Persistent homotopy theory

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Basic setup I

$X \subset Z$ finite subset, Z a metric space. $D(Z)$ = poset of finite subsets of Z. $s \geq 0$.

• $P_s(X)$ = poset of subsets $\sigma \subset X$ such that $d(x, y) \leq s$ for all $x, y \in \sigma$.

$P_s(X)$ is the poset of non-degenerate simplices of the Vietoris-Rips complex $V_s(X)$. $BP_s(X)$ is barycentric subdivision of $V_s(X)$.

We have poset inclusions

$$\sigma : P_s(X) \subset P_t(X), \ \ s \leq t,$$

$P_0(X) = X$, and $P_t(X) = \mathcal{P}(X)$ (all subsets of X) for t suff large.

• $k \geq 0$: $P_{s,k}(X) \subset P_s(X)$ subposet of simplices σ such that each element $x \in \sigma$ has at least k neighbours y such that $d(x, y) \leq s$.

$P_{s,k}(X)$ is the poset of non-degenerate simplices of the degree Rips complex $L_{s,k}(X)$.

Basic setup II

The usual inclusions: $s \leq t$

$$
\begin{tikzcd}
P_s(X) \arrow[r, "\sigma"] & P_t(X) \\
P_{s,k}(X) \arrow[r, "\sigma"] \arrow[u] & P_{t,k}(X) \arrow[u] \\
P_{s.k+1}(X) \arrow[r, "\sigma"'] \arrow[u] & P_{t.k+1}(X) \arrow[u]
\end{tikzcd}
$$

Also

- $P_{s,0}(X) = P_s(X)$ for all s,
- $P_{s,k}(X) = \emptyset$ for k suff. large.

Initial impression: $BP_s(X)$ is a huge model for $V_s(X)$, because all simplices of $V_s(X)$ are vertices of $BP_s(X)$.

Fundamental groupoid

$x_0, \dots, x_k$: list of elements of X such that $d(x_i, x_j) \leq s$ (may have repeats).

$$[x_0, \dots, x_k] = \{x_0\} \cup \dots \cup \{x_k\}.$$

Graph $Gr_s(X)$: vertices are elements of X, there is an edge $x \to y$ if $[x, y] \in P_s(X)$.

There is an edge $[x, y] : x \to y$ if and only if there is an edge $[y, x] : y \to x$. There is an edge $[x, x] : x \to x$.

$\Gamma_s(X)$ is **category** generated by $Gr_s(X)$, subject to relations defined by simplices $[x_0, x_1, x_2]$.

Lemma 1.

$\Gamma_s(X)$ is a groupoid, and $\Gamma_s(X) \simeq G(P_s(X)) \simeq \pi V_s(X)$.

$\pi V_s(X)$ is the fundamental groupoid of $V_s(X)$, $G(P_s(X))$ is the free groupoid on the poset $P_s(X)$.

Stability: setup

$D(Z)$ is the poset of finite subsets of Z (all data sets in Z), with Hausdorff metric d_H.

Hausdorff metric:

$r > 0$: Given $X \subset Y$ in $D(Z)$, $d_H(X, Y) < r$ if for all $y \in Y$ there is an $x \in X$ such that $d(y, x) < r$.

For arbitrary $X, Y \in D(Z)$: $d_H(X, Y) < r$ if and only if (equivalently)

1) $d_H(X, X \cup Y) < r$ and $d_H(Y, X \cup Y) < r$.
2) for all $x \in X$ there is a $y \in Y$ such that $d(x, y) < r$, and for all $y \in Y$ there is an $x \in X$ such that $d(y, x) < r$.

$X \subset Y$, $d_H(X, Y) < r$: Construct a function $\theta : Y \to X$ such that

$$\theta(y) = \begin{cases} y & \text{if } y \in X \\ x_y & \text{for some } x_y \in X \text{ with } d(y, x_y) < r. \end{cases}$$

If $\tau \in P_s(Y)$ then $\theta(\tau) \in P_{s+2r}(X)$. Have a diagram of poset morphisms

$$\begin{tikzcd} P_s(X) \arrow[r, "\sigma"] \arrow[d, "i"'] & P_{s+2r}(X) \arrow[d, "i"] \\ P_s(Y) \arrow[r, "\sigma"'] \arrow[ur, "\theta"] & P_{s+2r}(Y) \end{tikzcd}
\qquad
\begin{tikzcd} & y_1 \arrow[r, dash, "s"] \arrow[dl, dash, "r"'] & y_2 \arrow[dr, dash, "r"] & \\ \theta(y_1) \arrow[rrr, dash, dotted, "s+2r"'] & & & \theta(y_2) \end{tikzcd}$$

such that upper triangle commutes, and lower triangle commutes up to homotopy:

$$\sigma(\tau) \to \sigma(\tau) \cup i(\theta(\tau)) \leftarrow i(\theta(\tau)).$$

Stability results

Theorem 2 (Rips stability).

Suppose $X \subset Y$ in $D(Z)$ such that $d_H(X, Y) < r$. There is a homotopy commutative diagram (homotopy interleaving)

$$\begin{tikzcd} P_s(X) \arrow[r, "\sigma"] \arrow[d, "i"'] & P_{s+2r}(X) \arrow[d, "i"] \\ P_s(Y) \arrow[r, "\sigma"'] \arrow[ur, "\theta"] & P_{s+2r}(Y) \end{tikzcd}$$

Theorem 3.

Suppose $X \subset Y$ in $D(Z)$ such that $d_H(X^{k+1}_{dis}, Y^{k+1}_{dis}) < r$. There is a homotopy commutative diagram

$$\begin{tikzcd} P_{s,k}(X) \arrow[r, "\sigma"] \arrow[d, "i"'] & P_{s+2r,k}(X) \arrow[d, "i"] \\ P_{s,k}(Y) \arrow[r, "\sigma"'] \arrow[ur, "\theta"] & P_{s+2r,k}(Y) \end{tikzcd}$$

Blumberg-Lesnick Theorem

Theorem 4.

Suppose given $X, Y \subset Z$ *are data sets with* $d_H(X, Y) < r$.

Then there are maps $\phi : P_s(X) \to P_{s+2r}(Y)$ *and* $\psi : P_s(Y) \to P_{s+2r}(X)$ *such that*

$$\psi \cdot \phi \simeq \sigma : P_s(X) \to P_{s+4r}(X) \text{ and}$$
$$\phi \cdot \psi \simeq \sigma : P_s(Y) \to P_{s+4r}(Y).$$

Proof

Set

$$U = \{(x,y) \mid x \in X, y \in Y, d(x,y) < r\ \}.$$

$P_{s,X}(U) \subset \mathcal{P}(U)$: all subsets σ such that $d(x,x') \leq s$ for all $(x,y),(x',y') \in \sigma$. Define poset $P_{s,Y}(U)$ similarly.

1) The maps $P_{s,X}(U) \to P_s(X)$, $P_{s,Y}(U) \to P_s(Y)$ are weak equivalences (Quillen Theorem A).

2) There are inclusions

$$P_{s,X}(U) \subset P_{s+2r,Y}(U),\ \ P_{s,Y}(U) \subset P_{s+2r,X}(U),$$

(triangle inequality) and

$$\begin{gathered} P_{s,X}(U) \subset P_{s+2r,Y}(U) \subset P_{s+4r,X}(U) \\ P_{s,Y}(U) \subset P_{s+2r,X}(U) \subset P_{s+4r,Y}(U) \end{gathered}$$

Weak equivalences up to shift

Suppose that $X \subset Y$ in $D(Z)$ and we have a homotopy interleaving

$$\begin{tikzcd} V_s(X) \arrow[r, "\sigma"] \arrow[d, "i"'] & V_{s+r}(X) \arrow[d, "i"] \\ V_s(Y) \arrow[r, "\sigma"'] \arrow[ur, "\theta"] & V_{s+r}(Y) \end{tikzcd}$$

(as in stability theorem), where upper triangle commutes and lower triangle commutes up to homotopy fixing $\sigma : V_s(X) \to V_{s+r}(X)$.

1) $i : \pi_0 V_*(X) \to \pi_0 V_*(Y)$ is an r-**monomorphism**: if $i([x]) = i([y])$ in $\pi_0 V_s(Y)$ then $\sigma[x] = \sigma[y]$ in $\pi_0 V_{s+r}(X)$

2) $i : \pi_0 V_*(X) \to \pi_0 V_*(Y)$ is an r-**epimorphism**: given $[y] \in \pi_0 V_s(Y)$, $\sigma[y] = i[x]$ for some $[x] \in \pi_0 V_{s+r}(X)$.

3) All $i : \pi_n(V_*(X), x) \to \pi_n(V_*(Y), i(x))$ are r-**isomorphisms**.

Systems

A system of spaces is a functor $X : [0, \infty) \to s\mathbf{Set}$, aka. a diagram of simplicial sets with index category $[0, \infty)$.

A map of systems $X \to Y$ is a natural transformation of functors defined on $[0, \infty)$.

Examples

1) The functors $V_*(X), BP_*(X)$, $s \mapsto V_s(X), BP_s(X)$ are systems of spaces, for a data set $X \subset Z$.

2) If $X \subset Y \subset Z$ are data sets, the induced maps $P_s(X) \to P_s(Y)$, $V_s(X) \to V_s(Y)$ define maps of systems $P_*(X) \to P_*(Y)$ and $V_*(X) \to V_*(Y)$.

Homotopy types

There are many ways to discuss homotopy types of systems. The oldest is the **projective structure** (Bousfield-Kan):

A map $f : X \to Y$ is a **weak equivalence** (resp. **fibration**) if each map $X_s \to Y_s$ is a weak equiv. (resp. fibration) of simplicial sets.

A map $A \to B$ is a projective cofibration if it has the left lifting property with respect all maps which are trivial fibrations.

Example: $L_s(A)$ is the system with $L_s(A)_t = \emptyset$ for $t < s$ and $L_t(A) = A$ for $t \geq s$. If $A \subset B$ is an inclusion of simplicial sets, then $L_s(A) \to L_s(B)$ is a projective cofibration.

Lemma 5.

Suppose that $X \subset Y \subset Z$ are data sets. Then $V_*(X) \to V_*(Y)$ is a projective cofibration.

r-equivalences

Suppose that $f : X \to Y$ is a map of systems. Say that f is an r-**equivalence** if

1) the map $f : \pi_0(X) \to \pi_0(Y)$ is an r-isomorphism of systems of sets
2) the maps $f : \pi_k(X_s, x) \to \pi_k(Y_s.f(x))$ are r-isomorphisms of systems of groups, for all $s \geq 0$, $x \in X_s$.

Observation: Suppose given a diagram of systems

$$\begin{tikzcd} X_1 \arrow[r, "f_1"] \arrow[d, "\simeq"'] & Y_1 \arrow[d, "\simeq"] \\ X_2 \arrow[r, "f_2"'] & Y_2 \end{tikzcd}$$

Then f_1 is an r-equivalence iff f_2 is an r-equivalence.

Example (stability): Suppose that $X \overset{i}{\subset} Y \subset Z$ are data sets, and that $d_H(X, Y) < r$. Then the maps $i : V_*(X) \to V_*(Y)$ and $i : BP_*(X) \to BP_*(Y)$ are $2r$-equivalences.

Triangle axiom

Lemma 6.

Suppose given a commutative triangle

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "h"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

If one of the maps is an r-equivalence, a second is an s-equivalence, then the third map is a $(r+s)$-equivalence.

Proof.

Suppose X, Y, Z are systems of sets, h is an r-isomorphism and g is an s-isomorphism. Given $z \in Y_t$, $g(z) = h(w)$ for some $w \in X_{t+s}$. Then $g(z) = g(f(w))$ in Z_{t+s} so $z = f(w)$ in Y_{t+s+r}. $\square$

Lemma 7.

Suppose that $p : X \to Y$ is a sectionwise fibration of systems of Kan complexes and that p is an r-equivalence.
Then each lifting problem

$$
\begin{tikzcd}
\partial\Delta^n \arrow[r, "\alpha"] \arrow[d] & X_s \arrow[r, "\sigma"] \arrow[d] & X_{s+2r} \arrow[d, "p"] \\
\Delta^n \arrow[r, "\beta"'] \arrow[urr, dotted, "\theta"] & Y_s \arrow[r, "\sigma"'] & Y_{s+2r}
\end{tikzcd}
$$

can be solved **up to shift** $2r$.

Proof of Lemma 7

The original diagram can be replaced up to homotopy by a diagram

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "{(\alpha_0,*,\dots,*)}"] \arrow[d] & X_s \arrow[r, "\sigma"] \arrow[d, "p"] & X_{s+r} \arrow[d, "p"] \\ \Delta^n \arrow[r, "\beta"'] & Y_s \arrow[r, "\sigma"'] & Y_{s+r} \end{tikzcd} \tag{1}$$

$p_*([\alpha_0]) = 0$ in $\pi_{n-1}(Y_s,*)$, so $\sigma_*([\alpha_0]) = 0$ in $\pi_{n-1}(X_{s+r},*)$.
The trivializing homotopy for $\sigma(\alpha_0)$ in X_{s+r} defines a homotopy from (1) (outer) to the diagram

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "*"] \arrow[d] & X_{s+r} \arrow[d, "p"] \\ \Delta^n \arrow[r, "\omega"'] & Y_{s+r} \end{tikzcd} \tag{2}$$

$\sigma_*([\omega]) \in \pi_n(Y_{s+2r},*)$ lifts to an element of $\pi_n(X_{s+2r},*)$ up to homotopy, giving the desired lifting.

Lemma 8.

Suppose that $p : X \to Y$ is a sectionwise fibration of systems of Kan complexes, and that all lifting problems

$$\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[d] & X_s \arrow[r, "\sigma"] \arrow[d] & X_{s+r} \arrow[d, "p"] \\
\Delta^n \arrow[r] \arrow[urr, dotted, "\theta"] & Y_s \arrow[r, "\sigma"'] & Y_{s+r}
\end{tikzcd}$$

have solutions up to shift r, in the sense that the dotted arrow exists making the diagram commute. Then the map $p : X \to Y$ is an r-equivalence.

Proof.

If $p_*([\alpha]) = 0$ for $[\alpha] \in \pi_{n-1}(X_s, *)$, then there is a diagram on the left above. The existence of θ gives $\sigma_*([\alpha]) = 0$ in $\pi_{n-1}(X_{s+r}, *)$. $\square$

Fibrations III

Corollary 9.

Suppose given a pullback diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "p'"'] & X \arrow[d, "p"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

where p is a sectionwise fibration and an r-equivalence.
Then the map p' is a sectionwise fibration and a $2r$-equivalence.

Question: Is there a dual statement? Do maps which are cofibrations and r equivalences push out to $2r$-equivalences?

Homology

A map $f : A \to B$ of systems of simplicial abelian groups (chain complexes) is an r-equivalence if the induced maps $H_k(A) \to H_k(B)$ are r-isomorphisms for $k \geq 0$.

Example: Suppose that $X \subset Y \subset Z$ are data sets and that $d_H(X, Y) < r$. Then $\mathbb{Z}(X) \to \mathbb{Z}(Y)$ is a $2r$-equivalence (by the interleaving), so that $H_k(X) \to H_k(Y)$ is a $2r$-isomorphism for $k \geq 0$ (all coefficients).

Lemma 10.

1) *Suppose that $f : A \to B$ is an r-equivalence with homotopy cofibre $p : B \to C$. Then the map $C \to 0$ is a $2r$-equivalence.*
2) *Suppose that $C \to 0$ is an r-equivalence. Then $f : A \to B$ is an r-equivalence.*

Warning: There is no Hurewicz theorem. We can't say that if $X \to *$ is an r-equivalence then $H_*(X)$ is r-equvalent to $H_*(*)$.

Groupoids

Question: What does it mean for $X \to *$ to be an r-equivalence?

Facts: 1) If $X \to *$ is an r-equivalence, then all Postnikov sections P_nX and n-connected covers $X(n)$ are r-equivalent to a point.

2) If $X \to *$ is an r-equivalence, then

$$\sigma_* = 0 : \pi_k(X_s, *) \to \pi_k(X_{s+r}, *)$$

for $k \geq 1$. All $[x] \in \pi_0 X_s$ map to the same element of $\pi_0 X_{s+r}$.

Example: $P_1 X = B\pi(X)$, so fundamental groupoid $\pi(X)$ is r-equivalent to a point. We can discuss systems of groupoids G such that $G \to *$ are r-equivalences.

$P_0 G$ has same objects as G, and exactly one morphism $x \to y$ if $\hom_G(x, y) \neq \emptyset$. There is a natural functor $\pi : G \to P_0 G$.

Lemma 11.

Suppose that $G \to *$ is an r-equivalence. Then there is an interleaving

$$\begin{tikzcd} G_s \arrow[r, "\sigma"] \arrow[d, "\pi"'] & G_{s+r} \arrow[d, "\pi"] \\ P_0 G_s \arrow[r, "\sigma"'] \arrow[ur, "\theta"] & P_0 G_{s+r} \end{tikzcd}$$

and all elements of $\pi_0 G_s$ map to the same element of $\pi_0 G_{s+r}$.

Proof.

Any two morphisms $\alpha, \beta : x \to y$ of G_s map to the same morphism of G_{s+r}, so θ exists.

In effect, $\beta^{-1} \cdot \alpha \in G_s(x, x) = \pi_1(BG_s, x)$. $\square$

2-groupoids

A 2-groupoid H is a groupoid enriched in simplicial sets, such that each simplicial set $H(x,y)$ is the nerve of a groupoid.

Each H has a bisimplicial nerve BH which defines a homotopy type.

Every 2-groupoid H has an associated groupoid P_1H with a functorial map $\pi : H \to P_1H$, such that $P_1H(x,y) = P_0(H(x,y))$.

Fact: Every space X has a fundamental 2-groupoid $\pi_2 X$ such that $B\pi_2(X) \simeq P_2(X)$.

Lemma 12 (slightly conjectural).

Suppose that H is a system of 2-groupoids such that $BH \to *$ is an r-equivalence. Then $P_1H \to *$ is an r-equivalence, and there is an interleaving

$$\begin{tikzcd} H_s \arrow[r, "\sigma"] \arrow[d, "\pi"'] & H_{s+r} \arrow[d, "\pi"] \\ P_1H_s \arrow[r, "\sigma"'] \arrow[ur, "\theta"] & P_1H_{s+r} \end{tikzcd}$$

Homology

H a system of 2-groupoids s.t. $BH \to *$ is an r-equivalence.

0) $P_0H \to *$ is an r-isomorphism. P_0H is a system of disjoint unions of trivial groupoids (contractible spaces). $H_0(BP_0H) \to \mathbb{Z}$ is an r-isomorphism, and there are **no** non-trivial higher homology groups.

$H_0(BH) \cong H_0(BP_0H) \to \mathbb{Z}$ is an r-isomorphism.

1) $P_1H \to *$ is an r-equivalence. The interleaving

$$\begin{tikzcd} P_1H_s \arrow[r, "\sigma"] \arrow[d, "\pi"'] & P_1H_{s+r} \arrow[d, "\pi"] \\ P_0H_s \arrow[r, "\sigma"'] \arrow[ur, "\theta"] & P_0H_{s+r} \end{tikzcd}$$

forces $H_k(BP_1H_s) \to 0$ to be an r-isomorphism for $k \geq 1$, because all higher homology groups of BP_0H_s are trivial.

$H_1(BH) \cong H_1(BP_1H) \to 0$ is an r-isomorphism.

Homology II

2) $P_2H \to *$ is an r-equivalence. The interleaving

$$\begin{tikzcd} P_2H_s \arrow[r, "\sigma"] \arrow[d, "\pi"'] & P_2H_{s+r} \arrow[d, "\pi"] \\ P_1H_s \arrow[r, "\sigma"'] \arrow[ur, "\theta"] & P_1H_{s+r} \end{tikzcd}$$

forces $H_k(BP_2H) \to 0$ to be a $2r$-**isomorphism** for $k \geq 1$:

$\pi \cdot \sigma(\alpha) = \sigma \cdot \pi(\alpha) = 0$ for $\alpha \in H_k(BP_2H_s)$ since $H_k(BP_1H_s) \to 0$ is an r-isomorphism.

Then $\sigma \cdot \sigma(\alpha) = \theta \cdot \pi \cdot \sigma(\alpha) = 0$ in $H_k(BP_2H_{s+2r})$.

$H_2(BH) \cong H_2(BP_2H) \to 0$ is a $2r$-isomorphism.

We construct spaces from the poset of data sets $D(Z)$. There are two choices:

1) $D_s(Z) \subset BD(Z)$ consists of strings of simplices

$$\sigma : \sigma_0 \subset \sigma_1 \subset \cdots \subset \sigma_n$$

such that $d_H(\sigma_0, \sigma_n) \leq s$.

2) $P_s(Z) \subset \mathcal{P}(D(Z))$ is poset consisting of finite subsets σ such that $d_H(X, Y) \leq s$ for all $X, Y \in \sigma$.

Theorem 13.

There are weak equivalences

$$D_s(Z) \underset{\simeq}{\xleftarrow{\gamma}} BND_s(Z) \xrightarrow{\phi} BP_s(Z),$$

where $\phi(\sigma) = \{\sigma_0, \dots, \sigma_n\}$.

Proof I

- There is a functor $f : P_s(Z) \to D(Z)$ with $\sigma = \{X_0, \dots, X_k\} \mapsto X_0 \cup \dots \cup X_k$.

$f : BP_s(Z) \to BD(Z)$ takes simplices of $BP_s(Z)$ to simplices of $D_s(Z)$ and induces $f : BP_s(Z) \to D_s(Z)$.

The following diagram commutes:

$$\begin{tikzcd} BND_s(Z) \arrow[rr, "\phi"] \arrow[dr, "\gamma"', "\simeq"] & & BP_s(Z) \arrow[dl, "f"] \\ & D_s(Z) & \end{tikzcd}$$

- Show that f is a weak equivalence. Suppose that $\tau : Y_0 \subset \dots \subset Y_k$ is a non-degenerate simplex of $BD_s(Z)$. Show that $f : f^{-1}(\tau) \to \Delta^k$ is a weak equivalence.

Proof II

- $f^{-1}(\tau)$ is the nerve of a poset, with objects $\{Z_0, \dots, Z_m\}$ such that $\cup_i Z_i$ is some Y_j, with morphisms covering inclusions $Y_j \subset Y_k$.

- Given $\tau = \{Z_0, \dots, Z_m\}$ with $\cup_i Z_i = Y_j$, there are poset morphisms

$$\{Z_0, \dots, Z_m\} \to \{Z_0, \dots, Z_m\} \cup \{Y_0, \dots, Y_j\} \leftarrow \{Y_0, \dots, Y_j\}.$$

- There is a simplicial set map $\sigma : \Delta^k \to f^{-1}(\tau)$ defined by the string of inclusions

$$\{Y_0\} \subset \{Y_0, Y_1\} \subset \cdots \subset \{Y_0, \dots, Y_k\}$$

The map $f : f^{-1}(\tau) \to \Delta^k$ is a homotopy equivalence. $\square$

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