nLab

Introduction to Stable homotopy theory -- S

S4D2 – Graduate Seminar on Topology

Complex oriented cohomology

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Abstract. *The category of those generalized cohomology theories that are equipped with a universal "complex orientation" happens to unify within it the abstract structure theory of stable homotopy theory with the concrete richness of the differential topology of cobordism theory and of the arithmetic geometry of formal group laws, such as elliptic curves. In the seminar we work through classical results in algebraic topology, organized such as to give in the end a first glimpse of the modern picture of chromatic homotopy theory.*

Accompanying notes.

Main page: *Introduction to Stable homotopy theory*.

Contents

1. Seminar) Complex oriented cohomology

Outline. We start with two classical topics of algebraic topology that first run independently in parallel:

- S.1) Generalized cohomology
- S.2) Cobordism theory

The development of either of these happens to give rise to the concept of *spectra* and via this concept it turns out that both topics are intimately related. The unification of both is our third topic

- S.3) Complex oriented cohomology

Literature. (Kochman 96).

S.1) Generalized cohomology

Idea. The concept that makes algebraic topology be about methods of homological algebra applied to topology is that of generalized homology and generalized cohomology: these are covariant functors or contravariant functors, respectively,

$$\mathrm{Spaces} \longrightarrow \mathrm{Ab}^{\mathbb{Z}}$$

from (sufficiently nice) topological spaces to $\mathbb{Z}$-graded abelian groups, such that a few key properties of the homotopy types of topological spaces is preserved as one passes them from Ho(Top) to the much more tractable abelian category Ab.

Literature. (Aguilar-Gitler-Prieto 02, chapters 7,8 and 12, Kochman 96, 3.4, 4.2, Schwede 12, II.6)

Generalized cohomology functors

Idea. A generalized (Eilenberg-Steenrod) cohomology theory is such a contravariant functor which satisfies the key properties exhibited by ordinary cohomology (as computed for instance by singular cohomology), notably homotopy invariance and excision, *except* that its value on the point is not required to be concentrated in degree 0. Dually for generalized homology. There are two versions of the axioms, one for reduced cohomology, and they are equivalent if properly set up.

An important example of a generalised cohomology theory other than ordinary cohomology is topological K-theory. The other two examples of key relevance below are cobordism cohomology and stable cohomotopy.

Literature. (Switzer 75, section 7, Aguilar-Gitler-Prieto 02, section 12 and section 9, Kochman 96, 3.4).

Reduced cohomology

The traditional formulation of reduced generalized cohomology in terms of point-set topology is this:

Definition 1.1. A **reduced cohomology theory** is

1. a functor

$$\tilde E^\bullet \;:\; (\mathrm{Top}_{\mathrm{CW}}^{*/})^{\mathrm{op}} \longrightarrow \mathrm{Ab}^{\mathbb{Z}}$$

   from the opposite of pointed topological spaces (CW-complexes) to $\mathbb{Z}$-graded abelian groups ("cohomology groups"), in components

$$\tilde E \;:\; (X \overset{f}{\longrightarrow} Y) \mapsto (\tilde E^\bullet(Y) \overset{f^*}{\longrightarrow} \tilde E^\bullet(X))\,,$$

2. equipped with a natural isomorphism of degree +1, to be called the **suspension isomorphism**, of the form

$$\sigma_E \;:\; \tilde E^\bullet(-) \overset{\simeq}{\longrightarrow} \tilde E^{\bullet+1}(\Sigma -)$$

such that:

1. **(homotopy invariance)** If $f_1, f_2 : X \longrightarrow Y$ are two morphisms of pointed topological spaces such that there is a (base point preserving) homotopy $f_1 \simeq f_2$ between them, then the induced homomorphisms of abelian groups are equal

$$f_1^* = f_2^*\,.$$

2. **(exactness)** For $i : A \hookrightarrow X$ an inclusion of pointed topological spaces, with $j : X \longrightarrow \mathrm{Cone}(i)$ the induced mapping cone (def.), then this gives an exact sequence of graded abelian groups

$$\tilde E^\bullet(\mathrm{Cone}(i)) \overset{j^*}{\longrightarrow} \tilde E^\bullet(X) \overset{i^*}{\longrightarrow} \tilde E^\bullet(A)\,.$$

(e.g. AGP 02, def. 12.1.4)

This is equivalent (prop. 1.4 below) to the following more succinct homotopy-theoretic definition:

Definition 1.2. A **reduced generalized cohomology theory** is a functor

$$\tilde E^\bullet \;:\; \mathrm{Ho}(\mathrm{Top}^{*/})^{\mathrm{op}} \longrightarrow \mathrm{Ab}^{\mathbb{Z}}$$

from the opposite of the pointed classical homotopy category (def., def.), to $\mathbb{Z}$-graded abelian groups, and equipped with natural isomorphisms, to be called the **suspension isomorphism** of the form

$$\sigma \;:\; \tilde E^{\bullet+1}(\Sigma -) \overset{\simeq}{\longrightarrow} \tilde E^\bullet(-)$$

such that:

- **(exactness)** it takes homotopy cofiber sequences in $\mathrm{Ho}(\mathrm{Top}^{*/})$ (def.) to exact sequences.

As a consequence (prop. 1.4 below), we find yet another equivalent definition:

Definition 1.3. A **reduced generalized cohomology theory** is a functor

$$\tilde E^\bullet : (\mathrm{Top}^{*/})^{op} \longrightarrow \mathrm{Ab}^{\mathbb{Z}}$$

from the opposite of the category of pointed topological spaces to $\mathbb{Z}$-graded abelian groups, such that

- **(WHE)** it takes weak homotopy equivalences to isomorphisms

and equipped with natural isomorphism, to be called the **suspension isomorphism** of the form

$$\sigma : \tilde E^{\bullet+1}(\Sigma -) \overset{\simeq}{\Rightarrow} \tilde E^\bullet(-)$$

such that

- **(exactness)** it takes homotopy cofiber sequences in $\mathrm{Ho}(\mathrm{Top}^{*/})$ (def.), to exact sequences.

Proposition 1.4. *The three definitions*

- *def. 1.1*
- *def. 1.2*
- *def. 1.3*

are indeed equivalent.

Proof. Regarding the equivalence of def. 1.1 with def. 1.2:

By the existence of the classical model structure on topological spaces (thm.), the characterization of its homotopy category (cor.) and the existence of CW-approximations, the homotopy invariance axiom in def. 1.1 is equivalent to the functor passing to the classical pointed homotopy category. In view of this and since on CW-complexes the standard topological mapping cone construction is a model for the homotopy cofiber (prop.), this gives the equivalence of the two versions of the exactness axiom.

Regarding the equivalence of def. 1.2 with def. 1.3:

This is the universal property of the classical homotopy category (thm.) which identifies it with the localization (def.) of $\mathrm{Top}^{*/}$ at the weak homotopy equivalences (thm.), together with the existence of CW approximations (rmk.): jointly this says that, up to natural isomorphism, there is a bijection between functors F and $\tilde F$ in the following diagram (which is filled by a natural isomorphism itself):

$$\begin{array}{ccc} \mathrm{Top}^{op} & \overset{F}{\longrightarrow} & \mathrm{Ab}^{\mathbb{Z}} \\ {}^{\gamma_{\mathrm{Top}}}\downarrow & \nearrow_{\tilde F} & \\ \mathrm{Ho}(\mathrm{Top})^{op} \simeq (\mathrm{Top}_{CW})/_\sim & & \end{array}$$

where F sends weak homotopy equivalences to isomorphisms and where $(-)_\sim$ means identifying homotopic maps. ▮

Prop. 1.4 naturally suggests (e.g. Lurie 10, section 1.4) that the concept of generalized cohomology be formulated in the generality of any abstract homotopy theory (model category), not necessarily that of (pointed) topological spaces:

Definition 1.5. Let $\mathcal{C}$ be a model category (def.) with $\mathcal{C}^{*/}$ its pointed model category (prop.).

A **reduced additive generalized cohomology theory** on $\mathcal{C}$ is

1. a functor

$$\tilde E^\bullet : \mathrm{Ho}(\mathcal{C}^{*/})^{op} \longrightarrow \mathrm{Ab}^{\mathbb{Z}}$$

2. a natural isomorphism ("suspension isomorphisms") of degree +1

$$\sigma : \tilde E^\bullet \longrightarrow \tilde E^{\bullet+1}\circ\Sigma$$

such that

- **(exactness)** $\tilde E^\bullet$ takes homotopy cofiber sequences to exact sequences.

Finally we need the following terminology:

Definition 1.6. Let $\tilde E^\bullet$ be a reduced cohomology theory according to either of def. 1.1, def. 1.2, def. 1.3 or def. 1.5.

We say $\tilde E^\bullet$ is **additive** if in addition

- **(wedge axiom)** For $\{X_i\}_{i \in I}$ any set of pointed CW-complexes, then the canonical morphism

$$\tilde E^\bullet(\vee_{i \in I} X_i) \longrightarrow \prod_{i \in I} \tilde E^\bullet(X_i)$$

  from the functor applied to their wedge sum (def.), to the product of its values on the wedge summands, is an isomorphism.

We say $\tilde E^\bullet$ is **ordinary** if its value on the 0-sphere S^0 is concentrated in degree 0:

- **(Dimension)** $\tilde E^{\bullet \neq 0}(\mathbb{S}^0) \simeq 0$.

If $\tilde E^\bullet$ is not ordinary, one also says that it is **generalized** or **extraordinary**.

A **homomorphism of reduced cohomology theories**

$$\eta : \tilde E^\bullet \longrightarrow \tilde F^\bullet$$

is a natural transformation between the underlying functors which is compatible with the suspension isomorphisms in that all the following squares commute

$$\begin{array}{ccc} \tilde E^\bullet(X) & \overset{\eta_X}{\longrightarrow} & \tilde F^\bullet(X) \\ {}^{\mathllap{\sigma_E}}\downarrow && \downarrow^{\mathrlap{\sigma_F}} \\ \tilde E^{\bullet+1}(\Sigma X) & \underset{\eta_{\Sigma X}}{\longrightarrow} & \tilde F^{\bullet+1}(\Sigma X) \end{array} \,.$$

We now discuss some constructions and consequences implied by the concept of reduced cohomology theories:

Definition 1.7. Given a generalized cohomology theory $(E^\bullet, \delta)$ on some $\mathcal{C}$ as in def. 1.5, and given a homotopy cofiber sequence in $\mathcal{C}$ (prop.),

$$X \overset{f}{\longrightarrow} Y \overset{g}{\longrightarrow} Z \overset{\mathrm{coker}(g)}{\longrightarrow} \Sigma X \,,$$

then the corresponding **connecting homomorphism** is the composite

$$\partial \;:\; E^\bullet(X) \overset{\sigma}{\longrightarrow} E^{\bullet+1}(\Sigma X) \overset{\mathrm{coker}(g)^\ast}{\longrightarrow} E^{\bullet+1}(Z) \,.$$

Proposition 1.8. *The connecting homomorphisms of def. 1.7 are parts of long exact sequences*

$$\cdots \overset{\partial}{\longrightarrow} E^\bullet(Z) \longrightarrow E^\bullet(Y) \longrightarrow E^\bullet(X) \overset{\partial}{\longrightarrow} E^{\bullet+1}(Z) \to \cdots \,.$$

Proof. By the defining exactness of $E^\bullet$, def. 1.5, and the way this appears in def. 1.7, using that σ is by definition an isomorphism. ▮

Unreduced cohomology

Given a reduced generalized cohomology theory as in def. 1.1, we may “un-reduce” it and evaluate it on unpointed topological spaces X simply by evaluating it on X_+ (def.). It is conventional to further generalize to relative cohomology and evaluate on unpointed subspace inclusions $i : A \hookrightarrow X$, taken as placeholders for their mapping cones $\mathrm{Cone}(i_+)$ (prop.).

In the following a *pair* (X,U) refers to a subspace inclusion of topological spaces $U \hookrightarrow X$. Whenever only one space is mentioned, the subspace is assumed to be the empty set $(X, \emptyset)$. Write $\mathrm{Top}_{\mathrm{CW}}^{\hookrightarrow}$ for the category of such pairs (the full subcategory of the arrow category of $\mathrm{Top}_{\mathrm{CW}}$ on the inclusions). We identify $\mathrm{Top}_{\mathrm{CW}} \hookrightarrow \mathrm{Top}_{\mathrm{CW}}^{\hookrightarrow}$ by $X \mapsto (X,\emptyset)$.

Definition 1.9. A *cohomology theory* (unreduced, relative) is

1. a functor

$$E^\bullet \colon (\mathrm{Top}_{\mathrm{CW}}^{\hookrightarrow})^{\mathrm{op}} \to \mathrm{Ab}^{\mathbb{Z}}$$

to the category of $\mathbb{Z}$-graded abelian groups,

2. a natural transformation of degree +1, to be called the **connecting homomorphism**, of the form

$$\delta_{(X,A)} \; : \; E^\bullet(A, \emptyset) \to E^{\bullet+1}(X,A) \; .$$

such that:

1. **(homotopy invariance)** For $f \colon (X_1, A_1) \to (X_2, A_2)$ a homotopy equivalence of pairs, then

$$E^\bullet(f) \; : \; E^\bullet(X_2, A_2) \overset{\simeq}{\longrightarrow} E^\bullet(X_1, A_1)$$

is an isomorphism;

2. **(exactness)** For $A \hookrightarrow X$ the induced sequence

$$\cdots \to E^n(X,A) \longrightarrow E^n(X) \longrightarrow E^n(A) \overset{\delta}{\longrightarrow} E^{n+1}(X,A) \to \cdots$$

is a long exact sequence of abelian groups.

3. **(excision)** For $U \hookrightarrow A \hookrightarrow X$ such that $\overline{U} \subset \mathrm{Int}(A)$, then the natural inclusion of the pair $i \colon (X - U, A - U) \hookrightarrow (X, A)$ induces an isomorphism

$$E^\bullet(i) \; : \; E^n(X,A) \overset{\simeq}{\longrightarrow} E^n(X-U, A-U)$$

We say $E^\bullet$ is **additive** if it takes coproducts to products:

- **(additivity)** If $(X,A) = \coprod_i (X_i, A_i)$ is a coproduct, then the canonical comparison morphism

$$E^n(X,A) \overset{\simeq}{\longrightarrow} \prod_i E^n(X_i, A_i)$$

is an isomorphism from the value on (X,A) to the product of values on the summands.

We say $E^\bullet$ is **ordinary** if its value on the point is concentrated in degree 0

- **(Dimension)**: $E^{\bullet \neq 0}(*, \emptyset) = 0$.

A homomorphism of unreduced cohomology theories

$$\eta \; : \; E^\bullet \longrightarrow F^\bullet$$

is a natural transformation of the underlying functors that is compatible with the connecting homomorphisms, hence such that all these squares commute:

$$\begin{array}{ccc} E^\bullet(A,\emptyset) & \overset{\eta_{(A,\emptyset)}}{\longrightarrow} & F^\bullet(A,\emptyset) \\ {}^{\mathllap{\delta_E}}\downarrow && \downarrow^{\mathrlap{\delta_F}} \\ E^{\bullet+1}(X,A) & \underset{}{\overset{\eta_{(X,A)}}{\longrightarrow}} & F^{\bullet+1}(X,A) \end{array} \; .$$

e.g. (AGP 02, def. 12.1.1).

Lemma 1.10. The excision axiom in def. 1.9 is equivalent to the following statement:

For all $A, B \hookrightarrow X$ with $X = \mathrm{Int}(A) \cup \mathrm{Int}(B)$, then the inclusion

$$i \colon (A, A \cap B) \longrightarrow (X, B)$$

induces an isomorphism,

$$i^\ast \; : \; E^\bullet(X,B) \overset{\simeq}{\longrightarrow} E^\bullet(A, A\cap B)$$

(e.g Switzer 75, 7.2)

Proof. In one direction, suppose that $E^\bullet$ satisfies the original excision axiom. Given A, B with $X = \mathrm{Int}\,(A) \cup \mathrm{Int}(B)$, set $U \coloneqq X - A$ and observe that

$$
\begin{aligned}
\overline{U} &= \overline{X - A} \\
&= X - \mathrm{Int}(A) \\
&\subset \mathrm{Int}(B)
\end{aligned}
$$

and that

$$(X - U, B - U) = (A, A \cap B) \; .$$

Hence the excision axiom implies $E^\bullet(X,B) \xrightarrow{\simeq} E^\bullet(A, A \cap B)$.

Conversely, suppose $E^\bullet$ satisfies the alternative condition. Given $U \hookrightarrow A \hookrightarrow X$ with $\overline{U} \subset \mathrm{Int}(A)$, observe that we have a cover

$$
\begin{aligned}
\mathrm{Int}(X - U) \cup \mathrm{Int}(A) &= (X - \overline{U}) \cap \mathrm{Int}\,(A) \\
&\supset (X - \mathrm{Int}(A)) \cap \mathrm{Int}(A) \\
&= X
\end{aligned}
$$

and that

$$(X - U, (X - U) \cap A) = (X - U, A - U) \; .$$

Hence

$$E^\bullet(X - U, A - U) \simeq E^\bullet(X - U, (X - U) \cap A) \simeq E^\bullet(X, A) \; .$$

∎

The following lemma shows that the dependence in pairs of spaces in a generalized cohomology theory is really a stand-in for evaluation on homotopy cofibers of inclusions.

Lemma 1.11. *Let $E^\bullet$ be an cohomology theory, def. 1.9, and let $A \hookrightarrow X$. Then there is an isomorphism*

$$E^\bullet(X, A) \xrightarrow{\simeq} E^\bullet(X \cup \mathrm{Cone}(A), *)$$

between the value of $E^\bullet$ on the pair (X, A) and its value on the unreduced mapping cone of the inclusion (rmk.), relative to a basepoint.

If moreover $A \hookrightarrow X$ is (the retract of) a relative cell complex inclusion, then also the morphism in cohomology induced from the quotient map $p \,:\, (X, A) \longrightarrow (X/A, *)$ is an isomorphism:

$$E^\bullet(p) \,:\, E^\bullet(X/A, *) \longrightarrow E^\bullet(X, A) \; .$$

(e.g AGP 02, corollary 12.1.10)

Proof. Consider $U \coloneqq (\mathrm{Cone}(A) - A \times \{0\}) \hookrightarrow \mathrm{Cone}(A)$, the cone on A minus the base A. We have

$$(X \cup \mathrm{Cone}(A) - U, \mathrm{Cone}(A) - U) \simeq (X, A)$$

and hence the first isomorphism in the statement is given by the excision axiom followed by homotopy invariance (along the contraction of the cone to the point).

Next consider the quotient of the mapping cone of the inclusion:

$$(X \cup \mathrm{Cone}(A), \mathrm{Cone}(A)) \longrightarrow (X/A, *) \; .$$

If $A \hookrightarrow X$ is a cofibration, then this is a homotopy equivalence since $\mathrm{Cone}(A)$ is contractible and since by the dual factorization lemma (lem.) and by the invariance of homotopy fibers under weak equivalences (lem.), $X \cup \mathrm{Cone}(A) \to X/A$ is a weak homotopy equivalence, hence, by the universal property of the classical homotopy category (thm.) a homotopy equivalence on CW-complexes.

Hence now we get a composite isomorphism

$$E^\bullet(X/A, *) \xrightarrow{\simeq} E^\bullet(X \cup \mathrm{Cone}(A), \mathrm{Cone}(A)) \xrightarrow{\simeq} E^\bullet(X, A) \; .$$

∎

Example 1.12. As an important special case of : Let (X, x) be a pointed CW-complex. For $p : (\mathrm{Cone}(X), X) \to (\Sigma X, \{x\})$ the quotient map from the reduced cone on X to the reduced suspension, then

$$E^\bullet(p) \,:\, E^\bullet(\mathrm{Cone}(X), X) \xrightarrow{\simeq} E^\bullet(\Sigma X, \{x\})$$

is an isomorphism.

Proposition 1.13. ***(exact sequence of a triple)***

For $E^\bullet$ an unreduced generalized cohomology theory, def. 1.9, then every inclusion of two consecutive subspaces

$$Z \hookrightarrow Y \hookrightarrow X$$

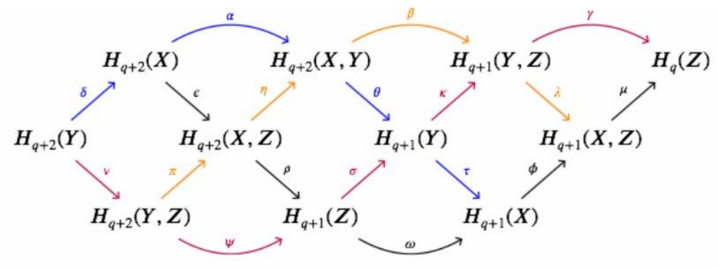
induces a long exact sequence of cohomology groups of the form

$$\cdots \to E^{q-1}(Y,Z) \overset{\bar\delta}{\longrightarrow} E^q(X,Y) \longrightarrow E^q(X,Z) \longrightarrow E^q(Y,Z) \to \cdots$$

where

$$\bar\delta \,:\, E^{q-1}(Y,Z) \longrightarrow E^{q-1}(Y) \overset{\delta}{\longrightarrow} E^q(X,Y) \,.$$

Proof. Apply the braid lemma to the interlocking long exact sequences of the three pairs (X,Y), (X,Z), (Y,Z):



(graphics from this Maths.SE comment, showing the dual situation for homology)

See here for details. ▮

Remark 1.14. The exact sequence of a triple in prop. 1.13 is what gives rise to the Cartan-Eilenberg spectral sequence for E-cohomology of a CW-complex X.

Example 1.15. For (X,x) a pointed topological space and $\mathrm{Cone}(X) = (X \wedge (I_+))/X$ its reduced cone, the long exact sequence of the triple $(\{x\}, X, \mathrm{Cone}(X))$, prop. 1.13,

$$0 \simeq E^q(\mathrm{Cone}(X),\{x\}) \longrightarrow E^q(X,\{x\}) \overset{\bar\delta}{\longrightarrow} E^{q+1}(\mathrm{Cone}(X),X) \longrightarrow E^{q+1}(\mathrm{Cone}(X),\{x\}) \simeq 0$$

exhibits the connecting homomorphism $\bar\delta$ here as an isomorphism

$$\bar\delta \,:\, E^q(X,\{x\}) \overset{\simeq}{\longrightarrow} E^{q+1}(\mathrm{Cone}(X),X) \,.$$

This is the *suspension isomorphism* extracted from the unreduced cohomology theory, see def. 1.17 below.

Proposition 1.16. ***(Mayer-Vietoris sequence)***

Given $E^\bullet$ an unreduced cohomology theory, def. 1.9. Given a topological space covered by the interior of two spaces as $X = \mathrm{Int}(A) \cup \mathrm{Int}(B)$, then for each $C \subset A \cap B$ there is a long exact sequence of cohomology groups of the form

$$\cdots \to E^{n-1}(A\cap B, C) \overset{\bar\delta}{\longrightarrow} E^n(X,C) \longrightarrow E^n(A,C) \oplus E^n(B,C) \longrightarrow E^n(A \cap B, C) \to \cdots \,.$$

e.g. (Switzer 75, theorem 7.19, Aguilar-Gitler-Prieto 02, theorem 12.1.22)

Relation between unreduced and reduced cohomology

Definition 1.17. **(unreduced to reduced cohomology)**

Let $E^\bullet$ be an unreduced cohomology theory, def. 1.9. Define a reduced cohomology theory, def. 1.1 $(\tilde E^\bullet, \sigma)$ as follows.

For $x \colon * \to X$ a pointed topological space, set

$$\tilde E^\bullet(X,x) \coloneqq E^\bullet(X,\{x\}) \,.$$

This is clearly functorial. Take the suspension isomorphism to be the composite

$$\sigma : \tilde{E}^{\bullet+1}(\Sigma X) = E^{\bullet+1}(\Sigma X, \{x\}) \xrightarrow{E^\bullet(p)} E^{\bullet+1}(\mathrm{Cone}(X), X) \xrightarrow{\bar{\delta}^{-1}} E^\bullet(X, \{x\}) = \tilde{E}^\bullet(X)$$

of the isomorphism $E^\bullet(p)$ from example 1.12 and the inverse of the isomorphism $\bar{\delta}$ from example 1.15.

Proposition 1.18. *The construction in def. 1.17 indeed gives a reduced cohomology theory.*

(e.g Switzer 75, 7.34)

Proof. We need to check the exactness axiom given any $A \hookrightarrow X$. By lemma 1.11 we have an isomorphism

$$\tilde{E}^\bullet(X \cup \mathrm{Cone}(A)) = E^\bullet(X \cup \mathrm{Cone}(A), \{*\}) \xrightarrow{\simeq} E^\bullet(X, A) \,.$$

Unwinding the constructions shows that this makes the following diagram commute:

$$\begin{array}{ccc} \tilde{E}^\bullet(X \cup \mathrm{Cone}(A)) & \xrightarrow{\simeq} & E^\bullet(X, A) \\ \downarrow && \downarrow \\ \tilde{E}^\bullet(X) & = & E^\bullet(X, \{x\}) \,, \\ \downarrow && \downarrow \\ \tilde{E}^\bullet(A) & = & E^\bullet(A, \{a\}) \end{array}$$

where the vertical sequence on the right is exact by prop. 1.13. Hence the left vertical sequence is exact. ▮

Definition 1.19. **(reduced to unreduced cohomology)**

Let $(\tilde{E}^\bullet, \sigma)$ be a reduced cohomology theory, def. 1.1. Define an unreduced cohomolog theory $E^\bullet$, def. 1.9, by

$$E^\bullet(X, A) \coloneqq \tilde{E}^\bullet(X_+ \cup \mathrm{Cone}(A_+))$$

and let the connecting homomorphism be as in def. 1.7.

Proposition 1.20. *The construction in def. 1.19 indeed yields an unreduced cohomology theory.*

e.g. (Switzer 75, 7.35)

Proof. Exactness holds by prop. 1.8. For excision, it is sufficient to consider the alternative formulation of lemma 1.10. For CW-inclusions, this follows immediately with lemma 1.11. ▮

Theorem 1.21. *The constructions of def. 1.19 and def. 1.17 constitute a pair of functors between then categories of reduced cohomology theories, def. 1.1 and unreduced cohomology theories, def. 1.9 which exhbit an equivalence of categories.*

Proof. (…careful with checking the respect for suspension iso and connecting homomorphism..)

To see that there are natural isomorphisms relating the two composites of these two functors to the identity:

One composite is

$$\begin{aligned} E^\bullet &\mapsto (\tilde{E}^\bullet : (X, x) \mapsto E^\bullet(X, \{x\})) \\ &\mapsto ((E')^\bullet : (X, A) \mapsto E^\bullet(X_+ \cup \mathrm{Cone}(A_+)), *) \end{aligned} \,,$$

where on the right we have, from the construction, the reduced mapping cone of the original inclusion $A \hookrightarrow X$ with a base point adjoined. That however is isomorphic to the unreduced mapping cone of the original inclusion (prop.- P#UnreducedMappingConeAsReducedConeOfBasedPointAdjoined)). With this the natural isomorphism is given by lemma 1.11.

The other composite is

$$\begin{aligned} \tilde{E}^\bullet &\mapsto (E^\bullet : (X, A) \mapsto \tilde{E}^\bullet(X_+ \cup \mathrm{Cone}(A_+))) \\ &\mapsto ((\tilde{E}')^\bullet : X \mapsto \tilde{E}^\bullet(X_+ \cup \mathrm{Cone}(*_+))) \end{aligned}$$

where on the right we have the reduced mapping cone of the point inclusion with a point adoined. As before, this is isomorphic to the unreduced mapping cone of the point inclusion. That finally is clearly homotopy equivalent to X, and so now the natural isomorphism follows with homotopy invariance. ▮

Finally we record the following basic relation between reduced and unreduced cohomology:

Proposition 1.22. *Let $E^\bullet$ be an unreduced cohomology theory, and $\tilde{E}^\bullet$ its reduced cohomology theory from* *def. 1.17. For $(X, *)$ a pointed topological space, then there is an identification*

$$E^\bullet(X) \simeq \tilde{E}^\bullet(X) \oplus E^\bullet(*)$$

of the unreduced cohomology of X with the direct sum of the reduced cohomology of X and the unreduced cohomology of the base point.

Proof. The pair $* \hookrightarrow X$ induces the sequence

$$\cdots \to E^{\bullet-1}(*) \overset{\delta}{\longrightarrow} \tilde{E}^\bullet(X) \longrightarrow E^\bullet(X) \longrightarrow E^\bullet(*) \overset{\delta}{\longrightarrow} \tilde{E}^{\bullet+1}(X) \to \cdots$$

which by the exactness clause in def. 1.9 is exact.

Now since the composite $* \to X \to *$ is the identity, the morphism $E^\bullet(X) \to E^\bullet(*)$ has a section and so is in particular an epimorphism. Therefore, by exactness, the connecting homomorphism vanishes, $\delta = 0$ and we have a short exact sequence

$$0 \to \tilde{E}^\bullet(X) \longrightarrow E^\bullet(X) \longrightarrow E^\bullet(*) \to 0$$

with the right map an epimorphism. Hence this is a split exact sequence and the statement follows. ∎

Generalized homology functors

All of the above has a dual version with generalized cohomology replaced by generalized homology. For ease of reference, we record these dual definitions:

Definition 1.23. A **reduced homology theory** is a functor

$$\tilde{E}_\bullet : (\mathrm{Top}_{\mathrm{CW}}^{*/}) \longrightarrow \mathrm{Ab}^{\mathbb{Z}}$$

from the category of pointed topological spaces (CW-complexes) to $\mathbb{Z}$-graded abelian groups ("homology groups"), in components

$$\tilde{E}_\bullet : (X \overset{f}{\longrightarrow} Y) \mapsto (\tilde{E}_\bullet(X) \overset{f_*}{\longrightarrow} \tilde{E}_\bullet(Y)),$$

and equipped with a natural isomorphism of degree +1, to be called the **suspension isomorphism**, of the form

$$\sigma : \tilde{E}_\bullet(-) \overset{\simeq}{\longrightarrow} \tilde{E}_{\bullet+1}(\Sigma -)$$

such that:

1. **(homotopy invariance)** If $f_1, f_2 : X \longrightarrow Y$ are two morphisms of pointed topological spaces such that there is a (base point preserving) homotopy $f_1 \simeq f_2$ between them, then the induced homomorphisms of abelian groups are equal
$$f_{1*} = f_{2*}\,.$$
2. **(exactness)** For $i : A \hookrightarrow X$ an inclusion of pointed topological spaces, with $j : X \longrightarrow \mathrm{Cone}(i)$ the induced mapping cone, then this gives an exact sequence of graded abelian groups
$$\tilde{E}_\bullet(A) \overset{i_*}{\longrightarrow} \tilde{E}_\bullet(X) \overset{j_*}{\longrightarrow} \tilde{E}_\bullet(\mathrm{Cone}(i))\,.$$

We say $\tilde{E}_\bullet$ is **additive** if in addition

- **(wedge axiom)** For $\{X_i\}_{i \in I}$ any set of pointed CW-complexes, then the canonical morphism
$$\oplus_{i \in I} \tilde{E}_\bullet(X_i) \longrightarrow \tilde{E}^\bullet(\vee_{i \in I} X_i)$$
from the direct sum of the value on the summands to the value on the wedge sum (prop.-P#WedgeSumAsCoproduct)), is an isomorphism.

We say $\tilde{E}_\bullet$ is **ordinary** if its value on the 0-sphere S^0 is concentrated in degree 0:

- **(Dimension)** $\tilde{E}_{\bullet \neq 0}(\mathbb{S}^0) \simeq 0$.

A homomorphism of reduced cohomology theories

$$\eta : \tilde{E}_\bullet \longrightarrow \tilde{F}_\bullet$$

is a natural transformation between the underlying functors which is compatible with the suspension isomorphisms in that all the following squares commute

$$\begin{array}{ccc} \tilde E_\bullet(X) & \overset{\eta_X}{\longrightarrow} & \tilde F_\bullet(X) \\ {}^{\mathllap{\sigma_E}}\downarrow && \downarrow^{\mathrlap{\sigma_F}} \\ \tilde E_{\bullet+1}(\Sigma X) & \overset{\eta_{\Sigma X}}{\longrightarrow} & \tilde F_{\bullet+1}(\Sigma X) \end{array} \,.$$

Definition 1.24. A **homology theory** (unreduced, relative) is a functor

$$E_\bullet : (\mathrm{Top}_{\mathrm{CW}}^{\hookrightarrow}) \longrightarrow \mathrm{Ab}^{\mathbb{Z}}$$

to the category of $\mathbb{Z}$-graded abelian groups, as well as a natural transformation of degree +1, to be called the **connecting homomorphism**, of the form

$$\delta_{(X,A)} \;:\; E_{\bullet+1}(X,A) \longrightarrow E^\bullet(A,\emptyset) \,.$$

such that:

1. **(homotopy invariance)** For $f : (X_1, A_1) \to (X_2, A_2)$ a homotopy equivalence of pairs, then

   $$E_\bullet(f) \;:\; E_\bullet(X_1, A_1) \overset{\simeq}{\longrightarrow} E_\bullet(X_2, A_2)$$

   is an isomorphism;

2. **(exactness)** For $A \hookrightarrow X$ the induced sequence

   $$\cdots \to E_{n+1}(X,A) \overset{\delta}{\longrightarrow} E_n(A) \longrightarrow E_n(X) \longrightarrow E_n(X,A) \to \cdots$$

   is a long exact sequence of abelian groups.

3. **(excision)** For $U \hookrightarrow A \hookrightarrow X$ such that $\overline{U} \subset \mathrm{Int}(A)$, then the natural inclusion of the pair $i : (X - U, A - U) \hookrightarrow (X, A)$ induces an isomorphism

   $$E_\bullet(i) \;:\; E_n(X - U, A - U) \overset{\simeq}{\longrightarrow} E_n(X, A)$$

We say $E^\bullet$ is **additive** if it takes coproducts to direct sums:

- **(additivity)** If $(X,A) = \coprod_i (X_i, A_i)$ is a coproduct, then the canonical comparison morphism

  $$\oplus_i E^n(X_i, A_i) \overset{\simeq}{\longrightarrow} E^n(X,A)$$

  is an isomorphismfrom the direct sum of the value on the summands, to the value on the total pair.

We say $E_\bullet$ is **ordinary** if its value on the point is concentrated in degree 0

- **(Dimension)**: $E_{\bullet \neq 0}(\ast, \emptyset) = 0$.

A homomorphism of unreduced homology theories

$$\eta \;:\; E_\bullet \longrightarrow F_\bullet$$

is a natural transformation of the underlying functors that is compatible with the connecting homomorphisms, hence such that all these squares commute:

$$\begin{array}{ccc} E_{\bullet+1}(X,A) & \overset{\eta_{(X,A)}}{\longrightarrow} & F_{\bullet+1}(X,A) \\ {}^{\mathllap{\delta_E}}\downarrow && \downarrow^{\mathrlap{\delta_F}} \\ E_\bullet(A,\emptyset) & \overset{\eta_{(A,\emptyset)}}{\longrightarrow} & F^\bullet(A,\emptyset) \end{array} \,.$$

Multiplicative cohomology theories

The generalized cohomology theories considered above assign *cohomology groups*. It is familiar from ordinary cohomology with coefficients not just in a group but in a ring, that also the cohomology groups inherit compatible ring structure. The generalization of this phenomenon to generalized cohomology theories is captured by the concept of multiplicative cohomology theories:

Definition 1.25. Let E_1, E_2, E_3 be three unreduced generalized cohomology theories (def.). A **pairing of cohomology theories**

$$\mu : E_1 \Box E_2 \longrightarrow E_3$$

is a natural transformation (of functors on $(\mathrm{Top}_{\mathrm{CW}}^{\hookrightarrow} \times \mathrm{Top}_{\mathrm{CW}}^{\hookrightarrow})^{\mathrm{op}}$) of the form

$$\mu_{n_1,n_2} : E_1^{n_1}(X,A) \otimes E_2^{n_2}(Y,B) \longrightarrow E_3^{n_1+n_2}(X \times Y\,,\; A \times Y \cup X \times B)$$

such that this is compatible with the connecting homomorphisms δ_i of E_i, in that the following are commuting squares

$$\begin{array}{ccc} E_1^{n_1}(A) \otimes E_2^{n_2}(Y,B) & \overset{\delta_1 \otimes \mathrm{id}_2}{\longrightarrow} & E_1^{n_1+1}(X,A) \otimes E_2^{n_2}(Y,B) \\ {}^{\mu_{n_1,n_2}}\downarrow && \downarrow^{\mu_{n_1+1,n_2}} \\ \array{ E_3^{n_1+n_2}(A \times Y, A \times B) \\ \simeq \\ E_3^{n_1+n_2}(A \times Y \cup X \times B, X \times B) } & \overset{\delta_3}{\longrightarrow} & E_3^{n_1+n_2+1}(X \times Y, A \times B) \end{array}$$

and

$$\begin{array}{ccc} E_1^{n_1}(X,A) \otimes E_2^{n_2}(B) & \overset{(-1)^{n_1} \mathrm{id}_1 \otimes \delta_2}{\longrightarrow} & E_1^{n_1+1}(X,A) \otimes E_2^{n_2}(Y,B) \\ {}^{\mu_{n_1,n_2}}\downarrow && \downarrow^{\mu_{n_1,n_2+1}} \\ \array{ E_3^{n_1+n_2}(X \times B, A \times B) \\ \simeq \\ E_3^{n_1+n_2}(A \times Y \cup X \times B, A \times Y) } & \overset{\delta_3}{\longrightarrow} & E_3^{n_1+n_2+1}(X \times Y, A \times B) \end{array} \,,$$

where the isomorphisms in the bottom left are the excision isomorphisms.

Definition 1.26. An (unreduced) **multiplicative cohomology theory** is an unreduced generalized cohomology theory theory E (def. 1.9) equipped with

1. (external multiplication) a pairing (def. 1.25) of the form $\mu : E \Box E \longrightarrow E$;
2. (unit) an element $1 \in E^0(\ast)$

such that

1. (associativity) $\mu \circ (\mathrm{id} \otimes \mu) = \mu \circ (\mu \otimes \mathrm{id})$;
2. (unitality) $\mu(1 \otimes x) = \mu(x \otimes 1) = x$ for all $x \in E^n(X,A)$.

The mulitplicative cohomology theory is called **commutative** (often considered by default) if in addition

- **(graded commutativity)**

$$\begin{array}{ccc} E^{n_1}(X,A) \otimes E^{n_2}(Y,B) & \overset{(u \otimes v) \mapsto (-1)^{n_1 n_2}(v \otimes u)}{\longrightarrow} & E^{n_2}(Y,B) \otimes E^{n_1}_{X,A} \\ {}^{\mu_{n_1,n_2}}\downarrow && \downarrow^{\mu_{n_2,n_1}} \\ E^{n_1+n_2}(X \times Y, A \times Y \cup X \times B) & \underset{(\mathrm{switch}_{(X,A),(Y,B)})^\ast}{\longrightarrow} & E^{n_1+n_2}(Y \times X, B \times X \cup Y \times A) \end{array} \,.$$

Given a multiplicative cohomology theory $(E, \mu, 1)$, its **cup product** is the composite of the above external multiplication with pullback along the diagonal maps $\Delta_{(X,A)} : (X,A) \longrightarrow (X \times X, A \times X \cup X \times A)$;

$$(-) \cup (-) : E^{n_1}(X,A) \otimes E^{n_2}(X,A) \overset{\mu_{n_1,n_2}}{\longrightarrow} E^{n_1+n_2}(X \times X,\; A \times X \cup X \times A) \overset{\Delta_{(X,A)}^\ast}{\longrightarrow} E^{n_1+n_2}(X,\; A \cup B)\,.$$

e.g. (Tamaki-Kono 06, II.6)

Proposition 1.27. *Let $(E,\mu,1)$ be a multiplicative cohomology theory, def. 1.26. Then*

1. *For every space X the cup product gives $E^\bullet(X)$ the structure of a $\mathbb{Z}$-graded ring, which is graded-commutative if $(E,\mu,1)$ is commutative.*
2. *For every pair (X,A) the external multiplication μ gives $E^\bullet(X,A)$ the structure of a left and right module over the graded ring $E^\bullet(\ast)$.*
3. *All pullback morphisms respect the left and right action of $E^\bullet(\ast)$ and the connecting homomorphisms respect the right action and the left action up to multiplication by $(-1)^{n_1}$*

Proof. Regarding the third point:

For pullback maps this is the naturality of the external product: let $f : (X,A) \longrightarrow (Y,B)$ be a morphism in $\mathrm{Top}_{\mathrm{CW}}^{\hookrightarrow}$

then naturality says that the following square commutes:

$$
\begin{array}{ccc}
E^{n_1}(\ast) \otimes E^{n_2}(Y,B) & \overset{\mu_{n_1,n_2}}{\longrightarrow} & E^{n_1+n_2}(Y,B) \\
{}^{(\mathrm{id},f^\ast)}\downarrow && \downarrow^{f^\ast} \\
E^{n_1}(\ast) \otimes E^{n_2}(X,A) & \overset{\mu_{n_1,n_2}}{\longrightarrow} & E^{n_1+n_2}(Y,B)
\end{array}
\,.
$$

For connecting homomorphisms this is the (graded) commutativity of the squares in def. 1.26:

$$
\begin{array}{ccc}
E^{n_1}(\ast) \otimes E^{n_2}(A) & \overset{(-1)^{n_1}(\mathrm{id},\delta)}{\longrightarrow} & E^{n_1}(\ast) \otimes E^{n_2+2}(X) \\
{}^{\mu_{n_1,n_2}}\downarrow && \downarrow^{\mu_{n_1,n_2}} \\
E^{n_1+n_2}(A) & \overset{\delta}{\longrightarrow} & E_3^{n_1+n_2+1}(X,B)
\end{array}
\,.
$$

▮

Brown representability theorem

Idea. Given any functor such as the generalized (co)homology functor above, an important question to ask is whether it is a *representable functor*. Due to the $\mathbb{Z}$-grading and the suspension isomorphisms, if a generalized (co)homology functor is representable at all, it must be represented by a $\mathbb{Z}$-indexed sequence of pointed topological spaces such that the reduced suspension of one is comparable to the next one in the list. This is a *spectrum* or more specifically: a *sequential spectrum* .

Whitehead observed that indeed every spectrum represents a generalized (co)homology theory. The *Brown representability theorem* states that, conversely, every generalized (co)homology theory is represented by a spectrum, subject to conditions of additivity.

As a first application, Eilenberg-MacLane spectra representing ordinary cohomology may be characterized via Brown representability.

Literature. (Switzer 75, section 9, Aguilar-Gitler-Prieto 02, section 12, Kochman 96, 3.4)

Traditional discussion

Write $\mathrm{Top}^{\ast/}_{\geq 1} \hookrightarrow \mathrm{Top}^{\ast/}$ for the full subcategory of *connected* pointed topological spaces. Write $\mathrm{Set}^{\ast/}$ for the category of pointed sets.

Definition 1.28. A **Brown functor** is a functor

$$F : \mathrm{Ho}(\mathrm{Top}^{\ast/}_{\geq 1})^{\mathrm{op}} \longrightarrow \mathrm{Set}^{\ast/}$$

(from the opposite of the classical homotopy category (def., def.) of connected pointed topological spaces) such that

1. **(additivity)** F takes small coproducts (wedge sums) to products;
2. **(Mayer-Vietoris)** If $X = \mathrm{Int}(A) \cup \mathrm{Int}(B)$ then for all $x_A \in F(A)$ and $x_B \in F(B)$ such that $(x_A)|_{A \cap B} = (x_B)|_{A\cap B}$ then there exists $x_X \in F(X)$ such that $x_A = (x_X)|_A$ and $x_B = (x_X)|_B$.

Proposition 1.29. *For every additive reduced cohomology theory $\tilde E^\bullet(-) : \mathrm{Ho}(\mathrm{Top}^{\ast/})^{\mathrm{op}} \to \mathrm{Set}^{\ast/}$ (def. 1.2) and for each degree $n \in \mathbb{N}$, the restriction of $\tilde E^n(-)$ to connected spaces is a Brown functor (def. 1.28).*

Proof. Under the relation between reduced and unreduced cohomology above, this follows from the exactness of the Mayer-Vietoris sequence of prop. 1.16. ▮

Theorem 1.30. ***(Brown representability)***

Every Brown functor F (def. 1.28) is representable, hence there exists $X \in \mathrm{Top}^{\ast/}_{\geq 1}$ and a natural isomorphism

$$[-, X]_\ast \overset{\simeq}{\longrightarrow} F(-)$$

(where $[-,-]_\ast$ denotes the hom-functor of $\mathrm{Ho}(\mathrm{Top}^{\ast/}_{\geq 1})$ (exmpl.)).

(e.g. AGP 02, theorem 12.2.22)

Remark 1.31. A key subtlety in theorem 1.30 is the restriction to *connected* pointed topological spaces in def. 1.28. This comes about since the proof of the theorem requires that continuous functions $f : X \longrightarrow Y$ that induce isomorphisms on pointed homotopy classes

$$[S^n, X]_* \longrightarrow [S^n, Y]_*$$

for all n are weak homotopy equivalences (For instance in AGP 02 this is used in the proof of theorem 12.2.19 there). But $[S^n, X]_* = \pi_n(X, x)$ gives the nth homotopy group of X *only* for the canonical basepoint, while for a weak homotopy equivalence in general one needs to consider the homotopy groups at all possible basepoints, at least one for each connected component. But so if one does assume that all spaces involved are connected, hence only have one connected component, then indeed weak homotopy equivalences are equivalently those maps $X \to Y$ making all the $[S^n, X]_* \longrightarrow [S^n, Y]_*$ into isomorphisms.

See also example 1.42 below.

The representability result applied degreewise to an additive reduced cohomology theory will yield (prop. 1.33 below) the following concept.

Definition 1.32. An **Omega-spectrum** X (def.) is

1. a sequence $\{X_n\}_{n \in \mathbb{N}}$ of pointed topological spaces $X_n \in \mathrm{Top}^{*/}$
2. weak homotopy equivalences

   $$\tilde\sigma_n \,:\, X_n \underset{\in W_{\mathrm{cl}}}{\overset{\tilde\sigma_n}{\longrightarrow}} \Omega X_{n+1}$$

   for each $n \in \mathbb{N}$, form each space to the loop space of the following space.

Proposition 1.33. *Every additive reduced cohomology theory $\tilde E^\bullet(-) : (\mathrm{Top}^*_{\mathrm{CW}})^{\mathrm{op}} \longrightarrow \mathrm{Ab}^{\mathbb{Z}}$ according to def. 1.2, is represented by an Omega-spectrum E (def. 1.32) in that in each degree $n \in \mathbb{N}$*

1. *$\tilde E^n(-)$ is represented by some $E_n \in \mathrm{Ho}(\mathrm{Top}^{*/})$;*
2. *the suspension isomorphism σ_n of $\tilde E^\bullet$ is represented by the structure map $\tilde\sigma_n$ of the Omega-spectrum in that for all $X \in \mathrm{Top}^{*/}$ the following diagram commutes:*

   $$\begin{array}{ccccc} \tilde E^n(X) & \overset{\sigma_n(X)}{\longrightarrow} & & \longrightarrow & \tilde E^{n+1}(\Sigma X) \\ {}^{\simeq}\downarrow & & & & \downarrow^{\simeq} \\ [X, E_n]_* & \overset{[X, \tilde\sigma_n]_*}{\longrightarrow} & [X, \Omega E_{n+1}]_* & \simeq & [\Sigma X, E_{n+1}]_* \end{array} ,$$

   where $[-,-]_* \coloneqq \mathrm{Hom}_{\mathrm{Ho}(\mathrm{Top}^{*/}_{\geq 1})}$ denotes the hom-sets in the classical pointed homotopy category (def.) and where in the bottom right we have the $(\Sigma \dashv \Omega)$-adjunction isomorphism (prop.).

Proof. If it were not for the connectedness clause in def. 1.28 (remark 1.31), then theorem 1.30 with prop. 1.29 would immediately give the existence of the $\{E_n\}_{n \in \mathbb{N}}$ and the remaining statement would follow immediately with the Yoneda lemma, which says in particular that morphisms between representable functors are in natural bijection with the morphisms of objects that represent them.

The argument with the connectivity condition in Brown representability taken into account is essentially the same, just with a little bit more care:

For X a pointed topological space, write $X^{(0)}$ for the connected component of its basepoint. Observe that the loop space of a pointed topological space only depends on this connected component:

$$\Omega X \simeq \Omega(X^{(0)}) \,.$$

Now for $n \in \mathbb{N}$, to show that $\tilde E^n(-)$ is representable by some $E_n \in \mathrm{Ho}(\mathrm{Top}^{*/})$, use first that the restriction of $\tilde E^{n+1}$ to connected spaces is represented by some $E^{(0)}_{n+1}$. Observe that the reduced suspension of any $X \in \mathrm{Top}^{*/}$ lands in $\mathrm{Top}^{*/}_{\geq 1}$. Therefore the $(\Sigma \dashv \Omega)$-adjunction isomorphism (prop.) implies that $\tilde E^{n+1}(\Sigma(-))$ is represented on *all* of $\mathrm{Top}^{*/}$ by $\Omega E^{(0)}_{n+1}$:

$$\tilde E^{n+1}(\Sigma X) \simeq [\Sigma X, E^{(0)}_{n+1}]_* \simeq [X, \Omega E^{(0)}_{n+1}]_* \simeq [X, \Omega E_{n+1}]_* \,,$$

where E_{n+1} is any pointed topological space with the given connected component $E^{(0)}_{n+1}$.

Now the suspension isomorphism of $\tilde{E}$ says that $E_n \in \mathrm{Ho}(\mathrm{Top}^{*/})$ representing $\tilde{E}^n$ exists and is given by $\Omega E_{n+1}^{(0)}$:

$$\tilde{E}^n(X) \simeq \tilde{E}^{n+1}(\Sigma, X) \simeq [X, \Omega E_{n+1}]$$

for any E_{n+1} with connected component $E_{n+1}^{(0)}$.

This completes the proof. Notice that running the same argument next for $(n+1)$ gives a representing space E_{n+1} such that its connected component of the base point is $E_{n+1}^{(0)}$ found before. And so on. ▮

Conversely:

Proposition 1.34. *Every Omega-spectrum E, def. 1.32, represents an additive reduced cohomology theory def. 1.1 $\tilde{E}^\bullet$ by*

$$\tilde{E}^n(X) := [X, E_n]_*$$

with suspension isomorphism given by

$$\sigma_n : \tilde{E}^n(X) = [X, E_n]_* \xrightarrow{[X, \tilde{\sigma}_n]} [X, \Omega E_{n+1}]_* \overset{\simeq}{\to} [\Sigma X, E_{n+1}] = \tilde{E}^{n+1}(\Sigma X) .$$

Proof. The additivity is immediate from the construction. The exactnes follows from the long exact sequences of homotopy cofiber sequences given by this prop.. ▮

Remark 1.35. If we consider the stable homotopy category $\mathrm{Ho}(\mathrm{Spectra})$ of spectra (def.) and consider any topological space X in terms of its suspension spectrum $\Sigma^\infty X \in \mathrm{Ho}(\mathrm{Spectra})$ (exmpl.), then the statement of prop. 1.34 is more succinctly summarized by saying that the graded reduced cohomology groups of a topological space X represented by an Omega-spectrum E are the hom-groups

$$\tilde{E}^\bullet(X) \;\simeq\; [\Sigma^\infty X, \Sigma^\bullet E]$$

in the stable homotopy category, into all the suspensions (thm.) of E.

This means that more generally, for $X \in \mathrm{Ho}(\mathrm{Spectra})$ any spectrum, it makes sense to consider

$$\tilde{E}^\bullet(X) \;:=\; [X, \Sigma^\bullet E]$$

to be the graded reduced generalized E-cohomology groups of the spectrum X.

See also in *part 1* this example.

Application to ordinary cohomology

Example 1.36. Let A be an abelian group. Consider singular cohomology $H^n(-, A)$ with coefficients in A. The corresponding reduced cohomology evaluated on n-spheres satisfies

$$\tilde{H}^n(S^q, A) \simeq \begin{cases} A & \text{if } q = n \\ 0 & \text{otherwise} \end{cases}$$

Hence singular cohomology is a generalized cohomology theory which is "ordinary cohomology" in the sense of def. 1.6.

Applying the Brown representability theorem as in prop. 1.33 hence produces an Omega-spectrum (def. 1.32) whose nth component space is characterized as having homotopy groups concentrated in degree n on A. These are called *Eilenberg-MacLane spaces* $K(A, n)$

$$\pi_q(K(A, n)) \simeq \begin{cases} A & \text{if } q = n \\ 0 & \text{otherwise} \end{cases} .$$

Here for $n > 0$ then $K(A, n)$ is connected, therefore with an essentially unique basepoint, while $K(A, 0)$ is (homotopy equivalent to) the underlying set of the group A.

Such spectra are called **Eilenberg-MacLane spectra** HA:

$$(HA)_n \simeq K(A, n) .$$

As a consequence of example 1.36 one obtains the uniqueness result of Eilenberg-Steenrod:

Proposition 1.37. *Let $\tilde{E}_1$ and $\tilde{E}_2$ be ordinary (def. 1.6) generalized (Eilenberg-Steenrod) cohomology* *theories. If there is an isomorphism*

$$\tilde{E}_1(S^0) \simeq \tilde{E}_2(S^0)$$

of cohomology groups of the 0-sphere, then there is an isomorphism of cohomology theories

$$\tilde{E}_1 \xrightarrow{\simeq} \tilde{E}_2 \,.$$

(e.g. Aguilar-Gitler-Prieto 02, theorem 12.3.6)

Homotopy-theoretic discussion

Using abstract homotopy theory in the guise of model category theory (see the *lecture notes on classical homotopy theory), the traditional proof and further discussion of the Brown representability theorem above becomes more transparent (Lurie 10, section 1.4.1, for exposition see also Mathew 11).*

This abstract homotopy-theoretic proof uses the general concept of homotopy colimits in model categories as well as the concept of derived hom-spaces ("∞-categories"). Even though in the accompanying Lecture notes on classical homotopy theory these concepts are only briefly indicated, the following is included for the interested reader.

Definition 1.38. Let $\mathcal{C}$ be a model category. A functor

$$F \;:\; \mathrm{Ho}(\mathcal{C})^{\mathrm{op}} \longrightarrow \mathrm{Set}$$

(from the opposite of the homotopy category of $\mathcal{C}$ to Set)

is called a **Brown functor** if

1. it sends small coproducts to products;
2. it sends homotopy pushouts in $\mathcal{C} \to \mathrm{Ho}(\mathcal{C})$ to weak pullbacks in Set (see remark 1.39).

Remark 1.39. A *weak pullback* is a diagram that satisfies the existence clause of a pullback, but not necessarily the uniqueness condition. Hence the second clause in def. 1.38 says that for a homotopy pushout square

$$\begin{array}{ccc} Z & \longrightarrow & X \\ \downarrow & \swArrow & \downarrow \\ Y & \longrightarrow & X \underset{Z}{\sqcup} Y \end{array}$$

in $\mathcal{C}$, then the induced universal morphism

$$F(X \underset{Z}{\sqcup} Y) \overset{\mathrm{epi}}{\longrightarrow} F(X) \underset{F(Z)}{\times} F(Y)$$

into the actual pullback is an epimorphism.

Definition 1.40. Say that a model category $\mathcal{C}$ is **compactly generated by cogroup objects closed under suspensions** if

1. $\mathcal{C}$ is generated by a set

   $$\{S_i \in \mathcal{C}\}_{i \in I}$$

   of compact objects (i.e. every object of $\mathcal{C}$ is a homotopy colimit of the objects S_i.)
2. each S_i admits the structure of a cogroup object in the homotopy category $\mathrm{Ho}(\mathcal{C})$;
3. the set $\{S_i\}$ is closed under forming reduced suspensions.

Example 1.41. (**suspensions are H-cogroup objects**)

Let $\mathcal{C}$ be a model category and $\mathcal{C}^{*/}$ its pointed model category (prop.) with zero object (rmk.). Write $\Sigma : X \mapsto 0 \coprod_X 0$ for the reduced suspension functor.

Then the fold map

$$\Sigma X \coprod \Sigma X \simeq 0 \underset{X}{\sqcup} 0 \underset{X}{\sqcup} 0 \longrightarrow 0 \underset{X}{\sqcup} X \underset{X}{\sqcup} 0 \simeq 0 \underset{X}{\sqcup} 0 \simeq \Sigma X$$

exhibits cogroup structure on the image of any suspension object ΣX in the homotopy category.

This is equivalently the group-structure of the first (fundamental) homotopy group of the values of functor co-represented by ΣX:

$$\mathrm{Ho}(\mathcal{C})(\Sigma X, -) : Y \mapsto \mathrm{Ho}(\mathcal{C})(\Sigma X, Y) \simeq \mathrm{Ho}(\mathcal{C})(X, \Omega Y) \simeq \pi_1 \, \mathrm{Ho}(\mathcal{C})(X,Y) \,.$$

Example 1.42. In bare pointed homotopy types $\mathcal{C} = \mathrm{Top}^{*/}_{\mathrm{Quillen}}$, the (homotopy types of) n-spheres S^n are cogroup objects for $n \geq 1$, but not for $n = 0$, by example 1.41. And of course they are compact objects.

So while $\{S^n\}_{n \in \mathbb{N}}$ generates all of the homotopy theory of $\mathrm{Top}^{*/}$, the latter is *not* an example of def. 1.40 due to the failure of S^0 to have cogroup structure.

Removing that generator, the homotopy theory generated by $\{S^n\}_{\substack{n \in \mathbb{N} \\ n \geq 1}}$ is $\mathrm{Top}^{*/}_{\geq 1}$, that of *connected* pointed homotopy types. This is one way to see how the connectedness condition in the classical version of Brown representability theorem arises. See also remark 1.31 above.

See also (Lurie 10, example 1.4.1.4)

In homotopy theories compactly generated by cogroup objects closed under forming suspensions, the following strenghtening of the Whitehead theorem holds.

Proposition 1.43. *In a homotopy theory compactly generated by cogroup objects $\{S_i\}_{i \in I}$ closed under forming suspensions, according to def. 1.40, a morphism $f : X \longrightarrow Y$ is an equivalence precisely if for each $i \in I$ the induced function of maps in the homotopy category*

$$\mathrm{Ho}(\mathcal{C})(S_i, f) : \mathrm{Ho}(\mathcal{C})(S_i, X) \longrightarrow \mathrm{Ho}(\mathcal{C})(S_i, Y)$$

is an isomorphism (a bijection).

(Lurie 10, p. 114, Lemma star)

Proof. By the ∞-Yoneda lemma, the morphism f is a weak equivalence precisely if for all objects $A \in \mathcal{C}$ the induced morphism of derived hom-spaces

$$\mathcal{C}(A, f) : \mathcal{C}(A, X) \longrightarrow \mathcal{C}(A, Y)$$

is an equivalence in $\mathrm{Top}_{\mathrm{Quillen}}$. By assumption of compact generation and since the hom-functor $\mathcal{C}(-,-)$ sends homotopy colimits in the first argument to homotopy limits, this is the case precisely already if it is the case for $A \in \{S_i\}_{i \in I}$.

Now the maps

$$\mathcal{C}(S_i, f) : \mathcal{C}(S_i, X) \longrightarrow \mathcal{C}(S_i, Y)$$

are weak equivalences in $\mathrm{Top}_{\mathrm{Quillen}}$ if they are weak homotopy equivalences, hence if they induce isomorphisms on all homotopy groups π_n for **all basepoints**.

It is this last condition of testing on all basepoints that the assumed cogroup structure on the S_i allows to do away with: this cogroup structure implies that $\mathcal{C}(S_i, -)$ has the structure of an H-group, and this implies (by group multiplication), that all connected components have the same homotopy groups, hence that all homotopy groups are independent of the choice of basepoint, up to isomorphism.

Therefore the above morphisms are equivalences precisely if they are so under applying π_n based on the connected component of the zero morphism

$$\pi_n \mathcal{C}(S_i, f) : \pi_n \mathcal{C}(S_i, X) \longrightarrow \pi_n \mathcal{C}(S_i, Y) \,.$$

Now in this pointed situation we may use that

$$\begin{aligned} \pi_n \mathcal{C}(-,-) &\simeq \pi_0 \mathcal{C}(-, \Omega^n(-)) \\ &\simeq \pi_0 \mathcal{C}(\Sigma^n(-), -) \\ &\simeq \mathrm{Ho}(\mathcal{C})(\Sigma^n(-), -) \end{aligned}$$

to find that f is an equivalence in $\mathcal{C}$ precisely if the induced morphisms

$$\mathrm{Ho}(\mathcal{C})(\Sigma^n S_i, f) : \mathrm{Ho}(\mathcal{C})(\Sigma^n S_i, X) \longrightarrow \mathrm{Ho}(\mathcal{C})(\Sigma^n S_i, Y)$$

are isomorphisms for all $i \in I$ and $n \in \mathbb{N}$.

Finally by the assumption that each suspension $\Sigma^n S_i$ of a generator is itself among the set of generators, the claim follows. ▮

Theorem 1.44. ***(Brown representability)***

Let $\mathcal{C}$ be a model category compactly generated by cogroup objects closed under forming suspensions, according to def. 1.40. Then a functor

$$F : \mathrm{Ho}(\mathcal{C})^{\mathrm{op}} \longrightarrow \mathrm{Set}$$

(from the opposite of the homotopy category of $\mathcal{C}$ to Set) is representable precisely if it is a Brown functor, def. 1.38.

(Lurie 10, theorem 1.4.1.2)

Proof. Due to the version of the Whitehead theorem of prop. 1.43 we are essentially reduced to showing that Brown functors F are representable on the S_i. To that end consider the following lemma. (In the following we notationally identify, via the Yoneda lemma, objects of $\mathcal{C}$, hence of $\mathrm{Ho}(\mathcal{C})$, with the functors they represent.)

Lemma ($\star$): *Given $X \in \mathcal{C}$ and $\eta \in F(X)$, hence $\eta : X \to F$, then there exists a morphism $f : X \to X'$ and an extension $\eta' : X' \to F$ of η which induces for each S_i a bijection $\eta' \circ (-) : \mathrm{PSh}(\mathrm{Ho}(\mathcal{C}))(S_i, X') \overset{\simeq}{\longrightarrow} \mathrm{Ho}(\mathcal{C})(S_i, F) \simeq F(S_i)$.*

To see this, first notice that we may directly find an extension η_0 along a map $X \to X_o$ such as to make a surjection: simply take X_0 to be the coproduct of **all** possible elements in the codomain and take

$$\eta_0 \;:\; X \sqcup \Big(\coprod_{\substack{i \in I, \\ \gamma : S_i \to F}} S_i \Big) \longrightarrow F$$

to be the canonical map. (Using that F, by assumption, turns coproducts into products, we may indeed treat the coproduct in $\mathcal{C}$ on the left as the coproduct of the corresponding functors.)

To turn the surjection thus constructed into a bijection, we now successively form quotients of X_0. To that end proceed by induction and suppose that $\eta_n : X_n \to F$ has been constructed. Then for $i \in I$ let

$$K_i \coloneqq \ker\Big(\mathrm{Ho}(\mathcal{C})(S_i, X_n) \overset{\eta_n \circ (-)}{\longrightarrow} F(S_i) \Big)$$

be the kernel of η_n evaluated on S_i. These K_i are the pieces that need to go away in order to make a bijection. Hence define X_{n+1} to be their joint homotopy cofiber

$$X_{n+1} \coloneqq \mathrm{coker}\left(\Big(\coprod_{\substack{i \in I, \\ \gamma \in K_i}} S_i \Big) \xrightarrow{(\gamma)_{\substack{i \in I \\ \gamma \in K_i}}} X_n \right) .$$

Then by the assumption that F takes this homotopy cokernel to a weak fiber (as in remark 1.39), there exists an extension η_{n+1} of η_n along $X_n \to X_{n+1}$:

Then by the assumption that F takes this homotopy cokernel to a weak fiber (as in remark 1.39), there exists an extension η_{n+1} of η_n along $X_n \to X_{n+1}$:

$$\begin{array}{ccccc} \Big(\coprod_{\substack{i \in I \\ \gamma \in K_i}} S_i \Big) & \xrightarrow{(\gamma)_{\substack{i \in I \\ \gamma \in K_i}}} & X_n & \overset{\eta_n}{\to} & F \\ \downarrow & (\mathrm{po}^h) & \downarrow & \nearrow_{\exists \eta_{n+1}} & \\ * & \longrightarrow & X_{n+1} & & \end{array} \quad \Leftrightarrow \quad \begin{array}{ccccc} & & F(X_{n+1}) & \longrightarrow & * \\ & {}^{\exists \eta_{n+1}} \nearrow & \downarrow^{\mathrm{epi}} & & \downarrow \\ * & \overset{\eta_n}{\to} & \ker\big((\gamma^*)_{\substack{i \in I \\ \gamma \in K_i}} \big) & \longrightarrow & * \\ & {}_{\eta_n} \searrow & \downarrow & (\mathrm{pb}) & \downarrow \\ & & F(X_n) & \underset{(\gamma^*)_{\substack{i \in I \\ \gamma \in K_i}}}{\longrightarrow} & \prod_{\substack{i \in I \\ \gamma \in K_i}} F(S_i) \end{array} .$$

It is now clear that we want to take

$$X' \coloneqq \underset{\longrightarrow}{\lim}_n X_n$$

and extend all the η_n to that colimit. Since we have no condition for evaluating F on colimits other than pushouts, observe that this sequential colimit is equivalent to the following pushout:

$$
\begin{array}{ccc}
\underset{n}{\sqcup} X_n & \longrightarrow & \underset{n}{\sqcup} X_{2n} \\
\downarrow && \downarrow \\
\underset{n}{\sqcup} X_{2n+1} & \longrightarrow & X'
\end{array}
\,,
$$

where the components of the top and left map alternate between the identity on X_n and the above successor maps $X_n \to X_{n+1}$. Now the excision property of F applies to this pushout, and we conclude the desired extension $\eta' : X' \to F$:

$$
\begin{array}{ccccc}
&& \underset{n}{\sqcup} X_n && \\
& \swarrow && \searrow & \\
\underset{n}{\sqcup} X_{2n+1} & \longrightarrow & X' & \longleftarrow & \underset{n}{\sqcup} X_{2n} \\
& {}_{(\eta_{2n+1})_n}\searrow & \downarrow^{\exists \eta} & \swarrow_{(\eta_{2n})_n} & \\
&& F &&
\end{array}
\;\;\;\;\Leftrightarrow\;\;\;\;
\begin{array}{ccccc}
&& F(X') && \\
& {}^{\exists \eta}\nearrow & \downarrow^{\mathrm{epi}} && \\
& \ast \overset{(\eta_n)_n}{\longrightarrow} & \underset{\longleftarrow_n}{\lim} F(X_n) && \\
& \swarrow && \searrow & \\
\prod_n F(X_{2n+1}) &&&& \prod_n (X_{2n}) \\
& \searrow && \swarrow & \\
&& \prod_n F(X_n) &&
\end{array}
\,,
$$

It remains to confirm that this indeed gives the desired bijection. Surjectivity is clear. For injectivity use that all the S_i are, by assumption, compact, hence they may be taken inside the sequential colimit:

$$
\begin{array}{ccc}
&& X_{n(\gamma)} \\
& {}^{\exists \hat \gamma}\nearrow & \downarrow \\
S_i & \overset{\gamma}{\longrightarrow} & X' = \underset{\longrightarrow_n}{\lim} X_n
\end{array}
\,.
$$

With this, injectivity follows because by construction we quotiented out the kernel at each stage. Because suppose that γ is taken to zero in $F(S_i)$, then by the definition of X_{n+1} above there is a factorization of γ through the point:

$$
\begin{array}{ccccc}
0 : & S_i & \overset{\hat \gamma}{\longrightarrow} & X_{n(\gamma)} & \overset{\eta_n}{\longrightarrow} & F \\
& \downarrow && \downarrow && \\
& \ast & \longrightarrow & X_{n(\gamma)+1} && \\
&&& \downarrow && \\
&&& X' &&
\end{array}
$$

This concludes the proof of Lemma ($\star$).

Now apply the construction given by this lemma to the case $X_0 \coloneqq 0$ and the unique $\eta_0 : 0 \overset{\exists !}{\to} F$. Lemma ($\star$) then produces an object X' which represents F on all the S_i, and we want to show that this X' actually represents F generally, hence that for every $Y \in \mathcal{C}$ the function

$$
\theta \coloneqq \eta' \circ (-) \;:\; \mathrm{Ho}(\mathcal{C})(Y, X') \longrightarrow F(Y)
$$

is a bijection.

First, to see that θ is surjective, we need to find a preimage of any $\rho : Y \to F$. Applying Lemma ($\star$) to $(\eta', \rho) : X' \sqcup Y \longrightarrow F$ we get an extension κ of this through some $X' \sqcup Y \longrightarrow Z$ and the morphism on the right of the following commuting diagram:

$$
\begin{array}{ccc}
\mathrm{Ho}(\mathcal{C})(-, X') && \longrightarrow && \mathrm{Ho}(\mathcal{C})(-, Z) \\
& {}_{\eta' \circ (-)}\searrow && \swarrow_{\kappa \circ (-)} & \\
&& F(-) &&
\end{array}
\,.
$$

Moreover, Lemma ($\star$) gives that evaluated on all S_i, the two diagonal morphisms here become isomorphisms. But then prop. 1.43 implies that $X' \longrightarrow Z$ is in fact an equivalence. Hence the component map $Y \to Z \simeq Z$ is a lift of κ through θ.

Second, to see that θ is injective, suppose $f, g : Y \to X'$ have the same image under θ. Then consider their homotopy pushout

$$
\begin{array}{ccc}
Y \sqcup Y & \stackrel{(f,g)}{\longrightarrow} & X' \\
\downarrow && \downarrow \\
Y & \longrightarrow & Z
\end{array}
$$

along the codiagonal of Y. Using that F sends this to a weak pullback by assumption, we obtain an extension $\bar \eta$ of η' along $X' \to Z$. Applying Lemma $(\star)$ to this gives a further extension $\bar \eta' : Z' \to Z$ which now makes the following diagram

$$
\begin{array}{ccc}
Ho(\mathcal{C})(-,X') && \longrightarrow && Ho(\mathcal{C})(-,Z) \\
& {}_{\eta' \circ (-)}\searrow && \swarrow_{\bar \eta' \circ (-)} \\
&& F(-)
\end{array}
$$

such that the diagonal maps become isomorphisms when evaluated on the S_i. As before, it follows via prop. 1.43 that the morphism $h : X' \longrightarrow Z'$ is an equivalence.

Since by this construction $h \circ f$ and $h \circ g$ are homotopic

$$
\begin{array}{ccccc}
Y \sqcup Y & \stackrel{(f,g)}{\longrightarrow} & X' \\
\downarrow && \downarrow & \searrow^{\mathrlap{h}}_{\simeq} \\
Y & \longrightarrow & Z & \longrightarrow & Z'
\end{array}
$$

it follows with h being an equivalence that already f and g were homotopic, hence that they represented the same element. ▮

Proposition 1.45. *Given a reduced additive cohomology functor $H^\bullet : Ho(\mathcal{C})^{op} \to Ab^{\mathbb{Z}}$, def. 1.5, its underlying Set-valued functors $H^n : Ho(\mathcal{C})^{op} \to Ab \to Set$ are Brown functors, def. 1.38.*

Proof. The first condition on a Brown functor holds by definition of $H^\bullet$. For the second condition, given a homotopy pushout square

$$
\begin{array}{ccc}
X_1 & \stackrel{f_1}{\longrightarrow} & Y_1 \\
\downarrow && \downarrow \\
X_2 & \stackrel{f_2}{\longrightarrow} & Y_2
\end{array}
$$

in $\mathcal{C}$, consider the induced morphism of the long exact sequences given by prop. 1.8

$$
\begin{array}{ccccccc}
H^\bullet(coker(f_2)) & \longrightarrow & H^\bullet(Y_2) & \stackrel{f_2^\ast}{\longrightarrow} & H^\bullet(X_2) & \longrightarrow & H^{\bullet+1}(\Sigma coker(f_2)) \\
{}^{\mathllap{\simeq}}\downarrow && \downarrow && \downarrow && \downarrow^{\mathrlap{\simeq}} \\
H^\bullet(coker(f_1)) & \longrightarrow & H^\bullet(Y_1) & \stackrel{f_1^\ast}{\longrightarrow} & H^\bullet(X_1) & \longrightarrow & H^{\bullet+1}(\Sigma coker(f_1))
\end{array}
$$

Here the outer vertical morphisms are isomorphisms, as shown, due to the pasting law (see also at *fiberwise recognition of stable homotopy pushouts*). This means that the four lemma applies to this diagram. Inspection shows that this implies the claim. ▮

Corollary 1.46. *Let $\mathcal{C}$ be a model category which satisfies the conditions of theorem 1.44, and let $(H^\bullet, \delta)$ be a reduced additive generalized cohomology functor on $\mathcal{C}$, def. 1.5. Then there exists a spectrum object $E \in Stab(\mathcal{C})$ such that*

1. *$H\bullet$ is degreewise represented by E:*

$$
H^\bullet \simeq Ho(\mathcal{C})(-,E_\bullet)\,,
$$

2. *the suspension isomorphism δ is given by the structure morphisms $\tilde \sigma_n : E_n \to \Omega E_{n+1}$ of the spectrum, in that*

$$
\delta : H^n(-) \simeq Ho(\mathcal{C})(-,E_n) \stackrel{Ho(\mathcal{C})(-,\tilde \sigma_n)}{\longrightarrow} Ho(\mathcal{C})(-,\Omega E_{n+1}) \simeq Ho(\mathcal{C})(\Sigma(-), E_{n+1}) \simeq H^{n+1}(\Sigma(-)) \,.
$$

Proof. Via prop. 1.45, theorem 1.44 gives the first clause. With this, the second clause follows by the Yoneda lemma. ▮

Milnor exact sequence

Idea. One tool for computing generalized cohomology groups via "inverse limits" are *Milnor exact* *sequences*. For instance the generalized cohomology of the classifying space $BU(1)$ plays a key role in the complex oriented cohomology-theory discussed below, and via the equivalence $BU(1) \simeq \mathbb{C}P^\infty$ to the homotopy type of the infinite complex projective space (def. 1.134), which is the direct limit of finite dimensional projective spaces $\mathbb{C}P^n$, this is an inverse limit of the generalized cohomology groups of the $\mathbb{C}P^n$s. But what really matters here is the derived functor of the limit-operation – the homotopy limit – and the Milnor exact sequence expresses how the naive limits receive corrections from higher "lim^1-terms". In practice one mostly proceeds by verifying conditions under which these corrections happen to disappear, these are the *Mittag-Leffler conditions*.

We need this for instance for the computation of Conner-Floyd Chern classes below.

Literature. (Switzer 75, section 7 from def. 7.57 on, Kochman 96, section 4.2, Goerss-Jardine 99, section VI.2,)

Lim^1

Definition 1.47. Given a tower $A_\bullet$ of abelian groups

$$\cdots \to A_3 \overset{f_2}{\to} A_2 \overset{f_1}{\to} A_1 \overset{f_0}{\to} A_0$$

write

$$\partial : \prod_n A_n \longrightarrow \prod_n A_n$$

for the homomorphism given by

$$\partial : (a_n)_{n\in\mathbb{N}} \mapsto (a_n - f_n(a_{n+1}))_{n\in\mathbb{N}} .$$

Remark 1.48. The limit of a sequence as in def. 1.47 – hence the group $\underset{\longleftarrow_n}{\lim} A_n$ universally equipped with morphisms $\underset{\longleftarrow_n}{\lim} A_n \overset{p_n}{\to} A_n$ such that all

$$\begin{array}{ccc} & \underset{\longleftarrow_n}{\lim} A_n & \\ {}^{p_{n+1}}\swarrow & & \searrow^{p_n} \\ A_{n+1} & \overset{f_n}{\longrightarrow} & A_n \end{array}$$

commute – is equivalently the kernel of the morphism ∂ in def. 1.47.

Definition 1.49. Given a tower $A_\bullet$ of abelian groups

$$\cdots \to A_3 \overset{f_2}{\to} A_2 \overset{f_1}{\to} A_1 \overset{f_0}{\to} A_0$$

then $\underset{\longleftarrow}{\lim}^1 A_\bullet$ is the cokernel of the map ∂ in def. 1.47, hence the group that makes a long exact sequence of the form

$$0 \to \underset{\longleftarrow_n}{\lim} A_n \longrightarrow \prod_n A_n \overset{\partial}{\longrightarrow} \prod_n A_n \longrightarrow \underset{\longleftarrow_n}{\lim}^1 A_n \to 0 \,,$$

Proposition 1.50. *The functor $\underset{\longleftarrow}{\lim}^1 : \mathrm{Ab}^{(\mathbb{N},\geq)} \longrightarrow \mathrm{Ab}$ (def. 1.49) satisfies*

1. *for every short exact sequence $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0 \quad \in \mathrm{Ab}^{(\mathbb{N},\geq)}$ then the induced sequence*

$$0 \to \underset{\longleftarrow_n}{\lim} A_n \to \underset{\longleftarrow_n}{\lim} B_n \to \underset{\longleftarrow_n}{\lim} C_n \to \underset{\longleftarrow_n}{\lim}^1 A_n \to \underset{\longleftarrow_n}{\lim}^1 B_n \to \underset{\longleftarrow_n}{\lim}^1 C_n \to 0$$

   is a long exact sequence of abelian groups;

2. *if $A_\bullet$ is a tower such that all maps are surjections, then $\underset{\longleftarrow_n}{\lim}^1 A_n \simeq 0$.*

(e.g. Switzer 75, prop. 7.63, Goerss-Jardine 96, section VI. lemma 2.11)

Proof. For the first property: Given $A_\bullet$ a tower of abelian groups, write

$$L^\bullet(A_\bullet) \coloneqq \left[0 \to \underbrace{\prod_n A_n}_{\deg 0} \overset{\partial}{\to} \underbrace{\prod_n A_n}_{\deg 1} \to 0 \right]$$

for the homomorphism from def. 1.47 regarded as the single non-trivial differential in a cochain complex of abelian groups. Then by remark 1.48 and def. 1.49 we have $H^0(L(A_\bullet)) \simeq \underset{\longleftarrow}{\lim} A_\bullet$ and $H^1(L(A_\bullet)) \simeq \underset{\longleftarrow}{\lim}^1 A_\bullet$.

With this, then for a short exact sequence of towers $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ the long exact sequence in question is the long exact sequence in homology of the corresponding short exact sequence of complexes

$$0 \to L^\bullet(A_\bullet) \longrightarrow L^\bullet(B_\bullet) \longrightarrow L^\bullet(C_\bullet) \to 0 \,.$$

For the second statement: If all the f_k are surjective, then inspection shows that the homomorphism ∂ in def. 1.47 is surjective. Hence its cokernel vanishes. ▮

Lemma 1.51. *The category $Ab^{(\mathbb{N},\geq)}$ of towers of abelian groups has enough injectives.*

Proof. The functor $(-)_n : Ab^{(\mathbb{N},\geq)} \to Ab$ that picks the n-th component of the tower has a right adjoint r_n, which sends an abelian group A to the tower

$$r_n \coloneqq \left[\cdots \overset{id}{\to} A \overset{id}{\to} \underbrace{A}_{=(r_n)_{n+1}} \overset{id}{\to} \underbrace{A}_{=(r_n)_n} \overset{id}{\to} \underbrace{0}_{=(r_n)_{n-1}} \to 0 \to \cdots \to 0 \to 0 \right] .$$

Since $(-)_n$ itself is evidently an exact functor, its right adjoint preserves injective objects (prop.).

So with $A_\bullet \in Ab^{(\mathbb{N},\geq)}$, let $A_n \hookrightarrow \tilde A_n$ be an injective resolution of the abelian group A_n, for each $n \in \mathbb{N}$. Then

$$A_\bullet \overset{(\eta_n)_{n\in\mathbb{N}}}{\longrightarrow} \prod_{n \in \mathbb{R}} r_n A_n \hookrightarrow \prod_{n\in\mathbb{N}} r_n \tilde A_n$$

is an injective resolution for $A_\bullet$. ▮

Proposition 1.52. *The functor $\underset{\longleftarrow}{\lim}^1 : Ab^{(\mathbb{N},\geq)} \longrightarrow Ab$ (def. 1.49) is the first right derived functor of the limit functor $\underset{\longleftarrow}{\lim} : Ab^{(\mathbb{N},\geq)} \longrightarrow Ab$.*

Proof. By lemma 1.51 there are enough injectives in $Ab^{(\mathbb{N},\geq)}$. So for $A_\bullet \in Ab^{(\mathbb{N},\geq)}$ the given tower of abelian groups, let

$$0 \to A_\bullet \overset{j^0}{\longrightarrow} J^0_\bullet \overset{j^1}{\longrightarrow} J^1_\bullet \overset{j^2}{\longrightarrow} J^2_\bullet \longrightarrow \cdots$$

be an injective resolution. We need to show that

$$\underset{\longleftarrow}{\lim}^1 A_\bullet \simeq \ker(\underset{\longleftarrow}{\lim}(j^2)) / \mathrm{im}(\underset{\longleftarrow}{\lim}(j^1)) \,.$$

Since limits preserve kernels, this is equivalently

$$\underset{\longleftarrow}{\lim}^1 A_\bullet \simeq (\underset{\longleftarrow}{\lim}(\ker(j^2)_\bullet)) / \mathrm{im}(\underset{\longleftarrow}{\lim}(j^1))$$

Now observe that each injective $J^q_\bullet$ is a tower of epimorphism. This follows by the defining right lifting property applied against the monomorphisms of towers of the following form

$$\begin{array}{ccccccccccccccc} \cdots & \to & 0 & \to & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \overset{id}{\to} & \cdots & \overset{id}{\to} & \mathbb{Z} & \overset{id}{\to} & \mathbb{Z} \\ \cdots && \downarrow^{id} && \downarrow^{id} && \downarrow^{incl} && \downarrow^{id} &&&& \downarrow^{id} && \downarrow^{id} \\ \cdots & \to & 0 & \to & 0 & \to & \mathbb{Z} & \underset{id}{\to} & \mathbb{Z} & \underset{id}{\to} & \cdots & \underset{id}{\to} & \mathbb{Z} & \underset{id}{\to} & \mathbb{Z} \end{array}$$

Therefore by the second item of prop. 1.50 the long exact sequence from the first item of prop. 1.50 applied to the short exact sequence

$$0 \to A_\bullet \overset{j^0}{\longrightarrow} J^0_\bullet \overset{j^1}{\longrightarrow} \ker(j^2)_\bullet \to 0$$

becomes

$$0 \to \underset{\longleftarrow}{\lim} A_\bullet \overset{\underset{\longleftarrow}{\lim} j^0}{\longrightarrow} \underset{\longleftarrow}{\lim} J^0_\bullet \overset{\underset{\longleftarrow}{\lim} j^1}{\longrightarrow} \underset{\longleftarrow}{\lim}(\ker(j^2)_\bullet) \longrightarrow \underset{\longleftarrow}{\lim}^1 A_\bullet \longrightarrow 0 \,.$$

Exactness of this sequence gives the desired identification $\underset{\longleftarrow}{\lim}^1 A_\bullet \simeq (\underset{\longleftarrow}{\lim}(\ker(j^2)_\bullet))/\mathrm{im}(\underset{\longleftarrow}{\lim}(j^1))$. ▮

Proposition 1.53. *The functor $\underset{\longleftarrow}{\lim}^1 : \mathrm{Ab}^{(\mathbb{N},\geq)} \longrightarrow \mathrm{Ab}$ (def. 1.49) is in fact the unique functor, up to natural isomorphism, satisfying the conditions in prop. 1.53.*

Proof. The proof of prop. 1.52 only used the conditions from prop. 1.50, hence any functor satisfying these conditions is the first right derived functor of $\underset{\longleftarrow}{\lim}$, up to natural isomorphism. ▮

The following is a kind of double dual version of the $\lim^1$ construction which is sometimes useful:

Lemma 1.54. *Given a cotower*

$$A_\bullet = (A_0 \overset{f_0}{\to} A_1 \overset{f_1}{\to} A_2 \to \cdots)$$

of abelian groups, then for every abelian group $B \in \mathrm{Ab}$ there is a short exact sequence of the form

$$0 \to \underset{\longleftarrow n}{\lim}^1 \mathrm{Hom}(A_n, B) \longrightarrow \mathrm{Ext}^1(\underset{\longrightarrow n}{\lim} A_n, B) \longrightarrow \underset{\longleftarrow n}{\lim} \mathrm{Ext}^1(A_n, B) \to 0 \,,$$

where $\mathrm{Hom}(-,-)$ denotes the hom-group, $\mathrm{Ext}^1(-,-)$ denotes the first Ext-group (and so $\mathrm{Hom}(-,-) = \mathrm{Ext}^0(-,-)$).

Proof. Consider the homomorphism

$$\tilde{\partial} \;:\; \underset{n}{\oplus} A_n \longrightarrow \underset{n}{\oplus} A_n$$

which sends $a_n \in A_n$ to $a_n - f_n(a_n)$. Its cokernel is the colimit over the cotower, but its kernel is trivial (in contrast to the otherwise formally dual situation in remark 1.48). Hence (as opposed to the long exact sequence in def. 1.49) there is a short exact sequence of the form

$$0 \to \underset{n}{\oplus} A_n \overset{\tilde{\partial}}{\longrightarrow} \underset{n}{\oplus} A_n \longrightarrow \underset{\longrightarrow n}{\lim} A_n \to 0 \,.$$

Every short exact sequence gives rise to a long exact sequence of derived functors (prop.) which in the present case starts out as

$$0 \to \mathrm{Hom}(\underset{\longrightarrow n}{\lim} A_n, B) \longrightarrow \prod_n \mathrm{Hom}(A_n, B) \overset{\partial}{\longrightarrow} \prod_n \mathrm{Hom}(A_n, B) \longrightarrow \mathrm{Ext}^1(\underset{\longrightarrow n}{\lim} A_n, B) \longrightarrow \prod_n \mathrm{Ext}^1(A_n, B) \overset{\partial}{\longrightarrow} \prod_n \mathrm{Ext}^1(A_n, B) \longrightarrow \cdots$$

where we used that direct sum is the coproduct in abelian groups, so that homs out of it yield a product, and where the morphism ∂ is the one from def. 1.47 corresponding to the tower

$$\mathrm{Hom}(A_\bullet, B) = (\cdots \to \mathrm{Hom}(A_2, B) \to \mathrm{Hom}(A_1, B) \to \mathrm{Hom}(A_0, B)) \,.$$

Hence truncating this long sequence by forming kernel and cokernel of ∂, respectively, it becomes the short exact sequence in question. ▮

Mittag-Leffler condition

Definition 1.55. A tower $A_\bullet$ of abelian groups

$$\cdots \to A_3 \to A_2 \to A_1 \to A_0$$

is said to satify the **Mittag-Leffler condition** if for all k there exists $i \geq k$ such that for all $j \geq i \geq k$ the image of the homomorphism $A_i \to A_k$ equals that of $A_j \to A_k$

$$\mathrm{im}(A_i \to A_k) \simeq \mathrm{im}(A_j \to A_k) \,.$$

(e.g. Switzer 75, def. 7.74)

Example 1.56. The Mittag-Leffler condition, def. 1.55, is satisfied in particular when all morphisms $A_{i+1} \to A_i$ are epimorphisms (hence surjections of the underlying sets).

Proposition 1.57. *If a tower $A_\bullet$ satisfies the Mittag-Leffler condition, def. 1.55, then its $\underset{\longleftarrow}{\lim}^1$ vanishes:*

$$\underset{\longleftarrow}{\lim}^1 A_\bullet = 0 \,.$$

e.g. (Switzer 75, theorem 7.75, Kochmann 96, prop. 4.2.3, Weibel 94, prop. 3.5.7)

Proof idea. One needs to show that with the Mittag-Leffler condition, then the cokernel of ∂ in def. 1.47

vanishes, hence that ∂ is an epimorphism in this case, hence that every $(a_n)_{n\in\mathbb{N}} \in \prod_n A_n$ has a preimage under ∂. So use the Mittag-Leffler condition to find pre-images of a_n by induction over n. ■

Mapping telescopes

Given a sequence

$$X_\bullet = \left(X_0 \overset{f_0}{\longrightarrow} X_1 \overset{f_1}{\longrightarrow} X_2 \overset{f_2}{\longrightarrow} \cdots \right)$$

of (pointed) topological spaces, then its *mapping telescope* is the result of forming the (reduced) mapping cylinder $\mathrm{Cyl}(f_n)$ for each n and then attaching all these cylinders to each other in the canonical way

Definition 1.58. For

$$X_\bullet = \left(X_0 \overset{f_0}{\longrightarrow} X_1 \overset{f_1}{\longrightarrow} X_2 \overset{f_2}{\longrightarrow} \cdots \right)$$

a sequence in Top, its **mapping telescope** is the quotient topological space of the disjoint union of product topological spaces

$$\mathrm{Tel}(X_\bullet) \coloneqq \left(\underset{n\in\mathbb{N}}{\sqcup} (X_n \times [n, n+1]) \right)/_\sim$$

where the equivalence relation quotiented out is

$$(x_n, n) \sim (f(x_n), n+1)$$

for all $n \in \mathbb{N}$ and $x_n \in X_n$.

Analogously for $X_\bullet$ a sequence of pointed topological spaces then use reduced cylinders (exmpl.) to set

$$\mathrm{Tel}(X_\bullet) \coloneqq \left(\underset{n\in\mathbb{N}}{\sqcup} (X_n \wedge [n, n+1]_+) \right)/_\sim \,.$$

Lemma 1.59. *For $X_\bullet$ the sequence of stages of a (pointed) CW-complex $X = \underset{\longleftarrow n}{\lim} X_n$, then the canonical map*

$$\mathrm{Tel}(X_\bullet) \longrightarrow X$$

from the mapping telescope, def. 1.58, is a weak homotopy equivalence.

Proof. Write in the following $\mathrm{Tel}(X)$ for $\mathrm{Tel}(X_\bullet)$ and write $\mathrm{Tel}(X_n)$ for the mapping telescop of the substages of the finite stage X_n of X. It is intuitively clear that each of the projections at finite stage

$$\mathrm{Tel}(X_n) \longrightarrow X_n$$

is a homotopy equivalence, hence (prop.) a weak homotopy equivalence. A concrete construction of a homotopy inverse is given for instance in (Switzer 75, proof of prop. 7.53).

Moreover, since spheres are compact, so that elements of homotopy groups $\pi_q(\mathrm{Tel}(X))$ are represented at some finite stage $\pi_q(\mathrm{Tel}(X_n))$ it follows that

$$\underset{\longrightarrow n}{\lim}\, \pi_q(\mathrm{Tel}(X_n)) \overset{\simeq}{\longrightarrow} \pi_q(\mathrm{Tel}(X))$$

are isomorphisms for all $q \in \mathbb{N}$ and all choices of basepoints (not shown).

Together these two facts imply that in the following commuting square, three morphisms are isomorphisms, as shown.

$$\begin{array}{ccc} \underset{\longleftarrow n}{\lim}\, \pi_q(\mathrm{Tel}(X_n)) & \overset{\simeq}{\longrightarrow} & \pi_q(\mathrm{Tel}(X)) \\ {}^{\simeq}\downarrow && \downarrow \\ \underset{\longleftarrow n}{\lim}\, \pi_q(X_n) & \underset{\simeq}{\longrightarrow} & \pi_q(X) \end{array} \,.$$

Therefore also the remaining morphism is an isomorphism (two-out-of-three). Since this holds for all q and all basepoints, it is a weak homotopy equivalence. ■

Milnor exact sequences

Proposition 1.60. ***(Milnor exact sequence for homotopy groups)***

Let

$$\cdots \to X_3 \overset{p_2}{\longrightarrow} X_2 \overset{p_1}{\longrightarrow} X_1 \overset{p_0}{\longrightarrow} X_0$$

be a tower of fibrations (Serre fibrations (def.)). Then for each $q \in \mathbb{N}$ there is a short exact sequence

$$0 \to \underset{\longleftarrow i}{\lim}^1 \pi_{q+1}(X_i) \longrightarrow \pi_q(\underset{\longleftarrow i}{\lim} X_i) \longrightarrow \underset{\longleftarrow i}{\lim} \pi_q(X_i) \to 0\,,$$

for $\pi_\bullet$ the homotopy group-functor (exact as pointed sets for $i = 0$, as groups for $i \geq 1$) which says that

1. *the failure of the limit over the homotopy groups of the stages of the tower to equal the homotopy groups of the limit of the tower is at most in the kernel of the canonical comparison map;*
2. *that kernel is the $\underset{\longleftarrow}{\lim}^1$ (def. 1.49) of the homotopy groups of the stages.*

An elementary but tedious proof is indicated in (Bousfield-Kan 72, chapter IX, theorem 3.1. The following is a neat model category-theoretic proof following (Goerss-Jardine 96, section VI. prop. 2.15), which however requires the concept of homotopy limit over towers.

Proof. With respect to the classical model structure on simplicial sets or the classical model structure on topological spaces, a tower of fibrations as stated is a fibrant object in the injective model structure on functors $[(\mathbb{N}, \geq), \mathrm{sSet}]_{\mathrm{inj}}$ ($[(\mathbb{N}, \geq), \mathrm{Top}]_{\mathrm{inj}}$) (prop). Hence the plain limit over this diagram represents the homotopy limit. By the discussion there, up to weak equivalence that homotopy limit is also the pullback in

$$\begin{array}{ccc} \mathrm{holim}\, X_\bullet & \longrightarrow & \prod_n \mathrm{Path}(X_n) \\ \downarrow & (\mathrm{pb}) & \downarrow \\ \prod_n X_n & \underset{(\mathrm{id}, p_n)_n}{\longrightarrow} & \prod_n X_n \times X_n \end{array}\,,$$

where on the right we have the product over all the canonical fibrations out of the path space objects. Hence also the left vertical morphism is a fibration, and so by taking its fiber over a basepoint, the pasting law gives a homotopy fiber sequence

$$\prod_n \Omega X_n \longrightarrow \mathrm{holim}\, X_\bullet \longrightarrow \prod_n X_n\,.$$

The long exact sequence of homotopy groups of this fiber sequence goes

$$\cdots \to \prod_n \pi_{q+1}(X_n) \longrightarrow \prod_n \pi_{q+1}(X_n) \longrightarrow \pi_q(\underset{\longleftarrow}{\lim} X_\bullet) \longrightarrow \prod_n \pi_q(X_n) \longrightarrow \prod_n \pi_q(X_n) \to \cdots\,.$$

Chopping that off by forming kernel and cokernel yields the claim for positive q. For $q = 0$ it follows by inspection. ▮

Proposition 1.61. ***(Milnor exact sequence for generalized cohomology)***

Let X be a pointed CW-complex, $X = \underset{\longrightarrow n}{\lim} X_n$ and let $\tilde E^\bullet$ an additive reduced cohomology theory, def. 1.1.

Then the canonical morphisms make a short exact sequence

$$0 \to \underset{\longleftarrow n}{\lim}^1 \tilde E^{\bullet - 1}(X_n) \longrightarrow \tilde E^\bullet(X) \longrightarrow \underset{\longleftarrow n}{\lim} \tilde E^\bullet(X_n) \to 0\,,$$

saying that

1. *the failure of the canonical comparison map $\tilde E^\bullet(X) \to \underset{\longleftarrow}{\lim} \tilde E^\bullet(X_n)$ to the limit of the cohomology groups on the finite stages to be an isomorphism is at most in a non-vanishing kernel;*
2. *this kernel is precisely the $\lim^1$ (def. 1.49) of the cohomology groups at the finite stages in one degree lower.*

e.g. (Switzer 75, prop. 7.66, Kochmann 96, prop. 4.2.2)

Proof. For

$$X_\bullet = \left(X_0 \overset{i_0}{\hookrightarrow} X_1 \overset{i_1}{\hookrightarrow} X_2 \overset{i_1}{\hookrightarrow} \cdots \right)$$

the sequence of stages of the (pointed) CW-complex $X = \underset{\longleftarrow n}{\lim} X_n$, write

$$
\begin{aligned}
A_X &\coloneqq \underset{n\in\mathbb{N}}{\sqcup} X_{2n} \times [2n, 2n+1];\\
B_X &\coloneqq \underset{n\in\mathbb{N}}{\sqcup} X_{(2n+1)} \times [2n+1, 2n+2]\,.
\end{aligned}
$$

for the disjoint unions of the cylinders over all the stages in even and all those in odd degree, respectively.

These come with canonical inclusion maps into the mapping telescope $\mathrm{Tel}(X_\bullet)$ (def.), which we denote by

$$
\begin{tikzcd}
A_X \arrow[dr, "\iota_{A_X}"'] & & B_X \arrow[dl, "\iota_{B_X}"] \\
& \mathrm{Tel}(X_\bullet) &
\end{tikzcd}
.
$$

Observe that

1. $A_X \cup B_X \simeq \mathrm{Tel}(X_\bullet)$;

2. $A_X \cap B_X \simeq \underset{n\in\mathbb{N}}{\sqcup} X_n$;

and that there are homotopy equivalences

1. $A_X \simeq \underset{n\in\mathbb{N}}{\sqcup} X_{2n+1}$

2. $B_X \simeq \underset{n\in\mathbb{N}}{\sqcup} X_{2n}$

3. $\mathrm{Tel}(X_\bullet) \simeq X$.

The first two are obvious, the third is this proposition.

This implies that the Mayer-Vietoris sequence (prop.) for $\tilde E^\bullet$ on the cover $A \sqcup B \to X$ is isomorphic to the bottom horizontal sequence in the following diagram:

$$
\begin{tikzcd}
\tilde E^{\bullet -1}(A_X) \oplus \tilde E^{\bullet -1}(B_X) \arrow[r] \arrow[d, "\simeq"] & \tilde E^{\bullet -1}(A_X \cap B_X) \arrow[r] \arrow[d, "\simeq"] & \tilde E^\bullet(X) \arrow[r, "(\iota_{A_X})^\ast - (\iota_{B_X})^\ast"] \arrow[d, "="] & \tilde E^\bullet(A_X) \oplus \tilde E^\bullet(B_X) \arrow[r] \arrow[d, "(\mathrm{id}, -\mathrm{id})"', "\simeq"] & \tilde E^\bullet(A_X \cap B_X) \arrow[d, "\simeq"] \\
\prod_n \tilde E^{\bullet -1}(X_n) \arrow[r, "\partial"'] & \prod_n \tilde E^{\bullet -1}(X_n) \arrow[r] & \tilde E^\bullet(X) \arrow[r, "(i_n^\ast)_n"] & \prod_n \tilde E^\bullet(X_n) \arrow[r, "\partial"'] & \prod_n \tilde E^\bullet(X_n)
\end{tikzcd}
,
$$

hence that the bottom sequence is also a long exact sequence.

To identify the morphism ∂, notice that it comes from pulling back E-cohomology classes along the inclusions $A \cap B \to A$ and $A \cap B \to B$. Comonentwise these are the inclusions of each X_n into the left and the right end of its cylinder inside the mapping telescope, respectively. By the construction of the mapping telescope, one of these ends is embedded via $i_n : X_n \hookrightarrow X_{n+1}$ into the cylinder over X_{n+1}. In conclusion, ∂ acts by

$$
\partial \,:\, (a_n)_{n\in\mathbb{N}} \mapsto (a_n - i_n^\ast(a_{n+1}))\,.
$$

(The relative sign is the one in $(\iota_{A_X})^\ast - (\iota_{B_X})^\ast$ originating in the definition of the Mayer-Vietoris sequence and properly propagated to the bottom sequence while ensuring that $\tilde E^\bullet(X) \to \prod_n \tilde E^\bullet(X_n)$ is really $(i_n^\ast)_n$ and not $(-1)^n (i_n^\ast)_n$, as needed for the statement to be proven.)

This is the morphism from def. 1.47 for the sequence

$$
\cdots \to \tilde E^\bullet(X_{n+1}) \overset{i_n^\ast}{\longrightarrow} \tilde E^\bullet(X_n) \overset{i_n^\ast}{\longrightarrow} \tilde E^\bullet(X_{n-1}) \to \cdots\,.
$$

Hence truncating the above long exact sequence by forming kernel and cokernel of ∂, the result follows via remark 1.48 and definition 1.49. ▮

In contrast:

Proposition 1.62. *Let X be a pointed CW-complex, $X = \underset{\longleftarrow_n}{\lim} X_n$.*

For $\tilde E_\bullet$ an additive reduced generalized homology theory, then

$$
\underset{\longrightarrow_n}{\lim} \tilde E_\bullet(X_n) \overset{\simeq}{\longrightarrow} \tilde E_\bullet(X)
$$

is an isomorphism.

(Switzer 75, prop. 7.53)

There is also a version for cohomology *of* spectra:

For $X, E \in \mathrm{Ho}(\mathrm{Spectra})$ two spectra, then the E-generalized cohomology of X is the graded group of homs in the stable homotopy category (def., exmpl.)

$$\begin{aligned} E^\bullet(X) &\coloneqq [X, E]_{-\bullet} \\ &\coloneqq [\Sigma^\bullet X, E] \end{aligned} .$$

The stable homotopy category is, in particular, the homotopy category of the stable model structure on orthogonal spectra, in that its localization at the stable weak homotopy equivalences is of the form

$$\gamma \,:\, \mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})_{\mathrm{stable}} \longrightarrow \mathrm{Ho}(\mathrm{Spectra}) \,.$$

In the following when considering an orthogonal spectrum $X \in \mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})$, we use, for brevity, the same symbol for its image under γ.

Proposition 1.63. *For $X, E \in \mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}})$ two orthogonal spectra (or two symmetric spectra such that X is a semistable symmetric spectrum) then there is a short exact sequence of the form*

$$0 \to \underset{\longleftarrow n}{\lim}^1 E^{\bullet + n - 1}(X_n) \longrightarrow E^\bullet(X) \longrightarrow \underset{\longleftarrow n}{\lim} \; E^{\bullet + n}(X_n) \to 0$$

where $\underset{\longleftarrow}{\lim}^1$ denotes the lim^1, and where this and the limit on the right are taken over the following structure morphisms

$$E^{\bullet + n + 1}(X_{n+1}) \overset{E^{\bullet + 1 n + 1}(\Sigma^X_n)}{\longrightarrow} E^{\bullet + n + 1}(X_n \wedge S^1) \overset{\simeq}{\to} E^{\bullet + n}(X_n) \,.$$

(Schwede 12, chapter II prop. 6.5 (ii)) (using that symmetric spectra underlying orthogonal spectra are semistable (Schwede 12, p. 40))

Corollary 1.64. *For $X, E \in \mathrm{Ho}(\mathrm{Spectra})$ two spectra such that the tower $n \mapsto E^{n-1}(X_n)$ satisfies the Mittag-Leffler condition (def. 1.55), then two morphisms of spectra $X \longrightarrow E$ are homotopic already if all their morphisms of component spaces $X_n \to E_n$ are.*

Proof. By prop. 1.57 the assumption implies that the $\lim^1$-term in prop. 1.63 vanishes, hence by exactness it follows that in this case there is an isomorphism

$$[X, E] \simeq E^0(X) \overset{\simeq}{\to} \underset{\longleftarrow n}{\lim} [X_n, E_n] \,.$$

∎

Serre-Atiyah-Hirzebruch spectral sequence

Idea. Another important tool for computing generalized cohomology is to reduce it to the computation of ordinary cohomology with coefficients. Given a generalized cohomology theory E, there is a spectral sequence known as the *Atiyah-Hirzebruch spectral sequence* (AHSS) which serves to compute E-cohomology of F-fiber bundles over a simplicial complex X in terms of ordinary cohomology with coefficients in the generalized cohomology $E^\bullet(F)$ of the fiber. For $E =$ HA this is known as the *Serre spectral sequence*.

The Atiyah-Hirzebruch spectral sequence in turn is a consequence of the "Cartan-Eilenberg spectral sequence" which arises from the exact couple of relative cohomology groups of the skeleta of the CW-complex, and whose first page is the relative cohomology groups for codimension-1 skeleta.

We need the AHSS for instance for the computation of Conner-Floyd Chern classes below.

Literature. (Kochman 96, section 2.2 and 4.2)

See also the accompanying *lecture notes on spectral sequences*.

Converging spectral sequences

Definition 1.65. A **cohomology spectral sequence** $\{E_r^{p,q}, d_r\}$ is

1. a sequence $\{E_r^{\bullet,\bullet}\}$ (for $r \in \mathbb{N}$, $r \geq 1$) of bigraded abelian groups (the "pages");
2. a sequence of linear maps (the "differentials")

$$\{d_r \,:\, E_r^{\bullet,\bullet} \longrightarrow E_r^{\bullet + r, \bullet - r + 1}\}$$

such that

- $H_{r+1}^{\bullet,\bullet}$ is the cochain cohomology of d_r, i.e. $E_{r+1}^{\bullet,\bullet} = H(E_r^{\bullet,\bullet}, d_r)$, for all $r \in \mathbb{N}$, $r \geq 1$.

Given a $\mathbb{Z}$-graded abelian group_ $C^\bullet$ equipped with a decreasing filtration

$$C^\bullet \supset \cdots \supset F^s C^\bullet \supset F^{s+1} C^\bullet \supset \cdots \supset 0$$

such that

$$C^\bullet = \underset{s}{\cup} F^s C^\bullet \quad \text{and} \quad 0 = \underset{s}{\cap} F^s C^\bullet$$

then the spectral sequence is said to **converge** to $C^\bullet$, denoted,

$$E_2^{\bullet,\bullet} \Rightarrow C^\bullet$$

if

1. in each bidegree (s,t) the sequence $\{E_r^{s,t}\}_r$ eventually becomes constant on a group

   $E_\infty^{s,t} \coloneqq E_{\gg 1}^{s,t}$;

2. $E_\infty^{\bullet,\bullet}$ is the associated graded of the filtered $C^\bullet$ in that

   $E_\infty^{s,t} \simeq F^s C^{s+t} / F^{s+1} C^{s+t}$.

The converging spectral sequence is called a **multiplicative spectral sequence** if

1. $\{E_2^{\bullet,\bullet}\}$ is equipped with the structure of a bigraded algebra;

2. $F^\bullet C^\bullet$ is equipped with the structure of a filtered graded algebra ($F^p C^k \cdot F^q C^l \subset F^{p+q} C^{k+l}$);

such that

1. each d_r is a derivation with respect to the (induced) algebra structure on $E_r^{\bullet,\bullet}$, graded of degree 1 with respect to total degree;

2. the multiplication on $E_\infty^{\bullet,\bullet}$ is compatible with that on $C^\bullet$.

Remark 1.66. The point of spectral sequences is that by subdividing the data in any graded abelian group $C^\bullet$ into filtration stages, with each stage itself subdivided into bidegrees, such that each consecutive stage depends on the previous one in way tightly controled by the bidegrees, then this tends to give much control on the computation of $C^\bullet$. For instance it often happens that one may argue that the differentials in some spectral sequence all vanish from some page on (one says that the spectral sequence *collapses* at that page) by pure degree reasons, without any further computation.

Example 1.67. The archetypical example of (co-)homology spectral sequences as in def. 1.65 are induced from a filtering on a (co-)chain complex, converging to the (co-)chain homology of the chain complex by consecutively computing relative (co-)chain homologies, relative to decreasing (increasing) filtering degrees. For more on such spectral sequences of filtered complexes see at *Interlude -- Spectral sequences* the section *For filtered complexes*.

A useful way to generate spectral sequences is via exact couples:

Definition 1.68. An **exact couple** is three homomorphisms of abelian groups of the form

$$\begin{array}{ccc} D & \overset{g}{\longrightarrow} & D \\ & {}_{f}\nwarrow \quad \swarrow_{h} & \\ & E & \end{array}$$

such that the image of one is the kernel of the next.

$$\mathrm{im}(h) = \ker(f)\,, \quad \mathrm{im}(f) = \ker(g)\,, \quad \mathrm{im}(g) = \ker(f)\;.$$

Given an exact couple, then its **derived exact couple** is

$$\begin{array}{ccc} \mathrm{im}(g) & \overset{g}{\longrightarrow} & \mathrm{im}(g) \\ & {}_{f}\nwarrow \quad \swarrow_{h \circ g^{-1}} & \\ & H(E, h \circ f) & \end{array} \,,$$

where g^{-1} denotes the operation of sending one equivalence class to the equivalenc class of any preimage under g of any of its representatives.

Proposition 1.69. ***(cohomological spectral sequence of an exact couple)***

Given an exact couple, def. 1.68,

$$\begin{matrix} D_1 & \stackrel{g_1}{\longrightarrow} & D_1 \\ & {}_{f_1}\nwarrow & \swarrow_{h_1} \\ & E_1 \end{matrix}$$

its derived exact couple

$$\begin{matrix} D_2 & \stackrel{g_2}{\longrightarrow} & D_2 \\ & {}_{f_2}\nwarrow & \swarrow_{h_2} \\ & E_2 \end{matrix}$$

is itself an exact couple. Accordingly there is induced a sequence of exact couples

$$\begin{matrix} D_r & \stackrel{g_r}{\longrightarrow} & D_r \\ & {}_{f_r}\nwarrow & \swarrow_{h_r} \\ & E_r \end{matrix} \,.$$

If the abelian groups D and E are equipped with bigrading such that

$$\deg(f) = (0,0)\,, \quad \deg(g) = (-1,1)\,, \quad \deg(h) = (1,0)$$

then $\{E_r^{\bullet,\bullet}, d_r\}$ with

$$\begin{aligned} d_r &\coloneqq h_r \circ f_r \\ &= h \circ g^{-r+1} \circ f \end{aligned}$$

is a cohomological spectral sequence, def. 1.65.

(As before in prop. 1.69, the notation g^{-n} with $n \in \mathbb{N}$ denotes the function given by choosing, on representatives, a preimage under $g^n = \underbrace{g \circ \cdots \circ g \circ g}_{n \text{ times}}$, with the implicit claim that all possible choices represent the same equivalence class.)

If for every bidegree (s,t) there exists $R_{s,t} \gg 1$ such that for all $r \geq R_{s,t}$

1. $g \colon D^{s+R,t-R} \stackrel{\simeq}{\longrightarrow} D^{s+R-1,t-R-1}$;
2. $g \colon D^{s-R+1,t+R-2} \stackrel{0}{\longrightarrow} D^{s-R,t+R-1}$

then this spectral sequence converges to the inverse limit group

$$G^\bullet \coloneqq \lim \left(\cdots \stackrel{g}{\to} D^{s,\bullet-s} \stackrel{g}{\longrightarrow} D^{s-1,\bullet-s+1} \stackrel{g}{\to} \cdots \right)$$

filtered by

$$F^p G^\bullet \coloneqq \ker(G^\bullet \to D^{p-1,\bullet-p+1}) \,.$$

(e.g. Kochmann 96, lemma 2.6.2)

Proof. We check the claimed form of the E_∞-page:

Since $\ker(h) = \mathrm{im}(g)$ in the exact couple, the kernel

$$\ker(d_{r-1}) \coloneqq \ker(h \circ g^{-r+2} \circ f)$$

consists of those elements x such that $g^{-r+2}(f(x)) = g(y)$, for some y, hence

$$\ker(d_{r-1})^{s,t} \simeq f^{-1}(g^{r-1}(D^{s+r-1,t-r+1})) \,.$$

By assumption there is for each (s,t) an $R_{s,t}$ such that for all $r \geq R_{s,t}$ then $\ker(d_{r-1})^{s,t}$ is independent of r.

Moreover, $\mathrm{im}(d_{r-1})$ consists of the image under h of those $x \in D^{s-1,t}$ such that $g^{r-2}(x)$ is in the image of f, hence (since $\mathrm{im}(f) = \ker(g)$ by exactness of the exact couple) such that $g^{r-2}(x)$ is in the kernel of g, hence such that x is in the kernel of g^{r-1}. If $r > R$ then by assumption $g^{r-1}|_{D^{s-1,t}} = 0$ and so then $\mathrm{im}(d_{r-1}) = \mathrm{im}(h)$.

(Beware this subtlety: while $g^{R_{s,t}}|_{D^{s-1,t}}$ vanishes by the convergence assumption, the expression $g^{R_{s,t}}|_{D^{s+r-1,t-r+1}}$ need not vanish yet. Only the higher power $g^{R_{s,t}+R_{s+1,t+2}+2}|_{D^{s+r-1,t-r+1}}$ is again guaranteed to vanish.)

It follows that

$$
\begin{aligned}
E_\infty^{p,n-p} & = \ker(d_R)/\mathrm{im}(d_R) \\
& \simeq f^{-1}(\mathrm{im}(g^{R-1}))/\mathrm{im}(h) \\
& \underset{\simeq}{\overset{f}{\longrightarrow}} \mathrm{im}(g^{R-1}) \cap \mathrm{im}(f) \\
& \simeq \mathrm{im}(g^{R-1}) \cap \ker(g)
\end{aligned}
$$

where in last two steps we used once more the exactness of the exact couple.

(Notice that the above equation means in particular that the E_∞-page is a sub-group of the image of the E_1-page under f.)

The last group above is that of elements $x \in G^n$ which map to zero in $D^{p-1,n-p+1}$ and where two such are identified if they agree in $D^{p,n-p}$, hence indeed

$$
E_\infty^{p,n-p} \simeq F^p G^n / F^{p+1} G^n \,.
$$

∎

Remark 1.70. Given a spectral sequence (def. 1.65), then even if it converges strongly, computing its infinity-page still just gives the associated graded of the filtered object that it converges to, not the filtered object itself. The latter is in each filter stage an extension of the previous stage by the corresponding stage of the infinity-page, but there are in general several possible extensions (the trivial extension or some twisted extensions). The problem of determining these extensions and hence the problem of actually determining the filtered object from a spectral sequence converging to it is often referred to as the **extension problem**.

More in detail, consider, for definiteness, a cohomology spectral sequence converging to some filtered $F^\bullet H^\bullet$

$$
E^{p,q} \;\Rightarrow\; H^\bullet \,.
$$

Then by definition of convergence there are isomorphisms

$$
E_\infty^{p,\bullet} \simeq F^p H^{p+\bullet} / F^{p+1} H^{p+\bullet} \,.
$$

Equivalently this means that there are short exact sequences of the form

$$
0 \to F^{p+1} H^{p+\bullet} \hookrightarrow F^p H^{p+\bullet} \longrightarrow E_\infty^{p,\bullet} \to 0 \,.
$$

for all p. The extension problem then is to inductively deduce $F^p H^\bullet$ from knowledge of $F^{p+1} H^\bullet$ and $E_\infty^{p,\bullet}$.

In good cases these short exact sequences happen to be split exact sequences, which means that the extension problem is solved by the direct sum

$$
F^p H^{p+\bullet} \simeq F^{p+1} H^{p+\bullet} \oplus E_\infty^{p,\bullet} \,.
$$

But in general this need not be the case.

One sufficient condition that these exact sequences split is that they consist of homomorphisms of R-modules, for some ring R, and that $E_\infty^{p,\bullet}$ are projective modules (for instance free modules) over R. Because then the Ext-group $\mathrm{Ext}^1_R(E_\infty^{p,\bullet}, -)$ vanishes, and hence all extensions are trivial, hence split.

So for instance for every spectral sequence in vector spaces the extension problem is trivial (since every vector space is a free module).

The AHSS

The following proposition requires, in general, to evaluate cohomology functors not just on CW-complexes, but on all topological spaces. Hence we invoke prop. 1.4 to regard a reduced cohomology theory as a contravariant functor on all pointed topological spaces, which sends weak homotopy equivalences to isomorphisms (def. 1.3).

Proposition 1.71. ***(Serre-Cartan-Eilenberg-Whitehead-Atiyah-Hirzebruch spectral sequence)***

Let $A^\bullet$ be a an additive unreduced generalized cohomology functor (def.). Let B be a CW-complex and let $X \overset{\pi}{\to} B$ be a Serre fibration (def.), such that all its fibers are weakly contractible or such that B is simply connected. In either case all fibers are identified with a typical fiber F up to weak homotopy equivalence by connectedness (this example), and well defined up to unique iso in the homotopy category by simply connectedness:

$$\begin{array}{ccc} F & \longrightarrow & X \\ && \downarrow^{\mathrlap{\in Fib_{cl}}} \\ && B \end{array}$$

If at least one of the following two conditions is met

- *B is finite-dimensional as a CW-complex;*
- *$A^\bullet(F)$ is bounded below in degree and the sequences $\cdots \to A^p(X_{n+1}) \to A^p(X_n) \to \cdots$ satisfy the Mittag-Leffler condition (def. 1.55) for all p;*

then there is a cohomology spectral sequence, def. 1.65, whose E_2-page is the ordinary cohomology $H^\bullet(B, A^\bullet(F))$ of B with coefficients in the A-cohomology groups $A^\bullet(F)$ of the fiber, and which converges to the A-cohomology groups of the total space

$$E_2^{p,q} = H^p(B, A^q(F)) \;\Rightarrow\; A^\bullet(X)$$

with respect to the filtering given by

$$F^p A^\bullet(X) \coloneqq \ker\big(A^\bullet(X) \to A^\bullet(X_{p-1})\big),$$

where $X_p \coloneqq \pi^{-1}(B_p)$ is the fiber over the pth stage of the CW-complex $B = \underset{\longleftarrow n}{\lim} B_n$.

Proof. The exactness axiom for A gives an exact couple, def. 1.68, of the form

$$\begin{array}{ccc} \prod_{s,t} A^{s+t}(X_s) & \longrightarrow & \prod_{s,t} A^{s+t}(X_s) \\ & {}_{\nwarrow} \quad {}_{\swarrow} & \\ & \prod_{s,t} A^{s+t}(X_s, X_{s-1}) & \end{array} \quad \left(\begin{array}{ccc} A^{s+t}(X_s) & \longrightarrow & A^{s+t}(X_{s-1}) \\ \uparrow && \downarrow_{\delta} \\ A^{s+t}(X_s, X_{s-1}) && A^{s+t+1}(X_s, X_{s-1}) \end{array} \right),$$

where we take $X_{\gg 1} = X$ and $X_{<0} = \emptyset$.

In order to determine the E_2-page, we analyze the E_1-page: By definition

$$E_1^{s,t} = A^{s+t}(X_s, X_{s-1})$$

Let $C(s)$ be the set of s-dimensional cells of B, and notice that for $\sigma \in C(s)$ then

$$(\pi^{-1}(\sigma), \pi^{-1}(\partial \sigma)) \simeq (D^n, S^{n-1}) \times F_\sigma,$$

where F_σ is weakly homotopy equivalent to F (exmpl.).

This implies that

$$\begin{aligned} E_1^{s,t} & \coloneqq A^{s+t}(X_s, X_{s-1}) \\ & \simeq \tilde A^{s+t}(X_s / X_{s-1}) \\ & \simeq \tilde A^{s+t}\Big(\underset{\sigma \in C(n)}{\vee} S^s \wedge F_+ \Big) \\ & \simeq \prod_{\sigma \in C(s)} \tilde A^{s+t}(S^s \wedge F_+), \\ & \simeq \prod_{\sigma \in C(s)} \tilde A^{t}(F_+) \\ & \simeq \prod_{\sigma \in C(s)} A^{t}(F) \\ & \simeq C^s_{cell}(B, A^t(F)) \end{aligned}$$

where we used the relation to reduced cohomology $\tilde A$, prop. 1.19 together with lemma 1.11, then the wedge axiom and the suspension isomorphism of the latter.

The last group $C^s_{cell}(B, A^t(F))$ appearing in this sequence of isomorphisms is that of cellular cochains (def.) of degree s on B with coefficients in the group $A^t(F)$.

Since cellular cohomology of a CW-complex agrees with its singular cohomology (thm.), hence with its ordinary cohomology, to conclude that the E_2-page is as claimed, it is now sufficient to show that the differential d_1 coincides with the differential in the cellular cochain complex (def.).

We discuss this now for $\pi = \mathrm{id}$, hence $X = B$ and $F = *$. The general case works the same, just with various factors of F appearing in the following:

Consider the following diagram, which commutes due to the naturality of the connecting homomorphism δ of $A^\bullet$:

$$
\begin{array}{ccccccc}
\partial^* \colon & C^{s-1}_{\mathrm{cell}}(X, A^t(*)) & = & \prod_{i \in I_{s-1}} A^t(*) & \longrightarrow & \prod_{i \in I_s} A^t(*) & = & C^s_{\mathrm{cell}}(X, A^t(*)) \\
&& & {}^{\simeq}\downarrow && \downarrow^{\simeq} \\
&& & \prod_{i \in I_{s-1}} \tilde A^{s+t-1}(S^{s-1}) && \prod_{i\in I_s} \tilde A^{s+t}(S^s) \\
&& & {}^{\simeq}\downarrow && \downarrow^{\simeq} \\
& d_1 \colon & & A^{s+t-1}(X_{s-1}, X_{s-2}) & \longrightarrow\; A^{s+t-1}(X_{s-1}) \;\overset{\delta}{\longrightarrow} & A^{s+t}(X_s, X_{s-1}) \\
&&& \downarrow & \downarrow & \downarrow \\
&&& A^{s+t-1}(S^{s-1}, \emptyset) & \longrightarrow\; A^{s+t-1}(S^{s-1}) \;\overset{\delta}{\longrightarrow} & A^{s+t}(D^s, S^{s-1})
\end{array}
.
$$

Here the bottom vertical morphisms are those induced from any chosen cell inclusion $(D^s, S^{s-1}) \hookrightarrow (X_s, X_{s-1})$.

The differential d_1 in the spectral sequence is the middle horizontal composite. From this the vertical isomorphisms give the top horizontal map. But the bottom horizontal map identifies this top horizontal morphism componentwise with the restriction to the boundary of cells. Hence the top horizontal morphism is indeed the coboundary operator ∂^* for the cellular cohomology of X with coefficients in $A^\bullet(*)$ (def.). This cellular cohomology coincides with singular cohomology of the CW-complex X (thm.), hence computes the ordinary cohomology of X.

Now to see the convergence. If B is finite dimensional then the convergence condition as stated in prop. 1.69 is met. Alternatively, if $A^\bullet(F)$ is bounded below in degree, then by the above analysis the E_1-page has a horizontal line below which it vanishes. Accordingly the same is then true for all higher pages, by each of them being the cohomology of the previous page. Since the differentials go right and down, eventually they pass beneath this vanishing line and become 0. This is again the condition needed in the proof of prop. 1.69 to obtain convergence.

By that proposition the convergence is to the inverse limit

$$
\underset{\longleftarrow}{\lim}(\cdots \to A^\bullet(X_{s+1}) \longrightarrow A^\bullet(X_s) \to \cdots) \,.
$$

If X is finite dimensional or more generally if the sequences that this limit is over satisfy the Mittag-Leffler condition (def. 1.55), then this limit is $A^\bullet(X)$, by prop. 1.57. ∎

Multiplicative structure

Proposition 1.72. *For $E^\bullet$ a multiplicative cohomology theory (def. 1.26), then the Atiyah-Hirzebruch spectral sequences (prop. 1.71) for $E^\bullet(X)$ are multiplicative spectral sequences.*

A decent proof is spelled out in (Kochman 96, prop. 4.2.9). Use the graded commutativity of smash products of spheres to get the sign in the graded derivation law for the differentials. See also the proof via Cartan-Eilenberg systems at *multiplicative spectral sequence – Examples – AHSS for multiplicative cohomology*.

Proposition 1.73. *Given a multiplicative cohomology theory $(A, \mu, 1)$ (def. 1.26), then for every Serre fibration $X \to B$ (def.) all the differentials in the corresponding Atiyah-Hirzebruch spectral sequence of prop. 1.71*

$$
H^\bullet(B, A^\bullet(F)) \;\Rightarrow\; A^\bullet(X)
$$

are linear over $A^\bullet(*)$.

Proof. By the proof of prop. 1.71, the differentials are those induced by the exact couple

$$
\begin{array}{ccccc}
\prod_{s,t} A^{s+t}(X_s) && \longrightarrow && \prod_{s,t} A^{s+t}(X_s) \\
& \nwarrow && \swarrow & \\
&& \prod_{s,t} A^{s+t}(X_s, X_{s-1}) &&
\end{array}
\left(
\begin{array}{ccc}
A^{s+t}(X_s) & \longrightarrow & A^{s+t}(X_{s-1}) \\
\uparrow && \downarrow_{\delta} \\
A^{s+t}(X_s, X_{s-1}) && A^{s+t+1}(X_s, X_{s-1})
\end{array}
\right).
$$

consisting of the pullback homomorphisms and the connecting homomorphisms of A.

By prop. 1.69 its differentials on page r are the composites of one pullback homomorphism, the preimage of $(r-1)$ pullback homomorphisms, and one connecting homomorphism of A. Hence the statement follows with prop. 1.27. ▮

Proposition 1.74. *For E a homotopy commutative ring spectrum (def.) and X a finite CW-complex, then the Kronecker pairing*

$$
\langle -,- \rangle_X : E^{\bullet_1}(X) \otimes E_{\bullet_2}(X) \longrightarrow \pi_{\bullet_2 - \bullet_1}(E)
$$

extends to a compatible pairing of Atiyah-Hirzebruch spectral sequences.

(Kochman 96, prop. 4.2.10)

S.2) Cobordism theory

Idea. As one passes from abelian groups to spectra, a miracle happens: even though the latter are just the proper embodiment of linear algebra in the context of homotopy theory ("higher algebra") their inspection reveals that spectra natively know about deep phenomena of differential topology, index theory and in fact string theory (for instance via a close relation between *genera and partition functions*).

A strong manifestation of this phenomenon comes about in complex oriented cohomology theory/chromatic homotopy theory that we eventually come to below. It turns out to be higher algebra over the complex Thom spectrum MU.

Here we first concentrate on its real avatar, the Thom spectrum MO. The seminal result of Thom's theorem says that the stable homotopy groups of MO form the cobordism ring of cobordism-equivalence classes of manifolds. In the course of discussing this *cobordism theory* one encounters various phenomena whose complex version also governs the complex oriented cohomology theory that we are interested in below.

Literature. (Kochman 96, chapter I and sections II.2, II6). A quick efficient account is in (Malkiewich 11). See also (Aguilar-Gitler-Prieto 02, section 11).

Classifying spaces and G-Structure

Idea. Every manifold X of dimension n carries a canonical vector bundle of rank n: its tangent bundle. There is a universal vector bundle of rank n, of which all others arise by pullback, up to isomorphism. The base space of this universal bundle is hence called the classifying space and denoted $B\,\mathrm{GL}(n) \simeq BO(n)$ (for $O(n)$ the orthogonal group). This may be realized as the homotopy type of a direct limit of Grassmannian manifolds. In particular the tangent bundle of a manifold X is classified by a map $X \longrightarrow BO(n)$, unique up to homotopy. For G a subgroup of $O(n)$, then a lift of this map through the canonical map $BG \longrightarrow BO(n)$ of classifying spaces is a *G-structure* on X

$$
\begin{array}{ccc}
& & BG \\
& \nearrow & \downarrow \\
X & \longrightarrow & BO(n)
\end{array}
$$

for instance an orientation for the inclusion $SO(n) \hookrightarrow O(n)$ of the special orthogonal group, or an almost complex structure for the inclusion $U(n) \hookrightarrow O(2n)$ of the unitary group.

All this generalizes, for instance from tangent bundles to normal bundles with respect to any embedding. It also behaves well with respect to passing to the boundary of manifolds, hence to bordism-classes of manifolds. This is what appears in Thom's theorem below.

Literature. (Kochman 96, 1.3-1.4), for stable normal structures also (Stong 68, beginning of chapter II)

Coset spaces

Proposition 1.75. *For X a smooth manifold and G a compact Lie group equipped with a free smooth action on X, then the quotient projection*

$$X \longrightarrow X/G$$

is a G-principal bundle (hence in particular a Serre fibration).

This is originally due to (Gleason 50). See e.g. (Cohen, theorem 1.3)

Corollary 1.76. *For G a Lie group and $H \subset G$ a compact subgroup, then the coset quotient projection*

$$G \longrightarrow G/H$$

is an H-principal bundle (hence in particular a Serre fibration).

Proposition 1.77. *For G a compact Lie group and $K \subset H \subset G$ closed subgroups, then the projection map on coset spaces*

$$p : G/K \longrightarrow G/H$$

is a locally trivial H/K-fiber bundle (hence in particular a Serre fibration).

Proof. Observe that the projection map in question is equivalently

$$G \times_H (H/K) \longrightarrow G/H\,,$$

(where on the left we form the Cartesian product and then divide out the diagonal action by H). This exhibits it as the H/K-fiber bundle associated to the H-principal bundle of corollary 1.76. ∎

Orthogonal and Unitary groups

Proposition 1.78. *The orthogonal group $O(n)$ is compact topological space, hence in particular a compact Lie group.*

Proposition 1.79. *The unitary group $U(n)$ is compact topological space, hence in particular a compact Lie group.*

Example 1.80. The n-spheres are coset spaces of orthogonal groups:

$$S^n \simeq O(n+1)/O(n)\,.$$

The odd-dimensional spheres are also coset spaces of unitary groups:

$$S^{2n+1} \simeq U(n+1)/U(n)$$

Proof. Regarding the first statement:

Fix a unit vector in $\mathbb{R}^{n+1}$. Then its orbit under the defining $O(n+1)$-action on $\mathbb{R}^{n+1}$ is clearly the canonical embedding $S^n \hookrightarrow \mathbb{R}^{n+1}$. But precisely the subgroup of $O(n+1)$ that consists of rotations around the axis formed by that unit vector stabilizes it, and that subgroup is isomorphic to $O(n)$, hence $S^n \simeq O(n+1)/O(n)$.

The second statement follows by the same kind of reasoning:

Clearly $U(n+1)$ acts transitively on the unit sphere S^{2n+1} in $\mathbb{C}^{n+1}$. It remains to see that its stabilizer subgroup of any point on this sphere is $U(n)$. If we take the point with coordinates $(1,0,0,\cdots,0)$ and regard elements of $U(n+1)$ as matrices, then the stabilizer subgroup consists of matrices of the block diagonal form

$$\begin{pmatrix} 1 & \vec{0} \\ \vec{0} & A \end{pmatrix}$$

where $A \in U(n)$. ∎

Proposition 1.81. *For $n, k \in \mathbb{N}$, $n \leq k$, then the canonical inclusion of orthogonal groups*

$$O(n) \hookrightarrow O(k)$$

is an (n-1)-equivalence, hence induces an isomorphism on homotopy groups in degrees $< n-1$ and a surjection in degree $n-1$.

Proof. Consider the coset quotient projection

$$O(n) \longrightarrow O(n+1) \longrightarrow O(n+1)/O(n)\,.$$

By prop. 1.78 and by corollary 1.76, the projection $O(n+1) \to O(n+1)/O(n)$ is a Serre fibration. Furthermore, example 1.80 identifies the coset with the n-sphere

$$S^n \simeq O(n+1)/O(n)\,.$$

Therefore the long exact sequence of homotopy groups (exmpl.)of the fiber sequence $O(n) \to O(n+1) \to S^n$ has the form

$$\cdots \to \pi_{\bullet+1}(S^n) \longrightarrow \pi_\bullet(O(n)) \longrightarrow \pi_\bullet(O(n+1)) \longrightarrow \pi_\bullet(S^n) \to \cdots$$

Since $\pi_{<n}(S^n) = 0$, this implies that

$$\pi_{<n-1}(O(n)) \overset{\simeq}{\longrightarrow} \pi_{<n-1}(O(n+1))$$

is an isomorphism and that

$$\pi_{n-1}(O(n)) \overset{\simeq}{\longrightarrow} \pi_{n-1}(O(n+1))$$

is surjective. Hence now the statement follows by induction over $k-n$. ▮

Similarly:

Proposition 1.82. *For $n,k \in \mathbb{N}$, $n \leq k$, then the canonical inclusion of unitary groups*

$$U(n) \hookrightarrow U(k)$$

is a 2n-equivalence, hence induces an isomorphism on homotopy groups in degrees $< 2n$ and a surjection in degree $2n$.

Proof. Consider the coset quotient projection

$$U(n) \longrightarrow U(n+1) \longrightarrow U(n+1)/U(n)\,.$$

By prop. 1.79 and corollary 1.76, the projection $U(n+1) \to U(n+1)/U(n)$ is a Serre fibration. Furthermore, example 1.80 identifies the coset with the (2n+1)-sphere

$$S^{2n+1} \simeq U(n+1)/U(n)\,.$$

Therefore the long exact sequence of homotopy groups (exmpl.)of the fiber sequence $U(n) \to U(n+1) \to S^{2n+1}$ is of the form

$$\cdots \to \pi_{\bullet+1}(S^{2n+1}) \longrightarrow \pi_\bullet(U(n)) \longrightarrow \pi_\bullet(U(n+1)) \longrightarrow \pi_\bullet(S^{2n+1}) \to \cdots$$

Since $\pi_{\leq 2n}(S^{2n+1}) = 0$, this implies that

$$\pi_{<2n}(U(n)) \overset{\simeq}{\longrightarrow} \pi_{<2n}(U(n+1))$$

is an isomorphism and that

$$\pi_{2n}(U(n)) \overset{\simeq}{\longrightarrow} \pi_{2n}(U(n+1))$$

is surjective. Hence now the statement follows by induction over $k-n$. ▮

Stiefel manifolds and Grassmannians

Throughout we work in the category $\mathrm{Top}_{\mathrm{cg}}$ of compactly generated topological spaces (def.). For these the Cartesian product $X \times (-)$ is a left adjoint (prop.) and hence preserves colimits.

Definition 1.83. For $n,k \in \mathbb{N}$ and $n \leq k$, then the nth **real Stiefel manifold** of $\mathbb{R}^k$ is the coset topological space.

$$V_n(\mathbb{R}^k) \coloneqq O(k)/O(k-n)\,,$$

where the action of $O(k-n)$ is via its canonical embedding $O(k-n) \hookrightarrow O(k)$.

Similarly the nth **complex Stiefel manifold** of $\mathbb{C}^k$ is

$$V_n(\mathbb{C}^k) \coloneqq U(k)/U(k-n)\,,$$

here the action of $U(k-n)$ is via its canonical embedding $U(k-n) \hookrightarrow U(k)$.

Definition 1.84. For $n, k \in \mathbb{N}$ and $n \leq k$, then the nth **real Grassmannian** of $\mathbb{R}^k$ is the coset topological space.

$$\mathrm{Gr}_n(\mathbb{R}^k) \coloneqq O(k)/(O(n) \times O(k-n)) \,,$$

where the action of the product group is via its canonical embedding $O(n) \times O(k-n) \hookrightarrow O(n)$ into the orthogonal group.

Similarly the nth **complex Grassmannian** of $\mathbb{C}^k$ is the coset topological space.

$$\mathrm{Gr}_n(\mathbb{C}^k) \coloneqq U(k)/(U(n) \times U(k-n)) \,,$$

where the action of the product group is via its canonical embedding $U(n) \times U(k-n) \hookrightarrow U(n)$ into the unitary group.

Example 1.85.

- $G_1(\mathbb{R}^{n+1}) \simeq \mathbb{R}P^n$ is real projective space of dimension n.
- $G_1(\mathbb{C}^{n+1}) \simeq \mathbb{C}P^n$ is complex projective space of dimension n (def. 1.134).

Proposition 1.86. *For all $n \leq k \in \mathbb{N}$, the canonical projection from the Stiefel manifold (def. 1.83) to the Grassmannian is a $O(n)$-principal bundle*

$$\begin{array}{ccc} O(n) & \hookrightarrow & V_n(\mathbb{R}^k) \\ && \downarrow \\ && \mathrm{Gr}_n(\mathbb{R}^k) \end{array}$$

and the projection from the complex Stiefel manifold to the Grassmannian us a $U(n)$-principal bundle:

$$\begin{array}{ccc} U(n) & \hookrightarrow & V_n(\mathbb{C}^k) \\ && \downarrow \\ && \mathrm{Gr}_n(\mathbb{C}^k) \end{array} \,.$$

Proof. By prop 1.76 and prop 1.77. ▮

Proposition 1.87. *The real Grassmannians $\mathrm{Gr}_n(\mathbb{R}^k)$ and the complex Grassmannians $\mathrm{Gr}_n(\mathbb{C}^k)$ of def. 1.84 admit the structure of CW-complexes. Moreover the canonical inclusions*

$$\mathrm{Gr}_n(\mathbb{R}^k) \hookrightarrow \mathrm{Gr}_n(\mathbb{R}^{k+1})$$

are subcomplex incusion (hence relative cell complex inclusions).

Accordingly there is an induced CW-complex structure on the classifying space (def. 1.91).

$$B O(n) \simeq \underset{\longrightarrow_k}{\lim} \mathrm{Gr}_n(\mathbb{R}^k) \,.$$

A proof is spelled out in (Hatcher, section 1.2 (pages 31-34)).

Proposition 1.88. *The Stiefel manifolds $V_n(\mathbb{R}^k)$ and $V_n(\mathbb{C}^k)$ from def. 1.83 admits the structure of a CW-complex.*

e.g. (James 59, p. 3, James 76, p. 5 with p. 21, Blaszczyk 07)

(And I suppose with that cell structure the inclusions $V_n(\mathbb{R}^k) \hookrightarrow V_n(\mathbb{R}^{k+1})$ are subcomplex inclusions.)

Proposition 1.89. *The real Stiefel manifold $V_n(\mathbb{R}^k)$ (def. 1.83) is (k-n-1)-connected.*

Proof. Consider the coset quotient projection

$$O(k-n) \longrightarrow O(k) \longrightarrow O(k)/O(k-n) = V_n(\mathbb{R}^k) \,.$$

By prop. 1.78 and by corollary 1.76, the projection $O(k) \to O(k)/O(k-n)$ is a Serre fibration. Therefore there is induced the long exact sequence of homotopy groups of this fiber sequence, and by prop. 1.81 it has the following form in degrees bounded by n:

$$\cdots \to \pi_{\bullet \leq k-n-1}(O(k-n)) \overset{\text{epi}}{\longrightarrow} \pi_{\bullet \leq k-n-1}(O(k)) \overset{0}{\longrightarrow} \pi_{\bullet \leq k-n-1}(V_n(\mathbb{R}^k)) \overset{0}{\longrightarrow} \pi_{\bullet -1 < k-n-1}(O(k)) \overset{\simeq}{\longrightarrow} \pi_{\bullet -1 < k-n-1}(O(k-n)) \to \cdots \,.$$

This implies the claim. (Exactness of the sequence says that every element in $\pi_{\bullet \leq n-1}(V_n(\mathbb{R}^k))$ is in the kernel of zero, hence in the image of 0, hence is 0 itself.) ▮

Similarly:

Proposition 1.90. *The complex Stiefel manifold $V_n(\mathbb{C}^k)$ (def. 1.83) is 2(k-n)-connected.*

Proof. Consider the coset quotient projection

$$U(k-n) \longrightarrow U(k) \longrightarrow U(k)/U(k-n) = V_n(\mathbb{C}^k) \;.$$

By prop. 1.79 and by corollary 1.76 the projection $U(k) \to U(k)/U(k-n)$ is a Serre fibration. Therefore there is induced the long exact sequence of homotopy groups of this fiber sequence, and by prop. 1.82 it has the following form in degrees bounded by n:

$$\cdots \to \pi_{\bullet \leq 2(k-n)}(U(k-n)) \overset{\text{epi}}{\longrightarrow} \pi_{\bullet \leq 2(k-n)}(U(k)) \overset{0}{\longrightarrow} \pi_{\bullet \leq 2(k-n)}(V_n(\mathbb{C}^k)) \overset{0}{\longrightarrow} \pi_{\bullet -1 < 2(k-n)}(U(k)) \overset{\simeq}{\longrightarrow} \pi_{\bullet -1 < 2(k-n)}(U(k-n)) \to \cdots \;.$$

This implies the claim. ▮

Classifying spaces

Definition 1.91. By def. 1.84 there are canonical inclusions

$$\mathrm{Gr}_n(\mathbb{R}^k) \hookrightarrow \mathrm{Gr}_n(\mathbb{R}^{k+1})$$

and

$$\mathrm{Gr}_n(\mathbb{C}^k) \hookrightarrow \mathrm{Gr}_n(\mathbb{C}^{k+1})$$

for all $k \in \mathbb{N}$. The colimit (in Top, see there, or rather in $\mathrm{Top}_{\mathrm{cg}}$, see this cor.) over these inclusions is denoted

$$BO(n) \coloneqq \underset{\longrightarrow_k}{\lim} \mathrm{Gr}_n(\mathbb{R}^k)$$

and

$$BU(n) \coloneqq \underset{\longrightarrow_k}{\lim} \mathrm{Gr}_n(\mathbb{C}^k) \,,$$

respectively.

Moreover, by def. 1.83 there are canonical inclusions

$$V_n(\mathbb{R}^k) \hookrightarrow V_n(\mathbb{R}^{k+1})$$

and

$$V_n(\mathbb{C}^k) \hookrightarrow V_n(\mathbb{C}^{k+1})$$

that are compatible with the $O(n)$-action and with the $U(n)$-action, respectively. The colimit (in Top, see there, or rather in $\mathrm{Top}_{\mathrm{cg}}$, see this cor.) over these inclusions, regarded as equipped with the induced $O(n)$-action, is denoted

$$EO(n) \coloneqq \underset{\longrightarrow_k}{\lim} V_n(\mathbb{R}^k)$$

and

$$EU(n) \coloneqq \underset{\longrightarrow_k}{\lim} V_n(\mathbb{C}^k) \,,$$

respectively.

The inclusions are in fact compatible with the bundle structure from prop. 1.86, so that there are induced projections

$$\left(\begin{array}{c} EO(n) \\ \downarrow \\ BO(n) \end{array} \right) \;\simeq\; \underset{\longrightarrow_k}{\lim} \left(\begin{array}{c} V_n(\mathbb{R}^k) \\ \downarrow \\ \mathrm{Gr}_n(\mathbb{R}^k) \end{array} \right)$$

and

$$\left(\begin{array}{c} EU(n) \\ \downarrow \\ BU(n) \end{array}\right) \simeq \underset{\longrightarrow_k}{\lim} \left(\begin{array}{c} V_n(\mathbb{C}^k) \\ \downarrow \\ \mathrm{Gr}_n(\mathbb{C}^k) \end{array}\right),$$

respectively. These are the standard models for the **universal principal bundles** for O and U, respectively. The corresponding associated vector bundles

$$EO(n) \underset{O(n)}{\times} \mathbb{R}^n$$

and

$$EU(n) \underset{U(n)}{\times} \mathbb{C}^n$$

are the corresponding **universal vector bundles**.

Since the Cartesian product $O(n) \times (-)$ in compactly generated topological spaces preserves colimits, it follows that the colimiting bundle is still an $O(n)$-principal bundle

$$\begin{aligned} (EO(n))/O(n) &\simeq (\underset{\longrightarrow_k}{\lim} V_n(\mathbb{R}^k))/O(n) \\ &\simeq \underset{\longrightarrow_k}{\lim} (V_n(\mathbb{R}^k)/O(n)) \\ &\simeq \underset{\longrightarrow_k}{\lim} \mathrm{Gr}_n(\mathbb{R}^k) \\ &\simeq BO(n) \end{aligned},$$

and anlogously for $EU(n)$.

As such this is the standard presentation for the $O(n)$-universal principal bundle and $U(n)$-universal principal bundle, respectively. Its base space $BO(n)$ is the corresponding **classifying space**.

Definition 1.92. There are canonical inclusions

$$\mathrm{Gr}_n(\mathbb{R}^k) \hookrightarrow \mathrm{Gr}_{n+1}(\mathbb{R}^{k+1})$$

and

$$\mathrm{Gr}_n(\mathbb{C}^k) \hookrightarrow \mathrm{Gr}_{n+1}(\mathbb{C}^{k+1})$$

given by adjoining one coordinate to the ambient space and to any subspace. Under the colimit of def. 1.91 these induce maps of classifying spaces

$$BO(n) \longrightarrow BO(n+1)$$

and

$$BU(n) \longrightarrow BU(n+1)\;.$$

Definition 1.93. There are canonical maps

$$\mathrm{Gr}_{n_1}(\mathbb{R}^{k_1}) \times \mathrm{Gr}_{n_2}(\mathbb{R}^{k_2}) \longrightarrow \mathrm{Gr}_{n_1+n_2}(\mathbb{R}^{k_1+k_2})$$

and

$$\mathrm{Gr}_{n_1}(\mathbb{C}^{k_1}) \times \mathrm{Gr}_{n_2}(\mathbb{C}^{k_2}) \longrightarrow \mathrm{Gr}_{n_1+n_2}(\mathbb{C}^{k_1+k_2})$$

given by sending ambient spaces and subspaces to their direct sum.

Under the colimit of def. 1.91 these induce maps of classifying spaces

$$BO(n_1) \times BO(n_2) \longrightarrow BO(n_1+n_2)$$

and

$$BU(n_1) \times BU(n_2) \longrightarrow BU(n_1+n_2)$$

Proposition 1.94. *The colimiting space $EO(n) = \underset{\longrightarrow_k}{\lim} V_n(\mathbb{R}^k)$ from def. 1.91 is weakly contractible.*

The colimiting space $EU(n) = \underset{\longrightarrow_k}{\lim} V_n(\mathbb{C}^k)$ from def. 1.91 is weakly contractible.

Proof. By propositions 1.89, and 1.90, the Stiefel manifolds are more and more highly connected as k increases. Since the inclusions are relative cell complex inclusions by prop. 1.88, the claim follows. ▮

Proposition 1.95. *The homotopy groups of the classifying spaces $B O(n)$ and $B U(n)$ (def. 1.91) are those of the orthogonal group $O(n)$ and of the unitary group $U(n)$, respectively, shifted up in degree: there are isomorphisms*

$$\pi_{\bullet+1}(B O(n)) \simeq \pi_\bullet O(n)$$

and

$$\pi_{\bullet+1}(B U(n)) \simeq \pi_\bullet U(n)$$

(for homotopy groups based at the canonical basepoint).

Proof. Consider the sequence

$$O(n) \longrightarrow E O(n) \longrightarrow B O(n)$$

from def. 1.91, with $O(n)$ the fiber. Since (by prop. 1.77) the second map is a Serre fibration, this is a fiber sequence and so it induces a long exact sequence of homotopy groups of the form

$$\cdots \to \pi_\bullet(O(n)) \longrightarrow \pi_\bullet(E O(n)) \longrightarrow \pi_\bullet(B O(n)) \longrightarrow \pi_{\bullet-1}(O(n)) \longrightarrow \pi_{\bullet-1}(E O(n)) \to \cdots .$$

Since by cor. 1.94 $\pi_\bullet(E O(n)) = 0$, exactness of the sequence implies that

$$\pi_\bullet(B O(n)) \overset{\simeq}{\longrightarrow} \pi_{\bullet-1}(O(n))$$

is an isomorphism.

The same kind of argument applies to the complex case. ▮

Proposition 1.96. *For $n \in \mathbb{N}$ there are homotopy fiber sequence (def.)*

$$S^n \longrightarrow B O(n) \longrightarrow B O(n+1)$$

and

$$S^{2n+1} \longrightarrow B U(n) \longrightarrow B U(n+1)$$

exhibiting the n-sphere ($(2n+1)$-sphere) as the homotopy fiber of the canonical maps from def. 1.92.

This means (thm.), that there is a replacement of the canonical inclusion $B O(n) \hookrightarrow B O(n+1)$ (induced via def. 1.91) by a Serre fibration

$$\begin{array}{ccc} B O(n) & \hookrightarrow & B O(n+1) \\ {}^{\text{weak homotopy} \atop \text{equivalence}}\downarrow & \nearrow_{\text{Serre fib.}} & \\ \tilde B O(n) & & \end{array}$$

such that S^n is the ordinary fiber of $B O(n) \to \tilde B O(n+1)$, and analogously for the complex case.

Proof. Take $\tilde B O(n) \coloneqq (E O(n+1))/O(n)$.

To see that the canonical map $B O(n) \longrightarrow (E O(n+1))/O(n)$ is a weak homotopy equivalence consider the commuting diagram

$$\begin{array}{ccc} O(n) & \overset{\text{id}}{\longrightarrow} & O(n) \\ \downarrow && \downarrow \\ E O(n) & \longrightarrow & E O(n+1) \\ \downarrow && \downarrow \\ B O(n) & \longrightarrow & (E O(n+1))/O(n) \end{array} .$$

By prop. 1.77 both bottom vertical maps are Serre fibrations and so both vertical sequences are fiber sequences. By prop. 1.95 part of the induced morphisms of long exact sequences of homotopy groups looks like this

$$
\begin{array}{ccc}
\pi_\bullet(BO(n)) & \longrightarrow & \pi_\bullet((EO(n+1))/O(n)) \\
{}^{\simeq}\downarrow & & \downarrow^{\simeq} \\
\pi_{\bullet-1}(O(n)) & \overset{=}{\longrightarrow} & \pi_{\bullet-1}(O(n))
\end{array}
\,,
$$

where the vertical and the bottom morphism are isomorphisms. Hence also the to morphisms is an isomorphism.

That $BO(n) \to \tilde{B}O(n+1)$ is indeed a Serre fibration follows again with prop. 1.77, which gives the fiber sequence

$$
O(n+1)/O(n) \longrightarrow (EO(n+1))/O(n) \longrightarrow (EO(n+1))/O(n+1)\,.
$$

The claim then follows with the identification

$$
O(n+1)/O(n) \simeq S^n
$$

of example 1.80.

The argument for the complex case is directly analogous, concluding instead with the identification

$$
U(n+1)/U(n) \simeq S^{2n+1}
$$

from example 1.80. ▮

G-Structure on the Stable normal bundle

Definition 1.97. Given a smooth manifold X of dimension n and equipped with an embedding

$$
i \colon X \hookrightarrow \mathbb{R}^k
$$

for some $k \in \mathbb{N}$, then the **classifying map of its normal bundle** is the function

$$
g_i \colon X \to \mathrm{Gr}_{k-n}(\mathbb{R}^k) \hookrightarrow BO(k-n)
$$

which sends $x \in X$ to the normal of the tangent space

$$
N_x X = (T_x X)^{\perp} \hookrightarrow \mathbb{R}^k
$$

regarded as a point in $G_{k-n}(\mathbb{R}^k)$.

The normal bundle of i itself is the subbundle of the tangent bundle

$$
T\mathbb{R}^k \simeq \mathbb{R}^k \times \mathbb{R}^k
$$

consisting of those vectors which are orthogonal to the tangent vectors of X:

$$
N_i \coloneqq \left\{ x \in X, v \in T_{i(x)}\mathbb{R}^k \;|\; v \perp i_\ast T_x X \subset T_{i(x)}\mathbb{R}^k \right\}\,.
$$

Definition 1.98. A (B,f)**-structure** is

1. for each $n \in \mathbb{N}$ a pointed CW-complex $B_n \in \mathrm{Top}^{*/}_{\mathrm{CW}}$
2. equipped with a pointed Serre fibration

$$
\begin{array}{c}
B_n \\
\downarrow^{f_n} \\
BO(n)
\end{array}
$$

   to the classifying space $BO(n)$ (def.);
3. for all $n_1 \leq n_2$ a pointed continuous function

   $$
   g_{n_1,n_2} \colon B_{n_1} \longrightarrow B_{n_2}
   $$

   which is the identity for $n_1 = n_2$;

such that for all $n_1 \leq n_2 \in \mathbb{N}$ these squares commute

$$\begin{array}{ccc} B_{n_1} & \overset{g_{n_1,n_2}}{\longrightarrow} & B_{n_2} \\ {}^{\mathllap{f_{n_1}}}\downarrow && \downarrow^{\mathrlap{f_{n_2}}} \\ B O(n_1) &\longrightarrow& B O(n_2) \end{array} \,,$$

where the bottom map is the canonical one from def. 1.92.

The (B,f)-structure is **multiplicative** if it is moreover equipped with a system of maps $\mu_{n_1,n_2} \colon B_{n_1}\times B_{n_2} \to B_{n_1+n_2}$ which cover the canonical multiplication maps (def.)

$$\begin{array}{ccc} B_{n_1} \times B_{n_2} & \overset{\mu_{n_1,n_2}}{\longrightarrow} & B_{n_1+n_2} \\ {}^{\mathllap{f_{n_1}\times f_{n_2}}}\downarrow && \downarrow^{\mathrlap{f_{n_1+n_2}}} \\ B O(n_1)\times B O(n_2) &\longrightarrow& B O(n_1+n_2) \end{array}$$

and which satisfy the evident associativity and unitality, for $B_0 = \ast$ the unit, and, finally, which commute with the maps g in that all $n_1,n_2,n_3 \in \mathbb{N}$ these squares commute:

$$\begin{array}{ccc} B_{n_1} \times B_{n_2} &\overset{id \times g_{n_2,n_2+n_3}}{\longrightarrow}& B_{n_1} \times B_{n_2+n_3} \\ {}^{\mathllap{\mu_{n_1,n_2}}}\downarrow && \downarrow^{\mathrlap{\mu_{n_1,n_2+n_3}}} \\ B_{n_1+n_2} &\underset{g_{n_1+n_2,n_1+n_2+n_3}}{\longrightarrow}& B_{n_1+n_2+n_3} \end{array}$$

and

$$\begin{array}{ccc} B_{n_1} \times B_{n_2} &\overset{g_{n_1,n_1+n_3} \times id}{\longrightarrow}& B_{n_1+n_3} \times B_{n_2} \\ {}^{\mathllap{\mu_{n_1,n_2}}}\downarrow && \downarrow^{\mathrlap{\mu_{n_1+n_3,n_2}}} \\ B_{n_1+n_2} &\underset{g_{n_1+n_2,n_1+n_2+n_3}}{\longrightarrow}& B_{n_1+n_2+n_3} \end{array}$$

Similarly, an S^2**-**(B,f)**-structure** is a compatible system

$$f_{2n} \colon B_{2n} \longrightarrow B O(2n)$$

indexed only on the even natural numbers.

Generally, an S^k**-**(B,f)**-structure** for $k \in \mathbb{N}$, $k \geq 1$ is a compatible system

$$f_{k n} \colon B_{k n} \longrightarrow B O(k n)$$

for all $n \in \mathbb{N}$, hence for all $k n \in k \mathbb{N}$.

Example 1.99. Examples of (B,f)-structures (def. 1.98) include the following:

1. $B_n = B O(n)$ and $f_n = id$ is **orthogonal structure** (or "no structure");
2. $B_n = E O(n)$ and f_n the universal principal bundle-projection is **framing-structure**;
3. $B_n = B SO(n) = E O(n)/SO(n)$ the classifying space of the special orthogonal group and f_n the canonical projection is **orientation structure**;
4. $B_n = B Spin(n) = E O(n)/Spin(n)$ the classifying space of the spin group and f_n the canonical projection is **spin structure**.

Examples of S^2-(B,f)-structures (def. 1.98) include

1. $B_{2n} = B U(n) = E O(2n)/U(n)$ the classifying space of the unitary group, and f_{2n} the canonical projection is **almost complex structure** (or rather: **almost Hermitian structure**).
2. $B_{2n} = B Sp(2n) = E O(2n)/Sp(2n)$ the classifying space of the symplectic group, and f_{2n} the canonical projection is **almost symplectic structure**.

Examples of S^4-(B,f)-structures (def. 1.98) include

1. $B_{4n} = B U_{\mathbb{H}}(n) = E O(4n)/U_{\mathbb{H}}(n)$ the classifying space of the quaternionic unitary group, and f_{4n} the canonical projection is **almost quaternionic structure**.

Definition 1.100. Given a smooth manifold X of dimension n, and given a (B,f)-structure as in def. 1.98, then a (B,f)**-structure on the stable normal bundle of the manifold** is an equivalence class of the following structure:

1. an embedding $i_X : X \hookrightarrow \mathbb{R}^k$ for some $k \in \mathbb{N}$;
2. a homotopy class of a lift $\hat{g}$ of the classifying map g of the normal bundle (def. 1.97)

$$\begin{array}{ccc} & & B_{k-n} \\ & {}^{\hat{g}}\nearrow & \downarrow^{f_{k-n}} \\ X & \stackrel{g}{\longrightarrow} & BO(k-n) \end{array} .$$

The equivalence relation on such structures is to be that generated by the relation $((i_X)_1, \hat{g}_1) \sim ((i_X)_, \hat{g}_2)$ if

1. $k_2 \geq k_1$
2. the second inclusion factors through the first as

$$(i_X)_2 : X \stackrel{(i_X)_1}{\hookrightarrow} \mathbb{R}^{k_1} \hookrightarrow \mathbb{R}^{k_2}$$

3. the lift of the classifying map factors accordingly (as homotopy classes)

$$\hat{g}_2 : X \stackrel{\hat{g}_1}{\longrightarrow} B_{k_1-n} \xrightarrow{g_{k_1-n,k_2-n}} B_{k_2-n} .$$

Thom spectra

Idea. Given a vector bundle V of rank n over a compact topological space, then its one-point compactification is equivalently the result of forming the bundle $D(V) \hookrightarrow V$ of unit n-balls, and identifying with one single point all the boundary unit n-spheres $S(V) \hookrightarrow V$. Generally, this construction $\mathrm{Th}(C) \coloneqq D(V)/S(V)$ is called the *Thom space* of V.

Thom spaces occur notably as codomains for would-be left inverses of embeddings of manifolds $X \hookrightarrow Y$. The *Pontrjagin-Thom collapse map* $Y \to \mathrm{Th}(NX)$ of such an embedding is a continuous function going the other way around, but landing not quite in X but in the Thom space of the normal bundle of X in Y. Composing this further with the classifying map of the normal bundle lands in the Thom space of the universal vector bundle over the classifying space $BO(k)$, denoted $MO(k)$. In particular in the case that $Y = S^n$ is an n-sphere (and every manifold embeds into a large enough n-sphere, see also at Whitney embedding theorem), the Pontryagin-Thom collapse map hence associates with every manifold an element of a homotopy group of a universal Thom space $MO(k)$.

This curious construction turns out to have excellent formal properties: as the dimension ranges, the universal Thom spaces arrange into a spectrum, called the *Thom spectrum*, and the homotopy groups defined by the Pontryagin-Thom collapse pass along to the stable homotopy groups of this spectrum.

Moreover, via Whitney sum of vector bundle the Thom spectrum naturally is a homotopy commutative ring spectrum (def.), and under the Pontryagin-Thom collapse the Cartesian product of manifolds is compatible with this ring structure.

Literature. (Kochman 96, 1.5, Schwede 12, chapter I, example 1.16)

Thom spaces

Definition 1.101. Let X be a topological space and let $V \to X$ be a vector bundle over X of rank n, which is associated to an O(n)-principal bundle. Equivalently this means that $V \to X$ is the pullback of the universal vector bundle $E_n \to BO(n)$ (def. 1.91) over the classifying space. Since $O(n)$ preserves the metric on $\mathbb{R}^n$, by definition, such V inherits the structure of a metric space-fiber bundle. With respect to this structure:

1. the **unit disk bundle** $D(V) \to X$ is the subbundle of elements of norm ≤ 1;
2. the **unit sphere bundle** $S(V) \to X$ is the subbundle of elements of norm $= 1$;

$$S(V) \stackrel{i_V}{\hookrightarrow} D(V) \hookrightarrow V;$$

3. the **Thom space** $\mathrm{Th}(V)$ is the cofiber (formed in Top (prop.)) of i_V

$$\mathrm{Th}(V) \coloneqq \mathrm{cofib}(i_V)$$

canonically regarded as a pointed topological space.

$$\begin{array}{ccc} S(V) & \overset{i_V}{\longrightarrow} & D(V) \\ \downarrow & (\mathrm{po}) & \downarrow \\ * & \longrightarrow & \mathrm{Th}(V) \end{array} .$$

If $V \to X$ is a general real vector bundle, then there exists an isomorphism to an $O(n)$-associated bundle and the Thom space of V is, up to based homeomorphism, that of this orthogonal bundle.

Remark 1.102. If the rank of V is positive, then $S(V)$ is non-empty and then the Thom space (def. 1.101) is the quotient topological space

$$\mathrm{Th}(V) \simeq D(V)/S(V) .$$

However, in the degenerate case that the rank of V vanishes, hence the case that $V = X \times \mathbb{R}^0 \simeq X$, then $D(V) \simeq V \simeq X$, but $S(V) = \emptyset$. Hence now the pushout defining the cofiber is

$$\begin{array}{ccc} \emptyset & \overset{i_V}{\longrightarrow} & X \\ \downarrow & (\mathrm{po}) & \downarrow \\ * & \longrightarrow & \mathrm{Th}(V) \simeq X_* \end{array} ,$$

which exhibits $\mathrm{Th}(V)$ as the coproduct of X with the point, hence as X with a basepoint freely adjoined.

$$\mathrm{Th}(X \times \mathbb{R}^0) = \mathrm{Th}(X) \simeq X_+ .$$

Proposition 1.103. *Let $V \to X$ be a vector bundle over a CW-complex X. Then the Thom space $\mathrm{Th}(V)$ (def. 1.101) is equivalently the homotopy cofiber (def.) of the inclusion $S(V) \longrightarrow D(V)$ of the sphere bundle into the disk bundle.*

Proof. The Thom space is defined as the ordinary cofiber of $S(V) \to D(V)$. Under the given assumption, this inclusion is a relative cell complex inclusion, hence a cofibration in the classical model structure on topological spaces (thm.). Therefore in this case the ordinary cofiber represents the homotopy cofiber (def.). ▮

The equivalence to the following alternative model for this homotopy cofiber is relevant when discussing Thom isomorphisms and orientation in generalized cohomology:

Proposition 1.104. *Let $V \to X$ be a vector bundle over a CW-complex X. Write $V - X$ for the complement of its 0-section. Then the Thom space $\mathrm{Th}(V)$ (def. 1.101) is homotopy equivalent to the mapping cone of the inclusion $(V - X) \hookrightarrow V$ (hence to the pair $(V, V - X)$ in the language of generalized (Eilenberg-Steenrod) cohomology).*

Proof. The mapping cone of any map out of a CW-complex represents the homotopy cofiber of that map (exmpl.). Moreover, transformation by (weak) homotopy equivalences between morphisms induces a (weak) homotopy equivalence on their homotopy fibers (prop.). But we have such a weak homotopy equivalence, given by contracting away the fibers of the vector bundle:

$$\begin{array}{ccc} V - X & \longrightarrow & V \\ {}^{\in W_{\mathrm{cl}}}\downarrow & & \downarrow^{\in W_{\mathrm{cl}}} \\ S(V) & \hookrightarrow & D(V) \end{array} .$$

▮

Proposition 1.105. *Let $V_1, V_2 \to X$ be two real vector bundles. Then the Thom space (def. 1.101) of the direct sum of vector bundles $V_1 \oplus V_2 \to X$ is expressed in terms of the Thom space of the pullbacks $V_2|_{D(V_1)}$ and $V_2|_{S(V_1)}$ of V_2 to the disk/sphere bundle of V_1 as*

$$\mathrm{Th}(V_1 \oplus V_2) \simeq \mathrm{Th}(V_2|_{D(V_1)})/\mathrm{Th}(V_2|_{S(V_1)}) .$$

Proof. Notice that

1. $D(V_1 \oplus V_2) \simeq D(V_2|_{\mathrm{Int}\,D(V_1)}) \cup S(V_1)$;
2. $S(V_1 \oplus V_2) \simeq S(V_2|_{\mathrm{Int}\,D(V_1)}) \cup \mathrm{Int}\,D(V_2|_{S(V_1)})$.

(Since a point at radius r in $V_1 \oplus V_2$ is a point of radius $r_1 \leq r$ in V_2 and a point of radius $\sqrt{r^2 - r_1^2}$ in V_1.) ▮

Proposition 1.106. *For V a vector bundle then the Thom space (def. 1.101) of $\mathbb{R}^n \oplus V$, the direct sum of vector bundles with the trivial rank n vector bundle, is homeomorphic to the smash product of the Thom space of V with the n-sphere (the n-fold reduced suspension).*

$$\mathrm{Th}(\mathbb{R}^n \oplus V) \simeq S^n \wedge \mathrm{Th}(V) = \Sigma^n \mathrm{Th}(V) \,.$$

Proof. Apply prop. 1.105 with $V_1 = \mathbb{R}^n$ and $V_2 = V$. Since V_1 is a trivial bundle, then

$$V_2|_{D(V_1)} \simeq V_2 \times D^n$$

(as a bundle over $X \times D^n$) and similarly

$$V_2|_{S(V_1)} \simeq V_2 \times S^n \,.$$

■

Example 1.107. By prop. 1.106 and remark 1.102 the Thom space (def. 1.101) of a trivial vector bundle of rank n is the n-fold suspension of the base space

$$\begin{aligned} \mathrm{Th}(X \times \mathbb{R}^n) &\simeq S^n \wedge \mathrm{Th}(X \times \mathbb{R}^0) \\ &\simeq S^n \wedge (X_+) \end{aligned} \,.$$

Therefore a general Thom space may be thought of as a “twisted suspension”, with twist encoded by a vector bundle (or rather by its underlying spherical fibration). See at *Thom spectrum – For infinity-module bundles* for more on this.

Correspondingly the *Thom isomorphism* (prop. 1.129 below) for a given Thom space is a twisted version of the *suspension isomorphism* (above).

Proposition 1.108. *For $V_1 \to X_1$ and $V_2 \to X_2$ to vector bundles, let $V_1 \boxtimes V_2 \to X_1 \times X_2$ be the direct sum of vector bundles of their pullbacks to $X_1 \times X_2$. The corresponding Thom space (def. 1.101) is the smash product of the individual Thom spaces:*

$$\mathrm{Th}(V_1 \boxtimes V_2) \simeq \mathrm{Th}(V_1) \wedge \mathrm{Th}(V_2) \,.$$

Remark 1.109. Given a vector bundle $V \to X$ of rank n, then the reduced ordinary cohomology of its Thom space $\mathrm{Th}(V)$ (def. 1.101) vanishes in degrees $< n$:

$$\tilde H^{\bullet < n}(\mathrm{Th}(V)) \simeq H^{\bullet < n}(D(V), S(V)) \simeq 0 \,.$$

Proof. Consider the long exact sequence of relative cohomology (from above)

$$\cdots \to H^{\bullet - 1}(D(V)) \overset{i^*}{\longrightarrow} H^{\bullet -1}(S(V)) \longrightarrow H^\bullet(D(V), S(V)) \longrightarrow H^\bullet(D(V)) \overset{i^*}{\longrightarrow} H^\bullet(S(V)) \to \cdots \,.$$

Since the cohomology in degree k only depends on the k-skeleton, and since for $k < n$ the k-skeleton of $S(V)$ equals that of X, and since $D(V)$ is even homotopy equivalent to X, the morhism i^* is an isomorphism in degrees lower than n. Hence by exactness of the sequence it follows that $H^{\bullet < n}(D(V), S(V)) = 0$. ■

Universal Thom spectra MG

Proposition 1.110. *For each $n \in \mathbb{N}$ the pullback of the rank-$(n+1)$ universal vector bundle to the classifying space of rank n vector bundles is the direct sum of vector bundles of the rank n universal vector bundle with the trivial rank-1 bundle: there is a pullback diagram of topological spaces of the form*

$$\begin{array}{ccc} \mathbb{R} \oplus (EO(n) \underset{O(n)}{\times} \mathbb{R}^n) & \longrightarrow & EO(n+1) \underset{O(n+1)}{\times} \mathbb{R}^{n+1} \\ \downarrow & (\mathrm{pb}) & \downarrow \\ BO(n) & \longrightarrow & BO(n+1) \end{array} \,,$$

where the bottom morphism is the canonical one (def.).

(e.g. Kochmann 96, p. 25)

Proof. For each $k \in \mathbb{N}$, $k \geq n$ there is such a pullback of the canonical vector bundles over Grassmannians

$$\begin{array}{ccc} \left\{ \begin{array}{c} V_n \subset \mathbb{R}^k, \\ v \in V_n, v_{n+1} \in \mathbb{R} \end{array} \right\} & \longrightarrow & \left\{ \begin{array}{c} V_{n+1} \subset \mathbb{R}^{k+1}, \\ v \in V_{n+1} \end{array} \right\} \\ \downarrow && \downarrow \\ \mathrm{Gr}_n(\mathbb{R}^k) & \longrightarrow & \mathrm{Gr}_{n+1}(\mathbb{R}^{k+1}) \end{array}$$

where the bottom morphism is the canonical inclusion (def.).

Now we claim that taking the colimit in each of the four corners of this system of pullback diagrams yields again a pullback diagram, and this proves the claim.

To see this, remember that we work in the category $\mathrm{Top}_{\mathrm{cg}}$ of compactly generated topological spaces (def.). By their nature, we may test the universal property of a would-be pullback space already by mapping compact topological spaces into it. Now observe that all the inclusion maps in the four corners of this system of diagrams are relative cell complex inclusions, by prop. 1.87. Together this implies (via this lemma) that we may test the universal property of the colimiting square at finite stages. And so this implies the claim by the above fact that at each finite stage there is a pullback diagram. ▮

Definition 1.111. The **universal real Thom spectrum** MO is the spectrum, which is represented by the sequential prespectrum (def.) whose nth component space is the Thom space (def. 1.101)

$$(MO)_n \coloneqq \mathrm{Th}(EO(n) \underset{O(n)}{\times} \mathbb{R}^n)$$

of the rank-n universal vector bundle, and whose structure maps are the image under the Thom space functor $\mathrm{Th}(-)$ of the top morphisms in prop. 1.110, via the homeomorphisms of prop. 1.106:

$$\sigma_n : \Sigma(MO)_n \simeq \mathrm{Th}(\mathbb{R} \oplus (EO(n) \underset{O(n)}{\times} \mathbb{R}^n)) \longrightarrow \mathrm{Th}(EO(n+1) \underset{O(n+1)}{\times} \mathbb{R}^{n+1}) = (MO)_{n+1} \,.$$

More generally, there are universal Thom spectra associated with any other tangent structure ("[[(B,f)]-structure]]"), notably for the orthogonal group replaced by the special orthogonal groups $\mathrm{SO}(n)$, or the spin groups $\mathrm{Spin}(n)$, or the string 2-group $\mathrm{String}(n)$, or the fivebrane 6-group $\mathrm{Fivebrane}(n)$,..., or any level in the Whitehead tower of $O(n)$. To any of these groups there corresponds a Thom spectrum (denoted, respectively, $M\,\mathrm{SO}$, MSpin, $M\,\mathrm{String}$, $M\,\mathrm{Fivebrane}$, etc.), which is in turn related to oriented cobordism, spin cobordism, string cobordism, et cetera.:

Definition 1.112. Given a (B,f)-structure $\mathcal{B}$ (def. 1.98), write $V_n^{\mathcal{B}}$ for the pullback of the universal vector bundle (def. 1.91) to the corresponding space of the (B,f)-structure and with

$$\begin{array}{ccc} V^{\mathcal{B}} & \longrightarrow & VO(n) \underset{O(n)}{\times} \mathbb{R}^n \\ \downarrow & (\mathrm{pb}) & \downarrow \\ B_n & \underset{f_n}{\longrightarrow} & BO(n) \end{array}$$

and we write e_{n_1,n_2} for the maps of total space of vector bundles over the g_{n_1,n_2}:

$$\begin{array}{ccc} V_{n_1}^{\mathcal{B}} & \overset{e_{n_1,n_2}}{\longrightarrow} & V_{n_2}^{\mathcal{B}} \\ \downarrow & (\mathrm{pb}) & \downarrow \\ B_{n_1} & \underset{g_{n_1,n_2}}{\longrightarrow} & B_{n_2} \end{array} \,.$$

Observe that the analog of prop. 1.110 still holds:

Proposiiton 1.113. *Given a (B,f)-structure $\mathcal{B}$ (def. 1.98), then the pullback of its rank-$(n+1)$ vector bundle $V_{n+1}^{\mathcal{B}}$ (def. 1.112) along the map $g_{n,n+1} : B_n \to B_{n+1}$ is the direct sum of vector bundles of the rank-n bundle $V_n^{\mathcal{B}}$ with the trivial rank-1-bundle: there is a pullback square*

$$\begin{array}{ccc} \mathbb{R} \oplus V_n^{\mathcal{B}} & \overset{e_{n,n+1}}{\longrightarrow} & V_{n+1}^{\mathcal{B}} \\ \downarrow & (\mathrm{pb}) & \downarrow \\ B_n & \underset{g_{n,n+1}}{\longrightarrow} & B_{n+1} \end{array} \,.$$

Proof. Unwinding the definitions, the pullback in question is

$$
\begin{aligned}
(g_{n,n+1})^* V^{\mathcal{B}}_{n+1} &= (g_{n,n+1})^* f^*_{n+1} (EO(n+1) \underset{O(n+1)}{\times} \mathbb{R}^{n+1}) \\
&\simeq (g_{n,n+1} \circ f_{n+1})^* (EO(n+1) \underset{O(n+1)}{\times} \mathbb{R}^{n+1}) \\
&\simeq (f_n \circ i_n)^* (EO(n+1) \underset{O(n+1)}{\times} \mathbb{R}^{n+1}) \\
&\simeq f^*_n i^*_n (EO(n+1) \underset{O(n+1)}{\times} \mathbb{R}^{n+1}) \\
&\simeq f^*_n (\mathbb{R} \oplus (EO(n) \underset{O(n)}{\times} \mathbb{R}^n)) \\
&\simeq \mathbb{R} \oplus V^{\mathcal{B}_n} ,
\end{aligned}
$$

where the second but last step is due to prop. 1.110. ▮

Definition 1.114. Given a (B,f)-structure $\mathcal{B}$ (def. 1.98), its **universal Thom spectrum** $M\mathcal{B}$ is, as a sequential prespectrum, given by component spaces being the Thom spaces (def. 1.101) of the $\mathcal{B}$-associated vector bundles of def. 1.112

$$(M\mathcal{B})_n \coloneqq \mathrm{Th}(V^{\mathcal{B}}_n)$$

and with structure maps given via prop. 1.106 by the top maps in prop. 1.113:

$$\sigma_n \,:\, \Sigma (M\mathcal{B})_n = \Sigma\, \mathrm{Th}(V^{\mathcal{E}}_n) \simeq \mathrm{Th}(\mathbb{R} \oplus V^{\mathcal{E}}_n) \xrightarrow{\mathrm{Th}(e_{n,n+1})} \mathrm{Th}(V^{\mathcal{B}}_{n+1}) = (M\mathcal{B})_{n+1} \,.$$

Similarly for an $S^k - (B,f)$-structure indexed on every kth natural number (such as almost complex structure, almost quaternionic structure, example 1.99), there is the corresponding Thom spectrum as a sequential S^k spectrum (def.).

If $B_n = BG_n$ for some natural system of groups $G_n \to O(n)$, then one usually writes MG for $M\mathcal{B}$. For instance $M\,\mathrm{SO}$, MSpin, MU, MSp etc.

If the (B,f)-structure is multiplicative (def. 1.98), then the Thom spectrum $M\mathcal{B}$ canonical becomes a ring spectrum (for more on this see Part 1-2 the section on orthogonal Thom spectra): the multiplication maps $B_{n_1} \times B_{n_2} \to B_{n_1+n_2}$ are covered by maps of vector bundles

$$V^{\mathcal{B}}_{n_1} \boxtimes V^{\mathcal{B}}_{n_2} \longrightarrow V^{\mathcal{B}}_{n_1+n_2}$$

and under forming Thom spaces this yields (via prop. 1.108) maps

$$(M\mathcal{B})_{n_1} \wedge (M\mathcal{B})_{n_2} \longrightarrow (M\mathcal{B})_{n_1+n_2}$$

which are associative by the associativity condition in a multiplicative (B,f)-structure. The unit is

$$(M\mathcal{B})_0 = \mathrm{Th}(V^{\mathcal{B}}_0) \simeq \mathrm{Th}(\ast) \simeq S^0 ,$$

by remark 1.102.

Example 1.115. The universal Thom spectrum (def. 1.114) for *framing structure* (exmpl.) is equivalently the sphere spectrum (def.)

$$M1 \simeq \mathbb{S} \,.$$

Because in this case $B_n \simeq \ast$ and so $E^{\mathcal{B}}_n \simeq \mathbb{R}^n$, whence $\mathrm{Th}(E^{\mathcal{B}}_n) \simeq S^n$.

Pontrjagin-Thom construction

Definition 1.116. For X a smooth manifold and $i{:}X \hookrightarrow \mathbb{R}^k$ an embedding, then a **tubular neighbourhood** of X is a subset of the form

$$\tau_i X \coloneqq \left\{ x \in \mathbb{R}^k \mid d(x, i(X)) < \epsilon \right\}$$

for some $\epsilon \in \mathbb{R}$, $\epsilon > 0$, small enough such that the map

$$N_i X \longrightarrow \tau_i X$$

from the normal bundle (def. 1.97) given by

$$(i(x), v) \mapsto (i(x), \epsilon(1 - e^{-|v|}) v)$$

is a diffeomorphism.

Proposition 1.117. ***(tubular neighbourhood theorem)***

For every embedding of smooth manifolds, there exists a tubular neighbourhood according to def. 1.116.

Remark 1.118. Given an embedding $i : X \hookrightarrow \mathbb{R}^k$ with a tubuluar neighbourhood $\tau_i X \text{hookrigtharrow} \mathbb{R}^k$ (def. 1.116) then by construction:

1. the Thom space (def. 1.101) of the normal bundle (def. 1.97) is homeomorphic to the quotient topological space of the topological closure of the tubular neighbourhood by its boundary:

   $\mathrm{Th}(N_i(X)) \simeq \overline{\tau_i(X)} / \partial \overline{\tau_i(X)}$;

2. there exists a continous function

   $$\mathbb{R}^k \longrightarrow \overline{\tau_i(X)} / \partial \overline{\tau_i(X)}$$

   which is the identity on $\tau_i(X) \subset \mathbb{R}^k$ and is constant on the basepoint of the quotient on all other points.

Definition 1.119. For X a smooth manifold of dimension n and for $i : X \hookrightarrow \mathbb{R}^k$ an embedding, then the **Pontrjagin-Thom collapse map** is, for any choice of tubular neighbourhood $\tau_i(X) \subset \mathbb{R}^k$ (def. 1.116) the composite map of pointed topological spaces

$$S^k \overset{\simeq}{\to} (\mathbb{R}^k)^* \longrightarrow \overline{\tau_i(X)} / \partial \overline{\tau_i(X)} \overset{\simeq}{\to} \mathrm{Th}(N_i X)$$

where the first map identifies the k-sphere as the one-point compactification of $\mathbb{R}^k$; and where the second and third maps are those of remark 1.118.

The **Pontrjagin-Thom construction** is the further composite

$$\xi_i \;:\; S^k \longrightarrow \mathrm{Th}(N_i X) \overset{\mathrm{Th}(e_i)}{\longrightarrow} \mathrm{Th}(E O(k-n) \underset{O(k-n)}{\times} \mathbb{R}^{k-n}) \simeq (M O)_{k-n}$$

with the image under the Thom space construction of the morphism of vector bundles

$$\begin{array}{ccc} \nu & \overset{e_i}{\longrightarrow} & E O(k-n) \underset{O(k-n)}{\times} \mathbb{R}^{k-n} \\ \downarrow & (pb) & \downarrow \\ X & \underset{g_i}{\longrightarrow} & B O(k-n) \end{array}$$

induced by the classifying map g_i of the normal bundle (def. 1.97).

This defines an element

$$[S^{n + (k-n)} \overset{\xi_i}{\to} (M O)_{k-n}] \in \pi_n M O$$

in the nth stable homotopy group (def.) of the Thom spectrum $M O$ (def. 1.111).

More generally, for X a smooth manifold with normal (B,f)-structure $(X, i, \hat g_i)$ according to def. 1.100, then its Pontrjagin-Thom construction is the composite

$$\xi_i \;:\; S^k \longrightarrow \mathrm{Th}(N_i X) \overset{\mathrm{Th}(\hat e_i)}{\longrightarrow} \mathrm{Th}(V^{\mathcal{B}}_{k-n}) \simeq (M \mathcal{B})_{k-n}$$

with

$$\begin{array}{ccc} \nu & \overset{\hat e_i}{\longrightarrow} & V^{\mathcal{B}}_{k-n} \\ \downarrow & (pb) & \downarrow \\ X & \underset{\hat g_i}{\longrightarrow} & B O(k-n) \end{array} \,.$$

Proposition 1.120. *The Pontrjagin-Thom construction (def. 1.119) respects the equivalence classes entering the definition of manifolds with stable normal $\mathcal{B}$-structure (def. 1.100) hence descends to a function (of sets)*

$$\xi \;:\; \left\{ \begin{matrix} n\text{-manifolds with stable} \\ \text{normal } \mathcal{B}\text{-structure} \end{matrix} \right\} \longrightarrow \pi_n(M\mathcal{B}) \,.$$

Proof. It is clear that the homotopies of classifying maps of $\mathcal{B}$-structures that are devided out in def. 1.100 map to homotopies of representatives of stable homotopy groups. What needs to be shown is that the construction respects the enlargement of the embedding spaces.

Given a embedded manifold $X \overset{i}{\hookrightarrow} \mathbb{R}^{k_1}$ with normal $\mathcal{B}$-structure

$$\begin{array}{ccc} & & B_{k_1 - n} \\ & {}^{\hat g_i}\nearrow & \downarrow^{f_{k-n}} \\ X & \underset{g_i}{\longrightarrow} & BO(k_1 - n) \end{array}$$

write

$$\alpha \;\colon\; S^{n + (k_1 - n)} \longrightarrow \mathrm{Th}(E^{\mathcal{B}_{k_1 - n}})$$

for its image under the Pontrjagin-Thom construction (def. 1.119). Now given $k_2 \in \mathbb{N}$, consider the induced embedding $X \overset{i}{\hookrightarrow} \mathbb{R}^{k_1} \hookrightarrow \mathbb{R}^{k_1 + k_2}$ with normal $\mathcal{B}$-structure given by the composite

$$\begin{array}{ccccc} & & B_{k_1 - n} & \overset{g_{k_1 - n, k_1 + k_2 - n}}{\longrightarrow} & B_{k_1 + k_2 - n} \\ & {}^{\hat g_i}\nearrow & \downarrow^{f_{k_1 - n} \times f_{k_2}} & & \downarrow^{f_{k_1 + k_2 - n}} \\ X & \underset{g_i}{\longrightarrow} & BO(k_1 - n) & \longrightarrow & BO(k_1 + k_2 - n) \end{array}$$

By prop. 1.113 and using the pasting law for pullbacks, the classifying map $\hat g'_i$ for the enlarged normal bundle sits in a diagram of the form

$$\begin{array}{ccccc} (\nu_i \oplus \mathbb{R}^{k_2}) & \overset{(\hat e_i \oplus \mathrm{id})}{\longrightarrow} & (V^{\mathcal{B}}_{k_1 - n} \oplus \mathbb{R}^{k_2}) & \overset{e_{k_1 - n, k_1 + k_2 - n}}{\longrightarrow} & V^{\mathcal{B}}_{k_1 + k_2 - n} \\ \downarrow & (\mathrm{pb}) & \downarrow & (\mathrm{pb}) & \downarrow \\ X & \underset{\hat g_i}{\longrightarrow} & B_{k_1 - n} & \underset{g_{k_1 - n, k_1 + k_2 - n}}{\longrightarrow} & B_{k_1 + k_2 - n} \end{array} \,.$$

Hence the Pontrjagin-Thom construction for the enlarged embedding space is (using prop. 1.106) the composite

$$\alpha_{k_2} \;\colon\; S^{n + (k_1 + k_2 - n)} \simeq \mathrm{Th}(\mathbb{R}^{k_2}) \wedge S^{n + (k_1 - n)} \longrightarrow \mathrm{Th}(\mathbb{R}^{k_2}) \wedge \mathrm{Th}(\nu_i) \overset{\mathrm{Th}(\mathrm{id}) \wedge \mathrm{Th}(\hat e_i)}{\longrightarrow} \mathrm{Th}(\mathbb{R}^{k_2}) \wedge \mathrm{Th}(E^{\mathcal{B}}_{k_1 - n})) \overset{\mathrm{Th}(e_{k_1 - n, k_1 + k_2 - n})}{\longrightarrow} \mathrm{Th}(V^{\mathcal{B}}_{k_1 + k_2 - n}$$

The composite of the first two morphisms here is $S^{k_k} \wedge \alpha$, while last morphism $\mathrm{Th}(\hat e_{k_1 - n, k_1 + k_2 - n})$ is the structure map in the Thom spectrum (by def. 1.114):

$$\alpha_{k_2} \;\colon\; S^{k_2} \wedge S^{n + (k_1 - n)} \overset{S^{k_2} \wedge \alpha}{\longrightarrow} S^{k_2} \wedge \mathrm{Th}(E^{\mathcal{B}}_{k_1 + k_2 - n}) \overset{\sigma^{M\mathcal{B}}_{k_1 - n, k_1 + k_2 - n}}{\longrightarrow} \mathrm{Th}(V^{\mathcal{B}}_{k_1 + k_2 - n})$$

This manifestly identifies α_{k_2} as being the image of α under the component map in the sequential colimit that defines the stable homotopy groups (def.). Therefore α and α_{k_2}, for all $k_2 \in \mathbb{N}$, represent the same element in $\pi_\bullet(M\mathcal{B})$. ▮

Bordism and Thom's theorem

Idea. By the Pontryagin-Thom collapse construction above, there is an assignment

$$n\,\mathrm{Manifolds} \longrightarrow \pi_n(MO)$$

which sends disjoint union and Cartesian product of manifolds to sum and product in the ring of stable homotopy groups of the Thom spectrum. One finds then that two manifolds map to the same element in the stable homotopy groups $\pi_\bullet(MO)$ of the universal Thom spectrum precisely if they are connected by a bordism. The bordism-classes $\Omega^O_\bullet$ of manifolds form a commutative ring under disjoint union and Cartesian product, called the *bordism ring*, and Pontrjagin-Thom collapse produces a ring homomorphism

$$\Omega^O_\bullet \longrightarrow \pi_\bullet(MO) \,.$$

Thom's theorem states that this homomorphism is an isomorphism.

More generally, for $\mathcal{B}$ a multiplicative (B,f)-structure, def. 1.98, there is such an identification

$$\Omega^{\mathcal{B}}_\bullet \simeq \pi_\bullet(M\mathcal{B})$$

between the ring of $\mathcal{B}$-cobordism classes of manifolds with $\mathcal{B}$-structure and the stable homotopy groups of the universal $\mathcal{B}$-Thom spectrum.

Literature. (Kochman 96, 1.5)

Bordism

Throughout, let $\mathcal{B}$ be a multiplicative (B,f)-structure (def. 1.98).

Definition 1.121. Write $I \coloneqq [0,1]$ for the standard interval, regarded as a smooth manifold with boundary. For $c \in \mathbb{R}_+$ Consider its embedding

$$e : I \hookrightarrow \mathbb{R} \oplus \mathbb{R}_{\geq 0}$$

as the arc

$$e : t \mapsto \cos(\pi t) \cdot e_1 + \sin(\pi t) \cdot e_2 \,,$$

where (e_1, e_2) denotes the canonical linear basis of $\mathbb{R}^2$, and equipped with the structure of a manifold with normal framing structure (example 1.99) by equipping it with the canonical framing

$$\mathrm{fr} : t \mapsto \cos(\pi t) \cdot e_1 + \sin(\pi t) \cdot e_2$$

of its normal bundle.

Let now $\mathcal{B}$ be a (B,f)-structure (def. 1.98). Then for $X \overset{i}{\hookrightarrow} \mathbb{R}^k$ any embedded manifold with $\mathcal{B}$-structure $\hat{g}: X \to B_{k-n}$ on its normal bundle (def. 1.100), define its **negative** or **orientation reversal** $-(X, i, \hat{g})$ of $(X, i, \hat{g})$ to be the restriction of the structured manifold

$$(X \times I \overset{(i,e)}{\hookrightarrow} \mathbb{R}^{k+2}, \hat{g} \times \mathrm{fr})$$

to $t = 1$.

Definition 1.122. Two closed manifolds of dimension n equipped with normal $\mathcal{B}$-structure $(X_1, i_1, \hat{g}_1)$ and $(X_2, i_2, \hat{g}_2)$ (def.) are called **bordant** if there exists a manifold with boundary W of dimension $n+1$ equipped with $\mathcal{B}$-strcuture $(W, i_W, \hat{g}_W)$ if its boundary with $\mathcal{B}$-structure restricted to that boundary is the disjoint union of X_1 with the negative of X_2, according to def. 1.121

$$\partial(W, i_W, \hat{g}_W) \simeq (X_1, i_1, \hat{g}_1) \sqcup -(X_2, i_2, \hat{g}_2) \,.$$

Proposition 1.123. *The relation of $\mathcal{B}$-bordism (def. 1.122) is an equivalence relation.*

Write $\Omega_\bullet^{\mathcal{B}}$ for the $\mathbb{N}$-graded set of $\mathcal{B}$-bordism classes of $\mathcal{B}$-manifolds.

Proposition 1.124. *Under disjoint union of manifolds, then the set of $\mathcal{B}$-bordism equivalence classes of def. 1.123 becomes an $\mathbb{Z}$-graded abelian group*

$$\Omega_\bullet^{\mathcal{B}} \in \mathrm{Ab}^{\mathbb{Z}}$$

*(that happens to be concentrated in non-negative degrees). This is called the $\mathcal{B}$**-bordism group**.*

*Moreover, if the (B,f)-structure $\mathcal{B}$ is multiplicative (def. 1.98), then Cartesian product of manifolds followed by the multiplicative composition operation of $\mathcal{B}$-structures makes the $\mathcal{B}$-bordism ring into a commutative ring, called the $\mathcal{B}$**-bordism ring**.*

$$\Omega_\bullet^{\mathcal{B}} \in \mathrm{CRing}^{\mathbb{Z}} \,.$$

e.g. (Kochmann 96, prop. 1.5.3)

Thom’s theorem

Recall that the Pontrjagin-Thom construction (def. 1.119) associates to an embbeded manifold $(X, i, \hat{g})$ with normal $\mathcal{B}$-structure (def. 1.100) an element in the stable homotopy group $\pi_{\dim(X)}(M\mathcal{B})$ of the universal $\mathcal{B}$-Thom spectrum in degree the dimension of that manifold.

Lemma 1.125. *For $\mathcal{B}$ be a multiplicative (B,f)-structure (def. 1.98), the $\mathcal{B}$-Pontrjagin-Thom construction (def. 1.119) is compatible with all the relations involved to yield a graded ring homomorphism*

$$\xi : \Omega_\bullet^{\mathcal{B}} \longrightarrow \pi_\bullet(M\mathcal{B})$$

from the $\mathcal{B}$-bordism ring (def. 1.124) to the stable homotopy groups of the universal $\mathcal{B}$-Thom spectrum equipped with the ring structure induced from the canonical ring spectrum structure (def. 1.114).

Proof. By prop. 1.120 the underlying function of sets is well-defined before dividing out the bordism relation (def. 1.122). To descend this further to a function out of the set underlying the bordism ring, we need to see that the Pontrjagin-Thom construction respects the bordism relation. But the definition of bordism is just so as to exhibit under ξ a left homotopy of representatives of homotopy groups.

Next we need to show that it is

1. a group homomorphism;

2. a ring homomorphism.

Regarding the first point:

The element 0 in the cobordism group is represented by the empty manifold. It is clear that the Pontrjagin-Thom construction takes this to the trivial stable homotopy now.

Given two n-manifolds with $\mathcal{B}$-structure, we may consider an embedding of their disjoint union into some $\mathbb{R}^k$ such that the tubular neighbourhoods of the two direct summands do not intersect. There is then a map from two copies of the k-cube, glued at one face

$$\square^k \underset{\square^{k-1}}{\sqcup} \square^k \longrightarrow \mathbb{R}^k$$

such that the first manifold with its tubular neighbourhood sits inside the image of the first cube, while the second manifold with its tubular neighbourhood sits indide the second cube. After applying the Pontryagin-Thom construction to this setup, each cube separately maps to the image under ξ of the respective manifold, while the union of the two cubes manifestly maps to the sum of the resulting elements of homotopy groups, by the very definition of the group operation in the homotopy groups (def.). This shows that ξ is a group homomorphism.

Regarding the second point:

The element 1 in the cobordism ring is represented by the manifold which is the point. Without restriction we may consoder this as embedded into $\mathbb{R}^0$, by the identity map. The corresponding normal bundle is of rank 0 and hence (by remark 1.102) its Thom space is S^0, the 0-sphere. Also $V_0^{\mathcal{B}}$ is the rank-0 vector bundle over the point, and hence $(M\mathcal{B})_0 \simeq S^0$ (by def. 1.114) and so $\xi(*) : (S^0 \overset{=}{\to} S^0)$ indeed represents the unit element in $\pi_\bullet(M\mathcal{B})$.

Finally regarding respect for the ring product structure: for two manifolds with stable normal $\mathcal{B}$-structure, represented by embeddings into $\mathbb{R}^{k_i}$, then the normal bundle of the embedding of their Cartesian product is the direct sum of vector bundles of the separate normal bundles bulled back to the product manifold. In the notation of prop. 1.108 there is a diagram of the form

$$\begin{array}{ccccc}
\nu_1 \boxtimes \nu_2 & \overset{\hat e_1 \boxtimes \hat e_2}{\longrightarrow} & V^{\mathcal{B}}_{n_1} \boxtimes V^{\mathcal{B}}_{n_2} & \overset{\kappa_{n_1,n_2}}{\longrightarrow} & V^{\mathcal{B}}_{n_1+n_2} \\
\downarrow & (pb) & \downarrow & (pb) & \downarrow \\
X_1 \times X_2 & \underset{\hat g_1 \times \hat g_2}{\longrightarrow} & B_{k_1-n_1} \times B_{k_2-n_2} & \underset{\mu_{k_1-n_1,k_2-n_2}}{\longrightarrow} & B_{k_1+k_2-n_1-n_2}
\end{array}
\,.$$

To the Pontrjagin-Thom construction of the product manifold is by definition the top composite in the diagram

$$\begin{array}{ccccccc}
S^{n_1+n_2+(k_1+k_2-n_1-n_2)} & \longrightarrow & \mathrm{Th}(\nu_1 \boxtimes \nu_2) & \overset{\mathrm{Th}(\hat e_1 \boxtimes \hat e_2)}{\longrightarrow} & \mathrm{Th}(V^{\mathcal{B}}_{k_1-n_1} \boxtimes V^{\mathcal{B}}_{k_2-n_2}) & \overset{\kappa_{k_1-n_1,k_2-n_2}}{\longrightarrow} & \mathrm{Th}(V^{\mathcal{B}}_{k_1+k_2-n_1-n_2}) \\
{}^{\simeq}\downarrow && \downarrow^{\simeq} && \downarrow^{\simeq} && \downarrow^{=} \\
S^{n_1+(k_1-n_1)} \wedge S^{n_2+(k_2-n_2)} & \longrightarrow & \mathrm{Th}(\nu_1) \wedge \mathrm{Th}(\nu_2) & \overset{\mathrm{Th}(\hat e_1) \wedge \mathrm{Th}(\hat e_2)}{\longrightarrow} & \mathrm{Th}(V_1^{\mathcal{B}}) \wedge \mathrm{Th}(V_2^{\mathcal{B}}) & \overset{\kappa_{k_1-n_1,k_2-n_2}}{\longrightarrow} & \mathrm{Th}(V^{\mathcal{B}}_{k_1+k_2-n_1-n_2})
\end{array}
\,,$$

which hence is equivalently the bottom composite, which in turn manifestly represents the product of the separate PT constructions in $\pi_\bullet(M\mathcal{B})$. ▮

Theorem 1.126. *The ring homomorphsim in lemma 1.125 is an isomorphism.*

Due to (Thom 54, Pontrjagin 55). See for instance (Kochmann 96, theorem 1.5.10).

Proof idea. Observe that given the result $\alpha : S^{n+(k-n)} \to \mathrm{Th}(V_{k-n})$ of the Pontrjagin-Thom construction map, the original manifold $X \overset{i}{\hookrightarrow} \mathbb{R}^k$ may be recovered as this pullback:

$$\begin{array}{ccc} X & \overset{i}{\longrightarrow} & S^{n+(k-n)} \\ {}^{\mathllap{g_i}}\downarrow & (\mathrm{pb}) & \downarrow^{\mathrlap{\alpha}} \\ BO(k-n) & \longrightarrow & \mathrm{Th}(V_{k-n}^{BO}) \end{array} \,.$$

To see this more explicitly, break it up into pieces:

$$\begin{array}{ccccc} X & \longrightarrow & X_+ & \hookrightarrow & S^{n+(k-n)} \\ \downarrow & (\mathrm{pb}) & \downarrow & (\mathrm{pb}) & \downarrow \\ X & \longrightarrow & X_+ \simeq \mathrm{Th}(X) & \overset{\mathrm{Th}(0)}{\longrightarrow} & \mathrm{Th}(\nu_i) \\ \downarrow & (\mathrm{pb}) & \downarrow & (\mathrm{pb}) & \downarrow \\ B_{k-n} & \longrightarrow & (B_{k-n})_+ \simeq \mathrm{Th}(B_{k-n}) & \underset{\mathrm{Th}(0)}{\longrightarrow} & \mathrm{Th}(V_{k-n}^{\mathcal{B}}) \\ \downarrow & (\mathrm{pb}) & \downarrow & (\mathrm{pb}) & \downarrow \\ BO(k-n) & \longrightarrow & (BO(k-n))_+ \simeq \mathrm{Th}(BO(k-n)) & \longrightarrow & \mathrm{Th}(V_{k-n}^{BO}) \end{array} \,.$$

Moreover, since the n-spheres are compact topological spaces, and since the classifying space $BO(n)$, and hence its universal Thom space, is a sequential colimit over relative cell complex inclusions, the right vertical map factors through some finite stage (by this lemma), the manifold X is equivalently recovered as a pullback of the form

$$\begin{array}{ccc} X & \longrightarrow & S^{n+(k-n)} \\ {}^{\mathllap{g_i}}\downarrow & (\mathrm{pb}) & \downarrow \\ \mathrm{Gr}_{k-n}(\mathbb{R}^k) & \overset{i}{\longrightarrow} & \mathrm{Th}(V_{k-n}(\mathbb{R}^k) \underset{O(k-n)}{\times} \mathbb{R}^{k-n}) \end{array} \,.$$

(Recall that $V_{k-n}^{\mathcal{B}}$ is our notation for the universal vector bundle with $\mathcal{B}$-structure, while $V_{k-n}(\mathbb{R}^k)$ denotes a Stiefel manifold.)

The idea of the proof now is to use this property as the blueprint of the construction of an inverse ζ to ξ: given an element in $\pi_n(M\mathcal{B})$ represented by a map as on the right of the above diagram, try to define X and the structure map g_i of its normal bundle as the pullback on the left.

The technical problem to be overcome is that for a general continuous function as on the right, the pullback has no reason to be a smooth manifold, and for two reasons:

1. the map $S^{n+(k-n)} \to \mathrm{Th}(V_{k-n})$ may not be smooth around the image of i;

2. even if it is smooth around the image of i, it may not be transversal to i, and the intersection of two non-transversal smooth functions is in general still not a smooth manifold.

The heart of the proof is in showing that for any α there are small homotopies relating it to an α' that is both smooth around the image of i and transversal to i.

The first condition is guaranteed by Sard's theorem, the second by Thom's transversality theorem.

(...) ∎

Thom isomorphism

Idea. If a vector bundle $E \overset{p}{\longrightarrow} X$ of rank n carries a cohomology class $\omega \in H^n(\mathrm{Th}(E), R)$ that looks fiberwise like a volume form – a Thom class – then the operation of pulling back from base space and then forming the cup product with this Thom class is an isomorphism on (reduced) cohomology

$$((-) \cup \omega) \circ p^\ast \;:\; H^\bullet(X, R) \overset{\simeq}{\longrightarrow} \tilde H^{\bullet + n}(\mathrm{Th}(E), R) \,.$$

This is the *Thom isomorphism*. It follows from the Serre spectral sequence (or else from the Leray-Hirsch theorem). A closely related statement gives the *Thom-Gysin sequence*.

In the special case that the vector bundle is trivial of rank n, then its Thom space coincides with the n-fold suspension of the base space (example 1.107) and the Thom isomorphism coincides with the suspension isomorphism. In this sense the Thom isomorphism may be regarded as a *twisted suspension isomorphism*.

We need this below to compute (co)homology of universal Thom spectra MU in terms of that of the classifying spaces BU.

Composed with pullback along the Pontryagin-Thom collapse map, the Thom isomorphism produces maps in cohomology that covariantly follow the underlying maps of spaces. These "Umkehr maps" have the interpretation of fiber integration against the Thom class.

Literature. (Kochman 96, 2.6)

Thom-Gysin sequence

The *Thom-Gysin sequence* is a type of long exact sequence in cohomology induced by a spherical fibration and expressing the cohomology groups of the total space in terms of those of the base plus correction. The sequence may be obtained as a corollary of the Serre spectral sequence for the given fibration. It induces, and is induced by, the Thom isomorphism.

Proposition 1.127. *Let R be a commutative ring and let*

$$\begin{array}{ccc} S^n & \longrightarrow & E \\ && \downarrow^{\pi} \\ && B \end{array}$$

be a Serre fibration over a simply connected CW-complex with typical fiber (exmpl.) the n-sphere.

*Then there exists an element $c \in H^{n+1}(E;R)$ (in the ordinary cohomology of the total space with coefficients in R, called the **Euler class** of π) such that the cup product operation $c \cup (-)$ sits in a long exact sequence of cohomology groups of the form*

$$\cdots \to H^k(B;R) \overset{\pi^*}{\longrightarrow} H^k(E;R) \longrightarrow H^{k-n}(B;R) \overset{c\cup(-)}{\longrightarrow} H^{k+1}(B;R) \to \cdots \,.$$

(e.g. Switzer 75, section 15.30, Kochman 96, corollary 2.2.6)

Proof. Under the given assumptions there is the corresponding Serre spectral sequence

$$E_2^{s,t} \;=\; H^s(B; H^t(S^n;R)) \;\Rightarrow\; H^{s+t}(E;R) \,.$$

Since the ordinary cohomology of the n-sphere fiber is concentrated in just two degees

$$H^t(S^n;R) = \begin{cases} R & \text{for}\; t = 0 \; \text{and} \; t = n \\ 0 & \text{otherwise} \end{cases}$$

the only possibly non-vanishing terms on the E_2 page of this spectral sequence, and hence on all the further pages, are in bidegrees $(\bullet,0)$ and $(\bullet,n)$:

$$E_2^{\bullet,0} \simeq H^\bullet(B;R) \,, \quad \text{and} \quad E_2^{\bullet,n} \simeq H^\bullet(B;R) \,.$$

As a consequence, since the differentials d_r on the rth page of the Serre spectral sequence have bidegree $(r+1,-r)$, the only possibly non-vanishing differentials are those on the $(n+1)$-page of the form

$$\begin{array}{ccc} E_{n+1}^{\bullet,n} & \simeq & H^\bullet(B;R) \\ {}^{\mathllap{d_{n+1}}}\downarrow && \\ E_{n+1}^{\bullet+n+1,0} & \simeq & H^{\bullet+n+1}(B;R) \end{array} \,.$$

Now since the coefficients R is a ring, the Serre spectral sequence is multiplicative under cup product and the differential is a derivation (of total degree 1) with respect to this product. (See at *multiplicative spectral sequence – Examples – AHSS for multiplicative cohomology*.)

To make use of this, write

$$\iota \coloneqq 1 \in H^0(B;R) \overset{\simeq}{\longrightarrow} E_{n+1}^{0,n}$$

for the unit in the cohomology ring $H^\bullet(B;R)$, but regarded as an element in bidegree $(0,n)$ on the $(n+1)$-page of the spectral sequence. (In particular ι does *not* denote the unit in bidegree $(0,0)$, and hence $d_{n+1}(\iota)$ need not vanish; while by the derivation property, it does vanish on the actual unit $1 \in H^0(B;R) \simeq E_{n+1}^{0,0}$.)

Write

$$c \coloneqq d_{n+1}(\iota) \quad \in E_{n+1}^{n+1,0} \overset{\simeq}{\longrightarrow} H^{n+1}(B;R)$$

for the image of this element under the differential. We will show that this is the Euler class in question.

To that end, notice that every element in $E_{n+1}^{\bullet,n}$ is of the form $\iota \cdot b$ for $b \in E_{n+1}^{\bullet,0} \simeq H^\bullet(B;R)$.

(Because the multiplicative structure gives a group homomorphism $\iota \cdot (-) \colon H^\bullet(B;R) \simeq E_{n+1}^{0,0} \to E_{n+1}^{0,n} \simeq H^\bullet(B;R)$, which is an isomorphism because the product in the spectral sequence does come from the cup product in the cohomology ring, see for instance (Kochman 96, first equation in the proof of prop. 4.2.9), and since hence ι does act like the unit that it is in $H^\bullet(B;R)$).

Now since d_{n+1} is a graded derivation and vanishes on $E_{n+1}^{\bullet,0}$ (by the above degree reasoning), it follows that its action on any element is uniquely fixed to be given by the product with c:

$$\begin{aligned} d_{n+1}(\iota \cdot b) &= d_{n+1}(\iota) \cdot b + (-1)^n \iota \cdot \underbrace{d_{n+1}(b)}_{=0} \\ &= c \cdot b \end{aligned} .$$

This shows that d_{n+1} is identified with the cup product operation in question:

$$\begin{array}{ccc} E_{n+1}^{s,n} & \simeq & H^s(B;R) \\ {}^{\mathllap{d_{n+1}}}\downarrow && \downarrow^{\mathrlap{c \cup (-)}} \\ E_{n+1}^{s+n+1,0} & \simeq & H^{s+n+1}(B;R) \end{array} .$$

In summary, the non-vanishing entries of the E_∞-page of the spectral sequence sit in exact sequences like so

$$\begin{array}{ccc} 0 \\ \downarrow \\ E_\infty^{s,n} \\ {}^{\mathllap{\ker(d_{n+1})}}\downarrow \\ E_{n+1}^{s,n} & \simeq & H^s(B;R) \\ {}^{\mathllap{d_{n+1}}}\downarrow && \downarrow^{\mathrlap{c \cup (-)}} \\ E_{n+1}^{s+n+1,0} & \simeq & H^{s+n+1}(B;R) \\ {}^{\mathllap{\mathrm{coker}(d_{n+1})}}\downarrow \\ E_\infty^{s+n+1,0} \\ \downarrow \\ 0 \end{array} .$$

Finally observe (lemma 1.128) that due to the sparseness of the E_∞-page, there are also short exact sequences of the form

$$0 \to E_\infty^{s,0} \longrightarrow H^s(E;R) \longrightarrow E_\infty^{s-n,n} \to 0 .$$

Concatenating these with the above exact sequences yields the desired long exact sequence. ■

Lemma 1.128. *Consider a cohomology spectral sequence converging to some filtered graded abelian group $F^\bullet C^\bullet$ such that*

1. $F^0 C^\bullet = C^\bullet$;
2. $F^s C^{<s} = 0$;
3. $E_\infty^{s,t} = 0$ *unless* $t = 0$ *or* $t = n$,

for some $n \in \mathbb{N}$, $n \geq 1$. Then there are short exact sequences of the form

$$0 \to E_\infty^{s,0} \longrightarrow C^s \longrightarrow E_\infty^{s-n,n} \to 0 .$$

(e.g. Switzer 75, p. 356)

Proof. By definition of convergence of a spectral sequence, the $E_\infty^{s,t}$ sit in short exact sequences of the form

$$0 \to F^{s+1} C^{s+t} \overset{i}{\longrightarrow} F^s C^{s+t} \longrightarrow E_\infty^{s,t} \to 0 .$$

So when $E_\infty^{s,t} = 0$ then the morphism i above is an isomorphism.

We may use this to either shift away the filtering degree

- if $t \geq n$ then $F^s C^{s+t} = F^{(s-1)+1} C^{(s-1)+(t+1)} \underset{\simeq}{\overset{i^{s-1}}{\longrightarrow}} F^0 C^{(s-1)+(t+1)} = F^0 C^{s+t} \simeq C^{s+t}$;

or to shift away the offset of the filtering to the total degree:

- if $0 \leq t-1 \leq n-1$ then $F^{s+1} C^{s+t} = F^{s+1} C^{(s+1)+(t-1)} \underset{\simeq}{\overset{i^{-(t-1)}}{\longrightarrow}} F^{s+t} C^{(s+1)+(t-1)} = F^{s+t} C^{s+t}$

Moreover, by the assumption that if $t < 0$ then $F^s C^{s+t} = 0$, we also get

$$F^s C^s \simeq E_\infty^{s,0} \,.$$

In summary this yields the vertical isomorphisms

$$\begin{array}{ccccccccc} 0 & \to & F^{s+1} C^{s+n} & \longrightarrow & F^s C^{s+n} & \longrightarrow & E_\infty^{s,n} & \to & 0 \\ && {}^{\mathllap{i^{-(n-1)}}}\downarrow^{\mathrlap{\simeq}} && {}^{\mathllap{i^{s-1}}}\downarrow^{\mathrlap{\simeq}} && \downarrow^{\mathrlap{=}} \\ 0 & \to & F^{s+n} C^{s+n} \simeq E_\infty^{s+n,0} & \longrightarrow & C^{s+n} & \longrightarrow & E_\infty^{s,n} & \to & 0 \end{array}$$

and hence with the top sequence here being exact, so is the bottom sequence. ▮

Thom isomorphism

Proposition 1.129. *Let $V \to B$ be a topological vector bundle of rank $n > 0$ over a simply connected CW-complex B. Let R be a commutative ring.*

There exists an element $c \in H^n(\mathrm{Th}(V); R)$ (in the ordinary cohomology, with coefficients in R, of the Thom space of V, called a ***Thom class****) such that forming the cup product with c induces an isomorphism*

$$H^\bullet(B; R) \overset{c \cup (-)}{\longrightarrow} \tilde H^{\bullet + n}(\mathrm{Th}(V); R)$$

of degree n from the unreduced cohomology group of B to the reduced cohomology of the Thom space of V.

Proof. Choose an orthogonal structure on V. Consider the *fiberwise* cofiber

$$E \coloneqq D(V)/_B S(V)$$

of the inclusion of the unit sphere bundle into the unit disk bundle of V (def. 1.101).

$$\begin{array}{ccccc} S^{n-1} & \hookrightarrow & D^n & \longrightarrow & S^n \\ \downarrow && \downarrow && \downarrow \\ S(V) & \hookrightarrow & D(V) & \longrightarrow & E \\ \downarrow && \downarrow && \downarrow^{\mathrlap{p}} \\ B & = & B & = & B \end{array}$$

Observe that this has the following properties

1. $E \overset{p}{\to} B$ is an n-sphere fiber bundle, hence in particular a Serre fibration;
2. the Thom space $\mathrm{Th}(V) \simeq E/B$ is the quotient of E by the base space, because of the pasting law applied to the following pasting diagram of pushout squares

$$\begin{array}{ccc} S(V) & \longrightarrow & D(V) \\ \downarrow & (\mathrm{po}) & \downarrow \\ B & \longrightarrow & D(V)/_B S(V) \\ \downarrow & (\mathrm{po}) & \downarrow \\ \ast & \longrightarrow & \mathrm{Th}(V) \end{array}$$

3. hence the reduced cohomology of the Thom space is (def.) the relative cohomology of E relative B

$$\tilde H^\bullet(\mathrm{Th}(V); R) \simeq H^\bullet(E, B; R) \,.$$

4. $E \overset{p}{\to} B$ has a global section $B \overset{s}{\to} E$ (given over any point $b \in B$ by the class of any point in the fiber of $S(V) \to B$ over b; or abstractly: induced via the above pushout by the commutation of the projections from $D(V)$ and from $S(V)$, respectively).

In the following we write $H^\bullet(-) \coloneqq H^\bullet(-;R)$, for short.

By the first point, there is the Thom-Gysin sequence (prop. 1.127), an exact sequence running vertically in the following diagram

$$\begin{array}{ccccc} && H^\bullet(B) \\ && {}^{\mathllap{p^\ast}}\downarrow & \searrow^{\simeq} \\ \tilde H^\bullet(\mathrm{Th}(V)) &\longrightarrow& H^\bullet(E) &\underset{s^\ast}{\longrightarrow}& H^\bullet(B) \\ && \downarrow \\ && H^{\bullet-n}(B) \end{array}.$$

By the second point above this is split, as shown by the diagonal isomorphism in the top right. By the third point above there is the horizontal exact sequence, as shown, which is the exact sequence in relative cohomology $\cdots \to H^\bullet(E,B) \to H^\bullet(E) \to H^\bullet(B) \to \cdots$ induced from the section $B \hookrightarrow E$.

Hence using the splitting to decompose the term in the middle as a direct sum, and then using horizontal and vertical exactness at that term yields

$$\begin{array}{ccccc} && H^\bullet(B) \\ && {}^{\mathllap{(0,\mathrm{id})}}\downarrow && \searrow^{\simeq} \\ \tilde H^\bullet(\mathrm{Th}(V)) &\overset{(\mathrm{id},0)}{\hookrightarrow}& \tilde H^\bullet(\mathrm{Th}(V)) \oplus H^\bullet(B) &\underset{(0,\mathrm{id})}{\longrightarrow}& H^\bullet(B) \\ && \downarrow^{\mathrlap{(\mathrm{id},0)}} \\ && H^{\bullet-n}(B) \end{array}$$

and hence an isomorphism

$$\tilde H^\bullet(\mathrm{Th}(V)) \overset{\simeq}{\longrightarrow} H^{\bullet-n}(B)\,.$$

To see that this is the inverse of a morphism of the form $c \cup (-)$, inspect the proof of the Gysin sequence. This shows that $H^{\bullet-n}(B)$ here is identified with elements that on the second page of the corresponding Serre spectral sequence are cup products

$$\iota \cup b$$

with ι fiberwise the canonical class $1 \in H^n(S^n)$ and with $b \in H^\bullet(B)$ any element. Since $H^\bullet(-;R)$ is a multiplicative cohomology theory (because the coefficients form a ring R), cup producs are preserved as one passes to the E_∞-page of the spectral sequence, and the morphism $H^\bullet(E) \to B^\bullet(B)$ above, hence also the isomorphism $\tilde H^\bullet(\mathrm{Th}(V)) \to H^\bullet(B)$, factors through the E_∞-page (see towards the end of the proof of the Gysin sequence). Hence the image of ι on the E_∞-page is the Thom class in question. ▮

Orientation in generalized cohomology

Idea. From the way the Thom isomorphism via a Thom class works in ordinary cohomology (as above), one sees what the general concept of orientation in generalized cohomology and of fiber integration in generalized cohomology is to be.

Specifically we are interested in complex oriented cohomology theories E, characterized by an orientation class on infinity complex projective space $\mathbb{C}P^\infty$ (def. 1.134), the classifying space for complex line bundles, which restricts to a generator on $S^2 \hookrightarrow \mathbb{C}P^\infty$.

(Another important application is given by taking $E =$ KU to be topological K-theory. Then orientation is spin^c structure and fiber integration with coefficients in E is fiber integration in K-theory. This is classical *index theory*.)

Literature. (Kochman 96, section 4.3, Adams 74, part III, section 10, Lurie 10, lecture 5)

- Riccardo Pedrotti, *Complex oriented cohomology – Orientation in generalized cohomology*, 2016 (pdf)

Universal E-orientation

Definition 1.130. Let E be a multiplicative cohomology theory (def. 1.26) and let $V \to X$ be a topological vector bundle of rank n. Then an E-**orientation** or E-**Thom class** on V is an element of degree n

$$u \in \tilde{E}^n(\mathrm{Th}(V))$$

in the reduced E-cohomology ring of the Thom space (def. 1.101) of V, such that for every point $x \in X$ its restriction $i_x^* u$ along

$$i_x : S^n \simeq \mathrm{Th}(\mathbb{R}^n) \overset{\mathrm{Th}(e_x)}{\longrightarrow} \mathrm{Th}(V)$$

(for $\mathbb{R}^n \overset{\mathrm{fib}_x}{\hookrightarrow} V$ the fiber of V over x) is a *generator*, in that it is of the form

$$i^* u = \epsilon \cdot \gamma_n$$

for

- $\epsilon \in \tilde{E}^0(S^0)$ a unit in $E^\bullet$;
- $\gamma_n \in \tilde{E}^n(S^n)$ the image of the multiplicative unit under the suspension isomorphism $\tilde{E}^0(S^0) \overset{\simeq}{\to} \tilde{E}^n(S^n)$.

(e.g. Kochmann 96, def. 4.3.4)

Remark 1.131. Recall that a *(B,f)-structure* $\mathcal{B}$ (def. 1.98) is a system of Serre fibrations $B_n \overset{f_n}{\to} BO(n)$ over the classifying spaces for orthogonal structure equipped with maps

$$g_{n,n+1} : B_n \longrightarrow B_{n+1}$$

covering the canonical inclusions of classifying spaces. For instance for $G_n \to O(n)$ a compatible system of topological group homomorphisms, then the (B,f)-structure given by the classifying spaces BG_n (possibly suitably resolved for the maps $BG_n \to BO(n)$ to become Serre fibrations) defines *G-structure*.

Given a (B,f)-structure, then there are the pullbacks $V_n^{\mathcal{B}} \coloneqq f_n^*(EO(n) \underset{O(n)}{\times} \mathbb{R}^n)$ of the universal vector bundles over $BO(n)$, which are the *universal vector bundles equipped with* (B,f)*-structure*

$$\begin{array}{ccc} V_n^{\mathcal{B}} & \longrightarrow & EO(n) \underset{O(n)}{\times} \mathbb{R}^n \\ \downarrow & (\mathrm{pb}) & \downarrow \\ B_n & \underset{f_n}{\longrightarrow} & BO(n) \end{array} .$$

Finally recall that there are canonical morphisms (prop.)

$$\phi_n : \mathbb{R} \oplus V_n^{\mathcal{B}} \longrightarrow V_{n+1}^{\mathcal{B}}$$

Definition 1.132. Let E be a multiplicative cohomology theory and let $\mathcal{B}$ be a multiplicative (B,f)-structure. Then a **universal E-orientation for vector bundles with $\mathcal{B}$-structure** is an E-orientation, according to def. 1.130, for each rank-n universal vector bundle with $\mathcal{B}$-structure:

$$\xi_n \in \tilde{E}^n(\mathrm{Th}(E_n^{\mathcal{B}})) \quad \forall n \in \mathbb{N}$$

such that these are compatible in that

1. for all $n \in \mathbb{N}$ then

   $$\xi_n = \phi_n^* \xi_{n+1} ,$$

   where

   $$\xi_n \in \tilde{E}^n(\mathrm{Th}(V_n)) \simeq \tilde{E}^{n+1}(\Sigma\, \mathrm{Th}(V_n)) \simeq \tilde{E}^{n+1}(\mathrm{Th}(\mathbb{R} \oplus V_n))$$

   (with the first isomorphism is the suspension isomorphism of E and the second exhibiting the homeomorphism of Thom spaces $\mathrm{Th}(\mathbb{R} \oplus V) \simeq \Sigma\, \mathrm{Th}(V)$ (prop. 1.106) and where

   $$\phi_n^* : \tilde{E}^{n+1}(\mathrm{Th}(V_{n+1})) \longrightarrow \tilde{E}^{n+1}(\mathrm{Th}(\mathbb{R} \oplus V_n))$$

   is pullback along the canonical $\phi_n : \mathbb{R} \oplus V_n \to V_{n+1}$ (prop. 1.110).

2. for all $n_1, n_2 \in \mathbb{N}$ then

   $$\xi_{n+1} \cdot \xi_{n+2} = \xi_{n_1 + n_2} .$$

Proposition 1.133. *A universal E-orientation, in the sense of def. 1.132, for vector bundles with (B,f)-structure $\mathcal{B}$, is equivalently (the homotopy class of) a homomorphism of ring spectra*

$$\xi : M\mathcal{B} \longrightarrow E$$

from the universal $\mathcal{B}$-Thom spectrum to a spectrum which via the Brown representability theorem (theorem 1.30) represents the given generalized (Eilenberg-Steenrod) cohomology theory E (and which we denote by the same symbol).

Proof. The Thom spectrum $M\mathcal{B}$ has a standard structure of a CW-spectrum. Let now E denote a sequential Omega-spectrum representing the multiplicative cohomology theory of the same name. Since, in the standard model structure on topological sequential spectra, CW-spectra are cofibrant (prop.) and Omega-spectra are fibrant (thm.) we may represent all morphisms in the stable homotopy category (def.) by actual morphisms

$$\xi : M\mathcal{B} \longrightarrow E$$

of sequential spectra (due to this lemma).

Now by definition (def.) such a homomorphism is precissely a sequence of base-point preserving continuous functions

$$\xi_n : (M\mathcal{B})_n = \mathrm{Th}(V_n^{\mathcal{B}}) \longrightarrow E_n$$

for $n \in \mathbb{N}$, such that they are compatible with the structure maps σ_n and equivalently with their $(S^1 \wedge (-) \dashv \mathrm{Maps}(S^1, -)_*)$-adjuncts $\tilde{\sigma}_n$, in that these diagrams commute:

$$\begin{array}{ccc} S^1 \wedge \mathrm{Th}(V_n^{\mathcal{B}}) & \overset{S^1 \wedge \xi_n}{\longrightarrow} & S^1 \wedge E_n \\ {}^{\sigma_n^{M\mathcal{B}}}\downarrow && \downarrow^{\sigma_n^E} \\ \mathrm{Th}(V_{n+1}^{\mathcal{B}}) & \underset{\xi_{n+1}}{\longrightarrow} & E_{n+1} \end{array} \quad \leftrightarrow \quad \begin{array}{ccc} \mathrm{Th}(V_n^{\mathcal{B}}) & \overset{\xi_n}{\longrightarrow} & E_n \\ {}^{\tilde{\sigma}_n^{M\mathcal{B}}}\downarrow && \downarrow^{\tilde{\sigma}_n^E} \\ \mathrm{Maps}(S^1, \mathrm{Th}(V_{n+1}^{\mathcal{B}})) & \underset{\mathrm{Maps}(S^1, \xi_{n+1})_*}{\longrightarrow} & \mathrm{Maps}(S^1, E_{n+1})_* \end{array}$$

for all $n \in \mathbb{N}$.

First of all this means (via the identification given by the Brown representability theorem, see prop. 1.33, that the components ξ_n are equivalently representatives of elements in the cohomology groups

$$\xi_n \in \tilde{E}^n(\mathrm{Th}(V_n^{\mathcal{B}}))$$

(which we denote by the same symbol, for brevity).

Now by the definition of universal Thom spectra (def. 1.111, def. 1.114), the structure map $\sigma_n^{M\mathcal{B}}$ is just the map $\phi_n : \mathbb{R} \oplus \mathrm{Th}(V_n^{\mathcal{B}}) \to \mathrm{Th}(V_{n+1}^{\mathcal{B}})$ from above.

Moreover, by the Brown representability theorem, the adjunct $\tilde{\sigma}_n^E \circ \xi_n$ (on the right) of $\sigma_n^E \circ S^1 \wedge \xi_n$ (on the left) is what represents (again by prop. 1.33) the image of

$$\xi_n \in E^n(\mathrm{Th}(V_n^{\mathcal{B}}))$$

under the suspension isomorphism. Hence the commutativity of the above squares is equivalently the first compatibility condition from def. 1.132: $\xi_n \simeq \phi_n^* \xi_{n+1}$ in $\tilde{E}^{n+1}(\mathrm{Th}(\mathbb{R} \oplus V_n^{\mathcal{B}}))$

Next, ξ being a homomorphism of ring spectra means equivalently (we should be modelling $M\mathcal{B}$ and E as structured spectra (here.) to be more precise on this point, but the conclusion is the same) that for all $n_1, n_2 \in \mathbb{N}$ then

$$\begin{array}{ccc} \mathrm{Th}(V_{n_1}^{\mathcal{B}}) \wedge \mathrm{Th}(V_{n_2}^{\mathcal{B}}) & \longrightarrow & \mathrm{Th}(V_{n_1+n_2}) \\ {}^{\xi_{n_1} \wedge \xi_{n_2}}\downarrow && \downarrow^{\xi_{n_1+n_2}} \\ E_{n_1} \wedge E_{n_2} & \underset{\cdot}{\longrightarrow} & E_{n_1+n_2} \end{array}$$

This is equivalently the condition $\xi_{n_1} \cdot \xi_{n_2} \simeq \xi_{n_1+n_2}$.

Finally, since $M\mathcal{B}$ is a ring spectrum, there is an essentially unique multiplicative homomorphism from the sphere spectrum

$$\mathbb{S} \overset{e}{\longrightarrow} M\mathcal{B} \,.$$

This is given by the component maps

$$e_n : S^n \simeq \mathrm{Th}(\mathbb{R}^n) \longrightarrow \mathrm{Th}(V_n^{\mathcal{B}})$$

that are induced by including the fiber of $V_n^{\mathcal{B}}$.

Accordingly the composite

$$\mathbb{S} \overset{e}{\longrightarrow} M\mathcal{B} \overset{\xi}{\longrightarrow} E$$

has as components the restrictions $i^* \xi_n$ appearing in def. 1.130. At the same time, also E is a ring spectrum, hence it also has an essentially unique multiplicative morphism $\mathbb{S} \to E$, which hence must agree with $i^*\xi$, up to homotopy. If we represent E as a symmetric ring spectrum, then the canonical such has the required property: e_0 is the identity element in degree 0 (being a unit of an ordinary ring, by definition) and hence e_n is necessarily its image under the suspension isomorphism, due to compatibility with the structure maps and using the above analysis. ▮

Complex projective space

For the fine detail of the discussion of complex oriented cohomology theories below, we recall basic facts about complex projective space.

Complex projective space $\mathbb{C}P^n$ is the projective space $\mathbb{A}P^n$ for $\mathbb{A} = \mathbb{C}$ being the complex numbers (and for $n \in \mathbb{N}$), a complex manifold of complex dimension n (real dimension $2n$). Equivalently, this is the complex Grassmannian $\mathrm{Gr}_1(\mathbb{C}^{n+1})$ (def. 1.84). For the special case $n = 1$ then $\mathbb{C}P^1 \simeq S^2$ is the Riemann sphere.

As n ranges, there are natural inclusions

$$* = \mathbb{C}P^0 \hookrightarrow \mathbb{C}P^1 \hookrightarrow \mathbb{C}P^2 \hookrightarrow \mathbb{C}P^3 \hookrightarrow \cdots \,.$$

The sequential colimit over this sequence is the infinite complex projective space $\mathbb{C}P^\infty$. This is a model for the classifying space $BU(1)$ of circle principal bundles/complex line bundles (an Eilenberg-MacLane space $K(\mathbb{Z},2)$).

Definition 1.134. For $n \in \mathbb{N}$, then **complex n-dimensional complex projective space** is the complex manifold (often just regarded as its underlying topological space) defined as the quotient

$$\mathbb{C}P^n \coloneqq (\mathbb{C}^{n+1} - \{0\})/_\sim$$

of the Cartesian product of $(n+1)$-copies of the complex plane, with the origin removed, by the equivalence relation

$$(z \sim w) \Leftrightarrow (z = \kappa \cdot w)$$

for some $\kappa \in \mathbb{C} - \{0\}$ and using the canonical multiplicative action of $\mathbb{C}$ on $\mathbb{C}^{n+1}$.

The canonical inclusions

$$\mathbb{C}^{n+1} \hookrightarrow \mathbb{C}^{n+2}$$

induce canonical inclusions

$$\mathbb{C}P^n \hookrightarrow \mathbb{C}P^{n+1} \,.$$

The sequential colimit over this sequence of inclusions is the *infinite complex projective space*

$$\mathbb{C}P^\infty \coloneqq \underset{\longleftarrow}{\lim}_n \, \mathbb{C}P^n \,.$$

The following equivalent characterizations are immediate but useful:

Proposition 1.135. *For $n \in \mathbb{N}$ then complex projective space, def. 1.134, is equivalently the complex Grassmannian*

$$\mathbb{C}P^n \simeq \mathrm{Gr}_1(\mathbb{C}^{n+1}) \,.$$

Proposition 1.136. *For $n \in \mathbb{N}$ then complex projective space, def. 1.134, is equivalently*

1. *the coset*

$$\mathbb{C}P^n \simeq U(n+1)/(U(n) \times U(1)) \,,$$

2. *the quotient of the (2n+1)-sphere by the circle group* $S^1 \simeq \{\kappa \in \mathbb{C} \,|\, |\kappa| = 1\}$

$$\mathbb{C}P^n \simeq S^{2n+1}/S^1 \,.$$

Proof. To see the second characterization from def. 1.134:

With $|-| : \mathbb{C}^n \longrightarrow \mathbb{R}$ the standard norm, then every element $\vec{z} \in \mathbb{C}^{n+1}$ is identified under the defining equivalence relation with

$$\frac{1}{|\vec{z}|}\vec{z} \in S^{2n-1} \hookrightarrow \mathbb{C}^{n+1}$$

lying on the unit $(2n-1)$-sphere. This fixes the action of $\mathbb{C} - 0$ up to a remaining action of complex numbers of unit absolute value. These form the circle group S^1.

The first characterization follows via prop. 1.135 from the general discusion at *Grassmannian*. With this the second characterization follows also with the coset identification of the $(2n+1)$-sphere: $S^{2n+1} \simeq U(n+1)/U(n)$ (exmpl.). ▮

Proposition 1.137. *There is a CW-complex structure on complex projective space $\mathbb{C}P^n$ (def. 1.134) for $n \in \mathbb{N}$, given by induction, where $\mathbb{C}P^{n+1}$ arises from $\mathbb{C}P^n$ by attaching a single cell of dimension $2(n+1)$ with attaching map the projection $S^{2n+1} \longrightarrow \mathbb{C}P^n$ from prop. 1.136:*

$$\begin{array}{ccc} S^{2n+1} & \longrightarrow & S^{2n+1}/S^1 \simeq \mathbb{C}P^n \\ {}^{\iota_{2n+2}}\downarrow & (\mathrm{po}) & \downarrow \\ D^{2n+2} & \longrightarrow & \mathbb{C}P^{n+1} \end{array} \,.$$

Proof. Given homogenous coordinates $(z_0, z_1, \cdots, z_n, z_{n+1}, z_{n+2}) \in \mathbb{C}^{n+2}$ for $\mathbb{C}P^{n+1}$, let

$$\phi \coloneqq -\arg(z_{n+2})$$

be the phase of z_{n+2}. Then under the equivalence relation defining $\mathbb{C}P^{n+1}$ these coordinates represent the same element as

$$\frac{1}{|\vec{z}|}(e^{i\phi}z_0, e^{i\phi}z_1, \cdots, e^{i\phi}z_{n+1}, r) \,,$$

where

$$r = |z_{n+2}| \in [0,1) \subset \mathbb{C}$$

is the absolute value of z_{n+2}. Representatives $\vec{z}'$ of this form ($|\vec{z}'| = 1$ and $z'_{n+2} \in [0,1]$) parameterize the 2n+2-disk D^{2n+2} ($2n+3$ real parameters subject to the one condition that the sum of their norm squares is unity) with boundary the $(2n+1)$-sphere at $r = 0$. The only remaining part of the action of $\mathbb{C} - \{0\}$ which fixes the form of these representatives is S^1 acting on the elements with $r = 0$ by phase shifts on the $z_0, \cdots, z_{n+1}$. The quotient of this remaining action on $D^{2(n+1)}$ identifies its boundary S^{2n+1}-sphere with $\mathbb{C}P^n$, by prop. 1.136. ▮

Proposition 1.138. *For $A \in$ Ab any abelian group, then the ordinary homology groups of complex projective space $\mathbb{C}P^n$ with coefficients in A are*

$$H_k(\mathbb{C}P^n, A) \simeq \begin{cases} A & \text{for } k \text{ even and } k \leq 2n \\ 0 & \text{otherwise} \end{cases} \,.$$

Similarly the ordinary cohomology groups of $\mathbb{C}P^n$ is

$$H^k(\mathbb{C}P^n, A) \simeq \begin{cases} A & \text{for } k \text{ even and } k \leq 2n \\ 0 & \text{otherwise} \end{cases} \,.$$

Moreover, if A carries the structure of a ring $R = (A, \cdot)$, then under the cup product the cohomology ring of $\mathbb{C}P^n$ is the the graded ring

$$H^\bullet(\mathbb{C}P^n, R) \simeq R[c_1]/(c_1^{n+1})$$

which is the quotient of the polynomial ring on a single generator c_1 in degree 2, by the relation that identifies cup products of more than n-copies of the generator c_1 with zero.

Finally, the cohomology ring of the infinite-dimensional complex projective space is the formal power series ring in one generator:

$$H^\bullet(\mathbb{C}P^\infty, R) \simeq R[[c_1]] \,.$$

(Or else the polynomial ring $R[c_1]$, see remark 1.139)

Proof. First consider the case that the coefficients are the integers $A = \mathbb{Z}$.

Since $\mathbb{C}P^n$ admits the structure of a CW-complex by prop. 1.137, we may compute its ordinary homology equivalently as its cellular homology (thm.). By definition (defn.) this is the chain homology of the chain complex of relative homology groups

$$\cdots \overset{\partial_{\text{cell}}}{\longrightarrow} H_{q+2}((\mathbb{C}P^n)_{q+2}, (\mathbb{C}P^n)_{q+1}) \overset{\partial_{\text{cell}}}{\longrightarrow} H_{q+1}((\mathbb{C}P^n)_{q+1}, (\mathbb{C}P^n)_{q}) \overset{\partial_{\text{cell}}}{\longrightarrow} H_{q}((\mathbb{C}P^n)_{q}, (\mathbb{C}P^n)_{q-1}) \overset{\partial_{\text{cell}}}{\longrightarrow} \cdots \,,$$

where $(-)_q$ denotes the qth stage of the CW-complex-structure. Using the CW-complex structure provided by prop. 1.137, then there are cells only in every second degree, so that

$$(\mathbb{C}P^n)_{2k+1} = (\mathbb{C}P)_{2k}$$

for all $k \in \mathbb{N}$. It follows that the cellular chain complex has a zero group in every second degree, so that all differentials vanish. Finally, since prop. 1.137 says that $(\mathbb{C}P^n)_{2k+2}$ arises from $(\mathbb{C}P^n)_{2k+1} = (\mathbb{C}P^n)_{2k}$ by attaching a single $2k+2$-cell it follows that (by passage to reduced homology)

$$H_{2k}(\mathbb{C}P^n, \mathbb{Z}) \simeq \tilde H_{2k}(S^{2k})((\mathbb{C}P^n)_{2k}/(\mathbb{C}P^n)_{2k-1}) \simeq \tilde H_{2k}(S^{2k}) \simeq \mathbb{Z} \,.$$

This establishes the claim for ordinary homology with integer coefficients.

In particular this means that $H_q(\mathbb{C}P^n, \mathbb{Z})$ is a free abelian group for all q. Since free abelian groups are the projective objects in Ab (prop.) it follows (with the discussion at *derived functors in homological algebra*) that the Ext-groups vanishe:

$$\mathrm{Ext}^1(H_q(\mathbb{C}P^n, \mathbb{Z}), A) = 0$$

and the Tor-groups vanishes:

$$\mathrm{Tor}_1(H_q(\mathbb{C}P^n), A) = 0 \,.$$

With this, the statement about homology and cohomology groups with general coefficients follows with the universal coefficient theorem for ordinary homology (thm.) and for ordinary cohomology (thm.).

Finally to see the action of the cup product: by definition this is the composite

$$\cup \;:\; H^\bullet(\mathbb{C}P^n, R) \otimes H^\bullet(\mathbb{C}P^n, R) \longrightarrow H^\bullet(\mathbb{C}P^n \times \mathbb{C}P^n, R) \overset{\Delta^\ast}{\longrightarrow} H^\bullet(\mathbb{C}P^n, R)$$

of the "cross-product" map that appears in the Kunneth theorem, and the pullback along the diagonal $\Delta : \mathbb{C}P^n \to \mathbb{C}P^n \times \mathbb{C}P^n$.

Since, by the above, the groups $H^{2k}(\mathbb{C}P^n, R) \simeq R[2k]$ and $H^{2k+1}(\mathbb{C}P^n, R) = 0$ are free and finitely generated, the Kunneth theorem in ordinary cohomology applies (prop.) and says that the cross-product map above is an isomorphism. This shows that under cup product pairs of generators are sent to a generator, and so the statement $H^\bullet(\mathbb{C}P^n, R) \simeq R[c_1](c_1^{n+1})$ follows.

This also implies that the projection maps

$$H^\bullet((\mathbb{C}P^\infty)_{2n+2}, R) = H^\bullet(\mathbb{C}P^{n+1}, R) \to H^\bullet(\mathbb{C}P^{n+}, R) = H^\bullet((\mathbb{C}P^\infty)_{2n}, R)$$

are all epimorphisms. Therefore this sequence satisfies the Mittag-Leffler condition (def. 1.55, example 1.56) and therefore the Milnor exact sequence for cohomology (prop. 1.61) implies the last claim to be proven:

$$\begin{aligned} H^\bullet(\mathbb{C}P^\infty, R) & \\ & \simeq H^\bullet(\underset{\longleftarrow_n}{\lim}\; \mathbb{C}P^n, R) \\ & \simeq \underset{\longrightarrow_n}{\lim}\; H^\bullet(\mathbb{C}P^n, R) \\ & \simeq \underset{\longrightarrow_n}{\lim}\; (R[c_1^E]/((c_1)^{n+1})) \\ & \simeq R[[c_1]] \,, \end{aligned}$$

where the last step is this prop.. ▮

Remark 1.139. There is in general a choice to be made in interpreting the cohomology groups of a multiplicative cohomology theory E (def. 1.26) as a ring:

a priori $E^\bullet(X)$ is a sequence

$$\{E^n(X)\}_{n \in \mathbb{Z}}$$

of abelian groups, together with a system of group homomorphisms

$$E^{n_1}(X) \otimes E^{n_2}(X) \longrightarrow E^{n_1+n_2}(X),$$

one for each pair $(n_1, n_2) \in \mathbb{Z} \times \mathbb{Z}$.

In turning this into a single ring by forming formal sums of elements in the groups $E^n(X)$, there is in general the choice of whether allowing formal sums of only finitely many elements, or allowing arbitrary formal sums.

In the former case the ring obtained is the direct sum

$$\oplus_{n \in \mathbb{N}} E^n(X)$$

while in the latter case it is the Cartesian product

$$\prod_{n \in \mathbb{N}} E^n(X) .$$

These differ in general. For instance if E is ordinary cohomology with integer coefficients and X is infinite complex projective space $\mathbb{C}P^\infty$, then (prop. 1.138))

$$E^n(X) = \begin{cases} \mathbb{Z} & n \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

and the product operation is given by

$$E^{2n_1}(X) \otimes E^{2n_2}(X) \longrightarrow E^{2(n_1+n_2)}(X)$$

for all n_1, n_2 (and zero in odd degrees, necessarily). Now taking the direct sum of these, this is the polynomial ring on one generator (in degree 2)

$$\oplus_{n \in \mathbb{N}} E^n(X) \;\simeq\; \mathbb{Z}[c_1] .$$

But taking the Cartesian product, then this is the formal power series ring

$$\prod_{n \in \mathbb{N}} E^n(X) \;\simeq\; \mathbb{Z}[[c_1]] .$$

A priori both of these are sensible choices. The former is the usual choice in traditional algebraic topology. However, from the point of view of regarding ordinary cohomology theory as a multiplicative cohomology theory right away, then the second perspective tends to be more natural:

The cohomology of $\mathbb{C}P^\infty$ is naturally computed as the inverse limit of the cohomolgies of the $\mathbb{C}P^n$, each of which unambiguously has the ring structure $\mathbb{Z}[c_1]/((c_1)^{n+1})$. So we may naturally take the limit in the category of commutative rings right away, instead of first taking it in $\mathbb{Z}$-indexed sequences of abelian groups, and then looking for ring structure on the result. But the limit taken in the category of rings gives the formal power series ring (see here).

See also for instance remark 1.1. in Jacob Lurie: *A Survey of Elliptic Cohomology*.

Complex orientation

Definition 1.140. A multiplicative cohomology theory E (def. 1.26) is called **complex orientable** if the the following equivalent conditions hold

1. The morphism

$$i^* : E^2(BU(1)) \longrightarrow E^2(S^2)$$

   is surjective.

2. The morphism

$$\tilde{\iota}^* : \tilde{E}^2(BU(1)) \longrightarrow \tilde{E}^2(S^2) \simeq \pi_0(E)$$

is surjective.

3. The element $1 \in \pi_0(E)$ is in the image of the morphism $\tilde{\iota}^*$.

A **complex orientation** on a multiplicative cohomology theory $E^\bullet$ is an element

$$c_1^E \in \tilde{E}^2(BU(1))$$

(the "first generalized Chern class") such that

$$i^* c_1^E = 1 \in \pi_0(E) \,.$$

Remark 1.141. Since $BU(1) \simeq K(\mathbb{Z}, 2)$ is the classifying space for complex line bundles, it follows that a complex orientation on $E^\bullet$ induces an E-generalization of the first Chern class which to a complex line bundle $\mathcal{L}$ on X classified by $\phi : X \to BU(1)$ assigns the class $c_1(\mathcal{L}) \coloneqq \phi^* c_1^E$. This construction extends to a general construction of E-Chern classes.

Proposition 1.142. *Given a complex oriented cohomology theory $(E^\bullet, c_1^E)$ (def. 1.140), then there is an isomorphism of graded rings*

$$E^\bullet(\mathbb{C}P^\infty) \simeq E^\bullet(\ast)[[c_1^E]]$$

between the E-cohomology ring of infinite-dimensional complex projective space (def. 1.134) and the formal power series (see remark 1.139) in one generator of even degree over the E-cohomology ring of the point.

Proof. Using the CW-complex-structure on $\mathbb{C}P^\infty$ from prop. 1.137, given by inductively identifying $\mathbb{C}P^{n+1}$ with the result of attaching a single $2n$-cell to $\mathbb{C}P^n$. With this structure, the unique 2-cell inclusion $i : S^2 \hookrightarrow \mathbb{C}P^\infty$ is identified with the canonical map $S^2 \to BU(1)$.

Then consider the Atiyah-Hirzebruch spectral sequence (prop. 1.71) for the E-cohomology of $\mathbb{C}P^n$.

$$H^\bullet(\mathbb{C}P^n, E^\bullet(\ast)) \;\Rightarrow\; E^\bullet(\mathbb{C}P^n) \,.$$

Since, by prop. 1.138, the ordinary cohomology with integer coefficients of complex projective space is

$$H^\bullet(\mathbb{C}P^n, \mathbb{Z}) \simeq \mathbb{Z}[c_1]/((c_1)^{n+1}) \,,$$

where c_1 represents a unit in $H^2(S^2, \mathbb{Z}) \simeq \mathbb{Z}$, and since similarly the ordinary homology of $\mathbb{C}P^n$ is a free abelian group, hence a projective object in abelian groups (prop.), the Ext-group vanishes in each degree ($\mathrm{Ext}^1(H_n(\mathbb{C}P^n), E^\bullet(\ast)) = 0$) and so the universal coefficient theorem (prop.) gives that the second page of the spectral sequence is

$$H^\bullet(\mathbb{C}P^n, E^\bullet(\ast)) \simeq E^\bullet(\ast)[c_1]/(c_1^{n+1}) \,.$$

By the standard construction of the Atiyah-Hirzebruch spectral sequence (here) in this identification the element c_1 is identified with a generator of the relative cohomology

$$E^2((\mathbb{C}P^n)_2, (\mathbb{C}P^n)_1) \simeq \tilde{E}^2(S^2)$$

(using, by the above, that this S^2 is the unique 2-cell of $\mathbb{C}P^n$ in the standard cell model).

This means that c_1 is a permanent cocycle of the spectral sequence (in the kernel of all differentials) precisely if it arises via restriction from an element in $E^2(\mathbb{C}P^n)$ and hence precisely if there exists a complex orientation c_1^E on E. Since this is the case by assumption on E, c_1 is a permanent cocycle. (For the fully detailed argument see (Pedrotti 16)).

The same argument applied to all elements in $E^\bullet(\ast)[c]$, or else the $E^\bullet(\ast)$-linearity of the differentials (prop. 1.73), implies that all these elements are permanent cocycles.

Since the AHSS of a multiplicative cohomology theory is a multiplicative spectral sequence (prop.) this implies that the differentials in fact vanish on all elements of $E^\bullet(\ast)[c_1]/(c_1^{n+1})$, hence that the given AHSS collapses on the second page to give

$$\mathcal{E}_\infty^{\bullet,\bullet} \simeq E^\bullet(\ast)[c_1^E]/((c_1^E)^{n+1})$$

or in more detail:

$$\mathcal{E}_\infty^{p,\bullet} \simeq \begin{cases} E^\bullet(\ast) & \text{if } p \leq 2n \text{ and even} \\ 0 & \text{otherwise} \end{cases} .$$

Moreover, since therefore all $\mathcal{E}_\infty^{p,\bullet}$ are free modules over $E^\bullet(\ast)$, and since the filter stage inclusions $F^{p+1}E^\bullet(X) \hookrightarrow F^p E^\bullet(X)$ are $E^\bullet(\ast)$-module homomorphisms (prop.) the extension problem (remark 1.70) trivializes, in that all the short exact sequences

$$0 \to F^{p+1}E^{p+\bullet}(X) \longrightarrow F^p E^{p+\bullet}(X) \longrightarrow \mathcal{E}_\infty^{p,\bullet} \to 0$$

split (since the Ext-group $\mathrm{Ext}^1_{E^\bullet(\ast)}(\mathcal{E}_\infty^{p,\bullet}, -) = 0$ vanishes on the free module, hence projective module $\mathcal{E}_\infty^{p,\bullet}$).

In conclusion, this gives an isomorphism of graded rings

$$E^\bullet(\mathbb{C}P^n) \simeq \underset{p}{\oplus} \mathcal{E}_\infty^{p,\bullet} \simeq E^\bullet(\ast)[c_1]/((c_1^E)^{n+1}) .$$

A first consequence is that the projection maps

$$E^\bullet((\mathbb{C}P^\infty)_{2n+2}) = E^\bullet(\mathbb{C}P^{n+1}) \to E^\bullet(\mathbb{C}P^{n+}) = E^\bullet((\mathbb{C}P^\infty)_{2n})$$

are all epimorphisms. Therefore this sequence satisfies the Mittag-Leffler condition (def., exmpl.) and therefore the Milnor exact sequence for generalized cohomology (prop.) finally implies the claim:

$$\begin{aligned} E^\bullet(BU(1)) & \simeq E^\bullet(\mathbb{C}P^\infty) \\ & \simeq E^\bullet(\underset{\longleftarrow n}{\lim}\, \mathbb{C}P^n) \\ & \simeq \underset{\longrightarrow n}{\lim}\, E^\bullet(\mathbb{C}P^n) \\ & \simeq \underset{\longrightarrow n}{\lim}\, (E^\bullet(\ast)[c_1^E]/((c_1^E)^{n+1})) \\ & \simeq E^\bullet(\ast)[[c_1^E]] , \end{aligned}$$

where the last step is this prop.. ▮

S.3) Complex oriented cohomology

Idea. Given the concept of orientation in generalized cohomology as above, it is clearly of interest to consider cohomology theories E such that there exists an orientation/Thom class on the universal vector bundle over any classifying space BG (or rather: on its induced spherical fibration), because then *all* G-associated vector bundles inherit an orientation.

Considering this for $G = U(n)$ the unitary groups yields the concept of *complex oriented cohomology theory*.

It turns out that a complex orientation on a generalized cohomology theory E in this sense is already given by demanding that there is a suitable generalization of the first Chern class of complex line bundles in E-cohomology. By the splitting principle, this already implies the existence of generalized Chern classes (Conner-Floyd Chern classes) of all degrees, and these are the required universal generalized Thom classes.

Where the ordinary first Chern class in ordinary cohomology is simply additive under tensor product of complex line bundles, one finds that the composite of generalized first Chern classes is instead governed by more general commutative formal group laws. This phenomenon governs much of the theory to follow.

Literature. (Kochman 96, section 4.3, Lurie 10, lectures 1-10, Adams 74, Part I, Part II, Pedrotti 16).

Chern classes

Idea. In particular ordinary cohomology HR is canonically a complex oriented cohomology theory. The behaviour of general Conner-Floyd Chern classes to be discussed below follows closely the behaviour of the ordinary Chern classes.

An ordinary Chern class is a characteristic class of complex vector bundles, and since there is the classifying space BU of complex vector bundles, the *universal* Chern classes are those of the universal complex vector bundle over the classifying space BU, which in turn are just the ordinary cohomology classes in $H^\bullet(BU)$

These may be computed inductively by iteratively applying to the spherical fibrations

$$S^{2n-1} \longrightarrow BU(n-1) \longrightarrow BU(n)$$

the Thom-Gysin exact sequence, a special case of the Serre spectral sequence.

Pullback of Chern classes along the canonical map $(BU(1))^n \longrightarrow BU(n)$ identifies them with the elementary symmetric polynomials in the first Chern class in $H^2(BU(1))$. This is the *splitting principle*.

Literature. (Kochman 96, section 2.2 and 2.3, Switzer 75, section 16, Lurie 10, lecture 5, prop. 6)

Existence

Proposition 1.143. *The cohomology ring of the classifying space $BU(n)$ (for the unitary group $U(n)$) is the polynomial ring on generators $\{c_k\}_{k=1}^n$ of degree 2, called the Chern classes*

$$H^\bullet(BU(n),\mathbb{Z}) \simeq \mathbb{Z}[c_1,\cdots,c_n]\;.$$

Moreover, for $Bi \colon BU(n_1) \longrightarrow \mathrm{BU}(n_2)$ the canonical inclusion for $n_1 \leq n_2 \in \mathbb{N}$, then the induced pullback map on cohomology

$$(Bi)^\ast \;\colon\; H^\bullet(BU(n_2)) \longrightarrow H^\bullet(BU(n_1))$$

is given by

$$(Bi)^\ast(c_k) \;=\; \left\{ \begin{array}{ll} c_k & \text{for}\; 1 \leq k \leq n_1 \\ 0 & \text{otherwise} \end{array} \right. \,.$$

(e.g. Kochmann 96, theorem 2.3.1)

Proof. For $n = 1$, in which case $BU(1) \simeq \mathbb{C}P^\infty$ is the infinite complex projective space, we have by prop. 1.138

$$H^\bullet(BU(1)) \simeq \mathbb{Z}[c_1]\,,$$

where c_1 is the first Chern class. From here we proceed by induction. So assume that the statement has been shown for $n-1$.

Observe that the canonical map $BU(n-1) \to BU(n)$ has as homotopy fiber the (2n-1)sphere (prop. 1.96) hence there is a homotopy fiber sequence of the form

$$S^{2n-1} \longrightarrow BU(n-1) \longrightarrow BU(n)\;.$$

Consider the induced Thom-Gysin sequence (prop. 1.127).

In odd degrees $2k+1 < 2n$ it gives the exact sequence

$$\cdots \to H^{2k}(BU(n-1)) \longrightarrow \underbrace{H^{2k+1-2n}(BU(n))}_{\simeq 0} \longrightarrow H^{2k+1}(BU(n)) \overset{(Bi)^\ast}{\longrightarrow} \underbrace{H^{2k+1}(BU(n-1))}_{\simeq 0} \to \cdots\,,$$

where the right term vanishes by induction assumption, and the middle term since ordinary cohomology vanishes in negative degrees. Hence

$$H^{2k+1}(BU(n)) \simeq 0 \quad \text{for}\; 2k+1 < 2n$$

Then for $2k+1 > 2n$ the Thom-Gysin sequence gives

$$\cdots \to H^{2k+1-2n}(BU(n)) \longrightarrow H^{2k+1}(BU(n)) \overset{(Bi)^\ast}{\longrightarrow} \underbrace{H^{2k+1}(BU(n-1))}_{\simeq 0} \to \cdots\,,$$

where again the right term vanishes by the induction assumption. Hence exactness now gives that

$$H^{2k+1-2n}(BU(n)) \longrightarrow H^{2k+1}(BU(n))$$

is an epimorphism, and so with the previous statement it follows that

$$H^{2k+1}(BU(n)) \simeq 0$$

for all k.

Next consider the Thom Gysin sequence in degrees $2k$

$$\cdots \to \underbrace{H^{2k-1}(BU(n-1))}_{\simeq 0} \longrightarrow H^{2k-2n}(BU(n)) \longrightarrow H^{2k}(BU(n)) \overset{(Bi)^*}{\longrightarrow} H^{2k}(BU(n-1)) \longrightarrow \underbrace{H^{2k+1-2n}(BU(n))}_{\simeq 0} \to \cdots .$$

Here the left term vanishes by the induction assumption, while the right term vanishes by the previous statement. Hence we have a short exact sequence

$$0 \to H^{2k-2n}(BU(n)) \longrightarrow H^{2k}(BU(n)) \overset{(Bi)^*}{\longrightarrow} H^{2k}(BU(n-1)) \to 0$$

for all k. In degrees $\bullet \leq 2n$ this says

$$0 \to \mathbb{Z} \overset{c_n \cup (-)}{\longrightarrow} H^{\bullet \leq 2n}(BU(n)) \overset{(Bi)^*}{\longrightarrow} (\mathbb{Z}[c_1, \cdots, c_{n-1}])_{\bullet \leq 2n} \to 0$$

for some Thom class $c_n \in H^{2n}(BU(n))$, which we identify with the next Chern class.

Since free abelian groups are projective objects in Ab, their extensions are all split (the Ext-group out of them vanishes), hence the above gives a direct sum decomposition

$$\begin{aligned} H^{\bullet \leq 2n}(BU(n)) & \simeq (\mathbb{Z}[c_1, \cdots, c_{n-1}])_{\bullet \leq 2n} \oplus \mathbb{Z}\langle 2n \rangle \\ & \simeq (\mathbb{Z}[c_1, \cdots, c_n])_{\bullet \leq 2n} \end{aligned} .$$

Now by another induction over these short exact sequences, the claim follows. ∎

Splitting principle

Lemma 1.144. *For $n \in \mathbb{N}$ let $\mu_n : B(U(1)^n) \longrightarrow BU(n)$ be the canonical map. Then the induced pullback operation on ordinary cohomology*

$$\mu_n^* : H^\bullet(BU(n); \mathbb{Z}) \longrightarrow H^\bullet(BU(1)^n; \mathbb{Z})$$

is a monomorphism.

A **proof** of lemma 1.144 via analysis of the Serre spectral sequence of $U(n)/U(1)^n \to BU(1)^n \to BU(n)$ is indicated in (Kochmann 96, p. 40). A proof via transfer of the Euler class of $U(n)/U(1)^n$ is indicated at *splitting principle* (here).

Proposition 1.145. *For $k \leq n \in \mathbb{N}$ let $Bi_n : B(U(1)^n) \longrightarrow BU(n)$ be the canonical map. Then the induced pullback operation on ordinary cohomology is of the form*

$$(Bi_n)^* : \mathbb{Z}[c_1, \cdots, c_k] \longrightarrow \mathbb{Z}[(c_1)_1, \cdots (c_1)_n]$$

and sends the kth Chern class c_k (def. 1.143) to the kth elementary symmetric polynomial in the n copies of the first Chern class:

$$(Bi_n)^* : c_k \mapsto \sigma_k((c_1)_1, \cdots, (c_1)_n) \coloneqq \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq n} (c_1)_{i_1} \cdots (c_1)_{i_n} .$$

Proof. First consider the case $n = 1$.

The classifying space $BU(1)$ (def. 1.91) is equivalently the infinite complex projective space $\mathbb{C}P^\infty$. Its ordinary cohomology is the polynomial ring on a single generator c_1, the first Chern class (prop. 1.138)

$$H^\bullet(BU(1)) \simeq \mathbb{Z}[c_1] .$$

Moreover, Bi_1 is the identity and the statement follows.

Now by the Künneth theorem for ordinary cohomology (prop.) the cohomology of the Cartesian product of n copies of $BU(1)$ is the polynomial ring in n generators

$$H^\bullet(BU(1)^n) \simeq \mathbb{Z}[(c_1)_1, \cdots, (c_1)_n] .$$

By prop. 1.143 the domain of $(Bi_n)^*$ is the polynomial ring in the Chern classes $\{c_i\}$, and by the previous statement the codomain is the polynomial ring on n copies of the first Chern class

$$(Bi_n)^* : \mathbb{Z}[c_1, \cdots, c_n] \longrightarrow \mathbb{Z}[(c_1)_1, \cdots, (c_1)_n] .$$

This allows to compute $(Bi_n)^*(c_k)$ by induction:

Consider $n \geq 2$ and assume that $(Bi_{n-1})^*_{n-1}(c_k) = \sigma_k((c_1)_1, \cdots, (c_1)_{(n-1)})$. We need to show that then also $(Bi_n)^*(c_k) = \sigma_k((c_1)_1, \cdots, (c_1)_n)$.

Consider then the commuting diagram

$$\begin{array}{ccc} BU(1)^{n-1} & \overset{Bi_{n-1}}{\longrightarrow} & BU(n-1) \\ {}^{Bj_{\hat{t}}}\downarrow & & \downarrow^{Bi_{\hat{t}}} \\ BU(1)^n & \underset{Bi_n}{\longrightarrow} & BU(n) \end{array}$$

where both vertical morphisms are induced from the inclusion

$$\mathbb{C}^{n-1} \hookrightarrow \mathbb{C}^n$$

which omits the tth coordinate.

Since two embeddings $i_{\hat{t}_1}, i_{\hat{t}_2} : U(n-1) \hookrightarrow U(n)$ differ by conjugation with an element in $U(n)$, hence by an inner automorphism, the maps $Bi_{\hat{t}_1}$ and $B_{\hat{i}_{t_2}}$ are homotopic, and hence $(Bi_{\hat{t}})^* = (Bi_{\hat{n}})^*$, which is the morphism from prop. 1.143.

By that proposition, $(Bi_{\hat{t}})^*$ is the identity on $c_{k<n}$ and hence by induction assumption

$$\begin{aligned} & (Bi_{n-1})^*(Bi_{\hat{t}})^* c_{k<n} = (Bi_{n-1})^* c_{k<n} \\ & = \sigma_k((c_1)_1, \cdots, \widehat{(c_1)_t}, \cdots, (c_1)_n) \end{aligned}.$$

Since pullback along the left vertical morphism sends $(c_1)_t$ to zero and is the identity on the other generators, this shows that

$$(Bi_n)^*(c_{k<n}) \simeq \sigma_{k<n}((c_1)_1, \cdots, \widehat{(c_1)_t}, \cdots, (c_1)_n) \mod (c_1)_t \,.$$

This implies the claim for $k < n$.

For the case $k = n$ the commutativity of the diagram and the fact that the right map is zero on c_n by prop. 1.143 shows that the element $(Bj_{\hat{t}})^*(Bi_n)^* c_n = 0$ for all $1 \leq t \leq n$. But by lemma 1.144 the morphism $(Bi_n)^*$, is injective, and hence $(Bi_n)^*(c_n)$ is non-zero. Therefore for this to be annihilated by the morphisms that send $(c_1)_t$ to zero, for all t, the element must be proportional to all the $(c_1)_t$. By degree reasons this means that it has to be the product of all of them

$$\begin{aligned} (Bi_n)^*(c_n) & = (c_1)_1 \otimes (c_1)_2 \otimes \cdots \otimes (c_1)_n \\ & = \sigma_n((c_1)_1, \cdots, (c_1)_n) \end{aligned}.$$

This completes the induction step, and hence the proof. ▮

Proposition 1.146. *For $k \leq n \in \mathbb{N}$, consider the canonical map*

$$\mu_{k,n-k} : BU(k) \times BU(n-k) \longrightarrow BU(n)$$

(which classifies the Whitney sum of complex vector bundles of rank k with those of rank $n-k$). Under pullback along this map the universal Chern classes (prop. 1.143) are given by

$$(\mu_{k,n-k})^*(c_t) \;=\; \sum_{i=0}^{t} c_i \otimes c_{t-i} \,,$$

where we take $c_0 = 1$ and $c_j = 0 \in H^\bullet(BU(r))$ if $j > r$.

So in particular

$$(\mu_{k,n-k})^*(c_n) \;=\; c_k \otimes c_{n-k} \,.$$

e.g. (Kochmann 96, corollary 2.3.4)

Proof. Consider the commuting diagram

$$
\begin{array}{ccc}
H^\bullet(BU(n)) & \overset{\mu^*_{k,n-k}}{\longrightarrow} & H^\bullet(BU(k)) \otimes H^\bullet(BU(n-k)) \\
{}^{\mu^*_k}\downarrow && \downarrow^{\mu^*_k \otimes \mu^*_{n-k}} \\
H^\bullet(BU(1)^n) & \simeq & H^\bullet(BU(1)^k) \otimes H^\bullet(BU(1)^{n-k})
\end{array}
.
$$

This says that for all t then

$$
\begin{aligned}
(\mu^*_k \otimes \mu^*_{n-k})\mu^*_{k,n-k}(c_t) & = \mu^*_n(c_t) \\
& = \sigma_t((c_1)_1, \cdots, (c_1)_n)
\end{aligned}
,
$$

where the last equation is by prop. 1.145.

Now the elementary symmetric polynomial on the right decomposes as required by the left hand side of this equation as follows:

$$
\sigma_t((c_1)_1, \cdots, (c_1)_n) \;=\; \sum_{r=0}^{t} \sigma_r((c_1)_1, \cdots, (c_1)_{n-k}) \cdot \sigma_{t-r}((c_1)_{n-k+1}, \cdots, (c_1)_n) \,,
$$

where we agree with $\sigma_q((c_1)_1, \cdots, (c_1)_p) = 0$ if $q > p$. It follows that

$$
(\mu^*_k \otimes \mu^*_{n-k})\mu^*_{k,n-k}(c_t) = (\mu^*_k \otimes \mu^*_{n-k})\left(\sum_{r=0}^{t} c_r \otimes c_{t-r} \right) .
$$

Since $(\mu^*_k \otimes \mu^*_{n-k})$ is a monomorphism by lemma 1.144, this implies the claim. ▮

Conner-Floyd Chern classes

Idea. For E a complex oriented cohomology theory, then the generators of the E-cohomology groups of the classifying space BU are called the *Conner-Floyd Chern classes*, in $E^\bullet(BU)$.

Using basic properties of the classifying space $BU(1)$ via its incarnation as the infinite complex projective space $\mathbb{C}P^\infty$, one finds that the Atiyah-Hirzebruch spectral sequences

$$
H^p(\mathbb{C}P^n, \pi_q(E)) \Rightarrow H^\bullet(\mathbb{C}P^n)
$$

collapse right away, and that the inverse system which they form satisfies the Mittag-Leffler condition. Accordingly the Milnor exact sequence gives that the ordinary first Chern class c_1 generates, over $\pi_\bullet(E)$, all Conner-Floyd classes over $BU(1)$:

$$
E^\bullet(BU(1)) \simeq \pi_\bullet(E)[[c_1]] \,.
$$

This is the key input for the discussion of formal group laws below.

Combining the Atiyah-Hirzebruch spectral sequence with the splitting principle as for ordinary Chern classes above yields, similarly, that in general Conner-Floyd classes are generated, over $\pi_\bullet(E)$, from the ordinary Chern classes.

Finally one checks that Conner-Floyd classes canonically serve as Thom classes for E-cohomology of the universal complex vector bundle, thereby showing that complex oriented cohomology theories are indeed canonically oriented on (spherical fibrations of) complex vector bundles.

Literature. (Kochman 96, section 4.3 Adams 74, part I.4, part II.2 II.4, part III.10, Lurie 10, lecture 5)

Proposition 1.147. *Given a complex oriented cohomology theory E with complex orientation c_1^E, then the E-generalized cohomology of the classifying space $BU(n)$ is freely generated over the graded commutative ring $\pi_\bullet(E)$ (prop.) by classes c_k^E for $0 \leq \; \leq n$ of degree $2k$, these are called the **Conner-Floyd-Chern classes***

$$
E^\bullet(BU(n)) \;\simeq\; \pi_\bullet(E)[[c_1^E, c_2^E, \cdots, c_n^E]] \,.
$$

Moreover, pullback along the canonical inclusion $BU(n) \to BU(n+1)$ is the identity on c_k^E for $k \leq n$ and sends c_{n+1}^E to zero.

For E being ordinary cohomology, this reduces to the ordinary Chern classes of prop. 1.143.

For details see (Pedrotti 16, prop. 3.1.14).

Formal group laws of first CF-Chern classes

Idea. The classifying space $BU(1)$ for complex line bundles is a homotopy type canonically equipped with commutative group structure (infinity-group-structure), corresponding to the tensor product of complex line bundles. By the above, for E a complex oriented cohomology theory the first Conner-Floyd Chern class of these complex line bundles generates the E-cohomology of $BU(1)$, it follows that the cohomology ring $E^\bullet(BU(1)) \simeq \pi_\bullet(E)[[c_1]]$ behaves like the ring of $\pi_\bullet(E)$-valued functions on a 1-dimensional commutative formal group equipped with a canonical coordinate function c_1. This is called a *formal group law* over the graded commutative ring $\pi_\bullet(E)$ (prop.).

On abstract grounds it follows that there exists a commutative ring L and a universal (1-dimensional commutative) formal group law ℓ over L. This is called the *Lazard ring*. Lazard's theorem identifies this ring concretely: it turns out to simply be the polynomial ring on generators in every even degree.

Further below this has profound implications on the structure theory for complex oriented cohomology. The Milnor-Quillen theorem on MU identifies the Lazard ring as the cohomology ring of the Thom spectrum MU, and then the Landweber exact functor theorem, implies that there are lots of complex oriented cohomology theories.

Literature. (Kochman 96, section 4.4, Lurie 10, lectures 1 and 2)

Formal group laws

Definition 1.148. An (commutative) adic ring is a (commutative) topological ring A and an ideal $I \subset A$ such that

1. the topology on A is the I-adic topology;

2. the canonical morphism

$$A \longrightarrow \underset{\longleftarrow n}{\lim} (A/I^n)$$

to the limit over quotient rings by powers of the ideal is an isomorphism.

A homomorphism of adic rings is a ring homomorphism that is also a continuous function (hence a function that preserves the filtering $A \supset \cdots \supset A/I^2 \supset A/I$). This gives a category $\mathrm{AdicRing}$ and a subcategory $\mathrm{AdicCRing}$ of commutative adic rings.

The opposite category of $\mathrm{AdicRing}$ (on Noetherian rings) is that of affine formal schemes.

Similarly, for R any fixed commutative ring, then adic rings under R are *adic R-algebras*. We write $\mathrm{Adic}\,A\,\mathrm{Alg}$ and $\mathrm{Adic}\,A\,\mathrm{CAlg}$ for the corresponding categories.

Example 1.149. For R a commutative ring and $n \in \mathbb{N}$ then the formal power series ring

$$R[[x_1, x_2, \cdots, x_n]]$$

in n variables with coefficients in R and equipped with the ideal

$$I = (x_1, \cdots, x_n)$$

is an adic ring (def. 1.148).

Proposition 1.150. *There is a fully faithful functor*

$$\mathrm{AdicRing} \hookrightarrow \mathrm{ProRing}$$

from adic rings (def. 1.148) to pro-rings, given by

$$(A, I) \mapsto ((A/I^\bullet)),$$

i.e. for $A, B \in \mathrm{AdicRing}$ two adic rings, then there is a natural isomorphism

$$\mathrm{Hom}_{\mathrm{AdicRing}}(A, B) \simeq \underset{\longleftarrow n_2}{\lim} \underset{\longrightarrow n_1}{\lim} \mathrm{Hom}_{\mathrm{Ring}}(A/I^{n_1}, B/I^{n_2}) .$$

Definition 1.151. For $R \in \mathrm{CRing}$ a commutative ring and for $n \in \mathbb{N}$, a **formal group law** of dimension n over R is the structure of a group object in the category $\mathrm{Adic}\,R\,\mathrm{CAlg}^{\mathrm{op}}$ from def. 1.148 on the object $R[[x_1, \cdots, x_n]]$ from example 1.149.

Hence this is a morphism

$$\mu : R[[x_1, \cdots, x_n]] \longrightarrow R[[x_1, \cdots, x_n, y_1, \cdots, y_n]]$$

in $\mathrm{Adic}\, R\, \mathrm{CAlg}$ satisfying unitality, associativity.

This is a **commutative formal group law** if it is an abelian group object, hence if it in addition satisfies the corresponding commutativity condition.

This is equivalently a set of n power series F_i of $2n$ variables $x_1, ..., x_n, y_1, ..., y_n$ such that (in notation $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n)$, $F(x,y) = (F_1(x,y), ..., F_n(x,y))$)

$$\begin{gathered} F(x, F(y,z)) = F(F(x,y),z) \\ F_i(x,y) = x_i + y_i + \text{ higher order terms} \end{gathered}$$

Example 1.152. A 1-dimensional commutative formal group law according to def. 1.151 is equivalently a formal power series

$$\mu(x,y) = \sum_{i,j \geq 0} a_{i,j} x^i y^j$$

(the image]under\muinR[x,y]oftheelementt \in R [t]$) such that

1. (unitality)

$$\mu(x,0) = x$$

2. (associativity)

$$\mu(x, \mu(y,z)) = \mu(\mu(x,y),z)\,;$$

3. (commutativity)

$$\mu(x,y) = \mu(y,x)\;.$$

The first condition means equivalently that

$$a_{i,0} = \begin{cases} 1 & \text{if}\, i = 0 \\ 0 & \text{otherwise} \end{cases} \quad , \quad a_{0,i} = \begin{cases} 1 & \text{if}\, i = 0 \\ 0 & \text{otherwise} \end{cases}\,.$$

Hence μ is necessarily of the form

$$\mu(x,y) \;=\; x + y + \sum_{i,j \geq 1} a_{i,j} x^i y^j\;.$$

The existence of inverses is no extra condition: by induction on the index i one finds that there exists a unique

$$\iota(x) = \sum_{i \geq 1} \iota(x)_i x^i$$

such that

$$\mu(x, \mathrm{iota}(x)) = x \quad , \quad \mu(\iota(x), x) = x\;.$$

Hence 1-dimensional formal group laws over R are equivalently monoids in $\mathrm{Adic}\, R\, \mathrm{CAlg}^{\mathrm{op}}$ on $R[[x]]$.

Formal group laws from complex orientation

Let again $B U(1)$ be the classifying space for complex line bundles, modeled, in particular, by infinite complex projective space $\mathbb{C}P^\infty$).

Lemma 1.153. *There is a continuous function*

$$\mu : \mathbb{C}P^\infty \times \mathbb{C}P \longrightarrow \mathbb{C}P^\infty$$

which represents the tensor product of line bundles in that under the defining equivalence, and for X any paracompact topological space, then

$$
\begin{array}{ccc}
[X, \mathbb{C}P^\infty \times \mathbb{C}P^\infty] & \simeq & \mathbb{C}\,\mathrm{LineBund}(X)_{/\sim} \times \mathbb{C}\,\mathrm{LineBund}(X)_{/\sim} \\
{}^{[X,\mu]}\downarrow && \downarrow^{\otimes} \\
[X, \mathbb{C}P^\infty] & \simeq & \mathbb{C}\,\mathrm{LineBund}(X)_{/\sim}
\end{array}
,
$$

where $[-,-]$ denotes the hom-sets in the (Serre-Quillen-)classical homotopy category and $\mathbb{C}\,\mathrm{LineBund}(X)_{/\sim}$ denotes the set of isomorphism classes of complex line bundles on X.

Together with the canonical point inclusion $* \to \mathbb{C}P^\infty$, this makes $\mathbb{C}P^\infty$ an abelian group object in the classical homotopy category.

Proof. By the Yoneda lemma (the fully faithfulness of the Yoneda embedding) there exists such a morphism $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \longrightarrow \mathbb{C}P^\infty$ in the classical homotopy category. But since $\mathbb{C}P^\infty$ admits the structure of a CW-complex (prop. 1.137)) it is cofibrant in the standard model structure on topological spaces (thm.), as is its Cartesian product with itself (prop.). Since moreover all spaces are fibrant in the classical model structure on topological spaces, it follows (by this lemma) that there is an actual continuous function representing that morphism in the homotopy category.

That this gives the structure of an abelian group object now follows via the Yoneda lemma from the fact that each $\mathbb{C}\,\mathrm{LineBund}(X)_{/\sim}$ has the structure of an abelian group under tensor product of line bundles, with the trivial line bundle (wich is classified by maps factoring through $* \to \mathbb{C}P^\infty$) being the neutral element, and that this group structure is natural in X. ▮

Remark 1.154. The space $BU(1) \simeq \mathbb{C}P^\infty$ has in fact more structure than that of a homotopy group from lemma 1.153. As an object of the homotopy theory represented by the classical model structure on topological spaces, it is a *2-group*, a 1-truncated infinity-group.

Proposition 1.155. *Let (E, c_1^E) be a complex oriented cohomology theory. Under the identification*

$$
E^\bullet(\mathbb{C}P^\infty) \simeq \pi_\bullet(E)[[c_1^E]] \quad , \quad E^\bullet(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \simeq \pi_\bullet(E)[[c_1^E \otimes 1, 1 \otimes c_1^E]]
$$

from prop. 1.142, the operation

$$
\pi_\bullet(E)[[c_1^E]] \simeq E^\bullet(\mathbb{C}P^\infty) \longrightarrow E^\bullet(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \simeq \pi_\bullet(E)[[c_1^E \otimes 1, 1 \otimes c_1^E]]
$$

of pullback in E-cohomology along the maps from lemma 1.153 constitutes a 1-dimensional graded-commutative formal group law (example 1.152)over the graded commutative ring $\pi_\bullet(E)$ (prop.). If we consider c_1^E to be in degree 2, then this formal group law is compatibly graded.

Proof. The associativity and commutativity conditions follow directly from the respective properties of the map μ in lemma 1.153. The grading follows from the nature of the identifications in prop. 1.142. ▮

Remark 1.156. That the grading of c_1^E in prop. 1.155 is in negative degree is because by definition

$$
\pi_\bullet(E) = E_\bullet = E^{-\bullet}
$$

(rmk.).

Under different choices of orientation, one obtains different but isomorphic formal group laws.

The universal 1d commutative formal group law and Lazard's theorem

It is immediate that there exists a ring carrying a universal formal group law. For observe that for $\sum_{i,j} a_{i,j} x_1^i x_1^j$ an element in a formal power series algebra, then the condition that it defines a formal group law is equivalently a sequence of polynomial equations on the coefficients a_k. For instance the commutativity condition means that

$$
a_{i,j} = a_{j,i}
$$

and the unitality constraint means that

$$
a_{i0} = \begin{cases} 1 & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases} .
$$

Similarly associativity is equivalently a condition on combinations of triple products of the coefficients. It is not necessary to even write this out, the important point is only that it is some polynomial equation.

This allows to make the following definition

Definition 1.157. The **Lazard ring** is the graded commutative ring generated by elenebts a_{ij} in degree $2(i+j-1)$ with $i, j \in \mathbb{N}$

$$L = \mathbb{Z}[a_{ij}]/(\text{relations } 1,2,3 \text{ below})$$

quotiented by the relations

1. $a_{ij} = a_{ji}$
2. $a_{10} = a_{01} = 1$; $\forall i \neq 1 : a_{i0} = 0$
3. the obvious associativity relation

for all i, j, k.

The **universal 1-dimensional commutative formal group law** is the formal power series with coefficients in the Lazard ring given by

$$\ell(x,y) \coloneqq \sum_{i,j} a_{ij} x^i y^j \in L[[x,y]] \;.$$

Remark 1.158. The grading is chosen with regards to the formal group laws arising from complex oriented cohomology theories (prop.) where the variable x naturally has degree -2. This way

$$\deg(a_{ij} x^i y^j) = \deg(a_i, j) + i \deg(x) + j \deg(y) = -2 \;.$$

The following is immediate from the definition:

Proposition 1.159. *For every ring R and 1-dimensional commutative formal group law μ over R (example 1.152), there exists a unique ring homomorphism*

$$f : L \longrightarrow R$$

from the Lazard ring (def. 1.157) to R, such that it takes the universal formal group law ℓ to μ

$$f_* \ell = \mu \;.$$

Proof. If the formal group law μ has coefficients $\{c_{i,j}\}$, then in order that $f_* \ell = \mu$, i.e. that

$$\sum_{i,j} f(a_{i,j}) x^i y^j = \sum_{i,j} c_{i,j} x^i y^j$$

it must be that f is given by

$$f(a_{i,j}) = c_{i,j}$$

where $a_{i,j}$ are the generators of the Lazard ring. Hence it only remains to see that this indeed constitutes a ring homomorphism. But this is guaranteed by the vary choice of relations imposed in the definition of the Lazard ring. ▮

What is however highly nontrivial is this statement:

Theorem 1.160. ***(Lazard's theorem)***

The Lazard ring L (def. 1.157) is isomorphic to a polynomial ring

$$L \simeq \mathbb{Z}[t_1, t_2, \cdots]$$

in countably many generators t_i in degree $2i$.

Remark 1.161. The Lazard theorem 1.160 first of all implies, via prop. 1.159, that there exists an abundance of 1-dimensional formal group laws: given any ring R then every choice of elements $\{t_i \in R\}$ defines a formal group law. (On the other hand, it is nontrivial to say which formal group law that is.)

Deeper is the fact expressed by the Milnor-Quillen theorem on MU: the Lazard ring in its polynomial incarnation of prop. 1.160 is canonically identieif with the graded commutative ring $\pi_\bullet(MU)$ of stable homotopy groups of the universal complex Thom spectrum MU. Moreover:

1. MU carries a universal complex orientation in that for E any homotopy commutative ring spectrum then homotopy classes of homotopy ring homomorphisms $MU \to E$ are in bijection to complex orientations on E;

2. every complex orientation on E induced a 1-dimensional commutative formal group law (prop.)

3. under forming stable homtopy groups every ring spectrum homomorphism $MU \to E$ induces a ring homomorphism

$$L \simeq \pi_\bullet(MU) \longrightarrow \pi_\bullet(E)$$

and hence, by the universality of L, a formal group law over $\pi_\bullet(E)$.

This is the formal group law given by the above complex orientation.

Hence the universal group law over the Lazard ring is a kind of decategorification of the universal complex orientation on MU.

Complex cobordism

Idea. There is a weak homotopy equivalence $\phi : BU(1) \xrightarrow{\simeq} MU(1)$ between the classifying space for complex line bundles and the Thom space of the universal complex line bundle. This gives an element $\pi_*(c_1) \in MU^2(BU(1))$ in the complex cobordism cohomology of $BU(1)$ which makes the universal complex Thom spectrum MU become a complex oriented cohomology theory.

This turns out to be a universal complex orientation on MU: for every other homotopy commutative ring spectrum E (def.) there is an equivalence between complex orientations on E and homotopy classes of homotopy ring spectrum homomorphisms

$$\{MU \longrightarrow E\}_{/\simeq} \;\simeq\; \{\text{complex orientations on } E\}\,.$$

Hence complex oriented cohomology theory is higher algebra over MU.

Literature. (Schwede 12, example 1.18, Kochman 96, section 1.4, 1.5, 4.4, Lurie 10, lectures 5 and 6)

Conner-Floyd-Chern classes are Thom classes

We discuss that for E a complex oriented cohomology theory, then the nth universal Conner-Floyd-Chern class c_n^E is in fact a universal Thom class for rank n complex vector bundles. On the one hand this says that the choice of a complex orientation on E indeed universally orients all complex vector bundles. On the other hand, we interpret this fact below as the unitality condition on a homomorphism of homotopy commutative ring spectra $MU \to E$ which represent that universal orienation.

Lemma 1.162. *For $n \in \mathbb{N}$, the fiber sequence (prop. 1.96)*

$$\begin{array}{ccc} S^{2n-1} & \longrightarrow & BU(n-1) \\ && \downarrow \\ && BU(n) \end{array}$$

exhibits $BU(n-1)$ as the sphere bundle of the universal complex vector bundle over $BU(n)$.

Proof. When exhibited by a fibration, here the vertical morphism is equivalently the quotient map

$$(EU(n))/U(n-1) \longrightarrow (EU(n))/U(n)$$

(by the proof of prop. 1.96).

Now the universal principal bundle $EU(n)$ is (def. \ref{EOn)}) equivalently the colimit

$$EU(n) \simeq \underset{\longrightarrow_k}{\lim}\, U(k)/U(k-n)\,.$$

Here each Stiefel manifold/coset spaces $U(k)/U(k-n)$ is equivalently the space of (complex) n-dimensional subspaces of $\mathbb{C}^k$ that are equipped with an orthonormal (hermitian) linear basis. The universal vector bundle

$$EU(n) \underset{U(n)}{\times} \mathbb{C}^n \simeq \underset{\longrightarrow_k}{\lim}\, U(k)/U(k-n) \underset{U(n)}{\times} \mathbb{C}^n$$

has as fiber precisely the linear span of any such choice of basis.

While the quotient $U(k)/(U(n-k) \times U(n))$ (the Grassmannian) divides out the entire choice of basis, the quotient $U(k)/(U(n-k) \times U(n-1))$ leaves the choice of precisely one unit vector. This is parameterized by the sphere S^{2n-1} which is thereby identified as the unit sphere in the respective fiber of $EU(n) \underset{U(n)}{\times} \mathbb{C}^n$. ▮

In particular:

Lemma 1.163. *The canonical map from the classifying space $BU(1) \simeq \mathbb{C}P^\infty$ (the inifnity complex projective space) to the Thom space of the universal complex line bundle is a weak homotopy equivalence*

$$BU(1) \overset{\in W_{\mathrm{cl}}}{\longrightarrow} MU(1) \coloneqq \mathrm{Th}(EU(1) \underset{U(1)}{\times} \mathbb{C}) \,.$$

Proof. Observe that the circle group $U(1)$ is naturally identified with the unit sphere in $\mathbb{C}$: $U(1) \simeq S(\mathbb{S})$. Therefore the sphere bundle of the universal complex line bundle is equivalently the $U(1)$-universal principal bundle

$$\begin{aligned} EU(1) \underset{U(1)}{\times} S(\mathbb{C}) & \simeq EU(1) \underset{U(1)}{\times} U(1) \\ & \simeq EU(1) \end{aligned} \,.$$

But the universal principal bundle is contractible

$$EU(1) \overset{\in W_{\mathrm{cl}}}{\longrightarrow} * \,.$$

(Alternatively this is the special case of lemma 1.162 for $n = 0$.)

Therefore the Thom space

$$\begin{aligned} \mathrm{Th}(EU(1) \underset{U(1)}{\times} \mathbb{B}) & \coloneqq D(EU(1) \underset{U(1)}{\times} \mathbb{B}) / S(EU(1) \underset{U(1)}{\times} \mathbb{B}) \\ & \overset{\in W_{\mathrm{cl}}}{\longrightarrow} D(EU(1) \underset{U(1)}{\times} \mathbb{B}) \\ & \overset{\in W_{\mathrm{cl}}}{\longrightarrow} BU(1) \end{aligned} \,.$$

■

Lemma 1.164. *For E a generalized (Eilenberg-Steenrod) cohomology theory, then the E-reduced cohomology of the Thom space of the complex universal vector bundle is equivalently the relative cohomology of $BU(n)$ relative $BU(n-1)$*

$$\tilde E^\bullet(\mathrm{Th}(EU(n) \underset{U(n)}{\times} \mathbb{C}^n)) \;\simeq\; E^\bullet(BU(n), BU(n-1)) \,.$$

If E is equipped with the structure of a complex oriented cohomology theory then

$$\tilde E^\bullet(\mathrm{Th}(EU(n) \underset{U(n)}{\times} \mathbb{C}^n)) \simeq c_n^E \cdot (\pi_\bullet(E))[[c_1^E, \cdots, c_n^E]] \,,$$

where the c_i are the universal E-Conner-Floyd-Chern classes.

Proof. Regarding the first statement:

In view of lemma 1.162 and using that the disk bundle is homotopy equivalent to the base space we have

$$\begin{aligned} \tilde E^\bullet(\mathrm{Th}(EU(n) \underset{U(n)}{\times} \mathbb{C}^n)) & = E^\bullet(D(EU(n) \underset{U(n)}{\times} \mathbb{C}^n), S(EU(n) \underset{U(n)}{\times} \mathbb{C}^n)) \\ & \simeq E^\bullet(EU(n), BU(n-1)) \end{aligned} \,.$$

Regarding the second statement: the Conner-Floyd classes freely generate the E-cohomology of $BU(n)$ for all n:

$$E^\bullet(BU(n)) \simeq \pi_\bullet(E)[[c_1^E, \cdots, c_n^E]] \,.$$

and the restriction morphism

$$E^\bullet(BU(n)) \longrightarrow E^\bullet(BU(n-1))$$

projects out c_n^E. Since this is in particular a surjective map, the relative cohomology $E^\bullet(BU(n), BU(n-1))$ is just the kernel of this map. ■

Proposition 1.165. *Let E be a complex oriented cohomology theory. Then the nth E-Conner-Floyd-Chern class*

$$c_n^E \in \tilde E(\mathrm{Th}(EU(n) \underset{U(n)}{\times} \mathbb{C}^n))$$

(using the identification of lemma 1.164) is a Thom class in that its restriction to the Thom space of any

fiber is a suspension of a unit in $\pi_0(E)$.

(Lurie 10, lecture 5, prop. 6)

Proof. Since $BU(n)$ is connected, it is sufficient to check the statement over the base point. Since that fixed fiber is canonically isomorphic to the direct sum of n complex lines, we may equivalently check that the restriction of c_n^E to the pullback of the universal rank n bundle along

$$i : BU(1)^n \longrightarrow BU(n)$$

satisfies the required condition. By the splitting principle, that restriction is the product of the n-copies of the first E-Conner-Floyd-Chern class

$$i^* c_n \simeq ((c_1^E)_1 \cdots (c_1^E)_n) \,.$$

Hence it is now sufficient to see that each factor restricts to a unit on the fiber, but that it precisely the condition that c_1^E is a complex orientaton of E. In fact by def. 1.166 the restriction is even to $1 \in \pi_0(E)$. ▮

Complex orientation as ring spectrum maps

For the present purpose:

Definition 1.166. For E a generalized (Eilenberg-Steenrod) cohomology theory, then a *complex orientation* on E is a choice of element

$$c_1^E \in E^2(BU(1))$$

in the cohomology of the classifying space $BU(1)$ (given by the infinite complex projective space) such that its image under the restriction map

$$\phi \,:\, \tilde E^2(BU(1)) \longrightarrow \tilde E^2(S^2) \simeq \pi_0(E)$$

is the unit

$$\phi(c_1^E) = 1 \,.$$

(Lurie 10, lecture 4, def. 2)

Remark 1.167. Often one just requires that $\phi(c_1^E)$ is *a* unit, i.e. an invertible element. However we are after identifying c_1^E with the degree-2 component $MU(1) \to E_2$ of homtopy ring spectrum morphisms $MU \to E$, and by unitality these necessarily send $S^2 \to MU(1)$ to the unit $\iota_2 : S^2 \to E$ (up to homotopy).

Lemma 1.168. *Let E be a homotopy commutative ring spectrum (def.) equipped with a complex orientation (def. 1.166) represented by a map*

$$c_1^E \,:\, BU(1) \longrightarrow E_2 \,.$$

Write $\{c_k^E\}_{k \in \mathbb{N}}$ for the induced Conner-Floyd-Chern classes. Then there exists a morphism of S^2-sequential spectra (def.)

$$MU \longrightarrow E$$

whose component map $MU_{2n} \longrightarrow E_{2n}$ represents c_n^E (under the identification of lemma 1.164), for all $n \in \mathbb{N}$.

Proof. Consider the standard model of MU as a sequential S^2-spectrum with component spaces the Thom spaces of the complex universal vector bundle

$$MU_{2n} \coloneqq \mathrm{Th}(EU(n) \times \mathbb{C}^n) \,.$$

Notice that this is a CW-spectrum (def., lemma).

In order to get a homomorphism of S^2-sequential spectra, we need to find representatives $f_{2n} \,:\, MU_{2n} \longrightarrow E_{2n}$ of c_n^E (under the identification of lemma 1.164) such that all the squares

$$\begin{array}{ccc} S^2 \wedge MU_{2n} & \overset{\mathrm{id} \wedge f_{2n}}{\longrightarrow} & S^2 \wedge E_{2n} \\ \downarrow && \downarrow \\ MU_{2(n+1)} & \underset{f_{2(n+1)}}{\longrightarrow} & E_{2n+1} \end{array}$$

commute strictly (not just up to homotopy).

To begin with, pick a map

$$f_0 : MU_0 \simeq S^0 \longrightarrow E_0$$

that represents $c_0 = 1$.

Assume then by induction that maps f_{2k} have been found for $k \leq n$. Observe that we have a homotopy-commuting diagram of the form

$$\begin{array}{ccc} S^2 \wedge MU_{2n} & \overset{\mathrm{id} \wedge f_{2n}}{\longrightarrow} & S^2 \wedge E_{2n} \\ \downarrow & \swArrow & \downarrow \\ MU_2 \wedge MU_{2n} & \overset{c_1 \wedge c_n}{\longrightarrow} & E_2 \wedge E_{2n} \, , \\ \downarrow & \swArrow & \downarrow^{\mu_{2,2n}} \\ MU_{2(n+1)} & \underset{c_{n+1}}{\longrightarrow} & E_{2(n+1)} \end{array}$$

where the maps denoted c_k are any representatives of the Chern classes of the same name, under the identification of lemma 1.164. Here the homotopy in the top square exhibits the fact that c_1^E is a complex orientation, while the homotopy in the bottom square exhibits the Whitney sum formula for Chern classes (prop. 1.146)).

Now since MU is a CW-spectrum, the total left vertical morphism here is a (Serre-)cofibration, hence a Hurewicz cofibration, hence satisfies the homotopy extension property. This means precisely that we may find a map $f_{2n+1} : MU_{2(n+1)} \longrightarrow E_{2(n+1)}$ homotopic to the given representative c_{n+1} such that the required square commutes strictly. ▮

Lemma 1.169. *For E a complex oriented homotopy commutative ring spectrum, the morphism of spectra*

$$c : MU \longrightarrow E$$

that represents the complex orientation by lemma 1.168 is a homomorphism of homotopy commutative ring spectra.

(Lurie 10, lecture 6, prop. 6)

Proof. The unitality condition demands that the diagram

$$\begin{array}{ccc} \mathbb{S} & \longrightarrow & MU \\ & \searrow & \downarrow^{c} \\ & & E \end{array}$$

commutes in the stable homotopy category $\mathrm{Ho}(\mathrm{Spectra})$. In components this means that

$$\begin{array}{ccc} S^{2n} & \longrightarrow & MU_{2n} \\ & \searrow & \downarrow^{c_n} \\ & & E_{2n} \end{array}$$

commutes up to homotopy, hence that the restriction of c_n to a fiber is the $2n$-fold suspension of the unit of E_{2n}. But this is the statement of prop. 1.165: the Chern classes are universal Thom classes.

Hence componentwise all these triangles commute up to some homotopy. Now we invoke the Milnor sequence for generalized cohomology of spectra (prop. 1.63). Observe that the tower of abelian groups $n \mapsto E^{n_1}(S^n)$ is actually constant (suspension isomorphism) hence trivially satisfies the Mittag-Leffler condition and so a homotopy of morphisms of spectra $\mathbb{S} \to E$ exists as soon as there are componentwise homotopies (cor. 1.64).

Next, the respect for the product demands that the square

$$\begin{array}{ccc} MU \wedge MU & \overset{c \wedge c}{\longrightarrow} & E \wedge E \\ \downarrow & & \downarrow \\ MU & \underset{c}{\longrightarrow} & E \end{array}$$

commutes in the stable homotopy category $\mathrm{Ho}(\mathrm{Spectra})$. In order to rephrase this as a condition on the components of the ring spectra, regard this as happening in the homotopy category $\mathrm{Ho}(\mathrm{OrthSpec}(\mathrm{Top}_{\mathrm{cg}}))_{\mathrm{stable}}$ of the model structure on orthogonal spectra, which is equivalent to the stable homotopy category (thm.).

Here the derived symmetric monoidal smash product of spectra is given by Day convolution (def.) and maps out of such a product are equivalently as in the above diagram is equivalent (cor.) to a suitably equivariant collection diagrams of the form

$$\begin{array}{ccc} MU_{2n_1} \wedge MU_{2n_2} & \overset{c_{n_1} \wedge c_{n_2}}{\longrightarrow} & E_{2n_1} \wedge E_{2n_2} \\ \downarrow && \downarrow \\ MU_{2(n_1+n_2)} & \underset{c_{(n_1+n_2)}}{\longrightarrow} & E_{2(n_1+n_2)} \end{array} \,,$$

where on the left we have the standard pairing operations for MU (def.) and on the right we have the given pairing on E.

That this indeed commutes up to homotopy is the Whitney sum formula for Chern classes (prop.).

Hence again we have componentwise homotopies. And again the relevant Mittag-Leffler condition on $n \mapsto E^{n-1}((\mathrm{MU} \wedge \mathrm{MU})_n)$-holds, by the nature of the universal *Conner-Floyd classes?*, prop. 1.147. Therefore these componentwise homotopies imply the required homotopy of morphisms of spectra (cor. 1.64). ▮

Theorem 1.170. *Let E be a homotopy commutative ring spectrum (def.). Then the map*

$$(MU \overset{c}{\longrightarrow} E) \;\mapsto\; (BU(1) \simeq MU_2 \overset{c_1}{\longrightarrow} E_2)$$

which sends a homomorphism c of homotopy commutative ring spectra to its component map in degree 2, interpreted as a class on $BU(1)$ via lemma 1.163, constitutes a bijection from homotopy classes of homomorphisms of homotopy commutative ring spectra to complex orientations (def. 1.166) on E.

(Lurie 10, lecture 6, theorem 8)

Proof. By lemma 1.168 and lemma 1.169 the map is surjective, hence it only remains to show that it is injective.

So let $c, c' : MU \to E$ be two morphisms of homotopy commutative ring spectra that have the same restriction, up to homotopy, to $c_1 \simeq {c_1}' : MU_2 \simeq BU(1)$. Since both are homotopy ring spectrum homomophisms, the restriction of their components $c_n, c'_n : MU_{2n} \to E_{2n}$ to $BU(1)^{\wedge^n}$ is a product of $c_1 \simeq c'_1$, hence c_n becomes homotopic to c_n' after this restriction. But by the splitting principle this restriction is injective on cohomology classes, hence c_n itself ist already homotopic to c'_n.

It remains to see that these homotopies may be chosen compatibly such as to form a single homotopy of maps of spectra

$$f \,:\, MU \wedge I_+ \longrightarrow E \,,$$

This follows due to the existence of the Milnor short exact sequence from prop. 1.63:

$$0 \to \underset{\longleftarrow n}{\lim}^1 E^{-1}(\Sigma^{-2n} MU_{2n}) \longrightarrow E^0(MU) \longrightarrow \underset{\longleftarrow n}{\lim}\, E^0(\Sigma^{-2n} MU_{2n}) \to 0 \,.$$

Here the Mittag-Leffler condition (def. 1.55) is clearly satisfied (by prop. 1.147 and lemma 1.164 all relevant maps are epimorphisms, hence the condition is satisfied by example 1.56). Hence the lim^1-term vanishes (prop. 1.57), and so by exactness the canonical morphism

$$E^0(MU) \overset{\simeq}{\longrightarrow} \underset{\longleftarrow n}{\lim}\, E^0(\Sigma^{-2n} MU_{2n})$$

is an isomorphism. This says that two homotopy classes of morphisms $MU \to E$ are equal precisely already if all their component morphisms are homotopic (represent the same cohomology class). ▮

Homology of MU

Idea. Since, by the above, every complex oriented cohomology theory E is indeed oriented over complex vector bundles, there is a Thom isomorphism which reduces the computation of the E-homology of MU, $E_\bullet(MU)$ to that of the classifying space BU. The homology of BU, in turn, may be determined by the duality with its cohomology (universal coefficient theorem) via Kronecker pairing and the induced duality of the corresponding Atiyah-Hirzebruch spectral sequences (prop. 1.74) from the Conner-Floyd classes above. Finally, via the Hurewicz homomorphism/Boardman homomorphism the homology of MU gives a first approximation to the homotopy groups of MU.

Literature. (Kochman 96, section 2.4, 4.3, Lurie 10, lecture 7)

Milnor-Quillen theorem on MU

Idea. From the computation of the homology of MU above and applying the Boardman homomorphism, one deduces that the stable homotopy groups $\pi_\bullet(\mathrm{MU})$ of MU are finitely generated. This implies that it is suffient to compute them over the p-adic integers for all primes p. Using the change of rings theorem, this finally is obtained from inspection of the filtration in the $H\mathbb{F}_p$-Adams spectral sequence for MU. This is Milnor's theorem wich together with Lazard's theorem shows that there is an isomorphism of rings $L \simeq \pi_\bullet(MU)$ with the Lazard ring. Finally Quillen's theorem on MU says that this isomorphism is exhibited by the universal ring homomorphism $L \longrightarrow \pi_\bullet(MU)$ which classifies the universal complex orientation on MU.

Literature. (Kochman 96, section 4.4, Lurie 10, lecture 10)

Landweber exact functor theorem

Idea. By the above, every complex oriented cohomology theory induces a formal group law from its first Conner-Floyd Chern class. Moreover, Quillen's theorem on MU together with Lazard's theorem say that the cohomology ring $\pi_\bullet(MU)$ of complex cobordism cohomology MU is the classifying ring for formal group laws.

The *Landweber exact functor theorem* says that, conversely, forming the tensor product of complex cobordism cohomology theory (MU) with a Landweber exact ring via some formal group law yields a cohomology theory, hence a complex oriented cohomology theory.

Literature. (Lurie 10, lectures 15,16)

Outlook: Geometry of $\mathrm{Spec}(\mathrm{MU})$

The grand conclusion of Quillen's theorem on MU (above): complex oriented cohomology theory is essentially the spectral geometry over $\mathrm{Spec}(MU)$, and the latter is a kind of derived version of the moduli stack of formal groups (1-dimensional commutative).

- Landweber-Novikov theorem
- Adams-Quillen theorem
- Adams-Novikov spectral sequence

(…)

Literature. (Kochman 96, sections 4.5-4.7 and section 5, Lurie 10, lectures 12-14)

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2. References

We follow in outline the textbook

- Stanley Kochman, chapters I - IV of *Bordism, Stable Homotopy and Adams Spectral Sequences*, AMS 1996

For some basics in algebraic topology see also

- Robert Switzer, *Algebraic Topology - Homotopy and Homology*, Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen, Vol. 212, Springer-Verlag, New York, N. Y., 1975.

Specifically for **S.1) Generalized cohomology** a neat account is in:

- Marcelo Aguilar, Samuel Gitler, Carlos Prieto, section 12 of *Algebraic topology from a homotopical viewpoint*, Springer (2002) (toc pdf)

For **S.2) Cobordism theory** an efficient collection of the highlights is in

- Cary Malkiewich, *Unoriented cobordism and MO*, 2011 (pdf)

except that it omits proof of the Leray-Hirsch theorem/Serre spectral sequence and that of the Thom isomorphism, but see the references there and see (Kochman 96, Aguilar-Gitler-Prieto 02, section 11.7) for details.

For **S.3) Complex oriented cohomology** besides (Kochman 96, chapter 4) have a look at

- Frank Adams, *Stable homotopy and generalized homology*, Chicago Lectures in mathematics, 1974

and

- Jacob Lurie, lectures 1-10 of *Chromatic Homotopy Theory*, 2010

See also

- Stefan Schwede, *Symmetric spectra*, 2012 (pdf)

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