nLab

Introduction to Homotopy Theory

This pages gives a detailed introduction to classical *homotopy theory*, starting with the concept of homotopy in topological spaces and motivating from this the "abstract homotopy theory" in general model categories.

For background on basic topology see at *Introduction to Topology*.

For application to homological algebra see at *Introduction to Homological algebra*.

For application to stable homotopy theory see at *Introduction to Stable homotopy theory*.

Contents

Context

Homotopy theory

While the field of algebraic topology clearly originates in topology, it is not actually interested in topological spaces regarded up to topological isomorphism, namely homeomorphism ("point-set topology"), but only in topological spaces regarded up to weak homotopy equivalence – hence it is interested only in the “weak homotopy types” of topological spaces. This is so notably because ordinary cohomology groups are invariants of the (weak) homotopy type of topological spaces but do not detect their homeomorphism class.

The category of topological spaces obtained by forcing weak homotopy equivalences to become isomorphisms is the “classical homotopy category” Ho(Top). This homotopy category however has forgotten a little too much information: homotopy theory really wants the weak homotopy equivalences not to become plain isomorphisms, but to become actual homotopy equivalences. The structure that reflects this is called a *model category* structure (short for “category of models for homotopy types”). For classical homotopy theory this is accordingly called the *classical model structure on topological spaces*. This we review here.

1. Topological homotopy theory

This section recalls relevant concepts from actual topology (“point-set topology”) and highlights facts that motivate the axiomatics of model categories below. We prove two technical lemmas (lemma 1.40 and lemma 1.52) that serve to establish the abstract homotopy theory of topological spaces further below.

Literature (Hirschhorn 15)

Throughout, let *Top* denote the category whose objects are topological spaces and whose morphisms are continuous functions between them. Its isomorphisms are the homeomorphisms.

(Further below we restrict attention to the full subcategory of compactly generated topological spaces.)

Universal constructions

To begin with, we recall some basics on universal constructions in Top: limits and colimits of diagrams of topological spaces; exponential objects.

Generally, recall:

Definition 1.1. A **diagram** in a category $\mathcal{C}$ is a small category I and a functor

$$\begin{aligned} X_\bullet \;:\; I &\longrightarrow \mathcal{C} \\ (i \overset{\phi}{\longrightarrow} j) &\mapsto (X_i \overset{X(\phi)}{\longrightarrow} X_j) \;. \end{aligned}$$

A **cone** over this diagram is an object Q equipped with morphisms $p_i : Q \longrightarrow X_i$ for all $i \in I$, such that all these triangles commute:

$$\begin{tikzcd} & Q \arrow[dl, "p_i"'] \arrow[dr, "p_j"] & \\ X_i \arrow[rr, "X(\phi)"'] & & X_j \end{tikzcd} \;.$$

Dually, a **co-cone** under the diagram is Q equipped with morphisms $q_i : X_i \longrightarrow Q$ such that all these triangles commute

$$\begin{tikzcd} X_i \arrow[rr, "X(\phi)"] \arrow[dr, "q_i"'] & & X_j \arrow[dl, "q_j"] \\ & Q & \end{tikzcd} \;.$$

A **limit** over the diagram is a universal cone, denoted $\underset{\longleftarrow i \in I}{\lim} X_i$, that is: a cone such that every other cone uniquely factors through it $Q \longrightarrow \underset{\longleftarrow}{\lim}_{i \in I} X_i$, making all the resulting triangles commute.

Dually, a **colimit** over the diagram is a universal co-cone, denoted $\underset{\longrightarrow}{\lim}_{i \in I} X_i$.

We now discuss limits and colimits in $\mathcal{C} =$ Top. The key for understanding these is the fact that there are initial and final topologies:

Definition 1.2. Let $\{X_i = (S_i, \tau_i) \in \mathrm{Top}\}_{i \in I}$ be a set of topological spaces, and let $S \in \mathrm{Set}$ be a bare set. Then

1. For $\{S \overset{f_i}{\to} S_i\}_{i \in I}$ a set of functions out of S, the **initial topology** $\tau_{\mathrm{initial}}(\{f_i\}_{i \in I})$ is the topology on S with the minimum collection of open subsets such that all $f_i : (S, \tau_{\mathrm{initial}}(\{f_i\}_{i \in I})) \to X_i$ are continuous.

2. For $\{S_i \overset{f_i}{\to} S\}_{i \in I}$ a set of functions into S, the **final topology** $\tau_{\mathrm{final}}(\{f_i\}_{i \in I})$ is the topology on S with the maximum collection of open subsets such that all $f_i : X_i \to (S, \tau_{\mathrm{final}}(\{f_i\}_{i \in I}))$ are continuous.

Example 1.3. For X a single topological space, and $\iota_S : S \hookrightarrow U(X)$ a subset of its underlying set, then the initial topology $\tau_{\mathrm{intial}}(\iota_S)$, def. 1.2, is the subspace topology, making

$$\iota_S \;:\; (S, \tau_{\mathrm{initial}}(\iota_S)) \hookrightarrow X$$

a topological subspace inclusion.

Example 1.4. Conversely, for $p_S : U(X) \longrightarrow S$ an epimorphism, then the final topology $\tau_{\mathrm{final}}(p_S)$ on S is the *quotient topology*.

Proposition 1.5. *Let I be a small category and let $X_\bullet : I \longrightarrow \mathrm{Top}$ be an I-diagram in Top (a functor from I to Top), with components denoted $X_i = (S_i, \tau_i)$, where $S_i \in \mathrm{Set}$ and τ_i a topology on S_i. Then:*

1. *The limit of $X_\bullet$ exists and is given by the topological space whose underlying set is the limit in Set of the underlying sets in the diagram, and whose topology is the initial topology, def. 1.2, for the functions p_i which are the limiting cone components:*

$$\begin{array}{ccccc} & & \underset{\longleftarrow}{\lim}_{i \in I} S_i & & \\ & {}^{p_i}\swarrow & & \searrow^{p_j} & . \\ S_i & & \longrightarrow & & S_j \end{array}$$

Hence

$$\underset{\longleftarrow}{\lim}_{i \in I} X_i \simeq \left(\underset{\longleftarrow}{\lim}_{i \in I} S_i,\ \tau_{\mathrm{initial}}(\{p_i\}_{i \in I})\right)$$

2. *The colimit of $X_\bullet$ exists and is the topological space whose underlying set is the colimit in Set of the underlying diagram of sets, and whose topology is the final topology, def. 1.2 for the component maps ι_i of the colimiting cocone*

$$\begin{array}{ccccc} S_i & & \longrightarrow & & S_j \\ & {}_{\iota_i}\searrow & & \swarrow_{\iota_j} & . \\ & & \underset{\longrightarrow}{\lim}_{i \in I} S_i & & \end{array}$$

Hence

$$\underset{\longrightarrow}{\lim}_{i \in I} X_i \simeq \left(\underset{\longrightarrow}{\lim}_{i \in I} S_i,\ \tau_{\mathrm{final}}(\{\iota_i\}_{i \in I}) \right)$$

(e.g. Bourbaki 71, section I.4)

Proof. The required universal property of $\left(\underset{\longleftarrow}{\lim}_{i \in I} S_i,\ \tau_{\mathrm{initial}}(\{p_i\}_{i \in I}) \right)$ (def. 1.1) is immediate: for

$$\begin{array}{ccc} & (S,\tau) & \\ {}^{f_i}\swarrow & & \searrow^{f_j} \\ X_i & \longrightarrow & X_j \end{array}$$

any cone over the diagram, then by construction there is a unique function of underlying sets $S \longrightarrow \underset{\longleftarrow}{\lim}_{i \in I} S_i$ making the required diagrams commute, and so all that is required is that this unique function is always continuous. But this is precisely what the initial topology ensures.

The case of the colimit is formally dual. ▮

Example 1.6. The limit over the empty diagram in Top is the point $*$ with its unique topology.

Example 1.7. For $\{X_i\}_{i \in I}$ a set of topological spaces, their coproduct $\underset{i \in I}{\sqcup} X_i \in \mathrm{Top}$ is their *disjoint union*.

In particular:

Example 1.8. For $S \in \mathrm{Set}$, the S-indexed coproduct of the point, $\coprod_{s \in S} *$ is the set S itself equipped with the final topology, hence is the discrete topological space on S.

Example 1.9. For $\{X_i\}_{i \in I}$ a set of topological spaces, their product $\prod_{i \in I} X_i \in \mathrm{Top}$ is the Cartesian product of the underlying sets equipped with the *product topology*, also called the *Tychonoff product*.

In the case that S is a finite set, such as for binary product spaces $X \times Y$, then a sub-basis for the product topology is given by the Cartesian products of the open subsets of (a basis for) each factor space.

Example 1.10. The equalizer of two continuous functions $f, g : X \rightrightarrows Y$ in Top is the equalizer of the underlying functions of sets

$$\mathrm{eq}(f,g) \hookrightarrow S_X \underset{g}{\overset{f}{\rightrightarrows}} S_Y$$

(hence the largets subset of S_X on which both functions coincide) and equipped with the subspace topology, example 1.3.

Example 1.11. The coequalizer of two continuous functions $f, g : X \rightrightarrows Y$ in Top is the coequalizer of the underlying functions of sets

$$S_X \underset{g}{\overset{f}{\rightrightarrows}} S_Y \longrightarrow \mathrm{coeq}(f,g)$$

(hence the quotient set by the equivalence relation generated by $f(x) \sim g(x)$ for all $x \in X$) and equipped with the quotient topology, example 1.4.

Example 1.12. For

$$\begin{array}{ccc} A & \overset{g}{\longrightarrow} & Y \\ {}^{\mathllap{f}}\downarrow \\ X \end{array}$$

two continuous functions out of the same domain, then the colimit under this diagram is also called the *pushout*, denoted

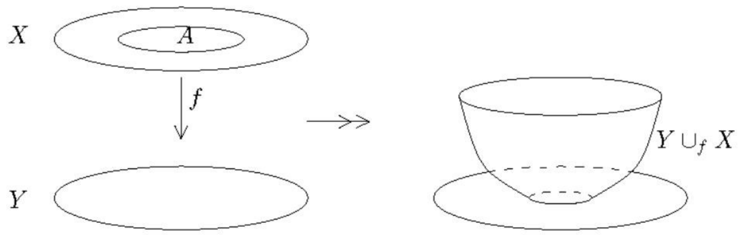
$$\begin{array}{ccc} A & \overset{g}{\longrightarrow} & Y \\ {}^{\mathllap{f}}\downarrow && \downarrow^{\mathrlap{g_\ast f}} & . \\ X & \longrightarrow & X \sqcup_A Y\,. \end{array}$$

(Here $g_\ast f$ is also called the pushout of f, or the *cobase change* of f along g.)

This is equivalently the coequalizer of the two morphisms from A to the coproduct of X with Y (example 1.7):

$$A \underoverset{\longrightarrow}{\longrightarrow}{} X \sqcup Y \longrightarrow X \sqcup_A Y\,.$$

If g is an inclusion, one also writes $X \cup_f Y$ and calls this the *attaching space*.



By example 1.11 the pushout/attaching space is the quotient topological space

$$X \sqcup_A Y \simeq (X \sqcup Y)/\sim$$

of the disjoint union of X and Y subject to the equivalence relation which identifies a point in X with a point in Y if they have the same pre-image in A.

(graphics from Aguilar-Gitler-Prieto 02)

Notice that the defining universal property of this colimit means that completing the span

$$\begin{array}{ccc} A & \longrightarrow & Y \\ \downarrow \\ X \end{array}$$

to a commuting square

$$\begin{array}{ccc} A & \longrightarrow & Y \\ \downarrow && \downarrow \\ X & \longrightarrow & Z \end{array}$$

is equivalent to finding a morphism

$$X \underset{A}{\sqcup} Y \longrightarrow Z\,.$$

Example 1.13. For $A \hookrightarrow X$ a topological subspace inclusion, example 1.3, then the pushout

$$\begin{array}{ccc} A & \hookrightarrow & X \\ \downarrow & (po) & \downarrow \\ \ast & \longrightarrow & X/A \end{array}$$

is the quotient space or *cofiber*, denoted X/A.

Example 1.14. An important special case of example 1.12:

For $n \in \mathbb{N}$ write

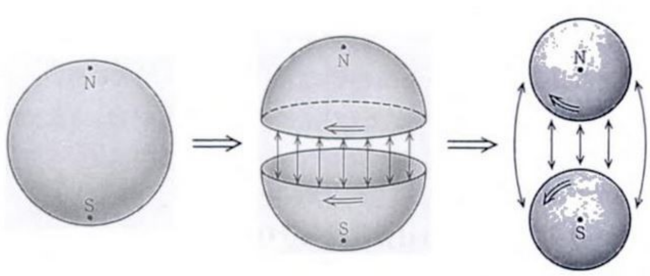
- $D^n \coloneqq \{\vec x \in \mathbb{R}^n \vert\; \vert \vec x\vert \leq 1\} \hookrightarrow \mathbb{R}^n$ for the standard topological n-disk (equipped with its subspace topology as a subset of Cartesian space);
- $S^{n-1} = \partial D^n \coloneqq \{\vec x \in \mathbb{R}^n \vert\; \vert \vec x\vert = 1\} \hookrightarrow \mathbb{R}^n$ for its boundary, the standard topological n-sphere.

Notice that $S^{-1} = \emptyset$ and that $S^0 = \ast \sqcup \ast$.

Let

$$i_n \colon S^{n-1} \longrightarrow D^n$$

be the canonical inclusion of the standard (n-1)-sphere as the boundary of the standard n-disk (both regarded as topological spaces with their subspace topology as subspaces of the Cartesian space $\mathbb{R}^n$).



Then the colimit in Top under the diagram

$$D^n \overset{i_n}{\longleftarrow} S^{n-1} \overset{i_n}{\longrightarrow} D^n \,,$$

i.e. the pushout of i_n along itself, is the n-sphere S^n:

$$\begin{array}{ccc} S^{n-1} & \overset{i_n}{\longrightarrow} & D^n \\ {}^{\mathllap{i_n}}\downarrow & (po) & \downarrow \\ D^n & \longrightarrow & S^n \end{array} \,.$$

(graphics from Ueno-Shiga-Morita 95)

Another kind of colimit that will play a role for certain technical constructions is transfinite composition. First recall

Definition 1.15. A *partial order* is a set S equipped with a relation $\leq$ such that for all elements $a,b,c \in S$

1) (reflexivity) $a \leq a$;

2) (transitivity) if $a \leq b$ and $b \leq c$ then $a \leq c$;

3) (antisymmetry) if $a \leq b$ and $\mathrm{b} \leq a$ then $a = b$.

This we may and will equivalently think of as a category with objects the elements of S and a unique morphism $a \to b$ precisely if $a \leq b$. In particular an order-preserving function between partially ordered sets is equivalently a functor between their corresponding categories.

A *bottom element* $\bot$ in a partial order is one such that $\bot \leq a$ for all a. A *top element* $\top$ is one for wich $a \leq \top$.

A partial order is a *total order* if in addition

4) (totality) either $a \leq b$ or $b \leq a$.

A total order is a *well order* if in addition

5) (well-foundedness) every non-empty subset has a least element.

An *ordinal* is the equivalence class of a well-order.

The *successor* of an ordinal is the class of the well-order with a top element freely adjoined.

A *limit ordinal* is one that is not a successor.

Example 1.16. The finite ordinals are labeled by $n \in \mathbb{N}$, corresponding to the well-orders $\{0 \leq 1 \leq 2 \cdots \leq n-1\}$. Here $(n+1)$ is the successor of n. The first non-empty limit ordinal is $\omega = [(\mathbb{N}, \leq)]$.

Definition 1.17. Let $\mathcal{C}$ be a category, and let $I \subset \mathrm{Mor}(\mathcal{C})$ be a class of its morphisms.

For α an ordinal (regarded as a category), an α-indexed *transfinite sequence* of elements in I is a diagram

$$X_\bullet : \alpha \longrightarrow \mathcal{C}$$

such that

1. $X_\bullet$ takes all successor morphisms $\beta \stackrel{\leq}{\to} \beta + 1$ in α to elements in I

$$X_{\beta,\beta+1} \in I$$

2. $X_\bullet$ is *continuous* in that for every nonzero limit ordinal $\beta < \alpha$, $X_\bullet$ restricted to the full-subdiagram $\{\gamma \mid \gamma \leq \beta\}$ is a colimiting cocone in $\mathcal{C}$ for $X_\bullet$ restricted to $\{\gamma \mid \gamma < \beta\}$.

The corresponding **transfinite composition** is the induced morphism

$$X_0 \longrightarrow X_\alpha \coloneqq \underset{\longrightarrow}{\lim} X_\bullet$$

into the colimit of the diagram, schematically:

$$\begin{array}{ccccccc} X_0 & \overset{X_{0,1}}{\longrightarrow} & X_1 & \overset{X_{1,2}}{\longrightarrow} & X_2 & \to & \cdots \\ & \searrow & \downarrow & \swarrow & \cdots & & \\ & & X_\alpha & & & & \end{array}.$$

We now turn to the discussion of mapping spaces/exponential objects.

Definition 1.18. For X a topological space and Y a locally compact topological space (in that for every point, every neighbourhood contains a compact neighbourhood), the **mapping space**

$$X^Y \in \mathrm{Top}$$

is the topological space

- whose underlying set is the set $\mathrm{Hom}_{\mathrm{Top}}(Y, X)$ of continuous functions $Y \to X$,
- whose open subsets are unions of finitary intersections of the following subbase elements of standard open subsets:

  the standard open subset $U^K \subset \mathrm{Hom}_{\mathrm{Top}}(Y, X)$ for

  - $K \hookrightarrow Y$ a compact topological space subset
  - $U \hookrightarrow X$ an open subset

  is the subset of all those continuous functions f that fit into a commuting diagram of the form

$$
\begin{array}{ccc}
K & \hookrightarrow & Y \\
\downarrow & & \downarrow^{f} \\
U & \hookrightarrow & X
\end{array}.
$$

Accordingly this is called the *compact-open topology* on the set of functions.

The construction extends to a functor

$$
(-)^{(-)} : \mathrm{Top}_{\mathrm{lc}}^{\mathrm{op}} \times \mathrm{Top} \longrightarrow \mathrm{Top}\,.
$$

Proposition 1.19. *For X a topological space and Y a locally compact topological space (in that for each point, each open neighbourhood contains a compact neighbourhood), the **topological mapping space** X^Y from def. 1.18 is an exponential object, i.e. the functor $(-)^Y$ is right adjoint to the product functor $Y \times (-)$: there is a natural bijection*

$$
\mathrm{Hom}_{\mathrm{Top}}(Z \times Y, X) \simeq \mathrm{Hom}_{\mathrm{Top}}(Z, X^Y)
$$

between continuous functions out of any product topological space of Y with any $Z \in \mathrm{Top}$ and continuous functions from Z into the mapping space.

A proof is spelled out here (or see e.g. Aguilar-Gitler-Prieto 02, prop. 1.3.1).

Remark 1.20. In the context of prop. 1.19 it is often assumed that Y is also a Hausdorff topological space. But this is not necessary. What assuming Hausdorffness only achieves is that all alternative definitions of “locally compact” become equivalent to the one that is needed for the proposition: for every point, every open neighbourhood contains a compact neighbourhood.

Remark 1.21. Proposition 1.19 fails in general if Y is not locally compact. Therefore the plain category Top of all topological spaces is not a Cartesian closed category.

This is no problem for the construction of the homotopy theory of topological spaces as such, but it becomes a technical nuisance for various constructions that one would like to perform within that homotopy theory. For instance on general pointed topological spaces the smash product is in general not associative.

On the other hand, without changing any of the following discussion one may just pass to a more convenient category of topological spaces such as notably the full subcategory of compactly generated topological spaces $\mathrm{Top}_{\mathrm{cg}} \hookrightarrow \mathrm{Top}$ (def. 3.35) which is Cartesian closed. This we turn to below.

Homotopy

The fundamental concept of homotopy theory is clearly that of *homotopy*. In the context of topological spaces this is about contiunous deformations of continuous functions parameterized by the standard interval:

Definition 1.22. Write

$$
I \coloneqq [0,1] \hookrightarrow \mathbb{R}
$$

for the standard topological interval, a compact connected topological subspace of the real line.

Equipped with the canonical inclusion of its two endpoints

$$
* \sqcup * \xrightarrow{(\delta_0, \delta_1)} I \xrightarrow{\exists !} *
$$

this is the standard interval object in Top.

For $X \in \mathrm{Top}$, the product topological space $X \times I$, example 1.9, is called the standard *cylinder object* over X. The endpoint inclusions of the interval make it factor the codiagonal on X

$$\nabla_X \;:\; X \sqcup X \xrightarrow{((\mathrm{id},\delta_0),(\mathrm{id},\delta_1))} X \times I \longrightarrow X \,.$$

Definition 1.23. For $f, g : X \longrightarrow Y$ two continuous functions between topological spaces X, Y, then a **left homotopy**

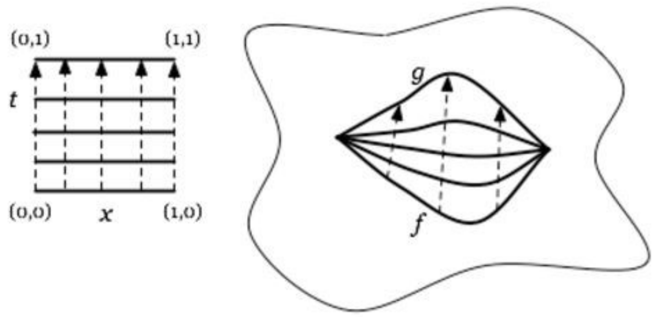
$$\eta : f \Rightarrow_L g$$

is a continuous function

$$\eta \;:\; X \times I \longrightarrow Y$$

out of the standard cylinder object over X, def. 1.22, such that this fits into a commuting diagram of the form

$$\begin{tikzcd} X \arrow[d, "(\mathrm{id},\delta_0)"'] \arrow[dr, "f"] & \\ X \times I \arrow[r, "\eta"] & Y \\ X \arrow[u, "(\mathrm{id},\delta_1)"] \arrow[ur, "g"'] & \end{tikzcd} \,.$$



(graphics grabbed from J. Tauber here)

Example 1.24. Let X be a topological space and let $x, y \in X$ be two of its points, regarded as functions $x, y : * \longrightarrow X$ from the point to X. Then a left homotopy, def. 1.23, between these two functions is a commuting diagram of the form

$$\begin{tikzcd} * \arrow[d, "\delta_0"'] \arrow[dr, "x"] & \\ I \arrow[r, "\eta"] & Y \\ * \arrow[u, "\delta_1"] \arrow[ur, "y"'] & \end{tikzcd} \,.$$

This is simply a continuous path in X whose endpoints are x and y.

For instance:

Example 1.25. Let

$$\mathrm{const}_0 \;:\; I \longrightarrow * \overset{\delta_0}{\longrightarrow} I$$

be the continuous function from the standard interval $I = [0,1]$ to itself that is constant on the value 0. Then there is a left homotopy, def. 1.23, from the identity function

$$\eta \;:\; \mathrm{id}_I \Rightarrow \mathrm{const}_0$$

given by

$$\eta(x,t) \coloneqq x(1-t) \,.$$

A key application of the concept of left homotopy is to the definition of homotopy groups:

Definition 1.26. For X a topological space, then its set $\pi_0(X)$ of *connected components*, also called the **0-th homotopy set**, is the set of left homotopy-equivalence classes (def. 1.23) of points $x : * \to X$, hence the set of path-connected components of X (example 1.24). By composition this extends to a functor

$$\pi_0 : \mathrm{Top} \longrightarrow \mathrm{Set}\,.$$

For $n \in \mathbb{N}$, $n \geq 1$ and for $x : * \to X$ any point, then the nth **homotopy group** $\pi_n(X,x)$ of X at x is the group

- whose underlying set is the set of left homotopy-equivalence classes of maps $I^n \longrightarrow X$ that take the boundary of I^n to x and where the left homotopies η are constrained to be constant on the boundary;
- whose group product operation takes $[\alpha : I^n \to X]$ and $[\beta : I^n \to X]$ to $[\alpha \cdot \beta]$ with

$$\alpha \cdot \beta \;:\; I^n \xrightarrow{\simeq} I^n \underset{I^{n-1}}{\sqcup} I^n \xrightarrow{(\alpha,\beta)} X\,,$$

where the first map is a homeomorphism from the unit n-cube to the n-cube with one side twice the unit length (e.g. $(x_1, x_2, x_3, \cdots) \mapsto (2x_1, x_2, x_3, \cdots)$).

By composition, this construction extends to a functor

$$\pi_{\bullet \geq 1} \;:\; \mathrm{Top}^{*/} \longrightarrow \mathrm{Grp}^{\mathbb{N}_{\geq 1}}$$

from pointed topological spaces to graded groups.

Notice that often one writes the value of this functor on a morphism f as $f_* = \pi_\bullet(f)$.

Remark 1.27. At this point we don't go further into the abstract reason why def. 1.26 yields group structure above degree 0, which is that positive dimension spheres are H-cogroup objects. But this is important, for instance in the proof of the Brown representability theorem. See the section *Brown representability theorem* in Part S.

Definition 1.28. A continuous function $f \;:\; X \longrightarrow Y$ is called a **homotopy equivalence** if there exists a continuous function the other way around, $g \;:\; Y \longrightarrow X$, and left homotopies, def. 1.23, from the two composites to the identity:

$$\eta_1 \;:\; f \circ g \Rightarrow_L \mathrm{id}_Y$$

and

$$\eta_2 \;:\; g \circ f \Rightarrow_L \mathrm{id}_X\,.$$

If here η_2 is constant along I, f is said to exhibit X as a **deformation retract** of Y.

Example 1.29. For X a topological space and $X \times I$ its standard cylinder object of def. 1.22, then the projection $p : X \times I \longrightarrow X$ and the inclusion $(\mathrm{id}, \delta_0) : X \longrightarrow X \times I$ are homotopy equivalences, def. 1.28, and in fact are homotopy inverses to each other:

The composition

$$X \xrightarrow{(\mathrm{id},\delta_0)} X \times I \xrightarrow{p} X$$

is immediately the identity on X (i.e. homotopic to the identity by a trivial homotopy), while the composite

$$X \times I \xrightarrow{p} X \xrightarrow{(\mathrm{id},\delta_0)} X \times I$$

is homotopic to the identity on $X \times I$ by a homotopy that is pointwise in X that of example 1.25.

Definition 1.30. A continuous function $f : X \longrightarrow Y$ is called a **weak homotopy equivalence** if its image under all the homotopy group functors of def. 1.26 is an isomorphism, hence if

$$\pi_0(f) \,:\, \pi_0(X) \overset{\simeq}{\longrightarrow} \pi_0(X)$$

and for all $x \in X$ and all $n \geq 1$

$$\pi_n(f) \,:\, \pi_n(X,x) \overset{\simeq}{\longrightarrow} \pi_n(Y, f(y)) \,.$$

Proposition 1.31. *Every homotopy equivalence, def. 1.28, is a weak homotopy equivalence, def. 1.30.*

In particular a deformation retraction, def. 1.28, is a weak homotopy equivalence.

Proof. First observe that for all $X \in$ Top the inclusion maps

$$X \overset{(\mathrm{id},\delta_0)}{\longrightarrow} X \times I$$

into the standard cylinder object, def. 1.22, are weak homotopy equivalences: by postcomposition with the contracting homotopy of the interval from example 1.25 all homotopy groups of $X \times I$ have representatives that factor through this inclusion.

Then given a general homotopy equivalence, apply the homotopy groups functor to the corresponding homotopy diagrams (where for the moment we notationally suppress the choice of basepoint for readability) to get two commuting diagrams

$$\begin{tikzcd}
\pi_\bullet(X) \arrow[d, "\pi_\bullet(\mathrm{id},\delta_0)"'] \arrow[dr, "\pi_\bullet(f)\circ\pi_\bullet(g)"] & \\
\pi_\bullet(X \times I) \arrow[r, "\pi_\bullet(\eta)"] & \pi_\bullet(Y) \\
\pi_\bullet(X) \arrow[u, "\pi_\bullet(\mathrm{id},\delta_1)"] \arrow[ur, "\pi_\bullet(\mathrm{id})"'] &
\end{tikzcd}
\quad , \quad
\begin{tikzcd}
\pi_\bullet(Y) \arrow[d, "\pi_\bullet(\mathrm{id},\delta_0)"'] \arrow[dr, "\pi_\bullet(g)\circ\pi_\bullet(f)"] & \\
\pi_\bullet(Y \times I) \arrow[r, "\pi_\bullet(\eta)"] & \pi_\bullet(X) \\
\pi_\bullet(Y) \arrow[u, "\pi_\bullet(\mathrm{id},\delta_1)"] \arrow[ur, "\pi_\bullet(\mathrm{id})"'] &
\end{tikzcd} \,.$$

By the previous observation, the vertical morphisms here are isomorphisms, and hence these diagrams exhibit $\pi_\bullet(f)$ as the inverse of $\pi_\bullet(g)$, hence both as isomorphisms. ▮

Remark 1.32. The converse of prop. 1.31 is not true generally: not every weak homotopy equivalence between topological spaces is a homotopy equivalence. (For an example with full details spelled out see for instance Fritsch, Piccinini: "Cellular Structures in Topology", p. 289-290).

However, as we will discuss below, it turns out that

1. every weak homotopy equivalence between CW-complexes is a homotopy equivalence (Whitehead's theorem, cor. 3.8);

2. every topological space is connected by a weak homotopy equivalence to a CW-complex (CW approximation, remark 3.12).

Example 1.33. For $X \in \mathrm{Top}$, the projection $X \times I \longrightarrow X$ from the cylinder object of X, def. 1.22, is a weak homotopy equivalence, def. 1.30. This means that the factorization

$$\nabla_X \,:\, X \sqcup X \hookrightarrow X \times I \overset{\simeq}{\longrightarrow} X$$

of the codiagonal ∇_X in def. 1.22, which in general is far from being a monomorphism, may be thought of as factoring it through a monomorphism after replacing X, up to weak homotopy equivalence, by $X \times I$.

In fact, further below (prop. 1.25) we see that $X \sqcup X \to X \times I$ has better properties than the generic monomorphism has, in particular better homotopy invariant properties: it has the left lifting property against all Serre fibrations $E \overset{p}{\longrightarrow} B$ that are also weak homotopy equivalences.

Of course the concept of left homotopy in def. 1.23 is accompanied by a concept of *right homotopy*. This we turn to now.

Definition 1.34. For X a topological space, its **standard topological path space object** is the topological mapping space X^I, prop. 1.19, out of the standard interval I of def. 1.22.

Example 1.35. The endpoint inclusion into the standard interval, def. 1.22, makes the path space X^I of def. 1.34 factor the diagonal on X through the inclusion of constant paths and the endpoint evaluation of paths:

$$\Delta_X \,:\, X \overset{X^{I \to *}}{\longrightarrow} X^I \overset{X^{* \sqcup * \to I}}{\longrightarrow} X \times X \,.$$

This is the formal dual to example 1.22. As in that example, below we will see (prop. 3.14) that this factorization has good properties, in that

1. $X^{I \to *}$ is a weak homotopy equivalence;

2. $X^{* \sqcup * \to I}$ is a Serre fibration.

So while in general the diagonal Δ_X is far from being an epimorphism or even just a Serre fibration, the factorization through the path space object may be thought of as replacing X, up to weak homotopy equivalence, by its path space, such as to turn its diagonal into a Serre fibration after all.

Definition 1.36. For $f, g : X \longrightarrow Y$ two continuous functions between topological spaces X, Y, then a **right homotopy** $f \Rightarrow_R g$ is a continuous function

$$\eta \,:\, X \longrightarrow Y^I$$

into the path space object of X, def. 1.34, such that this fits into a commuting diagram of the form

$$\begin{array}{ccc} && Y \\ & {}^{\mathllap{f}}\nearrow & \uparrow^{\mathrlap{X^{\delta_0}}} \\ X & \overset{\eta}{\longrightarrow} & Y^I \,. \\ & {}_{\mathllap{g}}\searrow & \downarrow^{\mathrlap{Y^{\delta_1}}} \\ && Y \end{array}$$

Cell complexes

We consider topological spaces that are built consecutively by attaching basic cells.

Definition 1.37. Write

$$I_{\mathrm{Top}} \coloneqq \left\{ S^{n-1} \overset{\iota_n}{\hookrightarrow} D^n \right\}_{n \in \mathbb{N}} \subset \mathrm{Mor}(\mathrm{Top})$$

for the set of canonical boundary inclusion maps of the standard n-disks, example 1.14. This going to be called the set of standard **topological generating cofibrations**.

Definition 1.38. For $X \in \mathrm{Top}$ and for $n \in \mathbb{N}$, an n**-cell attachment** to X is the pushout ("attaching space", example 1.12) of a generating cofibration, def. 1.37

$$
\begin{array}{ccc}
S^{n-1} & \overset{\phi}{\longrightarrow} & X \\
{}^{\iota_n}\downarrow & (\mathrm{po}) & \downarrow \\
D^n & \longrightarrow & X \underset{S^{n-1}}{\sqcup} D^n = X \cup_\phi D^n
\end{array}
$$

along some continuous function ϕ.

A continuous function $f : X \longrightarrow Y$ is called a **topological relative cell complex** if it is exhibited by a (possibly infinite) sequence of cell attachments to X, in that it is a transfinite composition (def. 1.17) of pushouts (example 1.12)

$$
\begin{array}{ccc}
\coprod_i S^{n_i - 1} & \longrightarrow & X_k \\
{}^{\coprod_i \iota_{n_i}}\downarrow & (\mathrm{po}) & \downarrow \\
\coprod_i D^{n_i} & \longrightarrow & X_{k+1}
\end{array}
$$

of coproducts (example 1.7) of generating cofibrations (def. 1.37).

A topological space X is a **cell complex** if $\emptyset \longrightarrow X$ is a relative cell complex.

A relative cell complex is called a **finite relative cell complex** if it is obtained from a finite number of cell attachments.

A (relative) cell complex is called a (relative) **CW-complex** if the above transfinite composition is countable

$$
\begin{array}{ccccccc}
X = X_0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & \cdots \\
& {}_f\searrow & \downarrow & \swarrow & & \cdots & \\
& & Y = \underset{\longrightarrow}{\lim} X_\bullet & & & &
\end{array}
$$

and if X_k is obtained from X_{k-1} by attaching cells precisely only of dimension k.

Remark 1.39. Strictly speaking a relative cell complex, def. 1.38, is a function $f : X \to Y$, *together* with its cell structure, hence together with the information of the pushout diagrams and the transfinite composition of the pushout maps that exhibit it.

In many applications, however, all that matters is that there is *some* (relative) cell decomosition, and then one tends to speak loosely and mean by a (relative) cell complex only a (relative) topological space that admits some cell decomposition.

The following lemma 1.40, together with lemma 1.52 below are the only two statements of the entire development here that involve the concrete particular nature of topological spaces ("point-set topology"), everything beyond that is general abstract homotopy theory.

Lemma 1.40. *Assuming the axiom of choice and the law of excluded middle, every compact subspace of a topological cell complex, def. 1.38, intersects the interior of a finite number of cells.*

(e.g. Hirschhorn 15, section 3.1)

Proof. So let Y be a topological cell complex and $C \hookrightarrow Y$ a compact subspace. Define a subset

$$
P \subset Y
$$

by *choosing* one point in the interior of the intersection with C of each cell of Y that intersects C.

It is now sufficient to show that P has no accumulation point. Because, by the compactness of X, every non-finite subset of C does have an accumulation point, and hence the lack of such shows that P is a finite set and hence that C intersects the interior of finitely many cells of Y.

To that end, let $c \in C$ be any point. If c is a 0-cell in Y, write $U_c \coloneqq \{c\}$. Otherwise write e_c for the unique cell of Y that contains c in its interior. By construction, there is exactly one point of P in the interior of e_c. Hence there is an open neighbourhood $c \in U_c \subset e_c$ containing no further points of P beyond possibly c itself, if c happens to be that single point of P in e_c.

It is now sufficient to show that U_c may be enlarged to an open subset $\tilde{U}_c$ of Y containing no point of P, except for possibly c itself, for that means that c is not an accumulation point of P.

To that end, let α_c be the ordinal that labels the stage Y_{α_c} of the transfinite composition in the cell complex-presentation of Y at which the cell e_c above appears. Let γ be the ordinal of the full cell complex. Then define the set

$$T \coloneqq \left\{ (\beta, U) \mid \alpha_c \leq \beta \leq \gamma \;,\; U \underset{\text{open}}{\subset} Y_\beta \;,\; U \cap Y_\alpha = U_c \;,\; U \cap P \in \{\emptyset, \{c\}\} \right\},$$

and regard this as a partially ordered set by declaring a partial ordering via

$$(\beta_1, U_1) < (\beta_2, U_2) \quad \Leftrightarrow \quad \beta_1 < \beta_2 \;,\; U_2 \cap Y_{\beta_1} = U_1 \;.$$

This is set up such that every element (β, U) of T with β the maximum value $\beta = \gamma$ is an extension $\tilde{U}_c$ that we are after.

Observe then that for $(\beta_s, U_s)_{s \in S}$ a chain in $(T, <)$ (a subset on which the relation $<$ restricts to a total order), it has an upper bound in T given by the union $(\cup_s \beta_s, \cup_s U_s)$. Therefore Zorn's lemma applies, saying that $(T, <)$ contains a maximal element $(\beta_{\max}, U_{\max})$.

Hence it is now sufficient to show that $\beta_{\max} = \gamma$. We argue this by showing that assuming $\beta_{\max} < \gamma$ leads to a contradiction.

So assume $\beta_{\max} < \gamma$. Then to construct an element of T that is larger than $(\beta_{\max}, U_{\max})$, consider for each cell d at stage $Y_{\beta_{\max}+1}$ its attaching map $h_d : S^{n-1} \to Y_{\beta_{\max}}$ and the corresponding preimage open set $h_d^{-1}(U_{\max}) \subset S^{n-1}$. Enlarging all these preimages to open subsets of D^n (such that their image back in $X_{\beta_{\max}+1}$ does not contain c), then $(\beta_{\max}, U_{\max}) < (\beta_{\max} + 1, \cup_d U_d)$. This is a contradiction. Hence $\beta_{\max} = \gamma$, and we are done. ▮

It is immediate and useful to generalize the concept of topological cell complexes as follows.

Definition 1.41. For $\mathcal{C}$ any category and for $K \subset \mathrm{Mor}(\mathcal{C})$ any sub-class of its morphisms, a **relative K-cell complexes** is a morphism in $\mathcal{C}$ which is a transfinite composition (def. 1.17) of pushouts of coproducts of morphsims in K.

Definition 1.42. Write

$$J_{\mathrm{Top}} \coloneqq \left\{ D^n \overset{(\mathrm{id}, \delta_0)}{\hookrightarrow} D^n \times I \right\}_{n \in \mathbb{N}} \subset \mathrm{Mor}(\mathrm{Top})$$

for the set of inclusions of the topological n-disks, def. 1.37, into their cylinder objects, def. 1.22, along (for definiteness) the left endpoint inclusion.

These inclusions are similar to the standard topological generating cofibrations I_{Top} of def. 1.37, but in contrast to these they are “acyclic” (meaning: trivial on homotopy classes of maps from “cycles” given by n-spheres) in that they are weak homotopy equivalences (by prop. 1.31).

Accordingly, J_{Top} is to be called the set of standard **topological generating acyclic cofibrations**.

Lemma 1.43. *For X a CW-complex (def. 1.38), then its inclusion $X \overset{(\mathrm{id},\delta_0)}{\longrightarrow} X \times I$ into its standard cylinder (def. 1.22) is a J_{Top}-relative cell complex (def. 1.41, def. 1.42).*

Proof. First erect a cylinder over all 0-cells

$$\begin{array}{ccc} \coprod_{x \in X_0} D^0 & \longrightarrow & X \\ \downarrow & (po) & \downarrow \\ \coprod_{x \in X_0} D^1 & \longrightarrow & Y_1 \end{array} \,.$$

Assume then that the cylinder over all n-cells of X has been erected using attachment from J_{Top}. Then the union of any $(n+1)$-cell σ of X with the cylinder over its boundary is homeomorphic to D^{n+1} and is like the cylinder over the cell “with end and interior removed”. Hence via attaching along $D^{n+1} \to D^{n+1} \times I$ the cylinder over σ is erected. ▮

Lemma 1.44. *The maps $D^n \hookrightarrow D^n \times I$ in def. 1.42 are finite relative cell complexes, def. 1.38. In other words, the elements of J_{Top} are I_{Top}-relative cell complexes.*

Proof. There is a homeomorphism

$$\begin{array}{ccc} D^n & = & D^n \\ {}^{\mathllap{(\mathrm{id},\delta_0)}}\downarrow && \downarrow \\ D^n \times I & \simeq & D^{n+1} \end{array}$$

such that the map on the right is the inclusion of one hemisphere into the boundary n-sphere of D^{n+1}. This inclusion is the result of attaching two cells:

$$\begin{array}{ccc} S^{n-1} & \overset{\iota_n}{\longrightarrow} & D^n \\ {}^{\mathllap{\iota_n}}\downarrow & (po) & \downarrow \\ D^n & \longrightarrow & S^n \\ && \downarrow^{=} \\ S^n & \overset{\mathrm{id}}{\longrightarrow} & S^n \\ {}^{\mathllap{\iota_{n+1}}}\downarrow & (po) & \downarrow \\ D^{n+1} & \underset{\mathrm{id}}{\longrightarrow} & D^{n+1} \end{array} \,.$$

here the top pushout is the one from example 1.14. ▮

Lemma 1.45. *Every J_{Top}-relative cell complex (def. 1.42, def. 1.41) is a weak homotopy equivalence, def. 1.30.*

Proof. Let $X \longrightarrow \hat X = \underset{\longleftarrow}{\lim}_{\beta \leq \alpha} X_\beta$ be a J_{Top}-relative cell complex.

First observe that with the elements $D^n \hookrightarrow D^n \times I$ of J_{Top} being homotopy equivalences for all $n \in \mathbb{N}$ (by example 1.29), each of the stages $X_\beta \longrightarrow X_{\beta+1}$ in the relative cell complex is also a homotopy equivalence. We make this fully explicit:

By definition, such a stage is a pushout of the form

$$\begin{array}{ccc} \underset{i \in I}{\sqcup} D^{n_i} & \longrightarrow & X_\beta \\ {}^{\mathllap{\underset{i \in I}{\sqcup}(\mathrm{id},\delta_0)}}\downarrow & (po) & \downarrow \\ \underset{i \in I}{\sqcup} D^{n_i} \times I & \longrightarrow & X_{\beta+1} \end{array} \,.$$

Then the fact that the projections $p_{n_i} : D^{n_i} \times I \to D^{n_i}$ are strict left inverses to the inclusions (id, δ_0) gives a commuting square of the form

$$
\begin{array}{ccc}
\underset{i \in I}{\sqcup} D^{n_i} & \longrightarrow & X_\beta \\
{}^{\underset{i \in I}{\sqcup} (\mathrm{id}, \delta_0)}\downarrow && \downarrow^{\mathrm{id}} \\
\underset{i \in I}{\sqcup} D^{n_i} \times I && \\
{}^{\underset{i \in I}{\sqcup} p_{n_i}}\downarrow && \downarrow \\
\underset{i \in I}{\sqcup} D^{n_i} & \longrightarrow & X_\beta
\end{array}
$$

and so the universal property of the colimit (pushout) $X_{\beta+1}$ gives a factorization of the identity morphism on the right through $X_{\beta+1}$

$$
\begin{array}{ccc}
\underset{i \in I}{\sqcup} D^{n_i} & \longrightarrow & X_\beta \\
{}^{\underset{i \in I}{\sqcup} (\mathrm{id}, \delta_0)}\downarrow && \downarrow \\
\underset{i \in I}{\sqcup} D^{n_i} \times I & \longrightarrow & X_{\beta+1} \\
{}^{\underset{i \in I}{\sqcup} p_{n_i}}\downarrow && \downarrow \\
\underset{i \in I}{\sqcup} D^{n_i} & \longrightarrow & X_\beta
\end{array}
$$

which exhibits $X_{\beta+1} \to X_\beta$ as a strict left inverse to $X_\beta \to X_{\beta+1}$. Hence it is now sufficient to show that this is also a homotopy right inverse.

To that end, let

$$
\eta_{n_i} : D^{n_i} \times I \longrightarrow D^{n_i} \times I
$$

be the left homotopy that exhibits p_{n_i} as a homotopy right inverse to p_{n_i} by example 1.29. For each $t \in [0,1]$ consider the commuting square

$$
\begin{array}{ccc}
\underset{i \in I}{\sqcup} D^{n_i} & \longrightarrow & X_\beta \\
\downarrow && \downarrow \\
\underset{i \in I}{\sqcup} D^{n_i} \times I && X_{\beta+1} \\
{}^{\eta_{n_i}(-,t)}\downarrow && \downarrow^{\mathrm{id}} \\
\underset{i \in I}{\sqcup} D^{n_i} \times I & \longrightarrow & X_{\beta+1}
\end{array}
\,.
$$

Regarded as a cocone under the span in the top left, the universal property of the colimit (pushout) $X_{\beta+1}$ gives a continuous function

$$
\eta(-,t) \;:\; X_{\beta+1} \longrightarrow X_{\beta+1}
$$

for each $t \in [0,1]$. For $t = 0$ this construction reduces to the provious one in that $\eta(-,0) : X_{\beta+1} \to X_\beta \to X_{\beta+1}$ is the composite which we need to homotope to the identity; while $\eta(-,1)$ is the identity. Since $\eta(-,t)$ is clearly also continuous in t it constitutes a continuous function

$$
\eta \;:\; X_{\beta+1} \times I \longrightarrow X_{\beta+1}
$$

which exhibits the required left homotopy.

So far this shows that each stage $X_\beta \to X_{\beta+1}$ in the transfinite composition defining $\hat{X}$ is a homotopy equivalence, hence, by prop. 1.31, a weak homotopy equivalence.

This means that all morphisms in the following diagram (notationally suppressing basepoints and showing only the finite stages)

$$\begin{array}{ccccccccc} \pi_n(X) & \overset{\simeq}{\to} & \pi_n(X_1) & \overset{\simeq}{\to} & \pi_n(X_2) & \overset{\simeq}{\to} & \pi_n(X_3) & \overset{\simeq}{\to} & \cdots \\ & {}_{\simeq}\searrow & \downarrow^{\simeq} & \swarrow_{\simeq} & \cdots \\ & & \underset{\longleftarrow}{\lim}_\alpha \pi_n(X_\alpha) \end{array}$$

are isomorphisms.

Moreover, lemma 1.40 gives that every representative and every null homotopy of elements in $\pi_n(\hat X)$ already exists at some finite stage X_k. This means that also the universally induced morphism

$$\underset{\longleftarrow}{\lim}_\alpha \pi_n(X_\alpha) \overset{\simeq}{\to} \pi_n(\hat X)$$

is an isomorphism. Hence the composite $\pi_n(X) \overset{\simeq}{\to} \pi_n(\hat X)$ is an isomorphism. ▮

Fibrations

Given a relative C-cell complex $\iota : X \to Y$, def. 1.41, it is typically interesting to study the extension problem along f, i.e. to ask which topological spaces E are such that every continuous function $f : X \longrightarrow E$ has an extension $\tilde f$ along ι

$$\begin{array}{ccc} X & \overset{f}{\longrightarrow} & E \\ {}^{\iota}\downarrow & \nearrow_{\exists \tilde f} \\ Y \end{array} .$$

If such extensions exists, it means that E is sufficiently "spread out" with respect to the maps in C. More generally one considers this extension problem fiberwise, i.e. with both E and Y (hence also X) equipped with a map to some base space B:

Definition 1.46. Given a category $\mathcal{C}$ and a sub-class $C \subset \mathrm{Mor}(\mathcal{C})$ of its morphisms, then a morphism $p : E \longrightarrow B$ in $\mathcal{C}$ is said to have the right lifting property against the morphisms in C if every commuting diagram in $\mathcal{C}$ of the form

$$\begin{array}{ccc} X & \longrightarrow & E \\ {}^{c}\downarrow & & \downarrow^{p} \\ Y & \longrightarrow & B \end{array},$$

with $c \in C$, has a lift h, in that it may be completed to a commuting diagram of the form

$$\begin{array}{ccc} X & \longrightarrow & E \\ {}^{c}\downarrow & {}^{h}\nearrow & \downarrow^{p} \\ Y & \longrightarrow & B \end{array}.$$

We will also say that f is a C-**injective morphism** if it satisfies the right lifting property against C.

Definition 1.47. A continuous function $p : E \longrightarrow B$ is called a **Serre fibration** if it is a J_{Top}-injective morphism; i.e. if it has the right lifting property, def. 1.46, against all topological generating acylic cofibrations, def. 1.42; hence if for every commuting diagram of continuous functions of the form

$$\begin{array}{ccc} D^n & \longrightarrow & E \\ {}^{(\mathrm{id},\delta_0)}\downarrow && \downarrow^{p}, \\ D^n \times I & \longrightarrow & B \end{array}$$

has a lift h, in that it may be completed to a commuting diagram of the form

$$\begin{array}{ccc} D^n & \longrightarrow & E \\ {}^{(\mathrm{id},\delta_0)}\downarrow & {}^{h}\nearrow & \downarrow^{p}. \\ D^n \times I & \longrightarrow & B \end{array}$$

Remark 1.48. Def. 1.47 says, in view of the definition of left homotopy, that a Serre fibration p is a map with the property that given a left homotopy, def. 1.23, between two functions into its codomain, and given a lift of one the two functions through p, then also the homotopy between the two lifts. Therefore the condition on a Serre fibration is also called the *homotopy lifting property* for maps whose domain is an n-disk.

More generally one may ask functions p to have such homotopy lifting property for functions with arbitrary domain. These are called *Hurewicz fibrations*.

Remark 1.49. The precise shape of D^n and $D^n \times I$ in def. 1.47 turns out not to actually matter much for the nature of Serre fibrations. We will eventually find below (prop. 3.5) that what actually matters here is only that the inclusions $D^n \hookrightarrow D^n \times I$ are relative cell complexes (lemma 1.44) and weak homotopy equivalences (prop. 1.31) and that all of these may be generated from them in a suitable way.

But for simple special cases this is readily seen directly, too. Notably we could replace the n-disks in def. 1.47 with any homeomorphic topological space. A choice important in the comparison to the classical model structure on simplicial sets (below) is to instead take the topological n-simplices Δ^n. Hence a Serre fibration is equivalently characterized as having lifts in all diagrams of the form

$$\begin{array}{ccc} \Delta^n & \longrightarrow & E \\ {}^{(\mathrm{id},\delta_0)}\downarrow && \downarrow^{p}. \\ \Delta^n \times I & \longrightarrow & B \end{array}$$

Other deformations of the n-disks are useful in computations, too. For instance there is a homeomorphism from the n-disk to its "cylinder with interior and end removed", formally:

$$\begin{array}{ccc} (D^n \times \{0\}) \cup (\partial D^n \times I) & \simeq & D^n \\ \downarrow && \downarrow \\ D^n \times I & \simeq & D^n \times I \end{array}$$

and hence f is a Serre fibration equivalently also if it admits lifts in all diagrams of the form

$$\begin{array}{ccc} (D^n \times \{0\}) \cup (\partial D^n \times I) & \longrightarrow & E \\ {}^{(\mathrm{id},\delta_0)}\downarrow && \downarrow^{p}. \\ \Delta^n \times I & \longrightarrow & B \end{array}$$

The following is a general fact about closure of morphisms defined by lifting properties which we prove in generality below as prop. 2.10.

Proposition 1.50. *A Serre fibration, def. 1.47 has the right lifting property against all retracts (see remark 2.12) of J_{Top}-relative cell complexes (def. 1.42, def. 1.38).*

The following statement is foreshadowing the long exact sequences of homotopy groups (below) induced by any fiber sequence, the full version of which we come to below (example 4.37) after having developed more of the abstract homotopy theory.

Proposition 1.51. *Let $f : X \longrightarrow Y$ be a Serre fibration, def. 1.47, let $y : * \to Y$ be any point and write*

$$F_y \overset{\iota}{\hookrightarrow} X \overset{f}{\longrightarrow} Y$$

for the fiber inclusion over that point. Then for every choice $x : * \to X$ of lift of the point y through f, the induced sequence of homotopy groups

$$\pi_\bullet(F_y, x) \overset{\iota_*}{\longrightarrow} \pi_\bullet(X, x) \overset{f_*}{\longrightarrow} \pi_\bullet(Y)$$

is exact, in that the kernel of f_* is canonically identified with the image of ι_*:

$$\ker(f_*) \simeq \mathrm{im}(\iota_*) \;.$$

Proof. It is clear that the image of ι_* is in the kernel of f_* (every sphere in $F_y \hookrightarrow X$ becomes constant on y, hence contractible, when sent forward to Y).

For the converse, let $[\alpha] \in \pi_\bullet(X, x)$ be represented by some $\alpha : S^{n-1} \to X$. Assume that $[\alpha]$ is in the kernel of f_*. This means equivalently that α fits into a commuting diagram of the form

$$\begin{array}{ccc} S^{n-1} & \overset{\alpha}{\longrightarrow} & X \\ \downarrow && \downarrow^{f} \\ D^n & \overset{\kappa}{\longrightarrow} & Y \end{array},$$

where κ is the contracting homotopy witnessing that $f_*[\alpha] = 0$.

Now since x is a lift of y, there exists a left homotopy

$$\eta \;:\; \kappa \Rightarrow \mathrm{const}_y$$

as follows:

$$\begin{array}{ccccc} && S^{n-1} & \overset{\alpha}{\longrightarrow} & X \\ && {}^{\iota_n}\downarrow && \downarrow^{f} \\ && D^n & \overset{\kappa}{\longrightarrow} & Y \\ && \downarrow^{(\mathrm{id},\delta_1)} && \downarrow^{\mathrm{id}} \\ D^n & \overset{(\mathrm{id},\delta_0)}{\longrightarrow} & D^n \times I & \overset{\eta}{\longrightarrow} & Y \\ \downarrow &&&& \downarrow \\ * && \overset{y}{\longrightarrow} && Y \end{array}$$

(for instance: regard D^n as embedded in $\mathbb{R}^n$ such that $0 \in \mathbb{R}^n$ is identified with the basepoint on the boundary of D^n and set $\eta(\vec{v}, t) \coloneqq \kappa(t \vec{v})$).

The pasting of the top two squares that have appeared this way is equivalent to the following commuting square

$$\begin{array}{ccccc} S^{n-1} & \longrightarrow & & \overset{\alpha}{\longrightarrow} & X \\ {}^{(\mathrm{id},\delta_1)}\downarrow &&&& \downarrow^{f} \\ S^{n-1} \times I & \overset{(\iota_n, \mathrm{id})}{\longrightarrow} & D^n \times I & \overset{\eta}{\longrightarrow} & Y \end{array}.$$

Because f is a Serre fibration and by lemma 1.43 and prop. 1.50, this has a lift

$$\tilde{\eta} : S^{n-1} \times I \longrightarrow X \,.$$

Notice that $\tilde{\eta}$ is a basepoint preserving left homotopy from $\alpha = \tilde{\eta}|_1$ to some $\alpha' \coloneqq \tilde{\eta}|_0$. Being homotopic, they represent the same element of $\pi_{n-1}(X,x)$:

$$[\alpha'] = [\alpha] \,.$$

But the new representative α' has the special property that its image in Y is not just trivializable, but trivialized: combining $\tilde{\eta}$ with the previous diagram shows that it sits in the following commuting diagram

$$\begin{array}{ccccc} \alpha' : & S^{n-1} & \overset{(\mathrm{id},\delta_0)}{\longrightarrow} & S^{n-1} \times I & \overset{\tilde{\eta}}{\longrightarrow} & X \\ & \downarrow^{\iota_n} & & \downarrow^{(\iota_n, \mathrm{id})} & & \downarrow^{f} \\ & D^n & \overset{(\mathrm{id},\delta_0)}{\longrightarrow} & D^n \times I & \overset{\eta}{\longrightarrow} & Y \\ & \downarrow & & & & \downarrow \\ & * & & \overset{y}{\longrightarrow} & & Y \end{array} \,.$$

The commutativity of the outer square says that $f_* \alpha'$ is constant, hence that α' is entirely contained in the fiber F_y. Said more abstractly, the universal property of fibers gives that α' factors through $F_y \overset{\iota}{\hookrightarrow} X$, hence that $[\alpha'] = [\alpha]$ is in the image of ι_*. ▮

The following lemma 1.52, together with lemma 1.40 above, are the only two statements of the entire development here that crucially involve the concrete particular nature of topological spaces ("point-set topology"), everything beyond that is general abstract homotopy theory.

Lemma 1.52. *The continuous functions with the right lifting property, def. 1.46 against the set $I_{\mathrm{Top}} = \{ S^{n-1} \hookrightarrow D^n \}$ of topological generating cofibrations, def. 1.37, are precisely those which are both weak homotopy equivalences, def. 1.30 as well as Serre fibrations, def. 1.47.*

Proof. We break this up into three sub-statements:

A) I_{Top}-injective morphisms are in particular weak homotopy equivalences

Let $p : \hat{X} \to X$ have the right lifting property against I_{Top}

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & \hat{X} \\ {}^{\iota_n}\downarrow & {}^{\exists}\nearrow & \downarrow^{p} \\ D^n & \longrightarrow & X \end{array}$$

We check that the lifts in these diagrams exhibit $\pi_\bullet(f)$ as being an isomorphism on all homotopy groups, def. 1.26:

For $n = 0$ the existence of these lifts says that every point of X is in the image of p, hence that $\pi_0(\hat{X}) \to \pi_0(X)$ is surjective. Let then $S^0 = * \coprod * \longrightarrow \hat{X}$ be a map that hits two connected components, then the existence of the lift says that if they have the same image in $\pi_0(X)$ then they were already the same connected component in $\hat{X}$. Hence $\pi_0(\hat{X}) \to \pi_0(X)$ is also injective and hence is a bijection.

Similarly, for $n \geq 1$, if $S^n \to \hat{X}$ represents an element in $\pi_n(\hat{X})$ that becomes trivial in $\pi_n(X)$, then the existence of the lift says that it already represented the trivial element itself. Hence $\pi_n(\hat{X}) \to \pi_n(X)$ has trivial kernel and so is injective.

Finally, to see that $\pi_n(\hat X) \to \pi_n(X)$ is also surjective, hence bijective, observe that every elements in $\pi_n(X)$ is equivalently represented by a commuting diagram of the form

$$\begin{array}{ccccc} S^{n-1} & \longrightarrow & * & \longrightarrow & \hat X \\ \downarrow && \downarrow && \downarrow \\ D^n & \longrightarrow & X & = & X \end{array}$$

and so here the lift gives a representative of a preimage in $\pi_n(\hat X)$.

B) I_{Top}-injective morphisms are in particular Serre fibrations

By an immediate closure property of lifting problems (we spell this out in generality as prop. 2.10, cor. 2.11 below) an I_{Top}-injective morphism has the right lifting property against all relative cell complexes, and hence, by lemma 1.44, it is also a J_{Top}-injective morphism, hence a Serre fibration.

C) Acyclic Serre fibrations are in particular I_{Top}-injective morphisms

(Hirschhorn 15, section 6).

Let $f : X \to Y$ be a Serre fibration that induces isomorphisms on homotopy groups. In degree 0 this means that f is an isomorphism on connected components, and this means that there is a lift in every commuting square of the form

$$\begin{array}{ccc} S^{-1} = \emptyset & \longrightarrow & X \\ \downarrow && \downarrow^{f} \\ D^0 = * & \longrightarrow & Y \end{array}$$

(this is $\pi_0(f)$ being surjective) and in every commuting square of the form

$$\begin{array}{ccc} S^0 & \longrightarrow & X \\ {}^{\iota_0}\downarrow && \downarrow^{f} \\ D^1 = * & \longrightarrow & Y \end{array}$$

(this is $\pi_0(f)$ being injective). Hence we are reduced to showing that for $n \geq 2$ every diagram of the form

$$\begin{array}{ccc} S^{n-1} & \overset{\alpha}{\longrightarrow} & X \\ {}^{\iota_n}\downarrow && \downarrow^{f} \\ D^n & \overset{\kappa}{\longrightarrow} & Y \end{array}$$

has a lift.

To that end, pick a basepoint on S^{n-1} and write x and y for its images in X and Y, respectively

Then the diagram above expresses that $f_*[\alpha] = 0 \in \pi_{n-1}(Y, y)$ and hence by assumption on f it follows that $[\alpha] = 0 \in \pi_{n-1}(X, x)$, which in turn mean that there is κ' making the upper triangle of our lifting problem commute:

$$\begin{array}{ccc} S^{n-1} & \overset{\alpha}{\longrightarrow} & X \\ {}^{\iota_n}\downarrow & \nearrow_{\kappa'} & \\ D^n && \end{array} .$$

It is now sufficient to show that any such κ' may be deformed to a ρ' which keeps making this upper triangle commute but also makes the remaining lower triangle commute.

To that end, notice that by the commutativity of the original square, we already have at least this commuting square:

$$\begin{array}{ccc} S^{n-1} & \overset{\iota_n}{\longrightarrow} & D^n \\ {}^{\iota_n}\downarrow && \downarrow^{f \circ \kappa'} \\ D^n & \underset{\kappa}{\longrightarrow} & Y \end{array} .$$

This induces the universal map $(\kappa, f \circ \kappa')$ from the pushout of its cospan in the top left, which is the n-sphere (see this example):

$$\begin{array}{ccccc} S^{n-1} & \overset{\iota_n}{\longrightarrow} & D^n \\ {}^{\iota_n}\downarrow & (\mathrm{po}) & \downarrow^{f \circ \kappa'} \\ D^n & \underset{\kappa}{\longrightarrow} & S^n \\ && & \searrow^{(\kappa, f \circ \kappa')} \\ && && Y \end{array} .$$

This universal morphism represents an element of the nth homotopy group:

$$[(\kappa, f \circ \kappa')] \in \pi_n(Y, y) .$$

By assumption that f is a weak homotopy equivalence, there is a $[\rho] \in \pi_n(X, x)$ with

$$f_*[\rho] = [(\kappa, f \circ \kappa')]$$

hence on representatives there is a lift up to homotopy

$$\begin{array}{ccc} && X \\ & {}^{\rho}\nearrow_{\Downarrow} & \downarrow^f \\ S^n & \underset{(\kappa, f \circ \kappa')}{\longrightarrow} & Y \end{array} .$$

Morever, we may always find ρ of the form (ρ', κ') for some $\rho' : D^n \to X$. ("Paste κ' to the reverse of ρ.")

Consider then the map

$$S^n \overset{(f \circ \rho', \kappa)}{\longrightarrow} Y$$

and observe that this represents the trivial class:

$$\begin{aligned} [(f \circ \rho', \kappa)] &= [(f \circ \rho', f \circ \kappa')] + [(f \circ \kappa', \kappa)] \\ &= f_* \underbrace{[(\rho', \kappa')]}_{= [\rho]} + [(f \circ \kappa', \kappa)] \\ &= [(\kappa, f \circ \kappa')] + [(f \circ \kappa', \kappa)] \\ &= 0 \end{aligned} .$$

This means equivalently that there is a homotopy

$$\phi : f \circ \rho' \Rightarrow \kappa$$

fixing the boundary of the n-disk.

Hence if we denote homotopy by double arrows, then we have now achieved the following situation

$$\begin{array}{ccc} S^{n-1} & \overset{\alpha}{\longrightarrow} & X \\ {}^{\iota_n}\downarrow & {}^{\rho'}\nearrow_{\Downarrow \phi} & \downarrow^{f} \\ D^n & \longrightarrow & Y \end{array}$$

and it now suffices to show that ϕ may be lifted to a homotopy of just ρ', fixing the boundary, for then the resulting homotopic ρ'' is the desired lift.

To that end, notice that the condition that $\phi : D^n \times I \to Y$ fixes the boundary of the n-disk means equivalently that it extends to a morphism

$$S^{n-1} \underset{S^{n-1}\times I}{\sqcup} D^n \times I \xrightarrow{(f\circ\alpha, \phi)} Y$$

out of the pushout that identifies in the cylinder over D^n all points lying over the boundary. Hence we are reduced to finding a lift in

$$\begin{array}{ccc} D^n & \overset{\rho'}{\longrightarrow} & X \\ \downarrow & & \downarrow^{f} \\ S^{n-1} \underset{S^{n-1}\times I}{\sqcup} D^n \times I & \xrightarrow{(f\circ\alpha,\phi)} & Y \end{array}.$$

But inspection of the left map reveals that it is homeomorphic again to $D^n \to D^n \times I$, and hence the lift does indeed exist. ▮

2. Abstract homotopy theory

In the above we discussed three classes of continuous functions between topological spaces

1. weak homotopy equivalences;
2. relative cell complexes;
3. Serre fibrations

and we saw first aspects of their interplay via lifting properties.

A fundamental insight due to (Quillen 67) is that in fact *all* constructions in homotopy theory are elegantly expressible via just the abstract interplay of these classes of morphisms. This was distilled in (Quillen 67) into a small set of axioms called a **model category structure** (because it serves to make all objects be *models* for homotopy types.)

This *abstract homotopy theory* is the royal road for handling any flavor of homotopy theory, in particular the stable homotopy theory that we are after in *Part 1*. Here we discuss the basic constructions and facts in abstract homotopy theory, then below we conclude section P1) by showing that the above system of classes of maps of topological spaces is indeed an example.

Literature (Dwyer-Spalinski 95)

Definition 2.1. A **category with weak equivalences** is

1. a category $\mathcal{C}$;
2. a sub-class $W \subset \mathrm{Mor}(\mathcal{C})$ of its morphisms;

such that

1. W contains all the isomorphisms of $\mathcal{C}$;

2. W is closed under **two-out-of-three**: in every commuting diagram in $\mathcal{C}$ of the form

$$\begin{array}{ccccc} & & Y & & \\ & \nearrow & & \searrow & \\ X & & \longrightarrow & & Z \end{array}$$

if two of the three morphisms are in W, then so is the third.

Remark 2.2. It turns out that a category with weak equivalences, def. 2.1, already determines a homotopy theory: the one given given by universally forcing weak equivalences to become actual homotopy equivalences. This may be made precise and is called the *simplicial localization* of a category with weak equivalences (Dwyer-Kan 80a, Dwyer-Kan 80b, Dwyer-Kan 80c). However, without further auxiliary structure, these simplicial localizations are in general intractable. The further axioms of a model category serve the sole purpose of making the universal homotopy theory induced by a category with weak equivalences be tractable:

Definition 2.3. A **model category** is

1. a category $\mathcal{C}$ with all limits and colimits (def. 1.1);
2. three sub-classes $W, \mathrm{Fib}, \mathrm{Cof} \subset \mathrm{Mor}(\mathcal{C})$ of its morphisms;

such that

1. the class W makes $\mathcal{C}$ into a **category with weak equivalences**, def. 2.1;
2. The pairs $(W \cap \mathrm{Cof}\ ,\ \mathrm{Fib})$ and $(\mathrm{Cap}\ ,\ W \cap \mathrm{Fib})$ are both weak factorization systems, def. 2.5.

One says:

- elements in W are *weak equivalences*,
- elements in Cof are *cofibrations*,
- elements in Fib are *fibrations*,
- elements in $W \cap \mathrm{Cof}$ are *acyclic cofibrations*,
- elements in $W \cap \mathrm{Fib}$ are *acyclic fibrations*.

The form of def. 2.3 is due to (Joyal, def. E.1.2). It implies various other conditions that (Quillen 67) demands explicitly, see prop. 2.10 and prop. 2.14 below.

We now dicuss the concept of weak factorization systems appearing in def. 2.3.

Factorization systems

Definition 2.4. Let $\mathcal{C}$ be any category. Given a diagram in $\mathcal{C}$ of the form

$$\begin{array}{ccc} X & \overset{f}{\longrightarrow} & Y \\ {}^{p}\downarrow & & \\ B & & \end{array}$$

then an *extension* of the morphism f along the morphism p is a completion to a commuting diagram of the form

$$\begin{array}{ccc} X & \stackrel{f}{\longrightarrow} & Y \\ {}^{\mathllap{p}}\downarrow & \nearrow_{\tilde f} & \\ B & & \end{array} \,.$$

Dually, given a diagram of the form

$$\begin{array}{ccc} && A \\ && \downarrow^{\mathrlap{p}} \\ X &\stackrel{f}{\longrightarrow}& Y \end{array}$$

then a **lift** of f through p is a completion to a commuting diagram of the form

$$\begin{array}{ccc} && A \\ & {}^{\tilde f}\nearrow & \downarrow^{\mathrlap{p}} \\ X &\stackrel{f}{\longrightarrow}& Y \end{array} \,.$$

Combining these cases: given a commuting square

$$\begin{array}{ccc} X_1 &\stackrel{f_1}{\longrightarrow}& Y_1 \\ {}^{\mathllap{p_l}}\downarrow && \downarrow^{\mathrlap{p_r}} \\ X_2 &\stackrel{f_1}{\longrightarrow}& Y_2 \end{array}$$

then a **lifting** in the diagram is a completion to a commuting diagram of the form

$$\begin{array}{ccc} X_1 &\stackrel{f_1}{\longrightarrow}& Y_1 \\ {}^{\mathllap{p_l}}\downarrow &\nearrow& \downarrow^{\mathrlap{p_r}} \\ X_2 &\stackrel{f_1}{\longrightarrow}& Y_2 \end{array} \,.$$

Given a sub-class of morphisms $K \subset \mathrm{Mor}(\mathcal{C})$, then

- a morphism p_r as above is said to have the **right lifting property against** K or to be a K**-injective morphism** if in all square diagrams with p_r on the right and any $p_l \in K$ on the left a lift exists.

dually:

- a morphism p_l is said to have the **left lifting property against** K or to be a K**-projective morphism** if in all square diagrams with p_l on the left and any $p_r \in K$ on the left a lift exists.

Definition 2.5. A **weak factorization system** (WFS) on a category $\mathcal{C}$ is a pair $(\mathrm{Proj}, \mathrm{Inj})$ of classes of morphisms of $\mathcal{C}$ such that

1. Every morphism $f : X \to Y$ of $\mathcal{C}$ may be factored as the composition of a morphism in Proj followed by one in Inj

$$f \;\colon\; X \stackrel{\in \mathrm{Proj}}{\longrightarrow} Z \stackrel{\in \mathrm{Inj}}{\longrightarrow} Y \,.$$

2. The classes are closed under having the lifting property, def. 2.4, against each other:
    1. Proj is precisely the class of morphisms having the left lifting property against every morphisms in Inj;

2. Inj is precisely the class of morphisms having the right lifting property against every morphisms in Proj.

Definition 2.6. For $\mathcal{C}$ a category, a **functorial factorization** of the morphisms in $\mathcal{C}$ is a functor

$$\mathrm{fact} : \mathcal{C}^{\Delta[1]} \longrightarrow \mathcal{C}^{\Delta[2]}$$

which is a section of the composition functor $d_1 : \mathcal{C}^{\Delta[2]} \to \mathcal{C}^{\Delta[1]}$.

Remark 2.7. In def. 2.6 we are using the following standard notation, see at *simplex category* and at *nerve of a category*:

Write $[1] = \{0 \to 1\}$ and $[2] = \{0 \to 1 \to 2\}$ for the ordinal numbers, regarded as posets and hence as categories. The arrow category $\mathrm{Arr}(\mathcal{C})$ is equivalently the functor category $\mathcal{C}^{\Delta[1]} \coloneqq \mathrm{Funct}(\Delta[1], \mathcal{C})$, while $\mathcal{C}^{\Delta[2]} \coloneqq \mathrm{Funct}(\Delta[2], \mathcal{C})$ has as objects pairs of composable morphisms in $\mathcal{C}$. There are three injective functors $\delta_i : [1] \to [2]$, where δ_i omits the index i in its image. By precomposition, this induces functors $d_i : \mathcal{C}^{\Delta[2]} \longrightarrow \mathcal{C}^{\Delta[1]}$. Here

- d_1 sends a pair of composable morphisms to their composition;
- d_2 sends a pair of composable morphisms to the first morphisms;
- d_0 sends a pair of composable morphisms to the second morphisms.

Definition 2.8. A weak factorization system, def. 2.5, is called a **functorial weak factorization system** if the factorization of morphisms may be chosen to be a functorial factorization fact, def. 2.6, i.e. such that $d_2 \circ \mathrm{fact}$ lands in Proj and $d_0 \circ \mathrm{fact}$ in Inj.

Remark 2.9. Not all weak factorization systems are functorial, def. 2.8, although most (including those produced by the small object argument (prop. 2.17 below), with due care) are.

Proposition 2.10. *Let $\mathcal{C}$ be a category and let $K \subset \mathrm{Mor}(\mathcal{C})$ be a class of morphisms. Write $K\,\mathrm{Proj}$ and $K\,\mathrm{Inj}$, respectively, for the sub-classes of K-projective morphisms and of K-injective morphisms, def. 2.4. Then:*

1. *Both classes contain the class of isomorphism of $\mathcal{C}$.*
2. *Both classes are closed under composition in $\mathcal{C}$.*

   $K\,\mathrm{Proj}$ is also closed under transfinite composition.
3. *Both classes are closed under forming retracts in the arrow category $\mathcal{C}^{\Delta[1]}$ (see remark 2.12).*
4. *$K\,\mathrm{Proj}$ is closed under forming pushouts of morphisms in $\mathcal{C}$ ("cobase change").*

   $K\,\mathrm{Inj}$ is closed under forming pullback of morphisms in $\mathcal{C}$ ("base change").
5. *$K\,\mathrm{Proj}$ is closed under forming coproducts in $\mathcal{C}^{\Delta[1]}$.*

   $K\,\mathrm{Inj}$ is closed under forming products in $\mathcal{C}^{\Delta[1]}$.

Proof. We go through each item in turn.

containing isomorphisms

Given a commuting square

$$\begin{array}{ccc} A & \overset{f}{\to} & X \\ {}^{\mathllap{i}}_{\mathllap{\in \mathrm{Iso}}}\downarrow && \downarrow^{\mathrlap{p}} \\ B & \underset{g}{\longrightarrow} & Y \end{array}$$

with the left morphism an isomorphism, then a lift is given by using the inverse of this isomorphism ${}^{f \circ i^{-1}}\nearrow$. Hence in particular there is a lift when $p \in K$ and so $i \in K\,\mathrm{Proj}$. The other case is formally dual.

closure under composition

Given a commuting square of the form

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow && \downarrow^{\mathrlap{p_1}}_{\mathrlap{\in K\,\mathrm{Inj}}} \\ {}^{\mathllap{i}}_{\mathllap{\in K}}\downarrow && \downarrow^{\mathrlap{p_2}}_{\mathrlap{\in K\,\mathrm{Inj}}} \\ B & \longrightarrow & Y \end{array}$$

consider its pasting decomposition as

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & \searrow & \downarrow^{\mathrlap{p_1}}_{\mathrlap{\in K\,\mathrm{Inj}}} \\ {}^{\mathllap{i}}_{\mathllap{\in K}}\downarrow && \downarrow^{\mathrlap{p_2}}_{\mathrlap{\in K\,\mathrm{Inj}}} \\ B & \longrightarrow & Y \end{array}$$

Now the bottom commuting square has a lift, by assumption. This yields another pasting decomposition

$$\begin{array}{ccc} A & \longrightarrow & X \\ {}^{\mathllap{i}}_{\mathllap{\in K}}\downarrow && \downarrow^{\mathrlap{p_1}}_{\mathrlap{\in K\,\mathrm{Inj}}} \\ \downarrow & \nearrow & \downarrow^{\mathrlap{p_2}}_{\mathrlap{\in K\,\mathrm{Inj}}} \\ B & \longrightarrow & Y \end{array}$$

and now the top commuting square has a lift by assumption. This is now equivalently a lift in the total diagram, showing that $p_1 \circ p_1$ has the right lifting property against K and is hence in $K\,\mathrm{Inj}$. The case of composing two morphisms in $K\,\mathrm{Proj}$ is formally dual. From this the closure of $K\,\mathrm{Proj}$ under transfinite composition follows since the latter is given by colimits of sequential composition and successive lifts against the underlying sequence as above constitutes a cocone, whence the extension of the lift to the colimit follows by its universal property.

closure under retracts

Let j be the retract of an $i \in K\,\mathrm{Proj}$, i.e. let there be a commuting diagram of the form.

$$\begin{array}{ccccccc} \mathrm{id}_A \colon & A & \longrightarrow & C & \longrightarrow & A \\ & \downarrow^{\mathrlap{j}} && \downarrow^{\mathrlap{i}}_{\mathrlap{\in K\,\mathrm{Proj}}} && \downarrow^{\mathrlap{j}} \\ \mathrm{id}_B \colon & B & \longrightarrow & D & \longrightarrow & B \end{array}.$$

Then for

$$\begin{array}{ccc} A & \longrightarrow & X \\ {}^{\mathllap{j}}\downarrow && \downarrow^{\mathrlap{f}}_{\mathrlap{\in K}} \\ B & \longrightarrow & Y \end{array}$$

a commuting square, it is equivalent to its pasting composite with that retract diagram

$$\begin{array}{ccccccc} A & \longrightarrow & C & \longrightarrow & A & \longrightarrow & X \\ \downarrow^{\mathrlap{j}} && \downarrow^{\mathrlap{i}}_{\mathrlap{\in K \mathrm{Proj}}} && \downarrow^{\mathrlap{j}} && \downarrow^{\mathrlap{f}}_{\mathrlap{\in K}} \\ B & \longrightarrow & D & \longrightarrow & B & \longrightarrow & Y \end{array}$$

Here the pasting composite of the two squares on the right has a lift, by assumption:

$$\begin{array}{ccccccc} A & \longrightarrow & C & \longrightarrow & A & \longrightarrow & X \\ \downarrow^{\mathrlap{j}} && \downarrow^{\mathrlap{i}} && \nearrow && \downarrow^{\mathrlap{f}}_{\mathrlap{\in K}} \\ B & \longrightarrow & D & \longrightarrow & B & \longrightarrow & Y \end{array}$$

By composition, this is also a lift in the total outer rectangle, hence in the original square. Hence j has the left lifting property against all $p \in K$ and hence is in $K\,\mathrm{Proj}$. The other case is formally dual.

closure under pushout and pullback

Let $p \in K\,\mathrm{Inj}$ and and let

$$\begin{array}{ccc} Z \times_f X & \longrightarrow & X \\ {}^{\mathllap{f^* p}}\downarrow && \downarrow^{\mathrlap{p}} \\ Z & \overset{f}{\longrightarrow} & Y \end{array}$$

be a pullback diagram in $\mathcal{C}$. We need to show that $f^* p$ has the right lifting property with respect to all $i \in K$. So let

$$\begin{array}{ccc} A & \longrightarrow & Z \times_f X \\ {}^{\mathllap{i}}_{\mathllap{\in K}}\downarrow && \downarrow^{\mathrlap{f^* p}} \\ B & \overset{g}{\longrightarrow} & Z \end{array}$$

be a commuting square. We need to construct a diagonal lift of that square. To that end, first consider the pasting composite with the pullback square from above to obtain the commuting diagram

$$\begin{array}{ccccc} A & \longrightarrow & Z \times_f X & \longrightarrow & X \\ {}^{\mathllap{i}}\downarrow && \downarrow^{\mathrlap{f^* p}} && \downarrow^{\mathrlap{p}} \\ B & \overset{g}{\longrightarrow} & Z & \overset{f}{\longrightarrow} & Y \end{array}.$$

By the right lifting property of p, there is a diagonal lift of the total outer diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow^{\mathrlap{i}} & {}^{\widehat{(f g)}}\nearrow & \downarrow^{\mathrlap{p}} \\ B & \overset{f g}{\longrightarrow} & Y \end{array}.$$

By the universal property of the pullback this gives rise to the lift $\hat g$ in

$$\begin{array}{ccccc} & & Z \times_f X & \longrightarrow & X \\ & {}^{\hat g}\nearrow & \downarrow^{f^* p} & & \downarrow^{p} \\ B & \stackrel{g}{\longrightarrow} & Z & \stackrel{f}{\longrightarrow} & Y \end{array}.$$

In order for $\hat g$ to qualify as the intended lift of the total diagram, it remains to show that

$$\begin{array}{ccc} A & \longrightarrow & Z \times_f X \\ \downarrow^{i} & {}^{\hat g}\nearrow & \\ B & & \end{array}$$

commutes. To do so we notice that we obtain two cones with tip A:

- one is given by the morphisms
  1. $A \to Z \times_f X \to X$
  2. $A \stackrel{i}{\to} B \stackrel{g}{\to} Z$

  with universal morphism into the pullback being

  - $A \to Z \times_f X$
- the other by
  1. $A \stackrel{i}{\to} B \stackrel{\hat g}{\to} Z \times_f X \to X$
  2. $A \stackrel{i}{\to} B \stackrel{g}{\to} Z$.

  with universal morphism into the pullback being

  - $A \stackrel{i}{\to} B \stackrel{\hat g}{\to} Z \times_f X$.

The commutativity of the diagrams that we have established so far shows that the first and second morphisms here equal each other, respectively. By the fact that the universal morphism into a pullback diagram is *unique* this implies the required identity of morphisms.

The other case is formally dual.

closure under (co-)products

Let $\{(A_s \stackrel{i_s}{\to} B_s) \in K\,\mathrm{Proj}\}_{s \in S}$ be a set of elements of $K\,\mathrm{Proj}$. Since colimits in the presheaf category $\mathcal{C}^{\Delta[1]}$ are computed componentwise, their coproduct in this arrow category is the universal morphism out of the coproduct of objects $\coprod_{s \in S} A_s$ induced via its universal property by the set of morphisms i_s:

$$\underset{s \in S}{\sqcup} A_s \overset{(i_s)_{s \in S}}{\longrightarrow} \underset{s \in S}{\sqcup} B_s \,.$$

Now let

$$\begin{array}{ccc} \underset{s \in S}{\sqcup} A_s & \longrightarrow & X \\ {}^{(i_s)_{s \in S}}\downarrow & & \downarrow^{f}_{\in K} \\ \underset{s \in S}{\sqcup} B_s & \longrightarrow & Y \end{array}$$

be a commuting square. This is in particular a cocone under the coproduct of objects, hence by the universal property of the coproduct, this is equivalent to a set of commuting diagrams

$$\left\{ \begin{array}{ccc} A_s & \longrightarrow & X \\ {}^{\mathllap{i_s}}_{\mathllap{\in K\,\mathrm{Proj}}}\downarrow && \downarrow^{\mathrlap{f}}_{\mathrlap{\in K}} \\ B_s & \longrightarrow & Y \end{array} \right\}_{s \in S} \,.$$

By assumption, each of these has a lift ℓ_s. The collection of these lifts

$$\left\{ \begin{array}{ccc} A_s & \longrightarrow & X \\ {}^{\mathllap{i_s}}_{\mathllap{\in \mathrm{Proj}}}\downarrow & {}^{\ell_s}\nearrow & \downarrow^{\mathrlap{f}}_{\mathrlap{\in K}} \\ B_s & \longrightarrow & Y \end{array} \right\}_{s \in S}$$

is now itself a compatible cocone, and so once more by the universal property of the coproduct, this is equivalent to a lift $(\ell_s)_{s\in S}$ in the original square

$$\begin{array}{ccc} \underset{s \in S}{\sqcup} A_s & \longrightarrow & X \\ {}^{\mathllap{(i_s)_{s \in S}}}\downarrow & {}^{(\ell_s)_{s\in S}}\nearrow & \downarrow^{\mathrlap{f}}_{\mathrlap{\in K}} \\ \underset{s \in S}{\sqcup} B_s & \longrightarrow & Y \end{array}$$

This shows that the coproduct of the i_s has the left lifting property against all $f \in K$ and is hence in $K\,\mathrm{Proj}$. The other case is formally dual. ▮

An immediate consequence of prop. 2.10 is this:

Corollary 2.11. *Let $\mathcal{C}$ be a category with all small colimits, and let $K \subset \mathrm{Mor}(\mathcal{C})$ be a sub-class of its morphisms. Then every K-injective morphism, def. 2.4, has the right lifting property, def. 2.4, against all K-relative cell complexes, def. 1.41 and their retracts, remark 2.12.*

Remark 2.12. By a retract of a morphism $X \overset{f}{\longrightarrow} Y$ in some category $\mathcal{C}$ we mean a retract of f as an object in the arrow category $\mathcal{C}^{\Delta[1]}$, hence a morphism $A \overset{g}{\longrightarrow} B$ such that in $\mathcal{C}^{\Delta[1]}$ there is a factorization of the identity on g through f

$$\mathrm{id}_g \;\colon\; g \longrightarrow f \longrightarrow g \,.$$

This means equivalently that in $\mathcal{C}$ there is a commuting diagram of the form

$$\begin{array}{cccccc} \mathrm{id}_A \colon & A &\longrightarrow& X &\longrightarrow& A \\ & \downarrow^{\mathrlap{g}} && \downarrow^{\mathrlap{f}} && \downarrow^{\mathrlap{g}} \\ \mathrm{id}_B \colon & B &\longrightarrow& Y &\longrightarrow& B \end{array} \,.$$

Lemma 2.13. *In every category $\mathcal{C}$ the class of isomorphisms is preserved under retracts in the sense of remark 2.12.*

Proof. For

$$\begin{array}{cccccc} \mathrm{id}_A \colon & A &\longrightarrow& X &\longrightarrow& A \\ & \downarrow^{\mathrlap{g}} && \downarrow^{\mathrlap{f}} && \downarrow^{\mathrlap{g}} \\ \mathrm{id}_B \colon & B &\longrightarrow& Y &\longrightarrow& B \end{array} \,.$$

a retract diagram and $X \overset{f}{\to} Y$ an isomorphism, the inverse to $A \overset{g}{\to} B$ is given by the composite

$$\begin{array}{ccc} && X & \longrightarrow & A \\ && \uparrow^{\mathrlap{f^{-1}}} \\ B & \longrightarrow & Y \end{array} \,.$$

▮

More generally:

Proposition 2.14. *Given a model category in the sense of def. 2.3, then its class of weak equivalences is closed under forming retracts (in the arrow category, see remark 2.12).*

(Joyal, prop. E.1.3)

Proof. Let

$$\begin{array}{cccccc} \mathrm{id}\colon & A & \longrightarrow & X & \longrightarrow & A \\ & {}^{\mathllap{f}}\downarrow && \downarrow^{\mathrlap{w}} && \downarrow^{\mathrlap{f}} \\ \mathrm{id}\colon & B & \longrightarrow & Y & \longrightarrow & B \end{array}$$

be a commuting diagram in the given model category, with $w \in W$ a weak equivalence. We need to show that then also $f \in W$.

First consider the case that $f \in \mathrm{Fib}$.

In this case, factor w as a cofibration followed by an acyclic fibration. Since $w \in W$ and by two-out-of-three (def. 2.1) this is even a factorization through an acyclic cofibration followed by an acyclic fibration. Hence we obtain a commuting diagram of the following form:

$$\begin{array}{cccccc} \mathrm{id}\colon & A & \to & X & \longrightarrow & A \\ & {}^{\mathllap{\mathrm{id}}}\downarrow && \downarrow^{\mathrlap{\in W \cap \mathrm{Cof}}} && \downarrow^{\mathrlap{\mathrm{id}}} \\ \mathrm{id}\colon & A' & \overset{s}{\to} & X' & \overset{t}{\longrightarrow} & A' \\ & {}^{\mathllap{f}}_{\mathllap{\in \mathrm{Fib}}}\downarrow && \downarrow^{\mathrlap{\in W \cap \mathrm{Fib}}} && \downarrow^{\mathrlap{f}}_{\mathrlap{\in \mathrm{Fib}}} \\ \mathrm{id}\colon & B & \to & Y & \longrightarrow & B \end{array}$$

where s is uniquely defined and where t is any lift of the top middle vertical acyclic cofibration against f. This now exhibits f as a retract of an acyclic fibration. These are closed under retract by prop. 2.10.

Now consider the general case. Factor f as an acyclic cofibration followed by a fibration and form the pushout in the top left square of the following diagram

$$\begin{array}{cccccc} \mathrm{id}\colon & A & \to & X & \longrightarrow & A \\ & {}^{\mathllap{\in W \cap \mathrm{Cof}}}\downarrow & (po) & \downarrow^{\mathrlap{\in W \cap \mathrm{Cof}}} && \downarrow^{\mathrlap{\in W \cap \mathrm{Cof}}} \\ \mathrm{id}\colon & A' & \to & X' & \longrightarrow & A' \\ & {}^{\mathllap{\in \mathrm{Fib}}}\downarrow && \downarrow^{\mathrlap{\in W}} && \downarrow^{\mathrlap{\in \mathrm{Fib}}} \\ \mathrm{id}\colon & B & \to & Y & \longrightarrow & B \end{array}$$

where the other three squares are induced by the universal property of the pushout, as is the identification of the middle horizontal composite as the identity on A'. Since acyclic cofibrations are closed under forming pushouts by prop. 2.10, the top middle vertical morphism is now an acyclic fibration, and hence by assumption and by two-out-of-three so is the middle bottom vertical morphism.

Thus the previous case now gives that the bottom left vertical morphism is a weak equivalence, and hence the total left vertical composite is. ▮

Lemma 2.15. ***(retract argument)***

Consider a composite morphism

$$f : X \overset{i}{\longrightarrow} A \overset{p}{\longrightarrow} Y \,.$$

1. *If f has the left lifting property against p, then f is a retract of i.*
2. *If f has the right lifting property against i, then f is a retract of p.*

Proof. We discuss the first statement, the second is formally dual.

Write the factorization of f as a commuting square of the form

$$\begin{array}{ccc} X & \overset{i}{\longrightarrow} & A \\ {}^{\mathllap{f}}\downarrow && \downarrow^{\mathrlap{p}} \\ Y & = & Y \end{array} .$$

By the assumed lifting property of f against p there exists a diagonal filler g making a commuting diagram of the form

$$\begin{array}{ccc} X & \overset{i}{\longrightarrow} & A \\ {}^{\mathllap{f}}\downarrow & {}^{g}\nearrow & \downarrow^{\mathrlap{p}} \\ Y & = & Y \end{array} .$$

By rearranging this diagram a little, it is equivalent to

$$\begin{array}{ccccc} & X & = & X \\ & {}^{\mathllap{f}}\downarrow && {}^{\mathllap{i}}\downarrow \\ \mathrm{id}_Y \colon & Y & \underset{g}{\longrightarrow} & A & \underset{p}{\longrightarrow} & Y \end{array} .$$

Completing this to the right, this yields a diagram exhibiting the required retract according to remark 2.12:

$$\begin{array}{cccccc} \mathrm{id}_X \colon & X & = & X & = & X \\ & {}^{\mathllap{f}}\downarrow && {}^{\mathllap{i}}\downarrow && {}^{\mathllap{f}}\downarrow \\ \mathrm{id}_Y \colon & Y & \underset{g}{\longrightarrow} & A & \underset{p}{\longrightarrow} & Y \end{array} .$$

▮

Small object argument

Given a set $C \subset \mathrm{Mor}(\mathcal{C})$ of morphisms in some category $\mathcal{C}$, a natural question is how to factor any given morphism $f \colon X \longrightarrow Y$ through a relative C-cell complex, def. 1.41, followed by a C-injective morphism, def. 1.46

$$f : X \overset{\in C\,\mathrm{cell}}{\longrightarrow} \hat X \overset{\in C\,\mathrm{inj}}{\longrightarrow} Y \,.$$

A first approximation to such a factorization turns out to be given simply by forming $\hat X = X_1$ by attaching **all** possible C-cells to X. Namely let

$$(C/f) \coloneqq \left\{ \begin{array}{ccc} \mathrm{dom}(c) & \longrightarrow & X \\ {}^{\mathllap{c \in C}}\downarrow && \downarrow^{\mathrlap{f}} \\ \mathrm{cod}(c) & \longrightarrow & Y \end{array} \right\}$$

be the set of **all** ways to find a C-cell attachment in f, and consider the pushout $\hat X$ of the coproduct of morphisms in C over all these:

$$
\begin{array}{ccc}
\coprod_{c \in (C/f)} \mathrm{dom}(c) & \longrightarrow & X \\
{}^{\coprod_{c \in (C/f)} c}\downarrow & (\mathrm{po}) & \downarrow \\
\coprod_{c \in (C/f)} \mathrm{cod}(c) & \longrightarrow & X_1
\end{array}
\,.
$$

This gets already close to producing the intended factorization:

First of all the resulting map $X \to X_1$ is a C-relative cell complex, by construction.

Second, by the fact that the coproduct is over all commuting squres to f, the morphism f itself makes a commuting diagram

$$
\begin{array}{ccc}
\coprod_{c \in (C/f)} \mathrm{dom}(c) & \longrightarrow & X \\
{}^{\coprod_{c \in (C/f)} c}\downarrow & & \downarrow^{f} \\
\coprod_{c \in (C/f)} \mathrm{cod}(c) & \longrightarrow & Y
\end{array}
$$

and hence the universal property of the colimit means that f is indeed factored through that C-cell complex X_1; we may suggestively arrange that factorizing diagram like so:

$$
\begin{array}{ccc}
\coprod_{c \in (C/f)} \mathrm{dom}(c) & \longrightarrow & X \\
{}^{\mathrm{id}}\downarrow & & \downarrow \\
\coprod_{c \in (C/f)} \mathrm{dom}(c) & & X_1 \\
{}^{\coprod_{c \in (C/f)} c}\downarrow & \nearrow & \downarrow \\
\coprod_{c \in (C/f)} \mathrm{cod}(c) & \longrightarrow & Y
\end{array}
\,.
$$

This shows that, finally, the colimiting co-cone map – the one that now appears diagonally – **almost** exhibits the desired right lifting of $X_1 \to Y$ against the $c \in C$. The failure of that to hold on the nose is only the fact that a horizontal map in the middle of the above diagram is missing: the diagonal map obtained above lifts not all commuting diagrams of $c \in C$ into f, but only those where the top morphism $\mathrm{dom}(c) \to X_1$ factors through $X \to X_1$.

The idea of the small object argument now is to fix this only remaining problem by iterating the construction: next factor $X_1 \to Y$ in the same way into

$$
X_1 \longrightarrow X_2 \longrightarrow Y
$$

and so forth. Since relative C-cell complexes are closed under composition, at stage n the resulting $X \longrightarrow X_n$ is still a C-cell complex, getting bigger and bigger. But accordingly, the failure of the accompanying $X_n \longrightarrow Y$ to be a C-injective morphism becomes smaller and smaller, for it now lifts against all diagrams where $\mathrm{dom}(c) \longrightarrow X_n$ factors through $X_{n-1} \longrightarrow X_n$, which intuitively is less and less of a condition as the X_{n-1} grow larger and larger.

The concept of *small object* is just what makes this intuition precise and finishes the small object argument. For the present purpose we just need the following simple version:

Definition 2.16. For $\mathcal{C}$ a category and $C \subset \mathrm{Mor}(\mathcal{C})$ a sub-set of its morphisms, say that these have *small domains* if there is an ordinal α (def. 1.15) such that for every $c \in C$ and for every C-relative cell complex given by a transfinite composition (def. 1.17)

$$
f : X \to X_1 \to X_2 \to \cdots \to X_\beta \to \cdots \longrightarrow \hat{X}
$$

every morphism $\mathrm{dom}(c) \longrightarrow \hat{X}$ factors through a stage $X_\beta \to \hat{X}$ of order $\beta < \alpha$:

$$\begin{array}{ccc} & & X_\beta \\ & \nearrow & \downarrow \\ \mathrm{dom}(c) & \longrightarrow & \hat X \end{array} .$$

The above discussion proves the following:

Proposition 2.17. ***(small object argument)***

Let $\mathcal{C}$ be a locally small category with all small colimits. If a set $C \subset \mathrm{Mor}(\mathcal{C})$ of morphisms has all small domains in the sense of def. 2.16, then every morphism $f : X \longrightarrow$ in $\mathcal{C}$ factors through a C-relative cell complex, def. 1.41, followed by a C-injective morphism, def. 1.46

$$f : X \overset{\in C \, \mathrm{cell}}{\longrightarrow} \hat X \overset{\in C \, \mathrm{inj}}{\longrightarrow} Y .$$

(Quillen 67, II.3 lemma)

Homotopy

We discuss how the concept of homotopy is abstractly realized in model categories, def. 2.3.

Definition 2.18. Let $\mathcal{C}$ be a model category, def. 2.3, and $X \in \mathcal{C}$ an object.

- A **path space object** $\mathrm{Path}(X)$ for X is a factorization of the diagonal $\Delta_X : X \to X \times X$ as

$$\Delta_X : X \underset{\in W}{\overset{i}{\longrightarrow}} \mathrm{Path}(X) \underset{\in \mathrm{Fib}}{\overset{(p_0, p_1)}{\longrightarrow}} X \times X .$$

where $X \to \mathrm{Path}(X)$ is a weak equivalence and $\mathrm{Path}(X) \to X \times X$ is a fibration.

- A **cylinder object** $\mathrm{Cyl}(X)$ for X is a factorization of the codiagonal (or "fold map") $\nabla_X : X \sqcup X \to X$ as

$$\nabla_X : X \sqcup X \underset{\in \mathrm{Cof}}{\overset{(i_0, i_1)}{\longrightarrow}} \mathrm{Cyl}(X) \underset{\in W}{\overset{p}{\longrightarrow}} X .$$

where $\mathrm{Cyl}(X) \to X$ is a weak equivalence. and $X \sqcup X \to \mathrm{Cyl}(X)$ is a cofibration.

Remark 2.19. For every object $X \in \mathcal{C}$ in a model category, a cylinder object and a path space object according to def. 2.18 exist: the factorization axioms guarantee that there exists

1. a factorization of the codiagonal as

$$\nabla_X : X \sqcup X \overset{\in \mathrm{Cof}}{\longrightarrow} \mathrm{Cyl}(X) \overset{\in W \cap \mathrm{Fib}}{\longrightarrow} X$$

2. a factorization of the diagonal as

$$\Delta_X : X \overset{\in W \cap \mathrm{Cof}}{\longrightarrow} \mathrm{Path}(X) \overset{\in \mathrm{Fib}}{\longrightarrow} X \times X .$$

The cylinder and path space objects obtained this way are actually better than required by def. 2.18: in addition to $\mathrm{Cyl}(X) \to X$ being just a weak equivalence, for these this is actually an acyclic fibration, and dually in addition to $X \to \mathrm{Path}(X)$ being a weak equivalence, for these it is actually an acyclic cofibrations.

Some authors call cylinder/path-space objects with this extra property "very good" cylinder/path-space objects, respectively.

One may also consider dropping a condition in def. 2.18: what mainly matters is the weak equivalence, hence some authors take cylinder/path-space objects to be defined as in def. 2.18 but without the condition that $X \sqcup X \to \mathrm{Cyl}(X)$ is a cofibration and without the condition that $\mathrm{Path}(X) \to X$ is a fibration. Such authors would then refer to the concept in def. 2.18 as "good" cylinder/path-space objects.

The terminology in def. 2.18 follows the original (Quillen 67, I.1 def. 4). With the induced concept of left/right homotopy below in def. 2.22, this admits a quick derivation of the key facts in the following, as we spell out below.

Lemma 2.20. *Let $\mathcal{C}$ be a model category. If $X \in \mathcal{C}$ is cofibrant, then for every cylinder object $\mathrm{Cyl}(X)$ of X, def. 2.18, not only is $(i_0, i_1) : X \sqcup X \to X$ a cofibration, but each*

$$i_0, i_1 : X \longrightarrow \mathrm{Cyl}(X)$$

is an acyclic cofibration separately.

Dually, if $X \in \mathcal{C}$ is fibrant, then for every path space object $\mathrm{Path}(X)$ of X, def. 2.18, not only is $(p_0, p_1) : \mathrm{Path}(X) \to X \times X$ a cofibration, but each

$$p_0, p_1 : \mathrm{Path}(X) \longrightarrow X$$

is an acyclic fibration separately.

Proof. We discuss the case of the path space object. The other case is formally dual.

First, that the component maps are weak equivalences follows generally: by definition they have a right inverse $\mathrm{Path}(X) \to X$ and so this follows by two-out-of-three (def. 2.1).

But if X is fibrant, then also the two projection maps out of the product $X \times X \to X$ are fibrations, because they are both pullbacks of the fibration $X \to *$

$$\begin{array}{ccc} X \times X & \longrightarrow & X \\ \downarrow & (\mathrm{pb}) & \downarrow \\ X & \longrightarrow & * \end{array} .$$

hence $p_i : \mathrm{Path}(X) \to X \times X \to X$ is the composite of two fibrations, and hence itself a fibration, by prop. 2.10. ▮

Path space objects are very non-unique as objects up to isomorphism:

Example 2.21. If $X \in \mathcal{C}$ is a fibrant object in a model category, def. 2.3, and for $\mathrm{Path}_1(X)$ and $\mathrm{Path}_2(X)$ two path space objects for X, def. 2.18, then the fiber product $\mathrm{Path}_1(X) \times_X \mathrm{Path}_2(X)$ is another path space object for X: the pullback square

$$\begin{array}{ccc} X & \overset{\Delta_X}{\longrightarrow} & X \times X \\ \downarrow && \downarrow \\ \mathrm{Path}_1(X) \underset{X}{\times} \mathrm{Path}_2(X) & \longrightarrow & \mathrm{Path}_1(X) \times \mathrm{Path}_2(X) \\ {}^{\in \mathrm{Fib}}\downarrow & (\mathrm{pb}) & \downarrow^{\in \mathrm{Fib}} \\ X \times X \times X & \overset{(\mathrm{id}, \Delta_X, \mathrm{id})}{\longrightarrow} & X \times X \times X \times X \\ \downarrow^{(\mathrm{pr}_1, \mathrm{pr}_3)}_{\in \mathrm{Fib}} && \downarrow^{(p_1, p_4)} \\ X \times X & = & X \times X \end{array}$$

gives that the induced projection is again a fibration. Moreover, using lemma 2.20 and two-out-of-three (def. 2.1) gives that $X \to \mathrm{Path}_1(X) \times_X \mathrm{Path}_2(X)$ is a weak equivalence.

For the case of the canonical topological path space objects of def 1.34, with $\mathrm{Path}_1(X) = \mathrm{Path}_2(X) = X^I = X^{[0,1]}$ then this new path space object is $X^{I \vee I} = X^{[0,2]}$, the mapping space out of the standard interval of length 2 instead of length 1.

Definition 2.22. Let $f, g : X \longrightarrow Y$ be two parallel morphisms in a model category.

- A **left homotopy** $\eta : f \Rightarrow_L g$ is a morphism $\eta : \mathrm{Cyl}(X) \longrightarrow Y$ from a cylinder object of X, def. 2.18, such that it makes this diagram commute:

$$\begin{array}{ccccc} X & \longrightarrow & \mathrm{Cyl}(X) & \longleftarrow & X \\ & {}_{f}\searrow & \downarrow^{\eta} & \swarrow_{g} & \\ && Y && \end{array} \,.$$

- A **right homotopy** $\eta : f \Rightarrow_R g$ is a morphism $\eta : X \to \mathrm{Path}(Y)$ to some path space object of X, def. 2.18, such that this diagram commutes:

$$\begin{array}{ccccc} && X && \\ & {}^{f}\swarrow & \downarrow^{\eta} & \searrow^{g} & \\ Y & \longleftarrow & \mathrm{Path}(Y) & \longrightarrow & Y \end{array} \,.$$

Lemma 2.23. *Let $f, g : X \to Y$ be two parallel morphisms in a model category.*

1. *Let X be cofibrant. If there is a left homotopy $f \Rightarrow_L g$ then there is also a right homotopy $f \Rightarrow_R g$ (def. 2.22) with respect to any chosen path space object.*

2. *Let X be fibrant. If there is a right homotopy $f \Rightarrow_R g$ then there is also a left homotopy $f \Rightarrow_L g$ with respect to any chosen cylinder object.*

In particular if X is cofibrant and Y is fibrant, then by going back and forth it follows that every left homotopy is exhibited by every cylinder object, and every right homotopy is exhibited by every path space object.

Proof. We discuss the first case, the second is formally dual. Let $\eta : \mathrm{Cyl}(X) \longrightarrow Y$ be the given left homotopy. Lemma 2.20 implies that we have a lift h in the following commuting diagram

$$\begin{array}{ccc} X & \overset{i \circ f}{\longrightarrow} & \mathrm{Path}(Y) \\ {}^{i_0}_{\in W \cap \mathrm{Cof}}\downarrow & {}^{h}\nearrow & \downarrow^{p_0, p_1}_{\in \mathrm{Fib}} \\ \mathrm{Cyl}(X) & \underset{(f \circ p, \eta)}{\longrightarrow} & Y \times Y \end{array} \,,$$

where on the right we have the chosen path space object. Now the composite $\tilde{\eta} \coloneqq h \circ i_1$ is a right homotopy as required:

$$\begin{array}{ccccc} &&&& \mathrm{Path}(Y) \\ &&& {}^{h}\nearrow & \downarrow^{p_0, p_1}_{\in \mathrm{Fib}} \\ X & \overset{i_1}{\longrightarrow} & \mathrm{Cyl}(X) & \underset{(f \circ p, \eta)}{\longrightarrow} & Y \times Y \end{array} \,.$$

■

Proposition 2.24. *For X a cofibrant object in a model category and Y a fibrant object, then the relations of left homotopy $f \Rightarrow_L g$ and of right homotopy $f \Rightarrow_R g$ (def. 2.22) on the hom set $\mathrm{Hom}(X, Y)$ coincide and are both equivalence relations.*

Proof. That both relations coincide under the (co-)fibrancy assumption follows directly from lemma 2.23.

The symmetry and reflexivity of the relation is obvious.

That right homotopy (hence also left homotopy) with domain X is a transitive relation follows from using example 2.21 to compose path space objects. ▮

The homotopy category

We discuss the construction that takes a model category, def. 2.3, and then universally forces all its weak equivalences into actual isomorphisms.

Definition 2.25. Let $\mathcal{C}$ be a model category, def. 2.3. Write $\mathrm{Ho}(\mathcal{C})$ for the category whose

- objects are those objects of $\mathcal{C}$ which are both fibrant and cofibrant;
- morphisms are the homotopy classes of morphisms of $\mathcal{C}$, hence the equivalence classes of morphism under the equivalence relation of prop. 2.24;

and whose composition operation is given on representatives by composition in $\mathcal{C}$.

This is, up to equivalence of categories, the **homotopy category of the model category** $\mathcal{C}$.

Proposition 2.26. *Def. 2.25 is well defined, in that composition of morphisms between fibrant-cofibrant objects in $\mathcal{C}$ indeed passes to homotopy classes.*

Proof. Fix any morphism $X \xrightarrow{f} Y$ between fibrant-cofibrant objects. Then for precomposition

$$(-) \circ [f] \; : \; \mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(Y, Z) \to \mathrm{Hom}_{\mathrm{Ho}(\mathcal{C}(X,Z))}$$

to be well defined, we need that with $(g \sim h) \; : \; Y \to Z$ also $(fg \sim fh) \; : \; X \to Z$. But by prop 2.24 we may take the homotopy $\sim$ to be exhibited by a right homotopy $\eta : Y \to \mathrm{Path}(Z)$, for which case the statement is evident from this diagram:

$$\begin{array}{ccccc} & & & & Z \\ & & & {}^{g}\nearrow & \uparrow^{p_1} \\ X & \overset{f}{\longrightarrow} & Y & \overset{\eta}{\longrightarrow} & \mathrm{Path}(Z) \\ & & & {}_{h}\searrow & \downarrow_{p_0} \\ & & & & Z \end{array} \,.$$

For postcomposition we may choose to exhibit homotopy by left homotopy and argue dually. ▮

We now spell out that def. 2.25 indeed satisfies the universal property that defines the localization of a category with weak equivalences at its weak equivalences.

Lemma 2.27. ***(Whitehead theorem in model categories)***

Let $\mathcal{C}$ be a model category. A weak equivalence between two objects which are both fibrant and cofibrant is a homotopy equivalence.

Proof. By the factorization axioms in the model category $\mathcal{C}$ and by two-out-of-three (def. 2.1), every weak equivalence $f : X \longrightarrow Y$ factors through an object Z as an acyclic cofibration followed by an acyclic fibration. In particular it follows that with X and Y both fibrant and cofibrant, so is Z, and hence it is sufficient to prove that acyclic (co-)fibrations between such objects are homotopy equivalences.

So let $f : X \longrightarrow Y$ be an acyclic fibration between fibrant-cofibrant objects, the case of acyclic cofibrations is formally dual. Then in fact it has a genuine right inverse given by a lift f^{-1} in the diagram

$$
\begin{array}{ccc}
\emptyset & \to & X \\
{}^{\in \mathrm{cof}}\downarrow & {}^{f^{-1}}\nearrow & \downarrow^{f}_{\in \mathrm{Fib} \cap W} \\
X & = & X
\end{array}
$$

To see that f^{-1} is also a left inverse up to left homotopy, let $\mathrm{Cyl}(X)$ be any cylinder object on X (def. 2.18), hence a factorization of the codiagonal on X as a cofibration followed by a an acyclic fibration

$$
X \sqcup X \overset{\iota_X}{\longrightarrow} \mathrm{Cyl}(X) \overset{p}{\longrightarrow} X
$$

and consider the commuting square

$$
\begin{array}{ccc}
X \sqcup X & \overset{(f^{-1} \circ f, \mathrm{id})}{\longrightarrow} & X \\
{}^{\iota_X}_{\in \mathrm{Cof}}\downarrow && \downarrow^{f}_{\in W \cap \mathrm{Fib}} \\
\mathrm{Cyl}(X) & \underset{f \circ p}{\longrightarrow} & Y
\end{array}
$$

which commutes due to f^{-1} being a genuine right inverse of f. By construction, this commuting square now admits a lift η, and that constitutes a left homotopy $\eta : f^{-1} \circ f \Rightarrow_L \mathrm{id}$. ▮

Definition 2.28. Given a model category $\mathcal{C}$, consider a *choice* for each object $X \in \mathcal{C}$ of

1. a factorization $\emptyset \underset{\in \mathrm{Cof}}{\overset{i_X}{\longrightarrow}} QX \underset{\in W \cap \mathrm{Fib}}{\overset{p_X}{\longrightarrow}} X$ of the initial morphism, such that when X is already cofibrant then $p_X = \mathrm{id}_X$;

2. a factorization $X \underset{\in W \cap \mathrm{Cof}}{\overset{j_X}{\longrightarrow}} PX \underset{\in \mathrm{Fib}}{\overset{q_X}{\longrightarrow}} *$ of the terminal morphism, such that when X is already fibrant then $j_X = \mathrm{id}_X$.

Write then

$$
\gamma_{P,Q} : \mathcal{C} \longrightarrow \mathrm{Ho}(\mathcal{C})
$$

for the functor to the homotopy category, def. 2.25, which sends an object X to the object PQX and sends a morphism $f : X \longrightarrow Y$ to the homotopy class of the result of first lifting in

$$
\begin{array}{ccc}
\emptyset & \longrightarrow & QY \\
{}^{i_X}\downarrow & {}^{Qf}\nearrow & \downarrow^{p_Y} \\
QX & \underset{f \circ p_X}{\longrightarrow} & Y
\end{array}
$$

and then lifting (here: extending) in

$$
\begin{array}{ccc}
QX & \overset{j_{QY} \circ Qf}{\longrightarrow} & PQY \\
{}^{j_{QX}}\downarrow & {}^{PQf}\nearrow & \downarrow^{q_{QY}} \\
PQX & \longrightarrow & *
\end{array}
$$

Lemma 2.29. *The construction in def. 2.28 is indeed well defined.*

Proof. First of all, the object PQX is indeed both fibrant and cofibrant (as well as related by a zig-zag of weak equivalences to X):

$$\begin{tikzcd} \emptyset \arrow[d, "\in \mathrm{Cof}"'] \arrow[dr, "\in \mathrm{Cof}"] & & \\ QX \arrow[r, "\in W \cap \mathrm{Cof}"'] \arrow[d, "\in W"'] & PQX \arrow[r, "\in \mathrm{Fib}"'] & * \\ X & & \end{tikzcd} .$$

Now to see that the image on morphisms is well defined. First observe that any two choices $(Qf)_i$ of the first lift in the definition are left homotopic to each other, exhibited by lifting in

$$\begin{tikzcd} QX \sqcup QX \arrow[r, "((Qf)_1, (Qf)_2)"] \arrow[d, "\in \mathrm{Cof}"'] & QY \arrow[d, "p_Y", "\in W \cap \mathrm{Fib}"'] \\ \mathrm{Cyl}(QX) \arrow[r, "f \circ p_X \circ \sigma_{QX}"'] & Y \end{tikzcd}$$

Hence also the composites $j_{QY} \circ (Q_f)_i$ are left homotopic to each other, and since their domain is cofibrant, then by lemma 2.23 they are also right homotopic by a right homotopy κ. This implies finally, by lifting in

$$\begin{tikzcd} QX \arrow[r, "\kappa"] \arrow[d, "\in W \cap \mathrm{Cof}"'] & \mathrm{Path}(PQY) \arrow[d, "\in \mathrm{Fib}"] \\ PQX \arrow[r, "(R(Qf)_1, P(Qf)_2)"'] & PQY \times PQY \end{tikzcd}$$

that also $P(Qf)_1$ and $P(Qf)_2$ are right homotopic, hence that indeed PQf represents a well-defined homotopy class.

Finally to see that the assignment is indeed functorial, observe that the commutativity of the lifting diagrams for Qf and PQf imply that also the following diagram commutes

$$\begin{tikzcd} X \arrow[d, "f"'] & QX \arrow[l, "p_X"'] \arrow[r, "j_{QX}"] \arrow[d, "Qf"] & PQX \arrow[d, "PQf"] \\ Y & QY \arrow[l, "p_y"] \arrow[r, "j_{QY}"'] & PQY \end{tikzcd} .$$

Now from the pasting composite

$$\begin{tikzcd} X \arrow[d, "f"'] & QX \arrow[l, "p_X"'] \arrow[r, "j_{QX}"] \arrow[d, "Qf"] & PQX \arrow[d, "PQf"] \\ Y \arrow[d, "g"'] & QY \arrow[l, "p_Y"] \arrow[r, "j_{QY}"'] \arrow[d, "Qg"] & PQY \arrow[d, "PQg"] \\ Z & QZ \arrow[l, "p_Z"] \arrow[r, "j_{QZ}"'] & PQZ \end{tikzcd}$$

one sees that $(PQg) \circ (PQf)$ is a lift of $g \circ f$ and hence the same argument as above gives that it is homotopic to the chosen $PQ(g \circ f)$. ▮

For the following, recall the concept of natural isomorphism between functors: for $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ two functors, then a *natural transformation* $\eta : F \Rightarrow G$ is for each object $c \in \mathrm{Obj}(\mathcal{C})$ a morphism $\eta_c : F(c) \longrightarrow G(c)$ in $\mathcal{D}$, such that for each morphism $f : c_1 \to c_2$ in $\mathcal{C}$ the following is a commuting square:

$$\begin{array}{ccc} F(c_1) & \overset{\eta_{c_1}}{\longrightarrow} & G(c_1) \\ {}^{\mathllap{F(f)}}\downarrow && \downarrow^{\mathrlap{G(f)}} \\ F(c_2) & \underset{\eta_{c_2}}{\longrightarrow} & G(c_2) \end{array}.$$

Such η is called a *natural isomorphism* if its η_c are isomorphisms for all objects c.

Definition 2.30. For $\mathcal{C}$ a category with weak equivalences, its **localization at the weak equivalences** is, if it exists,

1. a category denoted $\mathcal{C}[W^{-1}]$
2. a functor

$$\gamma : \mathcal{C} \longrightarrow \mathcal{C}[W^{-1}]$$

such that

1. γ sends weak equivalences to isomorphisms;
2. γ is universal with this property, in that:

   for $F : \mathcal{C} \longrightarrow D$ any functor out of $\mathcal{C}$ into any category D, such that F takes weak equivalences to isomorphisms, it factors through γ up to a natural isomorphism ρ

$$\begin{array}{ccccc} \mathcal{C} && \overset{F}{\longrightarrow} && D \\ & {}_{\mathllap{\gamma}}\searrow & \Downarrow^{\rho} & \nearrow_{\mathrlap{\tilde F}} \\ && \mathrm{Ho}(\mathcal{C}) \end{array}$$

   and this factorization is unique up to unique isomorphism, in that for $(\tilde F_1, \rho_1)$ and $(\tilde F_2, \rho_2)$ two such factorizations, then there is a unique natural isomorphism $\kappa : \tilde F_1 \Rightarrow \tilde F_2$ making the evident diagram of natural isomorphisms commute.

Theorem 2.31. *For $\mathcal{C}$ a model category, the functor $\gamma_{P,Q}$ in def. 2.28 (for any choice of P and Q) exhibits $\mathrm{Ho}(\mathcal{C})$ as indeed being the localization of the underlying category with weak equivalences at its weak equivalences, in the sense of def. 2.30:*

$$\begin{array}{ccc} \mathcal{C} & = & \mathcal{C} \\ {}^{\mathllap{\gamma_{P,Q}}}\downarrow && \downarrow^{\mathrlap{\gamma}} \\ \mathrm{Ho}(\mathcal{C}) & \simeq & \mathcal{C}[W^{-1}] \end{array}.$$

(Quillen 67, I.1 theorem 1)

Proof. First, to see that that $\gamma_{P,Q}$ indeed takes weak equivalences to isomorphisms: By two-out-of-three (def. 2.1) applied to the commuting diagrams shown in the proof of lemma 2.29, the morphism PQf is a weak equivalence if f is:

$$\begin{array}{ccccc} X & \underoverset{\simeq}{p_X}{\longleftarrow} & QX & \underoverset{\simeq}{j_{QX}}{\longrightarrow} & PQX \\ {}^{\mathllap{f}}\downarrow && \downarrow^{\mathrlap{Qf}} && \downarrow^{\mathrlap{PQf}} \\ Y & \underoverset{p_y}{\simeq}{\longleftarrow} & QY & \underoverset{j_{QY}}{\simeq}{\longrightarrow} & PQY \end{array}$$

With this the "Whitehead theorem for model categories", lemma 2.27, implies that PQf represents an isomorphism in $\mathrm{Ho}(\mathcal{C})$.

Now let $F : \mathcal{C} \longrightarrow D$ be any functor that sends weak equivalences to isomorphisms. We need to show that it factors as

$$\begin{array}{ccccc} \mathcal{C} && \overset{F}{\longrightarrow} && D \\ & {}_{\gamma}\searrow & \Downarrow^{\rho} & \nearrow_{\tilde F} \\ && \mathrm{Ho}(\mathcal{C}) \end{array}$$

uniquely up to unique natural isomorphism. Now by construction of P and Q in def. 2.28, $\gamma_{P,Q}$ is the identity on the full subcategory of fibrant-cofibrant objects. It follows that if $\tilde F$ exists at all, it must satisfy for all $X \overset{f}{\to} Y$ with X and Y both fibrant and cofibrant that

$$\tilde F([f]) \simeq F(f) \,,$$

(hence in particular $\tilde F(\gamma_{P,Q}(f)) = F(PQf)$).

But by def. 2.25 that already fixes $\tilde F$ on all of $\mathrm{Ho}(\mathcal{C})$, up to unique natural isomorphism. Hence it only remains to check that with this definition of $\tilde F$ there exists any natural isomorphism ρ filling the diagram above.

To that end, apply F to the above commuting diagram to obtain

$$\begin{array}{ccccc} F(X) & \underset{\text{iso}}{\overset{F(p_X)}{\longleftarrow}} & F(Q X) & \underset{\text{iso}}{\overset{F(j_{Q X})}{\longrightarrow}} & F(P Q X) \\ {}^{\mathllap{F(f)}}\downarrow && \downarrow^{\mathrlap{F(Q f)}} && \downarrow^{\mathrlap{F(P Q f)}} \\ F(Y) & \underset{F(p_y)}{\overset{\text{iso}}{\longleftarrow}} & F(Q Y) & \underset{F(j_{Q Y})}{\overset{\text{iso}}{\longrightarrow}} & F(P Q Y) \end{array}$$

Here now all horizontal morphisms are isomorphisms, by assumption on F. It follows that defining $\rho_X \coloneqq F(j_{Q X}) \circ F(p_X)^{-1}$ makes the required natural isomorphism:

$$\begin{array}{ccccccccc} \rho_X : & F(X) & \underset{\text{iso}}{\overset{F(p_X)^{-1}}{\longrightarrow}} & F(Q X) & \underset{\text{iso}}{\overset{F(j_{Q X})}{\longrightarrow}} & F(P Q X) & = & \tilde F(\gamma_{P,Q}(X)) \\ & {}^{\mathllap{F(f)}}\downarrow &&&& \downarrow^{\mathrlap{F(P Q f)}} && \downarrow^{\mathrlap{\tilde F(\gamma_{P,Q}(f))}} \\ \rho_Y : & F(Y) & \underset{F(p_y)^{-1}}{\overset{\text{iso}}{\longrightarrow}} & F(Q Y) & \underset{F(j_{Q Y})}{\overset{\text{iso}}{\longrightarrow}} & F(P Q Y) & = & \tilde F(\gamma_{P,Q}(X)) \end{array} \,.$$

∎

Remark 2.32. Due to theorem 2.31 we may suppress the choices of cofibrant Q and fibrant replacement P in def. 2.28 and just speak of the localization functor

$$\gamma \;:\; \mathcal{C} \longrightarrow \mathrm{Ho}(\mathcal{C})$$

up to natural isomorphism.

In general, the localization $\mathcal{C}[W^{-1}]$ of a category with weak equivalences $(\mathcal{C}, W)$ (def. 2.30) may invert *more* morphisms than just those in W. However, if the category admits the structure of a model category $(\mathcal{C}, W, \mathrm{Cof}, \mathrm{Fib})$, then its localiztion precisely only inverts the weak equivalences.

Proposition 2.33. *Let $\mathcal{C}$ be a model category (def. 2.3) and let $\gamma : \mathcal{C} \longrightarrow \mathrm{Ho}(\mathcal{C})$ be its localization functor (def. 2.28, theorem 2.31). Then a morphism f in $\mathcal{C}$ is a weak equivalence precisely if $\gamma(f)$ is an isomorphism in $\mathrm{Ho}(\mathcal{C})$.*

(e.g. Goerss-Jardine 96, II, prop 1.14)

While the construction of the homotopy category in def. 2.25 combines the restriction to good (fibrant/cofibrant) objects with the passage to homotopy classes of morphisms, it is often useful to consider intermediate stages:

Definition 2.34. Given a model category $\mathcal{C}$, write

$$\begin{array}{ccccc} & & \mathcal{C}_{fc} & & \\ & \swarrow & & \searrow & \\ \mathcal{C}_c & & & & \mathcal{C}_f \\ & \searrow & & \swarrow & \\ & & \mathcal{C} & & \end{array}$$

for the system of full subcategory inclusions of:

1. the category of fibrant objects $\mathcal{C}_f$
2. the category of cofibrant objects $\mathcal{C}_c$,
3. the category of fibrant-cofibrant objects $\mathcal{C}_{\mathrm{fc}}$,

all regarded a categories with weak equivalences (def. 2.1), via the weak equivalences inherited from $\mathcal{C}$, which we write $(\mathcal{C}_f, W_f)$, $(\mathcal{C}_c, W_c)$ and $(\mathcal{C}_{fc}, W_{fc})$.

Remark 2.35. Of course the subcategories in def. 2.34 inherit more structure than just that of categories with weak equivalences from $\mathcal{C}$. $\mathcal{C}_f$ and $\mathcal{C}_c$ each inherit "half" of the factorization axioms. One says that $\mathcal{C}_f$ has the structure of a "fibration category" called a "Brown-category of fibrant objects", while $\mathcal{C}_c$ has the structure of a "cofibration category".

We discuss properties of these categories of (co-)fibrant objects below in *Homotopy fiber sequences*.

The proof of theorem 2.31 immediately implies the following:

Corollary 2.36. *For $\mathcal{C}$ a model category, the restriction of the localization functor $\gamma : \mathcal{C} \longrightarrow \mathrm{Ho}(\mathcal{C})$ from def. 2.28 (using remark 2.32) to any of the sub-categories with weak equivalences of def. 2.34*

$$\begin{array}{ccccc} & & \mathcal{C}_{fc} & & \\ & \swarrow & & \searrow & \\ \mathcal{C}_c & & & & \mathcal{C}_f \\ & \searrow & & \swarrow & \\ & & \mathcal{C} & & \\ & & \downarrow^{\gamma} & & \\ & & \mathrm{Ho}(\mathcal{C}) & & \end{array}$$

exhibits $\mathrm{Ho}(\mathcal{C})$ equivalently as the localization also of these subcategories with weak equivalences, at their weak equivalences. In particular there are equivalences of categories

$$\mathrm{Ho}(\mathcal{C}) \simeq \mathcal{C}[W^{-1}] \simeq \mathcal{C}_f[W_f^{-1}] \simeq \mathcal{C}_c[W_c^{-1}] \simeq \mathcal{C}_{fc}[W_{fc}^{-1}] \,.$$

The following says that for computing the hom-sets in the homotopy category, even a mixed variant of the above will do; it is sufficient that the domain is cofibrant and the codomain is fibrant:

Lemma 2.37. *For $X, Y \in \mathcal{C}$ with X cofibrant and Y fibrant, and for P, Q fibrant/cofibrant replacement functors as in def. 2.28, then the morphism*

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(PX, QY) = \mathrm{Hom}_{\mathcal{C}}(PX, QY)/_\sim \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(j_X, p_Y)} \mathrm{Hom}_{\mathcal{C}}(X, Y)/_\sim$$

(on homotopy classes of morphisms, well defined by prop. 2.24) is a natural bijection.

(Quillen 67, I.1 lemma 7)

Proof. We may factor the morphism in question as the composite

$$\mathrm{Hom}_{\mathcal{C}}(PX, QY)/_\sim \overset{\mathrm{Hom}_{\mathcal{C}}(\mathrm{id}_{PX}, p_Y)/_\sim}{\longrightarrow} \mathrm{Hom}_{\mathcal{C}}(PX, Y)/_\sim \overset{\mathrm{Hom}_{\mathcal{C}}(j_X, \mathrm{id}_Y)/_\sim}{\longrightarrow} \mathrm{Hom}_{\mathcal{C}}(X, Y)/_\sim \,.$$

This shows that it is sufficient to see that for X cofibrant and Y fibrant, then

$$\mathrm{Hom}_{\mathcal{C}}(\mathrm{id}_X, p_Y)/_\sim \;:\; \mathrm{Hom}_{\mathcal{C}}(X, QY)/_\sim \to \mathrm{Hom}_{\mathcal{C}}(X, Y)/_\sim$$

is an isomorphism, and dually that

$$\mathrm{Hom}_{\mathcal{C}}(j_X, \mathrm{id}_Y)/_\sim \;:\; \mathrm{Hom}_{\mathcal{C}}(PX, Y)/_\sim \to \mathrm{Hom}_{\mathcal{C}}(X, Y)/_\sim$$

is an isomorphism. We discuss this for the former; the second is formally dual:

First, that $\mathrm{Hom}_{\mathcal{C}}(\mathrm{id}_X, p_Y)$ is surjective is the lifting property in

$$\begin{array}{ccc} \emptyset & \longrightarrow & QY \\ {}^{\mathllap{\in \mathrm{Cof}}}\downarrow && \downarrow^{\mathrlap{p_Y}}_{\mathrlap{\in W \cap \mathrm{Fib}}} \\ X & \overset{f}{\longrightarrow} & Y \end{array}$$

which says that any morphism $f : X \to Y$ comes from a morphism $\hat f : X \to QY$ under postcomposition with $QY \overset{p_Y}{\to} Y$.

Second, that $\mathrm{Hom}_{\mathcal{C}}(\mathrm{id}_X, p_Y)$ is injective is the lifting property in

$$\begin{array}{ccc} X \sqcup X & \overset{(f,g)}{\longrightarrow} & QY \\ {}^{\mathllap{\in \mathrm{Cof}}}\downarrow && \downarrow^{\mathrlap{p_Y}}_{\mathrlap{\in W \cap \mathrm{Fib}}} \\ \mathrm{Cyl}(X) & \underset{\eta}{\longrightarrow} & Y \end{array}$$

which says that if two morphisms $f, g : X \to QY$ become homotopic after postcomposition with $p_Y : QX \to Y$, then they were already homotopic before. ▮

We record the following fact which will be used in part 1.1 (here):

Lemma 2.38. *Let $\mathcal{C}$ be a model category (def. 2.3). Then every commuting square in its homotopy category $\mathrm{Ho}(\mathcal{C})$ (def. 2.25) is, up to isomorphism of squares, in the image of the localization functor $\mathcal{C} \longrightarrow \mathrm{Ho}(\mathcal{C})$ of a commuting square in $\mathcal{C}$ (i.e.: not just commuting up to homotopy).*

Proof. Let

$$\begin{array}{ccc} A & \overset{f}{\longrightarrow} & B \\ {}^{\mathllap{a}}\downarrow && \downarrow^{\mathrlap{b}} \\ A' & \underset{f'}{\longrightarrow} & B' \end{array} \quad \in \mathrm{Ho}(\mathcal{C})$$

be a commuting square in the homotopy category. Writing the same symbols for fibrant-cofibrant objects in $\mathcal{C}$ and for morphisms in $\mathcal{C}$ representing these, then this means that in $\mathcal{C}$ there is a left homotopy of the form

$$
\begin{array}{ccc}
A & \stackrel{f}{\longrightarrow} & B \\
{}^{\mathllap{i_1}}\downarrow && \downarrow^{\mathrlap{b}} \\
\mathrm{Cyl}(A) & \underset{\eta}{\longrightarrow} & B' \\
{}^{\mathllap{i_0}}\uparrow && \uparrow^{\mathrlap{f'}} \\
A & \underset{a}{\longrightarrow} & A'
\end{array}
\,.
$$

Consider the factorization of the top square here through the mapping cylinder of f

$$
\begin{array}{ccc}
A & \stackrel{f}{\longrightarrow} & B \\
{}^{\mathllap{i_1}}\downarrow & (po) & \downarrow^{\mathrlap{\in W}} \\
\mathrm{Cyl}(A) & \longrightarrow & \mathrm{Cyl}(f) \\
{}^{\mathllap{i_0}}\uparrow & {}_{\eta}\searrow & \downarrow \\
A && B' \\
& {}_{a}\searrow & \uparrow_{\mathrlap{f'}} \\
&& A'
\end{array}
$$

This exhibits the composite $A \overset{i_0}{\to} \mathrm{Cyl}(A) \to \mathrm{Cyl}(f)$ as an alternative representative of f in $\mathrm{Ho}(\mathcal{C})$, and $\mathrm{Cyl}(f) \to B'$ as an alternative representative for b, and the commuting square

$$
\begin{array}{ccc}
A & \longrightarrow & \mathrm{Cyl}(f) \\
{}^{\mathllap{a}}\downarrow && \downarrow \\
A' & \underset{f'}{\longrightarrow} & B'
\end{array}
$$

as an alternative representative of the given commuting square in $\mathrm{Ho}(\mathcal{C})$. ▮

Derived functors

Definition 2.39. For $\mathcal{C}$ and $\mathcal{D}$ two categories with weak equivalences, def. 2.1, then a functor $F \colon \mathcal{C} \longrightarrow \mathcal{D}$ is called a **homotopical functor** if it sends weak equivalences to weak equivalences.

Definition 2.40. Given a homotopical functor $F \colon \mathcal{C} \longrightarrow \mathcal{D}$ (def. 2.39) between categories with weak equivalences whose homotopy categories $\mathrm{Ho}(\mathcal{C})$ and $\mathrm{Ho}(\mathcal{D})$ exist (def. 2.30), then its ("total") *derived functor* is the functor $\mathrm{Ho}(F)$ between these homotopy categories which is induced uniquely, up to unique isomorphism, by their universal property (def. 2.30):

$$
\begin{array}{ccc}
\mathcal{C} & \stackrel{F}{\longrightarrow} & \mathcal{D} \\
{}^{\mathllap{\gamma_{\mathcal{C}}}}\downarrow & \swArrow_{\simeq} & \downarrow^{\mathrlap{\gamma_{\mathcal{D}}}} \\
\mathrm{Ho}(\mathcal{C}) & \underset{\exists\, \mathrm{Ho}(F)}{\longrightarrow} & \mathrm{Ho}(\mathcal{D})
\end{array}
\,.
$$

Remark 2.41. While many functors of interest between model categories are not homotopical in the sense of def. 2.39, many become homotopical after restriction to the full subcategories $\mathcal{C}_f$ of fibrant objects or $\mathcal{C}_c$ of cofibrant objects, def. 2.34. By corollary 2.36 this is just as good for the purpose of homotopy theory.

Therefore one considers the following generalization of def. 2.40:

Definition 2.42. Consider a functor $F \colon \mathcal{C} \longrightarrow \mathcal{D}$ out of a model category $\mathcal{C}$ (def. 2.3) into a category with weak equivalences $\mathcal{D}$ (def. 2.1).

1. If the restriction of F to the full subcategory $\mathcal{C}_f$ of fibrant object becomes a homotopical functor (def. 2.39), then the derived functor of that restriction, according to def. 2.40, is

called the *right derived functor* of F and denoted by $\mathbb{R}F$:

$$\begin{array}{ccccc} \mathcal{C}_f & \hookrightarrow & \mathcal{C} & \overset{F}{\longrightarrow} & \mathcal{D} \\ {}^{\gamma_{\mathcal{C}_f}}\downarrow && \swArrow_{\simeq} && \downarrow^{\gamma_{\mathcal{D}}} \\ \mathbb{R}F \colon \; \mathcal{C}_f[W^{-1}] & \simeq & \mathrm{Ho}(\mathcal{C}) & \underset{\mathrm{Ho}(F)}{\longrightarrow} & \mathrm{Ho}(\mathcal{D}) \end{array},$$

where we use corollary 2.36.

2. If the restriction of F to the full subcategory $\mathcal{C}_c$ of cofibrant object becomes a homotopical functor (def. 2.39), then the derived functor of that restriction, according to def. 2.40, is called the *left derived functor* of F and denoted by $\mathbb{L}F$:

$$\begin{array}{ccccc} \mathcal{C}_c & \hookrightarrow & \mathcal{C} & \overset{F}{\longrightarrow} & \mathcal{D} \\ {}^{\gamma_{\mathcal{C}_f}}\downarrow && \swArrow_{\simeq} && \downarrow^{\gamma_{\mathcal{D}}} \\ \mathbb{L}F \colon \; \mathcal{C}_c[W^{-1}] & \simeq & \mathrm{Ho}(\mathcal{C}) & \underset{\mathrm{Ho}(F)}{\longrightarrow} & \mathrm{Ho}(\mathcal{D}) \end{array},$$

where again we use corollary 2.36.

The key fact that makes def. 2.42 practically relevant is the following:

Proposition 2.43. ***(Ken Brown's lemma)***

Let $\mathcal{C}$ be a model category with full subcategories $\mathcal{C}_f, \mathcal{C}_c$ of fibrant objects and of cofibrant objects respectively (def. 2.34). Let $\mathcal{D}$ be a category with weak equivalences.

1. *A functor out of the category of fibrant objects*

$$F \colon \mathcal{C}_f \longrightarrow \mathcal{D}$$

is a homotopical functor, def. 2.39, already if it sends acylic fibrations to weak equivalences.

2. *A functor out of the category of cofibrant objects*

$$F \colon \mathcal{C}_c \longrightarrow \mathcal{D}$$

is a homotopical functor, def. 2.39, already if it sends acylic cofibrations to weak equivalences.

The following proof refers to the factorization lemma, whose full statement and proof we postpone to further below (lemma 4.9).

Proof. We discuss the case of a functor on a category of fibrant objects $\mathcal{C}_f$, def. 2.34. The other case is formally dual.

Let $f \colon X \longrightarrow Y$ be a weak equivalence in $\mathcal{C}_f$. Choose a path space object $\mathrm{Path}(X)$ (def. 2.18) and consider the diagram

$$\begin{array}{ccc} \mathrm{Path}(f) & \underset{\in W \cap \mathrm{Fib}}{\longrightarrow} & X \\ {}^{p_1^\ast f}_{\in W}\downarrow & (\mathrm{pb}) & \downarrow^{f}_{\in W} \\ \mathrm{Path}(Y) & \underset{\in W \cap \mathrm{Fib}}{\overset{p_1}{\longrightarrow}} & Y \,, \\ {}^{p_0}_{\in W \cap \mathrm{Fib}}\downarrow && \\ Y && \end{array}$$

where the square is a pullback and $\mathrm{Path}(f)$ on the top left is our notation for the universal cone object. (Below we discuss this in more detail, it is the *mapping cocone* of f, def. 4.1).

Here:

1. p_i are both acyclic fibrations, by lemma 2.20;

2. $\mathrm{Path}(f) \to X$ is an acyclic fibration because it is the pullback of p_1.

3. $p_1^* f$ is a weak equivalence, because the factorization lemma 4.9 states that the composite vertical morphism factors f through a weak equivalence, hence if f is a weak equivalence, then $p_1^* f$ is by two-out-of-three (def. 2.1).

Now apply the functor F to this diagram and use the assumption that it sends acyclic fibrations to weak equivalences to obtain

$$\begin{array}{ccc} F(\mathrm{Path}(f)) & \underset{\in W}{\longrightarrow} & F(X) \\ {}^{F(p_1^* f)}\downarrow && \downarrow^{F(f)} \\ F(\mathrm{Path}(Y)) & \underset{\in W}{\overset{F(p_1)}{\longrightarrow}} & F(Y) \\ {}^{F(p_0)}_{\in W}\downarrow \\ Y \end{array} \,.$$

But the factorization lemma 4.9, in addition says that the vertical composite $p_0 \circ p_1^* f$ is a fibration, hence an acyclic fibration by the above. Therefore also $F(p_0 \circ p_1^* f)$ is a weak equivalence. Now the claim that also $F(f)$ is a weak equivalence follows with applying two-out-of-three (def. 2.1) twice. ▮

Corollary 2.44. *Let $\mathcal{C}, \mathcal{D}$ be model categories and consider $F : \mathcal{C} \longrightarrow \mathcal{D}$ a functor. Then:*

1. *If F preserves cofibrant objects and acyclic cofibrations between these, then its left derived functor (def. 2.42) $\mathbb{L}F$ exists, fitting into a diagram*

$$\begin{array}{ccc} \mathcal{C}_c & \overset{F}{\longrightarrow} & \mathcal{D}_c \\ {}^{\gamma_{\mathcal{C}}}\downarrow & \swArrow_{\simeq} & \downarrow^{\gamma_{\mathcal{D}}} \\ \mathrm{Ho}(\mathcal{C}) & \overset{\mathbb{L}F}{\longrightarrow} & \mathrm{Ho}(\mathcal{D}) \end{array}$$

2. *If F preserves fibrant objects and acyclic fibrants between these, then its right derived functor (def. 2.42) $\mathbb{R}F$ exists, fitting into a diagram*

$$\begin{array}{ccc} \mathcal{C}_f & \overset{F}{\longrightarrow} & \mathcal{D}_f \\ {}^{\gamma_{\mathcal{C}}}\downarrow & \swArrow_{\simeq} & \downarrow^{\gamma_{\mathcal{D}}} \\ \mathrm{Ho}(\mathcal{C}) & \underset{\mathbb{R}F}{\longrightarrow} & \mathrm{Ho}(\mathcal{D}) \end{array} \,.$$

Proposition 2.45. Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor between two model categories (def. 2.3).

1. If F preserves fibrant objects and weak equivalences between fibrant objects, then the total right derived functor $\mathbb{R}F \coloneqq \mathbb{R}(\gamma_{\mathcal{D}} \circ F)$ (def. 2.42) in

$$\begin{array}{ccc} \mathcal{C}_f & \overset{F}{\longrightarrow} & \mathcal{D} \\ {}^{\gamma_{\mathcal{C}_f}}\downarrow & \swArrow_{\simeq} & \downarrow^{\gamma_{\mathcal{D}}} \\ \mathrm{Ho}(\mathcal{C}) & \underset{\mathbb{R}F}{\longrightarrow} & \mathrm{Ho}(\mathcal{D}) \end{array}$$

is given, up to isomorphism, on any object $X \in \mathcal{C} \overset{\gamma_{\mathcal{C}}}{\longrightarrow} \mathrm{Ho}(\mathcal{C})$ by appying F to a fibrant replacement $P X$ of X and then forming a cofibrant replacement $Q(F(P X))$ of the result:

$$\mathbb{R}F(X) \simeq Q(F(P X)) \;.$$

1. If F preserves cofibrant objects and weak equivalences between cofibrant objects, then the total left derived functor $\mathbb{L}F \coloneqq \mathbb{L}(\gamma_{\mathcal{D}} \circ F)$ (def. 2.42) in

$$\begin{array}{ccc} \mathcal{C}_c & \overset{F}{\longrightarrow} & \mathcal{D} \\ {}^{\mathllap{\gamma_{\mathcal{C}_c}}}\downarrow & \swArrow_{\simeq} & \downarrow^{\mathrlap{\gamma_{\mathcal{D}}}} \\ \mathrm{Ho}(\mathcal{C}) & \underset{\mathbb{L}F}{\longrightarrow} & \mathrm{Ho}(\mathcal{D}) \end{array}$$

is given, up to isomorphism, on any object $X \in \mathcal{C} \overset{\gamma_{\mathcal{C}}}{\longrightarrow} \mathrm{Ho}(\mathcal{C})$ by appying F to a cofibrant replacement $Q X$ of X and then forming a fibrant replacement $P(F(Q X))$ of the result:

$$\mathbb{L}F(X) \simeq P(F(Q X)) \;.$$

Proof. We discuss the first case, the second is formally dual. By the proof of theorem 2.31 we have

$$\begin{aligned} \mathbb{R}F(X) & \simeq \gamma_{\mathcal{D}}(F(\gamma_{\mathcal{C}})) \\ & \simeq \gamma_{\mathcal{D}} F(Q(P(X))) \end{aligned} \,.$$

But since F is a homotopical functor on fibrant objects, the cofibrant replacement morphism $F(Q(P(X))) \to F(P(X))$ is a weak equivalence in $\mathcal{D}$, hence becomes an isomorphism under $\gamma_{\mathcal{D}}$. Therefore

$$\mathbb{R}F(X) \simeq \gamma_{\mathcal{D}}(F(P(X))) \;.$$

Now since F is assumed to preserve fibrant objects, $F(P(X))$ is fibrant in $\mathcal{D}$, and hence $\gamma_{\mathcal{D}}$ acts on it (only) by cofibrant replacement. ▮

Quillen adjunctions

In practice it turns out to be useful to arrange for the assumptions in corollary 2.44 to be satisfied by pairs of adjoint functors. Recall that this is a pair of functors L and R going back and forth between two categories

$$\mathcal{C} \underoverset{\underset{R}{\longrightarrow}}{\overset{L}{\longleftarrow}}{} \mathcal{D}$$

such that there is a natural bijection between hom-sets with L on the left and those with R on the right:

$$\phi_{d,c} \;:\; \mathrm{Hom}_{\mathcal{C}}(L(d), c) \underset{\simeq}{\longrightarrow} \mathrm{Hom}_{\mathcal{D}}(d, R(c))$$

for all objects $d \in \mathcal{D}$ and $c \in \mathcal{C}$. This being natural means that $\phi \colon \mathrm{Hom}_{\mathcal{D}}(L(-),-) \Rightarrow \mathrm{Hom}_{\mathcal{C}}(-,R(-))$ is a natural transformation, hence that for all morphisms $g \colon d_2 \to d_1$ and $f \colon c_1 \to c_2$ the following is a commuting square:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(L(d_1), c_1) & \underoverset{\simeq}{\phi_{d_1,c_1}}{\longrightarrow} & \mathrm{Hom}_{\mathcal{D}}(d_1, R(c_1)) \\ {}^{\mathllap{L(f)\circ(-)\circ g}}\downarrow && \downarrow^{\mathrlap{g\circ(-)\circ R(g)}} \\ \mathrm{Hom}_{\mathcal{C}}(L(d_2), c_2) & \underoverset{\phi_{d_2,c_2}}{\simeq}{\longrightarrow} & \mathrm{Hom}_{\mathcal{D}}(d_2, R(c_2)) \end{array}$$

We write $(L \dashv R)$ to indicate an adjunction and call L the *left adjoint* and R the *right adjoint* of the adjoint pair.

The archetypical example of a pair of adjoint functors is that consisting of forming Cartesian products $Y \times (-)$ and forming mapping spaces $(-)^Y$, as in the category of compactly generated topological spaces of def. 3.35.

If $f : L(d) \to c$ is any morphism, then the image $\phi_{d,c}(f) : d \to R(c)$ is called its *adjunct*, and conversely. The fact that adjuncts are in bijection is also expressed by the notation

$$\frac{L(c) \overset{f}{\longrightarrow} d}{c \overset{\tilde f}{\longrightarrow} R(d)} \,.$$

For an object $d \in \mathcal{D}$, the adjunct of the identity on Ld is called the *adjunction unit* $\eta_d \,:\, d \longrightarrow RLd$.

For an object $c \in \mathcal{C}$, the adjunct of the identity on Rc is called the *adjunction counit* $\epsilon_c \,:\, LRc \longrightarrow c$.

Adjunction units and counits turn out to encode the adjuncts of all other morphisms by the formulas

- $\widetilde{(Ld \overset{f}{\to} c)} = (d \overset{\eta}{\to} RLd \overset{Rf}{\to} Rc)$
- $\widetilde{(d \overset{g}{\to} Rc)} = (Ld \overset{Lg}{\to} LRc \overset{\epsilon}{\to} c)$.

Definition 2.46. Let $\mathcal{C},\mathcal{D}$ be model categories. A pair of adjoint functors between them

$$(L \dashv R) \,:\, \mathcal{C} \underoverset{\underset{R}{\longrightarrow}}{\overset{L}{\longleftarrow}}{} \mathcal{D}$$

is called a **Quillen adjunction** (and L,R are called left/right **Quillen functors**, respectively) if the following equivalent conditions are satisfied

1. L preserves cofibrations and R preserves fibrations;
2. L preserves acyclic cofibrations and R preserves acyclic fibrations;
3. L preserves cofibrations and acylic cofibrations;
4. R preserves fibrations and acyclic fibrations.

Proposition 2.47. *The conditions in def. 2.46 are indeed all equivalent.*

(Quillen 67, I.4, theorem 3)

Proof. First observe that

- (i) *A left adjoint L between model categories preserves acyclic cofibrations precisely if its right adjoint R preserves fibrations.*
- (ii) *A left adjoint L between model categories preserves cofibrations precisely if its right adjoint R preserves acyclic fibrations.*

We discuss statement (i), statement (ii) is formally dual. So let $f : A \to B$ be an acyclic cofibration in $\mathcal{D}$ and $g : X \to Y$ a fibration in $\mathcal{C}$. Then for every commuting diagram as on the left of the following, its $(L \dashv R)$-adjunct is a commuting diagram as on the right here:

$$\begin{array}{ccc} A & \longrightarrow & R(X) \\ {}^{f}\downarrow && \downarrow^{R(g)} \\ B & \longrightarrow & R(Y) \end{array} \;\;\; , \;\;\; \begin{array}{ccc} L(A) & \longrightarrow & X \\ {}^{L(f)}\downarrow && \downarrow^{g} \\ L(B) & \longrightarrow & Y \end{array} .$$

If L preserves acyclic cofibrations, then the diagram on the right has a lift, and so the $(L \dashv R)$-adjunct of that lift is a lift of the left diagram. This shows that $R(g)$ has the right lifting property against all acylic cofibrations and hence is a fibration. Conversely, if R preserves fibrations, the same argument run from right to left gives that L preserves acyclic fibrations.

Now by repeatedly applying (i) and (ii), all four conditions in question are seen to be equivalent. ▮

Lemma 2.48. *Let $\mathcal{C} \underoverset{\underset{R}{\longrightarrow}}{\overset{L}{\longleftarrow}}{\bot} \mathcal{D}$ be a Quillen adjunction, def. 2.46.*

1. *For $X \in \mathcal{C}$ a fibrant object and $\mathrm{Path}(X)$ a path space object (def. 2.18), then $R(\mathrm{Path}(X))$ is a path space object for $R(X)$.*

2. *For $X \in \mathcal{C}$ a cofibrant object and $\mathrm{Cyl}(X)$ a cylinder object (def. 2.18), then $L(\mathrm{Cyl}(X))$ is a path space object for $L(X)$.*

Proof. Consider the second case, the first is formally dual.

First Observe that $L(Y \sqcup Y) \simeq LY \sqcup LY$ because L is left adjoint and hence preserves colimits, hence in particular coproducts.

Hence

$$L(\mathrm{X} \sqcup X \overset{\in \mathrm{Cof}}{\longrightarrow} \mathrm{Cyl}(X)) = (L(X) \sqcup L(X) \overset{\in \mathrm{Cof}}{\longrightarrow} L(\mathrm{Cyl}(X)))$$

is a cofibration.

Second, with Y cofibrant then also $Y \sqcup \mathrm{Cyl}(Y)$ is a cofibrantion, since $Y \to Y \sqcup Y$ is a cofibration (lemma 2.20). Therefore by Ken Brown's lemma (prop. 2.43) L preserves the weak equivalence $\mathrm{Cyl}(Y) \overset{\in W}{\longrightarrow} Y$. ▮

Proposition 2.49. *For $\mathcal{C} \underoverset{\underset{R}{\longrightarrow}}{\overset{L}{\longleftarrow}}{\bot} \mathcal{D}$ a Quillen adjunction, def. 2.46, then also the corresponding left and right derived functors, def. 2.42, via cor. 2.44, form a pair of adjoint functors*

$$\mathrm{Ho}(\mathcal{C}) \underoverset{\underset{\mathbb{R}R}{\longrightarrow}}{\overset{\mathbb{L}L}{\longleftarrow}}{\bot} \mathrm{Ho}(\mathcal{D}) .$$

(Quillen 67, I.4 theorem 3)

Proof. By def. 2.42 and lemma 2.37 it is sufficient to see that for $X, Y \in \mathcal{C}$ with X cofibrant and Y fibrant, then there is a natural bijection

$$\mathrm{Hom}_{\mathcal{C}}(LX, Y)/_\sim \simeq \mathrm{Hom}_{\mathcal{C}}(X, RY)/_\sim .$$

Since by the adjunction isomorphism for $(L \dashv R)$ such a natural bijection exists before passing to homotopy classes $(-)/_\sim$, it is sufficient to see that this respects homotopy classes. To that end, use from lemma 2.48 that with $\mathrm{Cyl}(Y)$ a cylinder object for Y, def. 2.18, then $L(\mathrm{Cyl}(Y))$ is a cylinder object for $L(Y)$. This implies that left homotopies

$$(f \Rightarrow_L g) : LX \longrightarrow Y$$

given by

$$\eta \,:\, \mathrm{Cyl}(LX) = L\,\mathrm{Cyl}(X) \longrightarrow Y$$

are in bijection to left homotopies

$$(\tilde f \Rightarrow_L \tilde g) \,:\, X \longrightarrow RY$$

given by

$$\tilde \eta \,:\, \mathrm{Cyl}(X) \longrightarrow RX \,.$$

▮

Definition 2.50. For $\mathcal{C},\mathcal{D}$ two model categories, a Quillen adjunction (def.2.46)

$$(L \dashv R) \,:\, \mathcal{C} \underoverset{\underset{R}{\longrightarrow}}{\overset{L}{\longleftarrow}}{\bot} \mathcal{D}$$

is called a **Quillen equivalence**, to be denoted

$$\mathcal{C} \underoverset{\underset{R}{\longrightarrow}}{\overset{L}{\longleftarrow}}{\simeq_Q} \mathcal{D} \,,$$

if the following equivalent conditions hold.

1. The right derived functor of R (via prop. 2.47, corollary 2.44) is an equivalence of categories

$$\mathbb{R}R : \mathrm{Ho}(\mathcal{C}) \overset{\simeq}{\longrightarrow} \mathrm{Ho}(\mathcal{D}) \,.$$

2. The left derived functor of L (via prop. 2.47, corollary 2.44) is an equivalence of categories

$$\mathbb{L}L : \mathrm{Ho}(\mathcal{D}) \overset{\simeq}{\longrightarrow} \mathrm{Ho}(\mathcal{C}) \,.$$

3. For every cofibrant object $d \in \mathcal{D}$, the "derived adjunction unit", hence the composite

$$d \overset{\eta}{\longrightarrow} R(L(d)) \overset{R(j_{L(d)})}{\longrightarrow} R(P(L(d)))$$

(of the adjunction unit with any fibrant replacement P as in def. 2.28) is a weak equivalence;

and for every fibrant object $c \in \mathcal{C}$, the "derived adjunction counit", hence the composite

$$L(Q(R(c))) \overset{L(p_{R(c)})}{\longrightarrow} L(R(c)) \overset{\epsilon}{\longrightarrow} c$$

(of the adjunction counit with any cofibrant replacement as in def. 2.28) is a weak equivalence in D.

4. For every cofibrant object $d \in \mathcal{D}$ and every fibrant object $c \in \mathcal{C}$, a morphism $d \longrightarrow R(c)$ is a weak equivalence precisely if its adjunct morphism $L(c) \to d$ is:

$$\frac{d \overset{\in W_{\mathcal{D}}}{\longrightarrow} R(c)}{L(d) \overset{\in W_{\mathcal{C}}}{\longrightarrow} c} \,.$$

Poposition 2.51. *The conditions in def. 2.50 are indeed all equivalent.*

(Quillen 67, I.4, theorem 3)

Proof. That $1) \Leftrightarrow 2)$ follows from prop. 2.49 (if in an adjoint pair one is an equivalence, then so is the other).

To see the equivalence $1), 2) \Leftrightarrow 3)$, notice (prop.) that a pair of adjoint functors is an equivalence of categories precisely if both the adjunction unit and the adjunction counit are natural isomorphisms. Hence it is sufficient to show that the morphisms called "derived adjunction (co-)units" above indeed represent the adjunction (co-)unit of $(\mathbb{L}L \dashv \mathbb{R}R)$ in the homotopy category. We show this now for the adjunction unit, the case of the adjunction counit is formally dual.

To that end, first observe that for $d \in \mathcal{D}_c$, then the defining commuting square for the left derived functor from def. 2.42

$$\begin{array}{ccc} \mathcal{D}_c & \overset{L}{\longrightarrow} & \mathcal{C} \\ {}^{\gamma_P}\downarrow & \swArrow_{\simeq} & \downarrow^{\gamma_{P,Q}} \\ \mathrm{Ho}(\mathcal{D}) & \underset{\mathbb{L}L}{\longrightarrow} & \mathrm{Ho}(\mathcal{C}) \end{array}$$

(using fibrant and fibrant/cofibrant replacement functors γ_P, $\gamma_{P,Q}$ from def. 2.28 with their universal property from theorem 2.31, corollary 2.36) gives that

$$(\mathbb{L}L)d \simeq PLPd \simeq PLd \quad \in \mathrm{Ho}(\mathcal{C}) \,,$$

where the second isomorphism holds because the left Quillen functor L sends the acyclic cofibration $j_d : d \to Pd$ to a weak equivalence.

The adjunction unit of $(\mathbb{L}L \dashv \mathbb{R}R)$ on $Pd \in \mathrm{Ho}(\mathcal{C})$ is the image of the identity under

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}((\mathbb{L}L)Pd, (\mathbb{L}L)Pd) \overset{\simeq}{\to} \mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(Pd, (\mathbb{R}R)(\mathbb{L}L)Pd) \,.$$

By the above and the proof of prop. 2.49, that adjunction isomorphism is equivalently that of $(L \dashv R)$ under the isomorphism

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(PLd, PLd) \xrightarrow{\mathrm{Hom}(j_{Ld}, \mathrm{id})} \mathrm{Hom}_{\mathcal{C}}(Ld, PLd)/_\sim$$

of lemma 2.37. Hence the derived adjunction unit is the $(L \dashv R)$-adjunct of

$$Ld \overset{j_{Ld}}{\longrightarrow} PLd \overset{\mathrm{id}}{\to} PLd \,,$$

which indeed (by the formula for adjuncts) is

$$X \overset{\eta}{\longrightarrow} RLd \xrightarrow{R(j_{Ld})} RPLd \,.$$

To see that $4) \Rightarrow 3)$:

Consider the weak equivalence $LX \overset{j_{LX}}{\longrightarrow} PLX$. Its $(L \dashv R)$-adjunct is

$$X \overset{\eta}{\longrightarrow} RLX \overset{R j_{LX}}{\longrightarrow} RPLX$$

by assumption 4) this is again a weak equivalence, which is the requirement for the derived unit in 3). Dually for derived counit.

To see $3) \Rightarrow 4)$:

Consider any $f : Ld \to c$ a weak equivalence for cofibrant d, firbant c. Its adjunct $\tilde{f}$ sits in a commuting diagram

$$\begin{array}{ccccc} \tilde f \colon & d & \stackrel{\eta}{\longrightarrow} & RLd & \stackrel{Rf}{\longrightarrow} & Rc \\ & {}^{=}\downarrow && \downarrow^{R j_{Ld}} && \downarrow^{R j_c} \\ & d & \underset{\in W}{\longrightarrow} & RPLd & \stackrel{RPf}{\longrightarrow} & RPc \end{array},$$

where Pf is any lift constructed as in def. 2.28.

This exhibits the bottom left morphism as the derived adjunction unit, hence a weak equivalence by assumption. But since f was a weak equivalence, so is Pf (by two-out-of-three). Thereby also RPf and Rj_Y, are weak equivalences by Ken Brown's lemma 2.43 and the assumed fibrancy of c. Therefore by two-out-of-three (def. 2.1) also the adjunct $\tilde f$ is a weak equivalence. ▮

In certain situations the conditions on a Quillen equivalence simplify. For instance:

Proposition 2.52. *If in a Quillen adjunction $\mathcal{C} \underoverset{\underset{R}{\longrightarrow}}{\overset{L}{\longleftarrow}}{\bot} \mathcal{D}$ (def. 2.46) the right adjoint R "creates weak equivalences" (in that a morphism f in $\mathcal{C}$ is a weak equivalence precisely if $U(f)$ is) then $(L \dashv R)$ is a Quillen equivalence (def. 2.50) precisely already if for all cofibrant objects $d \in \mathcal{D}$ the plain adjunction unit*

$$d \stackrel{\eta}{\longrightarrow} R(L(d))$$

is a weak equivalence.

Proof. By prop. 2.51, generally, $(L \dashv R)$ is a Quillen equivalence precisely if

1. for every cofibrant object $d \in \mathcal{D}$, the "derived adjunction unit"

   $$d \stackrel{\eta}{\longrightarrow} R(L(d)) \xrightarrow{R(j_{L(d)})} R(P(L(d)))$$

   is a weak equivalence;

2. for every fibrant object $c \in \mathcal{C}$, the "derived adjunction counit"

   $$L(Q(R(c))) \xrightarrow{L(p_{R(c)})} L(R(c)) \stackrel{\epsilon}{\longrightarrow} c$$

   is a weak equivalence.

Consider the first condition: Since R preserves the weak equivalence $j_{L(d)}$, then by two-out-of-three (def. 2.1) the composite in the first item is a weak equivalence precisely if η is.

Hence it is now sufficient to show that in this case the second condition above is automatic.

Since R also reflects weak equivalences, the composite in item two is a weak equivalence precisely if its image

$$R(L(Q(R(c)))) \xrightarrow{R(L(p_{R(c)}))} R(L(R(c))) \xrightarrow{R(\epsilon)} R(c)$$

under R is.

Moreover, assuming, by the above, that $\eta_{Q(R(c))}$ on the cofibrant object $Q(R(c))$ is a weak equivalence, then by two-out-of-three this composite is a weak equivalence precisely if the further composite with η is

$$Q(R(c)) \xrightarrow{\eta_{Q(R(c))}} R(L(Q(R(c)))) \xrightarrow{R(L(p_{R(c)}))} R(L(R(c))) \xrightarrow{R(\epsilon)} R(c)\,.$$

By the formula for adjuncts, this composite is the $(L \dashv R)$-adjunct of the original composite, which is just $p_{R(c)}$

$$\frac{L(Q(R(c))) \xrightarrow{L(p_{R(c)})} L(R(c)) \xrightarrow{\epsilon} c}{Q(R(C)) \xrightarrow{p_{R(c)}} R(c)}\,.$$

But $p_{R(c)}$ is a weak equivalence by definition of cofibrant replacement. ▮

3. The model structure on topological spaces

We now discuss how the category Top of topological spaces satisfies the axioms of abstract homotopy theory (model category) theory, def. 2.3.

Definition 3.1. Say that a continuous function, hence a morphism in Top, is

- a **classical weak equivalence** if it is a weak homotopy equivalence, def. 1.30;
- a **classical fibration** if it is a Serre fibration, def. 1.47;
- a **classical cofibration** if it is a retract (rem. 2.12) of a relative cell complex, def. 1.38.

and hence

- a **acyclic classical cofibration** if it is a classical cofibration as well as a classical weak equivalence;
- a **acyclic classical fibration** if it is a classical fibration as well as a classical weak equivalence.

Write

$$W_{\mathrm{cl}},\ \mathrm{Fib}_{\mathrm{cl}},\ \mathrm{Cof}_{\mathrm{cl}} \subset \mathrm{Mor}(\mathrm{Top})$$

for the classes of these morphisms, respectively.

We first prove now that the classes of morphisms in def. 3.1 satisfy the conditions for a model category structure, def. 2.3 (after some lemmas, this is theorem 3.7 below). Then we discuss the resulting classical homotopy category (below) and then a few variant model structures whose proof follows immediately along the line of the proof of $\mathrm{Top}_{\mathrm{Quillen}}$:

- The model structure on pointed topological spaces $\mathrm{Top}^{*/}_{\mathrm{Quillen}}$;
- The model structure on compactly generated topological spaces $(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{Quillen}}$ and $(\mathrm{Top}^{*/}_{\mathrm{cg}})_{\mathrm{Quillen}}$;
- The model structure on topologically enriched functors $[\mathcal{C}, (\mathrm{Top}_{\mathrm{cg}})_{\mathrm{Quillen}}]_{\mathrm{proj}}$ and $[\mathcal{C}, (\mathrm{Top}^{*}_{\mathrm{cg}})_{\mathrm{Quillen}}]_{\mathrm{proj}}$.

Proposition 3.2. *The classical weak equivalences, def. 3.1, satisfy two-out-of-three (def. 2.1).*

Proof. Since isomorphisms (of homotopy groups) satisfy 2-out-of-3, this property is directly inherited via the very definition of weak homotopy equivalence, def. 1.30. ▮

Lemma 3.3. *Every morphism $f : X \longrightarrow Y$ in Top factors as a classical cofibration followed by an acyclic classical fibration, def. 3.1:*

$$f \,:\, X \overset{\in \mathrm{Cof}_{\mathrm{cl}}}{\longrightarrow} \hat X \overset{\in W_{\mathrm{cl}} \cap \mathrm{Fib}_{\mathrm{cl}}}{\longrightarrow} Y \,.$$

Proof. By lemma 1.40 the set $I_{\mathrm{Top}} = \{S^{n-1} \hookrightarrow D^n\}$ of topological generating cofibrations, def. 1.37, has small domains, in the sense of def. 2.16 (the n-spheres are compact). Hence by the small object argument, prop. 2.17, f factors as an I_{Top}-relative cell complex, def. 1.41, hence just a plain relative cell complex, def. 1.38, followed by an I_{Top}-injective morphisms, def. 1.46:

$$f \,:\, X \overset{\in \mathrm{Cof}_{\mathrm{cl}}}{\longrightarrow} \hat X \overset{\in I_{\mathrm{Top}}\,\mathrm{Inj}}{\longrightarrow} Y \,.$$

By lemma 1.52 the map $\hat X \to Y$ is both a weak homotopy equivalence as well as a Serre fibration. ▮

Lemma 3.4. *Every morphism $f : X \longrightarrow Y$ in Top factors as an acyclic classical cofibration followed by a fibration, def. 3.1:*

$$f \,:\, X \overset{\in W_{\mathrm{cl}} \cap \mathrm{Cof}_{\mathrm{cl}}}{\longrightarrow} \hat X \overset{\in \mathrm{Fib}_{\mathrm{cl}}}{\longrightarrow} Y \,.$$

Proof. By lemma 1.40 the set $J_{\mathrm{Top}} = \{D^n \hookrightarrow D^n \times I\}$ of topological generating acyclic cofibrations, def. 1.42, has small domains, in the sense of def. 2.16 (the n-disks are compact). Hence by the small object argument, prop. 2.17, f factors as an J_{Top}-relative cell complex, def. 1.41, followed by a J_{top}-injective morphisms, def. 1.46:

$$f \,:\, X \overset{\in J_{\mathrm{Top}}\,\mathrm{Cell}}{\longrightarrow} \hat X \overset{\in J_{\mathrm{Top}}\,\mathrm{Inj}}{\longrightarrow} Y \,.$$

By definition this makes $\hat X \to Y$ a Serre fibration, hence a fibration.

By lemma 1.44 a relative J_{Top}-cell complex is in particular a relative I_{Top}-cell complex. Hence $X \to \hat X$ is a classical cofibration. By lemma 1.45 it is also a weak homotopy equivalence, hence a clasical weak equivalence. ▮

Lemma 3.5. *Every commuting square in Top with the left morphism a classical cofibration and the right morphism a fibration, def. 3.1*

$$\begin{array}{ccc} & \longrightarrow & \\ {}^{\mathllap{g \in \atop \mathrm{Cof}_{\mathrm{cl}}}}\downarrow & & \downarrow^{\mathrlap{f \in \atop \mathrm{Fib}_{\mathrm{cl}}}} \\ & \longrightarrow & \end{array}$$

admits a lift as soon as one of the two is also a classical weak equivalence.

Proof. **A)** If the fibration f is also a weak equivalence, then lemma 1.52 says that it has the right lifting property against the generating cofibrations I_{Top}, and cor. 2.11 implies the claim.

B) If the cofibration g on the left is also a weak equivalence, consider any factorization into a relative J_{Top}-cell complex, def. 1.42, def. 1.41, followed by a fibration,

$$g \,:\, \overset{\in J_{\mathrm{Top}}\,\mathrm{Cell}}{\longrightarrow} \overset{\in \mathrm{Fib}_{\mathrm{cl}}}{\longrightarrow} \,,$$

as in the proof of lemma 3.4. By lemma 1.45 the morphism $\overset{\in J_{\mathrm{Top}}\,\mathrm{Cell}}{\longrightarrow}$ is a weak homotopy equivalence, and so by two-out-of-three (prop. 3.2) the factorizing fibration is actually an acyclic fibration. By case A), this acyclic fibration has the right lifting property against the cofibration g itself, and so the retract argument, lemma 2.15 gives that g is a retract of a relative J_{Top}-cell complex. With this, finally cor. 2.11 implies that f has the right lifting property against g. ▮

Finally:

Proposition 3.6. *The systems $(\mathrm{Cof}_{\mathrm{cl}}, W_{\mathrm{cl}} \cap \mathrm{Fib}_{\mathrm{cl}})$ and $(W_{\mathrm{cl}} \cap \mathrm{Cof}_{\mathrm{cl}}, \mathrm{Fib}_{\mathrm{cl}})$ from def. 3.1 are weak factorization systems.*

Proof. Since we have already seen the factorization property (lemma 3.3, lemma 3.4) and the lifting properties (lemma 3.5), it only remains to see that the given left/right classes exhaust the class of morphisms with the given lifting property.

For the classical fibrations this is by definition, for the the classical acyclic fibrations this is by lemma 1.52.

The remaining statement for $\mathrm{Cof}_{\mathrm{cl}}$ and $W_{\mathrm{cl}} \cap \mathrm{Cof}_{\mathrm{cl}}$ follows from a general argument (here) for cofibrantly generated model categories (def. 3.9), which we spell out:

So let $f : X \longrightarrow Y$ be in $(I_{\mathrm{Top}}\,\mathrm{Inj})\mathrm{Proj}$, we need to show that then f is a retract (remark 2.12) of a relative cell complex. To that end, apply the small object argument as in lemma 3.3 to factor f as

$$f \;:\; X \overset{I_{\mathrm{Top}}\,\mathrm{Cell}}{\longrightarrow} \hat{Y} \overset{\in I_{\mathrm{Top}}\,\mathrm{Inj}}{\longrightarrow} Y \,.$$

It follows that f has the left lifting property against $\hat{Y} \to Y$, and hence by the retract argument (lemma 2.15) it is a retract of $X \overset{I\,\mathrm{Cell}}{\longrightarrow} \hat{Y}$. This proves the claim for $\mathrm{Cof}_{\mathrm{cl}}$.

The analogous argument for $W_{\mathrm{cl}} \cap \mathrm{Cof}_{\mathrm{cl}}$, using the small object argument for J_{Top}, shows that every $f \in (J_{\mathrm{Top}}\,\mathrm{Inj})\mathrm{Proj}$ is a retract of a J_{Top}-cell complex. By lemma 1.44 and lemma 1.45 a J_{Top}-cell complex is both an I_{Top}-cell complex and a weak homotopy equivalence. Retracts of the former are cofibrations by definition, and retracts of the latter are still weak homotopy equivalences by lemma 2.13. Hence such f is an acyclic cofibration. ▮

In conclusion, prop. 3.2 and prop. 3.6 say that:

Theorem 3.7. *The classes of morphisms in $\mathrm{Mor}(\mathrm{Top})$ of def. 3.1,*

- $W_{\mathrm{cl}} =$ *weak homotopy equivalences,*
- $\mathrm{Fib}_{\mathrm{cl}} =$ *Serre fibrations*
- $\mathrm{Cof}_{\mathrm{cl}} =$ *retracts of relative cell complexes*

define a model category structure (def. 2.3) $\mathrm{Top}_{\mathrm{Quillen}}$, the ***classical model structure on topological spaces*** *or* **Serre-Quillen model structure** *.*

In particular

1. every object in $\mathrm{Top}_{\mathrm{Quillen}}$ is fibrant;
2. the cofibrant objects in $\mathrm{Top}_{\mathrm{Quillen}}$ are the retracts of cell complexes.

Hence in particular the following classical statement is an immediate corollary:

Corollary 3.8. ***(Whitehead theorem)***

Every weak homotopy equivalence (def. 1.30) between topological spaces that are homeomorphic to a retract of a cell complex, in particular to a CW-complex (def. 1.38), is a homotopy equivalence (def. 1.28).

Proof. This is the "Whitehead theorem in model categories", lemma 2.27, specialized to $\mathrm{Top}_{\mathrm{Quillen}}$ via theorem 3.7. ▮

In proving theorem 3.7 we have in fact shown a bit more that stated. Looking back, all the structure of $\mathrm{Top}_{\mathrm{Quillen}}$ is entirely induced by the set I_{Top} (def. 1.37) of generating cofibrations and the set J_{Top} (def. 1.42) of generating acyclic cofibrations (whence the terminology). This phenomenon will keep recurring and will keep being useful as we construct further model categories, such as the classical model structure on pointed topological spaces (def. 3.31), the projective model structure on topological functors (thm. 3.76), and finally various model structures on spectra which we turn to in the section on stable homotopy theory.

Therefore we make this situation explicit:

Definition 3.9. A model category $\mathcal{C}$ (def. 2.3) is called **cofibrantly generated** if there exists two subsets

$$I, J \subset \mathrm{Mor}(\mathcal{C})$$

of its class of morphisms, such that

1. I and J have small domains according to def. 2.16,
2. the (acyclic) cofibrations of $\mathcal{C}$ are precisely the retracts, of I-relative cell complexes (J-relative cell complexes), def. 1.41.

Proposition 3.10. *For $\mathcal{C}$ a cofibrantly generated model category, def. 3.9, with generating (acylic) cofibrations I (J), then its classes $W, \mathrm{Fib}, \mathrm{Cof}$ of weak equivalences, fibrations and cofibrations are equivalently expressed as injective or projective morphisms (def. 2.4) this way:*

1. $\mathrm{Cof} = (I\,\mathrm{Inj})\mathrm{Proj}$
2. $W \cap \mathrm{Fib} = I\,\mathrm{Inj}$;
3. $W \cap \mathrm{Cof} = (J\,\mathrm{Inj})\mathrm{Proj}$;
4. $\mathrm{Fib} = J\,\mathrm{Inj}$;

Proof. It is clear from the definition that $I \subset (I\,\mathrm{Inj})\mathrm{Proj}$, so that the closure property of prop. 2.10 gives an inclusion

$$\mathrm{Cof} \subset (I\,\mathrm{Inj})\mathrm{Proj}\ .$$

For the converse inclusion, let $f \in (I\,\mathrm{Inj})\mathrm{Proj}$. By the small object argument, prop. 2.17, there is a factorization $f \colon \overset{\in I\,\mathrm{Cell}}{\longrightarrow} \overset{I\,\mathrm{Inj}}{\longrightarrow}$. Hence by assumption and by the retract argument lemma 2.15, f is a retract of an I-relative cell complex, hence is in Cof.

This proves the first statement. Together with the closure properties of prop. 2.10, this implies the second claim.

The proof of the third and fourth item is directly analogous, just with J replaced for I. ▮

The classical homotopy category

With the classical model structure on topological spaces in hand, we now have good control over the classical homotopy category:

Definition 3.11. The **Serre-Quillen classical homotopy category** is the homotopy category, def. 2.25, of the classical model structure on topological spaces $\mathrm{Top}_{\mathrm{Quillen}}$ from theorem 3.7: we write

$$\mathrm{Ho}(\mathrm{Top}) \coloneqq \mathrm{Ho}(\mathrm{Top}_{\mathrm{Quillen}})\ .$$

Remark 3.12. From just theorem 3.7, the definition 2.25 (def. 3.11) gives that

$$\mathrm{Ho}(\mathrm{Top}_{\mathrm{Quillen}}) \simeq (\mathrm{Top}_{\mathrm{Retract(Cell)}})/_{\sim}$$

is the category whose objects are retracts of cell complexes (def. 1.38) and whose morphisms are homotopy classes of continuous functions. But in fact more is true:

Theorem 3.7 in itself implies that every topological space is weakly equivalent to a retract of a cell complex, def. 1.38. But by the existence of CW approximations, this cell complex may even be taken to be a CW complex.

(Better yet, there is Quillen equivalence to the classical model structure on simplicial sets which implies a *functorial* CW approximation $|\mathrm{Sing}\, X| \overset{\in W_{\mathrm{cl}}}{\longrightarrow} X$ given by forming the geometric realization of the singular simplicial complex of X.)

Hence the Serre-Quillen classical homotopy category is also equivalently the category of just the CW-complexes whith homotopy classes of continuous functions between them

$$\begin{aligned} \mathrm{Ho}(\mathrm{Top}_{\mathrm{Quillen}}) &\simeq (\mathrm{Top}_{\mathrm{Retract(Cell)}})/_{\sim} \\ &\simeq (\mathrm{Top}_{\mathrm{CW}})/_{\sim} \end{aligned} \,.$$

It follows that the universal property of the homotopy category (theorem 2.31)

$$\mathrm{Ho}(\mathrm{Top}_{\mathrm{Quillen}}) \simeq \mathrm{Top}[W_{\mathrm{cl}}^{-1}]$$

implies that there is a bijection, up to natural isomorphism, between

1. functors out of $\mathrm{Top}_{\mathrm{CW}}$ which agree on homotopy-equivalent maps;

2. functors out of all of Top which send weak homotopy equivalences to isomorphisms.

This statement in particular serves to show that two different axiomatizations of generalized (Eilenberg-Steenrod) cohomology theories are equivalent to each other. See at *Introduction to Stable homotopy theory -- S* the section *generalized cohomology functors* (this prop.)

Beware that, by remark 1.32, what is **not** equivalent to $\mathrm{Ho}(\mathrm{Top}_{\mathrm{Quillen}})$ is the category

$$\mathrm{hTop} \coloneqq \mathrm{Top}/_{\sim}$$

obtained from *all* topological spaces with morphisms the homotopy classes of continuous functions. This category is “too large”, the correct homotopy category is just the genuine full subcategory

$$\mathrm{Ho}(\mathrm{Top}_{\mathrm{Quillen}}) \simeq (\mathrm{Top}_{\mathrm{Retract(Cell)}})/_{\sim} \simeq \mathrm{Top}/_{\sim} \; = \; \hookrightarrow \mathrm{hTop} \,.$$

Beware also the ambiguity of terminology: “classical homotopy category” some literature refers to hTop instead of $\mathrm{Ho}(\mathrm{Top}_{\mathrm{Quillen}})$. However, here we never have any use for hTop and will not mention it again.

Proposition 3.13. *Let X be a CW-complex, def. 1.38. Then the standard topological cylinder of def. 1.22*

$$X \sqcup X \overset{(i_0, i_1)}{\longrightarrow} X \times I \longrightarrow X$$

(obtained by forming the product with the standard topological intervall $I = [0,1]$) is indeed a cylinder object in the abstract sense of def. 2.18.

Proof. We describe the proof informally. It is immediate how to turn this into a formal proof, but the notation becomes tedious. (One place where it is spelled out completely is Ottina 14, prop.

2.9.)

So let $X_0 \to X_1 \to X_2 \to \cdots \to X$ be a presentation of X as a CW-complex. Proceed by induction on the cell dimension.

First observe that the cylinder $X_0 \times I$ over X_0 is a cell complex: First X_0 itself is a disjoint union of points. Adding a second copy for every point (i.e. attaching along $S^{-1} \to D^0$) yields $X_0 \sqcup X_0$, then attaching an inteval between any two corresponding points (along $S^0 \to D^1$) yields $X_0 \times I$.

So assume that for $n \in \mathbb{N}$ it has been shown that $X_n \times I$ has the structure of a CW-complex of dimension $(n+1)$. Then for each cell of X_{n+1}, attach it *twice* to $X_n \times I$, once at $X_n \times \{0\}$, and once at $X_n \times \{1\}$.

The result is X_{n+1} with a *hollow cylinder* erected over each of its $(n+1)$-cells. Now fill these hollow cylinders (along $S^{n+1} \to D^{n+1}$) to obtain $X_{n+1} \times I$.

This completes the induction, hence the proof of the CW-structure on $X \times I$.

The construction also manifestly exhibits the inclusion $X \sqcup X \overset{(i_0, i_1)}{\longrightarrow}$ as a relative cell complex.

Finally, it is clear (prop. 1.31) that $X \times I \to X$ is a weak homotopy equivalence. ▮

Conversely:

Proposition 3.14. *Let X be any topological space. Then the standard topological path space object (def. 1.34)*

$$X \longrightarrow X^I \overset{(X^{\delta_0}, X^{\delta_1})}{\longrightarrow} X \times X$$

(obtained by forming the mapping space, def. 1.18, with the standard topological intervall $I = [0,1]$) is indeed a path space object in the abstract sense of def. 2.18.

Proof. To see that $\mathrm{const} \colon X \to X^I$ is a weak homotopy equivalence it is sufficient, by prop. 1.31, to exhibit a homotopy equivalence. Let the homotopy inverse be $X^{\delta_0} \colon X^I \to X$. Then the composite

$$X \overset{\mathrm{const}}{\longrightarrow} X^I \overset{X^{\delta_0}}{\longrightarrow} X$$

is already equal to the identity. The other we round, the rescaling of paths provides the required homotopy

$$I \times X^I \overset{(t, \gamma) \mapsto \gamma(t \cdot (-))}{\longrightarrow} X^I \,.$$

To see that $X^I \to X \times X$ is a fibration, we need to show that every commuting square of the form

$$\begin{array}{ccc} D^n & \longrightarrow & X^I \\ {}^{\mathllap{i_0}}\downarrow && \downarrow \\ D^n \times I & \longrightarrow & X \times X \end{array}$$

has a lift.

Now first use the adjunction $(I \times (-)) \dashv (-)^I$ from prop. 1.19 to rewrite this equivalently as the following commuting square:

$$\begin{array}{ccc} D^n \sqcup D^n & \overset{(i_0, i_0)}{\longrightarrow} & (D^n \times I) \sqcup (D^n \times I) \\ {}^{\mathllap{(i_0, i_1)}}\downarrow && \downarrow \\ D^n \times I & \longrightarrow & X \end{array} \,.$$

This square is equivalently (example 1.12) a morphism out of the pushout

$$D^n \times I \underset{D^n \sqcup D^n}{\sqcup} ((D^n \times I) \sqcup (D^n \times I)) \longrightarrow X \,.$$

By the same reasoning, a lift in the original diagram is now equivalently a lifting in

$$\begin{array}{ccc} D^n \times I \underset{D^n \sqcup D^n}{\sqcup} ((D^n \times I) \sqcup (D^n \times I)) & \longrightarrow & X \\ \downarrow && \downarrow \\ (D^n \times I) \times I & \longrightarrow & * \end{array} \,.$$

Inspection of the component maps shows that the left vertical morphism here is the inclusion into the square times D^n of three of its faces times D^n. This is homeomorphic to the inclusion $D^{n+1} \to D^{n+1} \times I$ (as in remark 1.49). Therefore a lift in this square exsists, and hence a lift in the original square exists. ▮

Model structure on pointed spaces

A *pointed object* (X,x) is of course an object X equipped with a point $x \colon * \to X$, and a morphism of pointed objects $(X,x) \longrightarrow (Y,y)$ is a morphism $X \longrightarrow Y$ that takes x to y. Trivial as this is in itself, it is good to record some basic facts, which we do here.

Passing to pointed objects is also the first step in linearizing classical homotopy theory to stable homotopy theory. In particular, every category of pointed objects has a zero object, hence has zero morphisms. And crucially, if the original category had Cartesian products, then its pointed objects canonically inherit a non-cartesian tensor product: the smash product. These ingredients will be key below in the section on stable homotopy theory.

Definition 3.15. Let $\mathcal{C}$ be a category and let $X \in \mathcal{C}$ be an object.

The *slice category* $\mathcal{C}_{/X}$ is the category whose

* objects are morphisms $\begin{array}{c} A \\ \downarrow \\ X \end{array}$ in $\mathcal{C}$;
* morphisms are commuting triangles $\begin{array}{ccccc} A && \longrightarrow && B \\ & \searrow && \swarrow & \\ && X && \end{array}$ in $\mathcal{C}$.

Dually, the *coslice category* $\mathcal{C}^{X/}$ is the category whose

* objects are morphisms $\begin{array}{c} X \\ \downarrow \\ A \end{array}$ in $\mathcal{C}$;
* morphisms are commuting triangles $\begin{array}{ccccc} && X && \\ & \swarrow && \searrow & \\ A && \longrightarrow && B \end{array}$ in $\mathcal{C}$.

There are the canonical forgetful functors

$$U : \mathcal{C}_{/X}, \mathcal{C}^{X/} \longrightarrow \mathcal{C}$$

given by forgetting the morphisms to/from X.

We here focus on this class of examples:

Definition 3.16. For $\mathcal{C}$ a category with terminal object $*$, the coslice category (def. 3.15) $\mathcal{C}^{*/}$ is the corresponding *category of pointed objects*: its

- objects are morphisms in $\mathcal{C}$ of the form $* \xrightarrow{x} X$ (hence an object X equipped with a choice of point; i.e. a *pointed object*);
- morphisms are commuting triangles of the form

$$\begin{array}{ccccc} & & * & & \\ & {}^{x}\swarrow & & \searrow^{y} & \\ X & & \xrightarrow{f} & & Y \end{array}$$

(hence morphisms in $\mathcal{C}$ which preserve the chosen points).

Remark 3.17. In a category of pointed objects $\mathcal{C}^{*/}$, def. 3.16, the terminal object coincides with the initial object, both are given by $* \in \mathcal{C}$ itself, pointed in the unique way.

In this situation one says that $*$ is a *zero object* and that $\mathcal{C}^{*/}$ is a *pointed category*.

It follows that also all hom-sets $\mathrm{Hom}_{\mathcal{C}^{*/}}(X,Y)$ of $\mathcal{C}^{*/}$ are canonically pointed sets, pointed by the *zero morphism*

$$0 : X \overset{\exists !}{\longrightarrow} 0 \overset{\exists !}{\longrightarrow} Y \,.$$

Definition 3.18. Let $\mathcal{C}$ be a category with terminal object and finite colimits. Then the forgetful functor $U : \mathcal{C}^{*/} \to \mathcal{C}$ from its category of pointed objects, def. 3.16, has a left adjoint

$$\mathcal{C}^{*/} \underoverset{\underset{U}{\longrightarrow}}{\overset{(-)_+}{\longleftarrow}}{\bot} \mathcal{C}$$

given by forming the disjoint union (coproduct) with a base point ("adjoining a base point").

Proposition 3.19. *Let $\mathcal{C}$ be a category with all limits and colimits. Then also the category of pointed objects $\mathcal{C}^{*/}$, def. 3.16, has all limits and colimits.*

Moreover:

1. *the limits are the limits of the underlying diagrams in $\mathcal{C}$, with the base point of the limit induced by its universal property in $\mathcal{C}$;*
2. *the colimits are the limits in $\mathcal{C}$ of the diagrams with the basepoint adjoined.*

Proof. It is immediate to check the relevant universal property. For details see at *slice category – limits and colimits*. ▮

Example 3.20. Given two pointed objects (X,x) and (Y,y), then:

1. their product in $\mathcal{C}^{*/}$ is simply $(X \times Y, (x,y))$;
2. their coproduct in $\mathcal{C}^{*/}$ has to be computed using the second clause in prop. 3.19: since the point $*$ has to be adjoined to the diagram, it is given not by the coproduct in $\mathcal{C}$, but by the pushout in $\mathcal{C}$ of the form:

$$\begin{array}{ccc} * & \overset{x}{\longrightarrow} & X \\ {}^{y}\downarrow & (\mathrm{po}) & \downarrow \\ Y & \longrightarrow & X \vee Y \end{array} \,.$$

This is called the *wedge sum* operation on pointed objects.

Generally for a set $\{X_i\}_{i \in I}$ in $\mathrm{Top}^{*/}$

1. their product is formed in Top as in example 1.9, with the new basepoint canonically induced;
2. their coproduct is formed by the colimit in Top over the diagram with a basepoint adjoined, and is called the wedge sum $\vee_{i \in I} X_i$.

Example 3.21. For X a CW-complex, def. 1.38 then for every $n \in \mathbb{N}$ the quotient (example 1.13) of its n-skeleton by its $(n-1)$-skeleton is the wedge sum, def. 3.20, of n-spheres, one for each n-cell of X:

$$X^n / X^{n-1} \simeq \underset{i \in I_n}{\vee} S^n \,.$$

Definition 3.22. For $\mathcal{C}^{*/}$ a category of pointed objects with finite limits and finite colimits, the *smash product* is the functor

$$(-) \wedge (-) : \mathcal{C}^{*/} \times \mathcal{C}^{*/} \longrightarrow \mathcal{C}^{*/}$$

given by

$$X \wedge Y \;\coloneqq\; * \underset{X \sqcup Y}{\sqcup} (X \times Y)\,,$$

hence by the pushout in $\mathcal{C}$

$$\begin{array}{ccc} X \sqcup Y & \overset{(\mathrm{id}_X, y), (x, \mathrm{id}_Y)}{\longrightarrow} & X \times Y \\ \downarrow && \downarrow \\ * & \longrightarrow & X \wedge Y \end{array} \,.$$

In terms of the wedge sum from def. 3.20, this may be written concisely as

$$X \wedge Y = \frac{X \times Y}{X \vee Y} \,.$$

Remark 3.23. For a general category $\mathcal{C}$ in def. 3.22, the smash product need not be associative, namely it fails to be associative if the functor $(-) \times Z$ does not preserve the quotients involved in the definition.

In particular this may happen for $\mathcal{C} =$ Top.

A sufficient condition for $(-) \times Z$ to preserve quotients is that it is a left adjoint functor. This is the case in the smaller subcategory of compactly generated topological spaces, we come to this in prop. 3.44 below.

These two operations are going to be ubiquitous in stable homotopy theory:

symbol	name	category theory
$X \vee Y$	wedge sum	coproduct in $\mathcal{C}^{*/}$
$X \wedge Y$	smash product	tensor product in $\mathcal{C}^{*/}$

Example 3.24. For $X, Y \in \mathrm{Top}$, with $X_+, Y_+ \in \mathrm{Top}^{*/}$, def. 3.18, then

- $X_+ \vee Y_+ \simeq (X \sqcup Y)_+$;
- $X_+ \wedge Y_+ \simeq (X \times Y)_+$.

Proof. By example 3.20, $X_+ \vee Y_+$ is given by the colimit in Top over the diagram

$$\begin{tikzcd} & * \arrow[dl] \arrow[dr] & \\ X \quad * & & * \quad Y \end{tikzcd} .$$

This is clearly $X \sqcup * \sqcup Y$. Then, by definition 3.22

$$\begin{aligned} X_+ \wedge Y_+ &\simeq \frac{(X \sqcup *) \times (X \sqcup *)}{(X \sqcup *) \vee (Y \sqcup *)} \\ &\simeq \frac{X \times Y \sqcup X \sqcup Y \sqcup *}{X \sqcup Y \sqcup *} \\ &\simeq X \times Y \sqcup * \;. \end{aligned}$$

∎

Example 3.25. Let $\mathcal{C}^{*/} = \mathrm{Top}^{*/}$ be pointed topological spaces. Then

$$I_+ \in \mathrm{Top}^{*/}$$

denotes the standard interval object $I = [0,1]$ from def. 1.22, with a djoint basepoint adjoined, def. 3.18. Now for X any pointed topological space, then

$$X \wedge (I_+) = (X \times I)/(\{x_0\} \times I)$$

is the **reduced cylinder** over X: the result of forming the ordinary cyclinder over X as in def. 1.22, and then identifying the interval over the basepoint of X with the point.

(Generally, any construction in $\mathcal{C}$ properly adapted to pointed objects $\mathcal{C}^{*/}$ is called the "reduced" version of the unpointed construction. Notably so for "reduced suspension" which we come to below.)

Just like the ordinary cylinder $X \times I$ receives a canonical injection from the coproduct $X \sqcup X$ formed in Top, so the reduced cyclinder receives a canonical injection from the coproduct $X \sqcup X$ formed in $\mathrm{Top}^{*/}$, which is the wedge sum from example 3.20:

$$X \vee X \longrightarrow X \wedge (I_+) \;.$$

Example 3.26. For $(X,x), (Y,y)$ pointed topological spaces with Y a locally compact topological space, then the **pointed mapping space** is the topological subspace of the mapping space of def. 1.18

$$\mathrm{Maps}((Y,y),(X,x))_* \hookrightarrow (X^Y, \mathrm{const}_x)$$

on those maps which preserve the basepoints, and pointed by the map constant on the basepoint of X.

In particular, the **standard topological pointed path space object** on some pointed X (the pointed variant of def. 1.34) is the pointed mapping space $\mathrm{Maps}(I_+, X)_*$.

The pointed consequence of prop. 1.19 then gives that there is a natural bijection

$$\mathrm{Hom}_{\mathrm{Top}^{*/}}((Z,z) \wedge (Y,y), (X,x)) \simeq \mathrm{Hom}_{\mathrm{Top}^{*/}}((Z,z), \mathrm{Maps}((Y,y),(X,x))_*)$$

between basepoint-preserving continuous functions out of a smash product, def. 3.22, with pointed continuous functions of one variable into the pointed mapping space.

Example 3.27. Given a morphism $f : X \longrightarrow Y$ in a category of pointed objects $\mathcal{C}^{*/}$, def. 3.16, with finite limits and colimits,

1. its *fiber* or *kernel* is the pullback of the point inclusion

$$\begin{array}{ccc} \mathrm{fib}(f) & \longrightarrow & X \\ \downarrow & (\mathrm{pb}) & \downarrow^{f} \\ * & \longrightarrow & Y \end{array}$$

2. its *cofiber* or *cokernel* is the pushout of the point projection

$$\begin{array}{ccc} X & \overset{f}{\longrightarrow} & Y \\ \downarrow & (\mathrm{po}) & \downarrow \\ * & \longrightarrow & \mathrm{cofib}(f) \end{array} \,.$$

Remark 3.28. In the situation of example 3.27, both the pullback as well as the pushout are equivalently computed in $\mathcal{C}$. For the pullback this is the first clause of prop. 3.19. The second clause says that for computing the pushout in $\mathcal{C}$, first the point is to be adjoined to the diagram, and then the colimit over the larger diagram

$$\begin{array}{ccc} * & & \\ & \searrow & \\ & & X \overset{f}{\longrightarrow} Y \\ & & \downarrow \\ & & * \end{array}$$

be computed. But one readily checks that in this special case this does not affect the result. (The technical jargon is that the inclusion of the smaller diagram into the larger one in this case happens to be a final functor.)

Proposition 3.29. *Let $\mathcal{C}$ be a model category and let $X \in \mathcal{C}$ be an object. Then both the slice category $\mathcal{C}_{/X}$ as well as the coslice category $\mathcal{C}^{X/}$, def. 3.15, carry model structures themselves – the* ***model structure on a (co-)slice category****, where a morphism is a weak equivalence, fibration or cofibration iff its image under the forgetful functor U is so in $\mathcal{C}$.*

In particular the category $\mathcal{C}^{*/}$ of pointed objects, def. 3.16, in a model category $\mathcal{C}$ becomes itself a model category this way.

The corresponding homotopy category of a model category, def. 2.25, we call the ***pointed homotopy category*** *$\mathrm{Ho}(\mathcal{C}^{*/})$.*

Proof. This is immediate:

By prop. 3.19 the (co-)slice category has all limits and colimits. By definition of the weak equivalences in the (co-)slice, they satisfy two-out-of-three, def. 2.1, because the do in $\mathcal{C}$.

Similarly, the factorization and lifting is all induced by $\mathcal{C}$: Consider the coslice category $\mathcal{C}^{X/}$, the case of the slice category is formally dual; then if

$$\begin{array}{ccc} & X & \\ \swarrow & & \searrow \\ A & \underset{f}{\longrightarrow} & B \end{array}$$

commutes in $\mathcal{C}$, and a factorization of f exists in $\mathcal{C}$, it uniquely makes this diagram commute

$$\begin{array}{ccccc} & & X & & \\ & \swarrow & \downarrow & \searrow & \\ A & \longrightarrow & C & \longrightarrow & B \end{array} \,.$$

Similarly, if

$$\begin{array}{ccc} A & \longrightarrow & C \\ \downarrow & & \downarrow \\ B & \longrightarrow & D \end{array}$$

is a commuting diagram in $\mathcal{C}^{X/}$, hence a commuting diagram in $\mathcal{C}$ as shown, with all objects equipped with compatible morphisms from X, then inspection shows that any lift in the diagram necessarily respects the maps from X, too. ▮

Example 3.30. For $\mathcal{C}$ any model category, with $\mathcal{C}^{*/}$ its pointed model structure according to prop. 3.29, then the corresponding homotopy category (def. 2.25) is, by remark 3.17, canonically enriched in pointed sets, in that its hom-functor is of the form

$$[-,-]_* : \mathrm{Ho}(\mathcal{C}^{*/})^{\mathrm{op}} \times \mathrm{Ho}(\mathcal{C}^{*/}) \longrightarrow \mathrm{Set}^{*/}\,.$$

Definition 3.31. Write $\mathrm{Top}^{*/}_{\mathrm{Quillen}}$ for the **classical model structure on pointed topological spaces**, obtained from the classical model structure on topological spaces $\mathrm{Top}_{\mathrm{Quillen}}$ (theorem 3.7) via the induced coslice model structure of prop. 3.29.

Its homotopy category, def. 2.25,

$$\mathrm{Ho}(\mathrm{Top}^{*/}) \coloneqq \mathrm{Ho}(\mathrm{Top}^{*/}_{\mathrm{Quillen}})$$

we call the **classical pointed homotopy category**.

Remark 3.32. The fibrant objects in the pointed model structure $\mathcal{C}^{*/}$, prop. 3.29, are those that are fibrant as objects of $\mathcal{C}$. But the cofibrant objects in $\mathcal{C}^*$ are now those for which the basepoint inclusion is a cofibration in X.

For $\mathcal{C}^{*/} = \mathrm{Top}^{*/}_{\mathrm{Quillen}}$ from def. 3.31, then the corresponding cofibrant pointed topological spaces are tyically referred to as spaces **with non-degenerate basepoints** or . Notice that the point itself is cofibrant in $\mathrm{Top}_{\mathrm{Quillen}}$, so that cofibrant pointed topological spaces are in particular cofibrant topological spaces.

While the existence of the model structure on $\mathrm{Top}^{*/}$ is immediate, via prop. 3.29, for the discussion of topologically enriched functors (below) it is useful to record that this, too, is a cofibrantly generated model category (def. 3.9), as follows:

Definition 3.33. Write

$$I_{\mathrm{Top}^{*/}} = \left\{ S^{n-1}_+ \overset{(\iota_n)_+}{\longrightarrow} D^n_+ \right\} \quad \subset \mathrm{Mor}(\mathrm{Top}^{*/})$$

and

$$J_{\mathrm{Top}^{*/}} = \left\{ D^n_+ \overset{(\mathrm{id},\delta_0)_+}{\longrightarrow} (D^n \times I)_+ \right\} \quad \subset \mathrm{Mor}(\mathrm{Top}^{*/})\,,$$

respectively, for the sets of morphisms obtained from the classical generating cofibrations, def. 1.37, and the classical generating acyclic cofibrations, def. 1.42, under adjoining of basepoints (def. 3.18).

Theorem 3.34. *The sets $I_{\mathrm{Top}^{*/}}$ and $J_{\mathrm{Top}^{*/}}$ in def. 3.33 exhibit the classical model structure on pointed topological spaces $\mathrm{Top}^{*/}_{\mathrm{Quillen}}$ of def. 3.31 as a cofibrantly generated model category, def. 3.9.*

(This is also a special case of a general statement about cofibrant generation of coslice model structures, see this proposition.)

Proof. Due to the fact that in $J_{\mathrm{Top}^{*/}}$ a basepoint is freely adjoined, lemma 1.52 goes through verbatim for the pointed case, with J_{Top} replaced by $J_{\mathrm{Top}^{*/}}$, as do the other two lemmas above that depend on point-set topology, lemma 1.40 and lemma 1.45. With this, the rest of the proof follows by the same general abstract reasoning as above in the proof of theorem 3.7. ▮

Model structure on compactly generated spaces

The category Top has the technical inconvenience that mapping spaces X^Y (def. 1.18) satisfying the exponential property (prop. 1.19) exist in general only for Y a locally compact topological space, but fail to exist more generally. In other words: Top is not cartesian closed. But cartesian closure is necessary for some purposes of homotopy theory, for instance it ensures that

1. the smash product (def. 3.22) on pointed topological spaces is associative (prop. 3.44 below);
2. there is a concept of topologically enriched functors with values in topological spaces, to which we turn below;
3. geometric realization of simplicial sets preserves products.

The first two of these are crucial for the development of stable homotopy theory in the next section, the third is a great convenience in computations.

Now, since the homotopy theory of topological spaces only cares about the CW approximation to any topological space (remark 3.12), it is plausible to ask for a full subcategory of Top which still contains all CW-complexes, still has all limits and colimits, still supports a model category structure constructed in the same way as above, but which in addition is cartesian closed, and preferably such that the model structure interacts well with the cartesian closure.

Such a full subcategory exists, the category of compactly generated topological spaces. This we briefly describe now.

Literature (Strickland 09)

Definition 3.35. Let X be a topological space.

A subset $A \subset X$ is called **compactly closed** (or **k-closed**) if for every continuous function $f : K \longrightarrow X$ out of a compact Hausdorff space K, then the preimage $f^{-1}(A)$ is a closed subset of K.

The space X is called **compactly generated** if its closed subsets exhaust (hence coincide with) the k-closed subsets.

Write

$$\mathrm{Top}_{\mathrm{cg}} \hookrightarrow \mathrm{Top}$$

for the full subcategory of Top on the compactly generated topological spaces.

Definition 3.36. Write

$$\mathrm{Top} \overset{k}{\longrightarrow} \mathrm{Top}_{\mathrm{cg}} \hookrightarrow \mathrm{Top}$$

for the functor which sends any topological space $X = (S, \tau)$ to the topological space $(S, k\tau)$ with the same underlying set S, but with open subsets $k\tau$ the collection of all k-open subsets with respect to τ.

Lemma 3.37. *Let $X \in \mathrm{Top}_{\mathrm{cg}} \hookrightarrow \mathrm{Top}$ and let $Y \in \mathrm{Top}$. Then continuous functions*

$$X \longrightarrow Y$$

are also continuous when regarded as functions

$$X \longrightarrow k(Y)$$

with k from def. 3.36.

Proof. We need to show that for $A \subset X$ a k-closed subset, then the preimage $f^{-1}(A) \subset X$ is closed subset.

Let $\phi : K \longrightarrow X$ be any continuous function out of a compact Hausdorff space K. Since A is k-closed by assumption, we have that $(f \circ \phi)^{-1}(A) = \phi^{-1}(f^{-1}(A)) \subset K$ is closed in K. This means that $f^{-1}(A)$ is k-closed in X. But by the assumption that X is compactly generated, it follows that $f^{-1}(A)$ is already closed. ▮

Corollary 3.38. *For $X \in \mathrm{Top}_{\mathrm{cg}}$ there is a natural bijection*

$$\mathrm{Hom}_{\mathrm{Top}}(X, Y) \simeq \mathrm{Hom}_{\mathrm{Top}_{\mathrm{cg}}}(X, k(Y)) \; .$$

This means equivalently that the functor k (def. 3.36) together with the inclusion from def. 3.35 forms an pair of adjoint functors

$$\mathrm{Top}_{\mathrm{cg}} \underoverset{\underset{k}{\longleftarrow}}{\hookrightarrow}{\bot} \mathrm{Top} \; .$$

This in turn means equivalently that $\mathrm{Top}_{\mathrm{cg}} \hookrightarrow \mathrm{Top}$ is a coreflective subcategory with coreflector k. In particular k is idemotent in that there are natural homeomorphisms

$$k(k(X)) \simeq k(X) \; .$$

Hence colimits in $\mathrm{Top}_{\mathrm{cg}}$ exists and are computed as in Top. Also limits in $\mathrm{Top}_{\mathrm{cg}}$ exists, these are obtained by computing the limit in Top and then applying the functor k to the result.

The following is a slight variant of def. 1.18, appropriate for the context of $\mathrm{Top}_{\mathrm{cg}}$.

Definition 3.39. For $X, Y \in \mathrm{Top}_{\mathrm{cg}}$ (def. 3.35) the **compactly generated mapping space** $X^Y \in \mathrm{Top}_{\mathrm{cg}}$ is the compactly generated topological space whose underlying set is the set $C(Y, X)$ of continuous functions $f : Y \to X$, and for which a subbase for its topology has elements $U^{\phi(K)}$, for $U \subset X$ any open subset and $\phi : K \to Y$ a continuous function out of a compact Hausdorff space K given by

$$U^{\phi(\kappa)} \coloneqq \{ f \in C(Y, X) \,|\, f(\phi(K)) \subset U \} \; .$$

Remark 3.40. If Y is (compactly generated and) a Hausdorff space, then the topology on the compactly generated mapping space X^Y in def. 3.39 agrees with the compact-open topology of def. 1.18. Beware that it is common to say “compact-open topology” also for the topology of the compactly generated mapping space when Y is not Hausdorff. In that case, however, the two definitions in general disagree.

Proposition 3.41. *The category $\mathrm{Top}_{\mathrm{cg}}$ of def. 3.35 is cartesian closed:*

for every $X \in \mathrm{Top}_{\mathrm{cg}}$ then the operation $X \times (-) \times (-) \times X$ of forming the Cartesian product in $\mathrm{Top}_{\mathrm{cg}}$ (which by cor. 3.38 is k applied to the usual product topological space) together with the operation $(-)^X$ of forming the compactly generated mapping space (def. 3.39) forms a pair of adjoint functors

$$\mathrm{Top}_{\mathrm{cg}} \underoverset{\underset{(-)^X}{\longrightarrow}}{\overset{X \times (-)}{\longleftarrow}}{\bot} \mathrm{Top}_{\mathrm{cg}} \,.$$

For proof see for instance (Strickland 09, prop. 2.12).

Corollary 3.42. *For $X, Y \in \mathrm{Top}_{\mathrm{cg}}^{*/}$, the operation of forming the pointed mapping space (example 3.26) inside the compactly generated mapping space of def. 3.39*

$$\mathrm{Maps}(Y,X)_* \coloneqq \mathrm{fib}\left(X^Y \overset{\mathrm{ev}_y}{\longrightarrow} X \,, x\right)$$

is left adjoint to the smash product operation on pointed compactly generated topological spaces.

$$\mathrm{Top}_{\mathrm{cg}}^{*/} \underoverset{\underset{\mathrm{Maps}(Y,-)_*}{\longrightarrow}}{\overset{Y \wedge (-)}{\longleftarrow}}{\bot} \mathrm{Top}_{\mathrm{cg}}^{*/} \,.$$

Corollary 3.43. *For I a small category and $X_\bullet : I \to \mathrm{Top}_{\mathrm{cg}}^{*/}$ a diagram, then the compactly generated mapping space construction from def. 3.39 preserves limits in its covariant argument and sends colimits in its contravariant argument to limits:*

$$\mathrm{Maps}(X, \underset{\longleftarrow_i}{\lim} Y_i)_* \;\simeq\; \underset{\longleftarrow_i}{\lim} \mathrm{Maps}(X, Y_i)_*$$

and

$$\mathrm{Maps}(\underset{\longrightarrow_i}{\lim} X_i,\, Y)_* \simeq \underset{\longleftarrow_i}{\lim} \mathrm{Maps}(X_i, Y)_* \,.$$

Proof. The first statement is an immediate implication of $\mathrm{Maps}(X,-)_*$ being a right adjoint, according to cor. 3.42.

For the second statement, we use that by def. 3.35 a compactly generated topological space is uniquely determined if one knows all continuous functions out of compact Hausdorff spaces into it. Hence it is sufficient to show that there is a natural isomorphism

$$\mathrm{Hom}_{\mathrm{Top}_{\mathrm{cg}}^{*/}}\left(K,\, \mathrm{Maps}(\underset{\longrightarrow_i}{\lim} X_i,\, Y)_*\right) \simeq \mathrm{Hom}_{\mathrm{Top}_{\mathrm{cg}}^{*/}}\left(K,\, \underset{\longleftarrow_i}{\lim} \mathrm{Maps}(X_i, Y)_*\right)$$

for K any compact Hausdorff space.

With this, the statement follows by cor. 3.42 and using that ordinary hom-sets take colimits in the first argument and limits in the second argument to limits:

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Top}_{\mathrm{cg}}^{*/}}\left(K,\, \mathrm{Maps}(\underset{\longrightarrow_i}{\lim} X_i,\, Y)_*\right) & \simeq \mathrm{Hom}_{\mathrm{Top}_{\mathrm{cg}}^{*/}}\left(K \wedge \underset{\longrightarrow_i}{\lim} X_i,\, Y\right) \\ & \simeq \mathrm{Hom}_{\mathrm{Top}_{\mathrm{cg}}^{*/}}\left(\underset{\longrightarrow_i}{\lim} (K \wedge X_i),\, Y\right) \\ & \simeq \underset{\longleftarrow_i}{\lim} \left(\mathrm{Hom}_{\mathrm{Top}_{\mathrm{cg}}^{*/}}(K \wedge X_i,\, Y)\right) \\ & \simeq \underset{\longleftarrow_i}{\lim} \mathrm{Hom}_{\mathrm{Top}_{\mathrm{cg}}^{*/}}(K,\, \mathrm{Maps}(X_i, Y)_*) \\ & \simeq \mathrm{Hom}_{\mathrm{Top}_{\mathrm{cg}}^{*/}}\left(K,\, \underset{\longleftarrow_i}{\lim} \mathrm{Maps}(X_i, Y)_*\right) \end{aligned} \,.$$

▮

Moreover, compact generation fixes the associativity of the smash product (remark 3.23):

Proposition 3.44. *On pointed (def. 3.16) compactly generated topological spaces (def. 3.35) the smash product (def. 3.22)*

$$(-) \wedge (-) \, : \, \mathrm{Top}_{\mathrm{cg}}^{*/} \times \mathrm{Top}_{\mathrm{cg}}^{*/} \longrightarrow \mathrm{Top}_{\mathrm{cg}}^{*/}$$

is associative and the 0-sphere is a tensor unit for it.

Proof. Since $(-) \times X$ is a left adjoint by prop. 3.41, it presevers colimits and in particular quotient space projections. Therefore with $X, Y, Z \in \mathrm{Top}_{\mathrm{cg}}^{*/}$ then

$$\begin{aligned} (X \wedge Y) \wedge Z &= \frac{\frac{X \times Y}{X \times \{y\} \sqcup \{x\} \times Y} \times Z}{(X \wedge Y) \times \{z\} \sqcup \{[x] = [y]\} \times Z} \\ &\simeq \frac{\frac{X \times Y \times Z}{X \times \{y\} \times Z \sqcup \{x\} \times Y \times Z}}{X \times Y \times \{z\}} \\ &\simeq \frac{X \times Y \times Z}{X \vee Y \vee Z} \end{aligned} .$$

The analogous reasoning applies to yield also $X \wedge (Y \wedge Z) \simeq \frac{X \times Y \times Z}{X \vee Y \vee Z}$.

The second statement follows directly with prop. 3.41. ▮

Remark 3.45. Corollary 3.42 together with prop. 3.44 says that under the smash product the category of pointed compactly generated topological spaces is a closed symmetric monoidal category with tensor unit the 0-sphere.

$$(\mathrm{Top}_{\mathrm{cg}}^{*/}, \wedge, S^0), .$$

Notice that by prop. 3.41 also unpointed compactly generated spaces under Cartesian product form a closed symmetric monoidal category, hence a cartesian closed category

$$(\mathrm{Top}_{\mathrm{cg}}, \times, *) .$$

The fact that $\mathrm{Top}_{\mathrm{cg}}^{*/}$ is still closed symmetric monoidal but no longer Cartesian exhibits $\mathrm{Top}_{\mathrm{cg}}^{*/}$ as being "more linear" than $\mathrm{Top}_{\mathrm{cg}}$. The "full linearization" of $\mathrm{Top}_{\mathrm{cg}}$ is the closed symmteric monoidal category of structured spectra under smash product of spectra which we discuss in section 1.

Due to the idempotency $k \circ k \simeq k$ (cor. 3.38) it is useful to know plenty of conditions under which a given topological space is already compactly generated, for then applying k to it does not change it and one may continue working as in Top.

Example 3.46. Every CW-complex is compactly generated.

Proof. Since a CW-complex is a Hausdorff space, by prop. 3.53 and prop. 3.54 its k-closed subsets are precisely those whose intersection with every compact subspace is closed.

Since a CW-complex X is a colimit in Top over attachments of standard n-disks D^{n_i} (its cells), by the characterization of colimits in Top (prop.) a subset of X is open or closed precisely if its restriction to each cell is open or closed, respectively. Since the n-disks are compact, this implies one direction: if a subset A of X intersected with all compact subsets is closed, then A is closed.

For the converse direction, since a CW-complex is a Hausdorff space and since compact subspaces of Hausdorff spaces are closed, the intersection of a closed subset with a compact subset is closed. ▮

For completeness we record further classes of examples:

Example 3.47. The category $\mathrm{Top}_{\mathrm{cg}}$ of compactly generated topological spaces includes

1. all locally compact topological spaces,

2. all first-countable topological spaces,

hence in particular

1. all metrizable topological spaces,
2. all discrete topological spaces,
3. all codiscrete topological spaces.

(Lewis 78, p. 148)

Recall that by corollary 3.38, all colimits of compactly generated spaces are again compactly generated.

Example 3.48. The product topological space of a CW-complex with a compact CW-complex, and more generally with a locally compact CW-complex, is compactly generated.

(Hatcher "Topology of cell complexes", theorem A.6)

More generally:

Proposition 3.49. For X a compactly generated space and Y a locally compact Hausdorff space, then the product topological space $X \times Y$ is compactly generated.

e.g. (Strickland 09, prop. 26)

Finally we check that the concept of homotopy and homotopy groups does not change under passing to compactly generated spaces:

Proposition 3.50. *For every topological space X, the canonical function $k(X) \longrightarrow X$ (the adjunction unit) is a weak homotopy equivalence.*

Proof. By example 3.46, example 3.48 and lemma 3.37, continuous functions $S^n \to k(X)$ and their left homotopies $S^n \times I \to k(X)$ are in bijection with functions $S^n \to X$ and their homotopies $S^n \times I \to X$. ▮

Theorem 3.51. *The restriction of the model category structure on $\mathrm{Top}_{\mathrm{Quillen}}$ from theorem 3.7 along the inclusion $\mathrm{Top}_{\mathrm{cg}} \hookrightarrow \mathrm{Top}$ of def. 3.35 is still a model category structure, which is cofibrantly generated by the same sets I_{Top} (def. 1.37) and J_{Top} (def. 1.42) The coreflection of cor. 3.38 is a Quillen equivalence (def. 2.50)*

$$(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{Quillen}} \underset{k}{\overset{\hookrightarrow}{\underset{\longleftarrow}{\perp}}} \mathrm{Top}_{\mathrm{Quillen}} .$$

Proof. By example 3.46, the sets I_{Top} and J_{Top} are indeed in $\mathrm{Mor}(\mathrm{Top}_{\mathrm{cg}})$. By example 3.48 all arguments above about left homotopies between maps out of these basic cells go through verbatim in $\mathrm{Top}_{\mathrm{cg}}$. Hence the three technical lemmas above depending on actual point-set topology, topology, lemma 1.40, lemma 1.45 and lemma 1.52, go through verbatim as before. Accordingly, since the remainder of the proof of theorem 3.7 of $\mathrm{Top}_{\mathrm{Quillen}}$ follows by general abstract arguments from these, it also still goes through verbatim for $(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{Quillen}}$ (repeatedly use the small object argument and the retract argument to establish the two weak factorization systems).

Hence the (acyclic) cofibrations in $(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{Quillen}}$ are identified with those in $\mathrm{Top}_{\mathrm{Quillen}}$, and so the inclusion is a part of a Quillen adjunction (def. 2.46). To see that this is a Quillen equivalence (def. 2.50), it is sufficient to check that for X a compactly generated space then a continuous function $f : X \longrightarrow Y$ is a weak homotopy equivalence (def. 1.30) precisely if the adjunct $\tilde f : X \to k(Y)$ is a weak homotopy equivalence. But, by lemma 3.37, $\tilde f$ is the same function as f, just considered with different codomain. Hence the result follows with prop. 3.50. ▮

Compactly generated weakly Hausdorff topological spaces

While the inclusion $\mathrm{Top}_{\mathrm{cg}} \hookrightarrow \mathrm{Top}$ of def. 3.35 does satisfy the requirement that it gives a cartesian closed category with all limits and colimits and containing all CW-complexes, one may ask for yet smaller subcategories that still share all these properties but potentially exhibit further convenient properties still.

A popular choice introduced in (McCord 69) is to add the further restriction to topopological spaces which are not only compactly generated but also weakly Hausdorff. This was motivated from (Steenrod 67) where compactly generated Hausdorff spaces were used by the observation ((McCord 69, section 2)) that Hausdorffness is not preserved my many colimit operations, notably not by forming quotient spaces.

On the other hand, in above we wouldn't have imposed Hausdorffness in the first place. More intrinsic advantages of $\mathrm{Top}_{\mathrm{cgwH}}$ over $\mathrm{Top}_{\mathrm{cg}}$ are the following:

- every pushout of a morphism in $\mathrm{Top}_{\mathrm{cgwH}} \hookrightarrow \mathrm{Top}$ along a closed subspace inclusion in Top is again in $\mathrm{Top}_{\mathrm{cgwH}}$
- in $\mathrm{Top}_{\mathrm{cgwH}}$ quotient spaces are not only preserved by cartesian products (as is the case for all compactly generated spaces due to $X \times (-)$ being a left adjoint, according to cor. 3.38) but by all pullbacks
- in $\mathrm{Top}_{\mathrm{cgwH}}$ the regular monomorphisms are the closed subspace inclusions

We will not need this here or in the following sections, but we briefly mention it for completenes:

Definition 3.52. A topological space X is called **weakly Hausdorff** if for every continuous function

$$f : K \longrightarrow X$$

out of a compact Hausdorff space K, its image $f(K) \subset X$ is a closed subset of X.

Proposition 3.53. *Every Hausdorff space is a weakly Hausdorff space, def. 3.52.*

Proof. Since compact subspaces of Hausdorff spaces are closed. ▮

Proposition 3.54. *For X a weakly Hausdorff topological space, def. 3.52, then a subset $A \subset X$ is k-closed, def. 3.35, precisely if for every subset $K \subset X$ that is compact Hausdorff with respect to the subspace topology, then the intersection $K \cap A$ is a closed subset of X.*

e.g. (Strickland 09, lemma 1.4 (c))

Topological enrichment

So far the classical model structure on topological spaces which we established in theorem 3.7, as well as the projective model structures on topologically enriched functors induced from it in theorem 3.76, concern the hom-sets, but not the hom-spaces (def. 3.65), i.e. the model structure so far has not been related to the topology on hom-spaces. The following statements say that in fact the model structure and the enrichment by topology on the hom-spaces are compatible in a suitable sense: we have an "enriched model category". This implies in particular that the product/hom-adjunctions are Quillen adjunctions, which is crucial for a decent discusson of the derived functors of the suspension/looping adjunction below.

Definition 3.55. Let $i_1 : X_1 \to Y_1$ and $i_2 : X_2 \to Y_2$ be morphisms in $\mathrm{Top}_{\mathrm{cg}}$, def. 3.35. Their **pushout product**

$$i_1 \square i_2 \coloneqq ((\mathrm{id}, i_2), (i_1, \mathrm{id}))$$

is the universal morphism in the following diagram

$$\begin{tikzcd} & X_1 \times X_2 \arrow[dl, "(i_1,\mathrm{id})"'] \arrow[dr, "(\mathrm{id},i_2)"] & \\ Y_1 \times X_2 \arrow[dr] & (\mathrm{po}) & X_1 \times Y_2 \arrow[dl] \\ & (Y_1 \times X_2) \underset{X_1 \times X_2}{\sqcup} (X_1 \times Y_2) \arrow[d, "((\mathrm{id},i_2),(i_1,\mathrm{id}))"] & \\ & Y_1 \times Y_2 & \end{tikzcd}$$

Example 3.56. If $i_1 : X_1 \hookrightarrow Y_1$ and $i_2 : X_2 \hookrightarrow Y_2$ are inclusions, then their pushout product $i_1 \square i_2$ from def. 3.55 is the inclusion

$$(X_1 \times Y_2 \ \cup \ Y_1 \times X_2) \hookrightarrow Y_1 \times Y_2 \,.$$

For instance

$$(\{0\} \hookrightarrow I) \square (\{0\} \hookrightarrow I)$$

is the inclusion of two adjacent edges of a square into the square.

Example 3.57. The pushout product with an initial morphism is just the ordinary Cartesian product functor

$$(\emptyset \to X) \square (-) \simeq X \times (-) \,,$$

i.e.

$$(\emptyset \to X) \square (A \overset{f}{\to} B) \simeq (X \times A \overset{X \times f}{\longrightarrow} X \times B) \,.$$

Proof. The product topological space with the empty space is the empty space, hence the map $\emptyset \times A \overset{(\mathrm{id},f)}{\longrightarrow} \emptyset \times B$ is an isomorphism, and so the pushout in the pushout product is $X \times A$. From this one reads off the universal map in question to be $X \times f$:

$$\begin{tikzcd} & \emptyset \times A \arrow[dl] \arrow[dr, "\simeq"] & \\ X \times A \arrow[dr, "\simeq"'] & (\mathrm{po}) & \emptyset \times B \arrow[dl] \\ & X \times A \arrow[d, "((\mathrm{id},f),\exists !)"] & \\ & X \times B & \end{tikzcd} .$$

∎

Example 3.58. With

$$I_{\mathrm{Top}} : \{ S^{n-1} \overset{i_n}{\hookrightarrow} D^n \} \quad \text{and} \quad J_{\mathrm{Top}} : \{ D^n \overset{j_n}{\hookrightarrow} D^n \times I \}$$

the generating cofibrations (def. 1.37) and generating acyclic cofibrations (def. 1.42) of $(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{Quillen}}$ (theorem 3.51), then their pushout-products (def. 3.55) are

$$
\begin{aligned}
i_{n_1} \,\Box\, i_{n_2} &\simeq i_{n_1+n_2} \\
i_{n_1} \,\Box\, j_{n_2} &\simeq j_{n_1+n_2}
\end{aligned}
\,.
$$

Proof. To see this, it is profitable to model n-disks and n-spheres, up to homeomorphism, as n-cubes $D^n \simeq [0,1]^n \subset \mathbb{R}^n$ and their boundaries $S^{n-1} \simeq \partial [0,1]^n$. For the idea of the proof, consider the situation in low dimensions, where one readily sees pictorially that

$$
i_1 \,\Box\, i_1 : (\;=\; \cup \;||\;) \hookrightarrow \Box
$$

and

$$
i_1 \,\Box\, j_0 : (\;=\; \cup \;|\;) \hookrightarrow \Box \,.
$$

Generally, D^n may be represented as the space of n-tuples of elements in $[0,1]$, and S^n as the suspace of tuples for which at least one of the coordinates is equal to 0 or to 1.

Accordingly, $S^{n_1} \times D^{n_2} \hookrightarrow D^{n_1+n_2}$ is the subspace of (n_1+n_2)-tuples, such that at least one of the first n_1 coordinates is equal to 0 or 1, while $D^{n_1} \times S^{n_2} \hookrightarrow D^{n_1+n_2}$ is the subspace of (n_1+n_2)-tuples such that east least one of the last n_2 coordinates is equal to 0 or to 1. Therefore

$$
S^{n_1} \times D^{n_2} \cup D^{n_1} \times S^{n_2} \simeq S^{n_1+n_2} \,.
$$

And of course it is clear that $D^{n_1} \times D^{n_2} \simeq D^{n_1+n_2}$. This shows the first case.

For the second, use that $S^{n_1} \times D^{n_2} \times I$ is contractible to $S^{n_1} \times D^{n_2}$ in $D^{n_1} \times D^{n_2} \times I$, and that $S^{n_1} \times D^{n_2}$ is a subspace of $D^{n_1} \times D^{n_2}$. ▮

Definition 3.59. Let $i \colon A \to B$ and $p \colon X \to Y$ be two morphisms in $\mathrm{Top}_{\mathrm{cg}}$, def. 3.35. Their **pullback powering** is

$$
p^{\Box i} \coloneqq (p^B, X^i)
$$

being the universal morphism in

$$
\begin{tikzcd}
 & X^B \arrow[d, "(p^B, X^i)"] & \\
 & Y^B \underset{Y^A}{\times} X^A \arrow[dl] \arrow[dr] & \\
Y^B \arrow[dr, "Y^i"'] & (\mathrm{pb}) & X^A \arrow[dl, "p^A"] \\
 & Y^A &
\end{tikzcd}
$$

Proposition 3.60. *Let i_1, i_2, p be three morphisms in $\mathrm{Top}_{\mathrm{cg}}$, def. 3.35. Then for their pushout-products (def. 3.55) and pullback-powerings (def. 3.59) the following lifting properties are equivalent ("Joyal-Tierney calculus"):*

$$
\begin{array}{ccc}
 & i_1 \,\Box\, i_2 & \text{has LLP against} & p \\
\Leftrightarrow & i_1 & \text{has LLP against} & p^{\Box i_2} \\
\Leftrightarrow & i_2 & \text{has LLP against} & p^{\Box i_1}
\end{array}
\,.
$$

Proof. We claim that by the cartesian closure of $\mathrm{Top}_{\mathrm{cg}}$, and carefully collecting terms, one finds a natural bijection between commuting squares and their lifts as follows:

$$\begin{array}{ccc} Q & \overset{f}{\longrightarrow} & X^B \\ {}^{\mathllap{i_1}}\downarrow && \downarrow^{\mathrlap{p^{\Box i_2}}} \\ P & \underset{(g_1,g_2)}{\longrightarrow} & Y^B \underset{Y^A}{\times} X^A \end{array} \;\;\;\; \leftrightarrow \;\;\;\; \begin{array}{ccc} Q \times B \underset{Q\times A}{\sqcup} P \times A & \overset{(\tilde f, \tilde g_2)}{\longrightarrow} & X \\ {}^{\mathllap{i_1 \Box i_2}}\downarrow && \downarrow^{\mathrlap{p}} \\ P \times B & \underset{\tilde g_1}{\longrightarrow} & Y \end{array}$$

where the tilde denotes product/hom-adjuncts, for instance

$$\frac{P \overset{g_1}{\longrightarrow} Y^B}{P \times B \overset{\tilde g_1}{\longrightarrow} Y}$$

etc.

To see this in more detail, observe that both squares above each represent two squares from the two components into the fiber product and out of the pushout, respectively, as well as one more square exhibiting the compatibility condition on these components:

$$\begin{aligned} & \begin{array}{ccc} Q & \overset{f}{\longrightarrow} & X^B \\ {}^{\mathllap{i_1}}\downarrow && \downarrow^{\mathrlap{p^{\Box i_2}}} \\ P & \underset{(g_1,g_2)}{\longrightarrow} & Y^B \underset{Y^A}{\times} X^A \end{array} \\ \simeq & \left\{ \begin{array}{ccc} Q & \overset{f}{\longrightarrow} & X^B \\ {}^{\mathllap{i_1}}\downarrow && \downarrow^{\mathrlap{p^B}} \\ P & \underset{g_1}{\longrightarrow} & Y^B \end{array} \;,\; \begin{array}{ccc} Q & \overset{f}{\longrightarrow} & X^B \\ {}^{\mathllap{i_1}}\downarrow && \downarrow^{\mathrlap{X^{i_2}}} \\ P & \underset{g_1}{\longrightarrow} & X^A \end{array} \;,\; \begin{array}{ccc} P & \overset{g_2}{\longrightarrow} & X^A \\ {}^{\mathllap{g_1}}\downarrow && \downarrow^{\mathrlap{p^A}} \\ Y^B & \underset{Y^{i_2}}{\longrightarrow} & Y^A \end{array} \right\} \\ \leftrightarrow & \left\{ \begin{array}{ccc} Q \times B & \overset{\tilde f}{\longrightarrow} & X \\ {}^{\mathllap{(i_1,\mathrm{id})}}\downarrow && \downarrow^{\mathrlap{p}} \\ P \times B & \underset{\tilde g_2}{\longrightarrow} & Y \end{array} \;,\; \begin{array}{ccc} Q \times A & \overset{(\mathrm{id}, i_2)}{\longrightarrow} & Q \times B \\ {}^{\mathllap{(i_1,\mathrm{id})}}\downarrow && \downarrow^{\mathrlap{\tilde f}} \\ P \times A & \underset{\tilde g_2}{\longrightarrow} & X \end{array} \;,\; \begin{array}{ccc} P \times A & \overset{\tilde g_2}{\longrightarrow} & X \\ {}^{\mathllap{(\mathrm{id}, i_2)}}\downarrow && \downarrow^{\mathrlap{p}} \\ P \times B & \underset{\tilde g_1}{\longrightarrow} & Y \end{array} \right\} \\ \simeq & \begin{array}{ccc} Q \times B \underset{Q\times A}{\sqcup} P \times A & \overset{(\tilde f, \tilde g_2)}{\longrightarrow} & X \\ {}^{\mathllap{i_1 \Box i_2}}\downarrow && \downarrow^{\mathrlap{p}} \\ P \times B & \underset{\tilde g_1}{\longrightarrow} & Y \end{array} \end{aligned}$$

■

Proposition 3.61. *The pushout-product in $\mathrm{Top}_{\mathrm{cg}}$ (def. 3.35) of two classical cofibrations is a classical cofibration:*

$$\mathrm{Cof}_{\mathrm{cl}} \Box \mathrm{Cof}_{\mathrm{cl}} \subset \mathrm{Cof}_{\mathrm{cl}} \;.$$

If one of them is acyclic, then so is the pushout-product:

$$\mathrm{Cof}_{\mathrm{cl}} \Box (W_{\mathrm{cl}} \cap \mathrm{Cof}_{\mathrm{cl}}) \subset W_{\mathrm{cl}} \cap \mathrm{Cof}_{\mathrm{cl}} \;.$$

Proof. Regarding the first point:

By example 3.58 we have

$$I_{\mathrm{Top}} \Box I_{\mathrm{Top}} \subset I_{\mathrm{Top}}$$

Hence

$$
\begin{aligned}
& I_{\mathrm{Top}} \square I_{\mathrm{Top}} && \text{has LLP against} && W_{\mathrm{cl}} \cap \mathrm{Fib}_{\mathrm{cl}} \\
\Leftrightarrow\; & I_{\mathrm{Top}} && \text{has LLP against} && (W_{\mathrm{cl}} \cap \mathrm{Fib}_{\mathrm{cl}})^{\square I_{\mathrm{Top}}} \\
\Rightarrow\; & \mathrm{Cof}_{\mathrm{cl}} && \text{has LLP against} && (W_{\mathrm{cl}} \cap \mathrm{Fib}_{\mathrm{cl}})^{\square I_{\mathrm{Top}}} \\
\Leftrightarrow\; & I_{\mathrm{Top}} \square \mathrm{Cof}_{\mathrm{cl}} && \text{has LLP against} && W_{\mathrm{cl}} \cap \mathrm{Fib}_{\mathrm{cl}} \quad , \\
\Leftrightarrow\; & I_{\mathrm{Top}} && \text{has LLP against} && (W_{\mathrm{cl}} \cap \mathrm{Fib}_{\mathrm{cl}})^{\mathrm{Cof}_{\mathrm{cl}}} \\
\Rightarrow\; & \mathrm{Cof}_{\mathrm{cl}} && \text{has LLP against} && (W_{\mathrm{cl}} \cap \mathrm{Fib}_{\mathrm{cl}})^{\mathrm{Cof}_{\mathrm{cl}}} \\
\Leftrightarrow\; & \mathrm{Cof}_{\mathrm{cl}} \square \mathrm{Cof}_{\mathrm{cl}} && \text{has LLP against} && W_{\mathrm{cl}} \cap \mathrm{Fib}_{\mathrm{cl}}
\end{aligned}
$$

where all logical equivalences used are those of prop. 3.60 and where all implications appearing are by the closure property of lifting problems, prop. 2.10.

Regarding the second point: By example 3.58 we moreover have

$$
I_{\mathrm{Top}} \square J_{\mathrm{Top}} \subset J_{\mathrm{Top}}
$$

and the conclusion follows by the same kind of reasoning. ∎

Remark 3.62. In model category theory the property in proposition 3.61 is referred to as saying that the model category $(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{Quillen}}$ from theorem \ref{ModelStructureOnTopcg}

1. is a monoidal model category with respect to the Cartesian product on $\mathrm{Top}_{\mathrm{cg}}$;

2. is an enriched model category, over itself.

A key point of what this entails is the following:

Proposition 3.63. *For $X \in (\mathrm{Top}_{\mathrm{cg}})_{\mathrm{Quillen}}$ cofibrant (a retract of a cell complex) then the product-hom-adjunction for Y (prop. 3.41) is a Quillen adjunction*

$$
(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{Quillen}} \underset{(-)^X}{\overset{X\times(-)}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} (\mathrm{Top}_{\mathrm{cg}})_{\mathrm{Quillen}} \;.
$$

Proof. By example 3.57 we have that the left adjoint functor is equivalently the pushout product functor with the initial morphism of X:

$$
X \times (-) \simeq (\emptyset \to X) \square (-) \;.
$$

By assumption $(\emptyset \to X)$ is a cofibration, and hence prop. 3.61 says that this is a left Quillen functor. ∎

The statement and proof of prop. 3.63 has a direct analogue in pointed topological spaces

Proposition 3.64. *For $X \in (\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}$ cofibrant with respect to the classical model structure on pointed compactly generated topological spaces (theorem 3.51, prop. 3.29) (hence a retract of a cell complex with non-degenerate basepoint, remark 3.32) then the pointed product-hom-adjunction from corollary 3.42 is a Quillen adjunction (def. 2.46):*

$$
(\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}} \underset{\mathrm{Maps}(X,-)_*}{\overset{X\wedge(-)}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} (\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}} \;.
$$

Proof. Let now $\square_\wedge$ denote the **smash pushout product** and $(-)^{\square(-)}$ the **smash pullback powering** defined as in def. 3.55 and def. 3.59, but with Cartesian product replaced by smash product (def. 3.22) and compactly generated mapping space replaced by pointed mapping spaces (def. 3.26).

By theorem 3.34 $(\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}$ is cofibrantly generated by $I_{\mathrm{Top}^{*/}} = (I_{\mathrm{Top}})_+$ and $J_{\mathrm{Top}^{*/}} = (J_{\mathrm{Top}})_+$.

Example 3.24 gives that for $i_n \in I_{\mathrm{Top}}$ and $j_n \in J_{\mathrm{Top}}$ then

$$(i_{n_1})_+ \Box_\wedge (i_{n_2})_+ \simeq (i_{n_1+n_2})_+$$

and

$$(i_{n_1})_+ \wedge_\wedge (i_{n_2})_+ \simeq (i_{n_1+n_2})_+ \,.$$

Hence the pointed analog of prop. 3.61 holds and therefore so does the pointed analog of the conclusion in prop. 3.63. ▮

Model structure on topological functors

With classical topological homotopy theory in hand (theorem 3.7, theorem 3.51), it is straightforward now to generalize this to a homotopy theory of *topological diagrams*. This is going to be the basis for the stable homotopy theory of spectra, because spectra may be identified with certain topological diagrams (prop.).

Technically, "topological diagram" here means "Top-enriched functor". We now discuss what this means and then observe that as an immediate corollary of theorem 3.7 we obtain a model category structure on topological diagrams.

As a by-product, we obtain the model category theory of homotopy colimits in topological spaces, which will be useful.

In the following we say *Top-enriched category* and *Top-enriched functor* etc. for what often is referred to as "topological category" and "topological functor" etc. As discussed there, these latter terms are ambiguous.

Literature (Riehl, chapter 3) for basics of enriched category theory; (Piacenza 91) for the projective model structure on topological functors.

Definition 3.65. A **topologically enriched category** $\mathcal{C}$ is a $\mathrm{Top}_{\mathrm{cg}}$-enriched category, hence:

1. a class $\mathrm{Obj}(\mathcal{C})$, called the **class of objects**;
2. for each $a, b \in \mathrm{Obj}(\mathcal{C})$ a compactly generated topological space (def. 3.35)

   $$\mathcal{C}(a,b) \in \mathrm{Top}_{\mathrm{cg}}\,,$$

   called the **space of morphisms** or the **hom-space** between a and b;
3. for each $a, b, c \in \mathrm{Obj}(\mathcal{C})$ a continuous function

   $$\circ_{a,b,c} \;:\; \mathcal{C}(a,b) \times \mathcal{C}(b,c) \longrightarrow \mathcal{C}(a,c)$$

   out of the cartesian product (by cor. 3.38: the image under k of the product topological space), called the *composition* operation;
4. for each $a \in \mathrm{Obj}(\mathcal{C})$ a point $\mathrm{Id}_a \in \mathcal{C}(a,a)$, called the *identity* morphism on a

such that the composition is associative and unital.

Similarly a **pointed topologically enriched category** is such a structure with $\mathrm{Top}_{\mathrm{cg}}$ replaced by $\mathrm{Top}_{\mathrm{cg}}^{*/}$ (def. 3.16) and with the Cartesian product replaced by the smash product (def. 3.22) of pointed topological spaces.

Remark 3.66. Given a (pointed) topologically enriched category as in def. 3.65, then forgetting the topology on the hom-spaces (along the forgetful functor $U : \mathrm{Top}_{\mathrm{cg}} \to \mathrm{Set}$) yields an ordinary locally small category with

$$\mathrm{Hom}_{\mathcal{C}}(a, b) = U(\mathcal{C}(a, b)) \;.$$

It is in this sense that $\mathcal{C}$ is a category with extra structure, and hence "enriched".

The archetypical example is $\mathrm{Top}_{\mathrm{cg}}$ itself:

Example 3.67. The category $\mathrm{Top}_{\mathrm{cg}}$ (def. 3.35) canonically obtains the structure of a topologically enriched category, def. 3.65, with hom-spaces given by the compactly generated mapping spaces (def. 3.39)

$$\mathrm{Top}_{\mathrm{cg}}(X, Y) \coloneqq Y^X$$

and with composition

$$Y^X \times Z^Y \longrightarrow Z^X$$

given by the adjunct under the (product⊣ mapping-space)-adjunction from prop. 3.41 of the evaluation morphisms

$$X \times Y^X \times Z^Y \overset{(\mathrm{ev}, \mathrm{id})}{\longrightarrow} Y \times Z^Y \overset{\mathrm{ev}}{\longrightarrow} Z \;.$$

Similarly, pointed compactly generated topological spaces $\mathrm{Top}_k^{*/}$ form a pointed topologically enriched category, using the pointed mapping spaces from example 3.26:

$$\mathrm{Top}_{\mathrm{cg}}^{*/}(X, Y) \coloneqq \mathrm{Maps}(X, Y)_* \;.$$

Definition 3.68. A topologically enriched functor between two topologically enriched categories

$$F : \mathcal{C} \longrightarrow \mathcal{D}$$

is a $\mathrm{Top}_{\mathrm{cg}}$-enriched functor, hence:

1. a function

$$F_0 : \mathrm{Obj}(\mathcal{C}) \longrightarrow \mathrm{Obj}(\mathcal{D})$$

of objects;

2. for each $a, b \in \mathrm{Obj}(\mathcal{C})$ a continuous function

$$F_{a,b} : \mathcal{C}(a, b) \longrightarrow \mathcal{D}(F_0(a), F_0(b))$$

of hom-spaces,

such that this preserves composition and identity morphisms in the evident sense.

A homomorphism of topologically enriched functors

$$\eta : F \Rightarrow G$$

is a $\mathrm{Top}_{\mathrm{cg}}$-enriched natural transformation: for each $c \in \mathrm{Obj}(\mathcal{C})$ a choice of morphism $\eta_c \in \mathcal{D}(F(c), G(c))$ such that for each pair of objects $c, d \in \mathcal{C}$ the two continuous functions

$$\eta_d \circ F(-) : \mathcal{C}(c, d) \longrightarrow \mathcal{D}(F(c), G(d))$$

and

$$G(-) \circ \eta_c : \mathcal{C}(c, d) \longrightarrow \mathcal{D}(F(c), G(d))$$

agree.

We write $[\mathcal{C}, \mathcal{D}]$ for the resulting category of topologically enriched functors.

Remark 3.69. The condition on an enriched natural transformation in def. 3.68 is just that on an ordinary natural transformation on the underlying unenriched functors, saying that for every morphisms $f : c \to d$ there is a commuting square

$$f \quad \mapsto \quad \begin{array}{ccc} \mathcal{C}(c,c) \times X & \overset{\eta_c}{\longrightarrow} & F(c) \\ {}^{\mathcal{C}(c,f)}\downarrow && \downarrow^{F(f)} \\ \mathcal{C}(c,d) \times X & \underset{\eta_d}{\longrightarrow} & F(d) \end{array}$$

Example 3.70. For $\mathcal{C}$ any topologically enriched category, def. 3.65 then a topologically enriched functor (def. 3.68)

$$F \,:\, \mathcal{C} \longrightarrow \mathrm{Top}_{\mathrm{cg}}$$

to the archetical topologically enriched category from example 3.67 may be thought of as a topologically enriched copresheaf, at least if $\mathcal{C}$ is small (in that its class of objects is a proper set).

Hence the category of topologically enriched functors

$$[\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}]$$

according to def. 3.68 may be thought of as the (co-)presheaf category over $\mathcal{C}$ in the realm of topological enriched categories.

A functor $F \in [\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}]$ is equivalently

1. a compactly generated topological space $F_a \in \mathrm{Top}_{\mathrm{cg}}$ for each object $a \in \mathrm{Obj}(\mathcal{C})$;

2. a continuous function

$$F_a \times \mathcal{C}(a,b) \longrightarrow F_b$$

for all pairs of objects $a, b \in \mathrm{Obj}(\mathcal{C})$

such that composition is respected, in the evident sense.

For every object $c \in \mathcal{C}$, there is a topologically enriched representable functor, denoted $y(c)$ or $\mathcal{C}(c,-)$ which sends objects to

$$y(c)(d) = \mathcal{C}(c,d) \in \mathrm{Top}_{\mathrm{cg}}$$

and whose action on morphisms is, under the above identification, just the composition operation in $\mathcal{C}$.

Proposition 3.71. *For $\mathcal{C}$ any small topologically enriched category, def. 3.65 then the enriched functor category $[\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}]$ from example 3.70 has all limits and colimits, and they are computed objectwise:*

if

$$F_\bullet \,:\, I \longrightarrow [\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}]$$

is a diagram of functors and $c \in \mathcal{C}$ is any object, then

$$(\underset{\longleftarrow i}{\lim} F_i)(c) \simeq \underset{\longleftarrow i}{\lim} (F_i(c)) \quad \in \mathrm{Top}_{\mathrm{cg}}$$

and

$$(\underset{\longrightarrow}{\lim}_i F_i)(c) \simeq \underset{\longrightarrow}{\lim}_i (F_i(c)) \quad \in \mathrm{Top}_{\mathrm{cg}} \,.$$

Proof. First consider the underlying diagram of functors F_i° where the topology on the hom-spaces of $\mathcal{C}$ and of $\mathrm{Top}_{\mathrm{cg}}$ has been forgotten. Then one finds

$$(\underset{\longleftarrow}{\lim}_i F_i^\circ)(c) \simeq \underset{\longleftarrow}{\lim}_i (F_i^\circ(c)) \quad \in \mathrm{Set}$$

and

$$(\underset{\longrightarrow}{\lim}_i F_i^\circ)(c) \simeq \underset{\longrightarrow}{\lim}_i (F_i^\circ(c)) \quad \in \mathrm{Set}$$

by the universal property of limits and colimits. (Given a morphism of diagrams then a unique compatible morphism between their limits or colimits, respectively, is induced as the universal factorization of the morphism of diagrams regarded as a cone or cocone, respectvely, over the codomain or domain diagram, respectively).

Hence it only remains to see that equipped with topology, these limits and colimits in Set become limits and colimits in $\mathrm{Top}_{\mathrm{cg}}$. That is just the statement of prop. 1.5 with corollary 3.38. ▮

Definition 3.72. Let $\mathcal{C}$ be a topologically enriched category, def. 3.65, with $[\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}]$ its category of topologically enriched copresheaves from example 3.70.

1. Define a functor

   $$(-)\cdot(-) \,:\, [\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}] \times \mathrm{Top}_{\mathrm{cg}} \longrightarrow [\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}]$$

   by forming objectwise cartesian products (hence k of product topological spaces)

   $$F \cdot X \,:\, c \mapsto F(c) \times X \,.$$

   This is called the **tensoring** of $[\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}]$ over $\mathrm{Top}_{\mathrm{cg}}$.

2. Define a functor

   $$(-)^{(-)} \,:\, (\mathrm{Top}_{\mathrm{cg}})^{\mathrm{op}} \times [\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}] \longrightarrow [\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}]$$

   by forming objectwise compactly generated mapping spaces (def. 3.39)

   $$F^X \,:\, c \mapsto F(c)^X \,.$$

   This is called the **powering** of $[\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}]$ over $\mathrm{Top}_{\mathrm{cg}}$.

Analogously, for $\mathcal{C}$ a pointed topologically enriched category, def. 3.65, with $[\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}]$ its category of pointed topologically enriched copresheaves from example 3.70, then:

1. Define a functor

   $$(-)\wedge(-) \,:\, [\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}] \times \mathrm{Top}_{\mathrm{cg}}^{*/} \longrightarrow [\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}]$$

   by forming objectwise smash products (def. 3.22)

   $$F \wedge X \,:\, c \mapsto F(c) \wedge X \,.$$

   This is called the **smash tensoring** of $[\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}]$ over $\mathrm{Top}_{\mathrm{cg}}^{*/}$.

2. Define a functor

$$\mathrm{Maps}(-,-)_* : \mathrm{Top}_{\mathrm{cg}}^{*/} \times [\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}] \longrightarrow [\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}]$$

by forming objectwise pointed mapping spaces (example 3.26)

$$F^X : c \mapsto \mathrm{Maps}(X, F(c))_* \,.$$

This is called the **pointed powering** of $[\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}]$ over $\mathrm{Top}_{\mathrm{cg}}$.

There is a full blown $\mathrm{Top}_{\mathrm{cg}}$-enriched Yoneda lemma. The following records a slightly simplified version which is all that is needed here:

Proposition 3.73. ***(topologically enriched Yoneda-lemma)***

Let $\mathcal{C}$ be a topologically enriched category, def. 3.65, write $[\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}]$ for its category of topologically enriched (co-)presheaves, and for $c \in \mathrm{Obj}(\mathcal{C})$ write $y(c) = \mathcal{C}(c,-) \in [\mathcal{C}, \mathrm{Top}_k]$ for the topologically enriched functor that it represents, all according to example 3.70. Recall the tensoring operation $(F, X) \mapsto F \cdot X$ from def. 3.72.

For $c \in \mathrm{Obj}(\mathcal{C})$, $X \in \mathrm{Top}_{\mathrm{cg}}$ and $F \in [\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}]$, there is a natural bijection between

1. *morphisms $y(c) \cdot X \longrightarrow F$ in $[\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}]$;*

2. *morphisms $X \longrightarrow F(c)$ in $\mathrm{Top}_{\mathrm{cg}}$.*

In short:

$$\frac{y(c) \cdot X \longrightarrow F}{X \longrightarrow F(c)}$$

Proof. Given a morphism $\eta : y(c) \cdot X \longrightarrow F$ consider its component

$$\eta_c : \mathcal{C}(c,c) \times X \longrightarrow F(c)$$

and restrict that to the identity morphism $\mathrm{id}_c \in \mathcal{C}(c,c)$ in the first argument

$$\eta_c(\mathrm{id}_c, -) : X \longrightarrow F(c) \,.$$

We claim that just this $\eta_c(\mathrm{id}_c, -)$ already uniquely determines all components

$$\eta_d : \mathcal{C}(c,d) \times X \longrightarrow F(d)$$

of η, for all $d \in \mathrm{Obj}(\mathcal{C})$: By definition of the transformation η (def. 3.68), the two functions

$$F(-) \circ \eta_c : \mathcal{C}(c,d) \longrightarrow F(d)^{\mathcal{C}(c,c) \times X}$$

and

$$\eta_d \circ \mathcal{C}(c,-) \times X : \mathcal{C}(c,d) \longrightarrow F(d)^{\mathcal{C}(c,c) \times X}$$

agree. This means (remark 3.69) that they may be thought of jointly as a function with values in commuting squares in $\mathrm{Top}_{\mathrm{cg}}$ of this form:

$$f \quad \mapsto \quad \begin{array}{ccc} \mathcal{C}(c,c) \times X & \overset{\eta_c}{\longrightarrow} & F(c) \\ {}^{\mathcal{C}(c,f)}\downarrow && \downarrow^{F(f)} \\ \mathcal{C}(c,d) \times X & \underset{\eta_d}{\longrightarrow} & F(d) \end{array}$$

For any $f \in \mathcal{C}(c,d)$, consider the restriction of

$$\eta_d \circ \mathcal{C}(c,f) \in F(d)^{\mathcal{C}(c,c)\times X}$$

to $\mathrm{id}_c \in \mathcal{C}(c,c)$, hence restricting the above commuting squares to

$$f \quad \mapsto \quad \begin{array}{ccc} \{\mathrm{id}_c\} \times X & \overset{\eta_c}{\longrightarrow} & F(c) \\ {}^{\mathcal{C}(c,f)}\downarrow && \downarrow^{F(f)} \\ \{f\} \times X & \underset{\eta_d}{\longrightarrow} & F(d) \end{array}$$

This shows that η_d is fixed to be the function

$$\eta_d(f,x) = F(f) \circ \eta_c(\mathrm{id}_c, x)$$

and this is a continuous function since all the operations it is built from are continuous.

Conversely, given a continuous function $\alpha : X \longrightarrow F(c)$, define for each d the function

$$\eta_d : (f,x) \mapsto F(f) \circ \alpha \,.$$

Running the above analysis backwards shows that this determines a transformation $\eta : y(c) \times X \to F$. ▮

Definition 3.74. For $\mathcal{C}$ a small topologically enriched category, def. 3.65, write

$$I_{\mathrm{Top}}^{\mathcal{C}} \coloneqq \left\{ y(c) \cdot (S^{n-1} \overset{\iota_n}{\longrightarrow} D^n) \right\}_{\substack{n \in \mathbb{N}, \\ c \in \mathrm{Obj}(\mathcal{C})}}$$

and

$$J_{\mathrm{Top}}^{\mathcal{C}} \coloneqq \left\{ y(c) \cdot (D^n \overset{(\mathrm{id},\delta_0)}{\longrightarrow} D^n \times I) \right\}_{\substack{n \in \mathbb{N}, \\ c \in \mathrm{Obj}(\mathcal{C})}}$$

for the sets of morphisms given by tensoring (def. 3.72) the representable functors (example 3.70) with the generating cofibrations (def.1.37) and acyclic generating cofibrations (def. 1.42), respectively, of $(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{Quillen}}$ (theorem 3.51).

These are going to be called the **generating cofibrations** and **acyclic generating cofibrations** for the *projective model structure on topologically enriched functors* over $\mathcal{C}$.

Analgously, for $\mathcal{C}$ a pointed topologically enriched category, write

$$I_{\mathrm{Top}^{*/}}^{\mathcal{C}} \coloneqq \left\{ y(c) \wedge (S^{n-1}_+ \overset{(\iota_n)_+}{\longrightarrow} D^n_+) \right\}_{\substack{n \in \mathbb{N}, \\ c \in \mathrm{Obj}(\mathcal{C})}}$$

and

$$J_{\mathrm{Top}^{*/}}^{\mathcal{C}} \coloneqq \left\{ y(c) \wedge (D^n_+ \overset{(\mathrm{id},\delta_0)_+}{\longrightarrow} (D^n \times I)_+) \right\}_{\substack{n \in \mathbb{N}, \\ c \in \mathrm{Obj}(\mathcal{C})}}$$

for the analogous construction applied to the pointed generating (acyclic) cofibrations of def. 3.33.

Definition 3.75. Given a small (pointed) topologically enriched category $\mathcal{C}$, def. 3.65, say that a morphism in the category of (pointed) topologically enriched copresheaves $[\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}]$ ($[\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}]$), example 3.70, hence a natural transformation between topologically enriched functors, $\eta : F \to G$ is

- a **projective weak equivalence**, if for all $c \in \mathrm{Obj}(\mathcal{C})$ the component $\eta_c : F(c) \to G(c)$ is a weak homotopy equivalence (def. 1.30);

- a **projective fibration** if for all $c \in \mathrm{Obj}(\mathcal{C})$ the component $\eta_c : F(c) \to G(c)$ is a Serre fibration (def. 1.47);
- a **projective cofibration** if it is a retract (rmk. 2.12) of an $I_{\mathrm{Top}}^{\mathcal{C}}$-relative cell complex (def. 1.41, def. 3.74).

Write

$$[\mathcal{C}, (\mathrm{Top}_{\mathrm{cg}})_{\mathrm{Quillen}}]_{\mathrm{proj}}$$

and

$$[\mathcal{C}, (\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}]_{\mathrm{proj}}$$

for the categories of topologically enriched functors equipped with these classes of morphisms.

Theorem 3.76. *The classes of morphisms in def. 3.75 constitute a model category structure on $[\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}]$ and $[\mathcal{C}, \mathrm{Top}_{\mathrm{cg}}^{*/}]$, called the **projective model structure on enriched functors***

$$[\mathcal{C}, (\mathrm{Top}_{\mathrm{cg}})_{\mathrm{Quillen}}]_{\mathrm{proj}}$$

and

$$[\mathcal{C}, (\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}]_{\mathrm{proj}}$$

These are cofibrantly generated model category, def. 3.9, with set of generating (acyclic) cofibrations the sets $I_{\mathrm{Top}}^{\mathcal{C}}$, $J_{\mathrm{Top}}^{\mathcal{C}}$ and $I_{\mathrm{Top}^{*/}}^{\mathcal{C}}$, $J_{\mathrm{Top}^{*/}}^{\mathcal{C}}$ from def. 3.74, respectively.

(Piacenza 91, theorem 5.4)

Proof. By prop. 3.71 the category has all limits and colimits, hence it remains to check the model structure

But via the enriched Yoneda lemma (prop. 3.73) it follows that proving the model structure reduces objectwise to the proof of theorem 3.7, theorem 3.51. In particular, the technical lemmas 1.40, 1.45 and 1.52 generalize immediately to the present situation, with the evident small change of wording:

For instance, the fact that a morphism of topologically enriched functors $\eta : F \to G$ that has the right lifting property against the elements of $I_{\mathrm{Top}}^{\mathcal{C}}$ is a projective weak equivalence, follows by noticing that for fixed $\eta : F \to G$ the enriched Yoneda lemma prop. 3.73 gives a natural bijection of commuting diagrams (and their fillers) of the form

$$\left(\begin{array}{ccc} y(c) \cdot S^{n-1} & \longrightarrow & F \\ {}^{(\mathrm{id} \cdot \iota_n)}\downarrow && \downarrow^{\eta} \\ y(c) \cdot D^n & \longrightarrow & G \end{array} \right) \leftrightarrow \left(\begin{array}{ccc} S^{n-1} & \longrightarrow & F(c) \\ \downarrow && \downarrow^{\eta_c} \\ D^n & \longrightarrow & G(c) \end{array} \right),$$

and hence the statement follows with part A) of the proof of lemma 1.52.

With these three lemmas in hand, the remaining formal part of the proof goes through verbatim as above: repeatedly use the small object argument (prop. 2.17) and the retract argument (prop. 2.15) to establish the two weak factorization systems. (While again the structure of a category with weak equivalences is evident.) ▮

Example 3.77. Given examples 3.67 and 3.70, the next evident example of a pointed topologically enriched category besides $\mathrm{Top}_{\mathrm{cg}}^{*/}$ itself is the functor category

$$[\mathrm{Top}_{\mathrm{cg}}^{*/}, \mathrm{Top}_{\mathrm{cg}}^{*/}]\,.$$

The only technical problem with this is that $\mathrm{Top}_{\mathrm{cg}}^{*/}$ is not a small category (it has a proper class of objects), which means that the existence of all limits and colimits via prop. 3.71 may (and does) fail.

But so we just restrict to a small topologically enriched subcategory. A good choice is the full subcategory

$$\mathrm{Top}_{\mathrm{cg,fin}}^{*/} \hookrightarrow \mathrm{Top}_{\mathrm{cg}}^{*/}$$

of topological spaces homoemorphic to finite CW-complexes. The resulting projective model category (via theorem 3.76)

$$[\mathrm{Top}_{\mathrm{cg,fin}}^{*/}\,,\, (\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}]_{\mathrm{proj}}$$

is also also known as the **strict model structure for excisive functors**. (This terminology is the special case for $n = 1$ of the terminology "n-excisive functors" as used in "Goodwillie calculus", a homotopy-theoretic analog of differential calculus.) After enlarging its class of weak equivalences while keeping the cofibrations fixed, this will become Quillen equivalent to a model structure for spectra. This we discuss in part 1.2, in the section on pre-excisive functors.

One consequence of theorem 3.76 is the model category theoretic incarnation of the theory of *homotopy colimits*.

Observe that ordinary limits and colimits (def. 1.1) are equivalently characterized in terms of adjoint functors:

Let $\mathcal{C}$ be any category and let I be a small category. Write $[I, \mathcal{C}]$ for the corresponding functor category. We may think of its objects as I-shaped diagrams in $\mathcal{C}$, and of its morphisms as homomorphisms of these diagrams. There is a canonical functor

$$\mathrm{const}_I \,:\, \mathcal{C} \longrightarrow [I, \mathcal{C}]$$

which sends each object of $\mathcal{C}$ to the diagram that is constant on this object. Inspection of the definition of the universal properties of limits and colimits on one hand, and of left adjoint and right adjoint functors on the other hand, shows that

1. precisely when $\mathcal{C}$ has all colimits of shape I, then the functor const_I has a left adjoint functor, which is the operation of forming these colimits:

$$[I, \mathcal{C}] \underoverset{\underset{\mathrm{const}_I}{\longleftarrow}}{\overset{\underset{\longrightarrow}{\lim}_I}{\longrightarrow}}{\bot} \mathcal{C}$$

2. precisely when $\mathcal{C}$ has all limits of shape I, then the functor const_I has a right adjoint functor, which is the operation of forming these limits.

$$[I, \mathcal{C}] \underoverset{\underset{\underset{\longleftarrow}{\lim}_I}{\longrightarrow}}{\overset{\mathrm{const}_I}{\longleftarrow}}{\bot} \mathcal{C}$$

Proposition 3.78. *Let I be a small topologically enriched category (def. 3.65). Then the* $(\underset{\longrightarrow}{\lim}_I \dashv \mathrm{const}_I)$-*adjunction*

$$[I, (\mathrm{Top}_{\mathrm{cg}})_{\mathrm{Quillen}}]_{\mathrm{proj}} \underoverset{\underset{\mathrm{const}_I}{\longleftarrow}}{\overset{\underset{\longrightarrow}{\lim}_I}{\longrightarrow}}{\bot} (\mathrm{Top}_{\mathrm{cg}})_{\mathrm{Quillen}}$$

is a Quillen adjunction (def. 2.46) between the projective model structure on topological functors on I, from theorem 3.76, and the classical model structure on topological spaces from theorem 3.51.

Similarly, if I is enriched in pointed topological spaces, then for the classical model structure on pointed topological spaces (prop. 3.29, theorem 3.34) the adjunction

$$[I, (\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}]_{\mathrm{proj}} \underset{\mathrm{const}}{\overset{\underset{\longrightarrow}{\lim}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} (\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}$$

is a Quillen adjunction.

Proof. Since the fibrations and weak equivalences in the projective model structure (def. 3.75) on the functor category are objectwise those of $(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{Quillen}}$ and of $(\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}$, respectively, it is immediate that the functor const_I preserves these. In particular it preserves fibrations and acyclic fibrations and so the claim follows (prop. 2.47). ▮

Definition 3.79. In the situation of prop. 3.78 we say that the left derived functor (def. 2.42) of the colimit functor is the **homotopy colimit**

$$\mathrm{hocolim}_I \coloneqq \mathbb{L}\underset{\longrightarrow_I}{\lim} \;:\; \mathrm{Ho}([I, \mathrm{Top}]) \longrightarrow \mathrm{Ho}(\mathrm{Top})$$

and

$$\mathrm{hocolim}_I \coloneqq \mathbb{L}\underset{\longrightarrow_I}{\lim} \;:\; \mathrm{Ho}([I, \mathrm{Top}^{*/}]) \longrightarrow \mathrm{Ho}(\mathrm{Top}^{*/}) \;.$$

Remark 3.80. Since every object in $(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{Quillen}}$ and in $(\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}$ is fibrant, the homotopy colimit of any diagram $X_\bullet$, according to def. 3.79, is (up to weak homotopy equivalence) the result of forming the ordinary colimit of any projectively cofibrant replacement $\hat{X}_\bullet \overset{\in W_{\mathrm{proj}}}{\longrightarrow} X_\bullet$.

Example 3.81. Write $\mathbb{N}^{\leq}$ for the poset (def. 1.15) of natural numbers, hence for the small category (with at most one morphism from any given object to any other given object) that looks like

$$\mathbb{N}^{\leq} = \{0 \to 1 \to 2 \to 3 \to \cdots\} \;.$$

Regard this as a topologically enriched category with the, necessarily, discrete topology on its hom-sets.

Then a topologically enriched functor

$$X_\bullet \;:\; \mathbb{N}^{\leq} \longrightarrow \mathrm{Top}_{\mathrm{cg}}$$

is just a plain functor and is equivalently a sequence of continuous functions (morphisms in $\mathrm{Top}_{\mathrm{cg}}$) of the form (also called a *cotower*)

$$X_0 \overset{f_0}{\longrightarrow} X_1 \overset{f_1}{\longrightarrow} X_2 \overset{f_2}{\longrightarrow} X_3 \longrightarrow \cdots \;.$$

It is immediate to check that those sequences $X_\bullet$ which are cofibrant in the projective model structure (theorem 3.76) are precisely those for which

1. all component morphisms f_i are cofibrations in $(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{Quillen}}$ or $(\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}$, respectively, hence retracts (remark 2.12) of relative cell complex inclusions (def. 1.38);

2. the object X_0, and hence all other objects, are cofibrant, hence are retracts of cell complexes (def. 1.38).

By example 3.81 it is immediate that the operation of forming colimits sends projective (acyclic) cofibrations between sequences of topological spaces to (acyclic) cofibrations in the classical model structure on pointed topological spaces. On those projectively cofibrant sequences where every map is not just a retract of a relative cell complex inclusion, but a plain relative cell complex inclusion, more is true:

Proposition 3.82. *In the projective model structures on cotowers in topological spaces, $[\mathbb{N}^{\leq}, (\mathrm{Top}_{\mathrm{cg}})_{\mathrm{Quillen}}]_{\mathrm{proj}}$ and $[\mathbb{N}^{\leq}, (\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}]_{\mathrm{proj}}$ from def. 3.81, the following holds:*

1. *The colimit functor preserves fibrations between sequences of relative cell complex inclusions;*

2. *Let I be a finite category, let $D_\bullet(-) : I \to [\mathbb{N}^{\leq}, \mathrm{Top}_{\mathrm{cg}}]$ be a finite diagram of sequences of relative cell complexes. Then there is a weak homotopy equivalence*

$$\underset{\longrightarrow}{\lim}_n \left(\underset{\longleftarrow}{\lim}_i D_n(i) \right) \overset{\in W_{\mathrm{cl}}}{\longrightarrow} \underset{\longleftarrow}{\lim}_i \left(\underset{\longrightarrow}{\lim}_n D_n(i) \right)$$

   from the colimit over the limit sequnce to the limit of the colimits of sequences.

Proof. Regarding the first statement:

Use that both $(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{Quillen}}$ and $(\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}$ are cofibrantly generated model categories (theorem 3.34) whose generating acyclic cofibrations have compact topological spaces as domains and codomains. The colimit over a sequence of relative cell complexes (being a transfinite composition) yields another relative cell complex, and hence lemma 1.40 says that every morphism out of the domain or codomain of a generating acyclic cofibration into this colimit factors through a finite stage inclusion. Since a projective fibration is a degreewise fibration, we have the lifting property at that finite stage, and hence also the lifting property against the morphisms of colimits.

Regarding the second statement:

This is a model category theoretic version of a standard fact of plain category theory, which says that in the category Set of sets, filtered colimits commute with finite limits in that there is an isomorphism of sets of the form which we have to prove is a weak homotopy equivalence of topological spaces. But now using that weak homotopy equivalences are detected by forming homotopy groups (def. 1.26), hence hom-sets out of n-spheres, and since n-spheres are compact topological spaces, lemma 1.40 says that homming out of n-spheres commutes over the colimits in question. Moreover, generally homming out of anything commutes over limits, in particular finite limits (every hom functor is left exact functor in the second variable). Therefore we find isomorphisms of the form

$$\mathrm{Hom}\left(S^q, \underset{\longrightarrow}{\lim}_n \left(\underset{\longleftarrow}{\lim}_i D_n(i)\right)\right) \simeq \underset{\longrightarrow}{\lim}_n \left(\underset{\longleftarrow}{\lim}_i \mathrm{Hom}(S^q, D_n(i))\right) \overset{\sim}{\longrightarrow} \underset{\longleftarrow}{\lim}_i \left(\underset{\longrightarrow}{\lim}_n \mathrm{Hom}(S^q D_n(i))\right) \simeq \mathrm{Hom}\left(S^q, \underset{\longleftarrow}{\lim}_i \left(\underset{\longrightarrow}{\lim}_n D_n(i)\right)\right)$$

and similarly for the left homotopies $\mathrm{Hom}(S^q \times I, -)$ (and similarly for the pointed case). This implies the claimed isomorphism on homotopy groups. ▮

4. Homotopy fiber sequences

A key aspect of homotopy theory is that the universal constructions of category theory, such as limits and colimits, receive a refinement whereby their universal properties hold not just up to isomorphism but up to (weak) homotopy equivalence. One speaks of *homotopy limits* and *homotopy colimits*.

We consider this here just for the special case of homotopy fibers and homotopy cofibers, leading to the phenomenon of homotopy fiber sequences and their induced long exact sequences of homotopy groups which control much of the theory to follow.

Mapping cones

In the context of homotopy theory, a pullback diagram, such as in the definition of the fiber in example 3.27

$$
\begin{array}{ccc}
\mathrm{fib}(f) & \longrightarrow & X \\
\downarrow && \downarrow^{f} \\
* & \longrightarrow & Y
\end{array}
$$

ought to commute only up to a (left/right) homotopy (def. 2.22) between the outer composite morphisms. Moreover, it should satisfy its universal property up to such homotopies.

Instead of going through the full theory of what this means, we observe that this is plausibly modeled by the following construction, and then we check (below) that this indeed has the relevant abstract homotopy theoretic properties.

Definition 4.1. Let $\mathcal{C}$ be a model category, def. 2.3 with $\mathcal{C}^{*/}$ its model structure on pointed objects, prop. 3.29. For $f : X \longrightarrow Y$ a morphism between cofibrant objects (hence a morphism in $(\mathcal{C}^{*/})_c \hookrightarrow \mathcal{C}^{*/}$, def. 2.34), its **reduced mapping cone** is the object

$$
\mathrm{Cone}(f) \coloneqq * \underset{X}{\sqcup} \mathrm{Cyl}(X) \underset{X}{\sqcup} Y
$$

in the colimiting diagram

$$
\begin{array}{ccccc}
 && X & \overset{f}{\longrightarrow} & Y \\
 && \downarrow^{i_1} && \downarrow^{i} \\
X & \overset{i_0}{\longrightarrow} & \mathrm{Cyl}(X) && \\
\downarrow &&& \searrow^{\eta} & \downarrow \\
* & \longrightarrow && \longrightarrow & \mathrm{Cone}(f)
\end{array}
,
$$

where $\mathrm{Cyl}(X)$ is a cylinder object for X, def. 2.18.

Dually, for $f : X \longrightarrow Y$ a morphism between fibrant objects (hence a morphism in $(\mathcal{C}^{*})_f \hookrightarrow \mathcal{C}^{*/}$, def. 2.34), its **mapping cocone** is the object

$$
\mathrm{Path}_*(f) \coloneqq * \underset{Y}{\times} \mathrm{Path}(Y) \underset{Y}{\times} Y
$$

in the following limit diagram

$$
\begin{array}{ccccc}
\mathrm{Path}_*(f) & \longrightarrow && \longrightarrow & X \\
\downarrow & \searrow^{\eta} &&& \downarrow^{f} \\
 && \mathrm{Path}(Y) & \underset{p_1}{\longrightarrow} & Y \\
\downarrow && \downarrow^{p_0} && \\
* & \longrightarrow & Y &&
\end{array}
,
$$

where $\mathrm{Path}(Y)$ is a path space object for Y, def. 2.18.

Remark 4.2. When we write homotopies (def. 2.22) as double arrows between morphisms, then the limit diagram in def. 4.1 looks just like the square in the definition of fibers in example 3.27, except that it is filled by the right homotopy given by the component map denoted η:

$$\begin{tikzcd} \mathrm{Path}_*(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow_\eta"] & X \arrow[d, "f"] \\ * \arrow[r] & Y \end{tikzcd}.$$

Dually, the colimiting diagram for the mapping cone turns to look just like the square for the cofiber, except that it is filled with a left homotopy

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\swarrow_\eta"] & Y \arrow[d] \\ * \arrow[r] & \mathrm{Cone}(f) \end{tikzcd}$$

Proposition 4.3. *The colimit appearing in the definition of the reduced mapping cone in def. 4.1 is equivalent to three consecutive pushouts:*

$$\begin{tikzcd} & & X \arrow[r, "f"] \arrow[d, "i_1"] \arrow[dr, phantom, "(\mathrm{po})"] & Y \arrow[d, "i"] \\ X \arrow[r, "i_0"] \arrow[d] \arrow[dr, phantom, "(\mathrm{po})"] & \mathrm{Cyl}(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "(\mathrm{po})"] & \mathrm{Cyl}(f) \arrow[d] \\ * \arrow[r] & \mathrm{Cone}(X) \arrow[r] & \mathrm{Cone}(f) \end{tikzcd}.$$

The two intermediate objects appearing here are called

- *the plain **reduced cone*** $\mathrm{Cone}(X) \coloneqq * \underset{X}{\sqcup} \mathrm{Cyl}(X)$*;*
- *the **reduced mapping cylinder*** $\mathrm{Cyl}(f) \coloneqq \mathrm{Cyl}(X) \underset{X}{\sqcup} Y$*.*

Dually, the limit appearing in the definition of the mapping cocone in def. 4.1 is equivalent to three consecutive pullbacks:

$$\begin{tikzcd} \mathrm{Path}_*(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "(\mathrm{pb})"] & \mathrm{Path}(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "(\mathrm{pb})"] & X \arrow[d, "f"] \\ \mathrm{Path}_*(Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "(\mathrm{pb})"] & \mathrm{Path}(Y) \arrow[r, "p_1"'] \arrow[d, "p_0"] & Y \\ * \arrow[r] & Y \end{tikzcd}.$$

The two intermediate objects appearing here are called

- *the **based path space object*** $\mathrm{Path}_*(Y) \coloneqq * \prod_Y \mathrm{Path}(Y)$*;*
- *the **mapping path space** or **mapping co-cylinder*** $\mathrm{Path}(f) \coloneqq X \underset{Y}{\times} \mathrm{Path}(X)$*.*

Definition 4.4. Let $X \in \mathcal{C}^{*/}$ be any pointed object.

1. The mapping cone, def. 4.3, of $X \to *$ is called the **reduced suspension** of X, denoted

   $$\Sigma X = \mathrm{Cone}(X \to *)\ .$$

   Via prop. 4.3 this is equivalently the coproduct of two copies of the cone on X over their base:

$$
\begin{array}{ccccc}
 & & X & \longrightarrow & * \\
 & & \downarrow^{i_1} & (\mathrm{po}) & \downarrow \\
X & \overset{i_0}{\longrightarrow} & \mathrm{Cyl}(X) & \longrightarrow & \mathrm{Cone}(X) \\
\downarrow & (\mathrm{po}) & \downarrow & (\mathrm{po}) & \downarrow \\
* & \longrightarrow & \mathrm{Cone}(X) & \longrightarrow & \Sigma X
\end{array} \,.
$$

This is also equivalently the cofiber, example 3.27 of (i_0, i_1), hence (example 3.20) of the wedge sum inclusion:

$$
X \vee X = X \sqcup X \overset{(i_0,i_1)}{\longrightarrow} \mathrm{Cyl}(X) \overset{\mathrm{cofib}(i_0,i_1)}{\longrightarrow} \Sigma X \,.
$$

2. The mapping cocone, def. 4.3, of $* \to X$ is called the **loop space object** of X, denoted

$$
\Omega X = \mathrm{Path}_*(* \to X) \,.
$$

Via prop. 4.3 this is equivalently

$$
\begin{array}{ccccc}
\Omega X & \longrightarrow & \mathrm{Path}_*(X) & \longrightarrow & * \\
\downarrow & (\mathrm{pb}) & \downarrow & (\mathrm{pb}) & \downarrow \\
\mathrm{Path}_*(X) & \longrightarrow & \mathrm{Path}(X) & \underset{p_1}{\longrightarrow} & X \\
\downarrow & (\mathrm{pb}) & \downarrow^{p_0} & & \\
* & \longrightarrow & X & &
\end{array} \,.
$$

This is also equivalently the fiber, example 3.27 of (p_0, p_1):

$$
\Omega X \overset{\mathrm{fib}(p_0,p_1)}{\longrightarrow} \mathrm{Path}(X) \overset{(p_0,p_1)}{\longrightarrow} X \times X \,.
$$

Proposition 4.5. *In pointed topological spaces* $\mathrm{Top}^{*/}$,

- *the reduced suspension objects (def. 4.4) induced from the standard reduced cylinder* $(-) \wedge (I_+)$ *of example 3.25 are isomorphic to the smash product (def. 3.22) with the 1-sphere, for later purposes we choose to smash* ***on the left*** *and write*

$$
\mathrm{cofib}(X \vee X \to X \wedge (I_+)) \simeq S^1 \wedge X \,,
$$

Dually:

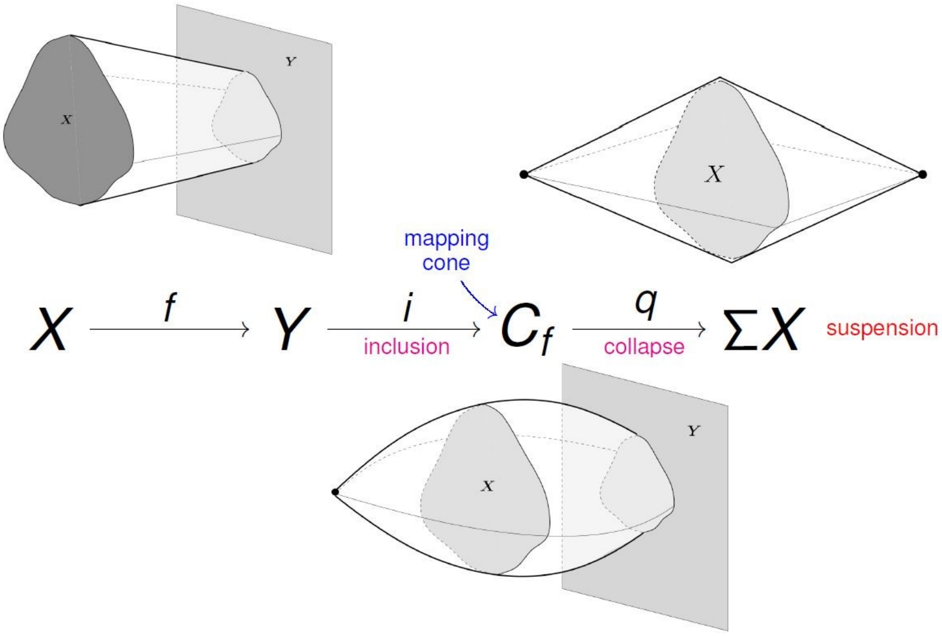
- *the loop space objects (def. 4.4) induced from the standard pointed path space object* $\mathrm{Maps}(I_+, -)_*$ *are isomorphic to the pointed mapping space (example 3.26) with the 1-sphere*

$$
\mathrm{fib}(\mathrm{Maps}(I_+, X)_* \to X \times X) \simeq \mathrm{Maps}(S^1, X)_* \,.
$$

Proof. By immediate inspection: For instance the fiber of $\mathrm{Maps}(I_+, X)_* \longrightarrow X \times X$ is clearly the subspace of the unpointed mapping space X^I on elements that take the endpoints of I to the basepoint of X. ■

Example 4.6. For $\mathcal{C} =$ Top with $\mathrm{Cyl}(X) = X \times I$ the standard cyclinder object, def. 1.22, then by example 1.12, the mapping cone, def. 4.1, of a continuous function $f : X \longrightarrow Y$ is obtained by

1. forming the cylinder over X;
2. attaching to one end of that cylinder the space Y as specified by the map f.
3. shrinking the other end of the cylinder to the point.



Accordingly the suspension of a topological space is the result of shrinking both ends of the cylinder on the object two the point. This is homeomoprhic to attaching two copies of the cone on the space at the base of the cone.

(graphics taken from Muro 10)

Below in example 4.19 we find the homotopy-theoretic interpretation of this standard topological mapping cone as a model for the *homotopy cofiber*.

Remark 4.7. The *formula* for the mapping cone in prop. 4.3 (as opposed to that of the mapping co-cone) does not require the presence of the basepoint: for $f : X \longrightarrow Y$ a morphism in $\mathcal{C}$ (as opposed to in $\mathcal{C}^{*/}$) we may still define

$$\mathrm{Cone}'(f) \coloneqq Y \underset{X}{\sqcup} \mathrm{Cone}'(X) \,,$$

where the prime denotes the *unreduced cone*, formed from a cylinder object in $\mathcal{C}$.

Proposition 4.8. *For $f : X \longrightarrow Y$ a morphism in Top, then its unreduced mapping cone, remark 4.7, with respect to the standard cylinder object $X \times I$ def. 1.22, is isomorphic to the reduced mapping cone, def. 4.1, of the morphism $f_+ : X_+ \to Y_+$ (with a basepoint adjoined, def. 3.18) with respect to the standard reduced cylinder (example 3.25):*

$$\mathrm{Cone}'(f) \simeq \mathrm{Cone}(f_+) \;.$$

Proof. By prop. 3.19 and example 3.24, $\mathrm{Cone}(f_+)$ is given by the colimit in Top over the following diagram:

$$\begin{array}{ccccc} * & \longrightarrow & X \sqcup * & \overset{(f,\mathrm{id})}{\longrightarrow} & Y \sqcup * \\ \downarrow && \downarrow && \downarrow \\ X \sqcup * & \longrightarrow & (X \times I) \sqcup * && \\ \downarrow &&&& \downarrow \\ * & \longrightarrow && \longrightarrow & \mathrm{Cone}(f_+) \end{array} \,.$$

We may factor the vertical maps to give

$$\begin{array}{ccccc} * & \longrightarrow & X \sqcup * & \overset{(f,\mathrm{id})}{\longrightarrow} & Y \sqcup * \\ \downarrow && \downarrow && \downarrow \\ X \sqcup * & \longrightarrow & (X \times I) \sqcup * && \\ \downarrow &&&& \downarrow \\ * \sqcup * & \longrightarrow && \longrightarrow & \mathrm{Cone}'(f)_+ \\ \downarrow &&&& \downarrow \\ * & \longrightarrow && \longrightarrow & \mathrm{Cone}'(f) \end{array} \,.$$

This way the top part of the diagram (using the pasting law to compute the colimit in two stages) is manifestly a cocone under the result of applying $(-)_+$ to the diagram for the unreduced cone. Since $(-)_+$ is itself given by a colimit, it preserves colimits, and hence gives the partial colimit $\mathrm{Cone}'(f)_+$ as shown. The remaining pushout then contracts the remaining copy of the point away. ▮

Example 4.6 makes it clear that every cycle $S^n \to Y$ in Y that happens to be in the image of X can be *continuously* translated in the cylinder-direction, keeping it constant in Y, to the other end of the cylinder, where it shrinks away to the point. This means that every homotopy group of Y, def. 1.26, in the image of f vanishes in the mapping cone. Hence in the mapping cone **the image of X under f in Y is removed up to homotopy**. This makes it intuitively clear how $\mathrm{Cone}(f)$ is a homotopy-version of the cokernel of f. We now discuss this formally.

Lemma 4.9. ***(factorization lemma)***

Let $\mathcal{C}_c$ be a category of cofibrant objects, def. 2.34. Then for every morphism $f : X \longrightarrow Y$ the mapping cylinder-construction in def. 4.3 provides a cofibration resolution of f, in that

1. *the composite morphism $X \overset{i_0}{\longrightarrow} \mathrm{Cyl}(X) \overset{(i_1)_* f}{\longrightarrow} \mathrm{Cyl}(f)$ is a cofibration;*

2. *f factors through this morphism by a weak equivalence left inverse to an acyclic cofibration*

$$f \,:\, X \underset{\in \mathrm{Cof}}{\overset{(i_1)_* f \circ i_0}{\longrightarrow}} \mathrm{Cyl}(f) \underset{\in W}{\longrightarrow} Y \,,$$

Dually:

Let $\mathcal{C}_f$ be a category of fibrant objects, def. 2.34. Then for every morphism $f : X \longrightarrow Y$ the mapping cocylinder-construction in def. 4.3 provides a fibration resolution of f, in that

1. *the composite morphism $\mathrm{Path}(f) \overset{p_1^* f}{\longrightarrow} \mathrm{Path}(Y) \overset{p_0}{\longrightarrow} Y$ is a fibration;*

2. *f factors through this morphism by a weak equivalence right inverse to an acyclic fibration:*

$$f \,:\, X \underset{\in W}{\longrightarrow} \mathrm{Path}(f) \underset{\in \mathrm{Fib}}{\overset{p_0 \circ p_1^* f}{\longrightarrow}} Y \,,$$

Proof. We discuss the second case. The first case is formally dual.

So consider the mapping cocylinder-construction from prop. 4.3

$$\begin{array}{ccc} \mathrm{Path}(f) & \overset{\in W \cap \mathrm{Fib}}{\longrightarrow} & X \\ {}^{\mathllap{p_1^* f}}\downarrow & (pb) & \downarrow^{\mathrlap{f}} \\ \mathrm{Path}(Y) & \underset{\in W \cap \mathrm{Fib}}{\overset{p_1}{\longrightarrow}} & Y \\ {}^{\mathllap{\in W \cap \mathrm{Fib}}}\downarrow^{\mathrlap{p_0}} \\ Y \end{array} \,.$$

To see that the vertical composite is indeed a fibration, notice that, by the pasting law, the above pullback diagram may be decomposed as a pasting of two pullback diagram as follows

$$
\begin{array}{ccccc}
\mathrm{Path}(f) & \underset{\in \mathrm{Fib}}{\overset{(f,\mathrm{id})^*(p_1,p_0)}{\longrightarrow}} & X \times Y & \overset{\mathrm{pr}_1}{\longrightarrow} & X \\
\downarrow && \downarrow^{(f,\mathrm{Id})} && \downarrow^f \\
\mathrm{Path}(Y) & \overset{(p_1,p_0)\in \mathrm{Fib}}{\longrightarrow} & Y \times Y & \overset{\mathrm{pr}_1}{\longrightarrow} & Y \,. \\
{}^{p_0}\downarrow & \swarrow_{\underset{\in \mathrm{Fib}}{\mathrm{pr}_2}} \\
Y
\end{array}
$$

Both squares are pullback squares. Since pullbacks of fibrations are fibrations by prop. 2.10, the morphism $\mathrm{Path}(f) \to X \times Y$ is a fibration. Similarly, since X is fibrant, also the projection map $X \times Y \to Y$ is a fibration (being the pullback of $X \to *$ along $Y \to *$).

Since the vertical composite is thereby exhibited as the composite of two fibrations

$$
\mathrm{Path}(f) \overset{(f,\mathrm{id})^*(p_1,p_0)}{\longrightarrow} X \times Y \overset{\mathrm{pr}_2 \circ (f,\mathrm{Id}) = \mathrm{pr}_2}{\longrightarrow} Y \,,
$$

it is itself a fibration.

Then to see that there is a weak equivalence as claimed:

The universal property of the pullback $\mathrm{Path}(f)$ induces a right inverse of $\mathrm{Path}(f) \to X$ fitting into this diagram

$$
\begin{array}{cccccc}
\mathrm{id}_X \colon & X & \underset{\in W}{\overset{\exists}{\longrightarrow}} & \mathrm{Path}(f) & \overset{\in W \cap \mathrm{Fib}}{\longrightarrow} & X \\
& {}^{f}\downarrow && \downarrow && \downarrow^f \\
\mathrm{id}_Y \colon & Y & \underset{\in W}{\overset{i}{\longrightarrow}} & \mathrm{Path}(Y) & \overset{p_1}{\to} & Y \,, \\
&& {}_{\mathrm{Id}}\searrow & \downarrow^{p_0} \\
&&& Y
\end{array}
$$

which is a weak equivalence, as indicated, by two-out-of-three (def. 2.1).

This establishes the claim. ▮

Categories of fibrant objects

Below we discuss the homotopy-theoretic properties of the mapping cone- and mapping cocone-constructions from above. Before we do so, we here establish a collection of general facts that hold in categories of fibrant objects and dually in categories of cofibrant objects, def. 2.34.

Literature (Brown 73, section 4).

Lemma 4.10. *Let $f : X \longrightarrow Y$ be a morphism in a category of fibrant objects, def. 2.34. Then given any choice of path space objects $\mathrm{Path}(X)$ and $\mathrm{Path}(Y)$, def. 2.18, there is a replacement of $\mathrm{Path}(X)$ by a path space object $\widehat{\mathrm{Path}(X)}$ along an acylic fibration, such that $\widehat{\mathrm{Path}(X)}$ has a morphism ϕ to $\mathrm{Path}(Y)$ which is compatible with the structure maps, in that the following diagram commutes*

$$\begin{array}{ccccc} & & X & \overset{f}{\longrightarrow} & Y \\ & \swarrow & \downarrow & & \downarrow \\ \mathrm{Path}(X) & \underset{\in W \cap \mathrm{Fib}}{\longleftarrow} & \widetilde{\mathrm{Path}(X)} & \overset{\phi}{\longrightarrow} & \mathrm{Path}(Y) \\ & {}^{(p_0^X, p_1^X)}\searrow & \downarrow^{(p_0^Y, p_1^Y)} & & \downarrow^{(\tilde p_0^X, \tilde p_1^X)} \\ & & X \times X & \overset{(f,f)}{\longrightarrow} & Y \times Y \end{array}.$$

(Brown 73, section 2, lemma 2)

Proof. Consider the commuting square

$$\begin{array}{ccccc} X & \overset{f}{\longrightarrow} & Y & \longrightarrow & \mathrm{Path}(Y) \\ \downarrow & & & & \downarrow^{(p_0^Y, p_1^Y)} \\ \mathrm{Path}(X) & \overset{(p_0^X, p_1^X)}{\longrightarrow} & X \times X & \overset{(f,f)}{\longrightarrow} & Y \times Y \end{array}.$$

Then consider its factorization through the pullback of the right morphism along the bottom morphism,

$$\begin{array}{ccccc} X & \longrightarrow & (f \circ p_0^X, f \circ p_1^X)^* \mathrm{Path}(Y) & \longrightarrow & \mathrm{Path}(Y) \\ & {}^{\in W}\searrow & \downarrow^{\in W \cap \mathrm{Fib}} & & \downarrow^{(p_0^Y, p_1^Y)}_{\in \mathrm{Fib}} \\ & & \mathrm{Path}(X) & \overset{(f \circ p_0^X, f \circ p_1^X)}{\longrightarrow} & Y \times Y \end{array}.$$

Finally use the factorization lemma 4.9 to factor the morphism $X \to (f \circ p_0^X, f \circ p_1^X)^* \mathrm{Path}(Y)$ through a weak equivalence followed by a fibration, the object this factors through serves as the desired path space resolution

$$\begin{array}{ccccc} X & \overset{\in W}{\longrightarrow} & \widetilde{\mathrm{Path}(X)} & \longrightarrow & \mathrm{Path}(Y) \\ & {}^{\in W}\searrow & \downarrow^{\in W \cap \mathrm{Fib}} & & \downarrow^{(p_0^Y, p_1^Y)} \\ & & \mathrm{Path}(X) & \overset{(f \circ p_0^X, f \circ p_1^X)}{\longrightarrow} & Y \times Y \end{array}.$$

▮

Lemma 4.11. *In a category of fibrant objects $\mathcal{C}_f$, def. 2.34, let*

$$\begin{array}{ccccc} A_1 & & \overset{f}{\longrightarrow} & & A_2 \\ & {}_{\in \mathrm{Fib}}\searrow & & \swarrow_{\in \mathrm{Fib}} & \\ & & B & & \end{array}$$

be a morphism over some object B in $\mathcal{C}_f$ and let $u : B' \to B$ be any morphism in $\mathcal{C}_f$. Let

$$\begin{array}{ccccc} u^* A_1 & & \overset{u^* f}{\longrightarrow} & & u^* A_2 \\ & {}_{\in \mathrm{Fib}}\searrow & & \swarrow_{\in \mathrm{Fib}} & \\ & & B' & & \end{array}$$

be the corresponding morphism pulled back along u.

Then

- *if f is a fibration then also $u^* f$ is a fibration;*

- *if f is a weak equivalence then also $u^* f$ is a weak equivalence.*

(Brown 73, section 4, lemma 1)

Proof. For $f \in \mathrm{Fib}$ the statement follows from the pasting law which says that if in

$$\begin{array}{ccc} B' \times_B A_1 & \longrightarrow & A_1 \\ \downarrow^{u^* f \in \mathrm{Fib}} && \downarrow^{f \in \mathrm{Fib}} \\ B' \times_B A_2 & \longrightarrow & A_2 \\ \downarrow^{\in \mathrm{Fib}} && \downarrow^{\in \mathrm{Fib}} \\ B' & \overset{u}{\longrightarrow} & B \end{array}$$

the bottom and the total square are pullback squares, then so is the top square. The same reasoning applies for $f \in W \cap \mathrm{Fib}$.

Now to see the case that $f \in W$:

Consider the full subcategory $(\mathcal{C}_{/B})_f$ of the slice category $\mathcal{C}_{/B}$ (def. 3.15) on its fibrant objects, i.e. the full subcategory of the slice category on the fibrations

$$\begin{array}{c} X \\ \downarrow^{p}_{\in \mathrm{Fib}} \\ B \end{array}$$

into B. By factorizing for every such fibration the diagonal morphisms into the fiber product $X \underset{B}{\times} X$ through a weak equivalence followed by a fibration, we obtain path space objects $\mathrm{Path}_B(X)$ relative to B:

$$(\Delta_X)/B \;:\; \begin{array}{ccccc} X & \overset{\in W}{\longrightarrow} & \mathrm{Path}_B(X) & \overset{\in \mathrm{Fib}}{\longrightarrow} & X \underset{B}{\times} X \\ & {}_{\in \mathrm{Fib}}\searrow & \downarrow & \swarrow_{\in \mathrm{Fib}} \\ && B \end{array} \,.$$

With these, the factorization lemma (lemma 4.9) applies in $(\mathcal{C}_{/B})_f$.

(Notice that for this we do need the restriction of $\mathcal{C}_{/B}$ to the fibrations, because this ensures that the projections $p_i \colon X_1 \times_B X_2 \to X_i$ are still fibrations, which is used in the proof of the factorization lemma (here).)

So now given any

$$\begin{array}{ccc} X && \underset{\in W}{\overset{f}{\longrightarrow}} && Y \\ & {}_{\in \mathrm{Fib}}\searrow && \swarrow_{\in \mathrm{Fib}} \\ && B \end{array}$$

apply the factorization lemma in $(\mathcal{C}_{/B})_f$ to factor it as

$$\begin{array}{ccccc} X & \overset{i \in W}{\longrightarrow} & \mathrm{Path}_B(f) & \overset{\in W \cap \mathrm{Fib}}{\longrightarrow} & Y \\ & {}_{\in \mathrm{Fib}}\searrow & \downarrow & \swarrow_{\in \mathrm{Fib}} \\ && B \end{array} \,.$$

By the previous discussion it is sufficient now to show that the base change of i to B' is still a weak equivalence. But by the factorization lemma in $(\mathcal{C}_{/B})_f$, the morphism i is right inverse to another acyclic fibration over B:

$$\array{ id_X \;\colon\; X &\overset{i \in W}{\longrightarrow}& Path_B(f) &\overset{\in W \cap Fib}{\longrightarrow}& X \\ & {}_{\in Fib}\searrow & \downarrow & \swarrow_{\in Fib} \\ && B } \,.$$

(Notice that if we had applied the factorization lemma of Δ_X in $\mathcal{C}_f$ instead of $(\Delta_X)/B$ in $(\mathcal{C}_{/B})$ then the corresponding triangle on the right here would not commute.)

Now we may reason as before: the base change of the top morphism here is exhibited by the following pasting composite of pullbacks:

$$\array{ B' \underset{B}{\times} X &\longrightarrow& X \\ \downarrow &(pb)& \downarrow \\ B' \underset{B}{\times} Path_B(f) &\longrightarrow& Path_B(f) \\ \downarrow &(pb)& \downarrow^{\mathrlap{\in W \cap Fib}} \\ B' \underset{B}{\times} X &\longrightarrow& X \\ \downarrow &(pb)& \downarrow \\ B' &\longrightarrow& B }$$

The acyclic fibration $Path_B(f)$ is preserved by this pullback, as is the identity $id_X \colon X \to Path_B(X) \to X$. Hence the weak equivalence $X \to Path_B(X)$ is preserved by two-out-of-three (def. 2.1).

Lemma 4.12. *In a category of fibrant objects, def. 2.34, the pullback of a weak equivalence along a fibration is again a weak equivalence.*

(Brown 73, section 4, lemma 2)

Proof. Let $u \colon B' \to B$ be a weak equivalence and $p \colon E \to B$ be a fibration. We want to show that the left vertical morphism in the pullback

$$\array{ E \times_B B' &\longrightarrow& B' \\ \downarrow^{\mathrlap{\Rightarrow \in W}} && \downarrow^{\mathrlap{\in W}} \\ E &\overset{\in Fib}{\longrightarrow}& B }$$

is a weak equivalence.

First of all, using the factorization lemma 4.9 we may factor $B' \to B$ as

$$B' \overset{\in W}{\longrightarrow} Path(u) \overset{\in W \cap F}{\longrightarrow} B$$

with the first morphism a weak equivalence that is a right inverse to an acyclic fibration and the right one an acyclic fibration.

Then the pullback diagram in question may be decomposed into two consecutive pullback diagrams

$$\array{ E \times_B B' &\to& B' \\ \downarrow && \downarrow \\ Q &\overset{\in Fib}{\longrightarrow}& Path(u) \,, \\ \downarrow^{\mathrlap{\in W \cap Fib}} && \downarrow^{\mathrlap{\in W \cap Fib}} \\ E &\overset{\in Fib}{\longrightarrow}& B }$$

where the morphisms are indicated as fibrations and acyclic fibrations using the stability of these under arbitrary pullback.

This means that the proof reduces to proving that weak equivalences $u : B' \overset{\in W}{\longrightarrow} B$ that are right inverse to some acyclic fibration $v : B \overset{\in W \cap F}{\longrightarrow} B'$ map to a weak equivalence under pullback along a fibration.

Given such u with right inverse v, consider the pullback diagram

$$
\begin{array}{ccccc}
 & & E & & \\
 & {}^{(p,\mathrm{id})}_{\in W} & \downarrow & & \searrow^{\mathrm{id}} \\
E_1 \coloneqq & & B \times_{B'} E & \overset{\in W \cap \mathrm{Fib}}{\longrightarrow} & E \\
 & & \downarrow^{\in \mathrm{Fib}} & & \downarrow^{p \in \mathrm{Fib}} \\
 & & & (\mathrm{pb}) & B \\
 & & \downarrow & & \downarrow^{v \in W \cap \mathrm{Fib}} \\
 & & B & \overset{v \in \mathrm{Fib} \cap W}{\longrightarrow} & B'
\end{array}
$$

Notice that the indicated universal morphism $p \times \mathrm{Id} : E \overset{\in W}{\longrightarrow} E_1$ into the pullback is a weak equivalence by two-out-of-three (def. 2.1).

The previous lemma 4.11 says that weak equivalences between fibrations over B are themselves preserved by base extension along $u : B' \to B$. In total this yields the following diagram

$$
\begin{array}{ccccc}
u^* E = B' \times_B E & \longrightarrow & E & & \\
{}^{u^*(p \times \mathrm{Id})}_{\in W} \downarrow & & {}^{p \times \mathrm{Id}}_{\in W} \downarrow & \searrow^{\mathrm{id}} & \\
u^* E_1 & \longrightarrow & E_1 & \overset{\in W \cap \mathrm{Fib}}{\longrightarrow} & E \\
\downarrow^{\in \mathrm{Fib}} & & \downarrow^{\in \mathrm{Fib}} & & \downarrow^{p \in \mathrm{Fib}} \\
 & & & & B \\
\downarrow & & \downarrow & & \downarrow^{v \in W \cap \mathrm{Fib}} \\
B' & \overset{u}{\longrightarrow} & B & \overset{v \in W \cap \mathrm{Fib}}{\longrightarrow} & B'
\end{array}
$$

so that with $p \times \mathrm{Id} : E \to E_1$ a weak equivalence also $u^*(p \times \mathrm{Id})$ is a weak equivalence, as indicated.

Notice that $u^* E = B' \times_B E \to E$ is the morphism that we want to show is a weak equivalence. By two-out-of-three (def. 2.1) for that it is now sufficient to show that $u^* E_1 \to E_1$ is a weak equivalence.

That finally follows now since, by assumption, the total bottom horizontal morphism is the identity. Hence so is the top horizontal morphism. Therefore $u^* E_1 \to E_1$ is right inverse to a weak equivalence, hence is a weak equivalence. ▮

Lemma 4.13. *Let $(\mathcal{C}^{*/})_f$ be a category of fibrant objects, def. 2.34 in a model structure on pointed objects (prop. 3.29). Given any commuting diagram in $\mathcal{C}$ of the form*

$$
\begin{array}{ccccc}
X'_1 & \underset{t}{\overset{\in W}{\longrightarrow}} & X_1 & \underset{g}{\overset{f}{\rightrightarrows}} & X_2 \\
 & & \downarrow^{p_1}_{\in \mathrm{Fib}} & & \downarrow^{p_2}_{\in \mathrm{Fib}} \\
 & & B & \overset{u}{\longrightarrow} & C
\end{array}
$$

(meaning: both squares commute and t equalizes f with g) then the localization functor $\gamma : (\mathcal{C}^{*/})_f \to \mathrm{Ho}(\mathcal{C}^{*/})$ (def. 2.28, cor 2.36) takes the morphisms $\mathrm{fib}(p_1) \rightrightarrows \mathrm{fib}(p_2)$ induced by f and g on fibers (example 3.27) to the same morphism, in the homotopy category.

(Brown 73, section 4, lemma 4)

Proof. First consider the pullback of p_2 along u: this forms the same kind of diagram but with the bottom morphism an identity. Hence it is sufficient to consider this special case.

Consider the full subcategory $(\mathcal{C}^{*/}_{/B})_f$ of the slice category $\mathcal{C}^{*/}_{/B}$ (def. 3.15) on its fibrant objects, i.e. the full subcategory of the slice category on the fibrations

$$\begin{array}{c} X \\ \downarrow^{p}_{\in \mathrm{Fib}} \\ B \end{array}$$

into B. By factorizing for every such fibration the diagonal morphisms into the fiber product $X \underset{B}{\times} X$ through a weak equivalence followed by a fibration, we obtain path space objects $\mathrm{Path}_B(X)$ relative to B:

$$(\Delta_X)/B : \begin{array}{ccccc} X & \overset{\in W}{\longrightarrow} & \mathrm{Path}_B(X) & \overset{\in \mathrm{Fib}}{\longrightarrow} & X \underset{B}{\times} X \\ & {}_{\in \mathrm{Fib}}\searrow & \downarrow & \swarrow_{\in \mathrm{Fib}} \\ && B \end{array} .$$

With these, the factorization lemma (lemma 4.9) applies in $(\mathcal{C}^{*/}_{/B})_f$.

Let then $X \overset{s}{\to} \mathrm{Path}_B(X_2) \xrightarrow{(p_0,p_1)} X_2 \times_B X_2$ be a path space object for X_2 in the slice over B and consider the following commuting square

$$\begin{array}{ccc} X'_1 & \overset{s f t}{\longrightarrow} & \mathrm{Path}_B(X_2) \\ {}^{t}_{\in W}\downarrow && \downarrow^{(p_0,p_1)}_{\in \mathrm{Fib}} \\ X_1 & \overset{(f,g)}{\longrightarrow} & X_2 \underset{B}{\times} X_2 \end{array} .$$

By factoring this through the pullback $(f,g)^*(p_0,p_1)$ and then applying the factorization lemma 4.9 and then two-out-of-three (def. 2.1) to the factoring morphisms, this may be replaced by a commuting square of the same form, where however the left morphism is an acyclic fibration

$$\begin{array}{ccc} X''_1 & \longrightarrow & \mathrm{Path}_B(X_2) \\ {}^{t}_{\in W \cap \mathrm{Fib}}\downarrow && \downarrow^{(p_0,p_1)}_{\in \mathrm{Fib}} \\ X_1 & \overset{(f,g)}{\longrightarrow} & X_2 \underset{B}{\times} X_2 \end{array} .$$

This makes also the morphism $X''_1 \to B$ be a fibration, so that the whole diagram may now be regarded as a diagram in the category of fibrant objects $(\mathcal{C}_{/B})_f$ of the slice category over B.

As such, the top horizontal morphism now exhibits a right homotopy which under localization $\gamma_B : (\mathcal{C}_{/B})_f \longrightarrow \mathrm{Ho}(\mathcal{C}_{/B})$ (def. 2.28) of the slice model structure (prop. 3.29) we have

$$\gamma_B(f) = \gamma_B(g) .$$

The result then follows by observing that we have a commuting square of functors

$$
\begin{array}{ccc}
(\mathcal{C}^{*/}_{/B})_f & \overset{\text{fib}}{\longrightarrow} & \mathcal{C}^{*/} \\
\downarrow^{\gamma_B} & \swArrow & \downarrow^{\gamma} \\
\text{Ho}(\mathcal{C}^{*/}_{/B}) & \longrightarrow & \text{Ho}(\mathcal{C}^{*/})
\end{array}
,
$$

because, by lemma 4.11, the top and right composite sends weak equivalences to isomorphisms, and hence the bottom filler exists by theorem 2.31. This implies the claim. ▮

Homotopy fibers

We now discuss the homotopy-theoretic properties of the mapping cone- and mapping cocone-constructions from above.

Literature (Brown 73, section 4).

Remark 4.14. The factorization lemma 4.9 with prop. 4.3 says that the mapping cocone of a morphism f, def. 4.1, is equivalently the plain fiber, example 3.27, of a fibrant resolution $\tilde f$ of f:

$$
\begin{array}{ccc}
\text{Path}_*(f) & \longrightarrow & \text{Path}(f) \\
\downarrow & (\text{pb}) & \downarrow^{\tilde f} \\
* & \longrightarrow & Y
\end{array}
.
$$

The following prop. 4.15 says that, up to equivalence, this situation is independent of the specific fibration resolution $\tilde f$ provided by the factorization lemma (hence by the prescription for the mapping cocone), but only depends on it being *some* fibration resolution.

Proposition 4.15. *In the category of fibrant objects $(\mathcal{C}^{*/})_f$, def. 2.34, of a model structure on pointed objects (prop. 3.29) consider a morphism of fiber-diagrams, hence a commuting diagram of the form*

$$
\begin{array}{ccccc}
\text{fib}(p_1) & \longrightarrow & X_1 & \underset{\in \text{Fib}}{\overset{p_1}{\longrightarrow}} & Y_1 \\
\downarrow^{h} && \downarrow^{g} && \downarrow^{f} \\
\text{fib}(p_2) & \longrightarrow & X_2 & \underset{\in \text{Fib}}{\overset{p_2}{\longrightarrow}} & Y_2
\end{array}
.
$$

If f and g weak equivalences, then so is h.

Proof. Factor the diagram in question through the pullback of p_2 along f

$$
\begin{array}{ccccc}
\text{fib}(p_1) & \longrightarrow & X_1 && \\
\downarrow^{h} && {}^{\in W}\downarrow & \searrow^{p_1} & \\
\text{fib}(f^* p_2) & \longrightarrow & f^* X_2 & \underset{\in \text{Fib}}{\overset{f^* p_2}{\longrightarrow}} & Y_1 \\
\downarrow^{\simeq} && \downarrow^{\in W} && \downarrow^{f}_{\in W} \\
\text{fib}(p_2) & \longrightarrow & X_2 & \underset{\in \text{Fib}}{\overset{p_2}{\longrightarrow}} & Y_2
\end{array}
$$

and observe that

1. $\text{fib}(f^* p_2) = \text{pt}^* f^* p_2 = \text{pt}^* p_2 = \text{fib}(p_2)$;
2. $f^* X_2 \to X_2$ is a weak equivalence by lemma 4.12;
3. $X_1 \to f^* X_2$ is a weak equivalence by assumption and by two-out-of-three (def. 2.1);

Moreover, this diagram exhibits $h : \mathrm{fib}(p_1) \to \mathrm{fib}(f^* p_2) = \mathrm{fib}(p_2)$ as the base change, along $* \to Y_1$, of $X_1 \to f^* X_2$. Therefore the claim now follows with lemma 4.11. ▮

Hence we say:

Definition 4.16. Let $\mathcal{C}$ be a model category and $\mathcal{C}^{*/}$ its model category of pointed objects, prop. 3.29. For $f : X \longrightarrow Y$ any morphism in its category of fibrant objects $(\mathcal{C}^{*/})_f$, def. 2.34, then its **homotopy fiber**

$$\mathrm{hofib}(f) \longrightarrow X$$

is the morphism in the homotopy category $\mathrm{Ho}(\mathcal{C}^{*/})$, def. 2.25, which is represented by the fiber, example 3.27, of any fibration resolution $\tilde{f}$ of f (hence any fibration $\tilde{f}$ such that f factors through a weak equivalence followed by $\tilde{f}$).

Dually:

For $f : X \longrightarrow Y$ any morphism in its category of cofibrant objects $(\mathcal{C}^{*/})_c$, def. 2.34, then its **homotopy cofiber**

$$Y \longrightarrow \mathrm{hocofib}(f)$$

is the morphism in the homotopy category $\mathrm{Ho}(\mathcal{C})$, def. 2.25, which is represented by the cofiber, example 3.27, of any cofibration resolution of f (hence any cofibration $\tilde{f}$ such that f factors as $\tilde{f}$ followed by a weak equivalence).

Proposition 4.17. *The homotopy fiber in def. 4.16 is indeed well defined, in that for f_1 and f_2 two fibration replacements of any morphisms f in $\mathcal{C}_f$, then their fibers are isomorphic in $\mathrm{Ho}(\mathcal{C}^{*/})$.*

Proof. It is sufficient to exhibit an isomorphism in $\mathrm{Ho}(\mathcal{C}^{*/})$ from the fiber of the fibration replacement given by the factorization lemma 4.9 (for any choice of path space object) to the fiber of any other fibration resolution.

Hence given a morphism $f : Y \longrightarrow X$ and a factorization

$$f : X \underset{\in W}{\longrightarrow} \hat{X} \underset{f_1}{\overset{\in \mathrm{Fib}}{\longrightarrow}} Y$$

consider, for any choice $\mathrm{Path}(Y)$ of path space object (def. 2.18), the diagram

$$\begin{array}{ccc} \mathrm{Path}(f) & \overset{\in W \cap \mathrm{Fib}}{\longrightarrow} & X \\ {}^{\in W}\downarrow & (\mathrm{pb}) & \downarrow^{\in W} \\ \mathrm{Path}(f_1) & \overset{\in W \cap \mathrm{Fib}}{\longrightarrow} & \hat{X} \\ {}^{\in \mathrm{Fib}}\downarrow & (\mathrm{pb}) & \downarrow^{\mathrm{\in Fib}}_{f_1} \\ \mathrm{Path}(Y) & \underset{\in W \cap \mathrm{Fib}}{\overset{p_1}{\longrightarrow}} & Y \\ {}^{p_0}_{\in W \cap \mathrm{Fib}}\downarrow \\ Y \end{array}$$

as in the proof of lemma 4.9. Now by repeatedly using prop. 4.15:

1. the bottom square gives a weak equivalence from the fiber of $\mathrm{Path}(f_1) \to \mathrm{Path}(Y)$ to the fiber of f_1;

2. The square

$$\begin{array}{ccc} \mathrm{Path}(f_1) & \stackrel{\mathrm{id}}{\longrightarrow} & \mathrm{Path}(f_1) \\ \downarrow && \downarrow \\ \mathrm{Path}(Y) & \underset{p_0}{\longrightarrow} & Y \end{array}$$

gives a weak equivalence from the fiber of $\mathrm{Path}(f_1) \to \mathrm{Path}(Y)$ to the fiber of $\mathrm{Path}(f_1) \to Y$.

3. Similarly the total vertical composite gives a weak equivalence via

$$\begin{array}{ccc} \mathrm{Path}(f) & \stackrel{\in W}{\longrightarrow} & \mathrm{Path}(f_1) \\ \downarrow && \downarrow \\ Y & \underset{\mathrm{id}}{\longrightarrow} & Y \end{array}$$

from the fiber of $\mathrm{Path}(f) \to Y$ to the fiber of $\mathrm{Path}(f_1) \to Y$.

Together this is a zig-zag of weak equivalences of the form

$$\mathrm{fib}(f_1) \stackrel{\in W}{\longleftarrow} \mathrm{fib}(\mathrm{Path}(f_1) \to \mathrm{Path}(Y)) \stackrel{\in W}{\longrightarrow} \mathrm{fib}(\mathrm{Path}(f_1) \to Y) \stackrel{\in W}{\longleftarrow} \mathrm{fib}(\mathrm{Path}(f) \to Y)$$

between the fiber of $\mathrm{Path}(f) \to Y$ and the fiber of f_1. This gives an isomorphism in the homotopy category. ▮

Example 4.18. (fibers of Serre fibrations)

In showing that Serre fibrations are abstract fibrations in the sense of model category theory, theorem 3.7 implies that the fiber F (example 3.27) of a Serre fibration, def. 1.47

$$\begin{array}{ccc} F & \longrightarrow & X \\ && \downarrow^{p} \\ && B \end{array}$$

over any point is actually a homotopy fiber in the sense of def. 4.16. With prop. 4.15 this implies that the weak homotopy type of the fiber only depends on the Serre fibration up to weak homotopy equivalence in that if $p' : X' \to B'$ is another Serre fibration fitting into a commuting diagram of the form

$$\begin{array}{ccc} X & \stackrel{\in W_{\mathrm{cl}}}{\longrightarrow} & X' \\ \downarrow^{p} && \downarrow^{p'} \\ B & \stackrel{\in W_{\mathrm{cl}}}{\longrightarrow} & B' \end{array}$$

then $F \stackrel{\in W_{\mathrm{cl}}}{\longrightarrow} F'$.

In particular this gives that the weak homotopy type of the fiber of a Serre fibration $p : X \to B$ does not change as the basepoint is moved in the same connected component. For let $\gamma : I \longrightarrow B$ be a path between two points

$$b_{0,1} \;:\; \ast \underset{\in W_{\mathrm{cl}}}{\overset{i_{0,1}}{\longrightarrow}} I \stackrel{\gamma}{\longrightarrow} B \;.$$

Then since all objects in $(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{Quillen}}$ are fibrant, and since the endpoint inclusions $i_{0,1}$ are weak equivalences, lemma 4.12 gives the zig-zag of top horizontal weak equivalences in the following diagram:

$$
\begin{array}{ccccccc}
F_{b_0} = & b_0^* p & \overset{\in W_{cl}}{\longrightarrow} & \gamma^* p & \overset{\in W_{cl}}{\longleftarrow} & b_1^* p & = F_{b_1} \\
& \downarrow & (pb) & \downarrow^{\gamma^* f}_{\underset{Fib}{\in}} & (pb) & \downarrow & \\
& * & \underset{i_0}{\overset{\in W_{cl}}{\longrightarrow}} & I & \underset{i_1}{\overset{\in W_{cl}}{\longleftarrow}} & * &
\end{array}
$$

and hence an isomorphism $F_{b_0} \simeq F_{b_1}$ in the classical homotopy category (def. 3.11).

The same kind of argument applied to maps from the square I^2 gives that if $\gamma_1, \gamma_2 : I \to B$ are two homotopic paths with coinciding endpoints, then the isomorphisms between fibers over endpoints which they induce are equal. (But in general the isomorphism between the fibers does depend on the choice of homotopy class of paths connecting the basepoints!)

The same kind of argument also shows that if B has the structure of a cell complex (def. 1.38) then the restriction of the Serre fibration to one cell D^n may be identified in the homotopy category with $D^n \times F$, and may be canonically identified so if the fundamental group of X is trivial. This is used when deriving the Serre-Atiyah-Hirzebruch spectral sequence for p (prop.).

Example 4.19. For every continuous function $f : X \longrightarrow Y$ between CW-complexes, def. 1.38, then the standard topological mapping cone is the attaching space (example 1.12)

$$Y \cup_f \mathrm{Cone}(X) \quad \in \mathrm{Top}$$

of Y with the standard cone $\mathrm{Cone}(X)$ given by collapsing one end of the standard topological cyclinder $X \times I$ (def. 1.22) as shown in example 4.6.

Equipped with the canonical continuous function

$$Y \longrightarrow Y \cup_f \mathrm{Cone}(X)$$

this represents the homotopy cofiber, def. 4.16, of f with respect to the classical model structure on topological spaces $\mathcal{C} = \mathrm{Top}_{\mathrm{Quillen}}$ from theorem 3.7.

Proof. By prop. 3.13, for X a CW-complex then the standard topological cylinder object $X \times I$ is indeed a cyclinder object in $\mathrm{Top}_{\mathrm{Quillen}}$. Therefore by prop. 4.3 and the factorization lemma 4.9, the mapping cone construction indeed produces first a cofibrant replacement of f and then the ordinary cofiber of that, hence a model for the homotopy cofiber. ▮

Example 4.20. The homotopy fiber of the inclusion of classifying spaces $BO(n) \hookrightarrow BO(n+1)$ is the n-sphere S^n. See this prop. at *Classifying spaces and G-structure*.

Example 4.21. Suppose a morphism $f : X \longrightarrow Y$ already happens to be a fibration between fibrant objects. The factorization lemma 4.9 replaces it by a fibration out of the mapping cocylinder $\mathrm{Path}(f)$, but such that the comparison morphism is a weak equivalence:

$$
\begin{array}{ccccc}
\mathrm{fib}(f) & \longrightarrow & X & \underset{\in Fib}{\overset{f}{\longrightarrow}} & Y \\
\downarrow^{\in W} & & \downarrow^{\in W} & & \downarrow^{\mathrm{id}} \\
\mathrm{fib}(\tilde f) & \longrightarrow & \mathrm{Path}(f) & \underset{\in Fib}{\overset{\tilde f}{\longrightarrow}} & Y
\end{array}
.
$$

Hence by prop. 4.15 in this case the ordinary fiber of f is weakly equivalent to the mapping cocone, def. 4.1.

We may now state the abstract version of the statement of prop. 1.51:

Proposition 4.22. *Let $\mathcal{C}$ be a model category. For $f : X \to Y$ any morphism of pointed objects, and for A a pointed object, def. 3.16, then the sequence*

$$[A, \mathrm{hofib}(f)]_* \overset{i_*}{\longrightarrow} [A, X]_* \overset{f_*}{\longrightarrow} [A, Y]_*$$

is exact as a sequence of pointed sets.

(Where the sequence here is the image of the homotopy fiber sequence of def. 4.16 under the hom-functor $[A, -]_* : \mathrm{Ho}(\mathcal{C}^{*/}) \longrightarrow \mathrm{Set}^{*/}$ from example 3.30.)

Proof. Let A, X and Y denote fibrant-cofibrant objects in $\mathcal{C}^{*/}$ representing the given objects of the same name in $\mathrm{Ho}(\mathcal{C}^{*/})$. Moreover, let f be a fibration in $\mathcal{C}^{*/}$ representing the given morphism of the same name in $\mathrm{Ho}(\mathcal{C}^{*/})$.

Then by def. 4.16 and prop. 4.17 there is a representative $\mathrm{hofib}(f) \in \mathcal{C}$ of the homotopy fiber which fits into a pullback diagram of the form

$$\begin{array}{ccc} \mathrm{hofib}(f) & \overset{i}{\longrightarrow} & X \\ \downarrow && \downarrow^{f} \\ * &\longrightarrow& Y \end{array}$$

With this the hom-sets in question are represented by genuine morphisms in $\mathcal{C}^{*/}$, modulo homotopy. From this it follows immediately that $\mathrm{im}(i_*)$ includes into $\ker(f_*)$. Hence it remains to show the converse: that every element in $\ker(f_*)$ indeed comes from $\mathrm{im}(i_*)$.

But an element in $\ker(f_*)$ is represented by a morphism $\alpha : A \to X$ such that there is a left homotopy as in the following diagram

$$\begin{array}{ccccc} && A & \overset{\alpha}{\longrightarrow} & X \\ && {}^{i_0}\downarrow & {}^{\tilde \eta}\nearrow & \downarrow^{f} \\ A & \overset{i_1}{\longrightarrow} & \mathrm{Cyl}(A) & \overset{\eta}{\longrightarrow} & Y \\ \downarrow &&&& \downarrow^{=} \\ * && \longrightarrow && Y \end{array} \,.$$

Now by lemma 2.20 the square here has a lift $\tilde \eta$, as shown. This means that $i_1 \circ \tilde \eta$ is left homotopic to α. But by the universal property of the fiber, $i_1 \circ \tilde \eta$ factors through $i : \mathrm{hofib}(f) \to X$. ▮

With prop. 4.15 it also follows notably that the loop space construction becomes well-defined on the homotopy category:

Remark 4.23. Given an object $X \in \mathcal{C}^{*/}_f$, and picking any path space object $\mathrm{Path}(X)$, def. 2.18 with induced loop space object ΩX, def. 4.4, write $\mathrm{Path}_2(X) = \mathrm{Path}(X) \underset{X}{\times} \mathrm{Path}(X)$ for the path space object given by the fiber product of $\mathrm{Path}(X)$ with itself, via example 2.21. From the pullback diagram there, the fiber inclusion $\Omega X \to \mathrm{Path}(X)$ induces a morphism

$$\Omega X \times \Omega X \longrightarrow (\Omega X)_2 \,.$$

In the case where $\mathcal{C}^{*/} = \mathrm{Top}^{*/}$ and Ω is induced, via def. 4.4, from the standard path space object (def. 1.34), i.e. in the case that

$$\Omega X = \mathrm{fib}(\mathrm{Maps}(I_+, X)_* \longrightarrow X \times X) \,,$$

then this is the operation of concatenating two loops parameterized by $I = [0,1]$ to a single loop parameterized by $[0,2]$.

Proposition 4.24. *Let $\mathcal{C}$ be a model category, def. 2.3. Then the construction of forming loop space objects $X \mapsto \Omega X$, def. 4.4 (which on $\mathcal{C}^{*/}_f$ depends on a choice of path space objects, def.*

2.18) becomes unique up to isomorphism in the homotopy category (def. 2.25) of the model structure on pointed objects (prop. 3.29) and extends to a functor:

$$\Omega : \mathrm{Ho}(\mathcal{C}^{*/}) \longrightarrow \mathrm{Ho}(\mathcal{C}^{*/}) \,.$$

Dually, the reduced suspension operation, def. 4.4, which on $\mathcal{C}^{*/}$ depends on a choice of cylinder object, becomes a functor on the homotopy category

$$\Sigma : \mathrm{Ho}(\mathcal{C}^{*/}) \longrightarrow \mathrm{Ho}(\mathcal{C}^{*/}) \,.$$

Moreover, the pairing operation induced on the objects in the image of this functor via remark 4.23 (concatenation of loops) gives the objects in the image of Ω group object structure, and makes this functor lift as

$$\Omega : \mathrm{Ho}(\mathcal{C}^{*/}) \longrightarrow \mathrm{Grp}(\mathrm{Ho}(\mathcal{C}^{*/})) \,.$$

(Brown 73, section 4, theorem 3)

Proof. Given an object $X \in \mathcal{C}^{*/}$ and given two choices of path space objects $\mathrm{Path}(X)$ and $\widetilde{\mathrm{Path}(X)}$, we need to produce an isomorphism in $\mathrm{Ho}(\mathcal{C}^{*/})$ between ΩX and $\tilde{\Omega} X$.

To that end, first lemma 4.10 implies that any two choices of path space objects are connected via a third path space by a span of morphisms compatible with the structure maps. By two-out-of-three (def. 2.1) every morphism of path space objects compatible with the inclusion of the base object is a weak equivalence. With this, lemma 4.11 implies that these morphisms induce weak equivalences on the corresponding loop space objects. This shows that all choices of loop space objects become isomorphic in the homotopy category.

Moreover, all the isomorphisms produced this way are actually equal: this follows from lemma 4.13 applied to

$$\begin{array}{ccccc} X & \overset{s}{\longrightarrow} & \mathrm{Path}(X) & \rightrightarrows & \widetilde{\mathrm{Path}(X)} \\ && \downarrow && \downarrow \\ && X \times X & \overset{\mathrm{id}}{\longrightarrow} & X \times X \end{array} \,.$$

This way we obtain a functor

$$\Omega : \mathcal{C}_f^{*/} \longrightarrow \mathrm{Ho}(\mathcal{C}^{*/}) \,.$$

By prop. 4.15 (and using that Cartesian product preserves weak equivalences) this functor sends weak equivalences to isomorphisms. Therefore the functor on homotopy categories now follows with theorem 2.31.

It is immediate to see that the operation of loop concatenation from remark 4.23 gives the objects $\Omega X \in \mathrm{Ho}(\mathcal{C}^{*/})$ the structure of monoids. It is now sufficient to see that these are in fact groups:

We claim that the inverse-assigning operation is given by the left map in the following pasting composite

$$\begin{array}{ccccc} \Omega' X & \longrightarrow & \mathrm{Path}'(X) & \longrightarrow & X \times X \\ \downarrow^{\simeq} && \downarrow^{\simeq} & (\mathrm{pb}) & \downarrow^{\mathrm{swap}} \\ \Omega X & \longrightarrow & \mathrm{Path}(X) & \underset{(p_0, p_1)}{\longrightarrow} & X \times X \end{array} \,,$$

(where $\mathrm{Path}'(X)$, thus defined, is the path space object obtained from $\mathrm{Path}(X)$ by "reversing the notion of source and target of a path").

To see that this is indeed an inverse, it is sufficient to see that the two morphisms

$$\Omega X \rightrightarrows (\Omega X)_2$$

induced from

$$\mathrm{Path}(X) \underset{(s\circ p_0, s\circ p_0)}{\overset{\Delta}{\rightrightarrows}} \mathrm{Path}(X) \times_X \mathrm{Path}'(X)$$

coincide in the homotopy category. This follows with lemma 4.13 applied to the following commuting diagram:

$$\begin{array}{ccccc} X & \overset{i}{\longrightarrow} & \mathrm{Path}(X) & \underset{(s\circ p_0, s\circ p_0)}{\overset{\Delta}{\rightrightarrows}} & \mathrm{Path}(X) \times_X \mathrm{Path}'(X) \\ & & {}^{(p_0,p_1)}\downarrow & & \downarrow \\ & & X \times X & \overset{\Delta\circ \mathrm{pr}_1}{\longrightarrow} & X \times X \end{array} \,.$$

∎

Homotopy pullbacks

The concept of homotopy fibers of def. 4.16 is a special case of the more general concept of homotopy pullbacks.

Definition 4.25. A model category $\mathcal{C}$ (def. 2.3) is called a **right proper model category** if pullback along fibrations preserves weak equivalences.

Example 4.26. By lemma 4.12, a model category $\mathcal{C}$ (def. 2.3) in which all objects are fibrant is a right proper model category (def. 4.25).

Definition 4.27. Let $\mathcal{C}$ be a right proper model category (def. 4.25). Then a commuting square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow^{g} \\ C & \underset{f}{\longrightarrow} & D \end{array}$$

in $\mathcal{C}_f$ is called a **homotopy pullback** (of f along g and equivalently of g along f) if the following equivalent conditions hold:

1. for some factorization of the form

$$g : B \overset{\in W}{\longrightarrow} \hat B \overset{\in \mathrm{Fib}}{\longrightarrow} D$$

the universally induced morphism from A into the pullback of $\hat B$ along f is a weak equivalence:

$$\begin{array}{ccc} A & \longrightarrow & B \\ {}^{\in W}\downarrow & & \downarrow^{\in W} \\ C \underset{D}{\times} \hat B & \longrightarrow & \hat B \\ \downarrow & (\mathrm{pb}) & \downarrow^{\in \mathrm{Fib}} \\ C & \longrightarrow & D \end{array} \,.$$

2. for some factorization of the form

$$f : C \overset{\in W}{\longrightarrow} \hat C \overset{\in \mathrm{Fib}}{\longrightarrow} D$$

the universally induced morphism from A into the pullback of $\hat{D}$ along g is a weak equivalence:

$$A \overset{\in W}{\longrightarrow} \hat{C} \underset{D}{\times} B \,.$$

3. the above two conditions hold for every such factorization.

(e.g. Goerss-Jardine 96, II (8.14))

Proposition 4.28. *The conditions in def. 4.27 are indeed equivalent.*

Proof. First assume that the first condition holds, in that

$$\array{ A &\longrightarrow& B \\ {}^{\mathllap{\in W}}\downarrow && \downarrow^{\mathrlap{\in W}} \\ C \underset{D}{\times} \hat{B} &\longrightarrow& \hat{B} \\ \downarrow &(pb)& \downarrow^{\mathrlap{\in Fib}} \\ C &\longrightarrow& D } \,.$$

Then let

$$f \colon C \overset{\in W}{\longrightarrow} \hat{C} \overset{\in Fib}{\longrightarrow} D$$

be any factorization of f and consider the pasting diagram (using the pasting law for pullbacks)

$$\array{ A &\longrightarrow& \hat{C} \underset{D}{\times} B &\longrightarrow& B \\ {}^{\mathllap{\in W}}\downarrow && \downarrow^{\mathrlap{\in W}} &(pb)& \downarrow^{\mathrlap{\in W}} \\ C \underset{D}{\times} \hat{B} &\overset{\in W}{\longrightarrow}& \hat{C} \underset{D}{\times} \hat{D} &\overset{\in Fib}{\longrightarrow}& \hat{B} \\ \downarrow &(pb)& \downarrow^{\mathrlap{\in Fib}} &(pb)& \downarrow^{\mathrlap{\in Fib}} \\ C &\underset{\in W}{\longrightarrow}& \hat{C} &\underset{\in Fib}{\longrightarrow}& D } \,,$$

where the inner morphisms are fibrations and weak equivalences, as shown, by the pullback stability of fibrations (prop. 2.10) and then since pullback along fibrations preserves weak equivalences by assumption of right properness (def. 4.25). Hence it follows by two-out-of-three (def. 2.1) that also the comparison morphism $A \to \hat{C} \underset{D}{\times} B$ is a weak equivalence.

In conclusion, if the homotopy pullback condition is satisfied for one factorization of g, then it is satisfied for all factorizations of f. Since the argument is symmetric in f and g, this proves the claim. ▮

Remark 4.29. In particular, an ordinary pullback square of fibrant objects, one of whose edges is a fibration, is a homotopy pullback square according to def. 4.27.

Proposition 4.30. *Let $\mathcal{C}$ be a right proper model category (def. 4.25). Given a diagram in $\mathcal{C}$ of the form*

$$\array{ A &\longrightarrow& B &\overset{\in Fib}{\longleftarrow}& C \\ \downarrow^{\mathrlap{\in W}} && \downarrow^{\mathrlap{\in W}} && \downarrow^{\mathrlap{\in W}} \\ D &\longrightarrow& E &\underset{\in Fib}{\longleftarrow}& F }$$

then the induced morphism on pullbacks is a weak equivalence

$$A \underset{B}{\times} C \overset{\in W}{\longrightarrow} D \underset{E}{\times} F \,.$$

Proof. (The reader should draw the 3-dimensional cube diagram which we describe in words now.)

First consider the universal morphism $C \to E \underset{F}{\times} C$ and observe that it is a weak equivalence by right properness (def. 4.25) and two-out-of-three (def. 2.1).

Then consider the universal morphism $A \underset{B}{\times} C \to A \underset{B}{\times} (E \underset{F}{\times} C)$ and observe that this is also a weak equivalence, since $A \underset{B}{\times} C$ is the limiting cone of a homotopy pullback square by remark 4.29, and since the morphism is the comparison morphism to the pullback of the factorization constructed in the first step.

Now by using the pasting law, then the commutativity of the "left" face of the cube, then the pasting law again, one finds that $A \underset{B}{\times} (E \underset{F}{\times} C) \simeq A \underset{D}{\times} (D \underset{E}{F} \times\,)$. Again by right properness this implies that $A \underset{B}{\times} (E \underset{F}{\times} C) \to D \underset{E}{\times} F$ is a weak equivalence.

With this the claim follows by two-out-of-three. ▮

Homotopy pullbacks satisfy the usual abstract properties of pullbacks:

Proposition 4.31. *Let $\mathcal{C}$ be a right proper model category (def. 4.25). If in a commuting square in $\mathcal{C}$ one edge is a weak equivalence, then the square is a homotopy pullback square precisely if the opposite edge is a weak equivalence, too.*

Proof. Consider a commuting square of the form

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow && \downarrow \\ C & \underset{\in W}{\longrightarrow} & D \end{array} \,.$$

To detect whether this is a homotopy pullback, by def. 4.27 and prop. 4.28, we are to choose any factorization of the right vertical morphism to obtain the pasting composite

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow && \downarrow^{\in W} \\ C \underset{D}{\times} \hat B & \overset{\in W}{\longrightarrow} & \hat B \\ \downarrow & (pb) & \downarrow^{\in Fib} \\ C & \underset{\in W}{\longrightarrow} & D \end{array} \,.$$

Here the morphism in the middle is a weak equivalence by right properness (def. 4.25). Hence it follows by two-out-of-three that the top left comparison morphism is a weak equivalence (and so the original square is a homotopy pullback) precisely if the top morphism is a weak equivalence. ▮

Proposition 4.32. *Let $\mathcal{C}$ be a right proper model category (def. 4.25).*

1. *(pasting law) If in a commuting diagram*

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C \\ \downarrow && \downarrow && \downarrow \\ D & \longrightarrow & E & \longrightarrow & F \end{array}$$

the square on the right is a homotoy pullback (def. 4.27) then the left square is, too, *precisely if the total rectangle is;*

2. *in the presence of functorial factorization (def. 2.6) through weak equivalences followed by fibrations:*

   every retract of a homotopy pullback square (in the category $\mathcal{C}_f^\square$ of commuting squares in $\mathcal{C}_f$) is itself a homotopy pullback square.

Proof. For the first statement: choose a factorization of $C \overset{\in W}{\longrightarrow} \hat F \overset{\in \mathrm{Fib}}{\longrightarrow} F$, pull it back to a factorization $B \to \hat B \overset{\in \mathrm{Fib}}{\longrightarrow} E$ and assume that $B \to \hat B$ is a weak equivalence, i.e. that the right square is a homotopy pullback. Now use the ordinary pasting law to conclude.

For the second statement: functorially choose a factorization of the two right vertical morphisms of the squares and factor the squares through the pullbacks of the corresponding fibrations along the bottom morphisms, respectively. Now the statement that the squares are homotopy pullbacks is equivalent to their top left vertical morphisms being weak equivalences. Factor these top left morphisms functorially as cofibrations followed by acyclic fibrations. Then the statement that the squares are homotopy pullbacks is equivalent to those top left cofibrations being acyclic. Now the claim follows using that the retract of an acyclic cofibration is an acyclic cofibration (prop. 2.10). ▮

Long sequences

The ordinary fiber, example 3.27, of a morphism has the property that taking it *twice* is always trivial:

$$* \simeq \mathrm{fib}(\mathrm{fib}(f)) \longrightarrow \mathrm{fib}(f) \longrightarrow X \overset{f}{\longrightarrow} Y \,.$$

This is crucially different for the homotopy fiber, def. 4.16. Here we discuss how this comes about and what the consequences are.

Proposition 4.33. *Let $\mathcal{C}_f$ be a category of fibrant objects of a model category, def. 2.34 and let $f : X \longrightarrow Y$ be a morphism in its category of pointed objects, def. 3.16. Then the homotopy fiber of its homotopy fiber, def. 4.16, is isomorphic, in $\mathrm{Ho}(\mathcal{C}^{*/})$, to the loop space object ΩY of Y (def. 4.4, prop. 4.24):*

$$\mathrm{hofib}(\mathrm{hofib}(X \overset{f}{\to} Y)) \simeq \Omega Y \,.$$

Proof. Assume without restriction that $f \colon X \longrightarrow Y$ is already a fibration between fibrant objects in $\mathcal{C}$ (otherwise replace and rename). Then its homotopy fiber is its ordinary fiber, sitting in a pullback square

$$\begin{array}{ccccc} \mathrm{hofib}(f) \simeq & F & \overset{i}{\longrightarrow} & X \\ & \downarrow && \downarrow^{\mathrlap{f}} \\ & * &\longrightarrow& Y \end{array} \,.$$

In order to compute $\mathrm{hofib}(\mathrm{hofib}(f))$, i.e. $\mathrm{hofib}(i)$, we need to replace the fiber inclusion i by a fibration. Using the factorization lemma 4.9 for this purpose yields, after a choice of path space object $\mathrm{Path}(X)$ (def. 2.18), a replacement of the form

$$\begin{array}{ccc} F &\overset{\in W}{\longrightarrow}& F \times_X \mathrm{Path}(X) \\ & {}_{\mathllap{i}}\searrow & \downarrow^{\mathrlap{\tilde i}}_{\mathrlap{\in \mathrm{Fib}}} \\ && X \end{array} \,.$$

Hence $\mathrm{hofib}(i)$ is the ordinary fiber of this map:

$$\mathrm{hofib}(\mathrm{hofib}(f)) \simeq F \times_X \mathrm{Path}(X) \times_X * \quad \in \mathrm{Ho}(\mathcal{C}^{*/}) \,.$$

Notice that

$$F \times_X \mathrm{Path}(X) \;\simeq\; * \times_Y \mathrm{Path}(X)$$

because of the pasting law:

$$\begin{array}{ccc} F \times_X \mathrm{Path}(X) & \longrightarrow & \mathrm{Path}(X) \\ \downarrow & (\mathrm{pb}) & \downarrow \\ F & \stackrel{i}{\longrightarrow} & X \\ \downarrow & (\mathrm{pb}) & \downarrow^{f} \\ * & \longrightarrow & Y \end{array} \,.$$

Hence

$$\mathrm{hofib}(\mathrm{hofib}(f)) \;\simeq\; * \times_Y \mathrm{Path}(X) \times_X * \,.$$

Now we claim that there is a choice of path space objects $\mathrm{Path}(X)$ and $\mathrm{Path}(Y)$ such that this model for the homotopy fiber (as an object in $\mathcal{C}^{*/}$) sits in a pullback diagram of the following form:

$$\begin{array}{ccc} * \times_Y \mathrm{Path}(X) \times_X * & \longrightarrow & \mathrm{Path}(X) \\ \downarrow & & \downarrow \in W \cap F \\ \Omega Y & \longrightarrow & \mathrm{Path}(Y) \times_Y X \\ \downarrow & (\mathrm{pb}) & \downarrow \\ * & \longrightarrow & Y \times X \end{array} \,.$$

By the pasting law and the pullback stability of acyclic fibrations, this will prove the claim.

To see that the bottom square here is indeed a pullback, check the universal property: A morphism out of any A into $* \underset{Y \times X}{\times} \mathrm{Path}(Y) \times_Y X$ is a morphism $a : A \to \mathrm{Path}(Y)$ and a morphism $b : A \to X$ such that $p_0(a) = *$, $p_1(a) = f(b)$ and $b = *$. Hence it is equivalently just a morphism $a : A \to \mathrm{Path}(Y)$ such that $p_0(a) = *$ and $p_1(a) = *$. This is the defining universal property of $\Omega Y \coloneqq * \underset{Y}{\times} \mathrm{Path}(Y) \underset{Y}{\times} *$.

Now to construct the right vertical morphism in the top square (Quillen 67, page 3.1): Let $\mathrm{Path}(Y)$ be any path space object for Y and let $\mathrm{Path}(X)$ be given by a factorization

$$(\mathrm{id}_X,\, i \circ f,\, \mathrm{id}_X) \,:\, X \stackrel{\in W}{\longrightarrow} \mathrm{Path}(X) \stackrel{\in \mathrm{Fib}}{\longrightarrow} X \times_Y \mathrm{Path}(Y) \times_Y X$$

and regarded as a path space object of X by further comoposing with

$$(\mathrm{pr}_1, \mathrm{pr}_3) : X \times_Y \mathrm{Path}(Y) \times_Y X \stackrel{\in \mathrm{Fib}}{\longrightarrow} X \times X \,.$$

We need to show that $\mathrm{Path}(X) \to \mathrm{Path}(Y) \times_Y X$ is an acyclic fibration.

It is a fibration because $X \times_Y \mathrm{Path}(Y) \times_Y X \to \mathrm{Path}(Y) \times_Y X$ is a fibration, this being the pullback of the fibration $X \stackrel{f}{\to} Y$.

To see that it is also a weak equivalence, first observe that $\mathrm{Path}(Y) \times_Y X \stackrel{\in W \cap \mathrm{Fib}}{\longrightarrow} X$, this being the pullback of the acyclic fibration of lemma 2.20. Hence we have a factorization of the identity as

$$\mathrm{id}_X \,:\, X \underset{\in W}{\stackrel{i}{\longrightarrow}} \mathrm{Path}(X) \longrightarrow \mathrm{Path}(Y) \times_Y X \underset{\in W \cap \mathrm{Fib}}{\longrightarrow} X$$

and so finally the claim follows by two-out-of-three (def. 2.1). ∎

Remark 4.34. There is a conceptual way to understand prop. 4.33 as follows: If we draw double arrows to indicate homotopies, then a homotopy fiber (def. 4.16) is depicted by the following filled square:

$$\begin{array}{ccc} \mathrm{hofib}(f) & \longrightarrow & * \\ \downarrow & \swArrow & \downarrow \\ X & \underset{f}{\longrightarrow} & Y \end{array}$$

just like the ordinary fiber (example 3.27) is given by a plain square

$$\begin{array}{ccc} \mathrm{fib}(f) & \longrightarrow & * \\ \downarrow & & \downarrow \\ X & \underset{f}{\longrightarrow} & Y \end{array} \,.$$

One may show that just like the fiber is the *universal* solution to making such a commuting square (a pullback limit cone def. 1.1), so the homotopy fiber is the universal solution up to homotopy to make such a commuting square up to homotopy – a homotopy pullback homotopy limit cone.

Now just like ordinary pullbacks satisfy the pasting law saying that attaching two pullback squares gives a pullback rectangle, the analogue is true for homotopy pullbacks. This implies that if we take the homotopy fiber of a homotopy fiber, thereby producing this double homotopy pullback square

$$\begin{array}{ccccc} \mathrm{hofib}(g) & \longrightarrow & \mathrm{hofib}(f) & \longrightarrow & * \\ \downarrow & \swArrow & \downarrow^{g} & \swArrow & \downarrow \\ * & \longrightarrow & X & \underset{f}{\longrightarrow} & Y \end{array}$$

then the total outer rectangle here is itself a homotopy pullback. But the outer rectangle exhibits the homotopy fiber of the point inclusion, which, via def. 4.4 and lemma 4.9, is the loop space object:

$$\begin{array}{ccc} \Omega Y & \longrightarrow & * \\ \downarrow & \swArrow & \downarrow \\ * & \longrightarrow & Y \end{array} \,.$$

Proposition 4.35. *Let $\mathcal{C}$ be a model category and let $f : X \to Y$ be morphism in the pointed homotopy category $\mathrm{Ho}(\mathcal{C}^{*/})$ (prop. 3.29). Then:*

1. *There is a long sequence to the left in $\mathcal{C}^{*/}$ of the form*

$$\cdots \longrightarrow \Omega X \overset{\overline{\Omega} f}{\longrightarrow} \Omega Y \longrightarrow \mathrm{hofib}(f) \longrightarrow X \overset{f}{\longrightarrow} Y \,,$$

where each morphism is the homotopy fiber (def. 4.16) of the following one: the ***homotopy fiber sequence*** *of f. Here $\overline{\Omega} f$ denotes Ωf followed by forming inverses with respect to the group structure on $\Omega(-)$ from prop. 4.24.*

Moreover, for $A \in \mathcal{C}^{*/}$ any object, then there is a long exact sequence

$$\cdots \to [A, \Omega^2 Y]_* \longrightarrow [A, \Omega\, \mathrm{hofib}(f)]_* \longrightarrow [A, \Omega X]_* \longrightarrow [A, \Omega Y] \longrightarrow [A, \mathrm{hofib}(f)]_* \longrightarrow [A, X]_* \longrightarrow [A, Y]_*$$

of pointed sets, where $[-,-]_*$ denotes the pointed set valued hom-functor of example 3.30.

1. *Dually, there is a long sequence to the right in $\mathcal{C}^{*/}$ of the form*

$$X \xrightarrow{f} Y \longrightarrow \mathrm{hocofib}(f) \longrightarrow \Sigma X \xrightarrow{\overline{\Sigma} f} \Sigma Y \to \cdots ,$$

where each morphism is the homotopy cofiber (def. 4.16) of the previous one: the ***homotopy cofiber sequence*** *of f. Moreover, for $A \in \mathcal{C}^{*/}$ any object, then there is a long exact sequence*

$$\cdots \to [\Sigma^2 X, A]_* \longrightarrow [\Sigma\, \mathrm{hocofib}(f), A]_* \longrightarrow [\Sigma Y, A]_* \longrightarrow [\Sigma X, A] \longrightarrow [\mathrm{hocofib}(f), A]_* \longrightarrow [Y, A]_* \longrightarrow [X, A]_*$$

of pointed sets, where $[-,-]_*$ denotes the pointed set valued hom-functor of example 3.30.

(Quillen 67, I.3, prop. 4)

Proof. That there are long sequences of this form is the result of combining prop. 4.33 and prop. 4.22.

It only remains to see that it is indeed the morphisms $\overline{\Omega} f$ that appear, as indicated.

In order to see this, it is convenient to adopt the following notation: for $f : X \to Y$ a morphism, then we denote the collection of generalized element of its homotopy fiber as

$$\mathrm{hofib}(f) = \left\{ (x, f(x) \overset{\gamma_1}{\rightsquigarrow} *) \right\}$$

indicating that these elements are pairs consisting of an element x of X and a “path” (an element of the given path space object) from $f(x)$ to the basepoint.

This way the canonical map $\mathrm{hofib}(f) \to X$ is $(x, f(x) \rightsquigarrow *) \mapsto x$. Hence in this notation the homotopy fiber of the homotopy fiber reads

$$\mathrm{hofib}(\mathrm{hofib}(f)) = \left\{ ((x, f(x) \overset{\gamma_1}{\rightsquigarrow} *), x \overset{\gamma_2}{\rightsquigarrow} *) \right\} .$$

This identifies with ΩY by forming the loops

$$\gamma_1 \cdot f(\overline{\gamma_2}) ,$$

where the overline denotes reversal and the dot denotes concatenation.

Then consider the next homotopy fiber

$$\mathrm{hofib}(\mathrm{hofib}(\mathrm{hofib}(f))) = \left\{ \left(((x, f(x) \overset{\gamma_1}{\rightsquigarrow} *), x \overset{\gamma_2}{\rightsquigarrow} *), \left(\begin{array}{ccc} x & \overset{\gamma_3}{\rightsquigarrow} & * \\ f(x) & \overset{f(\gamma_3)}{\rightsquigarrow} & * \\ {}^{\gamma_1}\searrow & \Rightarrow & \swarrow \\ & * & \end{array} \right) \right) \right\} ,$$

where on the right we have a path in $\mathrm{hofib}(f)$ from $(x, f(x) \overset{\gamma_1}{\rightsquigarrow} *)$ to the basepoint element. This is a path γ_3 together with a path-of-paths which connects f_1 to $f(\gamma_3)$.

By the above convention this is identified with the loop in X which is

$$\gamma_2 \cdot (\overline{\gamma}_3) .$$

But the map to $\mathrm{hofib}(\mathrm{hofib}(f))$ sends this data to $((x, f(x) \overset{\gamma_1}{\rightsquigarrow} *), x \overset{\gamma_2}{\rightsquigarrow} *)$, hence to the loop

$$
\begin{aligned}
\gamma_1 \cdot f(\overline{\gamma_2}) &\simeq f(\gamma_3) \cdot f(\overline{\gamma_2}) \\
&= f(\gamma_3 \cdot \overline{\gamma_2}) \\
&= f(\overline{\gamma_2 \cdot \overline{\gamma}_3}) \\
&= \overline{f(\gamma_2 \cdot \overline{\gamma}_3)}
\end{aligned}
,
$$

hence to the reveral of the image under f of the loop in X. ▮

Remark 4.36. In (Quillen 67, I.3, prop. 3, prop. 4) more is shown than stated in prop. 4.35: there the connecting homomorphism $\Omega Y \to \mathrm{hofib}(f)$ is not just shown to exist, but is described in detail via an action of ΩY on $\mathrm{hofib}(f)$ in $\mathrm{Ho}(\mathcal{C})$. This takes a good bit more work. For our purposes here, however, it is sufficient to know that such a morphism exists at all, hence that $\Omega Y \simeq \mathrm{hofib}(\mathrm{hofib}(f))$.

Example 4.37. Let $\mathcal{C} = (\mathrm{Top}_{\mathrm{cg}})_{\mathrm{Quillen}}$ be the classical model structure on topological spaces (compactly generated) from theorem 3.7, theorem 3.51. Then using the standard pointed topological path space objects $\mathrm{Maps}(I_+, X)$ from def. 1.34 and example 3.26 as the abstract path space objects in def. 2.18, via prop. 3.14, this gives that

$$
[*, \Omega^n X] \simeq \pi_n(X)
$$

is the nth homotopy group, def. 1.26, of X at its basepoint.

Hence using $A = *$ in the first item of prop. 4.35, the long exact sequence this gives is of the form

$$
\cdots \to \pi_3(X) \xrightarrow{f_*} \pi_3(Y) \longrightarrow \pi_2(\mathrm{hofib}(f)) \longrightarrow \pi_2(X) \xrightarrow{-f_*} \pi_2(Y) \longrightarrow \pi_1(\mathrm{hofib}(f)) \longrightarrow \pi_1(X) \xrightarrow{f_*} \pi_1(Y) \longrightarrow * \;.
$$

This is called the **long exact sequence of homotopy groups** induced by f.

Remark 4.38. As we pass to stable homotopy theory (in Part 1)), the long exact sequences in example 4.37 become long not just to the left, but also to the right. Given then a tower of fibrations, there is an induced sequence of such long exact sequences of homotopy groups, which organizes into an *exact couple*. For more on this see at *Interlude -- Spectral sequences* (this remark).

Example 4.39. Let again $\mathcal{C} = (\mathrm{Top}_{\mathrm{cg}})_{\mathrm{Quillen}}$ be the classical model structure on topological spaces (compactly generated) from theorem 3.7, theorem 3.51, as in example 4.37. For $E \in \mathrm{Top}_{\mathrm{cg}}^{*/}$ any pointed topological space and $i : A \hookrightarrow X$ an inclusion of pointed topological spaces, the exactness of the sequence in the second item of prop. 4.35

$$
\cdots \to [\mathrm{hocofib}(i), E] \longrightarrow [X, E]_* \longrightarrow [A, E]_* \to \cdots
$$

gives that the functor

$$
[-, E]_* \;:\; (\mathrm{Top}_{\mathrm{CW}}^{*/})^{\mathrm{op}} \longrightarrow \mathrm{Set}^{*/}
$$

behaves like one degree in an additive reduced cohomology theory (def.). The Brown representability theorem (thm.) implies that all additive reduced cohomology theories are degreewise representable this way (prop.).

5. The suspension/looping adjunction

We conclude this discussion of classical homotopy theory with the key statement that leads over to stable homotopy theory in *Introduction to Stable homotopy theory -- 1*: the suspension and looping adjunction on the classical pointed homotopy category.

Proposition 5.1. *The canonical loop space functor Ω and reduced suspension functor Σ from* *prop. 4.24 on the classical pointed homotopy category from def. 3.31 are adjoint functors, with Σ left adjoint and Ω right adjoint:*

$$(\Sigma \dashv \Omega) \;:\; \mathrm{Ho}(\mathrm{Top}^{*/}) \underset{\Omega}{\overset{\Sigma}{\leftrightarrows}} \mathrm{Ho}(\mathrm{Top}^{*/}) \,.$$

Moreover, this is equivalently the adjoint pair of derived functors, according to prop. 2.49, of the Quillen adjunction

$$(\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}} \underset{\mathrm{Maps}(S^1,-)_*}{\overset{S^1 \wedge (-)}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} (\mathrm{Top}_{\mathrm{cg}}^{*/})_{\mathrm{Quillen}}$$

of cor. 3.42.

Proof. By prop. 4.24 we may represent Σ and Ω by any choice of cylinder objects and path space objects (def. 2.18).

The standard topological path space $(-)^I$ is generally a path space object by prop. 3.14. With prop. 4.5 this shows that

$$\Omega \simeq \mathbb{R}\,\mathrm{Maps}(S^1, -)_* \,.$$

Moreover, by the existence of CW-approximations (remark 3.12) we may represent each object in the homotopy category by a CW-complex. On such, the standard topological cylinder $(-) \times I$ is a cylinder object by prop. 3.13. With prop. 4.5 this shows that

$$\Sigma \simeq \mathbb{L}(S^1 \wedge (-)) \,.$$

▮

Final remark 5.2. What is called *stable homotopy theory* is the result of universally forcing the $(\Sigma \dashv \Omega)$-adjunction of prop. 5.1 to become an equivalence of categories.

This is the topic of the next section at *Introduction to Stable homotopy theory -- 1*.

6. References

A concise and yet self-contained re-write of the proof (Quillen 67) of the classical model structure on topological spaces is provided in

- Philip Hirschhorn, *The Quillen model category of topological spaces* (arXiv:1508.01942).

For general model category theory a decent review is in

- William Dwyer, J. Spalinski, *Homotopy theories and model categories* (pdf) in Ioan Mackenzie James (ed.), *Handbook of Algebraic Topology* 1995

The equivalent definition of model categories that we use here is due to

- André Joyal, appendix E of *The theory of quasi-categories and its applications* (pdf)

The two originals are still a good source to turn to:

- Daniel Quillen, *Axiomatic homotopy theory* in *Homotopical algebra*, Lecture Notes in Mathematics, No. 43 43, Berlin (1967)

- Kenneth Brown, *Abstract Homotopy Theory and Generalized Sheaf Cohomology*, Transactions of the American Mathematical Society, Vol. 186 (1973), 419-458 (JSTOR)

For the restriction to the convenient category of compactly generated topological spaces good sources are

- Gaunce Lewis, *Compactly generated spaces* (pdf), appendix A of *The Stable Category and Generalized Thom Spectra* PhD thesis Chicago, 1978

- Neil Strickland, *The category of CGWH spaces*, 2009 (pdf)

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