nLab

Introduction to the Adams Spectral Sequence

This page gives a detailed introduction to the Adams spectral sequence in its general spectral form (Adams-Novikov spectral sequence).

For background on spectral sequences see *Introduction to Spectral Sequences*.

For background on stable homotopy theory see *Introduction to Stable homotopy theory*.

For background on complex oriented cohomology see *Introduction to Cobordism and Complex Oriented Cohomology*.

Contents

The main result of Part 1.1 was the construction of the stable homotopy category $\mathrm{Ho}(\mathrm{Spectra})$ (thm., def.) as a triangulated category (prop.) with graded abelian hom groups $[X,Y]_\bullet$ (def.).

These are the basic invariants of stable homotopy theory, the *stable homotopy groups*. They are as rich and interesting as they are, in general, hard to compute. The archetypical example for this phenomenon are the stable homotopy groups of spheres $\pi_\bullet(\mathbb{S})$. (We compute the first dozen of these, 2-locally, below.)

In order to get more control over $\mathrm{Ho}(\mathrm{Spectra})$, the main result of Part 1.2 was the construction of tensor triangulated category structure on $\mathrm{Ho}(\mathrm{Spectra})$ (prop.), induced form a symmetric monoidal smash product of spectra $\wedge$ (thm.)

$$(\mathrm{Ho}(\mathrm{Spectra}), \wedge, \mathbb{S}) \,.$$

As discussed in *Part I* (and briefly reviewed below), the tool of choice to break up the computation of stable homotopy groups in stable homotopy theory into tractable computations in homological algebra are *spectral sequences*. These break up computations of stable homotopy groups along chosen filtrations on spectra. Using the tensor triangulated structure, it turns out that every homotopy commutative ring spectrum E (def.) induces a well-adapted filtration on spectra that allows to compute the "formal neighbourhood around E" in any spectrum (called the E-*nilpotent completion*) via a spectral sequence. This is the E-*Adams spectral sequence* which we discuss here.

Where the Atiyah-Hirzebruch spectral sequence (see part S, this prop.) approximates $[X,Y]_\bullet$ via the ordinary cohomology $H^\bullet(X,\pi_\bullet(Y))$, the idea of the Adams spectral sequence is to make use of an auxiliary homotopy commutative ring spectrum E and approximate maps of spectra $X \to Y$ via their image $E_\bullet(X) \to E_\bullet(Y)$ in E-generalized homology (rmk).

But in order for maps of homology groups to have a chance to retain enough information, they should be forced to be equivariant with respect to extra structure inherited by forming E-homology.

For instance if $E = H\mathbb{F}_2$ then the dual Steenrod algebra $\mathcal{A}$ (co-)acts on $E_\bullet(X) = H_\bullet(X,\mathbb{F}_2)$ and a necessary condition for a morphism of homology groups to come from a morphism of spectra is that it is a homomorphism with respect to this co-action. The classical Adams spectral sequence (discussed below), accordingly, approximates $[X,Y]_\bullet$ by $\mathrm{Hom}_{\mathcal{A}}(H_\bullet(X,\mathbb{F}_2), H_\bullet(Y,\mathbb{F}_2))$.

More generally, since spectra are equivalently module spectra over the sphere spectrum $\mathbb{S}$, the operation of forming E-homology spectra $X \mapsto E \wedge S$ is equivalently the extension of scalars along the ring unit $\mathbb{S} \longrightarrow E$. This means that the extra structure inherited by E-homology groups contains the information given by the further extensions along the cosimplicial diagram

$$\mathbb{S} \longrightarrow E \rightrightarrows E \wedge E \overset{\longrightarrow}{\rightrightarrows} E \wedge E \wedge E \overset{\rightrightarrows}{\rightrightarrows} \cdots .$$

In good cases this gives $E_\bullet(X)$ the structure of a module over the Hopf algebroid $\pi_\bullet(E \wedge E) = E_\bullet(E) \leftleftarrows E_\bullet$ of "dual E-Steenrod operations". Accordingly the general E-Adams spectral sequence approximates $[X,Y]_\bullet$ by $\mathrm{Hom}_{E_\bullet(E)}(E_\bullet(X), E_\bullet(Y))$.

For $E =$ MU, BP, this is the *Adams-Novikov spectral sequence, considered below.*

We discuss first the

- General theory of E-Adams spectral sequences

and then consider the classical

- Examples and applications

First we set up the general theory of E-Adams spectral sequences. (We consider examples and applications further below.)

Literature (Adams 74, part III.15, Bousfield 79, sections 5 and 6, Ravenel 86, appendix A)

1. The spectral sequence

Filtered spectra

We introduce the types of spectral sequences of which the E-Adams spectral sequences (def. 1.14 below) are an example.

Definition 1.1. A **filtered spectrum** is a spectrum $Y \in \mathrm{Ho}(\mathrm{Spectra})$ equipped with a sequence $Y_\bullet : (\mathbb{N}, >) \longrightarrow \mathrm{Ho}(\mathrm{Spectra})$ in the stable homotopy category (def.) of the form

$$\cdots \longrightarrow Y_3 \overset{f_2}{\longrightarrow} Y_2 \overset{f_1}{\longrightarrow} Y_1 \overset{f_0}{\longrightarrow} Y_0 \coloneqq Y .$$

Remark 1.2. More generally a filtering on an object X in (stable or not) homotopy theory is a $\mathbb{Z}$-graded sequence $X_\bullet$ such that X is the homotopy colimit $X \simeq \underset{\longrightarrow}{\lim} X_\bullet$. But for the present purpose we stick with the simpler special case of def. 1.1.

Remark 1.3. There is *no* condition on the morphisms in def. 1.1. In particular, they are *not* required to be n-monomorphisms or n-epimorphisms for any n.

On the other hand, while they are also not explicitly required to have a presentation by cofibrations or fibrations, this follows automatically: by the existence of the model structure on topological sequential spectra (thm.) or equivalently (thm.) the model structure on orthogonal spectra (thm.), every filtering on a spectrum is equivalent to one in which all morphisms are represented by cofibrations or by fibrations.

This means that we may think of a filtration on a spectrum in the sense of def. 1.1 as equivalently being a tower of fibrations over that spectrum.

The following definition 1.4 unravels the structure encoded in a filtration on a spectrum, and motivates the concepts of exact couples and their spectral sequences from these.

Definition 1.4. **(exact couple of a filtered spectrum)**

Consider a spectrum $X \in \mathrm{Ho}(\mathrm{Spectra})$ and a filtered spectrum $Y_\bullet$ as in def. 1.1.

Write A_k for the homotopy cofiber of its kth stage, such as to obtain the diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & Y_3 & \overset{f_2}{\longrightarrow} & Y_2 & \overset{f_1}{\longrightarrow} & Y_1 & \overset{f_0}{\longrightarrow} & Y \\ && \downarrow^{g_3} && \downarrow^{g_2} && \downarrow^{g_1} && \downarrow^{g_0} \\ && A_3 && A_2 && A_1 && A_0 \end{array}$$

where each stage

$$\begin{array}{ccc} Y_{k+1} & \overset{f_k}{\longrightarrow} & Y_k \\ && \downarrow^{g_k} \\ && A_k \end{array}$$

is a homotopy cofiber sequence (def.), hence equivalently (prop.) a homotopy fiber sequence, hence where

$$Y_{k+1} \overset{f_k}{\longrightarrow} Y_k \overset{g_k}{\longrightarrow} A_k \overset{h_k}{\longrightarrow} \Sigma Y_{k+1}$$

is an exact triangle (prop.).

Apply the graded hom-group functor $[X,-]_\bullet$ (def.) to the above tower. This yields a diagram of $\mathbb{Z}$-graded abelian groups of the form

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & [X,Y_3]_\bullet & \overset{[X,f_2]_\bullet)}{\longrightarrow} & [X,Y_2]_\bullet & \overset{[X,f_1]_\bullet}{\longrightarrow} & [X,Y_1]_\bullet & \overset{[X,f_0]_\bullet}{\longrightarrow} & [X,Y_0]_\bullet \\ && \downarrow^{[X,g_3]_\bullet} && \downarrow^{[X,g_2]_\bullet} && \downarrow^{[X,g_1]_\bullet} && \downarrow^{[X,g_0]_\bullet} \\ && [X,A_3]_\bullet && [X,A_2]_\bullet && [X,A_1]_\bullet && [X,A_0]_\bullet \end{array}$$

where each hook at stage k extends to a long exact sequence of homotopy groups (prop.) via connecting homomorphisms $[X,h_k]_\bullet$

$$\cdots \to [X,A_k]_{\bullet+1} \overset{[X,h_k]_{\bullet+1}}{\longrightarrow} [X,Y_{k+1}]_\bullet \overset{[X,f_k]_\bullet}{\longrightarrow} [X,Y_k]_\bullet \overset{[X,g_k]_\bullet}{\longrightarrow} [X,A_k]_\bullet \overset{[X,h_k]_\bullet}{\longrightarrow} [X,Y_{k+1}]_{\bullet-1} \to \cdots \,.$$

If we regard the connecting homomorphism $[X,h_k]$ as a morphism of degree -1, then all this information fits into one diagram of the form

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & [X,Y_3]_\bullet & \overset{[X,f_2]_\bullet}{\longrightarrow} & [X,Y_2]_\bullet & \overset{[X,f_1]_\bullet}{\longrightarrow} & [X,Y_1]_\bullet & \overset{[X,f_0]_\bullet}{\longrightarrow} & [X,Y_0]_\bullet \\ && \downarrow & {}_{[X,h_2]_\bullet}\nwarrow & \downarrow & {}_{[X,h_1]_\bullet}\nwarrow & \downarrow & {}_{[X,h_0]_\bullet}\nwarrow & \downarrow^{[X,g_0]_\bullet} \\ && [X,A_3]_\bullet && [X,A_2]_\bullet && [X,A_1]_\bullet && [X,A_0]_\bullet \end{array}$$

where each triangle is a rolled-up incarnation of a long exact sequence of homotopy groups (and in particular is *not* a commuting diagram!).

If we furthermore consider the bigraded abelian groups $[X,Y_\bullet]_\bullet$ and $[X,A_\bullet]_\bullet$, then this information may further be rolled-up to a single diagram of the form

$$\begin{array}{ccc} [X,Y_\bullet]_\bullet & \overset{[X,f_\bullet]_\bullet}{\longrightarrow} & [X,Y_\bullet]_\bullet \\ {}_{[X,h_\bullet]_\bullet}\nwarrow && \downarrow^{[X,g_\bullet]_\bullet} \\ && [X,A_\bullet]_\bullet \end{array}$$

Specifically, regard the terms here as bigraded in the following way

$$\begin{aligned} D^{s,t}(X,Y) &\coloneqq [X,Y_s]_{t-s} \\ E^{s,t}(X,Y) &\coloneqq [X,A_s]_{t-s} \end{aligned} \,.$$

Then the bidegree of the morphisms is

morphism	bidegree
$[X,f]$	$(-1,-1)$
$[X,g]$	$(0,0)$
$[X,h]$	$(1,0)$

This way t counts the cycles of going around the triangles:

$$\cdots \to D^{s+1,t+1}(X,Y) \overset{[X,f]}{\longrightarrow} D^{s,t}(X,Y) \overset{[X,g]}{\longrightarrow} E^{s,t}(X,Y) \overset{[X,h]}{\longrightarrow} D^{s+1,t}(X,Y) \to \cdots$$

Data of this form is called an *exact couple*, def. 1.6 below.

Definition 1.5. An *unrolled* *exact couple* (of *Adams-type*) is a diagram of abelian groups of the form

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{D}^{3,\bullet} \arrow[r, "i_2"] \arrow[d] & \mathcal{D}^{2,\bullet} \arrow[r, "i_1"] \arrow[d, "j_2"] & \mathcal{D}^{1,\bullet} \arrow[r, "i_0"] \arrow[d, "j_1"] & \mathcal{D}^{0,\bullet} \arrow[d, "j_0"] \\
& \mathcal{E}^{3,\bullet} \arrow[ur, "k_2"'] & \mathcal{E}^{2,\bullet} \arrow[ur, "k_1"'] & \mathcal{E}^{1,\bullet} \arrow[ur, "k_0"'] & \mathcal{E}^{0,\bullet}
\end{tikzcd}$$

such that each triangle is a rolled-up long exact sequence of abelian groups of the form

$$\cdots \to \mathcal{D}^{s+1,t+1} \overset{i_s}{\longrightarrow} \mathcal{D}^{s,t} \overset{j_s}{\longrightarrow} \mathcal{E}^{s,t} \overset{k_s}{\longrightarrow} \mathcal{D}^{s+1,t} \to \cdots .$$

The collection of this "un-rolled" data into a single diagram of abelian groups is called the corresponding *exact couple*.

Definition 1.6. An *exact couple* is a diagram (non-commuting) of abelian groups of the form

$$\begin{tikzcd}
\mathcal{D} \arrow[r, "i"] & \mathcal{D} \arrow[d, "j"] \\
& \mathcal{E} \arrow[ul, "k"]
\end{tikzcd},$$

such that this is exact in each position, hence such that the kernel of every morphism is the image of the preceding one.

The concept of exact couple so far just collects the sequences of long exact sequences given by a filtration. Next we turn to extracting information from this sequence of sequences.

Remark 1.7. The sequence of long exact sequences in def. 1.4 is inter-locking, in that every $[X,Y_s]_{t-s}$ appears *twice*:

$$\begin{tikzcd}
& {} \arrow[dr] & & {} & & & \\
& & {[X,Y_{s+1}]_{t-s-1}} \arrow[ur] \arrow[dr, "{[X,g]}"] & & & & {} \\
& {[X,A_s]_{t-s}} \arrow[ur, "{[X,h]}"] \arrow[rr, "d_1"'] & & {[X,A_{s+1}]_{t-s-1}} \arrow[rr, "d_1"] \arrow[dr, "{[X,h]}"'] & & {[X,A_{s+2}]_{t-s-2}} \arrow[ur] & \\
{} \arrow[ur] & & & & {[X,Y_{s+2}]_{t-s-2}} \arrow[ur, "{[X,g]}"'] \arrow[dr] & & \\
& & & {} \arrow[ur] & & {} &
\end{tikzcd}$$

This gives rise to the horizontal ("splicing") composites d_1, as shown, and by the fact that the diagonal sequences are long exact, these are differentials in that they square to zero: $(d_1)^2 = 0$. Hence there is a cochain complex:

$$\cdots \longrightarrow [X,A_s]_{t-s} \overset{d_1}{\longrightarrow} [X,A_{s+1}]_{t-s-1} \overset{d_1}{\longrightarrow} [X,A_{s+2}]_{t-s-2} \longrightarrow \cdots .$$

We may read off from these interlocking long exact sequences what these differentials *mean*, as follows. An element $c \in [X,A_s]_{t-s}$ lifts to an element $\hat{c} \in [X,Y_{s+2}]_{t-s-1}$ precisely if $d_1 c = 0$:

$$\begin{tikzcd}
\hat{c} \in {[X,Y_{s+2}]_{t-s-1}} \arrow[dr, "{[X,f]}"] & & \\
& {[X,Y_{s+1}]_{t-s-1}} \arrow[dr, "{[X,g]}"] & \\
c \in {[X,A_s]_{t-s}} \arrow[ur, "{[X,h]}"] \arrow[rr, "d_1"'] & & {[X,A_{s+1}]_{t-s-1}}
\end{tikzcd}$$

In order to organize this observation, notice that in terms of the exact couple of remark 1.4, the differential

$$d_1 \;\coloneqq\; [X,g] \circ [X,h]$$

is the composite

$$d \coloneqq j \circ k \;.$$

Some terminology:

Definition 1.8. Given an exact couple, def. 1.6,

$$\begin{array}{ccc} \mathcal{D}^{\bullet,\bullet} & \overset{i}{\longrightarrow} & \mathcal{D}^{\bullet,\bullet} \\ & {}_{k}\nwarrow & \downarrow^{j} \\ && \mathcal{E}^{\bullet,\bullet} \end{array}$$

observe that the composite

$$d \coloneqq j \circ k$$

is a differential in that it squares to 0, due to the exactness of the exact couple:

$$\begin{aligned} d \circ d & = j \circ \underbrace{k \circ j}_{=0} \circ k \\ & = 0 \end{aligned} \;.$$

One says that the **page** of the exact couple is the graded chain complex

$$(\mathcal{E}^{\bullet,\bullet}, d \coloneqq j \circ k) \;.$$

Given a cochain complex like this, we are to pass to its cochain cohomology. Since the cochain complex here has the extra structure that it arises from an exact couple, its cohomology groups should still remember some of that extra structure. Indeed, the following says that the remaining extract structure on the cohomology of the page of an exact couple is itself again an exact couple, called the "derived exact couple".

Definition 1.9. Given an exact couple, def. 1.6, then its *derived exact couple* is the diagram

$$\begin{array}{ccc} \tilde{\mathcal{D}} & \overset{\tilde i}{\longrightarrow} & \tilde{\mathcal{D}} \\ & {}_{\tilde k}\nwarrow & \downarrow^{\tilde j} \\ && \tilde{\mathcal{E}} \end{array} \;\;\coloneqq\;\; \begin{array}{ccc} \mathrm{im}(i) & \overset{i}{\longrightarrow} & \mathrm{im}(i) \\ & {}_{k}\nwarrow & \downarrow j \circ i^{-1} \\ && H(\mathcal{E}, j \circ k) \end{array}$$

with

1. $\tilde{\mathcal{E}} \coloneqq \mathrm{ker}(d)/\mathrm{im}(d)$ (with $d \coloneqq j \circ k$ from def. 1.8);
2. $\tilde{\mathcal{D}} \coloneqq \mathrm{im}(i)$;
3. $\tilde{i} \coloneqq i|_{\mathrm{im}(i)}$;
4. $\tilde{j} \coloneqq j \circ i^{-1}$ (where i^{-1} is the operation of choosing any preimage under i);
5. $\tilde{k} \coloneqq k|_{\mathrm{ker}(d)}$.

Lemma 1.10. *The derived exact couple in def. 1.9 is well defined and is itself an exact couple, def. 1.6.*

Proof. This is straightforward to check. For completeness we spell it out:

First consider that the morphisms are well defined in the first place.

It is clear that $\tilde{i}$ is well-defined.

That $\tilde{j}$ lands in $\mathrm{ker}(d)$: it lands in the image of j which is in the kernel of k, by exactness, hence in the kernel of d by definition.

That $\tilde{j}$ is independent of the choice of preimage: For any $x \in \tilde{\mathcal{D}} = \mathrm{im}(i)$, let $y, y' \in \mathcal{D}$ be two preimages under i, hence $i(y) = i(y') = x$. This means that $i(y' - y) = 0$, hence that $y' - y \in \mathrm{ker}(i)$, hence that $y' - y \in \mathrm{im}(k)$, hence there exists $z \in \mathcal{E}$ such that $y' = y + k(z)$, hence $j(y') = j(y) + j(k(z)) = j(y) + d(z)$, but $d(z) = 0$ in $\tilde{\mathcal{E}}$.

That $\tilde{k}$ vanishes on $\mathrm{im}(d)$: because $\mathrm{im}(d) \subset \mathrm{im}(j)$ and hence by exactness.

That $\tilde{k}$ lands in $\mathrm{im}(i)$: since it is defined on $\ker(d) = \ker(j \circ k)$ it lands in $\ker(j)$. By exactness this is $\mathrm{im}(i)$.

That the sequence of maps is again exact:

The kernel of $\tilde{j}$ is those $x \in \mathrm{im}(i)$ such that their preimage $i^{-1}(x)$ is still in $\mathrm{im}(x)$ (by exactness of the original exact couple) hence such that $x \in \mathrm{im}(i|_{\mathrm{im}(i)})$, hence such that $x \in \mathrm{im}(\tilde{i})$.

The kernel of $\tilde{k}$ is the intersection of the kernel of k with the kernel of $d = j \circ k$, wich is still the kernel of k, hence the image of j, by exactness. Indeed this is also still the image of $\tilde{j}$, since for every $x \in \mathcal{D}$ then $\tilde{j}(i(x)) = j(x)$.

The kernel of $\tilde{i}$ is $\ker(i) \cap \mathrm{im}(i) \simeq \mathrm{im}(k) \cap \mathrm{im}(i)$, by exactness. Let $x \in \mathcal{E}$ such that $k(x) \in \mathrm{im}(i)$, then by exactness $k(x) \in \ker(j)$ hence $j(k(x)) = d(x) = 0$, hence $x \in \ker(d)$ and so $k(x) = \tilde{k}(x)$. ▮

Definition 1.11. Given an exact couple, def. 1.6, then the induced **spectral sequence of the exact couple** is the sequence of pages, def. 1.8, of the induced sequence of derived exact couples, def. 1.9, lemma 1.10.

The r**th page** of the spectral sequence is the page (def. 1.8) of the rth exact couple, denoted

$$\{\mathcal{E}_r, d_r\} \,.$$

Remark 1.12. So the spectral sequence of an exact couple (def. 1.11) is a sequence of cochain complexes $(\mathcal{E}_r, d_r)$, where the cohomology of one is the terms of the next one:

$$\mathcal{E}_{r+1} \simeq H(\mathcal{E}_r, d_r) \,.$$

In practice this is used as a successive stagewise approximation to the computation of a limiting term $\mathcal{E}_\infty$. What that limiting term is, if it exists at all, is the subject of *convergence* of the spectral sequence, we come to this below.

Def. 1.11 makes sense without a (bi-)grading on the terms of the exact couple, but much of the power of spectral sequences comes from the cases where such a bigrading is given, since together with the sequence of pages of the spectral sequence, this tends to organize computation of the successive cohomology groups in an efficient way. Therefore consider:

Definition 1.13. Given a filtered spectrum as in def. 1.1,

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & X_3 & \overset{f_2}{\longrightarrow} & X_2 & \overset{f_1}{\longrightarrow} & X_1 & \overset{f_0}{\longrightarrow} & X \\ && \downarrow^{g_3} && \downarrow^{g_2} && \downarrow^{g_1} && \downarrow^{g_0} \\ && A_3 && A_2 && A_1 && A_0 \end{array}$$

and given another spectrum $X \in \mathrm{Ho}(\mathrm{Spectra})$, the induced **spectral sequence of a filtered spectrum** is the spectral sequence that is induced, by def. 1.11 from the exact couple (def. 1.6) given by def. 1.4:

$$\begin{array}{ccc} \mathcal{D} & \overset{i}{\longrightarrow} & \mathcal{D} \\ & {}_{k}\nwarrow & \downarrow^{j} \\ && \mathcal{E} \end{array} \quad := \quad \begin{array}{ccc} \oplus_{s,t} D^{s,t}(X,Y) & \overset{[X,f]}{\longrightarrow} & \oplus_{s,t} D^{s,t}(X,Y) \\ & {}_{[X,h]}\nwarrow & \downarrow^{[X,g]} \\ && \oplus_{s,t} E^{s,t}(X,Y) \end{array} \quad := \quad \begin{array}{ccc} \oplus_{s,t} [X, Y_s]_{t-s} & \overset{[X,f]}{\longrightarrow} & \oplus_{s,t} [X, Y_s]_{t-s} \\ & {}_{[X,h]}\nwarrow & \downarrow^{[X,g]} \\ && [X, A_s]_{t-s} \end{array}$$

with the following bidegree of the differentials:

$$\mathrm{deg} = \begin{array}{ccc} \mathcal{D} & \overset{(-1,-1)}{\longrightarrow} & \mathcal{D} \\ & {}_{(1,0)}\nwarrow & \downarrow^{(0,0)} \\ && \mathcal{E} \end{array}$$

In particular the first page is

$$\begin{aligned} \mathcal{E}_1^{s,t} &= [X, A_s]_{t-s} \\ d_1 &= [X, g \circ h] \,. \end{aligned}$$

As we pass to derived exact couples, by def. 1.9, the bidegree of i and k is preserved, but that of j increases by $(1,1)$ with each page, since (by def. 1.8)

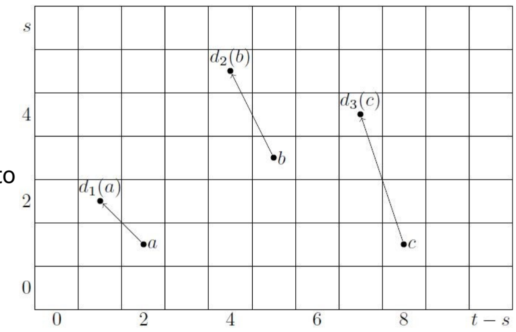
$$\begin{aligned} \mathrm{deg}(\tilde{j}) &= \mathrm{deg}(j \circ i^{-1}) \\ &= \mathrm{deg}(j) - \mathrm{deg}(i) \,. \\ &= \mathrm{deg}(j) + (1,1) \end{aligned}$$

Similarly the first differential has degree

$$\begin{aligned} \deg(j \circ k) &= \deg(j) + \deg(k) \\ &= (1,0) + (0,0) \\ &= (1,0) \end{aligned}$$

and so the differentials on the rth page are of the form

$$d_r : \mathcal{E}_r^{s,t} \longrightarrow \mathcal{E}_r^{s+r,t+r-1} \,.$$



It is conventional to depict this in tables where s increases vertically and upwards and $t - s$ increases horizontally and to the right, so that d_r goes up r steps and always one step to the left. This is the "Adams type" grading convention for spectral sequences (different from the Serre-Atiyah-Hirzebruch spectral sequence convention (prop.)). One also says that

* s is the *filtration degree*;
* $t - s$ is the *total degree*;
* t is the *internal degree*.

A priori all this is $\mathbb{N} \times \mathbb{Z}$-graded, but we regard it as being $\mathbb{Z} \times \mathbb{Z}$-graded by setting

$$\mathcal{D}^{s<0,\bullet} \coloneqq 0 \quad , \quad \mathcal{E}^{s<0,\bullet} \coloneqq 0$$

and trivially extending the definition of the differentials to these zero-groups.

E-Adams filtrations

Given a homotopy commutative ring spectrum (E, μ, e), then an *E-Adams spectral sequence* is a spectral sequence as in def. 1.13, where each cofiber is induced from the unit morphism $e : \mathbb{S} \longrightarrow E$:

Definition 1.14. Let $X, Y \in \mathrm{Ho}(\mathrm{Spectra})$ be two spectra (def.), and let $(E, \mu, e) \in \mathrm{CMon}(\mathrm{Ho}(\mathrm{Spectra}), \wedge, \mathbb{S})$ be a homotopy commutative ring spectrum (def.) in the tensor triangulated stable homotopy category $(\mathrm{Ho}(\mathrm{Spectra}), \wedge, \mathbb{S})$ (prop.).

Then the E-**Adams spectral sequence** for the computation of the graded abelian group

$$[X, Y]_\bullet \coloneqq [X, \Sigma^{-\bullet} Y]$$

of morphisms in the stable homotopy category (def.) is the spectral sequence of a filtered spectrum (def. 1.13) of the image under $[X, -]_\bullet$ of the tower

$$\begin{array}{ccccc} & \vdots & & & \\ & {}^{f_0}\downarrow & & & \\ & Y_3 & \overset{g_3}{\longrightarrow} & E \wedge Y_3 = A_3 & \\ & {}^{f_0}\downarrow & & & \\ & Y_2 & \overset{g_2}{\longrightarrow} & E \wedge Y_2 = A_2 \,, & \\ & {}^{f_0}\downarrow & & & \\ & Y_1 & \overset{g_1}{\longrightarrow} & E \wedge Y_1 = A_1 & \\ & {}^{f_0}\downarrow & & & \\ Y = & Y_0 & \overset{g_0}{\longrightarrow} & E \wedge Y_0 = A_0 & \end{array}$$

where each hook is a homotopy fiber sequence (equivalently a homotopy cofiber sequence, prop.), hence where each

$$Y_{n+1} \overset{f_n}{\longrightarrow} Y_n \overset{g_n}{\longrightarrow} A_n \overset{h_n}{\longrightarrow} \Sigma Y_{n+1}$$

is an exact triangle (prop.), where inductively

$$A_n \coloneqq E \wedge Y_n$$

is the derived smash product of spectra (corollary) of E with the stage Y_n (cor.), and where

$$g_n : Y_n \underset{\simeq}{\overset{\ell_{Y_n}^{-1}}{\longrightarrow}} \mathbb{S} \wedge Y_n \overset{e \wedge \mathrm{id}}{\longrightarrow} E \wedge Y_n$$

is the composition of the inverse derived unitor on Y_n (cor.) with the derived smash product of spectra of the unit e of E and the identity on Y_n.

Hence, by def 1.13, the first page is

$$E_1^{s,t}(X,Y) \coloneqq [X, A_s]_{t-s}$$
$$d_1 = [X, g \circ h]$$

and the differentials are of the form

$$d_r : E_r^{s,t}(X,Y) \longrightarrow E_r^{s+r,t+r-1}(X,Y) .$$

A priori $E_r^{\bullet,\bullet}(X,Y)$ is $\mathbb{N} \times \mathbb{Z}$-graded, but we regard it as being $\mathbb{Z} \times \mathbb{Z}$-graded by setting

$$E_r^{s<0,\bullet}(X,Y) \coloneqq 0$$

and trivially extending the definition of the differentials to these zero-groups.

(Adams 74, theorem 15.1 page 318)

Remark 1.15. The morphism

$$[X, g_k] : [X, Y_k]_\bullet \xrightarrow{[X, e \wedge \mathrm{id}_{Y_k}]} [X, E \wedge Y_k]_\bullet$$

in def. 1.14 is sometimes called the **Boardman homomorphism** (Adams 74, p. 58).

For $X = \mathbb{S}$ the sphere spectrum it reduces to a canonical morphism from stable homotopy to generalized homology (rmk.)

$$\pi_\bullet(g_k) : \pi_\bullet(Y_k) \longrightarrow E_\bullet(Y_k) .$$

For $E =$ HA an Eilenberg-MacLane spectrum (def.) this in turn reduces to the **Hurewicz homomorphism** for spectra.

This way one may think of the E-Adams filtration on Y in def. 1.14 as the result of filtering any spectrum Y by iteratively projecting out all its E-homology. This idea was historically the original motivation for the construction of the classical Adams spectral sequence by John Frank Adams, see the first pages of (Bruner 09) for a historical approach.

It is convenient to adopt the following notation for E-Adams spectral sequences (def. 1.14):

Definition 1.16. For $(E, \mu, e) \in \mathrm{CMon}(\mathrm{Ho}(\mathrm{Spectra}), \wedge, \mathbb{S})$ a homotopy commutative ring spectrum (def.), write $\overline{E}$ for the homotopy fiber of its unit $e : \mathbb{S} \to E$, i.e. such that there is a homotopy fiber sequence (equivalently a homotopy cofiber sequence, prop.) in the stable homotopy category $\mathrm{Ho}(\mathrm{Spectra})$ of the form

$$\overline{E} \longrightarrow \mathbb{S} \overset{e}{\longrightarrow} E ,$$

equivalently an exact triangle (prop.) of the form

$$\overline{E} \longrightarrow \mathbb{S} \overset{e}{\longrightarrow} E \longrightarrow \Sigma \overline{E} .$$

(Adams 74, theorem 15.1 page 319) Beware that for instance (Hopkins 99, proof of corollary 5.3) uses "$\overline{E}$" not for the homotopy fiber of $\mathbb{S} \overset{e}{\to} E$ but for its homotopy cofiber, hence for what is $\Sigma \overline{E}$ according to (Adams 74).

Lemma 1.17. *In terms of def. 1.16, the spectra entering the definition of the E-Adams spectral sequence in def. 1.14 are equivalently*

$$Y_p \;\simeq\; \overline{E}^p \wedge Y$$

and

$$A_p \;\simeq\; E \wedge \overline{E}^p \wedge Y .$$

where we write

$$\overline{E}^p \coloneqq \underbrace{\overline{E} \wedge \cdots \wedge \overline{E}}_{p \text{ factors}} \wedge Y .$$

Hence the first page of the E-Adams spectral sequence reads equivalently

$$E_1^{s,t}(X,Y) \simeq [X, E \wedge \overline{E}^s \wedge Y]_{t-s} .$$

(Adams 74, theorem 15.1 page 319)

Proof. By definition the statement holds for $p = 0$. Assume then by induction that it holds for some $p \geq 0$. Since the smash product of spectra-functor $(-) \wedge \overline{E}^p \wedge Y$ preserves homotopy cofiber sequences (lemma, this is part of the tensor triangulated structure of the stable homotopy category), its application to the homotopy cofiber sequence

$$\overline{E} \longrightarrow \mathbb{S} \overset{e}{\longrightarrow} E$$

from def. 1.16 yields another homotopy cofiber sequence, now of the form

$$\overline{E}^{p+1} \wedge Y \longrightarrow \overline{E}^p \wedge Y \overset{g_p}{\longrightarrow} E \wedge \overline{E}^p \wedge Y$$

where the morphism on the right is identified as g_p by the induction assumption, hence $A_{p+1} \simeq E \wedge \overline{E}^p \wedge Y$. Since Y_{p+1} is defined to be the homotopy fiber of g_p, it follows that $Y_{p+1} \simeq \overline{E}^{p+1} \wedge Y$. ▮

Remark 1.18. Terminology differs across authors. The filtration in def. 1.14 in the rewriting by lemma 1.17 is due to (Adams 74, theorem 15.1), where it is not give any name. In (Ravenel 84, p. 356) it is called the (canonical) **Adams tower** while in (Ravenel 86, def. 2.21) it is called the canonical **Adams resolution**. Several authors follow the latter usage, for instance (Rognes 12, def. 4.1). But (Hopkins 99) uses "Adams resolution" for the "E-injective resolutions" (see here) and uses "Adams tower" for yet another concept (def.).

We proceed now to analyze the first two pages and then the convergence properties of E-Adams spectral sequences of def. 1.14.

2. The first page

By lemma 1.17 the first page of an E-Adams spectral sequence (def. 1.14) looks like

$$E_1^{s,t}(X,Y) \simeq [X, E \wedge \overline{E}^s \wedge Y]_{s-t} \,.$$

We discuss now how, under favorable conditions, these hom-groups may alternatively be computed as morphisms of E-homology equipped with suitable comodule structure over a Hopf algebroid structure on the dual E-Steenrod operations $E_\bullet(E)$ (The E-generalized homology of E (rmk.)). Then below we discuss that, as a result, the d_1-cohomology of the first page computes the Ext-groups from the E-homology of Y to the E-homology of X, regarded as $E_\bullet(E)$-comodules.

The condition needed for this to work is the following.

Flat homotopy ring spectra

Definition 2.1. Call a homotopy commutative ring spectrum (E, μ, e) (def.) **flat** if the canonical right $\pi_\bullet(E)$-module structure on $E_\bullet(E)$ (prop.) (equivalently the canonical left module struture, see prop. 2.5 below) is a flat module.

The key consequence of the assumption that E is flat in the sense of def. 2.1 is the following.

Proposition 2.2. *Let (E, μ, e) be a homotopy commutative ring spectrum (def.) and let $X \in \mathrm{Ho}(\mathrm{Spectra})$ be any spectrum. Then there is a homomorphism of graded abelian groups of the form*

$$E_\bullet(E) \otimes_{\pi_\bullet(E)} E_\bullet(X) \longrightarrow [\mathbb{S}, E \wedge E \wedge X]_\bullet = \pi_\bullet(E \wedge E \wedge X)$$

(for $E_\bullet(-)$ the canonical $\pi_\bullet(E)$-modules from this prop.) given on elements

$$\Sigma^{n_1}\mathbb{S} \overset{\alpha_1}{\longrightarrow} E \wedge E \quad , \quad \Sigma^{n_2}\mathbb{S} \overset{\alpha_2}{\longrightarrow} E \wedge X$$

by

$$\alpha_1 \cdot \alpha_2 \,:\, \Sigma^{n_1+n_2}\mathbb{S} \overset{\simeq}{\longrightarrow} \Sigma^{n_1}\mathbb{S} \wedge \Sigma^{n_2}\mathbb{S} \overset{\alpha_1 \wedge \alpha_2}{\longrightarrow} E \wedge E \wedge E \wedge X \overset{\mathrm{id}_E \wedge \mu \wedge \mathrm{id}_X}{\longrightarrow} E \wedge E \wedge X \,.$$

If $E_\bullet(E)$ is a flat module over $\pi_\bullet(E)$ then this is an isomorphism.

(Adams 69, lecture 3, lemma 1 (p. 68), Adams 74, part III, lemma 12.5)

Proof. First of all, that the given pairing is a well defined homomorphism (descends from $E_\bullet(E) \times E_\bullet(X)$ to $E_\bullet(E) \otimes_{\pi_\bullet(E)} E_\bullet(X)$) follows from the associativity of μ.

We discuss that it is an isomorphism when $E_\bullet(E)$ is flat over $\pi_\bullet(E)$:

First consider the case that $X \simeq \Sigma^n \mathbb{S}$ is a suspension of the sphere spectrum. Then (by this example, using the tensor triangulated stucture on the stable homotopy category (prop.))

$$E_\bullet(X) = E_\bullet(\Sigma^n X) \simeq \pi_{\bullet - n}(E)$$

and

$$\pi_\bullet(E \wedge E \wedge X) = \pi_\bullet(E \wedge E \wedge \Sigma^n \mathbb{S}) \simeq E_{\bullet - n}(E)$$

and

$$E_\bullet(E) \otimes_{\pi_\bullet(E)} \pi_{\bullet - n}(E) \simeq E_{\bullet - n}(E)$$

Therefore in this case we have an isomorphism for all E.

For general X, we may without restriction assume that X is represented by a sequential CW-spectrum (prop.). Then the homotopy cofibers of its cell attachment maps are suspensions of the sphere spectrum (rmk.).

First consider the case that X is a CW-spectrum with finitely many cells. Consider the homotopy cofiber sequence of the $(k+1)$st cell attachment (by that remark):

$$\Sigma^{n_k - 1}\mathbb{S} \longrightarrow X_k \longrightarrow X_{k+1} \longrightarrow \Sigma^{n_k}\mathbb{S} \longrightarrow \Sigma X_k$$

and its image under the natural morphism $E_\bullet(E) \otimes_{\pi_\bullet(E)} E_\bullet(-) \to \pi_\bullet([\mathbb{S}, E \wedge E \wedge (-)])$, which is a commuting diagram of the form

$$\begin{array}{ccccccccc} E_\bullet(E) \otimes_{\pi_\bullet(E)} E_\bullet(\Sigma^{n_k - 1}\mathbb{S}) & \longrightarrow & E_\bullet(E) \otimes_{\pi_\bullet(E)} E_\bullet(X_k) & \longrightarrow & E_\bullet(E) \otimes_{\pi_\bullet(E)} E_\bullet(X_{k+1}) & \longrightarrow & E_\bullet(E) \otimes_{\pi_\bullet(E)} E_\bullet(\Sigma^{n_k}\mathbb{S}) & \longrightarrow & E_\bullet(E) \otimes_{\pi_\bullet(E)} E_\bullet(\Sigma X_k) \\ \downarrow && \downarrow && \downarrow && \downarrow && \downarrow \\ [\mathbb{S}, E \wedge E \wedge \Sigma^{n_k - 1}\mathbb{S}]_\bullet & \longrightarrow & [\mathbb{S}, E \wedge E \wedge X_k]_\bullet & \longrightarrow & [\mathbb{S}, E \wedge E \wedge X_{k+1}]_\bullet & \longrightarrow & [\mathbb{S}, E \wedge E \wedge \Sigma^{n_k}\mathbb{S}]_\bullet & \longrightarrow & [\mathbb{S}, E \wedge E \wedge \Sigma X_k]_\bullet \end{array}$$

Here the bottom row is a long exact sequence since $E \wedge E \wedge (-)$ preserves homotopy cofiber sequences (by this lemma, part of the tensor triangulated structure on $\mathrm{Ho}(\mathrm{Spectra})$ prop.), and since $[\mathbb{S}, -]_\bullet \simeq \pi_\bullet(-)$ sends homotopy cofiber sequences to long exact sequences (prop.). By the same reasoning, $E_\bullet(-)$ of the homotopy cofiber sequence is long exact; and by the assumption that $E_\bullet(E)$ is flat, the functor $E_\bullet(E) \otimes_{\pi_\bullet(E)} (-)$ preserves this exactness, so that also the top row is a long exact sequence.

Now by induction over the cells of X, the outer four vertical morphisms are isomorphisms. Hence the 5-lemma implies that also the middle morphism is an isomorphism.

This shows the claim inductively for all finite CW-spectra. For the general statement, now use that

1. every CW-spectrum is the filtered colimit over its finite CW-subspectra;
2. the symmetric monoidal smash product of spectra $\wedge$ (def.) preserves colimits in its arguments separately (since it has a right adjoint (prop.));
3. $[\mathbb{S}, -]_\bullet \simeq \pi_\bullet(-)$ commutes over filtered colimits of CW-spectrum inclusions (by this lemma, since spheres are compact);
4. $E_\bullet(E) \otimes_{\pi_\bullet(E)} (-)$ distributes over colimits (it being a left adjoint).

∎

Using prop. 2.2, we find below (theorem 2.34) that the first page of the E-Adams spectral sequence may be equivalently rewritten as hom-groups of comodules over $E_\bullet(E)$ regarded as a graded commutative Hopf algebroid. We now first discuss what this means.

The E-Steenrod algebra

We discuss here all the extra structure that exists on the E-self homology $E_\bullet(E)$ of a flat homotopy commutative ring spectrum. For $E = H\mathbb{F}_p$ the Eilenberg-MacLane spectrum on a prime field this reduces to the classical structure in algebraic topology called the *dual Steenrod algebra* $\mathcal{A}_p^*$. Therefore one may generally speak of $E_\bullet(E)$ as being the *dual E-Steenrod algebra*.

Without the qualifier "dual" then "E-Steenrod algebra" refers to the E-self-cohomology $E^\bullet(E)$. For $E = H\mathbb{F}_p$ this *Steenrod algebra* $\mathcal{A}_p$ (without "dual") is traditionally considered first, and the classical Adams spectral sequence was originally formulated in terms of $\mathcal{A}_p$ instead of $\mathcal{A}_p^*$. But one observes (Adams 74, p. 280) that the "dual" Steenrod algebra $E_\bullet(E)$ is much better behaved, at least as long as E is flat in the sense of def. 2.1.

Moreover, the dual E-Steenrod algebra $E_\bullet(E)$ is more fundamental in that it reflects a stacky geometry secretly underlying the E-Adams spectral sequence (Hopkins 99). This is the content of the concept of "commutative Hopf algebroid" (def. 2.9 below) which is equivalently the formal dual of a groupoid internal to affine schemes, def. 2.6.

A simple illustrative archetype of the following construction of commutative Hopf algebroids from homotopy commutative ring spectra is the following situation:

For X a finite set consider

$$\begin{array}{c} X \times X \times X \\ \downarrow^{\circ = (\mathrm{pr}_1, \mathrm{pr}_3)} \\ X \times X \\ {}^{s = \mathrm{pr}_1}\downarrow \uparrow \downarrow^{t = \mathrm{pr}_2} \\ X \end{array}$$

as the ("codiscrete") groupoid with X as objects and precisely one morphism from every object to every other. Hence the composition operation $\circ$, and the source and target maps are simply projections as shown. The identity morphism (going upwards in the above diagram) is the diagonal.

Then consider the image of this structure under forming the free abelian groups $\mathbb{Z}[X]$, regarded as commutative rings under pointwise multiplication.

Since

$$\mathbb{Z}[X \times X] \simeq \mathbb{Z}[X] \otimes \mathbb{Z}[X]$$

this yields a diagram of homomorphisms of commutative rings of the form

$$\begin{array}{c} (\mathbb{Z}[X] \otimes \mathbb{Z}[X]) \otimes_{\mathbb{Z}[X]} (\mathbb{Z}[X] \otimes \mathbb{Z}[X]) \\ \uparrow \\ \mathbb{Z}[X] \otimes \mathbb{Z}[X] \\ \uparrow \downarrow \uparrow \\ \mathbb{Z}[X] \end{array}$$

satisfying some obvious conditions. Observe that here

1. the two morphisms $\mathbb{Z}[X] \rightrightarrows \mathbb{Z}[X] \otimes \mathbb{Z}[X]$ are $f \mapsto f \otimes e$ and $f \mapsto e \otimes f$, respectively, where e denotes the unit element in $\mathbb{Z}[X]$;

2. the morphism $\mathbb{Z}[X] \otimes \mathbb{Z}[X] \to \mathbb{Z}[X]$ is the multiplication in the ring $\mathbb{Z}[X]$;

3. the morphism

$$\mathbb{Z}[X] \otimes \mathbb{Z}[X] \longrightarrow \mathbb{Z}[X] \otimes \mathbb{Z}[C] \otimes \mathbb{Z}[C] \overset{\simeq}{\longrightarrow} (\mathbb{Z}[X] \otimes \mathbb{Z}[X]) \otimes_{\mathbb{Z}[X]} (\mathbb{Z}[X] \otimes \mathbb{Z}[X])$$

is given by $f \otimes g \mapsto f \otimes e \otimes g$.

All of the following rich structure is directly modeled on this simplistic example. We simply

1. replace the commutative ring $\mathbb{Z}[X]$ with any flat homotopy commutative ring spectrum E,

2. replace tensor product of abelian groups $\otimes$ with derived smash product of spectra;

3. and form the stable homotopy groups $\pi_\bullet(-)$ of all resulting expressions.

Definition 2.3. Let (E, μ, e) be a homotopy commutative ring spectrum (def.) which is flat according to def. 2.1.

Then the **dual** E**-Steenrod algebra** is the pair of graded abelian groups

$$(E_\bullet(E), \pi_\bullet(E))$$

(rmk.) equipped with the following structure:

1. the graded commutative ring structure

$$\pi_\bullet(E) \otimes \pi_\bullet(E) \longrightarrow \pi_\bullet(E)$$

induced from E being a homotopy commutative ring spectrum (prop.);

2. the graded commutative ring structure

$$E_\bullet(E) \otimes E_\bullet(E) \longrightarrow E_\bullet(E)$$

induced from the fact that with E also $E \wedge E$ is canonically a homotopy commutative ring spectrum (exmpl.), so that also $E_\bullet(E) = \pi_\bullet(E \wedge E)$ is a graded commutative ring (prop.);

3. the homomorphism of graded commutative rings

$$\Psi : E_\bullet(E) \longrightarrow E_\bullet(E) \otimes_{\pi_\bullet(E)} E_\bullet(E)$$

induced under $\pi_\bullet(-)$ from

$$E \wedge E \overset{\mathrm{id} \wedge e \wedge \mathrm{id}}{\longrightarrow} E \wedge E \wedge E$$

via prop. 2.2;

4. the homomorphisms of graded commutative rings

$$\eta_L : \pi_\bullet(E) \longrightarrow E_\bullet(E)$$

and

$$\eta_R : \pi_\bullet(E) \longrightarrow E_\bullet(E)$$

induced under $\pi_\bullet(-)$ from the homomorphisms of commutative ring spectra

$$E \underset{\simeq}{\overset{r_E^{-1}}{\longrightarrow}} E \wedge \mathbb{S} \overset{\mathrm{id} \wedge e}{\longrightarrow} E \wedge E$$

and

$$E \underset{\simeq}{\overset{\ell_E^{-1}}{\longrightarrow}} \mathbb{S} \wedge E \overset{\mathrm{id} \wedge e}{\longrightarrow} E \wedge E \,,$$

respectively (exmpl.);

5. the homomorphism of graded commutative rings

$$\epsilon : E_\bullet(E) \longrightarrow \pi_\bullet(E)$$

induced under $\pi_\bullet(-)$ from

$$\mu : E \wedge E \longrightarrow E$$

regarded as a homomorphism of homotopy commutative ring spectra (exmpl.);

6. the homomorphisms graded commutative rings

$$c : E_\bullet(E) \longrightarrow E_\bullet(E)$$

induced under $\pi_\bullet(-)$ from the braiding

$$\tau_{E,E} : E \wedge E \longrightarrow E \wedge E$$

regarded as a homomorphism of homotopy commutative ring spectra (exmpl.).

(Adams 69, lecture 3, pages 66-68)

Notice that (as verified by direct unwinding of the definitions):

Lemma 2.4. *For (E, μ, e) a homotopy commutative ring spectrum (def.), consider $E_\bullet(E)$ with its canonical left and right $\pi_\bullet(E)$-module structure as in this prop.. These module structures coincide with those induced by the ring homomorphisms η_L and η_R from def. 2.3.*

These two actions need not strictly coincide, but they are isomorphic:

Proposition 2.5. *For (E, μ, e) a homotopy commutative ring spectrum (def.), consider $E_\bullet(E)$ with its canonical left and right $\pi_\bullet(E)$-module structure (prop.). Since E is a commutative monoid, this right module structure may equivalently be regarded as a left-module, too. Then the braiding*

$$E_\bullet(E) \simeq \pi_\bullet(E \wedge E) \overset{\pi_\bullet(\tau_{E,E})}{\longrightarrow} \pi_\bullet(E \wedge E) \simeq E_\bullet(E)$$

constitutes a module isomorphism (def.) between these two left module structures.

Proof. On representatives as in the proof of (this propo.), the original left action is given by (we are notationally suppressing associators throughout)

$$E \wedge E \wedge E \overset{\mu \wedge \mathrm{id}}{\longrightarrow} E \wedge E \,,$$

while the other left action, induced from the canonical right action, is given by

$$E \wedge E \wedge E \underoverset{\simeq}{\tau_{E, E \wedge E}}{\longrightarrow} E \wedge E \wedge E \overset{\mathrm{id} \wedge \mu}{\longrightarrow} E \wedge \;\; .$$

So in order that $\tau_{E,E}$ represents a module homomorphism under $\pi_\bullet(-)$, it is sufficient that the following diagram commutes (we write $E_i \coloneqq E$ for $i \in \{1,2,3\}$ to make the action of the braiding more manifest)

$$\array{ E_1 \wedge E_2 \wedge E_3 & \overset{\mathrm{id} \wedge \tau_{E_2,E_3}}{\longrightarrow} & E_1 \wedge E_3 \wedge E_2 \\ {}^{\mathrm{id}}\downarrow && \downarrow^{\tau_{E_1, E_3 \wedge E_2}} \\ && E_3 \wedge E_2 \wedge E_1 \\ {}^{\mu \wedge \mathrm{id}}\downarrow && \downarrow^{\mathrm{id} \wedge \mu} \\ E \wedge E_3 & \underset{\tau_{E,E_3}}{\longrightarrow} & E_3 \wedge E } \,.$$

But since (E,μ,e) is a commutative monoid (def.), it satisfies $\mu = \mu \circ \tau$ so that we may factor this diagram as follows:

$$\array{ E_1 \wedge E_2 \wedge E_3 & \overset{\mathrm{id} \wedge \tau_{E_2,E_3}}{\longrightarrow} & E_1 \wedge E_3 \wedge E_2 \\ {}^{\tau_{E_1,E_2} \wedge \mathrm{id}}\downarrow && \downarrow^{\tau_{E_1, E_3 \wedge E_2}} \\ E_2 \wedge E_1 \wedge E_3 & \overset{\tau_{E_2 \wedge E_1, E_3}}{\longrightarrow} & E_3 \wedge E_2 \wedge E_1 \\ {}^{\mu \wedge \mathrm{id}}\downarrow && \downarrow^{\mathrm{id} \wedge \mu} \\ E \wedge E_3 & \underset{\tau_{E,E_3}}{\longrightarrow} & E_3 \wedge E } \,.$$

Here the top square commutes by coherence of the braiding (rmk) since both composite morphisms correspond to the same permutation, while the bottom square commutesm due to the naturality of the braiding. Hence the total rectangle commutes. ∎

The dual E-Steenrod algebras of def. 2.3 evidently carry a lot of structure. The concept organizing this is that of_commutative Hopf algebroids_.

Definition 2.6. A **graded commutative Hopf algebroid** is an internal groupoid in the opposite category $\mathrm{gCRing}^{\mathrm{op}}$ of $\mathbb{Z}$-graded commutative rings, regarded with its cartesian monoidal category structure.

(e.g. Ravenel 86, def. A1.1.1)

Remark 2.7. We unwind def. 2.6. For $R \in \mathrm{gCRing}$, write $\mathrm{Spec}(R)$ for the same object, but regarded as an object in $\mathrm{gCRing}^{\mathrm{op}}$.

An internal category in $\mathrm{gCRing}^{\mathrm{op}}$ is a diagram in $\mathrm{gCRing}^{\mathrm{op}}$ of the form

$$\array{ \mathrm{Spec}(\Gamma) \underset{\mathrm{Spec}(A)}{\times} \mathrm{Spec}(\Gamma) \\ \downarrow^{\circ} \\ \mathrm{Spec}(\Gamma) \\ {}^{s}\downarrow \;\; \uparrow_{i} \downarrow^{t} \\ \mathrm{Spec}(A) } \,,$$

(where the fiber product at the top is over s on the left and t on the right) such that the pairing $\circ$ defines an associative composition over $\mathrm{Spec}(A)$, unital with respect to i. This is an internal groupoid if it is furthemore equipped with a morphism

$$\mathrm{inv} \;\colon\; \mathrm{Spec}(\Gamma) \longrightarrow \mathrm{Spec}(\Gamma)$$

acting as assigning inverses with respect to $\circ$.

The key basic fact to use in order to express this equivalently in terms of algebra is that tensor product of commutative rings exhibits the cartesian monoidal category structure on $\mathrm{CRing}^{\mathrm{op}}$, see at *CRing – Properties – Cocartesian comonoidal structure*:

$$\mathrm{Spec}(R_1) \underset{\mathrm{Spec}(R_3)}{\times} \mathrm{Spec}(R_2) \simeq \mathrm{Spec}(R_1 \otimes_{R_3} R_2) \,.$$

This means that the above is equivalently a diagram in gCRing of the form

$$
\begin{array}{c}
\Gamma \otimes_A \Gamma \\
\uparrow^{\Psi} \\
\Gamma \\
{}^{\eta_L}\uparrow \;\; \downarrow^{\epsilon} \;\; \uparrow^{\eta_R} \\
A
\end{array}
$$

as well as

$$c : \Gamma \longrightarrow \Gamma$$

and satisfying formally dual conditions, spelled out as def. 2.9 below. Here

* η_L, η_R are called the left and right *unit maps*;
* ϵ is called the *co-unit*;
* Ψ is called the *comultiplication*;
* c is called the *antipode* or *conjugation*

Remark 2.8. Generally, in a commutative Hopf algebroid, def. 2.6, the two morphisms $\eta_L, \eta_R : A \to \Gamma$ from remark 2.7 need not coincide, they make Γ genuinely into a bimodule over A, and it is the tensor product of bimodules that appears in remark 2.7. But it may happen that they coincide:

An internal groupoid $\mathcal{G}_1 \underset{t}{\overset{s}{\rightrightarrows}} \mathcal{G}_0$ for which the domain and codomain morphisms coincide, $s = t$, is euqivalently a group object in the slice category over $\mathcal{G}_0$.

Dually, a commutative Hopf algebroid $\Gamma \underset{\eta_R}{\overset{\eta_L}{\leftleftarrows}} A$ for which η_L and η_R happen to coincide is equivalently a commutative **Hopf algebra** Γ over A.

Writing out the formally dual axioms of an internal groupoid as in remark 2.7 yields the following equivalent but maybe more explicit definition of commutative Hopf algebroids, def. 2.6

Definition 2.9. A **commutative Hopf algebroid** is

1. two commutative rings, A and Γ;
2. ring homomorphisms
   1. (left/right unit)

      $\eta_L, \eta_R : A \longrightarrow \Gamma$;
   2. (comultiplication)

      $\Psi : \Gamma \longrightarrow \Gamma \otimes_A \Gamma$;
   3. (counit)

      $\epsilon : \Gamma \longrightarrow A$;
   4. (conjugation)

      $c : \Gamma \longrightarrow \Gamma$

such that

1. (co-unitality)
   1. (identity morphisms respect source and target)

      $\epsilon \circ \eta_L = \epsilon \circ \eta_R = \mathrm{id}_A$;
   2. (identity morphisms are units for composition)

      $(\mathrm{id}_\Gamma \otimes_A \epsilon) \circ \Psi = (\epsilon \otimes_A \mathrm{id}_\Gamma) \circ \Psi = \mathrm{id}_\Gamma$;
   3. (composition respects source and target)
      1. $\Psi \circ \eta_R = (\mathrm{id} \otimes_A \eta_R) \circ \eta_R$;

2. $\Psi \circ \eta_L = (\eta_L \otimes_A \mathrm{id}) \circ \eta_L$

2. (co-associativity) $(\mathrm{id}_\Gamma \otimes_A \Psi) \circ \Psi = (\Psi \otimes_A \mathrm{id}_\Gamma) \circ \Psi$;

3. (inverses)

   1. (inverting twice is the identity)

      $c \circ c = \mathrm{id}_\Gamma$;

   2. (inversion swaps source and target)

      $c \circ \eta_L = \eta_R$; $c \circ \eta_R = \eta_L$;

   3. (inverse morphisms are indeed left and right inverses for composition)

      the morphisms α and β induced via the coequalizer property of the tensor product from $(-) \cdot c(-)$ and $c(-) \cdot (-)$, respectively

$$\begin{tikzcd} \Gamma \otimes A \otimes \Gamma \arrow[r, shift left] \arrow[r, shift right] & \Gamma \otimes \Gamma \arrow[r, "\mathrm{coeq}"] \arrow[d, "(-)\cdot c(-)"'] & \Gamma \otimes_A \Gamma \arrow[dl, "\alpha"] \\ & \Gamma & \end{tikzcd}$$

      and

$$\begin{tikzcd} \Gamma \otimes A \otimes \Gamma \arrow[r, shift left] \arrow[r, shift right] & \Gamma \otimes \Gamma \arrow[r, "\mathrm{coeq}"] \arrow[d, "c(-)\cdot(-)"'] & \Gamma \otimes_A \Gamma \arrow[dl, "\beta"] \\ & \Gamma & \end{tikzcd}$$

      satisfy

      $\alpha \circ \Psi = \eta_L \circ \epsilon$

      and

      $\beta \circ \Psi = \eta_R \circ \epsilon$.

(Adams 69, lecture 3, pages 62-66, Ravenel 86, def. A1.1.1)

Remark 2.10. In (Adams 69, lecture 3, page 60) the terminology used is "Hopf algebra in a fully satisfactory sense" with emphasis that the left and right module structure may differ. According to (Ravenel 86, first page of appendix A1) the terminology "Hopf algebroid" for this situation is due to Haynes Miller.

Example 2.11. For R a commutative ring, then $R \otimes R$ becomes a commutative Hopf algebroid over R, formally dual (via def. 2.6) to the pair groupoid on $\mathrm{Spec}(R) \in \mathrm{CRing}^{\mathrm{op}}$.

For X a finite set and $R = \mathbb{Z}[X]$, then this reduces to the motivating example from above.

It is now straightforward, if somewhat tedious, to check that:

Proposition 2.12. *Let (E, μ, e) be a homotopy commutative ring spectrum (def.) which is flat according to def. 2.1, then the dual E-Steenrod algebra $(E_\bullet(E), \pi_\bullet(E))$ with the structure maps $(\eta_L, \eta_R, \epsilon, c, \Psi)$ from prop. 2.3 is a graded commutative Hopf algebroid according to def. 2.9:*

$$(E_\bullet(E), \pi_\bullet(E)) \in \mathrm{CommHopfAlgd}$$

(Adams 69, lecture 3, pages 67-71, Ravenel 86, chapter II, prop. 2.2.8)

Proof. One observes that $E \wedge E$ satisfies the axioms of a commutative Hopf algebroid object in homotopy commutative ring spectra, over E, by direct analogy to example 2.11 (one just has to verify that the symmetric braidings go along coherently, which works by use of the coherence theorem for symmetric monoidal categories (rmk.)). Applying the functor $\pi_\bullet(-)$ that forms stable homotopy groups to all structure morphisms of $E \wedge E$ yields the claimed structure morphisms of $E_\bullet(E)$. ▮

We close this subsection on commutative Hopf algebroids by discussion of their isomorphism classes, when regarded dually as affine groupoids:

Definition 2.13. Given an internal groupoid in $\mathrm{gCRing}^{\mathrm{op}}$ (def. 2.6, remark 2.7)

$$\begin{array}{c} \mathrm{Spec}(\Gamma) \underset{\mathrm{Spec}(A)}{\times} \mathrm{Spec}(\Gamma) \\ \downarrow^{\circ} \\ \mathrm{Spec}(\Gamma) \\ {}^{s}\downarrow \;\; \uparrow_{i} \downarrow^{t} \\ \mathrm{Spec}(A) \end{array} \,,$$

then its affife scheme $\mathrm{Spec}(A)_{/\sim}$ of **isomorphism classes of objects** is the *coequlizer?* of the source and target morphisms

$$\mathrm{Spec}(\mathrm{Gamma}) \underoverset{t}{s}{\rightrightarrows} \mathrm{Spec}(A) \overset{\mathrm{equ}}{\longrightarrow} \mathrm{Spec}(A)_{/\sim} \,.$$

Hence this is the formal dual of the equalizer of the left and right unit (def. 2.9)

$$A \underoverset{\eta_R}{\eta_L}{\rightrightarrows} \Gamma \,.$$

By example 2.11 every commutative ring gives rise to a commutative Hopf algebroid $R \otimes R$ over R. The *core* of R is the formal dual of the corresponding affine scheme of isomorphism classes according to def. 2.13:

Definition 2.14. For R a commutative ring, its **core** cR is the equalizer in

$$cR \longrightarrow R \rightrightarrows R \otimes R \,.$$

A ring which is isomorphic to its core is called a **solid ring**.

(Bousfield-Kan 72, §1, def. 2.1, Bousfield 79, 6.4)

Proposition 2.15. *The core of any ring R is solid (def. 2.14):*

$$ccR \simeq cR \,.$$

(Bousfield-Kan 72, prop. 2.2)

Proposition 2.16. *The following is the complete list of solid rings (def. 2.14) up to isomorphism:*

1. *The localization of the ring of integers at a set J of prime numbers (def. 4.11)*

   $$\mathbb{Z}[J^{-1}] \,;$$

2. *the cyclic rings*

   $$\mathbb{Z}/n\mathbb{Z}$$

   for $n \geq 2$;

3. *the product rings*

   $$\mathbb{Z}[J^{-1}] \times \mathbb{Z}/n\mathbb{Z} \,,$$

   for $n \geq 2$ such that each prime factor of n is contained in the set of primes J;

4. *the ring cores of product rings*

   $$c(\mathbb{Z}[J^{-1}] \times \prod_{p \in K} \mathbb{Z}/p^{e(p)}) \,,$$

   where $K \subset J$ are infinite sets of primes and $e(p)$ are positive natural numbers.

(Bousfield-Kan 72, prop. 3.5, Bousfield 79, p. 276)

Comodules over the E-Steenrod algebra

Definition 2.17. Let (E, μ, e) be a homotopy commutative ring spectrum (def.) which is flat according to def. 2.1.

For $X \in \mathrm{Ho}(\mathrm{Spectra})$ any spectrum, say that the **comodule structure** on $E_\bullet(X)$ (rmk.)) **over the dual E-Steenrod algebra** (def. 2.3) is

1. the canonical structure of a $\pi_\bullet(E)$-module (according to this prop.);

2. the homomorphism of $\pi_\bullet(E)$-modules

$$\Psi_{E_\bullet(X)} \,:\, E_\bullet(X) \longrightarrow E_\bullet(E) \otimes_{\pi_\bullet(E)} E_\bullet(X)$$

induced under $\pi_\bullet(-)$ and via prop. 2.2 from the morphism of spectra

$$E \wedge X \simeq E \wedge \mathbb{S} \wedge X \overset{\mathrm{id} \wedge e \wedge \mathrm{id}}{\longrightarrow} E \wedge E \wedge X \,.$$

Definition 2.18. Given a graded commutative Hopf algebroid Γ over A as in def. 2.9, hence an internal groupoid in $\mathrm{gCRing}^{\mathrm{op}}$, then a **comodule ring** over it is an action in $\mathrm{CRing}^{\mathrm{op}}$ of that internal groupoid.

In the same spirit, a **comodule** over a commutative Hopf algebroid (not necessarily a comodule ring) is a quasicoherent sheaf on the corresponding internal groupoid (regarded as a (algebraic) stack) (e.g. Hopkins 99, prop. 11.6). Explicitly in components:

Definition 2.19. Given a $\mathbb{Z}$-graded commutative Hopf algebroid Γ over A (def. 2.9) then a **left comodule** over Γ is

1. a $\mathbb{Z}$-graded A-module N;

2. (co-action) a homomorphism of graded A-modules

   $\Psi_N \,:\, N \longrightarrow \Gamma \otimes_A N$;

such that

1. (co-unitality)

   $(\epsilon \otimes_A \mathrm{id}_N) \circ \Psi_N = \mathrm{id}_N$;

2. (co-action property)

   $(\Psi \otimes_A \mathrm{id}_N) \circ \Psi_N = (\mathrm{id}_\Gamma \otimes_A \Psi_N) \circ \Psi_N$.

A homomorphism between graded comodules $f : N_1 \to N_2$ is a homomorphism of underlying graded A-modules such that the following diagram commutes

$$\begin{matrix} N_1 & \overset{f}{\longrightarrow} & N_1 \\ {}^{\mathllap{\Psi_{N_1}}}\downarrow && \downarrow^{\mathrlap{\Psi_{N_2}}} \\ \Gamma \otimes_A N_1 & \underset{\mathrm{id} \otimes_A f}{\longrightarrow} & \Gamma \otimes_A N_2 \end{matrix} \,.$$

We write

$$\Gamma \,\mathrm{CoMod}$$

for the resulting category of left comodules over Γ. Analogously for right comodules. The notation for the hom-sets in this category is abbreviated to

$$\mathrm{Hom}_\Gamma(-,-) \coloneqq \mathrm{Hom}_{\Gamma\,\mathrm{CoMod}}(-,-) \,.$$

A priori this is an Ab-enriched category, but it is naturally further enriched in graded abelian groups:

we may drop in the above definition of comodule homomorphisms $f : N_1 \to N_2$ the condition that the underlying morphism be grading-preserving. Say that f has degree n if it increases degree by n. This gives a $\mathbb{Z}$-graded hom-group

$$\mathrm{Hom}^\bullet_\Gamma(-,-) \,.$$

Example 2.20. For (Γ, A) a commutative Hopf algebroid, then A becomes a left Γ-comodule (def. 2.19) with coaction given by the right unit

$$A \overset{\eta_R}{\longrightarrow} \Gamma \simeq \Gamma \otimes_A A \,.$$

Proof. The required co-unitality property is the dual condition in def. 2.9

$$\epsilon \circ \eta_R = \mathrm{id}_A$$

of the fact in def. 2.6 that identity morphisms respect sources:

$$\mathrm{id} \,:\, A \overset{\eta_R}{\longrightarrow} \Gamma \simeq \Gamma \otimes_A A \overset{\epsilon \otimes_A \mathrm{id}}{\longrightarrow} A \otimes_A A \simeq A$$

The required co-action property is the dual condition

$$\Psi \circ \eta_R = (\mathrm{id} \otimes_A \eta_R) \circ \eta_R$$

of the fact in def. 2.6 that composition of morphisms in a groupoid respects sources

$$\begin{array}{ccc} A & \overset{\eta_R}{\longrightarrow} & \Gamma \\ {}^{\mathllap{\eta_R}}\downarrow && \downarrow^{\mathrlap{\Psi}} \\ \Gamma \simeq \Gamma \otimes_A A & \underset{\mathrm{id} \otimes_A \eta_R}{\longrightarrow} & \Gamma \otimes_A \Gamma \end{array} \,.$$

▮

Proposition 2.21. *Let (E, μ, e) be a homotopy commutative ring spectrum (def.) which is flat according to def. 2.1, and for $X \in \mathrm{Ho}(\mathrm{Spectra})$ any spectrum, then the morphism $\Psi_{E_\bullet(X)}$ from def. 2.17 makes $E_\bullet(X)$ into a comodule (def. 2.19) over the dual E-Steenrod algebra (def. 2.3)*

$$E_\bullet(X) \;\in\; E_\bullet(E)\mathrm{CoMod} \,.$$

(Adams 69, lecture 3, pages 67-71, Ravenel 86, chapter II, prop. 2.2.8)

Example 2.22. Given a commutative Hopf algebroid Γ over A, def. 2.9, then A itself becomes a left Γ-comodule (def. 2.19) with coaction given by

$$\Psi_A \;:\; A \overset{\eta_L}{\longrightarrow} \Gamma \simeq \Gamma \otimes_A A$$

and a right Γ-comodule with coaction given by

$$\Psi_A \;:\; A \overset{\eta_R}{\longrightarrow} \Gamma \simeq \Gamma \otimes_A A \,.$$

More generally:

Proposition 2.23. *Given a commutative Hopf algebroid Γ over A, there is a free-forgetful adjunction*

$$A\,\mathrm{Mod} \underoverset{\underset{\mathrm{co-free}}{\longrightarrow}}{\overset{\mathrm{forget}}{\longleftarrow}}{\bot} \Gamma\,\mathrm{CoMod}$$

between the category of Γ-comodules, def. 2.19 and the category of modules over A, where the cofree functor is right adjoint.

Moreover:

1. *The co-free Γ-comodule on an A-module C is $\Gamma \otimes_A C$ equipped with the coaction induced by the comultiplication Ψ in Γ.*

2. *The adjunct $\tilde f$ of a comodule homomorphism*

   $$N \overset{f}{\longrightarrow} \Gamma \otimes_A C$$

   is its composite with the counit ϵ of Γ

   $$\tilde f \;:\; N \overset{f}{\longrightarrow} \Gamma \otimes_A C \overset{\epsilon \otimes_A \mathrm{id}}{\longrightarrow} A \otimes_A C \simeq C \,.$$

The **proof** is formally dual to the proof that shows that constructing free modules is left adjoint to the forgetful functor from a category of modules to the underlying monoidal category (prop.). But since the details of the adjunction isomorphism are important for the following discussion, we spell it out:

Proof. A homomorphism into a co-free Γ-comodule is a morphism of A-modules of the form

$$f \;:\; N \longrightarrow \Gamma \otimes_A C$$

making the following diagram commute

$$\begin{array}{ccc} N & \overset{f}{\longrightarrow} & \Gamma \otimes_A C \\ {}^{\mathllap{\Psi_N}}\downarrow && \downarrow^{\mathrlap{\Psi \otimes_A \mathrm{id}}} \\ \Gamma \otimes_A N & \underset{\mathrm{id} \otimes_A f}{\longrightarrow} & \Gamma \otimes_A \Gamma \otimes_A C \end{array} \,.$$

Consider the composite

$$\tilde f \;:\; N \overset{f}{\longrightarrow} \Gamma \otimes_A C \overset{\epsilon \otimes_A \mathrm{id}}{\longrightarrow} A \otimes_A C \simeq C \,,$$

i.e. the “corestriction” of f along the counit of Γ. By definition this makes the following square commute

$$
\begin{array}{ccc}
\Gamma \otimes_A N & \overset{\mathrm{id} \otimes_A f}{\longrightarrow} & \Gamma \otimes_A \Gamma \otimes_A C \\
{}^{=}\downarrow && \downarrow^{\mathrm{id} \otimes_A \epsilon \otimes_A \mathrm{id}} \\
\Gamma \otimes_A N & \underset{\mathrm{id} \otimes_A \tilde f}{\longrightarrow} & \Gamma \otimes_A C
\end{array}
$$

Pasting this square onto the bottom of the previous one yields

$$
\begin{array}{ccc}
N & \overset{f}{\longrightarrow} & \Gamma \otimes_A C \\
{}^{\Psi_N}\downarrow && \downarrow^{\Psi \otimes_A \mathrm{id}} \\
\Gamma \otimes_A N & \underset{\mathrm{id} \otimes_A f}{\longrightarrow} & \Gamma \otimes_A \Gamma \otimes_A C \\
{}^{=}\downarrow && \downarrow^{\mathrm{id} \otimes_A \epsilon \otimes_A \mathrm{id}} \\
\Gamma \otimes_A N & \underset{\mathrm{id} \otimes_A \tilde f}{\longrightarrow} & \Gamma \otimes_A C
\end{array}
\,.
$$

Now due to co-unitality, the right right vertical composite is the identity on $\Gamma \otimes_A C$. But this means by the commutativity of the outer rectangle that f is uniquely fixed in terms of $\tilde f$ by the relation

$$
f = (\mathrm{id} \otimes_A f) \circ \Psi \,.
$$

This establishes a natural bijection

$$
\frac{N \overset{f}{\longrightarrow} \Gamma \otimes_A C}{N \overset{\tilde f}{\longrightarrow} C}
$$

and hence the adjunction in question. ▮

Proposition 2.24. *Consider a commutative Hopf algebroid Γ over A, def. 2.9. Any left comodule N over Γ (def. 2.19) becomes a right comodule via the coaction*

$$
N \overset{\Psi}{\longrightarrow} \Gamma \otimes_A N \overset{\simeq}{\to} N \otimes_A \Gamma \overset{\mathrm{id} \otimes_A c}{\longrightarrow} N \otimes_A \Gamma \,,
$$

where the isomorphism in the middle the is braiding in $A \mathrm{Mod}$ and where c is the conjugation map of Γ.

Dually, a right comodule N becoomes a left comodule with the coaction

$$
N \overset{\Psi}{\longrightarrow} N \otimes_A \Gamma \overset{\simeq}{\to} \Gamma \otimes_A N \overset{c \otimes_A \mathrm{id}}{\longrightarrow} \Gamma \otimes_A N \,.
$$

Definition 2.25. Given a commutative Hopf algebroid Γ over A, def. 2.9, and given N_1 a right Γ-comodule and N_2 a left comodule (def. 2.19), then their **cotensor product** $N_1 \Box_\Gamma N_2$ is the kernel of the difference of the two coaction morphisms:

$$
N_1 \Box_\Gamma N_2 \;\coloneqq\; \ker\left(N_1 \otimes_A N_2 \overset{\Psi_{N_1} \otimes_A \mathrm{id} - \mathrm{id} \otimes_A \Psi_{N_2}}{\longrightarrow} N_1 \otimes_A \Gamma \otimes_A N_2 \right) \,.
$$

If both N_1 and N_2 are left comodules, then their cotensor product is the cotensor product of N_2 with N_1 regarded as a right comodule via prop. 2.24.

e.g. (Ravenel 86, def. A1.1.4).

Example 2.26. Given a commutative Hopf algebroid Γ over A, (def.), and given N a left Γ-comodule (def.). Regard A itself canonically as a right Γ-comodule via example 2.22. Then the cotensor product

$$
\mathrm{Prim}(N) \coloneqq A \Box_\Gamma N
$$

is called the **primitive elements** of N:

$$
\mathrm{Prim}(N) = \{ n \in N \mid \Psi_N(n) = 1 \otimes n \} \,.
$$

Proposition 2.27. *Given a commutative Hopf algebroid Γ over A, def. 2.9, and given N_1, N_2 two left Γ-comodules (def. 2.19), then their cotensor product (def. 2.25) is commutative, in that there is an isomorphism*

$$
N_1 \Box N_2 \;\simeq\; N_2 \Box N_1 \,.
$$

(e.g. Ravenel 86, prop. A1.1.5)

Lemma 2.28. *Given a commutative Hopf algebroid Γ over A, def. 2.9, and given N_1, N_2 two left Γ-comodules (def. 2.19), such that N_1 is projective as an A-module, then*

1. *The morphism*

$$\mathrm{Hom}_A(N_1, A) \xrightarrow{f \mapsto (\mathrm{id} \otimes_A f) \circ \Psi_{N_1}} \mathrm{Hom}_A(N_1, \Gamma \otimes_A A) \simeq \mathrm{Hom}_A(N_1, \Gamma) \simeq \mathrm{Hom}_A(N_1, A) \otimes_A \Gamma$$

gives $\mathrm{Hom}_A(N_1, A)$ *the structure of a right* Γ*-comodule;*

2. *The cotensor product (def. 2.25) with respect to this right comodule structure is isomorphic to the hom of* Γ*-comodules:*

$$\mathrm{Hom}_A(N_1, A) \Box_\Gamma N_2 \simeq \mathrm{Hom}_\Gamma(N_1, N_2) \,.$$

Hence in particular

$$A \Box_\Gamma N_2 \;\simeq\; \mathrm{Hom}_\Gamma(A, N_2)$$

(e.g. Ravenel 86, lemma A1.1.6)

Remark 2.29. In computing the second page of E-Adams spectral sequences, the second statement in lemma 2.28 is the key translation that makes the comodule Ext-groups on the page be equivalent to a Cotor-groups. The latter lend themselves to computation, for instance via Lambda-algebra or via the May spectral sequence.

Universal coefficient theorem

The key use of the Hopf coalgebroid structure of prop. 2.3 for the present purpose is that it is extra structure inherited by morphisms in E-homology from morphisms of spectra. Namely forming E-homology $f_* : E_\bullet(X) \to E_\bullet(Y)$ of a morphism of a spectra $f : X \to Y$ does not just produce a morphism of E-homology groups

$$[X, Y]_\bullet \longrightarrow \mathrm{Hom}_{\mathrm{Ab}^{\mathbb{Z}}}(E_\bullet(X), E_\bullet(Y))$$

but in fact produces homomorphisms of comodules over $E_\bullet(E)$

$$\alpha \,:\, [X, Y]_\bullet \longrightarrow \mathrm{Hom}_{E_\bullet(E)}(E_\bullet(X), E_\bullet(Y)) \,.$$

This is the statement of lemma 2.30 below. The point is that $E_\bullet(E)$-comodule homomorphism are much more rigid than general abelian group homomorphisms and hence closer to reflecting the underlying morphism of spectra $f : X \to Y$.

In good cases such an approximation of *homotopy* by *homology* is in fact accurate, in that α is an isomorphism. In such a case (Adams 74, part III, section 13) speaks of a "universal coefficient theorem" (the coefficients here being E.)

One such case is exhibited by prop. 2.33 below. This allows to equivalently re-write the first page of the E-Adams spectral sequence in terms of E-homology homomorphisms in theorem 2.34 below.

Lemma 2.30. *For* $X, Y \in \mathrm{Ho}(\mathrm{Spectra})$ *any two spectra, the morphism (of* $\mathbb{Z}$*-graded abelian) generalized homology groups given by smash product with* E *(rmk.)*

$$\begin{array}{ccccc} \pi_\bullet(E \wedge -) & : & [X, Y]_\bullet & \longrightarrow & \mathrm{Hom}^\bullet_{\mathrm{Ab}^{\mathbb{Z}}}(E_\bullet(X), E_\bullet(Y)) \\ && (X \xrightarrow{f} Y) & \mapsto & \left(E_\bullet(X) \xrightarrow{f_*} E_\bullet(Y)\right) \end{array}$$

factors through the forgetful functor from $E_\bullet(E)$*-comodule homomorphisms (def. 2.19) over the dual* E*-Steenrod algebra (def. 2.3):*

$$\begin{array}{ccc} & & \mathrm{Hom}^\bullet_{E_\bullet(E)}(E_\bullet(X), E_\bullet(Y)) \\ & {}^{\exists}\nearrow & \downarrow^{\mathrm{forget}} \\ [X, N]_\bullet & \underset{\pi_\bullet(E \wedge -)}{\longrightarrow} & \mathrm{Hom}^\bullet_{\mathrm{Ab}^{\mathbb{Z}}}(E_\bullet(X), E_\bullet(Y)) \end{array} \,,$$

where $E_\bullet(X)$ *and* $E_\bullet(Y)$ *are regarded as* E*-Steenrod comodules according to def. 2.19, prop. 2.21.*

Proof. By def. 2.19 we need to show that for $X \xrightarrow{f} Y$ a morphism in $\mathrm{Ho}(\mathrm{Spectra})$ then the following diagram commutes

$$\begin{array}{ccc} E_\bullet(X) & \xrightarrow{f_*} & E_\bullet(Y) \\ {}^{\Psi_{E_\bullet(X)}}\downarrow && \downarrow^{\Psi_{E_\bullet(Y)}} \\ E_\bullet(E) \otimes_{\pi_\bullet(E)} E_\bullet(X) & \underset{\mathrm{id} \otimes_{\pi_\bullet(E)} f_*}{\longrightarrow} & E_\bullet(E) \otimes_{\pi_\bullet(E)} E_\bullet(Y) \end{array} \,.$$

By def. 2.19 and prop. 2.21 this is the image under foming stable homotopy groups $\pi_\bullet(-)$ of the following diagram in $\mathrm{Ho}(\mathrm{Spectra})$:

$$
\begin{array}{ccc}
E \wedge X & \overset{\mathrm{id}\wedge f}{\longrightarrow} & E \wedge Y \\
{}^{\simeq}\downarrow && \downarrow^{\simeq} \\
E \wedge \mathbb{S} \wedge X && E \wedge \mathbb{S} \wedge Y \\
{}^{\mathrm{id}\wedge e \wedge \mathrm{id}}\downarrow && \downarrow^{\mathrm{id}\wedge e\wedge \mathrm{id}} \\
E \wedge E \wedge X & \underset{\mathrm{id}\wedge\mathrm{id}\wedge f}{\longrightarrow} & E \wedge E \wedge Y
\end{array}
.
$$

But that this diagram commutes is simply the functoriality of the derived smash product of spectra as a functor on the product category $\mathrm{Ho}(\mathrm{Spectra}) \times \mathrm{Ho}(\mathrm{Spectra})$. ▮

Proposition 2.31. *Let (E,μ,e) be a homotopy commutative ring spectrum (def.), and let $X,Y \in \mathrm{Ho}(\mathrm{Spectra})$ be two spectra such that $E_\bullet(X)$ is a projective module over $\pi_\bullet(E)$ (via this prop.).*

Then the homomorphism of graded abelian groups

$$
\phi_{\mathrm{UC}} \;\colon\; [X, E\wedge Y]_\bullet \longrightarrow \mathrm{Hom}^\bullet_{\pi_\bullet(E)}(E_\bullet(X), E_\bullet(Y))_\bullet
$$

given by

$$
(X \overset{f}{\longrightarrow} E\wedge Y) \;\mapsto\; \pi_\bullet(E\wedge X \overset{\mathrm{id}\wedge f}{\longrightarrow} E \wedge E \wedge Y \overset{\mu\wedge \mathrm{id}}{\longrightarrow} E \wedge Y)
$$

is an isomorphism.

(Schwede 12, chapter II, prop. 6.20)

Proof. First of all we claim that the morphism in question factors as

$$
\beta \;\colon\; [X, E\wedge Y]_\bullet \overset{\simeq}{\longrightarrow} \mathrm{Hom}^\bullet_{E\,\mathrm{Mod}}(E\wedge X, E \wedge Y) \overset{\pi_\bullet}{\longrightarrow} \mathrm{Hom}^\bullet_{\pi_\bullet(E)}(E_\bullet(X), E_\bullet(Y))\,,
$$

where

1. $E\,\mathrm{Mod} = E\,\mathrm{Mod}(\mathrm{Ho}(\mathrm{Spectra}), \wedge, \mathbb{S})$ denotes the category of homotopy module spectra over E (def.)
2. the first morphisms is the free-forgetful adjunction isomorphism for forming free (prop.) E-homotopy module spectra
3. the second morphism is the respective component of the composite of the forgetful functor from E-homotopy module spectra back to $\mathrm{Ho}(\mathrm{Spectra})$ with the functor $\pi_\bullet$ that forms stable homotopy groups.

This is because (by this prop.) the first map is given by first smashing with E and then postcomposing with the E-action on the free module $E \wedge X$, which is the pairing $E \wedge E \overset{\mu}{\to} E$ (prop.).

Hence it is sufficient to show that the morphism on the right is an isomorphism.

We show more generally that for N_1, N_2 any two E-homotopy module spectra (def.) such that $\pi_\bullet(N_1)$ is a projective module over $\pi_\bullet(E)$, then

$$
\mathrm{Hom}^\bullet_{E\,\mathrm{Mod}}(N_1,N_2) \overset{\pi_\bullet}{\longrightarrow} \mathrm{Hom}^\bullet_{\pi_\bullet(E)}(\pi_\bullet(N_1), \pi_\bullet(N_2))
$$

is an isomorphism.

To see this, first consider the case that $\pi_\bullet(N_1)$ is in fact a $\pi_\bullet(E)$-free module.

This implies that there is a basis $\mathcal{B} = \{x_i\}_{i \in I}$ and a homomorphism

$$
\underset{i \in I}{\vee} \Sigma^{\vert x_i\vert} E \longrightarrow N_1
$$

of E-homotopy module spectra, such that this is a stable weak homotopy equivalence.

Observe that this sits in a commuting diagram of the form

$$
\begin{array}{ccc}
\mathrm{Hom}^\bullet_{E\,\mathrm{Mod}}(\underset{i\in I}{\vee} \Sigma^{\vert x_i\vert}E, N_2) & \overset{\pi_\bullet}{\longrightarrow} & \mathrm{Hom}^\bullet_{\pi_\bullet(E)}(\pi_\bullet(\underset{i \in I}{\vee} \Sigma^{\vert x_i\vert} E), \pi_\bullet(N_2)) \\
{}^{\simeq}\downarrow && \downarrow^{\simeq} \\
\prod_{i\in I} [\Sigma^{\vert x_i \vert}\mathbb{S}, N_2]_\bullet & \underset{\simeq}{\longrightarrow} & \prod_{i \in I} \pi_{\bullet + \vert x_i\vert}(N_2)
\end{array}
$$

where

1. the left vertical isomorphism exhibits wedge sum of spectra as the coproduct in the stable homotopy category (lemma);

2. the bottom isomorphism is from this prop.;

3. the right vertical isomorphism is that of the free-forgetful adjunction for modules over $\pi_\bullet(E)$.

Hence the top horizontal morphism is an isomorphism, which was to be shown.

Now consider the general case that $\pi_\bullet(N_1)$ is a projective module over $\pi_\bullet(E)$. Since (graded) projective modules are precisely the retracts of (graded) free modules (prop.), there exists a diagram of $\pi_\bullet(E)$-modules of the form

$$\mathrm{id} : \pi_\bullet(N_1) \longrightarrow \pi_\bullet(\underset{i \in I}{\vee} \Sigma^{|x_i|} E) \longrightarrow \pi_\bullet(N_1)$$

which induces the corresponding split idempotent of $\pi_\bullet(E)$-modules

$$\pi_\bullet(\underset{i \in I}{\vee} \Sigma^{|x_i|} E) \longrightarrow \pi_\bullet(N_1) \longrightarrow \pi_\bullet(\underset{i \in I}{\vee} \Sigma^{|x_i|} E) \,.$$

As before, by freeness this is actually the image under $\pi_\bullet$ of an idempotent of homotopy ring spectra

$$e_\bullet \;:\; \underset{i \in I}{\vee} \Sigma^{|x_i|} E \longrightarrow \underset{i \in I}{\vee} \Sigma^{|x_i|} E$$

and so in particular of spectra.

Now in the stable homotopy category $\mathrm{Ho}(\mathrm{Spectra})$ all idempotents split (prop.), hence there exists a diagram of spectra of the form

$$e \;:\; \underset{i \in I}{\vee} \Sigma^{|x_i|} E \longrightarrow X \longrightarrow \underset{i \in I}{\vee} \Sigma^{|x_i|} E$$

with $\pi_\bullet(e) = e_\bullet$.

Consider the composite

$$X \longrightarrow \underset{i \in I}{\vee} \Sigma^{|x_i|} E \longrightarrow N_1 \,.$$

Since $\pi_\bullet(e) = e_\bullet$ it follows that under $\pi_\bullet$ this is an isomorphism, then that $X \simeq N_1$ in the stable homotopy category.

In conclusion this exhibits N_1 as a retract of an free E-homotopy module spectrum

$$\mathrm{id} : N_1 \longrightarrow \underset{i \in I}{\vee} \Sigma^{|x_i|} E \longrightarrow N_1 \,,$$

hence of a spectrum for which the morphism in question is an isomorphism. Since the morphism in question is natural, its value on N_1 is a retract in the arrow category of an isomorphism, hence itself an isomorphism (lemma). ▮

Remark 2.32. A stronger version of the statement of prop. 2.31, with the free homotopy E-module spectrum $E \wedge Y$ replaced by any homotopy E-module spectrum F, is considered in (Adams 74, chapter III, prop. 13.5) ("universal coefficient theorem"). Strong conditions are considered that ensure that

$$F^\bullet(X) = [X, F]_\bullet \longrightarrow \mathrm{Hom}^\bullet_{\pi_\bullet(E)}(E_\bullet(X), \pi_\bullet(F))$$

is an isomormphism (expressing the F-cohomology of X as the $\pi_\bullet(E)$-linear dual of the E-homology of X).

For the following we need only the weaker but much more general statement of prop. 2.31, and in fact this is all that (Adams 74, p. 323) ends up using, too.

With this we finally get the following statement, which serves to identify maps of certain spectra with their induced maps on E-homology:

Proposition 2.33. *Let (E, μ, e) be a homotopy commutative ring spectrum (def.), and let $X, Y \in \mathrm{Ho}(\mathrm{Spectra})$ be two spectra such that*

1. E is flat according to def. 2.1;

2. $E_\bullet(X)$ is a projective module over $\pi_\bullet(E)$ (via this prop.).

Then the morphism from lemma 2.30

$$[X, E \wedge Y]_\bullet \overset{\pi_\bullet(E \wedge -)}{\longrightarrow} \mathrm{Hom}^\bullet_{E_\bullet(E)}(E_\bullet(X), E_\bullet(E \wedge Y))) \simeq \mathrm{Hom}^\bullet_{E_\bullet(E)}(E_\bullet(X), E_\bullet(E) \otimes_{\pi_\bullet(E)} E_\bullet(Y)))$$

is an isomorphism (where the isomophism on the right is that of prop. 2.2).

(Adams 74, part III, page 323)

Proof. Observe that the following diagram commutes:

$$
\begin{tikzcd}
{[X, E \wedge Y]_\bullet} \arrow[rr, "\pi_\bullet(E \wedge -)"] \arrow[dr, "\phi_{\mathrm{UC}}"'] & & \mathrm{Hom}^\bullet_{E_\bullet(E)}(E_\bullet(X), E_\bullet(E) \otimes_{\pi_\bullet(E)} E_\bullet(Y))) \arrow[dl, "\epsilon \otimes \mathrm{id} \circ (-)"] \\
& \mathrm{Hom}^\bullet_{\pi_\bullet(E)}(E_\bullet(X), E_\bullet(Y)) &
\end{tikzcd}
,
$$

where

1. the top morphism is the one from lemma 2.30;
2. the right vertical morphism is the adjunction isomorphism from prop. 2.23;
3. the left diagonal morphism is the one from prop. 2.31.

To see that this indeed commutes, notice that

1. the top morphism sends $(X \overset{f}{\to} E \wedge Y)$ to $E_\bullet(X) \overset{E_\bullet(f)}{\longrightarrow} E_\bullet(E \wedge Y) \simeq \pi_\bullet(E \wedge E \wedge Y)$ by definition;
2. the right vertical morphism sends this further to $E_\bullet(X) \overset{E_\bullet(f)}{\longrightarrow} \pi_\bullet(E \wedge E \wedge Y) \overset{\pi_\bullet(\mu \wedge \mathrm{id})}{\longrightarrow} \pi_\bullet(E \wedge Y)$, by the proof of prop. 2.23 (which says that the map is given by postcomposition with the counit of $E_\bullet(E)$) and def. 2.3 (which says that this counit is represented by μ);
3. by prop. 2.31 this is the same as the action of the left diagonal morphism.

But now

1. the right vertical morphism is an isomorphism by prop. 2.2;
2. the left diagonal morphism is an isomorphism by prop. 2.31

and so it follows that the top horizontal morphism is an isomorphism, too. ▮

In conclusion:

Theorem 2.34. *Let (E, μ, e) be a homotopy commutative ring spectrum (def.), and let $X, Y \in \mathrm{Ho}(\mathrm{Spectra})$ be two spectra such that*

1. *E is flat according to def. 2.1;*
2. *$E_\bullet(X)$ is a projective module over $\pi_\bullet(E)$ (via this prop.).*

Then the first page of the E-Adams spectral sequence, def. 1.14, for $[Y, X]_\bullet$ is isomorphic to the following chain complex of graded homs of comodules (def. 2.19) over the dual E-Steenrod algebra $(E_\bullet(E), \pi_\bullet(E))$ (prop. 2.3):

$$
E_1^{s,t}(X, Y) \;\simeq\; \mathrm{Hom}^t_{E_\bullet(E)}(E_\bullet(X), E_{\bullet - s}(A_s)) \quad , \qquad d_1 \;=\; \mathrm{Hom}_{E_\bullet(E)}(E_\bullet(X), E_\bullet(g \circ h))
$$

$$
0 \to \mathrm{Hom}^t_{E_\bullet(E)}(E_\bullet(X), E_\bullet(A_0)) \overset{d_1}{\longrightarrow} \mathrm{Hom}^t_{E_\bullet(E)}(E_\bullet(X), E_{\bullet-1}(A_1)) \overset{d_1}{\longrightarrow} \mathrm{Hom}^t_{E_\bullet(E)}(E_\bullet(X), E_{\bullet-2}(A_2)) \overset{d_1}{\longrightarrow} \cdots \,.
$$

(Adams 74, theorem 15.1 page 323)

Proof. This is prop. 2.33 applied to def. 1.14:

$$
\begin{aligned}
E_1^{s,t}(X, Y) & = [X, \underbrace{E \wedge Y_s}_{A_s}]_{t-s} \\
& \simeq \mathrm{Hom}^{t-s}_{E_\bullet(E)}(E_\bullet(X), E_\bullet(\underbrace{E \wedge Y_s}_{A_s})) \\
& \simeq \mathrm{Hom}^t_{E_\bullet(E)}(E_\bullet(X), E_{\bullet - s}(A_s))
\end{aligned}
$$

▮

3. The second page

Theorem 3.1. *Let (E, μ, e) be a homotopy commutative ring spectrum (def.), and let $X, Y \in \mathrm{Ho}(\mathrm{Spectra})$ be two spectra such that*

1. *E is flat according to def. 2.1;*
2. *$E_\bullet(X)$ is a projective module over $\pi_\bullet(E)$ (via this prop.).*

Then the entries of the second page of the E-Adams spectral sequence for $[X,Y]_\bullet$ (def. 1.14) are the Ext-groups of commutative Hopf algebroid-comodules (def. 2.19) over the commutative Hopf algebroid structure on the dual E-Steenrod algebra $E_\bullet(E)$ from prop. 2.3:

$$E_2^{s,t}(X,Y) \simeq \mathrm{Ext}^{s,t}_{E_\bullet(E)}(E_\bullet(X), E_\bullet(Y)) \,.$$

(On the right s is the degree that goes with any Ext-functor, and the "internal degree" t is the additional degree of morphisms between graded modules from def. 2.19.)

In the special case that $X = \mathbb{S}$ is the sphere spectrum, then (by prop. 2.28) these are equivalently Cotor-groups

$$E_2^{s,t}(X,Y) \simeq \mathrm{Cotor}^{s,t}_{E_\bullet(E)}(\pi_\bullet(E), E_\bullet(Y)) \,.$$

(Adams 74, theorem 15.1, page 323)

Proof. By theorem 2.34, under the given assumptions the first page reads

$$E_1^{s,t}(X,Y) \;\simeq\; \mathrm{Hom}^t_{E_\bullet(E)}(E_\bullet(X), E_{\bullet - s}(A_s)) \quad , \quad d_1 \;=\; \mathrm{Hom}_{E_\bullet(E)}(E_\bullet(X), E_\bullet(g \circ h))$$

$$0 \to \mathrm{Hom}^t_{E_\bullet(E)}(E_\bullet(X), E_\bullet(A_0)) \overset{d_1}{\longrightarrow} \mathrm{Hom}^t_{E_\bullet(E)}(E_\bullet(X), E_{\bullet-1}(A_1)) \overset{d_1}{\longrightarrow} \mathrm{Hom}^t_{E_\bullet(E)}(E_\bullet(X), E_{\bullet-2}(A_2)) \overset{d_1}{\longrightarrow} \cdots \,.$$

By remark 1.12 the second page is the cochain cohomology of this complex. Hence by the standard theory of derived functors in homological algebra (see the section *Via acyclic resolutions*), it is now sufficient to see that:

1. the category $E_\bullet(E)\mathrm{CoMod}$ (def. 2.19, prop. 2.12) is an abelian category with enough injectives (so that all right derived functors on $E_\bullet(E)\mathrm{CoMod}$ exist);

2. the first page graded chain complex $(E_1^{\bullet,t}(X,Y), d_1)$ is the image under the hom-functor $F \coloneqq \mathrm{Hom}_{E_\bullet(E)}(E_\bullet(Y), -)$ of an F-acyclic resolution of $E_\bullet(X)$ (so that its cohomology indeed computes the Ext-derived functor (theorem)).

That $E_\bullet(E)\mathrm{CoMod}$ is an abelian category is lemma 3.3 below, and that it has enough injectives is lemma 3.4.

Lemma 3.2 below shows that $E_\bullet(A_\bullet)$ is a resolution of $E_\bullet(Y)$ in $E_\bullet(E)\mathrm{CoMod}$. By prop. 2.2 it is a resolution by cofree comodules (def. 2.23). That these are F-acyclic is lemma 3.5 below. ▮

E-Adams resolutions

We discuss that the first page of the E-Adams spectral sequence indeed exhibits a resolution as required by the proof of theorem 3.1.

Lemma 3.2. *Given an E-Adams spectral sequence $(E_r^{s,t}(X,Y), d_r)$ as in def. 1.14, then the sequences of morphisms*

$$0 \to E_\bullet(Y_p) \overset{E_\bullet(g_p)}{\longrightarrow} E_\bullet(A_p) \overset{E_\bullet(h_p)}{\longrightarrow} E_{\bullet-1}(Y_{p+1}) \to 0$$

are short exact, hence their splicing of short exact sequences

$$\begin{array}{ccccccccccc} 0 & \to & E_\bullet(Y) & \overset{E_\bullet(g_0)}{\longrightarrow} & E_\bullet(A_0) && \overset{\partial}{\longrightarrow} && E_{\bullet-1}(A_1) && \overset{\partial}{\longrightarrow} && E_{\bullet-2}(A_2) & \longrightarrow & \cdots \\ &&&& & {}_{\mathllap{E_\bullet(h_0)}}\searrow && \nearrow_{\mathrlap{E_\bullet(g_1)}} & & {}_{\mathllap{E_\bullet(h_1)}}\searrow && \nearrow_{\mathrlap{E_\bullet(g_2)}} \\ &&&&&& E_{\bullet-1}(Y_1) &&&& E_{\bullet-2}(Y_2) \end{array}$$

is a long exact sequence, exhibiting the graded chain complex $(E_\bullet(A_\bullet), \partial)$ as a resolution of $E_\bullet(Y)$.

(Adams 74, theorem 15.1, page 322)

Proof. Consider the image of the defining homotopy cofiber sequence

$$Y_p \overset{g_p}{\longrightarrow} E \wedge Y_p \overset{h_p}{\longrightarrow} \Sigma Y_{p+1}$$

under the functor $E \wedge (-)$. This is itself a homotopy cofiber sequence of the form

$$E \wedge Y_p \overset{E \wedge g_p}{\longrightarrow} E \wedge E \wedge Y_p \overset{E \wedge h_p}{\longrightarrow} \Sigma E \wedge Y_{p+1}$$

(due to the tensor triangulated structure of the stable homotopy category, prop.).

Applying the stable homotopy groups functor $\pi_\bullet(-) \simeq [\mathbb{S}, -]_\bullet$ (lemma) to this yields a long exact sequence (prop.)

$$\cdots \longrightarrow E_\bullet(Y_{p+1}) \overset{E_\bullet(f_p)}{\longrightarrow} E_\bullet(Y_p) \overset{E_\bullet(g_p)}{\longrightarrow} E_\bullet(A_p) \overset{E_\bullet(h_p)}{\longrightarrow} E_{\bullet-1}(Y_{p+1}) \overset{E_{\bullet-1}(f_p)}{\longrightarrow} E_{\bullet-1}(Y_p) \overset{E_{\bullet-1}(g_p)}{\longrightarrow} E_{\bullet-1}(A_p) \longrightarrow \cdots \,.$$

But in fact this splits: by unitality of (E, μ, e), the product operation μ on the homotopy commutative ring spectrum E is a left inverse to g_p in that

$$\mathrm{id} : E \wedge Y_p \overset{E \wedge g_p}{\longrightarrow} E \wedge E \wedge Y_p \overset{\mu \wedge \mathrm{id}}{\longrightarrow} E \wedge Y_p \,.$$

Therefore $E_\bullet(g_p)$ is a monomorphism, hence its kernel is trivial, and so by exactness $E_\bullet(f_p) = 0$. This means that the above long exact sequence collapses to short exact sequences. ▮

Homological co-algebra

We discuss basic aspects of homological algebra in categories of comodules (def. 2.19) over commutative Hopf algebroids (def. 2.6), needed in the proof of theorem 3.1.

Lemma 3.3. *Let (Γ, A) be a commutative Hopf algebroid Γ over A (def. 2.6, 2.9), such that the right A-module structure on Γ induced by η_R is a flat module.*

Then the category $\Gamma\,\mathrm{CoMod}$ of comodules over Γ (def. 2.19) is an abelian category.

(e.g. Ravenel 86, theorem A1.1.3)

Proof. It is clear that, without any condition on the Hopf algebroid, $\Gamma\,\mathrm{CoMod}$ is an additive category.

Next we need to show if Γ is flat over A, that then this is also a pre-abelian category, in that kernels and cokernels exist.

To that end, let $f : (N_1, \Psi_{N_1}) \longrightarrow (N_2, \Psi_{N_2})$ be a morphism of comodules, hence a commuting diagram in AMod of the form

$$\begin{array}{ccc} N_1 & \overset{f}{\longrightarrow} & N_2 \\ \downarrow^{\Psi_{N_1}} && \downarrow^{\Psi_{N_2}} \\ \Gamma \otimes_A N_1 & \overset{\mathrm{id}_\Gamma \otimes_A f}{\longrightarrow} & \Gamma \otimes_A N_2 \end{array} \,.$$

Consider the kernel $\ker(f)$ of f in AMod and its image under $\Gamma \otimes_A (-)$

$$\begin{array}{ccccc} \ker(f) & \longrightarrow & N_1 & \overset{f}{\longrightarrow} & N_2 \\ \exists \downarrow && \downarrow^{\Psi_{N_1}} && \downarrow^{\Psi_{N_2}} \\ \Gamma \otimes_A \ker(f) & \longrightarrow & \Gamma \otimes_A N_1 & \overset{\mathrm{id}_\Gamma \otimes_A f}{\longrightarrow} & \Gamma \otimes_A N_2 \end{array} \,.$$

By the assumption that Γ is a flat module over A, also $\Gamma \otimes_A \ker(f) \simeq \ker(\Gamma \otimes_A f)$ is a kernel. Hence by the universal property of kernels and the commutativity of the square o the right, there exists a unique vertical morphism as shown on the left, making the left square commute. This means that the A-module $\ker(f)$ uniquely inherits the structure of a Γ-comodule such as to make $\ker(f) \to N_1$ a comodule homomorphism. By the same universal property it follows that $\ker(f)$ with this comodule structure is in fact the kernel of f in $\Gamma\,\mathrm{CoMod}$.

The argument for the existence of cokernels proceeds formally dually. Hence $\Gamma\,\mathrm{CoMod}$ is a pre-abelian category.

But it also follows from this construction that the comparison morphism

$$\mathrm{coker}(\ker(f)) \longrightarrow \ker(\mathrm{coker}(f))$$

formed in $\Gamma\,\mathrm{CoMod}$ has underlying it the corresponding comparison morphism in $A\,\mathrm{Mod}$. There this is an isomorphism by the fact that the category of modules $A\,\mathrm{Mod}$ is an abelian category, hence it is an isomorphism also in $\Gamma\,\mathrm{CoMod}$. So the latter is in fact an abelian category itself. ▮

Lemma 3.4. *Let (Γ, A) be a commutative Hopf algebroid Γ over A (def. 2.6, 2.9), such that the right A-module structure on Γ induced by η_R is a flat module.*

Then

1. *every co-free Γ-comodule (def. 2.23) on an injective module over A is an injective object in $\Gamma\,\mathrm{CoMod}$;*

2. *$\Gamma \,\mathrm{CoMod}$ has enough injectives (def.) if the axiom of choice holds in the ambient set theory.*

(e.g. Ravenel 86, lemma A1.2.2)

Proof. First of all, assuming the axiom of choice, then the category of modules $A\,\mathrm{Mod}$ has enough injectives (by this proposition).

Now by prop. 2.23 we have the adjunction

$$A\,\mathrm{Mod} \underset{\text{co}-\text{free}}{\overset{\text{forget}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \Gamma\,\mathrm{CoMod}\,.$$

Observe that the left adjoint is a faithful functor (being a forgetful functor) and that, by the proof of lemma 3.3, it is an exact functor. This implies that

1. for $I \in A\,\mathrm{Mod}$ an injective module, then the co-free comodule $\Gamma \otimes_A I$ is an injective object in $\Gamma\,\mathrm{CoMod}$ (by this lemma);
2. for $N \in \Gamma\,\mathrm{CoMod}$ any object, and for $i : \mathrm{forget}(N) \hookrightarrow I$ a monomorphism of A-modules into an injective A-module, then the adjunct $\tilde{i} : N \hookrightarrow \Gamma \otimes_A I$ is a monomorphism (by this lemma)) hence a monomorpism into an injective comodule, by the previous item.

Hence $\Gamma\,\mathrm{CoMod}$ has enough injective objects (def.). ▮

Lemma 3.5. *Let (Γ, A) be a commutative Hopf algebroid Γ over A (def. 2.6, 2.9), such that the right A-module structure on Γ induced by η_R is a flat module. Let $N \in \Gamma\,\mathrm{CoMod}$ be a Γ-comodule (def. 2.19) such that the underlying A-module is a projective module (a projective object in AMod).*

Then (assuming the axiom of choice in the ambient set theory) every co-free comodule (prop. 2.23) is an F-acyclic object for F the hom functor $\mathrm{Hom}_{\Gamma\,\mathrm{CoMod}}(N,-)$.

Proof. We need to show that the derived functors $\mathbb{R}^\bullet \mathrm{Hom}_\Gamma(N,-)$ vanish in positive degree on all co-free comodules, hence on $\Gamma \otimes_A K$, for all $K \in A\,\mathrm{Mod}$.

To that end, let $I^\bullet$ be an injective resolution of K in $A\,\mathrm{Mod}$. By lemma 3.4 then $\Gamma \otimes_A I^\bullet$ is a sequence of injective objects in $\Gamma\,\mathrm{CoMod}$ and by the assumption that Γ is flat over A it is an injective resolution of $\Gamma \otimes_A K$ in $\Gamma\,\mathrm{CoMod}$. Therefore the derived functor in question is given by

$$\begin{aligned} \mathbb{R}^{\bullet \geq 1}\mathrm{Hom}_\Gamma(N, \Gamma \otimes_A K) & \simeq H_{\bullet \geq 1}(\mathrm{Hom}_\Gamma(N, \Gamma \otimes_A I^\bullet)) \\ & \simeq H_{\bullet \geq 1}(\mathrm{Hom}_A(N, I^\bullet)) \\ & \simeq 0 \end{aligned} \,.$$

Here the second equivalence is the cofree/forgetful adjunction isomorphism of prop. 2.23, while the last equality then follows from the assumption that the A-module underlying N is a projective module (since hom functors out of projective objects are exact functors (here) and since derived functors of exact functors vanish in positive degree (here)). ▮

With lemma 3.5 the proof of theorem 3.1 is completed.

4. Convergence

We discuss the convergence of E-Adams spectral sequences (def. 1.14), i.e. the identification of the “limit”, in an appropriate sense, of the terms $E_r^{s,t}(X,Y)$ on the rth page of the spectral sequence as r goes to infinity.

If an E-Adams spectral sequence converges, then it converges not necessarily to the full stable homotopy groups $[X,Y]_\bullet$, but to some localization of them. This typically means, roughly, that only certain p-torsion subgroups in $[X,Y]_\bullet$ for some prime numbers p are retained. We give a precise discussion below in *Localization and adic completion of abelian groups*.

If one knows that $[X,Y]_q$ is a finitely generated abelian group (as is the case notably for $\pi_q^s = [\mathbb{S},\mathbb{S}]_q$ by the *Serre finiteness theorem*) then this allows to recover the full information from its pieces: by the *fundamental theorem of finitely generated abelian groups* (prop. 4.1 below) these groups are direct sums of powers $\mathbb{Z}^n$ of the infinite cyclic group with finite cyclic groups of the form $\mathbb{Z}/p^k\mathbb{Z}$, and so all one needs to compute is the powers k “one prime p at a time”. This we review below in *Primary decomposition of abelian groups*.

The deeper reason that E-Adams spectral sequences tend to converge to localizations of the abelian groups $[X,Y]_\bullet$ of morphisms of spectra, is that they really converges to the actual homotopy groups but of localizations of spectra. This is more than just a reformulation, because the localization at the level of spectra determies the filtration which controls the nature of the convergence. We discuss this localization of spectra below in *Localization and nilpotent completion of spectra*.

Then we state convergence properties of E-Adams spectral sequences below in *Convergence statements*.

Primary decomposition of abelian groups

An E-Adams spectral sequence *typically* converges (discussed below) not to the full stable homotopy groups $[X,Y]_\bullet$, but just to some piece which on the finite direct summands consists only of p-primary groups for some prime numbers p that depend on the nature of the homotopy ring spectrum E . Here we review basic facts about p-primary decomposition of finite abelian groups and introduce their graphical calculus (remark \ref{pprimarygraphical} below) which will allow to read off these p-primary pieces from the stable page of the E-Adams spectral sequnce.

Theorem 4.1. ***(fundamental theorem of finitely generated abelian groups)***

Every finitely generated abelian group A is isomorphic to a direct sum of p-primary cyclic groups $\mathbb{Z}/p^k\mathbb{Z}$ (for p a prime number and k a natural number) and copies of the infinite cyclic group $\mathbb{Z}$:

$$A \;\simeq\; \mathbb{Z}^n \oplus \bigoplus_i \mathbb{Z}/p_i^{k_i}\mathbb{Z} \,.$$

The summands of the form $\mathbb{Z}/p^k\mathbb{Z}$ are also called the p-primary components of A. Notice that the p_i need not all be distinct.

fundamental theorem of finite abelian groups*:*

In particular every finite abelian group is of this form for $n = 0$, hence is a direct sum of cyclic groups.

fundamental theorem of cyclic groups*:*

In particular every cyclic group $\mathbb{Z}/n\mathbb{Z}$ is a direct sum of cyclic groups of the form

$$\mathbb{Z}/n\mathbb{Z} \simeq \bigoplus_i \mathbb{Z}/p_i^{k_i}\mathbb{Z}$$

where all the p_i are distinct and k_i is the maximal power of the prime factor p_i in the prime decomposition of n.

Specifically, for each natural number d dividing n it contains $\mathbb{Z}/d\mathbb{Z}$ as the subgroup generated by $n/d \in \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$. In fact the lattice of subgroups of $\mathbb{Z}/n\mathbb{Z}$ is the formal dual of the lattice of natural numbers $\leq n$ ordered by inclusion.

(e.g. Roman 12, theorem 13.4, Navarro 03) for cyclic groups e.g. (Aluffi 09, pages 83-84)

This is a special case of the *structure theorem for finitely generated modules over a principal ideal domain*.

Example 4.2. For p a prime number, there are, up to isomorphism, two abelian groups of order p^2, namely

$$(\mathbb{Z}/p\mathbb{Z}) \oplus (\mathbb{Z}/p\mathbb{Z})$$

and

$$\mathbb{Z}/p^2\mathbb{Z} \,.$$

Example 4.3. For p_1 and p_2 two distinct prime numbers, $p_1 \neq p_2$, then there is, up to isomorphism, precisely one abelian group of order $p_1 p_2$, namely

$$\mathbb{Z}/p_1\mathbb{Z} \oplus \mathbb{Z}/p_2\mathbb{Z} \,.$$

This is equivalently the cyclic group

$$\mathbb{Z}/p_1 p_2\mathbb{Z} \simeq \mathbb{Z}/p_1\mathbb{Z} \oplus \mathbb{Z}/p_2\mathbb{Z} \,.$$

The isomorphism is given by sending 1 to (p_2, p_1).

Example 4.4. Moving up, for two distinct prime numbers p_1 and p_2, there are exactly two abelian groups of order $p_1^2 p_2$, namely $(\mathbb{Z}/p_1\mathbb{Z}) \oplus (\mathbb{Z}/p_1\mathbb{Z}) \oplus (\mathbb{Z}/p_2\mathbb{Z})$ and $(\mathbb{Z}/p_1^2\mathbb{Z}) \oplus (\mathbb{Z}/p_2\mathbb{Z})$. The latter is the cyclic group of order $p_1^2 p_2$. For instance, $\mathbb{Z}/12\mathbb{Z} \cong (\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z})$.

Example 4.5. Similarly, there are four abelian groups of order $p_1^2 p_2^2$, three abelian groups of order $p_1^3 p_2$, and so on.

More generally, theorem 4.1 may be used to compute exactly how many abelian groups there are of any finite order n (up to isomorphism): write down its prime factorization, and then for each prime power p^k appearing therein, consider how many ways it can be written as a product of positive powers of p. That is, each partition of k yields an abelian group of order p^k. Since the choices can be made independently for each p, the numbers of such partitions for each p are then multiplied.

Of all these abelian groups of order n, of course, one of them is the cyclic group of order n. The fundamental theorem of cyclic groups says it is the one that involves the one-element partitions $k = [k]$, i.e. the cyclic groups of order p^k for each p.

Remark 4.6. (graphical representation of p-primary decomposition)

Theorem 4.1 says that for any prime number p, the p-primary part of any finitely generated abelian group is determined uniquely up to isomorphism by

- a total number $k \in \mathbb{N}$ of powers of p;
- a partition $k = k_1 + k_2 + \cdots + k_q$.

The corresponding p-primary group is

$$\bigoplus_{i=1}^{q} \mathbb{Z}/p^{k_i}\mathbb{Z} \,.$$

In the context of Adams spectral sequences it is conventional to depict this information graphically by

- k dots;
- of which sequences of length k_i are connected by vertical lines, for $i \in \{1, \cdots, q\}$.

For example the graphical representation of the p-primary group

$$\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^3\mathbb{Z} \oplus \mathbb{Z}/p^4\mathbb{Z}$$

is

$$
\begin{array}{cccc}
 & & & \bullet \\
 & & & | \\
 & & \bullet & \bullet \\
 & & | & | \\
 & & \bullet & \bullet \\
 & & | & | \\
\bullet & \bullet & \bullet & \bullet
\end{array}
\,.
$$

This notation comes from the convention of drawing stable pages of multiplicative Adams spectral sequences and reading them as encoding the extension problem for computing the homotopy groups that the spectral sequence converges to:

- a dot at the top of a vertical sequence of dots denotes the group $\mathbb{Z}/p\mathbb{Z}$;
- inductively, a dot vetically below a sequence of dots denotes a group extension of $\mathbb{Z}/p\mathbb{Z}$ by the group represented by the sequence of dots above;
- a vertical line between two dots means that the the generator of the group corresponding to the upper dot is, regarded after inclusion into the group extension, the product by p of the generator of the group corresponding to the lower dot, regarded as represented by the generator of the group extension.

So for instance

$$
\begin{array}{c}
\bullet \\
| \\
\bullet
\end{array}
$$

stands for an abelian group A which forms a group extension of the form

$$\begin{array}{c} \mathbb{Z}/p\mathbb{Z} \\ \downarrow \\ A \\ \downarrow \\ \mathbb{Z}/p\mathbb{Z} \end{array}$$

such that multiplication by p takes the generator of the bottom copy of $\mathbb{Z}/p\mathbb{Z}$, regarded as represented by the generator of A, to the generator of the image of the top copy of $\mathbb{Z}/p\mathbb{Z}$.

This means that of the two possible choices of extensions (by example 4.2) A corresponds to the non-trivial extension $A = \mathbb{Z}/p^2\mathbb{Z}$. Because then in

$$\begin{array}{c} \mathbb{Z}/p\mathbb{Z} \\ \downarrow \\ \mathbb{Z}/p^2\mathbb{Z} \\ \downarrow \\ \mathbb{Z}/p\mathbb{Z} \end{array}$$

the image of the generator 1 of the first group in the middle group is $p = p \cdot 1$.

Conversely, the notation

$$\begin{array}{c} \bullet \\ \\ \bullet \end{array}$$

stands for an abelian group A which forms a group extension of the form

$$\begin{array}{c} \mathbb{Z}/p\mathbb{Z} \\ \downarrow \\ A \\ \downarrow \\ \mathbb{Z}/p\mathbb{Z} \end{array}$$

such that multiplication by p of the generator of the top group in the middle group does *not* yield the generator of the bottom group.

This means that of the two possible choices (by example 4.2) A corresponds to the *trivial* extension $A = \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$. Because then in

$$\begin{array}{c} \mathbb{Z}/p\mathbb{Z} \\ \downarrow \\ \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \\ \downarrow \\ \mathbb{Z}/p\mathbb{Z} \end{array}$$

the generator 1 of the top group maps to the element $(1,0)$ in the middle group, and multiplication of that by p is $(0,0)$ instead of $(0,1)$, where the latter is the generator of the bottom group.

Similarly

$$\begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array}$$

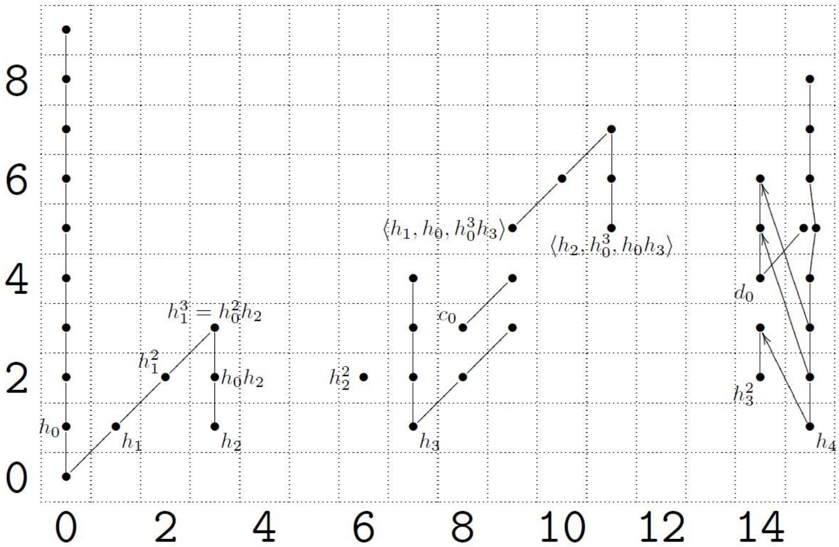
is to be read as the result of appending to the previous case a dot *below*, so that this now indicates a group extension of the form

$$\begin{array}{c} \mathbb{Z}/p^2\mathbb{Z} \\ \downarrow \\ A \\ \downarrow \\ \mathbb{Z}/p\mathbb{Z} \end{array}$$

such that p-times the generator of the bottom group, regarded as represented by the generator of the middle group, is the image of the generator of the top group. This is again the case for the unique non-trivial extension, and hence in this case the diagram stands for $A = \mathbb{Z}/p^3\mathbb{Z}$.

And so on.

For example the stable page of the $\mathbb{F}_2$-classical Adams spectral sequence for computation of the 2-primary part of the stable homotopy groups of spheres $\pi_{t-s}(\mathbb{S})$ has in ("internal") degree $t - s \leq 13$ the following non-trivial entries:



(graphics taken from (Schwede 12)))

Ignoring here the diagonal lines (which denote multiplication by the element h_1 that encodes the additional ring structure on $\pi_\bullet(\mathbb{S})$ which here we are not concerned with) and applying the above prescription, we read off for instance that $\pi_3(\mathbb{S}) \simeq \mathbb{Z}/8\mathbb{Z}$ (because all three dots are connected) while $\pi_8(\mathbb{S}) \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ (because here the two dots are not connected). In total

$k =$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**
$\pi_k(\mathbb{S})_{(2)} =$	$\mathbb{Z}_{(2)}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/8$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/16$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^3$	$\mathbb{Z}/2$	$\mathbb{Z}/8$	0	0

Here the only entry that needs further explanation is the one for $k = 0$. We discuss the relevant concepts for this below in the section *Localization and adic completion of abelian groups*, but for completeness, here is the quick idea:

The symbol $\mathbb{Z}_{(2)}$ refers to the 2-adic integers (def. 4.16), i.e. for the limit of abelian groups

$$\mathbb{Z}_{(2)} = \underset{\longleftarrow n \geq 1}{\lim} \mathbb{Z}/2^n\mathbb{Z}$$

This is not 2-primary, but it does arise when applying 2-adic completion of abelian groups (def. 4.15) to finitely generated abelian groups as in theorem 4.1. The 2-adic integers is the abelian group associated to the diagram

$$\begin{array}{c} \vdots \\ | \\ \bullet \\ | \\ \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array}$$

as in the above figure. Namely by the above prescrption, this infinite sequence should encode an abelian group A such that it is an extension of $\mathbb{Z}/p\mathbb{Z}$ by itself of the form

$$0 \to A \overset{p \cdot (-)}{\longrightarrow} A \longrightarrow \mathbb{Z}/p\mathbb{Z}$$

(Because it is supposed to encode an extension of $\mathbb{Z}/p\mathbb{Z}$ by the group corresponding to the result of chopping off the lowest dot, which however in this case does not change the figure.)

Indeed, by lemma 4.17 below we have a short exact sequence

$$0 \to \mathbb{Z}_{(p)} \overset{p\cdot(-)}{\longrightarrow} \mathbb{Z}_{(p)} \longrightarrow \mathbb{Z}/p\mathbb{Z} \to 0 \;.$$

Localization and adic completion of abelian groups

Remark 4.7. Recall that Ext-groups $\mathrm{Ext}^\bullet(A,B)$ between abelian groups $A, B \in$ Ab are concentrated in degrees 0 and 1 (prop.). Since

$$\mathrm{Ext}^0(A,B) \simeq \mathrm{Hom}(A,B)$$

is the plain hom-functor, this means that there is only one possibly non-vanishing Ext-group Ext^1, therefore often abbreviated to just "Ext":

$$\mathrm{Ext}(A,B) \coloneqq \mathrm{Ext}^1(A,B) \;.$$

Definition 4.8. Let K be an abelian group.

Then an abelian group A is called **K-local** if all the Ext-groups from K to A vanish:

$$\mathrm{Ext}^\bullet(K,A) \simeq 0$$

hence equivalently (remark 4.7) if

$$\mathrm{Hom}(K,A) \simeq 0 \quad \text{and} \quad \mathrm{Ext}(K,A) \simeq 0 \;.$$

A homomorphism of abelian groups $f : B \longrightarrow C$ is called **K-local** if for all K-local groups A the function

$$\mathrm{Hom}(f,A) \;:\; \mathrm{Hom}(B,A) \longrightarrow \mathrm{Hom}(A,A)$$

is a bijection.

> *(**Beware** that here it would seem more natural to use $\mathrm{Ext}^\bullet$ instead of Hom, but we do use Hom. See (Neisendorfer 08, remark 3.2).*

A homomorphism of abelian groups

$$\eta \;:\; A \longrightarrow L_K A$$

is called a **K-localization** if

1. $L_K A$ is K-local;
2. η is a K-local morphism.

We now discuss two classes of examples of localization of abelian groups

1. *Classical localization at/away from primes*;
2. *Formal completion at primes*.

Classical localization at/away from primes

For $n \in \mathbb{N}$, write $\mathbb{Z}/n\mathbb{Z}$ for the cyclic group of order n.

Lemma 4.9. *For $n \in \mathbb{N}$ and $A \in \mathrm{Ab}$ any abelian group, then*

1. *the hom-group out of $\mathbb{Z}/n\mathbb{Z}$ into A is the n-torsion subgroup of A*

$$\mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, A) \simeq \{a \in A \mid p \cdot a = 0\}$$

2. *the first Ext-group out of $\mathbb{Z}/n\mathbb{Z}$ into A is*

$$\mathrm{Ext}^1(\mathbb{Z}/n\mathbb{Z}, A) \simeq A/nA \;.$$

Proof. Regarding the first item: Since $\mathbb{Z}/p\mathbb{Z}$ is generated by its element 1 a morphism $\mathbb{Z}/p\mathbb{Z} \to A$ is fixed by the image a of this element, and the only relation on 1 in $\mathbb{Z}/p\mathbb{Z}$ is that $p \cdot 1 = 0$.

Regarding the second item:

Consider the canonical free resolution

$$0 \to \mathbb{Z} \overset{p\cdot(-)}{\longrightarrow} \mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z} \to 0 \,.$$

By the general discusson of derived functors in homological algebra this exhibits the Ext-group in degree 1 as part of the following short exact sequence

$$0 \to \mathrm{Hom}(\mathbb{Z}, A) \overset{\mathrm{Hom}(n\cdot(-), A)}{\longrightarrow} \mathrm{Hom}(\mathbb{Z}, A) \longrightarrow \mathrm{Ext}^1(\mathbb{Z}/n\mathbb{Z}, A) \to 0 \,,$$

where the morphism on the left is equivalently $A \overset{n\cdot(-)}{\longrightarrow} A$. ▮

Example 4.10. An abelian group A is $\mathbb{Z}/p\mathbb{Z}$-local precisely if the operation

$$p \cdot (-) \colon A \longrightarrow A$$

of multiplication by p is an isomorphism, hence if "p is invertible in A".

Proof. By the first item of lemma 4.9 we have

$$\mathrm{Hom}(\mathbb{Z}/p\mathbb{Z}, A) \simeq \{a \in A \mid p \cdot a = 0\}$$

By the second item of lemma 4.9 we have

$$\mathrm{Ext}^1(\mathbb{Z}/p\mathbb{Z}, A) \simeq A/pA \,.$$

Hence by def. 4.8 A is $\mathbb{Z}/p\mathbb{Z}$-local precisely if

$$\{a \in A \mid p \cdot a = 0\} \simeq 0$$

and if

$$A/pA \simeq 0 \,.$$

Both these conditions are equivalent to multiplication by p being invertible. ▮

Definition 4.11. For $J \subset \mathbb{N}$ a set of prime numbers, consider the direct sum $\oplus_{p \in J} \mathbb{Z}/p\mathbb{Z}$ of cyclic groups of order p.

The operation of $\otimes_{p \in J} \mathbb{Z}/p\mathbb{Z}$-localization of abelian groups according to def. 4.8 is called **inverting the primes** in J.

Specifically

1. for $J = \{p\}$ a single prime then $\mathbb{Z}/p\mathbb{Z}$-localization is called **localization away from** p;
2. for J the set of all primes except p then $\otimes_{p \in J} \mathbb{Z}/p\mathbb{Z}$-localization is called **localization at** p**;**
3. for J the set of all primes, then $\otimes_{p \in J} \mathbb{Z}/p\mathbb{Z}$-localizaton is called **rationalization**..

Definition 4.12. For $J \subset \mathrm{Primes} \subset \mathbb{N}$ a set of prime numbers, then

$$\mathbb{Z}[J^{-1}] \hookrightarrow \mathbb{Q}$$

denotes the subgroup of the rational numbers on those elements which have an expression as a fraction of natural numbers with denominator a product of elements in J.

This is the abelian group underlying the localization of a commutative ring of the ring of integers at the set J of primes.

If $J = \mathrm{Primes} - \{p\}$ is the set of all primes *except* p one also writes

$$\mathbb{Z}_{(p)} \coloneqq \mathbb{Z}[\mathrm{Primes} - \{p\}] \,.$$

Notice the parenthesis, to distinguish from the notation $\mathbb{Z}_p$ for the p-adic integers (def. 4.16 below).

Remark 4.13. The terminology in def. 4.11 is motivated by the following perspective of arithmetic geometry:

Generally for R a commutative ring, then an R-module is equivalently a quasicoherent sheaf on the spectrum of the ring $\mathrm{Spec}(R)$.

In the present case $R = \mathbb{Z}$ is the integers and abelian groups are identified with $\mathbb{Z}$-modules. Hence we may think of an abelian group A equivalently as a quasicoherent sheaf on Spec(Z).

The "ring of functions" on Spec(Z) is the integers, and a point in $\mathrm{Spec}(\mathbb{Z})$ is labeled by a prime number p, thought of as generating the ideal of functions on Spec(Z) which vanish at that point. The residue field at that point is $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$.

Inverting a prime means forcing p to become invertible, which, by this characterization, it is precisely *away* from that point which it labels. The localization of an abelian group at $\mathbb{Z}/p\mathbb{Z}$ hence corresponds to the restriction of the corresponding quasicoherent sheaf over $\mathrm{Spec}(\mathbb{Z})$ to the complement of the point labeled by p.

Similarly localization *at* p is localization away from all points except p.

See also at *function field analogy* for more on this.

Proposition 4.14. *For $J \subset \mathbb{N}$ a set of prime numbers, a homomorphism of abelian groups $f : A\text{lookrightarrow} B$ is local (def. 4.8) with respect to $\oplus_{p \in J} \mathbb{Z}/p\mathbb{Z}$ (def. 4.11) if under tensor product of abelian groups with $\mathbb{Z}[J^{-1}]$ (def. 4.12) it becomes an isomorphism*

$$f \otimes \mathbb{Z}[J^{-1}] : X \otimes \mathbb{Z}[J^{-1}] \xrightarrow{\simeq} Y \otimes \mathbb{Z}[J^{-1}] \;.$$

Moreover, for A any abelian group then its $\oplus_{p \in J} \mathbb{Z}/p\mathbb{Z}$-localization exists and is given by the canonical projection morphism

$$A \longrightarrow A \otimes \mathbb{Z}[J^{-1}] \;.$$

(e.g. Neisendorfer 08, theorem 4.2)

Formal completion at primes

We have seen above in remark 4.13 that classical localization of abelian groups at a prime number is geometrically interpreted as restricting a quasicoherent sheaf over Spec(Z) to a single point, the point that is labeled by that prime number.

Alternatively one may restrict to the "infinitesimal neighbourhood" of such a point. Technically this is called the *formal neighbourhood*, because its ring of functions is, by definition, the ring of formal power series (regarded as an adic ring or pro-ring). The corresponding operation on abelian groups is accordingly called *formal completion* or *adic completion* or just *completion*, for short, of abelian groups.

It turns out that if the abelian group is finitely generated then the operation of p-completion coincides with an operation of *localization* in the sense of def. 4.8, namely localization at the p-primary component $\mathbb{Z}(p^\infty)$ of the group $\mathbb{Q}/\mathbb{Z}$ (def. 4.22 below). On the one hand this equivalence is useful for deducing some key properties of p-completion, this we discuss below. On the other hand this situation is a shadow of the relation between localization of spectra and nilpotent completion of spectra, which is important for characterizing the convergence properties of Adams spectral sequences.

Definition 4.15. For p a prime number, then the **p-adic completion** of an abelian group A is the abelian group given by the limit

$$A_p^\wedge \coloneqq \underset{\longleftarrow}{\lim}(\cdots \longrightarrow A/p^3 A \longrightarrow A/p^2 A \longrightarrow A/pA)\,,$$

where the morphisms are the evident quotient morphisms. With these understood we often write

$$A_p^\wedge \coloneqq \underset{\longleftarrow}{\lim}_n A/p^n A$$

for short. Notice that here the indexing starts at $n = 1$.

Example 4.16. The p-adic completion (def. 4.15) of the integers $\mathbb{Z}$ is called the **p-adic integers**, often written

$$\mathbb{Z}_p \coloneqq \mathbb{Z}_p^\wedge \coloneqq \underset{\longleftarrow}{\lim}_n \mathbb{Z}/p^n\mathbb{Z}\,,$$

which is the original example that gives the general concept its name.

With respect to the canonical ring-structure on the integers, of course $p\mathbb{Z}$ is a prime ideal.

Compare this to the ring $\mathcal{O}_{\mathbb{C}}$ of holomorphic functions on the complex plane. For $x \in \mathbb{C}$ any point, it contains the prime ideal generated by $(z - x)$ (for z the canonical coordinate function on $\mathbb{z}$). The formal power series ring $\mathbb{C}[[(z.x)]]$ is the adic completion of $\mathcal{O}_{\mathbb{C}}$ at this ideal. It has the interpretation of functions defined on a formal neighbourhood of X in $\mathbb{C}$.

Analogously, the p-adic integers $\mathbb{Z}_p$ may be thought of as the functions defined on a formal neighbourhood of the point labeled by p in Spec(Z).

Lemma 4.17. *There is a short exact sequence*

$$0 \to \mathbb{Z}_p \overset{p\cdot(-)}{\longrightarrow} \mathbb{Z}_p \longrightarrow \mathbb{Z}/p\mathbb{Z} \to 0 \,.$$

Proof. Consider the following commuting diagram

$$\begin{array}{ccccc} \vdots && \vdots && \vdots \\ \downarrow && \downarrow && \downarrow \\ \mathbb{Z}/p^3\mathbb{Z} &\overset{p\cdot(-)}{\longrightarrow}& \mathbb{Z}/p^4\mathbb{Z} &\longrightarrow& \mathbb{Z}/p\mathbb{Z} \\ \downarrow && \downarrow && \downarrow \\ \mathbb{Z}/p^2\mathbb{Z} &\overset{p\cdot(-)}{\longrightarrow}& \mathbb{Z}/p^3\mathbb{Z} &\longrightarrow& \mathbb{Z}/p\mathbb{Z} \\ \downarrow && \downarrow && \downarrow \\ \mathbb{Z}/p\mathbb{Z} &\overset{p\cdot(-)}{\longrightarrow}& \mathbb{Z}/p^2\mathbb{Z} &\longrightarrow& \mathbb{Z}/p\mathbb{Z} \\ \downarrow && \downarrow && \downarrow \\ 0 &\longrightarrow& \mathbb{Z}/p\mathbb{Z} &\longrightarrow& \mathbb{Z}/p\mathbb{Z} \end{array} \,.$$

Each horizontal sequence is exact. Taking the limit over the vertical sequences yields the sequence in question. Since limits commute over limits, the result follows. ▮

We now consider a concept of p-completion that is in general different from def. 4.15, but turns out to coincide with it in finitely generated abelian groups.

Definition 4.18. For p a prime number, write

$$\mathbb{Z}[1/p] \coloneqq \underset{\longrightarrow}{\lim}\left(\mathbb{Z} \overset{p\cdot(-)}{\longrightarrow} \mathbb{Z} \overset{p\cdot(-)}{\longrightarrow} \mathbb{Z} \longrightarrow \cdots\right)$$

for the colimit (in Ab) over iterative applications of multiplication by p on the integers.

This is the abelian group generated by formal expressions $\frac{1}{p^k}$ for $k \in \mathbb{N}$, subject to the relations

$$(p\cdot n)\frac{1}{p^{k+1}} = n\frac{1}{p^k} \,.$$

Equivalently it is the abelian group underlying the ring localization $\mathbb{Z}[1/p]$.

Definition 4.19. For p a prime number, then localization of abelian groups (def. 4.8) at $\mathbb{Z}[1/p]$ (def. 4.18) is called **p-completion of abelian groups**.

Lemma 4.20. *An abelian group A is p-complete according to def. 4.19 precisely if both the limit as well as the lim^1 of the sequence*

$$\cdots \longrightarrow A \overset{p}{\longrightarrow} A \overset{p}{\longrightarrow} A \overset{p}{\longrightarrow} A$$

vanishes:

$$\underset{\longleftarrow}{\lim}\left(\cdots \longrightarrow A \overset{p}{\longrightarrow} A \overset{p}{\longrightarrow} A \overset{p}{\longrightarrow} A\right) \simeq 0$$

and

$$\underset{\longleftarrow}{\lim}^1\left(\cdots \longrightarrow A \overset{p}{\longrightarrow} A \overset{p}{\longrightarrow} A \overset{p}{\longrightarrow} A\right) \simeq 0 \,.$$

Proof. By def. 4.8 the group A is $\mathbb{Z}[1/p]$-local precisely if

$$\mathrm{Hom}(\mathbb{Z}[1/p], A) \simeq 0 \quad \text{and} \quad \mathrm{Ext}^1(\mathbb{Z}[1/p], A) \simeq 0 \,.$$

Now use the colimit definition $\mathbb{Z}[1/p] = \underset{\longrightarrow_n}{\lim}\, \mathbb{Z}$ (def. 4.18) and the fact that the hom-functor sends colimits in the first argument to limits to deduce that

$$\begin{aligned} \mathrm{Hom}(\mathbb{Z}[1/p], A) &= \mathrm{Hom}(\underset{\longrightarrow_n}{\lim}\, \mathbb{Z}, A) \\ &\simeq \underset{\longleftarrow_n}{\lim}\, \mathrm{Hom}(\mathbb{Z}, A) \\ &\simeq \underset{\longleftarrow_n}{\lim}\, A \end{aligned} \,.$$

This yields the first statement. For the second, use that for every cotower over abelian groups $B_\bullet$ there is a short exact sequence of the form

$$0 \to \underset{\longleftarrow n}{\lim}^1 \mathrm{Hom}(B_n, A) \longrightarrow \mathrm{Ext}^1(\underset{\longrightarrow n}{\lim} B_n, A) \longrightarrow \underset{\longleftarrow n}{\lim} \mathrm{Ext}^1(B_n, A) \to 0$$

(by this lemma).

In the present case all $B_n \simeq \mathbb{Z}$, which is a free abelian group, hence a projective object, so that all the Ext-groups out of it vannish. Hence by exactness there is an isomorphism

$$\mathrm{Ext}^1(\underset{\longrightarrow n}{\lim} \mathbb{Z}, A) \simeq \underset{\longleftarrow n}{\lim}^1 \mathrm{Hom}(\mathbb{Z}, A) \simeq \underset{\longleftarrow n}{\lim}^1 A \;.$$

This gives the second statement. ∎

Example 4.21. For p a prime number, the p-primary cyclic groups of the form $\mathbb{Z}/p^n\mathbb{Z}$ are p-complete (def. 4.19).

Proof. By lemma 4.20 we need to check that

$$\underset{\longleftarrow}{\lim}\left(\cdots \xrightarrow{p} \mathbb{Z}/p^n\mathbb{Z} \xrightarrow{p} \mathbb{Z}/p^n\mathbb{Z} \xrightarrow{p} \mathbb{Z}/p^n\mathbb{Z}\right) \simeq 0$$

and

$$\underset{\longleftarrow}{\lim}^1\left(\cdots \xrightarrow{p} \mathbb{Z}/p^n\mathbb{Z} \xrightarrow{p} \mathbb{Z}/p^n\mathbb{Z} \xrightarrow{p} \mathbb{Z}/p^n\mathbb{Z}\right) \simeq 0 \;.$$

For the first statement observe that n consecutive stages of the tower compose to the zero morphism. First of all this directly implies that the limit vanishes, secondly it means that the tower satisfies the Mittag-Leffler condition (def.) and this implies that the $\lim^1$ also vanishes (prop.). ∎

Definition 4.22. For p a prime number, write

$$\mathbb{Z}(p^\infty) \coloneqq \mathbb{Z}[1/p]/\mathbb{Z}$$

(the p-primary part of $\mathbb{Q}/\mathbb{Z}$), where $\mathbb{Z}[1/p] = \underset{\longrightarrow}{\lim}(\mathbb{Z} \overset{p}{\to} \mathbb{Z} \overset{p}{\to} \mathbb{Z} \to \cdots)$ from def. 4.18.

Since colimits commute over each other, this is equivalently

$$\mathbb{Z}(p^\infty) \simeq \underset{\longrightarrow}{\lim}(0 \hookrightarrow \mathbb{Z}/p\mathbb{Z} \hookrightarrow \mathbb{Z}/p^2\mathbb{Z} \hookrightarrow \cdots) \;.$$

Theorem 4.23. *For p a prime number, the $\mathbb{Z}[1/p]$-localization*

$$A \longrightarrow L_{\mathbb{Z}[1/p]} A$$

of an abelian group A (def. 4.18, def. 4.8), hence the p-completion of A according to def. 4.19, is given by the morphism

$$A \longrightarrow \mathrm{Ext}^1(\mathbb{Z}(p^\infty), A)$$

into the first Ext-group into A out of $\mathbb{Z}(p^\infty)$ (def. 4.22), which appears as the first connecting homomorphism δ in the long exact sequence of Ext-groups

$$0 \to \mathrm{Hom}(\mathbb{Z}(p^\infty), A) \longrightarrow \mathrm{Hom}(\mathbb{Z}[1/p], A) \longrightarrow \mathrm{Hom}(\mathbb{Z}, A) \overset{\delta)}{\longrightarrow} \mathrm{Ext}^1(\mathbb{Z}(p^\infty), A) \to \cdots \;.$$

that is induced (via this prop.) from the defining short exact sequence

$$0 \to \mathbb{Z} \longrightarrow \mathbb{Z}[1/p] \longrightarrow \mathbb{Z}(p^\infty) \to 0$$

(def. 4.22).

e.g. (Neisendorfer 08, p. 16)

Proposition 4.24. *If A is a finitely generated abelian group, then its p-completion (def. 4.19, for any prime number p) is equivalently its p-adic completion (def. 4.15)*

$$\mathbb{Z}[1/p]A \simeq A_p^\wedge \;.$$

Proof. By theorem 4.23 the p-completion is $\mathrm{Ext}^1(\mathbb{Z}(p^\infty), A)$. By def. 4.22 there is a colimit

$$\mathbb{Z}(p^\infty) = \underset{\longrightarrow}{\lim}(\mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p^3\mathbb{Z} \to \cdots) \;.$$

Together this implies, by this lemma, that there is a short exact sequence of the form

$$0 \to \underset{\longleftarrow}{\lim}^1 \mathrm{Hom}(\mathbb{Z}/p^n\mathbb{Z}, A) \longrightarrow X_p^\wedge \longrightarrow \underset{\longleftarrow n}{\lim} \mathrm{Ext}^1(\mathbb{Z}/p^n\mathbb{Z}, A) \to 0 \;.$$

By lemma 4.9 the lim^1 on the left is over the p^n-torsion subgroups of A, as n ranges. By the assumption that A is finitely generated, there is a maximum n such that all torsion elements in A are annihilated by p^n. This means that the Mittag-Leffler condition (def.) is satisfied by the tower of p-torsion subgroups, and hence the lim^1-term vanishes (prop.).

Therefore by exactness of the above sequence there is an isomorphism

$$\begin{aligned} L_{\mathbb{Z}[1/p]} X &\simeq \underset{\longleftarrow n}{\lim} \mathrm{Ext}^1(\mathbb{Z}/p^n\mathbb{Z}, A) \\ &\simeq \underset{\longleftarrow n}{\lim} A/p^n A \end{aligned},$$

where the second isomorphism is by lemma 4.9. ∎

Proposition 4.25. *If A is a p-divisible group in that $A \xrightarrow{p\cdot(-)} A$ is an isomorphism, then its p-completion (def. 4.19) vanishes.*

Proof. By theorem 4.23 the localization morphism δ sits in a long exact sequence of the form

$$0 \to \mathrm{Hom}(\mathbb{Z}(p^\infty), A) \longrightarrow \mathrm{Hom}(\mathbb{Z}[1/p], A) \overset{\phi}{\longrightarrow} \mathrm{Hom}(\mathbb{Z}, A) \overset{\delta}{\longrightarrow} \mathrm{Ext}^1(\mathbb{Z}(p^\infty), A) \to \cdots .$$

Here by def. 4.18 and since the hom-functor takes colimits in the first argument to limits, the second term is equivalently the limit

$$\mathrm{Hom}(\mathbb{Z}[1/p], A) \simeq \underset{\longleftarrow}{\lim}\left(\cdots \to A \overset{p\cdot(-)}{\longrightarrow} A \overset{p\cdot(-)}{\longrightarrow} A \right) .$$

But by assumption all these morphisms $p\cdot(-)$ that the limit is over are isomorphisms, so that the limit collapses to its first term, which means that the morphism ϕ in the above sequence is an isomorphism. But by exactness of the sequence this means that $\delta = 0$. ∎

Corollary 4.26. *Let p be a prime number. If A is a finite abelian group, then its p-completion (def. 4.19) is equivalently its p-primary part.*

Proof. By the fundamental theorem of finite abelian groups, A is a finite direct sum

$$A \simeq \underset{i}{\oplus} \mathbb{Z}/p_i^{k_i}\mathbb{Z}$$

of cyclic groups of ordr $p_i^{k_1}$ for p_i prime numbers and $k_i \in \mathbb{N}$ (thm.).

Since finite direct sums are equivalently products (biproducts: Ab is an additive category) this means that

$$\mathrm{Ext}^1(\mathbb{Z}(p^\infty), A) \simeq \prod_i \mathrm{Ext}^1(\mathbb{Z}(p^\infty), \mathbb{Z}/p_i^{k_1}\mathbb{Z}) .$$

By theorem 4.23 the ith factor here is the p-completion of $\mathbb{Z}/p_i^{k_i}\mathbb{Z}$, and since $p\cdot(-)$ is an isomorphism on $\mathbb{Z}/p_i^{k_i}\mathbb{Z}$ if $p_i \neq p$ (because its kernel evidently vanishes), prop. 4.25 says that in this case the factor vanishes, so that only the factors with $p_i = p$ remain. On these however $\mathrm{Ext}^1(\mathbb{Z}(p^\infty), -)$ is the identity by example 4.21. ∎

Localization and nilpotent completion of spectra

We discuuss

1. *Bousfield localization of spectra*
2. *Nilpotent completion of spectra*

which are the analogs in stable homotopy theory of the construction of localization of abelian groups discussed above.

Literature: (Bousfield 79)

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Localization of spectra

Definition 4.27. Let $E \in \mathrm{Ho}(\mathrm{Spectra})$ be be a spectrum. Say that

1. a spectrum X is E**-acyclic** if the smash product with E is zero, $E \wedge X \simeq 0$;
2. a morphism $f : X \to Y$ of spectra is an E**-equivalence** if $E \wedge f : E \wedge X \to E \wedge Y$ is an isomorphism in $\mathrm{Ho}(\mathrm{Spectra})$, hence if $E_\bullet(f)$ is an isomorphism in E-generalized homology;

3. a spectrum X is E-**local** if the following equivalent conditions hold

   1. for every E-equivalence f then $[f,X]_\bullet$ is an isomorphism;

   2. every morphism $Y \longrightarrow X$ out of an E-acyclic spectrum Y is zero in $\mathrm{Ho}(\mathrm{Spectra})$;

(Bousfield 79, §1) see also for instance (Lurie, Lecture 20, example 4)

Lemma 4.28. *The two conditions in the last item of def. 4.27 are indeed equivalent.*

Proof. Notice that $A \in \mathrm{Ho}(\mathrm{Spectra})$ being E-acyclic means equivalently that the unique morphism $0 \longrightarrow A$ is an E-equivalence.

Hence one direction of the claim is trivial. For the other direction we need to show that for $[-,X]_\bullet$ to give an isomorphism on all E-equivalences f, it is sufficient that it gives an isomorphism on all E-equivalences of the form $0 \to A$.

Given a morphism $f : A \to B$, write $B \longrightarrow B/A$ for its homotopy cofiber. Then since $\mathrm{Ho}(\mathrm{Spectra})$ is a triangulated category (prop.) the defining axioms of triangulated categories (def., lemma) give that there is a commuting diagram of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \overset{\mathrm{id}}{\longrightarrow} & A & \longrightarrow & 0 & \longrightarrow & \Sigma A \\ \downarrow && \downarrow^{\mathrm{id}} && \downarrow^{f} && \downarrow && \downarrow^{\mathrm{id}} \\ \Sigma^{-1} B/A & \longrightarrow & A & \underset{f}{\longrightarrow} & B & \longrightarrow & B/A & \longrightarrow & \Sigma A \end{array},$$

where both the top as well as the bottom are homotopy cofiber sequences. Hence applying $[-,X]_\bullet$ to this diagram in $\mathrm{Ho}(\mathrm{Spectra})$ yields a diagram of graded abelian groups of the form

$$\begin{array}{ccccccccc} 0 & \longleftarrow & [A,X]_\bullet & \longleftarrow & [A,X]_\bullet & \longleftarrow & 0 & \longleftarrow & [A,X]_{\bullet+1} \\ \uparrow && \uparrow^{\mathrm{id}} && \uparrow^{[f,X]_\bullet} && \uparrow && \uparrow^{\mathrm{id}} \\ [B/A,X]_{\bullet+1} & \longleftarrow & [A,X]_\bullet & \longleftarrow & [B,X]_\bullet & \longleftarrow & [B/A,X]_\bullet & \longleftarrow & [A,X]_{\bullet+1} \end{array},$$

where now both horizontal sequences are long exact sequences (prop.).

Hence if $[B/A,X]_\bullet \longrightarrow 0$ is an isomorphism, then all four outer vertical morphisms in this diagram are isomorphisms, and then the five-lemma implies that also $[f,X]_\bullet$ is an isomorphism.

Hence it is now sufficient to observe that with $f : A \to B$ an E-equivalence, then its homotopy cofiber B/A is E-acyclic.

To see this, notice that by the tensor triangulated structure on $\mathrm{Ho}(\mathrm{Spectra})$ (prop.) the smash product with E preserves homotopy cofiber sequences, so that there is a homotopy cofiber sequence

$$E \wedge A \overset{E \wedge f}{\longrightarrow} E \wedge B \longrightarrow E \wedge (B/A) \longrightarrow E \wedge \Sigma A \,.$$

But if the first morphism here is an isomorphism, then the axioms of a triangulated category (def.) imply that $E \wedge B/A \simeq 0$. In detail: by the axioms we may form the morphism of homotopy cofiber sequences

$$\begin{array}{ccccccc} E\wedge A & \overset{E\wedge f}{\longrightarrow} & E \wedge B & \longrightarrow & E \wedge B/A & \longrightarrow & E \wedge \Sigma A \\ \downarrow^{\mathrm{id}} && \downarrow^{(E\wedge f)^{-1}} && \downarrow && \downarrow^{\mathrm{id}} \\ E \wedge A & \underset{\mathrm{id}}{\longrightarrow} & E \wedge A & \longrightarrow & 0 & \longrightarrow & E \wedge \Sigma A \end{array}.$$

Then since two of the three vertical morphisms on the left are isomorphisms, so is the third (lemma). ▮

Definition 4.29. Given $E, X \in \mathrm{Ho}(\mathrm{Spectra})$, then an E-**Bousfield localization of spectra** of X is

1. an E-local spectrum $L_E X$

2. an E-equivalence $X \longrightarrow L_E X$.

according to def. 4.27.

We discuss now that E-Localizations always exist. The key to this is the following lemma 4.30, which asserts that a spectrum being E-local is equivalent to it being A-null, for some "small" spectrum A:

Lemma 4.30. *For every spectrum E there exists a spectrum A such that any spectrum X is E-local (def. 4.27) precisely if it is A-null, i.e.*

$$X \text{ is } E\text{-local} \quad \Leftrightarrow \quad [A, X]_* = 0$$

and such that

1. *A is E-acyclic (def. 4.27);*
2. *there exists an infinite cardinal number κ such that A is a κ-CW spectrum (hence a CW spectrum (def.) with at most κ many cells);*
3. *the class of E-acyclic spectra (def. 4.27) is the class generated by A under*
   1. *wedge sum*
   2. *the relation that if in a homotopy cofiber sequence $X_1 \to X_2 \to X_3$ two of the spectra are in the class, then so is the third.*

(Bousfield 79, lemma 1.13 with lemma 1.14) review includes (Bauer 11, p.2,3, VanKoughnett 13, p. 8)

Proposition 4.31. *For $E \in \mathrm{Ho}(\mathrm{Spectra})$ any spectrum, every spectrum X sits in a homotopy cofiber sequence of the form*

$$G_E(X) \longrightarrow X \overset{\eta_X}{\longrightarrow} L_E(X)\,,$$

and natural in X, such that

1. *$G_E(X)$ is E-acyclic,*
2. *$L_E(X)$ is E-local,*

according to def. 4.27.

(Bousfield 79, theorem 1.1) see also for instance (Lurie, Lecture 20, example 4)

Proof. Consider the κ-CW-spectrum spectrum A whose existence is asserted by lemma 4.30. Let

$$I_A \coloneqq \{A \to \mathrm{Cone}(A)\}$$

denote the set containing as its single element the canonical morphism (of sequential spectra) from A into the standard cone of A, i.e. the cofiber

$$\mathrm{Cone}(A) \coloneqq \mathrm{cofib}(A \to A \wedge I_+) \simeq A \wedge I$$

of the inclusion of A into its standard cylinder spectrum (def.).

Since the standard cylinder spectrum on a CW-spectrum is a good cylinder object (prop.) this means (lemma) that for X any fibrant sequential spectrum, and for $A \longrightarrow X$ any morphism, then an extension along the cone inclusion

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & \nearrow & \\ \mathrm{Cone}(A) & & \end{array}$$

equivalently exhibits a null-homotopy of the top morphism. Hence the $(A \to \mathrm{Cone}(A))$-injective objects in $\mathrm{Ho}(\mathrm{Spectra})$ are precisely those spectra X for which $[A, X]_\bullet \simeq 0$.

Moreover, due to the degreewise nature of the smash tensoring $\mathrm{Cone}(A) = A \wedge I$ (def), the inclusion morphism $A \to \mathrm{Cone}(A)$ is degreewise the inclusion of a CW-complex into its standard cone, which is a relative cell complex inclusion (prop.).

By this lemma the κ-cell spectrum A is κ-*small object* (def.) with respect to morphisms of spectra which are degreewise relative cell complex inclusion small object argument .

Hence the small object argument applies (prop.) and gives for every X a factorization of the terminal morphism $X \to *$ as an I_A-relative cell complex (def.) followed by an I_A-injective morphism (def.)

$$X \overset{I_A\,\mathrm{Cell}}{\longrightarrow} L_E X \overset{I_A\,\mathrm{Inj}}{\longrightarrow} *\,.$$

By the above, this means that $[A, L_E X] = 0$, hence by lemma 4.30 that $L_E X$ is E-local.

It remains to see that the homotopy fiber of $X \to L_E X$ is E-acyclic: By the tensor triangulated structure on $\mathrm{Ho}(\mathrm{Spectra})$ (prop.) it is sufficient to show that the homotopy cofiber is E-acyclic (since it differs from the homotopy fiber only by suspension). By the pasting law, the homotopy cofiber of a transfinite composition is the transfinite composition of a sequence of homotopy pushouts. By lemma 4.30 and applying the pasting law again, all these homotopy pushouts produce E-acyclic objects. Hence we conclude by observing that the the transfinite composition of the morphisms between these E-acyclic objects is E-acyclic. Since by construction all these morphisms are relative cell complex inclusions, this follows again with the compactness of the n-spheres (lemma). ▮

Lemma 4.32. *The morphism $X \to L_E(X)$ in prop. 4.31 exhibits an E-localization of X according to def. 4.29*

Proof. It only remains to show that $X \to L_E X$ is an E-equivalence. By the tensor triangulated structure on $\mathrm{Ho}(\mathrm{Spectra})$ (prop.) the smash product with E preserves homotopy cofiber sequences, so that

$$E \wedge G_E X \longrightarrow E \wedge X \overset{E \wedge \eta_X}{\longrightarrow} E \wedge L_E X \longrightarrow E \wedge \Sigma G_E X$$

is also a homotopy cofiber sequence. But now $E \wedge G_E X \simeq 0$ by prop. 4.31, and so the axioms (def.) of the triangulated structure on $\mathrm{Ho}(\mathrm{Spectra})$ (prop.) imply that $E \wedge \eta$ is an isomorphism. ▮

Nilpotent completion of spectra

Definition 4.33. Let (E, μ, e) be a homotopy commutative ring spectrum (def.) and $Y \in \mathrm{Ho}(\mathrm{Spectra})$ any spectrum. Write $\overline{E}$ for the homotopy fiber of the unit $\mathbb{S} \overset{e}{\to} E$ as in def. 1.16 such that the E-Adams filtration of Y (def. 1.14) reads (according to lemma 1.17)

$$\begin{array}{c} \vdots \\ \downarrow \\ \overline{E}^3 \wedge Y \\ \downarrow \\ \overline{E}^2 \wedge Y \\ \downarrow \\ \overline{E} \wedge Y \\ \downarrow \\ Y \end{array} \cdot$$

For $s \in \mathbb{N}$, write

$$\overline{E}_{s-1} \coloneqq \mathrm{hocof}(\overline{E}^s \overset{i^s}{\longrightarrow} \mathbb{S})$$

for the homotopy cofiber. Here $\overline{E}_{-1} \simeq 0$. By the tensor triangulated structure of $\mathrm{Ho}(\mathrm{Spectra})$ (prop.), this homotopy cofiber is preserved by forming smash product with Y, and so also

$$\overline{E}_n \wedge Y \simeq \mathrm{hocof}(\overline{E}^n \wedge Y \longrightarrow Y) \,.$$

Now let

$$\overline{E}_s \overset{p_{s-1}}{\longrightarrow} \overline{E}_{s-1}$$

be the morphism implied by the octahedral axiom of the triangulated category $\mathrm{Ho}(\mathrm{Spectra})$ (def., prop.):

$$\begin{array}{ccccccc} \overline{E}^{s+1} & \overset{i}{\longrightarrow} & \overline{E}^s & \longrightarrow & E \wedge \overline{E}^s & \longrightarrow & \Sigma \overline{E}^{s+1} \\ {}^{=}\downarrow && \downarrow^{i^s} && \downarrow && \downarrow \\ \overline{E}^{s+1} & \longrightarrow & \mathbb{S} & \longrightarrow & \overline{E}_s & \longrightarrow & \Sigma \overline{E}^{s+1} \\ && \downarrow && \downarrow^{p_{s-1}} && \\ && \overline{E}_{s-1} & \overset{=}{\longrightarrow} & \overline{E}_{s-1} && \\ && \downarrow && \downarrow && \\ && \Sigma \overline{E}^s & \longrightarrow & \Sigma E \wedge \overline{E}^s && \end{array} \,.$$

By the commuting square in the middle and using again the tensor triangulated structure, this yields an inverse sequence under Y:

$$Y \simeq \mathbb{S} \wedge Y \longrightarrow \cdots \overset{p_3 \wedge \mathrm{id}}{\longrightarrow} \overline{E}_3 \wedge Y \overset{p_2 \wedge \mathrm{id}}{\longrightarrow} \overline{E}_2 \wedge Y \overset{p_1 \wedge \mathrm{id}}{\longrightarrow} \overline{E}_1 \wedge Y$$

The **E-nilpotent completion** $Y^\wedge_E$ of Y is the homotopy limit over the resulting inverse sequence

$$Y^\wedge_E \coloneqq \mathbb{R}\underset{\longleftarrow n}{\lim} \, \overline{E}_n \wedge Y$$

or rather the canonical morphism into it

$$Y \longrightarrow Y_E^\wedge .$$

Concretely, if

$$Y \simeq \mathbb{S} \wedge Y \longrightarrow \cdots \overset{p_3 \wedge \mathrm{id}}{\longrightarrow} \overline{E}_3 \wedge Y \overset{p_2 \wedge \mathrm{id}}{\longrightarrow} \overline{E}_2 \wedge Y \overset{p_1 \wedge \mathrm{id}}{\longrightarrow} \overline{E}_1 \wedge Y$$

is presented by a tower of fibrations between fibrant spectra in the model structure on topological sequential spectra, then $Y_E^\wedge$ is represented by the ordinary sequential limit over this tower.

(Bousfield 79, top, middle and bottom of page 272)

Remark 4.34. In (Bousfield 79) the E-nilpotent completion of X (def. 4.33) is denoted "$E^\wedge X$". The notation "$X_E^\wedge$" which we use here is more common among modern authors. It emphasizes the conceptual relation to p-adic completion $A_p^\wedge$ of abelian groups (def. 4.15) and is less likely to lead to confusion with the smash product of E with X.

Remark 4.35. The nilpotent completion $X_E^\wedge$ is E-local. This induces a universal morphism

$$L_E X \longrightarrow X_E^\wedge$$

from the E-Bousfield localization of spectra of X into the E-nilmpotent completion

(Bousfield 79, top of page 273)

We consider now conditions for this morphism to be an equivalence.

Proposition 4.36. *Let E be a connective ring spectrum such that the core of $\pi_0(E)$, def. 2.14, is either of*

- *the localization of the integers at a set J of primes, $c\pi_0(E) \simeq \mathbb{Z}[J^{-1}]$;*
- *a cyclic ring $c\pi_0(E) \simeq \mathbb{Z}/n\mathbb{Z}$, for $n \geq 2$.*

Then the map in remark 4.35 is an equivalence

$$L_E X \overset{\simeq}{\longrightarrow} X_E^\wedge .$$

(Bousfield 79, theorem 6.5, theorem 6.6).

Convergence theorems

We state the two main versions of Bousfield's convergence theorems for the E-Adams spectral sequence, below as theorem 4.40 and theorem 4.41.

First we need to define the concepts that enter the convergence statement:

1. the infinity-page $E_\infty^{s,t}(X,Y)$ (def. 4.37),
2. a filtration on $[X, Y_E^\wedge]_\bullet$ (def. 4.38)
3. what it means for the former to converge to the latter (def. 4.39).

Broadly the statement will be that typically

1. the E-Adams spectral sequence $E_r^{s,t}(X,Y)$ computes the stable homotopy groups $[X, Y_E^\wedge]$ of maps from X into the E-nilpotent completion of Y;
2. these groups are localizations of the full groups $[X,Y]_\bullet$ depending on the core of $\pi_0(E)$.

Literature: (Bousfield 79)

Definition 4.37. Let (E, μ, e) be a homotopy commutative ring spectrum (def.) and $X, Y \in \mathrm{Ho}(\mathrm{Spectra})$ two spectra with associated E-Adams spectral sequence $\{E_r^{s,t}, d_r\}$ (def. 1.14).

Observe that

$$\text{if } r > s \text{ then } \; E_{r+1}^{s,\bullet}(X,Y) \simeq \ker(d_r|_{E_r^{s,\bullet}(X,Y)}) \subset E_r^{s,\bullet}(X,Y)$$

since the differential d_r on the rth page has bidegree $(r, r-1)$, and since $E_r^{s<0,\bullet(X,Y)} \simeq 0$, so that for $r > s$ the image of d_r in $E_r^{s,t}(X,Y)$ vanishes.

Thus define the bigraded abelian group

$$E_\infty^{s,t}(X,Y) \coloneqq \lim_{r>s} E_r^{s,t}(X,Y) = \bigcap_{r>s} E_r^{s,t}(X,Y)$$

called the "**infinity page**" of the E-Adams spectral sequence.

Definition 4.38. Let (E,μ,e) be a homotopy commutative ring spectrum (def.) and $X, Y \in \mathrm{Ho}(\mathrm{Spectra})$ two spectra with associated E-Adams spectral sequence $\{E_r^{s,t}, d_r\}$ (def. 1.14) and E-nilpotent completion $Y_E^\wedge$ (def. 4.33).

Define a filtration

$$\cdots \hookrightarrow F^3[X,Y_E^\wedge]_\bullet \hookrightarrow F^2[X,Y_E^\wedge]_\bullet \hookrightarrow F^1[X,Y_E^\wedge]_\bullet = [X,Y_E^\wedge]_\bullet$$

on the graded abelian group $[X,Y_E^\wedge]_\bullet$ by

$$F^s[X,Y_E^\wedge]_\bullet \;\coloneqq\; \ker(\,[X,Y_E^\wedge]_\bullet \overset{[X, Y_E^\wedge \to \overline{E}_{s-1}\wedge Y]}{\longrightarrow} [X, \overline{E}_{s-1}\wedge Y]_\bullet\,),$$

where the morphisms $Y_E^\wedge \to \overline{E}_{s-1} \wedge Y$ is the canonical one from def. 4.33.

Definition 4.39. Let (E,μ,e) be a homotopy commutative ring spectrum (def.) and $X, Y \in \mathrm{Ho}(\mathrm{Spectra})$ two spectra with associated E-Adams spectral sequence $\{E_r^{s,t}, d_r\}$ (def. 1.14) and E-nilpotent completion $Y_E^\wedge$ (def. 4.33).

Say that the E-Adams spectral sequence $\{E_r^{s,t}, d_r\}$ **converges completely** to the E-nilpotent completion $[X,Y_E^\wedge]_\bullet$ if the following two canonical morphisms are isomorphisms

1. $[X,Y_E^\wedge]_\bullet \longrightarrow \underset{\longleftarrow s}{\lim} [X,Y_E^\wedge]_\bullet / F^s[X,Y_E^\wedge]_\bullet$

   (where on the right we have the limit over the tower of quotients by the stages of the filtration from def. 4.38)

2. $F^s[X,Y_E^\wedge]_{t-s}/F^{s+1}[X,Y^\wedge]_{t-s} \longrightarrow E_\infty^{s,t}(X,Y) \qquad \forall s,t$

   (where $F^s[X,Y_E^\wedge]_\bullet$ is the filtration stage from def. 4.38 and $E_\infty^{s,t}(X,Y)$ is the infinity-page from def. 4.37).

Notice that the first morphism is always surjective, while the second is necessarily injective, hence the condition is equivalently that the first is also injective, and the second also surjective.

(Bousfield 79, §6)

Now we state sufficient conditions for complete convergence of the E-Adams spectral sequence. It turns out that convergence is controled by the core (def. 2.14) of the ring $\pi_0(E)$. By prop. 2.16 these cores are either localizations of the integers $\mathbb{Z}[J^{-1}]$ at a set J of primes (def. 4.11) or are cyclic rings, or cores of products of these. We discuss the first two cases.

Theorem 4.40. *Let (E,μ,e) be a homotopy commutative ring spectrum (def.) and let $X, Y \in \mathrm{Ho}(\mathrm{Spectra})$ be two spectra such that*

1. *the core (def. 2.14) of the 0-th stable homotopy group ring of E (prop.) is the localization of the integers at a set J of primes (def. 4.11)*

   $$c\pi_0(E) \simeq \mathbb{Z}[J^{-1}] \subset \mathbb{Q}$$

2. *X is a CW-spectrum (def.) with a finite number of cells (rmk.);*

then the E-Adams spectral sequence for $[X,Y]_\bullet$ (def. 1.14) converges completely (def. 4.39) to the localization

$$[X,Y_E^\wedge]_\bullet = \mathbb{Z}[J^{-1}] \otimes [X,Y]_\bullet$$

of $[X,Y]_\bullet$.

(Bousfield 79, theorem 6.5)

Theorem 4.41. *Let (E,μ,e) be a homotopy commutative ring spectrum (def.) and let $X, Y \in \mathrm{Ho}(\mathrm{Spectra})$ be two spectra such that*

1. *the core (def. 2.14) of the 0-th stable homotopy group ring of E (prop.) is a prime field*

$c\pi_0(E) \simeq \mathbb{F}_p$

for some prime number p;

2. *Y is a connective spectrum in that its stable homotopy groups $\pi_\bullet(Y)$ vanish in negative degree;*

3. *X is a CW-spectrum (def.) with a finite number of cells (rmk.);*

4. *$[X,Y]_\bullet$ is degreewise a finitely generated group*

then the E-Adams spectral sequence for $[X,Y]_\bullet$ (def. 1.14) converges completely (def. 4.39) to the p-adic completion (def. 4.15)

$$[X, Y_E^\wedge]_\bullet \simeq \underset{\longleftarrow n}{\lim} [X,Y]_\bullet / p^n [X,Y]_\bullet$$

of $[X,Y]_\bullet$.

(Bousfield 79, theorem 6.6)

Examples

We now consider examples applying the general theory of E-Adams spectral sequences above in special cases to the concrete computation of certain stable homotopy groups.

Example 4.42. Examples of commutative ring spectra that are flat according to def. 2.1 include $E =$

- $\mathbb{S}$ – the sphere spectrum;
- $H\mathbb{F}_p$ – Eilenberg-MacLane spectra for prime fields;
- MO, MU, MSp – Thom spectra;
- KO, KU – topological K-theory spectra.

(Adams 69, lecture 1, lemma 28 (p. 45))

Proof of the first two items. For $E = \mathbb{S}$ we have $\mathbb{S}_\bullet(\mathbb{S}) \coloneqq \pi_\bullet(\mathbb{S} \wedge \mathbb{S}) \simeq \pi_\bullet(\mathbb{S})$, since the sphere spectrum $\mathbb{S}$ is the tensor unit for the derived smash product of spectra (cor.). Hence the statement follows since every ring is, clearly, flat over itself.

For $E = H\mathbb{F}_p$ we have that $\pi_\bullet(H\mathbb{F}_p) \simeq \mathbb{F}_p$ (prop.), hence a field (a prime field). Every module over a field is a projective module (prop.) and every projective module is flat (prop.). ∎

Example 4.43. Examples of ring spectra that are *not* flat in the sense of def. 2.1 include HZ, and MSU.

Examples 4.44.

- For $X = \mathbb{S}$ and $E = H\mathbb{F}_p$, then theorem 3.1 and theorem \ref{ConvergenceOfEAdamsSpectralSequenceToECompletion} with example \ref{ExamplesOfEnilpotentLocalizations} gives a spectral sequence

  $$\mathrm{Ext}_{\mathcal{A}_p^*}(\mathbb{F}_p, \mathbb{F}_p) \;\Rightarrow\; \pi_\bullet(\mathbb{S}) \otimes Z_p^\wedge \,.$$

  This is the *classical Adams spectral sequence*.

- For $X = \mathbb{S}$ and $E =$ MU, then theorem 3.1 and theorem \ref{ConvergenceOfEAdamsSpectralSequenceToECompletion} with example \ref{ExamplesOfEnilpotentLocalizations} gives a spectral sequence

  $$\mathrm{Ext}_{\mathrm{MU}_*(\mathrm{MU})}(\mathrm{MU}_*, \mathrm{MU}_*) \;\Rightarrow\; \pi_\bullet(\mathbb{S}) \,.$$

  This is the *Adams-Novikov spectral sequence*.

5. Classical Adams spectral sequence ($E = H\mathbb{F}_2$, $X = \mathbb{S}$)

We consider now the example of the E-Adams spectral sequence $\{E_r^{s,t}(X,Y), d_r\}$ (def. 1.14) for the case that

1. $E = H\mathbb{F}_p$ is the Eilenberg-MacLane spectrum (def.) with coefficients in a prime field, regarded in $\mathrm{Ho}(\mathrm{Spectra})$ with its canonical struture of a homotopy commutative ring spectrum induced (via this corollary) from its canonical structure of an orthogonal ring spectrum (from this def.);

2. $X = Y = \mathbb{S}$ are both the sphere spectrum.

This example is called the *classical Adams spectral sequence*.

The $H\mathbb{F}_p$-dual Steenrod algebra according to the general definition 2.3 turns out to be the classical dual Steenrod algebra $\mathcal{A}_p^*$ recalled below .

Notice that $H\mathbb{F}_2$ satisfies the two assumptions needed to identify the second page of the $H\mathbb{F}_p$-Adams spectral sequence according to theorem 3.1:

Lemma 5.1. *The Eilenberg-MacLane spectrum $H\mathbb{F}_p$ is flat according to 2.1, and $H\mathbb{F}_p(\mathbb{S})$ is a projective module over $\pi_\bullet(H\mathbb{F}_p)$.*

Proof. The stable homotopy groups of $H\mathbb{F}_p$ is the prime field $\mathbb{F}_p$ itself, regarded as a graded commutative ring concentrated in degree 0 (prop.)

$$\pi_\bullet(H\mathbb{F}_p) \simeq \mathbb{F}_p \,.$$

Since this is a field, all modules over it are projective modules (prop.), hence in particular flat modules (prop.). ∎

Corollary 5.2. *The classical Adams spectral sequence, i.e. the E-Adams spectral sequence (def. 1.14) for $E = H\mathbb{F}_p$ (def.) and $X = Y = \mathbb{S}$, has on its second page the Ext-groups of classical dual Steenrod algebra comodules from $\mathbb{F}_p \simeq H\mathbb{F}_p(\mathbb{S})$ to itself, and converges completely (def. 4.39) to the p-adic completion (def. 4.15) of the stable homotopy groups of spheres, hence in degree 0 to the p-adic integers and in all other degrees to the p-primary part (theorem 4.1)*

$$E_2^{s,t}(\mathbb{S},\mathbb{S}) \;=\; \mathrm{Ext}^{s,t}_{\mathcal{A}_p^*}(\mathbb{F}_p, \mathbb{F}_p) \;\Rightarrow\; (\pi_\bullet(\mathbb{S}))_p \,.$$

Proof. By lemma 5.1 the conditions of theorem 3.1 are satisfied, which implies the form of the second page.

For the convergence statement, we check the assumptions in theorem 4.41:

1. By prop. 2.15 and prop. 2.16 the ring $\mathbb{F}_p = \pi_0(H\mathbb{F}_p)$ coincides with its core: $c\mathbb{F}_p \simeq \mathbb{F}_p$;

2. $\mathbb{S}$ is clearly a connective spectrum;

3. $\mathbb{S}$ is clearly a finite CW-spectrum;

4. the groups $\pi_\bullet(\mathbb{S}) \simeq [\mathbb{S},\mathbb{S}]_\bullet$ are degreewise finitely generated, by *Serre's finiteness theorem?*.

Hence theorem 4.41 applies and gives the convergence as stated.

Finally, by prop. 5.5 the dual E-Steenrod algebra in the present case is the classical dual Steenrod algebra. ∎

We now use the classical Adams spectral sequence from corollary 5.2 to compute the first dozen stable homotopy groups of spheres.

The dual Steenrod algebra

Definition 5.3. Let p be a prime number. Write $\mathbb{F}_p$ for the corresponding prime field.

The **mod p-Steenrod algebra** $\mathcal{A}_p$ is the graded co-commutative Hopf algebra over $\mathbb{F}_p$ which is

- for $p = 2$ generated by elements denoted Sq^n for $n \in \mathbb{N}$, $n \geq 1$;

- for $p > 2$ generated by elements denoted β and P^n for $\in \mathbb{N}$, $n \geq 1$

(called the **Serre-Cartan basis elements**)

whose product is subject to the following relations (called the **Ádem relations**):

for $p = 2$:

for $0 < h < 2k$ the

$$\mathrm{Sq}^h \mathrm{Sq}^k \;=\; \sum_{i=0}^{[h/2]} \binom{k-i-1}{h-2i} \mathrm{Sq}^{h+k-i} \mathrm{Sq}^i ,$$

for $p > 2$:

for $0 < h < pk$ then

$$P^h P^k \;=\; \sum_{i=0}^{[h/p]} (-1)^{h+i} \binom{(p-1)(k-i)-1}{h-pi} P^{h+k-i} P^i$$

and if $0 < h < pk$ then

$$\begin{aligned} P^h \beta P^k = & \sum_{[h/p]}^{i=0} (-1)^{h+i} \binom{(p-1)(k-i)}{h-pi} \beta P^{h+k-i} P^i \\ & + \sum_{[(h-1)/p]}^{i=0} (-1)^{h+i-1} \binom{(p-1)(k-i)-1}{h-pi-1} P^{h+k-i} \beta P^i \end{aligned}$$

and whose coproduct Ψ is subject to the following relations:

for $p = 2$:

$$\Psi(\mathrm{Sq}^n) \;=\; \sum_{k=0}^{n} \mathrm{Sq}^k \otimes \mathrm{Sq}^{n-k}$$

for $p > 2$:

$$\Psi(P^n) \;=\; \sum_{n}^{k=0} P^k \otimes P^{n-k}$$

and

$$\Psi(\beta) = \beta \otimes 1 + 1 \otimes \beta \;.$$

e.g. (Kochmann 96, p. 52)

Definition 5.4. The $\mathbb{F}_p$-linear dual of the mod p-Steenrod algebra (def. 5.3) is itself naturally a graded commutative Hopf algebra (with coproduct the linear dual of the original product, and vice versa), called the **dual Steenrod algebra** $\mathbb{A}^*_{\mathbb{F}_p}$.

Proposition 5.5. *There is an isomorphism*

$$\mathcal{A}^*_p \simeq H_\bullet(H\mathbb{F}_p, \mathbb{F}_p) = \pi_\bullet(H\mathbb{F}_p \wedge H\mathbb{F}_p) \;.$$

(e.g. Ravenel 86, p. 49, Rognes 12, remark 7.24)

We now give the generators-and-relations description of the dual Steenrod algebra $\mathcal{A}^*_p$ from def. 5.4, in terms of linear duals of the generators for $\mathcal{A}_p$ itself, according to def. 5.3.

Theorem 5.6. ***(Milnor's theorem)***

The dual mod 2-Steenrod algebra $\mathcal{A}^*_2$ (def. 5.4) is, as an associative algebra, the free graded commutative algebra

$$\mathcal{A}^*_p \simeq \mathrm{Sym}_{\mathbb{F}_p}(\xi_1, \xi_2, \cdots,)$$

on generators:

- ξ_n, $n \geq 1$ *being the linear dual to* $\mathrm{Sq}^{p^{n-1}} \mathrm{Sq}^{p^{n-2}} \cdots \mathrm{Sq}^p \mathrm{Sq}^1$,

  of degree $2^n - 1$.

The dual mod p-Steenrod algebra $\mathcal{A}^*_p$ (def. 5.4) is, as an associative algebra, the free graded commutative algebra

$$\mathcal{A}^*_p \simeq \mathrm{Sym}_{\mathbb{F}_p}(\xi_1, \xi_2, \cdots, \tau_0, \tau_1, \cdots)$$

on generators:

- ξ_n, $n \geq 1$ *being the linear dual to* $P^{p^{n-1}} P^{p^{n-2}} \cdots P^p P^1$, *of degree* $2(p^n - 1)$.

- τ_n *being linear dual to* $P^{p^{n-1}} P^{p^{n-2}} \cdots P^p P^1 \beta$.

Moreover, the coproduct on $\mathcal{A}_p^*$ *is given on generators by*

$$\Psi(\xi_n) = \sum_{k=0}^{n} \xi_{n-k}^{p^k} \otimes \xi_k$$

and

$$\Psi(\tau_n) = \tau_n \otimes 1 + \sum_{k=0}^{n} \xi_{n-k}^{p^k} \xi_{n-k}^{p^k} \otimes \tau_k \,,$$

where we set $\xi_0 \coloneqq 1$.

(This defines the coproduct on the full algbra by it being an algebra homomorphism.)

This is due to (Milnor 58). See for instance (Kochmann 96, theorem 2.5.1, Ravenel 86, chapter III, theorem 3.1.1)

The cobar complex

In order to compute the second page of the classical $H\mathbb{F}_p$-Adams spectral sequence (cor. 5.2) we consider a suitable cochain complex whose cochain cohomology gives the relevant Ext-groups.

Definition 5.7. Let (Γ, A) be a graded commutative Hopf algebra, hence a commutative Hopf algebroid for which the left and right units coincide $\eta : A \longrightarrow \Gamma$ (remark 2.8).

Then the **unit coideal** of Γ is the cokernel

$$\overline{\Gamma} \coloneqq \mathrm{coker}(A \overset{\eta}{\longrightarrow} \Gamma) \,.$$

Remark. By co-unitality of graded commutative Hopf algebras (def. 2.9) $\epsilon \circ \eta = \mathrm{id}_A$ the defining projection of the unit coideal (def. 5.7)

$$A \overset{\eta}{\longrightarrow} \Gamma \longrightarrow \overline{\Gamma}$$

forms a split exact sequence which exhibits a direct sum decomposition

$$\Gamma \simeq A \oplus \overline{\Gamma} \,.$$

Lemma 5.8. *Let* (Γ, A) *be a commutative Hopf algebra, hence a commutative Hopf algebroid for which the left and right units coincide* $\eta : A \longrightarrow \Gamma$.

Then the unit coideal $\overline{\Gamma}$ *(def. 5.7) carries the structure of an* A*-bimodule such that the projection morphism*

$$\Gamma \longrightarrow \overline{\Gamma}$$

is an A*-bimodule homomorphism. Moreover, the coproduct* $\Psi : \Gamma \longrightarrow \Gamma \otimes_A \Gamma$ *descends to a morphism* $\overline{\Gamma} : \overline{\Gamma} \longrightarrow \overline{\Gamma} \otimes_A \overline{\Gamma}$ *such that the projection intertwines the two coproducts.*

Proof. For the first statement, consider the commuting diagram

$$\begin{array}{ccccc} A \otimes A & \overset{A \otimes \eta}{\longrightarrow} & A \otimes \Gamma & \longrightarrow & A \otimes \overline{\Gamma} \\ \downarrow && \downarrow && \downarrow^{\exists} \\ A & \underset{\eta}{\longrightarrow} & \Gamma & \longrightarrow & \overline{\Gamma} \end{array} \,,$$

where the left commuting square exhibits the fact that η is a homomorphism of left A-modules.

Since the tensor product of abelian groups $\otimes$ is a right exact functor it preserves cokernels, hence $A \otimes \overline{\Gamma}$ is the cokernel of $A \otimes A \to A \otimes \Gamma$ and hence the right vertical morphisms exists by the universal property of cokernels. This is the compatible left module structure on $\overline{\Gamma}$. Similarly the right A-module structure is obtained.

For the second statement, consider the commuting diagram

$$
\begin{array}{ccccc}
A & \overset{\eta}{\longrightarrow} & \Gamma & \longrightarrow & \overline{\Gamma} \\
{}^{\eta}\downarrow & & \downarrow^{\Psi} & & \downarrow^{\exists} \\
\Gamma \simeq \Gamma \otimes_A A & \underset{\mathrm{id} \otimes_A \eta}{\longrightarrow} & \Gamma \otimes_A \Gamma & \longrightarrow & \overline{\Gamma} \otimes_A \overline{\Gamma}
\end{array}
\,.
$$

Here the left square commutes by one of the co-unitality conditions on (Γ, A), equivalently this is the co-action property of A regarded canonically as a Γ-comodule.

Since also the bottom morphism factors through zero, the universal property of the cokernel $\overline{\Gamma}$ implies the existence of the right vertical morphism as shown. ▮

Definition 5.9. **(cobar complex)**

Let (Γ, A) be a commutative Hopf algebra, hence a commutative Hopf algebroid for which the left and right units coincide $A \overset{\eta}{\longrightarrow} \Gamma$. Let N be a left Γ-comodule.

The **cobar complex** $C^\bullet_\Gamma(N)$ is the cochain complex of abelian groups with terms

$$
C^s_\Gamma(N) \coloneqq \underbrace{\overline{\Gamma} \otimes_A \cdots \otimes_A \overline{\Gamma}}_{s \text{ factors}} \otimes_A N
$$

(for $\overline{\Gamma}$ the unit coideal of def. 5.7, with its A-bimodule structure via lemma 5.8)

and with differentials $d_s \colon C^s_\Gamma(N) \longrightarrow C^{s+1}_\Gamma(N)$ given by the alternating sum of the coproducts via lemma 5.8.

(Ravenel 86, def. A1.2.11)

Proposition 5.10. *Let (Γ, A) be a commutative Hopf algebra, hence a commutative Hopf algebroid for which the left and right units coincide $A \overset{\eta}{\longrightarrow} \Gamma$. Let N be a left Γ-comodule.*

Then the cochain cohomology of the cobar complex $C^\bullet_\Gamma(N)$ (def. 5.9) is the Ext-groups of comodules from A (regarded as a left comodule via def. 2.20) into N

$$
H^\bullet(C^\bullet_\Gamma(N)) \;\simeq\; \mathrm{Ext}^\bullet_\Gamma(A, N) \,.
$$

(Ravenel 86, cor. A1.2.12, Kochman 96, prop. 5.2.1)

Proof idea. One first shows that there is a resolution of N by co-free comodules given by the complex

$$
D^\bullet_\Gamma(N) \coloneqq \Gamma \otimes_A \overline{\Gamma}^{\otimes_A^\bullet} \otimes_A N
$$

with differentials given by the alternating sum of the coproducts. This is called the *cobar resolution* of N.

To see that this is indeed a resolution, one observes that a contracting homotopy is given by

$$
s(\gamma\gamma_1 | \cdots | \gamma_s n) \coloneqq \epsilon(\gamma)\gamma_1 | \cdots | \gamma_s n
$$

for $s > 0$ and

$$
s(\gamma n) \coloneqq 0 \,.
$$

Now from lemma 3.5, in view of remark , and since A is trivially projective over itself, it follows that this is an F-acyclic resolution for $F \coloneqq \mathrm{Hom}_\Gamma(A, -)$.

This means that the resolution serves to compute the Ext-functor in question and we get

$$
\begin{aligned}
\mathrm{Ext}^\bullet_\Gamma(A, N) & \simeq H^\bullet(\mathrm{Hom}_\Gamma(A, D^\bullet_\Gamma(N))) \\
& = H^\bullet(\mathrm{Hom}_\Gamma(A, \Gamma \otimes_A \overline{\Gamma}^{\otimes_A^\bullet} \otimes_A N)) \\
& \simeq H^\bullet(\mathrm{Hom}_A(A, \overline{\Gamma}^{\otimes_A^\bullet} \otimes_A N)) \\
& \simeq H^\bullet(\overline{\Gamma}^{\otimes_A^\bullet} \otimes_A N) \,,
\end{aligned}
\,,
$$

where the second-but-last equivalence is the isomorphism of the co-free/forgetful adjunction

$$
A\,\mathrm{Mod} \underoverset{\underset{\mathrm{co-free}}{\longrightarrow}}{\overset{\mathrm{forget}}{\longleftarrow}}{\bot} \Gamma\,\mathrm{CoMod}
$$

from prop. 2.23, while the last equivalence is the isomorphism of the free/forgetful adjunction

$$
A\,\mathrm{Mod} \underoverset{\underset{\mathrm{forget}}{\longrightarrow}}{\overset{\mathrm{free}}{\longleftarrow}}{\bot} \mathrm{Ab}
$$

■

The May spectral sequence

The cobar complex (def. 5.9) realizes the second page of the classical Adams spectral sequence (cor. 5.2) as the cochain cohomology of a cochain complex. This is still hard to compute directly, but we now discuss that this cochain complex admits a filtration so that the induced spectral sequence of a filtered complex is computable and has trivial extension problem (rmk.). This is called the *May spectral sequence*.

We obtain this spectral sequence in prop. 5.16 below. First we need to consider some prerequisites.

Lemma 5.11. *Let (Γ, A) be a graded commutative Hopf algebra, i.e. a graded commutative Hopf algebroid with left and right unit coinciding for which the underlying A-algebra of Γ is a free graded commutative A-algebra on a set of generators $\{x_i\}_{i \in I}$*

such that

1. *all generators x_i are primitive elements;*
2. *A is in degree 0;*
3. *$(i < j) \Rightarrow (\deg(x_i) \leq \deg(x_j))$;*
4. *there are only finitely many x_i in a given degree,*

then the Ext of Γ-comodules from A to itself is the free graded commutative algebra on these generators

$$\mathrm{Ext}_\Gamma(A, A) \simeq A[\{x_i\}_{i \in I}] \,.$$

(Ravenel 86, lemma 3.1.9, Kochman 96, prop. 3.7.5)

Proof. Consider the co-free left Γ-comodule (prop.)

$$\Gamma \otimes_A A[\{y_i\}_{i \in I}]$$

and regard it as a chain complex of left comodules by defining a differential via

$$\begin{aligned} d &: x_i \mapsto y_i \\ d &: y_i \mapsto 0 \end{aligned}$$

and extending as a graded derivation.

We claim that d is a homomorphism of left comodules: Due to the assumption that all the x_i are primitive we have on generators that

$$\begin{aligned} (\mathrm{id}, d)(\Psi(x_i)) &= (\mathrm{id}, d)(x_i \otimes 1 + 1 \otimes x_i) \\ &= x_i \otimes \underbrace{(d1)}_{=0} + 1 \otimes \underbrace{(dx_i)}_{=y_i} \\ &= \Psi(dx_i) \end{aligned}$$

and

$$\begin{aligned} (\mathrm{id}, d)(\Psi(y_i)) &= (\mathrm{id}, d)(1, y_i) \\ &= (1, dy_i) \\ &= 0 \\ &= \Psi(0) \\ &= \Psi(dy_i) \end{aligned} \,.$$

Since d is a graded derivation on a free graded commutative algbra, and Ψ is an algebra homomorphism, this implies the statement for all other elements.

Now observe that the canonical chain map

$$(\Gamma \otimes_A A[\{y_i\}_{i \in I}],\ d) \overset{\simeq_{\mathrm{qi}}}{\longrightarrow} A$$

(which projects out the generators x_i and y_i and is the identity on A), is a quasi-isomorphism, by construction. Therefore it constitutes a co-free resolution of A in left Γ-comodules.

Since the counit η is assumed to be flat, and since $A[\{y_i\}_{i \in I}]$ is degreewise a free module over A, hence in particular a projective module, prop. 3.5 says that the above is an acyclic resolution with respect to the functor $\mathrm{Hom}_\Gamma(A,-) : \Gamma\,\mathrm{CoMod} \longrightarrow A\,\mathrm{Mod}$. Therefore it computes the Ext-functor. Using that forming co-free comodules is right adjoint to forgetting Γ-comodule structure over A (prop. 2.23), this yields:

$$\begin{aligned} \mathrm{Ext}^\bullet_\Gamma(A,A) & \simeq H_\bullet(\mathrm{Hom}_\Gamma(A, \Gamma \otimes_A A[\{y_i\}_{i\in I}]), d) \\ & \simeq H_\bullet(\mathrm{Hom}_A(A, A[\{y_i\}_{i\in I}]), d=0) \\ & \simeq \mathrm{Hom}_A(A, A[\{y_i\}_{i\in I}]) \\ & \simeq A[\{x_i\}_{i\in I}] \end{aligned} .$$

∎

Lemma 5.12. *If (Γ, A) as above is equipped with a filtering, then there is a spectral sequence*

$$\mathcal{E}_1 \;=\; \mathrm{Ext}_{\mathrm{gr}_\bullet \Gamma}(\mathrm{gr}_\bullet A, \mathrm{gr}_\bullet A) \;\Rightarrow\; \mathrm{Ext}_\Gamma(A,A)$$

converging to the Ext over Γ from A to itself, whose first page is the Ext over the associated graded Hopf algebra $\mathrm{gr}_\bullet \Gamma$.

(Ravenel 86, lemma 3.1.9, Kochman 96, prop. 3.7.5)

Proof. The filtering induces a filtering on the cobar complex (def. 5.9) which computes Ext_Γ (prop. 5.10). The spectral sequence in question is the corresponding spectral sequence of a filtered complex. Its first page is the homology of the associated graded complex (by this prop.), which hence is the homology of the cobar complex (def. 5.9) of the associated graded Hopf algebra $\mathrm{gr}_\bullet \Gamma$. By prop. 5.10 this is the Ext-groups as shown. ∎

Let now $A \coloneqq \mathbb{F}_2$, $\Gamma \coloneqq \mathcal{A}_2^\bullet$ be the mod 2 dual Steenrod algebra. By Milnor's theorem (prop. 5.6), as an $\mathbb{F}_2$-algebra this is

$$\mathcal{A}_2^\bullet = \mathrm{Sym}_{\mathbb{F}_2}(\xi_1, \xi_2, \cdots) \;.$$

and the coproduct is given by

$$\Psi(\xi_n) = \sum_{k=0}^{i} \xi_{i-k}^{2^k} \otimes \xi_k \,,$$

where we set $\xi_0 \coloneqq 1$.

Definition 5.13. Introduce new generators

$$h_{i,n} \coloneqq \begin{cases} \xi_i^{2^n} & \text{for}\ i \geq 1,\, k \geq 0 \\ 1 & \text{for}\ i = 0 \end{cases}$$

Remark 5.14. By binary expansion of powers, there is a unique way to express every monomial in $\mathbb{F}_2[\xi_1, \xi_2, \cdots]$ as a product of the new generators in def. 5.13 such that each such element appears at most once in the product. E.g.

$$\begin{aligned} \xi_i^5 \xi_j^7 & = \xi_i^{2^0+2^2} \xi_j^{2^0+2^1+2^2} \\ & = h_{i,0} h_{i,1} h_{j,0} h_{j,1} h_{j,2} \end{aligned} .$$

Proposition 5.15. *In terms of the generators $\{h_{i,n}\}$ from def. 5.13, the coproduct on the dual Steenrod algebra $\mathcal{A}_2^\ast$ takes the following simple form*

$$\Psi(h_{i,n}) \;=\; \sum_{k=0}^{i} h_{i-k,n+k} \otimes h_{k,n} \;.$$

Proof. Using that the coproduct of a bialgebra is a homomorphism for the algebra structure and using freshman's dream arithmetic over $\mathbb{F}_2$, one computes:

$$
\begin{aligned}
\Psi(h_{i,n}) &= \Psi\left(\xi_i^{2^n}\right) \\
&= (\Psi(\xi_i))^{2^n} \\
&= \left(\sum_{k=0}^i \xi_{i-k}^{2^k} \otimes \xi_k\right)^{2^n} \\
&= \sum_{k=0}^i \left(\xi_{i-k}^{2^k}\right)^{2^n} \otimes \xi_k^{2^n} \\
&= \sum_{k=0}^i \xi_{i-k}^{2^k \cdot 2^n} \otimes \xi_k^{2^n} \\
&= \sum_{k=0}^i \xi_{i-k}^{2^{(k+n)}} \otimes \xi_k^{2^n} \\
&= \sum_{k=0}^i h_{i-k,n+k} \otimes h_{k,n}
\end{aligned}
.
$$

∎

Proposition 5.16. *There exists a converging spectral sequence of graded $\mathbb{F}_2$-vector spaces of the form*

$$
E_1^{s,t,p} = \mathbb{F}_2[\{h_{i,n}\}_{\substack{i \geq 1, \\ n \geq 0}}] \;\Rightarrow\; \mathrm{Ext}^{s,t}_{\mathcal{A}_2^*}(\mathbb{F}_2, \mathbb{F}_2)\,,
$$

*called the **May spectral sequence** (where s and t are from the bigrading of the spectral sequence itself, while the index p is that of the graded $\mathbb{F}_2$-vector spaces), with*

1. *$h_{i,n} \in E_1^{1, 2^{2^{i+n}-2^n}-1, 2i-1}$*
2. *first differential given by*

$$
d_1(h_{i,n}) = \sum_{k=0}^i h_{i-k,n+k} \otimes h_{k,n}\,;
$$

3. *higher differentials of the form*

$$
d_r \,:\, E_r^{s,t,p} \longrightarrow E_r^{s+1,t-1,p-2r+1}\,,
$$

   where the filtration is by maximal degree.

Notice that since everything is $\mathbb{F}_2$-linear, the extension problem of this spectral sequence is trivial.

(Kochman 96, prop. 5.3.1)

Proof. Define a grading on the dual Steenrod algebra $\mathcal{A}_2^\bullet$ (theorem 5.6) by taking the degree of the generators from def.5.13 to be (this idea is due to (Ravenel 86, p.69))

$$
|h_{i,n}| \coloneqq 2i - 1
$$

and extending this additively to monomials, via the unique decomposition of remark 5.14.

For example

$$
\begin{aligned}
|\xi_i^5 \xi_j^7| &= |h_{i,0} h_{i,1} h_{j,0} h_{j,1} h_{j,2}| \\
&= 2(2i-1) + 3(2j-1)
\end{aligned}
.
$$

Consider the corresponding increasing filtration

$$
\cdots \subset F_p \mathcal{A}_2^* \subset F_{p+1}\mathcal{A}_2^* \subset \cdots \subset \mathcal{A}_2^*
$$

with filtering stage p containing all elements of total degree $\leq p$.

Observe via prop. 5.15 that

$$
\Psi(h_{i,n}) = \underbrace{h_{i,n} \otimes 1}_{\deg = 2i-1} + \sum_{0<k<i} \underbrace{h_{i-k,n+k} \otimes h_{k,n}}_{\deg = 2i-2} + \underbrace{1 \otimes h_{i,n}}_{\deg = 2i-1}\,.
$$

This means that after projection to the associated graded Hopf algebra

$$
F_\bullet \mathcal{A}_2^* \longrightarrow \mathrm{gr}_\bullet \mathcal{A}_2^* \coloneqq F_\bullet(\mathcal{A}_2^*)/F_{\bullet-1}(\mathcal{A}_2^*)
$$

all the generators $h_{i,n}$ become primitive elements:

$$
\Psi(h_{i,n}) = h_{i,n} \otimes 1 + 1 \otimes h_{i,n} \qquad \in \mathrm{gr}_\bullet \mathcal{A}_2^* \otimes \mathrm{gr}_\bullet \mathcal{A}_2^*\,.
$$

Hence lemma 5.11 applies and says that the Ext from $\mathbb{F}_2$ to itself over the associated graded Hopf algebra is the polynomial algebra in these generators:

$$\mathrm{Ext}_{\mathrm{gr}_\bullet \mathcal{A}_2^*}(\mathbb{F}_2, \mathbb{F}_2) \simeq \mathbb{F}_2[\{h_{i,n}\}_{\substack{i \geq 1, \\ n \geq 0}}] \,.$$

Moreover, lemma 5.12 says that this is the first page of a spectral sequence that converges to the Ext over the original Hopf algebra:

$$\mathcal{E}_1 = \mathbb{F}_2[\{h_{i,n}\}_{\substack{i \geq 1 \\ n \geq 0}}] \;\Rightarrow\; \mathrm{Ext}_{\mathcal{A}_2^*}(\mathbb{F}_2, \mathbb{F}_2) \,.$$

Moreover, again by lemma 5.12, the differentials on any r-page are the restriction of the differentials of the bar complex to the r-almost cycles (prop.). Now the differential of the cobar complex is the alternating sum of the coproduct on $\mathcal{A}_2^*$, hence by prop. 5.15 this is:

$$d_1(h_{i,n}) = \sum_{k=0}^{i} h_{i-k, n+k} \otimes h_{k,n} \,.$$

∎

The second page

Now we use the May spectral sequence (prop. 5.16) to compute the second page and in fact the stable page of the classical Adams spectral sequence (cor. 5.2) in low internal degrees $t - s$.

Lemma 5.17. ***(terms on the second page of May spectral sequence)***

In the range $t - s \leq 13$, the second page of the May spectral sequence for $\mathrm{Ext}_{\mathbb{A}^*_{\mathbb{F}_2}}(\mathbb{F}_2, \mathbb{F}_2)$ has as generators all the

- h_n
- $b_{i,n} \coloneqq (h_{i,n})^2$

as well as the element

- $x_7 \coloneqq h_{2,0} h_{2,1} + h_{1,1} h_{3,0}$

subject to the relations

- $h_n h_{n+1} = 0$
- $h_2 b_{2,0} = h_0 x_7$
- $h_2 x_7 = h_0 b_{2,1}$.

e.g. (Ravenel 86, lemma 3.2.8 and lemma 3.2.10, Kochman 96, lemma 5.3.2)

Proof. Remember that the differential in the cobar complex (def. 5.9) lands not in $\Gamma = \mathcal{A}_2^*$ itself, but in the unit coideal $\overline{\Gamma} \coloneqq \mathrm{coker}(\eta)$ where the generator $h_{0,n} = \xi_0 = 1$ disappears.

Using this we find for the differential d_1 of the generators in low degree on the first page of the May spectral sequence (prop. 5.16) via the formula for the differential from prop. 5.15, the following expressions:

$$\begin{aligned} d_1(h_n) & \coloneqq d_1(h_{1,n}) \\ & = \overline{\Psi}(h_{1,n}) \\ & = h_{1,n} \otimes \underbrace{h_{0,n}}_{=0} + \underbrace{h_{0,n+1}}_{=0} \otimes h_{1,n} \\ & = 0 \end{aligned}$$

and hence all the elements h_n are cocycles on the first page of the May spectral sequence.

Also, since d_1 is a derivation (by definition of the cobar complex, def. 5.9) and since the product of the image of the cobar complex in the first page of the May spectral sequence is graded commutative, we have for all n, k that

$$\begin{aligned} d_1(h_{n,k})^2 & = 2 h_{n,k}(d_1(h_{n,k})) \\ & = 0 \end{aligned}$$

(since $2 = 0 \bmod 2$).

Similarly we compute d_1 on the other generators. These terms do not vanish, but so they impose relations on products in the cobar complex:

$$
\begin{aligned}
d_1(h_{2,0}) &= h_{1,1} \otimes h_{1,0} \\
d_1(h_{2,1}) &= h_{1,2} \otimes h_{1,1} \\
d_1(h_{2,2}) &= h_{1,3} \otimes h_{1,2} \\
d_1(h_{2,3}) &= h_{1,4} \otimes h_{1,3} \\
d_1(h_{3,0}) &= h_{2,1} \otimes h_{1,0} + h_{1,2} \otimes h_{2,0}
\end{aligned}
$$

This shows that $h_n h_{n+1} = 0$ in the given range.

The remaining statements follow similarly. ∎

Remark 5.18. With lemma 5.17, so far we see the following picture in low degrees.

$$
\begin{array}{cccccc}
\vdots & \vdots & & & & \\
3 & h_0^4 & & & h_1^3,\; h_0^2 h_2 & \\
2 & h_0^2 & & h_1^2 & h_0 h_2 & \\
1 & h_0 & h_1 & & h_2 & \\
 & 0 & 1 & 2 & 3 & 4
\end{array}
$$

Here the relation

$$h_0 \otimes h_1 = 0$$

removes a vertical tower of elements above h_1.

So far there are two different terms in degree $(s, t-s) = (3,3)$. The next lemma shows that these become identified on the next page.

Lemma 5.19. ***(differentials on the second page of the May spectral sequence)***

The differentials on the second page of the May spectral sequence (prop. 5.16) relevant for internal degrees $t - s \leq 12$ are

1. $d_2(h_n) = 0$

2. $d_2(b_{2,n}) = h_n^2 h_{n+2} + h_{n+1}^3$

3. $d_2(x_7) = h_0 h_2^2$

4. $d_2(b_{3,0}) = h_1 b_{2,1} + h_3 b_{2,0}$

(Kochman 96, lemma 5.3.3)

Proof. The first point follows as before in lemma 5.17, in fact the h_n are infinite cycles in the May spectral sequence.

We spell out the computation for the second item:

We may represent $b_{2,k}$ by $\xi_2^{2^k} \times \xi_2^{2^k}$ plus terms of lower degree. Choose the representative

$$B_{2,k} = \xi_2^{2^k} \otimes \xi_2^{2^k} \;+\; \xi_1^{2^{k+1}} \otimes \xi_1^{2^k} \xi_2^{2^k} \;+\; \xi_1^{2^{k+1}} \xi^{2^k} \otimes \xi_1^{2^k}\,.$$

Then we compute $d B_{2,k}$, using the definition of the cobar complex (def. 5.9), the value of the coproduct on dual generators (theorem 5.6), remembering that the coproduct Ψ on a Hopf algebra is a homomorphism for the underlying commutative ring, and using freshman's dream arithmetic to evaluate prime-2 powers of sums. For the three summands we obtain

$$
\begin{aligned}
d(\xi_2^{2^k} \otimes \xi_2^{2^k}) &= \overline{\Psi}(\xi_2^{2^k}) \otimes \xi_2^{2^k} + \xi_2^{2^k} \otimes \overline{\Psi}(\xi_2^{2k}) \\
&= \underbrace{\xi_1^{2^{k+1}} \otimes \xi_1^{2^k} \otimes \xi_2^{2^k}}_{c_1} \;+\; \underbrace{\xi_2^{2^k} \otimes \xi_1^{2^{k+1}} \otimes \xi_1^{2^k}}_{c_2}
\end{aligned}
$$

and

$$
\begin{aligned}
d(\xi_1^{2^{k+1}} \otimes \xi_1^{2^k}\xi_2^{2^k}) &= \xi_1^{2^k} \otimes \overline{\Psi}(\xi_1^{2^k}\xi_2^{2^k}) \\
&= \xi_1^{2^{k+1}} \otimes \left(\xi_1^{2^k} \otimes 1 + 1 \otimes \xi_1^{2^k}\right)\left(\xi_2^{2^k} \otimes 1 + \xi_1^{2^{k+1}} \otimes \xi_1^{2^k} + 1 \otimes \xi_2^{2^k}\right) \\
&= \underbrace{\xi_1^{2^{k+1}} \otimes \xi_1^{2^{k+1}+2^k} \otimes \xi_1^{2^k}}_{b} + \underbrace{\xi_1^{2^{k+1}} \otimes \xi_1^{2^k} \otimes \xi_2^{2^k}}_{c_1} + \underbrace{\xi_1^{2^{k+1}} \otimes \xi_2^{2^k} \otimes \xi_1^{2^k}}_{a} + \xi_1^{2^{k+1}} \otimes \xi_1^{2^{k+1}} \otimes \xi_1^{2^{k+1}}
\end{aligned}
$$

and

$$
\begin{aligned}
d(\xi_1^{2^{k+1}}\xi^{2^k} \otimes \xi_1^{2^k}) &= \overline{\Psi}(\xi_1^{2^{k+1}}\xi^{2^k}) \otimes \xi_1^{2^k} \\
&= \left(\xi_1^{2^{k+1}} \otimes 1 + 1 \otimes \xi_1^{2^{k+1}}\right)\left(\xi_2^{2^k} \otimes 1 + \xi_1^{2^{k+1}} \otimes \xi_1^{2^k} + 1 \otimes \xi_2^{2^k}\right) \otimes \xi_1^{2^k} \\
&= \xi_1^{2^{k+2}} \otimes \xi_1^{2^k} \otimes \xi_1^{2^k} + \underbrace{\xi_1^{2^{k+1}} \otimes \xi_2^{2^k} \otimes \xi_1^{2^k}}_{a} + \underbrace{\xi_2^{2^k} \otimes \xi_1^{2^{k+1}} \otimes \xi_1^{2^k}}_{c_2} + \underbrace{\xi_1^{2^{k+1}} \otimes \xi_1^{2^{k+1}+2^k} \otimes \xi_1^{2^k}}_{b}
\end{aligned}
.
$$

The labeled summands appear twice in $dB_{2,k}$ hence vanish (mod 2). The remaining terms are

$$
dB_{2,k} = \xi_1^{2^{k+1}} \otimes \xi_1^{2^{k+1}} \otimes \xi_1^{2^{k+1}} + \xi_1^{2^{k+2}} \otimes \xi_1^{2^k} \otimes \xi_1^{2^k}
$$

and these indeed represent the claimed elements. ∎

Remark 5.20. With lemma 5.19 the picture from remark 5.18 is further refined:

For $k = 0$ the differentia $d_2(b_{2,n}) = h_n^2 h_{n+2} + h_{n+1}^3$ means that on the third page of the May spectral sequence there is an identification

$$
h_1^3 = h_0^2 h_2 \;.
$$

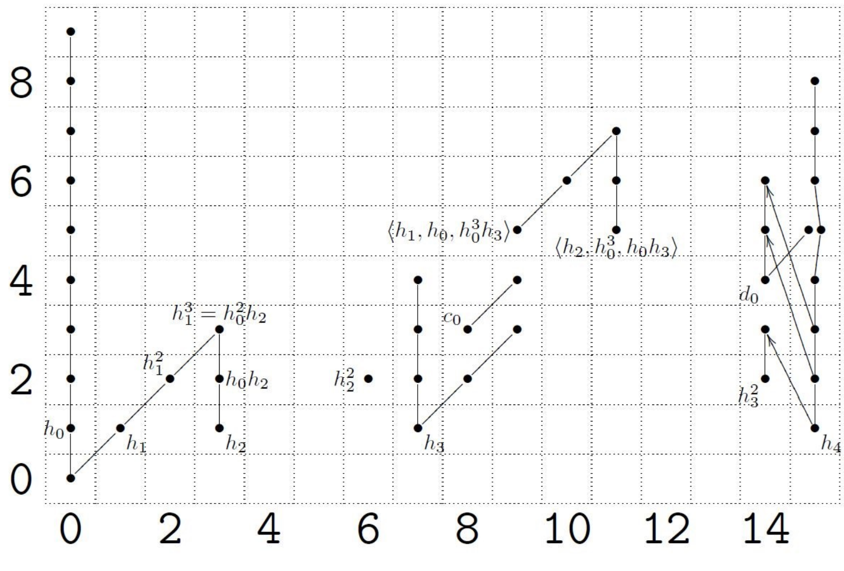
Hence where on page two we saw two distinct elements in bidegree $(s, t-s) = (3,3)$, on the next page these merge:

$$
\begin{array}{cccccc}
\vdots & \vdots & & & & \\
3 & h_0^4 & & & h_1^3 = h_0^2 h_2 & \\
2 & h_0^2 & & h_1^2 & h_0 h_2 & \\
1 & h_0 & h_1 & & h_2 & \\
 & 0 & 1 & 2 & 3 & 4
\end{array}
$$

Proceeding in this fashion, one keeps going until the 4-page of the May spectral sequence (Kochman 96, lemma 5.3.5). Inspection of degrees shows that this is sufficient, and one obtains:

Theorem 5.21. ***(stable page of classical Adams spectral sequence)***

In internal degree $t - s \leq 12$ the infinity page (def. 4.37) of the classical Adams spectral sequence (cor. 5.2) is spanned by the items in the following table



Here every dot is a generator for a copy of $\mathbb{Z}/2\mathbb{Z}$. Vertical edges denote multiplication with h_0 and diagonal edges denotes multiplication with h_1.

e.g. (Ravenel 86, theorem 3.2.11, Kochman 96, prop. 5.3.6), graphics taken from (Schwede 12))

The first dozen stable stems

Theorem 5.21 gives the stable page of the classical Adams spectral sequence in low degree. By corollary 5.2 and def. 4.39 we have that a vertical sequence of dots encodes an 2-primary part of the stable homotopy groups of spheres according to the graphical calculus of remark 4.6 (the rules for determining group extensions there is just the solution to the extension problem (rmk.) in view of def. 4.39):

$k =$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**
$\pi_k(\mathbb{S} \otimes \mathbb{Z}_{(2)}) =$	$\mathbb{Z}_{(2)}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/8$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/16$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^3$	$\mathbb{Z}/2$	$\mathbb{Z}/8$	0	0

The full answer in this range turns out to be this:

$k =$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	$\cdots$
$\pi_k(\mathbb{S}) =$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/240$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^3$	$\mathbb{Z}/6$	$\mathbb{Z}/504$	0	$\mathbb{Z}/3$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/480 \oplus \mathbb{Z}/2$	$\cdots$

And expanding the range yields this :

stable homotopy groups of spheres at 2

(graphics taken from Hatcher's website)

6. The case $E = H\mathbb{F}_p$ and $X = MU$

used to compute the stable homotopy groups of the complex Thom spectrum MU from the homology of MU

(hence, by Thom's theorem, equivalently the complex cobordism ring $\Omega^U_\bullet \simeq \pi_\bullet U$), see at *Seminar session: Milnor-Quillen theorem on MU*)

This is the *Milnor-Quillen theorem on MU*, see at *Seminar session: Milnor-Quillen theorem on MU*

(Adams 74, part II, around section 8, Lurie 10, around lecture 9)

7. Adams-Novikov spectral sequence ($E = MU$, $X = \mathbb{S}$)

this is the classical *Adams-Novikov spectral sequence* , converges faster than the classical choice $E = H\mathbb{F}_p$ to the stable homotopy groups of spheres, (…)

(Kochman 96, section 5)

---

8. References

For the general theory we follow the original

- John Frank Adams, section 2 of *Lectures on generalised cohomology*, in Peter Hilton (ed.) *Category Theory, Homology Theory and Their Applications III*, volume 99 of Lecture Notes in Mathematics (1969), Springer-Verlag Berlin-Heidelberg-New York.

- Frank Adams, section III.15 of *Stable homotopy and generalized homology*, Chicago Lectures in mathematics, 1974

- Aldridge Bousfield, sections 5 and 6 of *The localization of spectra with respect to homology*, Topology 18 (1979), no. 4, 257–281. (pdf)

For the homological algebra of comodules over Hopf algebroids we follow appendix A of

- Doug Ravenel, *Complex cobordism and stable homotopy groups of spheres*, 1986/2003

For the special case of the classical Adams spectral sequence and of the Adams-Novikov spectral sequence we follow

- Stanley Kochman, chapter 5 of *Bordism, Stable Homotopy and Adams Spectral Sequences*, AMS 1996

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