map** if the induced morphism on homotopy groups $\pi_\bullet(f) : \pi_\bullet(X, x) \to \pi_\bullet(Y, f(x))$ is

1. an isomorphism in degree $< n$;
2. an epimorphism in degree n.

(Hence an weak homotopy equivalence is an "∞-connected map".)

Lemma 2.14. *Let $f : A \longrightarrow X$ be a continuous function between topological spaces. Then there exists for each $n \in \mathbb{N}$ a relative CW-complex $\hat{f} : A \hookrightarrow \hat{Y}$ together with an extension $\phi : Y \to X$, i.e.*

$$\begin{array}{ccc} A & \overset{f}{\longrightarrow} & X \\ {}^{\hat{f}}\downarrow & \nearrow_{\phi} & \\ \hat{X} & & \end{array}$$

such that ϕ is n-connected.

Moreover:

- *if f itself is k-connected, then the relative CW-complex $\hat{f}$ may be chosen to have cells* *only of dimension $k+1 \leq \dim \leq n$.*

- *if A is already a CW-complex, then $\hat f : A \to X$ may be chosen to be a subcomplex inclusion.*

(tomDieck 08, theorem 8.6.1)

Proposition 2.15. *For every continuous function $f : A \longrightarrow X$ out of a CW-complex A, there exists a relative CW-complex $\hat f : A \longrightarrow \hat X$ that factors f followed by a weak homotopy equivalence*

$$\begin{array}{ccccc} A & & \overset{f}{\longrightarrow} & & X \\ & {}_{\hat f}\searrow & & \nearrow_{\underset{\in \mathrm{WHE}}{\phi}} \\ & & \hat X \end{array}$$

Proof. Apply lemma 2.14 iteratively for $n \in \mathbb{N}$ to produce a sequence with cocone of the form

$$\begin{array}{ccccccc} A & \overset{f_0}{\longrightarrow} & X_0 & \overset{f_2}{\longrightarrow} & X_1 & \longrightarrow & \cdots \\ & {}_{f}\searrow & \downarrow^{\phi_0} & \swarrow_{\phi_1} & \cdots \\ & & X \end{array} \,,$$

where each f_n is a relative CW-complex adding cells exactly of dimension n, and where ϕ_n in n-connected.

Let then $\hat X$ be the colimit over the sequence (its transfinite composition) and $\hat f : A \to X$ the induced component map. By definition of relative CW-complexes, this $\hat f$ is itself a relative CW-complex.

By the universal property of the colimit this factors f as

$$\begin{array}{ccccccc} A & \overset{f_0}{\longrightarrow} & X_0 & \overset{f_1}{\longrightarrow} & X_1 & \longrightarrow & \cdots \\ & \searrow & \downarrow & \swarrow & \cdots \\ & & \hat X \\ & & \downarrow^{\phi} \\ & & X \end{array} \,.$$

Finally to see that ϕ is a weak homotopy equivalence: since n-spheres are compact topological spaces, then every map $\alpha : S^n \to \hat X$ factors through a finite stage $i \in \mathbb{N}$ as $S^n \to X_i \to \hat X$ (by this lemma). By possibly including further into higher stages, we may choose $i > n$. But then the above says that further mapping along $\hat X \to X$ is the same as mapping along ϕ_i, which is $(i > n)$-connected and hence an isomorphism on the homotopy class of α. ▮

Proposition 2.16. *For X any topological sequential spectrum (def.1.1), then there exists a CW-spectrum $\hat X$ (def. 2.7) and a homomorphism*

$$\phi : \hat X \xrightarrow{\in W_{\mathrm{strict}}} X \,.$$

which is degreewise a weak homotopy equivalence, hence a weak equivalence in the strict *model structure of theorem 2.3.*

Proof. First let $\hat X_0 \longrightarrow X_0$ be a CW-approximation of the component space in degree 0, via prop. 2.15. Then proceed by induction: suppose that for $n \in \mathbb{N}$ a CW-approximation $\phi_{k \leq n} : \hat X_{k \leq n} \to X_{k \leq n}$ has been found such that all the structure maps in degrees $< n$ are respected. Consider then the composite continuous function

$$S^1 \wedge \hat X_n \overset{S^1 \wedge \phi_n}{\longrightarrow} S^1 \wedge X_n \overset{\sigma_n}{\longrightarrow} X_{n+1} \,.$$

Applying prop. 2.15 to this function factors it as

$$S^1 \wedge \hat X_n \hookrightarrow \hat X_{n+1} \overset{\phi_{n+1}}{\longrightarrow} X_{n+1} \,.$$

Hence we have obtained the next stage $\hat X_{n+1}$ of the CW-approximation. The respect for the structure maps is just this factorization property:

$$\begin{array}{ccc} S^1 \wedge \hat X_n & \overset{S^1 \wedge \phi_n}{\longrightarrow} & S^1 \wedge X_n \\ {}^{\mathllap{incl}}\downarrow && \downarrow^{\mathrlap{\sigma_n}} \\ \hat X_{n+1} & \underset{\phi_{n+1}}{\longrightarrow} & X_{n+1} \end{array} \,.$$

▮

Topological enrichment

We discuss here how the hom-set of homomorphisms between any two sequential spectra is naturally equipped with a topology, and how these *hom-spaces* interact well with the strict model structure on sequential spectra from theorem 2.3. This is in direct analogy to the compatibility of compactly generated mapping spaces (def.) with the classical model structure on compactly generated topological spaces discussed at *Classical homotopy theory – Topological enrichment*. It gives an improved handle on the analysis of morphisms of spectra below in the proof of the stable model structure and it paves the way to the discussion of fully fledge *mapping spectra* below in *part 1.2*. There we will give a fully general account of the principles underlying the following. Here we just consider a pragmatic minimum that allows us to proceed.

Definition 2.17. For $X,Y \in SeqSpec(Top_{cg})$ two sequential spectra (def. 1.1) let

$$SeqSpec(X,Y) \in Top_{cg}^{*/}$$

be the pointed topological space whose underlying set is the hom-set $Hom_{SeqSpec(Top_{cg})}(X,Y)$ of homomorphisms from X to Y, and which is equipped with the final topology (def.) generated by those functions

$$\phi \;\colon\; K \longrightarrow Hom_{SeqSpec(Top_{cg})}(X,Y)$$

out of compact Hausdorff spaces K, for which there exists a homomorphism of spectra

$$\tilde \phi \;\colon\; X \wedge K \longrightarrow Y$$

out of the smash tensoring of X with K (def. 1.6) such that for all $y \in K$, $n \in \mathbb{N}$, $x \in X_n$

$$\phi(y)_n(x) = \tilde \phi_n(x,y) \,.$$

By construction this makes $\mathrm{SeqSpec}(X,Y)$ indeed into a compactly generated topological space, and it gives a natural bijection

$$\mathrm{Hom}_{\mathrm{Top}^{*/}_{\mathrm{cg}}}(K, \mathrm{SeqSpec}(X,Y)) \simeq \mathrm{Hom}_{\mathrm{SeqSpec}(\mathrm{Top}^{*/}_{\mathrm{cg}})}(X \wedge K, Y) .$$

In *Prelude -- Classical homotopy theory* we discussed, in the section *Topological enrichment*, that the classical model structure on topological spaces (when restricted to compactly generated topological spaces) interacts well with forming smash products and pointed mapping spaces. Concretely, the smash pushout product of two classical cofibrations is a classical cofibration, and is acyclic if either of the factors is:

$$\mathrm{Cof}_{\mathrm{cl}} \square \mathrm{Cof}_{\mathrm{cl}} \subset \mathrm{Cof}_{\mathrm{cl}} \;, \qquad (\mathrm{Cof}_{\mathrm{cl}} \cap W_{\mathrm{cl}}) \square \mathrm{Cof}_{\mathrm{cl}} \subset \mathrm{Cof}_{\mathrm{cl}} \cap W_{\mathrm{cl}} \;.$$

We also saw that, by Joyal-Tierney calculus (prop.), this is equivalent to the *pullback powering* satisfying the dual relations

$$\mathrm{Fib}_{\mathrm{cl}}^{\square \mathrm{Cof}_{\mathrm{cl}}} \subset \mathrm{Fib}_{\mathrm{cl}} \;, \quad \mathrm{Fib}_{\mathrm{cl}}^{\square (\mathrm{Cof}_{\mathrm{cl}} \cap W_{\mathrm{cl}})} \subset \mathrm{Fib}_{\mathrm{cl}} \cap W_{\mathrm{cl}} \;, \quad (\mathrm{Fib}_{\mathrm{cl}} \cap W_{\mathrm{cl}})^{\square \mathrm{Cof}_{\mathrm{cl}}} \subset \mathrm{Fib}_{\mathrm{cl}} \cap W_{\mathrm{cl}} \;.$$

Now that we passed from spaces to spectra, def. 1.6 generalizes the smash product of spaces to the smash tensoring of sequential spectra by spaces, and generalizes the pointed mapping space construction for spaces to the powering of a space into a sequential spectrum. Accordingly there is now the analogous concept of *pushout product* with respect to smash tensoring, and of *pullback powering* with respect to smash powering.

From the way things are presented, it is immediate that these operations on spectra satisfy the analogous compatibility condition with the strict model structure on spectra from theorem 2.3, in fact this follows generally for topologically enriched functor categories and is inherited via prop. 1.23. But since this will be important for some of the discussion to follow, we here make it explicit:

Definition 2.18. Let $f : X \to Y$ be a morphism in $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$ (def. 1.1) and let $i : A \to B$ a morphism in $\mathrm{Top}^{*/}_{\mathrm{cg}}$.

Their **pushout product with respect to smash tensoring** is the universal morphism

$$f \square i \coloneqq ((\mathrm{id}, i), (f, \mathrm{id}))$$

in

$$\begin{array}{ccccc} & & X \wedge A & & \\ & {}^{(f,\mathrm{id})}\swarrow & & \searrow^{(\mathrm{id},i)} & \\ Y \wedge A & & (\mathrm{po}) & & X \wedge B \\ & \searrow & & \swarrow & \\ & & (Y \wedge A) \underset{X \wedge A}{\sqcup} (X \wedge B) & & \\ & & \downarrow^{((\mathrm{id},i),(f,\mathrm{id}))} & & \\ & & Y \wedge B & & \end{array} ,$$

where $(-) \wedge (-)$ denotes the smash tensoring from def. 1.6.

Dually, their **pullback powering** is the universal morphism

$$f^{\square i} \coloneqq (\mathrm{Maps}(B,f)_*, \mathrm{Maps}(i,X)_*)$$

in

$$
\begin{tikzcd}
 & \mathrm{Maps}(B,X)_* \arrow[d, "{(\mathrm{Maps}(B,f)_*, \mathrm{Maps}(i,X)_*)}"] & \\
 & \mathrm{Maps}(B,Y)_* \underset{\mathrm{Maps}(A,Y)_*}{\times} \mathrm{Maps}(A,X)_* \arrow[dl] \arrow[dr] & \\
\mathrm{Maps}(B,Y)_* \arrow[dr, "\mathrm{Maps}(i,Y)_*"'] & (\mathrm{pb}) & \mathrm{Maps}(A,X)_* \arrow[dl, "\mathrm{Maps}(A,p)_*"] \\
 & \mathrm{Maps}(A,Y)_* &
\end{tikzcd},
$$

where $\mathrm{Maps}(-,-)_*$ denotes the smash powering from def. 1.6.

Similarly, for $f : X \to Y$ and $i : A \to B$ both morphisms of sequential spectra, then their pullback powering is the universal morphism

$$f^{\square i} \coloneqq (\mathrm{SeqSpec}(B,f), \mathrm{SeqSpec}(i,X))$$

in

$$
\begin{tikzcd}
 & \mathrm{SeqSpec}(B,X)_* \arrow[d, "{(\mathrm{SeqSpec}(B,f)_*, \mathrm{SeqSpec}(i,X)_*)}"] & \\
 & \mathrm{SeqSpec}(B,Y)_* \underset{\mathrm{SeqSpec}(A,Y)_*}{\times} \mathrm{SeqSpec}(A,X)_* \arrow[dl] \arrow[dr] & \\
\mathrm{SeqSpec}(B,Y)_* \arrow[dr, "\mathrm{SeqSpec}(i,Y)_*"'] & (\mathrm{pb}) & \mathrm{SeqSpec}(A,X)_* \arrow[dl, "\mathrm{SeqSpec}(A,p)_*"] \\
 & \mathrm{SeqSpec}(A,Y)_* &
\end{tikzcd},
$$

where now $\mathrm{SeqSpec}(-,-)$ is the hom-space functor from def. 2.17.

Proposition 2.19. *The operation of forming pushout products with respect to smash tensoring in def. 2.18 is compatible with the strict model structure on sequential spectra from theorem 2.3 and with the classical model structure on compactly generated pointed topological spaces (thm., prop.) in that it takes two cofibrations to a cofibration, and to an acyclic cofibration if at least one of the inputs is acyclic:*

$$
\begin{aligned}
\mathrm{Cof}_{\mathrm{strict}} \square \mathrm{Cof}_{\mathrm{cl}} &\subset \mathrm{Cof}_{\mathrm{strict}} \\
\mathrm{Cof}_{\mathrm{strict}} \square (\mathrm{Cof}_{\mathrm{cl}} \square W_{\mathrm{cl}}) &\subset \mathrm{Cof}_{\mathrm{strict}} \cap W_{\mathrm{strict}} \\
(\mathrm{Cof}_{\mathrm{strict}} \cap W_{\mathrm{strict}}) \square \mathrm{Cof}_{\mathrm{cl}} &\subset \mathrm{Cof}_{\mathrm{strict}} \cap W_{\mathrm{strict}}
\end{aligned}
.
$$

Dually, the pullback powering satisfies

$$
\begin{aligned}
\mathrm{Fib}_{\mathrm{strict}}^{\square \mathrm{Cof}_{\mathrm{cl}}} &\subset \mathrm{Fib}_{\mathrm{strict}} \\
\mathrm{Fib}_{\mathrm{strict}}^{\square (\mathrm{Cof}_{\mathrm{cl}} \cap W_{\mathrm{cl}})} &\subset \mathrm{Fib}_{\mathrm{strict}} \cap W_{\mathrm{strict}} \\
(\mathrm{Fib}_{\mathrm{strict}} \cap W_{\mathrm{strict}})^{\square \mathrm{Cof}_{\mathrm{cl}}} &\subset \mathrm{Fib}_{\mathrm{strict}} \cap W_{\mathrm{strict}}
\end{aligned}
.
$$

Proof. The statement concering the pullback powering follows directly form the analogous statement for topological spaces (prop.) by the fact that via theorem 2.3 the fibrations and weak equivalences in $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}}$ are degree-wise those in $(\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}$. From this the statement about the pushout product follows dually by Joyal-Tierney calculus (prop.). ▮

Remark 2.20. In the language of model category-theory, prop. 2.19 says that $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}}$ is an *enriched model category*, the enrichment being over $(\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}$. This is often referred to simply as a "topological model category".

Proposition 2.21. *For $X \in \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$ a sequential spectrum, $f \in \mathrm{Mor}(\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}}))$ any morphism of sequential spectra, and for $g \in \mathrm{Mor}(\mathrm{Top}_{\mathrm{cpt}}^{*/})$ a morphism of compact Hausdorff spaces, then the hom-spaces of def. 2.17 interact with the pushout-product and pullback-powering from def. 2.18 in that there is a natural isomorphism*

$$\mathrm{SeqSpec}(f \,\square\, g, X) \simeq \mathrm{SeqSpec}(f, X)^{\square g}\,.$$

Proposition 2.22. *For $X, Y \in \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$ two sequential spectra with X a CW-spectrum (def. 2.7), then there is a natural bijection*

$$\pi_0\, \mathrm{SeqSpec}(X,Y) \simeq [X,Y]_{\mathrm{strict}}$$

between the connected components of the hom-space from def. 2.17 and the hom-set in the homotopy category (def.) of the strict model structure from theorem 2.3.

Proof. By def. 2.17 the path components of the hom-space are the left homotopy classes of morphisms of spectra with respect to the standard cylinder spectrum of def. 1.28:

$$\frac{I_+ \longrightarrow \mathrm{SeqSpec}(X,Y)}{X \wedge (I_+) \longrightarrow Y}\,.$$

By prop. 2.13, for X a CW-spectrum then the standard cylinder spectrum $X \wedge (I_+)$ is a good cyclinder object (def.) on a cofibrant object.

Since moreover every object in $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}}$ is fibrant, the statement follows (with this lemma). ∎

3. The stable model structure on sequential spectra

The actual spectrum objects of interest in stable homotopy theory are not the pre-spectra of def. 1.1, but the Omega-spectra of def. 1.16 among them. Hence we need to equip the category of sequential pre-spectra of def. 1.1 with a model structure (def.) whose fibrant-cofibrant objects are, in particular Omega-spectra. More in detail, it is plausible to require that every pre-spectrum is weakly equivalent to a fibrant-cofibrant one which is both an Omega-spectrum and a CW-spectrum as in def. 2.7. By prop. 2.12 this suggests to construct a model category structure on $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$ that has the same cofibrations as the strict model structure of theorem 2.3, but more weak equivalences (and hence less fibrations), such as to make every sequential pre-spectrum weakly equivalent to an Omega cell spectrum.

Such a situation is called a *Bousfield localization of a model category*.

Bousfield localization

In plain category theory, a *localization* of a category $\mathcal{C}$ is equivalently a full subcategory

$$i : \mathcal{C}_{\mathrm{loc}} \hookrightarrow \mathcal{C}$$

such that the inclusion functor has a left adjoint L

$$\mathcal{C}_{\mathrm{loc}} \underoverset{\underset{i}{\longrightarrow}}{\overset{L}{\longleftarrow}}{\bot} \mathcal{C} \,.$$

The adjunction unit $\eta_X : X \to L(X)$ "reflects" every object X of $\mathcal{C}$ into one in the $\mathcal{C}_{\mathrm{loc}}$, and therefore this is also called a *reflective subcategory* inclusion.

It is a classical fact (Gabriel-Zisman 67, prop.) that in this situation

$$\mathcal{C}_{\mathrm{loc}} \simeq \mathcal{C}[W_L^{-1}]$$

is equivalently the localization (def.) of $\mathcal{C}$ at the "L-equivalences", namely at those morphisms f such that $L(f)$ is an isomorphism. Hence one also speaks of *reflective localizations*.

The following concept of *Bousfield localization of model categories* is the evident lift of this concept of reflective localization from the realm of categories to the realm of model categories (def.), where isomorphism is generealized to weak equivalence and where adjoint functors are taken to exhibit Quillen adjunctions.

Definition 3.1. A **left Bousfield localization** $\mathcal{C}_{\mathrm{loc}}$ of a model category $\mathcal{C}$ (def.) is another model category structure on the same underlying category with the same cofibrations,

$$\mathrm{Cof}_{\mathrm{loc}} = \mathrm{Cof}$$

but more weak equivalences

$$W_{\mathrm{loc}} \supset W \,.$$

Notice that:

Proposition 3.2. *Given a left Bousfield localization $\mathcal{C}_{\mathrm{loc}}$ of $\mathcal{C}$ as in def. 3.1, then*

1. $\mathrm{Fib}_{\mathrm{loc}} \subset \mathrm{Fib}$*;*

2. $W_{\mathrm{loc}} \cap \mathrm{Fib}_{\mathrm{loc}} = W \cap \mathrm{Fib}$*;*

3. *the identity functors constitute a Quillen adjunction*

$$\mathcal{C}_{\mathrm{loc}} \underoverset{\underset{\mathrm{id}}{\longrightarrow}}{\overset{\mathrm{id}}{\longleftarrow}}{\bot} \mathcal{C} \,.$$

4. *the induced adjunction of derived functors (prop.) exhibits a reflective subcategory inclusion of homotopy categories (def.)*

$$\mathrm{Ho}(\mathcal{C}_{\mathrm{loc}}) \underoverset{\underset{\mathbb{R}\,\mathrm{id}}{\longrightarrow}}{\overset{\mathbb{L}\,\mathrm{id}}{\longleftarrow}}{\bot} \mathrm{Ho}(\mathcal{C}) \,.$$

Proof. Regarding the first two items:

Using the properties of the weak factorization systems (def.) of (acyclic cofibrations, fibrations) and (cofibrations, acyclic fibrations) for both model structures we get

$$\begin{aligned} \mathrm{Fib}_{\mathrm{loc}} &= (\mathrm{Cof}_{\mathrm{loc}} \cap W_{\mathrm{loc}})\mathrm{Inj} \\ &\subset (\mathrm{Cof}_{\mathrm{loc}} \cap W)\mathrm{Inj} \\ &= \mathrm{Fib} \end{aligned}$$

and

$$
\begin{aligned}
\mathrm{Fib}_{\mathrm{loc}} \cap W_{\mathrm{loc}} &= \mathrm{Cof}_{\mathrm{loc}}\mathrm{Inj} \\
&= \mathrm{Cof}\,\mathrm{Inj} \quad . \\
&= \mathrm{Fib} \cap W
\end{aligned}
$$

Regarding the third point:

By construction, $\mathrm{id} : \mathcal{C} \to \mathcal{C}_{\mathrm{loc}}$ preserves cofibrations and acyclic cofibrations, hence is a left Quillen functor.

Regarding the fourth point:

Since $\mathrm{Cof}_{\mathrm{loc}} = \mathrm{Cof}$ the notion of left homotopy in $\mathcal{C}_{\mathrm{loc}}$ is the same as that in $\mathcal{C}$, and hence the inclusion of the subcategory of local cofibrant-fibrant objects into the homotopy category of the original cofibrant-fibrant objects is clearly a full inclusion. Since $\mathrm{Fib}_{\mathrm{loc}} \subset \mathrm{Fib}$ by the first statement, on these cofibrant-fibrant objects the right derived functor of the identity is just the identity and hence does exhibit this inclusion. The left adjoint to this inclusion is given by $\mathbb{L}\mathrm{id}$, by the general properties of Quillen adjunctions (prop). ■

In plain category theory, given a reflective subcategory

$$
\mathcal{C}_{\mathrm{loc}} \underoverset{\underset{i}{\longrightarrow}}{\overset{L}{\longleftarrow}}{\bot} \mathcal{C}
$$

then the composite

$$
Q \coloneqq i \circ L : \mathcal{C} \longrightarrow \mathcal{C}
$$

is an idempotent monad on $\mathcal{C}$, hence, in particular, an endofunctor equipped with a natural transformation $\eta_X : X \to LX$ (the adjunction unit) – which “reflects” every object into one in the image of L – such that this reflection is a projection in that each $L(\eta_X)$ is an isomorphism. This characterizes the reflective subcategory $\mathcal{C}_{\mathrm{loc}} \hookrightarrow \mathcal{C}$ as the subcategory of those objects X for which η_X is an isomorphism.

The following is the lift of this alternative perspective of reflective localization via idempotent monads from category theory to model category theory.

Definition 3.3. Let $\mathcal{C}$ be a model category (def.) which is right proper (def.), in that pullback along fibrations preserves weak equivalences.

Say that a **Quillen idempotent monad** on $\mathcal{C}$ is

1. an endofunctor

   $$
   Q : \mathcal{C} \longrightarrow \mathcal{C}
   $$

2. a natural transformation

   $$
   \eta : \mathrm{id}_{\mathcal{C}} \longrightarrow Q
   $$

such that

1. (homotopical functor) Q preserves weak equivalences;

2. (idempotency) for all $X \in \mathcal{C}$ the morphisms

   $$
   Q(\eta_X) : Q(X) \overset{\in W}{\longrightarrow} Q(Q(X))
   $$

and

$$\eta_{Q(X)} : Q(X) \overset{\in W}{\longrightarrow} Q(Q(X))$$

are weak equivalences;

3. (right-properness of the localization) if in a pullback square in $\mathcal{C}$

$$\begin{array}{ccc} f^* Z & \overset{f^* h}{\longrightarrow} & X \\ \downarrow & (pb) & \downarrow^{f} \\ Z & \underset{h}{\longrightarrow} & Y \end{array}$$

we have that

   1. f is a fibration;

   2. η_X, η_Y, and $Q(h)$ are weak equivalences

then $Q(f^* h)$ is a weak equivalence.

Definition 3.4. For $Q : \mathcal{C} \longrightarrow \mathcal{C}$ a Quillen idempotent monad according to def. 3.3, say that a morphism f in $\mathcal{C}$ is

1. a **Q-weak equivalence** if $Q(f)$ is a weak equivalence;

2. **a Q-cofibation** if it is a cofibration.

3. **a Q-fibration** if it has the right lifting property against the morphisms that are both (Q-)cofibrations as well as Q-weak equivalences.

Write

$$\mathcal{C}_Q$$

for $\mathcal{C}$ equipped with these classes of morphisms.

Since Q preserves weak equivalences (by def. 3.3) then if the classes of morphisms in def. 3.4 do constitute a model category structure, then this is a left Bousfield localization of $\mathcal{C}$, according to def. 3.1.

We establish a couple of lemmas that will prove that the model structure indeed exists (prop. 3.7 below).

Lemma 3.5. *In the situation of def. 3.4, a morphism is an acyclic fibration in $\mathcal{C}_Q$ precisely if it is an acyclic fibration in $\mathcal{C}$.*

Proof. Let f be a fibration and a weak equivalence. Since Q preserves weak equivalences by condition 1 in def. 3.3, f is also a Q-weak equivalence. Since Q-cofibrations are cofibrations, the acyclic fibration f has right lifting against Q-cofibrations, hence in particular against against Q-acyclic Q-cofibrations, hence is a Q-fibration.

In the other direction, let $f : X \longrightarrow Y$ be a Q-acyclic Q-fibration. Consider its factorization into a cofibration followed by an acyclic fibration

$$f : X \underset{\in \mathrm{Cof}}{\overset{i}{\longrightarrow}} Z \underset{\in W \cap \mathrm{Fib}}{\overset{p}{\longrightarrow}} Y \,.$$

Observe that Q-equivalences satisfy two-out-of-three (def.), by functoriality and since the plain equivalences do. Now the assumption that Q preserves weak equivalences together with two-out-of-three implies that i is a Q-weak equivalence, hence a Q-acyclic Q-cofibration. This implies that f has the right lifting property against i (since f is assumed to be a Q-fibration, which is defined by this lifting property). Hence the retract argument (prop.) implies that f is a retract of the acyclic fibration p, and so is itself an acyclic fibration. ▮

Lemma 3.6. *In the situation of def. 3.4, if a morphism $f : X \longrightarrow Y$ is a fibration, and if η_X, η_Y are weak equivalences, then f is a Q-fibration.*

(e.g. Goerss-Jardine 96, chapter X, lemma 4.4)

Proof. We need to show under the given assumptions that for every commuting square of the form

$$\begin{array}{ccc} A & \overset{\alpha}{\longrightarrow} & X \\ {}^{\mathllap{\in W_Q \cap Cof_Q}}{}^{\mathllap{i}}\downarrow && \downarrow^{\mathrlap{f}} \\ B & \underset{\beta}{\longrightarrow} & Y \end{array}$$

there exists a lifting.

To that end, first consider a factorization of the image under Q of this square as follows:

$$\begin{array}{ccc} Q(A) & \overset{Q(\alpha)}{\longrightarrow} & Q(X) \\ {}^{\mathllap{Q(i)}}\downarrow && \downarrow^{\mathrlap{Q(f)}} \\ Q(B) & \underset{Q(\beta)}{\longrightarrow} & Q(Y) \end{array} \quad \simeq \quad \begin{array}{ccccc} Q(A) & \underoverset{\in W \cap Cof}{j_\alpha}{\longrightarrow} & Z & \underoverset{\in Fib}{p_\alpha}{\longrightarrow} & Q(X) \\ {}^{\mathllap{Q(i)}}\downarrow && \downarrow^{\mathrlap{\pi}} && \downarrow^{\mathrlap{Q(f)}} \\ Q(B) & \underoverset{j_\beta}{\in W \cap Cof}{\longrightarrow} & W & \underoverset{p_\beta}{\in Fib}{\longrightarrow} & Q(Y) \end{array}$$

(This exists even without assuming functorial factorization: factor the bottom morphism, form the pullback of the resulting p_β, observe that this is still a fibration, and then factor (through j_α) the universal morpism from the outer square into this pullback.)

Now consider the pullback of the right square above along the naturality square of $\eta : \mathrm{id} \to Q$, take this to be the right square in the following diagram

$$\begin{array}{ccccccc} \alpha \colon & A & \overset{(j_\alpha \circ \eta_A, \alpha)}{\longrightarrow} & Z \underset{Q(X)}{\times} X & \longrightarrow & X \\ & {}^{\mathllap{i}}\downarrow && \downarrow^{\mathrlap{(\pi,f)}} && \downarrow^{\mathrlap{f}} & , \\ \beta \colon & B & \underset{(j_\beta \circ \eta_B, \beta)}{\longrightarrow} & W \underset{Q(Y)}{\times} Y & \longrightarrow & Y \end{array}$$

where the left square is the universal morphism into the pullback which is induced from the naturality squares of η on α and β.

We claim that (π, f) here is a weak equivalence. This implies that we find the desired lift by factoring (π, f) into an acyclic cofibration followed by an acyclic fibration and then lifting consecutively as follows

$$
\begin{array}{ccccccc}
\alpha : & A & \overset{(j_\alpha \circ \eta_A, \alpha)}{\longrightarrow} & Z \underset{Q(X)}{\times} X & \longrightarrow & X \\
& {}^{\mathrm{id}}\downarrow & & {}^{\in W \cap \mathrm{Cof}}\downarrow & {}^{\exists}\nearrow & \downarrow^{f}_{\in \mathrm{Fib}} \\
& A & \longrightarrow & & \longrightarrow & Y \\
& {}^{i}_{\in \mathrm{Cof}}\downarrow & {}^{\exists}\nearrow & \downarrow^{\in W \cap \mathrm{Fib}} & & \downarrow^{\mathrm{id}} \\
\beta : & B & \underset{(j_\beta \circ \eta_B, \beta)}{\longrightarrow} & W \underset{Q(Y)}{\times} Y & \longrightarrow & Y
\end{array}
.
$$

To see that (ϕ, f) indeed is a weak equivalence:

Consider the diagram

$$
\begin{array}{ccccc}
Q(A) & \underset{\in W \cap \mathrm{Cof}}{\overset{j_\alpha}{\longrightarrow}} & Z & \underset{\in W}{\overset{\mathrm{pr}_1}{\longleftarrow}} & Z \underset{Q(X)}{\times} X \\
{}^{Q(i)}_{\in W}\downarrow & & \downarrow^{\pi} & & \downarrow^{(\pi, f)} \\
Q(B) & \underset{j_\beta}{\overset{\in W \cap \mathrm{Cof}}{\longrightarrow}} & Z & \underset{\mathrm{pr}_2}{\overset{\in W}{\longleftarrow}} & W \underset{Q(X)}{\times} X
\end{array}
.
$$

Here the projections are weak equivalences as shown, because by assumption in def. 3.3 the ambient model category is right proper and these projections are the pullbacks along the fibrations p_α and p_β of the morphisms η_X and η_Y, respectively, where the latter are weak equivalences by assumption. Moreover $Q(i)$ is a weak equivalence, since i is a Q-weak equivalence.

Hence now it follows by two-out-of-three (def.) that π and then (π, f) are weak equivalences. ▮

Proposition 3.7. ***(Bousfield-Friedlander theorem)***

Let $\mathcal{C}$ be a right proper model category. Let $Q : \mathcal{C} \longrightarrow \mathcal{C}$ be a Quillen idempotent monad on $\mathcal{C}$, according to def. 3.3.

Then the Bousfield localization model category $\mathcal{C}_Q$ (def. 3.1) at the Q-weak equivalences (def. 3.4) exists, in that the model structure on $\mathcal{C}$ with the classes of morphisms in def. 3.4 exists.

(Bousfield-Friedlander 78, theorem 8.7, Bousfield 01, theorem 9.3, Goerss-Jardine 96, chapter X, lemma 4.5, lemma 4.6, theorem 4.1)

Proof. The existence of limits and colimits is guaranteed since $\mathcal{C}$ is already assumed to be a model category. The two-out-of-three poperty for Q-weak equivalences is an immediate consequence of two-out-of-three for the original weak equivalences of $\mathcal{C}$. Moreover, according to lemma 3.5 the pair of classes $(\mathrm{Cof}_Q, W_Q \cap \mathrm{Fib}_Q)$ equals the pair $(\mathrm{Cof}, W \cap \mathrm{Fib})$, and this is a weak factorization system by the model structure $\mathcal{C}$.

Hence it remains to show that $(W_Q \cap \mathrm{Cof}_Q,\ \mathrm{Fib}_Q)$ is a weak factorization system. The condition $\mathrm{Fib}_Q = \mathrm{RLP}(W_Q \cap \mathrm{Cof}_Q)$ holds by definition of Fib_Q. Once we show that every morphism factors as $W_Q \cap \mathrm{Cof}_Q$ followed by Fib_Q, then the condition $W_Q \cap \mathrm{Cof}_Q = \mathrm{LLP}(\mathrm{Fib}_Q)$ follows from the retract argument (lemma) and the fact that the classes W_Q and Cof_Q are closed under retracts, because W and $\mathrm{Cof} = \mathrm{Cof}_Q$ are (by this prop. and this prop., respectively).

So we may conclude by showing the existence of $(W_Q \cap \mathrm{Cof}_Q,\ \mathrm{Fib}_Q)$ factorizations:

First we consider the case of morphisms of the form $f : Q(Y) \to Q(Y)$. These may be factored with respect to $\mathcal{C}$ as

$$f \;:\; Q(X) \underset{\in W \cap \mathrm{Cof}}{\overset{\in i}{\longrightarrow}} Z \underset{\in \mathrm{Fib}}{\overset{p}{\longrightarrow}} Q(Y) \;.$$

Here i is already a Q-acyclic Q-cofibration, since Q preserves weak equivalences by the first clause in def. 3.3. Now apply $\mathrm{id} \overset{\eta}{\to} Q$ to obtain

$$\begin{array}{ccccccc} f: & Q(X) & \underset{\in W \cap \mathrm{Cof}}{\overset{i}{\longrightarrow}} & Z & \underset{\in \mathrm{Fib}}{\overset{p}{\longrightarrow}} & Q(Y) \\ & \downarrow^{\eta_{Q(X)}}_{\in W} && \downarrow^{\eta_Z} && \downarrow^{\eta_{Q(Y)}}_{\in W} & , \\ & Q(Q(X)) & \underset{Q(i)}{\overset{\in W}{\longrightarrow}} & Q(Z) & \longrightarrow & Q(Q(Y)) \end{array}$$

where $\eta_{Q(X)}$ and $\eta_{Q(Y)}$ are weak equivalences by idempotency (the second clause in def. 3.3), and $Q(i)$ is a weak equivalence since Q preserves weak equivalences. Hence by two-out-of-three also η_Z is a weak equivalence. Therefore lemma 3.6 gives that p is a Q-fibration, and hence the above factorization is already as desired

$$f \;:\; Q(X) \underset{\in W_Q \cap \mathrm{Cof}_Q}{\overset{\in i}{\longrightarrow}} Z \underset{\in \mathrm{Fib}_Q}{\overset{p}{\longrightarrow}} Q(Y) \;.$$

Now for an arbitrary morphism $g : X \to Y$, form a factorization of $Q(g)$ as above and then decompose the naturality square for η on g into the pullback of the resulting Q-fibration along η_Y:

$$\begin{array}{ccccccc} g: & X & \overset{\tilde i}{\longrightarrow} & Z \underset{Q(Y)}{\times} Y & \overset{\tilde p \in \mathrm{Fib}_Q}{\longrightarrow} & Y \\ & {}^{\eta_X}_{\in W_Q}\downarrow && \downarrow^{\eta'} & (\mathrm{pb}) & \downarrow^{\eta_Y}_{\in W_Q} & . \\ Q(g): & Q(Y) & \underset{i}{\overset{\in W_Q}{\longrightarrow}} & Z & \underset{p}{\overset{\in \mathrm{Fib}_Q}{\longrightarrow}} & Q(Y) \end{array}$$

This exhibits η' as the pullback of a Q-weak equivalence along a fibration between objects on which η is a weak equivalence. Then the third clause in def. 3.3 says that η' is itself as a Q-weak equivalence. This way, two-out-of-three implies that $\tilde i$ is a Q-weak equivalence.

Observe that $\tilde p$ is a Q-fibration, because it is the pullback of a Q-fibration and because Q-fibrations are defined by a right lifting property (def. 3.4) and hence closed under pullback (prop.) Finally, apply factorization in $(\mathrm{Cof}, W \cap \mathrm{Fib})$ to $\tilde i$ to obtain the desired factorization

$$f \;:\; \underset{W_Q \cap \mathrm{Cof}}{\overset{\tilde i_L}{\longrightarrow}} \underset{W \cap \mathrm{Fib} = W_Q \cap \mathrm{Fib}_Q}{\overset{\tilde i_R}{\longrightarrow}} \underset{\mathrm{Fib}_Q}{\overset{\tilde p}{\longrightarrow}} \;.$$

▮

While this establishes the Q-model structure, so far this leaves open a more explicit description of the Q-fibrations. This is provided by the next statement.

Proposition 3.8. *For $Q : \mathcal{C} \longrightarrow \mathcal{C}$ a Quillen idempotent monad according to def. 3.3, then a morphism $f : X \to Y$ in $\mathcal{C}$ is a Q-fibration (def. 3.4) precisely if*

1. *f is a fibration;*

2. *the η-naturality square on f*

$$\begin{array}{ccc} X & \overset{\eta_X}{\longrightarrow} & Q(X) \\ {}^{\mathllap{f}}\downarrow & (\mathrm{pb})^h & \downarrow^{\mathrlap{Q(f)}} \\ Y & \underset{\eta_Y}{\longrightarrow} & Q(Y) \end{array}$$

exhibits a homotopy pullback in $\mathcal{C}$ (def.), in that for any factorization of $Q(f)$ through a weak equivalence followed by a fibration p, then the universally induced morphism

$$X \longrightarrow p^\ast Y$$

is weak equivalence (in $\mathcal{C}$).

(e.g. Goerss-Jardine 96, chapter X, theorem 4.8)

Proof. First consider the case that f is a fibration and that the square is a homotopy pullback. We need to show that then f is a Q-fibration.

Factor $Q(f)$ as

$$Q(f) \;\colon\; Q(X) \underset{\in W \cap \mathrm{Cof}}{\overset{i}{\longrightarrow}} Z \underset{\in \mathrm{Fib}}{\overset{p}{\longrightarrow}} Q(Y) \,.$$

By the proof of prop. 3.7, the morphism p is also a Q-fibration. Hence by the existence of the Q-local model structure, also due to prop. 3.7, its pullback $\tilde p$ is also a Q-fibration

$$\begin{array}{ccc} X & \overset{\eta_X}{\longrightarrow} & Q(X) \\ {}^{\mathllap{\tilde i}}_{\mathllap{\in W}}\downarrow && \downarrow^{\mathrlap{i}}_{\mathrlap{\in W}} \\ Y \underset{Q(Y)}{\times} Z & \overset{p^\ast \eta_Y}{\longrightarrow} & Z \\ {}^{\mathllap{\tilde p}}_{\mathllap{\in \mathrm{Fib}_Q}}\downarrow & (\mathrm{pb}) & \downarrow^{\mathrlap{p}}_{\mathrlap{\in \mathrm{Fib}_Q}} \\ Y & \underset{\eta_Y}{\longrightarrow} & Q(Y) \end{array} \,.$$

Here $\tilde i$ is a weak equivalence by assumption that the diagram exhibits a homotopy pullback. Hence it factors as

$$\tilde i \;\colon\; X \underset{\in W \cap \mathrm{Cof}}{\overset{j}{\longrightarrow}} \hat X \underset{\in W \cap \mathrm{Fib} = W_Q \cap \mathrm{Fib}_Q}{\overset{\pi}{\longrightarrow}} Y \underset{Q(Y)}{\times} Z \,.$$

This yields the situation

$$\begin{array}{ccc} X & \overset{=}{\longrightarrow} & X \\ {}^{\mathllap{j}}_{\mathllap{\in W \cap \mathrm{Cof}}}\downarrow & {}^{\exists}\nearrow & \downarrow^{\mathrlap{f}}_{\mathrlap{\in \mathrm{Fib}}} \\ \hat X & \underset{\tilde p \circ \pi}{\overset{\in \mathrm{Fib}_Q}{\longrightarrow}} & Y \end{array} \;\;\;\;\; \leftrightarrow \;\;\;\;\; \begin{array}{ccccc} X & \overset{j}{\longrightarrow} & \hat X & \overset{\exists}{\longrightarrow} & X \\ {}^{\mathllap{f}}\downarrow && \downarrow^{\mathrlap{\tilde p \circ \pi}} && \downarrow^{\mathrlap{f}} \\ Y &=& Y &=& Y \end{array} .$$

As in the retract argument (prop.) this diagram exhibits f as a retract (in the arrow category, rmk.) of the Q-fibration $\tilde p \circ \pi$. Hence by the existence of the Q-model structure (prop. 3.7) and by the closure properties for fibrations (prop.), also f is a Q-fibration.

Now for the converse. Assume that f is a Q-fibration. Since $\mathcal{C}_Q$ is a left Bousfield localization of $\mathcal{C}$ (prop. 3.7), f is also a fibration (prop. 3.2). We need to show that the η-naturality square on f exhibits a homotopy pullback.

So factor $Q(f)$ as before, and consider the pasting composite of the factorization of the given square with the naturality squares of η:

$$
\begin{array}{ccccc}
X & \underset{\in W_Q}{\overset{\eta_X}{\longrightarrow}} & Q(X) & \underset{\in W \subset W_Q}{\overset{\eta_{Q(X)}}{\longrightarrow}} & Q(Q(X)) \\
{}^{\mathllap{\in W_Q}}{}^{\tilde i}\downarrow && {}^{\mathllap{\in W \subset W_Q}}{}^{i}\downarrow && \downarrow^{Q(i)}_{\in W} \\
Y \underset{Q(Y)}{\times} Z & \underset{\in W_Q}{\overset{p^\ast \eta_Y}{\longrightarrow}} & Z & \underset{\in W}{\overset{\eta_Z}{\longrightarrow}} & Q(Z) \\
{}^{\mathllap{\in \mathrm{Fib}_Q}}{}^{\tilde p}\downarrow & (\mathrm{pb}) & \downarrow^p_{\in \mathrm{Fib}_Q \subset \mathrm{Fib}} && \downarrow^{Q(p)} \\
Y & \underset{\eta_Y}{\overset{\in W_Q}{\longrightarrow}} & Q(Y) & \underset{\eta_{Q(Y)}}{\overset{\in W \subset W_Q}{\longrightarrow}} & Q(Q(Y))
\end{array}
\,.
$$

Here the top and bottom horizontal morphisms are weak (Q-)equivalences by the idempotency of Q, and $Q(i)$ is a weak equivalence since Q preserves weak equivalences (first and second clause in def. 3.3). Hence by two-out-of-three also η_Z is a weak equivalence. From this, lemma 3.6 gives that p is a Q-fibration. Then $p^\ast \eta_Y$ is a Q-weak equivalence since it is the pullback of a Q-weak equivalence along a fibration between objects whose η is a weak equivalence, via the third clause in def. 3.3. Finally two-out-of-three implies that $\tilde i$ is a Q-weak equivalence.

In particular, the bottom right square is a homotopy pullback (since two opposite edges are weak equivalences, by this prop.), and since the left square is a genuine pullback of a fibration, hence a homotopy pullback, the total bottom rectangle here exhibits a homotopy pullback by the pasting law for homotopy pullbacks (prop.).

Now by naturality of η, that total bottom rectangle is the same as the following rectangle

$$
\begin{array}{ccccc}
Y \underset{Q(Y)}{\times} Z & \overset{\eta\left(Y \underset{Q(Y)}{\times} Z\right)}{\longrightarrow} & Q(Y \underset{Q(Y)}{\times} Z) & \underset{\in W}{\overset{Q(p^\ast \eta_Y)}{\longrightarrow}} & Q(Z) \\
{}^{\mathllap{\in \mathrm{Fib}_Q}}{}^{\tilde p}\downarrow && \downarrow^{Q(\tilde p)} && \downarrow^{Q(p)} \\
Y & \underset{\eta_Y}{\longrightarrow} & Q(Y) & \underset{Q(\eta_Y)}{\overset{\in W}{\longrightarrow}} & Q(Q(Y))
\end{array}
\,,
$$

where now $Q(p^\ast \eta_Y) \in W$ since $p^\ast \eta_Y \in W_Q$, as we had just established. This means again that the right square is a homotopy pullback (prop.), and since the total rectangle still is a homotopy pullback itself, by the previous remark, so is now also the left square, by the other direction of the pasting law for homotopy pullbacks (prop.).

So far this establishes that the η-naturality square of $\tilde p$ is a homotopy pullback. We still need to show that also the η-naturality square of f is a homotopy pullback.

Factor $\tilde i$ as a cofibration followed by an acyclic fibration. Since $\tilde i$ is also a Q-weak equivalence, by the above, two-out-of-three for Q-fibrations gives that this factorization is of the form

$$
X \underset{\in W_Q \cap \mathrm{Cof} = W_Q \cap \mathrm{Cof}_Q}{\overset{j}{\longrightarrow}} \hat X \underset{\in W \cap \mathrm{Fib} = W_Q \cap \mathrm{Fib}_Q}{\overset{\pi}{\longrightarrow}} Y \underset{Q(Y)}{\times} Z \,.
$$

As in the first part of the proof, but now with $(W \cap \mathrm{Cof}, \mathrm{Fib})$ replaced by $(W_Q \cap \mathrm{Cof}_Q, \mathrm{Fib}_Q)$ and using lifting in the Q-model structure, this yields the situation

$$
\begin{array}{ccc}
X & \overset{=}{\longrightarrow} & X \\
{}^{\in W_Q \cap \mathrm{Cof}_Q}{}^{j}\downarrow & {}^{\exists}\nearrow & \downarrow^{f}_{\in \mathrm{Fib}_Q} \\
\hat X & \underset{\tilde p \circ \pi}{\longrightarrow} & Y
\end{array}
\qquad \leftrightarrow \qquad
\begin{array}{ccccc}
X & \overset{j}{\longrightarrow} & \hat X & \overset{\exists}{\longrightarrow} & X \\
{}^{f}\downarrow && \downarrow^{\tilde p \circ \pi} && \downarrow^{f} \\
Y & = & Y & = & Y
\end{array}
\,.
$$

As in the retract argument (prop.) this diagram exhibits f as a retract (in the arrow category, rmk.) of $\tilde p \circ \pi$.

Observe that the η-naturality square of the weak equivalence π is a homotopy pullback, since Q preserves weak equivalences (first clause of def. 3.3) and since a square with two weak equivalences on opposite sides is a homotopy pullback (prop.). It follows that also the η-naturality square of $\tilde p \circ \pi$ is a homotopy pullback, by the pasting law for homotopy pullbacks (prop.).

In conclusion, we have exhibited f as a retract (in the arrow category, rmk.) of a morphism $\tilde p \circ \pi$ whose η-naturality square is a homotopy pullback. By naturality of η, this means that the whole η-naturality square of f is a retract (in the category of commuting squares in $\mathcal{C}$) of a homotopy pullback square. This means that it is itself a homotopy pullback square (prop.). ▮

Proof of the stable model structure

We show now that the operation of Omega-spectrification of topological sequental spectra, from def. 1.19, is a Quillen idempotent monad in the sense of def. 3.3. Via the Bousfield-Friedlander theorem (prop. 3.7) this establishes the stable model structure on topological sequential spectra in theorem 3.11 below.

Lemma 3.9. *The Omega-spectrification (Q,η) from def. 1.19 preserves homotopy pullbacks (def.) in the strict model structure $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}}$ from theorem 2.3.*

(Schwede 97, lemma 2.1.3 (e))

Proof. Since, by prop. 1.20, Q preserves weak equivalences, it is sufficient to show that every pullback square in $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$ of a fibration

$$
\begin{array}{ccc}
B \underset{Y}{\times} X & \longrightarrow & X \\
\downarrow & (\mathrm{pb}) & \downarrow^{\in \mathrm{Fib}} \\
B & \longrightarrow & Y
\end{array}
$$

is taken by Q to a homotopy pullback square. By prop. 1.25 we need to check that this is the case for the kth component space of the sequential spectra in the diagram, for all $k \in \mathbb{N}$.

Let $Z^X_{i,k}$, $Z^Y_{i,k}$ etc. denote the objects appearing in the definition of $(QX)_k \coloneqq \underset{\longrightarrow_i}{\lim} Z^X_{i,k}$, $(QY)_k \coloneqq \underset{\longrightarrow_i}{\lim} Z^Y_{i,k}$, etc. (def. 1.19).

Use the small object argument (prop.) for the set $J_{(\mathrm{Top}^{*/})}$ of acyclic generating cofibrations in $(\mathrm{Top}^{*/}_{\mathrm{cg}})_{\mathrm{Quillen}}$ (def.) to construct a functorial factorization (def.) through acyclic relative cell complex inclusions (def.) followed by Serre fibrations (def.) in each degree:

$$
Z^X_{i,k} \overset{\in J_{\mathrm{Top}}\,\mathrm{Cell}}{\longrightarrow} W_i \overset{\in \mathrm{Fib}_{\mathrm{cl}}}{\longrightarrow} Z^Y_{i,k} \,.
$$

Notice that by construction $Z^K_{\bullet,k}$ and $Z^Y_{\bullet,k}$ are sequences of relative cell complexes. This implies, by the way the small object argument works and by the commutativity of each

$$\begin{array}{ccc} Z^X_{i,k} & \overset{\in J_{(\mathrm{Top}^*/)}\,\mathrm{Cell}}{\longrightarrow} & W_i \\ {}^{\in I_{(\mathrm{Top}^*/)}\,\mathrm{Cell}}\downarrow && \downarrow \\ Z^X_{i+1,k} & \overset{\in J_{(\mathrm{Top}^*/)}\,\mathrm{Cell}}{\longrightarrow} & W_{i+1} \end{array},$$

that also $W_\bullet$ is a sequence of relative cell complex inclusions: a cell in W_i is given by the top square in the following diagram, and the total rectangle is the image of that cell as a cell in W_{i+1}:

$$\begin{array}{ccc} S^{n-1} & \overset{i_n}{\longrightarrow} & D^{n-1} \\ \downarrow && \downarrow \\ Z^X_{i,k} & \overset{\in J_{(\mathrm{Top}^*/)}\,\mathrm{Cell}}{\longrightarrow} & W_i \\ {}^{\in I_{(\mathrm{Top}^*/)}\,\mathrm{Cell}}\downarrow && \downarrow \\ Z^X_{i+1,k} & \overset{\in J_{(\mathrm{Top}^*/)}\,\mathrm{Cell}}{\longrightarrow} & W_{i+1} \end{array},$$

Therefore, forming the colimit over $i \in I$ of these sequences sends the degreewise Serre fibration to a Serre fibration (prop.): because we test for a Serre fibration by lifting against the morphism in $J_{\mathrm{Top}^*/}$, which have compact domain and codomain, and these may be taken inside the colimit over relative cell complex inclusions (by this lemma)). So we have a Serre fibration

$$\underset{\longrightarrow_i}{\lim} W_i \overset{\in W_{\mathrm{cl}}}{\longrightarrow} (QY)_k$$

for each $k \in \mathbb{N}$.

Consider then the commuting diagrams

$$\begin{array}{ccccccc} Z^B_{i,k} & \longrightarrow & Z^Y_{i,k} & \overset{\in \mathrm{Fib}_{\mathrm{cl}}}{\longleftarrow} & W_i & \overset{\in W_{\mathrm{cl}}\,\in \mathrm{Cof}_{\mathrm{cl}}}{\longleftarrow} & Z^X_{i,k} \\ \downarrow^{\phi^B}_{\in W_{\mathrm{cl}}} && \downarrow^{\phi^Y}_{\in W_{\mathrm{cl}}} && & {}^{\exists \in W_{\mathrm{cl}}}\searrow & \downarrow^{\phi^X}_{\in W_{\mathrm{cl}}} \\ \Omega^i B_{k+i} & \longrightarrow & \Omega^i Y_{k+i} & \longleftarrow & & \underset{\in \mathrm{Fib}_{\mathrm{cl}}}{\longleftarrow} & \Omega^i X_{k+i} \end{array}$$

where the vertical morphisms are composites of the weak equivalences $\phi_{i,k} \colon Z_{i+1,k} \overset{\phi_{i,k}}{\longrightarrow} \Omega Z_{i,k+1}$ from def. 1.19.

The diagonal is a chosen lift (where we use that $\Omega = \mathrm{Maps}(S^1, -)_*$ preserves Serre fibrations by prop. 0.2). This lift is a weak equivalence by two-out-of-three. On the left of the diagram this exhibits now a weak equivalence of cospan-diagrams with right leg a fibration. Therefore, since forming the limit over these cospan diagrams is a homotopy pullback (def., all objects here being fibrant), this induces a weak equivalence on these limits (prop.)

$$\kappa : Z^B_{i,k} \underset{Z^Y_{i,k}}{\times} W_i \overset{\in W_{\mathrm{cl}}}{\longrightarrow} \Omega^i B_{k+i} \underset{\Omega^i Y_{k+i}}{\times} \Omega^i X_{k+i} \simeq \Omega^i (B_{k+i} \underset{Y_{k+i}}{\times} X_{k+i}) \,.$$

By universality of the pullback there is a commuting triangle

$$
\begin{tikzcd}
Z_{i,k}^{B \times_Y X} \arrow[rr, "\rho_i"] \arrow[dr, "\phi \in W_{\mathrm{cl}}"'] & & Z_{i,k}^B \underset{Z_{i,k}^Y}{\times} W_i \arrow[dl, "\kappa \in W_{\mathrm{cl}}"] \\
& \Omega^i(B_{i+k} \underset{Y_{i+k}}{\times} X_{i+k}) &
\end{tikzcd}
$$

and hence by two-out-of-three also the top morphism is a weak equivalence.

Now observe that colimits over sequences of relative cell inclusions preserve finite limits up to weak equivalence (prop.). This follows again by using that n-spheres may be taken inside the colimits from the classical fact that filtered colimits preserve finite limits. In conclusion then, we have a weak equivalence of the form

$$
(Q(B \underset{Y}{\times} X))_k = \underset{\longrightarrow_i}{\lim} Z_{i,k}^{B \times_Y X} \underset{\in W_{\mathrm{cl}}}{\overset{\lim_i \rho_i}{\longrightarrow}} \underset{\longrightarrow_i}{\lim} \left(Z_{i,k}^B \underset{Z_{i,k}^Y}{\times} W_i \right) \overset{\in W_{\mathrm{cl}}}{\longrightarrow} \left(\underset{\longrightarrow_i}{\lim} Z_{i,k}^B \right) \underset{\lim_i Z_{i,k}^Y}{\times} \left(\underset{\longrightarrow_i}{\lim} W_i \right) = (QB)_k \underset{(QY)_k}{\times} \left(\underset{\longrightarrow_i}{\lim} W_i \right) .
$$

This exhibits (degreewise and hence globally) the homotopy pullback property to be show. ▮

Proposition 3.10. *The Omega-spectrification (Q,η) from def. 1.19 is a Quillen idempotent monad in the sense of def. 3.3 on the strict model structre theorem 2.3:*

$$
Q \;:\; \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}} \longrightarrow \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}} \,.
$$

(Schwede 97, prop. 2.1.5)

Proof. First notice that the strict model structure is indeed right proper, as demanded in def. 3.3: Since every object in $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$ is fibrant (this being so degreewise in $(\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}$) this follows from this lemma.

The first two conditions required on a Quillen idempotent monad in def. 3.3 are explicit in prop. 1.20.

The third condition follows from lemma 3.9: A pullback of a Q-equivalence along a fibration is a homotopy pullback and is hence sent by Q to another homotopy pullback square.

$$
\begin{array}{ccccccccc}
f^* Z & \overset{f^* h}{\longrightarrow} & X & & & Q(f^* Z) & \overset{Q(f^* h) \in W}{\longrightarrow} & Q(X) \\
\downarrow & (\mathrm{pb}) & \downarrow^{f \in \mathrm{Fib}} & & \Rightarrow & \downarrow & (\mathrm{pb})^h & \downarrow^{Q(f)} \\
Z & \underset{h \in W_Q}{\longrightarrow} & Y & & & Q(Z) & \underset{Q(h) \in W}{\longrightarrow} & Q(Y)
\end{array}
$$

By definition of Q-equivalence that resulting homotopy pullback square has the bottom edge a weak equivalence, and hence also the top edge is a weak equivalence (prop.). ▮

Theorem 3.11. *The left Bousfield localization of the strict model structure on sequential spectra (theorem 2.3) at the class of stable weak homotopy equivalences (def. 1.14) exists, called the **stable model structure** on topological sequential spectra*

$$
\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}} \underset{\underset{\mathrm{id}}{\longrightarrow}}{\overset{\overset{\mathrm{id}}{\longleftarrow}}{\bot}} \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}} \,.
$$

Moreover, its fibrant objects are precisely the Omega-spectra (def.1.16).

Proof. Let (Q,η) be the Omega-spectrification operation from def. 1.19. According to prop. 3.10 this is a Quillen-idempotent monad (def. 3.3) on $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}}$. Hence the

Bousfield-Friedlander theorem (prop. 3.7) asserts that the Bousfield localization of the strict model structure at the Q-equivalences exists. By prop. 1.20 these are precisely the stable weak homotopy equivalences.

Finally, by prop. 3.8 an object $X \in \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}$ is fibrant in $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}$ precisely if

$$\begin{array}{ccc} X & \overset{\eta_X}{\longrightarrow} & Q(X) \\ \downarrow && \downarrow \\ * & \longrightarrow & * \end{array}$$

exhibits a homotopy pullback in $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}}$. Since every object in $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}}$ is fibrant, the vertical morphisms here are fibrations. The pullback of $Q(X)$ along id_* is just $Q(X)$ itself, and the universally induced morphism into this pullback is just η_X itself. Hence the square is a homotopy pullback precisely if η_X is a weak equivalence in $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}}$, hence degreewise a weak homotopy equivalence. Since $Q(X)$ is an Omega-spectrum by prop. 1.20, this means precisely that X is an Omega-spectrum. ▮

Stability of the homotopy theory

We discuss that the stable model structure $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}$ of theorem 3.11 is indeed a *stable model category*, in that the canonical reduced suspension operation is an equivalence of categories from the stable homotopy category (def. 4.1) to itself. This is theorem 3.23 below.

Definition 3.12. A pointed model category $\mathcal{C}$ (exmpl.) is called a **stable model category** if the canonically induced reduced suspension and loop space object-functors (prop.) on its homotopy category (defn.) constitute an equivalence of categories

$$(\Sigma \dashv \Omega) : \mathrm{Ho}(\mathcal{C}) \underset{\Omega}{\overset{\Sigma}{\underset{\longrightarrow}{\overset{\longleftarrow}{\simeq}}}} \mathrm{Ho}(\mathcal{C}) \,.$$

Literature (Jardine 15, sections 10.3 and 10.4)

First we observe that the *alternative suspension* induces an equivalence of homotopy categories:

Lemma 3.13. *With Σ and Ω the alternative suspension and alternative looping functors from def. 1.32:*

1. *Ω preserves Omega-spectra (def. 1.16);*

2. *Σ preserves stable weak homotopy equivalences (def. 1.14).*

Proof. Regarding the first statement:

By prop. 0.2, Ω acts on component spaces and adjunct structure maps as the right Quillen functor

$$\mathrm{Maps}(S^1, -)_* : (\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}} \longrightarrow (\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}$$

on the classical model structure on pointed compactly generated topological spaces (thm., prop.). Since in this model structure all objects are fibrant, Ken Brown's lemma (prop.)

implies that with $\tilde{\sigma}_n^X$ a weak homotopy equivalence, so is $\tilde{\sigma}_n^{\Omega X} = \mathrm{Maps}(S^1, \tilde{\sigma}_n^X)$.

Regarding the second point:

Let $f : X \to Y$ be a stable weak homotopy equivalence. By the existence of the model structure $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}$ from theorem 3.11, Σf is a stable weak homotopy equivalence precisely if its image in the homotopy category $\mathrm{Ho}(\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}})$ is an isomorphism (prop.). By the Yoneda lemma (fully faithfulness of the Yoneda embedding), this is the case if for all $Z \in \mathrm{Ho}(\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}})$ the function

$$[\Sigma f, Z]_{\mathrm{stable}} \;:\; [\Sigma Y, Z]_{\mathrm{stable}} \longrightarrow [\Sigma X, Z]_{\mathrm{stable}}$$

is a bijection. By the fact that the stable model structure is a left Bousfield localization of the strict model structure with fibrant objects the Omega-spectra, this is the case equivalently (using this lemma) if

$$[\Sigma f, Z]_{\mathrm{strict}} \;:\; [\Sigma Y, Z]_{\mathrm{strict}} \longrightarrow [\Sigma X, Z]_{\mathrm{strict}}$$

is a bijection for all Omega-spectra Z. Now by the Quillen adjunction $\Sigma \dashv \Omega$ on the strict model category (prop. 2.5) this is equivalent to

$$[f, \Omega Z]_{\mathrm{strict}} \;:\; [Y, \Omega Z]_{\mathrm{strict}} \longrightarrow [X, \Omega Z]_{\mathrm{strict}}$$

being a bijection for all Omega-spectra Z. But since Ω preserves Omega-spectra by the first point above, this is still maps into a fibrant objects, hence is again equivalent (using again the property of the left Bousfield localization) to the hom in the strict model structure

$$[f, \Omega Z]_{\mathrm{stable}} \;:\; [Y, \Omega Z]_{\mathrm{stable}} \longrightarrow [X, \Omega Z]_{\mathrm{stable}}$$

being a bijection for all ΩZ. But this is indeed a bijection, since f is a stable weak homotopy equivalence, hence an isomorphism in the homotopy category. ▮

Lemma 3.14. *For X a sequential spectrum, then (using remark 1.35 to suppress parenthesis)*

1. *the structure maps constitute a homomorphism*

   $$\Sigma X[-1] \longrightarrow X$$

   (from the shift, def. 1.31, of the alternative suspension, def. 1.32) and this is a stable weak homotopy equivalence,

2. *the adjunct structure maps constitute a homomorphism*

   $$X \longrightarrow \Omega X[1]$$

   (to the shift, def. 1.31, of the alternative looping, def. 1.32)

   If X is an Omega-spectrum (def. 1.16) then this is a weak equivalence in the strict model structure (def. 2.1), hence in particular a stable weak homotopy equivalence.

Proof. The diagrams that need to commute for the structure maps to give a homomorphism as claimed are in degree 0 this one

$$
\begin{array}{ccc}
S^1 \wedge S^1 \wedge \ast & \overset{0}{\longrightarrow} & X_0 \\
{}^{S^1 \wedge 0}\downarrow && \downarrow^{\sigma_0} \\
S^1 \wedge X_0 & \underset{\sigma_0}{\longrightarrow} & X_1
\end{array}
$$

and in degree $n \geq 1$ these:

$$
\begin{array}{ccc}
S^1 \wedge S^1 \wedge X_{n-1} & \overset{S^1 \wedge \sigma_{n-1}}{\longrightarrow} & X_n \\
{}^{S^1 \wedge \sigma_{n-1}}\downarrow && \downarrow^{\sigma_n} \\
S^1 \wedge X_n & \underset{\sigma_n}{\longrightarrow} & X_{n+1}
\end{array}
.
$$

But in all these cases commutativity it trivially satisfied.

That the adjunct structure maps constitute a morphism $X \to \Omega X[1]$ follows dually.

If X is an Omega-spectrum, then by definition this last morphism is already a weak equivalence in the strict model structure, hence in particular a weak equivalence in the stable model structure.

From this it follows that also $\Sigma X[-1] \to X$ is a stable weak homotopy equivalence, because for every Omega-spectrum Y then by the adjunctions in prop. 1.36 we have a commuting diagram of the form

$$
\begin{array}{ccc}
[X,Y]_{\text{strict}} & \longrightarrow & [\Sigma X[-1], Y]_{\text{strict}} \\
{}^{\text{id}}\downarrow && \downarrow^{\simeq} \\
[X,Y]_{\text{strict}} & \underset{\simeq}{\longrightarrow} & [X, \Omega Y[1]]_{\text{strict}}
\end{array}
.
$$

(To see the commutativity of this diagram in detail, consider for any $[f] \in [X,Y]_{\text{strict}}$ chasing the element σ_n^Y in the two possible ways through the natural adjunction isomorphism:

$$
\begin{array}{ccc}
[S^1 \wedge Y_{n-1}, Y_n] & \simeq & [Y_{n-1}, \Omega Y_n] \\
{}^{[S^1 \wedge f_{n-1}, Y_n]}\downarrow && \downarrow^{[f_{n-1}, \Omega Y_n]} \\
[S^1 \wedge X_{n-1}, Y_n] & \simeq & [X_{n-1}, \Omega Y_n]
\end{array}
$$

Sending σ_n^Y down gives $\sigma_n^Y \circ S^1 \wedge f_{n-1}$ which equals (by the homomorphism property) $f_n \circ \sigma_n^X$. Instead sending σ_n^Y to the right yields $\tilde{\sigma}_n^Y$ and then down yields $\tilde{\sigma}_n^Y \circ f_{n-1}$. By commutativity this is adjunct to $f_n \circ \sigma_n^X$.)

Hence

$$
[X,Y]_{\text{strict}} \longrightarrow [\Sigma X[-1], Y]_{\text{strict}}
$$

is a bijection for all Omega-spectra Y, and so the conclusion that $\Sigma X[-1] \to X$ is a stable weak homotopy equivalence follows as in the proof of lemma 3.13. ▮

Lemma 3.15. *The total derived functor of the alternative suspension operation Σ of def. 1.32 exists and constitutes an equivalence of categories from the stable homotopy category to itself:*

$$
\Sigma \;:\; \mathrm{Ho}(\mathrm{SeqSpec}(\mathrm{Top})_{\text{stable}}) \overset{\simeq}{\longrightarrow} \mathrm{Ho}(\mathrm{SeqSpec}(\mathrm{Top})_{\text{stable}}) \;.
$$

Proof. The total derived functor of Σ exists, because by lemma 3.13 Σ preserves stable weak homotopy equivalences. Also the shift functor $[-1]$ from def. 1.31 clearly preserves stable equivalences, hence both descend to the homotopy category. There, by prop. 3.14 and remark 1.35, they are inverses of each other, up to isomorphism. ∎

Lemma 3.16. *The canonical suspension functor on the homotopy category of any model category (from this prop.) in the case of the stable homotopy category (def. 4.1) $\mathrm{Ho}(\mathrm{Spectra}) = \mathrm{Ho}(\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}})$ is represented by the "standard suspension" operation of def. 1.29.*

Proof. By CW-approximation (prop. 2.16), every object in the stable homotopy category is represented by a CW-spectrum. By prop. 2.13, on CW-spectra the canonical suspension functor on the homotopy category (from this prop.) is represented by the "standard suspension" operation of def. 1.29. ∎

The combination of lemma 3.15 with lemma 3.16 gives that in order to show that $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}$ is indeed a stable model category according to def. 3.12, we are reduced to showing that in the homotopy category the alternative suspension operation (which we know gives an equivalence) is naturally isomorphic to the standard suspension operation (which we know is the correct suspension operation). This we turn to now.

According to remark 1.34, both should be directly comparable and isomorphic in the homotopy category "in even degrees", but non-comparable in odd degree. In order to make this precise, we now introduce the concept of sequential spectra with components only in even degree and then use an adjunction back to ordinary sequential spectra.

Observe that the definition of the category $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$ of sequential spectra in def. 1.1 does not require anything specific of the circle S^1: the same kind of definition may be considered for any other pointed topological space T in place of S^1. The construction of the stable model structure $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}$ in theorem 3.11 does depend on the nature of S^1, but only in that it uses that the n-spheres $S^n = (S^1)^{\wedge n}$

1. co-represent homotopy groups in the classical pointed homotopy category: $[S^n, -]_* \simeq \pi_n(-)$;

2. are compact, so that maps out of them factor through finite stages of transfinite compositions of relative cell complex inclusions.

Both points still hold with S^1 replaced by $S^1 \wedge K_+$, for K any contractible compact topological space. Moreover, since only the stable homotopy groups matter for the construction of the stable model category, one could replace S^1 by any S^k: While the smash powers $(S^k)^{\wedge n}$ co-represent only every kth homotopy group, this is still sufficient for co-represent all the stable homotopy groups.

The following is an immediate variant of the definition 1.1 of sequential spectra:

Definition 3.17. Let $T = K_+ \in \mathrm{Top}_{\mathrm{cg}}^{*/}$ be a compact contractible topological space with a basepoint freely adjoined, and let $k \in \mathbb{N}$, $k \geq 1$.

A **sequential** $T \wedge S^k$**-spectrum** is a sequence of component spaces $X_{kn} \in \mathrm{Top}_{\mathrm{cg}}$ for $n \in \mathbb{N}$, and a sequence of structure maps of the form

$$\sigma_{k,n} : T \wedge S^k \wedge X_{kn} \longrightarrow X_{k(n+1)} \,.$$

A homomorphism of sequential $T \wedge S^k$-spectra $f : X \to Y$ is a sequence of component maps $f_{kn} : X_{kn} \to Y_{kn}$ such that all these diagrams commute:

$$
\begin{array}{ccc}
T \wedge S^k \wedge X_{kn} & \overset{T \wedge S^k \wedge f_{kn}}{\longrightarrow} & T \wedge S^k \wedge Y_{kn} \\
{}^{\mathllap{\sigma^X_{k,n}}}\downarrow && \downarrow^{\mathrlap{\sigma^Y_{k,n}}} \\
X_{k(n+1)} & \underset{f_{k(n+1)}}{\longrightarrow} & Y_{k(n+1)}
\end{array}
\,.
$$

Write

$$
\mathrm{Seq}_{T \wedge S^k}\mathrm{Spec}(\mathrm{Top}_{\mathrm{cg}})
$$

for the resulting category of sequential $T \wedge S^k$-spectra.

Proposition 3.18. *For any $T \wedge S^k$ as in def. 3.17, there exists a model category structure*

$$
\mathrm{Seq}_{T \wedge S^k}\mathrm{Spec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}
$$

on the category of sequential $T \wedge S^k$-spectra, where

- *the weak equivalences are the morphisms that induce isomorphisms under $\underset{\longrightarrow_{kn \in k\mathbb{N}}}{\lim} \pi_{kn}(-)$;*
- *the fibrations are the morphisms whose η_k-naturality square is a homotopy pullback, where $\eta_K \colon \mathrm{id} \to Q_k$ is the $K \wedge S^k$-spectrification functor defined as in def. 1.19 but with S^1 replaced by $T \wedge S^k$ throughout.*

Proof. The proof is verbatim that of theorem 3.11, with S^1 replaced by $T \wedge S^k$ throughout. ▮

Lemma 3.19. *For $k \in \mathbb{N}$, $k \geq 1$, there is a pair of adjoint functors*

$$
\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}}) \underoverset{\underset{R_k}{\longrightarrow}}{\overset{L_k}{\longleftarrow}}{\bot} \mathrm{Seq}_{S^k}\mathrm{Spec}(\mathrm{Top}_{\mathrm{cg}})
$$

between sequential spectra (def. 1.1) and sequential S^k-spectra (def. 3.17)

- *where $(R_k X)_{kn} \coloneqq X_{kn}$ and*

$$
\sigma_n^{R_k X} \;\colon\; S^k X_{kn} \simeq S^{k-1} \wedge S^1 \wedge X_{kn} \overset{S^1 \wedge \sigma^X_{kn}}{\longrightarrow} S^{k-1} \wedge X_{kn+1} \longrightarrow \cdots \longrightarrow S^1 \wedge X_{kn+(k-1)} \overset{\sigma^X_{kn+(k-1)}}{\longrightarrow} X_{k(n+1)}
$$

- *and where*

$$
(L_k \mathcal{X})_n \coloneqq \left\{ \begin{array}{ll} \mathcal{X}_n & \text{if}\; n \in k\mathbb{N} \\ S^q \wedge \mathcal{X}_{n-q} & \text{if}\; q \lt k \;\text{and}\; n-q \in k \mathbb{N} \end{array} \right.
$$

and

$$
\sigma_n^{L_k \mathcal{X}} = \left\{ \begin{array}{ll} \sigma^{\mathcal{X}}_{n-(k-1)} & \text{if}\; n+1 \in k\mathbb{N} \\ \mathrm{id}_{S^1 \wedge \mathcal{X}_n} & \text{otherwise} \end{array} \right.
\,.
$$

Moreover, for each $X \in \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$, the adjunction unit

$$
L_k R_k X \longrightarrow X
$$

is a stable weak homotopy equivalence (def. 1.14).

Proof. For ease of notation we discuss this for $k = 2$. The general case is directly analogous. To see that we have an adjunction, consider a homomorphism

$$f : L_2 \mathcal{X} \longrightarrow Y .$$

Given its even-graded component maps, then its odd-graded component maps f_{2n+1} need to fit into commuting squares of the form

$$\begin{array}{ccc} S^1 \wedge \mathcal{X}_{2n} & \overset{S^1 \wedge f_{2n}}{\longrightarrow} & S^1 \wedge Y_{2n} \\ {}^{\mathrm{id}}\downarrow && \downarrow^{\sigma^Y_{2n}} \\ S^1 \wedge \mathcal{X}_{2n} & \underset{f_{2n+1}}{\longrightarrow} & Y_{2n+1} \end{array} .$$

Since here the left map is an identity, this uniquely fixes the odd-graded components f_{2n+1} in terms of the even-graded components. Moreover, these components then make the following pasting rectangles comute

$$\begin{array}{ccc} S^2 \wedge \mathcal{X}_{2n} & \overset{S^2 \wedge f_{2n}}{\longrightarrow} & S^2 \wedge Y_{2n} \\ {}^{\simeq}\downarrow && \downarrow^{S^1 \wedge \sigma^Y_{2n}} \\ S^2 \wedge \mathcal{X}_{2n} & \overset{S^1 \wedge f_{2n+1}}{\longrightarrow} & S^1 \wedge Y_{2n+1} \\ {}^{\sigma^{\mathcal{X}}_{2n}}\downarrow && \downarrow^{\sigma^Y_{2n+1}} \\ \mathcal{X}_{2n+2} & \overset{f_{2n+2}}{\longrightarrow} & Y_{2n+2} \end{array} .$$

This equivalently exhibits f as a homomorphism of the form

$$\tilde f : \mathcal{X} \longrightarrow R_2 Y$$

and hence establishes the adjunction isomorphism.

Finally to see that the adjunction unit is a stable weak homotopy equivalence: for $X \in \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$ then the morphism of stable homotopy groups induced from

$$L_2 R_2 X \longrightarrow X$$

is in degree q given by

$$\begin{array}{ccccccccc} \underset{\longrightarrow}{\lim}(\cdots & \to & \pi_{q+2k}(X_{2k}) & & \longrightarrow & & \pi_{q+2k+2}(X_{q+2k+2}) & \to & \cdots) & = & \pi_q(L_2 R_2 X) \\ && {}^{\simeq}\downarrow &&&& {}^{\simeq}\downarrow &&&& \downarrow \\ \underset{\longrightarrow}{\lim}(\cdots & \to & \pi_{q+2k}(X_{2k}) & \longrightarrow & \pi_{2+2k+1}(X_{2k+1}) & \longrightarrow & \pi_{q+2k+2}(X_{q+2k+2}) & \to & \cdots) & = & \pi_q(X) \end{array} .$$

From this it is clear by inspection that the induced vertical map on the right is an isomorphism. Stated more abstractly: the inclusion of partially ordered sets $\mathbb{N}^{\leq}_{\mathrm{even}} \hookrightarrow \mathbb{N}^{\leq}$ is a cofinal functor and hence restriction along it preserves colimits. ▮

Definition 3.20. For

$$\alpha : T_1 \wedge S^k \longrightarrow T_2 \wedge S^k$$

any morphism, write

$$\alpha^\ast : \mathrm{Seq}_{T_2 \wedge S^k}\mathrm{Spect}(\mathrm{Top}_{\mathrm{cg}}) \longrightarrow \mathrm{Seq}_{T_1 \wedge S^k}\mathrm{Spect}(\mathrm{Top}_{\mathrm{cg}})$$

for the functor from the category of sequential $T_2 \wedge S^k$-spectra (def. 3.17) to that of $T_1 \wedge S^k$-spectra which sends any X to $\alpha^* X$ with

$$(\alpha^* X)_{kn} \coloneqq X_{kn}$$

and

$$\sigma^{\alpha^* X}_{k,n} \;\colon\; T_1 \wedge S^k \wedge X_{kn} \overset{\alpha \wedge \mathrm{id}}{\longrightarrow} T_2 \wedge S^k \wedge X_{kn} \overset{\sigma^X_{k,n}}{\longrightarrow} X_{k(n+1)} \,.$$

Lemma 3.21. *For $T \coloneqq K_+$ a compact contractible topological space with base point adjoined, and for $k \in \mathbb{N}$, write $i \colon S^k \longrightarrow T \wedge S^k$ for the canonical inclusion. Then the induced functor i^* from def. 3.20 is the right adjoint in a Quillen equivalence (def.)*

$$\mathrm{Seq}_{T \wedge S^1}\mathrm{Spec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}} \underoverset{\underset{i^*}{\longrightarrow}}{\overset{L}{\longleftarrow}}{\simeq_{\mathrm{Qu}}} \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}$$

between the stable model structures of sequential S^k-spectra and of sequential $T \wedge S^k$-spectra (prop. 3.18), respectively.

(Jardine 15, theorem 10.40)

Proof. Write $p \colon T \wedge S^1 \to S^1$ for the canonical projection.

A morphism

$$f \;\colon\; X \longrightarrow i^* Y$$

is given by components fitting into commuting squares of the form

$$\begin{array}{ccc} S^1 \wedge X_n & \overset{S^1 \wedge f_n}{\longrightarrow} & S^1 \wedge Y_n \\ {}^{\mathrm{id}}\downarrow && \downarrow^{i \wedge \mathrm{id}} \\ S^1 \wedge X_n && T \wedge S^1 \wedge Y_n \\ {}^{\sigma^X_n}\downarrow && \downarrow^{\sigma^Y_n} \\ X_{n+1} & \underset{f_{n+1}}{\longrightarrow} & Y_{n+1} \end{array} \,.$$

Since $p \circ i = \mathrm{id}$, every such diagram factors as

$$\begin{array}{ccc} S^1 \wedge X_n & \overset{S^1 \wedge f_n}{\longrightarrow} & S^1 \wedge Y_n \\ {}^{i \wedge \mathrm{id}}\downarrow && \downarrow^{i \wedge \mathrm{id}} \\ T \wedge S^1 \wedge X_n & \overset{T \wedge S^1 \wedge f_n}{\longrightarrow} & T \wedge S^1 \wedge Y_n \\ {}^{p \wedge \mathrm{id}}\downarrow && \downarrow \\ S^1 \wedge X_n && \\ {}^{\sigma^X_n}\downarrow && \downarrow^{\sigma^Y_n} \\ X_{n+1} & \underset{f_{n+1}}{\longrightarrow} & Y_{n+1} \end{array} \,.$$

Here the bottom square exhibits the components of a morphism

$$\tilde f \;\colon\; p^* X \longrightarrow Y$$

and this correspondence is clearly naturally bijective

This establishes the adjunction $p^* \dashv i^*$. This is a Quillen equivalence because for every $Z \in \mathrm{Top}_{\mathrm{cg}}^{*/}$ then by the contractibility of K there is an equivalence

$$[T \wedge S^q, Z]_* \simeq [S^q, Z]_*$$

and hence the concept of stable weak homotopy equivalences in both categories agrees. Hence any $\tilde{f} : p^* X \to Y$ is a stable weak homotopy equivalence precisely if $f : X \to i^* y$ is. ▮

With this in hand, we now finally state the comparison between standard and alternative suspension:

Lemma 3.22. *There is a natural isomorphism in the homotopy category $\mathrm{Ho}(\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}})$ of the stable model structure, between the total derived functors (prop.) of the standard suspension (def. 1.29) and of the alternative suspension (def. 1.32):*

$$\Sigma(-) \;\simeq\; (-) \wedge S^1 \qquad \in \mathrm{Ho}(\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}})$$

Notice that we agreed in Part P to suppress the notation $\mathbb{L}$ for left derived functors of the suspension functor, not to clutter the notation. If we re-instantiate this then the above says that there is a natural isomorphism

$$\mathbb{L}\Sigma \;\simeq\; \mathbb{L}((-) \wedge S^1)\,.$$

(Jardine 15, corollary 10.42, prop. 10.53)

Proof. Consider the adjunction $(L_2 \dashv R_2) : \mathrm{SeqSpec}(\mathrm{Top}) \leftrightarrow \mathrm{Seq}_2\mathrm{Spec}(\mathrm{Top})$ from lemma 3.19. We claim that there is a natural isomorphism

$$\tau \,:\, R_2(\Sigma(-)) \simeq R_2((-) \wedge S^1)\,,$$

in $\mathrm{Ho}(\mathrm{Seq}_{S^2}\mathrm{Spec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}})$.

This implies the statement, since by lemma 3.19 the adjunction unit is a stable weak equivalence, so that we get natural isomorphisms

$$\Sigma X \simeq L_2 R_2(\Sigma X) \underset{\simeq}{\overset{L_2 \tau}{\longrightarrow}} L_2 R_2(X \wedge S^1) \simeq X \wedge S^1$$

in $\mathrm{Ho}(\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}})$ (where we are using that R_2 evidently preserves cofibrant spectra, so that L_2 applied to τ represents the correct derived functor of L_2 and hence preserves this isomorphism).

Now to see that the isomorphism τ exists. Write

$$\tau_{S^2, S^1} \,:\, S^2 \wedge S^1 \overset{\simeq}{\longrightarrow} S^1 \wedge S^2$$

for the braiding isomorphism, which swaps the first two canonical coordinates with the third. Since the homotopy class of this map is trivial in that

$$[\tau_{S^2, S^1}] = 1 \in \mathbb{Z} \simeq \pi_3(S^3)$$

is the trivial element in the homotopy groups of spheres (and that is the point of passing to S^2-spectra here, because for S^1-spectra the analogous map τ_{S^1, S^1} has non-trivial class, remark 1.34) it follows that there is a left homotopy (def.) of the form

$$
\begin{tikzcd}
S^3 \arrow[r, "i_0"] \arrow[dr, "\mathrm{id}"'] & (I_+) \wedge S^3 \arrow[d] & S^3 \arrow[l, "i_1"'] \arrow[dl, "\tau_{S^2,S^1}"] \\
& S^3 &
\end{tikzcd}
$$

By forming the smash product of the entire diagram with X_{2n} and pasting on the right the naturality square for the braiding with S^1

$$
\begin{tikzcd}
S^1 \wedge S^2 \wedge X_{2n} \arrow[d, "S^1 \wedge (\sigma_{2n+1} \circ (S^1 \wedge \sigma_{2n}))"'] & S^2 \wedge X_{2n} \wedge S^1 \arrow[l, "\tau_{S^2 \wedge X_{2n}, S^1}"'] \arrow[d, "(\sigma_{2n+1} \circ (S^1 \wedge \sigma_{2n})) \wedge S^1"] \\
S^1 \wedge X_{2(n+1)} & X_{2n} \wedge S^1 \arrow[l, "\tau_{X_{2n}, S^1}"]
\end{tikzcd}
$$

this yields the diagram

$$
\begin{tikzcd}
S^3 \wedge X_{2n} \arrow[r, "i_0"] \arrow[dr, "\mathrm{id}"'] & (I_+) \wedge S^3 \wedge X_{2n} \arrow[d] & S^3 \wedge X_{2n} \arrow[l, "i_1"'] \arrow[dl, "\tau_{S^2,S^1} \wedge X_n"] & S^2 \wedge X_{2n} \wedge S^1 \arrow[l, "S^2 \wedge \tau_{X_{2n},S^1}"', "\simeq"] \arrow[d] \\
& S^3 \wedge X_{2n} \arrow[dr, "S^1 \wedge (\sigma_{2n+1} \circ (S^1 \wedge \sigma_{2n}))"'] & & \cdot \arrow[d, "(\sigma_{2n+1} \circ (S^1 \wedge \sigma_{2n})) \wedge S^1"] \\
& & S^1 \wedge X_{2n} & X_{2n} \wedge S^1 \arrow[l, "\simeq", "\tau_{X_{2n},S^1}"']
\end{tikzcd}
$$

Here the left diagonal composite is the structure map of $R_2(\Sigma X)$ in degree n, while the right vertical morphism is the structure map of $R_2(X \wedge S^1)$ in degree n. In the middle we have the structure map of an auxiliary $(I_+) \wedge S^2$-spectrum (def. 3.17)

$$
Z \in \mathrm{Seq}_{I_+ \wedge S^2}\mathrm{Spec}(\mathrm{Top}_{\mathrm{cg}}) ,
$$

and the horizontal morphisms exhibit the functors of def. 3.20 from $(I_+) \wedge S^2$-spectra to S^2-spectra with

$$
i_0^* Z = R_2(\Sigma X) \quad , \quad i_1^* Z = R_2(X \wedge S^1) \; .
$$

By lemma 3.21 and since I is contractible, these functors are equivalences of categories on the $\mathrm{Ho}(\mathrm{Seq}_{S^2}\mathrm{Spec}(\mathrm{Top}_{\mathrm{cg}}))$, and moreover they have the same inverse, namely p^* for $p : I_+ \wedge S^2 \to S^2$ the canonical projection. This implies the isomorphism.

Explicitly, due to the equivalence there exists V with $Z \simeq p^* V$ and with this we may form the composite isomorphism

$$
R_2(\Sigma X) \simeq i_0^* Z \simeq i_0^* p^* V \simeq V \simeq i_1^* p^* V \simeq i_1^* Z \simeq R_2(X \wedge S^1) \; .
$$

■

We conclude:

Theorem 3.23. *The stable model structure* $\mathrm{SeqSpec}(\mathrm{Top})_{\mathrm{stable}}$ *from theorem 3.11 indeed gives a stable model category in the sense of def. 3.12, in that the canonically induced reduced suspension functor (prop.) on its homotopy category is an equivalence of categories*

$$\Sigma : \mathrm{Ho}(\mathrm{SeqSpec}(\mathrm{Top})_{\mathrm{stable}}) \xrightarrow{\simeq} \mathrm{Ho}(\mathrm{SeqSpec}(\mathrm{Top})_{\mathrm{stable}}) \,.$$

Proof. By lemma 3.16, the canonical suspension functor is represented, on fibrant-cofibrant objects, by the standard suspension functor of def. 1.29. By prop. 3.22 this is naturally isomorphic – on the level of the homotopy category – to the alternative suspension operation of def. 1.32. Therefore the claim follows with prop. 3.15. ▮

In fact this lifts to a Quillen equivalence:

Proposition 3.24. *The $(\Sigma \dashv \Omega)$-adjunction from prop. 1.36 is a Quillen equivalence (def.) with respect to the stable model structure of theorem 3.11:*

$$\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}} \underoverset{\underset{\Omega}{\longrightarrow}}{\overset{\Sigma}{\longleftarrow}}{\simeq_Q} \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}} \,.$$

Its derived functors (prop.) exhibit the canonical reduced suspension and looping operation as an adjoint equivalence on the stable homotopy category

$$\mathrm{Ho}(\mathrm{Spectra}) \underoverset{\underset{\Omega}{\longrightarrow}}{\overset{\Sigma}{\longleftarrow}}{\simeq} \mathrm{Ho}(\mathrm{Spectra}) \,.$$

Proof. By prop. 2.5 and the fact that the stable model structure has the same cofibrations as the strict model structure, Σ preserves stable cofibrations. Moreover, by lemma 3.13 Σ preserves in fact all stable weak equivalences. Hence Σ is a left Quillen functor and so $(\Sigma \dashv \Omega)$ is a Quillen adjunction. Finally lemma 3.15 gives that this Quillen adjunction is a Quillen equivalence. ▮

In summary, this concludes the characterization of the stable homotopy category as the result of stabilizing the canonical $(\Sigma \dashv \Omega)$-adjunction on the classical homotopy category:

Theorem 3.25. *The classical model structure $(\mathrm{Top}^{*/}_{\mathrm{cg}})_{\mathrm{Quillen}}$ on pointed compactly generated topological spaces (thm., prop.) and the stable model structure on topological sequential spectra $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$ (theorem 3.11) sit in a commuting diagram of Quillen adjunctions of the form*

$$\begin{array}{ccc}
(\mathrm{Top}^{*/}_{\mathrm{cg}})_{\mathrm{Quillen}} & \underoverset{\underset{\Omega}{\longrightarrow}}{\overset{\Sigma}{\longleftarrow}}{\bot} & (\mathrm{Top}^{*/}_{\mathrm{cg}})_{\mathrm{Quillen}} \\
{}^{\Sigma^\infty}\downarrow \dashv \uparrow^{\Omega^\infty} && {}^{\Sigma^\infty}\downarrow \dashv \uparrow^{\Omega^\infty} \\
\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}} & \underoverset{\underset{\Omega}{\longrightarrow}}{\overset{\Sigma}{\longleftarrow}}{\bot} & \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}} \,, \\
{}^{\mathrm{id}}\downarrow \dashv \uparrow^{\mathrm{id}} && {}^{\mathrm{id}}\downarrow \dashv \uparrow^{\mathrm{id}} \\
\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}} & \underoverset{\underset{\Omega}{\longrightarrow}}{\overset{\Sigma}{\longleftarrow}}{\simeq_Q} & \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}
\end{array}$$

where the top parts is from corollary 2.6, the bottom vertical Quillen adjunction is the Bousfield localization of theorem 3.11 and the bottom horizontal adjunction is the Quillen equivalence of prop. 3.24.

Hence (by this prop.) the derived functors of the functors in this diagram yield a commuting square of adjoint functors between the classical homotopy category (def.) and the stable homotopy category (def. 4.1) of the form

$$
\begin{array}{ccc}
\mathrm{Ho}(\mathrm{Top}^{*/}) & \underoverset{\underset{\Omega}{\longrightarrow}}{\overset{\Sigma}{\longleftarrow}}{\bot} & \mathrm{Ho}(\mathrm{Top}^{*/}) \\
{}^{\Sigma^\infty}\downarrow \dashv \uparrow^{\Omega^\infty} && {}^{\Sigma^\infty}\downarrow \dashv \uparrow^{\Omega^\infty} \\
\mathrm{Ho}(\mathrm{Spectra}) & \underoverset{\underset{\Omega}{\longrightarrow}}{\overset{\Sigma}{\longleftarrow}}{\simeq} & \mathrm{Ho}(\mathrm{Spectra})
\end{array},
$$

where the horizontal adjunctions are the canonically induced (via this prop.)suspension/looping functors by prop. 0.2 and by lemma 3.16 and theorem 3.23.

Cofibrant generation

We show that the stable model structure $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}$ from theorem 3.11 is a cofibrantly generated model category (def.).

We will not use the result of this section in the remainder of part 1.1, but the following argument is the blueprint for the proof of the model structure on orthogonal spectra that we consider in part 1.2, in the section *The stable model structure on structured spectra*, and it will be used in the proof of the Quillen equivalence of $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}$ to the stable model structure on orthogonal spectra (thm.).

Moreover, that $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}$ is cofibrantly generated means that for $\mathcal{C}$ any topologically enriched category (def.) then there exists a projective model structure on functors $[\mathcal{C}, \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}]_{\mathrm{proj}}$ on the category of topologically enriched functors $\mathcal{C} \to \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$ (def.), in direct analogy to the projective model structure $[\mathcal{C}, (\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}]_{\mathrm{proj}}$ (thm.). This is the model structure for *parameterized stable homotopy theory*. Just as the stable homotopy theory discussed here is the natural home of generalized (Eilenberg-Steenrod) cohomology theories (example 4.6) so parameterized stable homotopy theory is the natural home of twisted cohomology theories.

In order to express the generating (acyclic) cofibrations, we need the following simple but important concept.

Definition 3.26. For $K \in \mathrm{Top}_{\mathrm{cg}}^{*/}$, and $n \in \mathbb{N}$, write $F_n K \in \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$ for the **free spectrum** on K at n, with components

$$
(F_n K)_q \coloneqq \begin{cases} * & \text{for } q < n \\ S^{q-n} \wedge K & \text{for } q \geq n \end{cases}
$$

and with structure maps σ_q the canonical identifications for $q \geq n$

$$
\sigma_q \;:\; S^1 \wedge (F_n K)_q = S^1 \wedge S^{q-n} \wedge K \overset{\simeq}{\longrightarrow} S^{q+1-n} \wedge K = (F_n K)_{q+1} \,.
$$

For $n \in \mathbb{N}$, write

$$
k_n \;:\; F_{n+1} S^1 \longrightarrow F_n S^0
$$

for the canonical morphisms of free sequential spectra with the following components

$$
\begin{array}{ccccc}
 & \vdots & & \vdots \\
(k_n)_{n+3} & S^3 & \overset{\mathrm{id}}{\longrightarrow} & S^3 \\
(k_n)_{n+2} & S^2 & \overset{\mathrm{id}}{\longrightarrow} & S^2 \\
(k_n)_{n+1} & S^1 & \overset{\mathrm{id}}{\longrightarrow} & S^1 \\
(k_n)_n : & * & \overset{0}{\longrightarrow} & S^0 \\
 & * & \longrightarrow & * \\
 & \vdots & & \vdots \\
 & * & \longrightarrow & * \\
 & \underbrace{} & & \underbrace{} \\
k_n : & F_{n+1}S^1 & \longrightarrow & F_n S^0
\end{array}
$$

Example 3.27. The free spectrum $F_0 S^0$ (def. 3.26) is the standard sequential sphere spectrum from def. 1.4

$$F_0 S^0 \simeq \mathbb{S}_{\mathrm{std}} \,.$$

Generally the free spectrum $F_0 K$ is the suspension spectrum (def. 1.3) on K:

$$F_0 K \simeq \Sigma^\infty K \,.$$

Just as forming suspension spectra is left adjoint to extracting the 0th component space of a sequential spectrum (prop. 1.10), so forming the nth free spectrum is left adjoint to extracting the nth component space:

Proposition 3.28. *For $n \in \mathbb{N}$, let*

$$\mathrm{Ev}_n \;:\; \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}}) \longrightarrow \mathrm{Top}_{\mathrm{cg}}^{*/}$$

be the functor from sequential spectra (def. 1.1) to pointed topological spaces given by extracting the nth component space

$$\mathrm{Ev}_n(X) \coloneqq X_n \,.$$

Then this functor is right adjoint to forming nth free spectra (def. 3.26):

$$(F_n \dashv \mathrm{Ev}_n) \;:\; \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}}) \underoverset{\underset{\mathrm{Ev}_n}{\longrightarrow}}{\overset{F_n}{\longleftarrow}}{\bot} \mathrm{Top}_{\mathrm{cg}}^{*/} \,.$$

Proof. The proof is verbatim as that of prop. 1.10, just with n zeros inserted at the bottom of the sequences of components maps. ▮

Definition 3.29. Write

$$I_{\mathrm{seq}}^{\mathrm{stable}} \coloneqq I_{\mathrm{seq}}^{\mathrm{strict}} \;\; \in \mathrm{SeqSpec}(\mathrm{Top})$$

for the set of morphisms appearing already in def. 2.2, and write

$$J_{\mathrm{seq}}^{\mathrm{stable}} \coloneqq J_{\mathrm{seq}}^{\mathrm{strict}} \;\sqcup\; \{k_n \Box i_+\}_{n \in \mathbb{N}, i_+ \in \left(I_{\mathrm{Top}^{*/}}\right)}$$

for the disjoint union of the other set of morphisms appearing in def. 2.2 with the set $\{k_n \Box i_+\}_{n, i_+}$ of pushout-products under smash tensoring (according to def. 2.18) of the morphisms k_n from def. 3.26 with the generating cofibrations of the classical model structure on pointed topological spaces (def.).

Theorem 3.30. *The stable model structure* $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}$ *from theorem 3.11 is cofibrantly generated (def.) with generating (acyclic) cofibrations the sets* $I_{\mathrm{seq}}^{\mathrm{stable}}$ *(and* $J_{\mathrm{seq}}^{\mathrm{stable}}$*) from def. 3.29.*

This is one of the cofibrantly model categories considered in (Mandell-May-Schwede-Shipley 01) .

Proof. It is clear (as in theorem 2.3) that the two classes have small domains (def.). Moreover, since $I_{\mathrm{seq}}^{\mathrm{stable}} = I_{\mathrm{seq}}^{\mathrm{strict}}$ and $\mathrm{Cof}_{\mathrm{stable}} = \mathrm{Cof}_{\mathrm{strict}}$ by definition, the fact that the ccofibrations are the retracts of relative $I_{\mathrm{seq}}^{\mathrm{stable}}$-cell complexes is part of theorem 2.3. It only remains to show that the stable acyclic cofibrations are precisely the retracts of relative $J_{\mathrm{seq}}^{\mathrm{stable}}$-cell complexes. This we is the statement of lemma 3.35 below. ▮

Lemma 3.31. *The morphisms of free spectra* $\{k_n\}_{n\in\mathbb{N}}$ *from def. 3.26 co-represent the adjunct structure maps of sequential spectra from def. 1.2, in that for* $X \in \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$*, then*

$$\begin{array}{ccc} \mathrm{SeqSpec}(F_n S^0, X) & \simeq & X_n \\ {}^{\mathllap{\mathrm{SeqSpec}(k_n, X)}}\downarrow && \downarrow^{\mathrlap{\tilde\sigma_n^X}} \\ \mathrm{SeqSpec}(F_{n+1}S^1, X) & \simeq & \Omega X_{n+1} \end{array} \,,$$

where on the left we have the hom-spaces of def. 2.21, and where the horizontal equivalences are via prop. 3.28.

Proof. Recall that we are precomposing with

$$\begin{array}{cccc} & \vdots && \vdots \\ (k_n)_{n+3} & S^3 & \overset{\mathrm{id}}{\longrightarrow} & S^3 \\ (k_n)_{n+2} & S^2 & \overset{\mathrm{id}}{\longrightarrow} & S^2 \\ (k_n)_{n+1} & S^1 & \overset{\mathrm{id}}{\longrightarrow} & S^1 \\ (k_n)_n : & \ast & \overset{0}{\longrightarrow} & S^0 \\ & \ast & \longrightarrow & \ast \\ & \vdots && \vdots \\ & \ast & \longrightarrow & \ast \\ & \underbrace{} && \underbrace{} \\ k_n : & F_{n+1}S^1 & \longrightarrow & F_n S^0 \end{array}$$

Now for X any sequential spectrum, then a morphism $f : F_n S^0 \to X$ is uniquely determined by its nth component $f_n : S^0 \to X_n$: the compatibility with the structure maps forces the next component, in particular, to be $\sigma_n^X \circ \Sigma f$:

$$\begin{array}{ccc} \Sigma S^0 & \overset{\Sigma f}{\longrightarrow} & \Sigma X_n \\ \downarrow^{\simeq} && \downarrow^{\sigma_n^X} \\ S^1 & \overset{\sigma_n^X \circ \Sigma f}{\longrightarrow} & X_n \end{array} .$$

But that $(n+1)$st component is just the component that similarly determines the precompositon of f with k_n, hence $f \circ k_n$ is uniquely determined by the map $\sigma_n^X \circ \Sigma f$. Therefore $\mathrm{SeqSpec}(k_n, -)$ is the function

$$\mathrm{SeqSpec}(k_n, -) \;:\; X_n = \mathrm{SeqSpec}(S^0, X_n) \overset{f \mapsto \sigma_n^X \circ \Sigma f}{\longrightarrow} \mathrm{Maps}\,(S^1, X_{n+1})_* = \Omega X_{n+1} \;.$$

It remains to see that this is indeed the $(\Sigma \dashv \Omega)$-adjunct of σ_n^X. By the general formula for adjuncts, this is

$$\tilde \sigma_n^X \;:\; X_n \overset{\eta}{\longrightarrow} \Omega \Sigma X_n \overset{\Omega \sigma_n^X}{\longrightarrow} \Omega X_{n+1} \;.$$

To compare to the above, we check what this does on points: $S^0 \overset{f}{\longrightarrow} X_n$ is sent to the composite

$$S^0 \overset{f}{\longrightarrow} X_n \overset{\eta}{\longrightarrow} \Omega \Sigma X_n \overset{\Omega \sigma_0^X}{\longrightarrow} \Omega X_{n+1} \;.$$

To identify this as a map $S^1 \to X_{n+1}$, we use the adjunction isomorphism once more to throw all the Ω-s on the right back to Σ-s the left, to finally find that this is indeed

$$\sigma_n^X \circ \Sigma f \;:\; S^1 = \Sigma S^0 \overset{\Sigma f}{\longrightarrow} \Sigma X_n \overset{\sigma_n^X}{\longrightarrow} X_{n+1} \;.$$

▮

Lemma 3.32. *Every element in $J^{\mathrm{stable}}_{\mathrm{seq}}$ (def. 3.29) is an acyclic cofibration in the model structure $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}$ from theorem 3.11.*

Proof. For the elements in $J^{\mathrm{strict}}_{\mathrm{seq}}$ this is part of theorem 2.3. It only remains to see that the morphisms $k_n \Box i_+$ are stable acyclic cofibrations.

To see that they are stable cofibrations, hence strict cofibrations:

By Joyal-Tierney calculus (prop.) $k_n \Box i_+$ has left lifting against any strict acyclic fibration f precisely if k_n has left lifting against the pullback powering $f^{\Box i_+}$ (def. 2.18). By prop. 2.19 the latter is still a strict acyclic fibration. Since k_n is evidently a strict cofibration, the lifting follows and hence also $k_n \Box i_+$ is a strict cofibration, hence a stable cofibration.

To see that they are stable weak equivalences: For each q the morphisms $k_n \wedge S^{q-1}$ are stable acyclic cofibrations, and since stable acyclic cofibrations are preserved under pushout, it follows by two-out-of-three that also $k_n \Box i_+$ is a stable weak equivalence. ▮

The reason for considering the set $\{k_n \Box i_+\}$ is to make the following true:

Lemma 3.33. *A morphism $f : X \to Y$ in $\mathrm{SeqSpec}(\mathrm{Top})$ is a $J^{\mathrm{stable}}_{\mathrm{seq}}$-injective morphism (def.) precisely if*

1. *it is fibration in the strict model structure (hence degreewise a fibration);*

2. *for all $n \in \mathbb{N}$ the commuting squares of structure map compatibilities on the underlying sequential spectra*

$$\begin{array}{ccc} X_n & \overset{\tilde \sigma_n^X}{\longrightarrow} & \Omega X_{n+1} \\ {}^{\mathllap{f_n}}\downarrow && \downarrow^{\mathrlap{\Omega f_{n+1}}} \\ Y_n & \underset{\tilde \sigma_n^Y}{\longrightarrow} & \Omega Y_{n+1} \end{array}$$

exhibit homotopy pullbacks (def.) in $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}}$, in that the comparison map

$$X_n \longrightarrow Y_n \underset{\Omega Y_{n+1}}{\times} \Omega X_{n-1}$$

is a weak homotopy equivalence (notice that Ωf_{n+1} is a fibration by the previous item and since $\Omega = \mathrm{Maps}(S^1, -)_*$ is a right Quillen functor by prop. 0.2).

In particular, the $J_{\mathrm{seq}}^{\mathrm{stable}}$-injective objects are precisely the Omega-spectra, def. 1.16.

Proof. By theorem 2.3, lifting against $J_{\mathrm{seq}}^{\mathrm{stric}}$ alone characterizes strict fibrations, hence degreewise fibrations. Lifting against the remaining pushout product morphism $k_n \square i_+$ is, by Joyal-Tierney calculus (prop.), equivalent to left lifting i_+ against the pullback powering $f^{\square k_n}$ from def. 2.18. Since the $\{i_+\}$ are the generating cofibrations in $\mathrm{Top}_{\mathrm{cg}}^{*/}$ such lifting means that $f^{\square k_n}$ is a weak equivalence in the strict model sructure. But by lemma 3.31, $f^{\square k_n}$ is precisely the comparison morphism in question. ▮

Lemma 3.34. *A morphism in $\mathrm{SeqSpec}(\mathrm{Top})$ which is both*

1. *a stable weak homotopy equivalence (def. 1.14);*
2. *a $J_{\mathrm{seq}}^{\mathrm{stable}}$-injective morphism (def. 3.29, def.)*

is an acyclic fibration in the strict model structure, hence is degreewise a weak homotopy equivalence and Serre fibration of topological spaces;

Proof. Let $f : X \to B$ be both a stable weak homotopy equivalence as well as a K-injective morphism. Since K contains the generating acyclic cofibrations for the strict model structure, f is in particular a strict fibration, hence a degreewise fibration.

Consider the fiber F of f, hence the morphism $F \to *$ which is the pullback of f along $* \to B$. Notice that since f is a strict fibration, this is the homotopy fiber (def.) of f in the strict model structure.

We claim that

1. F is an Omega-spectrum;
2. $F \to *$ is a stable weak homotopy equivalence.

The first item follows since F, being the pullback of a K-injective morphisms, is a K-injective object (prop.), so that, by lemma 3.33, F it is an Omega-spectrum.

For the second item:

Since $F \to X \overset{f}{\to} B$ is degreewise a homotopy fiber sequence, there are degreewise its long exact sequences of homotopy groups (exmpl.)

$$\cdots \to \pi_{\bullet+1}(B_n) \longrightarrow \pi_\bullet(F_n) \longrightarrow \pi_\bullet(X_n) \overset{(f_n)_*}{\longrightarrow} \pi_\bullet(B_n) \to \cdots \to \pi_1(B_n) \longrightarrow \pi_0(F_n) \longrightarrow \pi_0(X_n) \longrightarrow \pi_0(B)_n$$

Since in the category Ab of abelian group forming filtered colimits is an exact functor (prop.), it follows that after passing to stable homotopy groups the resulting sequence

$$\cdots \pi_{\bullet+1}(X) \overset{f_*}{\longrightarrow} \pi_{\bullet+1}(B) \longrightarrow \pi_\bullet(F) \longrightarrow \pi_\bullet(X) \overset{(f_*}{\longrightarrow} \pi_\bullet(B) \to \cdots$$

is still a long exact sequence.

Since, by assumption, f_* is an isomorphism, this exactness implies that $\pi_\bullet(F) = 0$, and hence that $F \to *$ is a stable weak homotopy equivalence. But since, by the first item above, F is an Omega-spectrum, it follows (via example 1.18) that $F \to *$ is even a degreewise weak homotopy equivalence, hence that $\pi_\bullet(F_n) \simeq 0$ for all $n \in \mathbb{N}$.

Feeding this back into the above degreewise long exact sequence of homotopy groups now implies that $\pi_{\bullet \geq 1}(f_n)$ is a weak homotopy equivalence for all n and for each homotopy group in positive degree.

To deduce the remaining case that also $\pi_0(f_0)$ is an isomorphism, observe that by assumption of K-injectivity, lemma 3.33 gives that f_0 is the pullback (in topological spaces) of $\Omega(f_1)$. But by the above Ωf_1 is a weak homotopy equivalence, and since $\Omega = \mathrm{Maps}(S^1, -)_*$ is a right Quillen functor (prop. 0.2) it is also a Serre fibration. Therefore f_0 is the pullback of an acyclic Serre fibration and hence itself a weak homotopy equivalence. ▮

Lemma 3.35. *The retracts (rmk.) of $J^{\mathrm{stable}}_{\mathrm{seq}}$-relative cell complexes are precisely the stable acyclic cofibrations.*

Proof. Since all elements of $J^{\mathrm{stable}}_{\mathrm{seq}}$ are stable weak equivalences and strict cofibrations by lemma 3.32, it follows that every retract of a relative $J^{\mathrm{stable}}_{\mathrm{seq}}$-cell complex has the same property.

In the other direction, let f be a stable acyclic cofibration. Apply the small object argument (prop.) to factor it

$$f : \underset{J^{\mathrm{stable}}_{\mathrm{seq}}\,\mathrm{Cell}}{\overset{i}{\longrightarrow}} \underset{J^{\mathrm{stable}}_{\mathrm{seq}}\,\mathrm{Inj}}{\overset{p}{\longrightarrow}}$$

as a $J^{\mathrm{stable}}_{\mathrm{seq}}$-relative cell complex i followed by a $J^{\mathrm{stable}}_{\mathrm{seq}}$-injective morphism p. By the previous statement i is a stable weak homotopy equivalence, and hence by assumption and by two-out-of-three so is p. Therefore lemma 3.34 implies that p is a strict acyclic fibration. But then the assumption that f is a strict cofibration means that it has the left lifting property against p, and so the retract argument (prop.) implies that f is a retract of the relative $J^{\mathrm{stable}}_{\mathrm{seq}}$-cell complex i. ▮

This completes the proof of theorem 3.30.

4. The stable homotopy category

Definition 4.1. Write

$$\mathrm{Ho}(\mathrm{Spectra}) \coloneqq \mathrm{Ho}(\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}})$$

for the homotopy category (defn.) of the stable model structure on topological sequential spectra from theorem 3.11.

This is called the **stable homotopy category**.

The stable homotopy category of def. 4.1 inherits particularly nice properties that are usefully axiomatized for themselves. This axiomatics is called *triangulated category* structure (def. 4.15 below) where the “triangles” are referring to the structure of the long fiber sequences and long cofiber sequences (prop.) which happen to coincide in stable homotopy theory.

Additivity

The stable homotopy category $\mathrm{Ho}(\mathrm{Spectra})$ is the analog in homotopy theory of the category Ab of abelian groups in homological algebra. While the stable homotopy category is *not* an *abelian category*, as Ab is, but a homotopy-theoretic version of that to which we turn below, it *is* an additive category.

Lemma 4.2. *The stable homotopy category (def. 4.1) has finite coproducts. They are represented by wedge sums (example 1.27) of CW-spectra (def. 2.7).*

Proof. Having finite coproducts means

1. having empty coproducts, hence initial objects,

2. and having binary coproducts.

Regarding the initial object:

The spectrum $\Sigma^\infty *$ (suspension spectrum (example 1.3) on the point) is both an initial object and a terminal object in $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$. This implies in particular that it is both fibrant and cofibrant. Finally its standard cylinder spectrum (example 1.28) is trivial $(\Sigma^\infty *) \wedge (I_+) \simeq \Sigma^\infty *$. All together with means that for X any fibrant-cofibrant spectrum, then

$$\mathrm{Hom}_{\mathrm{Ho}(\mathrm{Spectra})}(\Sigma^\infty *, Z) \simeq \mathrm{Hom}_{\mathrm{SeqSpec}}(\Sigma^\infty *, Z)/_\sim \simeq *$$

and so $\Sigma^\infty *$ also represents the initial object in the stable homotopy category.

Now regarding binary coproducts:

By prop. 2.16 and prop. 2.12, every spectrum has a cofibrant replacement by a CW-spectrum. By prop. 2.11 the wedge sum $X \vee Y$ of two CW-spectra is still a CW-spectrum, hence still cofibrant.

Let P and Q be fibrant and cofibrant replacement functors, respectively, as in the section Classical homotopy theory – The homotopy category.

We claim now that $P(X \vee Y) \in \mathrm{Ho}(\mathrm{Spectra})$ is the coproduct of PX with PY in $\mathrm{Ho}(\mathrm{Spectra})$. By definition of the homotopy category (def.) this is equivalent to claiming that for Z any stable fibrant spectrum (hence an Omega-spectrum by theorem 3.11) then there is a natural isomorphism

$$\mathrm{Hom}_{\mathrm{SeqSpec}}(P(X \vee Y), QZ)/_\sim \simeq \mathrm{Hom}_{\mathrm{SeqSpec}}(PX, QZ)/_\sim \times \mathrm{Hom}_{\mathrm{SeqSpec}}(PY, QZ)/_\sim$$

between left homotopy-classes of morphisms of sequential spectra.

But since $X \vee Y$ is cofibrant and Z is fibrant, there is a natural isomorphism (prop.)

$$\mathrm{Hom}_{\mathrm{SeqSpec}}(P(X \vee Y), QZ)/_\sim \overset{\simeq}{\longrightarrow} \mathrm{Hom}_{\mathrm{SeqSpec}}(X \vee Y, Z)/_\sim \,.$$

Now the wedge sum $X \vee Y$ is the coproduct in $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$, and hence morphisms out of it are indeed in natural bijection with pairs of morphisms out of the two summands. But we need this property to hold still after dividing out left homotopy. The key is that smash tensoring (def. 1.6) distributes over wedge sum

$$(X \vee Y) \wedge (I_+) \simeq (X \wedge (I_+)) \vee (Y \wedge (I_+))$$

(due to the fact that the smash product of compactly generated pointed topological spaces distributes this way over wedge sum of pointed spaces). This means that also left homotopies out of $X \vee Y$ are in natural bijection with pairs of left homotopies out of the summands separately, and hence that there is a natural isomorphism

$$\mathrm{Hom}_{\mathrm{SeqSpec}}(X \vee Y, Z)/_{\sim} \overset{\simeq}{\longrightarrow} \mathrm{Hom}_{\mathrm{SeqSpec}}(X, Z)/_{\sim} \times \mathrm{Hom}_{\mathrm{SeqSpec}}(Y, Z)/_{\sim} \;.$$

Finally we may apply the inverse of the natural isomorphism used before (prop.) to obtain in total

$$\mathrm{Hom}_{\mathrm{SeqSpec}}(X, Z)/_{\sim} \times \mathrm{Hom}_{\mathrm{SeqSpec}}(Y, Z)/_{\sim} \overset{\simeq}{\longrightarrow} \mathrm{Hom}_{\mathrm{SeqSpec}}(PX, QZ)/_{\sim} \times \mathrm{Hom}_{\mathrm{SeqSpec}}(PY, QZ)/_{\sim} \;.$$

The composite of all these isomorphisms proves the claim. ▮

Definition 4.3. Define group structure on the pointed hom-sets of the stable homotopy category (def. 4.1)

$$[X, Y] \in \mathrm{Grp}$$

induced from the fact (prop.) that the hom-sets of any homotopy category into an object in the image of the canonical loop space functor Ω inherit group structure, together with the fact (theorem 3.23) that on the stable homotopy category Ω and Σ are inverse to each other, so that

$$[X, Y] \simeq [X, \Omega\Sigma Y] \,,$$

Lemma 4.4. *The group structure on* $[X, Y]$ *in def. 4.3 is abelian and composition in* $\mathrm{Ho}(\mathrm{Spectra})$ *is bilinear with respect to this group structure. (Hence this makes* $\mathrm{Ho}(\mathrm{Spectra})$ *an Ab-enriched category.)*

Proof. Recall (prop, rmk.) that the group structure is given by concatenation of loops

$$X \overset{\Delta_X}{\to} X \times X \overset{(f,g)}{\longrightarrow} \Omega\Sigma X \times \Omega\Sigma X \longrightarrow \Omega\Sigma X \;.$$

That the group structure is abelian follows via the Eckmann-Hilton argument from the fact that there is always a compatible second (and indeed arbitrarily many compatible) further group structures, since, by stability

$$[X, Y] \simeq [X, \Omega\Sigma Y] \simeq [X, \Omega \circ (\Omega\Sigma) \circ \Sigma Y] = [X, \Omega^2\Sigma^2 Y] \;.$$

That composition of morphisms distributes over the operation in this group is evident for precomposition. Let $f : W \to X$ then clearly

$$f^\ast \;:\; [X, \Omega\Sigma Y] \longrightarrow [W, \Omega\Sigma Y]$$

preserves the group structure induced by the group structure on $\Omega\Sigma Y$. That the same holds for postcomposition may be immediately deduced from noticing that this group structure is also the same as that induced by the cogroup structure on $\Sigma\Omega X$, so that with $g : Y \to Z$ then

$$g_\ast \;:\; [\Sigma\Omega X, Y] \longrightarrow [\Sigma\Omega X, Z]$$

preserves group structure.

More explicitly, we may see the respect for groupstructure structure of the postcomposition opeation from the naturality of the loop composition map which is manifest when representing loop spectra via the standard topological loop space object $\Omega X = \mathrm{fib}(\mathrm{Maps}(I_+, X) \to X \times X)$ (rmk.) under smash powering (def. 1.6).

To make this fully explicit, consider the following diagram in $\mathrm{Ho}(\mathrm{Spectra})$:

$$
\begin{array}{ccccccccccc}
Y \times Y & \overset{\simeq}{\longrightarrow} & \Omega\Sigma Y \times \Omega \Sigma Y & \overset{\simeq}{\longrightarrow} & Q(\mathrm{Maps}(S^1, \Sigma Y)_* \times \mathrm{Maps}(S^1, \Sigma Y)_*) & \longrightarrow & Q(\mathrm{Maps}(S^1_{[0,2]}, \Sigma Y))_* & \simeq & \Omega \Sigma Y & \simeq & Y \\
{}^{g \times g}\downarrow && \downarrow^{\Omega\Sigma g \times \Omega \Sigma g} && \downarrow^{Q(\mathrm{Maps}(S^1, \Sigma g)_* \times \mathrm{Maps}(S^1, \Omega\Sigma g)_*)} && \downarrow^{Q(\mathrm{Maps}(S^1_{[0,2]}, \Sigma g)_*)} && \downarrow^{\Omega \Sigma g} && \downarrow^{g} \\
Z \times Z & \overset{\simeq}{\longrightarrow} & \Omega\Sigma Z \times \Omega \Sigma Z & \overset{\simeq}{\longrightarrow} & Q(\mathrm{Maps}(S^1, \Sigma Z)_* \times \mathrm{Maps}(S^1, \Sigma Z)_*) & \longrightarrow & Q(\mathrm{Maps}(S^1_{[0,2]}, \Sigma Z)_*) & \simeq & \Omega \Sigma Z & \simeq & Z
\end{array}
,
$$

where $S^1_{[0,2]}$ denotes the sphere of length 2.

Here the leftmost square and the rightmost square are the naturality squares of the equivalence of categories $(\Sigma \dashv \Omega)$ (theorem 3.23).

The second square from the left and the second square from the right exhibit the equivalent expression of Ω as the right derived functor of (either the standard or the alternative, by lemma 3.22) degreewise loop space functor. Here we let ΣX denote any fibrant representative, for notational brevity, and use that the derived functor of a right Quillen functor is given on fibrant objects by the original functor followed by cofibrant replacement (prop.).

The middle square is the image under Q of the evident naturality square for concatenation of loops. This is where we use that we have the standard model for forming loop spaces and concatenation of loops (rmk.): the diagram commutes because the loops are always poinwise pushed forward along the map f. ∎

It is conventional (Adams 74, p. 138) to furthermore make the following definition:

Definition 4.5. For $X, Y \in \mathrm{Ho}(\mathrm{Spectra})$ two spectra, define the $\mathbb{Z}$-graded abelian group

$$
[X,Y]_\bullet \in \mathrm{Ab}^{\mathbb{Z}}
$$

to be in degree n the abelian hom group of lemma 4.4 out of X into the n-fold suspension of Y (lemma 3.22):

$$
[X,Y]_n \coloneqq [X, \Sigma^{-n} Y] .
$$

Defining the composition of $f_1 \in [X,Y]_{n_1}$ with $f_2 \in [Y,Z]_{n_2}$ to be the composite

$$
X \overset{f_1}{\longrightarrow} \Sigma^{-n_1} Y \overset{\Sigma^{-n_2}(f_2)}{\longrightarrow} \Sigma^{-n_2}(\Sigma^{-n_1} Z) \simeq \Sigma^{-n_1 - n_2} Z
$$

gives the stable homotopy category the structure of an $\mathrm{Ab}^{\mathbb{Z}}$-enriched category.

Example 4.6. **(generalized cohomology groups)**

Let $E \in \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$ be an Omega-spectrum (def. 1.16) and let $X \in \mathrm{Top}_{\mathrm{cg}}^{*/}$ be a pointed topological space with $\Sigma^\infty X$ its suspension spectrum (example 1.3). Then the graded abelian group (by prop. 4.4, def. 4.5)

$$
\begin{aligned}
\tilde E^\bullet(X) &\coloneqq [\Sigma^\infty X, E]_{-\bullet} \\
&= [\Sigma^\infty X, \Sigma^\bullet E] \\
&\simeq [X, \Omega^\infty \Sigma^\bullet E]_* \\
&\simeq [X, E_\bullet]_*
\end{aligned}
$$

is also called the **reduced cohomology of** X in the **generalized (Eilenberg-Steenrod) cohomology theory** that is represented by E.

Here the equivalences used are

1. the adjunction isomorphism of $(\Sigma^\infty \dashv \Omega^\infty)$ from theorem 3.25;

2. the isomorphism $\Sigma \simeq [1]$ of suspension with the shift spectrum (def. 1.31) on $\mathrm{Ho}(\mathrm{Spectra})$ of lemma 3.14, together with the nature of Ω^∞ from prop. 1.10.

The latter expression

$$\tilde E^n(X) \simeq [X, E_n]_*$$

(on the right the hom in in the classical homotopy category $\mathrm{Ho}(\mathrm{Top}^{*/})$ of pointed topological spaces) is manifestly the definition of reduced generalized (Eilenberg-Steenrod) cohomology as discussed in part S in the section on the Brown representability theorem.

Suppose E here is not necessarily given as an Omega-spectrum. In general the hom-groups $[X,E] = [X,E]_{\mathrm{stable}}$ in the stable homotopy category are given by the naive homotopy classes of maps out of a cofibrant resolution of X into a fibrant resolution of E (by this lemma). By theorem 3.11 a fibrant replacement of E is given by Omega-spectrification $Q E$ (def. 1.19). Since the stable model structure of theorem 3.11 is a left Bousfield localization of the strict model structure from theorem 2.3, and since for the latter all objects are fibrant, it follows that

$$[X,E]_{\mathrm{stable}} \simeq [X, QE]_{\mathrm{strict}} \,,$$

and hence

$$\begin{aligned} E^0(X) & \coloneqq [\Sigma^\infty X, E]_{\mathrm{stable}} \\ & \simeq [\Sigma^\infty X, QE]_{\mathrm{strict}} \\ & \simeq [X, \Omega^\infty Q E]_* \\ & = [X, (QE)_0]_* \end{aligned} \,,$$

where the last two hom-sets are again those of the classical homotopy category. Now if E happens to be a CW-spectrum, then by remark 1.21 its Omega-spectrification is given simply by $(QE)_n \simeq \underset{\longrightarrow k}{\lim} \Omega^k E_{n+k})$ and hence in this case

$$E^0(X) \simeq [X, \underset{\longrightarrow k}{\lim} \Omega^k E_k]_* \,.$$

If X here is moreover a compact topological space, then it may be taken inside the colimit (e.g. Weibel 94, topology exercise 10.9.2), and using the $(\Sigma \dashv \Omega)$-adjunction this is rewritten as

$$\begin{aligned} E^0(X) & \simeq \underset{\longrightarrow k}{\lim} [X, \Omega^k E_k]_* \\ & \simeq \underset{\longrightarrow k}{\lim} [\Sigma^k X, E_k]_* \end{aligned} \,.$$

(e.g. Adams 74, prop. 2.8).

This last expression is sometimes used to define cohomology with coefficients in an arbitrary spectrum. For examples see in the part S the section *Orientation in generalized cohomology*.

More generally, it is immediate now that there is a concept of E-cohomology not only for spaces and their suspension spectra, but also for general spectra: for $X \in \mathrm{Ho}(\mathrm{Spectra})$ be any spectrum, then

$$\tilde E^\bullet(X) \coloneqq [X, \Sigma^\bullet E]$$

is called the reduced E-cohomology of the spectrum X.

Beware that here one usually drops the tilde sign.

In summary, lemma 4.2 and lemma 4.4 state that the stable homotopy category is an Ab-enriched category with finite coproducts. This is called an *additive category*:

Definition 4.7. An **additive category** is a category which is

1. an Ab-enriched category;

   (sometimes called a pre-additive category–this means that each hom-set carries the structure of an abelian group and composition is bilinear)

2. which admits finite coproducts

   (and hence, by prop. 4.8 below, finite products which coincide with the coproducts, hence finite biproducts).

Proposition 4.8. *In an Ab-enriched category, a finite product is also a coproduct, and dually.*

This statement includes the zero-ary case: any terminal object is also an initial object, hence a zero object (and dually), hence every additive category (def. 4.7) has a zero object.

More precisely, for $\{X_i\}_{i \in I}$ a finite set of objects in an Ab-enriched category, then the unique morphism

$$\coprod_{i \in I} X_i \longrightarrow \prod_{j \in I} X_j\,,$$

whose components are identities for $i = j$ and are zero otherwise, is an isomorphism.

Proof. Consider first the zero-ary case. Given an initial object $\emptyset$ and a terminal object $*$, observe that since the hom-sets $\mathrm{Hom}(\emptyset,\emptyset)$ and $\mathrm{Hom}(*,*)$ by definition contain a single element, this element has to be the zero element in the abelian group structure. But it also has to be the identity morphism, and hence $\mathrm{id}_\emptyset = 0$ and $\mathrm{id}_* = 0$. It follows that the 0-element in $\mathrm{Hom}(*,\emptyset)$ is a left and right inverse to the unique element in $\mathrm{Hom}(\emptyset,*)$, and so this is an isomorphism

$$0 : \emptyset \overset{\simeq}{\longrightarrow} * \,.$$

Consider now the case of binary (co-)products. Using the existence of the zero object, hence of zero morphisms, then in addition to its canonical projection maps $p_i : X_1 \times X_2 \to X_i$, any binary product also receives "injection" maps $X_i \to X_1 \times X_2$, and dually for the coproduct:

$$\begin{array}{ccccc} X_1 && && X_2 \\ & {}_{\mathllap{(\mathrm{id},0)}}\searrow && \swarrow_{\mathrlap{(0,\mathrm{id})}} & \\ {}^{\mathllap{\mathrm{id}_{X_1}}}\downarrow && X_1 \times X_2 && \downarrow^{\mathrlap{\mathrm{id}_{X_2}}} \\ & \swarrow_{\mathrlap{p_{X_1}}} && {}_{\mathllap{p_{X_2}}}\searrow & \\ X_1 && && X_2 \end{array} \quad , \quad \begin{array}{ccccc} X_1 && && X_2 \\ & \searrow^{\mathrlap{i_{X_1}}} && {}^{\mathllap{i_{X_2}}}\swarrow & \\ {}^{\mathllap{\mathrm{id}_{X_1}}}\downarrow && X_1 \sqcup X_2 && \downarrow^{\mathrlap{\mathrm{id}_{X_2}}} \\ & \swarrow_{\mathrlap{(\mathrm{id},0)}} && {}_{\mathllap{(0,\mathrm{id})}}\searrow & \\ X_1 && && X_2 \end{array}$$

Observe some basic compatibility of the Ab-enrichment with the product:

First, for $(\alpha_1, \beta_1), (\alpha_2, \beta_2) : R \to X_1 \times X_2$ then

$$(\star) \qquad (\alpha_1, \beta_1) + (\alpha_2, \beta_2) = (\alpha_1 + \alpha_2,\ \beta_1 + \beta_2)$$

(using that the projections p_1 and p_2 are linear and by the universal property of the porduct).

Second, $(\mathrm{id}, 0) \circ p_1$ and $(0, \mathrm{id}) \circ p_2$ are two projections on $X_1 \times X_2$ whose sum is the identity:

$$(\star\star) \qquad (\mathrm{id}, 0) \circ p_1 + (0, \mathrm{id}) \circ p_2 = \mathrm{id}_{X_1 \times X_2}\ .$$

(We may check this, via the Yoneda lemma on generalized elements: for $(\alpha, \beta) : R \to X_1 \times X_2$ any morphism, then $(\mathrm{id}, 0) \circ p_1 \circ (\alpha, \beta) = (\alpha, 0)$ and $(0, \mathrm{id}) \circ p_2 \circ (\alpha, \beta) = (0, \beta)$, so the statement follows with equation $(\star)$.)

Now observe that for $f_i : X_i \to Q$ any two morphisms, the sum

$$\phi \coloneqq f_1 \circ p_1 + f_2 \circ p_2 : X_1 \times X_2 \longrightarrow Q$$

gives a morphism of cocones

$$\begin{array}{ccccc} X_1 && && X_2 \\ & \searrow^{(\mathrm{id},0)} && {}^{(0,\mathrm{id})}\swarrow & \\ {}^{\mathrm{id}_{X_1}}\downarrow && X_1 \times X_2 && \downarrow^{\mathrm{id}_{X_2}} \\ X_1 && \downarrow^{\phi} && X_2 \\ & {}_{f_1}\searrow && \swarrow_{f_2} & \\ && Q && \end{array} .$$

Moreover, this is unique: suppose ϕ' is another morphism filling this diagram, then, by using equation $(\star\star)$, we get

$$\begin{aligned} (\phi - \phi') &= (\phi - \phi') \circ \mathrm{id}_{X_1 \times X_2} \\ &= (\phi - \phi') \circ ((\mathrm{id}_{X_1}, 0) \circ p_1 + (0, \mathrm{id}_{X_2}) \circ p_2) \\ &= \underbrace{(\phi - \phi') \circ (\mathrm{id}_{X_1}, 0)}_{=0} \circ p_1 + \underbrace{(\phi - \phi') \circ (0, \mathrm{id}_{X_2})}_{=0} \circ p_2 \\ &= 0 \end{aligned}$$

and hence $\phi = \phi'$. This means that $X_1 \times X_2$ satisfies the universal property of a coproduct.

By a dual argument, the binary coproduct $X_1 \sqcup X_2$ is seen to also satisfy the universal property of the binary product. By induction, this implies the statement for all finite (co-)products. ▮

Remark 4.9. Finite coproducts coinciding with products as in prop. 4.8 are also called *biproducts* or *direct sums*, denoted

$$X_1 \oplus X_2 \coloneqq X_1 \sqcup X_2 \simeq X_1 \times X_2\ .$$

The zero object is denoted "0", of course.

Conversely:

Definition 4.10. A **semiadditive category** is a category that has all finite products which, moreover, are biproducts in that they coincide with finite coproducts as in def. 4.8.

Proposition 4.11. *In a semiadditive category, def. 4.10, the hom-sets acquire the structure of commutative monoids by defining the sum of two morphisms $f, g : X \longrightarrow Y$ to be*

$$f + g \;\coloneqq\; X \overset{\Delta_X}{\to} X \times X \simeq X \oplus X \overset{f \oplus g}{\longrightarrow} Y \oplus Y \simeq Y \sqcup Y \overset{\nabla_X}{\to} Y \,.$$

With respect to this operation, composition is bilinear.

Proof. The associativity and commutativity of $+$ follows directly from the corresponding properties of $\oplus$. Bilinearity of composition follows from naturality of the diagonal Δ_X and codiagonal ∇_X:

$$\begin{array}{ccccccccccc} W & \overset{\Delta_W}{\to} & W \times W & \overset{\simeq}{\to} & W \oplus W \\ \downarrow^{e} && \downarrow^{e \times e} && \downarrow^{e \oplus e} \\ X & \overset{\Delta_X}{\to} & X \times X & \simeq & X \oplus X & \overset{f \oplus g}{\longrightarrow} & Y \oplus Y & \simeq & Y \sqcup Y & \overset{\nabla_X}{\to} & Y \\ &&&&&& \downarrow^{h \oplus h} && \downarrow^{h \sqcup h} && \downarrow^{h} \\ &&&&&& Z \oplus Z & \simeq & Z \sqcup Z & \overset{\nabla_Z}{\to} & Z \end{array}$$

■

Proposition 4.12. *Given an additive category according to def. 4.7, then the enrichement in commutative monoids which is induced on it via prop. 4.8 and prop. 4.11 from its underlying semiadditive category structure coincides with the original enrichment.*

Proof. By the proof of prop. 4.8, the codiagonal on any object in an additive category is the sum of the two projections:

$$\nabla_X \;:\; X \oplus X \overset{p_1 + p_2}{\longrightarrow} X \,.$$

Therefore (checking on generalized elements, as in the proof of prop. 4.8) for all morphisms $f, g : X \to Y$ we have commuting squares of the form

$$\begin{array}{ccc} X & \overset{f+g}{\longrightarrow} & Y \\ {}^{\Delta_X}\downarrow && \uparrow^{\nabla_Y = \atop p_1 + p_2} \\ X \oplus X & \underset{f \oplus g}{\longrightarrow} & Y \oplus Y \end{array}$$

■

Remark 4.13. Prop. 4.12 says that being an additive category is an extra property on a category, not extra structure. We may ask whether a given category is additive or not, without specifying with respect to which abelian group structure on the hom-sets.

In conclusion we have:

Proposition 4.14. *The stable homotopy category (def. 4.1) is an additive category (def. 4.7).*

Hence prop. 4.8 implies that in the stable homotopy category finite coproducts (wedge sums) and finite products agree, in that they are finite biproducts (direct sums).

$$\vee \simeq \times \simeq \oplus \quad \in \; \mathrm{Ho}(\mathrm{Spectra}) \,.$$

Proof. By lemma 4.2 and lemma 4.4. ■

Triangulated structure

We have seen above that the stable homotopy category $\mathrm{Ho}(\mathrm{Spectra})$ is an additive category. In the context of homological algebra, when faced with an additive category one next asks for the existence of kernels (fibers) and cokernels (cofibers) to yield a pre-abelian category, and then asks that these are suitably compatible, to yield an abelian category.

Now here in stable homotopy theory, the concept of kernels and cokernels is replaced by that of homotopy fibers and homotopy cofibers. That these certainly exist for homotopy theories presented by model categories was the topic of the general discussion in the section *Homotopy theory – Homotopy fibers*. Various of the properties they satisfy was the topic of the sections *Homotopy theory – Long sequences* and *Homotopy theory – Homotopy pullbacks.*. For the special case of *stable homotopy theory* we will find a crucial further property relating homotopy fibers to homotopy cofibers.

The axiomatic formulation of a subset of these properties of stable homotopy fibers and stable homotopy cofibers is called a *triangulated category* structure. This is the analog in stable homotopy theory of abelian category structure in homological algebra.

	category of abelian groups	**stable homotopy category**
direct sums and hom-abelian groups	additive category	additive category
(homotopy) fibers and cofibers exist	pre-additive category	homotopy category of a model category
(homotopy) fibers and cofibers are compatible	abelian category	triangulated category

Literature (Hubery, Schwede 12, II.2)

Definition 4.15. A **triangulated category** is

1. an additive category Ho (def. 4.7);

2. a functor, called the **suspension functor** or **shift functor**

   $$\Sigma : \mathrm{Ho} \overset{\simeq}{\longrightarrow} \mathrm{Ho}$$

   which is required to be an equivalence of categories;

3. a sub-class $\mathrm{CofSeq} \subset \mathrm{Mor}(\mathrm{Ho}^{\Delta[3]})$ of the class of triples of composable morphisms, called the class of **distinguished triangles**, where each element that starts at A ends at ΣA; we write these as

   $$A \longrightarrow B \longrightarrow B/A \longrightarrow \Sigma A ,$$

   or

   $$\begin{array}{ccccc} A && \overset{f}{\longrightarrow} && B \\ & {}_{[1]}\nwarrow && \swarrow & \\ && B/A && \end{array}$$

   (whence the name *triangle*);

such that the following conditions hold:

- **T0** For every morphism $f : A \to B$, there does exist a distinguished triangle of the form

  $$A \overset{f}{\longrightarrow} B \longrightarrow B/A \longrightarrow \Sigma A \,.$$

  If (f, g, h) is a distinguished triangle and there is a commuting diagram in Ho of the form

  $$\array{ A &\overset{f}{\longrightarrow}& B &\overset{g}{\longrightarrow}& B/A &\overset{h}{\longrightarrow}& \Sigma A \\ \downarrow^{\mathrlap{\in Iso}} && \downarrow^{\mathrlap{\in Iso}} && \downarrow^{\mathrlap{\in Iso}} && \downarrow^{\mathrlap{\in Iso}} \\ A' &\overset{f'}{\longrightarrow}& B' &\overset{g'}{\longrightarrow}& B'/A' &\overset{h'}{\longrightarrow}& \Sigma A' }$$

  (with all vertical morphisms being isomorphisms) then (f', g', h') is also a distinguished triangle.

- **T1** For every object $X \in \mathrm{Ho}$ then $(0, id_X, 0)$ is a distinguished triangle

  $$0 \longrightarrow X \overset{id_X}{\longrightarrow} X \longrightarrow 0 \,;$$

- **T2** If (f, g, h) is a distinguished triangle, then so is $(g, h, -\Sigma f)$; hence if

  $$A \overset{f}{\longrightarrow} B \overset{g}{\longrightarrow} B/A \overset{h}{\longrightarrow} \Sigma A$$

  is, then so is

  $$B \overset{g}{\longrightarrow} B/A \overset{h}{\longrightarrow} \Sigma A \overset{-\Sigma f}{\longrightarrow} \Sigma B \,.$$

- **T3** Given a commuting diagram in Ho of the form

  $$\array{ A &\longrightarrow& B &\longrightarrow& B/A &\longrightarrow& \Sigma A \\ \downarrow^{\mathrlap{\phi_A}} && \downarrow^{\mathrlap{\phi_B}} \\ A' &\longrightarrow& B' &\longrightarrow& B'/A' &\longrightarrow& \Sigma A' }$$

  where the top and bottom are distinguished triangles, then there exists a morphism $B/A \to B'/A'$ such as to make the completed diagram commute

  $$\array{ A &\longrightarrow& B &\longrightarrow& B/A &\longrightarrow& \Sigma A \\ \downarrow^{\mathrlap{\phi_A}} && \downarrow^{\mathrlap{\phi_B}} && \downarrow^{\mathrlap{\exists}} && \downarrow^{\mathrlap{\Sigma \phi_A}} \\ A' &\longrightarrow& B' &\longrightarrow& B'/A' &\longrightarrow& \Sigma A' }$$

- **T4 (octahedral axiom)** For every pair of composable morphisms $f : A \to B$ and $f' : B \to D$ then there is a commutative diagram of the form

$$
\begin{array}{ccccccc}
A & \overset{f}{\longrightarrow} & B & \overset{g}{\longrightarrow} & B/A & \overset{h}{\longrightarrow} & \Sigma A \\
{}^{=}\downarrow && {}^{f'}\downarrow && \downarrow^{x} && \downarrow^{=} \\
A & \underset{f' \circ f}{\longrightarrow} & D & \underset{g''}{\longrightarrow} & D/A & \underset{h''}{\longrightarrow} & \Sigma A \\
&& {}^{g'}\downarrow && \downarrow^{y} \\
&& D/B & \underset{\simeq}{\longrightarrow} & D/B \\
&& {}^{h'}\downarrow && \downarrow^{(\Sigma g)\circ h'} \\
&& \Sigma B & \underset{\Sigma g}{\longrightarrow} & \Sigma B/A
\end{array}
$$

such that the two top horizontal sequences and the two middle vertical sequences each are distinguished triangles.

Proposition 4.16. *The stable homotopy category $\mathrm{Ho}(\mathrm{Spectra})$ from def. 4.1, equipped with the canonical suspension functor $\Sigma \colon \mathrm{Ho}(\mathrm{Spectra}) \overset{\simeq}{\longrightarrow} \mathrm{Ho}(\mathrm{Spectra})$ (according to this prop.) is a triangulated category (def. 4.15) for the distinguished triangles being the closure under isomorphism of triangles of the images (under localization $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}} \to \mathrm{Ho}(\mathrm{Spectra})$ (prop.) of the stable model category of theorem 3.11) of the canonical long homotopy cofiber sequences (prop.)*

$$
A \overset{f}{\longrightarrow} B \longrightarrow \mathrm{hocofib}(f) \longrightarrow \Sigma A \,.
$$

(e.g. Schwede 12, chapter II, theorem 2.9)

Proof. By prop. 4.14 the stable homotopy category is additive, by theorem 3.23 the functor Σ is an equivalence.

The axioms T0 and T1 are immediate from the definition of homotopy cofiber sequences.

The axiom T2 is the very characterization of long homotopy cofiber sequences (from this prop.).

Regarding axiom T3:

By the factorization axioms of the model category we may represent the morphisms $A \to A'$ and $B \to B'$ in the homotopy category by cofibrations in the model category. Then $B \to B/A$ and $B' \to B'/A'$ are represented by their ordinary cofibers (def., prop.).

We may assume without restriction (lemma) that the commuting square

$$
\begin{array}{ccc}
A & \overset{f}{\longrightarrow} & B \\
{}^{\phi_A}\downarrow && \downarrow^{\phi_B} \\
A' & \underset{f'}{\longrightarrow} & B'
\end{array}
$$

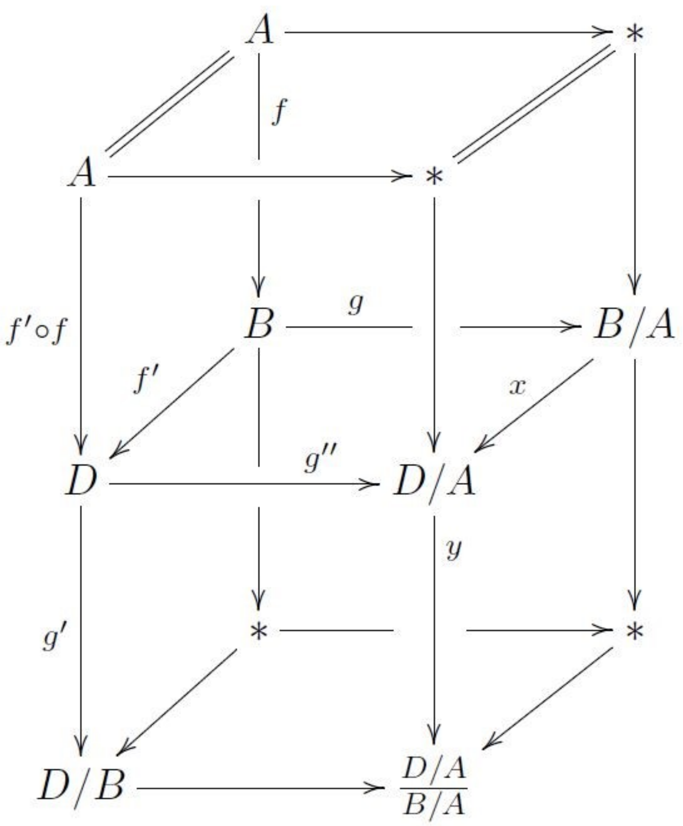
in the homotopy category is the image of a commuting square (not just commuting up to homotopy) in $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$. In this case then the morphism $B/A \to B'/A'$ is induced by the universal property of ordinary cofibers. To see that this also completes the last vertical morphism, observe that by the small object argument (prop.) we have functorial factorization (def.).

With this, again the universal property of the ordinary cofiber gives the fourth vertical morphism needed for T3.

Axiom T4 follows in the same fashion: we may represent all spectra by CW-spectra and represent f and f', hence also $f' \circ f$, by cofibrations. Then forming the functorial mapping cones as above produces the commuting diagram

$$\begin{array}{ccccccc} A & \overset{f}{\longrightarrow} & B & \overset{g}{\longrightarrow} & B/A & \overset{h}{\longrightarrow} & \Sigma A \\ {}^{=}\downarrow & (1) & {}^{f'}\downarrow & (2) & \downarrow^{x} && \downarrow^{=} \\ A & \underset{f'\circ f}{\longrightarrow} & D & \underset{g''}{\longrightarrow} & D/A & \underset{h''}{\longrightarrow} & \Sigma A \\ && {}^{g'}\downarrow & (3) & \downarrow^{y} && \\ && D/B & \underset{\simeq}{\longrightarrow} & D/B && \\ && {}^{h'}\downarrow && \downarrow^{(\Sigma g)\circ h'} && \\ && \Sigma B & \underset{\Sigma g}{\longrightarrow} & \Sigma B/A && \end{array}$$

The fact that the second horizontal morphism from below is indeed an isomorphism follows by applying the pasting law for homotopy pushouts twice (prop.):



Draw all homotopy cofibers as homotopy pushout squares (def.) with one edge going to the point. Then assemble the squares (1)-(3) in the pasting composite of two cubes on top of each other: (1) as the left face of the top cube, (2) as the middle face where the two cubes touch, and (3) as the front face of the bottom cube. All remaining edges are points. This way the rear and front face of the top cube and the left and right face of the bottom cube are homotopy pushouts by construction. Also the top face

$$\begin{array}{ccc} A & \longrightarrow & * \\ {}^{\simeq}\downarrow && \downarrow \\ A & \longrightarrow & * \end{array}$$

is a homotopy pushout, since two opposite edges of it are weak equivalences (prop.). From this the pasting law for homotopy pushouts (prop.) gives that also the middle square (2) is a homotopy pushout. Applying the pasting law once more this way, now for the bottom cube, gives that the bottom square

$$\begin{array}{ccc} * & \longrightarrow & * \\ \downarrow && \downarrow \\ D/B & \longrightarrow & (D/A)/(B/A) \end{array}$$

is a homotopy pushout. Since here the left edge is a weak equivalence, necessarily, so is the right edge (prop.), which hence exhibits the claimed identification

$$D/B \simeq (D/A)/(B/A) \,.$$

∎

Remark 4.17. All we used in the proof (of prop. 4.16) of the octahedral axiom (T4) is the existence and nature of homotopy pushouts. In fact one may show that the octahedral axiom is equivalent to the existence of homotopy pushouts, in the sense of axiom B in (Hubery).

Long fiber-cofiber sequences

In homotopy theory there are generally long homotopy fiber sequences to the left and long homotopy cofiber sequences to the right, as discussed in the section *Homotopy theory – Long sequences*. We prove now, in the generality of the axiomatics of triangulated categories (since the stable homotopy category is triangulated by prop. 4.16), that in stable homotopy theory both these sequences are long in both directions, and in fact coincide.

Literature (Schwede 12, II.2)

Lemma 4.18. *For* $(\mathrm{Ho}, \Sigma, \mathrm{CofSeq})$ *a triangulated category, def. 4.15, and*

$$A \overset{f}{\longrightarrow} B \overset{g}{\longrightarrow} B/A \overset{h}{\longrightarrow} \Sigma A$$

a distinguished triangle, then

$$g \circ f = 0$$

is the zero morphism.

Proof. Consider the commuting diagram

$$\begin{array}{ccccccc} A & \overset{\mathrm{id}}{\longrightarrow} & A & \longrightarrow & 0 & \longrightarrow & \Sigma A \\ \downarrow^{\mathrm{id}} && \downarrow^{f} &&&& \\ A & \overset{f}{\longrightarrow} & B & \overset{g}{\longrightarrow} & B/A & \overset{h}{\longrightarrow} & \Sigma A \end{array} .$$

Observe that the top part is a distinguished triangle by axioms T1 and T2 in def. 4.15. Hence by T3 there is an extension to a commuting diagram of the form

$$\begin{array}{ccccccc} A & \overset{\mathrm{id}}{\longrightarrow} & A & \longrightarrow & 0 & \longrightarrow & \Sigma A \\ \downarrow^{\mathrm{id}} && \downarrow^{f} && \downarrow && \downarrow^{\Sigma f} \\ A & \overset{f}{\longrightarrow} & B & \overset{g}{\longrightarrow} & B/A & \overset{h}{\longrightarrow} & \Sigma A \end{array} .$$

Now the commutativity of the middle square proves the claim. ▮

Proposition 4.19. *Let* $(\mathrm{Ho}, \Sigma, \mathrm{CofSeq})$ *be a triangulated category, def. 4.15, with hom-functor denoted by* $[-,-]_* : \mathrm{Ho}^{\mathrm{op}} \times \mathrm{Ho} \to \mathrm{Ab}$. *For* $X \in \mathrm{Ho}$ *any object, and for* $D \in \mathrm{CofSeq}$ *any distinguished triangle*

$$D = (A \overset{f}{\longrightarrow} B \overset{g}{\longrightarrow} B/A \overset{h}{\longrightarrow} \Sigma A)$$

then the sequences of abelian groups

1. *(long cofiber sequence)*

$$[\Sigma A, X]_* \overset{[h,X]_*}{\longrightarrow} [B/A, X]_* \overset{[g,X]_*}{\longrightarrow} [B, X]_* \overset{[f,X]_*}{\longrightarrow} [A, X]_*$$

2. *(long fiber sequence)*

$$[X,A]_* \overset{[X,f]_*}{\longrightarrow} [X,B]_* \overset{[X,g]_*}{\longrightarrow} [X,B/A]_* \overset{[X,h]_*}{\longrightarrow} [X,\Sigma A]_*$$

are long exact sequences.

Proof. Regarding the first case:

Since $g \circ f = 0$ by lemma 4.18, we have an inclusion $\mathrm{im}([g,X]_*) \subset \ker([f,X]_*)$. Hence it is sufficient to show that if $\psi : B \to X$ is in the kernel of $[f,X]_*$ in that $\psi \circ f = 0$, then there is $\phi : B/A \to X$ with $\phi \circ g = \psi$. To that end, consider the commuting diagram

$$\begin{array}{ccccccc} A & \overset{f}{\longrightarrow} & B & \overset{g}{\longrightarrow} & B/A & \overset{h}{\longrightarrow} & \Sigma A \\ \downarrow && {}^{\psi}\downarrow &&&& \\ 0 & \longrightarrow & X & \overset{\mathrm{id}}{\longrightarrow} & X & \longrightarrow & 0 \end{array} \,,$$

where the commutativity of the left square exhibits our assumption.

The top part of this diagram is a distinguished triangle by assumption, and the bottom part is by condition $T1$ in def. 4.15. Hence by condition T3 there exists ϕ fitting into a commuting diagram of the form

$$\begin{array}{ccccccc} A & \overset{f}{\longrightarrow} & B & \overset{g}{\longrightarrow} & B/A & \overset{h}{\longrightarrow} & \Sigma A \\ \downarrow && {}^{\psi}\downarrow && \downarrow^{\phi} && \downarrow \\ 0 & \longrightarrow & X & \overset{\mathrm{id}}{\longrightarrow} & X & \longrightarrow & 0 \end{array} \,.$$

Here the commutativity of the middle square exhibits the desired conclusion.

This shows that the first sequence in question is exact at $[B,X]_*$. Applying the same reasoning to the distinguished triangle $(g,h,-\Sigma f)$ provided by T2 yields exactness at $[B/A,X]_*$.

Regarding the second case:

Again, from lemma 4.18 it is immediate that

$$\mathrm{im}([X,f]_*) \subset \ker([X,g]_*)$$

so that we need to show that for $\psi : X \to B$ in the kernel of $[X,g]_*$, hence such that $g \circ \psi = 0$, then there exists $\phi : X \to A$ with $f \circ \phi = \psi$.

To that end, consider the commuting diagram

$$\begin{array}{ccccccc} X & \longrightarrow & 0 & \longrightarrow & \Sigma X & \overset{-\Sigma \mathrm{id}}{\longrightarrow} & \Sigma X \\ \downarrow^{\psi} && \downarrow &&&& \\ B & \overset{g}{\longrightarrow} & B/A & \overset{h}{\longrightarrow} & \Sigma A & \overset{-\Sigma f}{\longrightarrow} & \Sigma B \end{array} \,,$$

where the commutativity of the left square exhibits our assumption.

Now the top part of this diagram is a distinguished triangle by conditions T1 and T2 in def. 4.15, while the bottom part is a distinguished triangle by applying T2 to the given distinguished triangle. Hence by T3 there exists $\tilde{\phi} : \Sigma X \to \Sigma A$ such as to extend to a commuting diagram of the form

$$
\begin{array}{ccccccc}
X & \longrightarrow & 0 & \longrightarrow & \Sigma X & \overset{-\Sigma \mathrm{id}}{\longrightarrow} & \Sigma X \\
\downarrow^{\psi} && \downarrow && \downarrow^{\tilde \phi} && \downarrow^{\Sigma \psi} \\
B & \overset{g}{\longrightarrow} & B/A & \overset{h}{\longrightarrow} & \Sigma A & \overset{-\Sigma f}{\longrightarrow} & \Sigma B
\end{array}
,
$$

At this point we appeal to the condition in def. 4.15 that $\Sigma \colon \mathrm{Ho} \to \mathrm{Ho}$ is an equivalence of categories, so that in particular it is a fully faithful functor. It being a full functor implies that there exists $\phi \colon X \to A$ with $\tilde \phi = \Sigma \phi$. It being faithful then implies that the whole commuting square on the right is the image under Σ of a commuting square

$$
\begin{array}{ccc}
X & \overset{-\mathrm{id}}{\longrightarrow} & X \\
{}^{\phi}\downarrow && \downarrow^{\psi} \\
A & \underset{-f}{\longrightarrow} & B
\end{array}
.
$$

This concludes the exactness of the second sequence at $[X,B]_\ast$. As before, exactness at $[X,B/A]_\ast$ follows with the same argument applied to the shifted triangle, via T2. ▮

Lemma 4.20. *Consider a morphism of distinguished triangles in a triangulated category (def. 4.15):*

$$
\begin{array}{ccccccc}
A & \longrightarrow & B & \overset{g}{\longrightarrow} & B/A & \overset{h}{\longrightarrow} & \Sigma A \\
\downarrow^{a} && \downarrow^{b} && \downarrow^{c} && \downarrow^{\Sigma a} \\
A' & \longrightarrow & B' & \longrightarrow & B'/A' & \longrightarrow & \Sigma A'
\end{array}
.
$$

If two out of $\{a,b,c\}$ are isomorphisms, then so is the third.

Proof. Consider the image of the situation under the hom-functor $[X,-]_\ast$ out of any object X:

$$
\begin{array}{ccccccccc}
[X,A]_\ast & \longrightarrow & [X,B]_\ast & \overset{g}{\longrightarrow} & [X,B/A]_\ast & \overset{h}{\longrightarrow} & [X,\Sigma A]_\ast & \longrightarrow & [X,\Sigma B]_\ast \\
\downarrow^{a_\ast} && \downarrow^{b_\ast} && \downarrow^{c_\ast} && \downarrow^{(\Sigma a)_\ast} && \downarrow^{(\Sigma b)_\ast} \\
[X,A']_\ast & \longrightarrow & [X,B']_\ast & \longrightarrow & [X,B'/A']_\ast & \longrightarrow & [X,\Sigma A']_\ast & \longrightarrow & [X,\Sigma B']_\ast
\end{array}
,
$$

where we extended one step to the right using axiom T2 (def. 4.15).

By prop. 4.19 here the top and bottom are exact sequences.

So assume the case that a and b are isomorphisms, hence that $a_\ast$, $b_\ast$, $(\Sigma a)_\ast$ and $(\Sigma b)_\ast$ are isomorphisms. Then by exactness of the horizontal sequences, the five lemma implies that $c_\ast$ is an isomorphism. Since this holds naturally for all X, the Yoneda lemma (fully faithfulness of the Yoneda embedding) then implies that c is an isomorphism.

If instead b and c are isomorphisms, apply this same argument to the triple $(b,c,\Sigma a)$ to conclude that Σa is an isomorphism. Since Σ is an equivalence of categories, this implies then that a is an isomorphism.

Analogously for the third case. ▮

Lemma 4.21. *If $(g,h,-\Sigma f)$ is a distinguished triangle in a triangulated category (def. 4.15), then so is (f,g,h).*

Proof. By T0 there is some distinguished triangle of the form (f,g',h'). By T2 this gives a distinguished triangle $(-\Sigma f, -\Sigma g', -\Sigma h')$. By T3 there is a morphism c' giving a commuting diagram

$$\begin{array}{ccccccc} \Sigma A & \overset{-\Sigma f}{\longrightarrow} & \Sigma B & \overset{-\Sigma g}{\longrightarrow} & \Sigma C & \overset{-\Sigma h}{\longrightarrow} & \Sigma^2 A \\ {}^{=}\downarrow && {}^{=}\downarrow && {}^{c'}\downarrow && {}^{=}\downarrow \\ \Sigma A & \overset{-\Sigma f}{\longrightarrow} & \Sigma B & \overset{-\Sigma g'}{\longrightarrow} & \Sigma C & \overset{-\Sigma h'}{\longrightarrow} & \Sigma^2 A \end{array} \,.$$

Now lemma 4.20 gives that c' is an isomorphism. Since Σ is an equivalence of categories, there is an isomorphism c such that $c' = \Sigma c$. Since Σ is in particular a faithful functor, this c exhibits an isomorphism between (f, g, h) and (f, g', h'). Since the latter is distinguished, so is the former, by T0. ▮

In conclusion:

Proposition 4.22. *Let*

$$X \overset{f}{\longrightarrow} Y \overset{g}{\longrightarrow} Z$$

be a homotopy cofiber sequence (def.) of spectra in the stable homotopy category (def. 4.1) $\mathrm{Ho}(\mathrm{Spectra})$. Let $A \in \mathrm{Ho}(\mathrm{Spectra})$ be any other spectrum. Then the abelian hom-groups of the stable homotopy category (def. 4.3, lemma 4.4) sit in long exact sequences of the form

$$\cdots \longrightarrow [A, \Omega Y] \overset{-(\Omega g)_*}{\longrightarrow} [A, \Omega Z] \longrightarrow [A, X] \overset{f_*}{\longrightarrow} [A, Y] \overset{g_*}{\longrightarrow} [A, Z] \longrightarrow [A, \Sigma X] \overset{-(\Sigma f)_*}{\longrightarrow} [A, \Sigma Y] \longrightarrow \cdots \,.$$

Proof. By prop. 4.16 the above abstract reasoning in triangulated categories applies. By prop. 4.19 we have long exact sequences to the right as shown. By lemma 4.21 these also extend to the left as shown. ▮

This suggests that homotopy cofiber sequences coincide with homotopy fiber sequence in the stable homotopy category. This is indeed the case:

Proposition 4.23. *In the stable homotopy category, a sequence of morphisms is a homotopy cofiber sequence precisely if it is a homotopy fiber sequence.*

Specifically for $f : X \longrightarrow Y$ any morphism in $\mathrm{Ho}(\mathrm{Spectra})$, then there is an isomorphism

$$\phi \;:\; \mathrm{hofib}(f) \overset{\simeq}{\longrightarrow} \Omega\, \mathrm{hocof}(f)$$

between the homotopy fiber and the looping of the homotopy cofiber, which fits into a commuting diagram in the stable homotopy category $\mathrm{Ho}(\mathrm{Spectra})$ of the form

$$\begin{array}{ccccc} \Omega Y & \longrightarrow & \mathrm{hofib}(f) & \longrightarrow & X \\ {}^{=}\downarrow && \downarrow^{\phi}_{\simeq} && \downarrow^{\simeq} \\ \Omega Y & \longrightarrow & \Omega\, \mathrm{hocof}(f) & \longrightarrow & \Omega \Sigma X \end{array} \,,$$

where the top row is the homotopy fiber sequence of f, while the bottom row is the image under the looping functor Ω of the homotopy cofiber sequence of f.

(Lewis-May-Steinberger 86, chapter III, theorem 2.4)

Proof. Label the diagram in question as follows

$$\begin{array}{ccccc} \Omega Y & \overset{a}{\longrightarrow} & \mathrm{hofib}(f) & \overset{b}{\longrightarrow} & X \\ {}^{=}\downarrow & (1) & \downarrow^{\phi}_{\simeq} & (2) & \downarrow^{\simeq} \\ \Omega Y & \underset{c}{\longrightarrow} & \Omega \, \mathrm{hocof}(f) & \underset{d}{\longrightarrow} & \Omega \Sigma X \end{array} \,.$$

Let X be represented by a CW-spectrum (by prop. 2.16), hence in particular by a cofibrant sequential spectrum (by prop. 2.12). By prop. 2.13 and the factorization lemma (lemma) this implies that the standard mapping cone construction on f (def.) is a model for the homotopy cofiber of f (exmpl.):

$$\mathrm{hocof}(f) \simeq \mathrm{Cone}(f) \,.$$

By construction of mapping cones, this sits in the following commuting squares in $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$.

$$\begin{array}{ccc} X & \longrightarrow & \mathrm{Cone}(X) \\ \downarrow & (\mathrm{po}) & \downarrow \\ Y & \longrightarrow & \mathrm{Cone}(f) \\ \downarrow & (\mathrm{po}) & \downarrow \\ \ast & \longrightarrow & \Sigma X \end{array} \,.$$

Consider then the commuting diagram

$$\begin{array}{ccccccc} \Omega Y & \overset{a}{\longrightarrow} & \mathrm{hofib}(f) & \overset{\phi}{\longrightarrow} & \Omega \, \mathrm{hocof}(f) & \overset{d}{\longrightarrow} & \Omega \Sigma X \simeq X \\ \downarrow && \downarrow && \downarrow && \downarrow \\ \ast & \longrightarrow & X & \longrightarrow & \mathrm{Cone}(X) & \longrightarrow & \mathrm{Cone}(X) \\ \downarrow && \downarrow^{f} && \downarrow && \downarrow \\ Y & \underset{=}{\longrightarrow} & Y & \longrightarrow & \mathrm{Cone}(f) & \longrightarrow & \Sigma X \end{array} \,,$$

in the stable homotopy category $\mathrm{Ho}(\mathrm{Spectra})$ (def. 4.1). Here the bottom commuting squares are the images under localization $\gamma : \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}}) \longrightarrow \mathrm{Ho}(\mathrm{Spectra})$ (thm.) of the above commuting squares in the definition of the mapping cone, and the top row of squares are the morphisms induced via the universal property of fibers by forming homotopy fibers of the bottom vertical morphisms (fibers of fibration replacements, which may be chosen compatibly, either by pullback or by invoking the small object argument).

First of all, this exhibits the composition of the left two horizontal morphisms $\phi \circ a \simeq c$ in the above diagram as the left part (1) of the commuting diagram to be proven.

Now observe that the pasting composite of the two rectangles on the right of the previous diagram is isomorphic, in $\mathrm{Ho}(\mathrm{Spectra})$, to the following pasting composite:

$$\begin{array}{ccccc} \mathrm{hofib}(f) & \overset{b}{\longrightarrow} & X & \underset{\simeq}{\overset{\eta}{\longrightarrow}} & \Omega \Sigma X \simeq X \\ \downarrow && \downarrow && \downarrow \\ X & \overset{\mathrm{id}}{\longrightarrow} & X & \longrightarrow & \mathrm{Cone}(X) \\ \downarrow && \downarrow && \downarrow \\ Y & \longrightarrow & \ast & \longrightarrow & \Sigma X \end{array} \,.$$

This is because the pasting composite of the bottom squares is isomorphic already in $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$ by the above commuting diagrams for the mapping cone and the suspension, and then using again the universal property of homotopy fibers.

Hence the top composite morphisms coincide, by universality of homotopy fibers, with the previous top composite:

$$\eta \circ b \simeq d \circ \phi \,.$$

This is the commutativity of the right part (2) of the diagram to be proven.

So far we have shown that

$$\begin{array}{ccccc} \Omega Y & \longrightarrow & \mathrm{hofib}(f) & \longrightarrow & X \\ {}^{=}\downarrow && \downarrow^{\phi} && \downarrow^{=} \\ \Omega Y & \longrightarrow & \Omega\, \mathrm{hocof}(f) & \longrightarrow & X \end{array}$$

commutes in the stable homotopy category. It remains to see that ϕ is an isomorphism.

To that end, consider for any $A \in \mathrm{Ho}(\mathrm{Spectra})$ the image of this commuting diagram, prolonged to the left and right, under the hom-functor $[A,-]_\ast$ of the stable homotopy category:

$$\begin{array}{ccccccccc} [A,\Omega X] & \longrightarrow & [A,\Omega Y] & \longrightarrow & [A,\mathrm{hofib}(f)] & \longrightarrow & [A,X] & \longrightarrow & [A,Y] \\ {}^{=}\downarrow && {}^{=}\downarrow && \downarrow^{[A,\phi]} && \downarrow^{\simeq} && \downarrow^{\simeq} \\ [A,\Omega X] & \longrightarrow & [A,\Omega Y] & \longrightarrow & [A,\Omega\,\mathrm{hocof}(f)] & \longrightarrow & [A,\Omega \Sigma X] & \longrightarrow & [A,\Omega\Sigma Y] \end{array} \,.$$

Here the top row is long exact, since it is the long homotopy fiber sequence to the left that holds in the homotopy category of any model catgeory (prop.). Moreover, the bottom sequence is long exact by prop. 4.22. Hence the five lemma implies that $[A,\phi]_\ast$ is an isomorphism. Since this is the case for all A, the Yoneda lemma (faithfulness of the Yoneda embedding) implies that ϕ itself is an isomorphism. ▮

Remark 4.24. Prop. 4.23 is the homotopy theoretic analog of the clause that makes a pre-abelian category into an abelian category:

> A pre-abelian category is an additive category in which fibers (kernels) and cofibers (cokernels) exist. This is an abelian category if the cofiber of the fiber of any morphism equals coincides with the fiber of the cofiber of that morphism.
>
> Here we see that in stable homotopy theory, whose homotopy category is additive, and in which homotopy fibers and homotopy cofibers exist, the analogous statement is true even in a stronger form: the homotopy cofiber projection of the homotopy fiber inclusion of any morphism coincides with that morphism, and so does the homotopy fiber projection of the homotopy cofiber inclusion.

In particular there are long exact sequences of stable homotopy groups extending in both directions:

Lemma 4.25. *Let $X \in \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$ be any sequential spectrum, then there is an isomorphism*

$$\pi_0(X) \simeq [\mathbb{S}, X]$$

between its stable homotopy group in degree 0 (def. 1.11) and the hom-group (according to def. 4.7, prop. 4.14) in the stable homotopy category (def. 4.1) from the sphere spectrum (def. 1.4) into X.

Generally, with respect to the graded hom-groups of def. 4.5 we have

$$\pi_\bullet(X) \simeq [\mathbb{S}, X]_\bullet \,.$$

Proof. The hom-set in the homotopy category is equivalently given by the left homotopy-equivalence classes out of a cofibrant representative of $\mathbb{S}$ into a fibrant representative of X (lemma).

The standard sphere spectrum $\mathbb{S}_{\mathrm{std}} \coloneqq \Sigma^\infty S^0$ is a CW-spectrum and hence cofibrant, by prop. 2.12. Moreover, this implies by prop. 2.13 that left homotopies out of $\mathbb{S}_{\mathrm{str}}$ are represented by the standard sequential cylinder spectrum

$$\mathbb{S}_{\mathrm{std}} \wedge (I_+) \simeq \Sigma^\infty(I_+) \,.$$

By theorem 3.11, fibrant replacement for X is provided by its spectrification QX according to def. 1.19.

So it follows that $[\mathbb{S}, X]_*$ is given by left homotopy classes of morphisms

$$\Sigma^\infty S^0 = \mathbb{S}_{\mathrm{std}} \longrightarrow QX$$

in $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$. By the $(\Sigma^\infty \dashv \Omega^\infty)$-adjunction (prop. 1.10) these are equivalently morphisms

$$S^0 \longrightarrow (QX)_0$$

in $\mathrm{Top}^{*/}_{\mathrm{cg}}$. Hence equivalently morphisms

$$* \longrightarrow (QX)_0$$

in $\mathrm{Top}_{\mathrm{cg}}$, hence equivalently points in $(QX)_0$. Analogously, a left homotopy

$$\Sigma^\infty(I_+) \longrightarrow (QX)_0$$

in $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$ is equivalently a path

$$I \longrightarrow (QX)_0$$

in $\mathrm{Top}_{\mathrm{cg}}$.

In conclusion this establishes an isomorphism

$$[\mathbb{S}, X]_* \simeq \pi_0((QX)_0)$$

with π_0 of the 0-component of QX. With this the statement follows with example 1.18, since QX is an Omega-spectrum, by prop. 1.20.

From this the last statement follows from the identity

$$\pi_0(\Sigma^{-n}(-)) \simeq \pi_n(-) \,.$$

▮

As a consequence:

Proposition 4.26. *Let*

$$X \overset{f}{\longrightarrow} Y \overset{g}{\longrightarrow} Z$$

be a homotopy cofiber sequence (def.) in the stable homotopy category (def. 4.1). Then there is induced a long exact sequence of stable homotopy groups (def. 1.11) of the form

$$\cdots \longrightarrow \pi_{\bullet+1}(Z) \longrightarrow \pi_\bullet(X) \overset{f_*}{\longrightarrow} \pi_\bullet(Y) \overset{g_*}{\longrightarrow} \pi_\bullet(Z) \longrightarrow \pi_{\bullet-1}(X) \longrightarrow \cdots \,.$$

Proof. Via lemmma 4.25 this is a special case of prop. 4.22. ▮

As an example, we check explicitly what we already deduced abstractly in prop. 4.14, that in the stable homotopy category wedge sum and Cartesian product of spectra agree and constitute a biproduct/direct sum:

Example 4.27. For $X, Y \in \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$, then the canonical morphism

$$X \vee Y \longrightarrow X \times Y$$

out of the coproduct (wedge sum, example 1.27) into the product (via prop. 1.25), given by

$$\begin{array}{ccccc} X && && Y \\ & {}_{\searrow}{}^{i_X} && {}^{i_X}{}_{\swarrow} & \\ {}^{\mathrm{id}_X}\downarrow && X \sqcup Y && \downarrow^{\mathrm{id}_Y} \\ & {}^{\swarrow}{}_{(\mathrm{id},0)} && {}_{(0,\mathrm{id})}{}^{\searrow} & \\ X && && Y \\ & {}_{\searrow}{}^{(\mathrm{id},0)} && {}^{(0,\mathrm{id})}{}_{\swarrow} & \\ {}^{\mathrm{id}_X}\downarrow && X \times Y && \downarrow^{\mathrm{id}_Y} \\ & {}^{\swarrow}{}_{p_X} && {}_{p_Y}{}^{\searrow} & \\ X && && Y \end{array}$$

represents an isomorphism in the stable homotopy category.

Proof. By prop. 2.16, we may represent both X and Y by CW-spectra (def. 2.7) in $(\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}})_c[W_{\mathrm{st}}^{-1}]$. Then the canonical morphism

$$i_X : X \longrightarrow X \vee Y$$

is a cofibration according to theorem 2.3, because $X_{n+1} \underset{S^1 \wedge X_n}{\sqcup} S^1 (X \vee Y) \simeq X_{n+1} \vee S^1 \wedge Y_n$.

Hence its ordinary cofiber, which is Y, is its homotopy cofiber (def.), and so we have a homotopy cofiber sequence

$$X \longrightarrow X \vee Y \longrightarrow Y \,.$$

Moreover, under forming stable homotopy groups (def. 1.11), the inclusion map evidently gives an injection, and the projection map gives a surjection. Hence the long exact sequence of stable homotopy groups from prop. 4.26 gives the short exact sequence

$$0 \to \pi_\bullet(X) \longrightarrow \pi_\bullet(X \vee Y) \longrightarrow \pi_\bullet(Y) \to 0 \,.$$

Finally, due to the fact that the inclusion and projection for one of the two summands constitute a retraction, this is a split exact sequence, hence exhibits an isomorphism

$$\pi_k(X \vee Y) \overset{\simeq}{\longrightarrow} \pi_k(X) \oplus \pi_k(Y) \simeq \pi_k(X) \times \pi_k(Y) \simeq \pi_k(X \times Y)$$

for all k. But this just says that $X \vee Y \to X \times Y$ is a stable weak homotopy equivalence. ▮

Final Remark 4.28. For a tower of fibrations of spectra, the long sequences of stable homotopy groups associated with any (co-)fiber sequence of spectra, from prop. 4.26, combine to an exact couple. The induced spectral sequence of a tower of fibrations is the central tool of computation in stable homotopy theory.

We discuss how these spectral sequences arise in the section *Interlude -- Spectral sequences*.

We discuss in detail the special case of the Adams spectral sequences in the section *Part 2 -- Adams spectral sequences*.

But for handling any of these spectral sequences it is convenient, or, in many cases, necessary to have multiplicative structure available, induced from a symmetric monoidal smash product of spectra. This we turn to in *part 1.2 -- Structured spectra*.

5. References

We give the modern picture of the stable homotopy category, for which a quick survey may be found in

- Cary Malkiewich, *The stable homotopy category*, 2014 (pdf).

A classical textbook on stable homotopy theory for "unstructured" spectra is

- Frank Adams, part III sections 2, 4-7 of *Stable homotopy and generalized homology*, Chicago Lectures in mathematics, 1974

For establishing the stable model structure on spectra we use the Bousfield-Friedlander theorem as discussed in

- Paul Goerss, Rick Jardine, section X.4 of *Simplicial homotopy theory*, (1996)

and as applied for general Omega-spectrification functors in

- Stefan Schwede, *Spectra in model categories and applications to the algebraic cotangent complex*, Journal of Pure and Applied Algebra 120 (1997) 77-104 (pdf)

For the discussion of the stability of the homotopy theory of sequential spectra we follow

- John F. Jardine, sections 10.3 and 10.4 of *Local homotopy theory*, 2016

For the definition of triangulated categories and a discussion of various equivalent versions of the octahedral axiom the following brief note is useful:

- Andrew Hubery, *Notes on the octahedral axiom*, (pdf)

For the discussion of the triangulated structure of the stable homotopy category we follow

- Stefan Schwede, section II.2 of *Symmetric spectra*, 2012 (pdf)

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nLab

Introduction to Stable homotopy theory -- 1-2

We give an introduction to the stable homotopy category and to its key computational tool, the Adams spectral sequence. To that end we introduce the modern tools, such as model categories and highly structured ring spectra. In the accompanying seminar we consider applications to cobordism theory and complex oriented cohomology such as to converge in the end to a glimpse of the modern picture of chromatic homotopy theory.

Lecture notes.

Main page: *Introduction to Stable homotopy theory*.

Previous section: *Prelude -- Classical homotopy theory*

This section: *Part 1 -- Stable homotopy theory*

Previous subsection: *Part 1.1 -- Stable homotopy theory -- Sequential spectra*

This subsection: **Part 1.2 - Stable homotopy theory – Structured spectra**

Next section: *Part 2 -- Adams spectral sequences*

Stable homotopy theory – Structured spectra

The key result of part 1.1 was (thm.) the construction of a stable homotopy theory of spectra, embodied by a stable model structure on topological sequential spectra $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}$ (thm.) with its corresponding stable homotopy category $\mathrm{Ho}(\mathrm{Spectra})$, which stabilizes the canonical looping/suspension adjunction on pointed topological spaces in that it fits into a diagram of (Quillen-)adjunctions of the form

$$
\begin{array}{ccccccc}
(\mathrm{Top}^{*/}_{\mathrm{cg}})_{\mathrm{Quillen}} & \underoverset{\underset{\Omega}{\longrightarrow}}{\overset{\Sigma}{\longleftarrow}}{\bot} & (\mathrm{Top}^{*/}_{\mathrm{cg}})_{\mathrm{Quillen}} & & \mathrm{Ho}(\mathrm{Top}^{*/}) & \underoverset{\underset{\Omega}{\longrightarrow}}{\overset{\Sigma}{\longleftarrow}}{\bot} & \mathrm{Ho}(\mathrm{Top}^{*/}) \\
{}^{\mathllap{\Sigma^\infty}}\downarrow \dashv \uparrow^{\mathrlap{\Omega^\infty}} & & {}^{\mathllap{\Sigma^\infty}}\downarrow \dashv \uparrow^{\mathrlap{\Omega^\infty}} & \overset{\gamma}{\longrightarrow} & {}^{\mathllap{\Sigma^\infty}}\downarrow \dashv \uparrow^{\mathrlap{\Omega^\infty}} & & {}^{\mathllap{\Sigma^\infty}}\downarrow \dashv \uparrow^{\mathrlap{\Omega^\infty}} \\
\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}} & \underoverset{\underset{\Omega}{\longrightarrow}}{\overset{\Sigma}{\longleftarrow}}{\simeq_Q} & \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}} & & \mathrm{Ho}(\mathrm{Spectra}) & \underoverset{\underset{\Omega}{\longrightarrow}}{\overset{\Sigma}{\longleftarrow}}{\simeq} & \mathrm{Ho}(\mathrm{Spectra})
\end{array}
\,.
$$

But fitting into such a diagram does not yet uniquely characterize the stable homotopy category. For instance the trivial category on a single object would also form such a diagram. On the other hand, there is more canonical structure on the category of pointed topological spaces which is not yet reflected here.

Namely the smash product

$$\wedge \;:\; \mathrm{Ho}(\mathrm{Top}^{*/}) \longrightarrow \mathrm{Ho}(\mathrm{Top}^{*/})$$

of pointed topological spaces gives it the structure of a monoidal category (def. 1.1 below), and so it is natural to ask that the above stabilization diagram reflects and respects that extra structure. This means that there should be a smash product of spectra

$$\wedge \;:\; \mathrm{Ho}(\mathrm{Spectra}) \longrightarrow \mathrm{Ho}(\mathrm{Spectra})$$

such that $(\Sigma^\infty \dashv \Omega^\infty)$ is compatible, in that

$$\Sigma^\infty(X \wedge Y) \simeq (\Sigma^\infty X) \wedge (\Sigma^\infty Y)$$

(a "strong monoidal functor", def. 1.47 below).

We had already seen in part 1.1 that $\mathrm{Ho}(\mathrm{Spectra})$ is an additive category, where wedge sum of spectra is a direct sum operation $\oplus$. We discuss here that the smash product of spectra is the corresponding operation analogous to a tensor product of abelian groups.

abelian groups	**spectra**
$\oplus$ direct sum	$\vee$ wedge sum
$\otimes$ tensor product	$\wedge$ smash product

This further strenghtens the statement that spectra are the analog in homotopy theory of abelian groups. In particular, with respect to the smash product of spectra, the sphere spectrum becomes a ring spectrum that is the coresponding analog of the ring of integers.

With the analog of the tensor product in hand, we may consider doing algebra – the theory of rings and their modules – internal to spectra. This "higher algebra" accordingly is the theory of *ring spectra* and *module spectra*.

algebra	**homological algebra**	**higher algebra**
abelian group	chain complex	spectrum
ring	dg-ring	ring spectrum
module	dg-module	module spectrum

Where a ring is equivalently a monoid with respect to the tensor product of abelian groups, we are after a corresponding tensor product of spectra. This is to be the smash product of spectra, induced by the smash product on pointed topological spaces.

In particular the sphere spectrum becomes a ring spectrum with respect to this smash product and plays the role analogous to the ring of integers in abelian groups

abelian groups	**spectra**
$\mathbb{Z}$ integers	$\mathbb{S}$ sphere spectrum

Using this structure there is finally a full characterization of stable homotopy theory, we state (without proof) this *Schwede-Shipley uniqueness* as theorem 5.13 below.

There is a key point to be dealt with here: the smash product of spectra has to exhibit a certain *graded commutativity*. Informally, there are two ways to see this:

First, we have seen above that under the Dold-Kan correspondence chain complexes yield examples of spectra. But the tensor product of chain complexes is graded commutative.

Second, more fundamentally, we see in the discussion of the Brown representability theorem (here) that every (sequential) spectrum A induces a generalized homology theory given by the formula $X \mapsto \pi_\bullet(E \wedge X)$ (where the smash product is just the degreewise smash of pointed objects). By the suspension isomorphism this is such that for $X = S^n$ the n-sphere, then $\pi_{\bullet \geq 0}(E \wedge S^n) \simeq \pi_{\bullet \geq 0}(E_n)$. This means that instead of thinking of a sequential spectrum (def.) as indexed on the natural numbers equipped with addition $(\mathbb{N}, +)$, it may be more natural to think of sequential spectra as indexed on the n-spheres equipped with their smash product of pointed spaces $(\{S^n\}_n, \wedge)$.

Proposition 0.1. *There are homeomorphisms between n-spheres and their smash products*

$$\phi_{n_1,n_2} : S^{n_1} \wedge S^{n_2} \xrightarrow{\simeq} S^{n_1+n_2}$$

such that in Ho(Top) there are commuting diagrams like so:

$$\begin{array}{ccc} (S^{n_1} \wedge S^{n_2}) \wedge S^{n_3} & \overset{\simeq}{\longrightarrow} & S^{n_1} \wedge (S^{n_2} \wedge S^{n_3}) \\ {}^{\phi_{n_1,n_2} \wedge \mathrm{id}}\downarrow && \downarrow^{\mathrm{id} \wedge \phi_{n_2,n_3}} \\ S^{n_1+n_2} \wedge S^{n_3} && S^{n_1} \wedge S^{n_2+} \\ & {}_{\phi_{n_1+n_2,n_3}}\searrow \quad \swarrow_{\phi_{n_1,n_2+n_3}} & \\ & S^{n_1+n_2+n_3} & \end{array}$$

and

$$\begin{array}{ccc} S^{n_1} \wedge S^{n_2} & \overset{b_{n_1,n_2}}{\longrightarrow} & S^{n_2} \wedge S^{n_1} \\ {}^{\phi_{n_1,n_2}}\downarrow && \downarrow^{\phi_{n_2,n_1}} \\ S^{n_1+n_2} & \overset{(-1)^{n_1 n_2}}{\longrightarrow} & S^{n_1+n_2} \end{array}$$

where here $(-1)^n : S^n \to S^n$ denotes the homotopy class of a continuous function of degree $(-1)^n \in \mathbb{Z} \simeq [S^n, S^n]$.

Proof. With the n-sphere S^n realized as the one-point compactification of the Cartesian space $\mathbb{R}^n$, then ϕ_{n_1,n_2} is given by the identity on coordinates and the braiding homeomorphism

$$b_{n_1,n_2} : S^{n_1} \wedge S^{n_2} \overset{\sigma}{\longrightarrow} S^{n_2} \wedge S^{n_1}$$

is given by permuting the coordinates:

$$(x_1, \cdots, x_{n_1}, y_1, \cdots, y_{n_2}) \mapsto (y_1, \cdots, y_{n_2}, x_1, \cdots, x_{n_1}) \,.$$

This has degree $(-1)^{n_1 n_2}$. ▮

This phenomenon suggests that as we "categorify" the natural numbers to the n-spheres, hence the integers to the sphere spectrum, and as we think of the nth component space of a sequential spectrum as being the value assigned to the n-sphere

$$E_n \simeq E(S^n)$$

then there should be a possibly non-trivial action of the symmetric group Σ_n on E_n, due to the fact that there is such an action of S^n which is non-trivial according to prop. 0.1.

We discuss two ways of making this precise below in *Symmetric and orthogonal spectra*, and we discuss how these are unified by a concept of module objects over a monoid object representing the sphere spectrum below in *S-modules*.

The general abstract theory for handling this is *monoidal and enriched category theory*. We first develop the relevant basics in *Categorical algebra*.

1. Categorical algebra

When defining a commutative ring as an abelian group A equipped with an associative, commutative and untial bilinear pairing

$$A \otimes_{\mathbb{Z}} A \xrightarrow{(-)\cdot(-)} A$$

one evidently makes crucial use of the tensor product of abelian groups $\otimes_{\mathbb{Z}}$. That tensor product itself gives the category Ab of all abelian groups a structure similar to that of a ring, namely it equips it with a pairing

$$\mathrm{Ab} \times \mathrm{Ab} \xrightarrow{(-) \otimes_{\mathbb{Z}} (-)} \mathrm{Ab}$$

that is a functor out of the product category of Ab with itself, satisfying category-theoretic analogs of the properties of associativity, commutativity and unitality.

One says that a ring A is a *commutative monoid* in the category Ab of abelian groups, and that this concept makes sense since Ab itself is a *symmetric monoidal category*.

Now in stable homotopy theory, as we have seen above, the category Ab is improved to the stable homotopy category $\mathrm{Ho}(\mathrm{Spectra})$ (def. \ref{TheStableHomotopyCategory}), or rather to any stable model structure on spectra presenting it. Hence in order to correspondingly refine commutative monoids in Ab (namely commutative rings) to commutative monoids in Ho(Spectra) (namely commutative ring spectra), there needs to be a suitable symmetric monoidal category structure on the category of spectra. Its analog of the tensor product of abelian groups is to be called the *symmetric monoidal smash product of spectra*. The problem is how to construct it.

The theory for handling such a problem is *categorical algebra*. Here we discuss the minimum of categorical algebra that will allow us to elegantly construct the symmetric monoidal smash product of spectra.

Monoidal topological categories

We want to lift the concepts of *ring* and *module* from *abelian groups* to *spectra*. This requires a general idea of what it means to generalize these concepts at all. The abstract theory of such generalizations is that of *monoid in a monoidal category*.

We recall the basic definitions of monoidal categories and of monoids and modules internal to monoidal categories. We list archetypical examples at the end of this section, starting with example 1.9 below. These examples are all fairly immediate. The point of the present discussion is to construct the non-trivial example of Day convolution monoidal stuctures below.

Definition 1.1. A **(pointed) topologically enriched monoidal category** is a (pointed) topologically enriched category $\mathcal{C}$ (def.) equipped with

1. a (pointed) topologically enriched functor (def.)

$$\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$$

out of the (pointed) topologival product category of $\mathcal{C}$ with itself (def. 1.26), called the **tensor product**,

2. an object

$$1 \in \mathcal{C}$$

called the **unit object** or **tensor unit**,

3. a natural isomorphism (def.)

$$a : ((-) \otimes (-)) \otimes (-) \overset{\simeq}{\longrightarrow} (-) \otimes ((-) \otimes (-))$$

called the **associator**,

4. a natural isomorphism

$$\ell : (1 \otimes (-)) \overset{\simeq}{\longrightarrow} (-)$$

called the **left unitor**, and a natural isomorphism

$$r : (-) \otimes 1 \overset{\simeq}{\longrightarrow} (-)$$

called the **right unitor**,

such that the following two kinds of diagrams commute, for all objects involved:

1. **triangle identity**:

$$\begin{array}{ccc} (x \otimes 1) \otimes y & \overset{a_{x,1,y}}{\longrightarrow} & x \otimes (1 \otimes y) \\ {}_{\rho_x \otimes 1_y}\searrow && \swarrow_{1_x \otimes \lambda_y} \\ & x \otimes y & \end{array}$$

2. the **pentagon identity**:

$$\array{ && (w \otimes x) \otimes (y \otimes z) \\ & {}^{\mathllap{\alpha_{w \otimes x, y, z}}}\nearrow && \searrow^{\mathrlap{\alpha_{w,x,y \otimes z}}} \\ ((w \otimes x) \otimes y) \otimes z && && (w \otimes (x \otimes (y \otimes z))) \\ {}^{\mathllap{\alpha_{w,x,y}} \otimes id_z}\downarrow && && \uparrow^{\mathrlap{ id_w \otimes \alpha_{x,y,z} }} \\ (w \otimes (x \otimes y)) \otimes z && \underset{\alpha_{w,x \otimes y, z}}{\longrightarrow} && w \otimes ((x \otimes y) \otimes z) }$$

Lemma 1.2. ***(Kelly 64)***

Let $(\mathcal{C}, \otimes, 1)$ be a monoidal category, def. 1.1. Then the left and right unitors ℓ and r satisfy the following conditions:

1. $\ell_1 = r_1 \;:\; 1 \otimes 1 \overset{\simeq}{\longrightarrow} 1$;

2. *for all objects $x,y \in \mathcal{C}$ the following diagrams commutes:*

$$\array{ (1 \otimes x) \otimes y && \\ {}^{\mathllap{\alpha_{1,x,y}}}\downarrow & \searrow^{\mathrlap{\ell_x \otimes id_y}} \\ 1 \otimes (x \otimes y) & \underset{\ell_{x \otimes y}}{\longrightarrow} & x \otimes y } \;;$$

and

$$\array{ x \otimes (y \otimes 1) && \\ {}^{\mathllap{\alpha^{-1}_{1,x,y}}}\downarrow & \searrow^{\mathrlap{id_x \otimes r_y}} \\ (x \otimes y) \otimes 1 & \underset{r_{x \otimes y}}{\longrightarrow} & x \otimes y } \;;$$

For **proof** see at *monoidal category* this lemma and this lemma.

Remark 1.3. Just as for an associative algebra it is sufficient to demand $1 a = a$ and $a 1 = a$ and $(a b) c = a (b c)$ in order to have that expressions of arbitrary length may be re-bracketed at will, so there is a *coherence theorem for monoidal categories* which states that all ways of freely composing the unitors and associators in a monoidal category (def. 1.1) to go from one expression to another will coincide. Accordingly, much as one may drop the notation for the bracketing in an associative algebra altogether, so one may, with due care, reason about monoidal categories without always making all unitors and associators explicit.

(Here the qualifier "freely" means informally that we must not use any non-formal identification between objects, and formally it means that the diagram in question must be in the image of a strong monoidal functor from a *free* monoidal category. For example if in a particular monoidal category it so happens that the object $X \otimes (Y \otimes Z)$ is actually *equal* to $(X\otimes Y)\otimes Z$, then the various ways of going from one expression to another using only associators *and* this equality no longer need to coincide.)

Definition 1.4. A **(pointed) topological braided monoidal category**, is a (pointed) topological monoidal category $\mathcal{C}$ (def. 1.1) equipped with a natural isomorphism

$$\tau_{x,y} \colon x \otimes y \to y \otimes x$$

called the **braiding**, such that the following two kinds of diagrams commute for all objects involved ("hexagon identities"):

$$\array{ (x \otimes y) \otimes z &\stackrel{a_{x,y,z}}{\longrightarrow}& x \otimes (y \otimes z) &\stackrel{\tau_{x,y \otimes z}}{\longrightarrow}& (y \otimes z) \otimes x \\ \downarrow^{\mathrlap{\tau_{x,y}\otimes Id}} &&&& \downarrow^{\mathrlap{a_{y,z,x}}} \\ (y \otimes x) \otimes z &\stackrel{a_{y,x,z}}{\longrightarrow}& y \otimes (x \otimes z) &\stackrel{Id \otimes \tau_{x,z}}{\longrightarrow}& y \otimes (z \otimes x) }$$

and

$$\array{ x \otimes (y \otimes z) &\stackrel{a^{-1}_{x,y,z}}{\longrightarrow}& (x \otimes y) \otimes z &\stackrel{\tau_{x \otimes y, z}}{\longrightarrow}& z \otimes (x \otimes y) \\ \downarrow^{\mathrlap{Id \otimes \tau_{y,z}}} &&&& \downarrow^{\mathrlap{a^{-1}_{z,x,y}}} \\ x \otimes (z \otimes y) &\stackrel{a^{-1}_{x,z,y}}{\longrightarrow}& (x \otimes z) \otimes y &\stackrel{\tau_{x,z} \otimes Id}{\longrightarrow}& (z \otimes x) \otimes y } \,,$$

where $a_{x,y,z} \colon (x \otimes y) \otimes z \to x \otimes (y \otimes z)$ denotes the components of the associator of $\mathcal{C}^\otimes$.

Definition 1.5. A **(pointed) topological symmetric monoidal category** is a (pointed) topological braided monoidal category (def. 1.4) for which the braiding

$$\tau_{x,y} \colon x \otimes y \to y \otimes x$$

satisfies the condition:

$$\tau_{y,x} \circ \tau_{x,y} = 1_{x \otimes y}$$

for all objects x, y

Remark 1.6. In analogy to the coherence theorem for monoidal categories (remark 1.3) there is a coherence theorem for symmetric monoidal categories (def. 1.5), saying that every diagram built freely (see remark 1.6) from associators, unitors and braidings such that both sides of the diagram correspond to the same permutation of objects, coincide.

Definition 1.7. Given a (pointed) topological symmetric monoidal category $\mathcal{C}$ with tensor product $\otimes$ (def. 1.5) it is called a **closed monoidal category** if for each $Y \in \mathcal{C}$ the functor $Y \otimes (-) \simeq (-) \otimes Y$ has a right adjoint, denoted $\mathrm{hom}(Y, -)$

$$\mathcal{C} \underoverset{\underset{\mathrm{hom}(Y,-)}{\longrightarrow}}{\overset{(-)\otimes Y}{\longleftarrow}}{\bot} \mathcal{C} \,,$$

hence if there are natural bijections

$$\mathrm{Hom}_{\mathcal{C}}(X \otimes Y, Z) \;\simeq\; \mathrm{Hom}_{\mathcal{C}}\, C(X, \mathrm{hom}(Y, Z))$$

for all objects $X, Z \in \mathcal{C}$.

Since for the case that $X = 1$ is the tensor unit of $\mathcal{C}$ this means that

$$\mathrm{Hom}_{\mathcal{C}}(1, \mathrm{hom}(Y,Z)) \simeq \mathrm{Hom}_{\mathcal{C}}(Y, Z)\,,$$

the object $\mathrm{hom}(Y, Z) \in \mathcal{C}$ is an enhancement of the ordinary hom-set $\mathrm{Hom}_{\mathcal{C}}(Y, Z)$ to an object in $\mathcal{C}$. Accordingly, it is also called the **internal hom** between Y and Z.

In a closed monoidal category, the adjunction isomorphism between tensor product and internal hom even holds internally:

Proposition 1.8. *In a symmetric closed monoidal category (def. 1.7) there are natural isomorphisms*

$$\mathrm{hom}(X \otimes Y, Z) \;\simeq\; \mathrm{hom}(X, \mathrm{hom}(Y, Z))$$

whose image under $\mathrm{Hom}_{\mathcal{C}}(1, -)$ are the defining natural bijections of def. 1.7.

Proof. Let $A \in \mathcal{C}$ be any object. By applying the defining natural bijections twice, there are composite natural bijections

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(A, \mathrm{hom}(X \otimes Y, Z)) & \simeq \mathrm{Hom}_{\mathcal{C}}(A \otimes (X \otimes Y), Z) \\ & \simeq \mathrm{Hom}_{\mathcal{C}}((A \otimes X) \otimes Y, Z) \\ & \simeq \mathrm{Hom}_{\mathcal{C}}(A \otimes X, \mathrm{hom}(Y, Z)) \\ & \simeq \mathrm{Hom}_{\mathcal{C}}(A, \mathrm{hom}(X, \mathrm{hom}(Y, Z))) \end{aligned} \,.$$

Since this holds for all A, the Yoneda lemma (the fully faithfulness of the Yoneda embedding) says that there is an isomorphism $\mathrm{hom}(X \otimes Y, Z) \simeq \mathrm{hom}(X, \mathrm{hom}(Y, Z))$. Moreover, by taking $A = 1$ in the above and using the left unitor isomorphisms $A \otimes (X \otimes Y) \simeq X \otimes Y$ and $A \otimes X \simeq X$ we get a commuting diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(1, \mathrm{hom}(X \otimes Y,)) & \overset{\simeq}{\longrightarrow} & \mathrm{Hom}_{\mathcal{C}}(1, \mathrm{hom}(X, \mathrm{hom}(Y, Z))) \\ {}^{\simeq}\downarrow && \downarrow^{\simeq} \\ \mathrm{Hom}_{\mathcal{C}}(X \otimes Y, Z) & \overset{\simeq}{\longrightarrow} & \mathrm{Hom}_{\mathcal{C}}(X, \mathrm{hom}(Y, Z)) \end{array} \,.$$

∎

Example 1.9. The category Set of sets and functions between them, regarded as enriched in discrete topological spaces, becomes a symmetric monoidal category according to def. 1.5 with tensor product the Cartesian product $\times$ of sets. The associator, unitor and braiding isomorphism are the evident (almost unnoticable but nevertheless nontrivial) canonical identifications.

Similarly the category $\mathrm{Top}_{\mathrm{cg}}$ of compactly generated topological spaces (def.) becomes a symmetric monoidal category with tensor product the corresponding Cartesian products, hence the operation of forming k-ified (cor.) product topological spaces (exmpl.). The underlying functions of the associator, unitor and braiding isomorphisms are just those of the underlying sets, as above.

Symmetric monoidal categories, such as these, for which the tensor product is the Cartesian product are called *Cartesian monoidal categories*.

Both examples are closed monoidal categories (def. 1.7), with internal hom the mapping spaces (prop.).

Example 1.10. The category $\mathrm{Top}_{\mathrm{cg}}^{*/}$ of pointed compactly generated topological spaces with tensor product the smash product $\wedge$ (def.)

$$X \wedge Y \coloneqq \frac{X \times Y}{X \vee Y}$$

is a symmetric monoidal category (def. 1.5) with unit object the pointed 0-sphere S^0.

The components of the associator, the unitors and the braiding are those of Top as in example 1.9, descended to the quotient topological spaces which appear in the definition of the smash product. This works for pointed compactly generated spaces (but not for general pointed topological spaces) by this prop..

The category $\mathrm{Top}_{\mathrm{cg}}^{*/}$ is also a closed monoidal category (def. 1.7), with internal hom the pointed mapping space $\mathrm{Maps}(-,-)_*$ (exmpl.)

Example 1.11. The category Ab of abelian groups, regarded as enriched in discrete topological spaces, becomes a symmetric monoidal category with tensor product the actual tensor product of abelian groups $\otimes_{\mathbb{Z}}$ and with tensor unit the additive group $\mathbb{Z}$ of integers. Again the associator, unitor and braiding isomorphism are the evident ones coming from the underlying sets, as in example 1.9.

This is a closed monoidal category with internal hom $\mathrm{hom}(A,B)$ being the set of homomorphisms $\mathrm{Hom}_{\mathrm{Ab}}(A,B)$ equipped with the pointwise group structure for $\phi_1, \phi_2 \in \mathrm{Hom}_{\mathrm{Ab}}(A,B)$ then $(\phi_1 + \phi_2)(a) \coloneqq \phi_1(a) + \phi_2(b) \ \in B$.

This is the archetypical case that motivates the notation "$\otimes$" for the pairing operation in a monoidal category:

Example 1.12. The category category of chain complexes $\mathrm{Ch}_\bullet$, equipped with the tensor product of chain complexes is a symmetric monoidal category (def. 1.5).

In this case the braiding has a genuinely non-trivial aspect to it, beyond just the swapping of coordinates as in examples 1.9, 1.10 and def. 1.11, namely for $X, Y \in \mathrm{Ch}_\bullet$ then

$$(X \otimes Y)_n = \underset{n_1 + n_2 = n}{\otimes} X_{n_1} \otimes_{\mathbb{Z}} X_{n_2}$$

and in these components the braiding isomorphism is that of Ab, but with a minus sign thrown in whener two odd-graded components are commuted.

This is a first shadow of the graded-commutativity that also exhibited by spectra.

(e.g. Hovey 99, prop. 4.2.13)

Algebras and modules

Definition 1.13. Given a (pointed) topological monoidal category $(\mathcal{C}, \otimes, 1)$, then a **monoid internal to** $(\mathcal{C}, \otimes, 1)$ is

1. an object $A \in \mathcal{C}$;
2. a morphism $e : 1 \longrightarrow A$ (called the *unit*)
3. a morphism $\mu : A \otimes A \longrightarrow A$ (called the *product*);

such that

1. (associativity) the following diagram commutes

$$\begin{array}{ccccc} (A \otimes A) \otimes A & \underset{\simeq}{\overset{a_{A,A,A}}{\longrightarrow}} & A \otimes (A \otimes A) & \overset{A \otimes \mu}{\longrightarrow} & A \otimes A \\ {}^{\mu \otimes A}\downarrow & & & & \downarrow^{\mu} \\ A \otimes A & \longrightarrow & & \overset{\mu}{\longrightarrow} & A \end{array} ,$$

   where a is the associator isomorphism of $\mathcal{C}$;

2. (unitality) the following diagram commutes:

$$\begin{tikzcd} 1 \otimes A \arrow[r, "e \otimes \mathrm{id}"] \arrow[dr, "\ell"'] & A \otimes A \arrow[d, "\mu"] & A \otimes 1 \arrow[l, "\mathrm{id} \otimes e"'] \arrow[dl, "r"] \\ & A & \end{tikzcd},$$

where ℓ and r are the left and right unitor isomorphisms of $\mathcal{C}$.

Moreover, if $(\mathcal{C}, \otimes, 1)$ has the structure of a symmetric monoidal category (def. 1.5) $(\mathcal{C}, \otimes, 1, B)$ with symmetric braiding τ, then a monoid (A, μ, e) as above is called a **commutative monoid in** $(\mathcal{C}, \otimes, 1, B)$ if in addition

- (commutativity) the following diagram commutes

$$\begin{tikzcd} A \otimes A \arrow[rr, "\tau_{A,A}", "\simeq"'] \arrow[dr, "\mu"'] & & A \otimes A \arrow[dl, "\mu"] \\ & A & \end{tikzcd}.$$

A homomorphism of monoids $(A_1, \mu_1, e_1) \longrightarrow (A_2, \mu_2, f_2)$ is a morphism

$$f : A_1 \longrightarrow A_2$$

in $\mathcal{C}$, such that the following two diagrams commute

$$\begin{tikzcd} A_1 \otimes A_1 \arrow[r, "f \otimes f"] \arrow[d, "\mu_1"'] & A_2 \otimes A_2 \arrow[d, "\mu_2"] \\ A_1 \arrow[r, "f"'] & A_2 \end{tikzcd}$$

and

$$\begin{tikzcd} 1_c \arrow[r, "e_1"] \arrow[dr, "e_2"'] & A_1 \arrow[d, "f"] \\ & A_2 \end{tikzcd}.$$

Write $\mathrm{Mon}(\mathcal{C}, \otimes, 1)$ for the **category of monoids** in $\mathcal{C}$, and $\mathrm{CMon}(\mathcal{C}, \otimes, 1)$ for its subcategory of commutative monoids.

Example 1.14. Given a (pointed) topological monoidal category $(\mathcal{C}, \otimes, 1)$, then the tensor unit 1 is a monoid in $\mathcal{C}$ (def. 1.13) with product given by either the left or right unitor

$$\ell_1 = r_1 : 1 \otimes 1 \overset{\simeq}{\longrightarrow} 1 .$$

By lemma 1.2, these two morphisms coincide and define an associative product with unit the identity $\mathrm{id} : 1 \to 1$.

If $(\mathcal{C}, \otimes, 1)$ is a symmetric monoidal category (def. 1.5), then this monoid is a commutative monoid.

Example 1.15. Given a symmetric monoidal category $(\mathcal{C}, \otimes, 1)$ (def. 1.5), and given two commutative monoids (E_i, μ_i, e_i) $i \in \{1, 2\}$ (def. 1.13), then the tensor product $E_1 \otimes E_2$ becomes itself a commutative monoid with unit morphism

$$e : 1 \overset{\simeq}{\longrightarrow} 1 \otimes 1 \overset{e_1 \otimes e_2}{\longrightarrow} E_1 \otimes E_2$$

(where the first isomorphism is, $\ell_1^{-1} = r_1^{-1}$ (lemma 1.2)) and with product morphism given by

$$E_1 \otimes E_2 \otimes E_1 \otimes E_2 \overset{\mathrm{id} \otimes \tau_{E_2, E_1} \otimes \mathrm{id}}{\longrightarrow} E_1 \otimes E_1 \otimes E_2 \otimes E_2 \overset{\mu_1 \otimes \mu_2}{\longrightarrow} E_1 \otimes E_2$$

(where we are notationally suppressing the associators and where τ denotes the braiding of $\mathcal{C}$).

That this definition indeed satisfies associativity and commutativity follows from the corresponding properties of (E_i, μ_i, e_i), and from the hexagon identities for the braiding (def. 1.4) and from symmetry of the braiding.

Similarly one checks that for $E_1 = E_2 = E$ then the unit maps

$$E \simeq E \otimes 1 \overset{\mathrm{id} \otimes e}{\longrightarrow} E \otimes E$$
$$E \simeq 1 \otimes E \overset{e \otimes 1}{\longrightarrow} E \otimes E$$

and the product map

$$\mu : E \otimes E \longrightarrow E$$

and the braiding

$$\tau_{E,E} : E \otimes E \longrightarrow E \otimes E$$

are monoid homomorphisms, with $E \otimes E$ equipped with the above monoid structure.

Definition 1.16. Given a (pointed) topological monoidal category $(\mathcal{C}, \otimes, 1)$ (def. 1.1), and given (A, μ, e) a monoid in $(\mathcal{C}, \otimes, 1)$ (def. 1.13), then a **left module object** in $(\mathcal{C}, \otimes, 1)$ over (A, μ, e) is

1. an object $N \in \mathcal{C}$;
2. a morphism $\rho : A \otimes N \longrightarrow N$ (called the *action*);

such that

1. (unitality) the following diagram commutes:

$$\begin{tikzcd} 1 \otimes N \arrow[r, "e \otimes \mathrm{id}"] \arrow[dr, "\ell"'] & A \otimes N \arrow[d, "\rho"] \\ & N \end{tikzcd} ,$$

   where ℓ is the left unitor isomorphism of $\mathcal{C}$.

2. (action property) the following diagram commutes

$$\begin{tikzcd} (A \otimes A) \otimes N \arrow[r, "a_{A,A,N}", "\simeq"'] \arrow[d, "\mu \otimes N"'] & A \otimes (A \otimes N) \arrow[r, "A \otimes \rho"] & A \otimes N \arrow[d, "\rho"] \\ A \otimes N \arrow[rr, "\rho"] & & N \end{tikzcd} ,$$

A homomorphism of left A-module objects

$$(N_1, \rho_1) \longrightarrow (N_2, \rho_2)$$

is a morphism

$$f : N_1 \longrightarrow N_2$$

in $\mathcal{C}$, such that the following diagram commutes:

$$\begin{tikzcd} A \otimes N_1 \arrow[r, "A \otimes f"] \arrow[d, "\rho_1"'] & A \otimes N_2 \arrow[d, "\rho_2"] \\ N_1 \arrow[r, "f"'] & N_2 \end{tikzcd} .$$

For the resulting **category of modules** of left A-modules in $\mathcal{C}$ with A-module homomorphisms between them, we write

$$A\,\mathrm{Mod}(\mathcal{C}) \,.$$

This is naturally a (pointed) topologically enriched category itself.

Example 1.17. Given a monoidal category $(\mathcal{C}, \otimes, 1)$ (def. 1.1) with the tensor unit 1 regarded as a monoid in a monoidal category via example 1.14, then the left unitor

$$\ell_C : 1 \otimes C \longrightarrow C$$

makes every object $C \in \mathcal{C}$ into a left module, according to def. 1.16, over C. The action property holds due to lemma 1.2. This gives an equivalence of categories

$$\mathcal{C} \simeq 1\mathrm{Mod}(\mathcal{C})$$

of $\mathcal{C}$ with the category of modules over its tensor unit.

Example 1.18. The archetypical case in which all these abstract concepts reduce to the basic familiar ones is the symmetric monoidal category Ab of abelian groups from example 1.11.

1. A monoid in $(\mathrm{Ab}, \otimes_{\mathbb{Z}}, \mathbb{Z})$ (def. 1.13) is equivalently a ring.
2. A commutative monoid in in $(\mathrm{Ab}, \otimes_{\mathbb{Z}}, \mathbb{Z})$ (def. 1.13) is equivalently a commutative ring R.

3. An R-module object in $(\mathrm{Ab}, \otimes_{\mathbb{Z}}, \mathbb{Z})$ (def. 1.16) is equivalently an R-module;

4. The tensor product of R-module objects (def. 1.21) is the standard tensor product of modules.

5. The category of module objects $R\,\mathrm{Mod}(\mathrm{Ab})$ (def. 1.21) is the standard category of modules $R\,\mathrm{Mod}$.

Example 1.19. Closely related to the example 1.18, but closer to the structure we will see below for spectra, are monoids in the category of chain complexes $(\mathrm{Ch}_\bullet, \otimes, \mathbb{Z})$ from example 1.12. These monoids are equivalently differential graded algebras.

Proposition 1.20. *In the situation of def. 1.16, the monoid (A, μ, e) canonically becomes a left module over itself by setting $\rho \coloneqq \mu$. More generally, for $C \in \mathcal{C}$ any object, then $A \otimes C$ naturally becomes a left A-module by setting:*

$$\rho \,:\, A \otimes (A \otimes C) \underset{\simeq}{\overset{a^{-1}_{A,A,C}}{\longrightarrow}} (A \otimes A) \otimes C \overset{\mu \otimes \mathrm{id}}{\longrightarrow} A \otimes C \,.$$

The A-modules of this form are called ***free modules****.*

The free functor F constructing free A-modules is left adjoint to the forgetful functor U which sends a module (N, ρ) to the underlying object $U(N, \rho) \coloneqq N$.

$$A\,\mathrm{Mod}(\mathcal{C}) \underoverset{\underset{U}{\longrightarrow}}{\overset{F}{\longleftarrow}}{\bot} \mathcal{C} \,.$$

Proof. A homomorphism out of a free A-module is a morphism in $\mathcal{C}$ of the form

$$f \,:\, A \otimes C \longrightarrow N$$

fitting into the diagram (where we are notationally suppressing the associator)

$$\begin{array}{ccc} A \otimes A \otimes C & \overset{A \otimes f}{\longrightarrow} & A \otimes N \\ {}^{\mu \otimes \mathrm{id}}\downarrow && \downarrow^{\rho} \\ A \otimes C & \underset{f}{\longrightarrow} & N \end{array} \,.$$

Consider the composite

$$\tilde f \,:\, C \underset{\simeq}{\overset{\ell_C}{\longrightarrow}} 1 \otimes C \overset{e \otimes \mathrm{id}}{\longrightarrow} A \otimes C \overset{f}{\longrightarrow} N \,,$$

i.e. the restriction of f to the unit "in" A. By definition, this fits into a commuting square of the form (where we are now notationally suppressing the associator and the unitor)

$$\begin{array}{ccc} A \otimes C & \overset{\mathrm{id} \otimes \tilde f}{\longrightarrow} & A \otimes N \\ {}^{\mathrm{id} \otimes e \otimes \mathrm{id}}\downarrow && \downarrow^{=} \\ A \otimes A \otimes C & \underset{\mathrm{id} \otimes f}{\longrightarrow} & A \otimes N \end{array} \,.$$

Pasting this square onto the top of the previous one yields

$$\begin{array}{ccc} A \otimes C & \overset{\mathrm{id} \otimes \tilde f}{\longrightarrow} & A \otimes N \\ {}^{\mathrm{id} \otimes e \otimes \mathrm{id}}\downarrow && \downarrow^{=} \\ A \otimes A \otimes C & \overset{A \otimes f}{\longrightarrow} & A \otimes N \\ {}^{\mu \otimes \mathrm{id}}\downarrow && \downarrow^{\rho} \\ A \otimes C & \underset{f}{\longrightarrow} & N \end{array} \,,$$

where now the left vertical composite is the identity, by the unit law in A. This shows that f is uniquely determined by $\tilde f$ via the relation

$$f = \rho \circ (\mathrm{id}_A \otimes \tilde f) \,.$$

This natural bijection between f and $\tilde f$ establishes the adjunction. ▮

Definition 1.21. Given a (pointed) topological closed symmetric monoidal category $(\mathcal{C}, \otimes, 1)$ (def. 1.5, def. 1.7), given (A, μ, e) a commutative monoid in $(\mathcal{C}, \otimes, 1)$ (def. 1.13), and given (N_1, ρ_1) and (N_2, ρ_2) two left A-module objects (def.1.13), then

1. the **tensor product of modules** $N_1 \otimes_A N_2$ is, if it exists, the coequalizer

$$N_1 \otimes A \otimes N_2 \underset{\rho_1 \circ (\tau_{N_1,A} \otimes N_2)}{\overset{N_1 \otimes \rho_2}{\rightrightarrows}} N_1 \otimes N_1 \overset{\mathrm{coeq}}{\longrightarrow} N_1 \otimes_A N_2$$

and if $A \otimes (-)$ preserves these coequalizers, then this is equipped with the left A-action induced from the left A-action on N_1

2. the **function module** $\hom_A(N_1, N_2)$ is, if it exists, the equalizer

$$\hom_A(N_1, N_2) \overset{\mathrm{equ}}{\longrightarrow} \hom(N_1, N_2) \underset{\hom(A \otimes N_1, \rho_2) \circ (A \otimes (-))}{\overset{\hom(\rho_1, N_2)}{\rightrightarrows}} \hom(A \otimes N_1, N_2) \,.$$

equipped with the left A-action that is induced by the left A-action on N_2 via

$$\frac{A \otimes \hom(X, N_2) \longrightarrow \hom(X, N_2)}{A \otimes \hom(X, N_2) \otimes X \overset{\mathrm{id} \otimes \mathrm{ev}}{\longrightarrow} A \otimes N_2 \overset{\rho_2}{\longrightarrow} N_2} \,.$$

(e.g. Hovey-Shipley-Smith 00, lemma 2.2.2 and lemma 2.2.8)

Proposition 1.22. *Given a (pointed) topological closed symmetric monoidal category $(\mathcal{C}, \otimes, 1)$ (def. 1.5, def. 1.7), and given (A, μ, e) a commutative monoid in $(\mathcal{C}, \otimes, 1)$ (def. 1.13). If all coequalizers exist in $\mathcal{C}$, then the tensor product of modules $\otimes_A$ from def. 1.21 makes the category of modules $A\,\mathrm{Mod}(\mathcal{C})$ into a symmetric monoidal category, $(A\,\mathrm{Mod}, \otimes_A, A)$ with tensor unit the object A itself, regarded as an A-module via prop. 1.20.*

If moreover all equalizers exist, then this is a closed monoidal category (def. 1.7) with internal hom given by the function modules $\hom_A$ of def. 1.21.

(e.g. Hovey-Shipley-Smith 00, lemma 2.2.2, lemma 2.2.8)

Proof sketch. The associators and braiding for $\otimes_A$ are induced directly from those of $\otimes$ and the universal property of coequalizers. That A is the tensor unit for $\otimes_A$ follows with the same kind of argument that we give in the proof of example 1.23 below. ▮

Example 1.23. For (A, μ, e) a monoid (def. 1.13) in a symmetric monoidal category $(\mathcal{C}, \otimes, 1)$ (def. 1.1), the tensor product of modules (def. 1.21) of two free modules (def. 1.20) $A \otimes C_1$ and $A \otimes C_2$ always exists and is the free module over the tensor product in $\mathcal{C}$ of the two generators:

$$(A \otimes C_1) \otimes_A (A \otimes C_2) \simeq A \otimes (C_1 \otimes C_2) \,.$$

Hence if $\mathcal{C}$ has all coequalizers, so that the category of modules is a monoidal category $(A\,\mathrm{Mod}, \otimes_A, A)$ (prop. 1.22) then the free module functor (def. 1.20) is a strong monoidal functor (def. 1.47)

$$F \,:\, (\mathcal{C}, \otimes, 1) \longrightarrow (A\,\mathrm{Mod}, \otimes_A, A) \,.$$

Proof. It is sufficient to show that the diagram

$$A \otimes A \otimes A \underset{\mathrm{id} \otimes \mu}{\overset{\mu \otimes \mathrm{id}}{\rightrightarrows}} A \otimes A \overset{\mu}{\longrightarrow} A$$

is a coequalizer diagram (we are notationally suppressing the associators), hence that $A \otimes_A A \simeq A$, hence that the claim holds for $C_1 = 1$ and $C_2 = 1$.

To that end, we check the universal property of the coequalizer:

First observe that μ indeed coequalizes $\mathrm{id} \otimes \mu$ with $\mu \otimes \mathrm{id}$, since this is just the associativity clause in def. 1.13. So for $f : A \otimes A \longrightarrow Q$ any other morphism with this property, we need to show that there is a unique morphism $\phi : A \longrightarrow Q$ which makes this diagram commute:

$$\begin{array}{ccc} A \otimes A & \overset{\mu}{\longrightarrow} & A \\ {}^{\mathllap{f}}\downarrow & \swarrow_{\mathrlap{\phi}} & \\ Q & & \end{array} \,.$$

We claim that

$$\phi \,:\, A \underset{\simeq}{\overset{r^{-1}}{\longrightarrow}} A \otimes 1 \overset{\mathrm{id} \otimes e}{\longrightarrow} A \otimes A \overset{f}{\longrightarrow} Q \,,$$

where the first morphism is the inverse of the right unitor of $\mathcal{C}$.

First to see that this does make the required triangle commute, consider the following pasting composite of commuting diagrams

$$
\begin{array}{ccc}
A \otimes A & \overset{\mu}{\longrightarrow} & A \\
{}^{\mathrm{id}\otimes r^{-1}}_{\simeq}\downarrow && \downarrow^{r^{-1}}_{\simeq} \\
A \otimes A \otimes 1 & \overset{\mu\otimes \mathrm{id}}{\longrightarrow} & A \otimes 1 \\
{}^{\mathrm{id}\otimes e}\downarrow && \downarrow^{\mathrm{id}\otimes e} \\
A \otimes A \otimes A & \overset{\mu \otimes \mathrm{id}}{\longrightarrow} & A \otimes A \\
{}^{\mathrm{id}\otimes \mu}\downarrow && \downarrow^{f} \\
A \otimes A & \underset{f}{\longrightarrow} & Q
\end{array}
$$

Here the the top square is the naturality of the right unitor, the middle square commutes by the functoriality of the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ and the definition of the product category (def. 1.26), while the commutativity of the bottom square is the assumption that f coequalizes $\mathrm{id} \otimes \mu$ with $\mu \otimes \mathrm{id}$.

Here the right vertical composite is ϕ, while, by unitality of (A, μ, e), the left vertical composite is the identity on A, Hence the diagram says that $\phi \circ \mu = f$, which we needed to show.

It remains to see that ϕ is the unique morphism with this property for given f. For that let $q : A \to Q$ be any other morphism with $q \circ \mu = f$. Then consider the commuting diagram

$$
\begin{array}{ccc}
A \otimes 1 & \overset{\simeq}{\longleftarrow} & A \\
{}^{\mathrm{id}\otimes e}\downarrow & \searrow^{\simeq} & \downarrow^{=} \\
A \otimes A & \overset{\mu}{\longrightarrow} & A \,, \\
{}^{f}\downarrow & \swarrow_{q} & \\
Q & &
\end{array}
$$

where the top left triangle is the unitality condition and the two isomorphisms are the right unitor and its inverse. The commutativity of this diagram says that $q = \phi$. ▮

Definition 1.24. Given a monoidal category of modules $(A\,\mathrm{Mod}, \otimes_A, A)$ as in prop. 1.22, then a monoid (E, μ, e) in $(A\,\mathrm{Mod}, \otimes_A, A)$ (def. 1.13) is called an A-**algebra**.

Propposition 1.25. *Given a monoidal category of modules $(A\,\mathrm{Mod}, \otimes_A, A)$ in a monoidal category $(\mathcal{C}, \otimes, 1)$ as in prop. 1.22, and an A-algebra (E, μ, e) (def. 1.24), then there is an equivalence of categories*

$$
A\,\mathrm{Alg}_{\mathrm{comm}}(\mathcal{C}) \coloneqq \mathrm{CMon}(A\,\mathrm{Mod}) \simeq \mathrm{CMon}(\mathcal{C})^{A/}
$$

between the category of commutative monoids in $A\,\mathrm{Mod}$ and the coslice category of commutative monoids in $\mathcal{C}$ under A, hence between commutative A-algebras in $\mathcal{C}$ and commutative monoids E in $\mathcal{C}$ that are equipped with a homomorphism of monoids $A \longrightarrow E$.

(e.g. EKMM 97, VII lemma 1.3)

Proof. In one direction, consider a A-algebra E with unit $e_E : A \longrightarrow E$ and product $\mu_{E/A} : E \otimes_A E \longrightarrow E$. There is the underlying product μ_E

$$
\begin{array}{ccccc}
E \otimes A \otimes E & \underset{\longrightarrow}{\longrightarrow} & E \otimes E & \overset{\mathrm{coeq}}{\longrightarrow} & E \otimes_A E \\
&& & {}_{\mu_E}\searrow & \downarrow^{\mu_{E/A}} \\
&& & & E
\end{array}
$$

By considering a diagram of such coequalizer diagrams with middle vertical morphism $e_E \circ e_A$, one find that this is a unit for μ_E and that $(E, \mu_E, e_E \circ e_A)$ is a commutative monoid in $(\mathcal{C}, \otimes, 1)$.

Then consider the two conditions on the unit $e_E : A \longrightarrow E$. First of all this is an A-module homomorphism, which means that

$$
(\star) \quad
\begin{array}{ccc}
A \otimes A & \overset{\mathrm{id}\otimes e_E}{\longrightarrow} & A \otimes E \\
{}^{\mu_A}\downarrow && \downarrow^{\rho} \\
A & \underset{e_E}{\longrightarrow} & E
\end{array}
$$

commutes. Moreover it satisfies the unit property

$$
\begin{tikzcd}
A \otimes_A E \arrow[r, "e_A \otimes \mathrm{id}"] \arrow[dr, "\simeq"'] & E \otimes_A E \arrow[d, "\mu_{E/A}"] \\
 & E
\end{tikzcd}
$$

By forgetting the tensor product over A, the latter gives

$$
\begin{tikzcd}
A \otimes E \arrow[r, "e \otimes \mathrm{id}"] \arrow[d] & E \otimes E \arrow[d] \\
A \otimes_A E \arrow[r, "e_E \otimes \mathrm{id}"] \arrow[d, "\simeq"'] & E \otimes_A E \arrow[d, "\mu_{E/A}"] \\
E \arrow[r, equal] & E
\end{tikzcd}
\quad \simeq \quad
\begin{tikzcd}
A \otimes E \arrow[r, "e_E \otimes \mathrm{id}"] \arrow[d, "\rho"'] & E \otimes E \arrow[d, "\mu_E"] \\
E \arrow[r, "\mathrm{id}"'] & E
\end{tikzcd}
,
$$

where the top vertical morphisms on the left the canonical coequalizers, which identifies the vertical composites on the right as shown. Hence this may be pasted to the square $(\star)$ above, to yield a commuting square

$$
\begin{tikzcd}
A \otimes A \arrow[r, "\mathrm{id} \otimes e_E"] \arrow[d, "\mu_A"'] & A \otimes E \arrow[r, "e_E \otimes \mathrm{id}"] \arrow[d, "\rho"'] & E \otimes E \arrow[d, "\mu_E"] \\
A \arrow[r, "e_E"'] & E \arrow[r, "\mathrm{id}"'] & E
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \otimes A \arrow[r, "e_E \otimes e_E"] \arrow[d, "\mu_A"'] & E \otimes E \arrow[d, "\mu_E"] \\
A \arrow[r, "e_E"'] & E
\end{tikzcd}
.
$$

This shows that the unit e_A is a homomorphism of monoids $(A, \mu_A, e_A) \longrightarrow (E, \mu_E, e_E \circ e_A)$.

Now for the converse direction, assume that (A, μ_A, e_A) and (E, μ_E, e'_E) are two commutative monoids in $(\mathcal{C}, \otimes, 1)$ with $e_E : A \to E$ a monoid homomorphism. Then E inherits a left A-module structure by

$$\rho : A \otimes E \overset{e_A \otimes \mathrm{id}}{\longrightarrow} E \otimes E \overset{\mu_E}{\to} E \,.$$

By commutativity and associativity it follows that μ_E coequalizes the two induced morphisms $E \otimes A \otimes E \underset{\longrightarrow}{\overset{\longrightarrow}{}} E \otimes E$. Hence the universal property of the coequalizer gives a factorization through some $\mu_{E/A} : E \otimes_A E \longrightarrow E$. This shows that $(E, \mu_{E/A}, e_E)$ is a commutative A-algebra.

Finally one checks that these two constructions are inverses to each other, up to isomorphism. ▮

Topological ends and coends

For working with pointed topologically enriched functors, a certain shape of limits/colimits is particularly relevant: these are called (pointed topological enriched) *ends* and *coends*. We here introduce these and then derive some of their basic properties, such as notably the expression for topological left Kan extension in terms of coends (prop. 1.38 below). Further below it is via left Kan extension along the ordinary smash product of pointed topological spaces ("Day convolution") that the symmetric monoidal smash product of spectra is induced.

Definition 1.26. Let $\mathcal{C}, \mathcal{D}$ be pointed topologically enriched categories (def.), i.e. enriched categories over $(\mathrm{Top}_{\mathrm{cg}}^{*/}, \wedge, S^0)$ from example 1.10.

1. The **pointed topologically enriched opposite category** $\mathcal{C}^{\mathrm{op}}$ is the topologically enriched category with the same objects as $\mathcal{C}$, with hom-spaces

   $$\mathcal{C}^{\mathrm{op}}(X, Y) \coloneqq \mathcal{C}(Y, X)$$

   and with composition given by braiding followed by the composition in $\mathcal{C}$:

   $$\mathcal{C}^{\mathrm{op}}(X,Y) \wedge \mathcal{C}^{\mathrm{op}}(Y,Z) = \mathcal{C}(Y,X) \wedge \mathcal{C}(Z,Y) \underset{\simeq}{\overset{\tau}{\to}} \mathcal{C}(Z,Y) \wedge \mathcal{C}(Y,X) \overset{\circ_{Z,Y,X}}{\longrightarrow} \mathcal{C}(Z,X) = \mathcal{C}^{\mathrm{op}}(X,Z) \,.$$

2. the **pointed topological product category** $\mathcal{C} \times \mathcal{D}$ is the topologically enriched category whose objects are pairs of objects (c, d) with $c \in \mathcal{C}$ and $d \in \mathcal{D}$, whose hom-spaces are the smash product of the separate hom-spaces

   $$(\mathcal{C} \times \mathcal{D})((c_1, d_1), (c_2, d_2)) \coloneqq \mathcal{C}(c_1, c_2) \wedge \mathcal{D}(d_1, d_2)$$

   and whose composition operation is the braiding followed by the smash product of the separate composition operations:

$$
\begin{array}{ccc}
(\mathcal{C}\times\mathcal{D})((c_1,d_1),\ (c_2,d_2)) \wedge (\mathcal{C}\times\mathcal{D})((c_2,d_2),\ (c_3,d_3)) && \\
{}^{=}\downarrow && \\
(\mathcal{C}(c_1,c_2)\wedge\mathcal{D}(d_1,d_2))\wedge(\mathcal{C}(c_2,c_3)\wedge\mathcal{D}(d_2,d_3)) && \\
\downarrow^{\tau}_{\simeq} && \\
(\mathcal{C}(c_1,c_2)\wedge\mathcal{C}(c_2,c_3))\wedge(\mathcal{D}(d_1,d_2)\wedge\mathcal{D}(d_2,d_3)) & \xrightarrow{(\circ_{c_1,c_2,c_3})\wedge(\circ_{d_1,d_2,d_3})} & \mathcal{C}(c_1,c_3)\wedge\mathcal{D}(d_1,d_3) \\
&& \downarrow^{=} \\
&& (\mathcal{C}\times\mathcal{D})((c_1,d_1),\ (c_3,d_3))
\end{array}
.
$$

Example 1.27. A pointed topologically enriched functor (def.) into $\mathrm{Top}_{\mathrm{cg}}^{*/}$ (exmpl.) out of a pointed topological product category as in def. 1.26

$$F : \mathcal{C}\times\mathcal{D} \longrightarrow \mathrm{Top}_{\mathrm{cg}}^{*/}$$

(a "pointed topological bifunctor") has component maps of the form

$$F_{(c_1,d_1),(c_2,d_2)} : \mathcal{C}(c_1,c_2)\wedge\mathcal{D}(d_1,d_2) \longrightarrow \mathrm{Maps}(F_0((c_1,d_1)), F_0((c_2,d_2)))_* \,.$$

By functoriality and under passing to adjuncts (cor.) this is equivalent to two commuting *actions*

$$\rho_{c_1,c_2}(d) : \mathcal{C}(c_1,c_2)\wedge F_0((c_1,d)) \longrightarrow F_0((c_2,d))$$

and

$$\rho_{d_1,d_2}(c) : \mathcal{D}(d_1,d_2)\wedge F_0((c,d_1)) \longrightarrow F_0((c,d_2)) \,.$$

In the special case of a functor out of the product category of some $\mathcal{C}$ with its opposite category (def. 1.26)

$$F : \mathcal{C}^{\mathrm{op}}\times\mathcal{C} \longrightarrow \mathrm{Top}_{\mathrm{cg}}^{*/}$$

then this takes the form of a "pullback action" in the first variable

$$\rho_{c_2,c_1}(d) : \mathcal{C}(c_1,c_2)\wedge F_0((c_2,d)) \longrightarrow F_0((c_1,d))$$

and a "pushforward action" in the second variable

$$\rho_{d_1,d_2}(c) : \mathcal{C}(d_1,d_2)\wedge F_0((c,d_1)) \longrightarrow F_0((c,d_2)) \,.$$

Definition 1.28. Let $\mathcal{C}$ be a small pointed topologically enriched category (def.), i.e. an enriched category over $(\mathrm{Top}_{\mathrm{cg}}^{*/}, \wedge, S^0)$ from example 1.10. Let

$$F : \mathcal{C}^{\mathrm{op}}\times\mathcal{C} \longrightarrow \mathrm{Top}_{\mathrm{cg}}^{*/}$$

be a pointed topologically enriched functor (def.) out of the pointed topological product category of $\mathcal{C}$ with its opposite category, according to def. 1.26.

1. The **coend** of F, denoted $\int^{c\in\mathcal{C}} F(c,c)$, is the coequalizer in $\mathrm{Top}_{\mathrm{cg}}^{*}$ (prop., exmpl., prop., cor.) of the two actions encoded in F via example 1.27:

$$\coprod_{c,d\in\mathcal{C}} \mathcal{C}(c,d)\wedge F(d,c) \underset{\underset{c,d}{\sqcup}\rho_{(d,c)}(c)}{\overset{\underset{c,d}{\sqcup}\rho_{(c,d)}(d)}{\rightrightarrows}} \coprod_{c\in\mathcal{C}} F(c,c) \overset{\mathrm{coeq}}{\longrightarrow} \int^{c\in\mathcal{C}} F(c,c) \,.$$

2. The **end** of F, denoted $\int_{c\in\mathcal{C}} F(c,c)$, is the **equalizer** in $\mathrm{Top}_{\mathrm{cg}}^{*/}$ (prop., exmpl., prop., cor.) of the adjuncts of the two actions encoded in F via example 1.27:

$$\int_{c\in\mathcal{C}} F(c,c) \overset{\mathrm{equ}}{\longrightarrow} \prod_{c\in\mathcal{C}} F(c,c) \underset{\underset{c,d}{\sqcup}\tilde\rho_{(c,d)}(c)}{\overset{\underset{c,d}{\sqcup}\tilde\rho_{d,c}(d)}{\rightrightarrows}} \prod_{c\in\mathcal{C}} \mathrm{Maps}(\mathcal{C}(c,d),\ F(c,d))_* \,.$$

Example 1.29. Let G be a topological group. Write $\mathbf{B}(G_+)$ for the pointed topologically enriched category that has a single object $*$, whose single hom-space is G_+ (G with a basepoint freely adjoined (def.))

$$\mathbf{B}(G_+)(*,*) \coloneqq G_+$$

and whose composition operation is the product operation $(-)\cdot(-)$ in G under adjoining basepoints (exmpl.)

$$G_+ \wedge G_+ \simeq (G \times G)_+ \xrightarrow{((-)\cdot(-))_+} G_+ \,.$$

Then a topologically enriched functor

$$(X, \rho_l) \,:\, \mathbf{B}(G_+) \longrightarrow \mathrm{Top}^{*/}_{\mathrm{cg}}$$

is a pointed topological space $X \coloneqq F(\ast)$ equipped with a continuous function

$$\rho_l \,:\, G_+ \wedge X \longrightarrow X$$

satisfying the action property. Hence this is equivalently a continuous and basepoint-preserving left action (non-linear representation) of G on X.

The opposite category (def. 1.26) $(\mathbf{B}(G_+))^{\mathrm{op}}$ comes from the opposite group

$$(\mathbf{B}(G_+))^{\mathrm{op}} = \mathbf{B}(G_+^{\mathrm{op}}) \,.$$

(The canonical continuous isomorphism $G \simeq G^{\mathrm{op}}$ induces a canonical euqivalence of topologically enriched categories $(\mathbf{B}(G_+))^{\mathrm{op}} \simeq \mathbf{B}(G_+)$.)

So a topologically enriched functor

$$(Y, \rho_r) \,:\, (\mathbf{B}(G_+))^{\mathrm{op}} \longrightarrow \mathrm{Top}^{*}_{\mathrm{cg}}$$

is equivalently a basepoint preserving continuous *right* action of G.

Therefore the coend of two such functors (def. 1.28) coequalizes the relation

$$(xg,\ y) \sim (x,\ gy)$$

(where juxtaposition denotes left/right action) and hence is equivalently the canonical smash product of a right G-action with a left G-action, hence the quotient of the plain smash product by the diagonal action of the group G:

$$\int^{\ast \in \mathbf{B}(G_+)} (Y, \rho_r)(\ast) \wedge (X, \rho_l)(\ast) \;\simeq\; Y \wedge_G X \,.$$

Example 1.30. Let $\mathcal{C}$ be a small pointed topologically enriched category (def.). For $F, G \,:\, \mathcal{C} \longrightarrow \mathrm{Top}^{*/}_{\mathrm{cg}}$ two pointed topologically enriched functors, then the end (def. 1.28) of $\mathrm{Maps}(F(-), G(-))_*$ is a topological space whose underlying pointed set is the pointed set of natural transformations $F \to G$ (def.):

$$U\left(\int_{c \in \mathcal{C}} \mathrm{Maps}(F(c), G(c))_* \right) \;\simeq\; \mathrm{Hom}_{[\mathcal{C}, \mathrm{Top}^{*/}_{\mathrm{cg}}]}(F, G) \,.$$

Proof. The underlying pointed set functor $U \colon \mathrm{Top}^{*/}_{\mathrm{cg}} \to \mathrm{Set}^{*/}$ preserves all limits (prop., prop., prop.). Therefore there is an equalizer diagram in $\mathrm{Set}^{*/}$ of the form

$$U\left(\int_{c \in \mathcal{C}} \mathrm{Maps}(F(c), G(c))_* \right) \overset{\mathrm{equ}}{\longrightarrow} \prod_{c \in \mathcal{C}} \mathrm{Hom}_{\mathrm{Top}^{*/}_{\mathrm{cg}}}(F(c), G(c)) \underset{\underset{c,d}{\sqcup} U(\tilde\rho_{(c,d)}(c))}{\overset{\underset{c,d}{\sqcup} U(\tilde\rho_{d,c}(d))}{\rightrightarrows}} \prod_{c,d \in \mathcal{C}} \mathrm{Hom}_{\mathrm{Top}^{*/}_{\mathrm{cg}}}(\mathcal{C}(c,d), \mathrm{Maps}(F(c), G(d))_*) \,.$$

Here the object in the middle is just the set of collections of component morphisms $\left\{F(c) \overset{\eta_c}{\to} G(c)\right\}_{c \in \mathcal{C}}$. The two parallel maps in the equalizer diagram take such a collection to the functions which send any $c \overset{f}{\to} d$ to the result of precomposing

$$\begin{array}{ccc} F(c) & & \\ {}^{\mathllap{f(f)}}\downarrow & & \\ F(d) & \underset{\eta_d}{\longrightarrow} & G(d) \end{array}$$

and of postcomposing

$$\begin{array}{ccc} F(c) & \overset{\eta_c}{\longrightarrow} & G(c) \\ & & \downarrow^{\mathrlap{G(f)}} \\ & & G(d) \end{array}$$

each component in such a collection, respectively. These two functions being equal, hence the collection

$\{\eta_c\}_{c\in\mathcal{C}}$ being in the equalizer, means precisley that for all c, d and all $f : c \to d$ the square

$$\begin{array}{ccc} F(c) & \overset{\eta_c}{\longrightarrow} & G(c) \\ {}^{\mathllap{F(f)}}\downarrow && \downarrow^{\mathrlap{G(f)}} \\ F(d) & \underset{\eta_d}{\longrightarrow} & G(g) \end{array}$$

is a commuting square. This is precisley the condition that the collection $\{\eta_c\}_{c\in\mathcal{C}}$ be a natural transformation. ▮

Conversely, example 1.30 says that ends over bifunctors of the form $\mathrm{Maps}(F(-),G(-))_*$ constitute hom-spaces between pointed topologically enriched functors:

Definition 1.31. Let $\mathcal{C}$ be a small pointed topologically enriched category (def.). Define the structure of a pointed topologically enriched category on the category $[\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}]$ of pointed topologically enriched functors to $\mathrm{Top}_{\mathrm{cg}}^{*/}$ (exmpl.) by taking the hom-spaces to be given by the ends (def. 1.28) of example 1.30:

$$[\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}](F,G) \coloneqq \int_{c\in\mathcal{C}} \mathrm{Maps}(F(c),G(c))_*$$

The composition operation on these is defined to be the one induced by the composite maps

$$\left(\int_{c\in\mathcal{C}} \mathrm{Maps}(F(c),G(c))_*\right)\wedge\left(\int_{c\in\mathcal{C}} \mathrm{Maps}(G(c),H(c))_*\right) \longrightarrow \prod_{c\in\mathcal{C}} \mathrm{Maps}(F(c),G(c))_* \wedge \mathrm{Maps}(G(c),H(c))_* \xrightarrow{(\circ_{F(c),G(c),H(c)})_{c\in\mathcal{C}}} \prod_{c\in\mathcal{C}} \mathrm{Maps}(F(c),H(c$$

where the first, morphism is degreewise given by projection out of the limits that defined the ends. This composite evidently equalizes the two relevant adjunct actions (as in the proof of example 1.30) and hence defines a map into the end

$$\left(\int_{c\in\mathcal{C}} \mathrm{Maps}(F(c),G(c))_*\right)\wedge\left(\int_{c\in\mathcal{C}} \mathrm{Maps}(G(c),H(c))_*\right) \longrightarrow \int_{c\in\mathcal{C}} \mathrm{Maps}(F(c),H(c))_* \,.$$

The resulting pointed topologically enriched category $[\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}]$ is also called the $\mathrm{Top}_{\mathrm{cg}}^{*/}$-**enriched functor category** over $\mathcal{C}$ with coefficients in $\mathrm{Top}_{\mathrm{cg}}^{*/}$.

This yields an equivalent formulation in terms of ends of the pointed topologically enriched Yoneda lemma (prop.):

Proposition 1.32. ***(topologically enriched Yoneda lemma)***

Let $\mathcal{C}$ be a small pointed topologically enriched categories (def.). For $F : \mathcal{C} \to \mathrm{Top}_{\mathrm{cg}}^{*/}$ a pointed topologically enriched functor (def.) and for $c \in \mathcal{C}$ an object, there is a natural isomorphism

$$[\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}](\mathcal{C}(c,-),\, F) \;\simeq\; F(c)$$

between the hom-space of the pointed topological functor category, according to def. 1.31, from the functor represented by c to F, and the value of F on c.

In terms of the ends (def. 1.28) defining these hom-spaces, this means that

$$\int_{d\in\mathcal{C}} \mathrm{Maps}(\mathcal{C}(c,d), F(d))_* \;\simeq\; F(c)\,.$$

In this form the statement is also known as ***Yoneda reduction****.*

The **proof** of prop. 1.32 is formally dual to the proof of the next prop. 1.33.

Now that natural transformations are expressed in terms of ends (example 1.30), as is the Yoneda lemma (prop. 1.32), it is natural to consider the dual statement involving coends:

Proposition 1.33. ***(co-Yoneda lemma)***

Let $\mathcal{C}$ be a small pointed topologically enriched category (def.). For $F : \mathcal{C} \to \mathrm{Top}_{\mathrm{cg}}^{*/}$ a pointed topologically enriched functor (def.) and for $c \in \mathcal{C}$ an object, there is a natural isomorphism

$$F(-) \simeq \int^{c \in \mathcal{C}} \mathcal{C}(c,-) \wedge F(c) \,.$$

Moreover, the morphism that hence exhibits $F(c)$ as the coequalizer of the two morphisms in def. 1.28 is componentwise the canonical action

$$\mathcal{C}(c,d) \wedge F(c) \longrightarrow F(d)$$

which is adjunct to the component map $\mathcal{C}(d,c) \to \mathrm{Maps}(F(c),F(d))_*$ of the topologically enriched functor F.

(e.g. MMSS 00, lemma 1.6)

Proof. The coequalizer of pointed topological spaces that we need to consider has underlying it a coequalizer of underlying pointed sets (prop., prop., prop.). That in turn is the colimit over the diagram of underlying sets with the basepointe adjoined to the diagram (prop.). For a coequalizer diagram adding that extra point to the diagram clearly does not change the colimit, and so we need to consider the plain coequalizer of sets.

That is just the set of equivalence classes of pairs

$$(c \to c_0,\ x) \ \ \in \mathcal{C}(c,c_0) \wedge F(c) \,,$$

where two such pairs

$$(c \overset{f}{\to} c_0,\ x \in F(c)) \,, \quad (d \overset{g}{\to} c_0,\ y \in F(d))$$

are regarded as equivalent if there exists

$$c \overset{\phi}{\to} d$$

such that

$$f = g \circ \phi \,, \quad \text{and} \quad y = \phi(x) \,.$$

(Because then the two pairs are the two images of the pair (g,x) under the two morphisms being coequalized.)

But now considering the case that $d = c_0$ and $g = \mathrm{id}_{c_0}$, so that $f = \phi$ shows that any pair

$$(c \overset{\phi}{\to} c_0,\ x \in F(c))$$

is identified, in the coequalizer, with the pair

$$(\mathrm{id}_{c_0},\ \phi(x) \in F(c_0)) \,,$$

hence with $\phi(x) \in F(c_0)$.

This shows the claim at the level of the underlying sets. To conclude it is now sufficient (prop.) to show that the topology on $F(c_0) \in \mathrm{Top}_{\mathrm{cg}}^{*/}$ is the final topology (def.) of the system of component morphisms

$$\mathcal{C}(d,c) \wedge F(c) \longrightarrow \int^{c} \mathcal{C}(c,c_0) \wedge F(c)$$

which we just found. But that system includes

$$\mathcal{C}(c,c) \wedge F(c) \longrightarrow F(c)$$

which is a retraction

$$\mathrm{id} : F(c) \longrightarrow \mathcal{C}(c,c) \wedge F(c) \longrightarrow F(c)$$

and so if all the preimages of a given subset of the coequalizer under these component maps is open, it must have already been open in $F(c)$. ▮

Remark 1.34. The statement of the co-Yoneda lemma in prop. 1.33 is a kind of categorification of the following statement in analysis (whence the notation with the integral signs):

For X a topological space, $f : X \to \mathbb{R}$ a continuous function and $\delta(-,x_0)$ denoting the Dirac distribution, then

$$\int_{x \in X} \delta(x,x_0) f(x) = f(x_0) \,.$$

It is this analogy that gives the name to the following statement:

Proposition 1.35. ***(Fubini theorem for (co)-ends)***

For F a pointed topologically enriched bifunctor on a small pointed topological product category $\mathcal{C}_1 \times \mathcal{C}_2$ (def. 1.26), i.e.

$$F : (\mathcal{C}_1 \times \mathcal{C}_2)^{\mathrm{op}} \times (\mathcal{C}_1 \times \mathcal{C}_2) \longrightarrow \mathrm{Top}^{*/}_{\mathrm{cg}}$$

then its end and coend (def. 1.28) is equivalently formed consecutively over each variable, in either order:

$$\int^{(c_1,c_2)} F((c_1,c_2),(c_1,c_2)) \simeq \int^{c_1}\int^{c_2} F((c_1,c_2),(c_1,c_2)) \simeq \int^{c_2}\int^{c_1} F((c_1,c_2),(c_1,c_2))$$

and

$$\int_{(c_1,c_2)} F((c_1,c_2),(c_1,c_2)) \simeq \int_{c_1}\int_{c_2} F((c_1,c_2),(c_1,c_2)) \simeq \int_{c_2}\int_{c_1} F((c_1,c_2),(c_1,c_2)) \,.$$

Proof. Because limits commute with limits, and colimits commute with colimits. ▮

Remark 1.36. Since the pointed compactly generated mapping space functor (exmpl.)

$$\mathrm{Maps}(-,-)_* : \left(\mathrm{Top}^{*/}_{\mathrm{cg}}\right)^{\mathrm{op}} \times \mathrm{Top}^{*/}_{\mathrm{cg}} \longrightarrow \mathrm{Top}^{*/}_{\mathrm{cg}}$$

takes colimits in the first argument and limits in the second argument to limits (cor.), it in particular takes coends in the first argument and ends in the second argument, to ends (def. 1.28):

$$\mathrm{Maps}(X, \int_c F(c,c))_* \simeq \int_c \mathrm{Maps}(X, F(c,c)_*)$$

and

$$\mathrm{Maps}(\int^c F(c,c),\, Y)_* \simeq \int_c \mathrm{Maps}(F(c,c),\, Y)_* \,.$$

With this coend calculus in hand, there is an elegant proof of the defining universal property of the smash tensoring of topologically enriched functors $[\mathcal{C}, \mathrm{Top}^*_{\mathrm{cg}}]$ (def.)

Proposition 1.37. *For $\mathcal{C}$ a pointed topologically enriched category, there are natural isomorphisms*

$$[\mathcal{C}, \mathrm{Top}^{*/}_{\mathrm{cg}}](X \wedge K,\, Y) \;\simeq\; \mathrm{Maps}(K,\; [\mathcal{C}, \mathrm{Top}^{*/}_{\mathrm{cg}}](X,Y))_*$$

and

$$[\mathcal{C}, \mathrm{Top}^{*/}_{\mathrm{cg}}](X,\, \mathrm{Maps}(K,Y)_*) \;\simeq\; \mathrm{Maps}(K,\; [\mathcal{C}, \mathrm{Top}^{*/}_{\mathrm{cg}}](X,Y))$$

for all $X, Y \in [\mathcal{C}, \mathrm{Top}^{*/}_{\mathrm{cg}}]$ and all $K \in \mathrm{Top}^{*/}_{\mathrm{cg}}$.

In particular there is the combined natural isomorphism

$$[\mathcal{C}, \mathrm{Top}^{*/}_{\mathrm{cg}}](X \wedge K, Y) \;\simeq\; [\mathcal{C}, \mathrm{Top}^{*/}_{\mathrm{cg}}](X, \mathrm{Maps}(K,Y)_*)$$

exhibiting a pair of adjoint functors

$$[\mathcal{C}, \mathrm{Top}^{*/}_{\mathrm{cg}}] \underset{\mathrm{Maps}(K,-)_*}{\overset{(-)\wedge K}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} [\mathcal{C}, \mathrm{Top}^*_{\mathrm{cg}}] \,.$$

Proof. Via the end-expression for $[\mathcal{C}, \mathrm{Top}^{*/}_{\mathrm{cg}}](-,-)$ from def. 1.31 and the fact (remark 1.36) that the pointed mapping space construction $\mathrm{Maps}(-,-)_*$ preserves ends in the second variable, this reduces to the fact that $\mathrm{Maps}(-,-)_*$ is the internal hom in the closed monoidal category $\mathrm{Top}^{*/}_{\mathrm{cg}}$ (example 1.10) and hence satisfies the internal tensor/hom-adjunction isomorphism (prop. 1.8):

$$
\begin{aligned}
[\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}](X \wedge K, Y) & = \int_c \mathrm{Maps}((X \wedge K)(c), Y(c))_* \\
& \simeq \int_c \mathrm{Maps}(X(c) \wedge K, Y(x))_* \\
& \simeq \int_c \mathrm{Maps}(K, \mathrm{Maps}(X(c), Y(c))_*)_* \\
& \simeq \mathrm{Maps}(K, \int_c \mathrm{Maps}(X(c), Y(c)))_* \\
& = \mathrm{Maps}(K, [\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}](X, Y))_*
\end{aligned}
$$

and

$$
\begin{aligned}
[\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}](X, \mathrm{Maps}(K, Y)_*) & = \int_c \mathrm{Maps}(X(c), (\mathrm{Maps}(K, Y)_*)(c))_* \\
& \simeq \int_c \mathrm{Maps}(X(c), \mathrm{Maps}(K, Y(c))_*)_* \\
& \simeq \int_c \mathrm{Maps}(X(c) \wedge K, Y(c))_* \\
& \simeq \int_c \mathrm{Maps}(K, \mathrm{Maps}(X(c), Y(c))_*)_* \\
& \simeq \mathrm{Maps}(K, \int_c \mathrm{Maps}(X(c), Y(c))_*)_* \\
& \simeq \mathrm{Maps}(K, [\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}](X, Y))_* \,.
\end{aligned}
$$

▮

Proposition 1.38. ***(left Kan extension via coends)***

Let $\mathcal{C}, \mathcal{D}$ be small pointed topologically enriched categories (def.) and let

$$p : \mathcal{C} \longrightarrow \mathcal{D}$$

be a pointed topologically enriched functor (def.). Then precomposition with p constitutes a functor

$$p^* : [\mathcal{D}, \mathrm{Top}_{\mathrm{cg}}^{*/}] \longrightarrow [\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}]$$

*$G \mapsto G \circ p$. This functor has a left adjoint Lan_p, called **left Kan extension** along p*

$$[\mathcal{D}, \mathrm{Top}_{\mathrm{cg}}^{*/}] \underset{p^*}{\overset{\mathrm{Lan}_p}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} [\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}]$$

which is given objectwise by a coend (def. 1.28):

$$(\mathrm{Lan}_p F) : d \mapsto \int^{c \in \mathcal{C}} \mathcal{D}(p(c), d) \wedge F(c) \,.$$

Proof. Use the expression of natural transformations in terms of ends (example 1.30 and def. 1.31), then use the respect of $\mathrm{Maps}(-,-)_*$ for ends/coends (remark 1.36), use the smash/mapping space adjunction (cor.), use the Fubini theorem (prop. 1.35) and finally use Yoneda reduction (prop. 1.32) to obtain a sequence of natural isomorphisms as follows:

$$
\begin{aligned}
[\mathcal{D}, \mathrm{Top}_{\mathrm{cg}}^{*/}](\mathrm{Lan}_p F, G) & = \int_{d \in \mathcal{D}} \mathrm{Maps}((\mathrm{Lan}_p F)(d), G(d))_* \\
& = \int_{d \in \mathcal{D}} \mathrm{Maps}\Big(\int^{c \in \mathcal{C}} \mathcal{D}(p(c), d) \wedge F(c), G(d) \Big)_* \\
& \simeq \int_{d \in \mathcal{D}} \int_{c \in \mathcal{C}} \mathrm{Maps}(\mathcal{D}(p(c), d) \wedge F(c), G(d))_* \\
& \simeq \int_{c \in \mathcal{C}} \int_{d \in \mathcal{D}} \mathrm{Maps}(F(c), \mathrm{Maps}(\mathcal{D}(p(c), d), G(d))_*)_* \cdot \\
& \simeq \int_{c \in \mathcal{C}} \mathrm{Maps}(F(c), \int_{d \in \mathcal{D}} \mathrm{Maps}(\mathcal{D}(p(c), d), G(d))_*)_* \\
& \simeq \int_{c \in \mathcal{C}} \mathrm{Maps}(F(c), G(p(c)))_* \\
& = [\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}](F, p^* G)
\end{aligned}
$$

▮

Topological Day convolution

Given two functions $f_1, f_2 : G \longrightarrow \mathbb{C}$ on a group (or just a monoid) G, then their convolution product is, whenever well defined, given by the sum

$$f_1 \star f_2 : g \mapsto \sum_{g_1 \cdot g_2 = g} f_1(g_1) \cdot f_2(g_2) \,.$$

The operation of *Day convolution* is the categorification of this situation where functions are replaced by functors and monoids by monoidal categories. Further below we find the symmetric monoidal smash product of spectra as the Day convolution of topologically enriched functors over the monoidal category of finite pointed CW-complexes, or over sufficiently rich subcategories thereof.

Definition 1.39. Let $(\mathcal{C}, \otimes, 1)$ be a small pointed topological monoidal category (def. 1.1).

Then the **Day convolution tensor product** on the pointed topological enriched functor category $[\mathcal{C}, \mathrm{Top}^{*/}_{\mathrm{cg}}]$ (def. 1.31) is the functor

$$\otimes_{\mathrm{Day}} \;:\; [\mathcal{C}, \mathrm{Top}^{*/}_{\mathrm{cg}}] \times [\mathcal{C}, \mathrm{Top}^{*/}_{\mathrm{cg}}] \longrightarrow [\mathcal{C}, \mathrm{Top}^{*/}_{\mathrm{cg}}]$$

out of the pointed topological product category (def. 1.26) given by the following coend (def. 1.28)

$$X \otimes_{\mathrm{Day}} Y : c \;\mapsto\; \int^{(c_1,c_2) \in \mathcal{C} \times \mathcal{C}} \mathcal{C}(c_1 \otimes c_2, c) \wedge X(c_1) \wedge Y(c_2) \,.$$

Example 1.40. Let Seq denote the category with objects the natural numbers, and only the zero morphisms and identity morphisms on these objects (we consider this in a braoder context below in def. 2.4):

$$\mathrm{Seq}(n_1, n_2) \coloneqq \begin{cases} S^0 & \text{if}\; n_1 = n_2 \\ \ast & \text{otherwise} \end{cases} \,.$$

Regard this as a pointed topologically enriched category in the unique way. The operation of addition of natural numbers $\otimes = +$ makes this a monoidal category.

An object $X_\bullet \in [\mathrm{Seq}, \mathrm{Top}^{*/}_{\mathrm{cg}}]$ is an $\mathbb{N}$-sequence of pointed topological spaces. Given two such, then their Day convolution according to def. 1.39 is

$$\begin{aligned} (X \otimes_{\mathrm{Day}} Y)_n &= \int^{(n_1,n_2)} \mathrm{Seq}(n_1 + n_2, n) \wedge X_{n_1} \wedge X_{n_2} \\ &= \coprod_{\substack{n_1 + n_2 \\ = n}} \left(X_{n_1} \wedge X_{n_2}\right) \end{aligned} \,.$$

We observe now that Day convolution is equivalently a left Kan extension (def. 1.38). This will be key for understanding monoids and modules with respect to Day convolution.

Definition 1.41. Let $\mathcal{C}$ be a small pointed topologically enriched category (def.). Its **external tensor product** is the pointed topologically enriched functor

$$\bar{\wedge} \;:\; [\mathcal{C}, \mathrm{Top}^{*/}_{\mathrm{cg}}] \times [\mathcal{C}, \mathrm{Top}^{*/}_{\mathrm{cg}}] \longrightarrow [\mathcal{C} \times \mathcal{C}, \mathrm{Top}^{*/}_{\mathrm{cg}}]$$

from pairs of topologically enriched functors over mmathcal $\mathcal{C}$ to topologically enriched functors over the product category $\mathcal{C} \times \mathcal{C}$ (def. 1.26) given by

$$X \bar{\wedge} Y \;\coloneqq\; \wedge \circ (X, Y) \,,$$

i.e.

$$(X \bar{\wedge} Y)(c_1, c_2) = X(c_1) \wedge X(c_2) \,.$$

Proposition 1.42. *For $(\mathcal{C}, \otimes 1)$ a pointed topologically enriched monoidal category (def. 1.1) the Day convolution product (def. 1.39) of two functors is equivalently the left Kan extension (def. 1.38) of their external tensor product (def. 1.41) along the tensor product $\otimes : \mathcal{C} \times \mathcal{C}$: there is a natural isomorphism*

$$X \otimes_{\mathrm{Day}} Y \simeq \mathrm{Lan}_{\otimes} (X \bar{\wedge} Y) \,.$$

Hence the adjunction unit is a natural transformation of the form

$$\begin{tikzcd} \mathcal{C} \times \mathcal{C} \arrow[rr, "X \bar{\wedge} Y"] \arrow[dr, "\otimes"'] & \arrow[d, phantom, "\Downarrow"] & \mathrm{Top}^{*/}_{\mathrm{cg}} \\ & \mathcal{C} \arrow[ur, "X \otimes_{\mathrm{Day}} Y"'] & \end{tikzcd} .$$

This perspective is highlighted in (MMSS 00, p. 60).

Proof. By prop. 1.38 we may compute the left Kan extension as the following coend:

$$
\begin{aligned}
\mathrm{Lan}_{\otimes_{\mathcal{C}}}(X \bar{\wedge} Y)(c) & \simeq \int^{(c_1,c_2)} \mathcal{C}(c_1 \otimes_{\mathcal{C}} c_2, c) \wedge (X \bar{\wedge} Y)(c_1,c_2) \\
& = \int^{(c_1,c_2)} \mathcal{C}(c_1 \otimes c_2, c) \wedge X(c_1) \wedge X(c_2)
\end{aligned}
.
$$

∎

Proposition 1.42 implies the following fact, which is the key for the identification of "functors with smash product" below and then for the description of ring spectra further below.

Corollary 1.43. *The operation of Day convolution $\otimes_{\mathrm{Day}}$ (def. 1.39) is universally characterized by the property that there are natural isomorphisms*

$$
[\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}](X \otimes_{\mathrm{Day}} Y, Z) \simeq [\mathcal{C} \times \mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}](X \bar{\wedge} Y,\; Z \circ \otimes\,)\,,
$$

where $\bar{\wedge}$ is the external product of def. 1.41, hence that natural transformations of functors on $\mathcal{C}$ of the form

$$
(X \otimes_{\mathrm{Day}} Y)(c) \longrightarrow Z(c)
$$

are in natural bijection with natural transformations of functors on the product category mmathcal $\mathcal{C} \times \mathcal{C}$ (def. 1.26) of the form

$$
X(c_1) \wedge Y(c_2) \longrightarrow Z(c_1 \otimes c_2)\,.
$$

Write

$$
y \,:\, \mathcal{C}^{\mathrm{op}} \longrightarrow [\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}]
$$

for the $\mathrm{Top}_{\mathrm{cg}}^{*/}$-Yoneda embedding, so that for $c \in \mathcal{C}$ any object, $y(c)$ is the corepresented functor $y(c) : d \mapsto \mathcal{C}(c,d)$.

Proposition 1.44. *For $(\mathcal{C}, \otimes, 1)$ a small pointed topological monoidal category (def. 1.1), the Day convolution tensor product $\otimes_{\mathrm{Day}}$ of def. 1.39 makes the pointed topologically enriched functor category*

$$
([\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}],\; \otimes_{\mathrm{Day}}\,, y(1))
$$

into a pointed topological monoidal category (def. 1.1) with tensor unit $y(1)$ co-represented by the tensor unit 1 of $\mathcal{C}$.

Moreover, if $(\mathcal{C}, \otimes, 1)$ is equipped with a (symmetric) braiding $\tau^{\mathcal{C}}$ (def. 1.4), then so is $([\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}],\; \otimes_{\mathrm{Day}}\,, y(1))$.

Proof. Regarding associativity, observe that

$$
\begin{aligned}
(X \otimes_{\mathrm{Day}} (Y \otimes_{\mathrm{Day}} Z))(c) & \simeq \int^{(c_1,c_2)} \mathcal{C}(c_1 \otimes c_2,\, c) \wedge X(c_1) \wedge \int^{(d_1,d_2)} \mathcal{C}(d_1 \otimes d_2, c_2)(Y(d_1) \wedge Z(d_2)) \\
& \simeq \int^{c_1,d_1,d_2} \underbrace{\int^{c_2} \mathcal{C}(c_1 \otimes c_2, c) \wedge \mathcal{C}(d_1 \otimes d_2, c_2)}_{\simeq \mathcal{C}(c_1 \otimes (d_1 \otimes_{\mathcal{C}} d_2), c)} \wedge (X(c_1) \wedge (Y(d_1) \wedge Z(d_2))) \\
& \simeq \int^{c_1,d_1,d_2} \mathcal{C}(c_1 \otimes (d_1 \otimes d_2), c) \wedge (X(c_1) \wedge (Y(d_1) \wedge Z(d_2))) \\
& \simeq \int^{c_1,c_2,c_3} \mathcal{C}(c_1 \otimes (c_2 \otimes c_3), c) \wedge (X(c_1) \wedge (Y(c_2) \wedge Z(c_3)))
\end{aligned}
,
$$

where we used the Fubini theorem for coends (prop. 1.35) and then twice the co-Yoneda lemma (prop. 1.33). Similarly

$$
\begin{aligned}
((X \otimes_{\mathrm{Day}} Y) \otimes_{\mathrm{Day}} Z)(c) & \simeq \int^{(c_1,c_2)} \mathcal{C}(c_1 \otimes c_2, c) \wedge \int^{(d_1,d_2)} \mathcal{C}(d_1 \otimes d_2, c_1) \wedge (X(d_1) \wedge Y(d_2)) \wedge Y(c_2) \\
& \simeq \int^{c_2,d_1,d_2} \underbrace{\int^{c_1} \mathcal{C}(c_1 \otimes c_2, c) \wedge \mathcal{C}(d_1 \otimes d_2, c_1)}_{\simeq \mathcal{C}((d_1 \otimes d_2) \otimes c_2)} \wedge ((X(d_1) \wedge Y(d_2)) \wedge Z(c_2)) \\
& \simeq \int^{c_2,d_1,d_2} \mathcal{C}((d_1 \otimes d_2) \otimes c_2) \wedge ((X(d_1) \wedge Y(d_2)) \wedge Z(c_2)) \\
& \simeq \int^{c_1,c_2,c_3} \mathcal{C}((c_1 \otimes c_2) \otimes c_3) \wedge ((X(c_1) \wedge Y(c_2)) \wedge Z(c_3))
\end{aligned}
.
$$

So we obtain an associator by combining, in the integrand, the associator $\alpha^{\mathcal{C}}$ of $(\mathcal{C}, \otimes, 1)$ and $\tau^{\mathrm{Top}_{\mathrm{cg}}^{*/}}$ of $(\mathrm{Top}_{\mathrm{cg}}^{*/}, \wedge, S^0)$ (example 1.10):

$$
\begin{array}{ccc}
((X \otimes_{\mathrm{Day}} Y) \otimes_{\mathrm{Day}} Z)(c) & \simeq & \int^{c_1, c_2, c_3} \mathcal{C}((c_1 \otimes c_2) \otimes c_3) \wedge ((X(c_1) \wedge Y(c_2)) \wedge Z(c_3)) \\
{}^{\alpha^{\mathrm{Day}}_{X,Y,Z}(c)}\downarrow && \downarrow^{\int^{c_1,c_2,c_3} \mathcal{C}(\alpha^{\mathcal{C}}_{c_1,c_2,c_3}, c) \wedge \alpha^{\mathrm{Top}^{*/}_{\mathrm{cg}}}_{X(c_1),X(c_2),X(c_3)}} \\
(X \otimes_{\mathrm{Day}} (Y \otimes_{\mathrm{Day}} Z))(c) & \simeq & \int^{c_1, c_2, c_3} \mathcal{C}(c_1 \otimes (c_2 \otimes c_3), c) \wedge (X(c_1) \wedge (Y(c_2) \wedge Z(c_3)))
\end{array}
$$

It is clear that this satisfies the pentagon identity, since $\tau^{\mathcal{C}}$ and $\tau^{\mathrm{Top}^{*/}_{\mathrm{cg}}}$ do.

To see that $y(1)$ is the tensor unit for $\otimes_{\mathrm{Day}}$, use the Fubini theorem for coends (prop. 1.35) and then twice the co-Yoneda lemma (prop. 1.33) to get for any $X \in [\mathcal{C}, \mathrm{Top}^{*/}_{\mathrm{cg}}]$ that

$$
\begin{aligned}
X \otimes_{\mathrm{Day}} y(1) &= \int^{c_1, c_2 \in \mathcal{C}} \mathcal{C}(c_1 \otimes_{\mathcal{D}} c_2, -) \wedge X(c_1) \wedge \mathcal{C}(1, c_2) \\
&\simeq \int^{c_1 \in \mathcal{C}} \int^{c_2 \in \mathcal{C}} \mathcal{C}(c_1 \otimes_{\mathcal{C}} c_2, -) \wedge \mathcal{C}(1, c_2) \wedge X(c_1) \\
&\simeq \int^{c_1 \in \mathcal{C}} \mathcal{C}(c_1 \otimes_{\mathcal{C}} 1, -) \wedge X(c_1) \\
&\simeq \int^{c_1 \in \mathcal{C}} \mathcal{C}(c_1, -) \wedge X(c_1) \\
&\simeq X(-) \\
&\simeq X
\end{aligned}
\,.
$$

Hence the right unitor of Day convolution comes from the unitor of $\mathcal{C}$ under the integral sign:

$$
\begin{array}{ccc}
(X \otimes_{\mathrm{Day}} y(1))(c) & \simeq & \int^{c_1} \mathcal{C}(c_1 \otimes 1, c) \wedge X(c_1) \\
{}^{r^{\mathrm{Day}}_X(c)}\downarrow && \downarrow^{\int^{c_1} \mathcal{C}(r^{\mathcal{C}}_{c_1}, c) \wedge X(c_1)} \\
X(c) & \simeq & \int^{c_1} \mathcal{C}(c_1, c) \wedge X(c_1)
\end{array}
$$

Analogously for the left unitor. Hence the triangle identity for $\otimes_{\mathrm{Day}}$ follows from the triangle identity in $\mathcal{C}$ under the integral sign.

Similarly, if $\mathcal{C}$ has a braiding $\tau^{\mathcal{C}}$, it induces a braiding τ^{Day} under the integral sign:

$$
\begin{array}{ccc}
(X \otimes_{\mathrm{Day}} Y)(c) & = & \int^{c_1, c_2} \mathcal{C}(c_1 \otimes c_2, c) \wedge X(c_1) \wedge Y(c_2) \\
{}^{\tau^{\mathrm{Day}}_{X,Y}(c)}\downarrow && \downarrow^{\int^{c_1,c_2} \mathcal{C}(\tau^{\mathcal{C}}_{c_1,c_2}, c) \wedge \tau^{\mathrm{Top}^{*/}}_{X(c(1)),X(c_2)}} \\
(Y \otimes_{\mathrm{Day}} X)(c) & = & \int^{c_1, c_2} \mathcal{C}(c_2 \otimes c_1, c) \wedge Y(c_2) \wedge X(c_1)
\end{array}
$$

and the hexagon identity for τ^{Day} follows from that for $\tau^{\mathcal{C}}$ and $\tau^{\mathrm{Top}^{*/}_{\mathrm{cg}}}$ ▮

Moreover:

Proposition 1.45. *For $(\mathcal{C}, \otimes, 1)$ a small pointed topological symmetric monoidal category (def. 1.5), the monoidal category with Day convolution $([\mathcal{C}, \mathrm{Top}^{*/}_{\mathrm{cg}}], \otimes_{\mathrm{Day}}, y(1))$ from def. 1.44 is a closed monoidal category (def. 1.7). Its internal hom $[-,-]_{\mathrm{Day}}$ is given by the end (def. 1.28)*

$$
[X, Y]_{\mathrm{Day}}(c) \simeq \int_{c_1, c_2} \mathrm{Maps}(\mathcal{C}(c \otimes c_1, c_2),\ \mathrm{Maps}(X(c_1), Y(c_2))_*)_* \,.
$$

Proof. Using the Fubini theorem (def. 1.35) and the co-Yoneda lemma (def. 1.33) and in view of definition 1.31 of the enriched functor category, there is the following sequence of natural isomorphisms:

$$
\begin{aligned}
[\mathcal{C}, V](X, [Y, Z]_{\mathrm{Day}}) &\simeq \int_c \mathrm{Maps}\Big(X(c), \int_{c_1, c_2} \mathrm{Maps}(\mathcal{C}(c \otimes c_1, c_2), \mathrm{Maps}(Y(c_1), Z(c_2))_*)_*\Big)_* \\
&\simeq \int_c \int_{c_1, c_2} \mathrm{Maps}(\mathcal{C}(c \otimes c_1, c_2) \wedge X(c) \wedge Y(c_1),\ Z(c_2))_* \\
&\simeq \int_{c_2} \mathrm{Maps}\Big(\int^{c, c_1} \mathcal{C}(c \otimes c_1, c_2) \wedge X(c) \wedge Y(c_1),\ Z(c_2)\Big)_* \\
&\simeq \int_{c_2} \mathrm{Maps}\Big((X \otimes_{\mathrm{Day}} Y)(c_2), Z(c_2)\Big)_* \\
&\simeq [\mathcal{C}, V](X \otimes_{\mathrm{Day}} Y, Z)
\end{aligned}
\,.
$$

▮

Proposition 1.46. *In the situation of def. 1.44, the Yoneda embedding $c \mapsto \mathcal{C}(c,-)$ constitutes a strong monoidal functor (def. 1.47)*

$$(\mathcal{C}, \otimes, 1) \hookrightarrow ([\mathcal{C}, V], \otimes_{\mathrm{Day}}, y(1)) \,.$$

Proof. That the tensor unit is respected is part of prop. 1.44. To see that the tensor product is respected, apply the co-Yoneda lemma (prop. 1.33) twice to get the following natural isomorphism

$$\begin{aligned} (y(c_1) \otimes_{\mathrm{Day}} y(c_2))(c) & \simeq \int^{d_1, d_2} \mathcal{C}(d_1 \otimes d_2, c) \wedge \mathcal{C}(c_1, d_1) \wedge \mathcal{C}(c_2, d_2) \\ & \simeq \mathcal{C}(c_1 \otimes c_2, c) \\ & = y(c_1 \otimes c_2)(c) \end{aligned} \,.$$

▮

Functors with smash product

Since the symmetric monoidal smash product of spectra discussed below is an instance of Day convolution (def. 1.39), and since ring spectra are going to be the monoids (def. 1.13) with respect to this tensor product, we are interested in characterizing the monoids with respect to Day convolution. These turn out to have a particularly transparent expression as what is called *functors with smash product*, namely lax monoidal functors from the base monoidal category to $\mathrm{Top}_{\mathrm{cg}}^{*/}$. Their components are pairing maps of the form

$$R_{n_1} \wedge R_{n_2} \longrightarrow R_{n_1 + n_2}$$

satisfying suitable conditions. This is the form in which the structure of ring spectra usually appears in examples. It is directly analogous to how a dg-algebra, which is equivalently a monoid with respect to the tensor product of chain complexes (example 1.19), is given in components .

Here we introduce the concepts of monoidal functors and of functors with smash product and prove that they are equivalently the monoids with respect to Day convolution.

Definition 1.47. Let $(\mathcal{C}, \otimes_{\mathcal{C}}, 1_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, 1_{\mathcal{D}})$ be two (pointed) topologically enriched monoidal categories (def. 1.1). A topologically enriched **lax monoidal functor** between them is

1. a topologically enriched functor

$$F : \mathcal{C} \longrightarrow \mathcal{D} \,,$$

2. a morphism

$$\epsilon : 1_{\mathcal{D}} \longrightarrow F(1_{\mathcal{C}})$$

3. a natural transformation

$$\mu_{x,y} : F(x) \otimes_{\mathcal{D}} F(y) \longrightarrow F(x \otimes_{\mathcal{C}} y)$$

for all $x, y \in \mathcal{C}$

satisfying the following conditions:

1. **(associativity)** For all objects $x, y, z \in \mathcal{C}$ the following diagram commutes

$$\begin{array}{ccc} (F(x) \otimes_{\mathcal{D}} F(y)) \otimes_{\mathcal{D}} F(z) & \underoverset{\simeq}{a^{\mathcal{D}}_{F(x),F(y),F(z)}}{\longrightarrow} & F(x) \otimes_{\mathcal{D}} (F(y) \otimes_{\mathcal{D}} F(z)) \\ {}^{\mu_{x,y} \otimes \mathrm{id}}\downarrow && \downarrow^{\mathrm{id} \otimes \mu_{y,z}} \\ F(x \otimes_{\mathcal{C}} y) \otimes_{\mathcal{D}} F(z) && F(x) \otimes_{\mathcal{D}} (F(x \otimes_{\mathcal{C}} y)) \\ {}^{\mu_{x \otimes_{\mathcal{C}} y, z}}\downarrow && \downarrow^{\mu_{x, y \otimes_{\mathcal{C}} z}} \\ F((x \otimes_{\mathcal{C}} y) \otimes_{\mathcal{C}} z) & \underset{F(a^{\mathcal{C}}_{x,y,z})}{\longrightarrow} & F(x \otimes_{\mathcal{C}} (y \otimes_{\mathcal{C}} z)) \end{array} \,,$$

where $a^{\mathcal{C}}$ and $a^{\mathcal{D}}$ denote the associators of the monoidal categories;

2. **(unitality)** For all $x \in \mathcal{C}$ the following diagrams commutes

$$\begin{array}{ccc} 1_{\mathcal{D}} \otimes_{\mathcal{D}} F(x) & \overset{\epsilon \otimes \mathrm{id}}{\longrightarrow} & F(1_{\mathcal{C}}) \otimes_{\mathcal{D}} F(x) \\ {}^{\ell^{\mathcal{D}}_{F(x)}}\downarrow && \downarrow^{\mu_{1_{\mathcal{C}}, x}} \\ F(x) & \overset{F(\ell^{\mathcal{C}}_x)}{\longleftarrow} & F(1 \otimes_{\mathcal{C}} x) \end{array}$$

and

$$\begin{array}{ccc} F(x) \otimes_{\mathcal{D}} 1_{\mathcal{D}} & \overset{\mathrm{id}\otimes \epsilon}{\longrightarrow} & F(x) \otimes_{\mathcal{D}} F(1_{\mathcal{C}}) \\ {}^{r^{\mathcal{D}}_{F(x)}}\downarrow && \downarrow^{\mu_{x,1_{\mathcal{C}}}} \\ F(x) & \overset{F(r^{\mathcal{C}}_x)}{\longleftarrow} & F(x \otimes_{\mathcal{C}} 1) \end{array} \,,$$

where $\ell^{\mathcal{C}}$, $\ell^{\mathcal{D}}$, $r^{\mathcal{C}}$, $r^{\mathcal{D}}$ denote the left and right unitors of the two monoidal categories, respectively.

If ϵ and alll $\mu_{x,y}$ are isomorphisms, then F is called a **strong monoidal functor**.

If moreover $(\mathcal{C}, \otimes_{\mathcal{C}}, 1_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, 1_{\mathcal{D}})$ are equipped with the structure of braided monoidal categories (def. 1.4) with braidings $\tau^{\mathcal{C}}$ and $\tau^{\mathcal{D}}$, respectively, then the lax monoidal functor F is called a **braided monoidal functor** if in addition the following diagram commutes for all objects $x, y \in \mathcal{C}$

$$\begin{array}{ccc} F(x) \otimes_{\mathcal{C}} F(y) & \overset{\tau^{\mathcal{D}}_{F(x),F(y)}}{\longrightarrow} & F(y) \otimes_{\mathcal{D}} F(x) \\ {}^{\mu_{x,y}}\downarrow && \downarrow^{\mu_{y,x}} \\ F(x \otimes_{\mathcal{C}} y) & \underset{F(\tau^{\mathcal{C}}_{x,y})}{\longrightarrow} & F(y \otimes_{\mathcal{C}} x) \end{array} \,.$$

A homomorphism $f : (F_1, \mu_1, \epsilon_1) \longrightarrow (F_2, \mu_2, \epsilon_2)$ between two (braided) lax monoidal functors is a **monoidal natural transformation**, in that it is a natural transformation $f_x : F_1(x) \longrightarrow F_2(x)$ of the underlying functors

compatible with the product and the unit in that the following diagrams commute for all objects $x, y \in \mathcal{C}$:

$$\begin{array}{ccc} F_1(x) \otimes_{\mathcal{D}} F_1(y) & \overset{f(x)\otimes_{\mathcal{D}} f(y)}{\longrightarrow} & F_2(x) \otimes_{\mathcal{D}} F_2(y) \\ {}^{(\mu_1)_{x,y}}\downarrow && \downarrow^{(\mu_2)_{x,y}} \\ F_1(x \otimes_{\mathcal{C}} y) & \underset{f(x \otimes_{\mathcal{C}} y)}{\longrightarrow} & F_2(x \otimes_{\mathcal{C}} y) \end{array}$$

and

$$\begin{array}{ccccc} && 1_{\mathcal{D}} \\ & {}^{\epsilon_1}\swarrow && \searrow^{\epsilon_2} \\ F_1(1_{\mathcal{C}}) && \underset{f(1_{\mathcal{C}})}{\longrightarrow} && F_2(1_{\mathcal{C}}) \end{array} \,.$$

We write $\mathrm{MonFun}(\mathcal{C}, \mathcal{D})$ for the resulting category of lax monoidal functors between monoidal categories $\mathcal{C}$ and $\mathcal{D}$, similarly $\mathrm{BraidMonFun}(\mathcal{C}, \mathcal{D})$ for the category of braided monoidal functors between braided monoidal categories, and $\mathrm{SymMonFun}(\mathcal{C}, \mathcal{D})$ for the category of braided monoidal functors between symmetric monoidal categories.

Remark 1.48. In the literature the term “monoidal functor” often refers by default to what in def. 1.47 is called a *strong monoidal functor*. But for the purpose of the discussion of functors with smash product below, it is crucial to admit the generality of lax monoidal functors.

If $(\mathcal{C}, \otimes_{\mathcal{C}}, 1_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, 1_{\mathcal{D}})$ are symmetric monoidal categories (def. 1.5) then a braided monoidal functor (def. 1.47) between them is often called a **symmetric monoidal functor**.

Proposition 1.49. *For $\mathcal{C} \overset{F}{\longrightarrow} \mathcal{D} \overset{G}{\longrightarrow} \mathcal{E}$ two composable lax monoidal functors (def. 1.47) between monoidal categories, then their composite $F \circ G$ becomes a lax monoidal functor with structure morphisms*

$$\epsilon^{G\circ F} : 1_{\mathcal{E}} \overset{\epsilon^G}{\longrightarrow} G(1_{\mathcal{D}}) \overset{G(\epsilon^F)}{\longrightarrow} G(F(1_{\mathcal{C}}))$$

and

$$\mu^{G\circ F}_{c_1,c_2} : G(F(c_1)) \otimes_{\mathcal{E}} G(F(c_2)) \overset{\mu^G_{F(c_1),F(c_2)}}{\longrightarrow} G(F(c_1) \otimes_{\mathcal{D}} F(c_2)) \overset{G(\mu^F_{c_1,c_2})}{\longrightarrow} G(F(c_1 \otimes_{\mathcal{C}} c_2)) \,.$$

Proposition 1.50. *Let $(\mathcal{C}, \otimes_{\mathcal{C}}, 1_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, 1_{\mathcal{D}})$ be two monoidal categories (def. 1.1) and let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a lax monoidal functor (def. 1.47) between them.*

Then for (A, μ_A, e_A) a monoid in $\mathcal{C}$ (def. 1.13), its image $F(A) \in \mathcal{D}$ becomes a monoid $(F(A), \mu_{F(A)}, e_{F(A)})$ by setting

$$\mu_{F(A)} : F(A) \otimes_{\mathcal{C}} F(A) \longrightarrow F(A \otimes_{\mathcal{C}} A) \overset{F(\mu_A)}{\longrightarrow} F(A)$$

(where the first morphism is the structure morphism of F) and setting

$$e_{F(A)} \,:\, 1_{\mathcal{D}} \longrightarrow F(1_{\mathcal{C}}) \overset{F(e_A)}{\longrightarrow} F(A)$$

(where again the first morphism is the corresponding structure morphism of F).

This construction extends to a functor

$$\mathrm{Mon}(F) \,:\, \mathrm{Mon}(\mathcal{C}, \otimes_{\mathcal{C}}, 1_{\mathcal{C}}) \longrightarrow \mathrm{Mon}(\mathcal{D}, \otimes_{\mathcal{D}}, 1_{\mathcal{D}})$$

from the category of monoids of $\mathcal{C}$ (def. 1.13) to that of $\mathcal{D}$.

Moreover, if $\mathcal{C}$ and $\mathcal{D}$ are symmetric monoidal categories (def. 1.5) and F is a braided monoidal functor (def. 1.47) and A is a commutative monoid (def. 1.13) then so is $F(A)$, and this construction extends to a functor

$$\mathrm{CMon}(F) \,:\, \mathrm{CMon}(\mathcal{C}, \otimes_{\mathcal{C}}, 1_{\mathcal{C}}) \longrightarrow \mathrm{CMon}(\mathcal{D}, \otimes_{\mathcal{D}}, 1_{\mathcal{D}}) \,.$$

Proof. This follows immediately from combining the associativity and unitality (and symmetry) constraints of F with those of A. ▮

Definition 1.51. Let $(\mathcal{C}, \otimes_{\mathcal{C}}, 1_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, 1_{\mathcal{D}})$ be two (pointed) topologically enriched monoidal categories (def. 1.1), and let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a topologically enriched lax monoidal functor between them, with product operation μ.

Then a left **module over the lax monoidal functor** is

1. a topologically enriched functor

$$G \,:\, \mathcal{C} \longrightarrow \mathcal{D} \,;$$

2. a natural transformation

$$\rho_{x,y} \,:\, F(x) \otimes_{\mathcal{D}} G(y) \longrightarrow G(x \otimes_{\mathcal{C}} y)$$

such that

- **(action property)** For all objects $x, y, z \in \mathcal{C}$ the following diagram commutes

$$\begin{array}{ccc} (F(x) \otimes_{\mathcal{D}} F(y)) \otimes_{\mathcal{D}} G(z) & \underset{\simeq}{\overset{a^{\mathcal{D}}_{F(x),F(y),F(z)}}{\longrightarrow}} & F(x) \otimes_{\mathcal{D}} (F(y) \otimes_{\mathcal{D}} G(z)) \\ {}^{\mu_{x,y} \otimes \mathrm{id}}\downarrow && \downarrow^{\mathrm{id} \otimes \rho_{y,z}} \\ F(x \otimes_{\mathcal{C}} y) \otimes_{\mathcal{D}} G(z) && F(x) \otimes_{\mathcal{D}} (G(x \otimes_{\mathcal{C}} y)) \\ {}^{\rho_{x \otimes_{\mathcal{C}} y, z}}\downarrow && \downarrow^{\rho_{x, y \otimes_{\mathcal{C}} z}} \\ G((x \otimes_{\mathcal{C}} y) \otimes_{\mathcal{C}} z) & \underset{F(a^{\mathcal{C}}_{x,y,z})}{\longrightarrow} & G(x \otimes_{\mathcal{C}} (y \otimes_{\mathcal{C}} z)) \end{array} \,,$$

A homomorphism $f \,:\, (G_1, \rho_1) \longrightarrow (G_2, \rho_2)$ between two modules over a monoidal functor (F, μ, ϵ) is

- a natural transformation $f_x \,:\, G_1(x) \longrightarrow G_2(x)$ of the underlying functors

compatible with the action in that the following diagram commutes for all objects $x, y \in \mathcal{C}$:

$$\begin{array}{ccc} F(x) \otimes_{\mathcal{D}} G_1(y) & \overset{\mathrm{id} \otimes_{\mathcal{D}} f(y)}{\longrightarrow} & F(x) \otimes_{\mathcal{D}} G_2(y) \\ {}^{(\rho_1)_{x,y}}\downarrow && \downarrow^{(\mathrm{rhi}_2)_{x,y}} \\ G_1(x \otimes_{\mathcal{C}} y) & \underset{f(x \otimes_{\mathcal{C}} y)}{\longrightarrow} & G_2(x \otimes_{\mathcal{C}} y) \end{array}$$

We write $F\,\mathrm{Mod}$ for the resulting category of modules over the monoidal functor F.

Now we may finally state the main proposition on *functors with smash product*:

Proposition 1.52. *Let $(\mathcal{C}, \otimes, 1)$ be a pointed topologically enriched (symmetric) monoidal category (def. 1.1). Regard $(\mathrm{Top}^{*/}_{\mathrm{cg}}, \wedge, S^0)$ as a topological symmetric monoidal category as in example 1.10.*

Then (commutative) monoids in (def. 1.13) the Day convolution monoidal category $([\mathcal{C}, \mathrm{Top}^{*/}_{\mathrm{cg}}], \otimes_{\mathrm{Day}}, y(1_{\mathcal{C}}))$ of prop. 1.44 are equivalent to (braided) lax monoidal functors (def. 1.47) of the form

$$(\mathcal{C}, \otimes, 1) \longrightarrow (\mathrm{Top}^*_{\mathrm{cg}}, \wedge, S^0) \,,$$

*called **functors with smash products** on $\mathcal{C}$, i.e. there are equivalences of categories of the form*

$$\begin{aligned} \mathrm{Mon}([\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}], \otimes_{\mathrm{Day}}, y(1_{\mathcal{C}})) &\simeq \mathrm{MonFunc}(\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}) \\ \mathrm{CMon}([\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}], \otimes_{\mathrm{Day}}, y(1_{\mathcal{C}})) &\simeq \mathrm{SymMonFunc}(\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}) \end{aligned} \,.$$

Moreover, module objects over these monoid objects are equivalent to the corresponding modules over monoidal functors (def. 1.51).

This is stated in some form in (Day 70, example 3.2.2). It is highlighted again in (MMSS 00, prop. 22.1).

Proof. By definition 1.47, a lax monoidal functor $F : \mathcal{C} \to \mathrm{Top}_{\mathrm{cg}}^{*/}$ is a topologically enriched functor equipped with a morphism of pointed topological spaces of the form

$$S^0 \longrightarrow F(1_{\mathcal{C}})$$

and equipped with a natural system of maps of pointed topological spaces of the form

$$F(c_1) \wedge F(c_2) \longrightarrow F(c_1 \otimes_{\mathcal{C}} c_2)$$

for all $c_1, c_2 \in \mathcal{C}$.

Under the Yoneda lemma (prop. 1.32) the first of these is equivalently a morphism in $[\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}]$ of the form

$$y(S^0) \longrightarrow F \,.$$

Moreover, under the natural isomorphism of corollary 1.43 the second of these is equivalently a morphism in $[\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}]$ of the form

$$F \otimes_{\mathrm{Day}} F \longrightarrow F \,.$$

Translating the conditions of def. 1.47 satisfied by a lax monoidal functor through these identifications gives precisely the conditions of def. 1.13 on a (commutative) monoid in object F under $\otimes_{\mathrm{Day}}$.

Similarly for module objects and modules over monoidal functors. ∎

Proposition 1.53. *Let $f : \mathcal{C} \longrightarrow \mathcal{D}$ be a lax monoidal functor (def. 1.47) between pointed topologically enriched monoidal categories (def. 1.1). Then the induced functor*

$$f^* : [\mathcal{D}, \mathrm{Top}_{\mathrm{cg}}^{*/}] \longrightarrow [\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*}]$$

given by $(f^* X)(c) \coloneqq X(f(c))$ preserves monoids under Day convolution

$$f^* : \mathrm{Mon}([\mathcal{D}, \mathrm{Top}_{\mathrm{cg}}^{*/}], \otimes_{\mathrm{Day}}, y(1_{\mathcal{D}})) \longrightarrow \mathrm{Mon}([\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*}], \otimes_{\mathrm{Day}}, y(1_{\mathcal{C}})$$

Moreover, if $\mathcal{C}$ and $\mathcal{D}$ are symmetric monoidal categories (def. 1.5) and f is a braided monoidal functor (def. 1.47), then f^* also preserves commutative monoids

$$f^* : \mathrm{CMon}([\mathcal{D}, \mathrm{Top}_{\mathrm{cg}}^{*/}], \otimes_{\mathrm{Day}}, y(1_{\mathcal{D}})) \longrightarrow \mathrm{CMon}([\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*}], \otimes_{\mathrm{Day}}, y(1_{\mathcal{C}}) \,.$$

Similarly, for

$$A \in \mathrm{Mon}([\mathcal{D}, \mathrm{Top}_{\mathrm{cg}}^{*/}], \otimes_{\mathrm{Day}}, y(1_{\mathcal{D}}))$$

any fixed monoid, then f^* sends A-modules to $f^*(A)$-modules

$$f^* : A\,\mathrm{Mod}(\mathcal{D}) \longrightarrow (f^* A)\mathrm{Mod}(\mathcal{C}) \,.$$

Proof. This is an immediate corollary of prop. 1.52, since the composite of two (braided) lax monoidal functors is itself canonically a (braided) lax monoidal functor by prop. 1.49. ∎

2. $\mathbb{S}$-Modules

We give a unified discussion of the categories of

1. sequential spectra
2. symmetric spectra
3. orthogonal spectra
4. pre-excisive functors

(all in topological spaces) as categories of modules with respect to Day convolution monoidal structures on Top-enriched functor categories over restrictions to faithful sub-sites of the canonical representative of the sphere spectrum as a pre-excisive functor on $\mathrm{Top}^{*/}_{\mathrm{fin}}$.

This approach is due to (Mandell-May-Schwede-Shipley 00) following (Hovey-Shipley-Smith 00).

Pre-Excisive functors

We consider an almost tautological construction of a pointed topologically enriched category equipped with a closed symmetric monoidal product: the category of *pre-excisive functors*. Then we show that this tautological category restricts, in a certain sense, to the category of sequential spectra. However, under this restriction the symmetric monoidal product breaks, witnessing the lack of a functorial smash product of spectra on sequential spectra. However from inspection of this failure we see that there are categories of structured spectra "in between" those of all pre-excisive functors and plain sequential spectra, notably the categories of *orthogonal spectra* and of *symmetric spectra*. These intermediate categories retain the concrete tractable nature of sequential spectra, but are rich enough to also retain the symmetric monoidal product inherited from pre-excisive functors: this is the *symmetric monoidal smash product of spectra* that we are after.

Literature (MMSS 00, Part I and Part III)

Definition 2.1. Write

$$\iota_{\mathrm{fin}} : \mathrm{Top}^{*/}_{\mathrm{cg,fin}} \hookrightarrow \mathrm{Top}^{*/}_{\mathrm{cg}}$$

for the full subcategory of pointed compactly generated topological spaces (def.) on those that admit the structure of a finite CW-complex (a CW-complex (def.) with a finite number of cells).

We say that the pointed topological enriched functor category (def. 1.31)

$$\mathrm{Exc}(\mathrm{Top}_{\mathrm{cg}}) \coloneqq [\mathrm{Top}^{*/}_{\mathrm{cg,fin}}, \mathrm{Top}^{*/}_{\mathrm{cg}}]$$

is the category of **pre-excisive functors**. (We had previewed this in Part P, this example).

Write

$$\mathbb{S}_{\mathrm{exc}} \coloneqq y(S^0) \coloneqq \mathrm{Top}^{*/}_{\mathrm{cg,fin}}(S^0, -)$$

for the functor co-represented by 0-sphere. This is equivalently the inclusion ι_{fin} itself:

$$\mathbb{S}_{\mathrm{exc}} = \iota_{\mathrm{fin}} : K \mapsto K \,.$$

We call this the standard incarnation of the **sphere spectrum** as a pre-excisive functor.

By prop. 1.44 the smash product of pointed compactly generated topological spaces induces the structure of a closed (def. 1.7) symmetric monoidal category (def. 1.5)

$$\left(\mathrm{Exc}(\mathrm{Top}_{\mathrm{cg}}),\ \wedge \coloneqq \otimes_{\mathrm{Day}}\,,\ \mathbb{S}_{\mathrm{exc}}\right)$$

with

1. tensor unit the sphere spectrum $\mathbb{S}_{\mathrm{exc}}$;
2. tensor product the Day convolution product $\otimes_{\mathrm{Day}}$ from def. 1.39,

   called the **symmetric monoidal smash product of spectra** for the model of pre-excisive functors;
3. internal hom the dual operation $[-,-]_{\mathrm{Day}}$ from prop. 1.45,

   called the **mapping spectrum** construction for pre-excisive functors.

Remark 2.2. By example 1.14 the sphere spectrum incarnated as a pre-excisive functor $\mathbb{S}_{\mathrm{exc}}$ (according to def. 2.1) is canonically a commutative monoid in the category of pre-excisive functors (def. 1.13).

Moreover, by example 1.17, every object of $\mathrm{Exc}(\mathrm{Top}_{\mathrm{cg}})$ (def. 2.1) is canonically a module object over $\mathbb{S}_{\mathrm{exc}}$. We may therefore tautologically identify the category of pre-excisive functors with the module category over the sphere spectrum:

$$\mathrm{Exc}(\mathrm{Top}_{\mathrm{cg}}) \simeq \mathbb{S}_{\mathrm{exc}}\,\mathrm{Mod}\,.$$

Lemma 2.3. *Identified as a functor with smash product under prop. 1.52, the pre-excisive sphere spectrum $\mathbb{S}_{\mathrm{exc}}$ from def. 2.1 is given by the identity natural transformation*

$$\mu_{(K_1,K_2)} : \mathbb{S}_{\mathrm{exc}}(K_1) \wedge \mathbb{S}_{\mathrm{exc}}(K_2) = K_1 \wedge K_2 \overset{=}{\longrightarrow} K_1 \wedge K_2 = \mathbb{S}_{\mathrm{exc}}(K_1 \wedge K_2) \,.$$

Proof. We claim that this is in fact the unique structure of a monoidal functor that may be imposed on the canonical inclusion $\iota : \mathrm{Top}^{*/}_{\mathrm{cg,fin}} \hookrightarrow \mathrm{Top}^{*/}_{\mathrm{cg}}$, hence it must be the one in question. To see the uniqueness, observe that naturality of the matural transformation μ in particular says that there are commuting squares of the form

$$\begin{array}{ccc} S^0 \wedge S^0 & \overset{=}{\longrightarrow} & S^0 \wedge S^0 \\ {}^{x_1,x_2}\downarrow && \downarrow^{x_1,x_2} \\ K_1 \wedge K_2 & \underset{\mu_{K_1,K_2}}{\longrightarrow} & K_1 \wedge K_2 \end{array} \,,$$

where the vertical morphisms pick any two points in K_1 and K_2, respectively, and where the top morphism is necessarily the canonical identification since there is only one single isomorphism $S^0 \to S^0$, namely the identity. This shows that the bottom horizontal morphism has to be the identity on all points, hence has to be the identity. ▮

We now consider restricting the domain of the pre-excisive functors of def. 2.1.

Definition 2.4. Define the following pointed topologically enriched (def.) symmetric monoidal categories (def. 1.5):

1. Seq is the category whose objects are the natural numbers and which has only identity morphisms and zero morphisms on these objects, hence the hom-spaces are

   $$\mathrm{Seq}(n_1,n_2) \coloneqq \begin{cases} S^0 & \text{for}\ n_1 = n_2 \\ \ast & \text{otherwise} \end{cases}$$

   The tensor product is the addition of natural numbers, $\otimes = +$, and the tensor unit is 0. The braiding is, necessarily, the identity.

2. Sym is the standard skeleton of the core of FinSet with zero morphisms adjoined: its objects are the finite sets $\overline{n} \coloneqq \{1,\cdots,n\}$ for $n \in \mathbb{N}$ (hence $\overline{0}$ is the empty set), all non-zero morphisms are automorphisms and the automorphism group of $\{1,\cdots,n\}$ is the symmetric group $\Sigma(n)$ on n elements, hence the hom-spaces are the following discrete topological spaces:

   $$\mathrm{Sym}(n_1,n_2) \coloneqq \begin{cases} (\Sigma(n_1))_+ & \text{for}\ n_1 = n_2 \\ \ast & \text{otherwise} \end{cases}$$

   The tensor product is the disjoint union of sets, tensor unit is the empty set. The braiding

   $$\tau^{\mathrm{Sym}}_{n_1,n_2} : \overline{n_1} \cup \overline{n_2} \longrightarrow \overline{n_2} \cup \overline{n_1}$$

   is given by the canonical permutation in $\Sigma(n_1 + n_2)$ that shuffles the first n_1 elements past the remaining n_2 elements.

   (MMSS 00, example 4.2)

3. Orth has as objects the finite dimenional real linear inner product spaces $(\mathbb{R}^n, \langle -,-\rangle)$ and as non-zero morphisms the linear isometric isomorphisms between these; hence the automorphism group of the object $(\mathbb{R}^n, \langle -,-\rangle)$ is the orthogonal group $O(n)$; the monoidal product is direct sum of linear spaces, the tensor unit is the 0-vector space; again we turn this into a $\mathrm{Top}^{*/}_{\mathrm{cg}}$-enriched category by adjoining a basepoint to the hom-spaces;

   $$\mathrm{Orth}(V_1,V_2) \coloneqq \begin{cases} O(V_1)_+ & \text{for}\ \dim(V_1) = \dim(V_2) \\ \ast & \text{otherwise} \end{cases}$$

   The tensor product is the direct sum of linear inner product spaces, tensor unit is the 0-vector space. The braiding

   $$\tau^{\mathrm{Orth}}_{V_1,V_2} : V_1 \oplus V_2 \longrightarrow V_2 \oplus V_1$$

   is the canonical orthogonal transformation that switches the summands.

   (MMSS 00, example 4.4)

Notice that in the notation of example 1.29

1. the full subcategory of Orth on V is $\mathbf{B}(O(V)_+)$;

2. the full subcategory of Sym on $\{1,\cdots,n\}$ is $\mathbf{B}(\Sigma(n)_+)$;

3. the full subcategory of Seq on n is $\mathbf{B}(1_+)$.

Moreover, after discarding the zero morphisms, then these categories are the disjoint union of categories of the form $\mathbf{B}\,O(n)$, $\mathbf{B}\,\Sigma(n)$ and $\mathbf{B}1 = *$, respectively.

There is a sequence of canonical faithful pointed topological subcategory inclusions

$$\begin{array}{ccccccc} \mathrm{Seq} & \overset{\mathrm{seq}}{\hookrightarrow} & \mathrm{Sym} & \overset{\mathrm{sym}}{\hookrightarrow} & \mathrm{Orth} & \overset{\mathrm{orth}}{\hookrightarrow} & \mathrm{Top}^{*/}_{\mathrm{cg,fin}} \\ n & \mapsto & \{1,\cdots,n\} & \mapsto & \mathbb{R}^n & \mapsto & S^n \end{array},$$

into the pointed topological category of pointed compactly generated topological spaces of finite CW-type (def. 2.1).

Here S^V denotes the one-point compactification of V. On morphisms $\mathrm{sym}\colon (\Sigma_n)_+ \hookrightarrow (O(n))_+$ is the canonical inclusion of permutation matrices into orthogonal matrices and $\mathrm{orth}\colon O(V)_+ \hookrightarrow \mathrm{Aut}(S^V)$ is on $O(V)$ the topological subspace inclusions of the pointed homeomorphisms $S^V \to S^V$ that are induced under forming one-point compactification from linear isometries of V ("representation spheres").

Below we will often use these identifications to write just "n" for any of these objects, leaving implicit the identifications $n \mapsto \{1,\cdots,n\} \mapsto S^n$.

Consider the pointed topological diagram categries (def. 1.31, exmpl.) over these categories:

- $[\mathrm{Seq}, \mathrm{Top}^{*/}_{\mathrm{cg}}]$ is called the category of **sequences** of pointed topological spaces (e.g. HSS 00, def. 2.3.1);
- $[\mathrm{Sym}, \mathrm{Top}^{*/}_{\mathrm{cg}}]$ is called the category of **symmetric sequences** (e.g. HSS 00, def. 2.1.1);
- $[\mathrm{Orth}, \mathrm{Top}^{*/}_{\mathrm{cg}}]$ is called the category of **orthogonal sequences**.

Consider the sequence of restrictions of topological diagram categories, according to prop. 1.53 along the above inclusions:

$$\mathrm{Exc}(\mathrm{Top}_{\mathrm{cg}}) \overset{\mathrm{orth}^*}{\longrightarrow} [\mathrm{Orth}, \mathrm{Top}^{*/}_{\mathrm{cg}}] \overset{\mathrm{sym}^*}{\longrightarrow} [\mathrm{Sym}, \mathrm{Top}^{*/}_{\mathrm{cg}}] \overset{\mathrm{seq}^*}{\longrightarrow} [\mathrm{Seq}, \mathrm{Top}^{*/}_{\mathrm{cg}}] \,.$$

Write

$$\mathbb{S}_{\mathrm{orth}} \coloneqq \mathrm{orth}^* \mathbb{S}_{\mathrm{exc}}\,, \qquad \mathbb{S}_{\mathrm{sym}} \coloneqq \mathrm{sym}^* \mathbb{S}_{\mathrm{orth}}\,, \qquad \mathbb{S}_{\mathrm{seq}} \coloneqq \mathrm{seq}^* \mathbb{S}_{\mathrm{sym}}$$

for the restriction of the excisive functor incarnation of the sphere spectrum (from def. 2.1) along these inclusions.

Proposition 2.5. *The functors* seq, sym *and* orth *in def. 2.4 become strong monoidal functors (def. 1.47) when equipped with the canonical isomorphisms*

$$\mathrm{seq}(n_1) \cup \mathrm{seq}(n_2) = \{1,\cdots,n_1\} \cup \{1,\cdots,n_2\} \simeq \{1,\cdots,n_1+n_2\} = \mathrm{seq}(n_1+n_2)$$

and

$$\mathrm{sym}(\{1,\cdots,n_1\}) \oplus \mathrm{sym}(\{1,\cdots,n_2\}) = \mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2} \simeq \mathbb{R}^{n_1+n_2} = \mathrm{sym}(\{1,\cdots,n_1\} \cup \{1,\cdots,n_2\})$$

and

$$\mathrm{orth}(V_1) \wedge \mathrm{orth}(V_2) = S^{V_1} \wedge S^{V_2} \simeq S^{V_1 \oplus V_2} = \mathrm{orth}(V_1 \oplus V_2)\,.$$

Moreover, orth *and* sym *are braided monoidal functors (def. 1.47) (hence symmetric monoidal functors, remark 1.48). But* seq *is not braided monoidal.*

Proof. The first statement is clear from inspection.

For the second statement it is sufficient to observe that all the nontrivial braiding of n-spheres in $\mathrm{Top}^{*/}_{\mathrm{cg}}$ is given by the maps induced from exchanging coordinates in the realization of n-spheres as one-point compactifications of Cartesian spaces $S^n \simeq (\mathbb{R}^n)^*$. This corresponds precisely to the action of the symmetric group inside the orthogonal group acting via the canonical action of the orthogonal group on $\mathbb{R}^n$. This shows that sym and orth are braided, for they include precisely these objects (the n-spheres) with these braidings on them. Finally it is clear that seq is not braided, because the braiding on Seq is trivial, while that on Sym is not, so seq necessrily fails to preserve precisely these non-trivial isomorphisms. ▮

Remark 2.6. Since the standard excisive incarnation $\mathbb{S}_{\mathrm{exc}}$ of the sphere spectrum (def. 2.1) is the tensor unit with repect to the Day convolution product on pre-excisive functors, and since it is therefore canonically a commutative monoid, by example 1.14, prop. 1.53 says that the restricted sphere spectra

$\mathbb{S}_{\mathrm{orth}}$, $\mathbb{S}_{\mathrm{sym}}$ and $\mathbb{S}_{\mathrm{seq}}$ are still monoids, and that under restriction every pre-excisive functor, regarded as a $\mathbb{S}_{\mathrm{exc}}$-module via remark 2.2, canonically becomes a module under the restricted sphere spectrum:

$$\mathrm{orth}^* : \mathrm{Exc}(\mathrm{Top}_{\mathrm{cg}}) \simeq \mathbb{S}_{\mathrm{exc}}\,\mathrm{Mod} \longrightarrow \mathbb{S}_{\mathrm{orth}}\,\mathrm{Mod}$$
$$\mathrm{sym}^* : \mathrm{Exc}(\mathrm{Top}_{\mathrm{cg}}) \simeq \mathbb{S}_{\mathrm{exc}}\,\mathrm{Mod} \longrightarrow \mathbb{S}_{\mathrm{sym}}\,\mathrm{Mod}\,.$$
$$\mathrm{seq}^* : \mathrm{Exc}(\mathrm{Top}_{\mathrm{cg}}) \simeq \mathbb{S}_{\mathrm{exc}}\,\mathrm{Mod} \longrightarrow \mathbb{S}_{\mathrm{seq}}\,\mathrm{Mod}$$

Since all three functors orth, sym and seq are strong monoidal functors by prop. 2.5, all three restricted sphere spectra $\mathbb{S}_{\mathrm{orth}}$, $\mathbb{S}_{\mathrm{sym}}$ and $\mathbb{S}_{\mathrm{seq}}$ canonically are monoids, by prop. 1.53. Moreover, according to prop. 2.5, orth and sym are braided monoidal functors, while functor seq is not braided, therefore prop. 1.53 furthermore gives that $\mathbb{S}_{\mathrm{orth}}$ and $\mathbb{S}_{\mathrm{sym}}$ are commutative monoids, while $\mathbb{S}_{\mathrm{seq}}$ is not commutative:

sphere spectrum	$\mathbb{S}_{\mathrm{exc}}$	$\mathbb{S}_{\mathrm{orth}}$	$\mathbb{S}_{\mathrm{sym}}$	$\mathbb{S}_{\mathrm{seq}}$
monoid	yes	yes	yes	yes
commutative monoid	yes	yes	yes	no
tensor unit	yes	no	no	no

Explicitly:

Lemma 2.7. *The monoids $\mathbb{S}_{\mathrm{dia}}$ from def. 2.4 are, when identified as functors with smash product via prop. 1.52 given by assigning*

$$\begin{aligned} \mathbb{S}_{\mathrm{seq}} &: n \mapsto S^n \\ \mathbb{S}_{\mathrm{sym}} &: \overline{n} \mapsto S^n \\ \mathbb{S}_{\mathrm{orth}} &: V \mapsto S^V\,, \end{aligned}$$

respectively, with product given by the canonical isomorphisms

$$S^{V_1} \wedge S^{V_2} \longrightarrow S^{V_1 \oplus V_2}\,.$$

Proof. By construction these functors with smash products are the composites, according to prop. 1.49, of the monoidal functors seq, sym, orth, respectively, with the lax monoidal functor corresponding to $\mathbb{S}_{\mathrm{exc}}$. The former have as structure maps the canonical identifications by definition, and the latter has as structure map the canonical identifications by lemmma 2.3. ▮

Proposition 2.8. *There is an equivalence of categories*

$$(-)^{\mathrm{seq}} : \mathbb{S}_{\mathrm{seq}}\,\mathrm{Mod} \longrightarrow \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$$

which identifies the category of modules (def. 1.16) over the monoid $\mathbb{S}_{\mathrm{seq}}$ (remark 2.6) in the Day convolution monoidal structure (prop. 1.44) over the topological functor category $[\mathrm{Seq}, \mathrm{Top}_{\mathrm{cg}}^{*/}]$ from def. 2.4 with the category of sequential spectra (def.)

Under this equivalence, an $\mathbb{S}_{\mathrm{seq}}$-module X is taken to the sequential pre-spectrum X^{seq} whose component spaces are the values of the pre-excisive functor X on the standard n-sphere $S^n = (S^1)^{\wedge n}$

$$(X^{\mathrm{seq}})_n \coloneqq X(\mathrm{seq}(n)) = X(S^n)$$

and whose structure maps are the images of the action morphisms

$$\mathbb{S}_{\mathrm{seq}} \otimes_{\mathrm{Day}} X \longrightarrow X$$

under the isomorphism of corollary 1.43

$$\mathbb{S}_{\mathrm{seq}}(n_1) \wedge X(n_1) \longrightarrow X_{n_1+n_2}$$

evaluated at $n_1 = 1$

$$\begin{array}{ccc} \mathbb{S}_{\mathrm{seq}}(1) \wedge X(n) & \longrightarrow & X_{n+1} \\ {}^{\simeq}\downarrow & & \downarrow^{\simeq} \\ S^1 \wedge X_n & \longrightarrow & X_{n+1} \end{array}\,.$$

(Hovey-Shipley-Smith 00, prop. 2.3.4)

Proof. After unwinding the definitions, the only point to observe is that due to the action property,

$$\begin{array}{ccc} \mathbb{S}_{\mathrm{seq}} \otimes_{\mathrm{Day}} \mathbb{S}_{\mathrm{seq}} \otimes_{\mathrm{Day}} X & \overset{\mathrm{id} \otimes_{\mathrm{Day}} \rho}{\longrightarrow} & \mathbb{S}_{\mathrm{seq}} \otimes_{\mathrm{Day}} X \\ {}^{\mu \otimes_{\mathrm{Day}} \mathrm{id}}\downarrow & & \downarrow^{\rho} \\ \mathbb{S}_{\mathrm{seq}} \otimes_{\mathrm{Day}} X & \underset{\rho}{\longrightarrow} & X \end{array}$$

any $\mathbb{S}_{\mathrm{seq}}$-action

$$\rho : \mathbb{S}_{\mathrm{seq}} \otimes_{\mathrm{Day}} X \longrightarrow X$$

is indeed uniquely fixed by the components of the form

$$\mathbb{S}_{\mathrm{seq}}(1) \wedge X(n) \longrightarrow X(n) \ .$$

This is because under corollary 1.43 the action property is identified with the componentwise property

$$\begin{array}{ccc} S^{n_1} \wedge S^{n_2} \wedge X_{n_3} & \overset{\mathrm{id} \wedge \rho_{n_2,n_3}}{\longrightarrow} & S^{n_1} \wedge X_{n_2+n_3} \\ {}^{\simeq}\downarrow && \downarrow^{\rho_{n_1, n_2+n_3}} \\ S^{n_1+n_2} \wedge X_{n_3} & \underset{\rho_{n_1+n_2,n_3}}{\longrightarrow} & X_{n_1+n_2+n_3} \end{array}$$

where the left vertical morphism is an isomorphism by the nature of $\mathbb{S}_{\mathrm{seq}}$. Hence this fixes the components $\rho_{n',n}$ to be the n'-fold composition of the structure maps $\sigma_n \coloneqq \rho(1,n)$. ▮

However, since, by remark 2.8, $\mathbb{S}_{\mathrm{seq}}$ is not commutative, there is no tensor product induced on $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$ under the identification in prop. 2.8. But since $\mathbb{S}_{\mathrm{orth}}$ and $\mathbb{S}_{\mathrm{sym}}$ are commutative monoids by remark 2.8, it makes sense to consider the following definition.

Definition 2.9. In the terminology of remark 2.6 we say that

$$\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}}) \coloneqq \mathbb{S}_{\mathrm{orth}}\,\mathrm{Mod}$$

is the **category of orthogonal spectra**; and that

$$\mathrm{SymSpec}(\mathrm{Top}_{\mathrm{cg}}) \coloneqq \mathbb{S}_{\mathrm{sym}}\,\mathrm{Mod}$$

is the **category of symmetric spectra**.

By remark 2.6 and by prop. 1.22 these categories canonically carry a symmetric monoidal tensor product $\otimes_{\mathbb{S}_{\mathrm{orth}}}$ and $\otimes_{\mathbb{S}_{\mathrm{seq}}}$, respectively. This we call the **symmetric monoidal smash product of spectra**. We usually just write for short

$$\wedge \coloneqq \otimes_{\mathbb{S}_{\mathrm{orth}}} \ : \mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}}) \times \mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}}) \longrightarrow \mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})$$

and

$$\wedge \coloneqq \otimes_{\mathbb{S}_{\mathrm{sym}}} \ : \mathrm{SymSpec}(\mathrm{Top}_{\mathrm{cg}}) \times \mathrm{SymSpec}(\mathrm{Top}_{\mathrm{cg}}) \longrightarrow \mathrm{SymSpec}(\mathrm{Top}_{\mathrm{cg}})$$

In the next section we work out what these symmetric monoidal categories of orthogonal and of symmetric spectra look like more explicitly.

Symmetric and orthogonal spectra

We now define symmetric spectra and orthogonal spectra and their symmetric monoidal smash product. We proceed by giving the explicit definitions and then checking that these are equivalent to the abstract definition 2.9 from above.

Literature. (Hovey-Shipley-Smith 00, section 1, section 2, Schwede 12, chapter I)

Definition 2.10. A topological **symmetric spectrum** X is

1. a sequence $\{X_n \in \mathrm{Top}_{\mathrm{cg}}^{*/} \mid n \in \mathbb{N}\}$ of pointed compactly generated topological spaces;
2. a basepoint preserving continuous right action of the symmetric group $\Sigma(n)$ on X_n;
3. a sequence of morphisms $\sigma_n : S^1 \wedge X_n \longrightarrow X_{n+1}$

such that

- for all $n, k \in \mathbb{N}$ the composite

  $$S^k \wedge X_n \simeq S^{k-1} \wedge S^1 \wedge X_n \overset{\mathrm{id} \wedge \sigma_n}{\longrightarrow} S^{k-1} \wedge X_{n+1} \simeq S^{k-2} \wedge S^1 \wedge X_{n+2} \overset{\mathrm{id} \wedge \sigma_{n+1}}{\longrightarrow} \cdots \overset{\sigma_{n+k-1}}{\longrightarrow} X_{n+k}$$

  intertwines the $\Sigma(n) \times \Sigma(k)$-action.

A homomorphism of symmetric spectra $f : X \longrightarrow Y$ is

- a sequence of maps $f_n : X_n \longrightarrow Y_n$

such that

1. each f_n intetwines the $\Sigma(n)$-action;

2. the following diagrams commute

$$\begin{array}{ccc} S^1 \wedge X_n & \overset{f_n \wedge \mathrm{id}}{\longrightarrow} & S^1 \wedge Y_n \\ \downarrow^{\sigma_n^X} && \downarrow^{\sigma_n^Y} \\ X_{n+1} & \overset{f_{n+1}}{\longrightarrow} & Y_{n+1} \end{array} \,.$$

We write $\mathrm{SymSpec}(\mathrm{Top}_{\mathrm{cg}})$ for the resulting category of symmetric spectra.

(Hovey-Shipley-Smith 00, def. 1.2.2, Schwede 12, I, def. 1.1)

The definition of orthogonal spectra has the same structure, just with the symmetric groups replaced by the orthogonal groups.

Definition 2.11. A topological **orthogonal spectrum** X is

1. a sequence $\{X_n \in \mathrm{Top}_{\mathrm{cg}}^{*/} \mid n \in \mathbb{N}\}$ of pointed compactly generated topological spaces;

2. a basepoint preserving continuous right action of the orthogonal group $O(n)$ on X_n;

3. a sequence of morphisms $\sigma_n : S^1 \wedge X_n \longrightarrow X_{n+1}$

such that

- for all $n, k \in \mathbb{N}$ the composite

$$S^k \wedge X_n \simeq S^{k-1} \wedge S^1 \wedge X_n \overset{\mathrm{id} \wedge \sigma_n}{\longrightarrow} S^{k-1} \wedge X_{n+1} \simeq S^{k-2} \wedge S^1 \wedge X_{n+2} \overset{\mathrm{id} \wedge \sigma_{n+1}}{\longrightarrow} \cdots \overset{\sigma_{n+k-1}}{\longrightarrow} X_{n+k}$$

intertwines the $O(n) \times \mathrm{Ok}()$-action.

A homomorphism of orthogonal spectra $f : X \longrightarrow Y$ is

- a sequence of maps $f_n : X_n \longrightarrow Y_n$

such that

1. each f_n intetwines the $O(n)$-action;

2. the following diagrams commute

$$\begin{array}{ccc} S^1 \wedge X_n & \overset{f_n \wedge \mathrm{id}}{\longrightarrow} & S^1 \wedge Y_n \\ \downarrow^{\sigma_n^X} && \downarrow^{\sigma_n^Y} \\ X_{n+1} & \overset{f_{n+1}}{\longrightarrow} & Y_{n+1} \end{array} \,.$$

We write $\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})$ for the resulting category of orthogonal spectra.

(e.g. Schwede 12, I, def. 7.2)

Proposition 2.12. *Definitions 2.10 and 2.11 are indeed equivalent to def. 2.9:*

orthogonal spectra are euqivalently the module objects over the incarnation $\mathbb{S}_{\mathrm{orth}}$ of the sphere spectrum

$$\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}}) \simeq \mathbb{S}_{\mathrm{orth}}\,\mathrm{Mod}$$

and symmetric spectra sre equivalently the module objects over the incarnation $\mathbb{S}_{\mathrm{sym}}$ of the sphere spectrum

$$\mathrm{SymSpec}(\mathrm{Top}_{\mathrm{cg}}) \simeq \mathbb{S}_{\mathrm{sym}}\,\mathrm{Mod} \,.$$

(Hovey-Shipley-Smith 00, prop. 2.2.1)

Proof. We discuss this for symmetric spectra. The proof for orthogonal spectra is of the same form.

First of all, by example 1.29 an object in $[\mathrm{Sym}, \mathrm{Top}_{\mathrm{cg}}^{*/}]$ is equivalently a “symmetric sequence”, namely a sequence of pointed topological spaces X_k, for $k \in \mathbb{N}$, equipped with an action of $\Sigma(k)$ (def. 2.4).

By corollary 1.43 and lemma 2.7, the structure morphism of an $\mathbb{S}_{\mathrm{sym}}$-module object on X

$$\mathbb{S}_{\mathrm{sym}} \otimes_{\mathrm{Day}} X \longrightarrow X$$

is equivalently (as a functor with smash products) a natural transformation

$$S^{n_1} \wedge X_{n_2} \longrightarrow X_{n_1+n_2}$$

over $\mathrm{Sym} \times \mathrm{Sym}$. This means equivalently that there is such a morphism for all $n_1, n_2 \in \mathbb{N}$ and that it is $\Sigma(n_1) \times \Sigma(n_2)$-equivariant.

Hence it only remains to see that these natural transformations are uniquely fixed once the one for $n_1 = 1$ is given. To that end, observe that lemma 2.7 says that in the following commuting squares (exhibiting the action property on the level of functors with smash product, where we are notationally suppressing the associators) the left vertical morphisms are isomorphisms:

$$\begin{array}{ccc} S^{n_1} \wedge S^{n_2} \wedge X_{n_3} & \longrightarrow & S^{n_1} \wedge X_{n_2+n_3} \\ {}^{\simeq}\downarrow && \downarrow \\ S^{n_1+n_2} \wedge X_{n_3} & \longrightarrow & X_{n_1+n_2+n_3} \end{array} \,.$$

This says exactly that the action of $S^{n_1+n_2}$ has to be the composite of the actions of S^{n_2} followed by that of S^{n_1}. Hence the statement follows by induction.

Finally, the definition of homomorphisms on both sides of the equivalence are just so as to preserve precisely this structure, hence they conincide under this identification. ▮

Definition 2.13. Given $X, Y \in \mathrm{SymSpec}(\mathrm{Top}_{\mathrm{cg}})$ two symmetric spectra, def. 2.10, then their **smash product of spectra** is the symmetric spectrum

$$X \wedge Y \;\in \mathrm{SymSpec}(\mathrm{Top}_{\mathrm{cg}})$$

with component spaces the coequalizer

$$\underset{p+1+q=n}{\bigvee} \Sigma(p+1+q)_+ \underset{\Sigma_p \times \Sigma_1 \times \Sigma_q}{\wedge} X_p \wedge S^1 \wedge Y_q \underoverset{r}{\ell}{\rightrightarrows} \underset{p+q=n}{\bigvee} \Sigma(p+q)_+ \underset{\Sigma_p \times \Sigma_q}{\wedge} X_p \wedge Y_q \overset{\mathrm{coeq}}{\longrightarrow} (X \wedge Y)(n)$$

where ℓ has components given by the structure maps

$$X_p \wedge S^1 \wedge Y_q \overset{\mathrm{id} \wedge \sigma_q}{\longrightarrow} X_p \wedge Y_q$$

while r has components given by the structure maps conjugated by the braiding in $\mathrm{Top}_{\mathrm{cg}}^{*/}$ and the permutation action $\chi_{p,1}$ (that shuffles the element on the right to the left)

$$X_p \wedge S^1 \wedge X_q \overset{\tau^{\mathrm{Top}_{\mathrm{cg}}^{*/}}_{X_p, S^1} \wedge \mathrm{id}}{\longrightarrow} S^1 \wedge X_p \wedge X_q \overset{\sigma_p \wedge \mathrm{id}}{\longrightarrow} X_{p+1} \wedge X_q \overset{\chi_{p,1} \wedge \mathrm{id}}{\longrightarrow} X_{1+p} \wedge X_q \,.$$

Finally The structure maps of $X \wedge Y$ are those induced under the coequalizer by

$$S^1 \wedge (X_p \wedge Y_q \wedge) \simeq (S^1 \wedge X_p) \wedge Y_q \overset{\sigma_p^X \wedge \mathrm{id}}{\longrightarrow} X_{p+1} \wedge Y_q \,.$$

Analogously for orthogonal spectra.

(Schwede 12, p. 82)

Proposition 2.14. *Under the identification of prop. 2.12, the explicit smash product of spectra in def. 2.13 is equivalent to the abstractly defined tensor product in def. 2.9:*

in the case of symmetric spectra:

$$\wedge \;\simeq\; \otimes_{\mathbb{S}_{\mathrm{sym}}}$$

in the case of orthogonal spectra:

$$\wedge \;\simeq\; \otimes_{\mathbb{S}_{\mathrm{orth}}} \,.$$

(Schwede 12, E.1.16)

Proof. By def. 1.21 the abstractly defined tensor product of two $\mathbb{S}_{\mathrm{sym}}$-modules X and Y is the coequalizer

$$X \otimes_{\mathrm{Day}} \mathbb{S}_{\mathrm{sym}} \otimes_{\mathrm{Day}} Y \underset{\rho_1 \circ (\tau^{\mathrm{Day}}_{X,\mathbb{S}_{\mathrm{sym}}} \otimes \mathrm{id})}{\overset{X \otimes \rho_2}{\rightrightarrows}} X \otimes Y \overset{\mathrm{coeq}}{\longrightarrow} X \otimes_{\mathbb{S}_{\mathrm{sym}}} Y \,.$$

The Day convolution product appearing here is over the category Sym from def. 2.4. By example 1.29 and unwinding the definitions, this is for any two symmetric spectra A and B given degreewise by the wedge sum of component spaces summing to that total degree, smashed with the symmetric group with basepoint adjoined and then quotiented by the diagonal action of the symmetric group acting on the degrees separately:

$$\begin{aligned} (A \otimes_{\mathrm{Day}} B)(n) &= \int^{n_1, n_2} \underbrace{\Sigma(n_1+n_2, n)}_{= \left\{ \begin{array}{ll} \Sigma(n_1+n_2,n)_+ & \text{if } n_1+n_2 = n \\ * & \text{otherwise} \end{array} \right._+} \wedge A_{n_1} \wedge B_{n_1} \\ &\simeq \bigvee_{n_1+n_2=n} \Sigma(n_1+n_2)_+ \underset{O(n_1)\times O(n_2)}{\wedge} \left(A_{n_1} \wedge B_{n_2}\right) \end{aligned} \,.$$

This establishes the form of the coequalizer diagram. It remains to see that under this identification the two abstractly defined morphisms are the ones given in def. 2.13.

To see this, we apply the adjunction isomorphism between the Day convolution product and the external tensor product (cor. 1.43) twice, to find the following sequence of equivalent incarnations of morphisms:

$$\begin{array}{ccccc} (X \otimes_{\mathrm{Day}} (\mathbb{S}_{\mathrm{orth}} \otimes_{\mathrm{Day}} Y))(n) & \longrightarrow & (X \otimes_{\mathrm{Day}} Y)(n) & \longrightarrow & Z_n \\ \hline X_{n_1} \wedge (\mathbb{S}_{\mathrm{sym}} \otimes_{\mathrm{Day}} Y)(n'_2) & \longrightarrow & X_{n_1} \wedge Y(n'_2) & \longrightarrow & Z_{n_1+n'_2} \\ \hline (\mathbb{S}_{\mathrm{sym}} \otimes_{\mathrm{Day}} Y)(n'_2) & \longrightarrow & Y(n'_2) & \longrightarrow & \mathrm{Maps}(X_{n_1}, Z_{n_1+n'_2}) \\ \hline S^{n_2} \wedge Y_{n_3} & \longrightarrow & Y_{n_2+n_3} & \longrightarrow & \mathrm{Maps}(X_{n_1}, Z_{n_1+n_2+n_3}) \\ \hline X_{n_1} \wedge S^{n_2} \wedge Y_{n_3} & \longrightarrow & X_{n_1} \wedge Y_{n_2+n_3} & \longrightarrow & Z_{n_1+n_2+n_3} \end{array} \,.$$

This establishes the form of the morphism ℓ. By the same reasoning as in the proof of prop. 2.12, we may restrict the coequalizer to $n_2 = 1$ without changing it.

The form of the morphism r is obtained by the analogous sequence of identifications of morphisms, now with the parenthesis to the left. That it involves $\tau^{\mathrm{Top}^{*/}_{\mathrm{cg}}}$ and the permutation action τ^{sym} as shown above follows from the formula for the braiding of the Day convolution tensor product from the proof of prop. 1.44:

$$\tau^{\mathrm{Day}}_{A,B}(n) = \int^{n_1,n_2} \mathrm{Sym}(\tau^{\mathrm{Sym}}_{n_1,n_2}, n) \wedge \tau^{\mathrm{Top}^{*/}_{\mathrm{cg}}}_{A_{n_1}, B_{n_2}}$$

by translating it to the components of the precomposition

$$X \otimes_{\mathrm{Day}} \mathbb{S}_{\mathrm{sym}} \overset{\tau^{\mathrm{Day}}_{X,\mathbb{S}_{\mathrm{sym}}}}{\longrightarrow} \mathbb{S}_{\mathrm{sym}} \otimes_{\mathrm{Day}} X \longrightarrow X$$

via the formula from the proof of prop. 1.38 for the left Kan extension $A \otimes_{\mathrm{Day}} B \simeq \mathrm{Lan}_{\otimes} A \overline{\wedge} B$ (prop. 1.42):

$$\begin{aligned} [\mathrm{Sym}, \mathrm{Top}^{*/}_{\mathrm{cg}}](\tau^{\mathrm{Day}}_{X,\mathbb{S}_{\mathrm{sym}}}, X) &\simeq \int_n \mathrm{Maps}\left(\int^{n_1,n_2} \mathrm{Sym}(\tau^{\mathrm{sym}}_{n_1,n_2}, n) \wedge \tau^{\mathrm{Top}^{*/}_{\mathrm{cg}}}_{X_{n_1}, S^{n_2}}, X(n) \right)_* \\ &\simeq \int_{n_1,n_2} \mathrm{Maps}\left(\tau^{\mathrm{Top}^{*/}_{\mathrm{cg}}}_{X_{n_1}, S^{n_2}}, X(\tau^{\mathrm{sym}}_{n_1,n_2})\right)_* \end{aligned} \,.$$

This last expression is the function on morphisms which precomposes components under the coend with the braiding $\tau^{\mathrm{Top}^{*/}_{\mathrm{cg}}}_{X_{n_1}, S^{n_2}}$ in topological spaces and postcomposes them with the image of the functor X of the braiding in Sym. But the braiding in Sym is, by def. 2.4, given by the respective shuffle permutations $\tau^{\mathrm{sym}}_{n_1,n_2} = \chi_{n_1,n_2}$, and by prop. 2.12 the image of these under X is via the given $\Sigma_{n_1+n_2}$-action on $X_{n_1+n_2}$.

Finally to see that the structure map is as claimed: By prop. 2.12 the structure morphisms are the degree-1 components of the $\mathbb{S}_{\mathrm{sym}}$-action, and by prop. 1.21 the $\mathbb{S}_{\mathrm{sym}}$-action on a tensor product of $\mathbb{S}_{\mathrm{sym}}$-modules is induced via the action on the left tensor factor. ∎

Definition 2.15. A **commutative symmetric ring spectrum** E is

1. a sequence of component spaces $E_n \in \mathrm{Top}^{*/}_{\mathrm{cg}}$ for $n \in \mathbb{N}$;

2. a basepoint preserving continuous left action of the symmetric group $\Sigma(n)$ on E_n;

3. for all $n_1, n_2 \in \mathbb{N}$ a *multiplication map*

$$\mu_{n_1,n_2} : E_{n_1} \wedge E_{n_2} \longrightarrow E_{n_1+n_2}$$

(a morphism in $\mathrm{Top}_{\mathrm{cg}}^{*/}$)

4. two *unit maps*

$$\begin{aligned} \iota_0 &: S^0 \longrightarrow E_0 \\ \iota_1 &: S^1 \longrightarrow E_1 \end{aligned}$$

such that

1. (equivariance) μ_{n_1,n_2} intertwines the $\Sigma(n_1) \times \Sigma(n_2)$-action;

2. (associativity) for all $n_1, n_2, n_3 \in \mathbb{N}$ the following diagram commutes (where we are notationally suppressing the associators of $(\mathrm{Top}_{\mathrm{cg}}^{*/}, \wedge, S^0)$)

$$\begin{array}{ccc} E_{n_1} \wedge E_{n_2} \wedge E_{n_3} & \overset{\mathrm{id} \wedge \mu_{n_2,n_3}}{\longrightarrow} & E_{n_1} \wedge E_{n_2+n_3} \\ {}^{\mu_{n_1,n_2} \wedge \mathrm{id}}\downarrow && \downarrow^{\mu_{n_1,n_2+n_3}} \\ E_{n_1+n_2} \wedge E_{n_3} & \underset{\mu_{n_1+n_2,n_3}}{\longrightarrow} & E_{n_1+n_2+n_3} \end{array}$$

3. (unitality) for all $n \in \mathbb{N}$ the following diagram commutes

$$\begin{array}{ccc} S^0 \wedge E_n & \overset{\iota_0 \wedge \mathrm{id}}{\longrightarrow} & E_0 \wedge E_n \\ & {}_{\ell^{\mathrm{Top}_{\mathrm{cg}}^{*/}}_{E_n}}\searrow & \downarrow^{\mu_{0,n}} \\ && E_n \end{array}$$

and

$$\begin{array}{ccc} E_n \wedge S^0 & \overset{\mathrm{id} \wedge \iota_0}{\longrightarrow} & E_n \wedge E_0 \\ & {}_{r^{\mathrm{Top}_{\mathrm{cg}}^{*/}}_{E_n}}\searrow & \downarrow^{\mu_{n,0}} \\ && E_n \end{array},$$

where the diagonal morphisms ℓ and r are the left and right unitors in $(\mathrm{Top}_{\mathrm{cg}}^{*/}, \wedge, S^0)$, respectively.

4. (commutativity) for all $n_1, n_2 \in \mathbb{N}$ the following diagram commutes

$$\begin{array}{ccc} E_{n_1} \wedge E_{n_2} & \overset{\tau^{\mathrm{Top}_{\mathrm{cg}}^{*/}}_{E_{n_1},E_{n_2}}}{\longrightarrow} & E_{n_2} \wedge E_{n_1} \\ {}^{\mu_{n_1,n_2}}\downarrow && \downarrow^{\mu_{n_2,n_1}} \\ E_{n_1+n_2} & \underset{\chi_{n_1,n_2}}{\longrightarrow} & E_{n_2+n_1} \end{array}$$

where the top morphism τ is the braiding in $(\mathrm{Top}_{\mathrm{cg}}^{*/}, \wedge, S^0)$ (def. 1.10) and where $\chi_{n_1,n_2} \in \Sigma(n_1+n_2)$ denotes the permutation action which shuffles the first n_1 elements past the last n_2 elements.

A homomorphism of symmetric commutative ring spectra $f : E \longrightarrow E'$ is a sequence $f_n : E_n \longrightarrow E'_n$ of $\Sigma(n)$-equivariant pointed continuous functions such that the following diagrams commute for all $n_1, n_2 \in \mathbb{N}$

$$\begin{array}{ccc} E_{n_1} \wedge E_{n_2} & \overset{f_{n_1} \wedge f_{n_2}}{\longrightarrow} & E'_{n_1} \wedge E'_{n_2} \\ {}^{\mu_{n_1,n_2}}\downarrow && \downarrow^{\mu_{n_2,n_1}} \\ E_{n_1+n_2} & \underset{\chi_{n_1,n_2}}{\longrightarrow} & E_{n_2+n_1} \end{array}$$

and $f_0 \circ \iota_0 = \iota_0$ and $f_1 \circ \iota_1 = \iota_1$.

Write

$$\mathrm{CRing}(\mathrm{SymSpec}(\mathrm{Top}_{\mathrm{cg}}))$$

for the resulting category of symmetric commutative ring spectra.

We regard a symmetric ring spectrum in particular as a symmetric spectrum (def. 2.10) by taking the structure maps to be

$$\sigma_n : S^1 \wedge E_n \overset{\iota_1 \wedge \mathrm{id}}{\longrightarrow} E_1 \wedge E_n \overset{\mu_{1,n}}{\longrightarrow} E_{n+1} \,.$$

This defines a forgetful functor

$$\mathrm{CRing}(\mathrm{SymSpec}(\mathrm{Top}_{\mathrm{cg}})) \longrightarrow \mathrm{SymSpec}(\mathrm{Top}_{\mathrm{cg}})$$

There is an analogous definition of **orthogonal ring spectrum** and we write

$$\mathrm{CRing}(\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}}))$$

for the category that these form.

(e.g. Schwede 12, def. 1.3)

We discuss **examples** below in a dedicated section *Examples*.

Proposition 2.16. *The symmetric (orthogonal) commutative ring spectra in def. 2.15 are equivalently the commutative monoids in (def. 1.13) the symmetric monoidal category $\mathbb{S}_{\mathrm{sym}}\,\mathrm{Mod}$ ($\mathbb{S}_{\mathrm{orth}}\,\mathrm{Mod}$) of def. 2.9 with respect to the symmetric monoidal smash product of spectra $\wedge = \otimes_{\mathbb{S}_{\mathrm{sym}}}$ ($\wedge = \otimes_{\mathbb{S}_{\mathrm{orth}}}$). Hence there are equivalences of categories*

$$\mathrm{CRing}(\mathrm{SymSpec}(\mathrm{Top}_{\mathrm{cg}})) \;\simeq\; \mathrm{CMon}(\mathbb{S}_{\mathrm{sym}}\,\mathrm{Mod}, \otimes_{\mathbb{S}_{\mathrm{sym}}}, \mathbb{S}_{\mathrm{sym}})$$

and

$$\mathrm{CRing}(\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})) \;\simeq\; \mathrm{CMon}(\mathbb{S}_{\mathrm{orth}}\,\mathrm{Mod}, \otimes_{\mathbb{S}_{\mathrm{orth}}}, \mathbb{S}_{\mathrm{orth}}) \,.$$

Moreover, under these identifications the canonical forgetful functor

$$\mathrm{CMon}(\mathbb{S}_{\mathrm{sym}}\,\mathrm{Mod}, \otimes_{\mathbb{S}_{\mathrm{sym}}}, \mathbb{S}_{\mathrm{sym}}) \longrightarrow \mathrm{SymSpec}(\mathrm{Top}_{\mathrm{cg}})$$

and

$$\mathrm{CMon}(\mathbb{S}_{\mathrm{orth}}\,\mathrm{Mod}, \otimes_{\mathbb{S}_{\mathrm{orth}}}, \mathbb{S}_{\mathrm{orth}}) \longrightarrow \mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})$$

coincides with the forgetful functor defined in def. 2.15.

Proof. We discuss this for symmetric spectra. The proof for orthogonal spectra is directly analogous.

By prop. 1.25 and def. 2.9, the commutative monoids in $\mathbb{S}_{\mathrm{sym}}\,\mathrm{Mod}$ are equivalently commtutative monoids E in $([\mathrm{Sym}, \mathrm{Top}^{*/}_{\mathrm{cg}}], \otimes_{\mathrm{Day}}, y(0))$ equipped with a homomorphism of monoids $\mathbb{S}_{\mathrm{sym}} \longrightarrow E$. In turn, by prop. 1.52 this are equivalently braided lax monoidal functors (which we denote by the same symbols, for convenience) of the form

$$E : (\mathrm{Sym}, +, 0) \longrightarrow (\mathrm{Top}^{*/}_{\mathrm{cg}}, \wedge, S^0)$$

equipped with a monoidal natural transformation (def. 1.47)

$$\iota : \mathbb{S}_{\mathrm{sym}} \longrightarrow E \,.$$

The structure morphism of such a lax monoidal functor E has as components precisely the morphisms $\mu_{n_1,n_2} : E_{n_1} \wedge E_{n_2} \to E_{n_1+n_2}$. In terms of these, the associativity and braiding condition on the lax monoidal functor are manifestly the above associativity and commutativity conditions.

Moreover, by the proof of prop. 1.25 the $\mathbb{S}_{\mathrm{sym}}$-module structure on an an $\mathbb{S}_{\mathrm{sym}}$-algebra E has action given by

$$\mathbb{S}_{\mathrm{sym}} \wedge E \overset{e \wedge \mathrm{id}}{\longrightarrow} E \wedge E \overset{\mu}{\longrightarrow} E \,,$$

which shows, via the identification in prop 2.12, that the forgetful functors to underlying symmetric spectra coincide as claimed.

Hence it only remains to match the nature of the units. The above unit morphism ι has components

$$\iota_n : S^n \longrightarrow E_n$$

for all $n \in \mathbb{N}$, and the unitality condition for ι_0 and ι_1 is manifestly as in the statement above.

We claim that the other components are uniquely fixed by these:

By lemma 2.7, the product structure in $\mathbb{S}_{\mathrm{sym}}$ is by isomorphisms $S^{n_1} \wedge S^{n_2} \simeq S^{n_1+n_2}$, so that the commuting square for the coherence condition of this monoidal natural transformation

$$
\begin{array}{ccc}
S^{n_1} \wedge S^{n_2} & \overset{\iota_{n_1} \wedge \iota_{n_2}}{\longrightarrow} & E_{n_1} \wedge E_{n_2} \\
{}^{\simeq}\downarrow & & \downarrow^{\mu_{n_1,n_2}} \\
S^{n_1+n_2} & \underset{\iota_{n_1+n_2}}{\longrightarrow} & E_{n_1+n_2}
\end{array}
$$

says that $\iota_{n_1+n_2} = \mu_{n_1,n_2} \circ (\iota_{n_1} \wedge \iota_{n_2})$. This means that $\iota_{n \geq 2}$ is uniquely fixed once ι_0 and ι_1 are given.

Finally it is clear that homomorphisms on both sides of the equivalence precisely respect all this structure under both sides of the equivalence. ▮

Similarly:

Definition 2.17. Given a symmetric (orthogonal) commutative ring spectrum E (def. 2.15), then a left symmetric (orthogonal) **module spectrum** N over E is

1. a sequence of component spaces $N_n \in \mathrm{Top}_{\mathrm{cg}}^{*/}$ for $n \in \mathbb{N}$;

2. a basepoint preserving continuous left action of the symmetric group $\Sigma(n)$ on N_n;

3. for all $n_1, n_2 \in \mathbb{N}$ an *action map*

$$
\rho_{n_1,n_2} : E_{n_1} \wedge N_{n_2} \longrightarrow N_{n_1+n_2}
$$

(a morphism in $\mathrm{Top}_{\mathrm{cg}}^{*/}$)

such that

1. (equivariance) ρ_{n_1,n_2} intertwines the $\Sigma(n_1) \times \Sigma(n_2)$-action;

2. (action property) for all $n_1, n_2, n_3 \in \mathbb{N}$ the following diagram commutes (where we are notationally suppressing the associators of $(\mathrm{Top}_{\mathrm{cg}}^{*/}, \wedge, S^0)$)

$$
\begin{array}{ccc}
E_{n_1} \wedge E_{n_2} \wedge N_{n_3} & \overset{\mathrm{id} \wedge \rho_{n_2,n_3}}{\longrightarrow} & E_{n_1} \wedge N_{n_2+n_3} \\
{}^{\mu_{n_1,n_2} \wedge \mathrm{id}}\downarrow & & \downarrow^{\rho_{n_1,n_2+n_3}} \\
E_{n_1+n_2} \wedge N_{n_3} & \underset{\rho_{n_1+n_2,n_3}}{\longrightarrow} & N_{n_1+n_2+n_3}
\end{array}
$$

3. (unitality) for all $n \in \mathbb{N}$ the following diagram commutes

$$
\begin{array}{ccc}
S^0 \wedge N_n & \overset{\iota_0 \wedge \mathrm{id}}{\longrightarrow} & E_0 \wedge N_n \\
& {}_{\ell_{N_n}^{\mathrm{Top}_{\mathrm{cg}}^{*/}}}\searrow & \downarrow^{\mu_{0,n}} \\
& & N_n
\end{array}
.
$$

A homomorphism of left E-module spectra $f : N \longrightarrow N'$ is a sequence of pointed continuous functions $f_n : N_n \longrightarrow N'_n$ such that for all $n_1, n_2 \in \mathbb{N}$ the following diagrams commute:

$$
\begin{array}{ccc}
E_{n_1} \wedge N_{n_2} & \overset{\mathrm{id} \wedge f_{n_2}}{\longrightarrow} & E_{n_1} \wedge N'_{n_2} \\
{}^{\rho_{n_1,n_2}}\downarrow & & \downarrow^{\rho_{n_1,n_2}} \\
N_{n_1+n_2} & \underset{f_{n_1+n_2}}{\longrightarrow} & N'_{n_1+n_2}
\end{array}
.
$$

We write

$$
E\,\mathrm{Mod}(\mathrm{SymSpec}(\mathrm{Top}_{\mathrm{cg}})) \quad , \quad E\,\mathrm{Mod}(\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}}))
$$

for the resulting category of symmetric (orthogonal) E-module spectra.

(e.g. Schwede 12, I, def. 1.5)

Proposition 2.18. *Under the identification, from prop. 2.16, of commutative ring spectra with commutative monoids with respect to the symmetric monoidal smash product of spectra, the E-module spectra of def. 2.17 are equivalently the left module objects (def. 1.16) over the respective monoids, i.e. there are equivalences of categories*

$$
E\,\mathrm{Mod}(\mathrm{SymSpec}(\mathrm{Top}_{\mathrm{cg}})) \simeq E\,\mathrm{Mod}([\mathrm{Sym}, \mathrm{Top}_{\mathrm{cg}}^{*/}], \otimes_{\mathrm{Day}}, y(0))
$$

and

$$E\,\mathrm{Mod}(\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})) \;\simeq\; E\,\mathrm{Mod}([\mathrm{Orth}, \mathrm{Top}_{\mathrm{cg}}^{*/}],\, \otimes_{\mathrm{Day}}\,, y(0))\,,$$

where on the right we have the categories of modules from def. 1.16.

Proof. The proof is directly analogous to that of prop. 2.16. Now prop. 1.25 and prop. 1.52 give that the module objects in question are equivalently modules over a monoidal functor (def. 1.51) and writing these out in components yields precisely the above structures and properties. ▮

As diagram spectra

In *Introduction to Stable homotopy theory -- 1-1* we obtained the strict/level model structure on topological sequential spectra by identifying the category $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$ of sequential spectra with a category of topologically enriched functors with values in $\mathrm{Top}_{\mathrm{cg}}^{*/}$ (prop.) and then invoking the general existence of the projective model structure on functors (thm.).

Here we discuss the analogous construction for the more general structured spectra from above.

Proposition 2.19. *Let $(\mathcal{C}, \otimes_{\mathcal{C}}\,, 1_{\mathcal{C}})$ be a topologically enriched monoidal category (def. 1.1), and let $A \in \mathrm{Mon}([\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}],\, \otimes_{\mathrm{Day}}\,, y(1_{\mathcal{C}}))$ be a monoid in (def. 1.13) the pointed topological Day convolution monoidal category over $\mathcal{C}$ from prop. 1.44.*

Then the category of left A-modules (def. 1.16) is a pointed topologically enriched functor category category (exmpl.)

$$A\,\mathrm{Mod} \;\simeq\; [A\,\mathrm{Free}_{\mathcal{C}}\mathrm{Mod}^{\mathrm{op}},\, \mathrm{Top}_{\mathrm{cg}}^{*/}]\,,$$

over the category of free modules over A (prop. 1.20) on objects in $\mathcal{C}$ (under the Yoneda embedding $y \colon \mathcal{C}^{\mathrm{op}} \to [\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}]$). Hence the objects of $A\,\mathrm{Free}_{\mathcal{C}}\mathrm{Mod}$ are identified with those of $\mathcal{C}$, and its hom-spaces are

$$A\,\mathrm{Free}_{\mathcal{C}}\mathrm{Mod}(c_1, c_2) \;=\; A\,\mathrm{Mod}(A \otimes_{\mathrm{Day}} y(c_1),\, A \otimes_{\mathrm{Day}} y(c_2))\;.$$

(MMSS 00, theorem 2.2)

Proof. Use the identification from prop. 1.52 of A with a lax monoidal functor and of any A-module object N as a functor with the structure of a module over a monoidal functor, given by natural transformations

$$A(c_1) \otimes N(c_2) \overset{\rho_{c_1, c_2}}{\longrightarrow} N(c_1 \otimes c_2)\;.$$

Notice that these transformations have just the same structure as those of the enriched functoriality of N (def.) of the form

$$\mathcal{C}(c_1, c_2) \otimes N(c_1) \longrightarrow N(c_2)\;.$$

Hence we may unify these two kinds of transformations into a single kind of the form

$$\mathcal{C}(c_3 \otimes c_1, c_2) \otimes A(c_3) \otimes N(c_1) \overset{\mathrm{id} \otimes \rho_{c_3, c_1}}{\longrightarrow} \mathcal{C}(c_3 \otimes c_1, c_2) \otimes N(c_3 \otimes c_2) \longrightarrow N(c_2)$$

and subject to certain identifications.

Now observe that the hom-objects of $A\,\mathrm{Free}_{\mathcal{C}}\mathrm{Mod}$ have just this structure:

$$\begin{aligned} A\,\mathrm{Free}_{\mathcal{C}}\mathrm{Mod}(c_2, c_1) & = A\,\mathrm{Mod}(A \otimes_{\mathrm{Day}} y(c_2), A \otimes_{\mathrm{Day}} y(c_1)) \\ & \simeq [\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}](y(c_2), A \otimes_{\mathrm{Day}} y(c_1)) \\ & \simeq (A \otimes_{\mathrm{Day}} y(c_1))(c_2) \\ & \simeq \int^{c_3, c_4} \mathcal{C}(c_3 \otimes c_4, c_2) \wedge A(c_3) \wedge \mathcal{C}(c_1, c_4) \\ & \simeq \int^{c_3} \mathcal{C}(c_3 \otimes c_1, c_2) \wedge A(c_3) \end{aligned}\,.$$

Here we used first the free-forgetful adjunction of prop. 1.20, then the enriched Yoneda lemma (prop. 1.32), then the coend-expression for Day convolution (def. 1.39) and finally the co-Yoneda lemma (prop. 1.33).

Then define a topologically enriched category $\mathcal{D}$ to have objects and hom-spaces those of $A\,\mathrm{Free}_{\mathcal{C}}\mathrm{Mod}^{\mathrm{op}}$ as above, and whose composition operation is defined as follows:

$$
\begin{aligned}
\mathcal{D}(c_2, c3) \wedge \mathcal{D}(c_1, c_2) & \simeq \left(\int^{c_5} \mathcal{C}(c_5 \otimes_{\mathcal{C}} c_2, c_3) \wedge A(c_5)\right) \wedge \left(\int^{c_4} \mathcal{C}(c_4 \otimes_{\mathcal{C}} c_1, c_2) \wedge A(c_4)\right) \\
& \simeq \int^{c_4, c_5} \mathcal{C}(c_5 \otimes_{\mathcal{C}} c_2, c_3) \wedge \mathcal{C}(c_4 \otimes_{\mathcal{C}} c_1, c_2) \wedge A(c_5) \wedge A(c_4) \\
& \longrightarrow \int^{c_4, c_5} \mathcal{C}(c_5 \otimes_{\mathcal{C}} c_2, c_3) \wedge \mathcal{C}(c_5 \otimes_{\mathcal{C}} c_4 \otimes_{\mathcal{C}} c_1, c_5 \otimes_{\mathcal{C}} c_2) \wedge A(c_5 \otimes_{\mathcal{C}} c_4)' \\
& \longrightarrow \int^{c_4, c_5} \mathcal{C}(c_5 \otimes_{\mathcal{C}} c_4 \otimes_{\mathcal{C}} c_1, c_5 \otimes_{\mathcal{C}} c_2) \wedge A(c_5 \otimes_{\mathcal{C}} c_4) \\
& \longrightarrow \int^{c_4} \mathcal{C}(c_4 \otimes_{\mathcal{C}} c_1, c_3) \otimes_V A(c_4)
\end{aligned}
$$

where

1. the equivalence is braiding in the integrand (and the Fubini theorem, prop. 1.35);
2. the first morphism is, in the integrand, the smash product of
   1. forming the tensor product of hom-objects of $\mathcal{C}$ with the identity morphism on c_5;
   2. the monoidal functor incarnation $A(c_5) \wedge A(c_4) \longrightarrow A(c_5 \otimes_{\mathcal{C}} c_4)$ of the monoid structure on A;
3. the second morphism is, in the integrand, given by composition in $\mathcal{C}$;
4. the last morphism is the morphism induced on coends by regarding extranaturality in c_4 and c_5 separately as a special case of extranaturality in $c_6 \coloneqq c_4 \otimes c_5$ (and then renaming).

With this it is fairly straightforward to see that

$$A\,\mathrm{Mod} \simeq [\mathcal{D}, \mathrm{Top}_{\mathrm{cg}}^{*/}]\,,$$

because, by the above definition of composition, functoriality over $\mathcal{D}$ manifestly encodes the A-action property together with the functoriality over $\mathcal{C}$.

This way we are reduced to showing that actually $\mathcal{D} \simeq A\,\mathrm{Free}_{\mathcal{C}}\mathrm{Mod}^{\mathrm{op}}$.

But by construction, the image of the objects of $\mathcal{D}$ under the Yoneda embedding are precisely the free A-modules over objects of $\mathcal{C}$:

$$\mathcal{D}(c, -) \simeq A\,\mathrm{Free}_{\mathcal{C}}\mathrm{Mod}(-, c) \simeq (A \otimes_{\mathrm{Day}} y(c))(-)\ .$$

Since the Yoneda embedding is fully faithful, this shows that indeed

$$\mathcal{D}^{\mathrm{op}} \simeq A\,\mathrm{Free}_{\mathcal{C}}\mathrm{Mod} \hookrightarrow A\,\mathrm{Mod}\ .$$

▮

Example 2.20. For the sequential case $\mathrm{Dia} = \mathrm{Seq}$ in def. 2.4, then the opposite category of free modules on objects in Seq over $\mathbb{S}_{\mathrm{seq}}$ (def.) is identified as the category $\mathrm{StdSpheres}$ (def.):

$$\mathbb{S}_{\mathrm{seq}}\,\mathrm{Free}_{\mathrm{seq}}\mathrm{Mod}^{\mathrm{op}} \;\simeq\; \mathrm{StdSpheres}$$

Accordingly, in this case prop. 2.19 reduces to the identification (prop.) of sequential spectra as topological diagrams over $\mathrm{StdSpheres}$:

$$[\mathbb{S}_{\mathrm{seq}}\,\mathrm{Free}_{\mathrm{seq}}\mathrm{Mod}^{\mathrm{op}}, \mathrm{Top}_{\mathrm{cg}}^{*/}] \simeq [\mathrm{StdSpheres}, \mathrm{Top}_{\mathrm{cg}}^{*/}] \simeq \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})\ .$$

Proof. There is one object $y(n)$ for each $n \in \mathbb{N}$. Moreover, from the expression in the proof of prop. 2.19 we compute the hom-spaces between these to be

$$
\begin{aligned}
\mathbb{S}_{\mathrm{seq}}\,\mathrm{Free}_{\mathrm{seq}}\mathrm{Mod}(\mathbb{S}_{\mathrm{seq}} \otimes_{\mathrm{Day}} y_{k_2}, \mathbb{S}_{\mathrm{seq}} \otimes_{\mathrm{Day}} y_{k_1}) & \simeq \int^n \mathrm{Seq}(n + k_1, k_2) \wedge \mathbb{S}_{\mathrm{seq}}(n) \\
& \simeq \begin{cases} S^{k_2 - k_1} & \text{if } k_2 \geq k_1 \\ * & \text{otherwise} \end{cases}
\end{aligned}
.
$$

These are the objects and hom-spaces of the category $\mathrm{StdSpheres}$. It is straightforward to check that the definition of composition agrees, too. ▮

Stable weak homotopy equivalences

We consider the evident version of stable weak homotopy equivalences for structured spectra and prove a few technical lemmas about them that are needed in the proof of the stable model structure below

Definition 2.21. For $\mathrm{Dia} \in \{\mathrm{Top}_{\mathrm{cg,fin}}^{*/}, \mathrm{Orth}, \mathrm{Sym}, \mathrm{Seq}\}$ one of the shapes of structured spectra from def. 2.4, let

$\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}$ be the corresponding category of structured spectra (def. 2.1, prop. 2.8, def. 2.9).

1. The stable homotopy groups of an object $X \in \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}$ are those of the underlying sequential spectrum (def.):

$$\pi_\bullet(X) \coloneqq \pi_\bullet(\mathrm{seq}^* X)\ .$$

2. An object $X \in \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}$ is a **structured Omega-spectrum** if the underlying sequential spectrum $\mathrm{seq}^* X$ (def. 2.4) is a sequential Omega spectrum (def.)

3. A morphism f in $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}$ is a **stable weak homotopy equivalence** (or: **$\pi_\bullet$-isomorphism**) if the underlying morphism of sequential spectra $\mathrm{seq}^*(f)$ is a stable weak homotopy equivalence of sequential spectra (def.);

4. a morphism f is a **stable cofibration** if it is a cofibration in the strict model structure $\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}}$ from prop. 3.1.

(MMSS 00, def. 8.3 with the notation from p. 21, Mandell-May 02, III, def. 3.1, def. 3.2)

Lemma 2.22. *Given a morphism $f : X \longrightarrow Y$ in $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}$, then there are long exact sequences of stable homotopy groups (def. 2.21) of the form*

$$\cdots \longrightarrow \pi_{\bullet+1}(Y) \longrightarrow \pi_\bullet(\mathrm{Path}_*(f)) \longrightarrow \pi_\bullet(X) \overset{f_*}{\longrightarrow} \pi_\bullet(Y) \longrightarrow \pi_{\bullet-1}(\mathrm{Path}_*(f)) \longrightarrow \cdots$$

and

$$\cdots \longrightarrow \pi_{\bullet+1}(Y) \longrightarrow \pi_{\bullet+1}(\mathrm{Cone}(f)) \longrightarrow \pi_\bullet(X) \overset{f_*}{\longrightarrow} \pi_\bullet(Y) \longrightarrow \pi_\bullet(\mathrm{Cone}(f)) \longrightarrow \cdots,$$

where $\mathrm{Cone}(f)$ denotes the mapping cone and $\mathrm{Path}_*(f)$ the mapping cocone of f (def.) formed with respect to the standard cylinder spectrum $X \wedge (I_+)$ hence formed degreewise with respect to the standard reduced cylinder of pointed topological spaces.

(MMSS 00, theorem 7.4 (vi))

Proof. Since limits and colimits in the diagram category $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}$ are computed objectwise, the functor seq^* that restricts $\mathbb{S}_{\mathrm{dia}}$-modules to their underlying sequential spectra preserves both limits and colimits, hence it is sufficient to consider the statement for sequential spectra.

For the first case, there is degreewise the long exact sequence of homotopy groups to the left of pointed topological spaces (exmpl.)

$$\cdots \to \pi_2(Y) \longrightarrow \pi_1(\mathrm{Path}_*(f)) \longrightarrow \pi_1(X) \overset{f_*}{\longrightarrow} \pi_1(Y) \longrightarrow \pi_0(\mathrm{Path}_*(f)) \longrightarrow \pi_0(X_n) \overset{f_*}{\longrightarrow} \pi_0(Y_n)\ .$$

Observe that the sequential colimit that defines the stable homotopy groups (def.) preserves exact sequences of abelian groups, because generally filtered colimits in Ab are exact functors (prop.). This implies that by taking the colimit over n in the above sequences, we obtain a long exact sequence of stable homotopy groups as shown.

Now use that in sequential spectra the canonical morphism morphism $\mathrm{Path}_*(f) \longrightarrow \Omega\,\mathrm{Cone}(f)$ is a stable weak homotopy equivalence and is compatible with the map f (prop.) so that there is a commuting diagram of the form

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & \pi_{\bullet+1}(Y) & \longrightarrow & \pi_\bullet(\mathrm{Path}_*(f)) & \longrightarrow & \pi_\bullet(X) & \overset{f_*}{\longrightarrow} & \pi_\bullet(Y) & \longrightarrow & \pi_{\bullet-1}(\mathrm{Path}_*(f)) & \longrightarrow & \cdots \\
&& \downarrow^{=} && \downarrow^{\simeq} && \downarrow^{=} && \downarrow^{=} && \downarrow^{\simeq} \\
\cdots & \longrightarrow & \pi_{\bullet+1}(Y) & \longrightarrow & \pi_{\bullet+1}(\mathrm{Cone}(f)) & \longrightarrow & \pi_\bullet(X) & \overset{f_*}{\longrightarrow} & \pi_\bullet(Y) & \longrightarrow & \pi_\bullet(\mathrm{Cone}(f)) & \longrightarrow & \cdots
\end{array}\ .$$

Since the top sequence is exact, and since all vertical morphisms are isomorphisms, it follows that also the bottom sequence is exact. ▮

Lemma 2.23. *For $K \in \mathrm{Top}^{*/}_{\mathrm{cg,fin}}$ a CW-complex then the operation of smash tensoring $(-)\wedge K$ preserves stable weak homotopy equivalences in $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}$.*

Proof. Since limits and colimits in the diagram category $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}$ are computed objectwise, the functor seq^* that restricts $\mathbb{S}_{\mathrm{dia}}$-modules to their underlying sequential spectra preserves both limits and colimits, and it also preserves smash tensoring. Hence it is sufficient to consider the statement for sequential spectra.

Fist consider the case of a finite cell complex K.

Write

$$* = K_0 \hookrightarrow \cdots \hookrightarrow K_i \hookrightarrow K_{i+1} \hookrightarrow \cdots \hookrightarrow K$$

for the stages of the cell complex K, so that for each i there is a pushout diagram in $\mathrm{Top}_{\mathrm{cg}}$ of the form

$$\begin{array}{ccccc} S^{n_i - 1} & \longrightarrow & K_i & \longrightarrow & \ast \\ \downarrow & (\mathrm{po}) & \downarrow & (\mathrm{po}) & \downarrow \\ D^{n_i - 1} & \longrightarrow & K_{i+1} & \longrightarrow & S^{n_i} \end{array} .$$

Equivalently these are pushoutdiagrams in $\mathrm{Top}_{\mathrm{cg}}^{*/}$ of the form

$$\begin{array}{ccccc} S_+^{n_i - 1} & \longrightarrow & K_i & \longrightarrow & \ast \\ \downarrow & (\mathrm{po}) & \downarrow & (\mathrm{po}) & \downarrow \\ D_+^{n_i - 1} & \longrightarrow & K_{i+1} & \longrightarrow & S^{n_i} \end{array} .$$

Notice that it is indeed S^{n_i} that appears in the top right, not $S_+^{n_i}$.

Now forming the smash tensoring of any morphism $f : X \longrightarrow Y$ in $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(\mathrm{Top}_{\mathrm{cg}})$ by the morphisms in the pushout on the right yields a commuting diagram in $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}$ of the form

$$\begin{array}{ccccc} X \wedge K_i & \longrightarrow & X \wedge K_{i+1} & \longrightarrow & X \wedge S^{n_i} \\ \downarrow && \downarrow && \downarrow \\ Y \wedge K_i & \longrightarrow & Y \wedge K_{i+1} & \longrightarrow & Y \wedge S^{n_i} \end{array} .$$

Here the horizontal morphisms on the left are degreewise cofibrations in $\mathrm{Top}_{\mathrm{cg}}^{*/}$, hence the morphism on the right is degreewise their homotopy cofiber. This way lemma 2.22 implies that there are commuting diagrams

$$\begin{array}{ccccccccc} \pi_{\bullet+1}(X \wedge S^{n_i}) & \longrightarrow & \pi_\bullet(X \wedge K_i) & \longrightarrow & \pi_\bullet(X \wedge K_{i+1}) & \longrightarrow & \pi_\bullet(X \wedge S^{n_i}) & \longrightarrow & \pi_{\bullet-1}(X \wedge K_i) \\ \downarrow && \downarrow && \downarrow^{f \wedge K_{i+1}} && \downarrow && \downarrow \\ \pi_{\bullet+1}(Y \wedge S^{n_i}) & \longrightarrow & \pi_\bullet(Y \wedge K_i) & \longrightarrow & \pi_\bullet(Y \wedge K_{i+1}) & \longrightarrow & \pi_\bullet(Y \wedge S^{n_i}) & \longrightarrow & \pi_{\bullet-1}(X \wedge K_i) \end{array} ,$$

where the top and bottom are long exact sequences of stable homotopy groups.

Now proceed by induction. For $i = 0$ then clearly smash tensoring with $K_0 = \ast$ preserves stable weak homotopy equivalences. So assume that smash tensoring with K_i does, too. Observe that $(-) \wedge S^n$ preserves stable weak homotopy equivalences, since $\Sigma X[1] \to X$ is a stable weak homotopy equivalence (lemma). Hence in the above the two vertical morphisms on the left and the two on the right are isomorphism. Now the five lemma implies that also $f \wedge K_{i+1}$ is an isomorphism.

Finally, the statement for a non-finite cell complex follows with these arguments and then using that spheres are compact and hence maps out of them into a transfinite composition factor through some finite stage (prop.). ∎

Lemma 2.24. *The pushout in $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}$ of a stable weak homotopy equivalence along a morphism that is degreewise a cofibration in $(\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}$ is again a stable weak homotopy equivalence.*

Proof. Given a pushout square

$$\begin{array}{ccc} X & \overset{g}{\longrightarrow} & Z \\ {}^{f}\downarrow & (\mathrm{po}) & \downarrow \\ Y & \longrightarrow & Y \underset{X}{\sqcup} Z \end{array}$$

observe that the pasting law implies an isomorphism between the horizontal cofibers

$$\begin{array}{ccccc} X & \overset{g}{\longrightarrow} & Z & \longrightarrow & \mathrm{cofib}(g) \\ {}^{f}\downarrow & (\mathrm{po}) & \downarrow && \downarrow^{\simeq} \\ Y & \longrightarrow & Y \underset{X}{\sqcup} Z & \longrightarrow & \mathrm{cofib}(g) \end{array} .$$

Moreover, since cofibrations in $(\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}$ are preserves by pushout, and since pushout of spectra are computed degreewise, both the top and the bottom horizontal sequences here are degreewise homotopy cofiber sequence in $(\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}$. Hence lemma 2.22 applies and gives a commuting diagram

$$\begin{array}{ccccccccc} \pi_{\bullet+1}(\mathrm{cofib}(g)) & \longrightarrow & \pi_\bullet(X) & \longrightarrow & \pi_\bullet(Z) & \longrightarrow & \pi_\bullet(\mathrm{cofib}(g)) & \longrightarrow & \pi_{\bullet-1}(X) \\ \downarrow^{\simeq} && {}^{\pi_\bullet(f)}_{\simeq}\downarrow && \downarrow && \downarrow^{\simeq} && \downarrow^{\simeq} \\ \pi_{\bullet+1}(\mathrm{cofib}(g)) & \longrightarrow & \pi_\bullet(Y) & \longrightarrow & \pi_\bullet(Y \underset{X}{\sqcup} Z) & \longrightarrow & \pi_\bullet(\mathrm{cofib}(g)) & \longrightarrow & \pi_{\bullet-1}(Y) \end{array} ,$$

where the top and the bottom row are both long exact sequences of stable homotopy groups. Hence the claim now follows by the five lemma. ▮

Free spectra and Suspension spectra

The concept of *free spectrum* is a generalization of that of *suspension spectrum*. In fact the stable homotopy types of free spectra are precisely those of iterated loop space objects of suspension spectra. But for the development of the theory what matters is free spectra before passing to stable homotopy types, for as such they play the role of the basic cells for the stable model structures on spectra analogous to the role of the n-spheres in the classical model structure on topological spaces (def. 3.2 below).

Moreover, while free sequential spectra are just re-indexed suspension spectra, free symmetric spectra and free orthogonal spectra in addition come with suitably freely generated actions of the symmetric group and the orthogonal group. It turns out that this is not entirely trivial; it leads to a subtle issue (lemma 2.33 below) where the adjuncts of certain canonical inclusions of free spectra are stable weak homotopy equivalences for sequential and orthogonal spectra, but *not* for symmetric spectra.

Definition 2.25. For $\mathrm{Dia} \in \{\mathrm{Top}^{*/}_{\mathrm{fin}}, \mathrm{Orth}, \mathrm{Sym}, \mathrm{Seq}\}$ any one of the four diagram shapes of def. 2.4, and for each $n \in \mathbb{N}$, the functor

$$(-)_n \;:\; \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod} \overset{\mathrm{seq}^*}{\longrightarrow} \mathbb{S}_{\mathrm{seq}}\,\mathrm{Mod} \simeq \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}}) \overset{(-)_n}{\longrightarrow} \mathrm{Top}^{*/}_{\mathrm{cg}}$$

that sends a structured spectrum to the nth component space of its underlying sequential spectrum has, by prop. 1.38, a left adjoint

$$F^{\mathrm{dia}}_n \;:\; \mathrm{Top}^{*/} \longrightarrow \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}\,.$$

This is called the **free structured spectrum**-functor.

For the special case $n = 0$ it is also called the **structured suspension spectrum** functor and denoted

$$\Sigma^\infty_{\mathrm{dia}} K \;\coloneqq\; F^{\mathrm{dia}}_0 K$$

(Hovey-Shipley-Smith 00, def. 2.2.5, MMSS 00, section 8)

Lemma 2.26. *Let $\mathrm{Dia} \in \{\mathrm{Top}^{*/}_{\mathrm{fin}}, \mathrm{Orth}, \mathrm{Sym}, \mathrm{Seq}\}$ be any one of the four diagram shapes of def. 2.4. Then*

1. *the free spectrum on $K \in \mathrm{Top}^{*/}_{\mathrm{cg}}$ (def. 2.25) is equivalently the smash tensoring with K (def.) of the free module (def. 1.20) over $\mathbb{S}_{\mathrm{dia}}$ (remark 2.6) on the representable $y(n) \in [\mathrm{Dia}, \mathrm{Top}^{*/}_{\mathrm{cg}}]$*

$$\begin{aligned} F^{\mathrm{dia}}_n K &\simeq (\mathbb{S}_{\mathrm{dia}} \otimes_{\mathrm{Day}} y(n)) \wedge K \\ &\simeq \mathbb{S}_{\mathrm{dia}} \otimes_{\mathrm{Day}} (y(n) \wedge K) \end{aligned}\,;$$

2. *on $n' \in \mathrm{Dia}^{\mathrm{op}} \overset{y}{\hookrightarrow} [\mathrm{Dia}, \mathrm{Top}^{*/}_{\mathrm{cg}}]$ its value is given by the following coend expression (def. 1.28)*

$$(F^{\mathrm{dia}}_n K)(n') \;\simeq\; \int^{n_1 \in \mathrm{Dia}} \mathrm{Dia}(n_1 \otimes n, n') \wedge S^{n_1} \wedge K\,.$$

In particular the structured sphere spectrum is the free spectrum in degree 0 on the 0-sphere:

$$\mathbb{S}_{\mathrm{dia}} \simeq F^{\mathrm{dia}}_0 S^0$$

and generally for $K \in \mathrm{Top}^{*/}_{\mathrm{cg}}$ then

$$F^{\mathrm{dia}}_0 K \simeq \mathbb{S}_{\mathrm{dia}} \wedge K$$

is the smash tensoring of the strutured sphere spectrum with K.

(Hovey-Shipley-Smith 00, below def. 2.2.5, MMSS00, p. 7 with theorem 2.2)

Proof. Under the equivalence of categories

$$\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod} \simeq [\mathbb{S}_{\mathrm{dia}}\,\mathrm{Free}_{\mathrm{dia}}\mathrm{Mod}^{\mathrm{op}}, \mathrm{Top}^{*/}_{\mathrm{cg}}]$$

from prop. 2.19, the expression for $F^{\mathrm{dia}}_n K$ is equivalently the smash tensoring with K of the functor that n represents over $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Free}_{\mathrm{dia}}\mathrm{Mod}$:

$$\begin{aligned} F^{\mathrm{dia}}_n K &\simeq y_{\mathbb{S}_{\mathrm{dia}}\,\mathrm{Free}_{\mathrm{Dia}}\mathrm{Mod}}(n) \wedge K \\ &\simeq \mathbb{S}_{\mathrm{dia}}\,\mathrm{Free}_{\mathrm{dia}}\mathrm{Mod}(-, \mathbb{S}_{\mathrm{dia}} \wedge y_{\mathrm{Dia}}(n)) \wedge K \end{aligned}$$

(by fully faithfulness of the Yoneda embedding).

This way the first statement is a special case of the following general fact: For $\mathcal{C}$ a pointed topologically enriched category, and for $c \in \mathcal{C}$ any object, then there is an adjunction

$$[\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}] \underset{(-)_c}{\overset{y(c)\wedge(-)}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathrm{Top}_{\mathrm{cg}}^{*/}$$

(saying that evaluation at c is right adjoint to smash tensoring the functor represented by c) witnessed by the following composite natural isomorphism:

$$[\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}](y(c) \wedge K, F) \;\simeq\; \mathrm{Maps}(K, [\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}](y(c), F))_* \;\simeq\; \mathrm{Maps}(K, F(c))_* \;=\; \mathrm{Top}_{\mathrm{cg}}^{*/}(K, F(c)) \,.$$

The first is the characteristic isomorphism of tensoring from prop. 1.37, while the second is the enriched Yoneda lemma of prop. 1.32.

From this, the second statement follows by the proof of prop. 2.19.

For the last statement it is sufficient to observe that $y(0)$ is the tensor unit under Day convolution by prop. 1.44 (since 0 is the tensor unit in Dia), so that

$$\begin{aligned} F_0^{\mathrm{dia}} S^0 &= \mathbb{S}_{\mathrm{dia}} \otimes_{\mathrm{Day}} (y(0) \wedge S^0) \\ &\simeq \mathbb{S}_{\mathrm{dia}} \otimes y(S^0) \\ &\simeq \mathbb{S}_{\mathrm{dia}} \end{aligned} \,.$$

∎

Proposition 2.27. *Explicitly, the free spectra according to def. 2.25, look as follows:*

For sequential spectra:

$$(F_n^{\mathrm{Seq}} K)_q \simeq \left\{ \begin{array}{cc} S^{q-n} \wedge K & \text{if } q \geq n \\ * & \text{otherwise} \end{array} \right\}$$

for symmetric spectra:

$$(F_n^{\mathrm{Sym}} K)_q \simeq \begin{cases} \Sigma(q)_+ \wedge_{\Sigma(q-n)} S^{q-n} \wedge K & \text{if } q \geq n \\ * & \text{otherwise} \end{cases}$$

for orthogonal spectra:

$$(F_n^{\mathrm{Orth}} K)_q \simeq \begin{cases} O(q)_+ \wedge_{O(q-n)} \wedge S^{q-n} \wedge K & \text{if } q \geq n \\ * & \text{otherwise} \end{cases},$$

where "$\wedge_G$" is as in example 1.29.

(e.g. Schwede 12, example 3.20)

Proof. With the formula in item 2 of lemma 2.26 we have for the case of orthogonal spectra

$$\begin{aligned} (F_n^{\mathrm{Orth}} K)(\mathbb{R}^q) &\simeq \int^{n_1 \in \mathrm{Orth}} \underbrace{\mathrm{Orth}(n_1 + n, q)}_{= \begin{cases} O(q)_+ & \text{if } n_1 + n = q \\ * & \text{otherwise} \end{cases}} \wedge S^{n_1} \wedge K \\ &\simeq \begin{cases} \int^{n_1 = * \in \mathbf{B}(O(q-n))} O(q)_+ \underset{O(q-n)}{\wedge} S^{q-n} \wedge K & \text{if } q \geq n \\ * & \text{otherwise} \end{cases} \end{aligned},$$

where in the second line we used that the coend collapses to $n_1 = q - n$ ranging in the full subcategory

$$\mathbf{B}(O(q-n)_+) \hookrightarrow \mathrm{Orth}$$

on the object $\mathbb{R}^{q-n}$ and then we applied example 1.29. The case of symmetric spectra is verbatim the same, with the symmetric group replacing the orthogonal group, and the case of sequential spectra is again verbatim the same, with the orthogonal group replaced by the trivial group. ∎

Lemma 2.28. *For $\mathrm{Dia} \in \{\mathrm{Orth}, \mathrm{Sym}, \mathrm{Seq}\}$ the diagram shape for orthogonal spectra, symmetric spectra or sequential spectra, then the free structured spectra*

$$F_n^{\mathrm{dia}} S^0 \in \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}$$

from def. 2.25 have component spaces that admit the structure of CW-complexes.

Proof. We consider the case of orthogonal spectra. The case of symmetric spectra works verbatim the same, and the case of sequential spectra is tivial.

By prop. 2.27 we have to show that for all $q \geq n \in \mathbb{N}$ the topological spaces of the form

$$O(q)_+ \wedge_{O(q-n)} S^{q-n} \quad \in \mathrm{Top}_{\mathrm{cg}}^{*/}$$

admit the structure of CW-complexes.

To that end, use the homeomorphism

$$S^{q-n} \simeq D^{q-n} / \partial D^{q-n}$$

which is a diffeomorphism away from the basepoint and hence such that the action of the orthogonal group $O(q-n)$ induces a smooth action on D^{q-n} (which happens to be constant on ∂D^{q-n}).

Also observe that we may think of the above quotient by the group action

$$(x, g y) \simeq (x g, y)$$

as being the quotient by the diagonal action

$$O(q-n) \times (O(q)_+ \wedge S^{q-n}) \longrightarrow (O(q)_+ \wedge S^{q-n})$$

given by

$$(g, (x,y)) \mapsto (x g^{-1}, g y) \,.$$

Using this we may rewrite the space in question as

$$\begin{aligned} (O(q)_+ \wedge_{O(q-n)} S^{q-n}) & \simeq (O(q)_+ \wedge S^{q-n}) / O(q-n) \\ & \simeq \frac{O(q) \times D^{q-n}}{O(q) \times \partial D^{q-n}} / O(q-n) \\ & \simeq \frac{O(q) \times D^{q-n}}{\partial(O(q) \times D^{q-n})} / O(q-n) \\ & \simeq \frac{(O(q) \times D^{q-n})/O(q-n)}{(\partial(O(q) \times D^{q-n}))/O(q-n)} \end{aligned} \,.$$

Here $O(q) \times D^{q-n}$ has the structure of a smooth manifold with boundary and equipped with a smooth action of the compact Lie group $O(q-n)$. Under these conditions (Illman 83, corollary 7.2) states that $O(q) \times D^{q-n}$ admits the structure of a G-CW complex for $G = O(q-n)$, and moreover (Illman 83, line above theorem 7.1) states that this may be chosen such that the boundary is a G-CW subcomplex.

Now the quotient of a G-CW complex by G is a CW complex, and so the last expression above exhibits the quotient of a CW-complex by a subcomplex, hence exhibits CW-complex structure. ▮

Proposition 2.29. *Let $\mathrm{Dia} \in \{\mathrm{Top}_{\mathrm{cg,fin}}^{*/}, \mathrm{Orth}, \mathrm{Sym}\}$ be the diagram shape of either pre-excisive functors, orthogonal spectra or symmetric spectra. Then under the symmetric monoidal smash product of spectra (def. 2.1, def. 2.1, def.2.9) the free structured spectra of def. 2.25 behave as follows*

$$F_{n_1}^{\mathrm{dia}}(K_1) \otimes_{\mathbb{S}_{\mathrm{dia}}} F_{n_2}^{\mathrm{dia}}(K_2) \;\simeq\; F_{n_1+n_2}(K_1 \wedge K_2) \,.$$

In particular for structured suspension spectra $\Sigma_{\mathrm{dia}}^\infty \coloneqq F_0^{\mathrm{dia}}$ (def. 2.25) this gives isomorphisms

$$\Sigma_{\mathrm{dia}}^\infty(K_1) \otimes_{\mathbb{S}_{\mathrm{dia}}} \Sigma_{\mathrm{dia}}^\infty(K_2) \;\simeq\; \Sigma_{\mathrm{dia}}^\infty(K_1 \wedge K_2) \,.$$

Hence the structured suspension spectrum functor $\Sigma_{\mathrm{dia}}^\infty$ is a strong monoidal functor (def. 1.47) and in fact a braided monoidal functor (def. \ref{braided monoidal functor}) from pointed topological spaces equipped with the smash product of pointed objects, to structured spectra equipped with the symmetric monoidal smash product of spectra

$$\Sigma_{\mathrm{dia}}^\infty \;:\; (\mathrm{Top}_{\mathrm{cg}}^{*/}, \wedge, S^0) \longrightarrow (\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}, \otimes_{\mathbb{S}_{\mathrm{dia}}}, \mathbb{S}_{\mathrm{dia}}) \,.$$

More generally, for $X \in \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}$ then

$$X \otimes_{\mathbb{S}_{\mathrm{dia}}} (\Sigma_{\mathrm{dia}}^\infty K) \simeq X \wedge K \,,$$

where on the right we have the smash tensoring of X with $K \in \mathrm{Top}_{\mathrm{cg}}^{*/}$.

(MMSS 00, lemma 1.8 with theorem 2.2, Mandell-May 02, prop. 2.2.6)

Proof. By lemma 2.26 the free spectra are free modules over the structured sphere spectrum $\mathbb{S}_{\mathrm{dia}}$ of the form $F_n^{\mathrm{dia}}(K) \simeq \mathbb{S}_{\mathrm{dia}} \otimes_{\mathrm{Day}} (y(n) \wedge K)$. By example 1.23 the tensor product of such free modules is given by

$$\left(\mathbb{S}_{\mathrm{dia}} \otimes_{\mathrm{Day}} (y(n_1) \wedge K_1)\right) \otimes_{\mathrm{Day}} \left(\mathbb{S}_{\mathrm{dia}} \otimes_{\mathrm{Day}} (y(n_2) \wedge K_2)\right) \simeq \mathbb{S}_{\mathrm{dia}} \otimes_{\mathrm{Day}} (y(n_1) \wedge K) \otimes_{\mathrm{Day}} (y(n_2) \wedge K)\,.$$

Using the co-Yoneda lemma (prop. 1.33) the expression on the right is

$$\begin{aligned} \left((y(n_1) \wedge K_1) \otimes_{\mathrm{Day}} (y(n_2) \wedge K_2)\right)(c) &= \int^{c_{1,2}} \mathrm{Dia}(c_1 + c_2, c) \wedge y(n_1)(c_1) \wedge K_1 \wedge y(n_2)(c_2) \wedge K_2 \\ &\simeq \int^{c_1, c_2} \mathrm{Dia}(c_1 + c_2, c) \wedge \mathrm{Dia}(n_1, c_1) \wedge \mathrm{Dia}(n_2, c_2) \wedge K_1 \wedge K_2\,. \\ &\simeq \mathrm{Dia}(n_1 + n_2, c) \wedge K_1 \wedge K_2 \\ &\simeq (y(n_1 + n_2) \wedge (K_1 \wedge K_2))(c) \end{aligned}$$

For the last statement we may use that $\Sigma^\infty_{\mathrm{dia}} K \simeq \mathbb{S}_{\mathrm{dia}} \wedge K$, by lemma 2.26), and that $\mathbb{S}_{\mathrm{dia}}$ is the tensor unit for $\otimes_{\mathbb{S}_{\mathrm{dia}}}$ by prop. 1.22.

To see that $\Sigma^\infty_{\mathrm{dia}}$ is braided, write $\Sigma^\infty_{\mathrm{dia}} K \simeq \mathbb{S} \wedge K$. We need to see that

$$\begin{array}{ccc} (\mathbb{S} \wedge K_1) \otimes_{\mathbb{S}} (\mathbb{S} \wedge K_2) & \longrightarrow & (\mathbb{S} \wedge K_2) \otimes_{\mathbb{S}} (\mathbb{S} \wedge K_1) \\ \downarrow & & \downarrow \\ \mathbb{S} \wedge (K_1 \wedge K_2) & \longrightarrow & \mathbb{S} \wedge (K_2 \wedge K_1) \end{array}$$

commutes. Chasing the smash factors through this diagram and using symmetry (def. 1.5) and the hexagon identities (def. 1.4) shows that indeed it does. ▮

One use of free spectra is that they serve to co-represent adjuncts of structure morphisms of spectra. To this end, first consider the following general existence statement.

Lemma 2.30. *For each $n \in \mathbb{N}$ there exists a morphism*

$$\lambda_n : F^{\mathrm{dia}}_{n+1} S^1 \longrightarrow F^{\mathrm{dia}}_n S^0$$

between free spectra (def. 2.25) such that for every structured spectrum $X \in \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}$ precomposition λ_n^* forms a commuting diagram of the form

$$\begin{array}{ccccc} \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(F_n^{\mathrm{dia}} S^0, X) & \simeq & \mathrm{Top}^{*/}(S^0, X_n) & \simeq & X_n \\ \downarrow^{\lambda_n^*} & & & & \downarrow^{\tilde\sigma_n^X} \\ \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(F_{n+1}^{\mathrm{dia}} S^1, X) & \simeq & \mathrm{Top}^{*/}(S^1, X_{n+1}) & \simeq & \Omega X_{n+1} \end{array},$$

where the horizontal equivalences are the adjunction isomorphisms and the canonical identification, and where the right morphism is the $(\Sigma \dashv \Omega)$-adjunct of the structure map σ_n of the sequential spectrum $\mathrm{seq}^* X$ underlying X (def. 2.4).

Proof. Since all prescribed morphisms in the diagram are natural transformations, this is in fact a diagram of copresheaves on $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}$

$$\begin{array}{ccccc} \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(F_n^{\mathrm{dia}} S^0, -) & \simeq & \mathrm{Top}^{*/}(S^0, (-)_n) & \simeq & (-)_n \\ \downarrow & & & & \downarrow^{\tilde\sigma_n^{(-)}} \\ \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(F_{n+1}^{\mathrm{dia}} S^1, -) & \simeq & \mathrm{Top}^{*/}(S^1, (-)_{n+1}) & \simeq & \Omega(-)_{n+1} \end{array}.$$

With this the statement follows by the Yoneda lemma. ▮

Now we say explicitly what these maps are:

Definition 2.31. For $n \in \mathbb{N}$, write

$$\lambda_n : F_{n+1} S^1 \longrightarrow F_n S^0$$

for the adjunct under the (free structured spectrum $\dashv$ n-component)-adjunction in def. 2.25 of the composite morphism

$$S^1 \xrightarrow{=} (F_n^{\mathrm{Seq}}(S^0))_{n+1} \overset{(f_n^{\mathrm{Seq}})_{n+1}}{\hookrightarrow} (F_n^{\mathrm{dia}} S^0)_{n+1}\,,$$

where the first morphism is via prop. 2.27 and the second comes from the adjunction units according to def. 2.25.

(MMSS 00, def. 8.4, Schwede 12, example 4.26)

Lemma 2.32. *The morphisms of def. 2.31 are those whose existence is asserted by prop. 2.30.*

(MMSS 00, lemma 8.5, following Hovey-Shipley-Smith 00, remark 2.2.12)

Proof. Consider the case $\mathrm{Dia} = \mathrm{Seq}$ and $n = 0$. All other cases work analogously.

By lemma 2.27, in this case the morphism λ_0 has components like so:

$$\begin{array}{ccc} \vdots && \vdots \\ S^3 & \overset{\mathrm{id}}{\longrightarrow} & S^3 \\ S^2 & \overset{\mathrm{id}}{\longrightarrow} & S^2 \\ S^1 & \overset{\mathrm{id}}{\longrightarrow} & S^1 \\ \ast & \overset{0}{\longrightarrow} & S^0 \\ \\ \underbrace{} && \underbrace{} \\ F_1 S^1 & \overset{\lambda_0}{\longrightarrow} & F_0 S^0 \end{array} \,.$$

Now for X any sequential spectrum, then a morphism $f \colon F_0 S^0 \to X$ is uniquely determined by its 0th components $f_0 \colon S^0 \to X_0$ (that's of course the very free property of $F_0 S^0$); as the compatibility with the structure maps forces the first component, in particular, to be $\sigma_0^X \circ \Sigma f$:

$$\begin{array}{ccc} \Sigma S^0 & \overset{\Sigma f}{\longrightarrow} & \Sigma X_0 \\ \downarrow^{\simeq} && \downarrow^{\sigma_0^X} \\ S^1 & \overset{\sigma_0^X \circ \Sigma f}{\longrightarrow} & X_1 \end{array}$$

But that first component is just the component that similarly determines the precompositon of f with λ_0, hence $\lambda_0^\ast f$ is fully fixed as being the map $\sigma_0^X \circ \Sigma f$. Therefore $\lambda_0^\ast$ is the function

$$\lambda_0^\ast \;\colon\; X_0 = \mathrm{Maps}(S^0, X_0) \overset{f \mapsto \sigma_0^X \circ \Sigma f}{\longrightarrow} \mathrm{Maps}(S^1, X_1) = \Omega X_1 \;.$$

It remains to see that this is the $(\Sigma \dashv \Omega)$-adjunct of σ_0^X. By the general formula for adjuncts, this is

$$\tilde \sigma_0^X \;\colon\; X_0 \overset{\eta}{\longrightarrow} \Omega \Sigma X_0 \overset{\Omega \sigma_0^X}{\longrightarrow} \Omega X_1 \;.$$

To compare to the above, we check what this does on points: $S^0 \overset{f_0}{\longrightarrow} X_0$ is sent to the composite

$$S^0 \overset{f_0}{\longrightarrow} X_0 \overset{\eta}{\longrightarrow} \Omega \Sigma X_0 \overset{\Omega \sigma_0^X}{\longrightarrow} \Omega X_1 \;.$$

To identify this as a map $S^1 \to X_1$ we use the adjunction isomorphism once more to throw all the Ω-s on the right back to Σ-s the left, to finally find that this is indeed

$$\sigma_0^X \circ \Sigma f \;\colon\; S^1 = \Sigma S^0 \overset{\Sigma f}{\longrightarrow} \Sigma X_0 \overset{\sigma_0^X}{\longrightarrow} X_1 \;.$$

▮

Lemma 2.33. *The maps $\lambda_n \;\colon\; F_{n+1} S^1 \longrightarrow F_n S^0$ in def. 2.31 are*

1. *stable weak homotopy equivalences for sequential spectra, orthogonal spectra and pre-excisive functors, i.e. for $\mathrm{Dia} \in \{\mathrm{Top}^{\ast/}, \mathrm{Orth}, \mathrm{Seq}\}$;*

2. ***not** stable weak homotopy equivalences for the case of symmetric spectra $\mathrm{Dia} = \mathrm{Sym}$.*

(Hovey-Shipley-Smith 00, example 3.1.10, MMSS 00, lemma 8.6, Schwede 12, example 4.26)

Proof. This follows by inspection of the explicit form of the maps, via prop. 2.27. We discuss each case separately:

sequential case

Here the components of the morphism eventually stabilize to isomorphisms

$$
\begin{array}{cccc}
 & \vdots & & \vdots \\
(\lambda_n)_{n+3} & S^3 & \overset{\mathrm{id}}{\longrightarrow} & S^3 \\
(\lambda_n)_{n+2} & S^2 & \overset{\mathrm{id}}{\longrightarrow} & S^2 \\
(\lambda_n)_{n+1} & S^1 & \overset{\mathrm{id}}{\longrightarrow} & S^1 \\
(\lambda_n)_n : & * & \overset{0}{\longrightarrow} & S^0 \\
 & * & \longrightarrow & * \\
 & \vdots & & \vdots \\
 & * & \longrightarrow & * \\
 & \underbrace{} & & \underbrace{} \\
\lambda_n : & F_{n+1}S^1 & \longrightarrow & F_n S^0
\end{array}
$$

and this immediately gives that λ_n is an isomorphism on stable homotopy groups.

orthogonal case

Here for $q \geq n+1$ the q-component of λ_n is the quotient map

$$(\lambda_n)_q \;:\; O(q)_+ \wedge_{O(q-n-1)} S^{q-n} \simeq O(q)_+ \wedge_{O(q-n-1)} S^1 \wedge S^{q-n-1} \longrightarrow O(q)_+ \wedge_{O(q-n)} S^{q-n} \,.$$

By the suspension isomorphism for stable homotopy groups, λ_n is a stable weak homotopy equivalence precisely if any of its suspensions is. Hence consider instead $\Sigma^n \lambda_n \coloneqq S^n \wedge \lambda_n$, whose q-component is

$$(\Sigma^n \lambda_n)_q \;:\; O(q)_+ \wedge_{O(q-n-1)} S^q \longrightarrow O(q)_+ \wedge_{O(q-n)} S^q \,.$$

Now due to the fact that $O(q-k)$-action on S^q lifts to an $O(q)$-action, the quotients of the diagonal action of $O(q-k)$ equivalently become quotients of just the left action. Formally this is due to the existence of the commuting diagram

$$
\begin{array}{ccccc}
O(q)_+ \wedge S^q & \overset{\mathrm{id}}{\longrightarrow} & O(q)_+ \wedge S^q & \overset{\mathrm{id}}{\longrightarrow} & O(q)_+ \wedge S^q \\
\downarrow & & \downarrow & & \downarrow^{p_2} \\
Q(q)_+ \wedge_{Q(q-k)} S^q & \longrightarrow & Q(q)_+ \wedge_{Q(q)} S^q & \overset{\simeq}{\longrightarrow} & S^q
\end{array}
$$

which says that the image of any $(g,s) \in O(q)_+ \wedge S^q$ in the quotient $Q(q)_+ \wedge_{Q(q-k)} S^q$ is labeled by $([g], s)$.

It follows that $(\Sigma^n \lambda_n)_q$ is the smash product of a projection map of coset spaces with the identity on the sphere:

$$(\Sigma^n \lambda_n)_q \simeq \mathrm{proj}_+ \wedge \mathrm{id}_{S^q} \;:\; O(q)/O(q-n-1)_+ \wedge S^q \longrightarrow O(q)/O(q-n)_+ \wedge S^q \,.$$

Now finally observe that this projection function

$$\mathrm{proj} \;:\; O(q)/O(q-n-1) \longrightarrow O(q)/O(q-n)$$

is $(q-n-1)$-connected (see here). Hence its smash product with S^q is $(2q-n-1)$-connected.

The key here is the fast growth of the connectivity with q. This implies that for each s there exists q such that $\pi_{s+q}((\Sigma^n \lambda_n)_q)$ becomes an isomorphism. Hence $\Sigma^n \lambda_n$ is a stable weak homotopy equivalence and therefore so is λ_n.

symmetric case

Here the morphism λ_n has the same form as in the orthogonal case above, except that all occurences of orthogonal groups are replaced by just their sub-symmetric groups.

Accordingly, the analysis then proceeds entirely analogously, with the key difference that the projection

$$\Sigma(q)/\Sigma(q-n-1) \longrightarrow \Sigma(q)/\Sigma(q-n)$$

does *not* become highly connected as q increases, due to the discrete topological space underlying the symmetric group. Accordingly the conclusion now is the opposite: λ_n is not a stable weak homotopy equivalence in this case. ▮

Another use of free spectra is that their pushout products may be explicitly analyzed, and checking the pushout-product axiom for general cofibrations may be reduced to checking it on morphisms between free spectra.

Lemma 2.34. *The symmetric monoidal smash product of spectra of the free spectrum constructions (def.*

2.25) on the generating cofibrations $\{S^{n-1} \overset{i_n}{\hookrightarrow} D^n\}_{n \in \mathbb{B}}$ of the classical model structure on topological spaces is given by addition of indices

$$(F_k i_{n_1}) \square_{\mathbb{S}_{\mathrm{dia}}} (F_\ell i_{n_2}) \simeq F_{k+\ell}(i_{n_1+n_2}) \,.$$

Proof. By lemma 2.29 the commuting diagram defining the pushout product of free spectra

$$\begin{array}{ccccc} & & F_k S^{n_1-1}_+ \wedge_{\mathbb{S}_{\mathrm{dia}}} F_\ell S^{n_2-1}_+ & & \\ & \swarrow & & \searrow & \\ F_k D^{n_1}_+ \wedge_{\mathbb{S}_{\mathrm{dia}}} F_\ell S^{n_2-1}_+ & & & & F_k S^{n_1-1}_+ \wedge_{\mathbb{S}_{\mathrm{dia}}} F_\ell D^{n_2-1}_+ \\ & \searrow & & \swarrow & \\ & & F_k D^{n_1-1}_+ \wedge_{\mathbb{S}_{\mathrm{dia}}} F_k D^{n_2-1}_+ & & \end{array}$$

is equivalent to this diagram:

$$\begin{array}{ccccc} & & F_{k+\ell}((S^{n_1-1} \times S^{n_2-1})_+) & & \\ & \swarrow & & \searrow & \\ F_{k+\ell}((D^{n_1} \times S^{n_2-1})_+) & & & & F_{k+\ell}((S^{n_1-1} \times D^{n_2})_+) \\ & \searrow & & \swarrow & \\ & & F_{k+\ell}((D^{n_1} \times D^{n_2})_+) & & \end{array} \,.$$

Since the free spectrum construction is a left adjoint, it preserves pushouts, and so

$$(F_k i_{n_1}) \square_{\mathbb{S}_{\mathrm{dia}}} (F_\ell i_{n_2}) \simeq F_{k+\ell}(i_{n_1} \square i_{n_2}) \simeq F_{k+\ell}(i_{n_1+n_2}) \,,$$

where in the second step we used this lemma. ▮

3. The strict model structure on structured spectra

Theorem 3.1. *The four categories of*

1. *pre-excisive functors* $\mathrm{Exc}(\mathrm{Top}_{\mathrm{cg}})$*;*
2. *orthogonal spectra* $\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}}) = \mathbb{S}_{\mathrm{orth}}\,\mathrm{Mod}$*;*
3. *symmetric spectra* $\mathrm{SymSpec}(\mathrm{Top}_{\mathrm{cg}}) = \mathbb{S}_{\mathrm{sym}}\,\mathrm{Mod}$*;*
4. *sequential spectra* $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}}) = \mathbb{S}_{\mathrm{seq}}\,\mathrm{Mod}$

(from def. 2.1, prop. 2.8, def. 2.9) each admit a model category structure (def.) whose weak equivalences and fibrations are those morphisms which induce on all component spaces weak equivalences or fibrations, respectively, in the classical model structure on pointed topological spaces $(\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}$*. (thm., prop.). These are called the* ***strict model structures*** *(or* ***level model structures****) on structured spectra.*

Moreover, under the equivalences of categories of prop. 2.8 and prop. 2.12, the restriction functors in def. 2.4 constitute right adjoints of Quillen adjunctions (def.) between these model structures:

$$\begin{array}{ccccccc} \mathrm{Exc}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}} & & \mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}} & & \mathrm{SymSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}} & & \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}} \\ \downarrow^{\simeq} & & \downarrow^{\simeq} & & \downarrow^{\simeq} & & \downarrow^{\simeq} \\ \mathbb{S}\,\mathrm{Mod}_{\mathrm{strict}} & \underset{\mathrm{orth}^*}{\overset{\mathrm{orth}_!}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} & \mathbb{S}_{\mathrm{Orth}}\,\mathrm{Mod}_{\mathrm{strict}} & \underset{\mathrm{sym}^*}{\overset{\mathrm{sym}_!}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} & \mathbb{S}_{\mathrm{Sym}}\,\mathrm{Mod}_{\mathrm{strict}} & \underset{\mathrm{seq}^*}{\overset{\mathrm{seq}_!}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} & \mathbb{S}_{\mathrm{Seq}}\,\mathrm{Mod}_{\mathrm{strict}} \end{array} \,.$$

(MMSS 00, theorem 6.5)

Proof. By prop. 2.19 all four categories are equivalently categories of pointed topologically enriched functors

$$\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod} \simeq [\mathbb{S}_{\mathrm{dia}}\,\mathrm{Free}_{\mathrm{dia}}\mathrm{Mod}, \mathrm{Top}_{\mathrm{cg}}^{*/}]$$

and hence the existence of the model structures with componentwise weak equivalences and fibrations is a special case of the general existence of the projective model structure on enriched functors (thm.).

The three restriction functors dia^* each have a left adjoint $\mathrm{dia}_!$ by topological left Kan extension (prop. 1.38).

Moreover, the three right adjoint restriction functors are along inclusions of objects, hence evidently preserve componentwise weak equivalences and fibrations. Hence these are Quillen adjunctions. ▮

Definition 3.2. Recall the sets

$$I_{\mathrm{Top}^{*/}} \coloneqq \{S^{n-1}_+ \overset{(i_n)_+}{\hookrightarrow} D^n_+\}_{n \in \mathbb{N}}$$
$$J_{\mathrm{Top}^{*/}} \coloneqq \{D^n_+ \overset{(j_n)_+}{\hookrightarrow} (D^n \times I)_+\}_{n \in \mathbb{N}}$$

of generating cofibrations and generating acyclic cofibrations, respectively, of the classical model structure on pointed topological spaces (def.)

Write

$$I^{\mathrm{strict}}_{\mathrm{dia}} \;\coloneqq\; \left\{ F^{\mathrm{dia}}_c((i_n)_+) \right\}_{c \in \mathrm{Dia}, n \in \mathbb{N}}$$

for the set of images under forming free spectra, def. 2.25, on the morphisms in $I_{\mathrm{Top}^{*/}}$ from above. Similarly, write

$$J^{\mathrm{strict}}_{\mathrm{dia}} \;\coloneqq\; \left\{ F^{\mathrm{dia}}_c((j_n)_+) \right\},$$

for the set of images under forming free spectra of the morphisms in $J_{\mathrm{Top}^{*/}_{\mathrm{cg}}}$.

Proposition 3.3. *The sets $I^{\mathrm{strict}}_{\mathrm{dia}}$ and $J^{\mathrm{strict}}_{\mathrm{dia}}$ from def. 3.2 are, respectively sets of generating cofibrations and generating acyclic cofibrations that exhibit the strict model structure $\mathbb{S}_{\mathrm{Dia}}\,\mathrm{Mod}_{\mathrm{strict}}$ from theorem 3.1 as a cofibrantly generated model category (def.).*

(MMSS 00, theorem 6.5)

Proof. By theorem 3.1 the strict model structure is equivalently the projective pointed model structure on topologically enriched functors

$$\mathbb{S}_{\mathrm{Dia}}\,\mathrm{Mod}_{\mathrm{strict}} \simeq [\mathbb{S}_{\mathrm{Dia}}\,\mathrm{Free}_{\mathrm{Dia}}\mathrm{Mod}^{\mathrm{op}}, \mathrm{Top}^{*/}]_{\mathrm{proj}}$$

of the opposite of the category of free spectra on objects in $\mathcal{C} \hookrightarrow [\mathcal{C}, \mathrm{Top}^{*/}_{\mathrm{cg}}]$.

By the general discussion in *Part P -- Classical homotopy theory* (this theorem) the projective model structure on functors is cofibrantly generated by the smash tensoring of the representable functors with the elements in $I_{\mathrm{Top}^{*/}_{\mathrm{cg}}}$ and $J_{\mathrm{Top}^{*/}_{\mathrm{cg}}}$. By the proof of lemma 2.26, these are precisely the morphisms of free spectra in $I^{\mathrm{strict}}_{\mathrm{dia}}$ and $J^{\mathrm{strict}}_{\mathrm{dia}}$, respectively. ▮

Topological enrichment

By the general properties of the projective model structure on topologically enriched functors, theorem 3.1 implies that the strict model category of structured spectra inherits the structure of an enriched model category, enriched over the classical model structure on pointed topological spaces. This proceeds verbatim as for sequential spectra (in *part 1.1 – Topological enrichement*), but for ease of reference we here make it explicit again.

Definition 3.4. Let $\mathrm{Dia} \in \{\mathrm{Top}^{*/}_{\mathrm{cg,fin}}, \mathrm{Orth}, \mathrm{Sym}, \mathrm{Seq}\}$ one of the shapes for structured spectra from def. 2.4.

Let $f : X \to Y$ be a morphism in $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}$ (as in prop. 3.1) and let $i : A \to B$ a morphism in $\mathrm{Top}^{*/}_{\mathrm{cg}}$.

Their **pushout product with respect to smash tensoring** is the universal morphism

$$f \Box i \coloneqq ((\mathrm{id}, i), (f, \mathrm{id}))$$

in

$$\begin{array}{ccccc} && X \wedge A && \\ & {}^{(f,\mathrm{id})}\swarrow && \searrow^{(\mathrm{id},i)} & \\ Y \wedge A && (\mathrm{po}) && X \wedge B \\ & \searrow && \swarrow & \\ && (Y \wedge A) \underset{X \wedge A}{\sqcup} (X \wedge B) && \\ && \downarrow^{((\mathrm{id},i),(f,\mathrm{id}))} && \\ && Y \wedge B && \end{array},$$

where

$$(-)\wedge(-) : \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod} \times \mathrm{Top}^{*/}_{\mathrm{cg}} \simeq [\mathbb{S}_{\mathrm{dia}}\,\mathrm{Fre}_{\mathrm{dia}}\mathrm{Mod}^{\mathrm{op}},\ \mathrm{Top}^{*/}_{\mathrm{cg}}] \times \mathrm{Top}^{*/}_{\mathrm{cg}} \longrightarrow [\mathbb{S}_{\mathrm{dia}}\,\mathrm{Fre}_{\mathrm{dia}}\mathrm{Mod}^{\mathrm{op}},\ \mathrm{Top}^{*/}_{\mathrm{cg}}] \simeq \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}$$

denotes the smash tensoring of pointed topologically enriched functors with pointed topological spaces

(def.)

Dually, their **pullback powering** is the universal morphism

$$f^{\square i} \coloneqq (\mathrm{Maps}(B,f)_*, \mathrm{Maps}(i,X)_*)$$

in

$$\begin{array}{ccccc} && \mathrm{Maps}(B,X)_* && \\ && \downarrow^{(\mathrm{Maps}(B,f)_*, \mathrm{Maps}(i,X)_*)} && \\ && \mathrm{Maps}(B,Y)_* \underset{\mathrm{Maps}(A,Y)_*}{\times} \mathrm{Maps}(A,X)_* && \\ & \swarrow && \searrow & , \\ \mathrm{Maps}(B,Y)_* && (pb) && \mathrm{Maps}(A,X)_* \\ & {}_{\mathrm{Maps}(i,Y)_*}\searrow && \swarrow_{\mathrm{Maps}(A,p)_*} & \\ && \mathrm{Maps}(A,Y)_* && \end{array}$$

where

$$\mathrm{Maps}(-,-)_* \;:\; (\mathrm{Top}^*_{\mathrm{cg}})^{\mathrm{op}} \times \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod} \simeq (\mathrm{Top}^{*/}_{\mathrm{cg}})^{\mathrm{op}} \times [\mathbb{S}_{\mathrm{dia}}\,\mathrm{Free}_{\mathrm{Dia}}\mathrm{Mod}^{\mathrm{op}}, \mathrm{Top}^{*/}_{\mathrm{cg}}] \longrightarrow [\mathbb{S}_{\mathrm{dia}}\,\mathrm{Free}_{\mathrm{Dia}}\mathrm{Mod}^{\mathrm{op}}, \mathrm{Top}^{*/}_{\mathrm{cg}}] \simeq \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}$$

denotes the smash powering (def.).

Finally, for $f: X \to Y$ and $i: A \to B$ both morphisms in $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}$, then their pullback powering is the universal morphism

$$f^{\square i} \coloneqq (\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(B,f), \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(i,X))$$

in

$$\begin{array}{ccccc} && \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(B,X) && \\ && \downarrow^{(\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(B,f), \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(i,X))} && \\ && \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(B,Y) \underset{\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(A,Y)}{\times} \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(A,X) && \\ & \swarrow && \searrow & , \\ \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(B,Y) && (pb) && \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(A,X) \\ & {}_{\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(i,Y)}\searrow && \swarrow_{\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(A,p)} & \\ && \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(A,Y) && \end{array}$$

where now $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(-,-)$ is the hom-space functor of $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod} \simeq [\mathbb{S}_{\mathrm{dia}}\,\mathrm{Free}_{\mathrm{Dia}}\mathrm{Mod}^{\mathrm{op}}, \mathrm{Top}^{*/}_{\mathrm{cg}}]$ from def. 1.31.

Proposition 3.5. *The operations of forming pushout products and pullback powering with respect to smash tensoring in def. 3.4 is compatible with the strict model structure $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}_{\mathrm{strict}}$ on structured spectra from theorem 3.1 and with the classical model structure on pointed topological spaces $(\mathrm{Top}^{*/}_{\mathrm{cg}})_{\mathrm{Quillen}}$ (thm., prop.) in that pushout product takes two cofibrations to a cofibration, and to an acyclic cofibration if at least one of the inputs is acyclic, and pullback powering takes a fibration and a cofibration to a fibration, and to an acylic one if at least one of the inputs is acyclic:*

$$\begin{aligned} \mathrm{Cof}_{\mathrm{strict}} \square \mathrm{Cof}_{\mathrm{cl}} &\subset \mathrm{Cof}_{\mathrm{strict}} \\ \mathrm{Cof}_{\mathrm{strict}} \square (\mathrm{Cof}_{\mathrm{cl}} \cap W_{\mathrm{cl}}) &\subset \mathrm{Cof}_{\mathrm{strict}} \cap W_{\mathrm{strict}} \\ (\mathrm{Cof}_{\mathrm{strict}} \cap W_{\mathrm{strict}}) \square \mathrm{Cof}_{\mathrm{cl}} &\subset \mathrm{Cof}_{\mathrm{strict}} \cap W_{\mathrm{strict}} \end{aligned} .$$

Dually, the pullback powering (def. 3.4) satisfies

$$\begin{aligned} \mathrm{Fib}_{\mathrm{strict}}^{\square \mathrm{Cof}_{\mathrm{cl}}} &\subset \mathrm{Fib}_{\mathrm{strict}} \\ \mathrm{Fib}_{\mathrm{strict}}^{\square (\mathrm{Cof}_{\mathrm{cl}} \cap W_{\mathrm{cl}})} &\subset \mathrm{Fib}_{\mathrm{strict}} \cap W_{\mathrm{strict}} \\ (\mathrm{Fib}_{\mathrm{strict}} \cap W_{\mathrm{strict}})^{\square \mathrm{Cof}_{\mathrm{cl}}} &\subset \mathrm{Fib}_{\mathrm{strict}} \cap W_{\mathrm{strict}} \end{aligned} .$$

Proof. The statement concering the pullback powering follows directly from the analogous statement for topological spaces (prop.) by the fact that, via theorem 3.1, the fibrations and weak equivalences in $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}_{\mathrm{strict}}$ are degree-wise those in $(\mathrm{Top}^{*/}_{\mathrm{cg}})_{\mathrm{Quillen}}$, and since smash tensoring and powering is defined degreewise. From this the statement about the pushout product follows dually by Joyal-Tierney calculus (prop.). ▮

Remark 3.6. In the language of model category-theory, prop. 3.5 says that $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}_{\mathrm{strict}}$ is an *enriched model category*, the enrichment being over $(\mathrm{Top}^{*/}_{\mathrm{cg}})_{\mathrm{Quillen}}$. This is often referred to simply as a "topological model category".

We record some immediate consequences of prop. 3.5 that will be useful.

Proposition 3.7. *Let $K \in \mathrm{Top}^*_{\mathrm{cg}}$ be a retract of a cell complex (def.), then the smash-tensoring/powering adjunction from prop. 1.37 is a Quillen adjunction (def.) for the strict model structure from theorem 3.1*

$$\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}} \underoverset{\underset{\mathrm{Maps}(K,-)_*}{\longrightarrow}}{\overset{(-)\wedge K}{\longleftarrow}}{\bot} \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}} \,.$$

Proof. By assumption, K is a cofibrant object in the classical model structure on pointed topological spaces (thm., prop.), hence $* \to K$ is a cofibration in $(\mathrm{Top}^{*/}_{\mathrm{cg}})_{\mathrm{Quillen}}$. Observe then that the the pushout product of any morphism f with $* \to K$ is equivalently the smash tensoring of f with K:

$$f \,\Box\, (* \to K) \simeq f \wedge K \,.$$

This way prop. 3.5 implies that $(-)\wedge K$ preserves cofibrations and acyclic cofibrations, hence is a left Quillen functor. ∎

Lemma 3.8. *Let $X \in \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}_{\mathrm{strict}}$ be a structured spectrum, regarded in the strict model structure of theorem 3.1.*

1. *The smash powering of X with the standard topological interval I_+ (exmpl.) is a good path space object (def.)*

$$\Delta_X \,:\, X \overset{\in W_{\mathrm{strict}}}{\longrightarrow} X^{I_+} \overset{\in \mathrm{Fib}_{\mathrm{strict}}}{\longrightarrow} X \times X \,.$$

2. *If X is cofibrant, then its smash tensoring with the standard topological interval I_+ (exmpl.) is a good cylinder object (def.)*

$$\nabla_X \,:\, X \vee X \overset{\in \mathrm{Cof}_{\mathrm{strict}}}{\longrightarrow} X \wedge (I_+) \overset{\in W_{\mathrm{strict}}}{\longrightarrow} X \,.$$

Proof. It is clear that we have weak equivalences as shown ($I \to *$ is even a homotopy equivalence), what requires proof is that the path object is indeed good in that $X^{(I_+)} \to X \times X$ is a fibration, and the cylinder object is indeed good in that $X \vee X \to X \wedge (I_+)$ is indeed a cofibration.

For the first statement, notice that the pullback powering (def. 3.4) of $* \sqcup * \overset{(i_0,i_1)}{\longrightarrow} I$ into the terminal morphism $X \to *$ is the same as the powering $X^{(i_0,i_1)}$:

$$((X \to *)^{\Box (i_0,i_1)}) \;\simeq\; X^{(i_0,i_1)}, .$$

But since every object in $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}_{\mathrm{strict}}$ is fibrant, so that $X \to *$ is a fibration, and since (i_0,i_1) is a relative cell complex inclusion and hence a cofibration in $(\mathrm{Top}^{*/}_{\mathrm{cg}})_{\mathrm{Quilln}}$, prop. 3.5 says that $X^{(i_0,i_1)} : X^{I_+} \to X \times X$ is a fibration.

Dually, observe that

$$(* \to X) \,\Box\, (i_0,i_1) \;\simeq\; X \wedge (i_0,i_1) \,.$$

Hence if X is assumed to be cofibrant, so that $* \to X$ is a cofibration, then prop. 3.5 implies that $X \wedge (i_0,i_1) : X \wedge X \to X \wedge (I_+)$ is a cofibration. ∎

Proposition 3.9. *For $X \in \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}$ a structured spectrum, $f \in \mathrm{Mor}(\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod})$ any morphism of structured spectra, and for $g \in \mathrm{Mor}(\mathrm{Top}^{*/}_{\mathrm{cpt}})$ a morphism of pointed topological spaces, then the hom-spaces of def. 1.31 (via prop. 2.19) interact with the pushout-product and pullback-powering from def. 3.4 in that there is a natural isomorphism*

$$\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(f \,\Box\, g, X) \simeq (\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(f,X))^{\Box g} \,.$$

Proof. Since the pointed compactly generated mapping space functor (exmpl.)

$$\mathrm{Maps}(-,-)_* \,:\, \left(\mathrm{Top}^{*/}_{\mathrm{cg}}\right)^{\mathrm{op}} \times \mathrm{Top}^{*/}_{\mathrm{cg}} \longrightarrow \mathrm{Top}^{*/}_{\mathrm{cg}}$$

takes colimits in the first argument to limits (cor.) and ends in the second argument to ends (remark 1.36), and since limits and colimits in $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}$ are computed objectswise (this prop. via prop. 2.19) this follows with the end-formula for the mapping space (def. 1.31):

$$
\begin{aligned}
\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(f \,\square\, g, X) &= \int_c \mathrm{Maps}((f \,\square\, g)(c), X(c))_* \\
&\simeq \int_c \mathrm{Maps}(f(c) \,\square\, g, X(c))_* \\
&\simeq \int_c \mathrm{Maps}(f(c), X(c))_*^{\square g} \\
&\simeq \left(\int_c \mathrm{Maps}(f(c), X(c))_*\right)^{\square g} \\
&\simeq (\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(f, X))^{\square g}
\end{aligned}
.
$$

▮

Proposition 3.10. *For $X, Y \in \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(\mathrm{Top}_{\mathrm{cg}})$ two structured spectra with X cofibrant in the strict model structure of def. 3.1, then there is a natural bijection*

$$
\pi_0 \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(X, Y) \simeq [X, Y]_{\mathrm{strict}}
$$

between the connected components of the hom-space (def. 1.31 via prop. 2.19) and the hom-set in the homotopy category (def.) of the strict model structure from theorem 3.1.

Proof. By prop. 1.37 the path components of the hom-space are the left homotopy classes of morphisms of structured spectra with respect to the standard cylinder spectrum $X \wedge (I_+)$:

$$
\frac{I_+ \longrightarrow \mathrm{SeqSpec}(X, Y)}{X \wedge (I_+) \longrightarrow Y} .
$$

Moreover, by lemma 3.8 the degreewise standard reduced cylinder $X \wedge (I_+)$ of structured spectra is a good cylinder object on X in $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}_{\mathrm{strict}}$. Hence hom-sets in the strict homotopy category out of a cofibrant into a fibrant object are given by standard left homotopy classes of morphisms

$$
[X, Y]_{\mathrm{strict}} \simeq \mathrm{Hom}_{\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}}(X, Y)_{/\sim}
$$

(this lemma). Since X is cofibrant by assumption and since every object is fibrant in $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}_{\mathrm{strict}}$, this is the case. Hence the notion of left homotopy here is that seen by the standard interval, and so the claim follows. ▮

Monoidal model structure

We now combine the concepts of model category (def.) and monoidal category (def. 1.1).

Given a category $\mathcal{C}$ that is equipped both with the structure of a monoidal category and of a model category, then one may ask whether these two structures are compatible, in that the left derived functor (def.) of the tensor product exists to equip also the homotopy category with the structure of a monoidal category. If so, then one may furthermore ask if the localization functor $\gamma : \mathcal{C} \longrightarrow \mathrm{Ho}(\mathcal{C})$ is a monoidal functor (def. 1.47).

The axioms on a *monoidal model category* (def. 3.11 below) are such as to ensure that this is the case.

A key consequence is that, via prop. 1.50, for a monoidal model category the localization functor γ carries monoids to monoids. Applied to the stable model category of spectra established below, this gives that structured ring spectra indeed represent ring spectra in the homotopy category. (In fact much more is true, but requires further proof: there is also a model structure on monoids in the model structure of spectra, and with respect to that the structured ring spectra represent A-infinity rings/E-infinity rings.)

Definition 3.11. A (symmetric) **monoidal model category** is a model category $\mathcal{C}$ (def.) equipped with the structure of a closed (def. 1.7) symmetric (def. 1.5) monoidal category $(\mathcal{C}, \otimes, I)$ (def. 1.1) such that the following two compatibility conditions are satisfied

1. **(pushout-product axiom)** For every pair of cofibrations $f : X \to Y$ and $f' : X' \to Y'$, their pushout-product, hence the induced morphism out of the cofibered coproduct over ways of forming the tensor product of these objects

   $$
   f \,\square_{\otimes}\, g \;\coloneqq\; (X \otimes Y') \underset{X \otimes X'}{\sqcup} (Y \otimes X') \longrightarrow Y \otimes Y' ,
   $$

   is itself a cofibration, which, furthermore, is acyclic if at least one of f or f' is.

   (Equivalently this says that the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a *left Quillen bifunctor*.)

2. **(unit axiom)** For every cofibrant object X and every cofibrant resolution $\emptyset \overset{\in \mathrm{Cof}}{\longrightarrow} Q1 \underset{p_1}{\overset{\in W}{\longrightarrow}} 1$ of the tensor unit 1, the resulting morphism

   $$
   Q1 \otimes X \overset{p_1 \otimes X}{\longrightarrow} 1 \otimes X \underset{\ell}{\overset{\in \mathrm{Iso} \subset W}{\longrightarrow}} X
   $$

is a weak equivalence.

(Hovey 99, def. 4.2.6 Schwede-Shipley 00, def. 3.1, remark 3.2)

Observe some immediate consequences of these axioms:

Remark 3.12. Since a monoidal model category (def. 3.11) is assumed to be closed monoidal (def. 1.7), for every object X the tensor product $X \otimes (-) \simeq (-) \otimes X$ is a left adjoint and hence preserves all colimits. In particular it preserves the initial object $\emptyset$ (which is the colimit over the empty diagram).

If follows that the tensor-pushout-product axiom in def. 3.11 implies that for X a cofibrant object, then the functor $X \otimes (-)$ preserves cofibrations and acyclic cofibrations, since

$$f \,\square_{\otimes}\, (\emptyset \to X) \simeq f \otimes X \,.$$

This implies that if the tensor unit 1 happens to be cofibrant, then the unit axiom in def. 3.11 is already implied by the pushout-product axiom. This is because then we have a lift in

$$\begin{array}{ccc} \emptyset & \longrightarrow & Q1 \\ {}^{\in \mathrm{Cof}}\downarrow & \nearrow & \downarrow^{p_1}_{\in W} \\ 1 & = & 1 \end{array}$$

This lift is a weak equivalence by two-out-of-three (def.). Since it is hence a weak equivalence between cofibrant objects, it is preserved by the left Quillen functor $(-) \otimes X$ (for any cofibrant X) by Ken Brown's lemma (prop.). Hence now $p_1 \otimes X$ is a weak equivalence by two-out-of-three.

Since for all the categories of spectra that we are interested in here the tensor unit is always cofibrant (it is always a version of the sphere spectrum, being the image under the left Quillen functor $\Sigma^\infty_{\mathrm{dia}}$ of the cofibrant pointed space S^0, prop. 3.18), we may ignore the unit axiom.

Proposition 3.13. *Let $(\mathcal{C}, \otimes, I)$ be a monoidal model category (def. 3.11) with cofibrant tensor unit 1.*

Then the left derived functor $\otimes^L$ (def.) of the tensor product $\otimes$ exsists and makes the homotopy category (def.) into a monoidal category $(\mathrm{Ho}(\mathcal{C}), \otimes^L, \gamma(1))$ (def. 1.1) such that the localization functor $\gamma : \mathcal{C}_c \to \mathrm{Ho}(\mathcal{C})$ (thm.) on the category of cofibrant objects (def.) carries the structure of a strong monoidal functor (def. 1.47)

$$\gamma : (\mathcal{C}, \otimes, 1) \longrightarrow (\mathrm{Ho}(\mathcal{C}), \otimes^L, \gamma(1)) \,.$$

The first statement is also for instance in (Hovey 99, theorem 4.3.2).

Proof. For the left derived functor (def.) of the tensor product

$$\otimes \;\; \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$$

to exist, it is sufficient that its restriction to the subcategory

$$(\mathcal{C} \times \mathcal{C})_c \simeq \mathcal{C}_c \times \mathcal{C}_c$$

of cofibrant objects preserves acyclic cofibrations (by Ken Brown's lemma, here).

Every morphism (f,g) in the product category $\mathcal{C}_c \times \mathcal{C}_c$ (def. 1.26) may be written as a composite of a pairing with an identity morphisms

$$(f,g) \,:\, (c_1, d_1) \overset{(\mathrm{id}_{c_1}, g)}{\longrightarrow} (c_1, d_2) \overset{(f, \mathrm{id}_{c_2})}{\longrightarrow} (c_2, d_2) \,.$$

Now since the pushout product (with respect to tensor product) with the initial morphism $(\ast \to c_1)$ is equivalently the tensor product

$$(\ast \to c_1) \,\square_{\otimes}\, g \;\simeq\; \mathrm{id}_{c_1} \otimes g$$

and

$$f \,\square_{\otimes}\, (\ast \to c_2) \;\simeq\; f \otimes \mathrm{id}_{c_2}$$

the pushout-product axiom (def. 3.11) implies that on the subcategory of cofibrant objects the functor $\otimes$ preserves acyclic cofibrations. (This is why one speaks of a *Quillen bifunctor*, see also Hovey 99, prop. 4.3.1).

Hence $\otimes^L$ exists.

By the same decomposition and using the universal property of the localization of a category (def.) one finds that for $\mathcal{C}$ and $\mathcal{D}$ any two categories with weak equivalences (def.) then the localization of their product category is the product category of their localizations:

$$(\mathcal{C}\times\mathcal{D})[(W_{\mathcal{C}}\times W_{\mathcal{D}})^{-1}] \simeq (\mathcal{C}[W_{\mathcal{C}}^{-1}])\times(\mathcal{D}[W_{\mathcal{D}}^{-1}])\,.$$

With this, the universal property as a localization (def.) of the homotopy category of a model category (thm.) induces associators α^L and unitors ℓ^L, r^L on $(\mathrm{Ho}(\mathcal{C},\otimes^L))$:

First write

$$\mu : \gamma(-)\otimes^L\gamma(-) \overset{\simeq}{\longrightarrow} \gamma((-)\otimes(-))$$

for (the inverse of) the corresponding natural isomorphism in the localization diagram

$$\begin{tikzcd} \mathcal{C}\times\mathcal{C} \arrow[r, "\otimes"] \arrow[d, "\gamma\times\gamma"'] \arrow[dr, phantom, "\swArrow^{\mu^{-1}}"] & \mathcal{C} \arrow[d, "\gamma"] \\ \mathrm{Ho}(\mathcal{C})\times\mathrm{Ho}(\mathcal{C}) \arrow[r, "\otimes^L"'] & \mathrm{Ho}(\mathcal{C}) \end{tikzcd}\,.$$

Then consider the associators:

The essential uniqueness of derived functors shows that the left derived functor of $(-)\otimes((-)\otimes(-))$ and of $((-)\otimes(-))\otimes(-)$ is the composite of two applications of $\otimes^L$, due to the factorization

$$\begin{tikzcd} \mathcal{C}_c\times\mathcal{C}_c\times\mathcal{C}_c \arrow[r, "(-)\otimes((-)\otimes(-))"] \arrow[d, "\gamma\times\gamma\times\gamma"'] \arrow[dr, phantom, "\swArrow"] & \mathcal{C}_c \arrow[d, "\gamma"] \\ \mathrm{Ho}(\mathcal{C})\times\mathrm{Ho}(\mathcal{C})\times\mathrm{Ho}(\mathcal{C}) \arrow[r, "\mathbb{L}((-)\otimes((-)\otimes(-)))"'] & \mathrm{Ho}(\mathcal{C}) \end{tikzcd}$$

$$\simeq$$

$$\begin{tikzcd} \mathcal{C}_c\times\mathcal{C}_c\times\mathcal{C}_c \arrow[r, "\mathrm{id}\times\otimes"] \arrow[d, "\gamma\times\gamma\times\gamma"'] \arrow[dr, phantom, "\swArrow_{\mathrm{id}\times\mu^{-1}}"] & \mathcal{C}_c\times\mathcal{C}_c \arrow[r, "\otimes"] \arrow[d, "\gamma\times\gamma"'] \arrow[dr, phantom, "\swArrow_{\mu^{-1}}"] & \mathcal{C}_c \arrow[d, "\gamma"] \\ \mathrm{Ho}(\mathcal{C})\times\mathrm{Ho}(\mathcal{C})\times\mathrm{Ho}(\mathcal{C}) \arrow[r, "\mathrm{id}\times\otimes^L"'] & \mathrm{Ho}(\mathcal{C})\times\mathrm{Ho}(\mathcal{C}) \arrow[r, "\otimes^L"'] & \mathrm{Ho}(\mathcal{C}) \end{tikzcd}$$

and similarly for the case with the parenthesis to the left.

So let

$$\begin{tikzcd} \mathcal{C}_c\times\mathcal{C}_c\times\mathcal{C}_c \arrow[r, "((-)\otimes(-))\otimes(-)"] \arrow[d, "\gamma\times\gamma\times\gamma"'] \arrow[dr, phantom, "\swArrow_{\mu^{-1}\cdot(\mu^{-1}\times\mathrm{id})}"] & \mathcal{C} \arrow[d, "\gamma"] \\ \mathrm{Ho}(\mathcal{C})\times\mathrm{Ho}(\mathcal{C})\times\mathrm{Ho}(\mathcal{C}) \arrow[r, "((-)\otimes^L(-))\otimes^L(-)"] & \mathrm{Ho}(\mathcal{C}) \end{tikzcd}\,,\quad \begin{tikzcd} \mathcal{C}_c\times\mathcal{C}_c\times\mathcal{C}_c \arrow[r, "(-)\otimes((-)\otimes(-))"] \arrow[d, "\gamma\times\gamma\times\gamma"'] \arrow[dr, phantom, "\swArrow_{\mu^{-1}\cdot(\mathrm{id}\times\mu^{-1})}"] & \mathcal{C} \arrow[d, "\gamma"] \\ \mathrm{Ho}(\mathcal{C})\times\mathrm{Ho}(\mathcal{C})\times\mathrm{Ho}(\mathcal{C}) \arrow[r, "(-)\otimes^L((-)\otimes^L(-))"] & \mathrm{Ho}(\mathcal{C}) \end{tikzcd}$$

be the natural isomorphism exhibiting the derived functors of the two possible tensor products of three objects, as shown at the top. By pasting the second with the associator natural isomorphism of $\mathcal{C}$ we obtain another such factorization for the first, as shown on the left below,

$$(\star)\qquad \begin{tikzcd} \mathcal{C}_c\times\mathcal{C}_c\times\mathcal{C}_c \arrow[r, "((-)\otimes(-))\otimes(-)"] \arrow[d, "="'] \arrow[dr, phantom, "\swArrow_{\alpha}"] & \mathcal{C} \arrow[d, "="] \\ \mathcal{C}_c\times\mathcal{C}_c\times\mathcal{C}_c \arrow[r, "(-)\otimes((-)\otimes(-))"] \arrow[d, "\gamma\times\gamma\times\gamma"'] \arrow[dr, phantom, "\swArrow_{\mu^{-1}\cdot(\mathrm{id}\times\mu^{-1})}"] & \mathcal{C} \arrow[d, "\gamma"] \\ \mathrm{Ho}(\mathcal{C})\times\mathrm{Ho}(\mathcal{C})\times\mathrm{Ho}(\mathcal{C}) \arrow[r, "(-)\otimes^L((-)\otimes^L(-))"] & \mathrm{Ho}(\mathcal{C}) \end{tikzcd} \quad = \quad \begin{tikzcd} \mathcal{C}_c\times\mathcal{C}_c\times\mathcal{C}_c \arrow[r, "((-)\otimes(-))\otimes(-)"] \arrow[d, "\gamma\times\gamma\times\gamma"'] \arrow[dr, phantom, "\swArrow_{\mu^{-1}\cdot(\mathrm{id}\times\mu)}"] & \mathcal{C} \arrow[d, "\gamma"] \\ \mathrm{Ho}(\mathcal{C})\times\mathrm{Ho}(\mathcal{C})\times\mathrm{Ho}(\mathcal{C}) \arrow[r, "((-)\otimes^L(-))\otimes^L(-)"] \arrow[d, "="'] \arrow[dr, phantom, "\swArrow_{\alpha^L}"] & \mathrm{Ho}(\mathcal{C}) \arrow[d, "="] \\ \mathrm{Ho}(\mathcal{C})\times\mathrm{Ho}(\mathcal{C})\times\mathrm{Ho}(\mathcal{C}) \arrow[r, "(-)\otimes^L((-)\otimes^L(-))"'] & \mathrm{Ho}(\mathcal{C}) \end{tikzcd}$$

and hence by the universal property of the factorization through the derived functor, there exists a unique natural isomorphism α^L such as to make this composite of natural isomorphisms equal to the one shown on the right. Hence the pentagon identity satisfied by α implies a pentagon identity for α^L, and so α^L is an associator for $\otimes^L$.

Moreover, this equation of natural isomorphisms says that on components the following diagram commutes

$$\begin{tikzcd} (\gamma(X)\otimes^L\gamma(Y))\otimes^L\gamma(Z) \arrow[r, "\alpha^L_{\gamma(X),\gamma(Y),\gamma(Z)}"] & \gamma(X)\otimes^L(\gamma(Y)\otimes^L\gamma(Z)) \\ \gamma((X\otimes Y)\otimes Z) \arrow[u, "\mu^{-1}\cdot(\mu^{-1}\times\mathrm{id})"] \arrow[r, "\gamma(\alpha)"'] & \gamma(X\otimes(Y\otimes Z)) \arrow[u, "\mu^{-1}\cdot(\mathrm{id}\times\mu^{-1})"'] \end{tikzcd}\,.$$

This is just the coherence law for the the compatibility of the monoidal functor μ with the associators.

Similarly consider now the unitors.

The essential uniqueness of the derived functors gives that the left derived functor of $1 \otimes (-)$ is $\gamma(1) \otimes^L (-)$

$$
\begin{tikzcd}
\mathcal{C}_c \arrow[r, "1 \otimes (-)"] \arrow[d, "\gamma"'] & \mathcal{C}_c \arrow[d, "\gamma"] \\
\mathrm{Ho}(\mathcal{C}) \arrow[r, "\mathbb{L}(1 \otimes (-))"'] & \mathrm{Ho}(\mathcal{C})
\end{tikzcd}
\simeq
\begin{tikzcd}
\mathcal{C}_c \arrow[r, "(1,\mathrm{id})"] \arrow[d, "\gamma"'] & \mathcal{C}_c \times \mathcal{C}_c \arrow[r, "\otimes"] \arrow[d, "\gamma \times \gamma"'] \arrow[dr, phantom, "\swarrow_{\mu^{-1}}"] & \mathcal{C} \arrow[d, "\gamma"] \\
\mathrm{Ho}(\mathcal{C}) \arrow[r, "(\gamma(1),\mathrm{id})"'] & \mathrm{Ho}(\mathcal{C}) \times \mathrm{Ho}(\mathcal{C}) \arrow[r, "\otimes^L"'] & \mathrm{Ho}(\mathcal{C})
\end{tikzcd}
.
$$

Hence the left unitor ℓ of $\mathcal{C}$ induces a derived unitor ℓ^L by the following factorization

$$
\begin{tikzcd}
\mathcal{C}_c \arrow[r, "1 \otimes (-)"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\swarrow_{\ell}"] & \mathcal{C}_c \arrow[d, "\gamma"] \\
\mathcal{C}_c \arrow[r, "\mathrm{id}"] \arrow[d, "\gamma"'] & \mathcal{C}_c \arrow[d, "\gamma"] \\
\mathrm{Ho}(\mathcal{C}) \arrow[r, "\mathrm{id}"'] & \mathrm{Ho}(\mathcal{C})
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{C}_c \arrow[r, "1 \otimes (-)"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\swarrow_{\mu^{-1}_{1,(-)}}"] & \mathcal{C}_c \arrow[d, "\gamma"] \\
\mathrm{Ho}(\mathcal{C}) \arrow[r, "\gamma(1) \otimes^L (-)"] \arrow[d, "="'] \arrow[dr, phantom, "\swarrow_{\ell^L}"] & \mathrm{Ho}(\mathcal{C}) \arrow[d, "="] \\
\mathrm{Ho}(\mathcal{C}) \arrow[r, "\mathrm{id}"'] & \mathrm{Ho}(\mathcal{C})
\end{tikzcd}
.
$$

Moreover, in components this equation of natural isomorphism expresses the coherence law stating the compatibility of the monoidal functor μ with the unitors.

Similarly for the right unitors. ▮

The restriction to cofibrant objects in prop. 3.13 serves the purpose of giving explicit expressions for the associators and unitors of the derived tensor product $\otimes^L$ and hence to establish the monoidal category structure $(\mathrm{Ho}(\mathcal{C}), \otimes^L, \gamma(1))$ on the homotopy category of a monoidal model category. With that in hand, it is natural to ask how the localization functor on all of $\mathcal{C}$ interacts with the monoidal structure:

Proposition 3.14. *For $(\mathcal{C}, \otimes, 1)$ a monoidal model category (def. 3.11) then the localization functor to its monoidal homotopy category (prop. 3.13) is a lax monoidal functor*

$$
\gamma : (\mathcal{C}, \otimes, 1) \longrightarrow (\mathrm{Ho}(\mathcal{C}), \otimes^L, \gamma(1)) \,.
$$

The explicit **proof** of prop. 3.14 is tedious. An abstract proof using tools from homotopical 2-category theory is here.

Definition 3.15. Given monoidal model categories $(\mathcal{C}, \otimes_{\mathcal{C}}, 1_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, 1_{\mathcal{D}})$ (def. 3.11) with cofibrant tensor units $1_{\mathcal{C}}$ and $1_{\mathcal{D}}$, then a **strong monoidal Quillen adjunction** between them is a Quillen adjunction

$$
\mathcal{C} \underset{R}{\overset{L}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathcal{D}
$$

such that L (hence equivalently R) has the structure of a strong monoidal functor.

Proposition 3.16. *Given a strong monoidal Quillen adjunction (def. 3.15)*

$$
\mathcal{C} \underset{R}{\overset{L}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathcal{D}
$$

between monoidal model categories $(\mathcal{C}, \otimes_{\mathcal{C}}, 1_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, 1_{\mathcal{D}})$ with cofibrant tensor units $1_{\mathcal{C}}$ and $1_{\mathcal{D}}$, then the left derived functor of L canonically becomes a strong monoidal functor between homotopy categories

$$
\mathbb{L}L : (\mathrm{Ho}(\mathcal{C}), \otimes_{\mathcal{C}}, \gamma(1)_{\mathcal{C}}) \longrightarrow (\mathrm{Ho}(\mathcal{D}), \otimes_{\mathcal{D}}, \gamma(1)_{\mathcal{D}}) \,.
$$

Proof. As in the proof of prop. 3.13, consider the following pasting composite of commuting diagams:

$$
\begin{tikzcd}
\mathcal{D}_c \times \mathcal{D}_c \arrow[r, "\otimes_{\mathcal{D}}"] \arrow[d, "="'] & \mathcal{D}_c \arrow[r, "L"] \arrow[d, phantom, "\swarrow_{\simeq}"] & \mathcal{C}_c \arrow[d, "="] \\
\mathcal{D}_c \times \mathcal{D}_c \arrow[r, "L \times L"] \arrow[d, "\gamma_{\mathcal{D}} \times \gamma_{\mathcal{D}}"'] & \mathcal{C}_c \times \mathcal{C}_c \arrow[r, "\otimes_{\mathcal{C}}"] \arrow[d, "\gamma_{\mathcal{C}} \times \gamma_{\mathcal{C}}"] & \mathcal{C}_c \arrow[d, "\gamma_{\mathcal{C}}"] \\
\mathrm{Ho}(\mathcal{D}) \times \mathrm{Ho}(\mathcal{D}) \arrow[r, "\mathbb{L}L \times \mathbb{L}L"'] & \mathrm{Ho}(\mathcal{C}) \times \mathrm{Ho}(\mathcal{C}) \arrow[r, "\otimes^L_{\mathcal{C}}"'] & \mathrm{Ho}(\mathcal{C})
\end{tikzcd}
\simeq
\begin{tikzcd}
\mathcal{D}_c \times \mathcal{D}_c \arrow[r, "\otimes_{\mathcal{D}}"] \arrow[d, "\gamma_{\mathcal{D}} \times \gamma_{\mathcal{D}}"'] & \mathcal{D}_c \arrow[r, "L"] \arrow[d, "\gamma_{\mathcal{D}}"] & \mathcal{C}_c \arrow[d, "\gamma_{\mathcal{C}}"] \\
\mathrm{Ho}(\mathcal{D}) \times \mathrm{Ho}(\mathcal{D}) \arrow[r, "\otimes^L_{\mathcal{D}}"] \arrow[d, "="'] & \mathrm{Ho}(\mathcal{D}) \arrow[r, "\mathbb{L}L"] \arrow[d, phantom, "\swarrow_{\simeq}"] & \mathrm{Ho}(\mathcal{C}) \arrow[d, "="] \\
\mathrm{Ho}(\mathcal{D}) \times \mathrm{Ho}(\mathcal{D}) \arrow[r, "\mathbb{L}L \times \mathbb{L}L"'] & \mathrm{Ho}(\mathcal{C}) \times \mathrm{Ho}(\mathcal{C}) \arrow[r, "\otimes^L_{\mathcal{C}}"'] & \mathrm{Ho}(\mathcal{C})
\end{tikzcd}
.
$$

On the top left we have the natural transformation that exhibits L as a strong monoidal functor. By universality of localization and derived functors (def.) this induces the unique factorization through the natural transformation on the bottom right. This exhibits strong monoidal structure on the left derived functor $\mathbb{L}L$. ▮

With some general monoidal homotopy theory established, we now discuss that structured spectra indeed constitute an example. The version of the following theorem for the stable model structure of actual interest is theorem 4.14 further below.

Theorem 3.17.

1. *The classical model structure on pointed topological spaces equipped with the smash product is a monoidal model category*

$$((\mathrm{Top}^{*/}_{\mathrm{cg}})_{\mathrm{Quillen}}, \wedge, S^0) \,.$$

2. *Let* $\mathrm{Dia} \in \{\mathrm{Top}^{*/}_{\mathrm{cg,fin}}, \mathrm{Orth}, \mathrm{Sym}\}$. *The strict model structures on structured spectra modeled on* Dia *from theorem 3.1 equipped with the symmetric monoidal smash product of spectra (def. 2.1, def. 2.9) is a monoidal model category (def. 3.11)*

$$\left(\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}_{\mathrm{strict}},\ \wedge\ =\ \otimes_{\mathbb{S}_{\mathrm{dia}}},\ \mathbb{S}_{\mathrm{dia}}\right) \,.$$

(MMSS 00, theorem 12.1 (iii) with prop. 12.3)

Proof. By cofibrant generation of both model structures (this theorem and prop. 3.3) it is sufficient to check the pushout-product axiom on generating (acylic) cofibrations (this is as in the proof of this proposition).

Those of $\mathrm{Top}^{*/}_{\mathrm{cg}}$ are as recalled in def. 4.4. These satisfy (exmpl.) the relations

$$i_{k_1} \,\square\, i_{k_2} = i_{k_1+k_2}$$

and

$$i_{k_1} \,\square\, j_{k_2} = j_{k_1+k_2} \,.$$

This shows that

$$I_{\mathrm{Top}^{*/}} \,\square_{\otimes_{\mathbb{S}_{\mathrm{dia}}}}\, I_{\mathrm{Top}^{*/}} \subset I_{\mathrm{Top}^{*/}}$$

and

$$I_{\mathrm{Top}^{*/}} \,\square_{\otimes_{\mathbb{S}_{\mathrm{dia}}}}\, J_{\mathrm{Top}^{*/}} \subset J_{\mathrm{Top}^{*/}}$$

which implies the pushout-product axiom for $\mathrm{Top}^{*/}_{\mathrm{cg}}$. (However the monoid axiom (def.\ref{MonoidAxiom}) is problematic.)

Now by def. 3.2 the generating (acyclic) cofibrations of $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}_{\mathrm{strict}}$ are of the form $F_n^{\mathrm{dia}}(i_k)_+$ and $F_n^{\mathrm{dia}}(j_k)_+$, respectively. By prop. 2.29 these satisfy

$$F_{n_1}(i_{k_1})_+ \,\square_\wedge\, F_{n_2}(i_{k_2})_+ \;\simeq\; F_{n_1+n_2}(i_{k_1} \,\square_\wedge\, i_{k_2})_+$$

and

$$F_{n_1}(i_{k_1})_+ \,\square\, F_{n_2}(j_{k_2})_+ \;\simeq\; F_{n_1+n_2}(i_{k_1} \,\square\, j_{k_2})_+ \,.$$

Hence with the previous set of relations this shows that

$$I^{\mathrm{strict}}_{\mathrm{dia}} \,\square_{\otimes_{\mathbb{S}_{\mathrm{dia}}}}\, I^{\mathrm{strict}}_{\mathrm{dia}} \subset I^{\mathrm{strict}}_{\mathrm{dia}}$$

and

$$I^{\mathrm{strict}}_{\mathrm{dia}} \,\square_{\otimes_{\mathbb{S}_{\mathrm{dia}}}}\, J^{\mathrm{strict}}_{\mathrm{dia}} \subset J^{\mathrm{strict}}_{\mathrm{dia}}$$

and so the pushout-product axiom follows also for $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}_{\mathrm{strict}}$.

It is clear that in both cases the tensor unit is cofibrant: for $\mathrm{Top}^{*/}_{\mathrm{cg}}$ the tensor unit is the 0-sphere, which clearly is a CW-complex and hence cofibrant. For $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}$ the tensor unit is the standard sphere spectrum, which, by prop. 2.26 is the free structured spectrum (def. 2.25) on the 0-sphere

$$\mathbb{S}_{\mathrm{dia}} \simeq F_0^{\mathrm{dia}}(S^0) \,.$$

Now the free structured spectrum functor is a left Quillen functor (prop. 3.18) and hence $\mathbb{S}_{\mathrm{dia}}$ is cofibrant. ▮

Suspension and looping

For the strict model structure on topological sequential spectra, forming suspension spectra consitutes a Quillen adjunction $(\Sigma^\infty \dashv \Omega^\infty)$ with the classical model structure on pointed topological spaces (prop.) which is the precursor of the stabilization adjunction involving the stable model structure (thm.). Here we briefly discuss the lift of this strict adjunction to structured spectra.

Proposition 3.18. *Let $\mathrm{Dia} \in \{\mathrm{Top}^{*/}_{\mathrm{cg,fin}}, \mathrm{Orth}, \mathrm{Sym}, \mathrm{Seq}\}$ be one of the shapes of structured spectra from def. 2.4.*

For every $n \in \mathbb{N}$, the functors $\mathrm{Ev}^{\mathrm{dia}}_n$ of extracting the nth component space of a structured spectrum, and the functors F^{dia}_n of forming the free structured spectrum in degree n (def. 2.25) constitute a Quillen adjunction (def.) between the strict model structure on structured spectra from theorem 3.1 and the classical model structure on pointed topological spaces (thm., prop.):

$$\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}_{\mathrm{strict}} \underset{\mathrm{Ev}^{\mathrm{dia}}_n}{\overset{F^{\mathrm{dia}}_n}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} (\mathrm{Top}^{*/}_{\mathrm{cg}})_{\mathrm{Quillen}} \,.$$

For $n = 0$ and writing $\Sigma^\infty_{\mathrm{dia}} \coloneqq F^{\mathrm{dia}}_0$ and $\Omega^\infty_{\mathrm{dia}} \coloneqq \mathrm{Ev}^{\mathrm{dia}}_0$, $\Sigma^\infty_{\mathrm{dia}}$ this yields a strong monoidal Quillen adjunction (def. 3.15)

$$\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}_{\mathrm{strict}} \underset{\Omega^\infty_{\mathrm{dia}}}{\overset{\Sigma^\infty_{\mathrm{dia}}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} (\mathrm{Top}^{*/}_{\mathrm{cg}})_{\mathrm{Quillen}} \,.$$

Moreover, these Quillen adjunctions factor as

$$(\Sigma^\infty_{\mathrm{dia}} \dashv \Omega^\infty_{\mathrm{dia}}) \;:\; \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}} \underset{\mathrm{seq}^*}{\overset{\mathrm{seq}_!}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}} \underset{\Omega^\infty}{\overset{\Sigma^\infty}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} (\mathrm{Top}^{*/}_{\mathrm{cg}})$$

where the Quillen adjunction $(\mathrm{seq}_! \dashv \mathrm{seq}^*)$ is that from theorem 3.1 and where $(\Sigma^\infty \dashv \Omega^\infty)$ is the suspension spectrum adjunction for sequential spectra (prop.).

Proof. By the very definition of the projective model structure on functors (thm.) it is immediate that $\mathrm{Ev}^{\mathrm{dia}}_n$ preserves fibrations and weak equivalences, hence it is a right Quillen functor. F^{dia}_n is its left adjoint by definition.

That $\Sigma^\infty_{\mathrm{dia}}$ is a strong monoidal functor is part of the statement of prop. 2.29.

Moreover, it is clear from the definitions that

$$\Omega^\infty_{\mathrm{dia}} \simeq \Omega^\infty \circ \mathrm{seq}^* ,$$

hence the last statement follows by uniqueness of adjoints. ▮

Remark 3.19. In summary, we have established the following situation. There is a commuting diagram of Quillen adjunctions of the form

$$\begin{array}{ccc} (\mathrm{Top}^{*/}_{\mathrm{cg}})_{\mathrm{Quillen}} & \underset{\Omega}{\overset{\Sigma}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} & (\mathrm{Top}^{*/}_{\mathrm{cg}})_{\mathrm{Quillen}} \\ {}^{\Sigma^\infty}\downarrow \dashv \uparrow^{\Omega^\infty} && {}^{\Sigma^\infty}\downarrow \dashv \uparrow^{\Omega^\infty} \\ \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}} & \underset{\Omega}{\overset{\Sigma}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} & \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}} \\ {}^{\mathrm{dia}_!}\downarrow \dashv \uparrow^{\mathrm{dia}^*} && {}^{\mathrm{dia}_!}\downarrow \dashv \uparrow^{\mathrm{dia}^*} \\ \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}_{\mathrm{strict}} && \mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}_{\mathrm{strict}} \end{array} \,.$$

The top square stabilizes to the actual stable homotopy theory (thm.). On the other hand, the top square does not reflect the symmetric monoidal smash product of spectra (by remark 2.6). But the total vertical composite $\Sigma^\infty_{\mathrm{dia}} = \mathrm{dia}_! \Sigma^\infty$ does, in that it is a strong monoidal Quillen adjunction (def. 3.15) by prop. 3.18.

Hence to obtain a stable model category which is also a monoidal model category with respect to the symmetric monoidal smash product of spectra, it is now sufficient to find such a monoidal model structure on $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}$ such that $(\mathrm{seq}_! \dashv \mathrm{seq}^*)$ becomes a Quillen equivalence (def.)

This we now turn to in the section *The stable model structure on structured spectra*.

4. The stable model structure on structured spectra

Theorem 4.1. *The category $\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})$ of orthogonal spectra carries a model category structure (def.) where*

- *the weak equivalences W_{stable} are the stable weak homotopy equivalences (def. 2.21);*

- *the cofibrations $\mathrm{Cof}_{\mathrm{stable}}$ are the cofibrations of the strict model stucture of prop. 3.1;*
- *the fibrant objects are precisely the Omega-spectra (def. 2.21).*

Moreover, this is a cofibrantly generated model category (def.) with generating (acyclic) cofibrations the sets I^{stable} (J^{stable}) from def. 3.2.

(Mandell-May 02, theorem 4.2)

We give the **proof** below, after

Proof of the model structure

The generating cofibrations and acylic cofibrations are going to be the those induced via tensoring of representables from the classical model structure on topological spaces (giving the strict model structure), together with an additional set of morphisms to the generating acylic cofibrations that will force fibrant objects to be Omega-spectra. To that end we need the following little preliminary.

Definition 4.2. For $n \in \mathbb{N}$ let

$$\lambda_n : F_{n+1} S^1 \overset{k_n}{\longrightarrow} \mathrm{Cyl}(\lambda_n) \longrightarrow F_n S^0$$

be the factorization as in the factorization lemma of the morphism λ_n of lemma 2.30 through its mapping cylinder (prop.) formed with respect to the standard cylinder spectrum $(F_{n+1}S^1) \wedge (I_+)$:

Notice that:

Lemma 4.3. *The factorization in def. 4.2 is through a cofibration followed followed by a left homotopy equivalence in $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}}$*

Proof. Since the cell S^1 is cofibrant in $(\mathrm{Top}^{*/}_{\mathrm{cg}})_{\mathrm{Quillen}}$, and since $F_{n+1}(-)$ is a left Quillen functor by prop. 3.18, the free spectrum $F_{n+1}S^1$ is cofibrant in $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}}$. Therefore lemma 3.8 says that its standard cylinder spectrum is a good cylinder object and then the factorization lemma (lemma) says that k_n is a cofibration. Moreover, the morphism out of the standard mapping cylinder is a homotopy equivalence, with homotopies induced under tensoring from the standard homotopy contracting the standard cylinder. ▮

With this we may state the classes of morphisms that are going to be shown to be the classes of generating (acyclic) cofibrations for the stable model structures:

Definition 4.4. Recall the sets of generating (acyclic) cofibrations of the strict model structre def. 3.2. Set

$$I^{\mathrm{stable}}_{\mathbb{S}_{\mathrm{dia}}\mathrm{Mod}(\mathrm{Top}_{\mathrm{cg}})} \coloneqq I^{\mathrm{strict}}_{\mathbb{S}_{\mathrm{dia}}\mathrm{Mod}(\mathrm{Top}_{\mathrm{cg}})}$$

and

$$J^{\mathrm{stable}}_{\mathbb{S}_{\mathrm{dia}}\mathrm{Mod}(\mathrm{Top}_{\mathrm{cg}})} \coloneqq J^{\mathrm{strict}}_{\mathbb{S}_{\mathrm{dia}}\mathrm{Mod}(\mathrm{Top}_{\mathrm{cg}})} \sqcup \{k_n \,\Box\, i_+\}_{\substack{n \in \mathbb{N} \\ i \in I}}$$

for the disjoint union of the strict acyclic generating cofibration with the pushout products under smash tensoring of the resolved maps k_n from def. 4.2 with the elements in I.

(MMSS 00, def.6.2, def. 9.3)

Lemma 4.5. *Let $\mathrm{Dia} \in \{\mathrm{Top}^{*/}_{\mathrm{cg,fin}}, \mathrm{Orth}, \mathrm{Seq}\}$ (but not Sym). Then every element in $J^{\mathrm{stable}}_{\mathbb{S}_{\mathrm{dia}}\mathrm{Mod}(\mathrm{Top}_{\mathrm{cg}})}$ (def. 4.4) is both:*

1. *a cofibration with respect to the strict model structure (prop. 3.1);*
2. *a stable weak homotopy equivalence (def. 2.21).*

Proof. First regarding strict cofibrations:

By the Yoneda lemma, the elements in J have right lifting property against the strict fibrations, hence in particular they are strict cofibrations. Moreover, by Joyal-Tierney calculus (prop.), $k_n \,\Box\, i_+$ has left lifting against any acyclic strict fibration f precisely if k_n has left lifting against $f^{\Box i}$. By prop. 3.5 the latter is still a strict acyclic fibration. Since k_n by construction is a strict cofibration, the lifting follows and hence also $k_n \,\Box\, i_+$ is a strict cofibration.

Now regarding stable weak homotopy equivalences:

The morphisms in J^{strict} by design are strict weak equivalences, hence they are in particular stable weak homotopy equivalences. The morphisms k_n are stable weak homotopy equivalences by lemma 2.33 and by two-out-of-three.

To see that also the pushout products $k_n \square (i_n)_+$ are stable weak homotopy equivalences. (e.g. Mandell-May 02, p.46):

First $k_n \wedge (S^{n-1})_+$ is still a stable weak homotopy equivalence, by lemma. 2.23.

Moreover, observe that $\mathrm{dom}(k_n) \wedge i_+$ is degreewise a relative cell complex inclusion, hence degreewise a cofibration in the classical model structure on pointed topological spaces. This follows from lemma 2.28, which says that $\mathrm{dom}(k_n) \wedge i_+$ is degreewise the smash product of a CW complex with i_+, and from the fact that smashing with CW-complexes is a left Quillen functor $(\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}} \longrightarrow (\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}$ (prop.) and hence preserves cofibrations.

Altogether this implies by lemma 2.24 that the pushout of the stable weak homotopy equivalence $k_n \wedge (S^{n-1})_+$ along the degreewise cofibration $\mathrm{dom}(k_n) \wedge i_+$ is still a stable weak homtopy equivalence, and so the pushout product $k_n \square i_+$ is, too, by two-out-of-three. ▮

The point of the class K in def. 3.2 is to make the following true:

Lemma 4.6. *A morphism $f : X \to Y$ in $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}$ is a J^{stable}-injective morphism (for K from def. 4.4) precisely if*

1. *it is a fibration in the strict model structure (hence degreewise a fibration);*
2. *for all $n \in \mathbb{N}$ the commuting squares of structure map compatibility on the underlying sequential spectra*

$$\begin{array}{ccc} X_n & \overset{\tilde\sigma}{\longrightarrow} & \Omega X_{n+1} \\ \downarrow && \downarrow \\ Y_n & \underset{\tilde\sigma}{\longrightarrow} & \Omega Y_{n+1} \end{array}$$

are homotopy pullbacks (def.).

(MMSS 00, prop. 9.5)

Proof. By prop 3.3, lifting against J^{strict} alone characterizes strict fibrations, hence degreewise fibrations. Lifting against the remaining pushout product morphism $k_n \square i_+$ is, by Joyal-Tierney calculus, equivalent to left lifting i_+ against the dual pullback product of $f^{\square k_n}$, which means that $f^{\square k_n}$ is a weak homotopy equivalence. But by construction of k_n and by lemma 2.30, $f^{\square k_n}$ is the comparison morphism into the homotopy pullback under consideration. ▮

Corollary 4.7. *The J^{stable}-injective objects are precisely the Omega-spectra (def. 2.21).*

Lemma 4.8. *A morphism in $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}$ which is both*

1. *a stable weak homotopy equivalence (def. 2.21);*
2. *a J^{stable}-injective morphisms*

is an acyclic fibration in the strict model structure of prop. 3.1, hence is degreewise a weak homotopy equivalence and Serre fibration of topological spaces;

(MMSS 00, corollary 9.8)

Proof. Let $f : X \to B$ be both a stable weak homotopy equivalence as well as a J^{stable}-injective morphism. Since J^{stable} contains, by prop. 3.3, the generating acyclic cofibrations for the strict model structure of prop. 3.1, f is in particular a strict fibration, hence a degreewise fibration. Therefore the fiber F of f is its homotopy fiber in the strict model structure.

Hence by lemma 2.22 there is an exact sequence of stable homotopy groups of the form

$$\pi_{\bullet+1}(X) \xrightarrow{\pi_{\bullet+1}(f)} \pi_{\bullet+1}(Y) \longrightarrow \pi_\bullet(F) \longrightarrow \pi_\bullet(X) \xrightarrow{\pi_\bullet(f)} \pi_\bullet(Y)\,.$$

By exactness and by the assumption that $\pi_\bullet(f)$ is an isomorphism, this implies that $\pi_\bullet(F) \simeq 0$, hence that $F \to *$ is a stable weak homotopy equivalence.

Observe also that F, being the pullback of a J^{stable}-injective morphisms (by the standard closure properties) is a J^{stable}-injective object, so that by corollary 4.7 F is an Omega-spectrum. Since stable weak homotopy equivalences between Omega-spectra are already degreewise weak homotopy equivalences, together this says that $F \to *$ is a weak equivalence in the strict model structure, hence degreewise a weak homotopy equivalence. From this the long exact sequence of homotopy groups implies that $\pi_{\bullet \geq 1}(f_n)$ is a weak homotopy equivalence for all n and for each homotopy group in positive degree.

To deduce the remaining case that also $\pi_0(f_0)$ is an isomorphism, observe that, by assumption of

J^{stable}-injectivity, lemma 4.6 gives that f_n is a homotopy pullback (in topological spaces) of $\Omega(f_{n+1})$. But, by the above, $\Omega(f_{n+1})$ is a weak homotopy equivalence, since $\pi_\bullet(\Omega(-)) = \pi_{\bullet+1}(-)$. Therefore f_n is the homotopy pullback of a weak homotopy equivalence and hence itself a weak homotopy equivalence. ▮

Lemma 4.9. *The retracts of J^{stable}-relative cell complexes are precisely the morphisms which are*

1. *stable weak homotopy equivalences (def. 2.21),*
2. *as well as cofibrations with respect to the strict model structure of prop. 3.1.*

(MMSS 00, prop. 9.9 (i))

Proof. Since all elements of J^{stable} are stable weak homotopy equivalences as well as strict cofibrations by lemma 4.5, it follows that every retract of a relative K-cell complex has the same property.

In the other direction, if f is a stable weak homotopy equivalence and a strict cofibration, by the small object argument it factors $f \colon \overset{i}{\to} \overset{p}{\to}$ as a relative J^{stable}-cell complex i followed by a J^{stable}-injective morphism p. By the previous statement i is a stable weak homotopy equivalence, and so by assumption and by two-out-of-three so is p. Therefore lemma 4.8 implies that p is a strict acyclic fibration. But then the assumption that f is a strict cofibration means that it has the left lifting property against p, and so the retract argument implies that f is a retract of the relative K-cell complex i. ▮

Corollary 4.10. *The J^{stable}-injective morphisms are precisely those which are injective with respect to the cofibrations of the strict model structure that are also stable weak homotopy equivalences.*

(MMSS 00, prop. 9.9 (ii))

Lemma 4.11. *A morphism in $\mathbb{S}_{\mathrm{dia}}\,\mathrm{Mod}$ (for $\mathrm{Diq} \neq \mathrm{Sym}$) is both*

1. *a stable weak homotopy equivalence (def. \ref{StableEquivalencesForDiagramSpectra})*
2. *injective with respect to the cofibrations of the strict model structure that are also stable weak homotopy equivalences;*

precisely if it is an acylic fibration in the strict model structure of theorem 3.1.

(MMSS 00, prop. 9.9 (iii))

Proof. Every acyclic fibration in the strict model structure is injective with respect to strict cofibrations by the strict model structure; and it is a clearly a stable weak homotopy equivalence.

Conversely, a morphism injective with respect to strict cofibrations that are stable weak homotopy equivalences is a J^{stable}-injective morphism by corollary 4.10, and hence if it is also a stable equivalence then by lemma 4.8 it is a strict acylic fibration. ▮

Proof. (of theorem 4.1)

The non-trivial points to check are the two weak factorization systems.

That $(\mathrm{cof}_{\mathrm{stable}} \cap \mathrm{weq}_{\mathrm{stable}}\,,\ \mathrm{fib}_{\mathrm{stable}})$ is a weak factorization system follows from lemma 4.9 and the small object argument.

By lemma 4.11 the stable acyclic fibrations are equivalently the strict acyclic fibrations and hence the weak factorization system $(\mathrm{cof}_{\mathrm{stable}}\,,\ \mathrm{fib}_{\mathrm{stable}} \cap \mathrm{we}_{\mathrm{stable}})$ is identified with that of the strict model structure $(\mathrm{cof}_{\mathrm{strict}}\,,\ \mathrm{fib}_{\mathrm{strict}} \cap \mathrm{we}_{\mathrm{strict}})$. ▮

Stability of the homotopy theory

We show now that the model structure on orthogonal spectra $\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}$ from theorem 4.1 is Quillen equivalent (def.) to the stable model structure on topological sequential spectra $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}$ (thm.), hence that they model the same stable homotopy theory.

Theorem 4.12. *The free-forgetful adjunction $(\mathrm{seq}_! \dashv \mathrm{seq}^*)$ of def. 2.4 and theorem 3.1 is a Quillen equivalence (def.) between the stable model structure on topological sequential spectra (thm.) and the stable model structure on orthogonal spectra from theorem 4.1.*

$$\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}} \underset{\mathrm{seq}^*}{\overset{\mathrm{seq}_!}{\underset{\longrightarrow}{\overset{\longleftarrow}{\simeq_{\mathrm{Quillen}}}}}} \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}$$

(MMSS 00, theorem 10.4)

Proof. Since the forgetful functor seq^* “creates weak equivalences”, in that a morphism of orthogonal spectra is a weak equivalence precisely if the underlying morphism of sequential spectra is (by def. 2.21) it is sufficient to show (by this prop.) that for every cofibrant sequential spectrum X, the adjunction unit

$$X \longrightarrow \mathrm{seq}^* \mathrm{seq}_! X$$

is a stable weak homotopy equivalence.

By cofibrant generation of the stable model structure on topological sequential spectra $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}$ (thm.) every cofibrant sequential spectrum is a retract of an $I_{\mathrm{seq}}^{\mathrm{stable}}$-relative cell complex (def., def.), where

$$I_{\mathrm{seq}}^{\mathrm{stable}} = \left\{ F_{n_1} S_+^{n_2-1} \overset{F_{n_1}(i_{n_2})_+}{\longrightarrow} F_{n_1} D_+^{n_2} \right\} .$$

Since $\mathrm{seq}_!$ and seq^* both preserve colimits (seq^* because it evaluates at objects and colimits in the diagram category $\mathrm{OrthSpec}$ are computed objectwise, and $\mathrm{seq}_!$ because it is a left adjoint) we have for $X \simeq \underset{\longrightarrow i}{\lim} X_i$ a relative $I_{\mathrm{seq}}^{\mathrm{stable}}$-decompositon of X, that $\eta_X : X \to \mathrm{seq}^* \mathrm{seq}_! X$ is equivalently

$$\underset{\longrightarrow i}{\lim} \eta_{X_i} \;:\; \underset{\longrightarrow i}{\lim} X_i \longrightarrow \underset{\longrightarrow i}{\lim} \mathrm{seq}_! \mathrm{seq}^* X_i \;.$$

Now observe that the colimits involved in a relative $I_{\mathrm{seq}}^{\mathrm{stable}}$-complex (the coproducts, pushouts, transfinite compositions) are all homotopy colimits (def.): First, all objects involved are cofibrant. Now for the transfinite composition all the morphisms involved are cofibrations, so that their colimit is a homotopy colimit by this example, while for the pushout one of the morphisms out of the “top” objects is a cofibration, so that this is a homotopy pushout by (def.).

It follows that if all η_{X_i} are weak equivalences, then so is $\eta = \underset{\longrightarrow i}{\lim} \eta_{X_i}$.

Unwinding this, one finds that it is sufficient to show that

$$\eta_{F_{n_1} S^{n_2}} \;:\; F_{n_1} S_+^{n_2} \longrightarrow \mathrm{seq}^* \mathrm{seq}_! F_{n_1} S^{n_2}$$

is a stable weak homotopy equivalence for all $n_1, n_2 \in \mathbb{N}$.

Consider this for $n_2 \geq n_2$. Then there are canonical morphisms

$$F_{n_1} S^{n_2} \longrightarrow F_0 S^{n_2 - n_1}$$

whose components in degree $q \geq n_1$ are the identity. These are the composites of the maps $\lambda_k \wedge S^{k+n_2-n_1}$ for $k < n_1$ with λ_n from def. \reg{CorepresentationOfAdjunctsOfStructureMaps}. By prop. 2.33 also $\mathrm{seq}^* \mathrm{seq}_! \lambda_n$ are weak homotopy equivalences. Hence we have commuting diagrams of the form

$$\begin{array}{ccc} F_{n_1}^{\mathrm{seq}} S^{n_2} & \longrightarrow & F_0 S^{n_2-n_1} \\ {}^{\eta}\downarrow && \downarrow^{\simeq} \\ \mathrm{seq}^* F_{n_1}^{\mathrm{orth}} S^{n_2-n_1} & \longrightarrow & \mathrm{seq}^* F_0^{\mathrm{orth}} S^{n_2-n_1} \end{array} \;,$$

where the horizontal maps are stable weak homotopy equivalences by the previous argument and the right vertical morphism is an isomorphism by the formula in prop. 2.27.Hence the left vertical morphism is a stable weak homotopy equivalence by two-out-of-three.

If $n_2 < n_1$ then one reduces this to the above case by smashing with $S^{n_1-n_2}$. ▮

Remark 4.13. Theorem 4.12 means that the homotopy categories of $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}$ and $\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}$ are equivalent (prop.) via

$$\mathrm{Ho}(\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}) \underset{\mathbb{R}\,\mathrm{seq}^*}{\overset{\mathbb{L}\,\mathrm{seq}_!}{\underoverset{\longrightarrow}{\longleftarrow}{\simeq}}} \mathrm{Ho}(\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}) \;.$$

Since $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}$ is a stable model category (thm.) in that the derived suspension looping adjunction is an equivalence of categories, and and since this is a condition only on the homotopy categories, and since $\mathbb{R}\,\mathrm{seq}^{ast}$ manifestly preserves the construction of loop space objects, this implies that we have a commuting square of adjoint equivalences of homotopy categories

$$\begin{array}{ccc} \mathrm{Ho}(\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}) & \underset{\Omega}{\overset{\Sigma}{\underoverset{\longrightarrow}{\longleftarrow}{\simeq}}} & \mathrm{Ho}(\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}) \\ {}^{\mathbb{L}\,\mathrm{seq}_!}\downarrow \simeq \uparrow^{\mathbb{R}\,\mathrm{seq}^*} && {}^{\mathbb{L}\,\mathrm{seq}_!}\downarrow \simeq \uparrow^{\mathbb{R}\,\mathrm{seq}^*} \\ \mathrm{Ho}(\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}) & \underset{\Omega}{\overset{\Sigma}{\underoverset{\longrightarrow}{\longleftarrow}{\simeq}}} & \mathrm{Ho}(\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}) \end{array}$$

and so in particular also $\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}$ is a stable model category.

Due to the vertical equivalences here we will usually not distinguish between these homotopy categories and just speak of the *stable homotopy category* (def.)

$$\mathrm{Ho}(\mathrm{Spectra}) \coloneqq \mathrm{Ho}(\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}) \simeq \mathrm{Ho}(\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}) \,.$$

Monoidal model structure

We now discuss that the monoidal model category structure of the strict model structure on orthogonal spectra $\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{strict}}$ (theorem 3.17) remains intact as we pass to the stable model structure $\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}$ of theorem 4.1.

Theorem 4.14. *The stable model structure $\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}$ of theorem 4.1 equipped with the symmetric monoidal smash product of spectra (def. 2.9) is a monoidal model category (def. 3.11) with cofibrant tensor unit*

$$(\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}}),\ \wedge = \otimes_{\mathbb{S}_{\mathrm{orth}}},\ \mathbb{S}_{\mathrm{orth}}) \,.$$

(MMSS 00, prop. 12.6)

Proof. Since $\mathrm{Cof}_{\mathrm{stable}} = \mathrm{Cof}_{\mathrm{strict}}$, the fact that the pushout product of two stable cofibrations is again a stable cofibration is part of theorem 3.17.

It remains to show that if at least one of them is a stable weak homotopy equivalence (def. 2.21), then so is the pushout-product.

Since $\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})$ is a cofibrantly generated model category by theorem 4.1 and since it has internal homs (mapping spectra) with respect to $\otimes_{\mathbb{S}_{\mathrm{dia}}}$ (prop. 1.45), it suffices (as in the proof of this prop.) to check this on generating (acylic) cofibrations, i.e. to check that

$$I^{\mathrm{stable}} \Box_{\otimes} J^{\mathrm{stable}} \subset W_{\mathrm{stable}} \cap \mathrm{Cof}_{\mathrm{stable}} \,.$$

Now $I^{\mathrm{stable}} = I^{\mathrm{strict}}$ and $J^{\mathrm{stable}} = J^{\mathrm{strict}} \sqcup \{k_n \Box i_+\}$ so that the special case

$$\begin{aligned} I^{\mathrm{stable}} \Box_{\otimes} J^{\mathrm{strict}} &= I^{\mathrm{strict}} \Box_{\otimes} J^{\mathrm{strict}} \\ &\subset W_{\mathrm{strict}} \cap \mathrm{Cof}_{\mathrm{strict}} \\ &\subset W_{\mathrm{stable}} \cap \mathrm{Cof}_{\mathrm{stable}} \end{aligned}$$

follows again from the monoidal stucture on the strict model category of theorem 3.17.

It hence remains to see that

$$I^{\mathrm{strict}} \Box_{\otimes} (k_{n_1} \Box (i_{n_2})_+) \subset W_{\mathrm{stable}} \cap \mathrm{Cof}_{\mathrm{stable}}$$

for all $n_1, n_2 \in \mathbb{N}$.

By lemma 4.5 $k_n \Box i_+$ is in $\mathrm{Cof}_{\mathrm{strict}}$ and hence

$$I^{\mathrm{strict}} \Box_{\otimes} (k_{n_1} \Box (i_{n_2})_+) \subset \mathrm{Cof}_{\mathrm{strict}}$$

follows, once more, from the monoidalness of the strict model structure.

Hence it only remains to show that

$$I^{\mathrm{strict}} \Box_{\otimes} (k_{n_1} \Box (i_{n_2})_+) \subset W_{\mathrm{stable}} \,.$$

This we now prove by inspection:

By two-out-of-three applied to the definition of the pushout product, it is sufficient to show that for every $F_{n_3}(i_{n_4})_+$ in I^{strict}, the right vertical morphism in the pushout diagram

$$\begin{array}{ccc} & \xrightarrow{F_{n_3}(i_{n_4})_+ \otimes \mathrm{dom}(k_{n_1} \Box (i_{n_2})_+)} & \\ {}^{\mathrm{dom}(F_{n_3}(i_{n_4}) \otimes (k_{n_1} \Box (i_{n_2})_+)} \downarrow & (\mathrm{po}) & \downarrow \\ & \longrightarrow & \end{array}$$

is a stable weak homotopy equivalence. Since seq^* preserves pushouts, we may equivalently check this on the underlying sequential spectra.

Consider first the top horizontal morphism in this square.

We may rewrite it as

$$
\begin{aligned}
F_{n_3}(i_{n_4})_+ \otimes (\mathrm{dom}(k_{n_1}) \Box (i_{n_2})_+) &\simeq F_{n_3}(i_{n_4})_+ \otimes \left(F_{n_1} S^0 \wedge S_+^{n_2 - 1} \underset{F_{n_1+1} S^1 \wedge S_+^{n_2-1}}{\sqcup} F_{n_1+1} S^1 \wedge D_+^{n_2} \right) \\
&\simeq F_{n_3}(i_{n_4})_+ \otimes F_{n_1} S^0 \wedge S_+^{n_2-1} \underset{F_{n_3}(i_{n_4})_+ \otimes F_{n_1+1} S^1 \wedge S_+^{n_2-1}}{\sqcup} F_{n_3}(i_{n_4})_+ \otimes F_{n_1+1} S^1 \wedge D_+^{n_2}, \\
&\simeq F_{n_1+n_3}(i_{n_4})_+ \wedge S_+^{n_2-1} \underset{F_{n_1+n_3+1}(i_{n_4})_+ \wedge S_+^{n_2-1}}{\sqcup} F_{n_1+n_3+1}(i_{n_4})_+ \wedge S^1 \wedge D_+^{n_2}
\end{aligned}
$$

where we used that $X \otimes (-)$ is a left adjoint and hence preserves colimits, and we used prop. 2.29 to evaluate the smash product of free spectra.

Now by lemma 2.28 the morphism

$$
F_{n_1+n_3+1} S_+^{n_4-1} \wedge S^1 \wedge S_+^{n_2-1} \longrightarrow F_{n_1+n_3+1} S_+^{n_4-1} \wedge S^1 \wedge D_+^{n_2}
$$

is degreewise the smash product of a CW-complex with a relative cell complex inclusion, hence is itself degreewise a relative cell complex inclusion, and therefore its pushout

$$
F_{n+1+n_3} S_+^{n_4-1} \otimes F_{n_1} S^0 \wedge S_+^{n_2-1} \longrightarrow F_{n_3}(S^{n_4-1})_+ \otimes \mathrm{dom}(k_{n_1} \Box (i_{n_2})_+)
$$

is degreewise a retract of a relative cell complex inclusion. But since it is the identity on the smash factor $S_+^{n_4-1}$ in the argument of the free spectra as above, the morphism is degreewise the smash tensoring with S^{n_4-1} of a retract of a relative cell complex inclusion. Since the domain is degreewise a CW-complex by lemma 2.28, $F_{n_3}(S^{n_4-1})_+ \otimes \mathrm{dom}(k_{n_1} \Box (i_{n_2})_+)$ is degreewise the smash tensoring with $S_+^{n_4-1}$ of a retract of a cell complex.

The same argument applies to the domain of $F_{n_3}(i_4)_+ \otimes (\mathrm{dom}(k_n) \Box (i_2)_+)$, and so in conclusion this morphism is degreewise the smash product of a cofibration with a cofibrant object in $(\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}$, and hence is itself degreewise a cofibration.

Now consider the vertical morphism in the above square

The same argument that we just used shows that this is the smash tensoring of the stable weak homotopy equivalence $k_{n_1} \Box (i_{n_2})_+$ with a CW-complex. Hence by lemma 2.23 the left vertical morphism is a stable weak homotopy equivalence.

In conclusion, the right vertical morphism is the pushout of a stable weak homotopy equivalence along a degreewise cofibration of pointed topological spaces. Hence lemma 2.24 implies that it is itself a stable weak homotopy equivalence. ▮

Corollary 4.15. *The strong monoidal Quillen adjunction (def. 3.15) $(\Sigma^\infty_{\mathrm{orth}} \dashv \Omega^\infty_{\mathrm{orth}})$ on the strict model structure (prop. 3.18) descends to a strong monoidal Quillen adjunction on the stable monoidal model category from theorem 4.14:*

$$
\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}} \underset{\Omega^\infty_{\mathrm{orth}}}{\overset{\Sigma^\infty_{\mathrm{orth}}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} (\mathrm{Top}_{\mathrm{cg}}^{*/}, \wedge, S^0)_{\mathrm{Quillen}} \,.
$$

Proof. The stable model structure $\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}$ is a left Bousfield localization of the strict model structure (def.) in that it has the same cofibrations and a larger class of acyclic cofibrations. Hence $\Sigma^\infty_{\mathrm{orth}}$ is still a left Quillen functor also to the stable model structure. ▮

5. The monoidal stable homotopy category

We discuss now the consequences for the stable homotopy category (def.) of the fact that by theorem 4.12 and theorem 4.14 it is equivalently the homotopy category of a stable monoidal model category. This makes the stable homotopy category become a *tensor triangulated category* (def. 5.3) below. The abstract structure encoded by this governs much of stable homotopy theory (Hovey-Palmieri-Strickland 97). In particular it is this structure that gives rise to the E-Adams spectral sequences which we discuss in Part 2.

Corollary 5.1. *The stable homotopy category* $\mathrm{Ho}(\mathrm{Spectra})$ *(remark 4.13) inherits the structure of a symmetric monoidal category*

$$
(\mathrm{Ho}(\mathrm{Spectra}), \wedge^L, \mathbb{S} \coloneqq \gamma(\mathbb{S}_{\mathrm{orth}}))
$$

with tensor product the left derived functor $\wedge^L$ of the symmetric monoidal smash product of spectra (def. 2.9, def. 2.13, prop. 2.14) and with tensor unit the sphere spectrum $\mathbb{S}$ (the image in $\mathrm{Ho}(\mathrm{Spectra})$ of any of the structured sphere spectra from def. 2.4).

Moreover, the localization functor (def.) is a lax monoidal functor

$$\gamma \,:\, (\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}}), \wedge, \mathbb{S}_{\mathrm{orth}}) \longrightarrow (\mathrm{Ho}(\mathrm{Spectra}), \wedge^L, \gamma(\mathbb{S}))\;.$$

Proof. In view of theorem 4.14 this is a special case of prop. 3.13. ▮

Remark 5.2. Let $A, X \in \mathrm{Ho}(\mathrm{Spectra})$ be two spectra in the stable homotopy category, then the stable homotopy groups (def.) of their derived symmetric monoidal smash product of spectra (corollary 5.1) is also called the *generalized homology* of X with coefficients in A and denoted

$$A_\bullet(X) \coloneqq \pi_\bullet(A \wedge X)\;.$$

This is conceptually dual to the concept of generalized (Eilenberg-Steenrod) cohomology (example)

$$A^\bullet(X) \coloneqq [X, A]_\bullet\;.$$

Notice that (def., lemma)

$$\begin{aligned} A_\bullet(X) &= \pi_\bullet(A \wedge X) \\ &\simeq [\mathbb{S}, A \wedge X]_\bullet \end{aligned}\;.$$

In the special case that $X = \Sigma^\infty K$ is a suspension spectrum, then

$$A_\bullet(X) \simeq \pi_\bullet(A \wedge K)$$

(by prop. 2.29) and this is called the *generalized A-homology* of the topological space $K \in \mathrm{Top}_{\mathrm{cg}}^{*/}$.

Since the sphere spectrum $\mathbb{S}$ is the tensor unit for the derived smash product of spectra (corollary 5.1) we have

$$E_\bullet(\mathbb{S}) \simeq \pi_\bullet(E)\;.$$

For that reason often one also writes for short

$$E_\bullet \coloneqq \pi_\bullet(E)\;.$$

Notice that similarly the E-generalized cohomology (exmpl.) of the sphere spectrum is

$$\begin{aligned} E^\bullet &\coloneqq E^\bullet(\mathbb{S}) \\ &= [\mathbb{S}, E]_{-\bullet} \\ &\simeq \pi_{-\bullet}(E) \\ &\simeq E_{-\bullet} \end{aligned}\;.$$

(Beware that, as usual, here we are not displaying a tilde-symbol to indicate reduced cohomology).

Tensor triangulated structure

We discuss that the derived smash product of spectra from corollary 5.1 on the stable homotopy category interacts well with its structure of a triangulated category (def.).

Definition 5.3. A **tensor triangulated category** is a category Ho equipped with

1. the structure of a symmetric monoidal category $(\mathrm{Ho}, \otimes, 1, \tau)$ (def. 1.5);
2. the structure of a triangulated category $(\mathrm{Ho}, \Sigma, \mathrm{CofSeq})$ (def.);
3. for all objects $X, Y \in \mathrm{Ho}$ natural isomorphisms

$$e_{X,Y} \,:\, (\Sigma X) \otimes Y \overset{\simeq}{\longrightarrow} \Sigma(X \otimes Y)$$

such that

1. (tensor product is additive) for all $V \in \mathrm{Ho}$ the functors $V \otimes (-) \simeq (-) \otimes V$ preserve finite direct sums (are additive functors);
2. (tensor product is exact) for each object $V \in \mathrm{Ho}$ the functors $V \otimes (-) \simeq (-) \otimes V$ preserves distinguished triangles in that for

$$X \overset{f}{\longrightarrow} X \overset{g}{\longrightarrow} Y \overset{h}{\longrightarrow} \Sigma X$$

in CofSeq, then also

$$V \otimes X \overset{\mathrm{id}_V \otimes f}{\longrightarrow} V \otimes X \overset{\mathrm{id}_V \otimes g}{\longrightarrow} V \otimes Y \overset{\mathrm{id}_V \otimes h}{\longrightarrow} V \otimes (\Sigma X) \simeq \Sigma(V \otimes X)$$

is in CofSeq, where the equivalence at the end is $e_{X,V} \circ \tau_{V,\Sigma Y}$.

Jointly this says that for all objects V the equivalences e give $V \otimes (-)$ the structure of a *triangulated functor*.

(Balmer 05, def. 1.1)

In addition we ask that

1. (coherence) for all $X, Y, Z \in \mathrm{Ho}$ the following diagram commutes

$$\begin{array}{ccccc} (\Sigma(X) \otimes Y) \otimes Z & \overset{e_{X,Y} \otimes \mathrm{id}}{\longrightarrow} & (\Sigma(X \otimes Y)) \otimes Z & \overset{e_{X \otimes Y, Z}}{\longrightarrow} & \Sigma((X \otimes Y) \otimes Z) \\ {}^{\alpha_{\Sigma X, Y, Z}}\downarrow && && \downarrow^{\Sigma \alpha_{X,Y,Z}} \\ \Sigma(X) \otimes (Y \otimes Z) && \underset{e_{X, Y \otimes Z}}{\longrightarrow} && \Sigma(X \otimes (Y \otimes Z)) \end{array} \,,$$

where α is the associator of $(\mathrm{Ho}, \otimes, 1)$.

2. (graded commutativity) for all $n_1, n_2 \in \mathbb{Z}$ the following diagram commutes

$$\begin{array}{ccc} (\Sigma^{n_1} 1) \otimes (\Sigma^{n_2} 1) & \overset{\simeq}{\longrightarrow} & \Sigma^{n_1 + n_2} 1 \\ {}^{\tau_{\Sigma^{n_1} 1, \Sigma^{n_2} 1}}\downarrow && \downarrow^{(-1)^{n_1 \cdot n_2}} \\ (\Sigma^{n_2} 1) \otimes (\Sigma^{n_1} 1) & \underset{\simeq}{\longrightarrow} & \Sigma^{n_1 + n_2} 1 \end{array} \,,$$

where the horizontal isomorphisms are composites of the $e_{\cdot,\cdot}$ and the braidings.

(Hovey-Palmieri-Strickland 97, def. A.2.1)

Proposition 5.4. *The stable homotopy category* $\mathrm{Ho}(\mathrm{Spectra})$ *(def.) equipped with*

1. *its triangulated category structure* $(\mathrm{Ho}(\mathrm{Spectra}), \Sigma, \mathrm{CofSeq})$ *for distinguished triangles the homotopy cofiber sequences (prop.);*

2. *the derived symmetric monoidal smash product of spectra* $(\mathrm{Ho}(\mathrm{Spectra}), \wedge^L, \mathbb{S})$ *(corollary 5.1)*

is a tensor triangulated category in the sense of def. 5.3.

(e.g. Hovey-Palmieri-Strickland 97, 9.4)

We break up the **proof** into lemma 5.5, lemma 5.6, lemma 5.7 and lemma 5.9.

Lemma 5.5. *For* $V \in \mathrm{Ho}(\mathrm{Spectra})$ *any spectrum in the stable homotopy category (remark 4.13), then the derived symmetric monoidal smash product of spectra (corollary 5.1)*

$$V \wedge^L (-) : \mathrm{Ho}(\mathrm{Spectra}) \longrightarrow \mathrm{Ho}(\mathrm{Spectra})$$

preserves direct sums, in that for all $X, Y \in \mathrm{Ho}(\mathrm{Spectra})$ *then*

$$V \wedge^L (X \oplus Y) \simeq (V \wedge^L X) \oplus (V \wedge^L Y) \,.$$

Proof. The direct sum in $\mathrm{Ho}(\mathrm{Spectra})$ is represented by the wedge sum in $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$ (prop., prop.). Since wedge sum of sequential spectra is the coproduct in $\mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$ (exmpl.) and since the forgetful functor $\mathrm{seq}^\ast : \mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}}) \longrightarrow \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})$ preserves colimits (since by prop. 2.19 it acts by precomposition on functor categories, and since for these colimits are computed objectwise), it follows that also wedge sum of orthogonal spectra represents the direct sum operation in the stable homotopy category.

Now assume without restriction that V, X and Y are cofibrant orthogonal spectra representing the objects of the same name in the stable homotopy catgeory. Since wedge sum is coproduct, it follows that also the wedge sum $X \vee Y$ is cofibrant.

Since $V \wedge^L (-)$ is a left Quillen functor by theorem 4.14, it follows that the derived tensor product $V \wedge^L (X \oplus Y)$ is represented by the plain symmetric monoidal smash product of spectra $V \wedge (X \vee Y)$. By def. 2.9 (or more explicitly by prop. 2.14) this is the coequalizer

$$V \otimes_{\mathrm{Day}} \mathbb{S}_{\mathrm{orth}} \otimes_{\mathrm{Day}} (X \vee Y) \underoverset{\longrightarrow}{\longrightarrow}{} V \otimes_{\mathrm{Day}} (X \vee Y) \overset{\mathrm{coeq}}{\longrightarrow} V \otimes_{\mathbb{S}_{\mathrm{orth}}} (X \vee Y) \,.$$

Inserting the definition of Day convolution (def. 1.39), the middle term here is

$$\int^{c_1,c_2} \mathrm{Orth}(c_1 \otimes_{\mathrm{Orth}} c_2, -) \wedge V(c_1) \wedge (X \vee Y)(c_2) \simeq \int^{c_1,c_2} \mathrm{Orth}(c_1 \otimes_{\mathrm{Orth}} c_2, -) \wedge V(c_1) \wedge (X(c_2) \vee Y(c_2))$$
$$\int^{c_1,c_2} \mathrm{Orth}(c_1 \otimes_{\mathrm{Orth}} c_2, -) \wedge V(c_1) \wedge X(c_2) \;\vee\; \int^{c_1,c_2} \mathrm{Orth}(c_1 \otimes_{\mathrm{Orth}} c_2, -) \wedge V(c_1) \wedge Y(c_2)'$$
$$\simeq V \otimes_{\mathrm{Day}} X \;\vee\; V \otimes_{\mathrm{Day}} Y$$

where in the second but last step we used that the smash product in $\mathrm{Top}_{\mathrm{cg}}^{*/}$ distributes over wedge sum and that coends commute with wedge sums (both being colimits).

The analogous analysis applies to the left term in the coequalizer diagram. Hence the whole diagram splits as the wedge sum of the respective diagrams for $V \wedge X$ and $V \wedge Y$. ▮

Lemma 5.6. *For $X \in \mathrm{Ho}(\mathrm{Spectra})$ any spectrum in the stable homotopy category (remark 4.13), then the derived symmetric monoidal smash product of spectra (corollary 5.1)*

$$X \wedge^L (-) \,:\, \mathrm{Ho}(\mathrm{Spectra}) \longrightarrow \mathrm{Ho}(\mathrm{Spectra})$$

preserves homotopy cofiber sequences.

Proof. We may choose a cofibrant representative of X in $\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}$, which we denote by the same symbol. Then the functor

$$X \wedge (-) \,:\, \mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}} \longrightarrow \mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}\,\mathrm{stable}}$$

is a left Quillen functor in that it preserves cofibrations and acyclic cofibrations by theorem 4.14 and it is a left adjoint by prop. 1.22. Hence its left derived functor is equivalently its restriction to cofibrant objects followed by the localization functor.

But now every homotopy cofiber (def.) is represented by the ordinary cofiber of a cofibration. The left Quillen functor preserves both the cofibration as well as its cofiber. ▮

Lemma 5.7. *The canonical suspension functor on the stable homotopy category*

$$\Sigma \,:\, \mathrm{Ho}(\mathrm{Spectra}) \longrightarrow \mathrm{Ho}(\mathrm{Spectra})$$

commutes with forming the derived symmetric monoidal smash product of spectra $\wedge^L$ from corollary 5.1 in that for $X, Y \in \mathrm{Ho}(\mathrm{Spectra})$ any two spectra, then there are isomorphisms

$$\Sigma(X \wedge^L Y) \simeq (\Sigma X) \wedge^L Y \simeq X \wedge^L (\Sigma Y)\,.$$

Proof. By theorem 4.14 the symmetric monoidal smash product of spectra is a left Quillen functor, and by prop. 3.7 and lemma 3.8 the canonical suspension operation is the left derived functor of the left Quillen functor $(-) \wedge S^1$ of smash tensoring with S^1. Therefore all three expressions are represented by application of the underived functors on cofibrant representatives in $\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})$ (the fibrant replacement that is part of the derived functor construction is preserved by left Quillen functors).

So for X and Y cofibrant orthogonal spectra (which we denote by the same symbol as the objects in the homotopy category which they represent), by def. 2.9 (or more explicitly by prop. 2.14), the object $\Sigma(X \wedge^L Y) \in \mathrm{Ho}(\mathrm{Spectra})$ is represented by the coequalizer

$$(X \otimes_{\mathrm{Day}} \mathbb{S}_{\mathrm{orth}} \otimes Y) \wedge S^1 \underset{\longrightarrow}{\overset{\longrightarrow}{}} (X \otimes_{\mathrm{Day}} Y) \wedge S^1 \overset{\mathrm{coeq}}{\longrightarrow} (X \otimes_{\mathbb{S}_{\mathrm{orth}}} Y) \wedge S^1\,,$$

where the two morphisms bing coequalized are the images of those of def. 2.9 under smash tensoring with S^1. Now it is sufficient to observe that for any $K \in \mathrm{Top}_{\mathrm{cg}}^{*/}$ we have canonical isomorphisms

$$(X \otimes_{\mathrm{Day}} Y) \wedge K \simeq (X \otimes_{\mathrm{Day}} (Y \wedge K)) \simeq ((X \wedge K) \otimes_{\mathrm{Day}} Y)$$

and similarly for the triple Day tensor product.

This follows directly from the definition of the Day convolution product (def. 1.39)

$$((X \otimes_{\mathrm{Day}} Y) \wedge K)(V) = \int^{V_1, V_2} \mathrm{Orth}(V_1 \oplus V_2, V) \wedge X(V_1) \wedge Y(V_2) \wedge K$$

and the symmetry of the smash product on $\mathrm{Top}_{\mathrm{cg}}^{*/}$ (example 1.10). ▮

Example 5.8. For $A \in \mathrm{Ho}(\mathrm{Spectra})$ a spectrum, then the A-generalized homology (according to remark 5.2) of a suspension of the spectrum is the stable homotopy groups of A in shifted degree:

$$A_\bullet(\Sigma^n \mathbb{S}) \simeq \pi_{\bullet - n}(A)\,.$$

Proof. We compute

$$
\begin{aligned}
A_\bullet(\Sigma^n \mathbb{S}) & \coloneqq \pi_\bullet(A \wedge \Sigma^n \mathbb{S}) \\
& \simeq \pi_\bullet(\Sigma^n (A \wedge \mathbb{S})) \\
& \simeq \pi_\bullet(\Sigma^n A) \\
& \simeq [\mathbb{S}, \Sigma^n A] \\
& = [\mathbb{S}, A]_{-n} \\
& \simeq \pi_{\bullet - n}(A)
\end{aligned}
\,.
$$

Here we use

- first the definition (remark 5.2);
- then the fact that suspension commutes with smash product (lemma 5.7, part of the tensor triangulated structure of prop. 5.4);
- then the fact that the sphere spectrum is the tensor unit of the smash product of spectra (cor. 5.1);
- then the isomorphism of stable homotopy groups with graded homs out of the spjere spectrum (lemma).

■

Lemma 5.9. *For $n_1, n_2 \in \mathbb{Z}$ then the following diagram commutes in $\mathrm{Ho}(\mathrm{Spectra})$:*

$$
\begin{array}{ccc}
(\Sigma^{n_1} \mathbb{S}) \wedge^L (\Sigma^{n_2} \mathbb{S}) & \overset{\simeq}{\longrightarrow} & \Sigma^{n_1 + n_2} \mathbb{S} \\
{}^{\mathllap{\tau_{\Sigma^{n_1}\mathbb{S}, \Sigma^{n_2}\mathbb{S}}}}\downarrow && \downarrow^{\mathrlap{(-1)}} \\
(\Sigma^{n_2} \mathbb{S}) \wedge^L (\Sigma^{n_1} \mathbb{S}) & \underset{\simeq}{\longrightarrow} & \Sigma^{n_1 + n_2} \mathbb{S}
\end{array}
\,.
$$

Proof. It is sufficient to prove this for $n_1, n_2 \in \mathbb{N} \hookrightarrow \mathbb{Z}$. From this the general statement follows by looping and using lemma 5.7.

So assume $n_1, n_2 \geq 0$.

Observe that the sphere spectrum $\mathbb{S} = \gamma(\mathbb{S}_{\mathrm{orth}}) \in \mathrm{Ho}(\mathrm{Spectra})$ is represented by the orthogonal sphere spectrum $\mathbb{S}_{\mathrm{orth}} = \Sigma^\infty_{\mathrm{orth}} S^0$ (def. 2.25) and since $\Sigma^\infty_{\mathrm{orth}}$ is a left Quillen functor (prop. 3.18) and $S^0 \in (\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}$ is cofibrant, this is a cofibrant orthogonal spectrum. Hence, as in the proof of lemma 5.7, $\Sigma^{n_1}\mathbb{S}$ is represented by

$$
\mathbb{S} \wedge S^{n_1} \simeq \Sigma^\infty_{\mathrm{orth}} S^{n_1} \,.
$$

Since $\Sigma^\infty_{\mathrm{orth}}$ is a symmetric monoidal functor by prop. 2.29, it makes the following diagram commute

$$
\begin{array}{ccc}
(\mathbb{S} \wedge S^{n_1}) \otimes_{\mathbb{S}_{\mathrm{orth}}} (\mathbb{S} \wedge S^{n_2}) & \overset{\tau^{\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})}_{\mathbb{S}\wedge S^{n_1}, \mathbb{S} \wedge S^{n_2}}}{\longrightarrow} & (\mathbb{S} \wedge S^{n_2}) \otimes_{\mathbb{S}_{\mathrm{orth}}} (\mathbb{S} \wedge S^{n_1}) \\
\downarrow && \downarrow \\
\mathbb{S} \wedge (S^{n_1} \wedge S^{n_2}) & \underset{\mathbb{S}(\tau^{\mathrm{Top}_{\mathrm{cg}}^{*/}}_{S^{n_1}, S^{n_2}})}{\longrightarrow} & \mathbb{S} \wedge (S^{n_2} \wedge S^{n_1})
\end{array}
\,.
$$

Now the homotopy class of $\tau^{\mathrm{Top}_{\mathrm{cg}}^{*/}}_{S^{n_1}, S^{n_2}}$ in

$$
[S^{n_1 + n_2}, S^{n_2 + n_1}]_* \simeq \pi_{n_1 + n_2}(S^{n_1 + n_2}) \simeq \mathbb{Z}
$$

is

$$
[\tau^{\mathrm{Top}_{\mathrm{cg}}^{*/}}_{S^{n_1}, S^{n_2}}] = \left\{ \begin{array}{ll} 1 & \text{if } n_1 \cdot n_2 \text{ even} \\ -1 & \text{if } n_1 \cdot n_2 \text{ odd} \end{array} \right. \,.
$$

This translates to $\mathbb{S} \wedge \tau^{\mathrm{Top}_{\mathrm{cg}}^{*/}}_{S^{n_1}, S^{n_2}}$ under the identification (lemma)

$$
[\mathbb{S}, X]_\bullet \simeq \pi_\bullet(X)
$$

and using the adjunction $(-) \wedge (S^{n_1 + n_2}) \dashv \mathrm{Maps}(S^{n_1 + n_2}, -)_*$ from prop. 1.37:

$$
[\mathbb{S} \wedge (S^{n_1 + n_2}), \mathbb{S} \wedge (S^{n_1 + n_2})] \simeq [\mathbb{S}, \mathbb{S} \wedge \mathrm{Maps}(S^{n_1 + n_2}, S^{n_1 + n_2})] \,.
$$

■

Homotopy ring spectra

We discuss commutative monoids in the tensor triangulated stable homotopy category (prop. 5.4).

In this section the only tensor product that plays a role is the derived smash product of spectra from corollary 5.1. Therefore to ease notation, in this section (and in all of Part 2) we write for short

$$\wedge \coloneqq \wedge^L \,.$$

Definition 5.10. A commutative monoid (E, μ, e) (def. 1.13) in the monoidal stable homotopy category $(\mathrm{Ho}(\mathrm{Spectra}), \wedge, \mathbb{S})$ of corollary 5.1 is called a **homotopy commutative ring spectrum**.

A module object (def. 1.16) over E is accordingly called a **homotopy module spectrum**.

Proposition 5.11. *For (E, μ, e) a homotopy commutative ring spectrum (def. 5.10), its stable homotopy groups (def.)*

$$\pi_\bullet(E)$$

canonically inherit the structure of a $\mathbb{Z}$-graded-commutative ring.

Moreover, for $X \in \mathrm{Ho}(\mathrm{Spectra})$ any spectrum, then the generalized homology (remark 5.2)

$$E_\bullet(X) \coloneqq \pi_\bullet(E \wedge X)$$

(i.e. the stable homotopy groups of the free module over E on X (prop. 1.20)) canonically inherits the structure of a left graded $\pi_\bullet(E)$-module, and similarly

$$X_\bullet(E) \coloneqq \pi_\bullet(X \wedge E)$$

canonically inherits the structure of a right graded $\pi_\bullet(E)$-module.

Proof. Under the identification (lemma)

$$\begin{aligned} \pi_\bullet(E) &\simeq [\mathbb{S}, E]_\bullet \\ &\simeq [\mathbb{S}, \Sigma^{-\bullet} E] \\ &\simeq [\Sigma^\bullet \mathbb{S}, E] \end{aligned}$$

let

$$\alpha_i : \Sigma^{n_i} \mathbb{S} \longrightarrow E$$

for $i \in \{1, 2\}$ be two elements of $\pi_\bullet(E)$.

Observe that there is a canonical identification

$$\Sigma^{n_1 + n_2} \mathbb{S} \simeq \Sigma^{n_1} \mathbb{S} \wedge \Sigma^{n_2} \mathbb{S}$$

since $\mathbb{S} \simeq \mathbb{S} \wedge \mathbb{S}$ is the tensor unit (cor. 5.1, lemma 1.2) using lemma 5.7 (part of the tensor triangulated structure from prop. 5.4). With this we may form the composite

$$\alpha_1 \cdot \alpha_2 : \Sigma^{n_1 + n_2} \mathbb{S} \overset{\simeq}{\longrightarrow} \Sigma^{n_1} \mathbb{S} \wedge \Sigma^{n_2} \mathbb{S} \overset{\alpha_1 \wedge \alpha_2}{\longrightarrow} E \wedge E \overset{\mu}{\longrightarrow} E \,.$$

That this pairing is associative and unital follows directly from the associativity and unitality of μ and the coherence of the isomorphism on the left (prop. 5.4). Evidently the pairing is graded. That it is bilinear follows since addition of morphisms in the stable homotopy category is given by forming their direct sum (prop.) and since $\wedge$ distributes over direct sum (lemma 5.5, part of the tensor triangulated structure of prop. 5.4)).

It only remains to show graded-commutivity of the pairing. This is exhibited by the following commuting diagram:

$$\begin{array}{ccc} \Sigma^{n_1 + n_2} \mathbb{S} & \overset{(-1)^{n_1 \cdot n_2}}{\longrightarrow} & \Sigma^{n_1 + n_2} \mathbb{S} \\ {}^{\simeq}\downarrow && \downarrow^{\simeq} \\ \Sigma^{n_1} \mathbb{S} \wedge \Sigma^{n_2} \mathbb{S} & \overset{\tau_{\Sigma^{n_1} \mathbb{S}, \Sigma^{n_2} \mathbb{S}}}{\longrightarrow} & \Sigma^{n_2} \mathbb{S} \wedge \Sigma^{n_1} \mathbb{S} \\ {}^{\alpha_1 \wedge \alpha_2}\downarrow && \downarrow^{\alpha_2 \wedge \alpha_1} \\ E \wedge E & \overset{\tau_{E,E}}{\longrightarrow} & E \wedge E \\ & {}_{\mu}\searrow \quad \swarrow_{\mu} & \\ & E & \end{array} \,.$$

Here the top square is that of lemma 5.9 (part of the tensor triangulated structure of prop. 5.4)), the middle square is the naturality square of the braiding (def. 1.4, cor. 5.1), and the bottom triangle commutes by definition of (E,μ,e) being a commutative monoid (def. 1.13).

Similarly given

$$\alpha : \Sigma^{n_1}\mathbb{S} \longrightarrow E$$

as before and

$$\nu : \Sigma^{n_2}\mathbb{S} \longrightarrow E \wedge X\,,$$

then an action is defined by the composite

$$\alpha \cdot \nu : \Sigma^{n_1+n_2}\mathbb{S} \overset{\simeq}{\longrightarrow} \Sigma^{n_1}\mathbb{S} \wedge \Sigma^{n_2}\mathbb{S} \overset{\alpha\wedge\nu}{\longrightarrow} E \wedge E \wedge X \overset{\mu\wedge \mathrm{id}}{\longrightarrow} E \wedge X\,.$$

This is clearly a graded pairing, and the action property and unitality follow directly from the associativity and unitality, respectively, of (E,μ,e).

Analogously for the right action on $X_\bullet(E)$. ▮

Example 5.12. **(ring structure on the stable homotopy groups of spheres)**

The sphere spectrum $\mathbb{S} = \gamma(\mathbb{S}_{\mathrm{orth}})$ is a homotopy commutative ring spectrum (def. 5.10).

On the one hand this is because it is the tensor unit for the derived smash product of spectra (by cor. 5.1), and by example 1.14 every such is canonically a (commutative) monoid. On the other hand we have the explicit representation by the orthogonal ring spectrum (def. 2.15) $\mathbb{S}_{\mathrm{orth}}$, according to lemma 2.7, and the localization functor γ is a symmetric lax monoidal functor (prop. 3.14, and in fact a strong monoidal functor on cofibrant objects such as $\mathbb{S}_{\mathrm{orth}}$ according to prop. 3.13) and hence preserves commutative monoids (prop. 1.50).

The stable homotopy groups of the sphere spectrum are of course the stable homotopy groups of spheres (exmpl.)

$$\pi_\bullet^s \coloneqq \pi_\bullet(\mathbb{S}) \simeq \underset{\longrightarrow_k}{\lim}\, \pi_{\bullet+k}(S^k)\,.$$

Now prop. 5.11 gives the stable homotopy groups of spheres the structure of a graded commutative ring. By the proof of prop. 5.11, the product operation in that ring sends elements $\alpha_i : \Sigma^{n_i}\mathbb{S} \longrightarrow \mathbb{S}$ to

$$\Sigma^{n_1+n_2}\mathbb{S} \overset{\simeq}{\longrightarrow} \Sigma^{n_1}\mathbb{S} \wedge \Sigma^{n_2}\mathbb{S} \overset{\alpha_1\wedge\alpha_2}{\longrightarrow} \mathbb{S}\wedge\mathbb{S} \underset{\simeq}{\overset{\mu^{\mathbb{S}}}{\longrightarrow}} \mathbb{S}\,,$$

where now not only the first morphism, but also the last morphism is an isomorphism (the isomorphism from lemma 1.2). Hence up to isomorphism, the ring structure on the stable homotopy groups of spheres *is* the derived smash product of spectra.

This implies that for $X, Y \in \mathrm{Ho}(\mathrm{Spectra})$ any two spectra, then the graded abelian group $[X,Y]_\bullet$ (def.) of morphisms from X to Y in the stable homotopy category canonically becomes a module over the ring $\pi_\bullet^s$

$$\pi_\bullet^s \otimes [X,Y]_\bullet \longrightarrow [X,Y]_\bullet$$

by

$$(\Sigma^{n_1}\mathbb{S} \overset{\alpha}{\to} \mathbb{S}), (\Sigma^{n_2}X \overset{f}{\to} Y) \;\mapsto\; \left(\Sigma^{n_1+n_2}X \overset{\simeq}{\to} \Sigma^{n_1}\mathbb{S}\wedge\Sigma^{n_2}X \overset{\alpha\wedge f}{\longrightarrow} \mathbb{S}\wedge Y \overset{\simeq}{\to} Y \right)\,.$$

In particular for every spectrum $X \in \mathrm{Ho}(\mathrm{Spectra})$, its stable homotopy groups $\pi_\bullet(X) \simeq [\mathbb{S},X]_\bullet$ (lemma) canonically form a module over $\pi_\bullet^s$. If $X = E$ happens to carry the structure of a homotopy commutative ring spectrum, then this module structure coincides the one induced from the unit

$$\pi_\bullet(e) : \pi_\bullet^s = \pi_\bullet(\mathbb{S}) \longrightarrow \pi_\bullet(E)$$

under prop. 5.11.

(It is straightforward to unwind all this categorical algebra to concrete component expressions by proceeding as in the proof of this lemma).)

This finally allows to uniquely characterize the stable homotopy theory that we have been discussing:

Theorem 5.13. ***(Schwede-Shipley uniqueness theorem)***

The homotopy category $\mathrm{Ho}(\mathcal{C})$ (def.) of every stable homotopy category $\mathcal{C}$ (def.) canonically has graded hom-groups with the structure of modules over $\pi_\bullet^s = \pi_\bullet(\mathbb{S})$ (example 5.12). In terms of this, the following are equivalent:

1. *There is a zig-zag of Quillen equivalences (def.) between $\mathcal{C}$ and the stable model structure on topological sequential spectra (thm.) (equivalently (thm. 4.12) the stable model structure on orthogonal spectra)*

$$\mathcal{C} \underset{\longrightarrow}{\overset{\longleftarrow}{\simeq_{\mathrm{Qu}}}} \underset{\longleftarrow}{\overset{\longrightarrow}{\simeq_{\mathrm{Qu}}}} \cdots \underset{\longleftarrow}{\overset{\longrightarrow}{\simeq_{\mathrm{Qu}}}} \mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}} \underset{\longrightarrow}{\overset{\longleftarrow}{\simeq_{\mathrm{Qu}}}} \mathrm{SeqSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}$$

2. *there is an equivalence of categories between the homotopy category $\mathrm{Ho}(\mathcal{C})$ and the stable homotopy category $\mathrm{Ho}(\mathrm{Spectra})$ (def.)*

$$\mathrm{Ho}(\mathcal{C}) \simeq \mathrm{Ho}(\mathrm{Spectra})$$

which is $\pi_\bullet^s$-linear on all hom-groups.

(Schwede-Shipley 02, Uniqueness theorem)

6. Examples

For reference, we consider some basic examples of orthogonal ring spectra (def. 2.15) E. By prop. 2.16 and corollary 5.1 each of these examples in particular represents a homotopy commutative ring spectrum (def. 5.10) in the tensor triangulated stable homotopy category (prop. 5.4).

We make use of these examples of homotopy commutative ring spectra E in Part 2 in the computation of E-Adams spectral sequences.

For constructing representations as orthogonal ring spectra of spectra that are already known as sequential spectra (def.) two principles are usefully kept in mind:

1. by prop. 2.16 it is sufficient to give an equivariant multiplicative pairing $E_{n_1} \wedge E_{n_2} \to E_{n_1+n_2}$ and equivariant unit maps $S^0 \to E_0$, $S^1 \to E_1$, from these the structure maps $S^{n_1} \wedge E_{n_2} \to E_{n_1+n_2}$ are already uniquely induced;

2. the choice of $O(n)$-action on E_n is governed mainly by the demand that the unit map $S^n \to E_n$ has to be equivariant, with respect to the $O(n)$-action on S^n induced by regarding S^n as the one-point compactification of the defining $O(n)$-representation on $\mathbb{R}^n$ ("representation sphere").

Sphere spectrum

We already described the orthogonal sphere spectrum $\mathbb{S}$ as an orthogonal ring spectrum in lemma 2.7. The component spaces are the spheres S^n with their $O(n)$-action as representation spheres, and the multiplication maps are the canonical identifications

$$S^{n_1} \wedge S^{n_2} \longrightarrow S^{n_1+n_2} \,.$$

More generally, by prop. 2.29 the orthogonal suspension spectrum functor is a strong monoidal functor, and so by prop. 2.16 the suspension spectrum of a monoid in $\mathrm{Top}_{\mathrm{cg}}^{*/}$ (for instance G_+ for G a topological group) canonically carries the structure of an orthogonal ring spectrum.

The orthogonal sphere spectrum is the special case of this with $\mathbb{S}_{\mathrm{orth}} \simeq \Sigma^\infty_{\mathrm{orth}} S^0$ for S^0 the tensor unit in $\mathrm{Top}_{\mathrm{cg}}^{*/}$ (example 1.10) and hence a monoid by example 1.14.

Eilenberg-MacLane spectra

We discuss the model of Eilenberg-MacLane spectra as symmetric spectra and orthogonal spectra. To that end, notice the following model for Eilenberg-MacLane spaces.

Definition 6.1. For A an abelian group and $n \in \mathbb{N}$, the **reduced A-linearization** $A[S^n]_*$ of the n-sphere S^n is the topological space, whose underlying set is the quotient of the tensor product with A of the free abelian group on the underlying set of S^n,

$$A \otimes_{\mathbb{Z}} [S^n] = A[S^n] \longrightarrow A[S^n]_*$$

by the relation that identifies every formal linear combination of the basepoint of S^n with 0. The topology is the induced quotient topology

$$\underset{k \in \mathbb{N}}{\sqcup} A^k \times (S^n)^k \longrightarrow A[S^n]_*$$

(of the disjoint union of product topological spaces, where A is equipped with the discrete topology).

(Aguilar-Gitler-Prieto 02, def. 6.4.20)

Proposition 6.2. *For A a countable abelian group, then the reduced A-linearization $A[S^n]_*$ (def. 6.1) is an Eilenberg-MacLane space, in that its homotopy groups are*

$$\pi_q(A[S^n]_*) \simeq \begin{cases} A & \text{if } q = n \\ * & \text{otherwise} \end{cases}$$

(in particular for $n \geq 1$ then there is a unique connected component and hence we need not specify a basepoint for the homotopy group).

(Aguilar-Gitler-Prieto 02, corollary 6.4.23)

Definition 6.3. For A a countable abelian group, then the **orthogonal Eilenberg-MacLane spectrum** HA is the orthogonal spectrum (def. 2.11) with

- component spaces

  $$(HA)_n \coloneqq A[S^n]_*$$

  being the reduced A-linearization (def. 6.1) of the representation sphere S^n;

- $O(n)$-action on $A[S^n]_*$ induced from the canonical $O(n)$-action on S^n (representation sphere);

- structure maps

  $$\sigma_n \,:\, S^1 (HA)_n \longrightarrow (HA)_{n+1}$$

  hence

  $$S^1 \wedge A[S^n] \longrightarrow A[S^{n+1}]$$

  given by

  $$\left(x, \left(\sum_i a_i y_i\right), \right) \mapsto \sum_i a_i (x, y_i) \,.$$

The incarnation of HA as a symmetric spectrum is the same, with the group action of $O(n)$ replaced by the subgroup action of the symmetric group $\Sigma(n) \hookrightarrow O(n)$.

If R is a commutative ring, then the Eilenberg-MacLane spectrum HR becomes a commutative orthogonal ring spectrum or symmetric ring spectrum (def. 2.15) by

1. taking the multiplication

   $$(HR)_{n_1} \wedge (HR)_{n_2} = R[S^{n_1}]_* \wedge R[S^{n_2}]_* \longrightarrow R[S^{n_1+n_2}] = (HR)_{n_1+n_2}$$

   to be given by

   $$\left(\left(\sum_i a_i x_i\right), \left(\sum_j b_j y_j\right) \right) \mapsto \sum_{i,j} (a_i \cdot b_j)(x_i, y_j)$$

2. taking the unit maps

   $$S^n \longrightarrow A[S^n]_* = (HR)_n$$

   to be given by the canonical inclusion of generators

   $$x \mapsto 1x \,.$$

(Schwede 12, example I.1.14)

Proposition 6.4. *The stable homotopy groups (def. 2.21) of an Eilenberg-MacLane spectrum HA (def. 6.3) are*

$$\pi_q(HA) \simeq \begin{cases} A & \text{if } q = 0 \\ 0 & \text{otherwise} \end{cases}$$

Thom spectra

We discuss the realization of *Thom spectra* as orthogonal ring spectra. For background on Thom spectra realized as sequential spectra see Part S the section *Thom spectra*.

Definition 6.5. As an orthogonal ring spectrum (def. 2.15), the **universal Thom spectrum** MO has

- component spaces

  $$(MO)_n \coloneqq EO(n)_+ \underset{O(n)}{\wedge} S^n$$

  the Thom spaces (def.) of the universal vector bundle (def.) of rank n;

- left $O(n)$-action induced by the remaining canonical left action of $EO(n)$;

- canonical multiplication maps (def.)

  $$(EO(n_1)_+ \underset{O(n_1)}{\wedge} S^{n_1}) \wedge (EO(n_2)_+ \underset{O(n_2)}{\wedge} S^{n_2} \longrightarrow EO(n_1+n_2)_+ \underset{O(n_1+n_2)}{\wedge} S^{n_1+n_2}$$

- unit maps

  $$S^n \simeq O(n)_+ \wedge_{O(n)} S^n \longrightarrow EO(n)_+ \wedge_{O(n)} S^n$$

  induced by the fiber inclusion $O(V) \hookrightarrow EO(V)$.

(Schwede 12, I, example 1.16)

For the universal complex Thom spectrum MU the construction is a priori directly analogous, but with the real Cartesian space $\mathbb{R}^n$ replace by the complex vector space $\mathbb{C}^n$ thoughout. This makes the n-sphere $S^n = S^{(\mathbb{R}^n)}$ be replaced by the $2n$-sphere $S^{2n} \simeq S^{\mathbb{C}^n}$ throughout. Hence the construction requires a second step in which the resulting S^2-spectrum (def.) is turned into an actual orthogonal spectrum. This proceeds differently than for sequential spectra (lemma) due to the need to have compatible orthogonal group-action on all spaces.

Definition 6.6. The **universal complex Thom spectrum** MU is represented as an orthogonal ring spectrum (def. 2.15) as follows

First consider the component spaces

$$\overline{MU}_n \coloneqq EU(n)_+ \wedge_{U(n)} S^{(\mathbb{C}^n)}$$

given by the Thom spaces (def.) of the complex universal vector bundle (def.) of rank n, and equipped with the $O(n)$-action which is induced via the canonical inclusions

$$O(n) \hookrightarrow U(n) \hookrightarrow EU(n)\,.$$

Regard these as equipped with the canonical pairing maps (def.)

$$\overline{\mu}_{n_1,n_2} \;\colon\; \overline{MU}_{n_1} \wedge \overline{MU}_{n_2} \longrightarrow \overline{MU}_{n_1+n_2}\,.$$

These are $U(n)$-equivariant, hence in particular $O(n)$-equivariant.

Then take the actual components spaces to be loop spaces of these:

$$MU_n \coloneqq \mathrm{Maps}(S^n, \overline{MU}_n)$$

and regard these as equipped with the conjugation action by $O(n)$ induced by the above action on $\overline{MU}_n$ and the canonical action on $S^n \simeq S^{(\mathbb{R}^n)}$.

Define the actual pairing maps

$$\mu_{n_1,n_2} \;\colon\; MU_{n_1} \wedge MU_{n_2} \longrightarrow MU_{n_1+n_2}$$

via

$$\begin{aligned} \mathrm{Maps}(S^{n_1}, \overline{MU}_{n_1}) \wedge \mathrm{Maps}(S^{n_2}, \overline{MU}_{n_2}) &\longrightarrow \mathrm{Maps}(S^{n_1+n_2}, \overline{MU}_{n_1+n_2}) \\ (\alpha_1, \alpha_2) &\mapsto \overline{\mu}_{n_1,n_2} \circ (\alpha_1 \wedge \alpha_2) \end{aligned}\,.$$

Finally in order to define the unit maps, consider the isomorphism

$$S^{2n} \simeq S^{\mathbb{C}^n} \simeq S^{\mathbb{R}^n \oplus i \mathbb{R}^n} \simeq S^n \wedge S^n$$

and then take the unit maps

$$S^n \longrightarrow (MU)_n = \mathrm{Maps}(S^n, \overline{MU}_n)$$

to be the adjuncts of the canonical embeddings

$$S^n \wedge S^n \simeq S^{\mathbb{C}^n} \simeq U(n)_+ \wedge_{U(n)} S^{\mathbb{C}^n} \longrightarrow EU(n)_+ \wedge_{U(n)} S^{\mathbb{C}^n} \,.$$

(Schwede 12, I, example 1.18)

7. Conclusion

We summarize the results about stable homotopy theory obtained above.

First of all we have established a commuting diagram of Quillen adjunctions and Quillen equivalences of the form

$$\array{ (Top^{*/}_{cg})_{Quillen} & \underoverset{\underset{\Omega}{\longrightarrow}}{\overset{\Sigma}{\longleftarrow}}{\bot} & (Top^{*/}_{cg})_{Quillen} \\ {}^{\mathllap{\Sigma^\infty}}\downarrow \dashv \uparrow^{\mathrlap{\Omega^\infty}} && {}^{\mathllap{\Sigma^\infty}}\downarrow \dashv \uparrow^{\mathrlap{\Omega^\infty}} \\ SeqSpec(Top_{cg})_{strict} & \underoverset{\underset{\Omega}{\longrightarrow}}{\overset{\Sigma}{\longleftarrow}}{\bot} & SeqSpec(Top_{cg})_{strict} \\ {}^{\mathllap{id}}\downarrow \dashv \uparrow^{\mathrlap{id}} && {}^{\mathllap{id}}\downarrow \dashv \uparrow^{\mathrlap{id}} \\ SeqSpec(Top_{cg})_{stable} & \underoverset{\underset{\Omega}{\longrightarrow}}{\overset{\Sigma}{\longleftarrow}}{\simeq_Q} & SeqSpec(Top_{cg})_{stable} \\ {}^{\mathllap{seq_!}}\downarrow \simeq_Q \uparrow^{\mathrlap{seq^*}} && {}^{\mathllap{seq_!}}\downarrow \simeq_Q \uparrow^{\mathrlap{seq^*}} \\ OrthSpec(Top_{cg})_{stable} && OrthSpec(Top_{cg})_{stable} }$$

where

* $(Top^{*/}_{cg})_{Quillen}$ is the classical model structure on pointed topological spaces (thm., thm.);
* $SeqSpec(Top_{cg})_{stable}$ is the stable model structure on topological sequential spectra (thm.);
* $OrthSpec(Top_{cg})_{stable}$ is the stable model structure on orthogonal spectra from theorem 4.1.

Here the top part of the diagram is from remark 3.19, while the vertical Quillen equivalence $(seq_! \dashv seq^*)$ is from theorem 4.1.

Moreover, the top and bottom model categories are monoidal model categories (def. 3.11): $Top^{*/}_{cg}$ with respect to the smash product of pointed topological spaces (theorem 3.17) and $OrthSpec(Top_{cg})_{strict}$ as well as $OrthSpec(Top_{cg})_{stable}$ with respect to the symmetric monoidal smash product of spectra (theorem 3.17 and theorem 4.14); and the compsite vertical adjunction

$$\array{ (Top^{*/}_{cg}, \wedge, S^0) \\ {}^{\mathllap{\Sigma^\infty_{orth}}}\downarrow \dashv \uparrow^{\mathrlap{\Omega^\infty_{orth}}} \\ (OrthSpec(Top_{cg}), \wedge, \mathbb{S}_{orth}) }$$

is a strong monoidal Quillen adjunction (def. 3.15, corollary 4.15), and so also the induced adjunction of derived functors

$$\array{ (Ho(Top^{*/}), \wedge^L, S^0) \\ {}^{\mathllap{\Sigma^\infty}}\downarrow \dashv \uparrow^{\mathrlap{\Omega^\infty}} \\ (Ho(Spectra), \wedge^L, \mathbb{S}) }$$

is a strong monoidal adjunction (by prop. 3.16) from the the derived smash product of pointed topological spaces to the derived symmetric smash product of spectra.

Under passage to homotopy categories this yields a commuting diagram of derived adjoint functors

$$\array{ Ho(Top^{*/}) & \underoverset{\underset{\Omega}{\longrightarrow}}{\overset{\Sigma}{\longleftarrow}}{\bot} & Ho(Top^{*/}) \\ {}^{\mathllap{\Sigma^\infty}}\downarrow \dashv \uparrow^{\mathrlap{\Omega^\infty}} && {}^{\mathllap{\Sigma^\infty}}\downarrow \dashv \uparrow^{\mathrlap{\Omega^\infty}} \\ Ho(Spectra) & \underoverset{\underset{\Omega}{\longrightarrow}}{\overset{\Sigma}{\longleftarrow}}{\simeq} & Ho(Spectra) }$$

between the (Serre-Quillen-)classical homotopy category $Ho(Top^{*/})$ and the stable homotopy category $Ho(Spectra)$ (remark 4.13). The latter is an additive category (def.) with direct sum the wedge sum of spectra

$\oplus = \vee$ (lemma, lemma) and in fact a triangulated category (def.) with distinguished triangles the homotopy cofiber sequences of spectra (prop.).

While this is the situation already for sequential spectra (thm.), in addition we have now that both the classical homotopy category as well as the stable homotopy category are symmetric monoidal categories with respect to derived smash product of pointed topological spaces and the derived symmetric monoidal smash product of spectra, respectively (corollary 5.1).

Moreover, the derived smash product of spectra is compatible with the additive category structure (direct sums) and the triangulated category structure (homotopy cofiber sequences), this being a *tensor triangulated category* (prop. 5.4).

abelian groups	**spectra**
integers $\mathbb{Z}$	sphere spectrum $\mathbb{S}$
$\mathrm{Ab} \simeq \mathbb{Z}\mathrm{Mod}$	$\mathrm{Spectra} \simeq \mathbb{S}\mathrm{Mod}$
direct sum $\oplus$	wedge sum $\vee$
tensor product $\otimes_{\mathbb{Z}}$	smash product of spectra $\wedge_{\mathbb{S}}$
kernels/cokernels	homotopy fibers/homotopy cofibers

The commutative monoids with respect to this smash product of spectra are precisely the commutative orthogonal ring spectra (def. 2.15, prop. 2.16) and the module objects over these are precisely the orthogonal module spectra (def. 2.17, prop. 2.18).

algebra	**homological algebra**	**higher algebra**
abelian group	chain complex	spectrum
ring	dg-ring	ring spectrum
module	dg-module	module spectrum

The localization functors γ (def.) from the monoidal model categories to their homotopy categories are lax monoidal functors (cor. 5.1)

$$\begin{aligned} (\mathrm{Top}_{\mathrm{cg}}^{*/}, \wedge, S^0) &\longrightarrow (\mathrm{Ho}(\mathrm{Top}^{*/}), \wedge^L, \gamma(S^0)) \\ (\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}}), \wedge, \mathbb{S}_{\mathrm{orth}}) &\longrightarrow (\mathrm{Ho}(\mathrm{Spectra}), \wedge^L, \gamma(\mathbb{S})) \end{aligned} \,.$$

This implies that for $E \in \mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})$ a commutativeorthogonal ring spectrum, then its image $\gamma(E)$ in the stable homotopy category is a homotopy commutative ring spectrum (def. 5.10) and similarly for module spectra (prop. 1.50).

monoidal stable model category	**-localization→**	**tensor triangulated category**
stable model structure on orthogonal spectra $\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}}$		stable homotopy category $\mathrm{Ho}(\mathrm{Spectra})$
symmetric monoidal smash product of spectra		derived smash product of spectra
commutative orthogonal ring spectrum (E-infinity ring)		homotopy commutative ring spectrum

Finally, the graded hom-groups $[X,Y]_\bullet$ (def.) in the tensor triangulated stable homotopy category are canonically graded modules over the graded commutative ring of stable homotopy groups of spheres (exmpl. 5.12)

$$[X,Y]_\bullet \in \pi_\bullet(\mathbb{S})\mathrm{Mod} \,.$$

Hence the next question is how to actually compute any of these. This is the topic of *Part 2 -- The Adams spectral sequence*.

8. References

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nLab

Introduction to Spectral Sequences

This page is an introduction to spectral sequences. We motivate spectral sequences of filtered complexes from the computation of cellular cohomology via stratum-wise relative cohomology. In the end we generalize to spectral sequences of filtered spectra.

For background on homological algebra see at *Introduction to Homological algebra*.

For background on stable homotopy theory see at *Introduction to Stable homotopy theory*.

For application to complex oriented cohomology see at *Introduction to Cobordism and Complex Oriented Cohomology*.

For application to the Adams spectral sequence see *Introduction to Adams spectral sequences*.

Contents

In *Introduction to Stable homotopy theory* we have set up the concept of spectra X and their stable homotopy groups $\pi_\bullet(X)$ (def.). More generally for X and Y two spectra then there is the graded stable homotopy group $[X,Y]_\bullet$ of homotopy classes of maps bewteen them (def.). These may be thought of as generalized cohomology groups (exmpl.). Moreover, in *part 1.2* we discussed the symmetric monoidal smash product of spectra $X \wedge Y$. The stable homotopy groups of such a smash product spectrum may be thought of as generalized homology groups (rmk.).

These stable homotopy and generalized (co-)homology groups are the fundamental invariants in algebraic topology. In general they are as rich and interesting as they are hard to compute, as famously witnessed by the stable homotopy groups of spheres, some of which we compute in part 2.

In general the only practicable way to carry out such computations is by doing them along a decomposition of the given spectrum into a "sequence of stages" of sorts. The concept of *spectral sequence* is what formalizes this idea.

(Here the re-occurence of the root "spectr-" it is a historical coincidence, but a lucky one.)

Here we give a expository introduction to the concept of spectral sequences, building up in detail to the spectral sequence of a filtered complex.

We put these spectral sequences to use in

- *part 2 -- Adams spectral sequences.*
- *part S -- Complex oriented cohomology theory*

1. For filtered complexes

We begin with recalling basics of ordinary *relative homology* and then seamlessly derive the notion of *spectral sequences* from that as the natural way of computing the ordinary cohomology of a CW-complex stagewise from the relative cohomology of its skeleta. This is meant as motivation and warmup. What we are mostly going to use further below are spectral sequences induced by filtered spectra, this we turn to next.

Ordinary homology

Let X be a topological space and $A \hookrightarrow X$ a topological subspace. Write $C_\bullet(X)$ for the chain complex of singular homology on X and $C_\bullet(A) \hookrightarrow C_\bullet(X)$ for the chain map induced by the subspace inclusion.

Definition 1.1. The (degreewise) cokernel of this inclusion, hence the quotient $C_\bullet(X)/C_\bullet(A)$ of $C_\bullet(X)$ by the image of $C_\bullet(A)$ under the inclusion, is the **chain complex of A-relative singular chains**.

- A boundary in this quotient is called an **A-relative singular boundary**,
- a cycle is called an **A-relative singular cycle**.
- The chain homology of the quotient is the **A-relative singular homology of X**

$$H_n(X, A) \coloneqq H_n(C_\bullet(X)/C_\bullet(A)) .$$

Remark 1.2. This means that a singular $(n+1)$-chain $c \in C_{n+1}(X)$ is an A-relative cycle precisely if its boundary $\partial c \in C_n(X)$ is, while not necessarily 0, contained in the n-chains of A: $\partial c \in C_n(A) \hookrightarrow C_n(X)$. So the boundary vanishes possibly only "up to contributions coming from A".

We record two evident but important classes of long exact sequences that relative homology groups sit in:

Proposition 1.3. *Let $A \overset{i}{\hookrightarrow} X$ be a topological subspace inclusion. The corresponding relative singular homology, def. 1.1, sits in a long exact sequence of the form*

$$\cdots \to H_n(A) \xrightarrow{H_n(i)} H_n(X) \to H_n(X, A) \xrightarrow{\delta_{n-1}} H_{n-1}(A) \xrightarrow{H_{n-1}(i)} H_{n-1}(X) \to H_{n-1}(X, A) \to \cdots .$$

The connecting homomorphism $\delta_n : H_{n+1}(X, A) \to H_n(A)$ sends an element $[c] \in H_{n+1}(X, A)$ represented by an A-relative cycle $c \in C_{n+1}(X)$, to the class represented by the boundary $\partial^X c \in C_n(A) \hookrightarrow C_n(X)$.

Proof. This is the homology long exact sequence, induced by the defining short exact sequence $0 \to C_\bullet(A) \overset{i}{\hookrightarrow} C_\bullet(X) \to \mathrm{coker}(i) \simeq C_\bullet(X)/C_\bullet(A) \to 0$ of chain complexes. ▮

Proposition 1.4. *Let $B \hookrightarrow A \hookrightarrow X$ be a sequence of two topological subspace inclusions. Then there is a long exact sequence of relative singular homology groups of the form*

$$\cdots \to H_n(A, B) \to H_n(X, B) \to H_n(X, A) \to H_{n-1}(A, B) \to \cdots .$$

Proof. Observe that we have a short exact sequence of chain complexes, def. \ref{ShortExactSequenceOfChainComplexes}

$$0 \to C_\bullet(A)/C_\bullet(B) \to C_\bullet(X)/C_\bullet(B) \to C_\bullet(X)/C_\bullet(A) \to 0 .$$

The corresponding homology long exact sequence, prop. \ref{HomologyLongExactSequence}, is the long exact sequence in question. ▮

We look at some concrete fundamental examples in a moment. But first it is useful to make explicit the following general sub-notion of relative homology.

Let X still be a given topological space.

Definition 1.5. The **augmentation map** for the singular homology of X is the homomorphism of abelian groups

$$\epsilon : C_0(X) \to \mathbb{Z}$$

which adds up all the coefficients of all 0-chains:

$$\epsilon : : \sum_i n_i \sigma_i \mapsto \sum_i n_i .$$

Since the boundary of a 1-chain is in the kernel of this map, by example \ref{BasicExamplesOfChainBoundaries}, it constitutes a chain map

$$\epsilon : C_\bullet(X) \to \mathbb{Z},$$

where now $\mathbb{Z}$ is regarded as a chain complex concentrated in degree 0.

Definition 1.6. The **reduced singular chain complex** $\tilde{C}_\bullet(X)$ of X is the kernel of the augmentation map, the chain complex sitting in the short exact sequence

$$0 \to \tilde{C}_\bullet(C) \to C_\bullet(X) \overset{\epsilon}{\to} \mathbb{Z} \to 0 .$$

The **reduced singular homology** $\tilde{H}_\bullet(X)$ of X is the chain homology of the reduced singular chain complex

$$\tilde{H}_\bullet(X) \coloneqq H_\bullet(\tilde{C}_\bullet(X)) \; .$$

Equivalently:

Definition 1.7. The **reduced singular homology** of X, denoted $\tilde{H}_\bullet(X)$, is the chain homology of the augmented chain complex

$$\cdots \to C_2(X) \overset{\partial_1}{\to} C_1(X) \overset{\partial_0}{\to} C_0(X) \overset{\epsilon}{\to} \mathbb{Z} \to 0 \; .$$

Let X be a topological space, $H_\bullet(X)$ its singular homology and $\tilde{H}_\bullet(X)$ its reduced singular homology, def. 1.6.

Proposition 1.8. *For $n \in \mathbb{N}$ there is an isomorphism*

$$H_n(X) \simeq \begin{cases} \tilde{H}_n(X) & \text{for } n \geq 1 \\ \tilde{H}_0(X) \oplus \mathbb{Z} & \text{for } n = 0 \end{cases}$$

Proof. The homology long exact sequence, prop. \ref{HomologyLongExactSequence}, of the defining short exact sequence $\tilde{C}_\bullet(C) \to C_\bullet(X) \overset{\epsilon}{\to} \mathbb{Z}$ is, since $\mathbb{Z}$ here is concentrated in degree 0, of the form

$$\cdots \to \tilde{H}_n(X) \to H_n(X) \to 0 \to \cdots \to 0 \to \cdots \to \tilde{H}_1(X) \to H_1(X) \to 0 \to \tilde{H}_0(X) \to H_0(X) \overset{\epsilon}{\to} \mathbb{Z} \to 0 \; .$$

Here exactness says that all the morphisms $\tilde{H}_n(X) \to H_n(X)$ for positive n are isomorphisms. Moreover, since $\mathbb{Z}$ is a free abelian group, hence a projective object, the remaining short exact sequence

$$0 \to \tilde{H}_0(X) \to H_0(X) \to \mathbb{Z} \to 0$$

is split, by prop. \ref{SplittingLemma}, and hence $H_0(X) \simeq \tilde{H}_0(X) \oplus \mathbb{Z}$. ∎

Proposition 1.9. *For $X = *$ the point, the morphism*

$$H_0(\epsilon) : H_0(X) \to \mathbb{Z}$$

is an isomorphism. Accordingly the reduced homology of the point vanishes in every degree:

$$\tilde{H}_\bullet(*) \simeq 0 \; .$$

Proof. By the discussion in section 2) we have that

$$H_n(*) \simeq \begin{cases} \mathbb{Z} & \text{for } n = 0 \\ 0 & \text{otherwise} \end{cases} .$$

Moreover, it is clear that $\epsilon : C_0(*) \to \mathbb{Z}$ is the identity map. ∎

Now we can discuss the relation between reduced homology and relative homology.

Proposition 1.10. *For X an inhabited topological space, its reduced singular homology, def. 1.6, coincides with its relative singular homology relative to any base point $x : * \to X$:*

$$\tilde{H}_\bullet(X) \simeq H_\bullet(X, *) \; .$$

Proof. Consider the sequence of topological subspace inclusions

$$\emptyset \hookrightarrow * \overset{x}{\hookrightarrow} X \; .$$

By prop. 1.4 this induces a long exact sequence of the form

$$\cdots \to H_{n+1}(*) \to H_{n+1}(X) \to H_{n+1}(X,*) \to H_n(*) \to H_n(X) \to H_n(X,*) \to \cdots \to H_1(X) \to H_1(X,*) \to H_0(*) \xrightarrow{H_0(x)} H_0(X) \to H_n(X,*) \to$$

Here in positive degrees we have $H_n(*) \simeq 0$ and therefore exactness gives isomorphisms

$$H_n(X) \overset{\simeq}{\to} H_n(X,*) \quad \forall_{n \geq 1}$$

and hence with prop. 1.8 isomorphisms

$$\tilde{H}_n(X) \overset{\simeq}{\to} H_n(X,*) \quad \forall_{n \geq 1} \; .$$

It remains to deal with the case in degree 0. To that end, observe that $H_0(x) : H_0(*) \to H_0(X)$ is a monomorphism: for this notice that we have a commuting diagram

$$\begin{array}{ccc} H_0(*) & \overset{\mathrm{id}}{\to} & H_0(*) \\ {}^{H_0(x)}\downarrow & {}^{H_0(f)}\nearrow & \downarrow^{H_0(\epsilon)}_{\simeq} \\ H_0(X) & \overset{H_0(\epsilon)}{\longrightarrow} & \mathbb{Z} \end{array},$$

where $f : X \to *$ is the terminal map. That the outer square commutes means that $H_0(\epsilon) \circ H_0(x) = H_0(\epsilon)$ and hence the composite on the left is an isomorphism. This implies that $H_0(x)$ is an injection.

Therefore we have a short exact sequence as shown in the top of this diagram

$$\begin{array}{ccccccccc} 0 & \to & H_0(*) & \overset{H_0(x)}{\hookrightarrow} & H_0(X) & \to & H_0(X,*) & \to & 0 \\ & & & {}_{\simeq}\searrow & \downarrow^{H_0(\epsilon)} & & & & \\ & & & & \mathbb{Z} & & & & \end{array}.$$

Using this we finally compute

$$\begin{aligned} \tilde{H}_0(X) &\coloneqq \ker H_0(\epsilon) \\ &\simeq \operatorname{coker}(H_0(x)) \\ &\simeq H_0(X,*) \end{aligned}.$$

∎

With this understanding of homology *relative to a point* in hand, we can now characterize relative homology more generally. From its definition in def. 1.1, it is plausible that the relative homology group $H_n(X,A)$ provides information about the quotient topological space X/A. This is indeed true under mild conditions:

Definition 1.11. A topological subspace inclusion $A \hookrightarrow X$ is called a **good pair** if

1. A is closed inside X;

2. A has an neighbourhood $A \hookrightarrow U \hookrightarrow X$ such that $A \hookrightarrow U$ has a deformation retract.

Proposition 1.12. *If $A \hookrightarrow X$ is a topological subspace inclusion which is good in the sense of def. 1.11, then the A-relative singular homology of X coincides with the reduced singular homology, def. 1.6, of the quotient space X/A:*

$$H_n(X/A) \simeq \tilde{H}_n(X,A) .$$

The proof of this is spelled out at *Relative homology – relation to quotient topological spaces*. It needs the proof of the *Excision property* of relative homology. While important, here we will not further dwell on this. The interested reader can find more information behind the above links.

Cellular homology

With the general definition of relative homology in hand, we now consider the basic *cells* such that *cell complexes* built from such cells have tractable relative homology groups. Actually, up to weak homotopy equivalence, *every* Hausdorff topological space is given by such a cell complex and hence its relative homology, then called *cellular homology*, is a good tool for computing singular homology rather generally.

Definition 1.13. For $n \in \mathbb{N}$ write

- $D^n \hookrightarrow \mathbb{R}^n \in$ Top for the standard n-disk;

- $S^{n-1} \hookrightarrow \mathbb{R}^n \in$ Top for the standard $(n-1)$-sphere;

  (notice that the 0-sphere is the disjoint union of *two points*, $S^0 = * \coprod *$, and by definition the (-1)-sphere is the empty set)

- $S^{-1} \hookrightarrow D^n$ for the continuous function that includes the $(n-1)$-sphere as the boundary of the n-disk.

Example 1.14. The reduced singular homology of the n-sphere S^n equals the S^{n-1}-relative homology of the n-disk with respect to the canonical boundary inclusion $S^{n-1} \hookrightarrow D^n$: for all $n \in \mathbb{N}$

$$\tilde{H}_\bullet(S^n) \simeq H_\bullet(D^n, S^{n-1}) .$$

Proof. The n-sphere is homeomorphic to the n-disk with its entire boundary identified with a point:

$$S^n \simeq D^n / S^{n-1} .$$

Moreover the boundary inclusion is a *good pair* in the sense of def. 1.11. Therefore the example follows with prop. 1.12. ▮

When forming cell complexes from disks, then each relative dimension will be a wedge sum of disks:

Definition 1.15. For $\{x_i : * \to X_i\}_i$ a set of pointed topological spaces, their ***wedge sum*** $\vee_i X_i$ is the result of identifying all base points in their disjoint union, hence the quotient

$$\left(\coprod_i X_i\right) / \left(\coprod_i *\right) .$$

Example 1.16. The wedge sum of two pointed circles is the "figure 8"-topological space.

Proposition 1.17. *Let $\{* \to X_i\}_i$ be a set of pointed topological spaces. Write $\vee_i X_i \in \mathrm{Top}$ for their wedge sum and write $\iota_i : X_i \to \vee_i X_i$ for the canonical inclusion functions.*

Then for each $n \in \mathbb{N}$ the homomorphism

$$(\tilde H_n(\iota_i))_i : \oplus_i \tilde H_n(X_i) \to \tilde H_n(\vee_i X_i)$$

is an isomorphism.

Proof. By prop. 1.12 the reduced homology of the wedge sum is equivalently the relative homology of the disjoint union of spaces relative to their disjoint union of basepoints

$$\tilde H_n(\vee_i X_i) \simeq H_n(\coprod_i X_i, \coprod_i *) .$$

The relative homology preserves these coproducts (sends them to direct sums) and so

$$H_n(\coprod_i X_i, \coprod_i *) \simeq \oplus_i H_n(X_i, *) .$$

▮

The following defines topological spaces which are inductively built by gluing disks to each other.

Definition 1.18. A **CW complex of dimension** (-1) is the empty topological space.

By induction, for $n \in \mathbb{N}$ a **CW complex of dimension** n is a topological space X_n obtained from

1. a CW-complex X_{n-1} of dimension $n-1$;
2. an index set $\mathrm{Cell}(X)_n \in \mathrm{Set}$;
3. a set of continuous maps (the **attaching maps**) $\{f_i : S^{n-1} \to X_{n-1}\}_{i \in \mathrm{Cell}(X)_n}$

as the pushout

$$X_n \simeq \left(\coprod_{j \in \mathrm{Cell}(X)_n} D^n\right) \coprod_{j \in \mathrm{Cell}(X)_n S^{n-1}} X_n$$

in

$$\begin{array}{ccc} \coprod_{j \in \mathrm{Cell}(X)_n} S^{n-1} & \overset{(f_j)}{\longrightarrow} & X_{n-1} \\ \downarrow && \downarrow \\ \coprod_{j \in \mathrm{Cell}(X)_n} D^n & \to & X_n \end{array} ,$$

hence as the topological space obtained from X_{n-1} by gluing in n-disks D^n for each $j \in \mathrm{Cell}(X)_n$ along the given boundary inclusion $f_j : S^{n-1} \to X_{n-1}$.

By this construction, an n-dimensional CW-complex is canonically a *filtered topological space*, hence a sequence of topological subspace inclusions of the form

$$\emptyset \hookrightarrow X_0 \hookrightarrow X_1 \hookrightarrow \cdots \hookrightarrow X_{n-1} \hookrightarrow X_n$$

which are the right vertical morphisms in the above pushout diagrams.

A general **CW complex** X then is a topological space which is the limiting space of a possibly infinite such sequence, hence a topological space given as the sequential colimit over a tower diagram each of whose morphisms is such a filter inclusion

$$\emptyset \hookrightarrow X_0 \hookrightarrow X_1 \hookrightarrow \cdots \hookrightarrow X \,.$$

The following basic facts about the singular homology of CW complexes are important.

Now we can state a variant of singular homology adapted to CW complexes which admits a more systematic way of computing its homology groups. First we observe the following.

Proposition 1.19. *The relative singular homology, def. 1.1, of the filtering degrees of a CW complex X, def. 1.18, is*

$$H_n(X_k, X_{k-1}) \simeq \begin{cases} \mathbb{Z}[\mathrm{Cells}(X)_n] & \text{if } k = n \\ 0 & \text{otherwise} \end{cases},$$

where $\mathbb{Z}[\mathrm{Cells}(X)_n]$ denotes the free abelian group on the set of n-cells.

Proof. The inclusion $X_{k-1} \hookrightarrow X_k$ is a *good pair* in the sense of def. 1.11. The quotient X_k / X_{k-1} is by definition of CW-complexes a wedge sum, def. 1.15, of k-spheres, one for each element in $\mathrm{Cell}(X)_k$. Therefore by prop. 1.12 we have an isomorphism $H_n(X_k, X_{k-1}) \simeq \tilde{H}_n(X_k / X_{k-1})$ with the reduced homology of this wedge sum. The statement then follows by the respect of reduced homology for wedge sums, prop. 1.17. ∎

Proposition 1.20. *For X a CW complex with skeletal filtration $\{X_n\}_n$ as above, and with $k, n \in \mathbb{N}$ we have for the singular homology of X that*

$$(k > n) \Rightarrow (H_k(X_n) \simeq 0) \,.$$

In particular if X is a CW-complex of finite dimension $\dim X$ (the maximum degree of cells), then

$$(k > \dim X) \Rightarrow (H_k(X) \simeq 0) \,.$$

Moreover, for $k < n$ the inclusion

$$H_k(X_n) \overset{\simeq}{\to} H_k(X)$$

is an isomorphism and for $k = n$ we have an isomorphism

$$\mathrm{image}(H_n(X_n) \to H_n(X)) \simeq H_n(X) \,.$$

Proof. By the long exact sequence in relative homology, prop. 1.3 we have an exact sequence of the form

$$H_{k+1}(X_n, X_{n-1}) \to H_k(X_{n-1}) \to H_k(X_n) \to H_k(X_n, X_{n-1}) \,.$$

Now by prop. 1.19 the leftmost and rightmost homology groups here vanish when $k \neq n$ and $k \neq n-1$ and hence exactness implies that

$$H_k(X_{n-1}) \overset{\simeq}{\to} H_k(X_n)$$

is an isomorphism for $k \neq n, n-1$. This implies the first claims by induction on n.

Finally for the last claim use that the above exact sequence gives

$$H_{n-1+1}(X_n, X_{n-1}) \to H_{n-1}(X_{n-1}) \to H_{n-1}(X_n) \to 0$$

and hence that with the above the map $H_{n-1}(X_{n-1}) \to H_{n-1}(X)$ is surjective. ∎

We may now discuss the *cellular homology* of a CW complex.

Definition 1.21. For X a CW-complex, def. 1.18, its **cellular chain complex** $H_\bullet^{\mathrm{CW}}(X) \in \mathrm{Ch}_\bullet$ is the chain complex such that for $n \in \mathbb{N}$

- the abelian group of chains is the relative singular homology group, def. 1.1, of $X_n \hookrightarrow X$ relative to $X_{n-1} \hookrightarrow X$:

  $$H_n^{\mathrm{CW}}(X) := H_n(X_n, X_{n-1}) \,,$$

- the differential $\partial_{n+1}^{\mathrm{CW}} : H_{n+1}^{\mathrm{CW}}(X) \to H_n^{\mathrm{CW}}(X)$ is the composition

  $$\partial_n^{\mathrm{CW}} : H_{n+1}(X_{n+1}, X_n) \overset{\partial_n}{\to} H_n(X_n) \overset{i_n}{\to} H_n(X_n, X_{n-1}) \,,$$

  where ∂_n is the boundary map of the singular chain complex and where i_n is the morphism on relative homology induced from the canonical inclusion of pairs $(X_n, \emptyset) \to (X_n, X_{n-1})$.

Proposition 1.22. *The composition $\partial_n^{\mathrm{CW}} \circ \partial_{n+1}^{\mathrm{CW}}$ of two differentials in def. 1.21 is indeed zero, hence $H_\bullet^{\mathrm{CW}}(X)$ is indeed a chain complex.*

Proof. On representative singular chains the morphism i_n acts as the identity and hence $\partial_n^{\mathrm{CW}} \circ \partial_{n+1}^{\mathrm{CW}}$ acts as the double singular boundary, $\partial_n \circ \partial_{n+1} = 0$. ▮

Remark 1.23. This means that

- a **cellular n-chain** is a singular n-chain required to sit in filtering degree n, hence in $X_n \hookrightarrow X$;
- a **cellular n-cycle** is a singular n-chain whose singular boundary is not necessarily 0, but is contained in filtering degree $(n-2)$, hence in $X_{n-2} \hookrightarrow X$.
- a **cellular n-boundary** is a singular n-chain which is the boundary of a singular $(n+1)$-chain coming from filtering degree $(n+1)$.

This kind of situation – chains that are cycles only up to lower filtering degree and boundaries that come from specified higher filtering degree – has an evident generalization to higher relative filtering degrees. And in this greater generality the concept is of great practical relevance. Therefore before discussing cellular homology further now, we consider this more general "higher-order relative homology" that it suggests (namely the formalism of spectral sequences). After establishing a few fundamental facts about that we will come back in prop. 1.46 below to analyse the above cellular situation using this conceptual tool.

In theorem 1.48 we conclude that cellular homology and singular homology agree of CW-complexes agres.

First we abstract the structure on chain complexes that in the above example was induced by the CW-complex structure on the singular chain complex.

Filtered chain complexes

Definition 1.24. The structure of a **filtered chain complex** in a chain complex $C_\bullet$ is a sequence of chain map inclusions

$$\cdots \hookrightarrow F_{p-1}C_\bullet \hookrightarrow F_pC_\bullet \hookrightarrow \cdots \hookrightarrow C_\bullet \;.$$

The **associated graded complex** of a filtered chain complex, denoted $G_\bullet C_\bullet$, is the collection of quotient chain complexes

$$G_pC_\bullet \coloneqq F_pC_\bullet / F_{p-1}C_\bullet \;.$$

We say that element of $G_pC_\bullet$ are *in filtering degree* p.

Remark 1.25. In more detail this means that

1. $[\cdots \overset{\partial_n}{\to} C_n \xrightarrow{\partial_{n-1}} C_{n-1} \to \cdots]$ is a chain complex, hence $\{C_n\}$ are objects in $\mathcal{A}$ (R-modules) and $\{\partial_n\}$ are morphisms (module homomorphisms) with $\partial_{n+1} \circ \partial_n = 0$;
2. For each $n \in \mathbb{Z}$ there is a filtering $F_\bullet C_n$ on C_n and all these filterings are compatible with the differentials in that

$$\partial(F_pC_n) \subset F_pC_{n-1}$$

3. The grading associated to the filtering is such that the p-graded elements are those in the quotient

$$G_pC_n \coloneqq \frac{F_pC_n}{F_{p-1}C_n} \;.$$

Since the differentials respect the grading we have chain complexes $G_pC_\bullet$ in each filtering degree p.

Hence elements in a filtered chain complex are **bi-graded**: they carry a degree as elements of $C_\bullet$ as usual, but now they also carry a filtering degree: for $p, q \in \mathbb{Z}$ we therefore also write

$$C_{p,q} \coloneqq F_pC_{p+q}$$

and call this the collection of **(p,q)-chains** in the filtered chain complex.

Accordingly we have (p,q)-cycles and -boundaries. But for these we may furthermore refine to a notion where also the filtering degree of the boundaries is constrained:

Definition 1.26. Let $F_\bullet C_\bullet$ be a filtered chain complex. Its associated graded chain complex is the set of chain complexes

$$G_pC_\bullet \coloneqq F_pC_\bullet / F_{p-1}C_\bullet$$

for all p.

Then for $r, p, q \in \mathbb{Z}$ we say that

1. $G_p C_{p+q}$ is the module of (p,q)-***chains*** or of $(p+q)$-***chains in filtering degree*** p;

2. the module

$$\begin{aligned} Z^r_{p,q} &:= \{ c \in G_p C_{p+q} \mid \partial c = 0 \,\mathrm{mod}\, F_{p-r} C_\bullet \} \\ &= \{ c \in F_p C_{p+q} \mid \partial(c) \in F_{p-r} C_{p+q-1} \} / F_{p-1} C_{p+q} \end{aligned}$$

   is the module of r-***almost*** (p,q)-***cycles*** (the $(p+q)$-chains whose differential vanishes modulo terms of filtering degree $p-r$);

3. $B^r_{p,q} := \partial(F_{p+r-1} C_{p+q+1})$,

   is the module of r**-almost** (p,q)**-boundaries**.

Similarly we set

$$\begin{aligned} Z^\infty_{p,q} &:= \{ c \in F_p C_{p+q} \mid \partial c = 0 \} / F_{p-1} C_{p+q} = Z(G_p C_{p+q}) \\ B^\infty_{p,q} &:= \partial(F_p C_{p+q+1}) \,. \end{aligned}$$

From this definition we immediately have that the differentials $\partial : C_{p+q} \to C_{p+q-1}$ restrict to the r-almost cycles as follows:

Proposition 1.27. *The differentials of $C_\bullet$ restrict on r-almost cycles to homomorphisms of the form*

$$\partial^r : Z^r_{p,q} \to Z^r_{p-r, q+r-1} \,.$$

These are still differentials: $\partial^2 = 0$.

Proof. By the very definition of $Z^r_{p,q}$ it consists of elements in filtering degree p on which ∂ decreases the filtering degree to $p-r$. Also by definition of differential on a chain complex, ∂ decreases the actual degree $p+q$ by one. This explains that ∂ restricted to $Z^r_{p,q}$ lands in $Z^\bullet_{p-r,q+r-1}$. Now the image constists indeed of actual boundaries, not just r-almost boundaries. But since actual boundaries are in particular r-almost boundaries, we may take the codomain to be $Z^r_{p-r,q+r-1}$. ▮

As before, we will in general index these differentials by their codomain and hence write in more detail

$$\partial^r_{p,q} : Z^r_{p,q} \to Z^r_{p-r,q+r-1} \,.$$

Proposition 1.28. *We have a sequence of canonical inclusions*

$$B^0_{p,q} \hookrightarrow B^1_{p,q} \hookrightarrow \cdots B^\infty_{p,q} \hookrightarrow Z^\infty_{p,q} \hookrightarrow \cdots \hookrightarrow Z^1_{p,q} \hookrightarrow Z^0_{p,q} \,.$$

The following observation is elementary, and yet this is what drives the theory of spectral sequences, as it shows that almost cycles may be computed iteratively by homological means themselves.

Proposition 1.29. *The $(r+1)$-almost cycles are the ∂^r-kernel inside the r-almost cycles:*

$$Z^{r+1}_{p,q} \simeq \ker(Z^r_{p,q} \overset{\partial^r}{\to} Z^r_{p-r,q+r-1}) \,.$$

Proof. An element $c \in F_p C_{p+q}$ represents

1. an element in $Z^r_{p,q}$ if $\partial c \in F_{p-r} C_{p+q-1}$

2. an element in $Z^{r+1}_{p,q}$ if even $\partial c \in F_{p-r-1} C_{p+q-1} \hookrightarrow F_{p-r} C_{p+q-1}$.

The second condition is equivalent to ∂c representing the 0-element in the quotient $F_{p-r} C_{p+q-1} / F_{p-r-1} C_{p+q-1}$. But this is in turn equivalent to ∂c being 0 in $Z^r_{p-r,q+r-1} \subset F_{p-r} C_{p+q-1} / F_{p-r-1} C_{p+q-1}$. ▮

With a definition of almost-cycles and almost-boundaries, of course we are now interested in the corresponding homology groups:

Definition 1.30. For $r, p, q \in \mathbb{Z}$ define the r**-almost** (p,q)**-chain homology** of the filtered complex to be the quotient of the r-almost (p,q)-cycles by the r-almost (p,q)-boundaries, def. 1.26:

$$\begin{aligned} E^r_{p,q} &:= \frac{Z^r_{p,q}}{B^r_{p,q}} \\ &= \frac{\{ x \in F_p C_{p+q} \mid \partial x \in F_{p-r} C_{p+q-1} \}}{\partial(F_{p+r-1} C_{p+q+1}) \oplus F_{p-1} C_{p+q}} \end{aligned}$$

By prop. 1.27 the differentials of $C_\bullet$ restrict on the r-almost homology groups to maps

$$\partial^r : E^r_{p,q} \to E^r_{p-r,q+r-1} \,.$$

The central property of these r-almost homology groups now is their following iterative homological characterization.

Proposition 1.31. *With definition 1.30 we have that $E^{r+1}_{\bullet,\bullet}$ is the ∂^r-chain homology of $E^r_{\bullet,\bullet}$:*

$$E^{r+1}_{p,q} = \frac{\ker(\partial^r : E^r_{p,q} \to E^r_{p-r,q+r-1})}{\mathrm{im}(\partial^r : E^r_{p+r,q-r+1} \to E^r_{p,q})} \,.$$

Proof. By prop. 1.29. ∎

This structure on the collection of r-almost cycles of a filtered chain complex thus obtained is called a *spectral sequence*:

Definition 1.32. A **homology spectral sequence** of R-modules is

1. a set $\{E^r_{p,q}\}_{p,q,r\in\mathbb{Z}}$ of R-modules;
2. a set $\{\partial^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}\}_{r,p,q\in\mathbb{Z}}$ of homomorphisms

such that

1. the ∂^rs are differentials: $\forall_{p,q,r}(\partial^r_{p-r,q+r-1} \circ \partial^r_{p,q} = 0)$;
2. the modules $E^{r+1}_{p,q}$ are the ∂^r-homology of the modules in relative degree r:

$$\forall_{r,p,q}\left(E^{r+1}_{p,q} \simeq \frac{\ker(\partial^r_{p-r,q+r-1})}{\mathrm{im}(\partial^r_{p,q})}\right).$$

One says that $E^r_{\bullet,\bullet}$ is the r**-page** of the spectral sequence.

Since this turns out to be a useful structure to make explicit, as the above motivation should already indicate, one introduces the following terminology and basic facts to talk about spectral sequences.

Definition 1.33. Let $\{E^r_{p,q}\}_{r,p,q}$ be a spectral sequence, def. 1.32, such that for each p,q there is $r(p,q)$ such that for all $r \geq r(p,q)$ we have

$$E^{r\geq r(p,q)}_{p,q} \simeq E^{r(p,q)}_{p,q} \,.$$

Then one says that

1. the bigraded object

$$E^\infty \coloneqq \{E^\infty_{p,q}\}_{p,q} \coloneqq \{E^{r(p,q)}_{p,q}\}_{p,q}$$

is the **limit term** of the spectral sequence;

- the spectral sequence **abuts** to E^∞.

Example 1.34. If for a spectral sequence there is r_s such that all differentials on pages after r_s vanish, $\partial^{r\geq r_s} = 0$, then $\{E^{r_s}\}_{p,q}$ is a limit term for the spectral sequence. One says in this cases that the spectral sequence **degenerates** at r_s.

Proof. By the defining relation

$$E^{r+1}_{p,q} \simeq \ker(\partial^r_{p-r,q+r-1})/\mathrm{im}(\partial^r_{p,q}) = E^r_{\mathrm{pq}}$$

the spectral sequence becomes constant in r from r_s on if all the differentials vanish, so that $\ker(\partial^r_{p,q}) = E^r_{p,q}$ for all p,q. ∎

Example 1.35. If for a spectral sequence $\{E^r_{p,q}\}_{r,p,q}$ there is $r_s \geq 2$ such that the r_sth page is concentrated in a single row or a single column, then the spectral sequence degenerates on this pages, example 1.34, hence this page is a limit term, def. 1.33. One says in this case that the spectral sequence **collapses** on this page.

Proof. For $r \geq 2$ the differentials of the spectral sequence

$$\partial^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$$

have domain and codomain necessarily in different rows an columns (while for $r = 1$ both are in the same row and for $r = 0$ both coincide). Therefore if all but one row or column vanish, then all these differentials vanish. ▮

Definition 1.36. A spectral sequence $\{E^r_{p,q}\}_{r,p,q}$ is said to **converge** to a graded object $H_\bullet$ with filtering $F_\bullet H_\bullet$, traditionally denoted

$$E^r_{p,q} \Rightarrow H_\bullet ,$$

if the associated graded complex $\{G_p H_{p+q}\}_{p,q} \coloneqq \{F_p H_{p+q}/F_{p-1} H_{p+q}\}$ of H is the limit term of E, def. 1.33:

$$E^\infty_{p,q} \simeq G_p H_{p+q} \qquad \forall_{p,q} \; .$$

Remark 1.37. In practice spectral sequences are often referred to via their first non-trivial page, often also the page at which it collapses, def. 1.35, often already the second page. Then one tends to use notation such as

$$E^2_{p,q} \Rightarrow H_\bullet$$

to be read as "There is a spectral sequence whose second page is as shown on the left and which converges to a filtered object as shown on the right."

Definition 1.38. A spectral sequence $\{E^r_{p,q}\}$ is called a **bounded spectral sequence** if for all $n, r \in \mathbb{Z}$ the number of non-vanishing terms of total degree n, hence of the form $E^r_{k,n-k}$, is finite.

Definition 1.39. A spectral sequence $\{E^r_{p,q}\}$ is called

- a **first quadrant spectral sequence** if all terms except possibly for $p, q \geq 0$ vanish;
- a **third quadrant spectral sequence** if all terms except possibly for $p, q \leq 0$ vanish.

Such spectral sequences are bounded, def. 1.38.

Proposition 1.40. *A bounded spectral sequence, def. 1.38, has a limit term, def. 1.33.*

Proof. First notice that if a spectral sequence has at most N non-vanishing terms of total degree n on page r, then all the following pages have at most at these positions non-vanishing terms, too, since these are the homologies of the previous terms.

Therefore for a bounded spectral sequence for each n there is $L(n) \in \mathbb{Z}$ such that $E^r_{p,n-p} = 0$ for all $p \leq L(n)$ and all r. Similarly there is $T(n) \in \mathbb{Z}$ such $E^r_{n-q,q} = 0$ for all $q \leq T(n)$ and all r.

We claim then that the limit term of the bounded spectral sequence is in position (p, q) given by the value $E^r_{p,q}$ for

$$r > \max(p - L(p+q-1), q+1 - L(p+q+1)) \; .$$

This is because for such r we have

1. $E^r_{p-r,q+r-1} = 0$ because $p - r < L(p+q-1)$, and hence the kernel $\ker(\partial^r_{p-r,q+r-1}) = 0$ vanishes;
2. $E^r_{p+r,q-r+1} = 0$ because $q - r + 1 < T(p+q+1)$, and hence the image $\mathrm{im}(\partial^r_{p,q}) = 0$ vanishes.

Therefore

$$\begin{aligned} E^{r+1}_{p,q} &= \ker(\partial^r_{p-r,q+r-1})/\mathrm{im}(\partial^r_{p,q}) \\ &\simeq E^r_{p,q}/0 \\ &\simeq E^r_{p,q} \end{aligned} \; .$$

▮

The central statement about the notion of the spectral sequence of a filtered chain complex then is the following proposition. It says that the iterative computation of higher order relative homology indeed in the limit computes the genuine homology.

Definition 1.41. For $F_\bullet C_\bullet$ a filtered complex, write for $p \in \mathbb{Z}$

$$F_p H_\bullet(C) \coloneqq \mathrm{image}(H_\bullet(F_p C) \to H_\bullet(C)) \; .$$

This defines a filtering $F_\bullet H_\bullet(C)$ of the homology, regarded as a graded object.

Proposition 1.42. *If the spectral sequence of a filtered complex $F_\bullet C_\bullet$ of prop. 1.31 has a limit term, def. 1.33 then it converges, def. 1.36, to the chain homology of C.*

$$E^r_{p,q} \Rightarrow H_{p+q}(C_\bullet) \,,$$

i.e. for sufficiently large r we have

$$E^r_{p,q} \simeq G_p H_{p+q}(C) \,,$$

where on the right we have the associated graded object of the filtering of def. 1.41.

Proof. By assumption, there is for each p, q an $r(p,q)$ such that for all $r \geq r(p,q)$ the r-almost cycles and r-almost boundaries, def. 1.26, in $F_p C_{p+q}$ are the ordinary cycles and boundaries. Therefore for $r \geq r(p,q)$ def. 1.30 gives $E^r_{p,q} \simeq G_p H_{p+q}(C)$. ▮

This says what these spectral sequences are converging to. For computations it is also important to know how they start out for low r. We can generally characterize $E^r_{p,q}$ for very low values of r simply as follows:

Proposition 1.43. *We have*

- $E^0_{p,q} = G_p C_{p+q} = F_p C_{p+q} / F_{p-1} C_{p+q}$

  is the associated p-graded piece of C_{p+q};

- $E^1_{p,q} = H_{p+q}(G_p C_\bullet)$

Proof. For $r = 0$ def. 1.30 restricts to

$$E^0_{p,q} = \frac{F_p C_{p+q}}{F_{p-1} C_{p+q}} = G_p C_{p+q}$$

because for $c \in F_p C_{p+q}$ we automatically also have $\partial c \in F_p C_{p+q}$ since the differential respects the filtering degree by assumption.

For $r = 1$ def. 1.30 gives

$$E^1_{p,q} = \frac{\{c \in G_p C_{p+q} \,|\, \partial c = 0 \in G_p C_{p+q}\}}{\partial (F_p C_{p+q})} = H_{p+q}(G_p C_\bullet) \,.$$

▮

Remark 1.44. There is, in general, a decisive difference between the homology of the associated graded complex $H_{p+q}(G_p C_\bullet)$ and the associated graded piece of the genuine homology $G_p H_{p+q}(C_\bullet)$: in the former the differentials of cycles are required to vanish only up to terms in lower degree, but in the latter they are required to vanish genuinely. The latter expression is instead the value of the spectral sequence for $r \to \infty$, see prop. 1.42 below.

Comparing cellular and singular homology

These general facts now allow us, as a first simple example for the application of spectral sequences to see transparently that the cellular homology of a CW complex, def. 1.21, coincides with its genuine singular homology.

First notice that of course the structure of a CW-complex on a topological space X, def. 1.18 naturally induces on its singular simplicial complex $C_\bullet(X)$ the structure of a filtered chain complex, def. 1.24:

Definition 1.45. For $X_0 \hookrightarrow X_1 \hookrightarrow \cdots \hookrightarrow X$ a CW complex, and $p \in \mathbb{N}$, write

$$F_p C_\bullet(X) \coloneqq C_\bullet(X_p)$$

for the singular chain complex of $X_p \hookrightarrow X$. The given topological subspace inclusions $X_p \hookrightarrow X_{p+1}$ induce chain map inclusions $F_p C_\bullet(X) \hookrightarrow F_{p+1} C_\bullet(X)$ and these equip the singular chain complex $C_\bullet(X)$ of X with the structure of a bounded filtered chain complex

$$0 \hookrightarrow F_0 C_\bullet(X) \hookrightarrow F_1 C_\bullet(X) \hookrightarrow F_2 C_\bullet(X) \hookrightarrow \cdots \hookrightarrow F_\infty C_\bullet(X) \coloneqq C_\bullet(X) \,.$$

(If X is of finite dimension $\dim X$ then this is a bounded filtration.)

Write $\{E^r_{p,q}(X)\}$ for the spectral sequence of a filtered complex corresponding to this filtering.

Proposition 1.46. *The spectral sequence $\{E^r_{p,q}(X)\}$ of singular chains in a CW complex X, def. 1.45 converges, def. 1.36, to the singular homology of X:*

$$E^r_{p,q}(X) \Rightarrow H_\bullet(X) \,.$$

Proof. The spectral sequence $\{E^r_{p,q}(X)\}$ is clearly a first-quadrant spectral sequence, def. 1.39. Therefore it is a bounded spectral sequence, def. 1.38 and hence has a limit term, def. 1.40. So the statement follows with prop. 1.42. ▮

We now identify the low-degree pages of $\{E^r_{p,q}(X)\}$ with structures in singular homology theory.

Proposition 1.47.

- $r = 0$ – $E^0_{p,q}(X) \simeq C_{p+q}(X_p)/C_{p+q}(X_{p-1})$ *is the group of X_{p-1}-relative (p+q)-chains, def. 1.1, in X_p;*
- $r = 1$ – $E^1_{p,q}(X) \simeq H_{p+q}(X_p, X_{p-1})$ *is the X_{p-1}-relative singular homology, def. 1.1, of X_p;*
- $r = 2$ – $E^2_{p,q}(X) \simeq \begin{cases} H_p^{\mathrm{CW}}(X) & \text{for } q = 0 \\ 0 & \text{otherwise} \end{cases}$
- $r = \infty$ – $E^\infty_{p,q}(X) \simeq F_p H_{p+q}(X)/F_{p-1}H_{p+q}(X)$.

Proof. By straightforward and immediate analysis of the definitions. ▮

As a result of these general considerations we now obtain the promised isomorphism between the cellular homology and the singular homology of a CW-complex X:

Theorem 1.48. *For $X \in$ Top a CW complex, def. 1.18, its cellular homology, def. 1.21 $H_\bullet^{\mathrm{CW}}(X)$ coincides with its singular homology $H_\bullet(X)$:*

$$H_\bullet^{\mathrm{CW}}(X) \simeq H_\bullet(X) \,.$$

Proof. By the third item of prop. 1.47 the $(r=2)$-page of the spectral sequence $\{E^r_{p,q}(X)\}$ is concentrated in the $(q=0)$-row and hence it collapses there, def. 1.35. Accordingly we have

$$E^\infty_{p,q}(X) \simeq E^2_{p,q}(X)$$

for all p,q. By the third and fourth item of prop. 1.47 this non-trivial only for $q = 0$ and there it is equivalently

$$G_p H_p(X) \simeq H_p^{\mathrm{CW}}(X) \,.$$

Finally observe that $G_p H_p(X) \simeq H_p(X)$ by the definition of the filtering on the homology, def. 1.41, and using prop. 1.20. ▮

2. For filtered spectra

Definition 2.1. A filtered spectrum is a spectrum X equipped with a sequence $X_\bullet : (\mathbb{N}, >) \longrightarrow \mathrm{Spectra}$ of spectra of the form

$$\cdots \longrightarrow X_3 \overset{f_2}{\longrightarrow} X_2 \overset{f_1}{\longrightarrow} X_1 \overset{f_0}{\longrightarrow} X_0 = X \,.$$

Remark 2.2. More generally a filtering on an object X in (stable or not) homotopy theory is a $\mathbb{Z}$-graded sequence $X_\bullet$ such that X is the homotopy colimit $X \simeq \underset{\longrightarrow}{\lim} X_\bullet$. But for the present purpose we stick with the simpler special case of def. 2.1.

Remark 2.3. There is *no* condition on the morphisms in def. 2.1. In particular, they are *not* required to be n-monomorphisms or n-epimorphisms for any n.

On the other hand, while they are also not explicitly required to have a presentation by cofibrations or fibrations, this follows automatically: by the existence of model structures for spectra, every filtering on a spectrum is equivalent to one in which all morphisms are represented by cofibrations or by fibrations.

This means that we may think of a filtration on a spectrum X in the sense of def. 2.1 as equivalently being a tower of fibrations over X.

The following remark 2.4 unravels the structure encoded in a filtration on a spectrum, and motivates the concepts of exact couples and their spectral sequences from these.

Remark 2.4. Given a filtered spectrum as in def. 2.1, write A_k for the homotopy cofiber of its kth stage, such as to obtain the diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & X_3 & \overset{f_2}{\longrightarrow} & X_2 & \overset{f_2}{\longrightarrow} & X_1 & \overset{f_1}{\longrightarrow} & X \\ && \downarrow && \downarrow && \downarrow && \downarrow \\ && A_3 && A_2 && A_1 && A_0 \end{array}$$

where each stage

$$
\begin{array}{ccc}
X_{k+1} & \overset{f_k}{\longrightarrow} & X_k \\
 & & \downarrow^{\mathrm{cofib}(f_k)} \\
 & & A_k
\end{array}
$$

is a homotopy fiber sequence.

To break this down into invariants, apply the stable homotopy groups-functor (def.). This yields a diagram of $\mathbb{Z}$-graded abelian groups of the form

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & \pi_\bullet(X_3) & \overset{\pi_\bullet(f_2)}{\longrightarrow} & \pi_\bullet(X_2) & \overset{\pi_\bullet(f_2)}{\longrightarrow} & \pi_\bullet(X_1) & \overset{\pi_\bullet(f_1)}{\longrightarrow} & \pi_\bullet(X_0) \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & \pi_\bullet(A_3) & & \pi_\bullet(A_2) & & \pi_\bullet(A_1) & & \pi_\bullet(A_0)
\end{array}
\,.
$$

Each hook at stage k extends to a long exact sequence of homotopy groups (prop.) via connecting homomorphisms $\delta^k_\bullet$

$$
\cdots \to \pi_{\bullet+1}(A_k) \overset{\delta^k_{\bullet+1}}{\longrightarrow} \pi_\bullet(X_{k+1}) \overset{\pi_\bullet(f_k)}{\longrightarrow} \pi_\bullet(X_k) \longrightarrow \pi_\bullet(A_k) \overset{\delta^k_\bullet}{\longrightarrow} \pi_{\bullet-1}(X_{k+1}) \to \cdots \,.
$$

If we understand the connecting homomorphism

$$
\delta_k : \pi_\bullet(A_k) \longrightarrow \pi_\bullet(X_{k+1})
$$

as a morphism of degree -1, then all this information fits into one diagram of the form

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & \pi_\bullet(X_3) & \overset{\pi_\bullet(f_2)}{\longrightarrow} & \pi_\bullet(X_2) & \overset{\pi_\bullet(f_2)}{\longrightarrow} & \pi_\bullet(X_1) & \overset{\pi_\bullet(f_1)}{\longrightarrow} & \pi_\bullet(X_0) \\
 & & \downarrow & {}_{\delta_2}\nwarrow & \downarrow & {}_{\delta_1}\nwarrow & \downarrow & {}_{\delta_0}\nwarrow & \downarrow \\
 & & \pi_\bullet(A_3) & & \pi_\bullet(A_2) & & \pi_\bullet(A_1) & & \pi_\bullet(A_0)
\end{array}
\,,
$$

where each triangle is a rolled-up incarnation of a long exact sequence of homotopy groups (and in particular is *not* a commuting diagram!).

If we furthermore consider the bigraded abelian groups $\pi_\bullet(X_\bullet)$ and $\pi_\bullet(A_\bullet)$, then this information may further be rolled-up to a single diagram of the form

$$
\begin{array}{ccc}
\pi_\bullet(X_\bullet) & \overset{\pi_\bullet(f_\bullet)}{\longrightarrow} & \pi_\bullet(X_\bullet) \\
 & {}_{\delta}\nwarrow & \downarrow^{\pi_\bullet(\mathrm{cofib}(f_\bullet))} \\
 & & \pi_\bullet(A_\bullet)
\end{array}
$$

where the morphisms $\pi_\bullet(f_\bullet)$, $\pi_\bullet(\mathrm{cofib}(f_\bullet))$ and δ have bi-degree $(0,-1)$, $(0,0)$ and $(-1,1)$, respectively.

Here it is convenient to shift the bigrading, equivalently, by setting

$$
\begin{aligned}
\mathcal{D}^{s,t} &\coloneqq \pi_{t-s}(X_s) \\
\mathcal{E}^{s,t} &\coloneqq \pi_{t-s}(A_s)\,,
\end{aligned}
$$

because then t counts the cycles of going around the triangles:

$$
\cdots \to \mathcal{D}^{s+1,t+1} \overset{\pi_{t-s}(f_s)}{\longrightarrow} \mathcal{D}^{s,t} \overset{\pi_{t-s}(\mathrm{cofib}(f_s))}{\longrightarrow} \mathcal{E}^{s,t} \overset{\delta_s}{\longrightarrow} \mathcal{D}^{s+1,t} \to \cdots
$$

Data of this form is called an *exact couple*, def. 2.6 below.

Definition 2.5. An *unrolled* *exact couple* (of Adams-type) is a diagram of abelian groups of the form

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & \mathcal{D}^{3,\bullet} & \overset{i_2}{\longrightarrow} & \mathcal{D}^{2,\bullet} & \overset{i_1}{\longrightarrow} & \mathcal{D}^{1,\bullet} & \overset{i_0}{\longrightarrow} & \mathcal{D}^{0,\bullet} \\
 & & \downarrow & {}_{k_2}\nwarrow & {}^{j_2}\downarrow & {}_{k_1}\nwarrow & {}^{j_1}\downarrow & {}_{k_0}\nwarrow & {}_{j_0}\downarrow \\
 & & \mathcal{E}^{3,\bullet} & & \mathcal{E}^{2,\bullet} & & \mathcal{E}^{1,\bullet} & & \mathcal{E}^{0,\bullet}
\end{array}
$$

such that each triangle is a rolled-up long exact sequence of abelian groups of the form

$$
\cdots \to \mathcal{D}^{s+1,t+1} \overset{i_s}{\longrightarrow} \mathcal{D}^{s,t} \overset{j_s}{\longrightarrow} \mathcal{E}^{s,t} \overset{k_s}{\longrightarrow} \mathcal{D}^{s+1,t} \to \cdots \,.
$$

The collection of this “un-rolled” data into a single diagram of abelian groups is called the corresponding *exact couple*.

Definition 2.6. An *exact couple* is a diagram (non-commuting) of abelian groups of the form

$$
\begin{array}{ccc}
\mathcal{D} & \overset{i}{\longrightarrow} & \mathcal{D} \\
& {}_{k}\nwarrow & \downarrow^{j}, \\
& & \mathcal{E}
\end{array}
$$

such that this is exact sequence exact in each position, hence such that the kernel of every morphism is the image of the preceding one.

The concept of exact couple so far just collects the sequences of long exact sequences given by a filtration. Next we turn to extracting information from this sequence of sequences.

Remark 2.7. The sequence of long exact sequences in remark 2.4 is inter-locking, in that every $\pi_{t-s}(X_s)$ appears *twice*:

$$
\begin{array}{ccccccccc}
& \searrow & & & \nearrow & & & & \\
& & \pi_{t-s-1}(X_{s+1}) & & & & & & \\
& {}^{\delta^s_{t-s}}\nearrow & & & \searrow^{\pi_{t-s-1}(\mathrm{cofib}(f_{s+1}))} & & & & \nearrow \\
\pi_{t-s}(A_s) & & \underset{\text{def: } d_1^{s,t}}{\longrightarrow} & & \pi_{t-s-1}(A_{s+1}) & & \overset{\text{def: } d_1^{s+1,t}}{\longrightarrow} & & \pi_{t-s-2}(A_{s+2}) \\
\nearrow & & & & & {}^{\delta^{s+1}_{t-s-1}}\searrow & & \nearrow_{\pi_{t-s-2}(\mathrm{cofib}(f_{s+2}))} & \\
& & & & & & \pi_{t-s-2}(X_{s+2}) & & \\
& & & & & \nearrow & & \searrow &
\end{array}
$$

This gives rise to the horizontal composites $d_1^{s,t}$, as show above, and by the fact that the diagonal sequences are long exact, these are differentials: $d_1^2 = 0$, hence give a chain complex:

$$
\cdots \longrightarrow \pi_{t-s}(A_s) \overset{d_1^{s,t}}{\longrightarrow} \pi_{t-s-1}(A_{s+1}) \overset{d_1^{s+1,t}}{\longrightarrow} \pi_{t-s-2}(A_{s+2}) \longrightarrow \cdots .
$$

We read off from the interlocking long exact sequences what these differentials *mean*: an element $c \in \pi_{t-s}(A_s)$ lifts to an element $\hat{c} \in \pi_{t-s-1}(X_{s+2})$ precisely if $d_1 c = 0$:

$$
\begin{array}{cccccc}
\hat{c} \in & \pi_{t-s-1}(X_{s+2}) & & & & \\
& & \searrow^{\pi_{t-s-1}(f_{s+1})} & & & \\
& & & \pi_{t-s-1}(X_{s+1}) & & \\
& & {}^{\delta^s_{t-s}}\nearrow & & \searrow^{\pi_{t-s-1}(\mathrm{cofib}(f_{s+1}))} & \\
c \in & \pi_{t-s}(A_s) & & \underset{d_1^{s,t}}{\longrightarrow} & & \pi_{t-s-1}(A_{s+1})
\end{array}
$$

This means that the cochain cohomology of the complex $(\pi_\bullet(A_\bullet), d_1)$ produces elements of $\pi_\bullet(X_\bullet)$ and hence of $\pi_\bullet(X)$.

In order to organize this observation, notice that in terms of the exact couple of remark 2.4, the differential

$$
d_1^{s,t} \coloneqq \pi_{t-s-1}(\mathrm{cofib}(f_{s+1})) \circ \delta^s_{t-s}
$$

is a component of the composite

$$
d \coloneqq j \circ k .
$$

Some terminology:

Definition 2.8. Given an exact couple, def. 2.6,

$$
\begin{array}{ccc}
\mathcal{D}^{\bullet,\bullet} & \overset{i}{\longrightarrow} & \mathcal{D}^{\bullet,\bullet} \\
& {}_{k}\nwarrow & \downarrow^{j} \\
& & \mathcal{E}^{\bullet,\bullet}
\end{array}
$$

its *page* is the chain complex

$$
(E^{\bullet,\bullet}, d \coloneqq j \circ k) .
$$

Definition 2.9. Given an exact couple, def. 2.6, then the induced *derived exact couple* is the diagram

$$\begin{tikzcd} \tilde{\mathcal{D}} \arrow[r, "\tilde{i}"] & \tilde{\mathcal{D}} \arrow[d, "\tilde{j}"] \\ & \tilde{\mathcal{E}} \arrow[ul, "\tilde{k}"] \end{tikzcd}$$

with

1. $\tilde{\mathcal{E}} \coloneqq \ker(d)/\mathrm{im}(d)$;
2. $\tilde{\mathcal{D}} \coloneqq \mathrm{im}(i)$;
3. $\tilde{i} \coloneqq i|_{\mathrm{im}(i)}$;
4. $\tilde{j} \coloneqq j \circ (\mathrm{im}(i))^{-1}$;
5. $\tilde{k} \coloneqq k|_{\ker(d)}$.

Proposition 2.10. *A derived exact couple, def. 2.9, is again an exact couple, def. 2.6.*

Definition 2.11. Given an exact couple, def. 2.6, then the induced spectral sequence, def. 1.32, is the sequence of pages, def. 2.8, of the induced sequence of derived exact couples, def. 2.9, prop. 2.10.

Example 2.12. Consider a filtered spectrum, def. 2.1,

$$\begin{tikzcd} \cdots \arrow[r] & X_3 \arrow[r, "f_2"] \arrow[d] & X_2 \arrow[r, "f_2"] \arrow[d] & X_1 \arrow[r, "f_1"] \arrow[d] & X \arrow[d] \\ & A_3 & A_2 & A_1 & A_0 \end{tikzcd}$$

and its induced exact couple of stable homotopy groups, from remark 2.4

$$\begin{tikzcd} \mathcal{D} \arrow[r, "i"] & \mathcal{D} \arrow[d, "j"] \\ & \mathcal{E} \arrow[ul, "k"] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{D} \arrow[r, "{(-1,-1)}"] & \mathcal{D} \arrow[d, "{(0,0)}"] \\ & \mathcal{E} \arrow[ul, "{(1,0)}"] \end{tikzcd}$$

with bigrading as shown on the right.

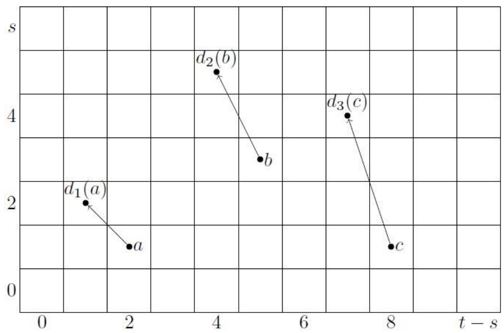
As we pass to derived exact couples, by def. 2.9, the bidegree of i and k is preserved, but that of j increases by $(1,1)$ in each step, since

$$\deg(\tilde{j}) = \deg(j \circ \mathrm{im}(i)^{-1}) = \deg(j) + (1,1) \; .$$

Therefore the induced spectral sequence has differentials of the form

$$d_r : \mathcal{E}_r^{s,t} \longrightarrow \mathcal{E}_r^{s+r,t+r-1} \; .$$

This is also called the Adams-type *spectral sequence of the tower of fibrations* $X_{n+1} \to X_n$.



This we discuss in detail in *part 2 -- Adams spectral sequences*.

3. References

A gentle exposition of the general idea of spectral sequences is in

- John McCleary, *A User's Guide to Spectral Sequences*, Cambridge University Press (1985, 2001)

A concise account streamlined for our purposes is in section 2 of

- John Rognes, *The Adams spectral sequence* (following Bruner), 2012 (pdf)

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nLab

Introduction to the Adams Spectral Sequence

This page gives a detailed introduction to the Adams spectral sequence in its general spectral form (Adams-Novikov spectral sequence).

For background on spectral sequences see *Introduction to Spectral Sequences*.

For background on stable homotopy theory see *Introduction to Stable homotopy theory*.

For background on complex oriented cohomology see *Introduction to Cobordism and Complex Oriented Cohomology*.

Contents

The main result of Part 1.1 was the construction of the stable homotopy category $\mathrm{Ho}(\mathrm{Spectra})$ (thm., def.) as a triangulated category (prop.) with graded abelian hom groups $[X,Y]_\bullet$ (def.).

These are the basic invariants of stable homotopy theory, the *stable homotopy groups*. They are as rich and interesting as they are, in general, hard to compute. The archetypical example for this phenonemon are the stable homotopy groups of spheres $\pi_\bullet(\mathbb{S})$. (We compute the first dozen of these, 2-locally, below.)

In order to get more control over $\mathrm{Ho}(\mathrm{Spectra})$, the main result of Part 1.2 was the construction of tensor triangulated category structure on $\mathrm{Ho}(\mathrm{Spectra})$ (prop.), induced form a symmetric monoidal smash product of spectra $\wedge$ (thm.)

$$(\mathrm{Ho}(\mathrm{Spectra}), \wedge, \mathbb{S}) .$$

As discussed in *Part I* (and briefly reviewed below), the tool of choice to break up the computation of stable homotopy groups in stable homotopy theory into tractable computations in homological algebra are *spectral sequences*. These break up computations of stable homotopy groups along chosen filtrations on spectra. Using the tensor triangulated structure, it turns out that every homotopy commutative ring spectrum E (def.) induces a well-adapted filtration on spectra that allows to compute the "formal neighbourhood around E" in any spectrum (called the E-*nilpotent completion*) via a spectral sequence. This is the E-*Adams spectral sequence* which we discuss here.

Where the Atiyah-Hirzebruch spectral sequence (see part S, this prop.) approximates $[X,Y]_\bullet$ via the ordinary cohomology $H^\bullet(X,\pi_\bullet(Y))$, the idea of the Adams spectral sequence is to make use of an auxiliary homotopy commutative ring spectrum E and approximate maps of spectra $X \to Y$ via their image $E_\bullet(X) \to E_\bullet(Y)$ in E-generalized homology (rmk).

But in order for maps of homology groups to have a chance to retain enough information, they should be forced to be equivariant with respect to extra structure inherited by forming E-homology.

For instance if $E = H\mathbb{F}_2$ then the dual Steenrod algebra $\mathcal{A}$ (co-)acts on $E_\bullet(X) = H_\bullet(X, \mathbb{F}_2)$ and a necessary condition for a morphism of homology groups to come from a morphism of spectra is that it is a homomorphism with respect to this co-action. The classical Adams spectral sequence (discussed below), accordingly, approximates $[X,Y]_\bullet$ by $\mathrm{Hom}_{\mathcal{A}}(H_\bullet(X,\mathbb{F}_2), H_\bullet(Y,\mathbb{F}_2))$.

More generally, since spectra are equivalently module spectra over the sphere spectrum $\mathbb{S}$, the operation of forming E-homology spectra $X \mapsto E \wedge S$ is equivalently the extension of scalars along the ring unit $\mathbb{S} \longrightarrow E$. This means that the extra structure inherited by E-homology groups contains the information given by the further extensions along the cosimplicial diagram

$$\mathbb{S} \longrightarrow E \overset{\longrightarrow}{\longrightarrow} E \wedge E \overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} E \wedge E \wedge E \overset{\overset{\longrightarrow}{\longrightarrow}}{\underset{\longrightarrow}{\longrightarrow}} \cdots .$$

In good cases this gives $E_\bullet(X)$ the structure of a module over the Hopf algebroid $\pi_\bullet(E \wedge E) = E_\bullet(E) \overset{\longleftarrow}{\longleftarrow} E_\bullet$ of "dual E-Steenrod operations". Accordingly the general E-Adams spectral sequence approximates $[X,Y]_\bullet$ by $\mathrm{Hom}_{E_\bullet(E)}(E_\bullet(X), E_\bullet(Y))$.

For $E =$ MU, BP, this is the *Adams-Novikov spectral sequence, considered below.*

We discuss first the

- General theory of E-Adams spectral sequences

and then consider the classical

- Examples and applications

First we set up the general theory of E-Adams spectral sequences. (We consider examples and applications further below.)

Literature (Adams 74, part III.15, Bousfield 79, sections 5 and 6, Ravenel 86, appendix A)

1. The spectral sequence

Filtered spectra

We introduce the types of spectral sequences of which the E-Adams spectral sequences (def. 1.14 below) are an example.

Definition 1.1. A **filtered spectrum** is a spectrum $Y \in \mathrm{Ho}(\mathrm{Spectra})$ equipped with a sequence $Y_\bullet \colon (\mathbb{N}, >) \longrightarrow \mathrm{Ho}(\mathrm{Spectra})$ in the stable homotopy category (def.) of the form

$$\cdots \longrightarrow Y_3 \overset{f_2}{\longrightarrow} Y_2 \overset{f_1}{\longrightarrow} Y_1 \overset{f_0}{\longrightarrow} Y_0 \coloneqq Y .$$

Remark 1.2. More generally a filtering on an object X in (stable or not) homotopy theory is a $\mathbb{Z}$-graded sequence $X_\bullet$ such that X is the homotopy colimit $X \simeq \underset{\longrightarrow}{\lim} X_\bullet$. But for the present purpose we stick with the simpler special case of def. 1.1.

Remark 1.3. There is *no* condition on the morphisms in def. 1.1. In particular, they are *not* required to be n-monomorphisms or n-epimorphisms for any n.

On the other hand, while they are also not explicitly required to have a presentation by cofibrations or fibrations, this follows automatically: by the existence of the model structure on topological sequential spectra (thm.) or equivalently (thm.) the model structure on orthogonal spectra (thm.), every filtering on a spectrum is equivalent to one in which all morphisms are represented by cofibrations or by fibrations.

This means that we may think of a filtration on a spectrum in the sense of def. 1.1 as equivalently being a tower of fibrations over that spectrum.

The following definition 1.4 unravels the structure encoded in a filtration on a spectrum, and motivates the concepts of exact couples and their spectral sequences from these.

Definition 1.4. (exact couple of a filtered spectrum)

Consider a spectrum $X \in \mathrm{Ho}(\mathrm{Spectra})$ and a filtered spectrum $Y_\bullet$ as in def. 1.1.

Write A_k for the homotopy cofiber of its kth stage, such as to obtain the diagram

$$
\begin{tikzcd}
\cdots \arrow[r] & Y_3 \arrow[r, "f_2"] \arrow[d, "g_3"] & Y_2 \arrow[r, "f_1"] \arrow[d, "g_2"] & Y_1 \arrow[r, "f_0"] \arrow[d, "g_1"] & Y \arrow[d, "g_0"] \\
& A_3 & A_2 & A_1 & A_0
\end{tikzcd}
$$

where each stage

$$
\begin{tikzcd}
Y_{k+1} \arrow[r, "f_k"] & Y_k \arrow[d, "g_k"] \\
& A_k
\end{tikzcd}
$$

is a homotopy cofiber sequence (def.), hence equivalently (prop.) a homotopy fiber sequence, hence where

$$Y_{k+1} \overset{f_k}{\longrightarrow} Y_k \overset{g_k}{\longrightarrow} A_k \overset{h_k}{\longrightarrow} \Sigma Y_{k+1}$$

is an exact triangle (prop.).

Apply the graded hom-group functor $[X,-]_\bullet$ (def.) to the above tower. This yields a diagram of $\mathbb{Z}$-graded abelian groups of the form

$$
\begin{tikzcd}
\cdots \arrow[r] & {[X,Y_3]_\bullet} \arrow[r, "{[X,f_2]_\bullet)}"] \arrow[d, "{[X,g_3]_\bullet}"] & {[X,Y_2]_\bullet} \arrow[r, "{[X,f_1]_\bullet}"] \arrow[d, "{[X,g_2]_\bullet}"] & {[X,Y_1]_\bullet} \arrow[r, "{[X,f_0]_\bullet}"] \arrow[d, "{[X,g_1]_\bullet}"] & {[X,Y_0]_\bullet} \arrow[d, "{[X,g_0]_\bullet}"] \\
& {[X,A_3]_\bullet} & {[X,A_2]_\bullet} & {[X,A_1]_\bullet} & {[X,A_0]_\bullet}
\end{tikzcd}
$$

where each hook at stage k extends to a long exact sequence of homotopy groups (prop.) via connecting homomorphisms $[X,h_k]_\bullet$

$$\cdots \to [X,A_k]_{\bullet+1} \overset{[X,h_k]_{\bullet+1}}{\longrightarrow} [X,Y_{k+1}]_\bullet \overset{[X,f_k]_\bullet}{\longrightarrow} [X,Y_k]_\bullet \overset{[X,g_k]_\bullet}{\longrightarrow} [X,A_k]_\bullet \overset{[X,h_k]_\bullet}{\longrightarrow} [X,Y_{k+1}]_{\bullet-1} \to \cdots .$$

If we regard the connecting homomorphism $[X,h_k]$ as a morphism of degree -1, then all this information fits into one diagram of the form

$$
\begin{tikzcd}
\cdots \arrow[r] & {[X,Y_3]_\bullet} \arrow[r, "{[X,f_2]_\bullet}"] \arrow[d] & {[X,Y_2]_\bullet} \arrow[r, "{[X,f_1]_\bullet}"] \arrow[d] & {[X,Y_1]_\bullet} \arrow[r, "{[X,f_0]_\bullet}"] \arrow[d] & {[X,Y_0]_\bullet} \arrow[d, "{[X,g_0]_\bullet}"] \\
& {[X,A_3]_\bullet} \arrow[ur, "{[X,h_2]_\bullet}"'] & {[X,A_2]_\bullet} \arrow[ur, "{[X,h_1]_\bullet}"'] & {[X,A_1]_\bullet} \arrow[ur, "{[X,h_0]_\bullet}"'] & {[X,A_0]_\bullet}
\end{tikzcd}
$$

where each triangle is a rolled-up incarnation of a long exact sequence of homotopy groups (and in particular is *not* a commuting diagram!).

If we furthermore consider the bigraded abelian groups $[X,Y_\bullet]_\bullet$ and $[X,A_\bullet]_\bullet$, then this information may further be rolled-up to a single diagram of the form

$$
\begin{tikzcd}
{[X,Y_\bullet]_\bullet} \arrow[r, "{[X,f_\bullet]_\bullet}"] & {[X,Y_\bullet]_\bullet} \arrow[d, "{[X,g_\bullet]_\bullet}"] \\
& {[X,A_\bullet]_\bullet} \arrow[ul, "{[X,h_\bullet]_\bullet}"]
\end{tikzcd}
$$

Specifically, regard the terms here as bigraded in the following way

$$
\begin{aligned}
D^{s,t}(X,Y) &:= [X,Y_s]_{t-s} \\
E^{s,t}(X,Y) &:= [X,A_s]_{t-s}
\end{aligned}
.
$$

Then the bidegree of the morphisms is

morphism	bidegree
$[X,f]$	$(-1,-1)$
$[X,g]$	$(0,0)$
$[X,h]$	$(1,0)$

This way t counts the cycles of going around the triangles:

$$\cdots \to D^{s+1,t+1}(X,Y) \overset{[X,f]}{\longrightarrow} D^{s,t}(X,Y) \overset{[X,g]}{\longrightarrow} E^{s,t}(X,Y) \overset{[X,h]}{\longrightarrow} D^{s+1,t}(X,Y) \to \cdots$$

Data of this form is called an *exact couple*, def. 1.6 below.

Definition 1.5. An *unrolled* *exact couple* (of *Adams-type*) is a diagram of abelian groups of the form

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{D}^{3,\bullet} \arrow[r, "i_2"] \arrow[d] & \mathcal{D}^{2,\bullet} \arrow[r, "i_1"] \arrow[d, "j_2"'] & \mathcal{D}^{1,\bullet} \arrow[r, "i_0"] \arrow[d, "j_1"'] & \mathcal{D}^{0,\bullet} \arrow[d, "j_0"] \\
 & \mathcal{E}^{3,\bullet} & \mathcal{E}^{2,\bullet} \arrow[ul, "k_2"] & \mathcal{E}^{1,\bullet} \arrow[ul, "k_1"] & \mathcal{E}^{0,\bullet} \arrow[ul, "k_0"]
\end{tikzcd}$$

such that each triangle is a rolled-up long exact sequence of abelian groups of the form

$$\cdots \to \mathcal{D}^{s+1,t+1} \overset{i_s}{\longrightarrow} \mathcal{D}^{s,t} \overset{j_s}{\longrightarrow} \mathcal{E}^{s,t} \overset{k_s}{\longrightarrow} \mathcal{D}^{s+1,t} \to \cdots .$$

The collection of this "un-rolled" data into a single diagram of abelian groups is called the corresponding *exact couple*.

Definition 1.6. An *exact couple* is a diagram (non-commuting) of abelian groups of the form

$$\begin{tikzcd}
\mathcal{D} \arrow[r, "i"] & \mathcal{D} \arrow[d, "j"] \\
 & \mathcal{E} \arrow[ul, "k"]
\end{tikzcd},$$

such that this is exact in each position, hence such that the kernel of every morphism is the image of the preceding one.

The concept of exact couple so far just collects the sequences of long exact sequences given by a filtration. Next we turn to extracting information from this sequence of sequences.

Remark 1.7. The sequence of long exact sequences in def. 1.4 is inter-locking, in that every $[X,Y_s]_{t-s}$ appears *twice*:

$$\begin{tikzcd}
 & {} \arrow[dr] & & {} & & & \\
 & & {[X,Y_{s+1}]_{t-s-1}} \arrow[dr, "{[X,g]}"] \arrow[ur] & & & & {} \\
 & {[X,A_s]_{t-s}} \arrow[ur, "{[X,h]}"] \arrow[rr, "d_1"] & & {[X,A_{s+1}]_{t-s-1}} \arrow[dr, "{[X,h]}"'] \arrow[rr, "d_1"] & & {[X,A_{s+2}]_{t-s-2}} \arrow[ur] & \\
{} \arrow[ur] & & & & {[X,Y_{s+2}]_{t-s-2}} \arrow[ur, "{[X,g]}"'] \arrow[dr] & & \\
 & & & {} \arrow[ur] & & {} &
\end{tikzcd}$$

This gives rise to the horizontal ("splicing") composites d_1, as shown, and by the fact that the diagonal sequences are long exact, these are differentials in that they square to zero: $(d_1)^2 = 0$. Hence there is a cochain complex:

$$\cdots \longrightarrow [X,A_s]_{t-s} \overset{d_1}{\longrightarrow} [X,A_{s+1}]_{t-s-1} \overset{d_1}{\longrightarrow} [X,A_{s+2}]_{t-s-2} \longrightarrow \cdots .$$

We may read off from these interlocking long exact sequences what these differentials *mean*, as follows. An element $c \in [X,A_s]_{t-s}$ lifts to an element $\hat{c} \in [X,Y_{s+2}]_{t-s-1}$ precisely if $d_1 c = 0$:

$$\begin{tikzcd}
\hat{c} \in {[X,Y_{s+2}]_{t-s-1}} \arrow[dr, "{[X,f]}"] & & \\
 & {[X,Y_{s+1}]_{t-s-1}} \arrow[dr, "{[X,g]}"] & \\
c \in {[X,A_s]_{t-s}} \arrow[ur, "{[X,h]}"] \arrow[rr, "d_1"] & & {[X,A_{s+1}]_{t-s-1}}
\end{tikzcd}$$

In order to organize this observation, notice that in terms of the exact couple of remark 1.4, the differential

$$d_1 \;\coloneqq\; [X,g] \circ [X,h]$$

is the composite

$$d \coloneqq j \circ k \;.$$

Some terminology:

Definition 1.8. Given an exact couple, def. 1.6,

$$\begin{array}{ccc} \mathcal{D}^{\bullet,\bullet} & \overset{i}{\longrightarrow} & \mathcal{D}^{\bullet,\bullet} \\ & {}_{k}\nwarrow & \downarrow^{j} \\ && \mathcal{E}^{\bullet,\bullet} \end{array}$$

observe that the composite

$$d \coloneqq j \circ k$$

is a differential in that it squares to 0, due to the exactness of the exact couple:

$$\begin{aligned} d \circ d &= j \circ \underbrace{k \circ j}_{=0} \circ k \\ &= 0 \end{aligned} \;.$$

One says that the **page** of the exact couple is the graded chain complex

$$(\mathcal{E}^{\bullet,\bullet}, d \coloneqq j \circ k) \;.$$

Given a cochain complex like this, we are to pass to its cochain cohomology. Since the cochain complex here has the extra structure that it arises from an exact couple, its cohomology groups should still remember some of that extra structure. Indeed, the following says that the remaining extract structure on the cohomology of the page of an exact couple is itself again an exact couple, called the "derived exact couple".

Definition 1.9. Given an exact couple, def. 1.6, then its *derived exact couple* is the diagram

$$\begin{array}{ccc} \widetilde{\mathcal{D}} & \overset{\tilde i}{\longrightarrow} & \widetilde{\mathcal{D}} \\ & {}_{\tilde k}\nwarrow & \downarrow^{\tilde j} \\ && \widetilde{\mathcal{E}} \end{array} \;\;\coloneqq\;\; \begin{array}{ccc} \mathrm{im}(i) & \overset{i}{\longrightarrow} & \mathrm{im}(i) \\ & {}_{k}\nwarrow & \downarrow j \circ i^{-1} \\ && H(\mathcal{E}, j \circ k) \end{array}$$

with

1. $\tilde{\mathcal{E}} \coloneqq \ker(d)/\mathrm{im}(d)$ (with $d \coloneqq j \circ k$ from def. 1.8);

2. $\tilde{\mathcal{D}} \coloneqq \mathrm{im}(i)$;

3. $\tilde{\imath} \coloneqq i|_{\mathrm{im}(i)}$;

4. $\tilde{\jmath} \coloneqq j \circ i^{-1}$ (where i^{-1} is the operation of choosing any preimage under i);

5. $\tilde{k} \coloneqq k|_{\ker(d)}$.

Lemma 1.10. *The derived exact couple in def. 1.9 is well defined and is itself an exact couple, def. 1.6.*

Proof. This is straightforward to check. For completeness we spell it out:

First consider that the morphisms are well defined in the first place.

It is clear that $\tilde{\imath}$ is well-defined.

That $\tilde{\jmath}$ lands in $\ker(d)$: it lands in the image of j which is in the kernel of k, by exactness, hence in the kernel of d by definition.

That $\tilde{\jmath}$ is independent of the choice of preimage: For any $x \in \tilde{\mathcal{D}} = \mathrm{im}(i)$, let $y, y' \in \mathcal{D}$ be two preimages under i, hence $i(y) = i(y') = x$. This means that $i(y' - y) = 0$, hence that $y' - y \in \ker(i)$, hence that $y' - y \in \mathrm{im}(k)$, hence there exists $z \in \mathcal{E}$ such that $y' = y + k(z)$, hence $j(y') = j(y) + j(k(z)) = j(y) + d(z)$, but $d(z) = 0$ in $\tilde{\mathcal{E}}$.

That $\tilde{k}$ vanishes on $\mathrm{im}(d)$: because $\mathrm{im}(d) \subset \mathrm{im}(j)$ and hence by exactness.

That $\tilde{k}$ lands in $\mathrm{im}(i)$: since it is defined on $\ker(d) = \ker(j \circ k)$ it lands in $\ker(j)$. By exactness this is $\mathrm{im}(i)$.

That the sequence of maps is again exact:

The kernel of $\tilde{j}$ is those $x \in \mathrm{im}\,(i)$ such that their preimage $i^{-1}(x)$ is still in $\mathrm{im}(x)$ (by exactness of the original exact couple) hence such that $x \in \mathrm{im}(i|_{\mathrm{im}(i)})$, hence such that $x \in \mathrm{im}(\tilde{\imath})$.

The kernel of $\tilde{k}$ is the intersection of the kernel of k with the kernel of $d = j \circ k$, wich is still the kernel of k, hence the image of j, by exactness. Indeed this is also still the image of $\tilde{j}$, since for every $x \in \mathcal{D}$ then $\tilde{j}(i(x)) = j(x)$.

The kernel of $\tilde{\imath}$ is $\ker(i) \cap \mathrm{im}(i) \simeq \mathrm{im}(k) \cap \mathrm{im}(i)$, by exactness. Let $x \in \mathcal{E}$ such that $k(x) \in \mathrm{im}(i)$, then by exactness $k(x) \in \ker(j)$ hence $j(k(x)) = d(x) = 0$, hence $x \in \ker(d)$ and so $k(x) = \tilde{k}(x)$. ▮

Definition 1.11. Given an exact couple, def. 1.6, then the induced **spectral sequence of the exact couple** is the sequence of pages, def. 1.8, of the induced sequence of derived exact couples, def. 1.9, lemma 1.10.

The r**th page** of the spectral sequence is the page (def. 1.8) of the rth exact couple, denoted

$$\{\mathcal{E}_r, d_r\}\,.$$

Remark 1.12. So the spectral sequence of an exact couple (def. 1.11) is a sequence of cochain complexes $(\mathcal{E}_r, d_r)$, where the cohomology of one is the terms of the next one:

$$\mathcal{E}_{r+1} \simeq H(\mathcal{E}_r, d_r)\,.$$

In practice this is used as a successive stagewise approximation to the computation of a limiting term $\mathcal{E}_\infty$. What that limiting term is, if it exists at all, is the subject of *convergence* of the spectral sequence, we come to this below.

Def. 1.11 makes sense without a (bi-)grading on the terms of the exact couple, but much of the power of spectral sequences comes from the cases where such a bigrading is given, since together with the sequence of pages of the spectral sequence, this tends to organize computation of the successive cohomology groups in an efficient way. Therefore consider:

Definition 1.13. Given a filtered spectrum as in def. 1.1,

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & X_3 & \overset{f_2}{\longrightarrow} & X_2 & \overset{f_1}{\longrightarrow} & X_1 & \overset{f_0}{\longrightarrow} & X \\ && \downarrow^{g_3} && \downarrow^{g_2} && \downarrow^{g_1} && \downarrow^{g_0} \\ && A_3 && A_2 && A_1 && A_0 \end{array}$$

and given another spectrum $X \in \mathrm{Ho}(\mathrm{Spectra})$, the induced **spectral sequence of a filtered spectrum** is the spectral sequence that is induced, by def. 1.11 from the exact couple (def. 1.6) given by def. 1.4:

$$\begin{array}{ccc} \mathcal{D} & \overset{i}{\longrightarrow} & \mathcal{D} \\ & {}_{k}\nwarrow & \downarrow^{j} \\ && \mathcal{E} \end{array} \quad \coloneqq \quad \begin{array}{ccc} \oplus_{s,t} D^{s,t}(X,Y) & \overset{[X,f]}{\longrightarrow} & \oplus_{s,t} D^{s,t}(X,Y) \\ & {}_{[X,h]}\nwarrow & \downarrow^{[X,g]} \\ && \oplus_{s,t} E^{s,t}(X,Y) \end{array} \quad \coloneqq \quad \begin{array}{ccc} \oplus_{s,t} [X,Y_s]_{t-s} & \overset{[X,f]}{\longrightarrow} & \oplus_{s,t} [X,Y_s]_{t-s} \\ & {}_{[X,h]}\nwarrow & \downarrow^{[X,g]} \\ && [X,A_s]_{t-s} \end{array}$$

with the following bidegree of the differentials:

$$\deg = \begin{array}{ccc} \mathcal{D} & \overset{(-1,-1)}{\longrightarrow} & \mathcal{D} \\ & {}_{(1,0)}\nwarrow & \downarrow^{(0,0)} \\ && \mathcal{E} \end{array}$$

In particular the first page is

$$\begin{aligned} \mathcal{E}_1^{s,t} &= [X, A_s]_{t-s} \\ d_1 &= [X, g \circ h]\,. \end{aligned}$$

As we pass to derived exact couples, by def. 1.9, the bidegree of i and k is preserved, but that of j increases by $(1,1)$ with each page, since (by def. 1.8)

$$\begin{aligned} \deg(\tilde{j}) &= \deg(j \circ i^{-1}) \\ &= \deg(j) - \deg(i)\,. \\ &= \deg(j) + (1,1) \end{aligned}$$

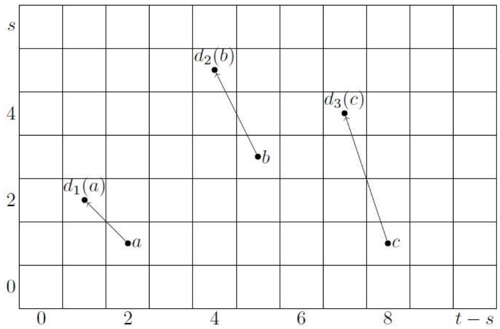
Similarly the first differential has degree

$$
\begin{aligned}
\deg(j \circ k) &= \deg(j) + \deg(k) \\
&= (1,0) + (0,0) \\
&= (1,0)
\end{aligned}
$$

and so the differentials on the rth page are of the form

$$d_r \,:\, \mathcal{E}_r^{s,t} \longrightarrow \mathcal{E}_r^{s+r,t+r-1} \,.$$

It is conventional to depict this in tables where s increases vertically and upwards and $t-s$ increases horizontally and to the right, so that d_r goes up r steps and always one step to the left. This is the "Adams type" grading convention for spectral sequences (different from the Serre-Atiyah-Hirzebruch spectral sequence convention (prop.)). One also says that



* s is the *filtration degree*;
* $t-s$ is the *total degree*;
* t is the *internal degree*.

A priori all this is $\mathbb{N} \times \mathbb{Z}$-graded, but we regard it as being $\mathbb{Z} \times \mathbb{Z}$-graded by setting

$$\mathcal{D}^{s<0,\bullet} \coloneqq 0 \quad , \quad \mathcal{E}^{s<0,\bullet} \coloneqq 0$$

and trivially extending the definition of the differentials to these zero-groups.

E-Adams filtrations

Given a homotopy commutative ring spectrum (E,μ,e), then an *E-Adams spectral sequence* is a spectral sequence as in def. 1.13, where each cofiber is induced from the unit morphism $e \,:\, \mathbb{S} \longrightarrow E$:

Definition 1.14. Let $X, Y \in \mathrm{Ho}(\mathrm{Spectra})$ be two spectra (def.), and let $(E,\mu,e) \in \mathrm{CMon}(\mathrm{Ho}(\mathrm{Spectra}), \wedge, \mathbb{S})$ be a homotopy commutative ring spectrum (def.) in the tensor triangulated stable homotopy category $(\mathrm{Ho}(\mathrm{Spectra}), \wedge, \mathbb{S})$ (prop.).

Then the E-**Adams spectral sequence** for the computation of the graded abelian group

$$[X,Y]_\bullet \coloneqq [X, \Sigma^{-\bullet} Y]$$

of morphisms in the stable homotopy category (def.) is the spectral sequence of a filtered spectrum (def. 1.13) of the image under $[X,-]_\bullet$ of the tower

$$
\begin{array}{ccccc}
 & \vdots & & & \\
 & {}^{f_0}\downarrow & & & \\
 & Y_3 & \overset{g_3}{\longrightarrow} & E \wedge Y_3 = A_3 & \\
 & {}^{f_0}\downarrow & & & \\
 & Y_2 & \overset{g_2}{\longrightarrow} & E \wedge Y_2 = A_2 \,, & \\
 & {}^{f_0}\downarrow & & & \\
 & Y_1 & \overset{g_1}{\longrightarrow} & E \wedge Y_1 = A_1 & \\
 & {}^{f_0}\downarrow & & & \\
Y = & Y_0 & \overset{g_0}{\longrightarrow} & E \wedge Y_0 = A_0 &
\end{array}
$$

where each hook is a homotopy fiber sequence (equivalently a homotopy cofiber sequence, prop.), hence where each

$$Y_{n+1} \overset{f_n}{\longrightarrow} Y_n \overset{g_n}{\longrightarrow} A_n \overset{h_n}{\longrightarrow} \Sigma Y_{n+1}$$

is an exact triangle (prop.), where inductively

$$A_n \coloneqq E \wedge Y_n$$

is the derived smash product of spectra (corollary) of E with the stage Y_n (cor.), and where

$$g_n \,:\, Y_n \underset{\simeq}{\overset{\ell_{Y_n}^{-1}}{\longrightarrow}} \mathbb{S} \wedge Y_n \overset{e \wedge \mathrm{id}}{\longrightarrow} E \wedge Y_n$$

is the composition of the inverse derived unitor on Y_n (cor.) with the derived smash product of spectra of the unit e of E and the identity on Y_n.

Hence, by def 1.13, the first page is

$$\begin{aligned} E_1^{s,t}(X,Y) &\coloneqq [X, A_s]_{t-s} \\ d_1 &= [X, g \circ h] \end{aligned}$$

and the differentials are of the form

$$d_r \,:\, E_r^{s,t}(X,Y) \longrightarrow E_r^{s+r,t+r-1}(X,Y)\,.$$

A priori $E_r^{\bullet,\bullet}(X,Y)$ is $\mathbb{N} \times \mathbb{Z}$-graded, but we regard it as being $\mathbb{Z} \times \mathbb{Z}$-graded by setting

$$E_r^{s<0,\bullet}(X,Y) \coloneqq 0$$

and trivially extending the definition of the differentials to these zero-groups.

(Adams 74, theorem 15.1 page 318)

Remark 1.15. The morphism

$$[X, g_k] \,:\, [X, Y_k]_\bullet \overset{[X, e \wedge \mathrm{id}_{Y_k}]}{\longrightarrow} [X, E \wedge Y_k]_\bullet$$

in def. 1.14 is sometimes called the **Boardman homomorphism** (Adams 74, p. 58).

For $X = \mathbb{S}$ the sphere spectrum it reduces to a canonical morphism from stable homotopy to generalized homology (rmk.)

$$\pi_\bullet(g_k) \,:\, \pi_\bullet(Y_k) \longrightarrow E_\bullet(Y_k)\,.$$

For $E =$ HA an Eilenberg-MacLane spectrum (def.) this in turn reduces to the **Hurewicz homomorphism** for spectra.

This way one may think of the E-Adams filtration on Y in def. 1.14 as the result of filtering any spectrum Y by iteratively projecting out all its E-homology. This idea was historically the original motivation for the construction of the classical Adams spectral sequence by John Frank Adams, see the first pages of (Bruner 09) for a historical approach.

It is convenient to adopt the following notation for E-Adams spectral sequences (def. 1.14):

Definition 1.16. For $(E, \mu, e) \in \mathrm{CMon}(\mathrm{Ho}(\mathrm{Spectra}), \wedge, \mathbb{S})$ a homotopy commutative ring spectrum (def.), write $\overline{E}$ for the homotopy fiber of its unit $e{:}\mathbb{S} \to E$, i.e. such that there is a homotopy fiber sequence (equivalently a homotopy cofiber sequence, prop.) in the stable homotopy category $\mathrm{Ho}(\mathrm{Spectra})$ of the form

$$\overline{E} \longrightarrow \mathbb{S} \overset{e}{\longrightarrow} E\,,$$

equivalently an exact triangle (prop.) of the form

$$\overline{E} \longrightarrow \mathbb{S} \overset{e}{\longrightarrow} E \longrightarrow \Sigma \overline{E}\,.$$

(Adams 74, theorem 15.1 page 319) Beware that for instance (Hopkins 99, proof of corollary 5.3) uses “$\overline{E}$” not for the homotopy fiber of $\mathbb{S} \overset{e}{\to} E$ but for its homotopy cofiber, hence for what is $\Sigma\overline{E}$ according to (Adams 74).

Lemma 1.17. *In terms of def. 1.16, the spectra entering the definition of the E-Adams spectral sequence in def. 1.14 are equivalently*

$$Y_p \;\simeq\; \overline{E}^p \wedge Y$$

and

$$A_p \;\simeq\; E \wedge \overline{E}^p \wedge Y\,.$$

where we write

$$\overline{E}^p \coloneqq \underbrace{\overline{E} \wedge \cdots \wedge \overline{E}}_{p \text{ factors}} \wedge Y\,.$$

Hence the first page of the E-Adams spectral sequence reads equivalently

$$E_1^{s,t}(X,Y) \simeq [X, E \wedge \overline{E}^s \wedge Y]_{t-s}\,.$$

(Adams 74, theorem 15.1 page 319)

Proof. By definition the statement holds for $p = 0$. Assume then by induction that it holds for some $p \geq 0$. Since the smash product of spectra-functor $(-) \wedge \overline{E}^p \wedge Y$ preserves homotopy cofiber sequences (lemma, this is part of the tensor triangulated structure of the stable homotopy category), its application to the homotopy cofiber sequence

$$\overline{E} \longrightarrow \mathbb{S} \overset{e}{\longrightarrow} E$$

from def. 1.16 yields another homotopy cofiber sequence, now of the form

$$\overline{E}^{p+1} \wedge Y \longrightarrow \overline{E}^p \wedge Y \overset{g_p}{\longrightarrow} E \wedge \overline{E}^p \wedge Y$$

where the morphism on the right is identified as g_p by the induction assumption, hence $A_{p+1} \simeq E \wedge \overline{E}^p \wedge Y$. Since Y_{p+1} is defined to be the homotopy fiber of g_p, it follows that $Y_{p+1} \simeq \overline{E}^{p+1} \wedge Y$. ▮

Remark 1.18. Terminology differs across authors. The filtration in def. 1.14 in the rewriting by lemma 1.17 is due to (Adams 74, theorem 15.1), where it is not give any name. In (Ravenel 84, p. 356) it is called the (canonical) **Adams tower** while in (Ravenel 86, def. 2.21) it is called the canonical **Adams resolution**. Several authors follow the latter usage, for instance (Rognes 12, def. 4.1). But (Hopkins 99) uses "Adams resolution" for the "E-injective resolutions" (see here) and uses "Adams tower" for yet another concept (def.).

We proceed now to analyze the first two pages and then the convergence properties of E-Adams spectral sequences of def. 1.14.

2. The first page

By lemma 1.17 the first page of an E-Adams spectral sequence (def. 1.14) looks like

$$E_1^{s,t}(X,Y) \simeq [X, E \wedge \overline{E}^s \wedge Y]_{s-t} \,.$$

We discuss now how, under favorable conditions, these hom-groups may alternatively be computed as morphisms of E-homology equipped with suitable comodule structure over a Hopf algebroid structure on the dual E-Steenrod operations $E_\bullet(E)$ (The E-generalized homology of E (rmk.)). Then below we discuss that, as a result, the d_1-cohomology of the first page computes the Ext-groups from the E-homology of Y to the E-homology of X, regarded as $E_\bullet(E)$-comodules.

The condition needed for this to work is the following.

Flat homotopy ring spectra

Definition 2.1. Call a homotopy commutative ring spectrum (E, μ, e) (def.) **flat** if the canonical right $\pi_\bullet(E)$-module structure on $E_\bullet(E)$ (prop.) (equivalently the canonical left module struture, see prop. 2.5 below) is a flat module.

The key consequence of the assumption that E is flat in the sense of def. 2.1 is the following.

Proposition 2.2. *Let (E,μ,e) be a homotopy commutative ring spectrum (def.) and let $X \in \mathrm{Ho}(\mathrm{Spectra})$ be any spectrum. Then there is a homomorphism of graded abelian groups of the form*

$$E_\bullet(E) \otimes_{\pi_\bullet(E)} E_\bullet(X) \longrightarrow [\mathbb{S}, E \wedge E \wedge X]_\bullet = \pi_\bullet(E \wedge E \wedge X)$$

(for $E_\bullet(-)$ the canonical $\pi_\bullet(E)$-modules from this prop.) given on elements

$$\Sigma^{n_1}\mathbb{S} \overset{\alpha_1}{\longrightarrow} E \wedge E \quad , \quad \Sigma^{n_2}\mathbb{S} \overset{\alpha_2}{\longrightarrow} E \wedge X$$

by

$$\alpha_1 \cdot \alpha_2 \,:\, \Sigma^{n_1+n_2}\mathbb{S} \overset{\simeq}{\longrightarrow} \Sigma^{n_1}\mathbb{S} \wedge \Sigma^{n_2}\mathbb{S} \overset{\alpha_1 \wedge \alpha_2}{\longrightarrow} E \wedge E \wedge E \wedge X \overset{\mathrm{id}_E \wedge \mu \wedge \mathrm{id}_X}{\longrightarrow} E \wedge E \wedge X \,.$$

If $E_\bullet(E)$ is a flat module over $\pi_\bullet(E)$ then this is an isomorphism.

(Adams 69, lecture 3, lemma 1 (p. 68), Adams 74, part III, lemma 12.5)

Proof. First of all, that the given pairing is a well defined homomorphism (descends from $E_\bullet(E) \times E_\bullet(X)$ to $E_\bullet(E) \otimes_{\pi_\bullet(E)} E_\bullet(X)$) follows from the associativity of μ.

We discuss that it is an isomorphism when $E_\bullet(E)$ is flat over $\pi_\bullet(E)$:

First consider the case that $X \simeq \Sigma^n \mathbb{S}$ is a suspension of the sphere spectrum. Then (by this example, using the tensor triangulated stucture on the stable homotopy category (prop.))

$$E_\bullet(X) = E_\bullet(\Sigma^n X) \simeq \pi_{\bullet - n}(E)$$

and

$$\pi_\bullet(E \wedge E \wedge X) = \pi_\bullet(E \wedge E \wedge \Sigma^n \mathbb{S}) \simeq E_{\bullet - n}(E)$$

and

$$E_\bullet(E) \otimes_{\pi_\bullet(E)} \pi_{\bullet - n}(E) \simeq E_{\bullet - n}(E)$$

Therefore in this case we have an isomorphism for all E.

For general X, we may without restriction assume that X is represented by a sequential CW-spectrum (prop.). Then the homotopy cofibers of its cell attachment maps are suspensions of the sphere spectrum (rmk.).

First consider the case that X is a CW-spectrum with finitely many cells. Consider the homotopy cofiber sequence of the $(k+1)$st cell attachment (by that remark):

$$\Sigma^{n_k - 1}\mathbb{S} \longrightarrow X_k \longrightarrow X_{k+1} \longrightarrow \Sigma^{n_k}\mathbb{S} \longrightarrow \Sigma X_k$$

and its image under the natural morphism $E_\bullet(E) \otimes_{\pi_\bullet(E)} E_\bullet(-) \to \pi_\bullet([\mathbb{S}, E \wedge E \wedge (-)])$, which is a commuting diagram of the form

$$\begin{array}{ccccccccc}
E_\bullet(E) \otimes_{\pi_\bullet(E)} E_\bullet(\Sigma^{n_k - 1}\mathbb{S}) & \longrightarrow & E_\bullet(E) \otimes_{\pi_\bullet(E)} E_\bullet(X_k) & \longrightarrow & E_\bullet(E) \otimes_{\pi_\bullet(E)} E_\bullet(X_{k+1}) & \longrightarrow & E_\bullet(E) \otimes_{\pi_\bullet(E)} E_\bullet(\Sigma^{n_k}\mathbb{S}) & \longrightarrow & E_\bullet(E) \otimes_{\pi_\bullet(E)} E_\bullet(\Sigma X
\\
\downarrow && \downarrow && \downarrow && \downarrow && \downarrow
\\
[\mathbb{S}, E \wedge E \wedge \Sigma^{n_k - 1}\mathbb{S}]_\bullet & \longrightarrow & [\mathbb{S}, E \wedge E \wedge X_k]_\bullet & \longrightarrow & [\mathbb{S}, E \wedge E \wedge X_{k+1}]_\bullet & \longrightarrow & [\mathbb{S}, E \wedge E \wedge \Sigma^{n_k}\mathbb{S}]_\bullet & \longrightarrow & [\mathbb{S}, E \wedge E \wedge \Sigma X_k]_\bullet
\end{array}$$

Here the bottom row is a long exact sequence since $E \wedge E \wedge (-)$ preserves homotopy cofiber sequences (by this lemma, part of the tensor triangulated structure on $\mathrm{Ho}(\mathrm{Spectra})$ prop.), and since $[\mathbb{S}, -]_\bullet \simeq \pi_\bullet(-)$ sends homotopy cofiber sequences to long exact sequences (prop.). By the same reasoning, $E_\bullet(-)$ of the homotopy cofiber sequence is long exact; and by the assumption that $E_\bullet(E)$ is flat, the functor $E_\bullet(E) \otimes_{\pi_\bullet(E)} (-)$ preserves this exactness, so that also the top row is a long exact sequence.

Now by induction over the cells of X, the outer four vertical morphisms are isomorphisms. Hence the 5-lemma implies that also the middle morphism is an isomorphism.

This shows the claim inductively for all finite CW-spectra. For the general statement, now use that

1. every CW-spectrum is the filtered colimit over its finite CW-subspectra;
2. the symmetric monoidal smash product of spectra $\wedge$ (def.) preserves colimits in its arguments separately (since it has a right adjoint (prop.));
3. $[\mathbb{S}, -]_\bullet \simeq \pi_\bullet(-)$ commutes over filtered colimits of CW-spectrum inclusions (by this lemma, since spheres are compact);
4. $E_\bullet(E) \otimes_{\pi_\bullet(E)} (-)$ distributes over colimits (it being a left adjoint).

∎

Using prop. 2.2, we find below (theorem 2.34) that the first page of the E-Adams spectral sequence may be equivalently rewritten as hom-groups of comodules over $E_\bullet(E)$ regarded as a graded commutative Hopf algebroid. We now first discuss what this means.

The E-Steenrod algebra

We discuss here all the extra structure that exists on the E-self homology $E_\bullet(E)$ of a flat homotopy commutative ring spectrum. For $E = H\mathbb{F}_p$ the Eilenberg-MacLane spectrum on a prime field this reduces to the classical structure in algebraic topology called the *dual Steenrod algebra* $\mathcal{A}_p^*$. Therefore one may generally speak of $E_\bullet(E)$ as being the *dual E-Steenrod algebra*.

Without the qualifier "dual" then "E-Steenrod algebra" refers to the E-self-cohomology $E^\bullet(E)$. For $E = H\mathbb{F}_p$ this *Steenrod algebra* $\mathcal{A}_p$ (without "dual") is traditionally considered first, and the classical Adams spectral sequence was originally formulated in terms of $\mathcal{A}_p$ instead of $\mathcal{A}_p^*$. But one observes (Adams 74, p. 280) that the "dual" Steenrod algebra $E_\bullet(E)$ is much better behaved, at least as long as E is flat in the sense of def. 2.1.

Moreover, the dual E-Steenrod algebra $E_\bullet(E)$ is more fundamental in that it reflects a stacky geometry secretly underlying the E-Adams spectral sequence (Hopkins 99). This is the content of the concept of "commutative Hopf algebroid" (def. 2.9 below) which is equivalently the formal dual of a groupoid internal to affine schemes, def. 2.6.

A simple illustrative archetype of the following construction of commutative Hopf algebroids from homotopy commutative ring spectra is the following situation:

For X a finite set consider

$$\begin{array}{c} X \times X \times X \\ \downarrow^{\circ = (\mathrm{pr}_1, \mathrm{pr}_3)} \\ X \times X \\ {}^{s = \mathrm{pr}_1}\downarrow \uparrow \downarrow^{t = \mathrm{pr}_2} \\ X \end{array}$$

as the ("codiscrete") groupoid with X as objects and precisely one morphism from every object to every other. Hence the composition operation $\circ$, and the source and target maps are simply projections as shown. The identity morphism (going upwards in the above diagram) is the diagonal.

Then consider the image of this structure under forming the free abelian groups $\mathbb{Z}[X]$, regarded as commutative rings under pointwise multiplication.

Since

$$\mathbb{Z}[X \times X] \simeq \mathbb{Z}[X] \otimes \mathbb{Z}[X]$$

this yields a diagram of homomorphisms of commutative rings of the form

$$\begin{array}{c} (\mathbb{Z}[X] \otimes \mathbb{Z}[X]) \otimes_{\mathbb{Z}[X]} (\mathbb{Z}[X] \otimes \mathbb{Z}[X]) \\ \uparrow \\ \mathbb{Z}[X] \otimes \mathbb{Z}[X] \\ \uparrow \downarrow \uparrow \\ \mathbb{Z}[X] \end{array}$$

satisfying some obvious conditions. Observe that here

1. the two morphisms $\mathbb{Z}[X] \rightrightarrows \mathbb{Z}[X] \otimes \mathbb{Z}[X]$ are $f \mapsto f \otimes e$ and $f \mapsto e \otimes f$, respectively, where e denotes the unit element in $\mathbb{Z}[X]$;

2. the morphism $\mathbb{Z}[X] \otimes \mathbb{Z}[X] \to \mathbb{Z}[X]$ is the multiplication in the ring $\mathbb{Z}[X]$;

3. the morphism

   $$\mathbb{Z}[X] \otimes \mathbb{Z}[X] \longrightarrow \mathbb{Z}[X] \otimes \mathbb{Z}[C] \otimes \mathbb{Z}[C] \xrightarrow{\simeq} (\mathbb{Z}[X] \otimes \mathbb{Z}[X]) \otimes_{\mathbb{Z}[X]} (\mathbb{Z}[X] \otimes \mathbb{Z}[X])$$

   is given by $f \otimes g \mapsto f \otimes e \otimes g$.

All of the following rich structure is directly modeled on this simplistic example. We simply

1. replace the commutative ring $\mathbb{Z}[X]$ with any flat homotopy commutative ring spectrum E,

2. replace tensor product of abelian groups $\otimes$ with derived smash product of spectra;

3. and form the stable homotopy groups $\pi_\bullet(-)$ of all resulting expressions.

Definition 2.3. Let (E, μ, e) be a homotopy commutative ring spectrum (def.) which is flat according to def. 2.1.

Then the **dual E-Steenrod algebra** is the pair of graded abelian groups

$$(E_\bullet(E), \pi_\bullet(E))$$

(rmk.) equipped with the following structure:

1. the graded commutative ring structure

   $$\pi_\bullet(E) \otimes \pi_\bullet(E) \longrightarrow \pi_\bullet(E)$$

   induced from E being a homotopy commutative ring spectrum (prop.);

2. the graded commutative ring structure

$$E_\bullet(E) \otimes E_\bullet(E) \longrightarrow E_\bullet(E)$$

induced from the fact that with E also $E \wedge E$ is canonically a homotopy commutative ring spectrum (exmpl.), so that also $E_\bullet(E) = \pi_\bullet(E \wedge E)$ is a graded commutative ring (prop.);

3. the homomorphism of graded commutative rings

$$\Psi : E_\bullet(E) \longrightarrow E_\bullet(E) \otimes_{\pi_\bullet(E)} E_\bullet(E)$$

induced under $\pi_\bullet(-)$ from

$$E \wedge E \overset{\mathrm{id} \wedge e \wedge \mathrm{id}}{\longrightarrow} E \wedge E \wedge E$$

via prop. 2.2;

4. the homomorphisms of graded commutative rings

$$\eta_L : \pi_\bullet(E) \longrightarrow E_\bullet(E)$$

and

$$\eta_R : \pi_\bullet(E) \longrightarrow E_\bullet(E)$$

induced under $\pi_\bullet(-)$ from the homomorphisms of commutative ring spectra

$$E \underset{\simeq}{\overset{r_E^{-1}}{\longrightarrow}} E \wedge \mathbb{S} \overset{\mathrm{id} \wedge e}{\longrightarrow} E \wedge E$$

and

$$E \underset{\simeq}{\overset{\ell_E^{-1}}{\longrightarrow}} \mathbb{S} \wedge E \overset{\mathrm{id} \wedge e}{\longrightarrow} E \wedge E \,,$$

respectively (exmpl.);

5. the homomorphism of graded commutative rings

$$\epsilon : E_\bullet(E) \longrightarrow \pi_\bullet(E)$$

induced under $\pi_\bullet(-)$ from

$$\mu : E \wedge E \longrightarrow E$$

regarded as a homomorphism of homotopy commutative ring spectra (exmpl.);

6. the homomorphisms graded commutative rings

$$c : E_\bullet(E) \longrightarrow E_\bullet(E)$$

induced under $\pi_\bullet(-)$ from the braiding

$$\tau_{E,E} : E \wedge E \longrightarrow E \wedge E$$

regarded as a homomorphism of homotopy commutative ring spectra (exmpl.).

(Adams 69, lecture 3, pages 66-68)

Notice that (as verified by direct unwinding of the definitions):

Lemma 2.4. *For (E, μ, e) a homotopy commutative ring spectrum (def.), consider $E_\bullet(E)$ with its canonical left and right $\pi_\bullet(E)$-module structure as in this prop.. These module structures coincide with those induced by the ring homomorphisms η_L and η_R from def. 2.3.*

These two actions need not strictly coincide, but they are isomorphic:

Proposition 2.5. *For (E, μ, e) a homotopy commutative ring spectrum (def.), consider $E_\bullet(E)$ with its canonical left and right $\pi_\bullet(E)$-module structure (prop.). Since E is a commutative monoid, this right module structure may equivalently be regarded as a left-module, too. Then the braiding*

$$E_\bullet(E) \simeq \pi_\bullet(E \wedge E) \overset{\pi_\bullet(\tau_{E,E})}{\longrightarrow} \pi_\bullet(E \wedge E) \simeq E_\bullet(E)$$

constitutes a module isomorphism (def.) between these two left module structures.

Proof. On representatives as in the proof of (this propo.), the original left action is given by (we are notationally suppressing associators throughout)

$$E \wedge E \wedge E \overset{\mu \wedge \mathrm{id}}{\longrightarrow} E \wedge E \,,$$

while the other left action, induced from the canonical right action, is given by

$$E \wedge E \wedge E \underoverset{\simeq}{\tau_{E, E \wedge E}}{\longrightarrow} E \wedge E \wedge E \overset{\mathrm{id} \wedge \mu}{\longrightarrow} E \wedge \;.$$

So in order that $\tau_{E,E}$ represents a module homomorphism under $\pi_\bullet(-)$, it is sufficient that the following diagram commutes (we write $E_i \coloneqq E$ for $i \in \{1,2,3\}$ to make the action of the braiding more manifest)

$$\begin{array}{ccc} E_1 \wedge E_2 \wedge E_3 & \overset{\mathrm{id} \wedge \tau_{E_2, E_3}}{\longrightarrow} & E_1 \wedge E_3 \wedge E_2 \\ {}^{\mathrm{id}}\downarrow && \downarrow^{\tau_{E_1, E_3 \wedge E_2}} \\ && E_3 \wedge E_2 \wedge E_1 \\ {}^{\mu \wedge \mathrm{id}}\downarrow && \downarrow^{\mathrm{id} \wedge \mu} \\ E \wedge E_3 & \underset{\tau_{E, E_3}}{\longrightarrow} & E_3 \wedge E \end{array} \,.$$

But since (E, μ, e) is a commutative monoid (def.), it satisfies $\mu = \mu \circ \tau$ so that we may factor this diagram as follows:

$$\begin{array}{ccc} E_1 \wedge E_2 \wedge E_3 & \overset{\mathrm{id} \wedge \tau_{E_2, E_3}}{\longrightarrow} & E_1 \wedge E_3 \wedge E_2 \\ {}^{\tau_{E_1, E_2} \wedge \mathrm{id}}\downarrow && \downarrow^{\tau_{E_1, E_3 \wedge E_2}} \\ E_2 \wedge E_1 \wedge E_3 & \overset{\tau_{E_2 \wedge E_1, E_3}}{\longrightarrow} & E_3 \wedge E_2 \wedge E_1 \\ {}^{\mu \wedge \mathrm{id}}\downarrow && \downarrow^{\mathrm{id} \wedge \mu} \\ E \wedge E_3 & \underset{\tau_{E, E_3}}{\longrightarrow} & E_3 \wedge E \end{array} \,.$$

Here the top square commutes by coherence of the braiding (rmk) since both composite morphisms correspond to the same permutation, while the bottom square commutesm due to the naturality of the braiding. Hence the total rectangle commutes. ▮

The dual E-Steenrod algebras of def. 2.3 evidently carry a lot of structure. The concept organizing this is that of commutative Hopf algebroids .

Definition 2.6. A **graded commutative Hopf algebroid** is an internal groupoid in the opposite category $\mathrm{gCRing}^{\mathrm{op}}$ of $\mathbb{Z}$-graded commutative rings, regarded with its cartesian monoidal category structure.

(e.g. Ravenel 86, def. A1.1.1)

Remark 2.7. We unwind def. 2.6. For $R \in \mathrm{gCRing}$, write $\mathrm{Spec}(R)$ for the same object, but regarded as an object in $\mathrm{gCRing}^{\mathrm{op}}$.

An internal category in $\mathrm{gCRing}^{\mathrm{op}}$ is a diagram in $\mathrm{gCRing}^{\mathrm{op}}$ of the form

$$\begin{array}{c} \mathrm{Spec}(\Gamma) \underset{\mathrm{Spec}(A)}{\times} \mathrm{Spec}(\Gamma) \\ \downarrow^{\circ} \\ \mathrm{Spec}(\Gamma) \\ {}^{s}\downarrow \;\; \uparrow_{i} \downarrow^{t} \\ \mathrm{Spec}(A) \end{array} \,,$$

(where the fiber product at the top is over s on the left and t on the right) such that the pairing $\circ$ defines an associative composition over $\mathrm{Spec}(A)$, unital with respect to i. This is an internal groupoid if it is furthemore equipped with a morphism

$$\mathrm{inv} : \mathrm{Spec}(\Gamma) \longrightarrow \mathrm{Spec}(\Gamma)$$

acting as assigning inverses with respect to $\circ$.

The key basic fact to use in order to express this equivalently in terms of algebra is that tensor product of commutative rings exhibits the cartesian monoidal category structure on $\mathrm{CRing}^{\mathrm{op}}$, see at *CRing – Properties – Cocartesian comonoidal structure*:

$$\mathrm{Spec}(R_1) \underset{\mathrm{Spec}(R_3)}{\times} \mathrm{Spec}(R_2) \simeq \mathrm{Spec}(R_1 \otimes_{R_3} R_2) \,.$$

This means that the above is equivalently a diagram in gCRing of the form

$$\begin{array}{c} \Gamma \otimes_A \Gamma \\ \uparrow^{\Psi} \\ \Gamma \\ {}^{\eta_L}\uparrow \;\; \downarrow^{\epsilon} \;\; \uparrow^{\eta_R} \\ A \end{array}$$

as well as

$$c : \Gamma \longrightarrow \Gamma$$

and satisfying formally dual conditions, spelled out as def. 2.9 below. Here

- η_L, η_R are called the left and right *unit maps*;
- ϵ is called the *co-unit*;
- Ψ is called the *comultiplication*;
- c is called the *antipode* or *conjugation*

Remark 2.8. Generally, in a commutative Hopf algebroid, def. 2.6, the two morphisms $\eta_L, \eta_R : A \to \Gamma$ from remark 2.7 need not coincide, they make Γ genuinely into a bimodule over A, and it is the tensor product of bimodules that appears in remark 2.7. But it may happen that they coincide:

An internal groupoid $\mathcal{G}_1 \underset{t}{\overset{s}{\rightrightarrows}} \mathcal{G}_0$ for which the domain and codomain morphisms coincide, $s = t$, is euqivalently a group object in the slice category over $\mathcal{G}_0$.

Dually, a commutative Hopf algebroid $\Gamma \underset{\eta_R}{\overset{\eta_L}{\leftleftarrows}} A$ for which η_L and η_R happen to coincide is equivalently a commutative **Hopf algebra** Γ over A.

Writing out the formally dual axioms of an internal groupoid as in remark 2.7 yields the following equivalent but maybe more explicit definition of commutative Hopf algebroids, def. 2.6

Definition 2.9. A **commutative Hopf algebroid** is

1. two commutative rings, A and Γ;
2. ring homomorphisms
   1. (left/right unit)

      $\eta_L, \eta_R : A \longrightarrow \Gamma$;
   2. (comultiplication)

      $\Psi : \Gamma \longrightarrow \Gamma \otimes_A \Gamma$;
   3. (counit)

      $\epsilon : \Gamma \longrightarrow A$;
   4. (conjugation)

      $c : \Gamma \longrightarrow \Gamma$

such that

1. (co-unitality)
   1. (identity morphisms respect source and target)

      $\epsilon \circ \eta_L = \epsilon \circ \eta_R = \mathrm{id}_A$;
   2. (identity morphisms are units for composition)

      $(\mathrm{id}_\Gamma \otimes_A \epsilon) \circ \Psi = (\epsilon \otimes_A \mathrm{id}_\Gamma) \circ \Psi = \mathrm{id}_\Gamma$;
   3. (composition respects source and target)
      1. $\Psi \circ \eta_R = (\mathrm{id} \otimes_A \eta_R) \circ \eta_R$;

2. $\Psi \circ \eta_L = (\eta_L \otimes_A \mathrm{id}) \circ \eta_L$

2. (co-associativity) $(\mathrm{id}_\Gamma \otimes_A \Psi) \circ \Psi = (\Psi \otimes_A \mathrm{id}_\Gamma) \circ \Psi$;

3. (inverses)

   1. (inverting twice is the identity)

      $c \circ c = \mathrm{id}_\Gamma$;

   2. (inversion swaps source and target)

      $c \circ \eta_L = \eta_R$; $c \circ \eta_R = \eta_L$;

   3. (inverse morphisms are indeed left and right inverses for composition)

      the morphisms α and β induced via the coequalizer property of the tensor product from $(-) \cdot c(-)$ and $c(-) \cdot (-)$, respectively

$$\begin{tikzcd} \Gamma \otimes A \otimes \Gamma \arrow[r, shift left] \arrow[r, shift right] & \Gamma \otimes \Gamma \arrow[r, "\mathrm{coeq}"] \arrow[d, "(-)\cdot c(-)"'] & \Gamma \otimes_A \Gamma \arrow[dl, "\alpha"] \\ & \Gamma & \end{tikzcd}$$

and

$$\begin{tikzcd} \Gamma \otimes A \otimes \Gamma \arrow[r, shift left] \arrow[r, shift right] & \Gamma \otimes \Gamma \arrow[r, "\mathrm{coeq}"] \arrow[d, "c(-)\cdot(-)"'] & \Gamma \otimes_A \Gamma \arrow[dl, "\beta"] \\ & \Gamma & \end{tikzcd}$$

satisfy

$\alpha \circ \Psi = \eta_L \circ \epsilon$

and

$\beta \circ \Psi = \eta_R \circ \epsilon$.

(Adams 69, lecture 3, pages 62-66, Ravenel 86, def. A1.1.1)

Remark 2.10. In (Adams 69, lecture 3, page 60) the terminology used is "Hopf algebra in a fully satisfactory sense" with emphasis that the left and right module structure may differ. According to (Ravenel 86, first page of appendix A1) the terminology "Hopf algebroid" for this situation is due to Haynes Miller.

Example 2.11. For R a commutative ring, then $R \otimes R$ becomes a commutative Hopf algebroid over R, formally dual (via def. 2.6) to the pair groupoid on $\mathrm{Spec}(R) \in \mathrm{CRing}^{\mathrm{op}}$.

For X a finite set and $R = \mathbb{Z}[X]$, then this reduces to the motivating example from above.

It is now straightforward, if somewhat tedious, to check that:

Proposition 2.12. *Let (E, μ, e) be a homotopy commutative ring spectrum (def.) which is flat according to def. 2.1, then the dual E-Steenrod algebra $(E_\bullet(E), \pi_\bullet(E))$ with the structure maps $(\eta_L, \eta_R, \epsilon, c, \Psi)$ from prop. 2.3 is a graded commutative Hopf algebroid according to def. 2.9:*

$$(E_\bullet(E), \pi_\bullet(E)) \in \mathrm{CommHopfAlgd}$$

(Adams 69, lecture 3, pages 67-71, Ravenel 86, chapter II, prop. 2.2.8)

Proof. One observes that $E \wedge E$ satisfies the axioms of a commutative Hopf algebroid object in homotopy commutative ring spectra, over E, by direct analogy to example 2.11 (one just has to verify that the symmetric braidings go along coherently, which works by use of the coherence theorem for symmetric monoidal categories (rmk.)). Applying the functor $\pi_\bullet(-)$ that forms stable homotopy groups to all structure morphisms of $E \wedge E$ yields the claimed structure morphisms of $E_\bullet(E)$. ∎

We close this subsection on commutative Hopf algebroids by discussion of their isomorphism classes, when regarded dually as affine groupoids:

Definition 2.13. Given an internal groupoid in $\mathrm{gCRing}^{\mathrm{op}}$ (def. 2.6, remark 2.7)

$$
\begin{array}{c}
\mathrm{Spec}(\Gamma) \underset{\mathrm{Spec}(A)}{\times} \mathrm{Spec}(\Gamma) \\
\downarrow^{\circ} \\
\mathrm{Spec}(\Gamma) \\
{}^{s}\downarrow \;\; \uparrow_{i} \downarrow^{t} \\
\mathrm{Spec}(A)
\end{array}
,
$$

then its affife scheme $\mathrm{Spec}(A)_{/\sim}$ of **isomorphism classes of objects** is the *coequlizer?* of the source and target morphisms

$$
\mathrm{Spec}(\mathrm{Gamma}) \underset{t}{\overset{s}{\rightrightarrows}} \mathrm{Spec}(A) \overset{\mathrm{equ}}{\longrightarrow} \mathrm{Spec}(A)_{/\sim} \,.
$$

Hence this is the formal dual of the equalizer of the left and right unit (def. 2.9)

$$
A \underset{\eta_R}{\overset{\eta_L}{\rightrightarrows}} \Gamma \,.
$$

By example 2.11 every commutative ring gives rise to a commutative Hopf algebroid $R \otimes R$ over R. The *core* of R is the formal dual of the corresponding affine scheme of isomorphism classes according to def. 2.13:

Definition 2.14. For R a commutative ring, its **core** cR is the equalizer in

$$
cR \longrightarrow R \rightrightarrows R \otimes R \,.
$$

A ring which is isomorphic to its core is called a **solid ring**.

(Bousfield-Kan 72, §1, def. 2.1, Bousfield 79, 6.4)

Proposition 2.15. *The core of any ring R is solid (def. 2.14):*

$$
ccR \simeq cR \,.
$$

(Bousfield-Kan 72, prop. 2.2)

Proposition 2.16. *The following is the complete list of solid rings (def. 2.14) up to isomorphism:*

1. *The localization of the ring of integers at a set J of prime numbers (def. 4.11)*

   $$\mathbb{Z}[J^{-1}] \,;$$

2. *the cyclic rings*

   $$\mathbb{Z}/n\mathbb{Z}$$

   for $n \geq 2$;

3. *the product rings*

   $$\mathbb{Z}[J^{-1}] \times \mathbb{Z}/n\mathbb{Z} \,,$$

   for $n \geq 2$ such that each prime factor of n is contained in the set of primes J;

4. *the ring cores of product rings*

   $$c(\mathbb{Z}[J^{-1}] \times \prod_{p \in K} \mathbb{Z}/p^{e(p)}) \,,$$

   where $K \subset J$ are infinite sets of primes and $e(p)$ are positive natural numbers.

(Bousfield-Kan 72, prop. 3.5, Bousfield 79, p. 276)

Comodules over the E-Steenrod algebra

Definition 2.17. Let (E, μ, e) be a homotopy commutative ring spectrum (def.) which is flat according to def. 2.1.

For $X \in \mathrm{Ho}(\mathrm{Spectra})$ any spectrum, say that the **comodule structure** on $E_\bullet(X)$ (rmk.) **over the dual E-Steenrod algebra** (def. 2.3) is

1. the canonical structure of a $\pi_\bullet(E)$-module (according to this prop.);

2. the homomorphism of $\pi_\bullet(E)$-modules

$$\Psi_{E_\bullet(X)} : E_\bullet(X) \longrightarrow E_\bullet(E) \otimes_{\pi_\bullet(E)} E_\bullet(X)$$

induced under $\pi_\bullet(-)$ and via prop. 2.2 from the morphism of spectra

$$E \wedge X \simeq E \wedge \mathbb{S} \wedge X \overset{\mathrm{id} \wedge e \wedge \mathrm{id}}{\longrightarrow} E \wedge E \wedge X \,.$$

Definition 2.18. Given a graded commutative Hopf algebroid Γ over A as in def. 2.9, hence an internal groupoid in $\mathrm{gCRing}^{\mathrm{op}}$, then a **comodule ring** over it is an action in $\mathrm{CRing}^{\mathrm{op}}$ of that internal groupoid.

In the same spirit, a **comodule** over a commutative Hopf algebroid (not necessarily a comodule ring) is a quasicoherent sheaf on the corresponding internal groupoid (regarded as a (algebraic) stack) (e.g. Hopkins 99, prop. 11.6). Explicitly in components:

Definition 2.19. Given a $\mathbb{Z}$-graded commutative Hopf algebroid Γ over A (def. 2.9) then a **left comodule** over Γ is

1. a $\mathbb{Z}$-graded A-module N;

2. (co-action) a homomorphism of graded A-modules

   $\Psi_N : N \longrightarrow \Gamma \otimes_A N$;

such that

1. (co-unitality)

   $(\epsilon \otimes_A \mathrm{id}_N) \circ \Psi_N = \mathrm{id}_N$;

2. (co-action property)

   $(\Psi \otimes_A \mathrm{id}_N) \circ \Psi_N = (\mathrm{id}_\Gamma \otimes_A \Psi_N) \circ \Psi_N$.

A homomorphism between graded comodules $f : N_1 \to N_2$ is a homomorphism of underlying graded A-modules such that the following diagram commutes

$$\begin{array}{ccc} N_1 & \overset{f}{\longrightarrow} & N_1 \\ {}^{\mathllap{\Psi_{N_1}}}\downarrow && \downarrow^{\mathrlap{\Psi_{N_2}}} \\ \Gamma \otimes_A N_1 & \underset{\mathrm{id} \otimes_A f}{\longrightarrow} & \Gamma \otimes_A N_2 \end{array} \,.$$

We write

$$\Gamma\,\mathrm{CoMod}$$

for the resulting category of left comodules over Γ. Analogously for right comodules. The notation for the hom-sets in this category is abbreviated to

$$\mathrm{Hom}_\Gamma(-,-) \coloneqq \mathrm{Hom}_{\Gamma\,\mathrm{CoMod}}(-,-) \,.$$

A priori this is an Ab-enriched category, but it is naturally further enriched in graded abelian groups:

we may drop in the above definition of comodule homomorphisms $f : N_1 \to N_2$ the condition that the underlying morphism be grading-preserving. Say that f has degree n if it increases degree by n. This gives a $\mathbb{Z}$-graded hom-group

$$\mathrm{Hom}^\bullet_\Gamma(-,-) \,.$$

Example 2.20. For (Γ, A) a commutative Hopf algebroid, then A becomes a left Γ-comodule (def. 2.19) with coaction given by the right unit

$$A \overset{\eta_R}{\longrightarrow} \Gamma \simeq \Gamma \otimes_A A \,.$$

Proof. The required co-unitality property is the dual condition in def. 2.9

$$\epsilon \circ \eta_R = \mathrm{id}_A$$

of the fact in def. 2.6 that identity morphisms respect sources:

$$\mathrm{id} : A \overset{\eta_R}{\longrightarrow} \Gamma \simeq \Gamma \otimes_A A \overset{\epsilon \otimes_A \mathrm{id}}{\longrightarrow} A \otimes_A A \simeq A$$

The required co-action property is the dual condition

$$\Psi \circ \eta_R = (\mathrm{id} \otimes_A \eta_R) \circ \eta_R$$

of the fact in def. 2.6 that composition of morphisms in a groupoid respects sources

$$\begin{array}{ccc} A & \overset{\eta_R}{\longrightarrow} & \Gamma \\ {}^{\mathllap{\eta_R}}\downarrow && \downarrow^{\mathrlap{\Psi}} \\ \Gamma \simeq \Gamma \otimes_A A & \underset{\mathrm{id} \otimes_A \eta_R}{\longrightarrow} & \Gamma \otimes_A \Gamma \end{array} \,.$$

▮

Proposition 2.21. *Let (E, μ, e) be a homotopy commutative ring spectrum (def.) which is flat according to def. 2.1, and for $X \in \mathrm{Ho}(\mathrm{Spectra})$ any spectrum, then the morphism $\Psi_{E_\bullet(X)}$ from def. 2.17 makes $E_\bullet(X)$ into a comodule (def. 2.19) over the dual E-Steenrod algebra (def. 2.3)*

$$E_\bullet(X) \;\in\; E_\bullet(E)\mathrm{CoMod} \,.$$

(Adams 69, lecture 3, pages 67-71, Ravenel 86, chapter II, prop. 2.2.8)

Example 2.22. Given a commutative Hopf algebroid Γ over A, def. 2.9, then A itself becomes a left Γ-comodule (def. 2.19) with coaction given by

$$\Psi_A \;\colon\; A \overset{\eta_L}{\longrightarrow} \Gamma \simeq \Gamma \otimes_A A$$

and a right Γ-comodule with coaction given by

$$\Psi_A \;\colon\; A \overset{\eta_R}{\longrightarrow} \Gamma \simeq \Gamma \otimes_A A \,.$$

More generally:

Proposition 2.23. *Given a commutative Hopf algebroid Γ over A, there is a free-forgetful adjunction*

$$A\,\mathrm{Mod} \underoverset{\underset{\text{co-free}}{\longrightarrow}}{\overset{\text{forget}}{\longleftarrow}}{\bot} \Gamma\,\mathrm{CoMod}$$

between the category of Γ-comodules, def. 2.19 and the category of modules over A, where the cofree functor is right adjoint.

Moreover:

1. *The co-free Γ-comodule on an A-module C is $\Gamma \otimes_A C$ equipped with the coaction induced by the comultiplication Ψ in Γ.*

2. *The adjunct $\tilde f$ of a comodule homomorphism*

   $$N \overset{f}{\longrightarrow} \Gamma \otimes_A C$$

   is its composite with the counit ϵ of Γ

   $$\tilde f \;\colon\; N \overset{f}{\longrightarrow} \Gamma \otimes_A C \overset{\epsilon \otimes_A \mathrm{id}}{\longrightarrow} A \otimes_A C \simeq C \,.$$

The **proof** is formally dual to the proof that shows that constructing free modules is left adjoint to the forgetful functor from a category of modules to the underlying monoidal category (prop.). But since the details of the adjunction isomorphism are important for the following discussion, we spell it out:

Proof. A homomorphism into a co-free Γ-comodule is a morphism of A-modules of the form

$$f \;\colon\; N \longrightarrow \Gamma \otimes_A C$$

making the following diagram commute

$$\begin{array}{ccc} N & \overset{f}{\longrightarrow} & \Gamma \otimes_A C \\ {}^{\mathllap{\Psi_N}}\downarrow && \downarrow^{\mathrlap{\Psi \otimes_A \mathrm{id}}} \\ \Gamma \otimes_A N & \underset{\mathrm{id} \otimes_A f}{\longrightarrow} & \Gamma \otimes_A \Gamma \otimes_A C \end{array} \,.$$

Consider the composite

$$\tilde f \;\colon\; N \overset{f}{\longrightarrow} \Gamma \otimes_A C \overset{\epsilon \otimes_A \mathrm{id}}{\longrightarrow} A \otimes_A C \simeq C \,,$$

i.e. the "corestriction" of f along the counit of Γ. By definition this makes the following square commute

$$\begin{array}{ccc} \Gamma \otimes_A N & \xrightarrow{\mathrm{id} \otimes_A f} & \Gamma \otimes_A \Gamma \otimes_A C \\ {}^{=}\downarrow && \downarrow^{\mathrm{id} \otimes_A \epsilon \otimes_A \mathrm{id}} \\ \Gamma \otimes_A N & \underset{\mathrm{id} \otimes_A \tilde f}{\longrightarrow} & \Gamma \otimes_A C \end{array}$$

Pasting this square onto the bottom of the previous one yields

$$\begin{array}{ccc} N & \overset{f}{\longrightarrow} & \Gamma \otimes_A C \\ {}^{\Psi_N}\downarrow && \downarrow^{\Psi \otimes_A \mathrm{id}} \\ \Gamma \otimes_A N & \underset{\mathrm{id} \otimes_A f}{\longrightarrow} & \Gamma \otimes_A \Gamma \otimes_A C \\ {}^{=}\downarrow && \downarrow^{\mathrm{id} \otimes_A \epsilon \otimes_A \mathrm{id}} \\ \Gamma \otimes_A N & \underset{\mathrm{id} \otimes_A \tilde f}{\longrightarrow} & \Gamma \otimes_A C \end{array} \,.$$

Now due to co-unitality, the right right vertical composite is the identity on $\Gamma \otimes_A C$. But this means by the commutativity of the outer rectangle that f is uniquely fixed in terms of $\tilde f$ by the relation

$$f = (\mathrm{id} \otimes_A f) \circ \Psi \,.$$

This establishes a natural bijection

$$\frac{N \overset{f}{\longrightarrow} \Gamma \otimes_A C}{N \overset{\tilde f}{\longrightarrow} C}$$

and hence the adjunction in question. ▮

Proposition 2.24. *Consider a commutative Hopf algebroid Γ over A, def. 2.9. Any left comodule N over Γ (def. 2.19) becomes a right comodule via the coaction*

$$N \overset{\Psi}{\longrightarrow} \Gamma \otimes_A N \overset{\simeq}{\longrightarrow} N \otimes_A \Gamma \overset{\mathrm{id} \otimes_A c}{\longrightarrow} N \otimes_A \Gamma \,,$$

where the isomorphism in the middle the is braiding in $A\mathrm{Mod}$ and where c is the conjugation map of Γ.

Dually, a right comodule N becoomes a left comodule with the coaction

$$N \overset{\Psi}{\longrightarrow} N \otimes_A \Gamma \overset{\simeq}{\longrightarrow} \Gamma \otimes_A N \overset{c \otimes_A \mathrm{id}}{\longrightarrow} \Gamma \otimes_A N \,.$$

Definition 2.25. Given a commutative Hopf algebroid Γ over A, def. 2.9, and given N_1 a right Γ-comodule and N_2 a left comodule (def. 2.19), then their **cotensor product** $N_1 \Box_\Gamma N_2$ is the kernel of the difference of the two coaction morphisms:

$$N_1 \Box_\Gamma N_2 \;\coloneqq\; \ker\left(N_1 \otimes_A N_2 \overset{\Psi_{N_1} \otimes_A \mathrm{id} - \mathrm{id} \otimes_A \Psi_{N_2}}{\longrightarrow} N_1 \otimes_A \Gamma \otimes_A N_2 \right) \,.$$

If both N_1 and N_2 are left comodules, then their cotensor product is the cotensor product of N_2 with N_1 regarded as a right comodule via prop. 2.24.

e.g. (Ravenel 86, def. A1.1.4).

Example 2.26. Given a commutative Hopf algebroid Γ over A, (def.), and given N a left Γ-comodule (def.). Regard A itself canonically as a right Γ-comodule via example 2.22. Then the cotensor product

$$\mathrm{Prim}(N) \coloneqq A \Box_\Gamma N$$

is called the **primitive elements** of N:

$$\mathrm{Prim}(N) = \{ n \in N \mid \Psi_N(n) = 1 \otimes n \} \,.$$

Proposition 2.27. *Given a commutative Hopf algebroid Γ over A, def. 2.9, and given N_1, N_2 two left Γ-comodules (def. 2.19), then their cotensor product (def. 2.25) is commutative, in that there is an isomorphism*

$$N_1 \Box N_2 \;\simeq\; N_2 \Box N_1 \,.$$

(e.g. Ravenel 86, prop. A1.1.5)

Lemma 2.28. *Given a commutative Hopf algebroid Γ over A, def. 2.9, and given N_1, N_2 two left Γ-comodules (def. 2.19), such that N_1 is projective as an A-module, then*

1. *The morphism*

$$\mathrm{Hom}_A(N_1, A) \xrightarrow{f \mapsto (\mathrm{id} \otimes_A f) \circ \Psi_{N_1}} \mathrm{Hom}_A(N_1, \Gamma \otimes_A A) \simeq \mathrm{Hom}_A(N_1, \Gamma) \simeq \mathrm{Hom}_A(N_1, A) \otimes_A \Gamma$$

gives $\mathrm{Hom}_A(N_1, A)$ the structure of a right Γ-comodule;

2. *The cotensor product (def. 2.25) with respect to this right comodule structure is isomorphic to the hom of Γ-comodules:*

$$\mathrm{Hom}_A(N_1, A) \,\square_\Gamma\, N_2 \simeq \mathrm{Hom}_\Gamma(N_1, N_2) \,.$$

Hence in particular

$$A \,\square_\Gamma\, N_2 \;\simeq\; \mathrm{Hom}_\Gamma(A, N_2)$$

(e.g. Ravenel 86, lemma A1.1.6)

Remark 2.29. In computing the second page of E-Adams spectral sequences, the second statement in lemma 2.28 is the key translation that makes the comodule Ext-groups on the page be equivalent to a Cotor-groups. The latter lend themselves to computation, for instance via Lambda-algebra or via the May spectral sequence.

Universal coefficient theorem

The key use of the Hopf coalgebroid structure of prop. 2.3 for the present purpose is that it is extra structure inherited by morphisms in E-homology from morphisms of spectra. Namely forming E-homology $f_* : E_\bullet(X) \to E_\bullet(Y)$ of a morphism of a spectra $f : X \to Y$ does not just produce a morphism of E-homology groups

$$[X, Y]_\bullet \longrightarrow \mathrm{Hom}_{\mathrm{Ab}^{\mathbb{Z}}}(E_\bullet(X), E_\bullet(Y))$$

but in fact produces homomorphisms of comodules over $E_\bullet(E)$

$$\alpha \,:\, [X, Y]_\bullet \longrightarrow \mathrm{Hom}_{E_\bullet(E)}(E_\bullet(X), E_\bullet(Y)) \,.$$

This is the statement of lemma 2.30 below. The point is that $E_\bullet(E)$-comodule homomorphism are much more rigid than general abelian group homomorphisms and hence closer to reflecting the underlying morphism of spectra $f : X \to Y$.

In good cases such an approximation of *homotopy* by *homology* is in fact accurate, in that α is an isomorphism. In such a case (Adams 74, part III, section 13) speaks of a "universal coefficient theorem" (the coefficients here being E.)

One such case is exhibited by prop. 2.33 below. This allows to equivalently re-write the first page of the E-Adams spectral sequence in terms of E-homology homomorphisms in theorem 2.34 below.

Lemma 2.30. *For $X, Y \in \mathrm{Ho}(\mathrm{Spectra})$ any two spectra, the morphism (of $\mathbb{Z}$-graded abelian) generalized homology groups given by smash product with E (rmk.)*

$$\begin{array}{ccccc} \pi_\bullet(E \wedge -) & : & [X, Y]_\bullet & \longrightarrow & \mathrm{Hom}^\bullet_{\mathrm{Ab}^{\mathbb{Z}}}(E_\bullet(X), E_\bullet(Y)) \\ & & (X \xrightarrow{f} Y) & \mapsto & \left(E_\bullet(X) \xrightarrow{f_*} E_\bullet(Y)\right) \end{array}$$

factors through the forgetful functor from $E_\bullet(E)$-comodule homomorphisms (def. 2.19) over the dual E-Steenrod algebra (def. 2.3):

$$\begin{array}{ccc} & & \mathrm{Hom}^\bullet_{E_\bullet(E)}(E_\bullet(X), E_\bullet(Y)) \\ & {}^{\exists} \nearrow & \downarrow^{\mathrm{forget}} \\ [X, N]_\bullet & \underset{\pi_\bullet(E \wedge -)}{\longrightarrow} & \mathrm{Hom}^\bullet_{\mathrm{Ab}^{\mathbb{Z}}}(E_\bullet(X), E_\bullet(Y)) \end{array} \,,$$

where $E_\bullet(X)$ and $E_\bullet(Y)$ are regarded as E-Steenrod comodules according to def. 2.19, prop. 2.21.

Proof. By def. 2.19 we need to show that for $X \xrightarrow{f} Y$ a morphism in $\mathrm{Ho}(\mathrm{Spectra})$ then the following diagram commutes

$$\begin{array}{ccc} E_\bullet(X) & \overset{f_*}{\longrightarrow} & E_\bullet(Y) \\ {}^{\Psi_{E_\bullet(X)}}\downarrow & & \downarrow^{\Psi_{E_\bullet(Y)}} \\ E_\bullet(E) \otimes_{\pi_\bullet(E)} E_\bullet(X) & \underset{\mathrm{id} \otimes_{\pi_\bullet(E)} f_*}{\longrightarrow} & E_\bullet(E) \otimes_{\pi_\bullet(E)} E_\bullet(Y) \end{array} \,.$$

By def. 2.19 and prop. 2.21 this is the image under foming stable homotopy groups $\pi_\bullet(-)$ of the following diagram in $\mathrm{Ho}(\mathrm{Spectra})$:

$$
\begin{array}{ccc}
E \wedge X & \overset{\mathrm{id}\wedge f}{\longrightarrow} & E \wedge Y \\
{}^{\simeq}\downarrow && \downarrow^{\simeq} \\
E \wedge \mathbb{S} \wedge X && E \wedge \mathbb{S} \wedge Y\,. \\
{}^{\mathrm{id}\wedge e \wedge \mathrm{id}}\downarrow && \downarrow^{\mathrm{id}\wedge e \wedge \mathrm{id}} \\
E \wedge E \wedge X & \underset{\mathrm{id}\wedge \mathrm{id}\wedge f}{\longrightarrow} & E \wedge E \wedge Y
\end{array}
$$

But that this diagram commutes is simply the functoriality of the derived smash product of spectra as a functor on the product category $\mathrm{Ho}(\mathrm{Spectra}) \times \mathrm{Ho}(\mathrm{Spectra})$. ▮

Proposition 2.31. *Let (E, μ, e) be a homotopy commutative ring spectrum (def.), and let $X, Y \in \mathrm{Ho}(\mathrm{Spectra})$ be two spectra such that $E_\bullet(X)$ is a projective module over $\pi_\bullet(E)$ (via this prop.).*

Then the homomorphism of graded abelian groups

$$\phi_{\mathrm{UC}} \;:\; [X, E \wedge Y]_\bullet \longrightarrow \mathrm{Hom}^\bullet_{\pi_\bullet(E)}(E_\bullet(X), E_\bullet(Y))_\bullet$$

given by

$$(X \overset{f}{\longrightarrow} E \wedge Y) \;\mapsto\; \pi_\bullet(E \wedge X \overset{\mathrm{id}\wedge f}{\longrightarrow} E \wedge E \wedge Y \overset{\mu \wedge \mathrm{id}}{\longrightarrow} E \wedge Y)$$

is an isomorphism.

(Schwede 12, chapter II, prop. 6.20)

Proof. First of all we claim that the morphism in question factors as

$$\beta \;:\; [X, E \wedge Y]_\bullet \overset{\simeq}{\longrightarrow} \mathrm{Hom}^\bullet_{E\,\mathrm{Mod}}(E \wedge X, E \wedge Y) \overset{\pi_\bullet}{\longrightarrow} \mathrm{Hom}^\bullet_{\pi_\bullet(E)}(E_\bullet(X), E_\bullet(Y))\,,$$

where

1. $E\,\mathrm{Mod} = E\,\mathrm{Mod}(\mathrm{Ho}(\mathrm{Spectra}), \wedge, \mathbb{S})$ denotes the category of homotopy module spectra over E (def.)
2. the first morphisms is the free-forgetful adjunction isomorphism for forming free (prop.) E-homotopy module spectra
3. the second morphism is the respective component of the composite of the forgetful functor from E-homotopy module spectra back to $\mathrm{Ho}(\mathrm{Spectra})$ with the functor $\pi_\bullet$ that forms stable homotopy groups.

This is because (by this prop.) the first map is given by first smashing with E and then postcomposing with the E-action on the free module $E \wedge X$, which is the pairing $E \wedge E \overset{\mu}{\to} E$ (prop.).

Hence it is sufficient to show that the morphism on the right is an isomorphism.

We show more generally that for N_1, N_2 any two E-homotopy module spectra (def.) such that $\pi_\bullet(N_1)$ is a projective module over $\pi_\bullet(E)$, then

$$\mathrm{Hom}^\bullet_{E\,\mathrm{Mod}}(N_1, N_2) \overset{\pi_\bullet}{\longrightarrow} \mathrm{Hom}^\bullet_{\pi_\bullet(E)}(\pi_\bullet(N_1), \pi_\bullet(N_2))$$

is an isomorphism.

To see this, first consider the case that $\pi_\bullet(N_1)$ is in fact a $\pi_\bullet(E)$-free module.

This implies that there is a basis $\mathcal{B} = \{x_i\}_{i \in I}$ and a homomorphism

$$\underset{i \in I}{\vee} \Sigma^{|x_i|} E \longrightarrow N_1$$

of E-homotopy module spectra, such that this is a stable weak homotopy equivalence.

Observe that this sits in a commuting diagram of the form

$$
\begin{array}{ccc}
\mathrm{Hom}^\bullet_{E\,\mathrm{Mod}}(\underset{i \in I}{\vee} \Sigma^{|x_i|}E, N_2) & \overset{\pi_\bullet}{\longrightarrow} & \mathrm{Hom}^\bullet_{\pi_\bullet(E)}(\pi_\bullet(\underset{i \in I}{\vee} \Sigma^{|x_i|}E), \pi_\bullet(N_2)) \\
{}^{\simeq}\downarrow && \downarrow^{\simeq} \\
\prod_{i \in I} [\Sigma^{|x_i|}\mathbb{S}, N_2]_\bullet & \underset{\simeq}{\longrightarrow} & \prod_{i \in I} \pi_{\bullet + |x_i|}(N_2)
\end{array}
$$

where

1. the left vertical isomorphism exhibits wedge sum of spectra as the coproduct in the stable homotopy category (lemma);

2. the bottom isomorphism is from this prop.;

3. the right vertical isomorphism is that of the free-forgetful adjunction for modules over $\pi_\bullet(E)$.

Hence the top horizontal morphism is an isomorphism, which was to be shown.

Now consider the general case that $\pi_\bullet(N_1)$ is a projective module over $\pi_\bullet(E)$. Since (graded) projective modules are precisely the retracts of (graded) free modules (prop.), there exists a diagram of $\pi_\bullet(E)$-modules of the form

$$\mathrm{id} : \pi_\bullet(N_1) \longrightarrow \pi_\bullet(\underset{i \in I}{\vee} \Sigma^{|x_i|} E) \longrightarrow \pi_\bullet(N_1)$$

which induces the corresponding split idempotent of $\pi_\bullet(E)$-modules

$$\pi_\bullet(\underset{i \in I}{\vee} \Sigma^{|x_i|} E) \longrightarrow \pi_\bullet(N_1) \longrightarrow \pi_\bullet(\underset{i \in I}{\vee} \Sigma^{|x_i|} E)\,.$$

As before, by freeness this is actually the image under $\pi_\bullet$ of an idempotent of homotopy ring spectra

$$e_\bullet : \underset{i \in I}{\vee} \Sigma^{|x_i|} E \longrightarrow \underset{i \in I}{\vee} \Sigma^{|x_i|} E$$

and so in particular of spectra.

Now in the stable homotopy category $\mathrm{Ho}(\mathrm{Spectra})$ all idempotents split (prop.), hence there exists a diagram of spectra of the form

$$e : \underset{i \in I}{\vee} \Sigma^{|x_i|} E \longrightarrow X \longrightarrow \underset{i \in I}{\vee} \Sigma^{|x_i|} E$$

with $\pi_\bullet(e) = e_\bullet$.

Consider the composite

$$X \longrightarrow \underset{i \in I}{\vee} \Sigma^{|x_i|} E \longrightarrow N_1\,.$$

Since $\pi_\bullet(e) = e_\bullet$ it follows that under $\pi_\bullet$ this is an isomorphism, then that $X \simeq N_1$ in the stable homotopy category.

In conclusion this exhibits N_1 as a retract of an free E-homotopy module spectrum

$$\mathrm{id} : N_1 \longrightarrow \underset{i \in I}{\vee} \Sigma^{|x_i|} E \longrightarrow N_1\,,$$

hence of a spectrum for which the morphism in question is an isomorphism. Since the morphism in question is natural, its value on N_1 is a retract in the arrow category of an isomorphism, hence itself an isomorphism (lemma). ∎

Remark 2.32. A stronger version of the statement of prop. 2.31, with the free homotopy E-module spectrum $E \wedge Y$ replaced by any homotopy E-module spectrum F, is considered in (Adams 74, chapter III, prop. 13.5) ("universal coefficient theorem"). Strong conditions are considered that ensure that

$$F^\bullet(X) = [X, F]_\bullet \longrightarrow \mathrm{Hom}^\bullet_{\pi_\bullet(E)}(E_\bullet(X), \pi_\bullet(F))$$

is an isomormphism (expressing the F-cohomology of X as the $\pi_\bullet(E)$-linear dual of the E-homology of X).

For the following we need only the weaker but much more general statement of prop. 2.31, and in fact this is all that (Adams 74, p. 323) ends up using, too.

With this we finally get the following statement, which serves to identify maps of certain spectra with their induced maps on E-homology:

Proposition 2.33. *Let (E, μ, e) be a homotopy commutative ring spectrum (def.), and let $X, Y \in \mathrm{Ho}(\mathrm{Spectra})$ be two spectra such that*

1. *E is flat according to def. 2.1;*

2. *$E_\bullet(X)$ is a projective module over $\pi_\bullet(E)$ (via this prop.).*

Then the morphism from lemma 2.30

$$[X, E \wedge Y]_\bullet \overset{\pi_\bullet(E \wedge -)}{\longrightarrow} \mathrm{Hom}^\bullet_{E_\bullet(E)}(E_\bullet(X), E_\bullet(E \wedge Y))) \simeq \mathrm{Hom}^\bullet_{E_\bullet(E)}(E_\bullet(X), E_\bullet(E) \otimes_{\pi_\bullet(E)} E_\bullet(Y)))$$

is an isomorphism (where the isomophism on the right is that of prop. 2.2).

(Adams 74, part III, page 323)

Proof. Observe that the following diagram commutes:

$$
\begin{tikzcd}
{[X, E \wedge Y]_\bullet} \arrow[rr, "\pi_\bullet(E\wedge -)"] \arrow[dr, "\phi_{UC}"'] && \mathrm{Hom}^\bullet_{E_\bullet(E)}(E_\bullet(X), E_\bullet(E) \otimes_{\pi_\bullet(E)} E_\bullet(Y))) \arrow[dl, "\epsilon \otimes \mathrm{id} \circ (-)"] \\
& \mathrm{Hom}^\bullet_{\pi_\bullet(E)}(E_\bullet(X), E_\bullet(Y))
\end{tikzcd}
\,,
$$

where

1. the top morphism is the one from lemma 2.30;
2. the right vertical morphism is the adjunction isomorphism from prop. 2.23;
3. the left diagonal morphism is the one from prop. 2.31.

To see that this indeed commutes, notice that

1. the top morphism sends $(X \overset{f}{\to} E \wedge Y)$ to $E_\bullet(X) \overset{E_\bullet(f)}{\longrightarrow} E_\bullet(E \wedge Y) \simeq \pi_\bullet(E \wedge E \wedge Y)$ by definition;
2. the right vertical morphism sends this further to $E_\bullet(X) \overset{E_\bullet(f)}{\longrightarrow} \pi_\bullet(E \wedge E \wedge Y) \overset{\pi_\bullet(\mu \wedge \mathrm{id})}{\longrightarrow} \pi_\bullet(E \wedge Y)$, by the proof of prop. 2.23 (which says that the map is given by postcomposition with the counit of $E_\bullet(E)$) and def. 2.3 (which says that this counit is represented by μ);
3. by prop. 2.31 this is the same as the action of the left diagonal morphism.

But now

1. the right vertical morphism is an isomorphism by prop. 2.2;
2. the left diagonal morphism is an isomorphism by prop. 2.31

and so it follows that the top horizontal morphism is an isomorphism, too. ▮

In conclusion:

Theorem 2.34. *Let (E, μ, e) be a homotopy commutative ring spectrum (def.), and let $X, Y \in \mathrm{Ho}(\mathrm{Spectra})$ be two spectra such that*

1. *E is flat according to def. 2.1;*
2. *$E_\bullet(X)$ is a projective module over $\pi_\bullet(E)$ (via this prop.).*

Then the first page of the E-Adams spectral sequence, def. 1.14, for $[Y, X]_\bullet$ is isomorphic to the following chain complex of graded homs of comodules (def. 2.19) over the dual E-Steenrod algebra $(E_\bullet(E), \pi_\bullet(E))$ (prop. 2.3):

$$
E_1^{s,t}(X,Y) \;\simeq\; \mathrm{Hom}^t_{E_\bullet(E)}(E_\bullet(X), E_{\bullet - s}(A_s)) \quad , \qquad d_1 \;=\; \mathrm{Hom}_{E_\bullet(E)}(E_\bullet(X), E_\bullet(g \circ h))
$$

$$
0 \to \mathrm{Hom}^t_{E_\bullet(E)}(E_\bullet(X), E_\bullet(A_0)) \overset{d_1}{\longrightarrow} \mathrm{Hom}^t_{E_\bullet(E)}(E_\bullet(X), E_{\bullet-1}(A_1)) \overset{d_1}{\longrightarrow} \mathrm{Hom}^t_{E_\bullet(E)}(E_\bullet(X), E_{\bullet-2}(A_2)) \overset{d_1}{\longrightarrow} \cdots \,.
$$

(Adams 74, theorem 15.1 page 323)

Proof. This is prop. 2.33 applied to def. 1.14:

$$
\begin{aligned}
E_1^{s,t}(X,Y) & = [X, \underbrace{E \wedge Y_s}_{A_s}]_{t-s} \\
& \simeq \mathrm{Hom}^{t-s}_{E_\bullet(E)}(E_\bullet(X), E_\bullet(\underbrace{E \wedge Y_s}_{A_s})) \\
& \simeq \mathrm{Hom}^t_{E_\bullet(E)}(E_\bullet(X), E_{\bullet-s}(A_s))
\end{aligned}
$$

▮

3. The second page

Theorem 3.1. *Let (E, μ, e) be a homotopy commutative ring spectrum (def.), and let $X, Y \in \mathrm{Ho}(\mathrm{Spectra})$ be two spectra such that*

1. *E is flat according to def. 2.1;*
2. *$E_\bullet(X)$ is a projective module over $\pi_\bullet(E)$ (via this prop.).*

Then the entries of the second page of the E-Adams spectral sequence for $[X,Y]_\bullet$ (def. 1.14) are the Ext-groups of commutative Hopf algebroid-comodules (def. 2.19) over the commutative Hopf algebroid structure on the dual E-Steenrod algebra $E_\bullet(E)$ from prop. 2.3:

$$E_2^{s,t}(X,Y) \simeq \mathrm{Ext}^{s,t}_{E_\bullet(E)}(E_\bullet(X), E_\bullet(Y)) \,.$$

(On the right s is the degree that goes with any Ext-functor, and the "internal degree" t is the additional degree of morphisms between graded modules from def. 2.19.)

In the special case that $X = \mathbb{S}$ is the sphere spectrum, then (by prop. 2.28) these are equivalently Cotor-groups

$$E_2^{s,t}(X,Y) \simeq \mathrm{Cotor}^{s,t}_{E_\bullet(E)}(\pi_\bullet(E), E_\bullet(Y)) \,.$$

(Adams 74, theorem 15.1, page 323)

Proof. By theorem 2.34, under the given assumptions the first page reads

$$E_1^{s,t}(X,Y) \;\simeq\; \mathrm{Hom}^t_{E_\bullet(E)}(E_\bullet(X), E_{\bullet-s}(A_s)) \quad , \quad d_1 \;=\; \mathrm{Hom}_{E_\bullet(E)}(E_\bullet(X), E_\bullet(g\circ h))$$

$$0 \to \mathrm{Hom}^t_{E_\bullet(E)}(E_\bullet(X), E_\bullet(A_0)) \overset{d_1}{\longrightarrow} \mathrm{Hom}^t_{E_\bullet(E)}(E_\bullet(X), E_{\bullet-1}(A_1)) \overset{d_1}{\longrightarrow} \mathrm{Hom}^t_{E_\bullet(E)}(E_\bullet(X), E_{\bullet-2}(A_2)) \overset{d_1}{\longrightarrow} \cdots \,.$$

By remark 1.12 the second page is the cochain cohomology of this complex. Hence by the standard theory of derived functors in homological algebra (see the section *Via acyclic resolutions*), it is now sufficient to see that:

1. the category $E_\bullet(E)\mathrm{CoMod}$ (def. 2.19, prop. 2.12) is an abelian category with enough injectives (so that all right derived functors on $E_\bullet(E)\mathrm{CoMod}$ exist);

2. the first page graded chain complex $(E_1^{\bullet,t}(X,Y), d_1)$ is the image under the hom-functor $F \coloneqq \mathrm{Hom}_{E_\bullet(E)}(E_\bullet(Y), -)$ of an F-acyclic resolution of $E_\bullet(X)$ (so that its cohomology indeed computes the Ext-derived functor (theorem)).

That $E_\bullet(E)\mathrm{CoMod}$ is an abelian category is lemma 3.3 below, and that it has enough injectives is lemma 3.4.

Lemma 3.2 below shows that $E_\bullet(A_\bullet)$ is a resolution of $E_\bullet(Y)$ in $E_\bullet(E)\mathrm{CoMod}$. By prop. 2.2 it is a resolution by cofree comodules (def. 2.23). That these are F-acyclic is lemma 3.5 below. ▮

E-Adams resolutions

We discuss that the first page of the E-Adams spectral sequence indeed exhibits a resolution as required by the proof of theorem 3.1.

Lemma 3.2. *Given an E-Adams spectral sequence $(E_r^{s,t}(X,Y), d_r)$ as in def. 1.14, then the sequences of morphisms*

$$0 \to E_\bullet(Y_p) \overset{E_\bullet(g_p)}{\longrightarrow} E_\bullet(A_p) \overset{E_\bullet(h_p)}{\longrightarrow} E_{\bullet-1}(Y_{p+1}) \to 0$$

are short exact, hence their splicing of short exact sequences

$$\begin{array}{ccccccccccc}
0 & \to & E_\bullet(Y) & \overset{E_\bullet(g_0)}{\longrightarrow} & E_\bullet(A_0) && \overset{\partial}{\longrightarrow} && E_{\bullet-1}(A_1) && \overset{\partial}{\longrightarrow} && E_{\bullet-2}(A_2) & \longrightarrow & \cdots \\
&&&&& {}_{E_\bullet(h_0)}\searrow && \nearrow_{E_\bullet(g_1)} && {}_{E_\bullet(h_1)}\searrow && \nearrow_{E_\bullet(g_2)} \\
&&&&&& E_{\bullet-1}(Y_1) &&&& E_{\bullet-2}(Y_2)
\end{array}$$

is a long exact sequence, exhibiting the graded chain complex $(E_\bullet(A_\bullet), \partial)$ as a resolution of $E_\bullet(Y)$.

(Adams 74, theorem 15.1, page 322)

Proof. Consider the image of the defining homotopy cofiber sequence

$$Y_p \overset{g_p}{\longrightarrow} E \wedge Y_p \overset{h_p}{\longrightarrow} \Sigma Y_{p+1}$$

under the functor $E \wedge (-)$. This is itself a homotopy cofiber sequence of the form

$$E \wedge Y_p \overset{E \wedge g_p}{\longrightarrow} E \wedge E \wedge Y_p \overset{E\wedge h_p}{\longrightarrow} \Sigma E \wedge Y_{p+1}$$

(due to the tensor triangulated structure of the stable homotopy category, prop.).

Applying the stable homotopy groups functor $\pi_\bullet(-) \simeq [\mathbb{S}, -]_\bullet$ (lemma) to this yields a long exact sequence (prop.)

$$\cdots \longrightarrow E_\bullet(Y_{p+1}) \overset{E_\bullet(f_p)}{\longrightarrow} E_\bullet(Y_p) \overset{E_\bullet(g_p)}{\longrightarrow} E_\bullet(A_p) \overset{E_\bullet(h_p)}{\longrightarrow} E_{\bullet-1}(Y_{p+1}) \overset{E_{\bullet-1}(f_p)}{\longrightarrow} E_{\bullet-1}(Y_p) \overset{E_{\bullet-1}(g_p)}{\longrightarrow} E_{\bullet-1}(A_p) \longrightarrow \cdots .$$

But in fact this splits: by unitality of (E, μ, e), the product operation μ on the homotopy commutative ring spectrum E is a left inverse to g_p in that

$$\mathrm{id} : E \wedge Y_p \overset{E \wedge g_p}{\longrightarrow} E \wedge E \wedge Y_p \overset{\mu \wedge \mathrm{id}}{\longrightarrow} E \wedge Y_p .$$

Therefore $E_\bullet(g_p)$ is a monomorphism, hence its kernel is trivial, and so by exactness $E_\bullet(f_p) = 0$. This means that the above long exact sequence collapses to short exact sequences. ∎

Homological co-algebra

We discuss basic aspects of homological algebra in categories of comodules (def. 2.19) over commutative Hopf algebroids (def. 2.6), needed in the proof of theorem 3.1.

Lemma 3.3. *Let (Γ, A) be a commutative Hopf algebroid Γ over A (def. 2.6, 2.9), such that the right A-module structure on Γ induced by η_R is a flat module.*

Then the category $\Gamma\,\mathrm{CoMod}$ of comodules over Γ (def. 2.19) is an abelian category.

(e.g. Ravenel 86, theorem A1.1.3)

Proof. It is clear that, without any condition on the Hopf algebroid, $\Gamma\,\mathrm{CoMod}$ is an additive category.

Next we need to show if Γ is flat over A, that then this is also a pre-abelian category, in that kernels and cokernels exist.

To that end, let $f : (N_1, \Psi_{N_1}) \longrightarrow (N_2, \Psi_{N_2})$ be a morphism of comodules, hence a commuting diagram in AMod of the form

$$\begin{array}{ccc} N_1 & \overset{f}{\longrightarrow} & N_2 \\ \downarrow^{\Psi_{N_1}} && \downarrow^{\Psi_{N_2}} \\ \Gamma \otimes_A N_1 & \overset{\mathrm{id}_\Gamma \otimes_A f}{\longrightarrow} & \Gamma \otimes_A N_2 \end{array} .$$

Consider the kernel $\ker(f)$ of f in AMod and its image under $\Gamma \otimes_A (-)$

$$\begin{array}{ccccc} \ker(f) & \longrightarrow & N_1 & \overset{f}{\longrightarrow} & N_2 \\ \exists \downarrow && \downarrow^{\Psi_{N_1}} && \downarrow^{\Psi_{N_2}} \\ \Gamma \otimes_A \ker(f) & \longrightarrow & \Gamma \otimes_A N_1 & \overset{\mathrm{id}_\Gamma \otimes_A f}{\longrightarrow} & \Gamma \otimes_A N_2 \end{array} .$$

By the assumption that Γ is a flat module over A, also $\Gamma \otimes_A \ker(f) \simeq \ker(\Gamma \otimes_A f)$ is a kernel. Hence by the universal property of kernels and the commutativity of the square o the right, there exists a unique vertical morphism as shown on the left, making the left square commute. This means that the A-module $\ker(f)$ uniquely inherits the structure of a Γ-comodule such as to make $\ker(f) \to N_1$ a comodule homomorphism. By the same universal property it follows that $\ker(f)$ with this comodule structure is in fact the kernel of f in $\Gamma\,\mathrm{CoMod}$.

The argument for the existence of cokernels proceeds formally dually. Hence $\Gamma\,\mathrm{CoMod}$ is a pre-abelian category.

But it also follows from this construction that the comparison morphism

$$\mathrm{coker}(\ker(f)) \longrightarrow \ker(\mathrm{coker}(f))$$

formed in $\Gamma\,\mathrm{CoMod}$ has underlying it the corresponding comparison morphism in $A\,\mathrm{Mod}$. There this is an isomorphism by the fact that the category of modules $A\,\mathrm{Mod}$ is an abelian category, hence it is an isomorphism also in $\Gamma\,\mathrm{CoMod}$. So the latter is in fact an abelian category itself. ∎

Lemma 3.4. *Let (Γ, A) be a commutative Hopf algebroid Γ over A (def. 2.6, 2.9), such that the right A-module structure on Γ induced by η_R is a flat module.*

Then

1. *every co-free Γ-comodule (def. 2.23) on an injective module over A is an injective object in $\Gamma\,\mathrm{CoMod}$;*

2. *$\Gamma\,\mathrm{CoMod}$ has enough injectives (def.) if the axiom of choice holds in the ambient set theory.*

(e.g. Ravenel 86, lemma A1.2.2)

Proof. First of all, assuming the axiom of choice, then the category of modules $A\,\mathrm{Mod}$ has enough injectives (by this proposition).

Now by prop. 2.23 we have the adjunction

$$A\,\mathrm{Mod} \underoverset{\underset{\text{co-free}}{\longrightarrow}}{\overset{\text{forget}}{\longleftarrow}}{\bot} \Gamma\,\mathrm{CoMod}\,.$$

Observe that the left adjoint is a faithful functor (being a forgetful functor) and that, by the proof of lemma 3.3, it is an exact functor. This implies that

1. for $I \in A\,\mathrm{Mod}$ an injective module, then the co-free comodule $\Gamma \otimes_A I$ is an injective object in $\Gamma\,\mathrm{CoMod}$ (by this lemma);

2. for $N \in \Gamma\,\mathrm{CoMod}$ any object, and for $i : \mathrm{forget}(N) \hookrightarrow I$ a monomorphism of A-modules into an injective A-module, then the adjunct $\tilde{i} : N \hookrightarrow \Gamma \otimes_A I$ is a monomorphism (by this lemma)) hence a monomorpism into an injective comodule, by the previous item.

Hence $\Gamma\,\mathrm{CoMod}$ has enough injective objects (def.). ▮

Lemma 3.5. *Let (Γ, A) be a commutative Hopf algebroid Γ over A (def. 2.6, 2.9), such that the right A-module structure on Γ induced by η_R is a flat module. Let $N \in \Gamma\,\mathrm{CoMod}$ be a Γ-comodule (def. 2.19) such that the underlying A-module is a projective module (a projective object in AMod).*

Then (assuming the axiom of choice in the ambient set theory) every co-free comodule (prop. 2.23) is an F-acyclic object for F the hom functor $\mathrm{Hom}_{\Gamma\,\mathrm{CoMod}}(N, -)$.

Proof. We need to show that the derived functors $\mathbb{R}^\bullet \mathrm{Hom}_\Gamma(N, -)$ vanish in positive degree on all co-free comodules, hence on $\Gamma \otimes_A K$, for all $K \in A\,\mathrm{Mod}$.

To that end, let $I^\bullet$ be an injective resolution of K in $A\,\mathrm{Mod}$. By lemma 3.4 then $\Gamma \otimes_A I^\bullet$ is a sequence of injective objects in $\Gamma\,\mathrm{CoMod}$ and by the assumption that Γ is flat over A it is an injective resolution of $\Gamma \otimes_A K$ in $\Gamma\,\mathrm{CoMod}$. Therefore the derived functor in question is given by

$$\begin{aligned} \mathbb{R}^{\bullet \geq 1} \mathrm{Hom}_\Gamma(N, \Gamma \otimes_A K) & \simeq H_{\bullet \geq 1}(\mathrm{Hom}_\Gamma(N, \Gamma \otimes_A I^\bullet)) \\ & \simeq H_{\bullet \geq 1}(\mathrm{Hom}_A(N, I^\bullet)) \\ & \simeq 0 \end{aligned} .$$

Here the second equivalence is the cofree/forgetful adjunction isomorphism of prop. 2.23, while the last equality then follows from the assumption that the A-module underlying N is a projective module (since hom functors out of projective objects are exact functors (here) and since derived functors of exact functors vanish in positive degree (here)). ▮

With lemma 3.5 the proof of theorem 3.1 is completed.

4. Convergence

We discuss the convergence of E-Adams spectral sequences (def. 1.14), i.e. the identification of the "limit", in an appropriate sense, of the terms $E_r^{s,t}(X,Y)$ on the rth page of the spectral sequence as r goes to infinity.

If an E-Adams spectral sequence converges, then it converges not necessarily to the full stable homotopy groups $[X,Y]_\bullet$, but to some localization of them. This typically means, roughly, that only certain p-torsion subgroups in $[X,Y]_\bullet$ for some prime numbers p are retained. We give a precise discussion below in *Localization and adic completion of abelian groups*.

If one knows that $[X,Y]_q$ is a finitely generated abelian group (as is the case notably for $\pi_q^s = [\mathbb{S}, \mathbb{S}]_q$ by the *Serre finiteness theorem*) then this allows to recover the full information from its pieces: by the *fundamental theorem of finitely generated abelian groups* (prop. 4.1 below) these groups are direct sums of powers $\mathbb{Z}^n$ of the infinite cyclic group with finite cyclic groups of the form $\mathbb{Z}/p^k\mathbb{Z}$, and so all one needs to compute is the powers k "one prime p at a time". This we review below in *Primary decomposition of abelian groups*.

The deeper reason that E-Adams spectral sequences tend to converge to localizations of the abelian groups $[X,Y]_\bullet$ of morphisms of spectra, is that they really converges to the actual homotopy groups but of localizations of spectra. This is more than just a reformulation, because the localization at the level of spectra determies the filtration which controls the nature of the convergence. We discuss this localization of spectra below in *Localization and nilpotent completion of spectra*.

Then we state convergence properties of E-Adams spectral sequences below in *Convergence statements*.

Primary decomposition of abelian groups

An E-Adams spectral sequence *typically* converges (discussed below) not to the full stable homotopy groups $[X,Y]_\bullet$, but just to some piece which on the finite direct summands consists only of p-primary groups for some prime numbers p that depend on the nature of the homotopy ring spectrum E . Here we review basic facts about p-primary decomposition of finite abelian groups and introduce their graphical calculus (remark \ref{pprimarygraphical} below) which will allow to read off these p-primary pieces from the stable page of the E-Adams spectral sequnce.

Theorem 4.1. ***(fundamental theorem of finitely generated abelian groups)***

Every finitely generated abelian group A is isomorphic to a direct sum of p-primary cyclic groups $\mathbb{Z}/p^k\mathbb{Z}$ (for p a prime number and k a natural number) and copies of the infinite cyclic group $\mathbb{Z}$:

$$A \;\simeq\; \mathbb{Z}^n \oplus \bigoplus_i \mathbb{Z}/p_i^{k_i}\mathbb{Z}\,.$$

The summands of the form $\mathbb{Z}/p^k\mathbb{Z}$ are also called the p-primary components of A. Notice that the p_i need not all be distinct.

fundamental theorem of finite abelian groups*:*

In particular every finite abelian group is of this form for $n = 0$, hence is a direct sum of cyclic groups.

fundamental theorem of cyclic groups*:*

In particular every cyclic group $\mathbb{Z}/n\mathbb{Z}$ is a direct sum of cyclic groups of the form

$$\mathbb{Z}/n\mathbb{Z} \simeq \bigoplus_i \mathbb{Z}/p_i^{k_i}\mathbb{Z}$$

where all the p_i are distinct and k_i is the maximal power of the prime factor p_i in the prime decomposition of n.

Specifically, for each natural number d dividing n it contains $\mathbb{Z}/d\mathbb{Z}$ as the subgroup generated by $n/d \in \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$. In fact the lattice of subgroups of $\mathbb{Z}/n\mathbb{Z}$ is the formal dual of the lattice of natural numbers $\leq n$ ordered by inclusion.

(e.g. Roman 12, theorem 13.4, Navarro 03) for cyclic groups e.g. (Aluffi 09, pages 83-84)

This is a special case of the *structure theorem for finitely generated modules over a principal ideal domain*.

Example 4.2. For p a prime number, there are, up to isomorphism, two abelian groups of order p^2, namely

$$(\mathbb{Z}/p\mathbb{Z}) \oplus (\mathbb{Z}/p\mathbb{Z})$$

and

$$\mathbb{Z}/p^2\mathbb{Z}\,.$$

Example 4.3. For p_1 and p_2 two distinct prime numbers, $p_1 \neq p_2$, then there is, up to isomorphism, precisely one abelian group of order $p_1 p_2$, namely

$$\mathbb{Z}/p_1\mathbb{Z} \oplus \mathbb{Z}/p_2\mathbb{Z}\,.$$

This is equivalently the cyclic group

$$\mathbb{Z}/p_1 p_2\mathbb{Z} \simeq \mathbb{Z}/p_1\mathbb{Z} \oplus \mathbb{Z}/p_2\mathbb{Z}\,.$$

The isomorphism is given by sending 1 to (p_2, p_1).

Example 4.4. Moving up, for two distinct prime numbers p_1 and p_2, there are exactly two abelian groups of order $p_1^2 p_2$, namely $(\mathbb{Z}/p_1\mathbb{Z}) \oplus (\mathbb{Z}/p_1\mathbb{Z}) \oplus (\mathbb{Z}/p_2\mathbb{Z})$ and $(\mathbb{Z}/p_1^2\mathbb{Z}) \oplus (\mathbb{Z}/p_2\mathbb{Z})$. The latter is the cyclic group of order $p_1^2 p_2$. For instance, $\mathbb{Z}/12\mathbb{Z} \cong (\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z})$.

Example 4.5. Similarly, there are four abelian groups of order $p_1^2 p_2^2$, three abelian groups of order $p_1^3 p_2$, and so on.

More generally, theorem 4.1 may be used to compute exactly how many abelian groups there are of any finite order n (up to isomorphism): write down its prime factorization, and then for each prime power p^k appearing therein, consider how many ways it can be written as a product of positive powers of p. That is, each partition of k yields an abelian group of order p^k. Since the choices can be made independently for each p, the numbers of such partitions for each p are then multiplied.

Of all these abelian groups of order n, of course, one of them is the cyclic group of order n. The fundamental theorem of cyclic groups says it is the one that involves the one-element partitions $k = [k]$, i.e. the cyclic groups of order p^k for each p.

Remark 4.6. **(graphical representation of p-primary decomposition)**

Theorem 4.1 says that for any prime number p, the p-primary part of any finitely generated abelian group is determined uniquely up to isomorphism by

- a total number $k \in \mathbb{N}$ of powers of p;
- a partition $k = k_1 + k_2 + \cdots + k_q$.

The corresponding p-primary group is

$$\bigoplus_{i=1}^{q} \mathbb{Z}/p^{k_i}\mathbb{Z} \,.$$

In the context of Adams spectral sequences it is conventional to depict this information graphically by

- k dots;
- of which sequences of length k_i are connected by vertical lines, for $i \in \{1, \cdots, q\}$.

For example the graphical representation of the p-primary group

$$\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^3\mathbb{Z} \oplus \mathbb{Z}/p^4\mathbb{Z}$$

is

$$\begin{array}{cccc} & & & \bullet \\ & & & \vert \\ & & \bullet & \bullet \\ & & \vert & \vert \\ & & \bullet & \bullet \\ & & \vert & \vert \\ \bullet & \bullet & \bullet & \bullet \end{array} \,.$$

This notation comes from the convention of drawing stable pages of multiplicative Adams spectral sequences and reading them as encoding the extension problem for computing the homotopy groups that the spectral sequence converges to:

- a dot at the top of a vertical sequence of dots denotes the group $\mathbb{Z}/p\mathbb{Z}$;
- inductively, a dot vetically below a sequence of dots denotes a group extension of $\mathbb{Z}/p\mathbb{Z}$ by the group represented by the sequence of dots above;
- a vertical line between two dots means that the the generator of the group corresponding to the upper dot is, regarded after inclusion into the group extension, the product by p of the generator of the group corresponding to the lower dot, regarded as represented by the generator of the group extension.

So for instance

$$\begin{array}{c} \bullet \\ \vert \\ \bullet \end{array}$$

stands for an abelian group A which forms a group extension of the form

$$\begin{array}{c} \mathbb{Z}/p\mathbb{Z} \\ \downarrow \\ A \\ \downarrow \\ \mathbb{Z}/p\mathbb{Z} \end{array}$$

such that multiplication by p takes the generator of the bottom copy of $\mathbb{Z}/p\mathbb{Z}$, regarded as represented by the generator of A, to the generator of the image of the top copy of $\mathbb{Z}/p\mathbb{Z}$.

This means that of the two possible choices of extensions (by example 4.2) A corresponds to the non-trivial extension $A = \mathbb{Z}/p^2\mathbb{Z}$. Because then in

$$\begin{array}{c} \mathbb{Z}/p\mathbb{Z} \\ \downarrow \\ \mathbb{Z}/p^2\mathbb{Z} \\ \downarrow \\ \mathbb{Z}/p\mathbb{Z} \end{array}$$

the image of the generator 1 of the first group in the middle group is $p = p \cdot 1$.

Conversely, the notation

$$\begin{array}{c} \bullet \\ \\ \bullet \end{array}$$

stands for an abelian group A which forms a group extension of the form

$$\begin{array}{c} \mathbb{Z}/p\mathbb{Z} \\ \downarrow \\ A \\ \downarrow \\ \mathbb{Z}/p\mathbb{Z} \end{array}$$

such that multiplication by p of the generator of the top group in the middle group does *not* yield the generator of the bottom group.

This means that of the two possible choices (by example 4.2) A corresponds to the *trivial* extension $A = \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$. Because then in

$$\begin{array}{c} \mathbb{Z}/p\mathbb{Z} \\ \downarrow \\ \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \\ \downarrow \\ \mathbb{Z}/p\mathbb{Z} \end{array}$$

the generator 1 of the top group maps to the element $(1,0)$ in the middle group, and multiplication of that by p is $(0,0)$ instead of $(0,1)$, where the latter is the generator of the bottom group.

Similarly

$$\begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array}$$

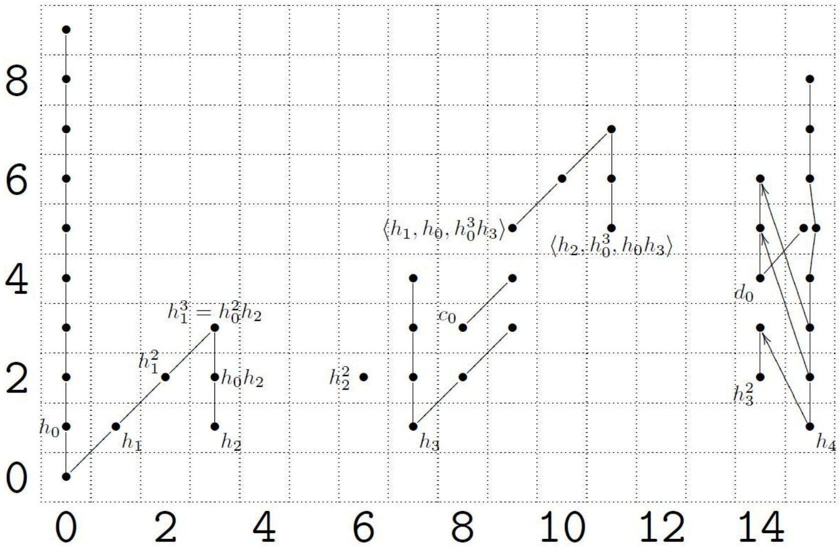
is to be read as the result of appending to the previous case a dot *below*, so that this now indicates a group extension of the form

$$\begin{array}{c} \mathbb{Z}/p^2\mathbb{Z} \\ \downarrow \\ A \\ \downarrow \\ \mathbb{Z}/p\mathbb{Z} \end{array}$$

such that p-times the generator of the bottom group, regarded as represented by the generator of the middle group, is the image of the generator of the top group. This is again the case for the unique non-trivial extension, and hence in this case the diagram stands for $A = \mathbb{Z}/p^3\mathbb{Z}$.

And so on.

For example the stable page of the $\mathbb{F}_2$-classical Adams spectral sequence for computation of the 2-primary part of the stable homotopy groups of spheres $\pi_{t-s}(\mathbb{S})$ has in ("internal") degree $t - s \leq 13$ the following non-trivial entries:



(graphics taken from (Schwede 12)))

Ignoring here the diagonal lines (which denote multiplication by the element h_1 that encodes the additional ring structure on $\pi_\bullet(\mathbb{S})$ which here we are not concerned with) and applying the above prescription, we read off for instance that $\pi_3(\mathbb{S}) \simeq \mathbb{Z}/8\mathbb{Z}$ (because all three dots are connected) while $\pi_8(\mathbb{S}) \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ (because here the two dots are not connected). In total

$k =$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**
$\pi_k(\mathbb{S})_{(2)} =$	$\mathbb{Z}_{(2)}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/8$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/16$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^3$	$\mathbb{Z}/2$	$\mathbb{Z}/8$	0	0

Here the only entry that needs further explanation is the one for $k = 0$. We discuss the relevant concepts for this below in the section *Localization and adic completion of abelian groups*, but for completeness, here is the quick idea:

The symbol $\mathbb{Z}_{(2)}$ refers to the 2-adic integers (def. 4.16), i.e. for the limit of abelian groups

$$\mathbb{Z}_{(2)} = \underset{\longleftarrow n \geq 1}{\lim} \mathbb{Z}/2^n\mathbb{Z}$$

This is not 2-primary, but it does arise when applying 2-adic completion of abelian groups (def. 4.15) to finitely generated abelian groups as in theorem 4.1. The 2-adic integers is the abelian group associated to the diagram

$$\begin{array}{c} \vdots \\ | \\ \bullet \\ | \\ \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array}$$

as in the above figure. Namely by the above prescrption, this infinite sequence should encode an abelian group A such that it is an extension of $\mathbb{Z}/p\mathbb{Z}$ by itself of the form

$$0 \to A \overset{p\cdot(-)}{\longrightarrow} A \longrightarrow \mathbb{Z}/p\mathbb{Z}$$

(Because it is supposed to encode an extension of $\mathbb{Z}/p\mathbb{Z}$ by the group corresponding to the result of chopping off the lowest dot, which however in this case does not change the figure.)

Indeed, by lemma 4.17 below we have a short exact sequence

$$0 \to \mathbb{Z}_{(p)} \xrightarrow{p\cdot(-)} \mathbb{Z}_{(p)} \longrightarrow \mathbb{Z}/p\mathbb{Z} \to 0 \;.$$

Localization and adic completion of abelian groups

Remark 4.7. Recall that Ext-groups $\mathrm{Ext}^\bullet(A,B)$ between abelian groups $A, B \in$ Ab are concentrated in degrees 0 and 1 (prop.). Since

$$\mathrm{Ext}^0(A,B) \simeq \mathrm{Hom}(A,B)$$

is the plain hom-functor, this means that there is only one possibly non-vanishing Ext-group Ext^1, therefore often abbreviated to just "Ext":

$$\mathrm{Ext}(A,B) \coloneqq \mathrm{Ext}^1(A,B) \;.$$

Definition 4.8. Let K be an abelian group.

Then an abelian group A is called K**-local** if all the Ext-groups from K to A vanish:

$$\mathrm{Ext}^\bullet(K,A) \simeq 0$$

hence equivalently (remark 4.7) if

$$\mathrm{Hom}(K,A) \simeq 0 \quad \text{and} \quad \mathrm{Ext}(K,A) \simeq 0 \;.$$

A homomorphism of abelian groups $f : B \longrightarrow C$ is called K**-local** if for all K-local groups A the function

$$\mathrm{Hom}(f,A) \,:\, \mathrm{Hom}(B,A) \longrightarrow \mathrm{Hom}(A,A)$$

is a bijection.

> (***Beware*** *that here it would seem more natural to use* $\mathrm{Ext}^\bullet$ *instead of* Hom*, but we do use* Hom*. See (Neisendorfer 08, remark 3.2).*

A homomorphism of abelian groups

$$\eta \,:\, A \longrightarrow L_K A$$

is called a K**-localization** if

1. $L_K A$ is K-local;
2. η is a K-local morphism.

We now discuss two classes of examples of localization of abelian groups

1. *Classical localization at/away from primes*;
2. *Formal completion at primes*.

Classical localization at/away from primes

For $n \in \mathbb{N}$, write $\mathbb{Z}/n\mathbb{Z}$ for the cyclic group of order n.

Lemma 4.9. *For $n \in \mathbb{N}$ and $A \in \mathrm{Ab}$ any abelian group, then*

1. *the hom-group out of $\mathbb{Z}/n\mathbb{Z}$ into A is the n-torsion subgroup of A*

$$\mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, A) \simeq \{a \in A \mid p \cdot a = 0\}$$

2. *the first Ext-group out of $\mathbb{Z}/n\mathbb{Z}$ into A is*

$$\mathrm{Ext}^1(\mathbb{Z}/n\mathbb{Z}, A) \simeq A/nA \;.$$

Proof. Regarding the first item: Since $\mathbb{Z}/p\mathbb{Z}$ is generated by its element 1 a morphism $\mathbb{Z}/p\mathbb{Z} \to A$ is fixed by the image a of this element, and the only relation on 1 in $\mathbb{Z}/p\mathbb{Z}$ is that $p \cdot 1 = 0$.

Regarding the second item:

Consider the canonical free resolution

$$0 \to \mathbb{Z} \overset{p\cdot(-)}{\longrightarrow} \mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z} \to 0 \,.$$

By the general discusson of derived functors in homological algebra this exhibits the Ext-group in degree 1 as part of the following short exact sequence

$$0 \to \mathrm{Hom}(\mathbb{Z}, A) \overset{\mathrm{Hom}(n\cdot(-), A)}{\longrightarrow} \mathrm{Hom}(\mathbb{Z}, A) \longrightarrow \mathrm{Ext}^1(\mathbb{Z}/n\mathbb{Z}, A) \to 0 \,,$$

where the morphism on the left is equivalently $A \overset{n\cdot(-)}{\longrightarrow} A$. ▮

Example 4.10. An abelian group A is $\mathbb{Z}/p\mathbb{Z}$-local precisely if the operation

$$p \cdot (-) \;:\; A \longrightarrow A$$

of multiplication by p is an isomorphism, hence if "p is invertible in A".

Proof. By the first item of lemma 4.9 we have

$$\mathrm{Hom}(\mathbb{Z}/p\mathbb{Z}, A) \simeq \{a \in A \;\vert\; p \cdot a = 0\}$$

By the second item of lemma 4.9 we have

$$\mathrm{Ext}^1(\mathbb{Z}/p\mathbb{Z}, A) \simeq A/pA \,.$$

Hence by def. 4.8 A is $\mathbb{Z}/p\mathbb{Z}$-local precisely if

$$\{a \in A \;\vert\; p \cdot a = 0\} \simeq 0$$

and if

$$A/pA \simeq 0 \,.$$

Both these conditions are equivalent to multiplication by p being invertible. ▮

Definition 4.11. For $J \subset \mathbb{N}$ a set of prime numbers, consider the direct sum $\oplus_{p \in J} \mathbb{Z}/p\mathbb{Z}$ of cyclic groups of order p.

The operation of $\otimes_{p\in J} \mathbb{Z}/p\mathbb{Z}$-localization of abelian groups according to def. 4.8 is called **inverting the primes** in J.

Specifically

1. for $J = \{p\}$ a single prime then $\mathbb{Z}/p\mathbb{Z}$-localization is called **localization away from** p;
2. for J the set of all primes except p then $\otimes_{p\in J} \mathbb{Z}/p\mathbb{Z}$-localization is called **localization at** p**;**
3. for J the set of all primes, then $\otimes_{p\in J} \mathbb{Z}/p\mathbb{Z}$-localizaton is called **rationalization**..

Definition 4.12. For $J \subset \mathrm{Primes} \subset \mathbb{N}$ a set of prime numbers, then

$$\mathbb{Z}[J^{-1}] \hookrightarrow \mathbb{Q}$$

denotes the subgroup of the rational numbers on those elements which have an expression as a fraction of natural numbers with denominator a product of elements in J.

This is the abelian group underlying the localization of a commutative ring of the ring of integers at the set J of primes.

If $J = \mathrm{Primes} - \{p\}$ is the set of all primes *except* p one also writes

$$\mathbb{Z}_{(p)} \coloneqq \mathbb{Z}[\mathrm{Primes} - \{p\}] \,.$$

Notice the parenthesis, to distinguish from the notation $\mathbb{Z}_p$ for the p-adic integers (def. 4.16 below).

Remark 4.13. The terminology in def. 4.11 is motivated by the following perspective of arithmetic geometry:

Generally for R a commutative ring, then an R-module is equivalently a quasicoherent sheaf on the spectrum of the ring $\mathrm{Spec}(R)$.

In the present case $R = \mathbb{Z}$ is the integers and abelian groups are identified with $\mathbb{Z}$-modules. Hence we may think of an abelian group A equivalently as a quasicoherent sheaf on Spec(Z).

The "ring of functions" on Spec(Z) is the integers, and a point in $\mathrm{Spec}(\mathbb{Z})$ is labeled by a prime number p, thought of as generating the ideal of functions on Spec(Z) which vanish at that point. The residue field at that point is $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$.

Inverting a prime means forcing p to become invertible, which, by this characterization, it is precisely *away* from that point which it labels. The localization of an abelian group at $\mathbb{Z}/p\mathbb{Z}$ hence corresponds to the restriction of the corresponding quasicoherent sheaf over $\mathrm{Spec}(\mathbb{Z})$ to the complement of the point labeled by p.

Similarly localization *at* p is localization away from all points except p.

See also at *function field analogy* for more on this.

Proposition 4.14. *For $J \subset \mathbb{N}$ a set of prime numbers, a homomorphism of abelian groups $f : A\mathrm{lookrightarrow} B$ is local (def. 4.8) with respect to $\oplus_{p \in J} \mathbb{Z}/p\mathbb{Z}$ (def. 4.11) if under tensor product of abelian groups with $\mathbb{Z}[J^{-1}]$ (def. 4.12) it becomes an isomorphism*

$$f \otimes \mathbb{Z}[J^{-1}] : X \otimes \mathbb{Z}[J^{-1}] \xrightarrow{\simeq} Y \otimes \mathbb{Z}[J^{-1}] \,.$$

Moreover, for A any abelian group then its $\oplus_{p \in J} \mathbb{Z}/p\mathbb{Z}$-localization exists and is given by the canonical projection morphism

$$A \longrightarrow A \otimes \mathbb{Z}[J^{-1}] \,.$$

(e.g. Neisendorfer 08, theorem 4.2)

Formal completion at primes

We have seen above in remark 4.13 that classical localization of abelian groups at a prime number is geometrically interpreted as restricting a quasicoherent sheaf over Spec(Z) to a single point, the point that is labeled by that prime number.

Alternatively one may restrict to the "infinitesimal neighbourhood" of such a point. Technically this is called the *formal neighbourhood*, because its ring of functions is, by definition, the ring of formal power series (regarded as an adic ring or pro-ring). The corresponding operation on abelian groups is accordingly called *formal completion* or *adic completion* or just *completion*, for short, of abelian groups.

It turns out that if the abelian group is finitely generated then the operation of p-completion coincides with an operation of *localization* in the sense of def. 4.8, namely localization at the p-primary component $\mathbb{Z}(p^\infty)$ of the group $\mathbb{Q}/\mathbb{Z}$ (def. 4.22 below). On the one hand this equivalence is useful for deducing some key properties of p-completion, this we discuss below. On the other hand this situation is a shadow of the relation between localization of spectra and nilpotent completion of spectra, which is important for characterizing the convergence properties of Adams spectral sequences.

Definition 4.15. For p a prime number, then the **p-adic completion** of an abelian group A is the abelian group given by the limit

$$A_p^\wedge \coloneqq \underset{\longleftarrow}{\lim}(\cdots \longrightarrow A/p^3 A \longrightarrow A/p^2 A \longrightarrow A/p A)\,,$$

where the morphisms are the evident quotient morphisms. With these understood we often write

$$A_p^\wedge \coloneqq \underset{\longleftarrow n}{\lim}\, A/p^n A$$

for short. Notice that here the indexing starts at $n = 1$.

Example 4.16. The p-adic completion (def. 4.15) of the integers $\mathbb{Z}$ is called the **p-adic integers**, often written

$$\mathbb{Z}_p \coloneqq \mathbb{Z}_p^\wedge \coloneqq \underset{\longleftarrow n}{\lim}\, \mathbb{Z}/p^n\mathbb{Z}\,,$$

which is the original example that gives the general concept its name.

With respect to the canonical ring-structure on the integers, of course $p\mathbb{Z}$ is a prime ideal.

Compare this to the ring $\mathcal{O}_{\mathbb{C}}$ of holomorphic functions on the complex plane. For $x \in \mathbb{C}$ any point, it contains the prime ideal generated by $(z-x)$ (for z the canonical coordinate function on $\mathbb{z}$). The formal power series ring $\mathbb{C}[[(z.x)]]$ is the adic completion of $\mathcal{O}_{\mathbb{C}}$ at this ideal. It has the interpretation of functions defined on a formal neighbourhood of X in $\mathbb{C}$.

Analogously, the p-adic integers $\mathbb{Z}_p$ may be thought of as the functions defined on a formal neighbourhood of the point labeled by p in Spec(Z).

Lemma 4.17. *There is a short exact sequence*

$$0 \to \mathbb{Z}_p \xrightarrow{p\cdot(-)} \mathbb{Z}_p \longrightarrow \mathbb{Z}/p\mathbb{Z} \to 0 \,.$$

Proof. Consider the following commuting diagram

$$\begin{array}{ccccc} \vdots && \vdots && \vdots \\ \downarrow && \downarrow && \downarrow \\ \mathbb{Z}/p^3\mathbb{Z} & \xrightarrow{p\cdot(-)} & \mathbb{Z}/p^4\mathbb{Z} & \longrightarrow & \mathbb{Z}/p\mathbb{Z} \\ \downarrow && \downarrow && \downarrow \\ \mathbb{Z}/p^2\mathbb{Z} & \xrightarrow{p\cdot(-)} & \mathbb{Z}/p^3\mathbb{Z} & \longrightarrow & \mathbb{Z}/p\mathbb{Z} \\ \downarrow && \downarrow && \downarrow \\ \mathbb{Z}/p\mathbb{Z} & \xrightarrow{p\cdot(-)} & \mathbb{Z}/p^2\mathbb{Z} & \longrightarrow & \mathbb{Z}/p\mathbb{Z} \\ \downarrow && \downarrow && \downarrow \\ 0 & \longrightarrow & \mathbb{Z}/p\mathbb{Z} & \longrightarrow & \mathbb{Z}/p\mathbb{Z} \end{array} \,.$$

Each horizontal sequence is exact. Taking the limit over the vertical sequences yields the sequence in question. Since limits commute over limits, the result follows. ▮

We now consider a concept of p-completion that is in general different from def. 4.15, but turns out to coincide with it in finitely generated abelian groups.

Definition 4.18. For p a prime number, write

$$\mathbb{Z}[1/p] \coloneqq \underset{\longrightarrow}{\lim}\left(\mathbb{Z} \xrightarrow{p\cdot(-)} \mathbb{Z} \xrightarrow{p\cdot(-)} \mathbb{Z} \longrightarrow \cdots \right)$$

for the colimit (in Ab) over iterative applications of multiplication by p on the integers.

This is the abelian group generated by formal expressions $\frac{1}{p^k}$ for $k \in \mathbb{N}$, subject to the relations

$$(p \cdot n)\frac{1}{p^{k+1}} = n\frac{1}{p^k} \,.$$

Equivalently it is the abelian group underlying the ring localization $\mathbb{Z}[1/p]$.

Definition 4.19. For p a prime number, then localization of abelian groups (def. 4.8) at $\mathbb{Z}[1/p]$ (def. 4.18) is called **p-completion of abelian groups**.

Lemma 4.20. *An abelian group A is p-complete according to def. 4.19 precisely if both the limit as well as the lim^1 of the sequence*

$$\cdots \longrightarrow A \xrightarrow{p} A \xrightarrow{p} A \xrightarrow{p} A$$

vanishes:

$$\underset{\longleftarrow}{\lim}\left(\cdots \longrightarrow A \xrightarrow{p} A \xrightarrow{p} A \xrightarrow{p} A \right) \simeq 0$$

and

$$\underset{\longleftarrow}{\lim}^1\left(\cdots \longrightarrow A \xrightarrow{p} A \xrightarrow{p} A \xrightarrow{p} A \right) \simeq 0 \,.$$

Proof. By def. 4.8 the group A is $\mathbb{Z}[1/p]$-local precisely if

$$\mathrm{Hom}(\mathbb{Z}[1/p], A) \simeq 0 \qquad \text{and} \qquad \mathrm{Ext}^1(\mathbb{Z}[1/p], A) \simeq 0 \,.$$

Now use the colimit definition $\mathbb{Z}[1/p] = \underset{\longrightarrow_n}{\lim}\, \mathbb{Z}$ (def. 4.18) and the fact that the hom-functor sends colimits in the first argument to limits to deduce that

$$\begin{aligned} \mathrm{Hom}(\mathbb{Z}[1/p], A) &= \mathrm{Hom}(\underset{\longrightarrow_n}{\lim}\, \mathbb{Z}, A) \\ &\simeq \underset{\longleftarrow_n}{\lim}\, \mathrm{Hom}(\mathbb{Z}, A) \\ &\simeq \underset{\longleftarrow_n}{\lim}\, A \end{aligned} \,.$$

This yields the first statement. For the second, use that for every cotower over abelian groups $B_\bullet$ there is a short exact sequence of the form

$$0 \to \underset{\longleftarrow n}{\lim}^1 \mathrm{Hom}(B_n, A) \longrightarrow \mathrm{Ext}^1(\underset{\longrightarrow n}{\lim} B_n, A) \longrightarrow \underset{\longleftarrow n}{\lim} \mathrm{Ext}^1(B_n, A) \to 0$$

(by this lemma).

In the present case all $B_n \simeq \mathbb{Z}$, which is a free abelian group, hence a projective object, so that all the Ext-groups out of it vannish. Hence by exactness there is an isomorphism

$$\mathrm{Ext}^1(\underset{\longrightarrow n}{\lim} \mathbb{Z}, A) \simeq \underset{\longleftarrow n}{\lim}^1 \mathrm{Hom}(\mathbb{Z}, A) \simeq \underset{\longleftarrow n}{\lim}^1 A \;.$$

This gives the second statement. ■

Example 4.21. For p a prime number, the p-primary cyclic groups of the form $\mathbb{Z}/p^n\mathbb{Z}$ are p-complete (def. 4.19).

Proof. By lemma 4.20 we need to check that

$$\underset{\longleftarrow}{\lim}\left(\cdots \xrightarrow{p} \mathbb{Z}/p^n\mathbb{Z} \xrightarrow{p} \mathbb{Z}/p^n\mathbb{Z} \xrightarrow{p} \mathbb{Z}/p^n\mathbb{Z}\right) \simeq 0$$

and

$$\underset{\longleftarrow}{\lim}^1\left(\cdots \xrightarrow{p} \mathbb{Z}/p^n\mathbb{Z} \xrightarrow{p} \mathbb{Z}/p^n\mathbb{Z} \xrightarrow{p} \mathbb{Z}/p^n\mathbb{Z}\right) \simeq 0 \;.$$

For the first statement observe that n consecutive stages of the tower compose to the zero morphism. First of all this directly implies that the limit vanishes, secondly it means that the tower satisfies the Mittag-Leffler condition (def.) and this implies that the $\lim^1$ also vanishes (prop.). ■

Definition 4.22. For p a prime number, write

$$\mathbb{Z}(p^\infty) \coloneqq \mathbb{Z}[1/p]/\mathbb{Z}$$

(the p-primary part of $\mathbb{Q}/\mathbb{Z}$), where $\mathbb{Z}[1/p] = \underset{\longrightarrow}{\lim}(\mathbb{Z} \overset{p}{\to} \mathbb{Z} \overset{p}{\to} \mathbb{Z} \to \cdots)$ from def. 4.18.

Since colimits commute over each other, this is equivalently

$$\mathbb{Z}(p^\infty) \simeq \underset{\longrightarrow}{\lim}(0 \hookrightarrow \mathbb{Z}/p\mathbb{Z} \hookrightarrow \mathbb{Z}/p^2\mathbb{Z} \hookrightarrow \cdots) \;.$$

Theorem 4.23. *For p a prime number, the $\mathbb{Z}[1/p]$-localization*

$$A \longrightarrow L_{\mathbb{Z}[1/p]} A$$

of an abelian group A (def. 4.18, def. 4.8), hence the p-completion of A according to def. 4.19, is given by the morphism

$$A \longrightarrow \mathrm{Ext}^1(\mathbb{Z}(p^\infty), A)$$

into the first Ext-group into A out of $\mathbb{Z}(p^\infty)$ (def. 4.22), which appears as the first connecting homomorphism δ in the long exact sequence of Ext-groups

$$0 \to \mathrm{Hom}(\mathbb{Z}(p^\infty), A) \longrightarrow \mathrm{Hom}(\mathbb{Z}[1/p], A) \longrightarrow \mathrm{Hom}(\mathbb{Z}, A) \overset{\delta)}{\longrightarrow} \mathrm{Ext}^1(\mathbb{Z}(p^\infty), A) \to \cdots \;.$$

that is induced (via this prop.) from the defining short exact sequence

$$0 \to \mathbb{Z} \longrightarrow \mathbb{Z}[1/p] \longrightarrow \mathbb{Z}(p^\infty) \to 0$$

(def. 4.22).

e.g. (Neisendorfer 08, p. 16)

Proposition 4.24. *If A is a finitely generated abelian group, then its p-completion (def. 4.19, for any prime number p) is equivalently its p-adic completion (def. 4.15)*

$$\mathbb{Z}[1/p]A \simeq A_p^\wedge \;.$$

Proof. By theorem 4.23 the p-completion is $\mathrm{Ext}^1(\mathbb{Z}(p^\infty), A)$. By def. 4.22 there is a colimit

$$\mathbb{Z}(p^\infty) = \underset{\longrightarrow}{\lim}(\mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p^3\mathbb{Z} \to \cdots) \;.$$

Together this implies, by this lemma, that there is a short exact sequence of the form

$$0 \to \underset{\longleftarrow}{\lim}^1 \mathrm{Hom}(\mathbb{Z}/p^n\mathbb{Z}, A) \longrightarrow X_p^\wedge \longrightarrow \underset{\longleftarrow n}{\lim} \mathrm{Ext}^1(\mathbb{Z}/p^n\mathbb{Z}, A) \to 0 \;.$$

By lemma 4.9 the lim^1 on the left is over the p^n-torsion subgroups of A, as n ranges. By the assumption that A is finitely generated, there is a maximum n such that all torsion elements in A are annihilated by p^n. This means that the Mittag-Leffler condition (def.) is satisfied by the tower of p-torsion subgroups, and hence the lim^1-term vanishes (prop.).

Therefore by exactness of the above sequence there is an isomorphism

$$\begin{aligned} L_{\mathbb{Z}[1/p]} X &\simeq \underset{\longleftarrow}{\lim}_n \mathrm{Ext}^1(\mathbb{Z}/p^n\mathbb{Z}, A) \\ &\simeq \underset{\longleftarrow}{\lim}_n A/p^n A \end{aligned} ,$$

where the second isomorphism is by lemma 4.9. ∎

Proposition 4.25. *If A is a p-divisible group in that $A \xrightarrow{p\cdot(-)} A$ is an isomorphism, then its p-completion (def. 4.19) vanishes.*

Proof. By theorem 4.23 the localization morphism δ sits in a long exact sequence of the form

$$0 \to \mathrm{Hom}(\mathbb{Z}(p^\infty), A) \longrightarrow \mathrm{Hom}(\mathbb{Z}[1/p], A) \overset{\phi}{\longrightarrow} \mathrm{Hom}(\mathbb{Z}, A) \overset{\delta}{\longrightarrow} \mathrm{Ext}^1(\mathbb{Z}(p^\infty), A) \to \cdots .$$

Here by def. 4.18 and since the hom-functor takes colimits in the first argument to limits, the second term is equivalently the limit

$$\mathrm{Hom}(\mathbb{Z}[1/p], A) \simeq \underset{\longleftarrow}{\lim}\left(\cdots \to A \overset{p\cdot(-)}{\longrightarrow} A \overset{p\cdot(-)}{\longrightarrow} A \right) .$$

But by assumption all these morphisms $p\cdot(-)$ that the limit is over are isomorphisms, so that the limit collapses to its first term, which means that the morphism ϕ in the above sequence is an isomorphism. But by exactness of the sequence this means that $\delta = 0$. ∎

Corollary 4.26. *Let p be a prime number. If A is a finite abelian group, then its p-completion (def. 4.19) is equivalently its p-primary part.*

Proof. By the fundamental theorem of finite abelian groups, A is a finite direct sum

$$A \simeq \underset{i}{\oplus} \mathbb{Z}/p_i^{k_i}\mathbb{Z}$$

of cyclic groups of ordr $p_i^{k_1}$ for p_i prime numbers and $k_i \in \mathbb{N}$ (thm.).

Since finite direct sums are equivalently products (biproducts: Ab is an additive category) this means that

$$\mathrm{Ext}^1(\mathbb{Z}(p^\infty), A) \simeq \prod_i \mathrm{Ext}^1(\mathbb{Z}(p^\infty), \mathbb{Z}/p_i^{k_1}\mathbb{Z}) .$$

By theorem 4.23 the ith factor here is the p-completion of $\mathbb{Z}/p_i^{k_i}\mathbb{Z}$, and since $p\cdot(-)$ is an isomorphism on $\mathbb{Z}/p_i^{k_i}\mathbb{Z}$ if $p_i \neq p$ (because its kernel evidently vanishes), prop. 4.25 says that in this case the factor vanishes, so that only the factors with $p_i = p$ remain. On these however $\mathrm{Ext}^1(\mathbb{Z}(p^\infty), -)$ is the identity by example 4.21. ∎

Localization and nilpotent completion of spectra

We discuuss

1. *Bousfield localization of spectra*
2. *Nilpotent completion of spectra*

which are the analogs in stable homotopy theory of the construction of localization of abelian groups discussed above.

Literature: (Bousfield 79)

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Localization of spectra

Definition 4.27. Let $E \in \mathrm{Ho}(\mathrm{Spectra})$ be be a spectrum. Say that

1. a spectrum X is E**-acyclic** if the smash product with E is zero, $E \wedge X \simeq 0$;
2. a morphism $f: X \to Y$ of spectra is an E**-equivalence** if $E \wedge f : E \wedge X \to E \wedge Y$ is an isomorphism in $\mathrm{Ho}(\mathrm{Spectra})$, hence if $E_\bullet(f)$ is an isomorphism in E-generalized homology;

3. a spectrum X is E**-local** if the following equivalent conditions hold

   1. for every E-equivalence f then $[f,X]_\bullet$ is an isomorphism;

   2. every morphism $Y \longrightarrow X$ out of an E-acyclic spectrum Y is zero in $\mathrm{Ho}(\mathrm{Spectra})$;

(Bousfield 79, §1) see also for instance (Lurie, Lecture 20, example 4)

Lemma 4.28. *The two conditions in the last item of def. 4.27 are indeed equivalent.*

Proof. Notice that $A \in \mathrm{Ho}(\mathrm{Spectra})$ being E-acyclic means equivalently that the unique morphism $0 \longrightarrow A$ is an E-equivalence.

Hence one direction of the claim is trivial. For the other direction we need to show that for $[-,X]_\bullet$ to give an isomorphism on all E-equivalences f, it is sufficient that it gives an isomorphism on all E-equivalences of the form $0 \to A$.

Given a morphism $f : A \to B$, write $B \longrightarrow B/A$ for its homotopy cofiber. Then since $\mathrm{Ho}(\mathrm{Spectra})$ is a triangulated category (prop.) the defining axioms of triangulated categories (def., lemma) give that there is a commuting diagram of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \overset{\mathrm{id}}{\longrightarrow} & A & \longrightarrow & 0 & \longrightarrow & \Sigma A \\ \downarrow && \downarrow^{\mathrm{id}} && \downarrow^{f} && \downarrow && \downarrow^{\mathrm{id}} \\ \Sigma^{-1} B/A & \longrightarrow & A & \underset{f}{\longrightarrow} & B & \longrightarrow & B/A & \longrightarrow & \Sigma A \end{array},$$

where both the top as well as the bottom are homotopy cofiber sequences. Hence applying $[-,X]_\bullet$ to this diagram in $\mathrm{Ho}(\mathrm{Spectra})$ yields a diagram of graded abelian groups of the form

$$\begin{array}{ccccccccc} 0 & \longleftarrow & [A,X]_\bullet & \longleftarrow & [A,X]_\bullet & \longleftarrow & 0 & \longleftarrow & [A,X]_{\bullet+1} \\ \uparrow && \uparrow^{\mathrm{id}} && \uparrow^{[f,X]_\bullet} && \uparrow && \uparrow^{\mathrm{id}} \\ [B/A,X]_{\bullet+1} & \longleftarrow & [A,X]_\bullet & \longleftarrow & [B,X]_\bullet & \longleftarrow & [B/A,X]_\bullet & \longleftarrow & [A,X]_{\bullet+1} \end{array},$$

where now both horizontal sequences are long exact sequences (prop.).

Hence if $[B/A,X]_\bullet \longrightarrow 0$ is an isomorphism, then all four outer vertical morphisms in this diagram are isomorphisms, and then the five-lemma implies that also $[f,X]_\bullet$ is an isomorphism.

Hence it is now sufficient to observe that with $f : A \to B$ an E-equivalence, then its homotopy cofiber B/A is E-acyclic.

To see this, notice that by the tensor triangulated structure on $\mathrm{Ho}(\mathrm{Spectra})$ (prop.) the smash product with E preserves homotopy cofiber sequences, so that there is a homotopy cofiber sequence

$$E \wedge A \overset{E \wedge f}{\longrightarrow} E \wedge B \longrightarrow E \wedge (B/A) \longrightarrow E \wedge \Sigma A \,.$$

But if the first morphism here is an isomorphism, then the axioms of a triangulated category (def.) imply that $E \wedge B/A \simeq 0$. In detail: by the axioms we may form the morphism of homotopy cofiber sequences

$$\begin{array}{ccccccc} E \wedge A & \overset{E \wedge f}{\longrightarrow} & E \wedge B & \longrightarrow & E \wedge B/A & \longrightarrow & E \wedge \Sigma A \\ \downarrow^{\mathrm{id}} && \downarrow^{(E \wedge f)^{-1}} && \downarrow && \downarrow^{\mathrm{id}} \\ E \wedge A & \underset{\mathrm{id}}{\longrightarrow} & E \wedge A & \longrightarrow & 0 & \longrightarrow & E \wedge \Sigma A \end{array}.$$

Then since two of the three vertical morphisms on the left are isomorphisms, so is the third (lemma). ▮

Definition 4.29. Given $E, X \in \mathrm{Ho}(\mathrm{Spectra})$, then an E-**Bousfield localization of spectra** of X is

1. an E-local spectrum $L_E X$

2. an E-equivalence $X \longrightarrow L_E X$.

according to def. 4.27.

We discuss now that E-Localizations always exist. The key to this is the following lemma 4.30, which asserts that a spectrum being E-local is equivalent to it being A-null, for some "small" spectrum A:

Lemma 4.30. *For every spectrum E there exists a spectrum A such that any spectrum X is E-local (def. 4.27) precisely if it is A-null, i.e.*

$$X \text{ is } E\text{-local} \quad \Leftrightarrow \quad [A, X]_* = 0$$

and such that

1. *A is E-acyclic (def. 4.27);*

2. *there exists an infinite cardinal number κ such that A is a κ-CW spectrum (hence a CW spectrum (def.) with at most κ many cells);*

3. *the class of E-acyclic spectra (def. 4.27) is the class generated by A under*

   1. *wedge sum*

   2. *the relation that if in a homotopy cofiber sequence $X_1 \to X_2 \to X_3$ two of the spectra are in the class, then so is the third.*

(Bousfield 79, lemma 1.13 with lemma 1.14) review includes (Bauer 11, p.2,3, VanKoughnett 13, p. 8)

Proposition 4.31. *For $E \in \mathrm{Ho}(\mathrm{Spectra})$ any spectrum, every spectrum X sits in a homotopy cofiber sequence of the form*

$$G_E(X) \longrightarrow X \overset{\eta_X}{\longrightarrow} L_E(X)\,,$$

and natural in X, such that

1. *$G_E(X)$ is E-acyclic,*

2. *$L_E(X)$ is E-local,*

according to def. 4.27.

(Bousfield 79, theorem 1.1) see also for instance (Lurie, Lecture 20, example 4)

Proof. Consider the κ-CW-spectrum spectrum A whose existence is asserted by lemma 4.30. Let

$$I_A \coloneqq \{A \to \mathrm{Cone}(A)\}$$

denote the set containing as its single element the canonical morphism (of sequential spectra) from A into the standard cone of A, i.e. the cofiber

$$\mathrm{Cone}(A) \coloneqq \mathrm{cofib}(A \to A \wedge I_+) \simeq A \wedge I$$

of the inclusion of A into its standard cylinder spectrum (def.).

Since the standard cylinder spectrum on a CW-spectrum is a good cylinder object (prop.) this means (lemma) that for X any fibrant sequential spectrum, and for $A \longrightarrow X$ any morphism, then an extension along the cone inclusion

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & \nearrow & \\ \mathrm{Cone}(A) & & \end{array}$$

equivalently exhibits a null-homotopy of the top morphism. Hence the $(A \to \mathrm{Cone}(A))$-injective objects in $\mathrm{Ho}(\mathrm{Spectra})$ are precisely those spectra X for which $[A, X]_\bullet \simeq 0$.

Moreover, due to the degreewise nature of the smash tensoring $\mathrm{Cone}(A) = A \wedge I$ (def), the inclusion morphism $A \to \mathrm{Cone}(A)$ is degreewise the inclusion of a CW-complex into its standard cone, which is a relative cell complex inclusion (prop.).

By this lemma the κ-cell spectrum A is κ-*small object* (def.) with respect to morphisms of spectra which are degreewise relative cell complex inclusion small object argument .

Hence the small object argument applies (prop.) and gives for every X a factorization of the terminal morphism $X \to *$ as an I_A-relative cell complex (def.) followed by an I_A-injective morphism (def.)

$$X \overset{I_A\,\mathrm{Cell}}{\longrightarrow} L_E X \overset{I_A\,\mathrm{Inj}}{\longrightarrow} *\,.$$

By the above, this means that $[A, L_E X] = 0$, hence by lemma 4.30 that $L_E X$ is E-local.

It remains to see that the homotopy fiber of $X \to L_E X$ is E-acyclic: By the tensor triangulated structure on $\mathrm{Ho}(\mathrm{Spectra})$ (prop.) it is sufficient to show that the homotopy cofiber is E-acyclic (since it differs from the homotopy fiber only by suspension). By the pasting law, the homotopy cofiber of a transfinite composition is the transfinite composition of a sequence of homotopy pushouts. By lemma 4.30 and applying the pasting law again, all these homotopy pushouts produce E-acyclic objects. Hence we conclude by observing that the the transfinite composition of the morphisms between these E-acyclic objects is E-acyclic. Since by construction all these morphisms are relative cell complex inclusions, this follows again with the compactness of the n-spheres (lemma). ∎

Lemma 4.32. *The morphism $X \to L_E(X)$ in prop. 4.31 exhibits an E-localization of X according to def. 4.29*

Proof. It only remains to show that $X \to L_E X$ is an E-equivalence. By the tensor triangulated structure on $\mathrm{Ho}(\mathrm{Spectra})$ (prop.) the smash product with E preserves homotopy cofiber sequences, so that

$$E \wedge G_E X \longrightarrow E \wedge X \overset{E \wedge \eta_X}{\longrightarrow} E \wedge L_E X \longrightarrow E \wedge \Sigma G_E X$$

is also a homotopy cofiber sequence. But now $E \wedge G_E X \simeq 0$ by prop. 4.31, and so the axioms (def.) of the triangulated structure on $\mathrm{Ho}(\mathrm{Spectra})$ (prop.) imply that $E \wedge \eta$ is an isomorphism. ∎

Nilpotent completion of spectra

Definition 4.33. Let (E, μ, e) be a homotopy commutative ring spectrum (def.) and $Y \in \mathrm{Ho}(\mathrm{Spectra})$ any spectrum. Write $\overline{E}$ for the homotopy fiber of the unit $\mathbb{S} \overset{e}{\to} E$ as in def. 1.16 such that the E-Adams filtration of Y (def. 1.14) reads (according to lemma 1.17)

$$\begin{array}{c} \vdots \\ \downarrow \\ \overline{E}^3 \wedge Y \\ \downarrow \\ \overline{E}^2 \wedge Y \\ \downarrow \\ \overline{E} \wedge Y \\ \downarrow \\ Y \end{array} \,.$$

For $s \in \mathbb{N}$, write

$$\overline{E}_{s-1} \coloneqq \mathrm{hocof}(\overline{E}^s \overset{i^s}{\longrightarrow} \mathbb{S})$$

for the homotopy cofiber. Here $\overline{E}_{-1} \simeq 0$. By the tensor triangulated structure of $\mathrm{Ho}(\mathrm{Spectra})$ (prop.), this homotopy cofiber is preserved by forming smash product with Y, and so also

$$\overline{E}_n \wedge Y \simeq \mathrm{hocof}(\overline{E}^n \wedge Y \longrightarrow Y) \,.$$

Now let

$$\overline{E}_s \overset{p_{s-1}}{\longrightarrow} \overline{E}_{s-1}$$

be the morphism implied by the octahedral axiom of the triangulated category $\mathrm{Ho}(\mathrm{Spectra})$ (def., prop.):

$$\begin{array}{ccccccc} \overline{E}^{s+1} & \overset{i}{\longrightarrow} & \overline{E}^s & \longrightarrow & E \wedge \overline{E}^s & \longrightarrow & \Sigma \overline{E}^{s+1} \\ {}^{=}\downarrow && \downarrow^{i^s} && \downarrow && \downarrow \\ \overline{E}^{s+1} & \longrightarrow & \mathbb{S} & \longrightarrow & \overline{E}_s & \longrightarrow & \Sigma \overline{E}^{s+1} \\ && \downarrow && \downarrow^{p_{s-1}} \\ && \overline{E}_{s-1} & \overset{=}{\longrightarrow} & \overline{E}_{s-1} \\ && \downarrow && \downarrow \\ && \Sigma \overline{E}^s & \longrightarrow & \Sigma E \wedge \overline{E}^s \end{array} \,.$$

By the commuting square in the middle and using again the tensor triangulated structure, this yields an inverse sequence under Y:

$$Y \simeq \mathbb{S} \wedge Y \longrightarrow \cdots \overset{p_3 \wedge \mathrm{id}}{\longrightarrow} \overline{E}_3 \wedge Y \overset{p_2 \wedge \mathrm{id}}{\longrightarrow} \overline{E}_2 \wedge Y \overset{p_1 \wedge \mathrm{id}}{\longrightarrow} \overline{E}_1 \wedge Y$$

The **E-nilpotent completion** $Y_E^\wedge$ of Y is the homotopy limit over the resulting inverse sequence

$$Y_E^\wedge \coloneqq \mathbb{R}\underset{\longleftarrow n}{\lim} \, \overline{E}_n \wedge Y$$

or rather the canonical morphism into it

$$Y \longrightarrow Y^\wedge_E \,.$$

Concretely, if

$$Y \simeq \mathbb{S} \wedge Y \longrightarrow \cdots \overset{p_3 \wedge \mathrm{id}}{\longrightarrow} \overline{E}_3 \wedge Y \overset{p_2 \wedge \mathrm{id}}{\longrightarrow} \overline{E}_2 \wedge Y \overset{p_1 \wedge \mathrm{id}}{\longrightarrow} \overline{E}_1 \wedge Y$$

is presented by a tower of fibrations between fibrant spectra in the model structure on topological sequential spectra, then $Y^\wedge_E$ is represented by the ordinary sequential limit over this tower.

(Bousfield 79, top, middle and bottom of page 272)

Remark 4.34. In (Bousfield 79) the E-nilpotent completion of X (def. 4.33) is denoted "$E^\wedge X$". The notation "$X^\wedge_E$" which we use here is more common among modern authors. It emphasizes the conceptual relation to p-adic completion $A^\wedge_p$ of abelian groups (def. 4.15) and is less likely to lead to confusion with the smash product of E with X.

Remark 4.35. The nilpotent completion $X^\wedge_E$ is E-local. This induces a universal morphism

$$L_E X \longrightarrow X^\wedge_E$$

from the E-Bousfield localization of spectra of X into the E-nilmpotent completion

(Bousfield 79, top of page 273)

We consider now conditions for this morphism to be an equivalence.

Proposition 4.36. *Let E be a connective ring spectrum such that the core of $\pi_0(E)$, def. 2.14, is either of*

- *the localization of the integers at a set J of primes, $c\pi_0(E) \simeq \mathbb{Z}[J^{-1}]$;*
- *a cyclic ring $c\pi_0(E) \simeq \mathbb{Z}/n\mathbb{Z}$, for $n \geq 2$.*

Then the map in remark 4.35 is an equivalence

$$L_E X \overset{\simeq}{\longrightarrow} X^\wedge_E \,.$$

(Bousfield 79, theorem 6.5, theorem 6.6).

Convergence theorems

We state the two main versions of Bousfield's convergence theorems for the E-Adams spectral sequence, below as theorem 4.40 and theorem 4.41.

First we need to define the concepts that enter the convergence statement:

1. the infinity-page $E_\infty^{s,t}(X,Y)$ (def. 4.37),
2. a filtration on $[X, Y^\wedge_E]_\bullet$ (def. 4.38)
3. what it means for the former to converge to the latter (def. 4.39).

Broadly the statement will be that typically

1. the E-Adams spectral sequence $E_r^{s,t}(X,Y)$ computes the stable homotopy groups $[X, Y^\wedge_E]$ of maps from X into the E-nilpotent completion of Y;
2. these groups are localizations of the full groups $[X,Y]_\bullet$ depending on the core of $\pi_0(E)$.

Literature: (Bousfield 79)

Definition 4.37. Let (E, μ, e) be a homotopy commutative ring spectrum (def.) and $X, Y \in \mathrm{Ho}(\mathrm{Spectra})$ two spectra with associated E-Adams spectral sequence $\{E_r^{s,t}, d_r\}$ (def. 1.14).

Observe that

$$\text{if } r > s \text{ then } \; E_{r+1}^{s,\bullet}(X,Y) \simeq \ker(d_r\vert_{E_r^{s,\bullet}(X,Y)}) \subset E_r^{s,\bullet}(X,Y)$$

since the differential d_r on the rth page has bidegree $(r, r-1)$, and since $E_r^{s<0, \bullet(X,Y)} \simeq 0$, so that for $r > s$ the image of d_r in $E_r^{s,t}(X,Y)$ vanishes.

Thus define the bigraded abelian group

$$E_\infty^{s,t}(X,Y) \coloneqq \lim_{r>s} E_r^{s,t}(X,Y) = \underset{r>s}{\cap} E_r^{s,t}(X,Y)$$

called the "**infinity page**" of the E-Adams spectral sequence.

Definition 4.38. Let (E,μ,e) be a homotopy commutative ring spectrum (def.) and $X,Y \in \mathrm{Ho}(\mathrm{Spectra})$ two spectra with associated E-Adams spectral sequence $\{E_r^{s,t}, d_r\}$ (def. 1.14) and E-nilpotent completion $Y_E^\wedge$ (def. 4.33).

Define a filtration

$$\cdots \hookrightarrow F^3[X,Y_E^\wedge]_\bullet \hookrightarrow F^2[X,Y_E^\wedge]_\bullet \hookrightarrow F^1[X,Y_E^\wedge]_\bullet = [X,Y_E^\wedge]_\bullet$$

on the graded abelian group $[X,Y_E^\wedge]_\bullet$ by

$$F^s[X,Y_E^\wedge]_\bullet \; \coloneqq \; \ker(\, [X,Y_E^\wedge]_\bullet \xrightarrow{[X, Y_E^\wedge \to \overline{E}_{s-1} \wedge Y]} [X, \overline{E}_{s-1} \wedge Y]_\bullet \,) \,,$$

where the morphisms $Y_E^\wedge \to \overline{E}_{s-1} \wedge Y$ is the canonical one from def. 4.33.

Definition 4.39. Let (E,μ,e) be a homotopy commutative ring spectrum (def.) and $X,Y \in \mathrm{Ho}(\mathrm{Spectra})$ two spectra with associated E-Adams spectral sequence $\{E_r^{s,t}, d_r\}$ (def. 1.14) and E-nilpotent completion $Y_E^\wedge$ (def. 4.33).

Say that the E-Adams spectral sequence $\{E_r^{s,t}, d_r\}$ **converges completely** to the E-nilpotent completion $[X,Y_E^\wedge]_\bullet$ if the following two canonical morphisms are isomorphisms

1. $[X,Y_E^\wedge]_\bullet \longrightarrow \underset{\longleftarrow s}{\lim} [X,Y_E^\wedge]_\bullet / F^s[X,Y_E^\wedge]_\bullet$

   (where on the right we have the limit over the tower of quotients by the stages of the filtration from def. 4.38)

2. $F^s[X,Y_E^\wedge]_{t-s} / F^{s+1}[X,Y^\wedge]_{t-s} \longrightarrow E_\infty^{s,t}(X,Y) \qquad \forall s,t$

   (where $F^s[X,Y_E^\wedge]_\bullet$ is the filtration stage from def. 4.38 and $E_\infty^{s,t}(X,Y)$ is the infinity-page from def. 4.37).

Notice that the first morphism is always surjective, while the second is necessarily injective, hence the condition is equivalently that the first is also injective, and the second also surjective.

(Bousfield 79, §6)

Now we state sufficient conditions for complete convergence of the E-Adams spectral sequence. It turns out that convergence is controled by the core (def. 2.14) of the ring $\pi_0(E)$. By prop. 2.16 these cores are either localizations of the integers $\mathbb{Z}[J^{-1}]$ at a set J of primes (def. 4.11) or are cyclic rings, or cores of products of these. We discuss the first two cases.

Theorem 4.40. *Let (E,μ,e) be a homotopy commutative ring spectrum (def.) and let $X,Y \in \mathrm{Ho}(\mathrm{Spectra})$ be two spectra such that*

1. *the core (def. 2.14) of the 0-th stable homotopy group ring of E (prop.) is the localization of the integers at a set J of primes (def. 4.11)*

   $$c\pi_0(E) \simeq \mathbb{Z}[J^{-1}] \subset \mathbb{Q}$$

2. *X is a CW-spectrum (def.) with a finite number of cells (rmk.);*

then the E-Adams spectral sequence for $[X,Y]_\bullet$ (def. 1.14) converges completely (def. 4.39) to the localization

$$[X,Y_E^\wedge]_\bullet = \mathbb{Z}[J^{-1}] \otimes [X,Y]_\bullet$$

of $[X,Y]_\bullet$.

(Bousfield 79, theorem 6.5)

Theorem 4.41. *Let (E,μ,e) be a homotopy commutative ring spectrum (def.) and let $X,Y \in \mathrm{Ho}(\mathrm{Spectra})$ be two spectra such that*

1. *the core (def. 2.14) of the 0-th stable homotopy group ring of E (prop.) is a prime field*

$c\pi_0(E) \simeq \mathbb{F}_p$

for some prime number p;

2. *Y is a connective spectrum in that its stable homotopy groups $\pi_\bullet(Y)$ vanish in negative degree;*

3. *X is a CW-spectrum (def.) with a finite number of cells (rmk.);*

4. *$[X,Y]_\bullet$ is degreewise a finitely generated group*

then the E-Adams spectral sequence for $[X,Y]_\bullet$ (def. 1.14) converges completely (def. 4.39) to the p-adic completion (def. 4.15)

$$[X,Y_E^\wedge]_\bullet \simeq \underset{\longleftarrow_n}{\lim} [X,Y]_\bullet / p^n [X,Y]_\bullet$$

of $[X,Y]_\bullet$.

(Bousfield 79, theorem 6.6)

Examples

We now consider examples applying the general theory of E-Adams spectral sequences above in special cases to the concrete computation of certain stable homotopy groups.

Example 4.42. Examples of commutative ring spectra that are flat according to def. 2.1 include $E =$

- $\mathbb{S}$ – the sphere spectrum;
- $H\mathbb{F}_p$ – Eilenberg-MacLane spectra for prime fields;
- MO, MU, MSp – Thom spectra;
- KO, KU – topological K-theory spectra.

(Adams 69, lecture 1, lemma 28 (p. 45))

Proof of the first two items. For $E = \mathbb{S}$ we have $\mathbb{S}_\bullet(\mathbb{S}) \coloneqq \pi_\bullet(\mathbb{S} \wedge \mathbb{S}) \simeq \pi_\bullet(\mathbb{S})$, since the sphere spectrum $\mathbb{S}$ is the tensor unit for the derived smash product of spectra (cor.). Hence the statement follows since every ring is, clearly, flat over itself.

For $E = H\mathbb{F}_p$ we have that $\pi_\bullet(H\mathbb{F}_p) \simeq \mathbb{F}_p$ (prop.), hence a field (a prime field). Every module over a field is a projective module (prop.) and every projective module is flat (prop.). ▮

Example 4.43. Examples of ring spectra that are *not* flat in the sense of def. 2.1 include HZ, and MSU.

Examples 4.44.

- For $X = \mathbb{S}$ and $E = H\mathbb{F}_p$, then theorem 3.1 and theorem \ref{ConvergenceOfEAdamsSpectralSequenceToECompletion} with example \ref{ExamplesOfEnilpotentLocalizations} gives a spectral sequence

  $$\mathrm{Ext}_{\mathcal{A}_p^*}(\mathbb{F}_p, \mathbb{F}_p) \;\Rightarrow\; \pi_\bullet(\mathbb{S}) \otimes Z_p^\wedge \,.$$

  This is the *classical Adams spectral sequence*.

- For $X = \mathbb{S}$ and $E =$ MU, then theorem 3.1 and theorem \ref{ConvergenceOfEAdamsSpectralSequenceToECompletion} with example \ref{ExamplesOfEnilpotentLocalizations} gives a spectral sequence

  $$\mathrm{Ext}_{\mathrm{MU}_*(\mathrm{MU})}(\mathrm{MU}_*, \mathrm{MU}_*) \;\Rightarrow\; \pi_\bullet(\mathbb{S}) \,.$$

  This is the *Adams-Novikov spectral sequence*.

5. Classical Adams spectral sequence ($E = H\mathbb{F}_2$, $X = \mathbb{S}$)

We consider now the example of the E-Adams spectral sequence $\{E_r^{s,t}(X,Y), d_r\}$ (def. 1.14) for the case that

1. $E = H\mathbb{F}_p$ is the Eilenberg-MacLane spectrum (def.) with coefficients in a prime field, regarded in $\mathrm{Ho(Spectra)}$ with its canonical struture of a homotopy commutative ring spectrum induced (via this corollary) from its canonical structure of an orthogonal ring spectrum (from this def.);

2. $X = Y = \mathbb{S}$ are both the sphere spectrum.

This example is called the *classical Adams spectral sequence*.

The $H\mathbb{F}_p$-dual Steenrod algebra according to the general definition 2.3 turns out to be the classical dual Steenrod algebra $\mathcal{A}_p^*$ recalled below .

Notice that $H\mathbb{F}_2$ satisfies the two assumptions needed to identify the second page of the $H\mathbb{F}_p$-Adams spectral sequence according to theorem 3.1:

Lemma 5.1. *The Eilenberg-MacLane spectrum $H\mathbb{F}_p$ is flat according to 2.1, and $H\mathbb{F}_p(\mathbb{S})$ is a projective module over $\pi_\bullet(H\mathbb{F}_p)$.*

Proof. The stable homotopy groups of $H\mathbb{F}_p$ is the prime field $\mathbb{F}_p$ itself, regarded as a graded commutative ring concentrated in degree 0 (prop.)

$$\pi_\bullet(H\mathbb{F}_p) \simeq \mathbb{F}_p \;.$$

Since this is a field, all modules over it are projective modules (prop.), hence in particular flat modules (prop.). ▮

Corollary 5.2. *The classical Adams spectral sequence, i.e. the E-Adams spectral sequence (def. 1.14) for $E = H\mathbb{F}_p$ (def.) and $X = Y = \mathbb{S}$, has on its second page the Ext-groups of classical dual Steenrod algebra comodules from $\mathbb{F}_p \simeq H\mathbb{F}_p(\mathbb{S})$ to itself, and converges completely (def. 4.39) to the p-adic completion (def. 4.15) of the stable homotopy groups of spheres, hence in degree 0 to the p-adic integers and in all other degrees to the p-primary part (theorem 4.1)*

$$E_2^{s,t}(\mathbb{S},\mathbb{S}) \;=\; \mathrm{Ext}^{s,t}_{\mathcal{A}_p^*}(\mathbb{F}_p, \mathbb{F}_p) \;\Rightarrow\; (\pi_\bullet(\mathbb{S}))_p \;.$$

Proof. By lemma 5.1 the conditions of theorem 3.1 are satisfied, which implies the form of the second page.

For the convergence statement, we check the assumptions in theorem 4.41:

1. By prop. 2.15 and prop. 2.16 the ring $\mathbb{F}_p = \pi_0(H\mathbb{F}_p)$ coincides with its core: $c\mathbb{F}_p \simeq \mathbb{F}_p$;

2. $\mathbb{S}$ is clearly a connective spectrum;

3. $\mathbb{S}$ is clearly a finite CW-spectrum;

4. the groups $\pi_\bullet(\mathbb{S}) \simeq [\mathbb{S},\mathbb{S}]_\bullet$ are degreewise finitely generated, by *Serre's finiteness theorem?*.

Hence theorem 4.41 applies and gives the convergence as stated.

Finally, by prop. 5.5 the dual E-Steenrod algebra in the present case is the classical dual Steenrod algebra. ▮

We now use the classical Adams spectral sequence from corollary 5.2 to compute the first dozen stable homotopy groups of spheres.

The dual Steenrod algebra

Definition 5.3. Let p be a prime number. Write $\mathbb{F}_p$ for the corresponding prime field.

The **mod p-Steenrod algebra** $\mathcal{A}_p$ is the graded co-commutative Hopf algebra over $\mathbb{F}_p$ which is

- for $p = 2$ generated by elements denoted Sq^n for $n \in \mathbb{N}$, $n \geq 1$;

- for $p > 2$ generated by elements denoted β and P^n for $\in \mathbb{N}$, $n \geq 1$

(called the **Serre-Cartan basis elements**)

whose product is subject to the following relations (called the **Ádem relations**):

for $p = 2$:

for $0 < h < 2k$ the

$$\mathrm{Sq}^h \mathrm{Sq}^k \;=\; \sum_{i=0}^{[h/2]} \binom{k-i-1}{h-2i} \mathrm{Sq}^{h+k-i} \mathrm{Sq}^i ,$$

for $p > 2$:

for $0 < h < pk$ then

$$P^h P^k \;=\; \sum_{i=0}^{[h/p]} (-1)^{h+i} \binom{(p-1)(k-i)-1}{h-pi} P^{h+k-i} P^i$$

and if $0 < h < pk$ then

$$P^h \beta P^k = \sum_{[h/p]}^{i=0} (-1)^{h+i} \binom{(p-1)(k-i)}{h-pi} \beta P^{h+k-i} P^i + \sum_{[(h-1)/p]}^{i=0} (-1)^{h+i-1} \binom{(p-1)(k-i)-1}{h-pi-1} P^{h+k-i} \beta P^i$$

and whose coproduct Ψ is subject to the following relations:

for $p = 2$:

$$\Psi(\mathrm{Sq}^n) \;=\; \sum_{k=0}^{n} \mathrm{Sq}^k \otimes \mathrm{Sq}^{n-k}$$

for $p > 2$:

$$\Psi(P^n) \;=\; \sum_{n}^{k=0} P^k \otimes P^{n-k}$$

and

$$\Psi(\beta) = \beta \otimes 1 + 1 \otimes \beta \;.$$

e.g. (Kochmann 96, p. 52)

Definition 5.4. The $\mathbb{F}_p$-linear dual of the mod p-Steenrod algebra (def. 5.3) is itself naturally a graded commutative Hopf algebra (with coproduct the linear dual of the original product, and vice versa), called the **dual Steenrod algebra** $\mathbb{A}^*_{\mathbb{F}_p}$.

Proposition 5.5. *There is an isomorphism*

$$\mathcal{A}^*_p \simeq H_\bullet(H\mathbb{F}_p, \mathbb{F}_p) = \pi_\bullet(H\mathbb{F}_p \wedge H\mathbb{F}_p) \;.$$

(e.g. Ravenel 86, p. 49, Rognes 12, remark 7.24)

We now give the generators-and-relations description of the dual Steenrod algebra $\mathcal{A}^*_p$ from def. 5.4, in terms of linear duals of the generators for $\mathcal{A}_p$ itself, according to def. 5.3.

Theorem 5.6. ***(Milnor's theorem)***

The dual mod 2-Steenrod algebra $\mathcal{A}^*_2$ (def. 5.4) is, as an associative algebra, the free graded commutative algebra

$$\mathcal{A}^*_p \simeq \mathrm{Sym}_{\mathbb{F}_p}(\xi_1, \xi_2, \cdots,)$$

on generators:

- ξ_n, $n \geq 1$ *being the linear dual to* $\mathrm{Sq}^{p^{n-1}} \mathrm{Sq}^{p^{n-2}} \cdots \mathrm{Sq}^p \mathrm{Sq}^1$,

  of degree $2^n - 1$.

The dual mod p-Steenrod algebra $\mathcal{A}^*_p$ (def. 5.4) is, as an associative algebra, the free graded commutative algebra

$$\mathcal{A}^*_p \simeq \mathrm{Sym}_{\mathbb{F}_p}(\xi_1, \xi_2, \cdots, \tau_0, \tau_1, \cdots)$$

on generators:

- ξ_n, $n \geq 1$ *being the linear dual to* $P^{p^{n-1}} P^{p^{n-2}} \cdots P^p P^1$, *of degree $2(p^n - 1)$.*

- *τ_n being linear dual to $P^{p^{n-1}} P^{p^{n-2}} \cdots P^p P^1 \beta$.*

Moreover, the coproduct on $\mathcal{A}_p^*$ is given on generators by

$$\Psi(\xi_n) = \sum_{k=0}^{n} \xi_{n-k}^{p^k} \otimes \xi_k$$

and

$$\Psi(\tau_n) = \tau_n \otimes 1 + \sum_{k=0}^{n} \xi_{n-k}^{p^k} \xi_{n-k}^{p^k} \otimes \tau_k \,,$$

where we set $\xi_0 \coloneqq 1$.

(This defines the coproduct on the full algbra by it being an algebra homomorphism.)

This is due to (Milnor 58). See for instance (Kochmann 96, theorem 2.5.1, Ravenel 86, chapter III, theorem 3.1.1)

The cobar complex

In order to compute the second page of the classical $H\mathbb{F}_p$-Adams spectral sequence (cor. 5.2) we consider a suitable cochain complex whose cochain cohomology gives the relevant Ext-groups.

Definition 5.7. Let (Γ, A) be a graded commutative Hopf algebra, hence a commutative Hopf algebroid for which the left and right units coincide $\eta : A \longrightarrow \Gamma$ (remark 2.8).

Then the **unit coideal** of Γ is the cokernel

$$\overline{\Gamma} \coloneqq \mathrm{coker}(A \overset{\eta}{\longrightarrow} \Gamma) \,.$$

Remark. By co-unitality of graded commutative Hopf algebras (def. 2.9) $\epsilon \circ \eta = \mathrm{id}_A$ the defining projection of the unit coideal (def. 5.7)

$$A \overset{\eta}{\longrightarrow} \Gamma \longrightarrow \overline{\Gamma}$$

forms a split exact sequence which exhibits a direct sum decomposition

$$\Gamma \simeq A \oplus \overline{\Gamma} \,.$$

Lemma 5.8. *Let (Γ, A) be a commutative Hopf algebra, hence a commutative Hopf algebroid for which the left and right units coincide $\eta : A \longrightarrow \Gamma$.*

Then the unit coideal $\overline{\Gamma}$ (def. 5.7) carries the structure of an A-bimodule such that the projection morphism

$$\Gamma \longrightarrow \overline{\Gamma}$$

is an A-bimodule homomorphism. Moreover, the coproduct $\Psi : \Gamma \longrightarrow \Gamma \otimes_A \Gamma$ descends to a morphism $\overline{\Gamma} : \overline{\Gamma} \longrightarrow \overline{\Gamma} \otimes_A \overline{\Gamma}$ such that the projection intertwines the two coproducts.

Proof. For the first statement, consider the commuting diagram

$$\begin{array}{ccccc} A \otimes A & \overset{A \otimes \eta}{\longrightarrow} & A \otimes \Gamma & \longrightarrow & A \otimes \overline{\Gamma} \\ \downarrow && \downarrow && \downarrow^{\exists} \\ A & \underset{\eta}{\longrightarrow} & \Gamma & \longrightarrow & \overline{\Gamma} \end{array} \,,$$

where the left commuting square exhibits the fact that η is a homomorphism of left A-modules.

Since the tensor product of abelian groups $\otimes$ is a right exact functor it preserves cokernels, hence $A \otimes \overline{\Gamma}$ is the cokernel of $A \otimes A \to A \otimes \Gamma$ and hence the right vertical morphisms exists by the universal property of cokernels. This is the compatible left module structure on $\overline{\Gamma}$. Similarly the right A-module structure is obtained.

For the second statement, consider the commuting diagram

$$\begin{array}{ccccc} A & \overset{\eta}{\longrightarrow} & \Gamma & \longrightarrow & \overline{\Gamma} \\ {}^{\eta}\downarrow && \downarrow^{\Psi} && \downarrow^{\exists} \\ \Gamma \simeq \Gamma \otimes_A A & \underset{\mathrm{id} \otimes_A \eta}{\longrightarrow} & \Gamma \otimes_A \Gamma & \longrightarrow & \overline{\Gamma} \otimes_A \overline{\Gamma} \end{array} \,.$$

Here the left square commutes by one of the co-unitality conditions on (Γ, A), equivalently this is the co-action property of A regarded canonically as a Γ-comodule.

Since also the bottom morphism factors through zero, the universal property of the cokernel $\overline{\Gamma}$ implies the existence of the right vertical morphism as shown. ▮

Definition 5.9. **(cobar complex)**

Let (Γ, A) be a commutative Hopf algebra, hence a commutative Hopf algebroid for which the left and right units coincide $A \overset{\eta}{\longrightarrow} \Gamma$. Let N be a left Γ-comodule.

The **cobar complex** $C^\bullet_\Gamma(N)$ is the cochain complex of abelian groups with terms

$$C^s_\Gamma(N) \coloneqq \underbrace{\overline{\Gamma} \otimes_A \cdots \otimes_A \overline{\Gamma}}_{s \text{ factors}} \otimes_A N$$

(for $\overline{\Gamma}$ the unit coideal of def. 5.7, with its A-bimodule structure via lemma 5.8)

and with differentials $d_s \colon C^s_\Gamma(N) \longrightarrow C^{s+1}_\Gamma(N)$ given by the alternating sum of the coproducts via lemma 5.8.

(Ravenel 86, def. A1.2.11)

Proposition 5.10. *Let (Γ, A) be a commutative Hopf algebra, hence a commutative Hopf algebroid for which the left and right units coincide $A \overset{\eta}{\longrightarrow} \Gamma$. Let N be a left Γ-comodule.*

Then the cochain cohomology of the cobar complex $C^\bullet_\Gamma(N)$ (def. 5.9) is the Ext-groups of comodules from A (regarded as a left comodule via def. 2.20) into N

$$H^\bullet(C^\bullet_\Gamma(N)) \;\simeq\; \mathrm{Ext}^\bullet_\Gamma(A, N) \,.$$

(Ravenel 86, cor. A1.2.12, Kochman 96, prop. 5.2.1)

Proof idea. One first shows that there is a resolution of N by co-free comodules given by the complex

$$D^\bullet_\Gamma(N) \coloneqq \Gamma \otimes_A \overline{\Gamma}^{\otimes_A^\bullet} \otimes_A N$$

with differentials given by the alternating sum of the coproducts. This is called the *cobar resolution* of N.

To see that this is indeed a resolution, one observes that a contracting homotopy is given by

$$s(\gamma\gamma_1 | \cdots | \gamma_s n) \coloneqq \epsilon(\gamma)\gamma_1 | \cdots | \gamma_s n$$

for $s > 0$ and

$$s(\gamma n) \coloneqq 0 \,.$$

Now from lemma 3.5, in view of remark , and since A is trivially projective over itself, it follows that this is an F-acyclic resolution for $F \coloneqq \mathrm{Hom}_\Gamma(A, -)$.

This means that the resolution serves to compute the Ext-functor in question and we get

$$\begin{aligned} \mathrm{Ext}^\bullet_\Gamma(A, N) & \simeq H^\bullet(\mathrm{Hom}_\Gamma(A, D^\bullet_\Gamma(N))) \\ & = H^\bullet(\mathrm{Hom}_\Gamma(A, \Gamma \otimes_A \overline{\Gamma}^{\otimes_A^\bullet} \otimes_A N)) \\ & \simeq H^\bullet(\mathrm{Hom}_A(A, \overline{\Gamma}^{\otimes_A^\bullet} \otimes_A N)) \\ & \simeq H^\bullet(\overline{\Gamma}^{\otimes_A^\bullet} \otimes_A N) \,, \end{aligned} \,,$$

where the second-but-last equivalence is the isomorphism of the co-free/forgetful adjunction

$$A\,\mathrm{Mod} \underoverset{\underset{\mathrm{co-free}}{\longrightarrow}}{\overset{\mathrm{forget}}{\longleftarrow}}{\bot} \Gamma\,\mathrm{CoMod}$$

from prop. 2.23, while the last equivalence is the isomorphism of the free/forgetful adjunction

$$A\,\mathrm{Mod} \underoverset{\underset{\mathrm{forget}}{\longrightarrow}}{\overset{\mathrm{free}}{\longleftarrow}}{\bot} \mathrm{Ab}$$

■

The May spectral sequence

The cobar complex (def. 5.9) realizes the second page of the classical Adams spectral sequence (cor. 5.2) as the cochain cohomology of a cochain complex. This is still hard to compute directly, but we now discuss that this cochain complex admits a filtration so that the induced spectral sequence of a filtered complex is computable and has trivial extension problem (rmk.). This is called the *May spectral sequence*.

We obtain this spectral sequence in prop. 5.16 below. First we need to consider some prerequisites.

Lemma 5.11. *Let (Γ, A) be a graded commutative Hopf algebra, i.e. a graded commutative Hopf algebroid with left and right unit coinciding for which the underlying A-algebra of Γ is a free graded commutative A-algebra on a set of generators $\{x_i\}_{i \in I}$*

such that

1. *all generators x_i are primitive elements;*
2. *A is in degree 0;*
3. *$(i < j) \Rightarrow (\deg(x_i) \leq \deg(x_j))$;*
4. *there are only finitely many x_i in a given degree,*

then the Ext of Γ-comodules from A to itself is the free graded commutative algebra on these generators

$$\mathrm{Ext}_\Gamma(A, A) \simeq A[\{x_i\}_{i \in I}] \,.$$

(Ravenel 86, lemma 3.1.9, Kochman 96, prop. 3.7.5)

Proof. Consider the co-free left Γ-comodule (prop.)

$$\Gamma \otimes_A A[\{y_i\}_{i \in I}]$$

and regard it as a chain complex of left comodules by defining a differential via

$$\begin{aligned} d &: x_i \mapsto y_i \\ d &: y_i \mapsto 0 \end{aligned}$$

and extending as a graded derivation.

We claim that d is a homomorphism of left comodules: Due to the assumption that all the x_i are primitive we have on generators that

$$\begin{aligned} (\mathrm{id}, d)(\Psi(x_i)) &= (\mathrm{id}, d)(x_i \otimes 1 + 1 \otimes x_i) \\ &= x_i \otimes \underbrace{(d1)}_{=0} + 1 \otimes \underbrace{(dx_i)}_{=y_i} \\ &= \Psi(dx_i) \end{aligned}$$

and

$$\begin{aligned} (\mathrm{id}, d)(\Psi(y_i)) &= (\mathrm{id}, d)(1, y_i) \\ &= (1, dy_i) \\ &= 0 \\ &= \Psi(0) \\ &= \Psi(dy_i) \end{aligned} \,.$$

Since d is a graded derivation on a free graded commutative algbra, and Ψ is an algebra homomorphism, this implies the statement for all other elements.

Now observe that the canonical chain map

$$(\Gamma \otimes_A A[\{y_i\}_{i \in I}],\ d) \overset{\simeq_{\mathrm{qi}}}{\longrightarrow} A$$

(which projects out the generators x_i and y_i and is the identity on A), is a quasi-isomorphism, by construction. Therefore it constitutes a co-free resolution of A in left Γ-comodules.

Since the counit η is assumed to be flat, and since $A[\{y_i\}_{i \in I}]$ is degreewise a free module over A, hence in particular a projective module, prop. 3.5 says that the above is an acyclic resolution with respect to the functor $\mathrm{Hom}_\Gamma(A,-) : \Gamma\,\mathrm{CoMod} \longrightarrow A\,\mathrm{Mod}$. Therefore it computes the Ext-functor. Using that forming co-free comodules is right adjoint to forgetting Γ-comodule structure over A (prop. 2.23), this yields:

$$\begin{aligned} \mathrm{Ext}^\bullet_\Gamma(A,A) & \simeq H_\bullet(\mathrm{Hom}_\Gamma(A, \Gamma \otimes_A A[\{y_i\}_{i \in I}]), d) \\ & \simeq H_\bullet(\mathrm{Hom}_A(A, A[\{y_i\}_{i \in I}]), d = 0) \\ & \simeq \mathrm{Hom}_A(A, A[\{y_i\}_{i \in I}]) \\ & \simeq A[\{x_i\}_{i \in I}] \end{aligned} .$$

■

Lemma 5.12. *If (Γ, A) as above is equipped with a filtering, then there is a spectral sequence*

$$\mathcal{E}_1 \;=\; \mathrm{Ext}_{\mathrm{gr}_\bullet \Gamma}(\mathrm{gr}_\bullet A, \mathrm{gr}_\bullet A) \;\Rightarrow\; \mathrm{Ext}_\Gamma(A,A)$$

converging to the Ext over Γ from A to itself, whose first page is the Ext *over the associated graded Hopf algebra $\mathrm{gr}_\bullet \Gamma$.*

(Ravenel 86, lemma 3.1.9, Kochman 96, prop. 3.7.5)

Proof. The filtering induces a filtering on the cobar complex (def. 5.9) which computes Ext_Γ (prop. 5.10). The spectral sequence in question is the corresponding spectral sequence of a filtered complex. Its first page is the homology of the associated graded complex (by this prop.), which hence is the homology of the cobar complex (def. 5.9) of the associated graded Hopf algebra $\mathrm{gr}_\bullet \Gamma$. By prop. 5.10 this is the Ext-groups as shown. ■

Let now $A \coloneqq \mathbb{F}_2$, $\Gamma \coloneqq \mathcal{A}_2^\bullet$ be the mod 2 dual Steenrod algebra. By Milnor's theorem (prop. 5.6), as an $\mathbb{F}_2$-algebra this is

$$\mathcal{A}_2^\bullet = \mathrm{Sym}_{\mathbb{F}_2}(\xi_1, \xi_2, \cdots) \,.$$

and the coproduct is given by

$$\Psi(\xi_n) = \sum_{k=0}^{i} \xi_{i-k}^{2^k} \otimes \xi_k \,,$$

where we set $\xi_0 \coloneqq 1$.

Definition 5.13. Introduce new generators

$$h_{i,n} \coloneqq \begin{cases} \xi_i^{2^n} & \text{for } i \geq 1,\, k \geq 0 \\ 1 & \text{for } i = 0 \end{cases}$$

Remark 5.14. By binary expansion of powers, there is a unique way to express every monomial in $\mathbb{F}_2[\xi_1, \xi_2, \cdots]$ as a product of the new generators in def. 5.13 such that each such element appears at most once in the product. E.g.

$$\begin{aligned} \xi_i^5 \xi_j^7 & = \xi_i^{2^0 + 2^2} \xi_j^{2^0 + 2^1 + 2^2} \\ & = h_{i,0} h_{i,1} h_{j,0} h_{j,1} h_{j,2} \end{aligned} .$$

Proposition 5.15. *In terms of the generators $\{h_{i,n}\}$ from def. 5.13, the coproduct on the dual Steenrod algebra $\mathcal{A}_2^\ast$ takes the following simple form*

$$\Psi(h_{i,n}) \;=\; \sum_{k=0}^{i} h_{i-k, n+k} \otimes h_{k,n} \,.$$

Proof. Using that the coproduct of a bialgebra is a homomorphism for the algebra structure and using freshman's dream arithmetic over $\mathbb{F}_2$, one computes:

$$
\begin{aligned}
\Psi(h_{i,n}) &= \Psi\left(\xi_i^{2^n}\right) \\
&= (\Psi(\xi_i))^{2^n} \\
&= \left(\sum_{k=0}^{i} \xi_{i-k}^{2^k} \otimes \xi_k\right)^{2^n} \\
&= \sum_{k=0}^{i} \left(\xi_{i-k}^{2^k}\right)^{2^n} \otimes \xi_k^{2^n} \\
&= \sum_{k=0}^{i} \xi_{i-k}^{2^k \cdot 2^n} \otimes \xi_k^{2^n} \\
&= \sum_{k=0}^{i} \xi_{i-k}^{2^{(k+n)}} \otimes \xi_k^{2^n} \\
&= \sum_{k=0}^{i} h_{i-k,n+k} \otimes h_{k,n}
\end{aligned}
.
$$

∎

Proposition 5.16. *There exists a converging spectral sequence of graded $\mathbb{F}_2$-vector spaces of the form*

$$
E_1^{s,t,p} = \mathbb{F}_2[\{h_{i,n}\}_{\substack{i \geq 1, \\ n \geq 0}}] \;\Rightarrow\; \mathrm{Ext}^{s,t}_{\mathcal{A}_2^*}(\mathbb{F}_2, \mathbb{F}_2)\,,
$$

called the ***May spectral sequence*** *(where s and t are from the bigrading of the spectral sequence itself, while the index p is that of the graded $\mathbb{F}_2$-vector spaces), with*

1. *$h_{i,n} \in E_1^{1, 2^{2^{i+n}} - 2^n - 1, 2i-1}$*
2. *first differential given by*

$$
d_1(h_{i,n}) = \sum_{k=0}^{i} h_{i-k,n+k} \otimes h_{k,n}\,;
$$

3. *higher differentials of the form*

$$
d_r \,:\, E_r^{s,t,p} \longrightarrow E_r^{s+1, t-1, p-2r+1}\,,
$$

where the filtration is by maximal degree.

Notice that since everything is $\mathbb{F}_2$-linear, the extension problem of this spectral sequence is trivial.

(Kochman 96, prop. 5.3.1)

Proof. Define a grading on the dual Steenrod algebra $\mathcal{A}_2^\bullet$ (theorem 5.6) by taking the degree of the generators from def.5.13 to be (this idea is due to (Ravenel 86, p.69))

$$
|h_{i,n}| \coloneqq 2i - 1
$$

and extending this additively to monomials, via the unique decomposition of remark 5.14.

For example

$$
\begin{aligned}
|\xi_i^5 \xi_j^7| &= |h_{i,0} h_{i,1} h_{j,0} h_{j,1} h_{j,2}| \\
&= 2(2i-1) + 3(2j-1)
\end{aligned}
.
$$

Consider the corresponding increasing filtration

$$
\cdots \subset F_p \mathcal{A}_2^* \subset F_{p+1} \mathcal{A}_2^* \subset \cdots \subset \mathcal{A}_2^*
$$

with filtering stage p containing all elements of total degree $\leq p$.

Observe via prop. 5.15 that

$$
\Psi(h_{i,n}) = \underbrace{h_{i,n} \otimes 1}_{\deg = 2i-1} + \sum_{0 < k < i} \underbrace{h_{i-k,n+k} \otimes h_{k,n}}_{\deg = 2i-2} + \underbrace{1 \otimes h_{i,n}}_{\deg = 2i-1}\,.
$$

This means that after projection to the associated graded Hopf algebra

$$
F_\bullet \mathcal{A}_2^* \longrightarrow \mathrm{gr}_\bullet \mathcal{A}_2^* \coloneqq F_\bullet(\mathcal{A}_2^*)/F_{\bullet -1}(\mathcal{A}_2^*)
$$

all the generators $h_{i,n}$ become primitive elements:

$$
\Psi(h_{i,n}) = h_{i,n} \otimes 1 + 1 \otimes h_{i,n} \qquad \in \mathrm{gr}_\bullet \mathcal{A}_2^* \otimes \mathrm{gr}_\bullet \mathcal{A}_2^*\,.
$$

Hence lemma 5.11 applies and says that the Ext from $\mathbb{F}_2$ to itself over the associated graded Hopf algebra is the polynomial algebra in these generators:

$$\mathrm{Ext}_{\mathrm{gr}_\bullet \mathcal{A}_2^*}(\mathbb{F}_2, \mathbb{F}_2) \simeq \mathbb{F}_2[\{h_{i,n}\}_{\substack{i \geq 1, \\ n \geq 0}}] \,.$$

Moreover, lemma 5.12 says that this is the first page of a spectral sequence that converges to the Ext over the original Hopf algebra:

$$\mathcal{E}_1 = \mathbb{F}_2[\{h_{i,n}\}_{\substack{i \geq 1 \\ n \geq 0}}] \;\Rightarrow\; \mathrm{Ext}_{\mathcal{A}_2^*}(\mathbb{F}_2, \mathbb{F}_2) \,.$$

Moreover, again by lemma 5.12, the differentials on any r-page are the restriction of the differentials of the bar complex to the r-almost cycles (prop.). Now the differential of the cobar complex is the alternating sum of the coproduct on $\mathcal{A}_2^*$, hence by prop. 5.15 this is:

$$d_1(h_{i,n}) = \sum_{k=0}^{i} h_{i-k,n+k} \otimes h_{k,n} \,.$$

∎

The second page

Now we use the May spectral sequence (prop. 5.16) to compute the second page and in fact the stable page of the classical Adams spectral sequence (cor. 5.2) in low internal degrees $t - s$.

Lemma 5.17. ***(terms on the second page of May spectral sequence)***

In the range $t - s \leq 13$, the second page of the May spectral sequence for $\mathrm{Ext}_{\mathbb{A}_{\mathbb{F}_2}^*}(\mathbb{F}_2, \mathbb{F}_2)$ has as generators all the

- h_n
- $b_{i,n} \coloneqq (h_{i,n})^2$

as well as the element

- $x_7 \coloneqq h_{2,0} h_{2,1} + h_{1,1} h_{3,0}$

subject to the relations

- $h_n h_{n+1} = 0$
- $h_2 b_{2,0} = h_0 x_7$
- $h_2 x_7 = h_0 b_{2,1}$.

e.g. (Ravenel 86, lemma 3.2.8 and lemma 3.2.10, Kochman 96, lemma 5.3.2)

Proof. Remember that the differential in the cobar complex (def. 5.9) lands not in $\Gamma = \mathcal{A}_2^*$ itself, but in the unit coideal $\overline{\Gamma} \coloneqq \mathrm{coker}(\eta)$ where the generator $h_{0,n} = \xi_0 = 1$ disappears.

Using this we find for the differential d_1 of the generators in low degree on the first page of the May spectral sequence (prop. 5.16) via the formula for the differential from prop. 5.15, the following expressions:

$$\begin{aligned} d_1(h_n) & \coloneqq d_1(h_{1,n}) \\ & = \overline{\Psi}(h_{1,n}) \\ & = h_{1,n} \otimes \underbrace{h_{0,n}}_{=0} + \underbrace{h_{0,n+1}}_{=0} \otimes h_{1,n} \\ & = 0 \end{aligned}$$

and hence all the elements h_n are cocycles on the first page of the May spectral sequence.

Also, since d_1 is a derivation (by definition of the cobar complex, def. 5.9) and since the product of the image of the cobar complex in the first page of the May spectral sequence is graded commutative, we have for all n, k that

$$\begin{aligned} d_1(h_{n,k})^2 & = 2 h_{n,k}(d_1(h_{n,k})) \\ & = 0 \end{aligned}$$

(since $2 = 0 \bmod 2$).

Similarly we compute d_1 on the other generators. These terms do not vanish, but so they impose relations on products in the cobar complex:

$$
\begin{aligned}
d_1(h_{2,0}) &= h_{1,1} \otimes h_{1,0} \\
d_1(h_{2,1}) &= h_{1,2} \otimes h_{1,1} \\
d_1(h_{2,2}) &= h_{1,3} \otimes h_{1,2} \\
d_1(h_{2,3}) &= h_{1,4} \otimes h_{1,3} \\
d_1(h_{3,0}) &= h_{2,1} \otimes h_{1,0} + h_{1,2} \otimes h_{2,0}
\end{aligned}
$$

This shows that $h_n h_{n+1} = 0$ in the given range.

The remaining statements follow similarly. ■

Remark 5.18. With lemma 5.17, so far we see the following picture in low degrees.

$$
\begin{array}{cccccc}
\vdots & \vdots & & & & \\
3 & h_0^4 & & & h_1^3,\ h_0^2 h_2 & \\
2 & h_0^2 & & h_1^2 & h_0 h_2 & \\
1 & h_0 & h_1 & & h_2 & \\
 & 0 & 1 & 2 & 3 & 4
\end{array}
$$

Here the relation

$$
h_0 \otimes h_1 = 0
$$

removes a vertical tower of elements above h_1.

So far there are two different terms in degree $(s, t-s) = (3,3)$. The next lemma shows that these become identified on the next page.

Lemma 5.19. ***(differentials on the second page of the May spectral sequence)***

The differentials on the second page of the May spectral sequence (prop. 5.16) relevant for internal degrees $t - s \leq 12$ are

1. $d_2(h_n) = 0$
2. $d_2(b_{2,n}) = h_n^2 h_{n+2} + h_{n+1}^3$
3. $d_2(x_7) = h_0 h_2^2$
4. $d_2(b_{3,0}) = h_1 b_{2,1} + h_3 b_{2,0}$

(Kochman 96, lemma 5.3.3)

Proof. The first point follows as before in lemma 5.17, in fact the h_n are infinite cycles in the May spectral sequence.

We spell out the computation for the second item:

We may represent $b_{2,k}$ by $\xi_2^{2^k} \times \xi_2^{2^k}$ plus terms of lower degree. Choose the representative

$$
B_{2,k} = \xi_2^{2^k} \otimes \xi_2^{2^k} \;+\; \xi_1^{2^{k+1}} \otimes \xi_1^{2^k} \xi_2^{2^k} \;+\; \xi_1^{2^{k+1}} \xi^{2^k} \otimes \xi_1^{2^k} \,.
$$

Then we compute $d B_{2,k}$, using the definition of the cobar complex (def. 5.9), the value of the coproduct on dual generators (theorem 5.6), remembering that the coproduct Ψ on a Hopf algebra is a homomorphism for the underlying commutative ring, and using freshman's dream arithmetic to evaluate prime-2 powers of sums. For the three summands we obtain

$$
\begin{aligned}
d(\xi_2^{2^k} \otimes \xi_2^{2^k}) &= \overline{\Psi}(\xi_2^{2^k}) \otimes \xi_2^{2^k} + \xi_2^{2^k} \otimes \overline{\Psi}(\xi_2^{2k}) \\
&= \underbrace{\xi_1^{2^{k+1}} \otimes \xi_1^{2^k} \otimes \xi_2^{2^k}}_{c_1} \;+\; \underbrace{\xi_2^{2^k} \otimes \xi_1^{2^{k+1}} \otimes \xi_1^{2^k}}_{c_2}
\end{aligned}
$$

and

$$
\begin{aligned}
d(\xi_1^{2^{k+1}} \otimes \xi_1^{2^k}\xi_2^{2^k}) &= \xi_1^{2^k} \otimes \overline{\Psi}(\xi_1^{2^k}\xi_2^{2^k}) \\
&= \xi_1^{2^{k+1}} \otimes \left(\xi_1^{2^k} \otimes 1 + 1 \otimes \xi_1^{2^k}\right)\left(\xi_2^{2^k} \otimes 1 + \xi_1^{2^{k+1}} \otimes \xi_1^{2^k} + 1 \otimes \xi_2^{2^k}\right) \\
&= \underbrace{\xi_1^{2^{k+1}} \otimes \xi_1^{2^{k+1}+2^k} \otimes \xi_1^{2^k}}_{b} + \underbrace{\xi_1^{2^{k+1}} \otimes \xi_1^{2^k} \otimes \xi_2^{2^k}}_{c_1} + \underbrace{\xi_1^{2^{k+1}} \otimes \xi_2^{2^k} \otimes \xi_1^{2^k}}_{a} + \xi_1^{2^{k+1}} \otimes \xi_1^{2^{k+1}} \otimes \xi_1^{2^{k+1}}
\end{aligned}
$$

and

$$
\begin{aligned}
d(\xi_1^{2^{k+1}}\xi^{2^k} \otimes \xi_1^{2^k}) &= \overline{\Psi}(\xi_1^{2^{k+1}}\xi^{2^k}) \otimes \xi_1^{2^k} \\
&= \left(\xi_1^{2^{k+1}} \otimes 1 + 1 \otimes \xi_1^{2^{k+1}}\right)\left(\xi_2^{2^k} \otimes 1 + \xi_1^{2^{k+1}} \otimes \xi_1^{2^k} + 1 \otimes \xi_2^{2^k}\right) \otimes \xi_1^{2^k} \\
&= \xi_1^{2^{k+2}} \otimes \xi_1^{2^k} \otimes \xi_1^{2^k} + \underbrace{\xi_1^{2^{k+1}} \otimes \xi_2^{2^k} \otimes \xi_1^{2^k}}_{a} + \underbrace{\xi_2^{2^k} \otimes \xi_1^{2^{k+1}} \otimes \xi_1^{2^k}}_{c_2} + \underbrace{\xi_1^{2^{k+1}} \otimes \xi_1^{2^{k+1}+2^k} \otimes \xi_1^{2^k}}_{b}
\end{aligned}
\,.
$$

The labeled summands appear twice in $dB_{2,k}$ hence vanish (mod 2). The remaining terms are

$$
dB_{2,k} = \xi_1^{2^{k+1}} \otimes \xi_1^{2^{k+1}} \otimes \xi_1^{2^{k+1}} + \xi_1^{2^{k+2}} \otimes \xi_1^{2^k} \otimes \xi_1^{2^k}
$$

and these indeed represent the claimed elements. ∎

Remark 5.20. With lemma 5.19 the picture from remark 5.18 is further refined:

For $k = 0$ the differentia $d_2(b_{2,n}) = h_n^2 h_{n+2} + h_{n+1}^3$ means that on the third page of the May spectral sequence there is an identification

$$
h_1^3 = h_0^2 h_2 \,.
$$

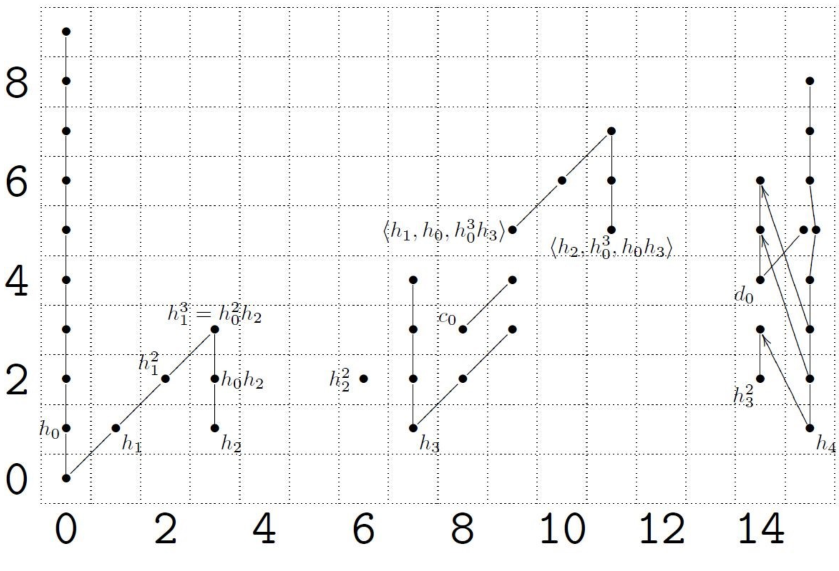
Hence where on page two we saw two distinct elements in bidegree $(s, t-s) = (3,3)$, on the next page these merge:

$$
\begin{array}{cccccc}
\vdots & \vdots & & & & \\
3 & h_0^4 & & & h_1^3 = h_0^2 h_2 & \\
2 & h_0^2 & & h_1^2 & h_0 h_2 & \\
1 & h_0 & h_1 & & h_2 & \\
 & 0 & 1 & 2 & 3 & 4
\end{array}
$$

Proceeding in this fashion, one keeps going until the 4-page of the May spectral sequence (Kochman 96, lemma 5.3.5). Inspection of degrees shows that this is sufficient, and one obtains:

Theorem 5.21. ***(stable page of classical Adams spectral sequence)***

In internal degree $t - s \leq 12$ the infinity page (def. 4.37) of the classical Adams spectral sequence (cor. 5.2) is spanned by the items in the following table



Here every dot is a generator for a copy of $\mathbb{Z}/2\mathbb{Z}$. Vertical edges denote multiplication with h_0 and diagonal edges denotes multiplication with h_1.

e.g. (Ravenel 86, theorem 3.2.11, Kochman 96, prop. 5.3.6), graphics taken from (Schwede 12))

The first dozen stable stems

Theorem 5.21 gives the stable page of the classical Adams spectral sequence in low degree. By corollary 5.2 and def. 4.39 we have that a vertical sequence of dots encodes an 2-primary part of the stable homotopy groups of spheres according to the graphical calculus of remark 4.6 (the rules for determining group extensions there is just the solution to the extension problem (rmk.) in view of def. 4.39):

$k =$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**
$\pi_k(\mathbb{S} \otimes \mathbb{Z}_{(2)}) =$	$\mathbb{Z}_{(2)}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/8$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/16$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^3$	$\mathbb{Z}/2$	$\mathbb{Z}/8$	0	0

The full answer in this range turns out to be this:

$k =$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	$\cdots$
$\pi_k(\mathbb{S}) =$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/240$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^3$	$\mathbb{Z}/6$	$\mathbb{Z}/504$	0	$\mathbb{Z}/3$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/480 \oplus \mathbb{Z}/2$	$\cdots$

And expanding the range yields this :

stable homotopy groups of spheres at 2

(graphics taken from Hatcher's website)

6. The case $E = H\mathbb{F}_p$ and $X = MU$

used to compute the stable homotopy groups of the complex Thom spectrum MU from the homology of MU

(hence, by Thom's theorem, equivalently the complex cobordism ring $\Omega^U_\bullet \simeq \pi_\bullet U$), see at *Seminar session: Milnor-Quillen theorem on MU*)

This is the *Milnor-Quillen theorem on MU*, see at *Seminar session: Milnor-Quillen theorem on MU*

(Adams 74, part II, around section 8, Lurie 10, around lecture 9)

7. Adams-Novikov spectral sequence ($E = MU$, $X = \mathbb{S}$)

this is the classical *Adams-Novikov spectral sequence* , converges faster than the classical choice $E = H\mathbb{F}_p$ to the stable homotopy groups of spheres, (…)

(Kochman 96, section 5)

---

8. References

For the general theory we follow the original

- John Frank Adams, section 2 of *Lectures on generalised cohomology*, in Peter Hilton (ed.) *Category Theory, Homology Theory and Their Applications III*, volume 99 of Lecture Notes in Mathematics (1969), Springer-Verlag Berlin-Heidelberg-New York.
- Frank Adams, section III.15 of *Stable homotopy and generalized homology*, Chicago Lectures in mathematics, 1974
- Aldridge Bousfield, sections 5 and 6 of *The localization of spectra with respect to homology*, Topology 18 (1979), no. 4, 257–281. (pdf)

For the homological algebra of comodules over Hopf algebroids we follow appendix A of

- Doug Ravenel, *Complex cobordism and stable homotopy groups of spheres*, 1986/2003

For the special case of the classical Adams spectral sequence and of the Adams-Novikov spectral sequence we follow

- Stanley Kochman, chapter 5 of *Bordism, Stable Homotopy and Adams Spectral Sequences*, AMS 1996

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nLab

Introduction to Stable homotopy theory -- S

S4D2 – Graduate Seminar on Topology

Complex oriented cohomology

Dr. Urs Schreiber

Abstract. *The category of those generalized cohomology theories that are equipped with a universal "complex orientation" happens to unify within it the abstract structure theory of stable homotopy theory with the concrete richness of the differential topology of cobordism theory and of the arithmetic geometry of formal group laws, such as elliptic curves. In the seminar we work through classical results in algebraic topology, organized such as to give in the end a first glimpse of the modern picture of chromatic homotopy theory.*

Accompanying notes.

Main page: *Introduction to Stable homotopy theory*.

Contents

1. Seminar) Complex oriented cohomology

Outline. We start with two classical topics of algebraic topology that first run independently in parallel:

- S.1) Generalized cohomology
- S.2) Cobordism theory

The development of either of these happens to give rise to the concept of *spectra* and via this concept it turns out that both topics are intimately related. The unification of both is our third topic

- S.3) Complex oriented cohomology

Literature. (Kochman 96).

S.1) Generalized cohomology

Idea. The concept that makes algebraic topology be about methods of homological algebra applied to topology is that of generalized homology and generalized cohomology: these are covariant functors or contravariant functors, respectively,

$$\mathrm{Spaces} \longrightarrow \mathrm{Ab}^{\mathbb{Z}}$$

from (sufficiently nice) topological spaces to $\mathbb{Z}$-graded abelian groups, such that a few key properties of the homotopy types of topological spaces is preserved as one passes them from Ho(Top) to the much more tractable abelian category Ab.

Literature. (Aguilar-Gitler-Prieto 02, chapters 7,8 and 12, Kochman 96, 3.4, 4.2, Schwede 12, II.6)

Generalized cohomology functors

Idea. A generalized (Eilenberg-Steenrod) cohomology theory is such a contravariant functor which satisfies the key properties exhibited by ordinary cohomology (as computed for instance by singular cohomology), notably homotopy invariance and excision, *except* that its value on the point is not required to be concentrated in degree 0. Dually for generalized homology. There are two versions of the axioms, one for reduced cohomology, and they are equivalent if properly set up.

An important example of a generalised cohomology theory other than ordinary cohomology is topological K-theory. The other two examples of key relevance below are cobordism cohomology and stable cohomotopy.

Literature. (Switzer 75, section 7, Aguilar-Gitler-Prieto 02, section 12 and section 9, Kochman 96, 3.4).

Reduced cohomology

The traditional formulation of reduced generalized cohomology in terms of point-set topology is this:

Definition 1.1. A **reduced cohomology theory** is

1. a functor

$$\tilde{E}^\bullet : (\mathrm{Top}_{\mathrm{CW}}^{*/})^{\mathrm{op}} \longrightarrow \mathrm{Ab}^{\mathbb{Z}}$$

   from the opposite of pointed topological spaces (CW-complexes) to $\mathbb{Z}$-graded abelian groups ("cohomology groups"), in components

$$\tilde{E} : (X \overset{f}{\longrightarrow} Y) \mapsto (\tilde{E}^\bullet(Y) \overset{f^*}{\longrightarrow} \tilde{E}^\bullet(X)),$$

2. equipped with a natural isomorphism of degree +1, to be called the **suspension isomorphism**, of the form

$$\sigma_E : \tilde{E}^\bullet(-) \overset{\simeq}{\longrightarrow} \tilde{E}^{\bullet+1}(\Sigma -)$$

such that:

1. **(homotopy invariance)** If $f_1, f_2 : X \longrightarrow Y$ are two morphisms of pointed topological spaces such that there is a (base point preserving) homotopy $f_1 \simeq f_2$ between them, then the induced homomorphisms of abelian groups are equal

$$f_1^* = f_2^* .$$

2. **(exactness)** For $i : A \hookrightarrow X$ an inclusion of pointed topological spaces, with $j : X \longrightarrow \mathrm{Cone}(i)$ the induced mapping cone (def.), then this gives an exact sequence of graded abelian groups

$$\tilde{E}^\bullet(\mathrm{Cone}(i)) \overset{j^*}{\longrightarrow} \tilde{E}^\bullet(X) \overset{i^*}{\longrightarrow} \tilde{E}^\bullet(A) .$$

(e.g. AGP 02, def. 12.1.4)

This is equivalent (prop. 1.4 below) to the following more succinct homotopy-theoretic definition:

Definition 1.2. A **reduced generalized cohomology theory** is a functor

$$\tilde{E}^\bullet : \mathrm{Ho}(\mathrm{Top}^{*/})^{\mathrm{op}} \longrightarrow \mathrm{Ab}^{\mathbb{Z}}$$

from the opposite of the pointed classical homotopy category (def., def.), to $\mathbb{Z}$-graded abelian groups, and equipped with natural isomorphisms, to be called the **suspension isomorphism** of the form

$$\sigma : \tilde{E}^{\bullet+1}(\Sigma -) \overset{\simeq}{\longrightarrow} \tilde{E}^\bullet(-)$$

such that:

* **(exactness)** it takes homotopy cofiber sequences in $\mathrm{Ho}(\mathrm{Top}^{*/})$ (def.) to exact sequences.

As a consequence (prop. 1.4 below), we find yet another equivalent definition:

Definition 1.3. A **reduced generalized cohomology theory** is a functor

$$\tilde E^\bullet : (\mathrm{Top}^{*/})^{\mathrm{op}} \longrightarrow \mathrm{Ab}^{\mathbb{Z}}$$

from the opposite of the category of pointed topological spaces to $\mathbb{Z}$-graded abelian groups, such that

* **(WHE)** it takes weak homotopy equivalences to isomorphisms

and equipped with natural isomorphism, to be called the **suspension isomorphism** of the form

$$\sigma : \tilde E^{\bullet+1}(\Sigma -) \overset{\simeq}{\longrightarrow} \tilde E^\bullet(-)$$

such that

* **(exactness)** it takes homotopy cofiber sequences in $\mathrm{Ho}(\mathrm{Top}^{*/})$ (def.), to exact sequences.

Proposition 1.4. *The three definitions*

* *def. 1.1*
* *def. 1.2*
* *def. 1.3*

are indeed equivalent.

Proof. Regarding the equivalence of def. 1.1 with def. 1.2:

By the existence of the classical model structure on topological spaces (thm.), the characterization of its homotopy category (cor.) and the existence of CW-approximations, the homotopy invariance axiom in def. 1.1 is equivalent to the functor passing to the classical pointed homotopy category. In view of this and since on CW-complexes the standard topological mapping cone construction is a model for the homotopy cofiber (prop.), this gives the equivalence of the two versions of the exactness axiom.

Regarding the equivalence of def. 1.2 with def. 1.3:

This is the universal property of the classical homotopy category (thm.) which identifies it with the localization (def.) of $\mathrm{Top}^{*/}$ at the weak homotopy equivalences (thm.), together with the existence of CW approximations (rmk.): jointly this says that, up to natural isomorphism, there is a bijection between functors F and $\tilde F$ in the following diagram (which is filled by a natural isomorphism itself):

$$\begin{array}{ccc} \mathrm{Top}^{\mathrm{op}} & \overset{F}{\longrightarrow} & \mathrm{Ab}^{\mathbb{Z}} \\ {}^{\mathllap{\gamma_{\mathrm{Top}}}}\downarrow & \nearrow_{\tilde F} & \\ \mathrm{Ho}(\mathrm{Top})^{\mathrm{op}} \simeq (\mathrm{Top}_{\mathrm{CW}})/_\sim & & \end{array}$$

where F sends weak homotopy equivalences to isomorphisms and where $(-)_\sim$ means identifying homotopic maps. ∎

Prop. 1.4 naturally suggests (e.g. Lurie 10, section 1.4) that the concept of generalized cohomology be formulated in the generality of any abstract homotopy theory (model category), not necessarily that of (pointed) topological spaces:

Definition 1.5. Let $\mathcal{C}$ be a model category (def.) with $\mathcal{C}^{*/}$ its pointed model category (prop.).

A **reduced additive generalized cohomology theory** on $\mathcal{C}$ is

1. a functor

$$\tilde E^\bullet : \mathrm{Ho}(\mathcal{C}^{*/})^{\mathrm{op}} \longrightarrow \mathrm{Ab}^{\mathbb{Z}}$$

2. a natural isomorphism ("suspension isomorphisms") of degree +1

$$\sigma : \tilde E^\bullet \longrightarrow \tilde E^{\bullet+1} \circ \Sigma$$

such that

* **(exactness)** $\tilde E^\bullet$ takes homotopy cofiber sequences to exact sequences.

Finally we need the following terminology:

Definition 1.6. Let $\tilde E^\bullet$ be a reduced cohomology theory according to either of def. 1.1, def. 1.2, def. 1.3 or def. 1.5.

We say $\tilde E^\bullet$ is **additive** if in addition

- **(wedge axiom)** For $\{X_i\}_{i \in I}$ any set of pointed CW-complexes, then the canonical morphism

$$\tilde E^\bullet(\vee_{i \in I} X_i) \longrightarrow \prod_{i \in I} \tilde E^\bullet(X_i)$$

  from the functor applied to their wedge sum (def.), to the product of its values on the wedge summands, is an isomorphism.

We say $\tilde E^\bullet$ is **ordinary** if its value on the 0-sphere S^0 is concentrated in degree 0:

- **(Dimension)** $\tilde E^{\bullet \neq 0}(\mathbb{S}^0) \simeq 0$.

If $\tilde E^\bullet$ is not ordinary, one also says that it is **generalized** or **extraordinary**.

A **homomorphism of reduced cohomology theories**

$$\eta : \tilde E^\bullet \longrightarrow \tilde F^\bullet$$

is a natural transformation between the underlying functors which is compatible with the suspension isomorphisms in that all the following squares commute

$$\begin{array}{ccc} \tilde E^\bullet(X) & \overset{\eta_X}{\longrightarrow} & \tilde F^\bullet(X) \\ {}^{\mathllap{\sigma_E}}\downarrow && \downarrow^{\mathrlap{\sigma_F}} \\ \tilde E^{\bullet+1}(\Sigma X) & \overset{\eta_{\Sigma X}}{\longrightarrow} & \tilde F^{\bullet+1}(\Sigma X) \end{array} \,.$$

We now discuss some constructions and consequences implied by the concept of reduced cohomology theories:

Definition 1.7. Given a generalized cohomology theory $(E^\bullet, \delta)$ on some $\mathcal{C}$ as in def. 1.5, and given a homotopy cofiber sequence in $\mathcal{C}$ (prop.),

$$X \overset{f}{\longrightarrow} Y \overset{g}{\longrightarrow} Z \overset{\mathrm{coker}(g)}{\longrightarrow} \Sigma X \,,$$

then the corresponding **connecting homomorphism** is the composite

$$\partial \;:\; E^\bullet(X) \overset{\sigma}{\longrightarrow} E^{\bullet+1}(\Sigma X) \overset{\mathrm{coker}(g)^\ast}{\longrightarrow} E^{\bullet+1}(Z) \,.$$

Proposition 1.8. *The connecting homomorphisms of def. 1.7 are parts of long exact sequences*

$$\cdots \overset{\partial}{\longrightarrow} E^\bullet(Z) \longrightarrow E^\bullet(Y) \longrightarrow E^\bullet(X) \overset{\partial}{\longrightarrow} E^{\bullet+1}(Z) \to \cdots \,.$$

Proof. By the defining exactness of $E^\bullet$, def. 1.5, and the way this appears in def. 1.7, using that σ is by definition an isomorphism. ▮

Unreduced cohomology

Given a reduced generalized cohomology theory as in def. 1.1, we may "un-reduce" it and evaluate it on unpointed topological spaces X simply by evaluating it on X_+ (def.). It is conventional to further generalize to relative cohomology and evaluate on unpointed subspace inclusions $i \colon A \hookrightarrow X$, taken as placeholders for their mapping cones $\mathrm{Cone}(i_+)$ (prop.).

In the following a *pair* (X,U) refers to a subspace inclusion of topological spaces $U \hookrightarrow X$. Whenever only one space is mentioned, the subspace is assumed to be the empty set $(X, \emptyset)$. Write $\mathrm{Top}_{\mathrm{CW}}^{\hookrightarrow}$ for the category of such pairs (the full subcategory of the arrow category of $\mathrm{Top}_{\mathrm{CW}}$ on the inclusions). We identify $\mathrm{Top}_{\mathrm{CW}} \hookrightarrow \mathrm{Top}_{\mathrm{CW}}^{\hookrightarrow}$ by $X \mapsto (X, \emptyset)$.

Definition 1.9. A *cohomology theory* (unreduced, relative) is

1. a functor

$$E^\bullet : (\mathrm{Top}_{\mathrm{CW}}^{\hookrightarrow})^{\mathrm{op}} \to \mathrm{Ab}^{\mathbb{Z}}$$

to the category of $\mathbb{Z}$-graded abelian groups,

2. a natural transformation of degree +1, to be called the **connecting homomorphism**, of the form

$$\delta_{(X,A)} \;:\; E^\bullet(A,\emptyset) \to E^{\bullet+1}(X,A)\;.$$

such that:

1. **(homotopy invariance)** For $f:(X_1,A_1) \to (X_2,A_2)$ a homotopy equivalence of pairs, then

$$E^\bullet(f) \;:\; E^\bullet(X_2,A_2) \overset{\simeq}{\longrightarrow} E^\bullet(X_1,A_1)$$

is an isomorphism;

2. **(exactness)** For $A \hookrightarrow X$ the induced sequence

$$\cdots \to E^n(X,A) \longrightarrow E^n(X) \longrightarrow E^n(A) \overset{\delta}{\longrightarrow} E^{n+1}(X,A) \to \cdots$$

is a long exact sequence of abelian groups.

3. **(excision)** For $U \hookrightarrow A \hookrightarrow X$ such that $\overline{U} \subset \mathrm{Int}(A)$, then the natural inclusion of the pair $i:(X-U,A-U) \hookrightarrow (X,A)$ induces an isomorphism

$$E^\bullet(i) \;:\; E^n(X,A) \overset{\simeq}{\longrightarrow} E^n(X-U,A-U)$$

We say $E^\bullet$ is **additive** if it takes coproducts to products:

- **(additivity)** If $(X,A) = \coprod_i (X_i,A_i)$ is a coproduct, then the canonical comparison morphism

$$E^n(X,A) \overset{\simeq}{\longrightarrow} \prod_i E^n(X_i,A_i)$$

is an isomorphism from the value on (X,A) to the product of values on the summands.

We say $E^\bullet$ is **ordinary** if its value on the point is concentrated in degree 0

- **(Dimension)**: $E^{\bullet\neq 0}(*,\emptyset) = 0$.

A homomorphism of unreduced cohomology theories

$$\eta \;:\; E^\bullet \longrightarrow F^\bullet$$

is a natural transformation of the underlying functors that is compatible with the connecting homomorphisms, hence such that all these squares commute:

$$\begin{array}{ccc} E^\bullet(A,\emptyset) & \overset{\eta_{(A,\emptyset)}}{\longrightarrow} & F^\bullet(A,\emptyset) \\ {}^{\mathllap{\delta_E}}\downarrow && \downarrow^{\mathrlap{\delta_F}} \\ E^{\bullet+1}(X,A) & \underset{\eta_{(X,A)}}{\longrightarrow} & F^{\bullet+1}(X,A) \end{array}\;.$$

e.g. (AGP 02, def. 12.1.1).

Lemma 1.10. The excision axiom in def. 1.9 is equivalent to the following statement:

For all $A,B \hookrightarrow X$ with $X = \mathrm{Int}(A) \cup \mathrm{Int}(B)$, then the inclusion

$$i:(A, A\cap B) \longrightarrow (X,B)$$

induces an isomorphism,

$$i^* \;:\; E^\bullet(X,B) \overset{\simeq}{\longrightarrow} E^\bullet(A,A\cap B)$$

(e.g Switzer 75, 7.2)

Proof. In one direction, suppose that $E^\bullet$ satisfies the original excision axiom. Given A,B with $X = \mathrm{Int}(A) \cup \mathrm{Int}(B)$, set $U \coloneqq X - A$ and observe that

$$
\begin{aligned}
\overline{U} &= \overline{X - A} \\
&= X - \mathrm{Int}(A) \\
&\subset \mathrm{Int}(B)
\end{aligned}
$$

and that

$$(X - U, B - U) = (A, A \cap B) \;.$$

Hence the excision axiom implies $E^\bullet(X, B) \xrightarrow{\simeq} E^\bullet(A, A \cap B)$.

Conversely, suppose $E^\bullet$ satisfies the alternative condition. Given $U \hookrightarrow A \hookrightarrow X$ with $\overline{U} \subset \mathrm{Int}(A)$, observe that we have a cover

$$
\begin{aligned}
\mathrm{Int}(X - U) \cup \mathrm{Int}(A) &= (X - \overline{U}) \cap \mathrm{Int}\,(A) \\
&\supset (X - \mathrm{Int}(A)) \cap \mathrm{Int}(A) \\
&= X
\end{aligned}
$$

and that

$$(X - U, (X - U) \cap A) = (X - U, A - U) \;.$$

Hence

$$E^\bullet(X - U, A - U) \simeq E^\bullet(X - U, (X - U) \cap A) \simeq E^\bullet(X, A) \;.$$

∎

The following lemma shows that the dependence in pairs of spaces in a generalized cohomology theory is really a stand-in for evaluation on homotopy cofibers of inclusions.

Lemma 1.11. *Let $E^\bullet$ be an cohomology theory, def. 1.9, and let $A \hookrightarrow X$. Then there is an isomorphism*

$$E^\bullet(X, A) \xrightarrow{\simeq} E^\bullet(X \cup \mathrm{Cone}(A), *)$$

between the value of $E^\bullet$ on the pair (X, A) and its value on the unreduced mapping cone of the inclusion (rmk.), relative to a basepoint.

If moreover $A \hookrightarrow X$ is (the retract of) a relative cell complex inclusion, then also the morphism in cohomology induced from the quotient map $p \,:\, (X, A) \longrightarrow (X/A, *)$ is an isomorphism:

$$E^\bullet(p) \,:\, E^\bullet(X/A, *) \longrightarrow E^\bullet(X, A) \;.$$

(e.g AGP 02, corollary 12.1.10)

Proof. Consider $U \coloneqq (\mathrm{Cone}(A) - A \times \{0\}) \hookrightarrow \mathrm{Cone}(A)$, the cone on A minus the base A. We have

$$(X \cup \mathrm{Cone}(A) - U, \mathrm{Cone}(A) - U) \simeq (X, A)$$

and hence the first isomorphism in the statement is given by the excision axiom followed by homotopy invariance (along the contraction of the cone to the point).

Next consider the quotient of the mapping cone of the inclusion:

$$(X \cup \mathrm{Cone}(A), \mathrm{Cone}(A)) \longrightarrow (X/A, *) \;.$$

If $A \hookrightarrow X$ is a cofibration, then this is a homotopy equivalence since $\mathrm{Cone}(A)$ is contractible and since by the dual factorization lemma (lem.) and by the invariance of homotopy fibers under weak equivalences (lem.), $X \cup \mathrm{Cone}(A) \to X/A$ is a weak homotopy equivalence, hence, by the universal property of the classical homotopy category (thm.) a homotopy equivalence on CW-complexes.

Hence now we get a composite isomorphism

$$E^\bullet(X/A, *) \xrightarrow{\simeq} E^\bullet(X \cup \mathrm{Cone}(A), \mathrm{Cone}(A)) \xrightarrow{\simeq} E^\bullet(X, A) \;.$$

∎

Example 1.12. As an important special case of : Let (X, x) be a pointed CW-complex. For $p \colon (\mathrm{Cone}(X), X) \to (\Sigma X, \{x\})$ the quotient map from the reduced cone on X to the reduced suspension, then

$$E^\bullet(p) \,:\, E^\bullet(\mathrm{Cone}(X), X) \xrightarrow{\simeq} E^\bullet(\Sigma X, \{x\})$$

is an isomorphism.

Proposition 1.13. ***(exact sequence of a triple)***

For $E^\bullet$ an unreduced generalized cohomology theory, def. 1.9, then every inclusion of two consecutive subspaces

$$Z \hookrightarrow Y \hookrightarrow X$$

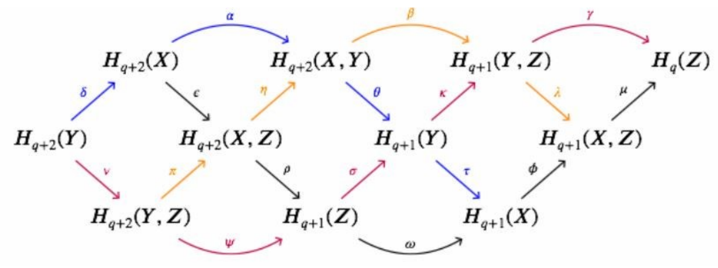
induces a long exact sequence of cohomology groups of the form

$$\cdots \to E^{q-1}(Y,Z) \overset{\bar\delta}{\longrightarrow} E^q(X,Y) \longrightarrow E^q(X,Z) \longrightarrow E^q(Y,Z) \to \cdots$$

where

$$\bar\delta \,:\, E^{q-1}(Y,Z) \longrightarrow E^{q-1}(Y) \overset{\delta}{\longrightarrow} E^q(X,Y)\;.$$

Proof. Apply the braid lemma to the interlocking long exact sequences of the three pairs (X,Y), (X,Z), (Y,Z):



(graphics from this Maths.SE comment, showing the dual situation for homology)

See here for details. ■

Remark 1.14. The exact sequence of a triple in prop. 1.13 is what gives rise to the Cartan-Eilenberg spectral sequence for E-cohomology of a CW-complex X.

Example 1.15. For (X,x) a pointed topological space and $\mathrm{Cone}(X) = (X \wedge (I_+))/X$ its reduced cone, the long exact sequence of the triple $(\{x\}, X, \mathrm{Cone}(X))$, prop. 1.13,

$$0 \simeq E^q(\mathrm{Cone}(X), \{x\}) \longrightarrow E^q(X,\{x\}) \overset{\bar\delta}{\longrightarrow} E^{q+1}(\mathrm{Cone}(X), X) \longrightarrow E^{q+1}(\mathrm{Cone}(X),\{x\}) \simeq 0$$

exhibits the connecting homomorphism $\bar\delta$ here as an isomorphism

$$\bar\delta \,:\, E^q(X,\{x\}) \overset{\simeq}{\longrightarrow} E^{q+1}(\mathrm{Cone}(X), X)\;.$$

This is the *suspension isomorphism* extracted from the unreduced cohomology theory, see def. 1.17 below.

Proposition 1.16. ***(Mayer-Vietoris sequence)***

Given $E^\bullet$ an unreduced cohomology theory, def. 1.9. Given a topological space covered by the interior of two spaces as $X = \mathrm{Int}(A) \cup \mathrm{Int}(B)$, then for each $C \subset A \cap B$ there is a long exact sequence of cohomology groups of the form

$$\cdots \to E^{n-1}(A \cap B, C) \overset{\bar\delta}{\longrightarrow} E^n(X,C) \longrightarrow E^n(A,C) \oplus E^n(B,C) \longrightarrow E^n(A\cap B, C) \to \cdots\;.$$

e.g. (Switzer 75, theorem 7.19, Aguilar-Gitler-Prieto 02, theorem 12.1.22)

Relation between unreduced and reduced cohomology

Definition 1.17. **(unreduced to reduced cohomology)**

Let $E^\bullet$ be an unreduced cohomology theory, def. 1.9. Define a reduced cohomology theory, def. 1.1 $(\tilde E^\bullet, \sigma)$ as follows.

For $x \colon * \to X$ a pointed topological space, set

$$\tilde E^\bullet(X,x) \coloneqq E^\bullet(X,\{x\})\;.$$

This is clearly functorial. Take the suspension isomorphism to be the composite

$$\sigma : \tilde E^{\bullet+1}(\Sigma X) = E^{\bullet+1}(\Sigma X, \{x\}) \overset{E^\bullet(p)}{\longrightarrow} E^{\bullet+1}(\mathrm{Cone}(X), X) \overset{\bar\delta^{-1}}{\longrightarrow} E^\bullet(X, \{x\}) = \tilde E^\bullet(X)$$

of the isomorphism $E^\bullet(p)$ from example 1.12 and the inverse of the isomorphism $\bar\delta$ from example 1.15.

Proposition 1.18. *The construction in def. 1.17 indeed gives a reduced cohomology theory.*

(e.g Switzer 75, 7.34)

Proof. We need to check the exactness axiom given any $A \hookrightarrow X$. By lemma 1.11 we have an isomorphism

$$\tilde E^\bullet(X \cup \mathrm{Cone}(A)) = E^\bullet(X \cup \mathrm{Cone}(A), \{*\}) \overset{\simeq}{\longrightarrow} E^\bullet(X, A) \; .$$

Unwinding the constructions shows that this makes the following diagram commute:

$$\begin{array}{ccc} \tilde E^\bullet(X \cup \mathrm{Cone}(A)) & \overset{\simeq}{\longrightarrow} & E^\bullet(X, A) \\ \downarrow && \downarrow \\ \tilde E^\bullet(X) & = & E^\bullet(X, \{x\}) \\ \downarrow && \downarrow \\ \tilde E^\bullet(A) & = & E^\bullet(A, \{a\}) \end{array} \,,$$

where the vertical sequence on the right is exact by prop. 1.13. Hence the left vertical sequence is exact. ▮

Definition 1.19. **(reduced to unreduced cohomology)**

Let $(\tilde E^\bullet, \sigma)$ be a reduced cohomology theory, def. 1.1. Define an unreduced cohomolog theory $E^\bullet$, def. 1.9, by

$$E^\bullet(X, A) \coloneqq \tilde E^\bullet(X_+ \cup \mathrm{Cone}(A_+))$$

and let the connecting homomorphism be as in def. 1.7.

Proposition 1.20. *The construction in def. 1.19 indeed yields an unreduced cohomology theory.*

e.g. (Switzer 75, 7.35)

Proof. Exactness holds by prop. 1.8. For excision, it is sufficient to consider the alternative formulation of lemma 1.10. For CW-inclusions, this follows immediately with lemma 1.11. ▮

Theorem 1.21. *The constructions of def. 1.19 and def. 1.17 constitute a pair of functors between then categories of reduced cohomology theories, def. 1.1 and unreduced cohomology theories, def. 1.9 which exhbit an equivalence of categories.*

Proof. (…careful with checking the respect for suspension iso and connecting homomorphism..)

To see that there are natural isomorphisms relating the two composites of these two functors to the identity:

One composite is

$$\begin{aligned} E^\bullet & \mapsto (\tilde E^\bullet : (X, x) \mapsto E^\bullet(X, \{x\})) \\ & \mapsto ((E')^\bullet : (X, A) \mapsto E^\bullet(X_+ \cup \mathrm{Cone}(A_+)), *) \end{aligned} \,,$$

where on the right we have, from the construction, the reduced mapping cone of the original inclusion $A \hookrightarrow X$ with a base point adjoined. That however is isomorphic to the unreduced mapping cone of the original inclusion (prop.- P#UnreducedMappingConeAsReducedConeOfBasedPointAdjoined)). With this the natural isomorphism is given by lemma 1.11.

The other composite is

$$\begin{aligned} \tilde E^\bullet & \mapsto (E^\bullet : (X, A) \mapsto \tilde E^\bullet(X_+ \cup \mathrm{Cone}(A_+))) \\ & \mapsto ((\tilde E')^\bullet : X \mapsto \tilde E^\bullet(X_+ \cup \mathrm{Cone}(*_+))) \end{aligned}$$

where on the right we have the reduced mapping cone of the point inclusion with a point adoined. As before, this is isomorphic to the unreduced mapping cone of the point inclusion. That finally is clearly homotopy equivalent to X, and so now the natural isomorphism follows with homotopy invariance. ▮

Finally we record the following basic relation between reduced and unreduced cohomology:

Proposition 1.22. *Let $E^\bullet$ be an unreduced cohomology theory, and $\tilde E^\bullet$ its reduced cohomology theory from* *def. 1.17. For $(X, *)$ a pointed topological space, then there is an identification*

$$E^\bullet(X) \simeq \tilde{E}^\bullet(X) \oplus E^\bullet(*)$$

of the unreduced cohomology of X with the direct sum of the reduced cohomology of X and the unreduced cohomology of the base point.

Proof. The pair $* \hookrightarrow X$ induces the sequence

$$\cdots \to E^{\bullet -1}(*) \overset{\delta}{\longrightarrow} \tilde{E}^\bullet(X) \longrightarrow E^\bullet(X) \longrightarrow E^\bullet(*) \overset{\delta}{\longrightarrow} \tilde{E}^{\bullet+1}(X) \to \cdots$$

which by the exactness clause in def. 1.9 is exact.

Now since the composite $* \to X \to *$ is the identity, the morphism $E^\bullet(X) \to E^\bullet(*)$ has a section and so is in particular an epimorphism. Therefore, by exactness, the connecting homomorphism vanishes, $\delta = 0$ and we have a short exact sequence

$$0 \to \tilde{E}^\bullet(X) \longrightarrow E^\bullet(X) \longrightarrow E^\bullet(*) \to 0$$

with the right map an epimorphism. Hence this is a split exact sequence and the statement follows. ▮

Generalized homology functors

All of the above has a dual version with generalized cohomology replaced by generalized homology. For ease of reference, we record these dual definitions:

Definition 1.23. A **reduced homology theory** is a functor

$$\tilde{E}_\bullet \;\colon\; (\mathrm{Top}_{\mathrm{CW}}^{*/}) \longrightarrow \mathrm{Ab}^{\mathbb{Z}}$$

from the category of pointed topological spaces (CW-complexes) to $\mathbb{Z}$-graded abelian groups ("homology groups"), in components

$$\tilde{E}_\bullet \;\colon\; (X \overset{f}{\longrightarrow} Y) \mapsto (\tilde{E}_\bullet(X) \overset{f_*}{\longrightarrow} \tilde{E}_\bullet(Y)) \,,$$

and equipped with a natural isomorphism of degree +1, to be called the **suspension isomorphism**, of the form

$$\sigma \;\colon\; \tilde{E}_\bullet(-) \overset{\simeq}{\longrightarrow} \tilde{E}_{\bullet+1}(\Sigma -)$$

such that:

1. **(homotopy invariance)** If $f_1, f_2 : X \longrightarrow Y$ are two morphisms of pointed topological spaces such that there is a (base point preserving) homotopy $f_1 \simeq f_2$ between them, then the induced homomorphisms of abelian groups are equal

$$f_{1*} = f_{2*} \,.$$

2. **(exactness)** For $i : A \hookrightarrow X$ an inclusion of pointed topological spaces, with $j : X \longrightarrow \mathrm{Cone}(i)$ the induced mapping cone, then this gives an exact sequence of graded abelian groups

$$\tilde{E}_\bullet(A) \overset{i_*}{\longrightarrow} \tilde{E}_\bullet(X) \overset{j_*}{\longrightarrow} \tilde{E}_\bullet(\mathrm{Cone}(i)) \,.$$

We say $\tilde{E}_\bullet$ is **additive** if in addition

- **(wedge axiom)** For $\{X_i\}_{i \in I}$ any set of pointed CW-complexes, then the canonical morphism

$$\oplus_{i \in I} \tilde{E}_\bullet(X_i) \longrightarrow \tilde{E}^\bullet(\vee_{i \in I} X_i)$$

  from the direct sum of the value on the summands to the value on the wedge sum (prop.-P#WedgeSumAsCoproduct)), is an isomorphism.

We say $\tilde{E}_\bullet$ is **ordinary** if its value on the 0-sphere S^0 is concentrated in degree 0:

- **(Dimension)** $\tilde{E}_{\bullet \neq 0}(\mathbb{S}^0) \simeq 0$.

A homomorphism of reduced cohomology theories

$$\eta \;\colon\; \tilde{E}_\bullet \longrightarrow \tilde{F}_\bullet$$

is a natural transformation between the underlying functors which is compatible with the suspension isomorphisms in that all the following squares commute

$$\begin{array}{ccc} \tilde E_\bullet(X) & \overset{\eta_X}{\longrightarrow} & \tilde F_\bullet(X) \\ {}^{\mathllap{\sigma_E}}\downarrow && \downarrow^{\mathrlap{\sigma_F}} \\ \tilde E_{\bullet+1}(\Sigma X) & \overset{\eta_{\Sigma X}}{\longrightarrow} & \tilde F_{\bullet +1}(\Sigma X) \end{array} \,.$$

Definition 1.24. A **homology theory** (unreduced, relative) is a functor

$$E_\bullet : (\mathrm{Top}_{\mathrm{CW}}^{\hookrightarrow}) \longrightarrow \mathrm{Ab}^{\mathbb{Z}}$$

to the category of $\mathbb{Z}$-graded abelian groups, as well as a natural transformation of degree +1, to be called the **connecting homomorphism**, of the form

$$\delta_{(X,A)} \;:\; E_{\bullet+1}(X,A) \longrightarrow E^\bullet(A, \emptyset) \,.$$

such that:

1. **(homotopy invariance)** For $f : (X_1,A_1) \to (X_2,A_2)$ a homotopy equivalence of pairs, then

$$E_\bullet(f) \;:\; E_\bullet(X_1,A_1) \overset{\simeq}{\longrightarrow} E_\bullet(X_2,A_2)$$

   is an isomorphism;

2. **(exactness)** For $A \hookrightarrow X$ the induced sequence

$$\cdots \to E_{n+1}(X,A) \overset{\delta}{\longrightarrow} E_n(A) \longrightarrow E_n(X) \longrightarrow E_n(X,A) \to \cdots$$

   is a long exact sequence of abelian groups.

3. **(excision)** For $U \hookrightarrow A \hookrightarrow X$ such that $\overline{U} \subset \mathrm{Int}(A)$, then the natural inclusion of the pair $i : (X-U, A-U) \hookrightarrow (X,A)$ induces an isomorphism

$$E_\bullet(i) \;:\; E_n(X-U,A-U) \overset{\simeq}{\longrightarrow} E_n(X,A)$$

We say $E^\bullet$ is **additive** if it takes coproducts to direct sums:

* **(additivity)** If $(X,A) = \coprod_i (X_i,A_i)$ is a coproduct, then the canonical comparison morphism

$$\oplus_i E^n(X_i,A_i) \overset{\simeq}{\longrightarrow} E^n(X,A)$$

  is an isomorphismfrom the direct sum of the value on the summands, to the value on the total pair.

We say $E_\bullet$ is **ordinary** if its value on the point is concentrated in degree 0

* **(Dimension)**: $E_{\bullet \neq 0}(\ast, \emptyset) = 0$.

A homomorphism of unreduced homology theories

$$\eta \;:\; E_\bullet \longrightarrow F_\bullet$$

is a natural transformation of the underlying functors that is compatible with the connecting homomorphisms, hence such that all these squares commute:

$$\begin{array}{ccc} E_{\bullet+1}(X,A) & \overset{\eta_{(X,A)}}{\longrightarrow} & F_{\bullet+1}(X,A) \\ {}^{\mathllap{\delta_E}}\downarrow && \downarrow^{\mathrlap{\delta_F}} \\ E_\bullet(A,\emptyset) & \overset{\eta_{(A,\emptyset)}}{\longrightarrow} & F^\bullet(A,\emptyset) \end{array} \,.$$

Multiplicative cohomology theories

The generalized cohomology theories considered above assign *cohomology groups*. It is familiar from ordinary cohomology with coefficients not just in a group but in a ring, that also the cohomology groups inherit compatible ring structure. The generalization of this phenomenon to generalized cohomology theories is captured by the concept of multiplicative cohomology theories:

Definition 1.25. Let E_1, E_2, E_3 be three unreduced generalized cohomology theories (def.). A **pairing of cohomology theories**

$$\mu : E_1 \square E_2 \longrightarrow E_3$$

is a natural transformation (of functors on $(\mathrm{Top}_{\mathrm{CW}}^{\hookrightarrow} \times \mathrm{Top}_{\mathrm{CW}}^{\hookrightarrow})^{\mathrm{op}}$) of the form

$$\mu_{n_1,n_2} : E_1^{n_1}(X,A) \otimes E_2^{n_2}(Y,B) \longrightarrow E_3^{n_1+n_2}(X \times Y\,,\ A \times Y \cup X \times B)$$

such that this is compatible with the connecting homomorphisms δ_i of E_i, in that the following are commuting squares

$$\begin{array}{ccc} E_1^{n_1}(A) \otimes E_2^{n_2}(Y,B) & \overset{\delta_1 \otimes \mathrm{id}_2}{\longrightarrow} & E_1^{n_1+1}(X,A) \otimes E_2^{n_2}(Y,B) \\ {}^{\mu_{n_1,n_2}}\downarrow && \downarrow^{\mu_{n_1+1,n_2}} \\ \array{ E_3^{n_1+n_2}(A\times Y, A \times B) \\ \simeq \\ E_3^{n_1+n_2}(A \times Y \cup X \times B, X \times B) } & \overset{\delta_3}{\longrightarrow} & E_3^{n_1+n_2+1}(X \times Y, A \times B) \end{array}$$

and

$$\begin{array}{ccc} E_1^{n_1}(X,A) \otimes E_2^{n_2}(B) & \overset{(-1)^{n_1}\mathrm{id}_1 \otimes \delta_2}{\longrightarrow} & E_1^{n_1+1}(X,A) \otimes E_2^{n_2}(Y,B) \\ {}^{\mu_{n_1,n_2}}\downarrow && \downarrow^{\mu_{n_1,n_2+1}} \\ \array{ E_3^{n_1+n_2}(X\times B, A \times B) \\ \simeq \\ E_3^{n_1+n_2}(A \times Y \cup X \times B, A \times Y) } & \overset{\delta_3}{\longrightarrow} & E_3^{n_1+n_2+1}(X \times Y, A \times B) \end{array} \,,$$

where the isomorphisms in the bottom left are the excision isomorphisms.

Definition 1.26. An (unreduced) **multiplicative cohomology theory** is an unreduced generalized cohomology theory theory E (def. 1.9) equipped with

1. (external multiplication) a pairing (def. 1.25) of the form $\mu : E \square E \longrightarrow E$;
2. (unit) an element $1 \in E^0(\ast)$

such that

1. (associativity) $\mu \circ (\mathrm{id} \otimes \mu) = \mu \circ (\mu \otimes \mathrm{id})$;
2. (unitality) $\mu(1 \otimes x) = \mu(x \otimes 1) = x$ for all $x \in E^n(X,A)$.

The mulitplicative cohomology theory is called **commutative** (often considered by default) if in addition

- **(graded commutativity)**

$$\begin{array}{ccc} E^{n_1}(X,A) \otimes E^{n_2}(Y,B) & \overset{(u \otimes v) \mapsto (-1)^{n_1 n_2}(v \otimes u)}{\longrightarrow} & E^{n_2}(Y,B) \otimes E^{n_1}_{X,A} \\ {}^{\mu_{n_1,n_2}}\downarrow && \downarrow^{\mu_{n_2,n_1}} \\ E^{n_1+n_2}(X \times Y, A \times Y \cup X \times B) & \underset{(\mathrm{switch}_{(X,A),(Y,B)})^\ast}{\longrightarrow} & E^{n_1+n_2}(Y \times X, B \times X \cup Y \times A) \end{array} \,.$$

Given a multiplicative cohomology theory $(E, \mu, 1)$, its **cup product** is the composite of the above external multiplication with pullback along the diagonal maps $\Delta_{(X,A)} : (X,A) \longrightarrow (X \times X, A \times X \cup X \times A)$;

$$(-) \cup (-) : E^{n_1}(X,A) \otimes E^{n_2}(X,A) \overset{\mu_{n_1,n_2}}{\longrightarrow} E^{n_1+n_2}(X \times X,\ A \times X \cup X \times A) \overset{\Delta^\ast_{(X,A)}}{\longrightarrow} E^{n_1+n_2}(X,\ A \cup B)\,.$$

e.g. (Tamaki-Kono 06, II.6)

Proposition 1.27. *Let $(E, \mu, 1)$ be a multiplicative cohomology theory, def. 1.26. Then*

1. *For every space X the cup product gives $E^\bullet(X)$ the structure of a $\mathbb{Z}$-graded ring, which is graded-commutative if $(E,\mu,1)$ is commutative.*
2. *For every pair (X,A) the external multiplication μ gives $E^\bullet(X,A)$ the structure of a left and right module over the graded ring $E^\bullet(\ast)$.*
3. *All pullback morphisms respect the left and right action of $E^\bullet(\ast)$ and the connecting homomorphisms respect the right action and the left action up to multiplication by $(-1)^{n_1}$*

Proof. Regarding the third point:

For pullback maps this is the naturality of the external product: let $f : (X,A) \longrightarrow (Y,B)$ be a morphism in $\mathrm{Top}_{\mathrm{CW}}^{\hookrightarrow}$

then naturality says that the following square commutes:

$$\begin{array}{ccc} E^{n_1}(*) \otimes E^{n_2}(Y,B) & \overset{\mu_{n_1,n_2}}{\longrightarrow} & E^{n_1+n_2}(Y,B) \\ {}^{(\mathrm{id},f^*)}\downarrow && \downarrow^{f^*} \\ E^{n_1}(*) \otimes E^{n_2}(X,A) & \overset{\mu_{n_1,n_2}}{\longrightarrow} & E^{n_1+n_2}(Y,B) \end{array} \,.$$

For connecting homomorphisms this is the (graded) commutativity of the squares in def. 1.26:

$$\begin{array}{ccc} E^{n_1}(*) \otimes E^{n_2}(A) & \overset{(-1)^{n_1}(\mathrm{id},\delta)}{\longrightarrow} & E^{n_1}(*) \otimes E^{n_2+2}(X) \\ {}^{\mu_{n_1,n_2}}\downarrow && \downarrow^{\mu_{n_1,n_2}} \\ E^{n_1+n_2}(A) & \overset{\delta}{\longrightarrow} & E_3^{n_1+n_2+1}(X,B) \end{array} \,.$$

∎

Brown representability theorem

Idea. Given any functor such as the generalized (co)homology functor above, an important question to ask is whether it is a *representable functor*. Due to the $\mathbb{Z}$-grading and the suspension isomorphisms, if a generalized (co)homology functor is representable at all, it must be represented by a $\mathbb{Z}$-indexed sequence of pointed topological spaces such that the reduced suspension of one is comparable to the next one in the list. This is a *spectrum* or more specifically: a *sequential spectrum* .

Whitehead observed that indeed every spectrum represents a generalized (co)homology theory. The *Brown representability theorem* states that, conversely, every generalized (co)homology theory is represented by a spectrum, subject to conditions of additivity.

As a first application, Eilenberg-MacLane spectra representing ordinary cohomology may be characterized via Brown representability.

Literature. (Switzer 75, section 9, Aguilar-Gitler-Prieto 02, section 12, Kochman 96, 3.4)

Traditional discussion

Write $\mathrm{Top}^{*/}_{\geq 1} \hookrightarrow \mathrm{Top}^{*/}$ for the full subcategory of *connected* pointed topological spaces. Write $\mathrm{Set}^{*/}$ for the category of pointed sets.

Definition 1.28. A **Brown functor** is a functor

$$F \,:\, \mathrm{Ho}(\mathrm{Top}^{*/}_{\geq 1})^{\mathrm{op}} \longrightarrow \mathrm{Set}^{*/}$$

(from the opposite of the classical homotopy category (def., def.) of connected pointed topological spaces) such that

1. **(additivity)** F takes small coproducts (wedge sums) to products;
2. **(Mayer-Vietoris)** If $X = \mathrm{Int}(A) \cup \mathrm{Int}(B)$ then for all $x_A \in F(A)$ and $x_B \in F(B)$ such that $(x_A)|_{A\cap B} = (x_B)|_{A \cap B}$ then there exists $x_X \in F(X)$ such that $x_A = (x_X)|_A$ and $x_B = (x_X)|_B$.

Proposition 1.29. *For every additive reduced cohomology theory $\tilde E^\bullet(-) \colon \mathrm{Ho}(\mathrm{Top}^{*/})^{\mathrm{op}} \to \mathrm{Set}^{*/}$ (def. 1.2) and for each degree $n \in \mathbb{N}$, the restriction of $\tilde E^n(-)$ to connected spaces is a Brown functor (def. 1.28).*

Proof. Under the relation between reduced and unreduced cohomology above, this follows from the exactness of the Mayer-Vietoris sequence of prop. 1.16. ∎

Theorem 1.30. ***(Brown representability)***

Every Brown functor F (def. 1.28) is representable, hence there exists $X \in \mathrm{Top}^{*/}_{\geq 1}$ and a natural isomorphism

$$[-,X]_* \overset{\simeq}{\longrightarrow} F(-)$$

(where $[-,-]_*$ denotes the hom-functor of $\mathrm{Ho}(\mathrm{Top}^{*/}_{\geq 1})$ (exmpl.)).

(e.g. AGP 02, theorem 12.2.22)

Remark 1.31. A key subtlety in theorem 1.30 is the restriction to *connected* pointed topological spaces in def. 1.28. This comes about since the proof of the theorem requires that continuous functions $f : X \longrightarrow Y$ that induce isomorphisms on pointed homotopy classes

$$[S^n, X]_* \longrightarrow [S^n, Y]_*$$

for all n are weak homotopy equivalences (For instance in AGP 02 this is used in the proof of theorem 12.2.19 there). But $[S^n, X]_* = \pi_n(X, x)$ gives the nth homotopy group of X *only* for the canonical basepoint, while for a weak homotopy equivalence in general one needs to consider the homotopy groups at all possible basepoints, at least one for each connected component. But so if one does assume that all spaces involved are connected, hence only have one connected component, then indeed weak homotopy equivalences are equivalently those maps $X \to Y$ making all the $[S^n, X]_* \longrightarrow [S^n, Y]_*$ into isomorphisms.

See also example 1.42 below.

The representability result applied degreewise to an additive reduced cohomology theory will yield (prop. 1.33 below) the following concept.

Definition 1.32. An **Omega-spectrum** X (def.) is

1. a sequence $\{X_n\}_{n \in \mathbb{N}}$ of pointed topological spaces $X_n \in \mathrm{Top}^{*/}$

2. weak homotopy equivalences

   $$\tilde\sigma_n : X_n \underset{\in W_{\mathrm{cl}}}{\overset{\tilde\sigma_n}{\longrightarrow}} \Omega X_{n+1}$$

   for each $n \in \mathbb{N}$, form each space to the loop space of the following space.

Proposition 1.33. *Every additive reduced cohomology theory $\tilde E^\bullet(-) : (\mathrm{Top}^*_{\mathrm{CW}})^{\mathrm{op}} \longrightarrow \mathrm{Ab}^{\mathbb{Z}}$ according to def. 1.2, is represented by an Omega-spectrum E (def. 1.32) in that in each degree $n \in \mathbb{N}$*

1. *$\tilde E^n(-)$ is represented by some $E_n \in \mathrm{Ho}(\mathrm{Top}^{*/})$;*

2. *the suspension isomorphism σ_n of $\tilde E^\bullet$ is represented by the structure map $\tilde\sigma_n$ of the Omega-spectrum in that for all $X \in \mathrm{Top}^{*/}$ the following diagram commutes:*

   $$\begin{array}{ccccc} \tilde E^n(X) & \overset{\sigma_n(X)}{\longrightarrow} & & \longrightarrow & \tilde E^{n+1}(\Sigma X) \\ {}^{\simeq}\downarrow & & & & \downarrow^{\simeq} \\ [X, E_n]_* & \overset{[X, \tilde\sigma_n]_*}{\longrightarrow} & [X, \Omega E_{n+1}]_* & \simeq & [\Sigma X, E_{n+1}]_* \end{array} ,$$

   where $[-,-]_* \coloneqq \mathrm{Hom}_{\mathrm{Ho}(\mathrm{Top}^{*/}_{\geq 1})}$ denotes the hom-sets in the classical pointed homotopy category (def.) and where in the bottom right we have the $(\Sigma \dashv \Omega)$-adjunction isomorphism (prop.).

Proof. If it were not for the connectedness clause in def. 1.28 (remark 1.31), then theorem 1.30 with prop. 1.29 would immediately give the existence of the $\{E_n\}_{n \in \mathbb{N}}$ and the remaining statement would follow immediately with the Yoneda lemma, which says in particular that morphisms between representable functors are in natural bijection with the morphisms of objects that represent them.

The argument with the connectivity condition in Brown representability taken into account is essentially the same, just with a little bit more care:

For X a pointed topological space, write $X^{(0)}$ for the connected component of its basepoint. Observe that the loop space of a pointed topological space only depends on this connected component:

$$\Omega X \simeq \Omega(X^{(0)}) \,.$$

Now for $n \in \mathbb{N}$, to show that $\tilde E^n(-)$ is representable by some $E_n \in \mathrm{Ho}(\mathrm{Top}^{*/})$, use first that the restriction of $\tilde E^{n+1}$ to connected spaces is represented by some $E^{(0)}_{n+1}$. Observe that the reduced suspension of any $X \in \mathrm{Top}^{*/}$ lands in $\mathrm{Top}^{*/}_{\geq 1}$. Therefore the $(\Sigma \dashv \Omega)$-adjunction isomorphism (prop.) implies that $\tilde E^{n+1}(\Sigma(-))$ is represented on *all* of $\mathrm{Top}^{*/}$ by $\Omega E^{(0)}_{n+1}$:

$$\tilde E^{n+1}(\Sigma X) \simeq [\Sigma X, E^{(0)}_{n+1}]_* \simeq [X, \Omega E^{(0)}_{n+1}]_* \simeq [X, \Omega E_{n+1}]_* ,$$

where E_{n+1} is any pointed topological space with the given connected component $E^{(0)}_{n+1}$.

Now the suspension isomorphism of $\tilde{E}$ says that $E_n \in \mathrm{Ho}(\mathrm{Top}^{*/})$ representing $\tilde{E}^n$ exists and is given by $\Omega E^{(0)}_{n+1}$:

$$\tilde{E}^n(X) \simeq \tilde{E}^{n+1}(\Sigma, X) \simeq [X, \Omega E_{n+1}]$$

for any E_{n+1} with connected component $E^{(0)}_{n+1}$.

This completes the proof. Notice that running the same argument next for $(n+1)$ gives a representing space E_{n+1} such that its connected component of the base point is $E^{(0)}_{n+1}$ found before. And so on. ▮

Conversely:

Proposition 1.34. *Every Omega-spectrum E, def. 1.32, represents an additive reduced cohomology theory def. 1.1 $\tilde{E}^\bullet$ by*

$$\tilde{E}^n(X) \coloneqq [X, E_n]_*$$

with suspension isomorphism given by

$$\sigma_n : \tilde{E}^n(X) = [X, E_n]_* \overset{[X,\tilde{\sigma}_n]}{\longrightarrow} [X, \Omega E_{n+1}]_* \overset{\simeq}{\to} [\Sigma X, E_{n+1}] = \tilde{E}^{n+1}(\Sigma X) .$$

Proof. The additivity is immediate from the construction. The exactnes follows from the long exact sequences of homotopy cofiber sequences given by this prop.. ▮

Remark 1.35. If we consider the stable homotopy category $\mathrm{Ho}(\mathrm{Spectra})$ of spectra (def.) and consider any topological space X in terms of its suspension spectrum $\Sigma^\infty X \in \mathrm{Ho}(\mathrm{Spectra})$ (exmpl.), then the statement of prop. 1.34 is more succinctly summarized by saying that the graded reduced cohomology groups of a topological space X represented by an Omega-spectrum E are the hom-groups

$$\tilde{E}^\bullet(X) \;\simeq\; [\Sigma^\infty X, \Sigma^\bullet E]$$

in the stable homotopy category, into all the suspensions (thm.) of E.

This means that more generally, for $X \in \mathrm{Ho}(\mathrm{Spectra})$ any spectrum, it makes sense to consider

$$\tilde{E}^\bullet(X) \;\coloneqq\; [X, \Sigma^\bullet E]$$

to be the graded reduced generalized E-cohomology groups of the spectrum X.

See also in *part 1* this example.

Application to ordinary cohomology

Example 1.36. Let A be an abelian group. Consider singular cohomology $H^n(-,A)$ with coefficients in A. The corresponding reduced cohomology evaluated on n-spheres satisfies

$$\tilde{H}^n(S^q, A) \simeq \begin{cases} A & \text{if } q = n \\ 0 & \text{otherwise} \end{cases}$$

Hence singular cohomology is a generalized cohomology theory which is “ordinary cohomology” in the sense of def. 1.6.

Applying the Brown representability theorem as in prop. 1.33 hence produces an Omega-spectrum (def. 1.32) whose nth component space is characterized as having homotopy groups concentrated in degree n on A. These are called *Eilenberg-MacLane spaces* $K(A,n)$

$$\pi_q(K(A,n)) \simeq \begin{cases} A & \text{if } q = n \\ 0 & \text{otherwise} \end{cases} .$$

Here for $n > 0$ then $K(A,n)$ is connected, therefore with an essentially unique basepoint, while $K(A,0)$ is (homotopy equivalent to) the underlying set of the group A.

Such spectra are called **Eilenberg-MacLane spectra** HA:

$$(HA)_n \simeq K(A,n) .$$

As a consequence of example 1.36 one obtains the uniqueness result of Eilenberg-Steenrod:

Proposition 1.37. *Let $\tilde{E}_1$ and $\tilde{E}_2$ be ordinary (def. 1.6) generalized (Eilenberg-Steenrod) cohomology* *theories. If there is an isomorphism*

$$\tilde{E}_1(S^0) \simeq \tilde{E}_2(S^0)$$

of cohomology groups of the 0-sphere, then there is an isomorphism of cohomology theories

$$\tilde{E}_1 \xrightarrow{\simeq} \tilde{E}_2 \,.$$

(e.g. Aguilar-Gitler-Prieto 02, theorem 12.3.6)

Homotopy-theoretic discussion

Using abstract homotopy theory in the guise of model category theory (see the *lecture notes on classical homotopy theory), the traditional proof and further discussion of the Brown representability theorem above becomes more transparent (Lurie 10, section 1.4.1, for exposition see also Mathew 11).*

This abstract homotopy-theoretic proof uses the general concept of homotopy colimits in model categories as well as the concept of derived hom-spaces ("∞-categories"). Even though in the accompanying Lecture notes on classical homotopy theory these concepts are only briefly indicated, the following is included for the interested reader.

Definition 1.38. Let $\mathcal{C}$ be a model category. A functor

$$F \,:\, \mathrm{Ho}(\mathcal{C})^{\mathrm{op}} \longrightarrow \mathrm{Set}$$

(from the opposite of the homotopy category of $\mathcal{C}$ to Set)

is called a **Brown functor** if

1. it sends small coproducts to products;
2. it sends homotopy pushouts in $\mathcal{C} \to \mathrm{Ho}(\mathcal{C})$ to weak pullbacks in Set (see remark 1.39).

Remark 1.39. A *weak pullback* is a diagram that satisfies the existence clause of a pullback, but not necessarily the uniqueness condition. Hence the second clause in def. 1.38 says that for a homotopy pushout square

$$\begin{array}{ccc} Z & \longrightarrow & X \\ \downarrow & \swArrow & \downarrow \\ Y & \longrightarrow & X \underset{Z}{\sqcup} Y \end{array}$$

in $\mathcal{C}$, then the induced universal morphism

$$F(X \underset{Z}{\sqcup} Y) \overset{\mathrm{epi}}{\longrightarrow} F(X) \underset{F(Z)}{\times} F(Y)$$

into the actual pullback is an epimorphism.

Definition 1.40. Say that a model category $\mathcal{C}$ is **compactly generated by cogroup objects closed under suspensions** if

1. $\mathcal{C}$ is generated by a set

   $$\{S_i \in \mathcal{C}\}_{i \in I}$$

   of compact objects (i.e. every object of $\mathcal{C}$ is a homotopy colimit of the objects S_i.)
2. each S_i admits the structure of a cogroup object in the homotopy category $\mathrm{Ho}(\mathcal{C})$;
3. the set $\{S_i\}$ is closed under forming reduced suspensions.

Example 1.41. (**suspensions are H-cogroup objects**)

Let $\mathcal{C}$ be a model category and $\mathcal{C}^{*/}$ its pointed model category (prop.) with zero object (rmk.). Write $\Sigma : X \mapsto 0 \coprod_X 0$ for the reduced suspension functor.

Then the fold map

$$\Sigma X \coprod \Sigma X \simeq 0 \underset{X}{\sqcup} 0 \underset{X}{\sqcup} 0 \longrightarrow 0 \underset{X}{\sqcup} X \underset{X}{\sqcup} 0 \simeq 0 \underset{X}{\sqcup} 0 \simeq \Sigma X$$

exhibits cogroup structure on the image of any suspension object ΣX in the homotopy category.

This is equivalently the group-structure of the first (fundamental) homotopy group of the values of functor co-represented by ΣX:

$$\mathrm{Ho}(\mathcal{C})(\Sigma X, -) : Y \mapsto \mathrm{Ho}(\mathcal{C})(\Sigma X, Y) \simeq \mathrm{Ho}(\mathcal{C})(X, \Omega Y) \simeq \pi_1 \mathrm{Ho}(\mathcal{C})(X, Y) \,.$$

Example 1.42. In bare pointed homotopy types $\mathcal{C} = \mathrm{Top}^{*/}_{\mathrm{Quillen}}$, the (homotopy types of) n-spheres S^n are cogroup objects for $n \geq 1$, but not for $n = 0$, by example 1.41. And of course they are compact objects.

So while $\{S^n\}_{n \in \mathbb{N}}$ generates all of the homotopy theory of $\mathrm{Top}^{*/}$, the latter is *not* an example of def. 1.40 due to the failure of S^0 to have cogroup structure.

Removing that generator, the homotopy theory generated by $\{S^n\}_{\substack{n \in \mathbb{N} \\ n \geq 1}}$ is $\mathrm{Top}^{*/}_{\geq 1}$, that of *connected* pointed homotopy types. This is one way to see how the connectedness condition in the classical version of Brown representability theorem arises. See also remark 1.31 above.

See also (Lurie 10, example 1.4.1.4)

In homotopy theories compactly generated by cogroup objects closed under forming suspensions, the following strenghtening of the Whitehead theorem holds.

Proposition 1.43. *In a homotopy theory compactly generated by cogroup objects $\{S_i\}_{i \in I}$ closed under forming suspensions, according to def. 1.40, a morphism $f : X \longrightarrow Y$ is an equivalence precisely if for each $i \in I$ the induced function of maps in the homotopy category*

$$\mathrm{Ho}(\mathcal{C})(S_i, f) : \mathrm{Ho}(\mathcal{C})(S_i, X) \longrightarrow \mathrm{Ho}(\mathcal{C})(S_i, Y)$$

is an isomorphism (a bijection).

(Lurie 10, p. 114, Lemma star)

Proof. By the ∞-Yoneda lemma, the morphism f is a weak equivalence precisely if for all objects $A \in \mathcal{C}$ the induced morphism of derived hom-spaces

$$\mathcal{C}(A, f) : \mathcal{C}(A, X) \longrightarrow \mathcal{C}(A, Y)$$

is an equivalence in $\mathrm{Top}_{\mathrm{Quillen}}$. By assumption of compact generation and since the hom-functor $\mathcal{C}(-,-)$ sends homotopy colimits in the first argument to homotopy limits, this is the case precisely already if it is the case for $A \in \{S_i\}_{i \in I}$.

Now the maps

$$\mathcal{C}(S_i, f) : \mathcal{C}(S_i, X) \longrightarrow \mathcal{C}(S_i, Y)$$

are weak equivalences in $\mathrm{Top}_{\mathrm{Quillen}}$ if they are weak homotopy equivalences, hence if they induce isomorphisms on all homotopy groups π_n for **all basepoints**.

It is this last condition of testing on all basepoints that the assumed cogroup structure on the S_i allows to do away with: this cogroup structure implies that $\mathcal{C}(S_i, -)$ has the structure of an H-group, and this implies (by group multiplication), that all connected components have the same homotopy groups, hence that all homotopy groups are independent of the choice of basepoint, up to isomorphism.

Therefore the above morphisms are equivalences precisely if they are so under applying π_n based on the connected component of the zero morphism

$$\pi_n \mathcal{C}(S_i, f) : \pi_n \mathcal{C}(S_i, X) \longrightarrow \pi_n \mathcal{C}(S_i, Y) \,.$$

Now in this pointed situation we may use that

$$\begin{aligned} \pi_n \mathcal{C}(-,-) &\simeq \pi_0 \mathcal{C}(-, \Omega^n(-)) \\ &\simeq \pi_0 \mathcal{C}(\Sigma^n(-), -) \\ &\simeq \mathrm{Ho}(\mathcal{C})(\Sigma^n(-), -) \end{aligned}$$

to find that f is an equivalence in $\mathcal{C}$ precisely if the induced morphisms

$$\mathrm{Ho}(\mathcal{C})(\Sigma^n S_i, f) : \mathrm{Ho}(\mathcal{C})(\Sigma^n S_i, X) \longrightarrow \mathrm{Ho}(\mathcal{C})(\Sigma^n S_i, Y)$$

are isomorphisms for all $i \in I$ and $n \in \mathbb{N}$.

Finally by the assumption that each suspension $\Sigma^n S_i$ of a generator is itself among the set of generators, the claim follows. ▮

Theorem 1.44. ***(Brown representability)***

Let $\mathcal{C}$ be a model category compactly generated by cogroup objects closed under forming suspensions, according to def. 1.40. Then a functor

$$F \,:\, \mathrm{Ho}(\mathcal{C})^{\mathrm{op}} \longrightarrow \mathrm{Set}$$

(from the opposite of the homotopy category of $\mathcal{C}$ to Set) is representable precisely if it is a Brown functor, def. 1.38.

(Lurie 10, theorem 1.4.1.2)

Proof. Due to the version of the Whitehead theorem of prop. 1.43 we are essentially reduced to showing that Brown functors F are representable on the S_i. To that end consider the following lemma. (In the following we notationally identify, via the Yoneda lemma, objects of $\mathcal{C}$, hence of $\mathrm{Ho}(\mathcal{C})$, with the functors they represent.)

Lemma ($\star$): *Given $X \in \mathcal{C}$ and $\eta \in F(X)$, hence $\eta : X \to F$, then there exists a morphism $f : X \to X'$ and an extension $\eta' : X' \to F$ of η which induces for each S_i a bijection $\eta' \circ (-) : \mathrm{PSh}(\mathrm{Ho}(\mathcal{C}))(S_i, X') \overset{\simeq}{\longrightarrow} \mathrm{Ho}(\mathcal{C})(S_i, F) \simeq F(S_i)$.*

To see this, first notice that we may directly find an extension η_0 along a map $X \to X_o$ such as to make a surjection: simply take X_0 to be the coproduct of **all** possible elements in the codomain and take

$$\eta_0 \;:\; X \sqcup \Big(\coprod_{\substack{i \in I, \\ \gamma : S_i \to F}} S_i \Big) \longrightarrow F$$

to be the canonical map. (Using that F, by assumption, turns coproducts into products, we may indeed treat the coproduct in $\mathcal{C}$ on the left as the coproduct of the corresponding functors.)

To turn the surjection thus constructed into a bijection, we now successively form quotients of X_0. To that end proceed by induction and suppose that $\eta_n : X_n \to F$ has been constructed. Then for $i \in I$ let

$$K_i \coloneqq \ker\Big(\mathrm{Ho}(\mathcal{C})(S_i, X_n) \overset{\eta_n \circ (-)}{\longrightarrow} F(S_i) \Big)$$

be the kernel of η_n evaluated on S_i. These K_i are the pieces that need to go away in order to make a bijection. Hence define X_{n+1} to be their joint homotopy cofiber

$$X_{n+1} \coloneqq \mathrm{coker}\left(\Big(\coprod_{\substack{i \in I, \\ \gamma \in K_i}} S_i \Big) \xrightarrow{(\gamma)_{\substack{i \in I \\ \gamma \in K_i}}} X_n \right).$$

Then by the assumption that F takes this homotopy cokernel to a weak fiber (as in remark 1.39), there exists an extension η_{n+1} of η_n along $X_n \to X_{n+1}$:

Then by the assumption that F takes this homotopy cokernel to a weak fiber (as in remark 1.39), there exists an extension η_{n+1} of η_n along $X_n \to X_{n+1}$:

$$
\begin{tikzcd}
\Big(\coprod_{\substack{i \in I \\ \gamma \in K_i}} S_i \Big) \arrow[r, "(\gamma)_{\substack{i \in I \\ \gamma \in K_i}}"] \arrow[d] \arrow[dr, phantom, "(\mathrm{po}^h)"] & X_n \arrow[r, "\eta_n"] \arrow[d] & F \\
{*} \arrow[r] & X_{n+1} \arrow[ur, "\exists \eta_{n+1}"'] &
\end{tikzcd}
\quad \Leftrightarrow \quad
\begin{tikzcd}
& F(X_{n+1}) \arrow[r] \arrow[d, "\mathrm{epi}"] & {*} \arrow[d] \\
{*} \arrow[ur, "\exists \eta_{n+1}"] \arrow[r, "\eta_n"] \arrow[dr, "\eta_n"'] & \ker\big((\gamma^*)_{\substack{i \in I \\ \gamma \in K_i}}\big) \arrow[r] \arrow[d] \arrow[dr, phantom, "(\mathrm{pb})"] & {*} \arrow[d] \\
& F(X_n) \arrow[r, "(\gamma^*)_{\substack{i \in I \\ \gamma \in K_i}}"'] & \prod_{\substack{i \in I \\ \gamma \in K_i}} F(S_i)
\end{tikzcd}
.
$$

It is now clear that we want to take

$$X' \coloneqq \underset{\longrightarrow_n}{\lim} X_n$$

and extend all the η_n to that colimit. Since we have no condition for evaluating F on colimits other than pushouts, observe that this sequential colimit is equivalent to the following pushout:

$$\begin{array}{ccc} \underset{n}{\sqcup} X_n & \longrightarrow & \underset{n}{\sqcup} X_{2n} \\ \downarrow && \downarrow \\ \underset{n}{\sqcup} X_{2n+1} & \longrightarrow & X' \end{array} \,,$$

where the components of the top and left map alternate between the identity on X_n and the above successor maps $X_n \to X_{n+1}$. Now the excision property of F applies to this pushout, and we conclude the desired extension $\eta' : X' \to F$:

$$\array{ && \underset{n}{\sqcup} X_n \\ & \swarrow && \searrow \\ \underset{n}{\sqcup} X_{2n+1} &\longrightarrow& X' &\longleftarrow& \underset{n}{\sqcup} X_{2n} \\ & {}_{\mathllap{(\eta_{2n+1})_n}}\searrow & \downarrow^{\exists \eta} & \swarrow_{\mathrlap{(\eta_{2n})_n}} \\ && F } \;\;\;\; \Leftrightarrow \;\;\;\; \array{ && F(X') \\ & {}^{\exists \eta}\nearrow & \downarrow^{\mathrm{epi}} \\ & \ast \overset{(\eta_n)_n}{\longrightarrow} & \underset{\longleftarrow_n}{\lim} F(X_n) \\ & \swarrow && \searrow \\ \prod_n F(X_{2n+1}) &&&& \prod_n (X_{2n}) \\ & \searrow && \swarrow \\ && \prod_n F(X_n) } \,,$$

It remains to confirm that this indeed gives the desired bijection. Surjectivity is clear. For injectivity use that all the S_i are, by assumption, compact, hence they may be taken inside the sequential colimit:

$$\array{ && X_{n(\gamma)} \\ & {}^{\exists \hat \gamma}\nearrow & \downarrow \\ S_i &\overset{\gamma}{\longrightarrow}& X' = \underset{\longrightarrow_n}{\lim} X_n } \,.$$

With this, injectivity follows because by construction we quotiented out the kernel at each stage. Because suppose that γ is taken to zero in $F(S_i)$, then by the definition of X_{n+1} above there is a factorization of γ through the point:

$$\array{ 0 \colon & S_i &\overset{\hat \gamma}{\longrightarrow}& X_{n(\gamma)} &\overset{\eta_n}{\longrightarrow}& F \\ & \downarrow && \downarrow \\ & \ast &\longrightarrow& X_{n(\gamma)+1} \\ &&& \downarrow \\ &&& X' }$$

This concludes the proof of Lemma ($\star$).

Now apply the construction given by this lemma to the case $X_0 \coloneqq 0$ and the unique $\eta_0 : 0 \overset{\exists !}{\to} F$. Lemma ($\star$) then produces an object X' which represents F on all the S_i, and we want to show that this X' actually represents F generally, hence that for every $Y \in \mathcal{C}$ the function

$$\theta \coloneqq \eta' \circ (-) \;\colon\; \mathrm{Ho}(\mathcal{C})(Y,X') \longrightarrow F(Y)$$

is a bijection.

First, to see that θ is surjective, we need to find a preimage of any $\rho : Y \to F$. Applying Lemma ($\star$) to $(\eta', \rho) : X' \sqcup Y \longrightarrow F$ we get an extension κ of this through some $X' \sqcup Y \longrightarrow Z$ and the morphism on the right of the following commuting diagram:

$$\array{ \mathrm{Ho}(\mathcal{C})(-,X') && \longrightarrow && \mathrm{Ho}(\mathcal{C})(-,Z) \\ & {}_{\mathllap{\eta' \circ (-)}}\searrow && \swarrow_{\mathrlap{\kappa \circ (-)}} \\ && F(-) } \,.$$

Moreover, Lemma ($\star$) gives that evaluated on all S_i, the two diagonal morphisms here become isomorphisms. But then prop. 1.43 implies that $X' \longrightarrow Z$ is in fact an equivalence. Hence the component map $Y \to Z \simeq Z$ is a lift of κ through θ.

Second, to see that θ is injective, suppose $f, g : Y \to X'$ have the same image under θ. Then consider their homotopy pushout

$$\begin{array}{ccc} Y \sqcup Y & \overset{(f,g)}{\longrightarrow} & X' \\ \downarrow && \downarrow \\ Y & \longrightarrow & Z \end{array}$$

along the codiagonal of Y. Using that F sends this to a weak pullback by assumption, we obtain an extension $\bar\eta$ of η' along $X' \to Z$. Applying Lemma $(\star)$ to this gives a further extension $\bar\eta' : Z' \to Z$ which now makes the following diagram

$$\begin{array}{ccc} Ho(\mathcal{C})(-,X') && \longrightarrow && Ho(\mathcal{C})(-,Z) \\ & {}_{\eta' \circ (-)}\searrow && \swarrow_{\bar\eta' \circ (-)} \\ && F(-) \end{array}$$

such that the diagonal maps become isomorphisms when evaluated on the S_i. As before, it follows via prop. 1.43 that the morphism $h : X' \longrightarrow Z'$ is an equivalence.

Since by this construction $h \circ f$ and $h \circ g$ are homotopic

$$\begin{array}{ccccc} Y \sqcup Y & \overset{(f,g)}{\longrightarrow} & X' \\ \downarrow && \downarrow & \searrow^{h}_{\simeq} \\ Y & \longrightarrow & Z & \longrightarrow & Z' \end{array}$$

it follows with h being an equivalence that already f and g were homotopic, hence that they represented the same element. ▮

Proposition 1.45. *Given a reduced additive cohomology functor $H^\bullet : Ho(\mathcal{C})^{op} \to Ab^{\mathbb{Z}}$, def. 1.5, its underlying Set-valued functors $H^n : Ho(\mathcal{C})^{op} \to Ab \to Set$ are Brown functors, def. 1.38.*

Proof. The first condition on a Brown functor holds by definition of $H^\bullet$. For the second condition, given a homotopy pushout square

$$\begin{array}{ccc} X_1 & \overset{f_1}{\longrightarrow} & Y_1 \\ \downarrow && \downarrow \\ X_2 & \overset{f_2}{\longrightarrow} & Y_2 \end{array}$$

in $\mathcal{C}$, consider the induced morphism of the long exact sequences given by prop. 1.8

$$\begin{array}{ccccccc} H^\bullet(coker(f_2)) & \longrightarrow & H^\bullet(Y_2) & \overset{f_2^*}{\longrightarrow} & H^\bullet(X_2) & \longrightarrow & H^{\bullet+1}(\Sigma coker(f_2)) \\ {}^{\simeq}\downarrow && \downarrow && \downarrow && \downarrow^{\simeq} \\ H^\bullet(coker(f_1)) & \longrightarrow & H^\bullet(Y_1) & \overset{f_1^*}{\longrightarrow} & H^\bullet(X_1) & \longrightarrow & H^{\bullet+1}(\Sigma coker(f_1)) \end{array}$$

Here the outer vertical morphisms are isomorphisms, as shown, due to the pasting law (see also at *fiberwise recognition of stable homotopy pushouts*). This means that the four lemma applies to this diagram. Inspection shows that this implies the claim. ▮

Corollary 1.46. *Let $\mathcal{C}$ be a model category which satisfies the conditions of theorem 1.44, and let $(H^\bullet, \delta)$ be a reduced additive generalized cohomology functor on $\mathcal{C}$, def. 1.5. Then there exists a spectrum object $E \in Stab(\mathcal{C})$ such that*

1. *$H\bullet$ is degreewise represented by E:*

$$H^\bullet \simeq Ho(\mathcal{C})(-, E_\bullet) ,$$

2. *the suspension isomorphism δ is given by the structure morphisms $\tilde\sigma_n : E_n \to \Omega E_{n+1}$ of the spectrum, in that*

$$\delta : H^n(-) \simeq Ho(\mathcal{C})(-, E_n) \overset{Ho(\mathcal{C})(-,\tilde\sigma_n)}{\longrightarrow} Ho(\mathcal{C})(-, \Omega E_{n+1}) \simeq Ho(\mathcal{C})(\Sigma(-), E_{n+1}) \simeq H^{n+1}(\Sigma(-)) .$$

Proof. Via prop. 1.45, theorem 1.44 gives the first clause. With this, the second clause follows by the Yoneda lemma. ▮

Milnor exact sequence

Idea. One tool for computing generalized cohomology groups via “inverse limits” are *Milnor exact* *sequences*. For instance the generalized cohomology of the classifying space $BU(1)$ plays a key role in the complex oriented cohomology-theory discussed below, and via the equivalence $BU(1) \simeq \mathbb{C}P^\infty$ to the homotopy type of the infinite complex projective space (def. 1.134), which is the direct limit of finite dimensional projective spaces $\mathbb{C}P^n$, this is an inverse limit of the generalized cohomology groups of the $\mathbb{C}P^n$s. But what really matters here is the derived functor of the limit-operation – the homotopy limit – and the Milnor exact sequence expresses how the naive limits receive corrections from higher "lim^1-terms". In practice one mostly proceeds by verifying conditions under which these corrections happen to disappear, these are the *Mittag-Leffler conditions*.

We need this for instance for the computation of Conner-Floyd Chern classes below.

Literature. (Switzer 75, section 7 from def. 7.57 on, Kochman 96, section 4.2, Goerss-Jardine 99, section VI.2,)

Lim^1

Definition 1.47. Given a tower $A_\bullet$ of abelian groups

$$\cdots \to A_3 \overset{f_2}{\to} A_2 \overset{f_1}{\to} A_1 \overset{f_0}{\to} A_0$$

write

$$\partial : \prod_n A_n \longrightarrow \prod_n A_n$$

for the homomorphism given by

$$\partial : (a_n)_{n\in\mathbb{N}} \mapsto (a_n - f_n(a_{n+1}))_{n\in\mathbb{N}} \,.$$

Remark 1.48. The limit of a sequence as in def. 1.47 – hence the group $\underset{\longleftarrow_n}{\lim} A_n$ universally equipped with morphisms $\underset{\longleftarrow_n}{\lim} A_n \overset{p_n}{\to} A_n$ such that all

$$\begin{array}{ccc} & \underset{\longleftarrow_n}{\lim} A_n & \\ {}^{p_{n+1}}\swarrow & & \searrow^{p_n} \\ A_{n+1} & \overset{f_n}{\longrightarrow} & A_n \end{array}$$

commute – is equivalently the kernel of the morphism ∂ in def. 1.47.

Definition 1.49. Given a tower $A_\bullet$ of abelian groups

$$\cdots \to A_3 \overset{f_2}{\to} A_2 \overset{f_1}{\to} A_1 \overset{f_0}{\to} A_0$$

then $\underset{\longleftarrow}{\lim}^1 A_\bullet$ is the cokernel of the map ∂ in def. 1.47, hence the group that makes a long exact sequence of the form

$$0 \to \underset{\longleftarrow_n}{\lim} A_n \longrightarrow \prod_n A_n \overset{\partial}{\longrightarrow} \prod_n A_n \longrightarrow \underset{\longleftarrow_n}{\lim}^1 A_n \to 0 \,,$$

Proposition 1.50. *The functor $\underset{\longleftarrow}{\lim}^1 : \mathrm{Ab}^{(\mathbb{N},\geq)} \longrightarrow \mathrm{Ab}$ (def. 1.49) satisfies*

1. *for every short exact sequence $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0 \quad \in \mathrm{Ab}^{(\mathbb{N},\geq)}$ then the induced sequence*

$$0 \to \underset{\longleftarrow_n}{\lim} A_n \to \underset{\longleftarrow_n}{\lim} B_n \to \underset{\longleftarrow_n}{\lim} C_n \to \underset{\longleftarrow_n}{\lim}^1 A_n \to \underset{\longleftarrow_n}{\lim}^1 B_n \to \underset{\longleftarrow_n}{\lim}^1 C_n \to 0$$

   is a long exact sequence of abelian groups;

2. *if $A_\bullet$ is a tower such that all maps are surjections, then $\underset{\longleftarrow_n}{\lim}^1 A_n \simeq 0$.*

(e.g. Switzer 75, prop. 7.63, Goerss-Jardine 96, section VI. lemma 2.11)

Proof. For the first property: Given $A_\bullet$ a tower of abelian groups, write

$$L^\bullet(A_\bullet) \coloneqq \left[0 \to \underbrace{\prod_n A_n}_{\deg 0} \overset{\partial}{\to} \underbrace{\prod_n A_n}_{\deg 1} \to 0 \right]$$

for the homomorphism from def. 1.47 regarded as the single non-trivial differential in a cochain complex of abelian groups. Then by remark 1.48 and def. 1.49 we have $H^0(L(A_\bullet)) \simeq \underset{\longleftarrow}{\lim} A_\bullet$ and $H^1(L(A_\bullet)) \simeq \underset{\longleftarrow}{\lim}^1 A_\bullet$.

With this, then for a short exact sequence of towers $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ the long exact sequence in question is the long exact sequence in homology of the corresponding short exact sequence of complexes

$$0 \to L^\bullet(A_\bullet) \longrightarrow L^\bullet(B_\bullet) \longrightarrow L^\bullet(C_\bullet) \to 0\,.$$

For the second statement: If all the f_k are surjective, then inspection shows that the homomorphism ∂ in def. 1.47 is surjective. Hence its cokernel vanishes. ▮

Lemma 1.51. *The category $\mathrm{Ab}^{(\mathbb{N},\geq)}$ of towers of abelian groups has enough injectives.*

Proof. The functor $(-)_n : \mathrm{Ab}^{(\mathbb{N},\geq)} \to \mathrm{Ab}$ that picks the n-th component of the tower has a right adjoint r_n, which sends an abelian group A to the tower

$$r_n \coloneqq \left[\cdots \overset{\mathrm{id}}{\to} A \overset{\mathrm{id}}{\to} \underbrace{A}_{=(r_n)_{n+1}} \overset{\mathrm{id}}{\to} \underbrace{A}_{=(r_n)_n} \overset{\mathrm{id}}{\to} \underbrace{0}_{=(r_n)_{n-1}} \to 0 \to \cdots \to 0 \to 0 \right].$$

Since $(-)_n$ itself is evidently an exact functor, its right adjoint preserves injective objects (prop.).

So with $A_\bullet \in \mathrm{Ab}^{(\mathbb{N},\geq)}$, let $A_n \hookrightarrow \tilde A_n$ be an injective resolution of the abelian group A_n, for each $n \in \mathbb{N}$. Then

$$A_\bullet \overset{(\eta_n)_{n \in \mathbb{N}}}{\longrightarrow} \prod_{n \in \mathbb{R}} r_n A_n \hookrightarrow \prod_{n \in \mathbb{N}} r_n \tilde A_n$$

is an injective resolution for $A_\bullet$. ▮

Proposition 1.52. *The functor $\underset{\longleftarrow}{\lim}^1 : \mathrm{Ab}^{(\mathbb{N},\geq)} \longrightarrow \mathrm{Ab}$ (def. 1.49) is the first right derived functor of the limit functor $\underset{\longleftarrow}{\lim} : \mathrm{Ab}^{(\mathbb{N},\geq)} \longrightarrow \mathrm{Ab}$.*

Proof. By lemma 1.51 there are enough injectives in $\mathrm{Ab}^{(\mathbb{N},\geq)}$. So for $A_\bullet \in \mathrm{Ab}^{(\mathbb{N},\geq)}$ the given tower of abelian groups, let

$$0 \to A_\bullet \overset{j^0}{\longrightarrow} J^0_\bullet \overset{j^1}{\longrightarrow} J^1_\bullet \overset{j^2}{\longrightarrow} J^2_\bullet \longrightarrow \cdots$$

be an injective resolution. We need to show that

$$\underset{\longleftarrow}{\lim}^1 A_\bullet \simeq \ker(\underset{\longleftarrow}{\lim}(j^2)) / \mathrm{im}(\underset{\longleftarrow}{\lim}(j^1))\,.$$

Since limits preserve kernels, this is equivalently

$$\underset{\longleftarrow}{\lim}^1 A_\bullet \simeq (\underset{\longleftarrow}{\lim}(\ker(j^2)_\bullet)) / \mathrm{im}(\underset{\longleftarrow}{\lim}(j^1))$$

Now observe that each injective $J^q_\bullet$ is a tower of epimorphism. This follows by the defining right lifting property applied against the monomorphisms of towers of the following form

$$\begin{array}{ccccccccccccc}
\cdots & \to & 0 & \to & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \overset{\mathrm{id}}{\longrightarrow} & \cdots & \overset{\mathrm{id}}{\longrightarrow} & \mathbb{Z} & \overset{\mathrm{id}}{\longrightarrow} & \mathbb{Z} \\
\cdots & & \downarrow^{\mathrm{id}} & & \downarrow^{\mathrm{id}} & & \downarrow^{\mathrm{incl}} & & \downarrow^{\mathrm{id}} & & & & \downarrow^{\mathrm{id}} & & \downarrow^{\mathrm{id}} \\
\cdots & \to & 0 & \to & 0 & \to & \mathbb{Z} & \underset{\mathrm{id}}{\longrightarrow} & \mathbb{Z} & \underset{\mathrm{id}}{\longrightarrow} & \cdots & \underset{\mathrm{id}}{\longrightarrow} & \mathbb{Z} & \underset{\mathrm{id}}{\longrightarrow} & \mathbb{Z}
\end{array}$$

Therefore by the second item of prop. 1.50 the long exact sequence from the first item of prop. 1.50 applied to the short exact sequence

$$0 \to A_\bullet \overset{j^0}{\longrightarrow} J^0_\bullet \overset{j^1}{\longrightarrow} \ker(j^2)_\bullet \to 0$$

becomes

$$0 \to \underset{\longleftarrow}{\lim} A_\bullet \overset{\underset{\longleftarrow}{\lim} j^0}{\longrightarrow} \underset{\longleftarrow}{\lim} J^0_\bullet \overset{\underset{\longleftarrow}{\lim} j^1}{\longrightarrow} \underset{\longleftarrow}{\lim}(\ker(j^2)_\bullet) \longrightarrow \underset{\longleftarrow}{\lim}^1 A_\bullet \longrightarrow 0\,.$$

Exactness of this sequence gives the desired identification $\underset{\longleftarrow}{\lim}^1 A_\bullet \simeq (\underset{\longleftarrow}{\lim}(\ker(j^2)_\bullet))/\mathrm{im}(\underset{\longleftarrow}{\lim}(j^1))$. ▮

Proposition 1.53. *The functor $\underset{\longleftarrow}{\lim}^1 : \mathrm{Ab}^{(\mathbb{N},\geq)} \longrightarrow \mathrm{Ab}$ (def. 1.49) is in fact the unique functor, up to natural isomorphism, satisfying the conditions in prop. 1.53.*

Proof. The proof of prop. 1.52 only used the conditions from prop. 1.50, hence any functor satisfying these conditions is the first right derived functor of $\underset{\longleftarrow}{\lim}$, up to natural isomorphism. ▮

The following is a kind of double dual version of the $\lim^1$ construction which is sometimes useful:

Lemma 1.54. *Given a cotower*

$$A_\bullet = (A_0 \overset{f_0}{\to} A_1 \overset{f_1}{\to} A_2 \to \cdots)$$

of abelian groups, then for every abelian group $B \in \mathrm{Ab}$ there is a short exact sequence of the form

$$0 \to \underset{\longleftarrow n}{\lim}^1 \mathrm{Hom}(A_n, B) \longrightarrow \mathrm{Ext}^1(\underset{\longrightarrow n}{\lim} A_n, B) \longrightarrow \underset{\longleftarrow n}{\lim} \mathrm{Ext}^1(A_n, B) \to 0 \,,$$

where $\mathrm{Hom}(-,-)$ denotes the hom-group, $\mathrm{Ext}^1(-,-)$ denotes the first Ext-group (and so $\mathrm{Hom}(-,-) = \mathrm{Ext}^0(-,-)$).

Proof. Consider the homomorphism

$$\tilde{\partial} : \underset{n}{\oplus} A_n \longrightarrow \underset{n}{\oplus} A_n$$

which sends $a_n \in A_n$ to $a_n - f_n(a_n)$. Its cokernel is the colimit over the cotower, but its kernel is trivial (in contrast to the otherwise formally dual situation in remark 1.48). Hence (as opposed to the long exact sequence in def. 1.49) there is a short exact sequence of the form

$$0 \to \underset{n}{\oplus} A_n \overset{\tilde{\partial}}{\longrightarrow} \underset{n}{\oplus} A_n \longrightarrow \underset{\longrightarrow n}{\lim} A_n \to 0 \,.$$

Every short exact sequence gives rise to a long exact sequence of derived functors (prop.) which in the present case starts out as

$$0 \to \mathrm{Hom}(\underset{\longrightarrow n}{\lim} A_n, B) \longrightarrow \prod_n \mathrm{Hom}(A_n, B) \overset{\partial}{\longrightarrow} \prod_n \mathrm{Hom}(A_n, B) \longrightarrow \mathrm{Ext}^1(\underset{\longrightarrow n}{\lim} A_n, B) \longrightarrow \prod_n \mathrm{Ext}^1(A_n, B) \overset{\partial}{\longrightarrow} \prod_n \mathrm{Ext}^1(A_n, B) \longrightarrow \cdots$$

where we used that direct sum is the coproduct in abelian groups, so that homs out of it yield a product, and where the morphism ∂ is the one from def. 1.47 corresponding to the tower

$$\mathrm{Hom}(A_\bullet, B) = (\cdots \to \mathrm{Hom}(A_2, B) \to \mathrm{Hom}(A_1, B) \to \mathrm{Hom}(A_0, B)) \,.$$

Hence truncating this long sequence by forming kernel and cokernel of ∂, respectively, it becomes the short exact sequence in question. ▮

Mittag-Leffler condition

Definition 1.55. A tower $A_\bullet$ of abelian groups

$$\cdots \to A_3 \to A_2 \to A_1 \to A_0$$

is said to satify the **Mittag-Leffler condition** if for all k there exists $i \geq k$ such that for all $j \geq i \geq k$ the image of the homomorphism $A_i \to A_k$ equals that of $A_j \to A_k$

$$\mathrm{im}(A_i \to A_k) \simeq \mathrm{im}(A_j \to A_k) \,.$$

(e.g. Switzer 75, def. 7.74)

Example 1.56. The Mittag-Leffler condition, def. 1.55, is satisfied in particular when all morphisms $A_{i+1} \to A_i$ are epimorphisms (hence surjections of the underlying sets).

Proposition 1.57. *If a tower $A_\bullet$ satisfies the Mittag-Leffler condition, def. 1.55, then its $\underset{\longleftarrow}{\lim}^1$ vanishes:*

$$\underset{\longleftarrow}{\lim}^1 A_\bullet = 0 \,.$$

e.g. (Switzer 75, theorem 7.75, Kochmann 96, prop. 4.2.3, Weibel 94, prop. 3.5.7)

Proof idea. One needs to show that with the Mittag-Leffler condition, then the cokernel of ∂ in def. 1.47

vanishes, hence that ∂ is an epimorphism in this case, hence that every $(a_n)_{n\in\mathbb{N}} \in \prod_n A_n$ has a preimage under ∂. So use the Mittag-Leffler condition to find pre-images of a_n by induction over n. ∎

Mapping telescopes

Given a sequence

$$X_\bullet = \left(X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots\right)$$

of (pointed) topological spaces, then its *mapping telescope* is the result of forming the (reduced) mapping cylinder $\mathrm{Cyl}(f_n)$ for each n and then attaching all these cylinders to each other in the canonical way

Definition 1.58. For

$$X_\bullet = \left(X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots\right)$$

a sequence in Top, its **mapping telescope** is the quotient topological space of the disjoint union of product topological spaces

$$\mathrm{Tel}(X_\bullet) \coloneqq \left(\underset{n\in\mathbb{N}}{\sqcup} (X_n \times [n, n+1])\right)/_\sim$$

where the equivalence relation quotiented out is

$$(x_n, n) \sim (f(x_n), n+1)$$

for all $n \in \mathbb{N}$ and $x_n \in X_n$.

Analogously for $X_\bullet$ a sequence of pointed topological spaces then use reduced cylinders (exmpl.) to set

$$\mathrm{Tel}(X_\bullet) \coloneqq \left(\underset{n\in\mathbb{N}}{\sqcup} (X_n \wedge [n, n+1]_+)\right)/_\sim \;.$$

Lemma 1.59. *For $X_\bullet$ the sequence of stages of a (pointed) CW-complex $X = \underset{\longleftarrow n}{\lim} X_n$, then the canonical map*

$$\mathrm{Tel}(X_\bullet) \longrightarrow X$$

from the mapping telescope, def. 1.58, is a weak homotopy equivalence.

Proof. Write in the following $\mathrm{Tel}(X)$ for $\mathrm{Tel}(X_\bullet)$ and write $\mathrm{Tel}(X_n)$ for the mapping telescop of the substages of the finite stage X_n of X. It is intuitively clear that each of the projections at finite stage

$$\mathrm{Tel}(X_n) \longrightarrow X_n$$

is a homotopy equivalence, hence (prop.) a weak homotopy equivalence. A concrete construction of a homotopy inverse is given for instance in (Switzer 75, proof of prop. 7.53).

Moreover, since spheres are compact, so that elements of homotopy groups $\pi_q(\mathrm{Tel}(X))$ are represented at some finite stage $\pi_q(\mathrm{Tel}(X_n))$ it follows that

$$\underset{\longrightarrow n}{\lim} \pi_q(\mathrm{Tel}(X_n)) \xrightarrow{\simeq} \pi_q(\mathrm{Tel}(X))$$

are isomorphisms for all $q \in \mathbb{N}$ and all choices of basepoints (not shown).

Together these two facts imply that in the following commuting square, three morphisms are isomorphisms, as shown.

$$\begin{array}{ccc} \underset{\longleftarrow n}{\lim} \pi_q(\mathrm{Tel}(X_n)) & \xrightarrow{\simeq} & \pi_q(\mathrm{Tel}(X)) \\ {}^{\simeq}\downarrow && \downarrow \\ \underset{\longleftarrow n}{\lim} \pi_q(X_n) & \underset{\simeq}{\longrightarrow} & \pi_q(X) \end{array} \;.$$

Therefore also the remaining morphism is an isomorphism (two-out-of-three). Since this holds for all q and all basepoints, it is a weak homotopy equivalence. ∎

Milnor exact sequences

Proposition 1.60. ***(Milnor exact sequence for homotopy groups)***

Let

$$\cdots \to X_3 \overset{p_2}{\longrightarrow} X_2 \overset{p_1}{\longrightarrow} X_1 \overset{p_0}{\longrightarrow} X_0$$

be a tower of fibrations (Serre fibrations (def.)). Then for each $q \in \mathbb{N}$ there is a short exact sequence

$$0 \to \underset{\longleftarrow_i}{\lim}^1 \pi_{q+1}(X_i) \longrightarrow \pi_q(\underset{\longleftarrow_i}{\lim} X_i) \longrightarrow \underset{\longleftarrow_i}{\lim} \pi_q(X_i) \to 0\,,$$

for $\pi_\bullet$ the homotopy group-functor (exact as pointed sets for $i = 0$, as groups for $i \geq 1$) which says that

1. *the failure of the limit over the homotopy groups of the stages of the tower to equal the homotopy groups of the limit of the tower is at most in the kernel of the canonical comparison map;*
2. *that kernel is the $\underset{\longleftarrow}{\lim}^1$ (def. 1.49) of the homotopy groups of the stages.*

An elementary but tedious proof is indicated in (Bousfield-Kan 72, chapter IX, theorem 3.1. The following is a neat model category-theoretic proof following (Goerss-Jardine 96, section VI. prop. 2.15), which however requires the concept of homotopy limit over towers.

Proof. With respect to the classical model structure on simplicial sets or the classical model structure on topological spaces, a tower of fibrations as stated is a fibrant object in the injective model structure on functors $[(\mathbb{N}, \geq), \mathrm{sSet}]_{\mathrm{inj}}$ ($[(\mathbb{N}, \geq), \mathrm{Top}]_{\mathrm{inj}}$) (prop). Hence the plain limit over this diagram represents the homotopy limit. By the discussion there, up to weak equivalence that homotopy limit is also the pullback in

$$\begin{array}{ccc} \mathrm{holim}\, X_\bullet & \longrightarrow & \prod_n \mathrm{Path}(X_n) \\ \downarrow & (\mathrm{pb}) & \downarrow \\ \prod_n X_n & \underset{(\mathrm{id}, p_n)_n}{\longrightarrow} & \prod_n X_n \times X_n \end{array}\,,$$

where on the right we have the product over all the canonical fibrations out of the path space objects. Hence also the left vertical morphism is a fibration, and so by taking its fiber over a basepoint, the pasting law gives a homotopy fiber sequence

$$\prod_n \Omega X_n \longrightarrow \mathrm{holim}\, X_\bullet \longrightarrow \prod_n X_n\,.$$

The long exact sequence of homotopy groups of this fiber sequence goes

$$\cdots \to \prod_n \pi_{q+1}(X_n) \longrightarrow \prod_n \pi_{q+1}(X_n) \longrightarrow \pi_q(\underset{\longleftarrow}{\lim} X_\bullet) \longrightarrow \prod_n \pi_q(X_n) \longrightarrow \prod_n \pi_q(X_n) \to \cdots\,.$$

Chopping that off by forming kernel and cokernel yields the claim for positive q. For $q = 0$ it follows by inspection. ∎

Proposition 1.61. ***(Milnor exact sequence for generalized cohomology)***

Let X be a pointed CW-complex, $X = \underset{\longrightarrow_n}{\lim} X_n$ and let $\tilde E^\bullet$ an additive reduced cohomology theory, def. 1.1.

Then the canonical morphisms make a short exact sequence

$$0 \to \underset{\longleftarrow_n}{\lim}^1 \tilde E^{\bullet - 1}(X_n) \longrightarrow \tilde E^\bullet(X) \longrightarrow \underset{\longleftarrow_n}{\lim} \tilde E^\bullet(X_n) \to 0\,,$$

saying that

1. *the failure of the canonical comparison map $\tilde E^\bullet(X) \to \underset{\longleftarrow}{\lim} \tilde E^\bullet(X_n)$ to the limit of the cohomology groups on the finite stages to be an isomorphism is at most in a non-vanishing kernel;*
2. *this kernel is precisely the $\lim^1$ (def. 1.49) of the cohomology groups at the finite stages in one degree lower.*

e.g. (Switzer 75, prop. 7.66, Kochmann 96, prop. 4.2.2)

Proof. For

$$X_\bullet = \left(X_0 \overset{i_0}{\hookrightarrow} X_1 \overset{i_1}{\hookrightarrow} X_2 \overset{i_1}{\hookrightarrow} \cdots \right)$$

the sequence of stages of the (pointed) CW-complex $X = \underset{\longleftarrow_n}{\lim} X_n$, write

$$
\begin{aligned}
A_X &\coloneqq \underset{n \in \mathbb{N}}{\sqcup} X_{2n} \times [2n, 2n+1]; \\
B_X &\coloneqq \underset{n \in \mathbb{N}}{\sqcup} X_{(2n+1)} \times [2n+1, 2n+2] .
\end{aligned}
$$

for the disjoint unions of the cylinders over all the stages in even and all those in odd degree, respectively.

These come with canonical inclusion maps into the mapping telescope $\mathrm{Tel}(X_\bullet)$ (def.), which we denote by

$$
\begin{tikzcd}
A_X \arrow[dr, "\iota_{A_X}"'] & & B_X \arrow[dl, "\iota_{B_X}"] \\
& \mathrm{Tel}(X_\bullet) &
\end{tikzcd}
.
$$

Observe that

1. $A_X \cup B_X \simeq \mathrm{Tel}(X_\bullet)$;
2. $A_X \cap B_X \simeq \underset{n \in \mathbb{N}}{\sqcup} X_n$;

and that there are homotopy equivalences

1. $A_X \simeq \underset{n \in \mathbb{N}}{\sqcup} X_{2n+1}$
2. $B_X \simeq \underset{n \in \mathbb{N}}{\sqcup} X_{2n}$
3. $\mathrm{Tel}(X_\bullet) \simeq X$.

The first two are obvious, the third is this proposition.

This implies that the Mayer-Vietoris sequence (prop.) for $\tilde E^\bullet$ on the cover $A \sqcup B \to X$ is isomorphic to the bottom horizontal sequence in the following diagram:

$$
\begin{tikzcd}
\tilde E^{\bullet-1}(A_X) \oplus \tilde E^{\bullet-1}(B_X) \arrow[r] \arrow[d, "\simeq"] & \tilde E^{\bullet-1}(A_X \cap B_X) \arrow[r] \arrow[d, "\simeq"] & \tilde E^\bullet(X) \arrow[r, "(\iota_{A_X})^* - (\iota_{B_X})^*"] \arrow[d, "="] & \tilde E^\bullet(A_X) \oplus \tilde E^\bullet(B_X) \arrow[r] \arrow[d, "{(\mathrm{id}, -\mathrm{id})}"', "\simeq"] & \tilde E^\bullet(A_X \cap B_X) \arrow[d, "\simeq"] \\
\prod_n \tilde E^{\bullet-1}(X_n) \arrow[r, "\partial"'] & \prod_n \tilde E^{\bullet-1}(X_n) \arrow[r] & \tilde E^\bullet(X) \arrow[r, "(i_n^*)_n"] & \prod_n \tilde E^\bullet(X_n) \arrow[r, "\partial"'] & \prod_n \tilde E^\bullet(X_n)
\end{tikzcd}
,
$$

hence that the bottom sequence is also a long exact sequence.

To identify the morphism ∂, notice that it comes from pulling back E-cohomology classes along the inclusions $A \cap B \to A$ and $A \cap B \to B$. Comonentwise these are the inclusions of each X_n into the left and the right end of its cylinder inside the mapping telescope, respectively. By the construction of the mapping telescope, one of these ends is embedded via $i_n : X_n \hookrightarrow X_{n+1}$ into the cylinder over X_{n+1}. In conclusion, ∂ acts by

$$
\partial \,:\, (a_n)_{n \in \mathbb{N}} \mapsto (a_n - i_n^*(a_{n+1})) \,.
$$

(The relative sign is the one in $(\iota_{A_X})^* - (\iota_{B_X})^*$ originating in the definition of the Mayer-Vietoris sequence and properly propagated to the bottom sequence while ensuring that $\tilde E^\bullet(X) \to \prod_n \tilde E^\bullet(X_n)$ is really $(i_n^*)_n$ and not $(-1)^n (i_n^*)_n$, as needed for the statement to be proven.)

This is the morphism from def. 1.47 for the sequence

$$
\cdots \to \tilde E^\bullet(X_{n+1}) \overset{i_n^*}{\longrightarrow} \tilde E^\bullet(X_n) \overset{i_n^*}{\longrightarrow} \tilde E^\bullet(X_{n-1}) \to \cdots \,.
$$

Hence truncating the above long exact sequence by forming kernel and cokernel of ∂, the result follows via remark 1.48 and definition 1.49. ▮

In contrast:

Proposition 1.62. *Let X be a pointed CW-complex, $X = \underset{\longleftarrow_n}{\lim} X_n$.*

For $\tilde E_\bullet$ an additive reduced generalized homology theory, then

$$
\underset{\longrightarrow_n}{\lim} \tilde E_\bullet(X_n) \overset{\simeq}{\longrightarrow} \tilde E_\bullet(X)
$$

is an isomorphism.

(Switzer 75, prop. 7.53)

There is also a version for cohomology *of* spectra:

For $X, E \in \mathrm{Ho}(\mathrm{Spectra})$ two spectra, then the E-generalized cohomology of X is the graded group of homs in the stable homotopy category (def., exmpl.)

$$\begin{aligned} E^\bullet(X) &\coloneqq [X,E]_{-\bullet} \\ &\coloneqq [\Sigma^\bullet X, E] \end{aligned} .$$

The stable homotopy category is, in particular, the homotopy category of the stable model structure on orthogonal spectra, in that its localization at the stable weak homotopy equivalences is of the form

$$\gamma \,:\, \mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}} \longrightarrow \mathrm{Ho}(\mathrm{Spectra}) \,.$$

In the following when considering an orthogonal spectrum $X \in \mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})$, we use, for brevity, the same symbol for its image under γ.

Proposition 1.63. *For $X, E \in \mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})$ two orthogonal spectra (or two symmetric spectra such that X is a semistable symmetric spectrum) then there is a short exact sequence of the form*

$$0 \to \underset{\longleftarrow n}{\lim}^1 E^{\bullet+n-1}(X_n) \longrightarrow E^\bullet(X) \longrightarrow \underset{\longleftarrow n}{\lim}\; E^{\bullet+n}(X_n) \to 0$$

where $\underset{\longleftarrow}{\lim}^1$ denotes the lim^1, and where this and the limit on the right are taken over the following structure morphisms

$$E^{\bullet+n+1}(X_{n+1}) \overset{E^{\bullet+1n+1}(\Sigma^X_n)}{\longrightarrow} E^{\bullet+n+1}(X_n \wedge S^1) \overset{\simeq}{\longrightarrow} E^{\bullet+n}(X_n) \,.$$

(Schwede 12, chapter II prop. 6.5 (ii)) (using that symmetric spectra underlying orthogonal spectra are semistable (Schwede 12, p. 40))

Corollary 1.64. *For $X, E \in \mathrm{Ho}(\mathrm{Spectra})$ two spectra such that the tower $n \mapsto E^{n-1}(X_n)$ satisfies the Mittag-Leffler condition (def. 1.55), then two morphisms of spectra $X \longrightarrow E$ are homotopic already if all their morphisms of component spaces $X_n \to E_n$ are.*

Proof. By prop. 1.57 the assumption implies that the $\lim^1$-term in prop. 1.63 vanishes, hence by exactness it follows that in this case there is an isomorphism

$$[X,E] \simeq E^0(X) \overset{\simeq}{\longrightarrow} \underset{\longleftarrow n}{\lim}\,[X_n, E_n] \,.$$

▮

Serre-Atiyah-Hirzebruch spectral sequence

Idea. Another important tool for computing generalized cohomology is to reduce it to the computation of ordinary cohomology with coefficients. Given a generalized cohomology theory E, there is a spectral sequence known as the *Atiyah-Hirzebruch spectral sequence* (AHSS) which serves to compute E-cohomology of F-fiber bundles over a simplicial complex X in terms of ordinary cohomology with coefficients in the generalized cohomology $E^\bullet(F)$ of the fiber. For $E = $ HA this is known as the *Serre spectral sequence*.

The Atiyah-Hirzebruch spectral sequence in turn is a consequence of the "Cartan-Eilenberg spectral sequence" which arises from the exact couple of relative cohomology groups of the skeleta of the CW-complex, and whose first page is the relative cohomology groups for codimension-1 skeleta.

We need the AHSS for instance for the computation of Conner-Floyd Chern classes below.

Literature. (Kochman 96, section 2.2 and 4.2)

See also the accompanying *lecture notes on spectral sequences*.

Converging spectral sequences

Definition 1.65. A **cohomology spectral sequence** $\{E_r^{p,q}, d_r\}$ is

1. a sequence $\{E_r^{\bullet,\bullet}\}$ (for $r \in \mathbb{N}$, $r \geq 1$) of bigraded abelian groups (the "pages");

2. a sequence of linear maps (the "differentials")

$$\{d_r \,:\, E_r^{\bullet,\bullet} \longrightarrow E_r^{\bullet+r,\bullet-r+1}\}$$

such that

- $H_{r+1}^{\bullet,\bullet}$ is the cochain cohomology of d_r, i.e. $E_{r+1}^{\bullet,\bullet} = H(E_r^{\bullet,\bullet}, d_r)$, for all $r \in \mathbb{N}$, $r \geq 1$.

Given a $\mathbb{Z}$-graded abelian group_ $C^\bullet$ equipped with a decreasing filtration

$$C^\bullet \supset \cdots \supset F^s C^\bullet \supset F^{s+1} C^\bullet \supset \cdots \supset 0$$

such that

$$C^\bullet = \underset{s}{\cup} F^s C^\bullet \quad \text{and} \quad 0 = \underset{s}{\cap} F^s C^\bullet$$

then the spectral sequence is said to **converge** to $C^\bullet$, denoted,

$$E_2^{\bullet,\bullet} \Rightarrow C^\bullet$$

if

1. in each bidegree (s,t) the sequence $\{E_r^{s,t}\}_r$ eventually becomes constant on a group

   $E_\infty^{s,t} \coloneqq E_{\gg 1}^{s,t}$;

2. $E_\infty^{\bullet,\bullet}$ is the associated graded of the filtered $C^\bullet$ in that

   $E_\infty^{s,t} \simeq F^s C^{s+t} / F^{s+1} C^{s+t}$.

The converging spectral sequence is called a **multiplicative spectral sequence** if

1. $\{E_2^{\bullet,\bullet}\}$ is equipped with the structure of a bigraded algebra;

2. $F^\bullet C^\bullet$ is equipped with the structure of a filtered graded algebra ($F^p C^k \cdot F^q C^l \subset F^{p+q} C^{k+l}$);

such that

1. each d_r is a derivation with respect to the (induced) algebra structure on $E_r^{\bullet,\bullet}$, graded of degree 1 with respect to total degree;

2. the multiplication on $E_\infty^{\bullet,\bullet}$ is compatible with that on $C^\bullet$.

Remark 1.66. The point of spectral sequences is that by subdividing the data in any graded abelian group $C^\bullet$ into filtration stages, with each stage itself subdivided into bidegrees, such that each consecutive stage depends on the previous one in way tightly controled by the bidegrees, then this tends to give much control on the computation of $C^\bullet$. For instance it often happens that one may argue that the differentials in some spectral sequence all vanish from some page on (one says that the spectral sequence *collapses* at that page) by pure degree reasons, without any further computation.

Example 1.67. The archetypical example of (co-)homology spectral sequences as in def. 1.65 are induced from a filtering on a (co-)chain complex, converging to the (co-)chain homology of the chain complex by consecutively computing relative (co-)chain homologies, relative to decreasing (increasing) filtering degrees. For more on such spectral sequences of filtered complexes see at *Interlude -- Spectral sequences* the section *For filtered complexes*.

A useful way to generate spectral sequences is via exact couples:

Definition 1.68. An **exact couple** is three homomorphisms of abelian groups of the form

$$\begin{array}{ccc} D & \overset{g}{\longrightarrow} & D \\ {}_{f}\nwarrow & & \swarrow_{h} \\ & E & \end{array}$$

such that the image of one is the kernel of the next.

$$\mathrm{im}(h) = \ker(f)\,, \quad \mathrm{im}(f) = \ker(g)\,, \quad \mathrm{im}(g) = \ker(f)\;.$$

Given an exact couple, then its **derived exact couple** is

$$\begin{array}{ccc} \mathrm{im}(g) & \overset{g}{\longrightarrow} & \mathrm{im}(g) \\ {}_{f}\nwarrow & & \swarrow_{h\circ g^{-1}} \\ & H(E, h\circ f) & \end{array} \,,$$

where g^{-1} denotes the operation of sending one equivalence class to the equivalenc class of any preimage under g of any of its representatives.

Proposition 1.69. ***(cohomological spectral sequence of an exact couple)***

Given an exact couple, def. 1.68,

$$\begin{array}{ccc} D_1 & \overset{g_1}{\longrightarrow} & D_1 \\ & {}_{f_1}\nwarrow & \swarrow_{h_1} \\ & E_1 & \end{array}$$

its derived exact couple

$$\begin{array}{ccc} D_2 & \overset{g_2}{\longrightarrow} & D_2 \\ & {}_{f_2}\nwarrow & \swarrow_{h_2} \\ & E_2 & \end{array}$$

is itself an exact couple. Accordingly there is induced a sequence of exact couples

$$\begin{array}{ccc} D_r & \overset{g_r}{\longrightarrow} & D_r \\ & {}_{f_r}\nwarrow & \swarrow_{h_r} \\ & E_r & \end{array} \,.$$

If the abelian groups D and E are equipped with bigrading such that

$$\deg(f) = (0,0)\,, \quad \deg(g) = (-1,1)\,, \quad \deg(h) = (1,0)$$

then $\{E_r^{\bullet,\bullet}, d_r\}$ with

$$\begin{aligned} d_r &\coloneqq h_r \circ f_r \\ &= h \circ g^{-r+1} \circ f \end{aligned}$$

is a cohomological spectral sequence, def. 1.65.

(As before in prop. 1.69, the notation g^{-n} with $n \in \mathbb{N}$ denotes the function given by choosing, on representatives, a preimage under $g^n = \underbrace{g \circ \cdots \circ g \circ g}_{n \text{ times}}$, with the implicit claim that all possible choices represent the same equivalence class.)

If for every bidegree (s,t) there exists $R_{s,t} \gg 1$ such that for all $r \geq R_{s,t}$

1. $g \colon D^{s+R,t-R} \overset{\simeq}{\longrightarrow} D^{s+R-1,t-R-1}$;

2. $g \colon D^{s-R+1,t+R-2} \overset{0}{\longrightarrow} D^{s-R,t+R-1}$

then this spectral sequence converges to the inverse limit group

$$G^\bullet \coloneqq \lim \left(\cdots \overset{g}{\to} D^{s,\bullet-s} \overset{g}{\longrightarrow} D^{s-1,\bullet-s+1} \overset{g}{\to} \cdots \right)$$

filtered by

$$F^p G^\bullet \coloneqq \ker(G^\bullet \to D^{p-1,\bullet-p+1}) \,.$$

(e.g. Kochmann 96, lemma 2.6.2)

Proof. We check the claimed form of the E_∞-page:

Since $\ker(h) = \operatorname{im}(g)$ in the exact couple, the kernel

$$\ker(d_{r-1}) \coloneqq \ker(h \circ g^{-r+2} \circ f)$$

consists of those elements x such that $g^{-r+2}(f(x)) = g(y)$, for some y, hence

$$\ker(d_{r-1})^{s,t} \simeq f^{-1}(g^{r-1}(D^{s+r-1,t-r+1})) \,.$$

By assumption there is for each (s,t) an $R_{s,t}$ such that for all $r \geq R_{s,t}$ then $\ker(d_{r-1})^{s,t}$ is independent of r.

Moreover, $\operatorname{im}(d_{r-1})$ consists of the image under h of those $x \in D^{s-1,t}$ such that $g^{r-2}(x)$ is in the image of f, hence (since $\mathrm{im}(f) = \ker(g)$ by exactness of the exact couple) such that $g^{r-2}(x)$ is in the kernel of g, hence such that x is in the kernel of g^{r-1}. If $r > R$ then by assumption $g^{r-1}|_{D^{s-1,t}} = 0$ and so then $\mathrm{im}(d_{r-1}) = \mathrm{im}(h)$.

(Beware this subtlety: while $g^{R_{s,t}}|_{D^{s-1,t}}$ vanishes by the convergence assumption, the expression $g^{R_{s,t}}|_{D^{s+r-1,t-r+1}}$ need not vanish yet. Only the higher power $g^{R_{s,t}+R_{s+1,t+2}+2}|_{D^{s+r-1,t-r+1}}$ is again guaranteed to vanish.)

It follows that

$$
\begin{aligned}
E_\infty^{p,n-p} &= \ker(d_R)/\mathrm{im}(d_R) \\
&\simeq f^{-1}(\mathrm{im}(g^{R-1}))/\mathrm{im}(h) \\
&\underset{\simeq}{\overset{f}{\longrightarrow}} \mathrm{im}(g^{R-1}) \cap \mathrm{im}(f) \\
&\simeq \mathrm{im}(g^{R-1}) \cap \ker(g)
\end{aligned}
$$

where in last two steps we used once more the exactness of the exact couple.

(Notice that the above equation means in particular that the E_∞-page is a sub-group of the image of the E_1-page under f.)

The last group above is that of elements $x \in G^n$ which map to zero in $D^{p-1,n-p+1}$ and where two such are identified if they agree in $D^{p,n-p}$, hence indeed

$$
E_\infty^{p,n-p} \simeq F^p G^n / F^{p+1} G^n \,.
$$

∎

Remark 1.70. Given a spectral sequence (def. 1.65), then even if it converges strongly, computing its infinity-page still just gives the associated graded of the filtered object that it converges to, not the filtered object itself. The latter is in each filter stage an extension of the previous stage by the corresponding stage of the infinity-page, but there are in general several possible extensions (the trivial extension or some twisted extensions). The problem of determining these extensions and hence the problem of actually determining the filtered object from a spectral sequence converging to it is often referred to as the **extension problem**.

More in detail, consider, for definiteness, a cohomology spectral sequence converging to some filtered $F^\bullet H^\bullet$

$$
E^{p,q} \;\Rightarrow\; H^\bullet \,.
$$

Then by definition of convergence there are isomorphisms

$$
E_\infty^{p,\bullet} \simeq F^p H^{p+\bullet} / F^{p+1} H^{p+\bullet} \,.
$$

Equivalently this means that there are short exact sequences of the form

$$
0 \to F^{p+1} H^{p+\bullet} \hookrightarrow F^p H^{p+\bullet} \longrightarrow E_\infty^{p,\bullet} \to 0 \,.
$$

for all p. The extension problem then is to inductively deduce $F^p H^\bullet$ from knowledge of $F^{p+1} H^\bullet$ and $E_\infty^{p,\bullet}$.

In good cases these short exact sequences happen to be split exact sequences, which means that the extension problem is solved by the direct sum

$$
F^p H^{p+\bullet} \simeq F^{p+1} H^{p+\bullet} \oplus E_\infty^{p,\bullet} \,.
$$

But in general this need not be the case.

One sufficient condition that these exact sequences split is that they consist of homomorphisms of R-modules, for some ring R, and that $E_\infty^{p,\bullet}$ are projective modules (for instance free modules) over R. Because then the Ext-group $\mathrm{Ext}^1_R(E_\infty^{p,\bullet}, -)$ vanishes, and hence all extensions are trivial, hence split.

So for instance for every spectral sequence in vector spaces the extension problem is trivial (since every vector space is a free module).

The AHSS

The following proposition requires, in general, to evaluate cohomology functors not just on CW-complexes, but on all topological spaces. Hence we invoke prop. 1.4 to regard a reduced cohomology theory as a contravariant functor on all pointed topological spaces, which sends weak homotopy equivalences to isomorphisms (def. 1.3).

Proposition 1.71. ***(Serre-Cartan-Eilenberg-Whitehead-Atiyah-Hirzebruch spectral sequence)***

Let $A^\bullet$ be a an additive unreduced generalized cohomology functor (def.). Let B be a CW-complex and let $X \overset{\pi}{\to} B$ be a Serre fibration (def.), such that all its fibers are weakly contractible or such that B is simply connected. In either case all fibers are identified with a typical fiber F up to weak homotopy equivalence by connectedness (this example), and well defined up to unique iso in the homotopy category by simply connectedness:

$$\begin{array}{ccc} F & \longrightarrow & X \\ && \downarrow^{\mathrlap{\in Fib_{cl}}} \\ && B \end{array}$$

If at least one of the following two conditions is met

- *B is finite-dimensional as a CW-complex;*
- *$A^\bullet(F)$ is bounded below in degree and the sequences $\cdots \to A^p(X_{n+1}) \to A^p(X_n) \to \cdots$ satisfy the Mittag-Leffler condition (def. 1.55) for all p;*

then there is a cohomology spectral sequence, def. 1.65, whose E_2-page is the ordinary cohomology $H^\bullet(B, A^\bullet(F))$ of B with coefficients in the A-cohomology groups $A^\bullet(F)$ of the fiber, and which converges to the A-cohomology groups of the total space

$$E_2^{p,q} = H^p(B, A^q(F)) \;\Rightarrow\; A^\bullet(X)$$

with respect to the filtering given by

$$F^p A^\bullet(X) \coloneqq \ker\left(A^\bullet(X) \to A^\bullet(X_{p-1})\right) \,,$$

where $X_p \coloneqq \pi^{-1}(B_p)$ is the fiber over the pth stage of the CW-complex $B = \underset{\longleftarrow_n}{\lim} B_n$.

Proof. The exactness axiom for A gives an exact couple, def. 1.68, of the form

$$\begin{array}{ccccc} \prod_{s,t} A^{s+t}(X_s) && \longrightarrow && \prod_{s,t} A^{s+t}(X_s) \\ & \nwarrow && \swarrow \\ && \prod_{s,t} A^{s+t}(X_s, X_{s-1}) \end{array} \quad \left(\begin{array}{ccc} A^{s+t}(X_s) & \longrightarrow & A^{s+t}(X_{s-1}) \\ \uparrow && \downarrow_{\delta} \\ A^{s+t}(X_s, X_{s-1}) && A^{s+t+1}(X_s, X_{s-1}) \end{array} \right) \,,$$

where we take $X_{\gg 1} = X$ and $X_{<0} = \emptyset$.

In order to determine the E_2-page, we analyze the E_1-page: By definition

$$E_1^{s,t} = A^{s+t}(X_s, X_{s-1})$$

Let $C(s)$ be the set of s-dimensional cells of B, and notice that for $\sigma \in C(s)$ then

$$(\pi^{-1}(\sigma), \pi^{-1}(\partial \sigma)) \simeq (D^n, S^{n-1}) \times F_\sigma \,,$$

where F_σ is weakly homotopy equivalent to F (exmpl.).

This implies that

$$\begin{aligned} E_1^{s,t} & \coloneqq A^{s+t}(X_s, X_{s-1}) \\ & \simeq \tilde A^{s+t}(X_s / X_{s-1}) \\ & \simeq \tilde A^{s+t}\Big(\underset{\sigma \in C(n)}{\vee} S^s \wedge F_+\Big) \\ & \simeq \prod_{\sigma \in C(s)} \tilde A^{s+t}(S^s \wedge F_+) \,, \\ & \simeq \prod_{\sigma \in C(s)} \tilde A^{t}(F_+) \\ & \simeq \prod_{\sigma \in C(s)} A^t(F) \\ & \simeq C^s_{cell}(B, A^t(F)) \end{aligned}$$

where we used the relation to reduced cohomology $\tilde A$, prop. 1.19 together with lemma 1.11, then the wedge axiom and the suspension isomorphism of the latter.

The last group $C^s_{cell}(B, A^t(F))$ appearing in this sequence of isomorphisms is that of cellular cochains (def.) of degree s on B with coefficients in the group $A^t(F)$.

Since cellular cohomology of a CW-complex agrees with its singular cohomology (thm.), hence with its ordinary cohomology, to conclude that the E_2-page is as claimed, it is now sufficient to show that the differential d_1 coincides with the differential in the cellular cochain complex (def.).

We discuss this now for $\pi = \mathrm{id}$, hence $X = B$ and $F = *$. The general case works the same, just with various factors of F appearing in the following:

Consider the following diagram, which commutes due to the naturality of the connecting homomorphism δ of $A^\bullet$:

$$\begin{array}{ccccccc} \partial^* : & C^{s-1}_{\mathrm{cell}}(X, A^t(*)) & = & \prod_{i \in I_{s-1}} A^t(*) & \longrightarrow & \prod_{i \in I_s} A^t(*) & = & C^s_{\mathrm{cell}}(X, A^t(*)) \\ & & & {}^{\simeq}\downarrow & & \downarrow^{\simeq} \\ & & & \prod_{i \in I_{s-1}} \tilde A^{s+t-1}(S^{s-1}) & & \prod_{i \in I_s} \tilde A^{s+t}(S^s) \\ & & & {}^{\simeq}\downarrow & & \downarrow^{\simeq} \\ d_1 : & A^{s+t-1}(X_{s-1}, X_{s-2}) & \longrightarrow & A^{s+t-1}(X_{s-1}) & \overset{\delta}{\longrightarrow} & A^{s+t}(X_s, X_{s-1}) \\ & \downarrow & & \downarrow & & \downarrow \\ & A^{s+t-1}(S^{s-1}, \emptyset) & \longrightarrow & A^{s+t-1}(S^{s-1}) & \overset{\delta}{\longrightarrow} & A^{s+t}(D^s, S^{s-1}) \end{array} \,.$$

Here the bottom vertical morphisms are those induced from any chosen cell inclusion $(D^s, S^{s-1}) \hookrightarrow (X_s, X_{s-1})$.

The differential d_1 in the spectral sequence is the middle horizontal composite. From this the vertical isomorphisms give the top horizontal map. But the bottom horizontal map identifies this top horizontal morphism componentwise with the restriction to the boundary of cells. Hence the top horizontal morphism is indeed the coboundary operator ∂^* for the cellular cohomology of X with coefficients in $A^\bullet(*)$ (def.). This cellular cohomology coincides with singular cohomology of the CW-complex X (thm.), hence computes the ordinary cohomology of X.

Now to see the convergence. If B is finite dimensional then the convergence condition as stated in prop. 1.69 is met. Alternatively, if $A^\bullet(F)$ is bounded below in degree, then by the above analysis the E_1-page has a horizontal line below which it vanishes. Accordingly the same is then true for all higher pages, by each of them being the cohomology of the previous page. Since the differentials go right and down, eventually they pass beneath this vanishing line and become 0. This is again the condition needed in the proof of prop. 1.69 to obtain convergence.

By that proposition the convergence is to the inverse limit

$$\underset{\longleftarrow}{\lim}(\cdots \to A^\bullet(X_{s+1}) \longrightarrow A^\bullet(X_s) \to \cdots) \,.$$

If X is finite dimensional or more generally if the sequences that this limit is over satisfy the Mittag-Leffler condition (def. 1.55), then this limit is $A^\bullet(X)$, by prop. 1.57. ▮

Multiplicative structure

Proposition 1.72. *For $E^\bullet$ a multiplicative cohomology theory (def. 1.26), then the Atiyah-Hirzebruch spectral sequences (prop. 1.71) for $E^\bullet(X)$ are multiplicative spectral sequences.*

A decent proof is spelled out in (Kochman 96, prop. 4.2.9). Use the graded commutativity of smash products of spheres to get the sign in the graded derivation law for the differentials. See also the proof via Cartan-Eilenberg systems at *multiplicative spectral sequence – Examples – AHSS for multiplicative cohomology*.

Proposition 1.73. *Given a multiplicative cohomology theory $(A, \mu, 1)$ (def. 1.26), then for every Serre fibration $X \to B$ (def.) all the differentials in the corresponding Atiyah-Hirzebruch spectral sequence of prop. 1.71*

$$H^\bullet(B, A^\bullet(F)) \Rightarrow A^\bullet(X)$$

are linear over $A^\bullet(*)$.

Proof. By the proof of prop. 1.71, the differentials are those induced by the exact couple

$$
\begin{array}{ccccc}
\prod_{s,t} A^{s+t}(X_s) & & \longrightarrow & & \prod_{s,t} A^{s+t}(X_s) \\
& \nwarrow & & \swarrow & \\
& & \prod_{s,t} A^{s+t}(X_s, X_{s-1}) & &
\end{array}
\;
\left(
\begin{array}{ccc}
A^{s+t}(X_s) & \longrightarrow & A^{s+t}(X_{s-1}) \\
\uparrow & & \downarrow_{\delta} \\
A^{s+t}(X_s, X_{s-1}) & & A^{s+t+1}(X_s, X_{s-1})
\end{array}
\right)
.
$$

consisting of the pullback homomorphisms and the connecting homomorphisms of A.

By prop. 1.69 its differentials on page r are the composites of one pullback homomorphism, the preimage of $(r-1)$ pullback homomorphisms, and one connecting homomorphism of A. Hence the statement follows with prop. 1.27. ▮

Proposition 1.74. *For E a homotopy commutative ring spectrum (def.) and X a finite CW-complex, then the Kronecker pairing*

$$
\langle -, - \rangle_X : E^{\bullet_1}(X) \otimes E_{\bullet_2}(X) \longrightarrow \pi_{\bullet_2 - \bullet_1}(E)
$$

extends to a compatible pairing of Atiyah-Hirzebruch spectral sequences.

(Kochman 96, prop. 4.2.10)

S.2) Cobordism theory

Idea. As one passes from abelian groups to spectra, a miracle happens: even though the latter are just the proper embodiment of linear algebra in the context of homotopy theory ("higher algebra") their inspection reveals that spectra natively know about deep phenomena of differential topology, index theory and in fact string theory (for instance via a close relation between *genera and partition functions*).

A strong manifestation of this phenomenon comes about in complex oriented cohomology theory/chromatic homotopy theory that we eventually come to below. It turns out to be higher algebra over the complex Thom spectrum MU.

Here we first concentrate on its real avatar, the Thom spectrum MO. The seminal result of Thom's theorem says that the stable homotopy groups of MO form the cobordism ring of cobordism-equivalence classes of manifolds. In the course of discussing this *cobordism theory* one encounters various phenomena whose complex version also governs the complex oriented cohomology theory that we are interested in below.

Literature. (Kochman 96, chapter I and sections II.2, II6). A quick efficient account is in (Malkiewich 11). See also (Aguilar-Gitler-Prieto 02, section 11).

Classifying spaces and G-Structure

Idea. Every manifold X of dimension n carries a canonical vector bundle of rank n: its tangent bundle. There is a universal vector bundle of rank n, of which all others arise by pullback, up to isomorphism. The base space of this universal bundle is hence called the classifying space and denoted $B\,\mathrm{GL}(n) \simeq BO(n)$ (for $O(n)$ the orthogonal group). This may be realized as the homotopy type of a direct limit of Grassmannian manifolds. In particular the tangent bundle of a manifold X is classified by a map $X \longrightarrow BO(n)$, unique up to homotopy. For G a subgroup of $O(n)$, then a lift of this map through the canonical map $BG \longrightarrow BO(n)$ of classifying spaces is a *G-structure* on X

$$
\begin{array}{ccc}
& & BG \\
& \nearrow & \downarrow \\
X & \longrightarrow & BO(n)
\end{array}
$$

for instance an orientation for the inclusion $\mathrm{SO}(n) \hookrightarrow O(n)$ of the special orthogonal group, or an almost complex structure for the inclusion $U(n) \hookrightarrow O(2n)$ of the unitary group.

All this generalizes, for instance from tangent bundles to normal bundles with respect to any embedding. It also behaves well with respect to passing to the boundary of manifolds, hence to bordism-classes of manifolds. This is what appears in Thom's theorem below.

Literature. (Kochman 96, 1.3-1.4), for stable normal structures also (Stong 68, beginning of chapter II)

Coset spaces

Proposition 1.75. *For X a smooth manifold and G a compact Lie group equipped with a free smooth action on X, then the quotient projection*

$$X \longrightarrow X/G$$

is a G-principal bundle (hence in particular a Serre fibration).

This is originally due to (Gleason 50). See e.g. (Cohen, theorem 1.3)

Corollary 1.76. *For G a Lie group and $H \subset G$ a compact subgroup, then the coset quotient projection*

$$G \longrightarrow G/H$$

is an H-principal bundle (hence in particular a Serre fibration).

Proposition 1.77. *For G a compact Lie group and $K \subset H \subset G$ closed subgroups, then the projection map on coset spaces*

$$p : G/K \longrightarrow G/H$$

is a locally trivial H/K-fiber bundle (hence in particular a Serre fibration).

Proof. Observe that the projection map in question is equivalently

$$G \times_H (H/K) \longrightarrow G/H\,,$$

(where on the left we form the Cartesian product and then divide out the diagonal action by H). This exhibits it as the H/K-fiber bundle associated to the H-principal bundle of corollary 1.76. ▮

Orthogonal and Unitary groups

Proposition 1.78. *The orthogonal group $O(n)$ is compact topological space, hence in particular a compact Lie group.*

Proposition 1.79. *The unitary group $U(n)$ is compact topological space, hence in particular a compact Lie group.*

Example 1.80. The n-spheres are coset spaces of orthogonal groups:

$$S^n \simeq O(n+1)/O(n)\,.$$

The odd-dimensional spheres are also coset spaces of unitary groups:

$$S^{2n+1} \simeq U(n+1)/U(n)$$

Proof. Regarding the first statement:

Fix a unit vector in $\mathbb{R}^{n+1}$. Then its orbit under the defining $O(n+1)$-action on $\mathbb{R}^{n+1}$ is clearly the canonical embedding $S^n \hookrightarrow \mathbb{R}^{n+1}$. But precisely the subgroup of $O(n+1)$ that consists of rotations around the axis formed by that unit vector stabilizes it, and that subgroup is isomorphic to $O(n)$, hence $S^n \simeq O(n+1)/O(n)$.

The second statement follows by the same kind of reasoning:

Clearly $U(n+1)$ acts transitively on the unit sphere S^{2n+1} in $\mathbb{C}^{n+1}$. It remains to see that its stabilizer subgroup of any point on this sphere is $U(n)$. If we take the point with coordinates $(1,0,0,\cdots,0)$ and regard elements of $U(n+1)$ as matrices, then the stabilizer subgroup consists of matrices of the block diagonal form

$$\left(\begin{array}{cc} 1 & \vec{0} \\ \vec{0} & A \end{array}\right)$$

where $A \in U(n)$. ▮

Proposition 1.81. *For $n,k \in \mathbb{N}$, $n \leq k$, then the canonical inclusion of orthogonal groups*

$$O(n) \hookrightarrow O(k)$$

is an (n-1)-equivalence, hence induces an isomorphism on homotopy groups in degrees $< n-1$ and a surjection in degree $n-1$.

Proof. Consider the coset quotient projection

$$O(n) \longrightarrow O(n+1) \longrightarrow O(n+1)/O(n)\,.$$

By prop. 1.78 and by corollary 1.76, the projection $O(n+1) \to O(n+1)/O(n)$ is a Serre fibration. Furthermore, example 1.80 identifies the coset with the n-sphere

$$S^n \simeq O(n+1)/O(n)\,.$$

Therefore the long exact sequence of homotopy groups (exmpl.)of the fiber sequence $O(n) \to O(n+1) \to S^n$ has the form

$$\cdots \to \pi_{\bullet+1}(S^n) \longrightarrow \pi_\bullet(O(n)) \longrightarrow \pi_\bullet(O(n+1)) \longrightarrow \pi_\bullet(S^n) \to \cdots$$

Since $\pi_{<n}(S^n) = 0$, this implies that

$$\pi_{<n-1}(O(n)) \overset{\simeq}{\longrightarrow} \pi_{<n-1}(O(n+1))$$

is an isomorphism and that

$$\pi_{n-1}(O(n)) \overset{\simeq}{\longrightarrow} \pi_{n-1}(O(n+1))$$

is surjective. Hence now the statement follows by induction over $k-n$. ▮

Similarly:

Proposition 1.82. *For $n,k \in \mathbb{N}$, $n \leq k$, then the canonical inclusion of unitary groups*

$$U(n) \hookrightarrow U(k)$$

is a 2n-equivalence, hence induces an isomorphism on homotopy groups in degrees $< 2n$ and a surjection in degree $2n$.

Proof. Consider the coset quotient projection

$$U(n) \longrightarrow U(n+1) \longrightarrow U(n+1)/U(n)\,.$$

By prop. 1.79 and corollary 1.76, the projection $U(n+1) \to U(n+1)/U(n)$ is a Serre fibration. Furthermore, example 1.80 identifies the coset with the (2n+1)-sphere

$$S^{2n+1} \simeq U(n+1)/U(n)\,.$$

Therefore the long exact sequence of homotopy groups (exmpl.)of the fiber sequence $U(n) \to U(n+1) \to S^{2n+1}$ is of the form

$$\cdots \to \pi_{\bullet+1}(S^{2n+1}) \longrightarrow \pi_\bullet(U(n)) \longrightarrow \pi_\bullet(U(n+1)) \longrightarrow \pi_\bullet(S^{2n+1}) \to \cdots$$

Since $\pi_{\leq 2n}(S^{2n+1}) = 0$, this implies that

$$\pi_{<2n}(U(n)) \overset{\simeq}{\longrightarrow} \pi_{<2n}(U(n+1))$$

is an isomorphism and that

$$\pi_{2n}(U(n)) \overset{\simeq}{\longrightarrow} \pi_{2n}(U(n+1))$$

is surjective. Hence now the statement follows by induction over $k-n$. ▮

Stiefel manifolds and Grassmannians

Throughout we work in the category $\mathrm{Top}_{\mathrm{cg}}$ of compactly generated topological spaces (def.). For these the Cartesian product $X \times (-)$ is a left adjoint (prop.) and hence preserves colimits.

Definition 1.83. For $n,k \in \mathbb{N}$ and $n \leq k$, then the nth **real Stiefel manifold** of $\mathbb{R}^k$ is the coset topological space.

$$V_n(\mathbb{R}^k) \coloneqq O(k)/O(k-n)\,,$$

where the action of $O(k-n)$ is via its canonical embedding $O(k-n) \hookrightarrow O(k)$.

Similarly the nth **complex Stiefel manifold** of $\mathbb{C}^k$ is

$$V_n(\mathbb{C}^k) \coloneqq U(k)/U(k-n)\,,$$

here the action of $U(k-n)$ is via its canonical embedding $U(k-n) \hookrightarrow U(k)$.

Definition 1.84. For $n, k \in \mathbb{N}$ and $n \leq k$, then the nth **real Grassmannian** of $\mathbb{R}^k$ is the coset topological space.

$$\mathrm{Gr}_n(\mathbb{R}^k) \coloneqq O(k)/(O(n) \times O(k-n)) \,,$$

where the action of the product group is via its canonical embedding $O(n) \times O(k-n) \hookrightarrow O(n)$ into the orthogonal group.

Similarly the nth **complex Grassmannian** of $\mathbb{C}^k$ is the coset topological space.

$$\mathrm{Gr}_n(\mathbb{C}^k) \coloneqq U(k)/(U(n) \times U(k-n)) \,,$$

where the action of the product group is via its canonical embedding $U(n) \times U(k-n) \hookrightarrow U(n)$ into the unitary group.

Example 1.85.

- $G_1(\mathbb{R}^{n+1}) \simeq \mathbb{R}P^n$ is real projective space of dimension n.
- $G_1(\mathbb{C}^{n+1}) \simeq \mathbb{C}P^n$ is complex projective space of dimension n (def. 1.134).

Proposition 1.86. *For all $n \leq k \in \mathbb{N}$, the canonical projection from the Stiefel manifold (def. 1.83) to the Grassmannian is a $O(n)$-principal bundle*

$$\begin{array}{ccc} O(n) & \hookrightarrow & V_n(\mathbb{R}^k) \\ && \downarrow \\ && \mathrm{Gr}_n(\mathbb{R}^k) \end{array}$$

and the projection from the complex Stiefel manifold to the Grassmannian us a $U(n)$-principal bundle:

$$\begin{array}{ccc} U(n) & \hookrightarrow & V_n(\mathbb{C}^k) \\ && \downarrow \\ && \mathrm{Gr}_n(\mathbb{C}^k) \end{array} \,.$$

Proof. By prop 1.76 and prop 1.77. ■

Proposition 1.87. *The real Grassmannians $\mathrm{Gr}_n(\mathbb{R}^k)$ and the complex Grassmannians $\mathrm{Gr}_n(\mathbb{C}^k)$ of def. 1.84 admit the structure of CW-complexes. Moreover the canonical inclusions*

$$\mathrm{Gr}_n(\mathbb{R}^k) \hookrightarrow \mathrm{Gr}_n(\mathbb{R}^{k+1})$$

are subcomplex incusion (hence relative cell complex inclusions).

Accordingly there is an induced CW-complex structure on the classifying space (def. 1.91).

$$BO(n) \simeq \underset{\longrightarrow_k}{\lim} \mathrm{Gr}_n(\mathbb{R}^k) \,.$$

A proof is spelled out in (Hatcher, section 1.2 (pages 31-34)).

Proposition 1.88. *The Stiefel manifolds $V_n(\mathbb{R}^k)$ and $V_n(\mathbb{C}^k)$ from def. 1.83 admits the structure of a CW-complex.*

e.g. (James 59, p. 3, James 76, p. 5 with p. 21, Blaszczyk 07)

(And I suppose with that cell structure the inclusions $V_n(\mathbb{R}^k) \hookrightarrow V_n(\mathbb{R}^{k+1})$ are subcomplex inclusions.)

Proposition 1.89. *The real Stiefel manifold $V_n(\mathbb{R}^k)$ (def. 1.83) is (k-n-1)-connected.*

Proof. Consider the coset quotient projection

$$O(k-n) \longrightarrow O(k) \longrightarrow O(k)/O(k-n) = V_n(\mathbb{R}^k) \,.$$

By prop. 1.78 and by corollary 1.76, the projection $O(k) \to O(k)/O(k-n)$ is a Serre fibration. Therefore there is induced the long exact sequence of homotopy groups of this fiber sequence, and by prop. 1.81 it has the following form in degrees bounded by n:

$$\cdots \to \pi_{\bullet \leq k-n-1}(O(k-n)) \overset{\text{epi}}{\longrightarrow} \pi_{\bullet \leq k-n-1}(O(k)) \overset{0}{\longrightarrow} \pi_{\bullet \leq k-n-1}(V_n(\mathbb{R}^k)) \overset{0}{\longrightarrow} \pi_{\bullet -1 < k-n-1}(O(k)) \overset{\simeq}{\longrightarrow} \pi_{\bullet -1 < k-n-1}(O(k-n)) \to \cdots \,.$$

This implies the claim. (Exactness of the sequence says that every element in $\pi_{\bullet \leq n-1}(V_n(\mathbb{R}^k))$ is in the kernel of zero, hence in the image of 0, hence is 0 itself.) ∎

Similarly:

Proposition 1.90. *The complex Stiefel manifold $V_n(\mathbb{C}^k)$ (def. 1.83) is 2(k-n)-connected.*

Proof. Consider the coset quotient projection

$$U(k-n) \longrightarrow U(k) \longrightarrow U(k)/U(k-n) = V_n(\mathbb{C}^k) \;.$$

By prop. 1.79 and by corollary 1.76 the projection $U(k) \to U(k)/U(k-n)$ is a Serre fibration. Therefore there is induced the long exact sequence of homotopy groups of this fiber sequence, and by prop. 1.82 it has the following form in degrees bounded by n:

$$\cdots \to \pi_{\bullet \leq 2(k-n)}(U(k-n)) \overset{\text{epi}}{\longrightarrow} \pi_{\bullet \leq 2(k-n)}(U(k)) \overset{0}{\longrightarrow} \pi_{\bullet \leq 2(k-n)}(V_n(\mathbb{C}^k)) \overset{0}{\longrightarrow} \pi_{\bullet -1 < 2(k-n)}(U(k)) \overset{\simeq}{\longrightarrow} \pi_{\bullet -1 < 2(k-n)}(U(k-n)) \to \cdots \;.$$

This implies the claim. ∎

Classifying spaces

Definition 1.91. By def. 1.84 there are canonical inclusions

$$\mathrm{Gr}_n(\mathbb{R}^k) \hookrightarrow \mathrm{Gr}_n(\mathbb{R}^{k+1})$$

and

$$\mathrm{Gr}_n(\mathbb{C}^k) \hookrightarrow \mathrm{Gr}_n(\mathbb{C}^{k+1})$$

for all $k \in \mathbb{N}$. The colimit (in Top, see there, or rather in $\mathrm{Top}_{\mathrm{cg}}$, see this cor.) over these inclusions is denoted

$$BO(n) \coloneqq \underset{\longrightarrow_k}{\lim} \mathrm{Gr}_n(\mathbb{R}^k)$$

and

$$BU(n) \coloneqq \underset{\longrightarrow_k}{\lim} \mathrm{Gr}_n(\mathbb{C}^k)\,,$$

respectively.

Moreover, by def. 1.83 there are canonical inclusions

$$V_n(\mathbb{R}^k) \hookrightarrow V_n(\mathbb{R}^{k+1})$$

and

$$V_n(\mathbb{C}^k) \hookrightarrow V_n(\mathbb{C}^{k+1})$$

that are compatible with the $O(n)$-action and with the $U(n)$-action, respectively. The colimit (in Top, see there, or rather in $\mathrm{Top}_{\mathrm{cg}}$, see this cor.) over these inclusions, regarded as equipped with the induced $O(n)$-action, is denoted

$$EO(n) \coloneqq \underset{\longrightarrow_k}{\lim} V_n(\mathbb{R}^k)$$

and

$$EU(n) \coloneqq \underset{\longrightarrow_k}{\lim} V_n(\mathbb{C}^k)\,,$$

respectively.

The inclusions are in fact compatible with the bundle structure from prop. 1.86, so that there are induced projections

$$\left(\begin{matrix} EO(n) \\ \downarrow \\ BO(n) \end{matrix} \right) \;\simeq\; \underset{\longrightarrow_k}{\lim} \left(\begin{matrix} V_n(\mathbb{R}^k) \\ \downarrow \\ \mathrm{Gr}_n(\mathbb{R}^k) \end{matrix} \right)$$

and

$$\left(\begin{array}{c} EU(n) \\ \downarrow \\ BU(n) \end{array}\right) \simeq \underset{\longrightarrow_k}{\lim} \left(\begin{array}{c} V_n(\mathbb{C}^k) \\ \downarrow \\ \mathrm{Gr}_n(\mathbb{C}^k) \end{array}\right),$$

respectively. These are the standard models for the **universal principal bundles** for O and U, respectively. The corresponding associated vector bundles

$$EO(n) \underset{O(n)}{\times} \mathbb{R}^n$$

and

$$EU(n) \underset{U(n)}{\times} \mathbb{C}^n$$

are the corresponding **universal vector bundles**.

Since the Cartesian product $O(n) \times (-)$ in compactly generated topological spaces preserves colimits, it follows that the colimiting bundle is still an $O(n)$-principal bundle

$$\begin{aligned} (EO(n))/O(n) &\simeq (\underset{\longrightarrow_k}{\lim} V_n(\mathbb{R}^k))/O(n) \\ &\simeq \underset{\longrightarrow_k}{\lim} (V_n(\mathbb{R}^k)/O(n)) \\ &\simeq \underset{\longrightarrow_k}{\lim} \mathrm{Gr}_n(\mathbb{R}^k) \\ &\simeq BO(n) \end{aligned},$$

and anlogously for $EU(n)$.

As such this is the standard presentation for the $O(n)$-universal principal bundle and $U(n)$-universal principal bundle, respectively. Its base space $BO(n)$ is the corresponding **classifying space**.

Definition 1.92. There are canonical inclusions

$$\mathrm{Gr}_n(\mathbb{R}^k) \hookrightarrow \mathrm{Gr}_{n+1}(\mathbb{R}^{k+1})$$

and

$$\mathrm{Gr}_n(\mathbb{C}^k) \hookrightarrow \mathrm{Gr}_{n+1}(\mathbb{C}^{k+1})$$

given by adjoining one coordinate to the ambient space and to any subspace. Under the colimit of def. 1.91 these induce maps of classifying spaces

$$BO(n) \longrightarrow BO(n+1)$$

and

$$BU(n) \longrightarrow BU(n+1)\;.$$

Definition 1.93. There are canonical maps

$$\mathrm{Gr}_{n_1}(\mathbb{R}^{k_1}) \times \mathrm{Gr}_{n_2}(\mathbb{R}^{k_2}) \longrightarrow \mathrm{Gr}_{n_1+n_2}(\mathbb{R}^{k_1+k_2})$$

and

$$\mathrm{Gr}_{n_1}(\mathbb{C}^{k_1}) \times \mathrm{Gr}_{n_2}(\mathbb{C}^{k_2}) \longrightarrow \mathrm{Gr}_{n_1+n_2}(\mathbb{C}^{k_1+k_2})$$

given by sending ambient spaces and subspaces to their direct sum.

Under the colimit of def. 1.91 these induce maps of classifying spaces

$$BO(n_1) \times BO(n_2) \longrightarrow BO(n_1+n_2)$$

and

$$BU(n_1) \times BU(n_2) \longrightarrow BU(n_1+n_2)$$

Proposition 1.94. *The colimiting space $EO(n) = \underset{\longrightarrow_k}{\lim} V_n(\mathbb{R}^k)$ from def. 1.91 is weakly contractible.*

The colimiting space $EU(n) = \underset{\longrightarrow_k}{\lim} V_n(\mathbb{C}^k)$ from def. 1.91 is weakly contractible.

Proof. By propositions 1.89, and 1.90, the Stiefel manifolds are more and more highly connected as k increases. Since the inclusions are relative cell complex inclusions by prop. 1.88, the claim follows. ▮

Proposition 1.95. *The homotopy groups of the classifying spaces $BO(n)$ and $BU(n)$ (def. 1.91) are those of the orthogonal group $O(n)$ and of the unitary group $U(n)$, respectively, shifted up in degree: there are isomorphisms*

$$\pi_{\bullet+1}(BO(n)) \simeq \pi_\bullet O(n)$$

and

$$\pi_{\bullet+1}(BU(n)) \simeq \pi_\bullet U(n)$$

(for homotopy groups based at the canonical basepoint).

Proof. Consider the sequence

$$O(n) \longrightarrow EO(n) \longrightarrow BO(n)$$

from def. 1.91, with $O(n)$ the fiber. Since (by prop. 1.77) the second map is a Serre fibration, this is a fiber sequence and so it induces a long exact sequence of homotopy groups of the form

$$\cdots \to \pi_\bullet(O(n)) \longrightarrow \pi_\bullet(EO(n)) \longrightarrow \pi_\bullet(BO(n)) \longrightarrow \pi_{\bullet-1}(O(n)) \longrightarrow \pi_{\bullet-1}(EO(n)) \to \cdots .$$

Since by cor. 1.94 $\pi_\bullet(EO(n)) = 0$, exactness of the sequence implies that

$$\pi_\bullet(BO(n)) \overset{\simeq}{\longrightarrow} \pi_{\bullet-1}(O(n))$$

is an isomorphism.

The same kind of argument applies to the complex case. ▮

Proposition 1.96. *For $n \in \mathbb{N}$ there are homotopy fiber sequence (def.)*

$$S^n \longrightarrow BO(n) \longrightarrow BO(n+1)$$

and

$$S^{2n+1} \longrightarrow BU(n) \longrightarrow BU(n+1)$$

exhibiting the n-sphere ($(2n+1)$-sphere) as the homotopy fiber of the canonical maps from def. 1.92.

This means (thm.), that there is a replacement of the canonical inclusion $BO(n) \hookrightarrow BO(n+1)$ (induced via def. 1.91) by a Serre fibration

$$\begin{tikzcd} BO(n) \arrow[r, hook] \arrow[d, "\text{weak homotopy equivalence}"'] & BO(n+1) \\ \tilde{B}O(n) \arrow[ur, "\text{Serre fib.}"'] & \end{tikzcd}$$

such that S^n is the ordinary fiber of $BO(n) \to \tilde{B}O(n+1)$, and analogously for the complex case.

Proof. Take $\tilde{B}O(n) \coloneqq (EO(n+1))/O(n)$.

To see that the canonical map $BO(n) \longrightarrow (EO(n+1))/O(n)$ is a weak homotopy equivalence consider the commuting diagram

$$\begin{array}{ccc} O(n) & \overset{\mathrm{id}}{\longrightarrow} & O(n) \\ \downarrow && \downarrow \\ EO(n) & \longrightarrow & EO(n+1) \\ \downarrow && \downarrow \\ BO(n) & \longrightarrow & (EO(n+1))/O(n) \end{array} .$$

By prop. 1.77 both bottom vertical maps are Serre fibrations and so both vertical sequences are fiber sequences. By prop. 1.95 part of the induced morphisms of long exact sequences of homotopy groups looks like this

$$\begin{array}{ccc} \pi_\bullet(BO(n)) & \longrightarrow & \pi_\bullet((EO(n+1))/O(n)) \\ {}^{\simeq}\downarrow && \downarrow^{\simeq} \\ \pi_{\bullet-1}(O(n)) & \stackrel{=}{\longrightarrow} & \pi_{\bullet-1}(O(n)) \end{array} \,,$$

where the vertical and the bottom morphism are isomorphisms. Hence also the to morphisms is an isomorphism.

That $BO(n) \to \tilde{B}O(n+1)$ is indeed a Serre fibration follows again with prop. 1.77, which gives the fiber sequence

$$O(n+1)/O(n) \longrightarrow (EO(n+1))/O(n) \longrightarrow (EO(n+1))/O(n+1)\,.$$

The claim then follows with the identification

$$O(n+1)/O(n) \simeq S^n$$

of example 1.80.

The argument for the complex case is directly analogous, concluding instead with the identification

$$U(n+1)/U(n) \simeq S^{2n+1}$$

from example 1.80. ▮

G-Structure on the Stable normal bundle

Definition 1.97. Given a smooth manifold X of dimension n and equipped with an embedding

$$i : X \hookrightarrow \mathbb{R}^k$$

for some $k \in \mathbb{N}$, then the **classifying map of its normal bundle** is the function

$$g_i : X \to \mathrm{Gr}_{k-n}(\mathbb{R}^k) \hookrightarrow BO(k-n)$$

which sends $x \in X$ to the normal of the tangent space

$$N_x X = (T_x X)^\perp \hookrightarrow \mathbb{R}^k$$

regarded as a point in $G_{k-n}(\mathbb{R}^k)$.

The normal bundle of i itself is the subbundle of the tangent bundle

$$T\mathbb{R}^k \simeq \mathbb{R}^k \times \mathbb{R}^k$$

consisting of those vectors which are orthogonal to the tangent vectors of X:

$$N_i \coloneqq \left\{ x \in X, v \in T_{i(x)}\mathbb{R}^k \;|\; v \perp i_\ast T_x X \subset T_{i(x)}\mathbb{R}^k \right\}\,.$$

Definition 1.98. A (B,f)**-structure** is

1. for each $n \in \mathbb{N}$ a pointed CW-complex $B_n \in \mathrm{Top}^{*/}_{\mathrm{CW}}$
2. equipped with a pointed Serre fibration

   $$\begin{array}{c} B_n \\ \downarrow^{f_n} \\ BO(n) \end{array}$$

   to the classifying space $BO(n)$ (def.);
3. for all $n_1 \leq n_2$ a pointed continuous function

   $$g_{n_1,n_2} : B_{n_1} \longrightarrow B_{n_2}$$

   which is the identity for $n_1 = n_2$;

such that for all $n_1 \leq n_2 \in \mathbb{N}$ these squares commute

$$\begin{matrix} B_{n_1} & \overset{g_{n_1,n_2}}{\longrightarrow} & B_{n_2} \\ {}^{\mathllap{f_{n_1}}}\downarrow && \downarrow^{\mathrlap{f_{n_2}}} \\ B O(n_1) &\longrightarrow& B O(n_2) \end{matrix} \,,$$

where the bottom map is the canonical one from def. 1.92.

The (B,f)-structure is **multiplicative** if it is moreover equipped with a system of maps $\mu_{n_1,n_2} \colon B_{n_1} \times B_{n_2} \to B_{n_1+n_2}$ which cover the canonical multiplication maps (def.)

$$\begin{matrix} B_{n_1} \times B_{n_2} &\overset{\mu_{n_1,n_2}}{\longrightarrow}& B_{n_1+n_2} \\ {}^{\mathllap{f_{n_1} \times f_{n_2}}}\downarrow && \downarrow^{\mathrlap{f_{n_1+n_2}}} \\ B O(n_1) \times B O(n_2) &\longrightarrow& B O(n_1 + n_2) \end{matrix}$$

and which satisfy the evident associativity and unitality, for $B_0 = \ast$ the unit, and, finally, which commute with the maps g in that all $n_1, n_2, n_3 \in \mathbb{N}$ these squares commute:

$$\begin{matrix} B_{n_1} \times B_{n_2} &\overset{id \times g_{n_2, n_2+n_3}}{\longrightarrow}& B_{n_1} \times B_{n_2 + n_3} \\ {}^{\mathllap{\mu_{n_1,n_2}}}\downarrow && \downarrow^{\mathrlap{\mu_{n_1,n_2+n_3}}} \\ B_{n_1+n_2} &\underset{g_{n_1+n_2, n_1+n_2+n_3}}{\longrightarrow}& B_{n_1+n_2+n_3} \end{matrix}$$

and

$$\begin{matrix} B_{n_1} \times B_{n_2} &\overset{g_{n_1, n_1+n_3} \times id}{\longrightarrow}& B_{n_1+n_3} \times B_{n_2} \\ {}^{\mathllap{\mu_{n_1,n_2}}}\downarrow && \downarrow^{\mathrlap{\mu_{n_1+n_3,n_2}}} \\ B_{n_1+n_2} &\underset{g_{n_1+n_2, n_1+n_2+n_3}}{\longrightarrow}& B_{n_1+n_2+n_3} \end{matrix}$$

Similarly, an S^2**-**(B,f)**-structure** is a compatible system

$$f_{2n} \colon B_{2n} \longrightarrow B O(2n)$$

indexed only on the even natural numbers.

Generally, an S^k**-**(B,f)**-structure** for $k \in \mathbb{N}$, $k \geq 1$ is a compatible system

$$f_{k n} \colon B_{k n} \longrightarrow B O(k n)$$

for all $n \in \mathbb{N}$, hence for all $k n \in k \mathbb{N}$.

Example 1.99. Examples of (B,f)-structures (def. 1.98) include the following:

1. $B_n = B O(n)$ and $f_n = id$ is **orthogonal structure** (or "no structure");
2. $B_n = E O(n)$ and f_n the universal principal bundle-projection is **framing-structure**;
3. $B_n = B SO(n) = E O(n)/SO(n)$ the classifying space of the special orthogonal group and f_n the canonical projection is **orientation structure**;
4. $B_n = B Spin(n) = E O(n)/Spin(n)$ the classifying space of the spin group and f_n the canonical projection is **spin structure**.

Examples of S^2-(B,f)-structures (def. 1.98) include

1. $B_{2n} = B U(n) = E O(2n)/U(n)$ the classifying space of the unitary group, and f_{2n} the canonical projection is **almost complex structure** (or rather: **almost Hermitian structure**).
2. $B_{2n} = B Sp(2n) = E O(2n)/Sp(2n)$ the classifying space of the symplectic group, and f_{2n} the canonical projection is **almost symplectic structure**.

Examples of S^4-(B,f)-structures (def. 1.98) include

1. $B_{4n} = B U_{\mathbb{H}}(n) = E O(4n)/U_{\mathbb{H}}(n)$ the classifying space of the quaternionic unitary group, and f_{4n} the canonical projection is **almost quaternionic structure**.

Definition 1.100. Given a smooth manifold X of dimension n, and given a (B,f)-structure as in def. 1.98, then a (B,f)**-structure on the stable normal bundle of the manifold** is an equivalence class of the following structure:

1. an embedding $i_X : X \hookrightarrow \mathbb{R}^k$ for some $k \in \mathbb{N}$;
2. a homotopy class of a lift $\hat{g}$ of the classifying map g of the normal bundle (def. 1.97)

$$\begin{array}{ccc} & & B_{k-n} \\ & {}^{\hat{g}}\nearrow & \downarrow^{f_{k-n}} \\ X & \xrightarrow{g} & BO(k-n) \end{array}.$$

The equivalence relation on such structures is to be that generated by the relation $((i_X)_1, \hat{g}_1) \sim ((i_X)_, \hat{g}_2)$ if

1. $k_2 \geq k_1$
2. the second inclusion factors through the first as

$$(i_X)_2 : X \overset{(i_X)_1}{\hookrightarrow} \mathbb{R}^{k_1} \hookrightarrow \mathbb{R}^{k_2}$$

3. the lift of the classifying map factors accordingly (as homotopy classes)

$$\hat{g}_2 : X \overset{\hat{g}_1}{\longrightarrow} B_{k_1-n} \overset{g_{k_1-n,k_2-n}}{\longrightarrow} B_{k_2-n}.$$

Thom spectra

Idea. Given a vector bundle V of rank n over a compact topological space, then its one-point compactification is equivalently the result of forming the bundle $D(V) \hookrightarrow V$ of unit n-balls, and identifying with one single point all the boundary unit n-spheres $S(V) \hookrightarrow V$. Generally, this construction $\mathrm{Th}(C) \coloneqq D(V)/S(V)$ is called the *Thom space* of V.

Thom spaces occur notably as codomains for would-be left inverses of embeddings of manifolds $X \hookrightarrow Y$. The *Pontrjagin-Thom collapse map* $Y \to \mathrm{Th}(NX)$ of such an embedding is a continuous function going the other way around, but landing not quite in X but in the Thom space of the normal bundle of X in Y. Composing this further with the classifying map of the normal bundle lands in the Thom space of the universal vector bundle over the classifying space $BO(k)$, denoted $MO(k)$. In particular in the case that $Y = S^n$ is an n-sphere (and every manifold embeds into a large enough n-sphere, see also at Whitney embedding theorem), the Pontryagin-Thom collapse map hence associates with every manifold an element of a homotopy group of a universal Thom space $MO(k)$.

This curious construction turns out to have excellent formal properties: as the dimension ranges, the universal Thom spaces arrange into a spectrum, called the *Thom spectrum*, and the homotopy groups defined by the Pontryagin-Thom collapse pass along to the stable homotopy groups of this spectrum.

Moreover, via Whitney sum of vector bundle the Thom spectrum naturally is a homotopy commutative ring spectrum (def.), and under the Pontryagin-Thom collapse the Cartesian product of manifolds is compatible with this ring structure.

Literature. (Kochman 96, 1.5, Schwede 12, chapter I, example 1.16)

Thom spaces

Definition 1.101. Let X be a topological space and let $V \to X$ be a vector bundle over X of rank n, which is associated to an O(n)-principal bundle. Equivalently this means that $V \to X$ is the pullback of the universal vector bundle $E_n \to BO(n)$ (def. 1.91) over the classifying space. Since $O(n)$ preserves the metric on $\mathbb{R}^n$, by definition, such V inherits the structure of a metric space-fiber bundle. With respect to this structure:

1. the **unit disk bundle** $D(V) \to X$ is the subbundle of elements of norm ≤ 1;
2. the **unit sphere bundle** $S(V) \to X$ is the subbundle of elements of norm $= 1$;

$$S(V) \overset{i_V}{\hookrightarrow} D(V) \hookrightarrow V;$$

3. the **Thom space** $\mathrm{Th}(V)$ is the cofiber (formed in Top (prop.)) of i_V

$$\mathrm{Th}(V) \coloneqq \mathrm{cofib}(i_V)$$

canonically regarded as a pointed topological space.

$$\begin{array}{ccc} S(V) & \overset{i_V}{\longrightarrow} & D(V) \\ \downarrow & (\mathrm{po}) & \downarrow \\ * & \longrightarrow & \mathrm{Th}(V) \end{array} .$$

If $V \to X$ is a general real vector bundle, then there exists an isomorphism to an $O(n)$-associated bundle and the Thom space of V is, up to based homeomorphism, that of this orthogonal bundle.

Remark 1.102. If the rank of V is positive, then $S(V)$ is non-empty and then the Thom space (def. 1.101) is the quotient topological space

$$\mathrm{Th}(V) \simeq D(V)/S(V) \,.$$

However, in the degenerate case that the rank of V vanishes, hence the case that $V = X \times \mathbb{R}^0 \simeq X$, then $D(V) \simeq V \simeq X$, but $S(V) = \emptyset$. Hence now the pushout defining the cofiber is

$$\begin{array}{ccc} \emptyset & \overset{i_V}{\longrightarrow} & X \\ \downarrow & (\mathrm{po}) & \downarrow \\ * & \longrightarrow & \mathrm{Th}(V) \simeq X_* \end{array} ,$$

which exhibits $\mathrm{Th}(V)$ as the coproduct of X with the point, hence as X with a basepoint freely adjoined.

$$\mathrm{Th}(X \times \mathbb{R}^0) = \mathrm{Th}(X) \simeq X_+ \,.$$

Proposition 1.103. *Let $V \to X$ be a vector bundle over a CW-complex X. Then the Thom space $\mathrm{Th}(V)$ (def. 1.101) is equivalently the homotopy cofiber (def.) of the inclusion $S(V) \longrightarrow D(V)$ of the sphere bundle into the disk bundle.*

Proof. The Thom space is defined as the ordinary cofiber of $S(V) \to D(V)$. Under the given assumption, this inclusion is a relative cell complex inclusion, hence a cofibration in the classical model structure on topological spaces (thm.). Therefore in this case the ordinary cofiber represents the homotopy cofiber (def.). ∎

The equivalence to the following alternative model for this homotopy cofiber is relevant when discussing Thom isomorphisms and orientation in generalized cohomology:

Proposition 1.104. *Let $V \to X$ be a vector bundle over a CW-complex X. Write $V - X$ for the complement of its 0-section. Then the Thom space $\mathrm{Th}(V)$ (def. 1.101) is homotopy equivalent to the mapping cone of the inclusion $(V - X) \hookrightarrow V$ (hence to the pair $(V, V - X)$ in the language of generalized (Eilenberg-Steenrod) cohomology).*

Proof. The mapping cone of any map out of a CW-complex represents the homotopy cofiber of that map (exmpl.). Moreover, transformation by (weak) homotopy equivalences between morphisms induces a (weak) homotopy equivalence on their homotopy fibers (prop.). But we have such a weak homotopy equivalence, given by contracting away the fibers of the vector bundle:

$$\begin{array}{ccc} V - X & \longrightarrow & V \\ {}^{\in W_{\mathrm{cl}}}\downarrow & & \downarrow^{\in W_{\mathrm{cl}}} \\ S(V) & \hookrightarrow & D(V) \end{array} .$$

∎

Proposition 1.105. *Let $V_1, V_2 \to X$ be two real vector bundles. Then the Thom space (def. 1.101) of the direct sum of vector bundles $V_1 \oplus V_2 \to X$ is expressed in terms of the Thom space of the pullbacks $V_2|_{D(V_1)}$ and $V_2|_{S(V_1)}$ of V_2 to the disk/sphere bundle of V_1 as*

$$\mathrm{Th}(V_1 \oplus V_2) \simeq \mathrm{Th}(V_2|_{D(V_1)}) / \mathrm{Th}(V_2|_{S(V_1)}) \,.$$

Proof. Notice that

1. $D(V_1 \oplus V_2) \simeq D(V_2|_{\mathrm{Int}\, D(V_1)}) \cup S(V_1)$;

2. $S(V_1 \oplus V_2) \simeq S(V_2|_{\mathrm{Int}\, D(V_1)}) \cup \mathrm{Int}\, D(V_2|_{S(V_1)})$.

(Since a point at radius r in $V_1 \oplus V_2$ is a point of radius $r_1 \leq r$ in V_2 and a point of radius $\sqrt{r^2 - r_1^2}$ in V_1.) ∎

Proposition 1.106. *For V a vector bundle then the Thom space (def. 1.101) of $\mathbb{R}^n \oplus V$, the direct sum of vector bundles with the trivial rank n vector bundle, is homeomorphic to the smash product of the Thom space of V with the n-sphere (the n-fold reduced suspension).*

$$\mathrm{Th}(\mathbb{R}^n \oplus V) \simeq S^n \wedge \mathrm{Th}(V) = \Sigma^n \mathrm{Th}(V) \,.$$

Proof. Apply prop. 1.105 with $V_1 = \mathbb{R}^n$ and $V_2 = V$. Since V_1 is a trivial bundle, then

$$V_2|_{D(V_1)} \simeq V_2 \times D^n$$

(as a bundle over $X \times D^n$) and similarly

$$V_2|_{S(V_1)} \simeq V_2 \times S^n \,.$$

∎

Example 1.107. By prop. 1.106 and remark 1.102 the Thom space (def. 1.101) of a trivial vector bundle of rank n is the n-fold suspension of the base space

$$\begin{aligned} \mathrm{Th}(X \times \mathbb{R}^n) &\simeq S^n \wedge \mathrm{Th}(X \times \mathbb{R}^0) \\ &\simeq S^n \wedge (X_+) \end{aligned} \,.$$

Therefore a general Thom space may be thought of as a “twisted suspension”, with twist encoded by a vector bundle (or rather by its underlying spherical fibration). See at *Thom spectrum – For infinity-module bundles* for more on this.

Correspondingly the *Thom isomorphism* (prop. 1.129 below) for a given Thom space is a twisted version of the *suspension isomorphism* (above).

Proposition 1.108. *For $V_1 \to X_1$ and $V_2 \to X_2$ to vector bundles, let $V_1 \boxtimes V_2 \to X_1 \times X_2$ be the direct sum of vector bundles of their pullbacks to $X_1 \times X_2$. The corresponding Thom space (def. 1.101) is the smash product of the individual Thom spaces:*

$$\mathrm{Th}(V_1 \boxtimes V_2) \simeq \mathrm{Th}(V_1) \wedge \mathrm{Th}(V_2) \,.$$

Remark 1.109. Given a vector bundle $V \to X$ of rank n, then the reduced ordinary cohomology of its Thom space $\mathrm{Th}(V)$ (def. 1.101) vanishes in degrees $< n$:

$$\tilde H^{\bullet < n}(\mathrm{Th}(V)) \simeq H^{\bullet < n}(D(V), S(V)) \simeq 0 \,.$$

Proof. Consider the long exact sequence of relative cohomology (from above)

$$\cdots \to H^{\bullet - 1}(D(V)) \overset{i^\ast}{\longrightarrow} H^{\bullet -1}(S(V)) \longrightarrow H^\bullet(D(V), S(V)) \longrightarrow H^\bullet(D(V)) \overset{i^\ast}{\longrightarrow} H^\bullet(S(V)) \to \cdots \,.$$

Since the cohomology in degree k only depends on the k-skeleton, and since for $k < n$ the k-skeleton of $S(V)$ equals that of X, and since $D(V)$ is even homotopy equivalent to X, the morhism $i^\ast$ is an isomorphism in degrees lower than n. Hence by exactness of the sequence it follows that $H^{\bullet < n}(D(V), S(V)) = 0$. ∎

Universal Thom spectra MG

Proposition 1.110. *For each $n \in \mathbb{N}$ the pullback of the rank-$(n+1)$ universal vector bundle to the classifying space of rank n vector bundles is the direct sum of vector bundles of the rank n universal vector bundle with the trivial rank-1 bundle: there is a pullback diagram of topological spaces of the form*

$$\begin{array}{ccc} \mathbb{R} \oplus (E O(n) \underset{O(n)}{\times} \mathbb{R}^n) & \longrightarrow & E O(n+1) \underset{O(n+1)}{\times} \mathbb{R}^{n+1} \\ \downarrow & (pb) & \downarrow \\ B O(n) & \longrightarrow & B O(n+1) \end{array} \,,$$

where the bottom morphism is the canonical one (def.).

(e.g. Kochmann 96, p. 25)

Proof. For each $k \in \mathbb{N}$, $k \geq n$ there is such a pullback of the canonical vector bundles over Grassmannians

$$\begin{array}{ccc} \left\{ \begin{array}{c} V_n \subset \mathbb{R}^k, \\ v \in V_n, v_{n+1} \in \mathbb{R} \end{array} \right\} & \longrightarrow & \left\{ \begin{array}{c} V_{n+1} \subset \mathbb{R}^{k+1}, \\ v \in V_{n+1} \end{array} \right\} \\ \downarrow && \downarrow \\ \mathrm{Gr}_n(\mathbb{R}^k) & \longrightarrow & \mathrm{Gr}_{n+1}(\mathbb{R}^{k+1}) \end{array}$$

where the bottom morphism is the canonical inclusion (def.).

Now we claim that taking the colimit in each of the four corners of this system of pullback diagrams yields again a pullback diagram, and this proves the claim.

To see this, remember that we work in the category $\mathrm{Top}_{\mathrm{cg}}$ of compactly generated topological spaces (def.). By their nature, we may test the universal property of a would-be pullback space already by mapping compact topological spaces into it. Now observe that all the inclusion maps in the four corners of this system of diagrams are relative cell complex inclusions, by prop. 1.87. Together this implies (via this lemma) that we may test the universal property of the colimiting square at finite stages. And so this implies the claim by the above fact that at each finite stage there is a pullback diagram. ▮

Definition 1.111. The **universal real Thom spectrum** MO is the spectrum, which is represented by the sequential prespectrum (def.) whose nth component space is the Thom space (def. 1.101)

$$(MO)_n \coloneqq \mathrm{Th}(EO(n) \underset{O(n)}{\times} \mathbb{R}^n)$$

of the rank-n universal vector bundle, and whose structure maps are the image under the Thom space functor $\mathrm{Th}(-)$ of the top morphisms in prop. 1.110, via the homeomorphisms of prop. 1.106:

$$\sigma_n : \Sigma (MO)_n \simeq \mathrm{Th}(\mathbb{R} \oplus (EO(n) \underset{O(n)}{\times} \mathbb{R}^n)) \longrightarrow \mathrm{Th}(EO(n+1) \underset{O(n+1)}{\times} \mathbb{R}^{n+1}) = (MO)_{n+1} \,.$$

More generally, there are universal Thom spectra associated with any other tangent structure ("[[(B,f)]-structure]]"), notably for the orthogonal group replaced by the special orthogonal groups $\mathrm{SO}(n)$, or the spin groups $\mathrm{Spin}(n)$, or the string 2-group $\mathrm{String}(n)$, or the fivebrane 6-group $\mathrm{Fivebrane}(n)$,..., or any level in the Whitehead tower of $O(n)$. To any of these groups there corresponds a Thom spectrum (denoted, respectively, $M\mathrm{SO}$, MSpin, $M\mathrm{String}$, $M\mathrm{Fivebrane}$, etc.), which is in turn related to oriented cobordism, spin cobordism, string cobordism, et cetera.:

Definition 1.112. Given a (B,f)-structure $\mathcal{B}$ (def. 1.98), write $V_n^{\mathcal{B}}$ for the pullback of the universal vector bundle (def. 1.91) to the corresponding space of the (B,f)-structure and with

$$\begin{array}{ccc} V^{\mathcal{B}} & \longrightarrow & VO(n) \underset{O(n)}{\times} \mathbb{R}^n \\ \downarrow & (\mathrm{pb}) & \downarrow \\ B_n & \underset{f_n}{\longrightarrow} & BO(n) \end{array}$$

and we write e_{n_1,n_2} for the maps of total space of vector bundles over the g_{n_1,n_2}:

$$\begin{array}{ccc} V_{n_1}^{\mathcal{B}} & \overset{e_{n_1,n_2}}{\longrightarrow} & V_{n_2}^{\mathcal{B}} \\ \downarrow & (\mathrm{pb}) & \downarrow \\ B_{n_1} & \underset{g_{n_1,n_2}}{\longrightarrow} & B_{n_2} \end{array} \,.$$

Observe that the analog of prop. 1.110 still holds:

Proposiiton 1.113. *Given a (B,f)-structure $\mathcal{B}$ (def. 1.98), then the pullback of its rank-$(n+1)$ vector bundle $V_{n+1}^{\mathcal{B}}$ (def. 1.112) along the map $g_{n,n+1} : B_n \to B_{n+1}$ is the direct sum of vector bundles of the rank-n bundle $V_n^{\mathcal{B}}$ with the trivial rank-1-bundle: there is a pullback square*

$$\begin{array}{ccc} \mathbb{R} \oplus V_n^{\mathcal{B}} & \overset{e_{n,n+1}}{\longrightarrow} & V_{n+1}^{\mathcal{B}} \\ \downarrow & (\mathrm{pb}) & \downarrow \\ B_n & \underset{g_{n,n+1}}{\longrightarrow} & B_{n+1} \end{array} \,.$$

Proof. Unwinding the definitions, the pullback in question is

$$
\begin{aligned}
(g_{n,n+1})^* V_{n+1}^{\mathcal{B}} &= (g_{n,n+1})^* f_{n+1}^* (EO(n+1) \underset{O(n+1)}{\times} \mathbb{R}^{n+1}) \\
&\simeq (g_{n,n+1} \circ f_{n+1})^* (EO(n+1) \underset{O(n+1)}{\times} \mathbb{R}^{n+1}) \\
&\simeq (f_n \circ i_n)^* (EO(n+1) \underset{O(n+1)}{\times} \mathbb{R}^{n+1}) \\
&\simeq f_n^* i_n^* (EO(n+1) \underset{O(n+1)}{\times} \mathbb{R}^{n+1}) \\
&\simeq f_n^* (\mathbb{R} \oplus (EO(n) \underset{O(n)}{\times} \mathbb{R}^n)) \\
&\simeq \mathbb{R} \oplus V^{\mathcal{B}_n} ,
\end{aligned}
$$

where the second but last step is due to prop. 1.110. ▮

Definition 1.114. Given a (B,f)-structure $\mathcal{B}$ (def. 1.98), its **universal Thom spectrum** $M\mathcal{B}$ is, as a sequential prespectrum, given by component spaces being the Thom spaces (def. 1.101) of the $\mathcal{B}$-associated vector bundles of def. 1.112

$$
(M\mathcal{B})_n \coloneqq \mathrm{Th}(V_n^{\mathcal{B}})
$$

and with structure maps given via prop. 1.106 by the top maps in prop. 1.113:

$$
\sigma_n \,:\, \Sigma (M\mathcal{B})_n = \Sigma\, \mathrm{Th}(V_n^{\mathcal{E}}) \simeq \mathrm{Th}(\mathbb{R} \oplus V_n^{\mathcal{E}}) \overset{\mathrm{Th}(e_{n,n+1})}{\longrightarrow} \mathrm{Th}(V_{n+1}^{\mathcal{B}}) = (M\mathcal{B})_{n+1} \,.
$$

Similarly for an $S^k-(B,f)$-structure indexed on every kth natural number (such as almost complex structure, almost quaternionic structure, example 1.99), there is the corresponding Thom spectrum as a sequential S^k spectrum (def.).

If $B_n = BG_n$ for some natural system of groups $G_n \to O(n)$, then one usually writes MG for $M\mathcal{B}$. For instance $M\,\mathrm{SO}$, MSpin, MU, MSp etc.

If the (B,f)-structure is multiplicative (def. 1.98), then the Thom spectrum $M\mathcal{B}$ canonical becomes a ring spectrum (for more on this see Part 1-2 the section on orthogonal Thom spectra): the multiplication maps $B_{n_1} \times B_{n_2} \to B_{n_1+n_2}$ are covered by maps of vector bundles

$$
V_{n_1}^{\mathcal{B}} \boxtimes V_{n_2}^{\mathcal{B}} \longrightarrow V_{n_1+n_2}^{\mathcal{B}}
$$

and under forming Thom spaces this yields (via prop. 1.108) maps

$$
(M\mathcal{B})_{n_1} \wedge (M\mathcal{B})_{n_2} \longrightarrow (M\mathcal{B})_{n_1+n_2}
$$

which are associative by the associativity condition in a multiplicative (B,f)-structure. The unit is

$$
(M\mathcal{B})_0 = \mathrm{Th}(V_0^{\mathcal{B}}) \simeq \mathrm{Th}(\ast) \simeq S^0 ,
$$

by remark 1.102.

Example 1.115. The universal Thom spectrum (def. 1.114) for *framing* *structure* (exmpl.) is equivalently the sphere spectrum (def.)

$$
M1 \simeq \mathbb{S} \,.
$$

Because in this case $B_n \simeq \ast$ and so $E_n^{\mathcal{B}} \simeq \mathbb{R}^n$, whence $\mathrm{Th}(E_n^{\mathcal{B}}) \simeq S^n$.

Pontrjagin-Thom construction

Definition 1.116. For X a smooth manifold and $i{:}X \hookrightarrow \mathbb{R}^k$ an embedding, then a **tubular neighbourhood** of X is a subset of the form

$$
\tau_i X \coloneqq \left\{ x \in \mathbb{R}^k \mid d(x, i(X)) < \epsilon \right\}
$$

for some $\epsilon \in \mathbb{R}$, $\epsilon > 0$, small enough such that the map

$$
N_i X \longrightarrow \tau_i X
$$

from the normal bundle (def. 1.97) given by

$$
(i(x), v) \mapsto (i(x), \epsilon(1 - e^{-|v|}) v)
$$

is a diffeomorphism.

Proposition 1.117. ***(tubular neighbourhood theorem)***

For every embedding of smooth manifolds, there exists a tubular neighbourhood according to def. 1.116.

Remark 1.118. Given an embedding $i : X \hookrightarrow \mathbb{R}^k$ with a tubuluar neighbourhood $\tau_i X \text{hookrigtharrow} \mathbb{R}^k$ (def. 1.116) then by construction:

1. the Thom space (def. 1.101) of the normal bundle (def. 1.97) is homeomorphic to the quotient topological space of the topological closure of the tubular neighbourhood by its boundary:

   $\mathrm{Th}(N_i(X)) \simeq \overline{\tau_i(X)} / \partial \overline{\tau_i(X)}$;

2. there exists a continous function

   $$\mathbb{R}^k \longrightarrow \overline{\tau_i(X)} / \partial \overline{\tau_i(X)}$$

   which is the identity on $\tau_i(X) \subset \mathbb{R}^k$ and is constant on the basepoint of the quotient on all other points.

Definition 1.119. For X a smooth manifold of dimension n and for $i : X \hookrightarrow \mathbb{R}^k$ an embedding, then the **Pontrjagin-Thom collapse map** is, for any choice of tubular neighbourhood $\tau_i(X) \subset \mathbb{R}^k$ (def. 1.116) the composite map of pointed topological spaces

$$S^k \overset{\simeq}{\to} (\mathbb{R}^k)^* \longrightarrow \overline{\tau_i(X)} / \partial \overline{\tau_i(X)} \overset{\simeq}{\to} \mathrm{Th}(N_i X)$$

where the first map identifies the k-sphere as the one-point compactification of $\mathbb{R}^k$; and where the second and third maps are those of remark 1.118.

The **Pontrjagin-Thom construction** is the further composite

$$\xi_i \,:\, S^k \longrightarrow \mathrm{Th}(N_i X) \overset{\mathrm{Th}(e_i)}{\longrightarrow} \mathrm{Th}(E O(k-n) \underset{O(k-n)}{\times} \mathbb{R}^{k-n}) \simeq (M O)_{k-n}$$

with the image under the Thom space construction of the morphism of vector bundles

$$\begin{array}{ccc} \nu & \overset{e_i}{\longrightarrow} & E O(k-n) \underset{O(k-n)}{\times} \mathbb{R}^{k-n} \\ \downarrow & (\mathrm{pb}) & \downarrow \\ X & \underset{g_i}{\longrightarrow} & B O(k-n) \end{array}$$

induced by the classifying map g_i of the normal bundle (def. 1.97).

This defines an element

$$[S^{n+(k-n)} \overset{\xi_i}{\to} (M O)_{k-n}] \in \pi_n M O$$

in the nth stable homotopy group (def.) of the Thom spectrum $M O$ (def. 1.111).

More generally, for X a smooth manifold with normal (B,f)-structure $(X, i, \hat g_i)$ according to def. 1.100, then its Pontrjagin-Thom construction is the composite

$$\xi_i \,:\, S^k \longrightarrow \mathrm{Th}(N_i X) \overset{\mathrm{Th}(\hat e_i)}{\longrightarrow} \mathrm{Th}(V^{\mathcal{B}}_{k-n}) \simeq (M\mathcal{B})_{k-n}$$

with

$$\begin{array}{ccc} \nu & \overset{\hat e_i}{\longrightarrow} & V^{\mathcal{B}}_{k-n} \\ \downarrow & (\mathrm{pb}) & \downarrow \\ X & \underset{\hat g_i}{\longrightarrow} & B O(k-n) \end{array} \,.$$

Proposition 1.120. *The Pontrjagin-Thom construction (def. 1.119) respects the equivalence classes entering the definition of manifolds with stable normal $\mathcal{B}$-structure (def. 1.100) hence descends to a function (of sets)*

$$\xi \,:\, \left\{ \begin{array}{c} n\text{-manifolds with stable} \\ \text{normal } \mathcal{B}\text{-structure} \end{array} \right\} \longrightarrow \pi_n(M\mathcal{B}) \,.$$

Proof. It is clear that the homotopies of classifying maps of $\mathcal{B}$-structures that are devided out in def. 1.100 map to homotopies of representatives of stable homotopy groups. What needs to be shown is that the construction respects the enlargement of the embedding spaces.

Given a embedded manifold $X \overset{i}{\hookrightarrow} \mathbb{R}^{k_1}$ with normal $\mathcal{B}$-structure

$$\begin{array}{ccc} & & B_{k_1-n} \\ & {}^{\mathllap{\hat g_i}}\nearrow & \downarrow^{\mathrlap{f_{k-n}}} \\ X & \underset{g_i}{\longrightarrow} & BO(k_1-n) \end{array}$$

write

$$\alpha \,:\, S^{n+(k_1-n)} \longrightarrow \mathrm{Th}(E^{\mathcal{B}_{k_1-n}})$$

for its image under the Pontrjagin-Thom construction (def. 1.119). Now given $k_2 \in \mathbb{N}$, consider the induced embedding $X \overset{i}{\hookrightarrow} \mathbb{R}^{k_1} \hookrightarrow \mathbb{R}^{k_1+k_2}$ with normal $\mathcal{B}$-structure given by the composite

$$\begin{array}{ccccc} & & B_{k_1-n} & \overset{g_{k_1-n,k_1+k_2-n}}{\longrightarrow} & B_{k_1+k_2-n} \\ & {}^{\mathllap{\hat g_i}}\nearrow & \downarrow^{\mathrlap{f_{k_1-n}\times f_{k_2}}} & & \downarrow^{\mathrlap{f_{k_1+k_2-n}}} \\ X & \underset{g_i}{\longrightarrow} & BO(k_1-n) & \longrightarrow & BO(k_1+k_2-n) \end{array}$$

By prop. 1.113 and using the pasting law for pullbacks, the classifying map $\hat g'_i$ for the enlarged normal bundle sits in a diagram of the form

$$\begin{array}{ccccc} (\nu_i \oplus \mathbb{R}^{k_2}) & \overset{(\hat e_i \oplus \mathrm{id})}{\longrightarrow} & (V^{\mathcal{B}}_{k_1-n} \oplus \mathbb{R}^{k_2}) & \overset{e_{k_1-n,k_1+k_2-n}}{\longrightarrow} & V^{\mathcal{B}}_{k_1+k_2-n} \\ \downarrow & (\mathrm{pb}) & \downarrow & (\mathrm{pb}) & \downarrow \\ X & \underset{\hat g_i}{\longrightarrow} & B_{k_1-n} & \underset{g_{k_1-n,k_1+k_2-n}}{\longrightarrow} & B_{k_1+k_2-n} \end{array} \,.$$

Hence the Pontrjagin-Thom construction for the enlarged embedding space is (using prop. 1.106) the composite

$$\alpha_{k_2} \,:\, S^{n+(k_1+k_2-n)} \simeq \mathrm{Th}(\mathbb{R}^{k_2}) \wedge S^{n+(k_1-n)} \longrightarrow \mathrm{Th}(\mathbb{R}^{k_2}) \wedge \mathrm{Th}(\nu_i) \overset{\mathrm{Th}(\mathrm{id})\wedge \mathrm{Th}(\hat e_i)}{\longrightarrow} \mathrm{Th}(\mathbb{R}^{k_2}) \wedge \mathrm{Th}(E^{\mathcal{B}}_{k_1-n})) \overset{\mathrm{Th}(e_{k_1-n,k_1+k_2-n})}{\longrightarrow} \mathrm{Th}(V^{\mathcal{B}}_{k_1+k_2-n}$$

The composite of the first two morphisms here is $S^{k_k} \wedge \alpha$, while last morphism $\mathrm{Th}(\hat e_{k_1-n,k_1+k_2-n})$ is the structure map in the Thom spectrum (by def. 1.114):

$$\alpha_{k_2} \,:\, S^{k_2} \wedge S^{n+(k_1-n)} \overset{S^{k_2}\wedge \alpha}{\longrightarrow} S^{k_2} \wedge \mathrm{Th}(E^{\mathcal{B}}_{k_1+k_2-n}) \overset{\sigma^{M\mathcal{B}}_{k_1-n,k_1+k_2-n}}{\longrightarrow} \mathrm{Th}(V^{\mathcal{B}}_{k_1+k_2-n})$$

This manifestly identifies α_{k_2} as being the image of α under the component map in the sequential colimit that defines the stable homotopy groups (def.). Therefore α and α_{k_2}, for all $k_2 \in \mathbb{N}$, represent the same element in $\pi_\bullet(M\mathcal{B})$. ▮

Bordism and Thom's theorem

Idea. By the Pontryagin-Thom collapse construction above, there is an assignment

$$n\,\mathrm{Manifolds} \longrightarrow \pi_n(MO)$$

which sends disjoint union and Cartesian product of manifolds to sum and product in the ring of stable homotopy groups of the Thom spectrum. One finds then that two manifolds map to the same element in the stable homotopy groups $\pi_\bullet(MO)$ of the universal Thom spectrum precisely if they are connected by a bordism. The bordism-classes $\Omega^O_\bullet$ of manifolds form a commutative ring under disjoint union and Cartesian product, called the *bordism ring*, and Pontrjagin-Thom collapse produces a ring homomorphism

$$\Omega^O_\bullet \longrightarrow \pi_\bullet(MO) \,.$$

Thom's theorem states that this homomorphism is an isomorphism.

More generally, for $\mathcal{B}$ a multiplicative (B,f)-structure, def. 1.98, there is such an identification

$$\Omega^{\mathcal{B}}_\bullet \simeq \pi_\bullet(M\mathcal{B})$$

between the ring of $\mathcal{B}$-cobordism classes of manifolds with $\mathcal{B}$-structure and the stable homotopy groups of the universal $\mathcal{B}$-Thom spectrum.

Literature. (Kochman 96, 1.5)

Bordism

Throughout, let $\mathcal{B}$ be a multiplicative (B,f)-structure (def. 1.98).

Definition 1.121. Write $I \coloneqq [0,1]$ for the standard interval, regarded as a smooth manifold with boundary. For $c \in \mathbb{R}_+$ Consider its embedding

$$e : I \hookrightarrow \mathbb{R} \oplus \mathbb{R}_{\geq 0}$$

as the arc

$$e : t \mapsto \cos(\pi t) \cdot e_1 + \sin(\pi t) \cdot e_2 \,,$$

where (e_1, e_2) denotes the canonical linear basis of $\mathbb{R}^2$, and equipped with the structure of a manifold with normal framing structure (example 1.99) by equipping it with the canonical framing

$$\mathrm{fr} : t \mapsto \cos(\pi t) \cdot e_1 + \sin(\pi t) \cdot e_2$$

of its normal bundle.

Let now $\mathcal{B}$ be a (B,f)-structure (def. 1.98). Then for $X \overset{i}{\hookrightarrow} \mathbb{R}^k$ any embedded manifold with $\mathcal{B}$-structure $\hat g : X \to B_{k-n}$ on its normal bundle (def. 1.100), define its **negative** or **orientation reversal** $-(X, i, \hat g)$ of $(X, i, \hat g)$ to be the restriction of the structured manifold

$$(X \times I \overset{(i,e)}{\hookrightarrow} \mathbb{R}^{k+2}, \hat g \times \mathrm{fr})$$

to $t = 1$.

Definition 1.122. Two closed manifolds of dimension n equipped with normal $\mathcal{B}$-structure $(X_1, i_1, \hat g_1)$ and $(X_2, i_2, \hat g_2)$ (def.) are called **bordant** if there exists a manifold with boundary W of dimension $n+1$ equipped with $\mathcal{B}$-strcuture $(W, i_W, \hat g_W)$ if its boundary with $\mathcal{B}$-structure restricted to that boundary is the disjoint union of X_1 with the negative of X_2, according to def. 1.121

$$\partial(W, i_W, \hat g_W) \simeq (X_1, i_1, \hat g_1) \sqcup -(X_2, i_2, \hat g_2) \,.$$

Proposition 1.123. *The relation of $\mathcal{B}$-bordism (def. 1.122) is an equivalence relation.*

Write $\Omega_\bullet^{\mathcal{B}}$ for the $\mathbb{N}$-graded set of $\mathcal{B}$-bordism classes of $\mathcal{B}$-manifolds.

Proposition 1.124. *Under disjoint union of manifolds, then the set of $\mathcal{B}$-bordism equivalence classes of def. 1.123 becomes an $\mathbb{Z}$-graded abelian group*

$$\Omega_\bullet^{\mathcal{B}} \in \mathrm{Ab}^{\mathbb{Z}}$$

*(that happens to be concentrated in non-negative degrees). This is called the $\mathcal{B}$-**bordism group**.*

*Moreover, if the (B,f)-structure $\mathcal{B}$ is multiplicative (def. 1.98), then Cartesian product of manifolds followed by the multiplicative composition operation of $\mathcal{B}$-structures makes the $\mathcal{B}$-bordism ring into a commutative ring, called the $\mathcal{B}$-**bordism ring**.*

$$\Omega_\bullet^{\mathcal{B}} \in \mathrm{CRing}^{\mathbb{Z}} \,.$$

e.g. (Kochmann 96, prop. 1.5.3)

Thom's theorem

Recall that the Pontrjagin-Thom construction (def. 1.119) associates to an embbeded manifold $(X, i, \hat g)$ with normal $\mathcal{B}$-structure (def. 1.100) an element in the stable homotopy group $\pi_{\dim(X)}(M\mathcal{B})$ of the universal $\mathcal{B}$-Thom spectrum in degree the dimension of that manifold.

Lemma 1.125. *For $\mathcal{B}$ be a multiplicative (B,f)-structure (def. 1.98), the $\mathcal{B}$-Pontrjagin-Thom construction (def. 1.119) is compatible with all the relations involved to yield a graded ring homomorphism*

$$\xi : \Omega_\bullet^{\mathcal{B}} \longrightarrow \pi_\bullet(M\mathcal{B})$$

from the $\mathcal{B}$-bordism ring (def. 1.124) to the stable homotopy groups of the universal $\mathcal{B}$-Thom spectrum equipped with the ring structure induced from the canonical ring spectrum structure (def. 1.114).

Proof. By prop. 1.120 the underlying function of sets is well-defined before dividing out the bordism relation (def. 1.122). To descend this further to a function out of the set underlying the bordism ring, we need to see that the Pontrjagin-Thom construction respects the bordism relation. But the definition of bordism is just so as to exhibit under ξ a left homotopy of representatives of homotopy groups.

Next we need to show that it is

1. a group homomorphism;

2. a ring homomorphism.

Regarding the first point:

The element 0 in the cobordism group is represented by the empty manifold. It is clear that the Pontrjagin-Thom construction takes this to the trivial stable homotopy now.

Given two n-manifolds with $\mathcal{B}$-structure, we may consider an embedding of their disjoint union into some $\mathbb{R}^k$ such that the tubular neighbourhoods of the two direct summands do not intersect. There is then a map from two copies of the k-cube, glued at one face

$$\square^k \underset{\square^{k-1}}{\sqcup} \square^k \longrightarrow \mathbb{R}^k$$

such that the first manifold with its tubular neighbourhood sits inside the image of the first cube, while the second manifold with its tubular neighbourhood sits indide the second cube. After applying the Pontryagin-Thom construction to this setup, each cube separately maps to the image under ξ of the respective manifold, while the union of the two cubes manifestly maps to the sum of the resulting elements of homotopy groups, by the very definition of the group operation in the homotopy groups (def.). This shows that ξ is a group homomorphism.

Regarding the second point:

The element 1 in the cobordism ring is represented by the manifold which is the point. Without restriction we may consoder this as embedded into $\mathbb{R}^0$, by the identity map. The corresponding normal bundle is of rank 0 and hence (by remark 1.102) its Thom space is S^0, the 0-sphere. Also $V_0^{\mathcal{B}}$ is the rank-0 vector bundle over the point, and hence $(M\mathcal{B})_0 \simeq S^0$ (by def. 1.114) and so $\xi(*) : (S^0 \overset{\simeq}{\to} S^0)$ indeed represents the unit element in $\pi_\bullet(M\mathcal{B})$.

Finally regarding respect for the ring product structure: for two manifolds with stable normal $\mathcal{B}$-structure, represented by embeddings into $\mathbb{R}^{k_i}$, then the normal bundle of the embedding of their Cartesian product is the direct sum of vector bundles of the separate normal bundles bulled back to the product manifold. In the notation of prop. 1.108 there is a diagram of the form

$$\begin{array}{ccccc} \nu_1 \boxtimes \nu_2 & \overset{\hat e_1 \boxtimes \hat e_2}{\longrightarrow} & V^{\mathcal{B}}_{n_1} \boxtimes V^{\mathcal{B}}_{n_2} & \overset{\kappa_{n_1,n_2}}{\longrightarrow} & V^{\mathcal{B}}_{n_1+n_2} \\ \downarrow & (pb) & \downarrow & (pb) & \downarrow \\ X_1 \times X_2 & \underset{\hat g_1 \times \hat g_2}{\longrightarrow} & B_{k_1-n_1} \times B_{k_2-n_2} & \underset{\mu_{k_1-n_1,k_2-n_2}}{\longrightarrow} & B_{k_1+k_2-n_1-n_2} \end{array} .$$

To the Pontrjagin-Thom construction of the product manifold is by definition the top composite in the diagram

$$\begin{array}{ccccccc} S^{n_1+n_2+(k_1+k_2-n_1-n_2)} & \longrightarrow & Th(\nu_1 \boxtimes \nu_2) & \overset{Th(\hat e_1 \boxtimes \hat e_2)}{\longrightarrow} & Th(V^{\mathcal{B}}_{k_1-n_1} \boxtimes V^{\mathcal{B}}_{k_2-n_2}) & \overset{\kappa_{k_1-n_1,k_2-n_2}}{\longrightarrow} & Th(V^{\mathcal{B}}_{k_1+k_2-n_1-n_2}) \\ {}^{\simeq}\downarrow && \downarrow^{\simeq} && \downarrow^{\simeq} && \downarrow^{=} \\ S^{n_1+(k_1-n_1)} \wedge S^{n_2+(k_2-n_2)} & \longrightarrow & Th(\nu_1) \wedge Th(\nu_2) & \overset{Th(\hat e_1) \wedge Th(\hat e_2)}{\longrightarrow} & Th(V_1^{\mathcal{B}}) \wedge Th(V_2^{\mathcal{B}}) & \overset{\kappa_{k_1-n_1,k_2-n_2}}{\longrightarrow} & Th(V^{\mathcal{B}}_{k_1+k_2-n_1-n_2}) \end{array} ,$$

which hence is equivalently the bottom composite, which in turn manifestly represents the product of the separate PT constructions in $\pi_\bullet(M\mathcal{B})$. ▮

Theorem 1.126. *The ring homomorphsim in lemma 1.125 is an isomorphism.*

Due to (Thom 54, Pontrjagin 55). See for instance (Kochmann 96, theorem 1.5.10).

Proof idea. Observe that given the result $\alpha : S^{n+(k-n)} \to Th(V_{k-n})$ of the Pontrjagin-Thom construction map, the original manifold $X \overset{i}{\hookrightarrow} \mathbb{R}^k$ may be recovered as this pullback:

$$\begin{array}{ccc} X & \overset{i}{\longrightarrow} & S^{n+(k-n)} \\ {}^{g_i}\downarrow & (pb) & \downarrow^{\alpha} \\ BO(k-n) & \longrightarrow & Th(V^{BO}_{k-n}) \end{array} \,.$$

To see this more explicitly, break it up into pieces:

$$\begin{array}{ccccc} X & \longrightarrow & X_+ & \hookrightarrow & S^{n+(k-n)} \\ \downarrow & (pb) & \downarrow & (pb) & \downarrow \\ X & \longrightarrow & X_+ \simeq Th(X) & \overset{Th(0)}{\longrightarrow} & Th(\nu_i) \\ \downarrow & (pb) & \downarrow & (pb) & \downarrow \\ B_{k-n} & \longrightarrow & (B_{k-n})_+ \simeq Th(B_{k-n}) & \underset{Th(0)}{\longrightarrow} & Th(V^{\mathcal{B}}_{k-n}) \\ \downarrow & (pb) & \downarrow & (pb) & \downarrow \\ BO(k-n) & \longrightarrow & (BO(k-n))_+ \simeq Th(BO(k-n)) & \longrightarrow & Th(V^{BO}_{k-n}) \end{array} \,.$$

Moreover, since the n-spheres are compact topological spaces, and since the classifying space $BO(n)$, and hence its universal Thom space, is a sequential colimit over relative cell complex inclusions, the right vertical map factors through some finite stage (by this lemma), the manifold X is equivalently recovered as a pullback of the form

$$\begin{array}{ccc} X & \longrightarrow & S^{n+(k-n)} \\ {}^{g_i}\downarrow & (pb) & \downarrow \\ Gr_{k-n}(\mathbb{R}^k) & \overset{i}{\longrightarrow} & Th(V_{k-n}(\mathbb{R}^k) \underset{O(k-n)}{\times} \mathbb{R}^{k-n}) \end{array} \,.$$

(Recall that $V^{\mathcal{B}}_{k-n}$ is our notation for the universal vector bundle with $\mathcal{B}$-structure, while $V_{k-n}(\mathbb{R}^k)$ denotes a Stiefel manifold.)

The idea of the proof now is to use this property as the blueprint of the construction of an inverse ζ to ξ: given an element in $\pi_n(M\mathcal{B})$ represented by a map as on the right of the above diagram, try to define X and the structure map g_i of its normal bundle as the pullback on the left.

The technical problem to be overcome is that for a general continuous function as on the right, the pullback has no reason to be a smooth manifold, and for two reasons:

1. the map $S^{n+(k-n)} \to Th(V_{k-n})$ may not be smooth around the image of i;
2. even if it is smooth around the image of i, it may not be transversal to i, and the intersection of two non-transversal smooth functions is in general still not a smooth manifold.

The heart of the proof is in showing that for any α there are small homotopies relating it to an α' that is both smooth around the image of i and transversal to i.

The first condition is guaranteed by Sard's theorem, the second by Thom's transversality theorem.

(...) ∎

Thom isomorphism

Idea. If a vector bundle $E \overset{p}{\longrightarrow} X$ of rank n carries a cohomology class $\omega \in H^n(Th(E), R)$ that looks fiberwise like a volume form – a Thom class – then the operation of pulling back from base space and then forming the cup product with this Thom class is an isomorphism on (reduced) cohomology

$$((-) \cup \omega) \circ p^\ast : H^\bullet(X, R) \overset{\simeq}{\longrightarrow} \tilde H^{\bullet + n}(Th(E), R) \,.$$

This is the *Thom isomorphism*. It follows from the Serre spectral sequence (or else from the Leray-Hirsch theorem). A closely related statement gives the *Thom-Gysin sequence*.

In the special case that the vector bundle is trivial of rank n, then its Thom space coincides with the n-fold suspension of the base space (example 1.107) and the Thom isomorphism coincides with the suspension isomorphism. In this sense the Thom isomorphism may be regarded as a *twisted suspension isomorphism*.

We need this below to compute (co)homology of universal Thom spectra MU in terms of that of the classifying spaces BU.

Composed with pullback along the Pontryagin-Thom collapse map, the Thom isomorphism produces maps in cohomology that covariantly follow the underlying maps of spaces. These "Umkehr maps" have the interpretation of fiber integration against the Thom class.

Literature. (Kochman 96, 2.6)

Thom-Gysin sequence

The *Thom-Gysin sequence* is a type of long exact sequence in cohomology induced by a spherical fibration and expressing the cohomology groups of the total space in terms of those of the base plus correction. The sequence may be obtained as a corollary of the Serre spectral sequence for the given fibration. It induces, and is induced by, the Thom isomorphism.

Proposition 1.127. *Let R be a commutative ring and let*

$$\begin{array}{ccc} S^n & \longrightarrow & E \\ && \downarrow^{\pi} \\ && B \end{array}$$

be a Serre fibration over a simply connected CW-complex with typical fiber (exmpl.) the n-sphere.

*Then there exists an element $c \in H^{n+1}(E;R)$ (in the ordinary cohomology of the total space with coefficients in R, called the **Euler class** of π) such that the cup product operation $c \cup (-)$ sits in a long exact sequence of cohomology groups of the form*

$$\cdots \to H^k(B;R) \overset{\pi^*}{\longrightarrow} H^k(E;R) \longrightarrow H^{k-n}(B;R) \overset{c\cup(-)}{\longrightarrow} H^{k+1}(B;R) \to \cdots \,.$$

(e.g. Switzer 75, section 15.30, Kochman 96, corollary 2.2.6)

Proof. Under the given assumptions there is the corresponding Serre spectral sequence

$$E_2^{s,t} \;=\; H^s(B; H^t(S^n;R)) \;\Rightarrow\; H^{s+t}(E;R) \,.$$

Since the ordinary cohomology of the n-sphere fiber is concentrated in just two degees

$$H^t(S^n;R) = \begin{cases} R & \text{for}\; t = 0 \;\text{and}\; t = n \\ 0 & \text{otherwise} \end{cases}$$

the only possibly non-vanishing terms on the E_2 page of this spectral sequence, and hence on all the further pages, are in bidegrees $(\bullet, 0)$ and $(\bullet, n)$:

$$E_2^{\bullet,0} \simeq H^\bullet(B;R)\,, \quad \text{and} \quad E_2^{\bullet,n} \simeq H^\bullet(B;R) \,.$$

As a consequence, since the differentials d_r on the rth page of the Serre spectral sequence have bidegree $(r+1, -r)$, the only possibly non-vanishing differentials are those on the $(n+1)$-page of the form

$$\begin{array}{ccc} E_{n+1}^{\bullet,n} & \simeq & H^\bullet(B;R) \\ {}^{d_{n+1}}\downarrow && \\ E_{n+1}^{\bullet+n+1,0} & \simeq & H^{\bullet+n+1}(B;R) \end{array} \,.$$

Now since the coefficients R is a ring, the Serre spectral sequence is multiplicative under cup product and the differential is a derivation (of total degree 1) with respect to this product. (See at *multiplicative spectral sequence – Examples – AHSS for multiplicative cohomology*.)

To make use of this, write

$$\iota \coloneqq 1 \in H^0(B;R) \overset{\simeq}{\longrightarrow} E_{n+1}^{0,n}$$

for the unit in the cohomology ring $H^\bullet(B;R)$, but regarded as an element in bidegree $(0,n)$ on the $(n+1)$-page of the spectral sequence. (In particular ι does *not* denote the unit in bidegree $(0,0)$, and hence $d_{n+1}(\iota)$ need not vanish; while by the derivation property, it does vanish on the actual unit $1 \in H^0(B;R) \simeq E_{n+1}^{0,0}$.)

Write

$$c \coloneqq d_{n+1}(\iota) \;\; \in E_{n+1}^{n+1,0} \overset{\simeq}{\longrightarrow} H^{n+1}(B;R)$$

for the image of this element under the differential. We will show that this is the Euler class in question.

To that end, notice that every element in $E_{n+1}^{\bullet,n}$ is of the form $\iota \cdot b$ for $b \in E_{n+1}^{\bullet,0} \simeq H^\bullet(B;R)$.

(Because the multiplicative structure gives a group homomorphism $\iota \cdot (-) : H^\bullet(B;R) \simeq E_{n+1}^{0,0} \to E_{n+1}^{0,n} \simeq H^\bullet(B;R)$, which is an isomorphism because the product in the spectral sequence does come from the cup product in the cohomology ring, see for instance (Kochman 96, first equation in the proof of prop. 4.2.9), and since hence ι does act like the unit that it is in $H^\bullet(B;R)$).

Now since d_{n+1} is a graded derivation and vanishes on $E_{n+1}^{\bullet,0}$ (by the above degree reasoning), it follows that its action on any element is uniquely fixed to be given by the product with c:

$$\begin{aligned} d_{n+1}(\iota \cdot b) &= d_{n+1}(\iota) \cdot b + (-1)^n \iota \cdot \underbrace{d_{n+1}(b)}_{=0} \\ &= c \cdot b \end{aligned} .$$

This shows that d_{n+1} is identified with the cup product operation in question:

$$\begin{array}{ccc} E_{n+1}^{s,n} & \simeq & H^s(B;R) \\ {}^{\mathllap{d_{n+1}}}\downarrow && \downarrow^{\mathrlap{c \cup (-)}} \\ E_{n+1}^{s+n+1,0} & \simeq & H^{s+n+1}(B;R) \end{array} .$$

In summary, the non-vanishing entries of the E_∞-page of the spectral sequence sit in exact sequences like so

$$\begin{array}{ccc} 0 \\ \downarrow \\ E_\infty^{s,n} \\ {}^{\mathllap{\ker(d_{n+1})}}\downarrow \\ E_{n+1}^{s,n} & \simeq & H^s(B;R) \\ {}^{\mathllap{d_{n+1}}}\downarrow && \downarrow^{\mathrlap{c \cup (-)}} \\ E_{n+1}^{s+n+1,0} & \simeq & H^{s+n+1}(B;R) \\ {}^{\mathllap{\mathrm{coker}(d_{n+1})}}\downarrow \\ E_\infty^{s+n+1,0} \\ \downarrow \\ 0 \end{array} .$$

Finally observe (lemma 1.128) that due to the sparseness of the E_∞-page, there are also short exact sequences of the form

$$0 \to E_\infty^{s,0} \longrightarrow H^s(E;R) \longrightarrow E_\infty^{s-n,n} \to 0 .$$

Concatenating these with the above exact sequences yields the desired long exact sequence. ∎

Lemma 1.128. *Consider a cohomology spectral sequence converging to some filtered graded abelian group $F^\bullet C^\bullet$ such that*

1. $F^0 C^\bullet = C^\bullet$;
2. $F^s C^{<s} = 0$;
3. $E_\infty^{s,t} = 0$ *unless* $t = 0$ *or* $t = n$,

for some $n \in \mathbb{N}$, $n \geq 1$. Then there are short exact sequences of the form

$$0 \to E_\infty^{s,0} \longrightarrow C^s \longrightarrow E_\infty^{s-n,n} \to 0 .$$

(e.g. Switzer 75, p. 356)

Proof. By definition of convergence of a spectral sequence, the $E_\infty^{s,t}$ sit in short exact sequences of the form

$$0 \to F^{s+1} C^{s+t} \overset{i}{\longrightarrow} F^s C^{s+t} \longrightarrow E_\infty^{s,t} \to 0 .$$

So when $E_\infty^{s,t} = 0$ then the morphism i above is an isomorphism.

We may use this to either shift away the filtering degree

- if $t \geq n$ then $F^s C^{s+t} = F^{(s-1)+1} C^{(s-1)+(t+1)} \underset{\simeq}{\overset{i^{s-1}}{\longrightarrow}} F^0 C^{(s-1)+(t+1)} = F^0 C^{s+t} \simeq C^{s+t}$;

or to shift away the offset of the filtering to the total degree:

- if $0 \leq t-1 \leq n-1$ then $F^{s+1} C^{s+t} = F^{s+1} C^{(s+1)+(t-1)} \underset{\simeq}{\overset{i^{-(t-1)}}{\longrightarrow}} F^{s+t} C^{(s+1)+(t-1)} = F^{s+t} C^{s+t}$

Moreover, by the assumption that if $t < 0$ then $F^s C^{s+t} = 0$, we also get

$$F^s C^s \simeq E_\infty^{s,0} \,.$$

In summary this yields the vertical isomorphisms

$$\begin{array}{ccccccccc} 0 & \to & F^{s+1} C^{s+n} & \longrightarrow & F^s C^{s+n} & \longrightarrow & E_\infty^{s,n} & \to & 0 \\ && {}^{\mathllap{i^{-(n-1)}}}\downarrow^{\simeq} && {}^{\mathllap{i^{s-1}}}\downarrow^{\simeq} && \downarrow^{=} \\ 0 & \to & F^{s+n} C^{s+n} \simeq E_\infty^{s+n,0} & \longrightarrow & C^{s+n} & \longrightarrow & E_\infty^{s,n} & \to & 0 \end{array}$$

and hence with the top sequence here being exact, so is the bottom sequence. ▮

Thom isomorphism

Proposition 1.129. *Let $V \to B$ be a topological vector bundle of rank $n > 0$ over a simply connected CW-complex B. Let R be a commutative ring.*

*There exists an element $c \in H^n(\mathrm{Th}(V); R)$ (in the ordinary cohomology, with coefficients in R, of the Thom space of V, called a **Thom class**) such that forming the cup product with c induces an isomorphism*

$$H^\bullet(B; R) \overset{c \cup (-)}{\longrightarrow} \tilde H^{\bullet + n}(\mathrm{Th}(V); R)$$

of degree n from the unreduced cohomology group of B to the reduced cohomology of the Thom space of V.

Proof. Choose an orthogonal structure on V. Consider the *fiberwise* cofiber

$$E \coloneqq D(V)/_B S(V)$$

of the inclusion of the unit sphere bundle into the unit disk bundle of V (def. 1.101).

$$\begin{array}{ccccc} S^{n-1} & \hookrightarrow & D^n & \longrightarrow & S^n \\ \downarrow && \downarrow && \downarrow \\ S(V) & \hookrightarrow & D(V) & \longrightarrow & E \\ \downarrow && \downarrow && \downarrow^{p} \\ B & = & B & = & B \end{array}$$

Observe that this has the following properties

1. $E \overset{p}{\to} B$ is an n-sphere fiber bundle, hence in particular a Serre fibration;
2. the Thom space $\mathrm{Th}(V) \simeq E/B$ is the quotient of E by the base space, because of the pasting law applied to the following pasting diagram of pushout squares

$$\begin{array}{ccc} S(V) & \longrightarrow & D(V) \\ \downarrow & (\mathrm{po}) & \downarrow \\ B & \longrightarrow & D(V)/_B S(V) \\ \downarrow & (\mathrm{po}) & \downarrow \\ * & \longrightarrow & \mathrm{Th}(V) \end{array}$$

3. hence the reduced cohomology of the Thom space is (def.) the relative cohomology of E relative B

$$\tilde H^\bullet(\mathrm{Th}(V); R) \simeq H^\bullet(E, B; R) \,.$$

4. $E \overset{p}{\to} B$ has a global section $B \overset{s}{\to} E$ (given over any point $b \in B$ by the class of any point in the fiber of $S(V) \to B$ over b; or abstractly: induced via the above pushout by the commutation of the projections from $D(V)$ and from $S(V)$, respectively).

In the following we write $H^\bullet(-) \coloneqq H^\bullet(-;R)$, for short.

By the first point, there is the Thom-Gysin sequence (prop. 1.127), an exact sequence running vertically in the following diagram

$$\begin{array}{ccccc} & & H^\bullet(B) & & \\ & & {}^{p^*}\downarrow & \searrow^{\simeq} & \\ \tilde H^\bullet(\mathrm{Th}(V)) & \longrightarrow & H^\bullet(E) & \underset{s^*}{\longrightarrow} & H^\bullet(B) \\ & & \downarrow & & \\ & & H^{\bullet - n}(B) & & \end{array} .$$

By the second point above this is split, as shown by the diagonal isomorphism in the top right. By the third point above there is the horizontal exact sequence, as shown, which is the exact sequence in relative cohomology $\cdots \to H^\bullet(E,B) \to H^\bullet(E) \to H^\bullet(B) \to \cdots$ induced from the section $B \hookrightarrow E$.

Hence using the splitting to decompose the term in the middle as a direct sum, and then using horizontal and vertical exactness at that term yields

$$\begin{array}{ccccc} & & H^\bullet(B) & & \\ & & {}^{(0,\mathrm{id})}\downarrow & \searrow^{\simeq} & \\ \tilde H^\bullet(\mathrm{Th}(V)) & \overset{(\mathrm{id},0)}{\hookrightarrow} & \tilde H^\bullet(\mathrm{Th}(V)) \oplus H^\bullet(B) & \underset{(0,\mathrm{id})}{\longrightarrow} & H^\bullet(B) \\ & & \downarrow^{(\mathrm{id},0)} & & \\ & & H^{\bullet - n}(B) & & \end{array}$$

and hence an isomorphism

$$\tilde H^\bullet(\mathrm{Th}(V)) \overset{\simeq}{\longrightarrow} H^{\bullet - n}(B) \,.$$

To see that this is the inverse of a morphism of the form $c \cup (-)$, inspect the proof of the Gysin sequence. This shows that $H^{\bullet - n}(B)$ here is identified with elements that on the second page of the corresponding Serre spectral sequence are cup products

$$\iota \cup b$$

with ι fiberwise the canonical class $1 \in H^n(S^n)$ and with $b \in H^\bullet(B)$ any element. Since $H^\bullet(-;R)$ is a multiplicative cohomology theory (because the coefficients form a ring R), cup producs are preserved as one passes to the E_∞-page of the spectral sequence, and the morphism $H^\bullet(E) \to B^\bullet(B)$ above, hence also the isomorphism $\tilde H^\bullet(\mathrm{Th}(V)) \to H^\bullet(B)$, factors through the E_∞-page (see towards the end of the proof of the Gysin sequence). Hence the image of ι on the E_∞-page is the Thom class in question. ▮

Orientation in generalized cohomology

Idea. From the way the Thom isomorphism via a Thom class works in ordinary cohomology (as above), one sees what the general concept of orientation in generalized cohomology and of fiber integration in generalized cohomology is to be.

Specifically we are interested in complex oriented cohomology theories E, characterized by an orientation class on infinity complex projective space $\mathbb{C}P^\infty$ (def. 1.134), the classifying space for complex line bundles, which restricts to a generator on $S^2 \hookrightarrow \mathbb{C}P^\infty$.

(Another important application is given by taking $E =$ KU to be topological K-theory. Then orientation is spin^c structure and fiber integration with coefficients in E is fiber integration in K-theory. This is classical *index theory*.)

Literature. (Kochman 96, section 4.3, Adams 74, part III, section 10, Lurie 10, lecture 5)

- Riccardo Pedrotti, *Complex oriented cohomology – Orientation in generalized cohomology*, 2016 (pdf)

Universal E-orientation

Definition 1.130. Let E be a multiplicative cohomology theory (def. 1.26) and let $V \to X$ be a topological vector bundle of rank n. Then an E-**orientation** or E-**Thom class** on V is an element of degree n

$$u \in \tilde{E}^n(\mathrm{Th}(V))$$

in the reduced E-cohomology ring of the Thom space (def. 1.101) of V, such that for every point $x \in X$ its restriction $i_x^* u$ along

$$i_x : S^n \simeq \mathrm{Th}(\mathbb{R}^n) \overset{\mathrm{Th}(e_x)}{\longrightarrow} \mathrm{Th}(V)$$

(for $\mathbb{R}^n \overset{\mathrm{fib}_x}{\hookrightarrow} V$ the fiber of V over x) is a *generator*, in that it is of the form

$$i^* u = \epsilon \cdot \gamma_n$$

for

- $\epsilon \in \tilde{E}^0(S^0)$ a unit in $E^\bullet$;
- $\gamma_n \in \tilde{E}^n(S^n)$ the image of the multiplicative unit under the suspension isomorphism $\tilde{E}^0(S^0) \overset{\simeq}{\to} \tilde{E}^n(S^n)$.

(e.g. Kochmann 96, def. 4.3.4)

Remark 1.131. Recall that a *(B,f)-structure* $\mathcal{B}$ (def. 1.98) is a system of Serre fibrations $B_n \overset{f_n}{\to} BO(n)$ over the classifying spaces for orthogonal structure equipped with maps

$$g_{n,n+1} : B_n \longrightarrow B_{n+1}$$

covering the canonical inclusions of classifying spaces. For instance for $G_n \to O(n)$ a compatible system of topological group homomorphisms, then the (B,f)-structure given by the classifying spaces BG_n (possibly suitably resolved for the maps $BG_n \to BO(n)$ to become Serre fibrations) defines *G-structure*.

Given a (B,f)-structure, then there are the pullbacks $V_n^{\mathcal{B}} \coloneqq f_n^*(EO(n) \underset{O(n)}{\times} \mathbb{R}^n)$ of the universal vector bundles over $BO(n)$, which are the *universal vector bundles equipped with* (B,f)*-structure*

$$\begin{array}{ccc} V_n^{\mathcal{B}} & \longrightarrow & EO(n) \underset{O(n)}{\times} \mathbb{R}^n \\ \downarrow & (\mathrm{pb}) & \downarrow \\ B_n & \underset{f_n}{\longrightarrow} & BO(n) \end{array} .$$

Finally recall that there are canonical morphisms (prop.)

$$\phi_n : \mathbb{R} \oplus V_n^{\mathcal{B}} \longrightarrow V_{n+1}^{\mathcal{B}}$$

Definition 1.132. Let E be a multiplicative cohomology theory and let $\mathcal{B}$ be a multiplicative (B,f)-structure. Then a **universal E-orientation for vector bundles with $\mathcal{B}$-structure** is an E-orientation, according to def. 1.130, for each rank-n universal vector bundle with $\mathcal{B}$-structure:

$$\xi_n \in \tilde{E}^n(\mathrm{Th}(E_n^{\mathcal{B}})) \quad \forall n \in \mathbb{N}$$

such that these are compatible in that

1. for all $n \in \mathbb{N}$ then

   $$\xi_n = \phi_n^* \xi_{n+1} ,$$

   where

   $$\xi_n \in \tilde{E}^n(\mathrm{Th}(V_n)) \simeq \tilde{E}^{n+1}(\Sigma\, \mathrm{Th}(V_n)) \simeq \tilde{E}^{n+1}(\mathrm{Th}(\mathbb{R} \oplus V_n))$$

   (with the first isomorphism is the suspension isomorphism of E and the second exhibiting the homeomorphism of Thom spaces $\mathrm{Th}(\mathbb{R} \oplus V) \simeq \Sigma\, \mathrm{Th}(V)$ (prop. 1.106) and where

   $$\phi_n^* : \tilde{E}^{n+1}(\mathrm{Th}(V_{n+1})) \longrightarrow \tilde{E}^{n+1}(\mathrm{Th}(\mathbb{R} \oplus V_n))$$

   is pullback along the canonical $\phi_n : \mathbb{R} \oplus V_n \to V_{n+1}$ (prop. 1.110).

2. for all $n_1, n_2 \in \mathbb{N}$ then

   $$\xi_{n+1} \cdot \xi_{n+2} = \xi_{n_1 + n_2} .$$

Proposition 1.133. *A universal E-orientation, in the sense of def. 1.132, for vector bundles with (B,f)-structure $\mathcal{B}$, is equivalently (the homotopy class of) a homomorphism of ring spectra*

$$\xi : M\mathcal{B} \longrightarrow E$$

from the universal $\mathcal{B}$-Thom spectrum to a spectrum which via the Brown representability theorem (theorem 1.30) represents the given generalized (Eilenberg-Steenrod) cohomology theory E (and which we denote by the same symbol).

Proof. The Thom spectrum $M\mathcal{B}$ has a standard structure of a CW-spectrum. Let now E denote a sequential Omega-spectrum representing the multiplicative cohomology theory of the same name. Since, in the standard model structure on topological sequential spectra, CW-spectra are cofibrant (prop.) and Omega-spectra are fibrant (thm.) we may represent all morphisms in the stable homotopy category (def.) by actual morphisms

$$\xi : M\mathcal{B} \longrightarrow E$$

of sequential spectra (due to this lemma).

Now by definition (def.) such a homomorphism is precissely a sequence of base-point preserving continuous functions

$$\xi_n : (M\mathcal{B})_n = \mathrm{Th}(V_n^{\mathcal{B}}) \longrightarrow E_n$$

for $n \in \mathbb{N}$, such that they are compatible with the structure maps σ_n and equivalently with their $(S^1 \wedge (-) \dashv \mathrm{Maps}(S^1, -)_*)$-adjuncts $\tilde{\sigma}_n$, in that these diagrams commute:

$$\begin{array}{ccc} S^1 \wedge \mathrm{Th}(V_n^{\mathcal{B}}) & \overset{S^1 \wedge \xi_n}{\longrightarrow} & S^1 \wedge E_n \\ {}^{\mathllap{\sigma_n^{M\mathcal{B}}}}\downarrow && \downarrow^{\mathrlap{\sigma_n^E}} \\ \mathrm{Th}(V_{n+1}^{\mathcal{B}}) & \underset{\xi_{n+1}}{\longrightarrow} & E_{n+1} \end{array} \;\;\;\;\; \leftrightarrow \;\;\;\;\; \begin{array}{ccc} \mathrm{Th}(V_n^{\mathcal{B}}) & \overset{\xi_n}{\longrightarrow} & E_n \\ {}^{\mathllap{\tilde\sigma_n^{M\mathcal{B}}}}\downarrow && \downarrow^{\mathrlap{\tilde\sigma_n^E}} \\ \mathrm{Maps}(S^1, \mathrm{Th}(V_{n+1}^{\mathcal{B}})) & \underset{\mathrm{Maps}(S^1, \xi_{n+1})_*}{\longrightarrow} & \mathrm{Maps}(S^1, E_{n+1})_* \end{array}$$

for all $n \in \mathbb{N}$.

First of all this means (via the identification given by the Brown representability theorem, see prop. 1.33, that the components ξ_n are equivalently representatives of elements in the cohomology groups

$$\xi_n \in \tilde{E}^n(\mathrm{Th}(V_n^{\mathcal{B}}))$$

(which we denote by the same symbol, for brevity).

Now by the definition of universal Thom spectra (def. 1.111, def. 1.114), the structure map $\sigma_n^{M\mathcal{B}}$ is just the map $\phi_n : \mathbb{R} \oplus \mathrm{Th}(V_n^{\mathcal{B}}) \to \mathrm{Th}(V_{n+1}^{\mathcal{B}})$ from above.

Moreover, by the Brown representability theorem, the adjunct $\tilde{\sigma}_n^E \circ \xi_n$ (on the right) of $\sigma_n^E \circ S^1 \wedge \xi_n$ (on the left) is what represents (again by prop. 1.33) the image of

$$\xi_n \in E^n(\mathrm{Th}(V_n^{\mathcal{B}}))$$

under the suspension isomorphism. Hence the commutativity of the above squares is equivalently the first compatibility condition from def. 1.132: $\xi_n \simeq \phi_n^* \xi_{n+1}$ in $\tilde{E}^{n+1}(\mathrm{Th}(\mathbb{R} \oplus V_n^{\mathcal{B}}))$

Next, ξ being a homomorphism of ring spectra means equivalently (we should be modelling $M\mathcal{B}$ and E as structured spectra (here.) to be more precise on this point, but the conclusion is the same) that for all $n_1, n_2 \in \mathbb{N}$ then

$$\begin{array}{ccc} \mathrm{Th}(V_{n_1}^{\mathcal{B}}) \wedge \mathrm{Th}(V_{n_2}^{\mathcal{B}}) & \longrightarrow & \mathrm{Th}(V_{n_1+n_2}) \\ {}^{\mathllap{\xi_{n_1} \wedge \xi_{n_2}}}\downarrow && \downarrow^{\mathrlap{\xi_{n_1+n_2}}} \\ E_{n_1} \wedge E_{n_2} & \underset{\cdot}{\longrightarrow} & E_{n_1+n_2} \end{array}$$

This is equivalently the condition $\xi_{n_1} \cdot \xi_{n_2} \simeq \xi_{n_1+n_2}$.

Finally, since $M\mathcal{B}$ is a ring spectrum, there is an essentially unique multiplicative homomorphism from the sphere spectrum

$$\mathbb{S} \overset{e}{\longrightarrow} M\mathcal{B} \,.$$

This is given by the component maps

$$e_n : S^n \simeq \mathrm{Th}(\mathbb{R}^n) \longrightarrow \mathrm{Th}(V_n^{\mathcal{B}})$$

that are induced by including the fiber of $V_n^{\mathcal{B}}$.

Accordingly the composite

$$\mathbb{S} \overset{e}{\longrightarrow} M\mathcal{B} \overset{\xi}{\longrightarrow} E$$

has as components the restrictions $i^*\xi_n$ appearing in def. 1.130. At the same time, also E is a ring spectrum, hence it also has an essentially unique multiplicative morphism $\mathbb{S} \to E$, which hence must agree with $i^*\xi$, up to homotopy. If we represent E as a symmetric ring spectrum, then the canonical such has the required property: e_0 is the identity element in degree 0 (being a unit of an ordinary ring, by definition) and hence e_n is necessarily its image under the suspension isomorphism, due to compatibility with the structure maps and using the above analysis. ▮

Complex projective space

For the fine detail of the discussion of complex oriented cohomology theories below, we recall basic facts about complex projective space.

Complex projective space $\mathbb{C}P^n$ is the projective space $\mathbb{A}P^n$ for $\mathbb{A} = \mathbb{C}$ being the complex numbers (and for $n \in \mathbb{N}$), a complex manifold of complex dimension n (real dimension $2n$). Equivalently, this is the complex Grassmannian $\mathrm{Gr}_1(\mathbb{C}^{n+1})$ (def. 1.84). For the special case $n = 1$ then $\mathbb{C}P^1 \simeq S^2$ is the Riemann sphere.

As n ranges, there are natural inclusions

$$* = \mathbb{C}P^0 \hookrightarrow \mathbb{C}P^1 \hookrightarrow \mathbb{C}P^2 \hookrightarrow \mathbb{C}P^3 \hookrightarrow \cdots \,.$$

The sequential colimit over this sequence is the infinite complex projective space $\mathbb{C}P^\infty$. This is a model for the classifying space $BU(1)$ of circle principal bundles/complex line bundles (an Eilenberg-MacLane space $K(\mathbb{Z},2)$).

Definition 1.134. For $n \in \mathbb{N}$, then **complex n-dimensional complex projective space** is the complex manifold (often just regarded as its underlying topological space) defined as the quotient

$$\mathbb{C}P^n \coloneqq (\mathbb{C}^{n+1} - \{0\})/_\sim$$

of the Cartesian product of $(n+1)$-copies of the complex plane, with the origin removed, by the equivalence relation

$$(z \sim w) \Leftrightarrow (z = \kappa \cdot w)$$

for some $\kappa \in \mathbb{C} - \{0\}$ and using the canonical multiplicative action of $\mathbb{C}$ on $\mathbb{C}^{n+1}$.

The canonical inclusions

$$\mathbb{C}^{n+1} \hookrightarrow \mathbb{C}^{n+2}$$

induce canonical inclusions

$$\mathbb{C}P^n \hookrightarrow \mathbb{C}P^{n+1} \,.$$

The sequential colimit over this sequence of inclusions is the *infinite complex projective space*

$$\mathbb{C}P^\infty \coloneqq \underset{\longleftarrow_n}{\lim}\, \mathbb{C}P^n \,.$$

The following equivalent characterizations are immediate but useful:

Proposition 1.135. *For $n \in \mathbb{N}$ then complex projective space, def. 1.134, is equivalently the complex Grassmannian*

$$\mathbb{C}P^n \simeq \mathrm{Gr}_1(\mathbb{C}^{n+1}) \,.$$

Proposition 1.136. *For $n \in \mathbb{N}$ then complex projective space, def. 1.134, is equivalently*

1. *the coset*

$$\mathbb{C}P^n \simeq U(n+1)/(U(n) \times U(1)) \,,$$

2. *the quotient of the (2n+1)-sphere by the circle group* $S^1 \simeq \{\kappa \in \mathbb{C} \,|\, |\kappa| = 1\}$

$$\mathbb{C}P^n \simeq S^{2n+1}/S^1 \,.$$

Proof. To see the second characterization from def. 1.134:

With $|-| : \mathbb{C}^n \longrightarrow \mathbb{R}$ the standard norm, then every element $\vec{z} \in \mathbb{C}^{n+1}$ is identified under the defining equivalence relation with

$$\frac{1}{|\vec{z}|}\vec{z} \in S^{2n-1} \hookrightarrow \mathbb{C}^{n+1}$$

lying on the unit $(2n-1)$-sphere. This fixes the action of $\mathbb{C} - 0$ up to a remaining action of complex numbers of unit absolute value. These form the circle group S^1.

The first characterization follows via prop. 1.135 from the general discusion at *Grassmannian*. With this the second characterization follows also with the coset identification of the $(2n+1)$-sphere: $S^{2n+1} \simeq U(n+1)/U(n)$ (exmpl.). ▮

Proposition 1.137. *There is a CW-complex structure on complex projective space $\mathbb{C}P^n$ (def. 1.134) for $n \in \mathbb{N}$, given by induction, where $\mathbb{C}P^{n+1}$ arises from $\mathbb{C}P^n$ by attaching a single cell of dimension $2(n+1)$ with attaching map the projection $S^{2n+1} \longrightarrow \mathbb{C}P^n$ from prop. 1.136:*

$$\begin{array}{ccc} S^{2n+1} & \longrightarrow & S^{2n+1}/S^1 \simeq \mathbb{C}P^n \\ {}^{\iota_{2n+2}}\downarrow & (\mathrm{po}) & \downarrow \\ D^{2n+2} & \longrightarrow & \mathbb{C}P^{n+1} \end{array} \,.$$

Proof. Given homogenous coordinates $(z_0, z_1, \cdots, z_n, z_{n+1}, z_{n+2}) \in \mathbb{C}^{n+2}$ for $\mathbb{C}P^{n+1}$, let

$$\phi \coloneqq -\arg(z_{n+2})$$

be the phase of z_{n+2}. Then under the equivalence relation defining $\mathbb{C}P^{n+1}$ these coordinates represent the same element as

$$\frac{1}{|\vec{z}|}(e^{i\phi}z_0, e^{i\phi}z_1, \cdots, e^{i\phi}z_{n+1}, r) \,,$$

where

$$r = |z_{n+2}| \in [0,1) \subset \mathbb{C}$$

is the absolute value of z_{n+2}. Representatives $\vec{z}'$ of this form ($|\vec{z}'| = 1$ and $z'_{n+2} \in [0,1]$) parameterize the 2n+2-disk D^{2n+2} ($2n+3$ real parameters subject to the one condition that the sum of their norm squares is unity) with boundary the $(2n+1)$-sphere at $r=0$. The only remaining part of the action of $\mathbb{C} - \{0\}$ which fixes the form of these representatives is S^1 acting on the elements with $r = 0$ by phase shifts on the $z_0, \cdots, z_{n+1}$. The quotient of this remaining action on $D^{2(n+1)}$ identifies its boundary S^{2n+1}-sphere with $\mathbb{C}P^n$, by prop. 1.136. ▮

Proposition 1.138. *For $A \in$ Ab any abelian group, then the ordinary homology groups of complex projective space $\mathbb{C}P^n$ with coefficients in A are*

$$H_k(\mathbb{C}P^n, A) \simeq \begin{cases} A & \text{for } k \text{ even and } k \leq 2n \\ 0 & \text{otherwise} \end{cases} \,.$$

Similarly the ordinary cohomology groups of $\mathbb{C}P^n$ is

$$H^k(\mathbb{C}P^n, A) \simeq \begin{cases} A & \text{for } k \text{ even and } k \leq 2n \\ 0 & \text{otherwise} \end{cases} \,.$$

Moreover, if A carries the structure of a ring $R = (A, \cdot)$, then under the cup product the cohomology ring of $\mathbb{C}P^n$ is the the graded ring

$$H^\bullet(\mathbb{C}P^n, R) \simeq R[c_1]/(c_1^{n+1})$$

which is the quotient of the polynomial ring on a single generator c_1 in degree 2, by the relation that identifies cup products of more than n-copies of the generator c_1 with zero.

Finally, the cohomology ring of the infinite-dimensional complex projective space is the formal power series ring in one generator:

$$H^\bullet(\mathbb{C}P^\infty, R) \simeq R[[c_1]] \;.$$

(Or else the polynomial ring $R[c_1]$, see remark 1.139)

Proof. First consider the case that the coefficients are the integers $A = \mathbb{Z}$.

Since $\mathbb{C}P^n$ admits the structure of a CW-complex by prop. 1.137, we may compute its ordinary homology equivalently as its cellular homology (thm.). By definition (defn.) this is the chain homology of the chain complex of relative homology groups

$$\cdots \overset{\partial_{\mathrm{cell}}}{\longrightarrow} H_{q+2}((\mathbb{C}P^n)_{q+2}, (\mathbb{C}P^n)_{q+1}) \overset{\partial_{\mathrm{cell}}}{\longrightarrow} H_{q+1}((\mathbb{C}P^n)_{q+1}, (\mathbb{C}P^n)_{q}) \overset{\partial_{\mathrm{cell}}}{\longrightarrow} H_{q}((\mathbb{C}P^n)_{q}, (\mathbb{C}P^n)_{q-1}) \overset{\partial_{\mathrm{cell}}}{\longrightarrow} \cdots \,,$$

where $(-)_q$ denotes the qth stage of the CW-complex-structure. Using the CW-complex structure provided by prop. 1.137, then there are cells only in every second degree, so that

$$(\mathbb{C}P^n)_{2k+1} = (\mathbb{C}P)_{2k}$$

for all $k \in \mathbb{N}$. It follows that the cellular chain complex has a zero group in every second degree, so that all differentials vanish. Finally, since prop. 1.137 says that $(\mathbb{C}P^n)_{2k+2}$ arises from $(\mathbb{C}P^n)_{2k+1} = (\mathbb{C}P^n)_{2k}$ by attaching a single $2k+2$-cell it follows that (by passage to reduced homology)

$$H_{2k}(\mathbb{C}P^n, \mathbb{Z}) \simeq \tilde H_{2k}(S^{2k})((\mathbb{C}P^n)_{2k}/(\mathbb{C}P^n)_{2k-1}) \simeq \tilde H_{2k}(S^{2k}) \simeq \mathbb{Z} \;.$$

This establishes the claim for ordinary homology with integer coefficients.

In particular this means that $H_q(\mathbb{C}P^n, \mathbb{Z})$ is a free abelian group for all q. Since free abelian groups are the projective objects in Ab (prop.) it follows (with the discussion at *derived functors in homological algebra*) that the Ext-groups vanishe:

$$\mathrm{Ext}^1(H_q(\mathbb{C}P^n, \mathbb{Z}), A) = 0$$

and the Tor-groups vanishes:

$$\mathrm{Tor}_1(H_q(\mathbb{C}P^n), A) = 0 \;.$$

With this, the statement about homology and cohomology groups with general coefficients follows with the universal coefficient theorem for ordinary homology (thm.) and for ordinary cohomology (thm.).

Finally to see the action of the cup product: by definition this is the composite

$$\cup \;:\; H^\bullet(\mathbb{C}P^n, R) \otimes H^\bullet(\mathbb{C}P^n, R) \longrightarrow H^\bullet(\mathbb{C}P^n \times \mathbb{C}P^n, R) \overset{\Delta^\ast}{\longrightarrow} H^\bullet(\mathbb{C}P^n, R)$$

of the "cross-product" map that appears in the Kunneth theorem, and the pullback along the diagonal $\Delta \colon \mathbb{C}P^n \to \mathbb{C}P^n \times \mathbb{C}P^n$.

Since, by the above, the groups $H^{2k}(\mathbb{C}P^n, R) \simeq R[2k]$ and $H^{2k+1}(\mathbb{C}P^n, R) = 0$ are free and finitely generated, the Kunneth theorem in ordinary cohomology applies (prop.) and says that the cross-product map above is an isomorphism. This shows that under cup product pairs of generators are sent to a generator, and so the statement $H^\bullet(\mathbb{C}P^n, R) \simeq R[c_1](c_1^{n+1})$ follows.

This also implies that the projection maps

$$H^\bullet((\mathbb{C}P^\infty)_{2n+2}, R) = H^\bullet(\mathbb{C}P^{n+1}, R) \to H^\bullet(\mathbb{C}P^{n+}, R) = H^\bullet((\mathbb{C}P^\infty)_{2n}, R)$$

are all epimorphisms. Therefore this sequence satisfies the Mittag-Leffler condition (def. 1.55, example 1.56) and therefore the Milnor exact sequence for cohomology (prop. 1.61) implies the last claim to be proven:

$$\begin{aligned} & H^\bullet(\mathbb{C}P^\infty, R) \\ & \simeq H^\bullet(\underset{\longleftarrow n}{\lim}\; \mathbb{C}P^n, R) \\ & \simeq \underset{\longrightarrow n}{\lim}\; H^\bullet(\mathbb{C}P^n, R) \\ & \simeq \underset{\longrightarrow n}{\lim}\; (R[c_1^E]/((c_1)^{n+1})) \\ & \simeq R[[c_1]] \,, \end{aligned}$$

where the last step is this prop.. ▮

Remark 1.139. There is in general a choice to be made in interpreting the cohomology groups of a multiplicative cohomology theory E (def. 1.26) as a ring:

a priori $E^\bullet(X)$ is a sequence

$$\{E^n(X)\}_{n\in\mathbb{Z}}$$

of abelian groups, together with a system of group homomorphisms

$$E^{n_1}(X)\otimes E^{n_2}(X)\longrightarrow E^{n_1+n_2}(X)\,,$$

one for each pair $(n_1,n_2)\in\mathbb{Z}\times\mathbb{Z}$.

In turning this into a single ring by forming formal sums of elements in the groups $E^n(X)$, there is in general the choice of whether allowing formal sums of only finitely many elements, or allowing arbitrary formal sums.

In the former case the ring obtained is the direct sum

$$\oplus_{n\in\mathbb{N}} E^n(X)$$

while in the latter case it is the Cartesian product

$$\prod_{n\in\mathbb{N}} E^n(X)\;.$$

These differ in general. For instance if E is ordinary cohomology with integer coefficients and X is infinite complex projective space $\mathbb{C}P^\infty$, then (prop. 1.138))

$$E^n(X)=\begin{cases}\mathbb{Z} & n \text{ even}\\ 0 & \text{otherwise}\end{cases}$$

and the product operation is given by

$$E^{2n_1}(X)\otimes E^{2n_2}(X)\longrightarrow E^{2(n_1+n_2)}(X)$$

for all n_1,n_2 (and zero in odd degrees, necessarily). Now taking the direct sum of these, this is the polynomial ring on one generator (in degree 2)

$$\oplus_{n\in\mathbb{N}} E^n(X)\;\simeq\;\mathbb{Z}[c_1]\;.$$

But taking the Cartesian product, then this is the formal power series ring

$$\prod_{n\in\mathbb{N}} E^n(X)\;\simeq\;\mathbb{Z}[[c_1]]\;.$$

A priori both of these are sensible choices. The former is the usual choice in traditional algebraic topology. However, from the point of view of regarding ordinary cohomology theory as a multiplicative cohomology theory right away, then the second perspective tends to be more natural:

The cohomology of $\mathbb{C}P^\infty$ is naturally computed as the inverse limit of the cohomolgies of the $\mathbb{C}P^n$, each of which unambiguously has the ring structure $\mathbb{Z}[c_1]/((c_1)^{n+1})$. So we may naturally take the limit in the category of commutative rings right away, instead of first taking it in $\mathbb{Z}$-indexed sequences of abelian groups, and then looking for ring structure on the result. But the limit taken in the category of rings gives the formal power series ring (see here).

See also for instance remark 1.1. in Jacob Lurie: *A Survey of Elliptic Cohomology*.

Complex orientation

Definition 1.140. A multiplicative cohomology theory E (def. 1.26) is called **complex orientable** if the the following equivalent conditions hold

1. The morphism

   $$i^*\,:\,E^2(BU(1))\longrightarrow E^2(S^2)$$

   is surjective.

2. The morphism

$$\tilde{\iota}^* : \tilde{E}^2(BU(1)) \longrightarrow \tilde{E}^2(S^2) \simeq \pi_0(E)$$

is surjective.

3. The element $1 \in \pi_0(E)$ is in the image of the morphism $\tilde{\iota}^*$.

A **complex orientation** on a multiplicative cohomology theory $E^\bullet$ is an element

$$c_1^E \in \tilde{E}^2(BU(1))$$

(the “first generalized Chern class”) such that

$$i^* c_1^E = 1 \in \pi_0(E) \,.$$

Remark 1.141. Since $BU(1) \simeq K(\mathbb{Z},2)$ is the classifying space for complex line bundles, it follows that a complex orientation on $E^\bullet$ induces an E-generalization of the first Chern class which to a complex line bundle $\mathcal{L}$ on X classified by $\phi : X \to BU(1)$ assigns the class $c_1(\mathcal{L}) \coloneqq \phi^* c_1^E$. This construction extends to a general construction of E-Chern classes.

Proposition 1.142. *Given a complex oriented cohomology theory $(E^\bullet, c_1^E)$ (def. 1.140), then there is an isomorphism of graded rings*

$$E^\bullet(\mathbb{C}P^\infty) \simeq E^\bullet(*)[[c_1^E]]$$

between the E-cohomology ring of infinite-dimensional complex projective space (def. 1.134) and the formal power series (see remark 1.139) in one generator of even degree over the E-cohomology ring of the point.

Proof. Using the CW-complex-structure on $\mathbb{C}P^\infty$ from prop. 1.137, given by inductively identifying $\mathbb{C}P^{n+1}$ with the result of attaching a single $2n$-cell to $\mathbb{C}P^n$. With this structure, the unique 2-cell inclusion $i : S^2 \hookrightarrow \mathbb{C}P^\infty$ is identified with the canonical map $S^2 \to BU(1)$.

Then consider the Atiyah-Hirzebruch spectral sequence (prop. 1.71) for the E-cohomology of $\mathbb{C}P^n$.

$$H^\bullet(\mathbb{C}P^n, E^\bullet(*)) \;\Rightarrow\; E^\bullet(\mathbb{C}P^n) \,.$$

Since, by prop. 1.138, the ordinary cohomology with integer coefficients of complex projective space is

$$H^\bullet(\mathbb{C}P^n, \mathbb{Z}) \simeq \mathbb{Z}[c_1]/((c_1)^{n+1}) \,,$$

where c_1 represents a unit in $H^2(S^2,\mathbb{Z}) \simeq \mathbb{Z}$, and since similarly the ordinary homology of $\mathbb{C}P^n$ is a free abelian group, hence a projective object in abelian groups (prop.), the Ext-group vanishes in each degree ($\mathrm{Ext}^1(H_n(\mathbb{C}P^n), E^\bullet(*)) = 0$) and so the universal coefficient theorem (prop.) gives that the second page of the spectral sequence is

$$H^\bullet(\mathbb{C}P^n, E^\bullet(*)) \simeq E^\bullet(*)[c_1]/(c_1^{n+1}) \,.$$

By the standard construction of the Atiyah-Hirzebruch spectral sequence (here) in this identification the element c_1 is identified with a generator of the relative cohomology

$$E^2((\mathbb{C}P^n)_2, (\mathbb{C}P^n)_1) \simeq \tilde{E}^2(S^2)$$

(using, by the above, that this S^2 is the unique 2-cell of $\mathbb{C}P^n$ in the standard cell model).

This means that c_1 is a permanent cocycle of the spectral sequence (in the kernel of all differentials) precisely if it arises via restriction from an element in $E^2(\mathbb{C}P^n)$ and hence precisely if there exists a complex orientation c_1^E on E. Since this is the case by assumption on E, c_1 is a permanent cocycle. (For the fully detailed argument see (Pedrotti 16)).

The same argument applied to all elements in $E^\bullet(*)[c]$, or else the $E^\bullet(*)$-linearity of the differentials (prop. 1.73), implies that all these elements are permanent cocycles.

Since the AHSS of a multiplicative cohomology theory is a multiplicative spectral sequence (prop.) this implies that the differentials in fact vanish on all elements of $E^\bullet(*)[c_1]/(c_1^{n+1})$, hence that the given AHSS collapses on the second page to give

$$\mathcal{E}_\infty^{\bullet,\bullet} \simeq E^\bullet(*)[c_1^E]/((c_1^E)^{n+1})$$

or in more detail:

$$\mathcal{E}_\infty^{p,\bullet} \simeq \begin{cases} E^\bullet(*) & \text{if } p \leq 2n \text{ and even} \\ 0 & \text{otherwise} \end{cases} .$$

Moreover, since therefore all $\mathcal{E}_\infty^{p,\bullet}$ are free modules over $E^\bullet(*)$, and since the filter stage inclusions $F^{p+1}E^\bullet(X) \hookrightarrow F^p E^\bullet(X)$ are $E^\bullet(*)$-module homomorphisms (prop.) the extension problem (remark 1.70) trivializes, in that all the short exact sequences

$$0 \to F^{p+1}E^{p+\bullet}(X) \longrightarrow F^p E^{p+\bullet}(X) \longrightarrow \mathcal{E}_\infty^{p,\bullet} \to 0$$

split (since the Ext-group $\mathrm{Ext}^1_{E^\bullet(*)}(\mathcal{E}_\infty^{p,\bullet}, -) = 0$ vanishes on the free module, hence projective module $\mathcal{E}_\infty^{p,\bullet}$).

In conclusion, this gives an isomorphism of graded rings

$$E^\bullet(\mathbb{C}P^n) \simeq \underset{p}{\oplus} \mathcal{E}_\infty^{p,\bullet} \simeq E^\bullet(*)[c_1]/((c_1^E)^{n+1}) .$$

A first consequence is that the projection maps

$$E^\bullet((\mathbb{C}P^\infty)_{2n+2}) = E^\bullet(\mathbb{C}P^{n+1}) \to E^\bullet(\mathbb{C}P^{n+}) = E^\bullet((\mathbb{C}P^\infty)_{2n})$$

are all epimorphisms. Therefore this sequence satisfies the Mittag-Leffler condition (def., exmpl.) and therefore the Milnor exact sequence for generalized cohomology (prop.) finally implies the claim:

$$\begin{aligned} E^\bullet(BU(1)) &\simeq E^\bullet(\mathbb{C}P^\infty) \\ &\simeq E^\bullet(\underset{\longleftarrow n}{\lim} \; \mathbb{C}P^n) \\ &\simeq \underset{\longrightarrow n}{\lim} \; E^\bullet(\mathbb{C}P^n) \\ &\simeq \underset{\longrightarrow n}{\lim} \; (E^\bullet(*)[c_1^E]/((c_1^E)^{n+1})) \\ &\simeq E^\bullet(*)[[c_1^E]] , \end{aligned}$$

where the last step is this prop.. ▮

S.3) Complex oriented cohomology

Idea. Given the concept of orientation in generalized cohomology as above, it is clearly of interest to consider cohomology theories E such that there exists an orientation/Thom class on the universal vector bundle over any classifying space BG (or rather: on its induced spherical fibration), because then *all* G-associated vector bundles inherit an orientation.

Considering this for $G = U(n)$ the unitary groups yields the concept of *complex oriented cohomology theory*.

It turns out that a complex orientation on a generalized cohomology theory E in this sense is already given by demanding that there is a suitable generalization of the first Chern class of complex line bundles in E-cohomology. By the splitting principle, this already implies the existence of generalized Chern classes (Conner-Floyd Chern classes) of all degrees, and these are the required universal generalized Thom classes.

Where the ordinary first Chern class in ordinary cohomology is simply additive under tensor product of complex line bundles, one finds that the composite of generalized first Chern classes is instead governed by more general commutative formal group laws. This phenomenon governs much of the theory to follow.

Literature. (Kochman 96, section 4.3, Lurie 10, lectures 1-10, Adams 74, Part I, Part II, Pedrotti 16).

Chern classes

Idea. In particular ordinary cohomology HR is canonically a complex oriented cohomology theory. The behaviour of general Conner-Floyd Chern classes to be discussed below follows closely the behaviour of the ordinary Chern classes.

An ordinary Chern class is a characteristic class of complex vector bundles, and since there is the classifying space BU of complex vector bundles, the *universal* Chern classes are those of the universal complex vector bundle over the classifying space BU, which in turn are just the ordinary cohomology classes in $H^\bullet(BU)$

These may be computed inductively by iteratively applying to the spherical fibrations

$$S^{2n-1} \longrightarrow BU(n-1) \longrightarrow BU(n)$$

the Thom-Gysin exact sequence, a special case of the Serre spectral sequence.

Pullback of Chern classes along the canonical map $(BU(1))^n \longrightarrow BU(n)$ identifies them with the elementary symmetric polynomials in the first Chern class in $H^2(BU(1))$. This is the *splitting principle*.

Literature. (Kochman 96, section 2.2 and 2.3, Switzer 75, section 16, Lurie 10, lecture 5, prop. 6)

Existence

Proposition 1.143. *The cohomology ring of the classifying space $BU(n)$ (for the unitary group $U(n)$) is the polynomial ring on generators $\{c_k\}_{k=1}^n$ of degree 2, called the Chern classes*

$$H^\bullet(BU(n), \mathbb{Z}) \simeq \mathbb{Z}[c_1, \cdots, c_n] \;.$$

Moreover, for $Bi : BU(n_1) \longrightarrow \mathrm{BU}(n_2)$ the canonical inclusion for $n_1 \leq n_2 \in \mathbb{N}$, then the induced pullback map on cohomology

$$(Bi)^\ast \;:\; H^\bullet(BU(n_2)) \longrightarrow H^\bullet(BU(n_1))$$

is given by

$$(Bi)^\ast(c_k) \;=\; \begin{cases} c_k & \text{for}\; 1 \leq k \leq n_1 \\ 0 & \text{otherwise} \end{cases} \;.$$

(e.g. Kochmann 96, theorem 2.3.1)

Proof. For $n = 1$, in which case $BU(1) \simeq \mathbb{C}P^\infty$ is the infinite complex projective space, we have by prop. 1.138

$$H^\bullet(BU(1)) \simeq \mathbb{Z}[c_1] \,,$$

where c_1 is the first Chern class. From here we proceed by induction. So assume that the statement has been shown for $n-1$.

Observe that the canonical map $BU(n-1) \to BU(n)$ has as homotopy fiber the (2n-1)sphere (prop. 1.96) hence there is a homotopy fiber sequence of the form

$$S^{2n-1} \longrightarrow BU(n-1) \longrightarrow BU(n) \;.$$

Consider the induced Thom-Gysin sequence (prop. 1.127).

In odd degrees $2k+1 < 2n$ it gives the exact sequence

$$\cdots \to H^{2k}(BU(n-1)) \longrightarrow \underbrace{H^{2k+1-2n}(BU(n))}_{\simeq 0} \longrightarrow H^{2k+1}(BU(n)) \overset{(Bi)^\ast}{\longrightarrow} \underbrace{H^{2k+1}(BU(n-1))}_{\simeq 0} \to \cdots \,,$$

where the right term vanishes by induction assumption, and the middle term since ordinary cohomology vanishes in negative degrees. Hence

$$H^{2k+1}(BU(n)) \simeq 0 \quad \text{for}\; 2k+1 < 2n$$

Then for $2k+1 > 2n$ the Thom-Gysin sequence gives

$$\cdots \to H^{2k+1-2n}(BU(n)) \longrightarrow H^{2k+1}(BU(n)) \overset{(Bi)^\ast}{\longrightarrow} \underbrace{H^{2k+1}(BU(n-1))}_{\simeq 0} \to \cdots \,,$$

where again the right term vanishes by the induction assumption. Hence exactness now gives that

$$H^{2k+1-2n}(BU(n)) \longrightarrow H^{2k+1}(BU(n))$$

is an epimorphism, and so with the previous statement it follows that

$$H^{2k+1}(BU(n)) \simeq 0$$

for all k.

Next consider the Thom Gysin sequence in degrees $2k$

$$\cdots \to \underbrace{H^{2k-1}(BU(n-1))}_{\simeq 0} \longrightarrow H^{2k-2n}(BU(n)) \longrightarrow H^{2k}(BU(n)) \overset{(Bi)^*}{\longrightarrow} H^{2k}(BU(n-1)) \longrightarrow \underbrace{H^{2k+1-2n}(BU(n))}_{\simeq 0} \to \cdots .$$

Here the left term vanishes by the induction assumption, while the right term vanishes by the previous statement. Hence we have a short exact sequence

$$0 \to H^{2k-2n}(BU(n)) \longrightarrow H^{2k}(BU(n)) \overset{(Bi)^*}{\longrightarrow} H^{2k}(BU(n-1)) \to 0$$

for all k. In degrees $\bullet \leq 2n$ this says

$$0 \to \mathbb{Z} \overset{c_n \cup (-)}{\longrightarrow} H^{\bullet \leq 2n}(BU(n)) \overset{(Bi)^*}{\longrightarrow} (\mathbb{Z}[c_1, \cdots, c_{n-1}])_{\bullet \leq 2n} \to 0$$

for some Thom class $c_n \in H^{2n}(BU(n))$, which we identify with the next Chern class.

Since free abelian groups are projective objects in Ab, their extensions are all split (the Ext-group out of them vanishes), hence the above gives a direct sum decomposition

$$\begin{aligned} H^{\bullet \leq 2n}(BU(n)) & \simeq (\mathbb{Z}[c_1, \cdots, c_{n-1}])_{\bullet \leq 2n} \oplus \mathbb{Z}\langle 2n \rangle \\ & \simeq (\mathbb{Z}[c_1, \cdots, c_n])_{\bullet \leq 2n} \end{aligned} .$$

Now by another induction over these short exact sequences, the claim follows. ▮

Splitting principle

Lemma 1.144. *For $n \in \mathbb{N}$ let $\mu_n : B(U(1)^n) \longrightarrow BU(n)$ be the canonical map. Then the induced pullback operation on ordinary cohomology*

$$\mu_n^* : H^\bullet(BU(n); \mathbb{Z}) \longrightarrow H^\bullet(BU(1)^n; \mathbb{Z})$$

is a monomorphism.

A **proof** of lemma 1.144 via analysis of the Serre spectral sequence of $U(n)/U(1)^n \to BU(1)^n \to BU(n)$ is indicated in (Kochmann 96, p. 40). A proof via transfer of the Euler class of $U(n)/U(1)^n$ is indicated at *splitting principle* (here).

Proposition 1.145. *For $k \leq n \in \mathbb{N}$ let $Bi_n : B(U(1)^n) \longrightarrow BU(n)$ be the canonical map. Then the induced pullback operation on ordinary cohomology is of the form*

$$(Bi_n)^* : \mathbb{Z}[c_1, \cdots, c_k] \longrightarrow \mathbb{Z}[(c_1)_1, \cdots (c_1)_n]$$

and sends the kth Chern class c_k (def. 1.143) to the kth elementary symmetric polynomial in the n copies of the first Chern class:

$$(Bi_n)^* : c_k \mapsto \sigma_k((c_1)_1, \cdots, (c_1)_n) \coloneqq \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq n} (c_1)_{i_1} \cdots (c_1)_{i_n} .$$

Proof. First consider the case $n = 1$.

The classifying space $BU(1)$ (def. 1.91) is equivalently the infinite complex projective space $\mathbb{C}P^\infty$. Its ordinary cohomology is the polynomial ring on a single generator c_1, the first Chern class (prop. 1.138)

$$H^\bullet(BU(1)) \simeq \mathbb{Z}[c_1] .$$

Moreover, Bi_1 is the identity and the statement follows.

Now by the Künneth theorem for ordinary cohomology (prop.) the cohomology of the Cartesian product of n copies of $BU(1)$ is the polynomial ring in n generators

$$H^\bullet(BU(1)^n) \simeq \mathbb{Z}[(c_1)_1, \cdots, (c_1)_n] .$$

By prop. 1.143 the domain of $(Bi_n)^*$ is the polynomial ring in the Chern classes $\{c_i\}$, and by the previous statement the codomain is the polynomial ring on n copies of the first Chern class

$$(Bi_n)^* : \mathbb{Z}[c_1, \cdots, c_n] \longrightarrow \mathbb{Z}[(c_1)_1, \cdots, (c_1)_n] .$$

This allows to compute $(Bi_n)^*(c_k)$ by induction:

Consider $n \geq 2$ and assume that $(Bi_{n-1})^*_{n-1}(c_k) = \sigma_k((c_1)_1, \cdots, (c_1)_{(n-1)})$. We need to show that then also $(Bi_n)^*(c_k) = \sigma_k((c_1)_1, \cdots, (c_1)_n)$.

Consider then the commuting diagram

$$\begin{array}{ccc} BU(1)^{n-1} & \overset{Bi_{n-1}}{\longrightarrow} & BU(n-1) \\ {}^{Bj_{\hat{t}}}\downarrow && \downarrow^{Bi_{\hat{t}}} \\ BU(1)^n & \underset{Bi_n}{\longrightarrow} & BU(n) \end{array}$$

where both vertical morphisms are induced from the inclusion

$$\mathbb{C}^{n-1} \hookrightarrow \mathbb{C}^n$$

which omits the tth coordinate.

Since two embeddings $i_{\hat{t}_1}, i_{\hat{t}_2} : U(n-1) \hookrightarrow U(n)$ differ by conjugation with an element in $U(n)$, hence by an inner automorphism, the maps $Bi_{\hat{t}_1}$ and $B_{\hat{i}_{t_2}}$ are homotopic, and hence $(Bi_{\hat{t}})^* = (Bi_{\hat{n}})^*$, which is the morphism from prop. 1.143.

By that proposition, $(Bi_{\hat{t}})^*$ is the identity on $c_{k<n}$ and hence by induction assumption

$$\begin{aligned} (Bi_{n-1})^*(Bi_{\hat{t}})^* c_{k<n} = (Bi_{n-1})^* c_{k<n} \\ = \sigma_k((c_1)_1, \cdots, \widehat{(c_1)_t}, \cdots, (c_1)_n) \end{aligned}.$$

Since pullback along the left vertical morphism sends $(c_1)_t$ to zero and is the identity on the other generators, this shows that

$$(Bi_n)^*(c_{k<n}) \simeq \sigma_{k<n}((c_1)_1, \cdots, \widehat{(c_1)_t}, \cdots, (c_1)_n) \mod (c_1)_t \,.$$

This implies the claim for $k < n$.

For the case $k = n$ the commutativity of the diagram and the fact that the right map is zero on c_n by prop. 1.143 shows that the element $(Bj_{\hat{t}})^*(Bi_n)^* c_n = 0$ for all $1 \leq t \leq n$. But by lemma 1.144 the morphism $(Bi_n)^*$, is injective, and hence $(Bi_n)^*(c_n)$ is non-zero. Therefore for this to be annihilated by the morphisms that send $(c_1)_t$ to zero, for all t, the element must be proportional to all the $(c_1)_t$. By degree reasons this means that it has to be the product of all of them

$$\begin{aligned} (Bi_n)^*(c_n) &= (c_1)_1 \otimes (c_1)_2 \otimes \cdots \otimes (c_1)_n \\ &= \sigma_n((c_1)_1, \cdots, (c_1)_n) \end{aligned}.$$

This completes the induction step, and hence the proof. ▮

Proposition 1.146. *For $k \leq n \in \mathbb{N}$, consider the canonical map*

$$\mu_{k,n-k} : BU(k) \times BU(n-k) \longrightarrow BU(n)$$

(which classifies the Whitney sum of complex vector bundles of rank k with those of rank $n-k$). Under pullback along this map the universal Chern classes (prop. 1.143) are given by

$$(\mu_{k,n-k})^*(c_t) \;=\; \sum_{i=0}^{t} c_i \otimes c_{t-i} \,,$$

where we take $c_0 = 1$ and $c_j = 0 \in H^\bullet(BU(r))$ if $j > r$.

So in particular

$$(\mu_{k,n-k})^*(c_n) \;=\; c_k \otimes c_{n-k} \,.$$

e.g. (Kochmann 96, corollary 2.3.4)

Proof. Consider the commuting diagram

$$\begin{array}{ccc} H^\bullet(BU(n)) & \overset{\mu_{k,n-k}^*}{\longrightarrow} & H^\bullet(BU(k)) \otimes H^\bullet(BU(n-k)) \\ {}^{\mu_k^*}\downarrow && \downarrow^{\mu_k^* \otimes \mu_{n-k}^*} \\ H^\bullet(BU(1)^n) & \simeq & H^\bullet(BU(1)^k) \otimes H^\bullet(BU(1)^{n-k}) \end{array} \,.$$

This says that for all t then

$$\begin{aligned} (\mu_k^* \otimes \mu_{n-k}^*)\mu_{k,n-k}^*(c_t) & = \mu_n^*(c_t) \\ & = \sigma_t((c_1)_1, \cdots, (c_1)_n) \end{aligned} \,,$$

where the last equation is by prop. 1.145.

Now the elementary symmetric polynomial on the right decomposes as required by the left hand side of this equation as follows:

$$\sigma_t((c_1)_1, \cdots, (c_1)_n) \;=\; \sum_{r=0}^{t} \sigma_r((c_1)_1, \cdots, (c_1)_{n-k}) \cdot \sigma_{t-r}((c_1)_{n-k+1}, \cdots, (c_1)_n) \,,$$

where we agree with $\sigma_q((c_1)_1, \cdots, (c_1)_p) = 0$ if $q > p$. It follows that

$$(\mu_k^* \otimes \mu_{n-k}^*)\mu_{k,n-k}^*(c_t) = (\mu_k^* \otimes \mu_{n-k}^*)\left(\sum_{r=0}^{t} c_r \otimes c_{t-r} \right) \,.$$

Since $(\mu_k^* \otimes \mu_{n-k}^*)$ is a monomorphism by lemma 1.144, this implies the claim. ▮

Conner-Floyd Chern classes

Idea. For E a complex oriented cohomology theory, then the generators of the E-cohomology groups of the classifying space BU are called the *Conner-Floyd Chern classes*, in $E^\bullet(BU)$.

Using basic properties of the classifying space $BU(1)$ via its incarnation as the infinite complex projective space $\mathbb{C}P^\infty$, one finds that the Atiyah-Hirzebruch spectral sequences

$$H^p(\mathbb{C}P^n, \pi_q(E)) \Rightarrow H^\bullet(\mathbb{C}P^n)$$

collapse right away, and that the inverse system which they form satisfies the Mittag-Leffler condition. Accordingly the Milnor exact sequence gives that the ordinary first Chern class c_1 generates, over $\pi_\bullet(E)$, all Conner-Floyd classes over $BU(1)$:

$$E^\bullet(BU(1)) \simeq \pi_\bullet(E)[[c_1]] \,.$$

This is the key input for the discussion of formal group laws below.

Combining the Atiyah-Hirzebruch spectral sequence with the splitting principle as for ordinary Chern classes above yields, similarly, that in general Conner-Floyd classes are generated, over $\pi_\bullet(E)$, from the ordinary Chern classes.

Finally one checks that Conner-Floyd classes canonically serve as Thom classes for E-cohomology of the universal complex vector bundle, thereby showing that complex oriented cohomology theories are indeed canonically oriented on (spherical fibrations of) complex vector bundles.

Literature. (Kochman 96, section 4.3 Adams 74, part I.4, part II.2 II.4, part III.10, Lurie 10, lecture 5)

Proposition 1.147. *Given a complex oriented cohomology theory E with complex orientation c_1^E, then the E-generalized cohomology of the classifying space $BU(n)$ is freely generated over the graded commutative ring $\pi_\bullet(E)$ (prop.) by classes c_k^E for $0 \leq \leq n$ of degree $2k$, these are called the **Conner-Floyd-Chern classes***

$$E^\bullet(BU(n)) \;\simeq\; \pi_\bullet(E)[[c_1^E, c_2^E, \cdots, c_n^E]] \,.$$

Moreover, pullback along the canonical inclusion $BU(n) \to BU(n+1)$ is the identity on c_k^E for $k \leq n$ and sends c_{n+1}^E to zero.

For E being ordinary cohomology, this reduces to the ordinary Chern classes of prop. 1.143.

For details see (Pedrotti 16, prop. 3.1.14).

Formal group laws of first CF-Chern classes

Idea. The classifying space $BU(1)$ for complex line bundles is a homotopy type canonically equipped with commutative group structure (infinity-group-structure), corresponding to the tensor product of complex line bundles. By the above, for E a complex oriented cohomology theory the first Conner-Floyd Chern class of these complex line bundles generates the E-cohomology of $BU(1)$, it follows that the cohomology ring $E^\bullet(BU(1)) \simeq \pi_\bullet(E)[[c_1]]$ behaves like the ring of $\pi_\bullet(E)$-valued functions on a 1-dimensional commutative formal group equipped with a canonical coordinate function c_1. This is called a *formal group law* over the graded commutative ring $\pi_\bullet(E)$ (prop.).

On abstract grounds it follows that there exists a commutative ring L and a universal (1-dimensional commutative) formal group law ℓ over L. This is called the *Lazard ring*. Lazard's theorem identifies this ring concretely: it turns out to simply be the polynomial ring on generators in every even degree.

Further below this has profound implications on the structure theory for complex oriented cohomology. The Milnor-Quillen theorem on MU identifies the Lazard ring as the cohomology ring of the Thom spectrum MU, and then the Landweber exact functor theorem, implies that there are lots of complex oriented cohomology theories.

Literature. (Kochman 96, section 4.4, Lurie 10, lectures 1 and 2)

Formal group laws

Definition 1.148. An (commutative) adic ring is a (commutative) topological ring A and an ideal $I \subset A$ such that

1. the topology on A is the I-adic topology;

2. the canonical morphism

$$A \longrightarrow \underset{\longleftarrow n}{\lim} (A/I^n)$$

to the limit over quotient rings by powers of the ideal is an isomorphism.

A homomorphism of adic rings is a ring homomorphism that is also a continuous function (hence a function that preserves the filtering $A \supset \cdots \supset A/I^2 \supset A/I$). This gives a category $\mathrm{AdicRing}$ and a subcategory $\mathrm{AdicCRing}$ of commutative adic rings.

The opposite category of $\mathrm{AdicRing}$ (on Noetherian rings) is that of affine formal schemes.

Similarly, for R any fixed commutative ring, then adic rings under R are *adic R-algebras*. We write $\mathrm{Adic}\,A\,\mathrm{Alg}$ and $\mathrm{Adic}\,A\,\mathrm{CAlg}$ for the corresponding categories.

Example 1.149. For R a commutative ring and $n \in \mathbb{N}$ then the formal power series ring

$$R[[x_1, x_2, \cdots, x_n]]$$

in n variables with coefficients in R and equipped with the ideal

$$I = (x_1, \cdots, x_n)$$

is an adic ring (def. 1.148).

Proposition 1.150. *There is a fully faithful functor*

$$\mathrm{AdicRing} \hookrightarrow \mathrm{ProRing}$$

from adic rings (def. 1.148) to pro-rings, given by

$$(A, I) \mapsto ((A/I^\bullet)),$$

i.e. for $A, B \in \mathrm{AdicRing}$ two adic rings, then there is a natural isomorphism

$$\mathrm{Hom}_{\mathrm{AdicRing}}(A, B) \simeq \underset{\longleftarrow n_2}{\lim} \underset{\longrightarrow n_1}{\lim} \mathrm{Hom}_{\mathrm{Ring}}(A/I^{n_1}, B/I^{n_2}) \,.$$

Definition 1.151. For $R \in \mathrm{CRing}$ a commutative ring and for $n \in \mathbb{N}$, a **formal group law** of dimension n over R is the structure of a group object in the category $\mathrm{Adic}\,R\,\mathrm{CAlg}^{\mathrm{op}}$ from def. 1.148 on the object $R[[x_1, \cdots, x_n]]$ from example 1.149.

Hence this is a morphism

$$\mu : R[[x_1, \cdots, x_n]] \longrightarrow R[[x_1, \cdots, x_n, y_1, \cdots, y_n]]$$

in $\mathrm{Adic} R \mathrm{CAlg}$ satisfying unitality, associativity.

This is a **commutative formal group law** if it is an abelian group object, hence if it in addition satisfies the corresponding commutativity condition.

This is equivalently a set of n power series F_i of $2n$ variables $x_1, \ldots, x_n, y_1, \ldots, y_n$ such that (in notation $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, $F(x, y) = (F_1(x,y), \ldots, F_n(x,y))$)

$$F(x, F(y,z)) = F(F(x,y),z)$$
$$F_i(x,y) = x_i + y_i + \text{ higher order terms}$$

Example 1.152. A 1-dimensional commutative formal group law according to def. 1.151 is equivalently a formal power series

$$\mu(x,y) = \sum_{i,j \geq 0} a_{i,j} x^i y^j$$

(the image]$\mathrm{under\mu in R[x,y] of the element t}$ \in R [t]$) such that

1. (unitality)

$$\mu(x, 0) = x$$

2. (associativity)

$$\mu(x, \mu(y,z)) = \mu(\mu(x,y),z) \,;$$

3. (commutativity)

$$\mu(x,y) = \mu(y,x) \;.$$

The first condition means equivalently that

$$a_{i,0} = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases} \quad , \quad a_{0,i} = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases} \,.$$

Hence μ is necessarily of the form

$$\mu(x,y) \;=\; x + y + \sum_{i,j \geq 1} a_{i,j} x^i y^j \;.$$

The existence of inverses is no extra condition: by induction on the index i one finds that there exists a unique

$$\iota(x) = \sum_{i \geq 1} \iota(x)_i x^i$$

such that

$$\mu(x, \mathrm{iota}(x)) = x \quad , \quad \mu(\iota(x), x) = x \;.$$

Hence 1-dimensional formal group laws over R are equivalently monoids in $\mathrm{Adic} R \mathrm{CAlg}^{\mathrm{op}}$ on $R[[x]]$.

Formal group laws from complex orientation

Let again $BU(1)$ be the classifying space for complex line bundles, modeled, in particular, by infinite complex projective space $\mathbb{C}P^\infty$).

Lemma 1.153. *There is a continuous function*

$$\mu : \mathbb{C}P^\infty \times \mathbb{C}P \longrightarrow \mathbb{C}P^\infty$$

which represents the tensor product of line bundles in that under the defining equivalence, and for X any paracompact topological space, then

$$
\begin{array}{ccc}
[X, \mathbb{C}P^\infty \times \mathbb{C}P^\infty] & \simeq & \mathbb{C}\,\mathrm{LineBund}(X)_{/\sim} \times \mathbb{C}\,\mathrm{LineBund}(X)_{/\sim} \\
{}^{[X,\mu]}\downarrow && \downarrow^{\otimes} \\
[X, \mathbb{C}P^\infty] & \simeq & \mathbb{C}\,\mathrm{LineBund}(X)_{/\sim}
\end{array}
,
$$

where $[-,-]$ denotes the hom-sets in the (Serre-Quillen-)classical homotopy category and $\mathbb{C}\,\mathrm{LineBund}(X)_{/\sim}$ denotes the set of isomorphism classes of complex line bundles on X.

Together with the canonical point inclusion $* \to \mathbb{C}P^\infty$, this makes $\mathbb{C}P^\infty$ an abelian group object in the classical homotopy category.

Proof. By the Yoneda lemma (the fully faithfulness of the Yoneda embedding) there exists such a morphism $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \longrightarrow \mathbb{C}P^\infty$ in the classical homotopy category. But since $\mathbb{C}P^\infty$ admits the structure of a CW-complex (prop. 1.137)) it is cofibrant in the standard model structure on topological spaces (thm.), as is its Cartesian product with itself (prop.). Since moreover all spaces are fibrant in the classical model structure on topological spaces, it follows (by this lemma) that there is an actual continuous function representing that morphism in the homotopy category.

That this gives the structure of an abelian group object now follows via the Yoneda lemma from the fact that each $\mathbb{C}\,\mathrm{LineBund}(X)_{/\sim}$ has the structure of an abelian group under tensor product of line bundles, with the trivial line bundle (wich is classified by maps factoring through $* \to \mathbb{C}P^\infty$) being the neutral element, and that this group structure is natural in X. ▮

Remark 1.154. The space $BU(1) \simeq \mathbb{C}P^\infty$ has in fact more structure than that of a homotopy group from lemma 1.153. As an object of the homotopy theory represented by the classical model structure on topological spaces, it is a *2-group*, a 1-truncated infinity-group.

Proposition 1.155. *Let (E, c_1^E) be a complex oriented cohomology theory. Under the identification*

$$
E^\bullet(\mathbb{C}P^\infty) \simeq \pi_\bullet(E)[[c_1^E]] \quad , \quad E^\bullet(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \simeq \pi_\bullet(E)[[c_1^E \otimes 1, 1 \otimes c_1^E]]
$$

from prop. 1.142, the operation

$$
\pi_\bullet(E)[[c_1^E]] \simeq E^\bullet(\mathbb{C}P^\infty) \longrightarrow E^\bullet(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \simeq \pi_\bullet(E)[[c_1^E \otimes 1, 1 \otimes c_1^E]]
$$

of pullback in E-cohomology along the maps from lemma 1.153 constitutes a 1-dimensional graded-commutative formal group law (example 1.152)over the graded commutative ring $\pi_\bullet(E)$ (prop.). If we consider c_1^E to be in degree 2, then this formal group law is compatibly graded.

Proof. The associativity and commutativity conditions follow directly from the respective properties of the map μ in lemma 1.153. The grading follows from the nature of the identifications in prop. 1.142. ▮

Remark 1.156. That the grading of c_1^E in prop. 1.155 is in negative degree is because by definition

$$
\pi_\bullet(E) = E_\bullet = E^{-\bullet}
$$

(rmk.).

Under different choices of orientation, one obtains different but isomorphic formal group laws.

The universal 1d commutative formal group law and Lazard's theorem

It is immediate that there exists a ring carrying a universal formal group law. For observe that for $\sum_{i,j} a_{i,j} x_1^i x_1^j$ an element in a formal power series algebra, then the condition that it defines a formal group law is equivalently a sequence of polynomial equations on the coefficients a_k. For instance the commutativity condition means that

$$
a_{i,j} = a_{j,i}
$$

and the unitality constraint means that

$$
a_{i0} = \begin{cases} 1 & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases} .
$$

Similarly associativity is equivalently a condition on combinations of triple products of the coefficients. It is not necessary to even write this out, the important point is only that it is some polynomial equation.

This allows to make the following definition

Definition 1.157. The **Lazard ring** is the graded commutative ring generated by elenebts a_{ij} in degree $2(i+j-1)$ with $i, j \in \mathbb{N}$

$$L = \mathbb{Z}[a_{ij}]/(\text{relations } 1, 2, 3 \text{ below})$$

quotiented by the relations

1. $a_{ij} = a_{ji}$

2. $a_{10} = a_{01} = 1;\ \forall i \neq 1 : a_{i0} = 0$

3. the obvious associativity relation

for all i, j, k.

The **universal 1-dimensional commutative formal group law** is the formal power series with coefficients in the Lazard ring given by

$$\ell(x, y) \coloneqq \sum_{i,j} a_{ij} x^i y^j \in L[[x, y]] \,.$$

Remark 1.158. The grading is chosen with regards to the formal group laws arising from complex oriented cohomology theories (prop.) where the variable x naturally has degree -2. This way

$$\deg(a_{ij} x^i y^j) = \deg(a_i, j) + i \deg(x) + j \deg(y) = -2 \,.$$

The following is immediate from the definition:

Proposition 1.159. *For every ring R and 1-dimensional commutative formal group law μ over R (example 1.152), there exists a unique ring homomorphism*

$$f : L \longrightarrow R$$

from the Lazard ring (def. 1.157) to R, such that it takes the universal formal group law ℓ to μ

$$f_* \ell = \mu \,.$$

Proof. If the formal group law μ has coefficients $\{c_{i,j}\}$, then in order that $f_* \ell = \mu$, i.e. that

$$\sum_{i,j} f(a_{i,j}) x^i y^j = \sum_{i,j} c_{i,j} x^i y^j$$

it must be that f is given by

$$f(a_{i,j}) = c_{i,j}$$

where $a_{i,j}$ are the generators of the Lazard ring. Hence it only remains to see that this indeed constitutes a ring homomorphism. But this is guaranteed by the vary choice of relations imposed in the definition of the Lazard ring. ▮

What is however highly nontrivial is this statement:

Theorem 1.160. ***(Lazard's theorem)***

The Lazard ring L (def. 1.157) is isomorphic to a polynomial ring

$$L \simeq \mathbb{Z}[t_1, t_2, \cdots]$$

in countably many generators t_i in degree $2i$.

Remark 1.161. The Lazard theorem 1.160 first of all implies, via prop. 1.159, that there exists an abundance of 1-dimensional formal group laws: given any ring R then every choice of elements $\{t_i \in R\}$ defines a formal group law. (On the other hand, it is nontrivial to say which formal group law that is.)

Deeper is the fact expressed by the Milnor-Quillen theorem on MU: the Lazard ring in its polynomial incarnation of prop. 1.160 is canonically identieif with the graded commutative ring $\pi_\bullet(MU)$ of stable homotopy groups of the universal complex Thom spectrum MU. Moreover:

1. MU carries a universal complex orientation in that for E any homotopy commutative ring spectrum then homotopy classes of homotopy ring homomorphisms $MU \to E$ are in bijection to complex orientations on E;

2. every complex orientation on E induced a 1-dimensional commutative formal group law (prop.)

3. under forming stable homtopy groups every ring spectrum homomorphism $MU \to E$ induces a ring homomorphism

$$L \simeq \pi_\bullet(MU) \longrightarrow \pi_\bullet(E)$$

and hence, by the universality of L, a formal group law over $\pi_\bullet(E)$.

This is the formal group law given by the above complex orientation.

Hence the universal group law over the Lazard ring is a kind of decategorification of the universal complex orientation on MU.

Complex cobordism

Idea. There is a weak homotopy equivalence $\phi : BU(1) \xrightarrow{\simeq} MU(1)$ between the classifying space for complex line bundles and the Thom space of the universal complex line bundle. This gives an element $\pi_*(c_1) \in MU^2(BU(1))$ in the complex cobordism cohomology of $BU(1)$ which makes the universal complex Thom spectrum MU become a complex oriented cohomology theory.

This turns out to be a universal complex orientation on MU: for every other homotopy commutative ring spectrum E (def.) there is an equivalence between complex orientations on E and homotopy classes of homotopy ring spectrum homomorphisms

$$\{MU \longrightarrow E\}_{/\simeq} \;\; \simeq \;\; \{\text{complex orientations on } E\}\,.$$

Hence complex oriented cohomology theory is higher algebra over MU.

Literature. (Schwede 12, example 1.18, Kochman 96, section 1.4, 1.5, 4.4, Lurie 10, lectures 5 and 6)

Conner-Floyd-Chern classes are Thom classes

We discuss that for E a complex oriented cohomology theory, then the nth universal Conner-Floyd-Chern class c_n^E is in fact a universal Thom class for rank n complex vector bundles. On the one hand this says that the choice of a complex orientation on E indeed universally orients all complex vector bundles. On the other hand, we interpret this fact below as the unitality condition on a homomorphism of homotopy commutative ring spectra $MU \to E$ which represent that universal orienation.

Lemma 1.162. *For $n \in \mathbb{N}$, the fiber sequence (prop. 1.96)*

$$\begin{array}{ccc} S^{2n-1} & \longrightarrow & BU(n-1) \\ && \downarrow \\ && BU(n) \end{array}$$

exhibits $BU(n-1)$ as the sphere bundle of the universal complex vector bundle over $BU(n)$.

Proof. When exhibited by a fibration, here the vertical morphism is equivalently the quotient map

$$(EU(n))/U(n-1) \longrightarrow (EU(n))/U(n)$$

(by the proof of prop. 1.96).

Now the universal principal bundle $EU(n)$ is (def. \ref{EOn)}) equivalently the colimit

$$EU(n) \simeq \underset{\longrightarrow_k}{\lim}\, U(k)/U(k-n)\,.$$

Here each Stiefel manifold/coset spaces $U(k)/U(k-n)$ is equivalently the space of (complex) n-dimensional subspaces of $\mathbb{C}^k$ that are equipped with an orthonormal (hermitian) linear basis. The universal vector bundle

$$EU(n) \underset{U(n)}{\times} \mathbb{C}^n \simeq \underset{\longrightarrow_k}{\lim}\, U(k)/U(k-n) \underset{U(n)}{\times} \mathbb{C}^n$$

has as fiber precisely the linear span of any such choice of basis.

While the quotient $U(k)/(U(n-k) \times U(n))$ (the Grassmannian) divides out the entire choice of basis, the quotient $U(k)/(U(n-k) \times U(n-1))$ leaves the choice of precisly one unit vector. This is parameterized by the sphere S^{2n-1} which is thereby identified as the unit sphere in the respective fiber of $EU(n) \underset{U(n)}{\times} \mathbb{C}^n$. ▮

In particular:

Lemma 1.163. *The canonical map from the classifying space $BU(1) \simeq \mathbb{C}P^\infty$ (the inifnity complex projective space) to the Thom space of the universal complex line bundle is a weak homotopy equivalence*

$$BU(1) \overset{\in W_{\mathrm{cl}}}{\longrightarrow} MU(1) \coloneqq \mathrm{Th}(EU(1) \underset{U(1)}{\times} \mathbb{C}) \,.$$

Proof. Observe that the circle group $U(1)$ is naturally identified with the unit sphere in $\mathbb{C}$: $U(1) \simeq S(\mathbb{S})$. Therefore the sphere bundle of the universal complex line bundle is equivalently the $U(1)$-universal principal bundle

$$\begin{aligned} EU(1) \underset{U(1)}{\times} S(\mathbb{C}) & \simeq EU(1) \underset{U(1)}{\times} U(1) \\ & \simeq EU(1) \end{aligned} \,.$$

But the universal principal bundle is contractible

$$EU(1) \overset{\in W_{\mathrm{cl}}}{\longrightarrow} * \,.$$

(Alternatively this is the special case of lemma 1.162 for $n = 0$.)

Therefore the Thom space

$$\begin{aligned} \mathrm{Th}(EU(1) \underset{U(1)}{\times} \mathbb{B}) & \coloneqq D(EU(1) \underset{U(1)}{\times} \mathbb{B}) / S(EU(1) \underset{U(1)}{\times} \mathbb{B}) \\ & \overset{\in W_{\mathrm{cl}}}{\longrightarrow} D(EU(1) \underset{U(1)}{\times} \mathbb{B}) \\ & \overset{\in W_{\mathrm{cl}}}{\longrightarrow} BU(1) \end{aligned} \,.$$

▮

Lemma 1.164. *For E a generalized (Eilenberg-Steenrod) cohomology theory, then the E-reduced cohomology of the Thom space of the complex universal vector bundle is equivalently the relative cohomology of $BU(n)$ relative $BU(n-1)$*

$$\tilde E^\bullet(\mathrm{Th}(EU(n) \underset{U(n)}{\times} \mathbb{C}^n)) \;\simeq\; E^\bullet(BU(n), BU(n-1)) \,.$$

If E is equipped with the structure of a complex oriented cohomology theory then

$$\tilde E^\bullet(\mathrm{Th}(EU(n) \underset{U(n)}{\times} \mathbb{C}^n)) \simeq c_n^E \cdot (\pi_\bullet(E))[[c_1^E, \cdots, c_n^E]] \,,$$

where the c_i are the universal E-Conner-Floyd-Chern classes.

Proof. Regarding the first statement:

In view of lemma 1.162 and using that the disk bundle is homotopy equivalent to the base space we have

$$\begin{aligned} \tilde E^\bullet(\mathrm{Th}(EU(n) \underset{U(n)}{\times} \mathbb{C}^n)) & = E^\bullet(D(EU(n) \underset{U(n)}{\times} \mathbb{C}^n), S(EU(n) \underset{U(n)}{\times} \mathbb{C}^n)) \\ & \simeq E^\bullet(EU(n), BU(n-1)) \end{aligned} \,.$$

Regarding the second statement: the Conner-Floyd classes freely generate the E-cohomology of $BU(n)$ for all n:

$$E^\bullet(BU(n)) \simeq \pi_\bullet(E)[[c_1^E, \cdots, c_n^E]] \,.$$

and the restriction morphism

$$E^\bullet(BU(n)) \longrightarrow E^\bullet(BU(n-1))$$

projects out c_n^E. Since this is in particular a surjective map, the relative cohomology $E^\bullet(BU(n), BU(n-1))$ is just the kernel of this map. ▮

Proposition 1.165. *Let E be a complex oriented cohomology theory. Then the nth E-Conner-Floyd-Chern class*

$$c_n^E \in \tilde E(\mathrm{Th}(EU(n) \underset{U(n)}{\times} \mathbb{C}^n))$$

(using the identification of lemma 1.164) is a Thom class in that its restriction to the Thom space of any

fiber is a suspension of a unit in $\pi_0(E)$.

(Lurie 10, lecture 5, prop. 6)

Proof. Since $BU(n)$ is connected, it is sufficient to check the statement over the base point. Since that fixed fiber is canonically isomorphic to the direct sum of n complex lines, we may equivalently check that the restriction of c_n^E to the pullback of the universal rank n bundle along

$$i : BU(1)^n \longrightarrow BU(n)$$

satisfies the required condition. By the splitting principle, that restriction is the product of the n-copies of the first E-Conner-Floyd-Chern class

$$i^* c_n \simeq ((c_1^E)_1 \cdots (c_1^E)_n) \;.$$

Hence it is now sufficient to see that each factor restricts to a unit on the fiber, but that it precisely the condition that c_1^E is a complex orientaton of E. In fact by def. 1.166 the restriction is even to $1 \in \pi_0(E)$. ▮

Complex orientation as ring spectrum maps

For the present purpose:

Definition 1.166. For E a generalized (Eilenberg-Steenrod) cohomology theory, then a *complex orientation* on E is a choice of element

$$c_1^E \in E^2(BU(1))$$

in the cohomology of the classifying space $BU(1)$ (given by the infinite complex projective space) such that its image under the restriction map

$$\phi : \tilde E^2(BU(1)) \longrightarrow \tilde E^2(S^2) \simeq \pi_0(E)$$

is the unit

$$\phi(c_1^E) = 1 \;.$$

(Lurie 10, lecture 4, def. 2)

Remark 1.167. Often one just requires that $\phi(c_1^E)$ is *a* unit, i.e. an invertible element. However we are after identifying c_1^E with the degree-2 component $MU(1) \to E_2$ of homtopy ring spectrum morphisms $MU \to E$, and by unitality these necessarily send $S^2 \to MU(1)$ to the unit $\iota_2 : S^2 \to E$ (up to homotopy).

Lemma 1.168. *Let E be a homotopy commutative ring spectrum (def.) equipped with a complex orientation (def. 1.166) represented by a map*

$$c_1^E : BU(1) \longrightarrow E_2 \;.$$

Write $\{c_k^E\}_{k \in \mathbb{N}}$ for the induced Conner-Floyd-Chern classes. Then there exists a morphism of S^2-sequential spectra (def.)

$$MU \longrightarrow E$$

whose component map $MU_{2n} \longrightarrow E_{2n}$ represents c_n^E (under the identification of lemma 1.164), for all $n \in \mathbb{N}$.

Proof. Consider the standard model of MU as a sequential S^2-spectrum with component spaces the Thom spaces of the complex universal vector bundle

$$MU_{2n} \coloneqq \mathrm{Th}(EU(n) \times \mathbb{C}^n) \;.$$

Notice that this is a CW-spectrum (def., lemma).

In order to get a homomorphism of S^2-sequential spectra, we need to find representatives $f_{2n} : MU_{2n} \longrightarrow E_{2n}$ of c_n^E (under the identification of lemma 1.164) such that all the squares

$$\begin{array}{ccc} S^2 \wedge MU_{2n} & \overset{\mathrm{id}\wedge f_{2n}}{\longrightarrow} & S^2 \wedge E_{2n} \\ \downarrow && \downarrow \\ MU_{2(n+1)} & \underset{f_{2(n+1)}}{\longrightarrow} & E_{2n+1} \end{array}$$

commute strictly (not just up to homotopy).

To begin with, pick a map

$$f_0 : MU_0 \simeq S^0 \longrightarrow E_0$$

that represents $c_0 = 1$.

Assume then by induction that maps f_{2k} have been found for $k \leq n$. Observe that we have a homotopy-commuting diagram of the form

$$\begin{array}{ccc} S^2 \wedge MU_{2n} & \overset{\mathrm{id} \wedge f_{2n}}{\longrightarrow} & S^2 \wedge E_{2n} \\ \downarrow & \swarrow\!\!\!\!\swarrow & \downarrow \\ MU_2 \wedge MU_{2n} & \overset{c_1 \wedge c_n}{\longrightarrow} & E_2 \wedge E_{2n} \\ \downarrow & \swarrow\!\!\!\!\swarrow & \downarrow^{\mu_{2,2n}} \\ MU_{2(n+1)} & \underset{c_{n+1}}{\longrightarrow} & E_{2(n+1)} \end{array},$$

where the maps denoted c_k are any representatives of the Chern classes of the same name, under the identification of lemma 1.164. Here the homotopy in the top square exhibits the fact that c_1^E is a complex orientation, while the homotopy in the bottom square exhibits the Whitney sum formula for Chern classes (prop. 1.146)).

Now since MU is a CW-spectrum, the total left vertical morphism here is a (Serre-)cofibration, hence a Hurewicz cofibration, hence satisfies the homotopy extension property. This means precisely that we may find a map $f_{2n+1} : MU_{2(n+1)} \longrightarrow E_{2(n+1)}$ homotopic to the given representative c_{n+1} such that the required square commutes strictly. ∎

Lemma 1.169. *For E a complex oriented homotopy commutative ring spectrum, the morphism of spectra*

$$c : MU \longrightarrow E$$

that represents the complex orientation by lemma 1.168 is a homomorphism of homotopy commutative ring spectra.

(Lurie 10, lecture 6, prop. 6)

Proof. The unitality condition demands that the diagram

$$\begin{array}{ccc} \mathbb{S} & \longrightarrow & MU \\ & \searrow & \downarrow^{c} \\ & & E \end{array}$$

commutes in the stable homotopy category $\mathrm{Ho}(\mathrm{Spectra})$. In components this means that

$$\begin{array}{ccc} S^{2n} & \longrightarrow & MU_{2n} \\ & \searrow & \downarrow^{c_n} \\ & & E_{2n} \end{array}$$

commutes up to homotopy, hence that the restriction of c_n to a fiber is the $2n$-fold suspension of the unit of E_{2n}. But this is the statement of prop. 1.165: the Chern classes are universal Thom classes.

Hence componentwise all these triangles commute up to some homotopy. Now we invoke the Milnor sequence for generalized cohomology of spectra (prop. 1.63). Observe that the tower of abelian groups $n \mapsto E^{n_1}(S^n)$ is actually constant (suspension isomorphism) hence trivially satisfies the Mittag-Leffler condition and so a homotopy of morphisms of spectra $\mathbb{S} \to E$ exists as soon as there are componentwise homotopies (cor. 1.64).

Next, the respect for the product demands that the square

$$\begin{array}{ccc} MU \wedge MU & \overset{c \wedge c}{\longrightarrow} & E \wedge E \\ \downarrow & & \downarrow \\ MU & \underset{c}{\longrightarrow} & E \end{array}$$

commutes in the stable homotopy category $\mathrm{Ho}(\mathrm{Spectra})$. In order to rephrase this as a condition on the components of the ring spectra, regard this as happening in the homotopy category $\mathrm{Ho}(\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}}))_{\mathrm{stable}}$ of the model structure on orthogonal spectra, which is equivalent to the stable homotopy category (thm.).

Here the derived symmetric monoidal smash product of spectra is given by Day convolution (def.) and maps out of such a product are equivalently as in the above diagram is equivalent (cor.) to a suitably equivariant collection diagrams of the form

$$\begin{array}{ccc} MU_{2n_1} \wedge MU_{2n_2} & \overset{c_{n_1} \wedge c_{n_2}}{\longrightarrow} & E_{2n_1} \wedge E_{2n_2} \\ \downarrow && \downarrow \\ MU_{2(n_1+n_2)} & \underset{c_{(n_1+n_2)}}{\longrightarrow} & E_{2(n_1+n_2)} \end{array} \,,$$

where on the left we have the standard pairing operations for MU (def.) and on the right we have the given pairing on E.

That this indeed commutes up to homotopy is the Whitney sum formula for Chern classes (prop.).

Hence again we have componentwise homotopies. And again the relevant Mittag-Leffler condition on $n \mapsto E^{n-1}((\mathrm{MU} \wedge \mathrm{MU})_n)$-holds, by the nature of the universal *Conner-Floyd classes?*, prop. 1.147. Therefore these componentwise homotopies imply the required homotopy of morphisms of spectra (cor. 1.64). ▮

Theorem 1.170. *Let E be a homotopy commutative ring spectrum (def.). Then the map*

$$(MU \overset{c}{\longrightarrow} E) \;\mapsto\; (BU(1) \simeq MU_2 \overset{c_1}{\longrightarrow} E_2)$$

which sends a homomorphism c of homotopy commutative ring spectra to its component map in degree 2, interpreted as a class on $BU(1)$ via lemma 1.163, constitutes a bijection from homotopy classes of homomorphisms of homotopy commutative ring spectra to complex orientations (def. 1.166) on E.

(Lurie 10, lecture 6, theorem 8)

Proof. By lemma 1.168 and lemma 1.169 the map is surjective, hence it only remains to show that it is injective.

So let $c, c' : MU \to E$ be two morphisms of homotopy commutative ring spectra that have the same restriction, up to homotopy, to $c_1 \simeq c_1' : MU_2 \simeq BU(1)$. Since both are homotopy ring spectrum homomophisms, the restriction of their components $c_n, c'_n : MU_{2n} \to E_{2n}$ to $BU(1)^{\wedge^n}$ is a product of $c_1 \simeq c'_1$, hence c_n becomes homotopic to c_n' after this restriction. But by the splitting principle this restriction is injective on cohomology classes, hence c_n itself ist already homotopic to c'_n.

It remains to see that these homotopies may be chosen compatibly such as to form a single homotopy of maps of spectra

$$f \,:\, MU \wedge I_+ \longrightarrow E \,,$$

This follows due to the existence of the Milnor short exact sequence from prop. 1.63:

$$0 \to \underset{\longleftarrow n}{\lim}^1 E^{-1}(\Sigma^{-2n} MU_{2n}) \longrightarrow E^0(MU) \longrightarrow \underset{\longleftarrow n}{\lim} \, E^0(\Sigma^{-2n} MU_{2n}) \to 0 \,.$$

Here the Mittag-Leffler condition (def. 1.55) is clearly satisfied (by prop. 1.147 and lemma 1.164 all relevant maps are epimorphisms, hence the condition is satisfied by example 1.56). Hence the lim^1-term vanishes (prop. 1.57), and so by exactness the canonical morphism

$$E^0(MU) \overset{\simeq}{\longrightarrow} \underset{\longleftarrow n}{\lim} \, E^0(\Sigma^{-2n} MU_{2n})$$

is an isomorphism. This says that two homotopy classes of morphisms $MU \to E$ are equal precisely already if all their component morphisms are homotopic (represent the same cohomology class). ▮

Homology of MU

Idea. Since, by the above, every complex oriented cohomology theory E is indeed oriented over complex vector bundles, there is a Thom isomorphism which reduces the computation of the E-homology of MU, $E_\bullet(MU)$ to that of the classifying space BU. The homology of BU, in turn, may be determined by the duality with its cohomology (universal coefficient theorem) via Kronecker pairing and the induced duality of the corresponding Atiyah-Hirzebruch spectral sequences (prop. 1.74) from the Conner-Floyd classes above. Finally, via the Hurewicz homomorphism/Boardman homomorphism the homology of MU gives a first approximation to the homotopy groups of MU.

Literature. (Kochman 96, section 2.4, 4.3, Lurie 10, lecture 7)

Milnor-Quillen theorem on MU

Idea. From the computation of the homology of MU above and applying the Boardman homomorphism, one deduces that the stable homotopy groups $\pi_\bullet(\mathrm{MU})$ of MU are finitely generated. This implies that it is suffient to compute them over the p-adic integers for all primes p. Using the change of rings theorem, this finally is obtained from inspection of the filtration in the $H\mathbb{F}_p$-Adams spectral sequence for MU. This is Milnor's theorem wich together with Lazard's theorem shows that there is an isomorphism of rings $L \simeq \pi_\bullet(MU)$ with the Lazard ring. Finally Quillen's theorem on MU says that this isomorphism is exhibited by the universal ring homomorphism $L \longrightarrow \pi_\bullet(MU)$ which classifies the universal complex orientation on MU.

Literature. (Kochman 96, section 4.4, Lurie 10, lecture 10)

Landweber exact functor theorem

Idea. By the above, every complex oriented cohomology theory induces a formal group law from its first Conner-Floyd Chern class. Moreover, Quillen's theorem on MU together with Lazard's theorem say that the cohomology ring $\pi_\bullet(MU)$ of complex cobordism cohomology MU is the classifying ring for formal group laws.

The *Landweber exact functor theorem* says that, conversely, forming the tensor product of complex cobordism cohomology theory (MU) with a Landweber exact ring via some formal group law yields a cohomology theory, hence a complex oriented cohomology theory.

Literature. (Lurie 10, lectures 15,16)

Outlook: Geometry of $\mathrm{Spec}(\mathrm{MU})$

The grand conclusion of Quillen's theorem on MU (above): complex oriented cohomology theory is essentially the spectral geometry over $\mathrm{Spec}(MU)$, and the latter is a kind of derived version of the moduli stack of formal groups (1-dimensional commutative).

- Landweber-Novikov theorem
- Adams-Quillen theorem
- Adams-Novikov spectral sequence

(…)

Literature. (Kochman 96, sections 4.5-4.7 and section 5, Lurie 10, lectures 12-14)

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2. References

We follow in outline the textbook

- Stanley Kochman, chapters I - IV of *Bordism, Stable Homotopy and Adams Spectral Sequences*, AMS 1996

For some basics in algebraic topology see also

- Robert Switzer, *Algebraic Topology - Homotopy and Homology*, Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen, Vol. 212, Springer-Verlag, New York, N. Y., 1975.

Specifically for **S.1) Generalized cohomology** a neat account is in:

- Marcelo Aguilar, Samuel Gitler, Carlos Prieto, section 12 of *Algebraic topology from a homotopical viewpoint*, Springer (2002) (toc pdf)

For **S.2) Cobordism theory** an efficient collection of the highlights is in

- Cary Malkiewich, *Unoriented cobordism and MO*, 2011 (pdf)

except that it omits proof of the Leray-Hirsch theorem/Serre spectral sequence and that of the Thom isomorphism, but see the references there and see (Kochman 96, Aguilar-Gitler-Prieto 02, section 11.7) for details.

For **S.3) Complex oriented cohomology** besides (Kochman 96, chapter 4) have a look at

- Frank Adams, *Stable homotopy and generalized homology*, Chicago Lectures in mathematics, 1974

and

- Jacob Lurie, lectures 1-10 of *Chromatic Homotopy Theory*, 2010

See also

- Stefan Schwede, *Symmetric spectra*, 2012 (pdf)

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