NOTES ON QUASI-CATEGORIES

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To the memory of Jon Beck

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Introduction

The notion of quasi-category was introduced by Boardman and Vogt in their work on homotopy invariant algebraic structures [BV]. A Kan complex and the nerve of a category are basic examples. The following notes are a collection of assertions on quasi-categories, many of which have not yet been formally proved. Our goal is to show that category theory has a natural extension to quasi-categories, The extended theory has applications to homotopy theory, homotopical algebra, higher category theory and higher topos theory.

A first draft of the notes was written in 2004 in view of its publication in the Proceedings of the Conference on higher categories held at the IMA in Minneapolis. An expanded version was used in a course given at the Fields Institute in January 2007. The latest version was used in a course at the CRM in Barcelona in February 2008.

Remarks on terminology: a quasi-category is sometime called a *weak Kan complex* in the literature [KP]. The term "quasi-category" was introduced to suggest a similarity with categories. We shall use the term *quategory* as an abreviation.

Quategories abound. The coherent nerve of a category enriched over Kan complexes is a quategory. The quasi-localisation of a model category is a quategory. A quategory can be large. For example, the coherent nerve of the category of Kan complexes is a large quategory $\mathcal{K}$. The coherent nerve of the category of (small) quategories is a large quategory $\mathcal{Q}_1$.

Quategories are examples of $(\infty, 1)$-categories in the sense of Baez and Dolan. Other examples are simplicial categories, Segal categories and complete Segal spaces (here called Rezk categories). Simplicial categories were introduced by Dwyer and Kan in their work on simplicial localisation. Segal categories by Schwnzel and Vogt under the name of Δ-categories [ScVo] and rediscovered by Hirschowitz and Simpson in their work on higher stacks. Complete Segal spaces (Rezk categories) were introduced by Rezk in his work on homotopy theories. To each of these examples is associated a model category and the four model categories are Quillen equivalent. The equivalence between simplicial categories, Segal categories and Rezk categories was established by Bergner [B2]. The equivalence between Rezk categories and quategories was established by Tierney and the author [JT2]. The equivalence between simplicial categories and quategories was established by Lurie [Lu1] and independantly by the author [J4]. Many aspects of category theory were extended to simplicial categories by Bousfield, Dwyer and Kan. The theory of homotopical categories of Dwyer, Hirschhorn, Kan and Smith is closely related to that of quategories [DHKS]. Many aspects of category theory were extended to Segal categories by Hirschowitz, Simpson, Toen and Vezzosi. Jacob Lurie has recently formulated his work on homotopoi in the language of quategories. In doing so, he has developped a formidable amount of quategory theory and our notes may serve as an introduction to his work. Many notions introduced here are due to Charles Rezk. The notion of homotopoi is an example. The notion of reduced category object is another.

Remark: the list $(\infty, 1)$-categories given above is not exhaustive and our account of the history of the subject is incomplete. The notion of A_∞-space introduced by Stasheff is a seminal idea in the whole subject. A theory of A_∞-categories was developped by Batanin [Bat1]. A theory of homotopy coherent diagrams was developped by Cordier and Porter[CP2].

The theory of quategories depends on homotopical algebra. A basic result states that the category of simplicial sets **S** admits a Quillen model structure in which the fibrant objects are the quategories (and the cofibration are the monomorphisms). This defines the *model structure for quategories*. The classical model structure on the category **S** is a Bousfield localisation of this model structure.

Many aspects of category theory can be formulated in the language of homotopical algebra. The category of small categories **Cat** admits a model structure in which the weak equivalences are the equivalence of categories; it is the *natural* model structure on **Cat**. Homotopy limits in the natural model structure are closely related to the pseudo-limits introduced by category theorists.

Many aspects of homotopical algebra can be formulated in the language of quategories. This is true for example of the theory of homotopy limits and colimits. Many results of homotopical algebra becomes simpler when formulated in the language of quategories. We hope a similar simplification of the proofs. But this is not be entirely clear at present, since the theory of quategories is presently in its infancy. A mathematical theory is a kind of social construction, and the complexity of a proof depends on the degree of maturity of the subject. What is considered to be "obvious" is the result of an implicit agreement between the experts based on their knowledge and experience.

The quategory $\mathcal{K}$ has many properties in common with the category of sets. It is the archetype of a *homotopos*. A *prestack* on a simplicial set A is defined to be a map $A^o \to \mathcal{K}$. A general homotopos is a left exact reflection of a quategory of prestacks. Homotopoi can be described abstractly by a system of axioms similar to the those of Giraud for a Grothendieck topos [Lu1]. They also admit an elegant characterization (due to Lurie) in terms of a strong descent property discovered by Rezk.

All the machinery of universal algebra can be extended to quategories. An *algebraic theory* is defined to be a small quategory with finite products T, and a *model* of T to be a map $T \to \mathcal{K}$ which preserves finite products. The models of T form a large quategory Model(T) which is complete an cocomplete. A *variety of homotopy algebras*, or an *homotopy variety* is defined to be a quategory equivalent to a quategory Mod(T) for some algebraic theory T. Homotopy varieties can be characterized by system of axioms closely related to those of Rosicky [Ros]. The notion of algebraic structure was extended by Ehresman to include the *essentially algebraic structures* defined by a *limit sketch*. For example, the notions of groupoid object and of category object in a category are essentially algebraic. The classical theory of limit sketches and of essentially algebraic structures is easily extended to quategories. A *category object* in a quategory X is defined to be a simplicial object $C : \Delta^o \to X$ satisfying the *Segal condition*. The theory of limit sketches is a natural framework for studying homotopy coherent algebraic structures in general and higher weak categories in particular. The quategory of models of a limit sketch is locally presentable and conversely, every locally presentable quategory is equivalent to the quategory of models of a limit sketch. The theory of accessible categories and of locally presentable categories was extended to quategories by Lurie.

A *para-variety* is defined to be a left exact reflection of a variety of homotopy algebras. For example, a homotopos is a para-variety. The quategories of spectra and of ring spectra are also examples. Para-varieties can be characterized by a system of axioms closely related to those of Vitale [Vi].

Factorisation systems are playing an important role in the theory of quategories. We introduce a general notion of homotopy factorisation system in a model category with examples in category theory, in classical homotopy theory and in the theory of quategories. A basic example is provided by the theory of Dwyer-Kan localisations. This is true also of the theory of prestacks.

The theory of quategories can analyse phenomena which belong properly to homotopy theory. The notion of stable quategory is an example. The notion of meta-stable quategory introduced in the notes is another. We give a proof that the quategory of parametrized spectra is a homotopos (joint work with Georg Biedermann). We sketch a new proof of the stabilisation hypothesis of Breen-Baez-Dolan [Si2]. We give a characterisation of homotopy varieties which improves a result of Rosicky.

There are important differences between category theory and the theory of quategories. An important difference lies in the fact that in a quategory a section of a morphism is not necessarly monic. For example, the diagonal of an object in a quategory is not necessarly monic. The notion of equivalence relation is affected accordingly and it becomes less restrictive. For example, in the quategory $\mathcal{K}$ every groupoid is effective. This is true in particular if the groupoid is a group. The quotient of the terminal object 1 by the action of a group G is the *classyfying space* of G,

$$BG = 1/G.$$

Of course, this sounds like a familiar idea in homotopy theory, since $BG = E/G$, where E is a contractible space on which G is acting freely. The fact that every groupoid in $\mathcal{K}$ is an equivalence groupoid has important consequences. In algebra, important mental simplications are obtained by quotienting a structure by a congruence relation. For example, we may wish to identify two objects of a category when these objects are isomorphic. But the quotient category does not exist unless we can identify isomorphic objects coherently. However, the quotient category always exists in $\mathcal{K}$: if $J(C)$ denotes the groupoid of isomorphisms of C, then the quotient C' can be constructed by a pushout square of category objects in $\mathcal{K}$:

$$\begin{array}{ccc} J(C) & \longrightarrow & C \\ \downarrow & & \downarrow \\ BJ(C) & \longrightarrow & C', \end{array}$$

where $BJ(C)$ is the quotient of C_0 by the groupoid $J(C)$. The category C' satisfies the *Rezk condition*: every isomorphism of C' is a unit; we shall say that it is *reduced*. Moreover, the canonical functor $C \to C'$ is an equivalence of categories! An important simplification is obtained by working with reduced categories, since a functor between reduced categories $f : C \to D$ is an equivalence iff it is an isomorphism! The notion of reduced category object is essentially algebraic. It turns out that the quategory of reduced category objects in $\mathcal{K}$ is equivalent to $\mathcal{Q}_1$. This follows from the Quillen equivalence between the model category for quategories and the model category for Rezk categories [JT2]. Hence a quategory is essentially the same thing as a reduced category object in $\mathcal{K}$.

In the last sections we venture a few steps in the theory of (∞, n)-categories for every $n \geq 1$. There is a notion of n-fold category object for every $n \geq 1$. The quategory of n-fold category objects in $\mathcal{K}$ is denoted by $Cat^n(\mathcal{K})$. By definition, we have

$$Cat^{n+1}(\mathcal{K}) = Cat(Cat^n(\mathcal{K})).$$

There is also a notion of n-category object for every $n \geq 1$. The quategory $Cat_n(\mathcal{K})$ of n-category objects in $\mathcal{K}$ is a full sub-quategory of $Cat^n(\mathcal{K})$. A n-category C is *reduced* if every invertible cell of C is a unit. The notion of reduced n-category object is essentially algebraic. The quategory of reduced n-category objects in $\mathcal{K}$ is denoted by $\mathcal{Q}_n$. The quategory $\mathcal{Q}_n$ is locally presentable, since the notion of reduced n-category object is essentially algebraic. It follows that $\mathcal{Q}_n$ is the homotopy localisation of a combinatorial model category. For example, it can be represented by a regular Cisinski model $(\hat{A}, W)$. Such a representation is determined by a map $r : A \to \mathcal{Q}_n$ whose left Kan extension $r_! : \hat{A} \to \mathcal{Q}_n$ induces an equivalence between the homotopy localisation of $(\hat{A}, W)$ and $\mathcal{Q}_n$. The class W is also determined by r, since a map $f : X \to Y$ in $\hat{A}$ belongs to W iff the morphism $r_!(f) : r_!X \to r_!Y$ is invertible in $\mathcal{Q}_n$. The notion of *n-quategory* is obtained by taking A to be a certain full subcategory Θ_n of the category of strict n-categories and by taking r to be the inclusion $\Theta_n \subset \mathcal{Q}_n$. In this case W the class of *weak categorical n-equivalences* Wcat_n. The model category $(\hat{\Theta}_n, \mathrm{Wcat}_n)$ is cartesian closed and its subcategory of fibrant objects $\mathbf{QCat}_n$ has the structure of a simplicial category enriched over Kan complexes. The coherent nerve of $\mathbf{QCat}_n$ is equivalent to $\mathcal{Q}_n$.

Note: The category Θ_n was first defined by the author as the opposite of the category of finite n-disks $\mathcal{D}_n$. It follows from this definition that the topos $\hat{\Theta}_n$ is classifying n-disks and that the geometric realisation functor $\hat{\Theta}_n \to \mathbf{Top}$ introduced by the author preserves finite limits (where **Top** is the category of compactly generated topological spaces). See [Ber] for a proof of these results. It was conjectured (jointly by Batanin, Street and the author) that Θ_n is isomorphic to a category T_n^* introduced by Batanin in his theory of higher operads [Bat3]. The conjecture was proved by Makkai and Zawadowski in [MZ] and by Berger in [Ber]. It shows that Θ_n is a full subcategory of the category of strict n-categories.

Note: It is conjectured by Cisinski and the author that the localiser Wcat_n is generated by a certain set of *spine inclusions* $S[t] \subseteq \Theta[t]$.

We close this introduction with a few general remarks on the notion of weak higher category. There are essentially three approaches for defining this notion: operadic, Segalian and Kanian. In the first approach, a weak higher category is viewed as an algebraic structure defined by a system of operations satisfying certain coherence conditions which are themselve expressed by higher operations, possibly at infinitum. The first algebraic definition of a weak higher groupoid is due to Grothendieck in his "Pursuing Stacks" [Gro] [**?**]. The first general definition of a weak higher category by Baez and Dolan is using operads. The definition by Batanin is using the higher operads introduced for this purpose. The Segalian approach has its origin in the work of Graeme Segal on infinite loop spaces [S1]. A homotopy coherent algebraic structure is defined to be a commutative diagram of spaces satisfying certain exactness conditions, called the *Segal conditions*. The spaces can be simplicial sets, and more generally the objects of a Quillen model category. The approach has the immense advantage of pushing the coherence conditions out of the way. The notions of Segal category, of Segal space and of Rezk category (ie complete Segal space) are explicitly Segalian. The Kanian approach has its origin in the work of Kan and in the work of Boardman and Vogt. The notion of quategory is *Kanian*, since it is defined by a cell filling condition (the Boardman condition). In the Kanian approach, a weak higher groupoid is the same thing as a Kan complex. We are thus liberated from the need to represent a homotopy type by an algebraic structure, since the homotopy type can now represents itself! Of course, it is always instructive to model homotopy types algebraically, since it is the purpose of algebraic topology to study spaces from an algebraic point of view. For example, a 2-type can be modeled by a categorical group and a simply-connected 3-type by a braided categorical group. In these examples, the homotopy type is fully described by the algebraic model, Partial models are also important as in rational homotopy theory. The different approaches to higher categories are not in conflict but complementary. The Kanian approach is heuristically stronger and more effective at the foundational level. It suggests that a weak higher category is the combinatorial representation of a space of a new kind, possibly a higher moduli stack. The nature of these spaces is presently unclear, but like categories, they should admit *irreversible paths*. Grothendieck topoi are not general enough, even in their higher incarnations, the homotopoi. For example, I do not know how to associate a higher topos to a 2-category. For this we need a notion of 2-*prestack*.

But this notion depends on what we choose to be the archetype of an $(\infty, 2)$-*topos*. The idea that there is a connection between the notion of weak category and that of space is very potent. It was a guiding principle, a *fil d'Ariane*, in the Pursuing Stacks of Grothendieck. It has inspired the notion of braided monoidal category and many conjectures by Baez and Dolan. It suggests that the category of weak categories has properties similar to that of spaces, for example, that it should be cartesian closed. It suggests the existence of classifying higher categories, in analogy with classifying spaces. Classifying spaces are often equipped with a natural algebraic structure. Operads were originally introduced for studying these structures and the corresponding algebra of operations in (co)homology. Many new invariants of topology, like the Jones polynomial, have not yet been explained within the classical setting of algebraic topology. Topological quantum field theory is pushing for an extension of algebraic topology and the operadic approach to higher categories may find its full meaning in the extension.

Notes: A notion of higher category based on the notion of complicial set was introduced by Street and Verity. The Segalian approach to universal algebra was developed by Badzioch [Bad2]. There many approaches to higher operads. A theory based on cartesian monads was developed by Leinster. A theory based on parametric right adjoints was developed by Batanin and Weber. A notion of quasi-operads (or multi-quategories) was recently introduced by Moerdijk and Weiss.

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I am indebted to Jon Beck for guiding my first steps in homotopy theory more than thirty years ago. Jon was deeply aware of the unity between homotopy theory and category theory and he contributed to both fields. He had the dream of using simplicial sets for the foundation of mathematics (including computer science and calculus!). I began to read Boardmann and Vogt after attending the beautiful talk that Jon gave on their work at the University of Durham in July 1977. I dedicate these notes to his memory.

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1. Elementary aspects

In this section we formally introduce the notion of quategory and describe a few basic properties. We introduce the notion of equivalence between quategories.

1.1. For terminology and notation about categories and simplicial sets, see appendix 46 and 48. We denote the category of simplicial sets by $\mathbf{S}$ and the category of small categories by $\mathbf{Cat}$.

1.2. The category Δ is a full subcategory of $\mathbf{Cat}$. Recall that the *nerve* of a small category C is the simplicial set NC obtained by putting

$$(NC)_n = \mathbf{Cat}([n], C)$$

for every $n \geq 0$. The nerve functor $N : \mathbf{Cat} \to \mathbf{S}$ is fully faithful. We shall regard it as an inclusion $N : \mathbf{Cat} \subset \mathbf{S}$ by adopting the same notation for a category and its nerve. The nerve functor has a left adjoint

$$\tau_1 : \mathbf{S} \to \mathbf{Cat}$$

which associates to a simplicial set X its *fundamental category* $\tau_1 X$. The classical fundamental groupoid $\pi_1 X$ is obtained by formally inverting the arrows of $\tau_1 X$. If X is a simplicial set, the canonical map $X \to N\tau_1 X$ is denoted as a map $X \to \tau_1 X$.

1.3. Recall that a simplicial set X is said to be a *Kan complex* if it satisfies the *Kan condition*: every horn $\Lambda^k[n] \to X$ has a filler $\Delta[n] \to X$,

$$\begin{array}{ccc} \Lambda^k[n] & \xrightarrow{\forall} & X \\ \downarrow & \nearrow_{\exists} & \\ \Delta[n]. & & \end{array}$$

The singular complex of a space and the nerve of a groupoid are examples. We shall denote by **Kan** the full subcategory of $\mathbf{S}$ spanned by the Kan complexes. If X is a Kan complex, then so is the simplicial set X^A for any simplicial set A. It follows that the category **Kan** is cartesian closed. A simplicial set X is (isomorphic to the nerve of) a groupoid iff every horn $\Lambda^k[n] \to X$ has a *unique* filler.

1.4. Let us say that a horn $\Lambda^k[n]$ is *inner* if $0 < k < n$. A simplicial set X is (isomorphic to the nerve of) a category iff every inner horn $\Lambda^k[n] \to X$ has a *unique* filler. We shall say that a simplicial set X is a *quasi-category*, in short a *quategory*, if it satisfies the *Boardman condition*: every inner horn $\Lambda^k[n] \to X$ has a filler $\Delta[n] \to X$. A Kan complex and the nerve of a category are examples. We shall say that a quategory with a single object is a *quasi-monoid*. If X is a quategory, we shall say that an element of X_0 is an *object* of X and that an element of X_1 is a *morphism*. A *map* of quategories $f : X \to Y$ is just a map of simplicial sets; we may say that it is a *functor*. We shall denote by **QCat** the full subcategory of $\mathbf{S}$ spanned by the quategories. If X is a quategory then so is the simplicial set X^A for any simplicial set A. Hence the category **QCat** is cartesian closed.

1.5. A quategory can be large. We say that quategory X is *locally small* if the simplicial set X is locally small (this means that the vertex map $X_n \to X_0^{n+1}$ has small fibers for every $n \geq 0$). Most quategories considered in these notes are small or locally small.

1.6. [J2] The notion of quategory has many equivalent descriptions. Recall that a map of simplicial sets is a called a *trivial fibration* if it has the right lifting property with respect to the inclusion $\partial\Delta[n] \subset \Delta[n]$ for every $n \geq 0$. Let us denote by $I[n]$ the simplicial subset of $\Delta[n]$ generated by the edges $(i, i+1)$ for $0 \leq i \leq n-1$ (by convention, $I[0] = \Delta[0]$). The simplicial set $I[n]$ is a chain of n arrows and we shall say that it is the *spine* of $\Delta[n]$. Notice that $I[2] = \Lambda^1[2]$ and that $X^{I[2]} = X^I \times_{s=t} X^I$. A simplicial set X is a quategory iff the projection

$$X^{\Delta[2]} \to X^{I[2]}$$

defined from the inclusion $I[2] \subset \Delta[2]$ is a trivial fibration iff the projection $X^{\Delta[n]} \to X^{I[n]}$ defined from the inclusion $I[n] \subset \Delta[n]$ is a trivial fibration for every $n \geq 0$.

1.7. If X is a simplicial set, we shall denote by $X(a, b)$ the fiber at $(a, b) \in X_0 \times X_0$ of the projection

$$(s, t) : X^I \to X^{\{0,1\}} = X \times X$$

defined by the inclusion $\{0, 1\} \subset I$. A vertex of $X(a, b)$ is an arrow $a \to b$ in X. If X is a quategory, then the simplicial set $X(a, b)$ is a Kan complex for every pair (a, b). Moreover, the projection $X^{\Delta[2]} \to X^I \times_{s=t} X^I$ defined from the inclusion $I[2] \subset \Delta[2]$ has a section, since it is a trivial fibration by 1.6. If we compose this section with the map $X^{d_1} : X^{\Delta[2]} \to X^I$, we obtain a "composition law"

$$X^I \times_{s=t} X^I \to X^I$$

well defined up to homotopy. It induces a "composition law"

$$X(b, c) \times X(a, b) \to X(a, c)$$

for each triple $(a, b, c) \in X_0 \times X_0 \times X_0$.

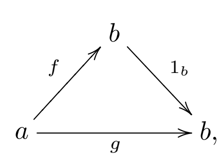
1.8. The fundamental category $\tau_1 X$ of a simplicial set X has a simple construction when X is a quategory. In this case we have

$$\tau_1 X = hoX,$$

where hoX is the *homotopy category* of X introduced by Boardman and Vogt in [BV]. By construction, $(hoX)(a, b) = \pi_0 X(a, b)$ and the composition law

$$hoX(b, c) \times hoX(a, b) \to hoX(a, c)$$

is induced by the "composition law" of 1.7. If $f, g : a \to b$ are two arrows in X, we shall say that a 2-simplex $u : \Delta[2] \to X$ with boundary $\partial u = (1_b, g, f)$,



is a *right homotopy* between f and g and we shall write $u : f \Rightarrow_R g$. Dually, we shall say that a 2-simplex $v : \Delta[2] \to X$ with boundary $\partial v = (g, f, 1_a)$,

$$\begin{array}{ccc} & a & \\ {\scriptstyle 1_a}\nearrow & & \searrow{\scriptstyle g} \\ a & \xrightarrow[f]{} & b. \end{array}$$

is a *left homotopy* between f and g and we shall write $v : f \Rightarrow_L g$. Two arrows $f, g : a \to b$ in a quategory X are homotopic in $X(a, b)$ iff there exists a right homotopy $u : f \Rightarrow_R g$ iff there exists a left homotopy $v : f \Rightarrow_L g$. Let us denote by $[f] : a \to b$ the homotopy class of an arrow $f : a \to b$. The composite of a class $[f] : a \to b$ with a class $[g] : b \to c$ is the class $[wd_1] : a \to c$, where w is any 2-simplex $\Delta[2] \to X$ filling the horn $(g, \star, f) : \Lambda^1[2] \to X$,

$$\begin{array}{ccc} & b & \\ {\scriptstyle f}\nearrow & & \searrow{\scriptstyle g} \\ a & \xrightarrow[wd_1]{} & c. \end{array}$$

1.9. There is an analogy between Kan complexes and groupoids. The nerve of) a category is a Kan complex iff the category is a groupoid. Hence the following commutative square is a pullback,

$$\begin{array}{ccc} \mathbf{Gpd} & \xrightarrow{in} & \mathbf{Kan} \\ {\scriptstyle in}\downarrow & & \downarrow{\scriptstyle in} \\ \mathbf{Cat} & \xrightarrow{in} & \mathbf{QCat}, \end{array}$$

where **Gpd** denotes the category of small groupoids and where the horizontal inclusions are induced by the nerve functor. The inclusion $\mathbf{Gpd} \subset \mathbf{Kan}$ has a left adjoint $\pi_1 : \mathbf{Kan} \to \mathbf{Gpd}$ and the inclusion $\mathbf{Cat} \subset \mathbf{QCat}$ has a left adjoint $\tau_1 : \mathbf{QCat} \to \mathbf{Cat}$. Moreover, the following square commutes up to a natural isomorphism,

$$\begin{array}{ccc} \mathbf{Gpd} & \xleftarrow{\pi_1} & \mathbf{Kan} \\ {\scriptstyle in}\downarrow & & \downarrow{\scriptstyle in} \\ \mathbf{Cat} & \xleftarrow{\tau_1} & \mathbf{QCat} \end{array}$$

1.10. We say that two vertices of a simplicial set X are *isomorphic* if they are isomorphic in the category $\tau_1 X$. We shall say that an arrow in X is *invertible*, or that it is an *isomorphism*, if its image by the canonical map $X \to \tau_1 X$ is invertible in the category $\tau_1 X$. When X is a quategory, two objects $a, b \in X$ are isomorphic iff there exists an isomorphism $f : a \to b$. In this case, there exists an arrow $g : b \to a$ together with two homotopies $gf \Rightarrow 1_a$ and $fg \Rightarrow 1_b$. A quategory X is a Kan complex iff the category hoX is a groupoid [J1]. Let J be the groupoid generated by one isomorphism $0 \to 1$. Then an arrow $f : a \to b$ in a quategory X is invertible iff the map $f : I \to X$ can be extended along the inclusion $I \subset J$. The inclusion functor $\mathbf{Gpd} \subset \mathbf{Cat}$ has a right adjoint $J : \mathbf{Cat} \to \mathbf{Gpd}$, where $J(C)$ is the groupoid of isomorphisms of a category C. Similarly, the inclusion functor

Kan $\subset$ **QCat** has a right adjoint $J : \mathbf{QCat} \to \mathbf{Kan}$ by [J1]. The simplicial set $J(X)$ is the largest Kan subcomplex of a quategory X. It is constructed by the following pullback square

$$\begin{array}{ccc} J(X) & \longrightarrow & J(hoX) \\ \downarrow & & \downarrow \\ X & \xrightarrow{h} & hoX, \end{array}$$

where h is the canonical map. Moreover, the following square commutes up to a natural isomorphism,

$$\begin{array}{ccc} \mathbf{Gpd} & \xleftarrow{\pi_1} & \mathbf{Kan} \\ {\scriptstyle J}\uparrow & & \uparrow{\scriptstyle J} \\ \mathbf{Cat} & \xleftarrow{\tau_1} & \mathbf{QCat}. \end{array}$$

1.11. The functor $\tau_1 : \mathbf{S} \to \mathbf{Cat}$ preserves finite products by a result of Gabriel and Zisman. For any pair (X, Y) of simplicial sets, let us put

$$\tau_1(X, Y) = \tau_1(Y^X).$$

If we apply the functor τ_1 to the composition map $Z^Y \times Y^X \to Z^X$ we obtain a composition law

$$\tau_1(Y, Z) \times \tau_1(X, Y) \to \tau_1(X, Z)$$

for a 2-category $\mathbf{S}^{\tau_1}$, where we put $\mathbf{S}^{\tau_1}(X, Y) = \tau_1(X, Y)$. By definition, a 1-cell of $\mathbf{S}^{\tau_1}$ is a map of simplicial sets $f : X \to Y$, and a 2-cell $f \to g : X \to Y$ is a morphism of the category $\tau_1(X, Y)$; we shall say that it is a *natural transformation* $f \to g$. Recall that a *homotopy* between two maps $f, g : X \to Y$ is an arrow $\alpha : f \to g$ in the simplicial set Y^X; it can be represented as a map $X \times I \to Y$ or as a map $X \to Y^I$. To a homotopy $\alpha : f \to g$ is associated a natural transformation $[\alpha] : f \to g$. When Y is a quategory, a natural transformation $[\alpha] : f \to g$ is invertible in $\tau_1(X, Y)$ iff the arrow $\alpha(a) : f(a) \to g(a)$ is invertible in Y for every vertex $a \in X$.

1.12. We call a map of simplicial sets $X \to Y$ a *categorical equivalence* if it is an equivalence in the 2-category $\mathbf{S}^{\tau_1}$. For example, a trivial fibration (as defined in 48.4) is a categorical equivalence. The functor $\tau_1 : \mathbf{S} \to \mathbf{Cat}$ takes a categorical equivalence to an equivalence of categories. If X and Y are quategories, we shall say that a categorical equivalence $X \to Y$ is an *equivalence of quategories*, or just an *equivalence* if the context is clear. A map between quategories $f : X \to Y$ is an equivalence iff there exists a map $g : Y \to X$ together with two isomorphisms $gf \to 1_X$ and $fg \to 1_Y$.

1.13. We say that a map of simplicial sets $u : A \to B$ is *essentially surjective* if the functor $\tau_1 A \to \tau_1 B$ is essentially surjective. We say that a map between quategories $f : X \to Y$ is *fully faithful* if the map $X(a, b) \to Y(fa, fb)$ induced by f is a homotopy equivalence for every pair $a, b \in X_0$. A map between quategories is an equivalence iff it is fully faithful and essentially surjective.

2. The model structure for quategories

The category of simplicial sets admits a model structure in which the fibrant objects are the quategories. The classical model structure on the category of simplicial sets is a Bousfield localisation of this model structure.

2.1. Recall that a map of simplicial sets $f : X \to Y$ is said to be a *Kan fibration* if it has the right lifting property with respect to the inclusion $\Lambda^k[n] \subset \Delta[n]$ for every $n > 0$ and $k \in [n]$. Recall that a map of simplicial sets is said to be *anodyne* if it belongs to the saturated class generated by the inclusions $\Lambda^k[n] \subset \Delta[n]$ $(n > 0, k \in [n])$ [GZ]. The category **S** admits a weak factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of anodyne maps and $\mathcal{B}$ is the class of Kan fibrations.

2.2. Let **Top** be the category of compactly generated topological spaces. We recall that the singular complex functor $r^! : \mathbf{Top} \to \mathbf{S}$ has a left adjoint $r_!$ which associates to a simplicial set its *geometric realisation*. A map of simplicial sets $u : A \to B$ is said to be a *weak homotopy equivalence* if the map $r_!(u) : r_!A \to r_!B$ is a homotopy equivalence of topological spaces. The notion of weak homotopy equivalence in **S** can be defined combinatorially by using Kan complexes instead of geometric realisation. To see this, we recall the construction of the homotopy category $\mathbf{S}^{\pi_0}$ by Gabriel and Zisman [GZ]. The functor $\pi_0 : \mathbf{S} \to \mathbf{Set}$ preserves finite products. For any pair (A, B) of simplicial sets, let us put

$$\pi_0(A, B) = \pi_0(B^A).$$

If we apply the functor π_0 to the composition map $C^B \times B^A \to C^A$ we obtain a composition law $\pi_0(B, C) \times \pi_0(A, B) \to \pi_0(A, C)$ for a category $\mathbf{S}^{\pi_0}$, where we put $\mathbf{S}^{\pi_0}(A, B) = \pi_0(A, B)$. A map of simplicial sets is called a *simplicial homotopy equivalence* if it is invertible in the category $\mathbf{S}^{\pi_0}$. A map of simplicial sets $u : A \to B$ is a weak homotopy equivalence iff the map

$$\pi_0(u, X) : \pi_0(B, X) \to \pi_0(A, X)$$

is bijective for every Kan complex X. Every simplicial homotopy equivalence is a weak homotopy equivalence and the converse holds for a map between Kan complexes.

2.3. Recall that the category **S** admits a Quillen model structure in which a weak equivalence is a weak homotopy equivalence and a cofibration is a monomorphism [Q]. The fibrant objects are the Kan complexes. The model structure is cartesian closed and proper. We shall say that it is the *classical model structure* on **S** and we shall denote it shortly by $(\mathbf{S}, \mathrm{Who})$, where Who denotes the class of weak homotopy equivalences. The fibrations are the Kan fibrations. A map is an acyclic cofibration iff it is anodyne.

2.4. We shall say that a functor $p : X \to Y$ between two categories is an *iso-fibration* if for every object $x \in X$ and every isomorphism $g \in Y$ with target $p(x)$, there exists an isomorphism $f \in X$ with target x such that $p(f) = g$. This notion is self dual:a functor $p : X \to Y$ is an iso-fibration iff the opposite functor $p^o : X^o \to Y^o$ is. The category **Cat** admits a model structure in which a weak equivalence is an equivalence of categories and a fibration is an iso-fibration [JT1]. The model structure is cartesian closed and proper. We shall say that it is the *natural* model structure on **Cat** and we shall denote it shortly by $(\mathbf{Cat}, Eq)$, where

Eq denotes the class of equivalences between categories. A functor $u : A \to B$ is a cofibration iff the map $Ob(u) : ObA \to ObB$ is monic. Every object is fibrant and cofibrant. A functor is an acyclic fibration iff it is fully faithful and surjective on objects.

2.5. For any simplicial set A, let us denote by $\tau_0 A$ the set of isomorphism classes of objects of the category $\tau_1 A$. The functor $\tau_0 : \mathbf{S} \to \mathbf{Set}$ preserves finite products, since the functor τ_1 preserves finite products. For any pair (A, B) of simplicial sets, let us put

$$\tau_0(A, B) = \tau_0(B^A).$$

If we apply the functor τ_0 to the the composition map $C^B \times B^A \to C^A$ we obtain the composition law $\tau_0(B, C) \times \tau_0(A, B) \to \tau_0(A, C)$ of a category $\mathbf{S}^{\tau_0}$, where we put $\mathbf{S}^{\tau_0}(A, B) = \tau_0(A, B)$. A map of simplicial sets is a categorical equivalence iff it is invertible in the category $\mathbf{S}^{\tau_0}$. We shall say that a map of simplicial sets $u : A \to B$ is a *weak categorical equivalence* if the map

$$\tau_0(u, X) : \tau_0(B, X) \to \tau_0(A, X)$$

is bijective for every quategory X. A map $u : A \to B$ is a weak categorical equivalence iff the functor

$$\tau_1(u, X) : \tau_1(B, X) \to \tau_1(A, X)$$

is an equivalence of categories for every quategory X.

2.6. The category $\mathbf{S}$ admits a model structure in which a weak equivalence is a weak categorical equivalence and a cofibration is a monomorphism [J2]. The fibrant objects are the quategories. The model structure is cartesian closed and left proper. We shall say that it is the *model structure for quategories* and we denote it shortly by $(\mathbf{S}, \mathrm{Wcat})$, where Who denotes the class of weak categorical equivalences. A fibration is called a *pseudo-fibration* The functor $X \mapsto X^o$ is an automorphism of the model structure $(\mathbf{S}, \mathrm{Wcat})$. . .

2.7. The cofibrations of the model structure $(\mathbf{S}/B, \mathrm{Wcat})$ are the monomorphisms. Hence the model structure is determined by its fibrant objects, that is, by the quategories, by 50.10.

2.8. The pair of adjoint functors

$$\tau_1 : \mathbf{S} \leftrightarrow \mathbf{Cat} : N$$

is a Quillen adjunction between the model categories $(\mathbf{S}, \mathrm{Wcat})$ and $(\mathbf{Cat}, Eq)$. A functor $u : A \to B$ in $\mathbf{Cat}$ is an equivalence (resp. an iso-fibration) iff the map $Nu : NA \to NB$ is a (weak) categorical equivalence (resp. a pseudo-fibration).

2.9. The classical model structure on $\mathbf{S}$ is a Bousfield localisation of the model structure for quategories. Hence a weak categorical equivalence is a weak homotopy equivalence and the converse holds for a map between Kan complexes. A Kan fibration is a pseudo-fibration and the converse holds for a map between Kan complexes. A simplicial set A is weakly categorically equivalent to a Kan complex iff its fundamental category $\tau_1 A$ is a groupoid.

2.10. We say that a map of simplicial sets is *mid anodyne* if it belongs to the saturated class generated by the inclusions $\Lambda^k[n] \subset \Delta[n]$ with $0 < k < n$. Every mid anodyne map is a weak categorical equivalence, monic and biunivoque (ie bijective on vertices). We do not have an example of a monic biunivoque weak categorical equivalence which is not mid anodyne.

2.11. We shall say that a map of simplicial sets is a *mid fibration* if it has the right lifting propery with respect to the inclusion $\Lambda^k[n] \subset \Delta[n]$ for every $0 < k < n$. A simplicial set X is a quategory iff the map $X \to 1$ is a mid fibration. If X is a quategory and C is a category, then every map $X \to C$ is a mid fibration. In particular, every functor in **Cat** is a mid fibration. The category **S** admits a weak factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of mid anodyne maps and $\mathcal{B}$ is the class of mid fibrations.

2.12. Recall that a *reflexive graph* is a 1-truncated simplicial set. If G is a reflexive graph, then the canonical map $G \to \tau_1 G$ is mid anodyne. It is thus a weak categorical equivalence. Hence the category $\tau_1 G$ is a fibrant replacement of the graph G in the model category $(\mathbf{S}, \mathrm{Wcat})$.

2.13. A pseudo-fibration is a mid fibration. Conversely, a mid fibration between quategories $p : X \to Y$ is a pseudo-fibration iff the following equivalent conditions are satisfied:

- the functor $ho(p) : hoX \to hoY$ is an isofibration;
- for every object $x \in X$ and every isomorphism $g \in Y$ with target $p(x)$, there exists an isomorphism $f \in X$ with target x such that $p(f) = g$;
- p has the right lifting property with respect to the inclusion $\{1\} \subset J$

2.14. Let J be the groupoid generated by one isomorphism $0 \to 1$. Then a map between quategories $p : X \to Y$ is a pseudo-fibration iff the map

$$\langle j_0, p \rangle : X^J \to Y^J \times_Y X$$

obtained from the square

$$\begin{array}{ccc} X^J & \xrightarrow{X^{j_0}} & X \\ {\scriptstyle p^I}\downarrow & & \downarrow{\scriptstyle p} \\ Y^I & \xrightarrow{Y^{j_0}} & Y, \end{array}$$

is a trivial fibration, where j_0 denotes the inclusion $\{0\} \subset J$.

2.15. Consider the functor $k : \Delta \to \mathbf{S}$ defined by putting $k[n] = \Delta'[n]$ for every $n \geq 0$, where $\Delta'[n]$ denotes the (nerve of) the groupoid freely generated by the category $[n]$. If $X \in \mathbf{S}$, let us put

$$k^!(X)_n = \mathbf{S}(\Delta'[n], X).$$

The functor $k^! : \mathbf{S} \to \mathbf{S}$ has a left adjoint $k_!$. The pair of adjoint functors

$$k_! : (\mathbf{S}, \mathrm{Who}) \leftrightarrow (\mathbf{S}, \mathrm{Wcat}) : k^!$$

is a Quillen adjunction and a homotopy coreflection (this means that the left derived functor of $k_!$ is fully faithful). If X is a quategory, then the canonical map $k^!(X) \to X$ factors through the inclusion $J(X) \subseteq X$ and the induced map $k^!(X) \to J(X)$ is a trivial fibration.

2.16. Recall that a simplicial set is said to be *finite* if it has a finite number of non-degenerate simplices. A *presentation* of a quategory X by a simplicial set A is a weak categorical equivalence $A \to X$. The presentation is *finite* if A is finite. We shall say that a quategory X is *finitely presentable* if it admits a finite presentation. A Kan complex X is finitely presentable iff there exists a weak homotopy equivalence $A \to X$ with A a finite simplicial set iff X has finite homotopy type. The nerve of the monoid freely generated by one idempotent is not finitely presentable. The nerve of a finite group is finitely presentable iff it is the trivial group.

2.17. Recall that a *reflexive graph* is a simplicial set of dimension ≤ 1. If A is a reflexive graph, then the canonical map $A \to \tau_1 A$ is mid anodyne; it is thus a presentation of the quategory $\tau_1 A$.

2.18. Let *Split* be the category with two objects 0 and 1 and two arrows $s : 0 \to 1$ and $r : 1 \to 0$ such that $rs = id$. If K is the simplicial set defined by the pushout square

$$\begin{array}{ccc} \Delta[1] & \xrightarrow{d_1} & \Delta[2] \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & K \end{array}$$

then the obvious map $K \to Split$ is mid anodyne. Hence the category *Split* is finitely presentable as a quategory. Observe that *Split* contains the monoid freely generated by one idempotent as a full subcategory. Hence a full subcategory of a finitely presentable quategory is not necessarily finitely presentable.

3. Equivalence with simplicial categories

Simplicial categories were introduced by Dwyer and Kan in their work on simplicial localisation. The category of simplicial categories admits a Quillen model structure, called the Bergner-Dwyer-Kan model structure. The coherent nerve of a fibrant simplicial category is a quategory. The coherent nerve functor induces a Quillen equivalence between simplicial categories and quategories .

3.1. Recall that a *simplicial category* is a category enriched over simplicial sets and that a *simplicial functor* is a functor enriched over simplicial sets. We denote by **SCat** the category of small simplicial categories and simplicial functors. The category **SCat** of small simplicial categories and simplicial functors admits a Quillen model structure in which the weak equivalences are the Dwyer-Kan equivalences and the fibrations are the Dwyer-Kan fibrations [B1], see 51.5. The model structure is left proper and the fibrant objects are the categories enriched over Kan complexes. We say that it is the *Bergner model structure* or the *model structure for simplicial categories*. We shall denote it by $(\mathbf{SCat}, DK)$, where DK denotes the class of Dwyer-Kan equivalences.

3.2. Recall that a *reflexive graph* is a 1-truncated simplicial set. Let **Grph** be the category of reflexive graphs. The obvious forgetful functor $U : \mathbf{Cat} \to \mathbf{Grph}$ has a left adjoint F. The composite $C = FU$ is a comonad on **Cat**. It follows that for any small category A, the sequence of categories $C_n A = C^{n+1}(A)$ $(n \geq 0)$ has the structure of a simplicial object $C_*(A)$ in **Cat**. The simplicial set $n \mapsto Ob(C_n A)$

is constant with value $Ob(A)$. It follows that $C_*(A)$ can be viewed as a simplicial category instead of a simplicial object in **Cat**. This defines a functor

$$C_* : \mathbf{Cat} \to \mathbf{SCat}.$$

If A is a category then the augmentation $C_*(A) \to A$ is a cofibrant replacement of A in the model category **SCat**. If X is a simplicial category, then a simplicial functor $C_*(A) \to X$ is said to be a *homotopy coherent diagram* $A \to X$. This notion was introduced by Vogt in [V].

3.3. The simplicial category $C_\star[n]$ has the following description. The objects of $C_\star[n]$ are the elements of $[n]$. If $i, j \in [n]$ and $i > j$, then $C_\star[n](i,j) = \emptyset$; if $i \leq j$, then the simplicial set $C_\star[n](i,j)$ is (the nerve of) the poset of subsets $S \subseteq [i.j]$ such that $\{i,j\} \subseteq S$. If $i \leq j \leq k$, the composition operation

$$C_\star[n](j,k) \times C_\star[n](i,j) \to C_\star[n](i,k)$$

is the union $(T,S) \mapsto T \cup S$.

3.4. The *coherent nerve* of a simplicial category X is the simplicial set $C^!X$ obtained by putting

$$(C^!X)_n = \mathbf{SCat}(C_\star[n], X)$$

for every $n \geq 0$. This notion was introduced by Cordier in [C]. The simplicial set $C^!(X)$ is a quategory when X is enriched over Kan complexes [**?**]. The functor $C^! : \mathbf{SCat} \to \mathbf{S}$ has a left adjoint $C_!$ and we have $C_!A = C_\star A$ when A is a category [J4]. Thus, a homotopy coherent diagram $A \to X$ with values in a simplicial category X is the same thing as a map of simplicial sets $A \to C^!X$.

3.5. The pair of adjoint functors

$$C_! : \mathbf{S} \leftrightarrow \mathbf{SCat} : C^!$$

is a Quillen equivalence between the model category $(\mathbf{S}, \mathrm{Wcat})$ and the model category $(\mathbf{SCat}, DK)$ [Lu1][J4].

3.6. A simplicial category can be large. For example, the *quategory of Kan complexes* $\mathbf{U}$ is defined to be the coherent nerve of the simplicial category **Kan**. The quategory $\mathbf{U}$ is large but locally small. It plays an important role in the theory of quategories, where it is the analog of the category of sets. It is the archetype of a *homotopos*, also called an ∞-*topos*.

3.7. The category **QCat** becomes enriched over Kan complexes if we put

$$Hom(X,Y) = J(Y^X)$$

for $X, Y \in \mathbf{QCat}$. For example, the *quategory of small quategories* $\mathbf{U}_1$ is defined to be the coherent nerve of the simplicial category **QCat**. The quategory $\mathbf{U}_1$ is large but locally small. It plays an important role in the theory of quategories where it is the analog of the category of small categories.

4. Equivalence with Rezk categories

Rezk categories were introduced by Charles Rezk under the name of complete Segal spaces. They are the fibrant objects of a model structure on the category of simplicial spaces. The first row of a Rezk category is a quategory. The first row functor induces a Quillen equivalence between Rezk categories and quategories.

4.1. Recal that a *bisimplicial set* is defined to be a contravariant functor $\Delta \times \Delta \to$ **Set** and that a *simplicial space* to be a contravariant functor $\Delta \to \mathbf{S}$. We can regard a simplicial space X as a bisimplicial set by putting $X_{mn} = (X_m)_n$ for every $m, n \geq 0$. Conversely, we can regard a bisimplicial set X as a simplicial space by putting $X_m = X_{m\star}$ for every $m \geq 0$. We denote the category of bisimplicial sets by $\mathbf{S}^{(2)}$. The *box product* $A\Box B$ of two simplicial sets A and B is the bisimplicial set $A\Box B$ obtained by putting

$$(A\Box B)_{mn} = A_m \times B_n$$

for every $m, n \geq 0$. This defines a functor of two variables $\Box : \mathbf{S} \times \mathbf{S} \to \mathbf{S}^{(2)}$. The box product funtor $\Box : \mathbf{S} \times \mathbf{S} \to \mathbf{S}^{(2)}$ is divisible on each side. This means that the functor $A\Box(-) : \mathbf{S} \to \mathbf{S}^{(2)}$ admits a right adjoint $A\backslash(-) : \mathbf{S}^{(2)} \to \mathbf{S}$ for every simplicial set A, and that the functor $(-)\Box B : \mathbf{S} \to \mathbf{S}^{(2)}$ admits a right adjoint $(-)/B : \mathbf{S}^{(2)} \to \mathbf{S}$ for every simplicial set B. For any pair of simplicial spaces X and Y, let us put

$$Hom(X, Y) = (Y^X)_0$$

This defines an enrichment of the category $\mathbf{S}^{(2)}$ over the category $\mathbf{S}$. For any simplicial set A we have $A\backslash X = Hom(A\Box 1, X)$.

4.2. We recall that the category of simplicial spaces $[\Delta^o, \mathbf{S}]$ admits a Reedy model structure in which the weak equivalences are the term-wise weak homotopy equivalences and the cofibrations are the monomorphisms. The model structure is simplicial if we put $Hom(X, Y) = (Y^X)_0$. It is cartesian closed and proper.

4.3. Let $I[n] \subseteq \Delta[n]$ be the n-chain. For any simplicial space X we have a canonical bijection

$$I[n]\backslash X = X_1 \times_{\partial_0 = \partial_1} X_1 \times \cdots \times_{\partial_0 = \partial_1} X_1,$$

where the successive fiber products are calculated by using the face maps $\partial_0, \partial_1 : X_1 \to X_0$. We say that a simplicial space X satisfies the *Segal condition* if the map

$$\Delta[n]\backslash X \longrightarrow I[n]\backslash X$$

obtained from the inclusion $I[n] \subseteq \Delta[n]$ is a weak homotopy equivalence for every $n \geq 2$ (the condition is trivially satisfied if $n < 2$). A *Segal space* is a Reedy fibrant simplicial space which satisfies the Segal condition.

4.4. The Reedy model structure on the category $[\Delta^o, \mathbf{S}]$ admits a Bousfield localisation with respect to the set of maps $I[n]\Box 1 \to \Delta[n]\Box 1$ for $n \geq 0$. The fibrant objects of the local model structure are the Segal spaces. The local model structure is simplicial, cartesian closed and left proper. We say that it is the *model structure for Segal spaces.*

4.5. Let J be the groupoid generated by one isomorphism $0 \to 1$. We regard J as a simplicial set via the nerve functor. A Segal space X is said to be *complete*, if it satisfies the *Rezk condition*: the map

$$1\backslash X \longrightarrow J\backslash X$$

obtained from the map $J \to 1$ is a weak homotopy equivalence. We shall say that a complete Segal space is a *Rezk category.*

4.6. The model structure for Segal spaces admits a Bousfield localisation with respect to the map $J\Box 1 \to 1\Box 1$. The fibrant objects of the local model structure are the Rezk categories. The local model structure is simplicial, cartesian closed and left proper. We say that it is the *model structure for Rezk categories.*

4.7. The *first row* of a simplicial space X is the simplicial set $r(X)$ obtained by putting $rw(X)_n = X_{n0}$ for every $n \geq 0$. The functor $rw : \mathbf{S}^{(2)} \to \mathbf{S}$ has a left adjoint c obtained by putting $c(A) = A\Box 1$ for every simplicial set A. The pair of adjoint functors

$$c : \mathbf{S} \leftrightarrow \mathbf{S}^{(2)} : rw$$

is a Quillen equivalence between the model category for quategories and the model category for Rezk categories [JT2].

4.8. Consider the functor $t^! : \mathbf{S} \to \mathbf{S}^{(2)}$ defined by putting

$$t^!(X)_{mn} = \mathbf{S}(\Delta[m] \times \Delta'[n], X)$$

for every $X \in \mathbf{S}$ and every $m, n \geq 0$, where $\Delta'[n]$ denotes the (nerve of the) groupoid freely generated by the category $[n]$. The functor $t^!$ has a left adjoint $t_!$ and the pair

$$t_! : \mathbf{S}^{(2)} \leftrightarrow \mathbf{S} : t^!$$

is a Quillen equivalence between the model category for Rezk categories and the model category for quategories [JT2].

5. Equivalence with Segal categories

Segal categories and precategories were introduced by Hirschowitz and Simpson in their work on higher stacks. The category of precategories admits a model structure in which the fibrant objects are the Reedy fibrant Segal categories. The first row of a fibrant Segal category is a quategory. The first row functor induces a Quillen equivalence between Segal categories and quategories.

5.1. A simplicial space $X : \Delta^o \to \mathbf{S}$ is called a *precategory* if the simplicial set X_0 is discrete. We shall denote by $\mathbf{PCat}$ the full subcategory of $\mathbf{S}^{(2)}$ spanned by the precategories. The category $\mathbf{PCat}$ is a presheaf category and the inclusion functor $p^* : \mathbf{PCat} \subset \mathbf{S}^{(2)}$ has a left adjoint $p_!$ and a right adjoint p_*.

5.2. If X is a precategory and $n \geq 1$, then the vertex map $v_n : X_n \to X_0^{n+1}$ takes its values in a discrete simplicial set. We thus have a decomposition

$$X_n = \bigsqcup_{a \in X_0^{[n]_0}} X(a),$$

where $X(a) = X(a_0, \ldots, a_n)$ denotes the fiber of v_n at $a = (a_0, \cdots, a_n)$. A precategory X satisfies the Segal condition iff he canonical map

$$X(a_0, a_1, \ldots, a_n) \to X(a_0, a_1) \times \cdots \times X(a_{n-1}, a_n)$$

is a weak homotopy equivalence for every $a \in X_0^{[n]_0}$ and $n \geq 2$. A precategory which satisfies the Segal condition is called a *Segal category.*

5.3. If C is a small category, then the bisimplicial set $N(C) = C\Box 1$ is a Segal category. The functor $N : \mathbf{Cat} \to \mathbf{PCat}$ has a left adjoint

$$\tau_1 : \mathbf{PCat} \to \mathbf{Cat}$$

which associates to a precategory X its *fundamental category* $\tau_1 X$. A map of precategories $f : X \to Y$ is said to be *essentially surjective* if the functor $\tau_1(f) : \tau_1 X \to \tau_1 Y$ is essentially surjective. A map of precategories $f : X \to Y$ is said to be *fully faithful* if the map

$$X(a, b) \to Y(fa, fb)$$

is a weak homotopy equivalence for every pair $a, b \in X_0$. We say that $f : X \to Y$ is a *categorical equivalence* if it is fully faithful and essentially surjective.

5.4. In [HS], Hirschowitz and Simpson construct a completion functor

$$S : \mathbf{PCat} \to \mathbf{PCat}$$

which associates to a precategory X a Segal category $S(X)$ "generated" by X. A map of precategories $f : X \to Y$ is called a *weak categorical equivalence* if the map $S(f) : S(X) \to S(Y)$ is a categorical equivalence. The category $\mathbf{PCat}$ admits a left proper model structure in which a a weak equivalence is a weak categorical equivalence and a cofibration is a monomorphism. It is the *Hirschowitz-Simpson model structure* or the *model structure for Segal categories*. The model structure is cartesian closed [P].

5.5. We recall that the category of simplicial spaces $[\Delta^o, \mathbf{S}]$ admits a Reedy model structure in which the weak equivalences are the term-wise weak homotopy equivalences and the cofibrations are the monomorphisms. A precategory is fibrant in the Hirschowitz-Simpson model structure iff it is a Reedy fibrant Segal category [B3].

5.6. The *first row* of a precategory X is the simplicial set $r(X)$ obtained by putting $r(X)_n = X_{n0}$ for every $n \geq 0$. The functor $r : \mathbf{PCat} \to \mathbf{S}$ has a left adjoint h obtained by putting $h(A) = A\Box 1$ for every simplicial set A. It was conjectured in [T1] (and proved in [JT2]) that the pair of adjoint functors

$$h : \mathbf{S} \leftrightarrow \mathbf{PCat} : r$$

is a Quillen equivalence between the model category for quategories and the model category for Segal categories.

5.7. The *diagonal* $d^*(X)$ of a precategory X is defined to be the diagonal of the bisimplicial set X. The functor $d^* : \mathbf{PCat} \to \mathbf{S}$ admits a right adjoint d_* and the pair of adjoint functors

$$d^* : \mathbf{PCat} \leftrightarrow \mathbf{S} : d_*$$

is a Quillen equivalence between the model category for Segal categories and the model category for quategories [JT2].

6. Minimal quategories

The theory of minimal Kan complexes can be extended to quategories. Every quategory has a minimal model which is unique up to isomorphism. A category is minimal iff it is skeletal.

6.1. Recall that a sub-Kan complex S of a Kan complex X is said to be a *(sub)model* of X if the inclusion $S \subseteq X$ is a homotopy equivalence. Recall that a Kan complex is said to be *minimal* if it has no proper (sub)model. Every Kan complex has a minimal model and that any two minimal models are isomorphic. Two Kan complexes are homotopy equivalent iff their minimal models are isomorphic.

6.2. We shall say that a subcategory S of a category C is a *model* of C if the inclusion $S \subseteq C$ is an equivalence. We say that a category C is *skeletal* iff it has no proper model.

6.3. A subcategory S of a category C is a model of C iff it is full and

$$\forall a \in ObC \quad \exists b \in ObS \quad a \simeq b,$$

where $a \simeq b$ means that a and b are isomorphic objects. A category C is skeletal iff

$$\forall a, b \in ObC \quad a \simeq b \quad \Rightarrow \quad a = b$$

6.4. Let $f : C \to D$ be an equivalence of categories. If C is skeletal, then f is monic on objects and morphisms. If D is skeletal, then f is surjective on objects and morphisms. If C and D are skeletal, then f is an isomorphism.

6.5. Every category has a skeletal model and any two skeletal models are isomorphic. Two categories are equivalent iff their skeletal models are isomorphic.

6.6. (Definition) If X is a quategory, we shall say that a sub-quategory $S \subseteq X$ is a *(sub)model* of X if the inclusion $S \subseteq X$ is an equivalence. We say that a quategory is *minimal* or *skeletal* if it has no proper (sub)model.

6.7. (Lemma) Let $S \subseteq X$ be model of a quategory X. Then the inclusion $u : S \subseteq X$ admits a retraction $r : X \to S$ and there exists an isomorphism $\alpha : ur \simeq 1_X$ such that $\alpha \circ u = 1_u$.

6.8. (Notationj) If X be a simplicial set and $n \geq 0$, consider the projection

$$\partial : X^{\Delta[n]} \to X^{\partial\Delta[n]}$$

defined by the inclusion $\partial\Delta[n] \subset \Delta[n]$. Its fiber at a vertex $x \in X^{\partial\Delta[n]}$ is a simplicial set $X\langle x\rangle$. If $n = 1$ we have $x = (a, b) \in X_0 \times X_0$ and $X\langle x\rangle = X(a, b)$. The simplicial set $X\langle x\rangle$ is a Kan complex when X is a quategory and $n > 0$. If $n > 0$, we say that two simplices $a, b : \Delta[n] \to X$ are *homotopic with fixed boundary*, and we write $a \simeq b$, if we have $\partial a = \partial b$ and a and b are homotopic in the simplicial set $X(\partial a) = X(\partial b)$. If $a, b \in X_0$, we shall write $a \simeq b$ to indicate that the vertices a and b are isomorphic.

6.9. (Proposition) If S is a simplicial subset of a simplicial set X, then for every simplex $x \in X_n$ we shall write $\partial x \in S$ to indicate that the map $\partial x : \partial\Delta[n] \to X$ factors through the inclusion $S \subseteq X$. If X is a quat, then the simplicial subset S is a model of X iff

$$\forall n \geq 0 \quad \forall a \in X_n \quad \big(\partial a \in S \quad \Rightarrow \quad \exists b \in S \quad a \simeq b\big).$$

A quategory X is a minimal iff

$$\forall n \geq 0 \quad \forall a, b \in X_n \quad \big(a \simeq b \quad \Rightarrow \quad a = b\big).$$

6.10. Let $f : X \to Y$ be an equivalence of quategories. If X is minimal, then f is monic. If Y is minimal, then f is a trivial fibration. If X and Y are minimal, then f is an isomorphism.

6.11. Every quat has a minimal model and any two minimal models are isomorphic. Two quategories are equivalent iff their minimal models are isomorphic.

7. Discrete fibrations and covering maps

We introduce a notion of discrete fibration between simplicial sets. It extends the notion of covering space map and the notion of discrete fibration between categories. The results of this section are taken from [J2].

7.1. Recall that a functor $p : E \to C$ between small categories is said to be a *discrete fibration*, but we shall say a *discrete right fibration*, if for every object $x \in E$ and every arrow $g \in C$ with target $p(x)$, there exists a unique arrow $f \in E$ with target x such that $p(f) = g$. For example, if $el(F)$ denotes the category of elements of a presheaf $F \in \hat{C}$, then the natural projection $el(F) \to C$ is a discrete right fibration. The functor $F \mapsto el(F)$ induces an equivalence between the category of presheaves $\hat{C}$ and the full subcategory of $\mathbf{Cat}/C$ spanned by the discrete right fibrations $E \to C$. Recall that a functor $u : A \to B$ is said to be *final*, but we shall say *0-final*, if the category $b\backslash A$ defined by the pullback square

$$\begin{array}{ccc} b\backslash A & \longrightarrow & A \\ \downarrow & & \downarrow u \\ b\backslash B & \longrightarrow & B \end{array}$$

is connected for every object $b \in B$. The category $\mathbf{Cat}$ admits a factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of 0-final functors and $\mathcal{B}$ is the class of discrete right fibrations.

7.2. A functor $p : E \to C$ is a discrete right fibration iff it is right orthogonal to the inclusion $\{n\} \subseteq \Delta[n]$ for every $n \geq 0$. We shall say that a map of simplicial sets a *discrete right fibration* if it is right orthogonal to the inclusion $\{n\} \subseteq \Delta[n]$ for every $n \geq 0$. We shall say that a map of simplicial sets $u : A \to B$ is *0-final* if the functor $\tau_1(u) : \tau_1 A \to \tau_1 B$ is 0-final. The category $\mathbf{S}$ admits a factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of 0-final maps and $\mathcal{B}$ is the class of discrete right fibrations.

7.3. For any simplicial set B, the functor $\tau_1 : \mathbf{S} \to \mathbf{Cat}$ induces an equivalence between the full subcategory of $\mathbf{S}/B$ spanned by the discrete right fibrations with target B and the full subcategory of $\mathbf{Cat}/B$ spanned by the discrete right fibrations with target $\tau_1 B$. The inverse equivalence associates to a discrete right fibration with target $\tau_1 B$ its base change along the canonical map $B \to \tau_1 B$.

7.4. Dually, a functor $p : E \to C$ is said to be a *discrete opfibration*, but we shall say a *discrete left fibration*, if for every object $x \in E$ and every arrow $g \in C$ with source $p(x)$, there exists a unique arrow $f \in E$ with source x such that $p(f) = g$. A functor $p : E \to C$ is a discrete left fibration iff the opposite functor $p^o : E^o \to B^o$

is a discrete right fibration. Recall that a functor $u : A \to B$ is said to be *initial*, but we shall say *0-initial*, if the category A/b defined by the pullback square

$$\begin{array}{ccc} A/b & \longrightarrow & A \\ \downarrow & & \downarrow u \\ B/b & \longrightarrow & B \end{array}$$

is connected for every object $b \in B$. The category **Cat** admits a factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of 0-initial functors and $\mathcal{B}$ is the class of discrete left fibrations.

7.5. A functor $p : E \to C$ is a discrete left fibration iff it is right orthogonal to the inclusion $\{0\} \subseteq \Delta[n]$ for every $n \geq 0$. We say that a map of simplicial sets is a *discrete left fibration* if it is right orthogonal to the inclusion $\{0\} \subseteq \Delta[n]$ for every $n \geq 0$. We say that a map of simplicial sets $u : A \to B$ is *0-initial* if the functor $\tau_1(u) : \tau_1 A \to \tau_1 B$ is 0-initial. The category **S** admits a factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of 0-initial maps and $\mathcal{B}$ is the class of discrete left fibrations.

7.6. For any simplicial set B, the functor $\tau_1 : \mathbf{S} \to \mathbf{Cat}$ induces an equivalence between the full subcategory of $\mathbf{S}/B$ spanned by the discrete left fibrations with target B and the full subcategory of $\mathbf{Cat}/B$ spanned by the discrete left fibrations with target $\tau_1 B$. The inverse equivalence associates to a discrete left fibration with target $\tau_1 B$ its base change along the canonical map $B \to \tau_1 B$.

7.7. We say that functor $p : E \to C$ is a *0-covering* if it is both a discrete fibration and a discrete opfibration. For example, if F is a presheaf on C, then the natural projection $el(F) \to C$ is a 0-covering iff the functor F takes every arrow in C to a bijection. If $c : C \to \pi_1 C$ is the canonical functor, then the functor $F \mapsto el(Fc)$ induces an equivalence between the category of presheaves on $\pi_1 C$ and the full subcategory of $\mathbf{Cat}/C$ spanned by the 0-coverings $E \to C$. We say that a functor $u : A \to B$ is *0-connected* if the functor $\pi_1(u) : \pi_1 A \to \pi_1 B$ is essentially surjective and full. The category **Cat** admits a factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of 0-connected functors and $\mathcal{B}$ is the class of 0-coverings.

7.8. We say that a map of simplicial sets $E \to B$ is a *0-covering* if it is a discrete left fibration and a discrete right fibration. A map is a 0-covering if it is right orthogonal to every map $\Delta[m] \to \Delta[n]$ in Δ. Recall that a map of simplicial sets is said to be *0-connected* if its homotopy fibers are connected. A map $u : A \to B$ is 0-connected iff the functor $\pi_1(u) : \pi_1 A \to \pi_1 B$ is 0-connected. The category **S** admits a factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of 0-connected maps and $\mathcal{B}$ is the class of 0-coverings.

7.9. If B is a simplicial set, then the functor $\pi_1 : \mathbf{S} \to \mathbf{Gpd}$ induces an equivalence between the category of 0-coverings of B and the category of 0-coverings of $\pi_1 B$. The inverse equivalence associates to a 0-covering with target $\pi_1 B$ its base change along the canonical map $B \to \pi_1 B$.

8. Left and right fibrations

We introduce the notions of left fibration and of right fibration. We also introduce the notions of initial map and of final map. The right fibrations with a fixed codomain B are the *prestacks* over B. The results of the section are taken from [J2].

8.1. Recall [GZ] that a map of simplicial sets is said to be a *Kan fibration* if it has the right lifting property with respect to every horn inclusion $h_n^k : \Lambda^k[n] \subset \Delta[n]$ ($n > 0$ and $k \in [n]$). Recall that a map of simplicial sets is said to be *anodyne* if it belongs to the saturated class generated by the inclusions h_n^k. A map is anodyne iff it is an acyclic cofibration in the model category $(\mathbf{S}, \mathrm{Who})$. Hence the category $\mathbf{S}$ admits a weak factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of anodyne maps and $\mathcal{B}$ is the class of Kan fibrations.

8.2. We say that a map of simplicial sets is a *right fibration* if it has the right lifting property with respect to the horn inclusions $h_n^k : \Lambda^k[n] \subset \Delta[n]$ with $0 < k \leq n$. Dually, we say that a map is a *left fibration* if it has the right lifting property with respect to the inclusions h_n^k with $0 \leq k < n$. A map $p : X \to Y$ is a left fibration iff the opposite map $p^o : X^o \to Y^o$ is a right fibration. A map is a Kan fibration iff it is both a left and a right fibration.

8.3. Our terminology is consistent with 7.2: every discrete right (resp. left) fibration is a right (resp. left) fibration.

8.4. The fibers of a right (resp. left) fibration are Kan complexes. Every right (resp. left) fibration is a pseudo-fibration.

8.5. A functor $p : E \to B$ is a right fibration iff it is 1-fibration.

8.6. A map of simplicial sets $f : X \to Y$ is a right fibration iff the map

$$\langle i_1, f \rangle : X^I \to Y^I \times_Y X$$

obtained from the square

$$\begin{array}{ccc} X^I & \xrightarrow{X^{i_1}} & X \\ {\scriptstyle f^I}\downarrow & & \downarrow{\scriptstyle f} \\ Y^I & \xrightarrow{Y^{i_1}} & Y \end{array}$$

is a trivial fibration, where i_1 denotes the inclusion $\{1\} \subset I$. Dually, a map $f : X \to Y$ is a left fibration iff the map $\langle i_0, f \rangle$ is a trivial fibration, where i_0 denotes the inclusion $\{0\} \subset I$.

8.7. A right fibration is discrete iff it is right orthogonal the inclusion $h_n^k : \Lambda^k[n] \subset \Delta[n]$ for every $0 < k \leq n$. A functor $A \to B$ in **Cat** is a right fibration iff it is a Grothendieck fibration whose fibers are groupoids.

8.8. We say that a map of simplicial sets is *right anodyne* if it belongs to the saturated class generated by the inclusions $h_n^k : \Lambda^k[n] \subset \Delta[n]$ with $0 < k \leq n$. Dually, we say that a map is *left anodyne* if it belongs to the saturated class generated by the inclusions h_n^k with $0 \leq k < n$. A map of simplicial sets $u : A \to B$ is left anodyne iff the opposite map $u^o : A^o \to B^o$ is right anodyne. The category $\mathbf{S}$ admits a weak factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of right (resp. left) anodyne maps and $\mathcal{B}$ is the class of right (resp. left) fibrations.

8.9. If the composite of two monomorphisms $u : A \to B$ and $v : B \to C$. is left (resp. right) anodyne and u is left (resp. right) anodyne, then v is left (resp. right) anodyne.

8.10. Let $\mathcal{E}$ be a category equipped with a class $\mathcal{W}$ of "weak equivalences" satisfying "three-for-two". We say that a class of maps $\mathcal{M} \subseteq \mathcal{E}$ is *invariant under weak equivalences* if for every commutative square

$$\begin{array}{ccc} A & \longrightarrow & A' \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u'} \\ B & \longrightarrow & B' \end{array}$$

in which the horizontal maps are weak equivalences, $u \in \mathcal{M} \Leftrightarrow u' \in \mathcal{M}$.

8.11. We say that a map of simplicial sets $u : A \to B$ is *final* if it admits a factorisation $u = wi : A \to B' \to B$ with i a right anodyne map and w a weak categorical equivalence. The class of final maps is invariant under weak categorical equivalences. A monomorphism is final iff it is right anodyne. The base change of a final map along a left fibration is final. A map $u : A \to B$ is final iff the simplicial set $L \times_B A$ is weakly contractible for every left fibration $L \to B$. For each vertex $b \in B$, let us choose a factorisation $1 \to Lb \to B$ of the map $b : 1 \to B$ as a left anodyne map $1 \to Lb$ followed by a left fibration $Lb \to B$. Then a map $u : A \to B$ is final iff the simplicial set $Lb \times_B A$ is weakly contractible for every vertex $b : 1 \to B$. When B is a quategory, we can take $Lb = b\backslash B$ (see **??**) and a map $u : A \to B$ is final iff the simplicial set $b\backslash A$ defined by the pullback square

$$\begin{array}{ccc} b\backslash A & \longrightarrow & A \\ \downarrow & & \downarrow{\scriptstyle u} \\ b\backslash B & \longrightarrow & B \end{array}$$

is weakly contractible for every object $b \in B$.

8.12. Dually, we say that a map of simplicial sets $u : A \to B$ is *initial* if the opposite map $u^o : A^o \to B^o$ is final. A map $u : A \to B$ is initial iff it admits a factorisation $u = wi : A \to B' \to B$ with i a left anodyne map and w a weak categorical equivalence. The class of initial maps is invariant under weak categorical equivalences. A monomorphism is initial iff it is left anodyne. The base change of an initial map along a right fibration is initial. A map $u : A \to B$ is initial iff the simplicial set $R \times_B A$ is weakly contractible for every right fibration $R \to B$. For each vertex $b \in B$, let us choose a factorisation $1 \to Rb \to B$ of the map $b : 1 \to B$ as a right anodyne map $1 \to Rb$ followed by a right fibration $Rb \to B$. Then a map $u : A \to B$ is initial iff the simplicial set $Rb \times_B A$ is weakly contractible for every vertex $b : 1 \to B$. When B is a quategory, we can take $Rb = B/b$ (see 9.7) and a map $u : A \to B$ is initial iff the simplicial set $b\backslash A$ defined by the pullback square

$$\begin{array}{ccc} A/b & \longrightarrow & A \\ \downarrow & & \downarrow u \\ B/b & \longrightarrow & B \end{array}$$

is weakly contractible for every object $b \in B$.

8.13. The base change of a weak categorical equivalence along a left or a right fibration is a weak categorical equivalence.

8.14. If $f : X \to Y$ is a right fibration, then so is the map

$$\langle u, f\rangle : X^B \to Y^B \times_{Y^A} X^A$$

obtained from the square

$$\begin{array}{ccc} X^B & \longrightarrow & X^A \\ \downarrow & & \downarrow \\ Y^B & \longrightarrow & Y^A, \end{array}$$

for any monomorphism of simplicial sets $u : A \to B$. Moreover, the map $\langle u, f\rangle$ is a trivial fibration if u is right anodyne. There are dual results for left fibrations and left anodyne maps.

8.15. To every left fibration $X \to B$ we can associate a functor

$$D(X) : \tau_1 B \to Ho(\mathbf{S}, \mathrm{Who})$$

called the *homotopy diagram* of X. To see, we first observe that the category $\mathbf{S}/B$ is enriched over $\mathbf{S}$; let us denote by $[X, Y]$ the simplicial set of maps $X \to Y$ between two objects of $\mathbf{S}/B$. The simplicial set $[X, Y]$ is a Kan complex when the structure map $Y \to B$ is a left or a right fibration. For every vertex $b \in B_0$, the map $b : 1 \to B$ is an object of $\mathbf{S}/B$ and the simplicial set $[b, X]$ is the fiber $X(b)$ of X at b. Let us put $D(X)(b) = [b, X]$. let us see that this defines a functor

$$D(X) : \tau_1 B \to Ho(\mathbf{S}, \mathrm{Who})$$

called the *homotopy diagram* of X. If $f : a \to b$ is an arrow in B, then the map $f : I \to B$ is an object of $\mathbf{S}/B$. From the inclusion $i_0 : \{0\} \to I$ we obtain a map $i_0 : a \to f$ and the inclusion $i_1 : \{1\} \to I$ a map $i_1 : b \to f$. We thus have a diagram of simplicial sets

$$[a, X] \xleftarrow{p_0} [f, X] \xrightarrow{p_1} [b, X],$$

where $p_0 = [i_0, X]$ and $p_1 = [i_1, X]$. The map p_0 is a trivial fibration by 8.14, since the structure map $X \to B$ is a left fibration and i_0 is left anodyne. It thus admits a section s_0. By composing p_1 with s_0 we obtain a map

$$f_! : X(a) \to X(b)$$

well defined up to homotopy. The homotopy class of f only depends on the homotopy class of f. Moreover, if $g : b \to c$, then the map $g_! f_!$ is homotopic to the map $(gf)_!$. This defines the functor $D(X)$ if we put $D(X)(a) = X(a) = [a, X]$ and $D(X)(f) = f_!$. Dually, to a right fibration $X \to B$ we associate a functor

$$D(X) : \tau_1 B^o \to Ho(\mathbf{S}, \text{Who})$$

called the *(contravariant) homotopy diagram* of X. If $f : a \to b$ is an arrow in B, then the inclusion $i_1 : a \to f$ is right anodyne. It follows that the map p_1 in the diagram

$$[a, X] \xleftarrow{p_0} [f, X] \xrightarrow{p_1} [b, X],$$

is a trivial fibration. It thus admits a section s_1. By composing p_0 with s_1 we obtain a map

$$f^* : X(b) \to X(a)$$

well defined up to homotopy. This defines the functor $D(X)$ if we put $D(X)(a) = X(a) = [a, X]$ and $D(X)(f) = f^*$.

9. Join and slice

For any object b of a category C there is a category C/b of objects of C *over* b. Similarly, for any vertex b of a simplicial set X there is a simplicial set X/b. More generally, we construct a simplicial set X/b for any map of simplicial sets $b : B \to X$. The construction uses the join of simplicial sets. The results of this section are taken from [J1] and [J2].

9.1. Recall that the *join* of two categories A and B is the category $C = A \star B$ obtained as follows: $Ob(C) = Ob(A) \sqcup Ob(B)$ and for any pair of objects $x, y \in Ob(A) \sqcup Ob(B)$ we have

$$C(x,y) = \begin{cases} A(x,y) & \text{if } x \in A \text{ and } y \in A \\ B(x,y) & \text{if } x \in B \text{ and } y \in B \\ 1 & \text{if } x \in A \text{ and } y \in B \\ \emptyset & \text{if } x \in B \text{ and } y \in A. \end{cases}$$

Composition of arrows is obvious. Notice that the category $A \star B$ is a poset if A and B are posets: it is the *ordinal sum* of the posets A and B. The operation $(A, B) \mapsto A \star B$ is functorial and coherently associative. It defines a monoidal structure on **Cat**, with the empty category as the unit object. The monoidal category $(\mathbf{Cat}, \star)$ is not symmetric but there is a natural isomorphism

$$(A \star B)^o = B^o \star A^o.$$

The category $1 \star A$ is called the *projective cone with base* A and the category $A \star 1$ the *inductive cone with cobase* A. The object 1 is terminal in $A \star 1$ and initial in $1 \star A$. The category $A \star B$ is equipped with a natural augmentation $A \star B \to I$ obtained by joining the functors $A \to 1$ and $B \to 1$. The resulting functor $\star : \mathbf{Cat} \times \mathbf{Cat} \to \mathbf{Cat}/I$ is right adjoint to the functor

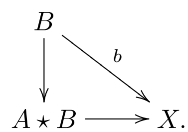
$$i^* : \mathbf{Cat}/I \to \mathbf{Cat} \times \mathbf{Cat},$$

where i denotes the inclusion $\{0, 1\} = \partial I \subset I$.

9.2. The monoidal category $(\mathbf{Cat}, \star)$ is not closed. But for every category $B \in \mathbf{Cat}$, the functor

$$(-) \star B : \mathbf{Cat} \to B\backslash\mathbf{Cat}$$

which associates to a category A the inclusion $A \subseteq A \star B$ has a right adjoint which takes a functor $b : B \to X$ to a category that we shall denote by X/b. We shall say that X/b is the *lower slice* of X by b. For any category A, there is a bijection between the functors $A \to X/b$ and the functors $A \star B \to X$ which extend b along the inclusion $B \subseteq A \star B$,



In particular, an object of X/b is a functor $c : 1 \star B \to X$ which extends b; it is a *projective cone with base* b. Dually, the functor $A \star (-) : \mathbf{Cat} \to A\backslash\mathbf{Cat}$ which associates to a category B the inclusion $B \subseteq A \star B$ has a right adjoint which takes a functor $a : A \to X$ to a category that we shall denote $a\backslash X$. We shall say that $a\backslash X$ is the *upper slice* of X by a. An object of $a\backslash X$ is a functor $c : A \star 1 \to C$ which extends a; it is an *inductive cone with cobase* a.

9.3. We shall denote by Δ_+ the category of all finite ordinals and order preserving maps, *including* the empty ordinal 0. We shall denote the ordinal n by n, so that we have $n = [n-1]$ for $n \geq 1$. We may occasionally denote the ordinal 0 by $[-1]$. Notice the isomorphism of categories $1 \star \Delta = \Delta_+$. The ordinal sum $(m, n) \mapsto m+n$ is functorial with respect to order preserving maps. This defines a monoidal structure on Δ_+,

$$+ : \Delta_+ \times \Delta_+ \to \Delta_+,$$

with 0 as the unit object.

9.4. Recall that an *augmented simplicial set* is defined to be a contravariant functor $\Delta_+ \to \mathbf{Set}$. We shall denote by $\mathbf{S}_+$ the category of augmented simplicial sets. By a general procedure due to Brian Day [Da], the monoidal structure of Δ_+ can be extended to $\mathbf{S}_+$ as a closed monoidal structure

$$\star : \mathbf{S}_+ \times \mathbf{S}_+ \to \mathbf{S}_+$$

with $0 = y(0)$ as the unit object. We call $X \star Y$ the *join* of the augmented simplicial sets X and Y. We have

$$(X \star Y)(n) = \bigsqcup_{i+j=n} X(i) \times Y(j)$$

for every $n \geq 0$.

9.5. From the inclusion $t : \Delta \subset \Delta_+$ we obtain a pair of adjoint functors

$$t^* : \mathbf{S}_+ \leftrightarrow \mathbf{S} : t_*.$$

The functor t^* removes the augmentation of an augmented simplicial set. The functor t_* gives a simplicial set A the trivial augmentation $A_0 \to 1$. Notice that $t_*(\emptyset) = 0 = y(0)$, where y is the Yoneda map $\Delta_+ \to \mathbf{S}_+$. The functor t_* is fully faithful and we shall regard it as an inclusion $t_* : \mathbf{S} \subset \mathbf{S}_+$. The operation $\star$ on $\mathbf{S}_+$ induces a monoidal structure on $\mathbf{S}$,

$$\star : \mathbf{S} \times \mathbf{S} \to \mathbf{S}.$$

By definition, $t_*(A \star B) = t_*(A) \star t_*(B)$ for any pair $A, B \in \mathbf{S}$. We call $A \star B$ the *join* of the simplicial sets A and B. It follows from the formula above, that we have

$$(A \star B)_n = A_n \sqcup B_n \sqcup \bigsqcup_{i+1+j=n} A_i \times A_j.$$

for every $n \geq 0$. Notice that we have

$$A \star \emptyset = A = \emptyset \star A$$

for any simplicial set A, since $t_*(\emptyset) = 0$ is the unit object for the operation $\star$ on $\mathbf{S}_+$. Hence the empty simplicial set is the unit object for the join operation on $\mathbf{S}$. The monoidal category $(\mathbf{S}, \star)$ is not symmetric but there is a natural isomorphism

$$(A \star B)^o = B^o \star A^o.$$

For every pair $m, n \geq 0$ we have

$$\Delta[m] \star \Delta[n] = \Delta[m+1+n]$$

since we have $[m] + [n] = [m+n+1]$. In particular,

$$1 \star 1 = \Delta[0] \star \Delta[0] = \Delta[1] = I.$$

The simplicial set $1 \star A$ is the *projective cone with base* A A and the simplicial set $A \star 1$ the *inductive cone with cobase* A.

9.6. If i denotes the inclusion $\{0,1\} = \partial I \subset I$, then the functor $i^* : \mathbf{S}/I \to \mathbf{S}/\partial I = \mathbf{S} \times \mathbf{S}$ has a right adjoint i_* which associates to a pair of simplicial sets (A, B) the simplicial set $A \star B$ equipped with the map $A \star B \to I$ obtained by joining the maps $A \to 1$ and $B \to 1$. It follows that we have

$$A \star B = (A \star 1) \times_I (1 \star B)$$

since we have $(A, B) = (A, 1) \times (1, B)$ in $\mathbf{S} \times \mathbf{S}$.

9.7. The monoidal category $(\mathbf{S}, \star)$ is not closed. But for any simplicial set B, the functor

$$(-) \star B : \mathbf{S} \to B \backslash \mathbf{S}$$

which associates to a simplicial set A the inclusion $B \subseteq A \star B$ has a right adjoint which takes a map of simplicial set $b : B \to X$ to a simplicial set X/b called the *lower slice* of X by b. For any simplicial set A, there is a bijection between the maps $A \to X/b$ and the maps $A \star B \to X$ which extend b along the inclusion $B \subseteq A \star B$,

$$\begin{array}{ccc} B & & \\ \downarrow & \searrow^{b} & \\ A \star B & \longrightarrow & X. \end{array}$$

In particular, a vertex $1 \to X/b$ is a map $c : 1 \star B \to X$ which extends b; it is a *projective cone with base* b in X. The simplicial set X/b is a quategory when X is a quategory. If $B = 1$ and $b \in X_0$, then a simplex $\Delta[n] \to X/b$ is a map $x : \Delta[n+1] \to X$ such that $x(n+1) = b$. Dually, for any simplicial set A, the functor $A \star (-) : \mathbf{S} \to A \backslash \mathbf{S}$ has a right adjoint which takes a map $a : A \to X$ to a simplicial set $a \backslash X$ called the *upper slice* of X by a. A vertex $1 \to a \backslash X$ is a map $c : A \star 1 \to X$ which extends a; it is an *inductive cone with cobase* a in X. The simplicial set $a \backslash X$ is a quategory when X is a quategory. If $A = 1$ and $a \in X_0$, then a simplex $\Delta[n] \to a \backslash X$ is a map $x : \Delta[n+1] \to X$ such that $x(0) = a$.

9.8. If A and B are simplicial sets, consider the functor

$$A \star (-) \star B : \mathbf{S} \to (A \star B)\backslash \mathbf{S}$$

which associates to X the inclusion $A \star B \subseteq A \star X \star B$ obtained by joining the maps $1_A : A \to A$, $\emptyset \to X$ and $1_B : B \to B$. The functor $A \star (-) \star B$ has a right adjoint which takes a map $f : A \star B \to Y$ to a simplicial set that we shall denote $Fact(f, Y)$. By construction, a vertex $1 \to Fact(f, Y)$ is a map $g : A \star 1 \star B \to Y$ which extends f. When $A = B = 1$, a vertex $1 \to Fact(f, Y)$ it is a *factorisation* of the arrow $f : I \to X$. If f is an arrow $a \to b$ then $Fact(f, X) = f\backslash(X/b) = (a\backslash X)/f$.

9.9. Recall that a model structure on a category $\mathcal{E}$ induces a model structure on the slice category $\mathcal{E}/B$ for each object $B \in \mathcal{E}$. In particular, we have a model category $(B\backslash\mathbf{S}, \mathrm{Wcat})$ for each simplicial set B. The pair of adjoint functors $X \mapsto X \star B$ and $(X, b) \mapsto X/b$ is a Quillen pair between the model categories $(\mathbf{S}, \mathrm{Wcat})$ and $(B\backslash\mathbf{S}, \mathrm{Wcat})$.

9.10. If $u : A \to B$ and $v : S \to T$ are two maps in $\mathbf{S}$, we shall denote by $u \star' v$ the map

$$(A \star T) \sqcup_{A \star S} (B \star S) \to B \star T$$

obtained from the commutative square

$$\begin{array}{ccc} A \star S & \xrightarrow{u \star S} & B \star S \\ {\scriptstyle A\star v}\downarrow & & \downarrow{\scriptstyle B \star v} \\ A \star T & \xrightarrow{u \star T} & B \star T. \end{array}$$

If u is an inclusion $A \subseteq B$ and v an inclusion $S \subseteq T$, then the map $u \star' v$ is the inclusion

$$(A \star T) \cup (B \star S) \subseteq B \star T.$$

If $u : A \to B$ and $v : S \to T$ are monomorphisms of simplicial sets, then

- $u \star' v$ is mid anodyne if u is right anodyne or v left anodyne;
- $u \star' v$ is left anodyne if u is anodyne;
- $u \star' v$ is right anodyne if v is anodyne.

9.11. [J1] [J2] (Lemma) Suppose that we have a commutative square

$$\begin{array}{ccc} (\{0\} \star T) \cup (I \star S) & \xrightarrow{u} & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ I \star T & \xrightarrow{v} & Y, \end{array}$$

where p is a mid fibration between quategories. If the arrow $u(I) \in X$ is invertible, then the square has a diagonal filler.

9.12. [J1] [J2] Suppose that we have a commutative square

$$\begin{array}{ccc} \Lambda^0[n] & \xrightarrow{x} & X \\ \downarrow & & \downarrow p \\ \Delta[n] & \longrightarrow & Y, \end{array}$$

in which p is a mid fibration between quategories. If $n > 1$ and the arrow $x(0,1) \in X$ is invertible, then the square has a diagonal filler. This follows from the lemma above if we use the decompositions $\Delta[n] = I \star \Delta[n-2]$ and $\Lambda^0[n] = (\{0\} \star \Delta[n-2]) \cup (I \star \partial\Delta[n-2])$.

9.13. [J1] [J2] A quategory X is a Kan complex iff the category hoX is a groupoid. This follows from the result above.

9.14. The simplicial set X/b depends functorially on the map $b : B \to X$. More precisely, to every commutative diagram

$$\begin{array}{ccc} B & \xleftarrow{u} & A \\ b \downarrow & & \downarrow a \\ X & \xrightarrow{f} & Y \end{array}$$

we can associate a map

$$f/u : X/b \to Y/a.$$

By definition. if $x : \Delta[n] \to X/b$, then the simplex $(f/u)(x) : \Delta[n] \to Y/a$ is obtained by composing the maps

$$\Delta[n] \star A \xrightarrow{\Delta[n]\star u} \Delta[n] \star B \xrightarrow{x} X \xrightarrow{f} Y.$$

9.15. A map $u : (M,p) \to (N,q)$ in the category $\mathbf{S}/B$ is a contravariant equivalence iff the map $1_X/u : dq\backslash X \to dp\backslash X$ is an equivalence of quategories for any map $d : B \to X$ with values in a quategory X. In particular, a map $u : A \to B$ is final iff the map $1_X/u : d\backslash X \to du\backslash X$ is an equivalence of quategories for any map $d : B \to X$ with values in a quategory X.

9.16. For any chain of three maps

$$S \xrightarrow{s} T \xrightarrow{t} X \xrightarrow{f} Y$$

we shall denote by $\langle s,t,f \rangle$ the map

$$X/t \to Y/ft \times_{Y/fts} X/ts$$

obtained from the commutative square

$$\begin{array}{ccc} X/t & \longrightarrow & X/ts \\ \downarrow & & \downarrow \\ Y/ft & \longrightarrow & Y/fts, \end{array}$$

Let us suppose that s is monic. Then the map $\langle s,t,f \rangle$ is a right fibration when f is a mid fibration, a Kan fibration when f is a left fibration, and it is a trivial fibration in each of the following cases:

- f is a trivial fibration;
- f is a right fibration and s is anodyne:
- f is a mid fibration and s is left anodyne.

9.17. The *fat join* of two simplicial sets A and B is the simplicial set $A \diamond B$ defined by the pushout square

$$\begin{array}{ccc} (A \times 0 \times B) \sqcup (A \times 1 \times B) & \longrightarrow & A \sqcup B \\ \downarrow & & \downarrow \\ A \times I \times B & \longrightarrow & A \diamond B. \end{array}$$

We have $A \sqcup B \subseteq A \diamond B$ and there is a canonical map $A \diamond B \to I$. This defines a continuous functor $\diamond : \mathbf{S} \times \mathbf{S} \to \mathbf{S}/I$ and we have

$$X \diamond Y = (X \diamond 1) \times_I (1 \diamond Y).$$

For a fixed $B \in \mathbf{S}$, the functor $(-) \diamond B : \mathbf{S} \to B \backslash \mathbf{S}$ which takes a simplicial set A to the inclusion $B \subseteq A \diamond B$ has a right adjoint which takes a map $b : B \to X$ to a simplicial set $X /\!\!/ b$ called the *fat lower slice* of X by b. If $B = 1$ and $b \in X_0$, then $X /\!\!/ b$ is the fiber at b of the target map $X^I \to X$. The simplicial set $X /\!\!/ b$ is a quategory when X is a quategory. Dually, there is also a *fat upper slice* $a \backslash\!\!\backslash X$ for any map $a : A \to X$. The simplicial set $a \backslash\!\!\backslash X$ is a quategory when X is a quategory.

9.18. For any pair of simplicial sets A and B, the square

$$\begin{array}{ccc} A \sqcup B & \longrightarrow & A \star B \\ \downarrow & & \downarrow \\ A \diamond B & \longrightarrow & I. \end{array}$$

has a unique diagonal filler

$$\theta_{AB} : A \diamond B \to A \star B.$$

and θ_{AB} is weak categorical equivalence. By adjointness, we obtain a map

$$\rho(b) : X/b \to X /\!\!/ b$$

for any simplicial set X and any map $b : B \to X$. The map $\rho(b)$ is an equivalence of quategories when X is a quategory.

9.19. The pair of adjoint functors $X \mapsto X \diamond B$ and $(X, b) \mapsto X /\!\!/ b$ is a Quillen adjoint pair between the model categories $(\mathbf{S}, \mathrm{Wcat})$ and $(B \backslash \mathbf{S}, \mathrm{Wcat})$.

10. Initial and terminal objects

We introduce the notions of inital, terminal and null objects. We also introduce a strict version of these notions and a corresponding model category..

10.1. If A is a simplicial set, we shall say that a vertex $a \in A$ is *terminal* if the map $a : 1 \to A$ is final (or equivalently right anodyne). Dually, we shall say that a vertex $a \in A$ is *initial* iff the map $a : 1 \to A$ is initial (or equivalently left anodyne). A vertex $a \in A$ is initial if the opposite vertex $a^o \in A^o$ is terminal.

10.2. The notion of terminal vertex is invariant under weak categorical equivalence. More precisely, if $u : A \to B$ is a weak categorical equivalence, then a vertex $a \in A$ is terminal in A iff the vertex $u(a)$ is terminal in B. If A is a simplicial set, then the vertex $1_a \in A/a$ is terminal in A/a for any vertex $a \in A$. Similarly for the vertex $1_a \in A/\!\!/a$.

10.3. If X is a quategory, then an object $a \in X$ is terminal iff the following equivalent conditions are satisfied:

- the simplicial set $X(x,a)$ is contractible for every object $x \in X$;
- every simplical sphere $x : \partial\Delta[n] \to X$ with $x(n) = a$ can be filled;
- the projection $X/a \to X$(resp. $X/\!\!/a \to X$) is a weak categorical equivalence.

Moreover, the projection $X/a \to X$ (resp. $X/\!\!/a \to X$) is a trivial fibration in this case. Dually, an object $a \in X$ is initial iff the following equivalent conditions are satisfied:

- the simplicial set $X(a,x)$ is contractible for every object $x \in X$;
- every simplical sphere $x : \partial\Delta[n] \to X$ with $x(0) = a$ can be filled;
- the projection $a\backslash X \to X$ (resp. $a\backslash\!\!\backslash X \to X$) is a weak categorical equivalence (resp. a trivial fibration).

Moreover, the projection $a\backslash X \to X$ (resp. $a\backslash\!\!\backslash X \to X$) is a weak categorical equivalence (resp. a trivial fibration) in this case.

10.4. The full simplicial subset spanned by the terminal (resp. initial) objects of a quategory is a contractible Kan complex when non-empty.

10.5. If A is a simplicial set, then a vertex $a \in A$ which is terminal in A is also terminal in the category $\tau_1 A$. The converse is true when A admits a terminal vertex.

10.6. Let B be a simplicial set. Then a vertex $b \in B$ is terminal iff the inclusion $E(b) \subseteq E$ is a weak homotopy equivalence for every left fibration $p : E \to B$, where $E(b) = p^{-1}(b)$. Recall from 12.1 that the category $\mathbf{S}/B$ is enriched over $\mathbf{S}$. For any object E of $\mathbf{S}/B$, let us denote by $\Gamma_B(E)$ the simplicial set $[B,E]$ of global sections of E. Then a vertex $b \in B$ is terminal iff the canonical projection $\Gamma_B(E) \to E(b)$ is a homotopy equivalence for every right fibration $E \to B$.

10.7. If b is a terminal object of a quategory X, then the projection $X/b \to X$ admits a section $s : X \to X/b$ such that $s(1_b) = b$. The section is homotopy unique and we shall say that it is a *terminal flow* on X. A terminal flow on (X,b) can be defined to be a map $r : X \star 1 \to X$ which extends the identity $X \to X$ along the inclusion $X \subset X \star 1$ and such that $r(b \star 1) = 1_b$. More generally, if (A,a) is a pointed simplicial set we shall say that a map $r : A \star 1 \to A$ is a *terminal flow* if it extends the identity $A \to A$ along the inclusion $A \subset A\star 1$ and we have $r(a\star 1) = 1_a$. The vertex a is then terminal in A. If A is a category with terminal object $a \in A$, then there is a unique terminal flow $r : A\star 1 \to A$ such that $r(1) = a$. In particular, the simplex $\Delta[n]$ is equipped with a unique erminal flow, since the category $[n]$ has a unique terminal object $n \in [n]$. The map $1 \star 1 \to 1$ gives the simplicial set 1 the structure of a monoid in the monoidal category $(\mathbf{S}, \star)$. We shall say that a terminal flow $r : A \star 1 \to A$ is *strict* if it is associative as a right action of the monoid 1 on A. A *morphism* of strict terminal flows $(A,r) \to (B,s)$ is a map $u : A \to B$ which respects the right actions. This defines a category $\mathbf{S}(t)$ whose objects are the strict terminal flows. The functor $U : \mathbf{S}(t) \to 1\backslash\mathbf{S}$ which associates to a terminal flow (A, r) the pointed simplicial set $(A, r(1))$ has a left adjoint F which associates to a pointed simplicial set (A, a) the simplicial set $A \star_a 1$ described by the pushout square

$$\begin{array}{ccc} 1 \star 1 & \longrightarrow & 1 \\ {\scriptstyle a\star 1}\downarrow & & \downarrow \\ A \star 1 & \longrightarrow & A \star_a 1 \end{array}$$

The flow on $A \star_a 1$ is induced by the canonical flow on $A \star 1$. Let us denote by $\Delta(t)$ the subcategory of Δ whose morphisms are the maps $f : [m] \to [n]$ which preserves the top elements. A strict terminal flow $A = (A, r)$ has a *nerve* $N(A) : \Delta(t)^o \to \mathbf{Set}$ defined by putting

$$N(A)_n = \mathbf{S}(t)(\Delta[n], A)$$

for every $n \geq 0$. The nerve functor

$$N : \mathbf{S}(t) \to [\Delta(t)^o, \mathbf{Set}]$$

is fully faithful and its image is the full subcategory of $[\Delta(t)^o, \mathbf{Set}]$ spanned by the presheaves X with $X_0 = 1$. We shall say that a map of strict terminal flows $f : (A, r) \to (B, s)$ is a *weak categorical equivalence* if the map $f : A \to B$ is a weak categorical equivalence. The category $\mathbf{S}(t)$ admits a model structure in which a weak equivalence is a weak categorical equivalence and a cofibration is a monomorphism. A strict terminal flow (A, r) is fibrant for this model structure iff the simplicial set A is a quategory. We shall denote this model structure shortly by $(\mathbf{S}(t), \mathrm{Wcat})$. The pair of adjoint functors

$$F : 1\backslash\mathbf{S} \leftrightarrow \mathbf{S}(t) : U$$

is a Quillen adjunction between the model categories $(1\backslash\mathbf{S}, \mathrm{Wcat})$ and $(\mathbf{S}(t), \mathrm{Wcat})$. The functor F is a homotopy reflection since the right derived functor U^R is fully faithful. A pointed simplicial set (A, a) belongs to the essential image of U^R iff the vertex a is terminal in A. Dually, an *initial flow* on a pointed simplicial set (A, a) is defined to be a map $l : 1 \star A \to A$ which extends the identity map $A \to A$ and such that $l(a \star 1) = 1_a$. An initial flow $l : 1 \star A \to A$ is *strict* if it is associative as a left action of the monoid 1 on A. There is then a category $\mathbf{S}(i)$ of strict initial flows and a model category $(\mathbf{S}(i), \mathrm{Wcat})$. . . .

10.8. We shall say that a vertex in a simplicial set A is *null* if it is both initial and terminal in A. We shall say that a simplicial set A is *null-pointed* if it admits a null vertex $0 \in A$. If a quategory X is null-pointed, then the projection $X^I \to X \times X$ admits a section which associates to a pair of objects $x, y \in X$ a *null morphism* $0 : x \to y$ obtained by composing the morphisms $x \to 0 \to y$. Moreover, the section is homotopy unique. Similarly, the codiagonal $X \sqcup X \to X$ admits an extension $m : X \star X \to X$ which associates to a pair of objects $x, y \in X$ a null morphism $m(x \star y) = 0 : x \to y$. Moreover, the map m is homotopy unique. More generally, if (A, a) is a pointed simplicial set, we shall say that a map $m : A \star A \to A$ is a *null flow* if it extends the codiagonal $A \sqcup A \to A$ and we have $m(a \star a) = 1_a$. The vertex a is then null in A. If A is a category with null object $a \in A$, then there is a unique null flow $m : A \star A \to A$ such that $m(a, a) = 1_a$. We shall say that a null flow $m : A \star A \to A$ is *strict* if it is associative. A *morphism* of null flows $(A, m) \to (B, n)$ is a map $u : A \to B$ which respects m and n This defines a category $\mathbf{S}(n)$ whose objects are the strict null flows. Let us say that a map of null flows $f : (A, m) \to (B, n)$ is a *weak categorical equivalence* (resp. a *pseudo-fibration*) if the map $f : A \to B$ is a weak categorical equivalence (resp. a *pseudo-fibration*). Then the category $\mathbf{S}(n)$ admits a model structure in which a weak equivalence is a weak categorical equivalence and a fibration is a pseudo-fibration. We shall denote this model structure shortly by $(\mathbf{S}(n), \mathrm{Wcat})$. The forgeful functor $U : \mathbf{S}(n) \to 1\backslash\mathbf{S}$ which associates to a null flow $(A, m, 0)$ the pointed simplicial set $(A, 0)$ has a left adjoint F and the pair of adjoint functors

$$F : 1\backslash\mathbf{S} \leftrightarrow \mathbf{S}(n) : U$$

is a Quillen adjunction between the model category $(1\backslash\mathbf{S}, \mathrm{Wcat})$ and the model category $(\mathbf{S}(n), \mathrm{Wcat})$. The functor F is a homotopy reflection and a pointed simplicial set (A, a) belongs to the essential image of the right derived functor U^R iff the vertex a is null in A. . .

11. Homotopy factorisation systems

The notion of homotopy factorisation system was introduced by Bousfield in his work on localisation. We introduce a more general notion and give examples. Most results of the section are taken from [J2].

11.1. Let $\mathcal{E}$ be a category equipped with a class of maps $\mathcal{W}$ satisfying "three-for-two". We shall say that a class of maps $\mathcal{M} \subseteq \mathcal{E}$ is *invariant under weak equivalences* if for every commutative square

$$\begin{array}{ccc} A & \longrightarrow & A' \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u'} \\ B & \longrightarrow & B' \end{array}$$

in which the horizontal maps are in $\mathcal{W}$, we have $u \in \mathcal{M} \Leftrightarrow u' \in \mathcal{M}$.

11.2. We shall say that a class of maps $\mathcal{M}$ in a category $\mathcal{E}$ has the *right cancellation property* if the implication

$$vu \in \mathcal{M} \text{ and } u \in \mathcal{M} \;\Rightarrow\; v \in \mathcal{M}$$

is true for any pair of maps $u : A \to B$ and $v : B \to C$. Dually, we shall say that $\mathcal{M}$ has the *left cancellation property* if the implication

$$vu \in \mathcal{M} \text{ and } v \in \mathcal{M} \;\Rightarrow\; u \in \mathcal{M}$$

is true for any pair of maps $u : A \to B$ and $v : B \to C$.

11.3. If $\mathcal{E}$ is a Quillen model category, we shall denote by $\mathcal{E}_f$ (resp. $\mathcal{E}_c$) the full subcategory of fibrant (resp. cofibrant) objects of $\mathcal{E}$ and we shall put $\mathcal{E}_{fc} = \mathcal{E}_f \cap \mathcal{E}_c$. For any class of maps $\mathcal{M} \subseteq \mathcal{E}$ we shall put

$$\mathcal{M}_f = \mathcal{M} \cap \mathcal{E}_f, \qquad \mathcal{M}_c = \mathcal{M} \cap \mathcal{E}_c \quad \text{and} \quad \mathcal{M}_{fc} = \mathcal{M} \cap \mathcal{E}_{fc}.$$

11.4. Let $\mathcal{E}$ be a model category with model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$. We say that a pair $(\mathcal{A}, \mathcal{B})$ of classes of maps in $\mathcal{E}$ is a *homotopy factorisation system* if the following conditions are satisfied:

- the classes $\mathcal{A}$ and $\mathcal{B}$ are invariant under weak equivalences;
- the pair $(\mathcal{A}_{fc} \cap \mathcal{C}, \mathcal{B}_{fc} \cap \mathcal{F})$ is a weak factorisation system in $\mathcal{E}_{fc}$;
- the class $\mathcal{A}$ has the right cancellation property;
- the class $\mathcal{B}$ has the left cancellation property.

The last two conditions are equivalent in the presence of the others. The class $\mathcal{A}$ is said to be the *left class* of the system and $\mathcal{B}$ to be the *right class*. We say that a system $(\mathcal{A}, \mathcal{B})$ is uniform if the pair $(\mathcal{A} \cap \mathcal{C}, \mathcal{B} \cap \mathcal{F})$ is a weak factorisation system.

11.5. The notions of homotopy factorisation systems and of factorisation systems coincide if the model structure is discrete (ie when $\mathcal{W}$ is the class of isomorphisms). The pairs $(\mathcal{E}, \mathcal{W})$ and $(\mathcal{W}, \mathcal{E})$ are trivial examples of homotopy factorisation systems.

11.6. A homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ is determined by each of the following 24 classes,

$$
\begin{array}{cccc}
\mathcal{A} & \mathcal{A}_c & \mathcal{A}_f & \mathcal{A}_{fc} \\
\\
\mathcal{A} \cap \mathcal{C} & \mathcal{A}_c \cap \mathcal{C} & \mathcal{A}_f \cap \mathcal{C} & \mathcal{A}_{fc} \cap \mathcal{C} \\
\\
\mathcal{A} \cap \mathcal{F} & \mathcal{A}_c \cap \mathcal{F} & \mathcal{A}_f \cap \mathcal{F} & \mathcal{A}_{fc} \cap \mathcal{F} \\
\\
\mathcal{B} & \mathcal{B}_c & \mathcal{B}_f & \mathcal{B}_{fc} \\
\\
\mathcal{B} \cap \mathcal{C} & \mathcal{B}_c \cap \mathcal{C} & \mathcal{B}_f \cap \mathcal{C} & \mathcal{B}_{fc} \cap \mathcal{C} \\
\\
\mathcal{B} \cap \mathcal{F} & \mathcal{B}_c \cap \mathcal{F} & \mathcal{B}_f \cap \mathcal{F} & \mathcal{B}_{fc} \cap \mathcal{F}.
\end{array}
$$

This property is useful in specifying a homotopy factorisation system.

11.7. Every homotopy factorisation system in a proper model category is uniform. This is true in particular for the homotopy factorisations systems in the model categories $(\mathbf{S}, Who)$ and $(\mathbf{Cat}, Eq)$.

11.8. If $\mathcal{E}$ is a model category we shall denote by $Ho(\mathcal{M})$ the image of a class of maps $\mathcal{M} \subseteq \mathcal{E}$ by the canonical functor $\mathcal{E} \to Ho(\mathcal{E})$. If $(\mathcal{A}, \mathcal{B})$ is a homotopy factorisation system $\mathcal{E}$, then the pair $(Ho(\mathcal{A}), Ho(\mathcal{B}))$ is a weak factorisation system in $Ho(\mathcal{E})$. Notice that the pair $(Ho(\mathcal{A}), Ho(\mathcal{B}))$ is not a factorisation system in general. The class $Ho(\mathcal{A})$ has the right cancellation property and the class $Ho(\mathcal{B})$ the left cancellation property. The system $(\mathcal{A}, \mathcal{B})$ is determined by the system $(Ho(\mathcal{A}), Ho(\mathcal{B}))$.

11.9. The intersection of the classes of a homotopy factorisation system is the class of weak equivalences. Each class of a homotopy factorisation system is closed under composition and retracts. The left class is closed under homotopy cobase change and the right class is closed under homotopy base change.

11.10. Let $(\mathcal{A}, \mathcal{B})$ be a homotopy factorisation system in a model category $\mathcal{E}$. Then we have $u \pitchfork p$ for every $u \in \mathcal{A}_c \cap \mathcal{C}$ and $p \in \mathcal{B}_f \cap \mathcal{F}$. If $A \in \mathcal{E}_c$ and $X \in \mathcal{E}_f$, then every map $f : A \to X$ admits a factorisation $f = pu$ with $u \in \mathcal{A}_c \cap \mathcal{C}$ and $p \in \mathcal{B}_c \cap \mathcal{F}$.

11.11. If $\mathcal{E}$ is a model category, then so is the category $\mathcal{E}/C$ for any object $C \in \mathcal{E}$. If $\mathcal{M}$ is a class of maps in $\mathcal{E}$, let us denote by $\mathcal{M}_C$ the class of maps in $\mathcal{E}/C$ whose underlying map belongs to $\mathcal{M}$. If $(\mathcal{A}, \mathcal{B})$ is a homotopy factorisation system in $\mathcal{E}$ and C is fibrant, then the pair $(\mathcal{A}_C, \mathcal{B}_C)$ is a homotopy factorisation system in $\mathcal{E}/C$. This true without restriction on C when the system $(\mathcal{A}, \mathcal{B})$ is uniform.

11.12. Dually, if $\mathcal{E}$ is a model category, then so is the category $C\backslash\mathcal{E}$ for any object $C \in \mathcal{E}$. If $\mathcal{M}$ is a class of maps in $\mathcal{E}$, let us denote by ${}_C\mathcal{M}$ the class of maps in $C\backslash\mathcal{E}$ whose underlying map belongs to $\mathcal{M}$. If $(\mathcal{A}, \mathcal{B})$ is a homotopy factorisation system in $\mathcal{E}$ and C is cofibrant, then the pair $({}_C\mathcal{A}, {}_C\mathcal{B})$ is a homotopy factorisation system in $C\backslash\mathcal{E}$. This is true without restriction C when the system $(\mathcal{A}, \mathcal{B})$ is uniform.

11.13. The model category $(\mathbf{Cat}, Eq)$ admits a (uniform) homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of essentially surjective functors and $\mathcal{B}$ the class of fully faithful functors.

11.14. We call a functor $u : A \to B$ a *localisation* (resp. *iterated localisation*) iff it admits a factorisation $u = wu' : A \to B' \to B$ with u' a strict localisation (resp. iterated strict localisation) and w an equivalence of categories. The model category $(\mathbf{Cat}, Eq)$ admits a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the the class of iterated localisations and $\mathcal{B}$ is the class of conservative functors.

11.15. The model category $(\mathbf{Cat}, Eq)$ admits a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ the class of 0-final functors. A functor $u : A \to B$ belongs to $\mathcal{B}$ iff it admits a factorisation $u = pw : A \to E \to B$ with w an equivalence and p a discrete right fibration. Dually, the model category $(\mathbf{Cat}, Eq)$ admits a homotopy factorisation system $(\mathcal{A}', \mathcal{B}')$ in which $\mathcal{A}'$ is the class of 0-initial functors. A functor $u : A \to B$ belongs to $\mathcal{B}$ iff it admits a factorisation $u = pw : A \to E \to B$ with w an equivalence and p a discrete left fibration.

11.16. The model category $(\mathbf{Cat}, Eq)$ admits a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ the class of 1-final functors. A functor $u : A \to B$ belongs to $\mathcal{B}$ iff it admits a factorisation $u = pw : A \to E \to B$ with w an equivalence and p a 1-fibration.

11.17. Recall that a functor $u : A \to B$ is said to be *0-connected* if the functor $\pi_1(u) : \pi_1 A \to \pi_1 B$ is essentially surjective and full . The category $\mathbf{Cat}$ admits a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of 0-connected functors. A functor $u : A \to B$ belongs to $\mathcal{B}$ iff it admits a factorisation $u = pw : A \to E \to B$ with w an equivalence and p a 0-covering,

11.18. We say that a map of simplicial sets

is *homotopy monic* if its homotopy fibers are empty or contractible. We say that a map of simplicial sets is *homotopy surjective* if its homotopy fibers are non-empty. A map $u : A \to B$ is homotopy surjective iff the map $\pi_0(u) : \pi_0 A \to \pi_0 B$ is surjective. The model category $(\mathbf{S}, \mathit{Who})$ admits a uniform homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of homotopy surjections and $\mathcal{B}$ the class of homotopy monomorphisms.

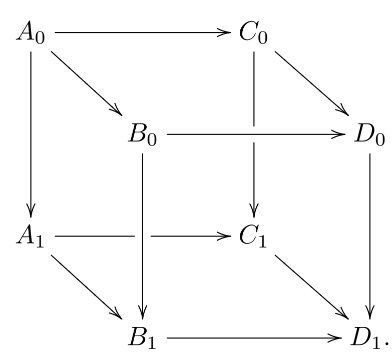
11.19. Recall from 1.13 that a map of simplicial sets $u : A \to B$ is said to be *essentially surjective* if the map $\tau_0(u) : \tau_0(A) \to \tau_0(B)$ is surjective. The model category $(\mathbf{S}, \mathrm{Wcat})$ admits a (non-uniform) homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of essentially surjective maps. A map in the class $\mathcal{B}$ is said to be *fully faithful*. A map between quategories $f : X \to Y$ is fully faithful iff the map $X(a, b) \to Y(fa, fb)$ induced by f is a weak homotopy equivalence for every pair of objects $a, b \in X_0$.

11.20. We say that a map of simplicial sets $u : A \to B$ is *conservative* if the functor $\tau_1(u) : \tau_1 A \to \tau_1 B$ is conservative. The model category $(\mathbf{S}, \mathrm{Wcat})$ admits a (non-uniform) homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{B}$ is the class of conservative maps. A map in the class $\mathcal{A}$ is an *iterated homotopy localisation*. See 18.2 for this notion.

11.21. The model category $(\mathbf{S}, \mathrm{Wcat})$ admits a uniform homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of final maps. A map $p : X \to Y$ belongs to $\mathcal{B}$ iff it admits a factorisation $p'w : X \to X' \to Y$ with p' a right fibration and w a weak categorical equivalence. The intersection $\mathcal{B} \cap \mathcal{F}$ is the class of right fibrations and the intersection $\mathcal{A} \cap \mathcal{C}$ the class of right anodyne maps. Dually, the model category $(\mathbf{S}, \mathrm{Wcat})$ admits a uniform homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of initial maps.

11.22. The model category $(\mathbf{S}, \mathrm{Wcat})$ admits a uniform homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of weak homotopy equivalences. A map $p : X \to Y$ belongs to $\mathcal{B}$ iff it admits a factorisation $p'w : X \to X' \to Y$ with p' a Kan fibration and w a weak categorical equivalence. The intersection $\mathcal{B} \cap \mathcal{F}$ is the class of Kan fibrations and the intersection $\mathcal{A} \cap \mathcal{C}$ the class of anodyne maps.

11.23. Let $(\mathcal{A}, \mathcal{B})$ be a homotopy factorisation system in a model category $\mathcal{E}$. Suppose that we have a commutative cube



in which the top and the bottom faces are homotopy cocartesian. If the arrows $A_0 \to A_1$, $B_0 \to B_1$ and $C_0 \to C_1$ belong to $\mathcal{A}$, then so does the arrow $D_0 \to D_1$.

11.24. [JT3] If $n \geq -1$, we shall say that a simplicial set X is a *n-object* if we have $\pi_i(X, x) = 1$ for every $x \in X$ and every $i > n$. If $n = -1$, this means that X is contractible or empty. If $n = 0$, this means that X is is homotopically equivalent to a discrete simplicial set. A Kan complex X is a n-object iff every simplicial sphere $\partial\Delta[m] \to X$ with $m > n + 1$ has a filler. We say that a map of simplicial sets $f : X \to Y$ is a *n-cover* if its homotopy fibers are n-objects. If $n = -1$, this means that f is homotopy monic. A Kan fibration is a n-cover iff it has the right lifting property with respect to the inclusion $\partial\Delta[m] \subset \Delta[m]$ for every $m > n + 1$. We shall say that a simplicial set X is *n-connected* if $X \neq \emptyset$ and we have $\pi_i(X, x) = 1$ for every $x \in X$ and every $i \leq n$. If $n = -1$, this means that $X \neq \emptyset$. If $n = 0$, this means that X is connected. We shall say that a map $f : X \to Y$ is *n-connected* if its homotopy fibers are n-connected. If $n = -1$, this means that f is homotopy surjective. A map $f : X \to Y$ is n-connected iff the map $\pi_i(X, x) \to \pi_i(Y, fx)$ induced by f is bijective for every $0 \leq i \leq n$ and $x \in X$ and a surjection for $i = n + 1$. If $\mathcal{A}_n$ is the class of n-connected maps and $\mathcal{B}_n$ the class of n-covers, then the pair $(\mathcal{A}_n, \mathcal{B}_n)$ is a uniform homotopy factorisation system on the model category $(\mathbf{S}, \mathit{Who})$. We say that it is the *n-factorisation system* on $(\mathbf{S}, \mathit{Who})$.

11.25. A simplicial set X is a n-object iff the diagonal map $X \to X \times X$ is $(n-1)$-cover (if $n = 0$ this means that the diagonal is homotopy monic). A simplicial set X is a n-connected iff it is non-empty and the diagonal $X \to X \times X$ is a $(n-1)$-connected. (if $n = 0$ this means that the diagonal is homotopy surjective).

11.26. The model category $(\mathbf{S}, \mathrm{Wcat})$ admits a uniform homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of n-connected maps. The intersection $\mathcal{B} \cap \mathcal{F}$ is the class of Kan n-covers.

11.27. If $n \geq -1$, we shall say that a right fibration $f : X \to Y$ is a *right n-fibration* if its fibers are n-objects. If $n = -1$, this means that f is fully faithful. If $n = 0$, this means that f is fiberwise homotopy equivalent to a right covering. A right fibration is a right n-fibration iff it has the right lifting property with respect to the inclusion $\partial\Delta[m] \subset \Delta[m]$ for every $m > n+1$. The model category $(\mathbf{S}, \mathrm{Wcat})$ admits a uniform homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which the intersection $\mathcal{B} \cap \mathcal{F}$ is the class of right n-fibrations. We say that a map in the class $\mathcal{A}$ is *n-final*. A map between quategories $u : A \to B$ is n-final iff the simplicial set $b\backslash A$ is n-connected for every object $b \in B$. indexAfibration!right n-fibration—textbf

11.28. Let $F : \mathcal{E} \leftrightarrow \mathcal{E}' : G$ be a Quillen pair between two model categories. If $(\mathcal{A}, \mathcal{B})$ is a homotopy factorisation system in $\mathcal{E}$ and $(\mathcal{A}', \mathcal{B}')$ a homotopy factorisation system in $\mathcal{E}'$, then the conditions $F(\mathcal{A}_c) \subseteq \mathcal{A}'_c$ and $G(\mathcal{B}'_f) \subseteq \mathcal{B}_f$ are equivalent. If the pair (F, G) is a Quillen equivalence, then the conditions $\mathcal{A}_c = F^{-1}(\mathcal{A})_c$ and $\mathcal{B}'_f = G^{-1}(\mathcal{B})_f$ are equivalent. In this case we shall say that $(\mathcal{A}', \mathcal{B}')$ is obtained by *transporting* $(\mathcal{A}, \mathcal{B})$ *across* the Quillen equivalence. Every homotopy factorisation system can be transported across a Quillen equivalence.

11.29. We shall say that a simplicial functor $f : X \to Y$ in **SCat** is *conservative* if the functor $ho(f) : hoX \to hoY$ is conservative. The Bergner model category **SCat** admits a (non-uniform) homotopy factorisation system in which the right class is the class of conservative functors. A map in the left class is an *iterated Dwyer-Kan localisation*. We saw in 3.5 that the adjoint pair of functors

$$C_! : \mathbf{S} \leftrightarrow \mathbf{SCat} : C^!$$

is a Quillen equivalence between the model category for quategories and the model category for simplicial categories. A map of simplicial sets $X \to Y$ is a homotopy localisation iff the functor $C_!(f) : C_!(X) \to C_!(Y)$ is a Dwyer-Kan localisation.

12. The covariant and contravariant model structures

In this section we introduce the covariant and the contravariant model structures on the category $\mathbf{S}/B$ for any simplicial set B. In the covariant structure, the fibrant objects are the left fibrations $X \to B$, and in the contravariant structure they are the right fibrations $X \to B$. The results of this section are taken from [J2].

12.1. The category $\mathbf{S}/B$ is enriched over $\mathbf{S}$ for any simplicial set B. We shall denote by $[X, Y]$ the simplicial set of maps $X \to Y$ between two objects of $\mathbf{S}/B$. If we apply the functor π_0 to the composition map $[Y, Z] \times [X, Y] \to [X, Z]$ we obtain a composition law

$$\pi_0[Y, Z] \times \pi_0[X, Y] \to \pi_0[X, Z]$$

for a category $(\mathbf{S}/B)^{\pi_0}$ if we put

$$(\mathbf{S}/B)^{\pi_0}(X, Y) = \pi_0[X, Y].$$

We shall say that a map in $\mathbf{S}/B$ is a *fibrewise homotopy equivalence* if the map is invertible in the category $(\mathbf{S}/B)^{\pi_0}$.

12.2. Let $\mathbf{R}(B)$ be the full subcategory of $\mathbf{S}/B$ spanned by the right fibrations $X \to B$. If $X \in \mathbf{R}(B)$, then the simplicial set $[A, X]$ is a Kan complex for every object $A \in \mathbf{S}/B$. In particular, the fiber $[b, X] = X(b)$ is a Kan complex for every vertex $b : 1 \to B$. A map $u : X \to Y$ in $\mathbf{R}(B)$ is a fibrewise homotopy equivalence iff the induced map between the fibers $X(b) \to Y(b)$ is a homotopy equivalence for every vertex $b \in B$.

12.3. We shall say that a map $u : M \to N$ in $\mathbf{S}/B$ is a *contravariant equivalence* if the map

$$\pi_0[u, X] : \pi_0[N, X] \to \pi_0[N, X]$$

is bijective for every object $X \in \mathbf{R}(B)$. A fibrewise homotopy equivalence in $\mathbf{S}/B$ is a contravariant equivalence and the converse holds for a map in $\mathbf{R}(B)$. A final map $M \to N$ in $\mathbf{S}/B$ is a contravariant equivalence and the converse holds if $N \in \mathbf{R}(B)$. A map $u : X \to Y$ in $\mathbf{S}/B$ is a contravariant equivalence iff its base change $L \times_B u : L \times_B X \to L \times_B Y$ along any left fibration $L \to B$ is a weak homotopy equivalence. For each vertex $b \in B$, let us choose a factorisation $1 \to Lb \to B$ of the map $b : 1 \to B$ as a left anodyne map $1 \to Lb$ followed by a left fibration $Lb \to B$. Then a map $u : M \to N$ in $\mathbf{S}/B$ is a contravariant equivalence iff the map $Lb \times_B u : Lb \times_B M \to Lb \times_B N$ is a weak homotopy equivalence for every vertex $b \in B$. When B is a quategory, we can take $Lb = b\backslash B$. In which case a map $u : M \to N$ is a contravariant equivalence iff the map $b\backslash u = b\backslash M \to b\backslash N$ is a weak homotopy equivalence for every object $b \in B$.

12.4. For any simplicial set B, the category $\mathbf{S}/B$ admits a model structure in which the weak equivalences are the contravariant equivalences and the cofibrations are the monomorphisms, We shall say that it is the *contravariant model structure* in $\mathbf{S}/B$. The fibrations are called *contravariant fibrations* and the fibrant objects are the right fibrations $X \to B$. The model structure is simplicial and we shall denote it shortly by $(\mathbf{S}/B, \mathrm{Wcont}(B))$, or more simply by $(\mathbf{S}/B, \mathrm{Wcont})$, where $\mathrm{Wcont}(B)$ denotes the class of contravariant equivalences in $\mathbf{S}/B$.

Every contravariant fibration in $\mathbf{S}/B$ is a right fibration and the converse holds for a map in $\mathbf{R}(B)$.

12.5. The cofibrations of the model structure $(\mathbf{S}/B, \mathrm{Wcont})$ are the monomorphisms. Hence the model structure is determined by its fibrant objects, that is, by the right fibrations $X \to B$, by 50.10.

12.6. Recall that the category $[A^o, \mathbf{S}]$ of simplicial presheaves on simplicial category A admits a model structure, called the *projective model structure*, in which a weak equivalence is a term-wise weak homotopy equivalence and a fibration is a term-wise Kan fibrations [Hi]. It then follows from 51.13 that if $B = C_!A$, then the projective model category $[A^o, \mathbf{S}]$ is equivalent to the model category $(\mathbf{S}/B, \mathrm{Wcat})$.

12.7. The contravariant model structure $(\mathbf{S}/B, \mathrm{Wcont})$ is a Bousfield localisation of the model structure $(\mathbf{S}/B, \mathrm{Wcat})$ induced by the model structure $(\mathbf{S}, \mathrm{Wcat})$ on $\mathbf{S}/B$. It follows that a weak categorical equivalence in $\mathbf{S}/B$ is a contravariant equivalence and that the converse holds for a map in $\mathbf{R}(B)$. Every contravariant fibration in $\mathbf{S}/B$ is a pseudo-fibration and the converse holds for a map in $\mathbf{R}(B)$.

12.8. A map $u : (M,p) \to (N,q)$ in $\mathbf{S}/B$ is a contravariant equivalence iff the map $bq\backslash X \to bp\backslash X$ induced by u is an equivalence of quategories of any map $b : B \to X$ with values in a quategory X.

12.9. Dually, we say that a map $u : M \to N$ in $\mathbf{S}/B$ is a *covariant equivalence* if the opposite map $u^o : M^o \to N^o$ in $\mathbf{S}/B^o$ is a contravariant equivalence. Let $\mathbf{L}(B)$ be the full subcategory of $\mathbf{S}/B$ spanned by the left fibrations $X \to B$. Then a map $u : M \to N$ in $\mathbf{S}/B$ is a covariant equivalence iff the map

$$\pi_0[u, X] : \pi_0[N, X] \to \pi_0[N, X]$$

is bijective for every object $X \in \mathbf{L}(B)$. A fibrewise homotopy equivalence in $\mathbf{S}/B$ is a covariant equivalence and the converse holds for a map in $\mathbf{L}(B)$. An initial map $M \to N$ in $\mathbf{S}/B$ is a covariant equivalence and the converse holds if $N \in \mathbf{N}(B)$. A map $u : M \to N$ in $\mathbf{S}/B$ is a covariant equivalence iff its base change $R \times_B u : R \times_B M \to R \times_B N$ along any right fibration $R \to B$ is a weak homotopy equivalence. For each vertex $b \in B$, let us choose a factorisation $1 \to Lb \to B$ of the map $b : 1 \to B$ as a right anodyne map $1 \to Rb$ followed by a right fibration $Rb \to B$. Then a map $u : M \to N$ in $\mathbf{S}/B$ is a covariant equivalence iff the map $Rb \times_B u : Rb \times_B X \to Rb \times_B Y$ is a weak homotopy equivalence for every vertex $b \in B$. When B is a quategory, we can take $Rb = B/b$. In this case a map $u : M \to N$ is a covariant equivalence iff the map $u/b = M/b \to N/b$ is a weak homotopy equivalence for every object $b \in B$.

12.10. The category $\mathbf{S}/B$ admits a model structure in which the weak equivalences are the covariant equivalences and the cofibrations are the monomorphisms, We shall say that it is the *covariant model structure* in $\mathbf{S}/B$. The fibrations are called *covariant fibrations* and fibrant objects are the left fibrations $X \to B$. The model structure is simplicial and we shall denote it shortly by $(\mathbf{S}/B, \mathrm{Wcov}(B))$, or more simply by $(\mathbf{S}/B, \mathrm{Wcov})$, where $\mathrm{Wcov}(B)$ denotes the class of covariant equivalences in $\mathbf{S}/B$.

12.11. Every covariant fibration in $\mathbf{S}/B$ is a left fibration and the converse holds for a map in $\mathbf{L}(B)$.

12.12. For any simplicial set B, we shall put

$$\mathcal{R}(B) = Ho(\mathbf{S}/B, \mathrm{Wcont}) \quad \text{and} \quad \mathcal{L}(B) = Ho(\mathbf{S}/B, \mathrm{Wcov}).$$

The functor $X \mapsto X^o$ induces an isomorphism of model categories

$$(\mathbf{S}/B, \mathrm{Wcont}) \simeq (\mathbf{S}/B^o, \mathrm{Wcov}),$$

hence also of categories $\mathcal{R}(B) \simeq \mathcal{L}(B^o)$.

12.13. The base change of a contravariant equivalence in $\mathbf{S}/B$ along a left fibration $A \to B$ is a contravariant equivalence in $\mathbf{S}/A$. Dually, the base change of a covariant equivalence in $\mathbf{S}/B$ along a right fibration $A \to B$ is a covariant equivalence in $\mathbf{S}/B$.

12.14. When the category $\tau_1 B$ is a groupoid, the two classes $\mathrm{Wcont}(B)$ and $\mathrm{Wcov}(B)$ coincide with the class of weak homotopy equivalences in $\mathbf{S}/B$. In particular, the model categories $(\mathbf{S}, \mathrm{Wcont})$, $(\mathbf{S}, \mathrm{Wcov})$ and $(\mathbf{S}, \mathit{Who})$, coincide. Thus,

$$\mathcal{L}(1) = \mathcal{R}(1) = Ho(\mathbf{S}, \mathit{Who}).$$

12.15. If $X, Y \in \mathbf{S}/B$, let us put

$$\langle X \mid Y \rangle = X \times_B Y.$$

This defines a functor of two variables

$$\langle - \mid - \rangle : \mathbf{S}/B \times \mathbf{S}/B \to \mathbf{S}.$$

If $X \in \mathbf{L}(B)$, then the functor $\langle X \mid - \rangle$ is a left Quillen functor between the model categories $(\mathbf{S}/B, \mathrm{Wcont})$ and $(\mathbf{S}, \mathit{Who})$. Dually, if $Y \in \mathbf{R}(B)$, then the functor $\langle - \mid Y \rangle$ is a left Quillen functor between the model categories $(\mathbf{S}/B, \mathrm{Wcov})$ and $(\mathbf{S}, \mathit{Who})$. It follows that the functor $\langle - \mid - \rangle$ induces a functor of two variables,

$$\langle - \mid - \rangle : \mathcal{L}(B) \times \mathcal{R}(B) \to Ho(\mathbf{S}, \mathit{Who}).$$

A morphism $v : Y \to Y'$ in $\mathcal{R}(B)$ is invertible iff the morphism

$$\langle X|v \rangle : \langle X \mid Y \rangle \to \langle X \mid Y' \rangle$$

is invertible for every $X \in \mathcal{L}(B)$. Dually, a morphism $u : X \to X'$ in $\mathcal{L}(B)$ is invertible iff the morphism

$$\langle u|Y \rangle : \langle X \mid Y \rangle \to \langle X' \mid Y \rangle$$

is invertible for every $Y \in \mathcal{R}(B)$.

12.16. We say that an object $X \to B$ in $\mathbf{S}/B$ is *finite* if X is a finite simplicial set. We shall say that a right fibration $X \to B$ is *finitely generated* if it is isomorphic to a finite object in the homotopy category $\mathcal{R}(B)$. A right fibration $X \to B$ is finitely generated iff there exists a final map $F \to X$ with codomain a finite object of $\mathbf{S}/B$. The base change $u^*(X) \to A$ of a finitely generated right fibration $X \to B$ along a weak categorical equivalence is finitely generated.

12.17. We shall say that a map $f : A \to B$ in $\mathbf{S}^I$ is a *contravariant equivalence*

$$\begin{array}{ccc} A_0 & \xrightarrow{f_0} & B_0 \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ A_1 & \xrightarrow{f_1} & B_1 \end{array}$$

if f_1 is a weak categorical equivalence and the map $(f_1)_!(A_0) \to B_1$ induced by f_0 is a contravariant equivalence in $\mathbf{S}/B_1$. The category $\mathbf{S}^I$ admits a cartesian closed model structure in which the weak equivalences are contravariant equivalences and the cofibrations are the monomorphisms. We shall denote it shortly by $(\mathbf{S}^I, Wcont)$. The fibrant objects are the right fibrations between quategories. The target functor

$$t : \mathbf{S}^I \to \mathbf{S}$$

is a Grothendieck bifibration and both a left and a right Quillen functor between the model categories $(\mathbf{S}^I, Wcont)$ and $(\mathbf{S}, \mathrm{Wcat})$. It gives the model category $(\mathbf{S}^I, Wcont)$ the structure of a bifibered model category over the model category $(\mathbf{S}, \mathrm{Wcat})$. We shall say that it is the *fibered model category for right fibrations.* It induces the contravariant model structure on each fiber $\mathbf{S}/B$. See 50.32 for the notion of bifibered model category. There is a dual *fibered model category for left fibrations* $(\mathbf{S}^I, \mathrm{Wcov})$

12.18. The model category $(\mathbf{S}/B, \mathrm{Wcont})$ admits a uniform homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of weak homotopy equivalences in $\mathbf{S}/B$. A contravariant fibration belongs to $\mathcal{B}$ iff it is a Kan fibration. It follows from 18.12 that a map $X \to Y$ in $\mathbf{R}(B)$ belongs to $\mathcal{B}$ iff the following square of fibers

$$\begin{array}{ccc} X(b) & \xrightarrow{u^*} & X(a) \\ \downarrow & & \downarrow \\ Y(b) & \xrightarrow{u^*} & Y(a) \end{array}$$

is homotopy cartesian in $(\mathbf{S}, Who)$ for every arrow $u : a \to b$ in B. We shall say that a map in $\mathcal{B}$ is *term-wise cartesian.*

13. Base changes

In this section, we study base changes between contravariant model structures. We introduce the notion of dominant map. The results of the section are taken from [J2].

13.1. A functor $u : A \to B$ between two small categories induces a pair of adjoint functors between the presheaf categories,

$$u_! : \hat{A} \to \hat{B} : u^*.$$

The functor $u_!$ is fully faithful iff the functor u is fully faithful. A functor u is said to be a *Morita equivalence* if the adjoint pair $(u_!, u^*)$ is an equivalence of categories. By a classical result, a functor u is a Morita equivalence iff it is fully faithful and every object $b \in B$ is a retract of an object in the image of u. A functor $u : A \to B$ is said to be *dominant*, but we shall say *0-dominant*, if the functor u^* is fully faithful. A functor $u : A \to B$ is 0-dominant iff the category $\mathrm{Fact}(f, A)$ defined by the pullback square

$$\begin{array}{ccc} \mathrm{Fact}(f, A) & \longrightarrow & A \\ \downarrow & & \downarrow{\scriptstyle u} \\ \mathrm{Fact}(f, B) & \longrightarrow & B \end{array}$$

is connected for every arrow $f \in B$, where $\mathrm{Fact}(f, B) = f\backslash(B/b) = (a\backslash B)/f$ is the category of factorisations of the arrow $f : a \to b$. We notice that the functor u is 0-final iff we have $u_!(1) = 1$, where 1 denotes terminal objects.

13.2. For any map of simplicial sets $u : A \to B$, the adjoint pair

$$u_! : \mathbf{S}/A \to \mathbf{S}/B : u^*$$

is a Quillen adjunction with respect to the contravariant model structures on these categories. It induces an adjoint pair of derived functors

$$\mathcal{R}_!(u) : \mathcal{R}(A) \leftrightarrow \mathcal{R}(B) : \mathcal{R}^*(u),$$

The adjunction is a Quillen equivalence when u is a weak categorical equivalence. We shall see in 21.5 that the functor $\mathcal{R}^*(u)$ has a right adjoint $\mathcal{R}_*(u)$.

13.3. If $u : A \to B$ is a map of simplicial sets, then the functor $u_! : \mathbf{S}/A \to \mathbf{S}/B$ takes a covariant equivalence to a covariant equivalence. Hence we have a strictly commutative square of functors

$$\begin{array}{ccc} \mathbf{S}/A & \longrightarrow & \mathbf{S}/B \\ \downarrow & & \downarrow{\scriptstyle u_!} \\ \mathcal{R}(A) & \xrightarrow{\mathcal{R}_!(u)} & \mathcal{R}(B). \end{array}$$

It follows that we have $\mathcal{R}_!(vu) = \mathcal{R}_!(v)\mathcal{R}_!(u)$ for any pair of maps $u : A \to B$ and $v : B \to C$. This defines a functor

$$\mathcal{R}_! : \mathbf{S} \to \mathbf{CAT},$$

where $\mathbf{CAT}$ is the category of large categories. It follows by adjointness that $\mathcal{R}^*$ has the structure of a contravariant (pseudo-) functor,

$$\mathcal{R}^* : \mathbf{S} \to \mathbf{CAT}.$$

13.4. Dually, for any map of simplicial sets $u : A \to B$, the adjoint pair

$$u_! : \mathbf{S}/A \to \mathbf{S}/B : u^*$$

is a Quillen adjunction with respect to the covariant model structures on these categories, (and it is a Quillen equivalence when u is a weak categorical equivalence). It induces an adjoint pair of derived functors

$$\mathcal{L}_!(u) : \mathcal{L}(A) \leftrightarrow \mathcal{L}(B) : \mathcal{L}^*(u).$$

The functor $\mathcal{L}^*(u)$ has a right adjoint $\mathcal{L}_*(u)$ by 21.5. If $u : A \to B$ and $v : B \to C$, then we have $\mathcal{L}_!(vu) = \mathcal{L}_!(v)\mathcal{L}_!(u)$. This defines a functor

$$\mathcal{L}_! : \mathbf{S} \to \mathbf{CAT},$$

where **CAT** is the category of large categories. It follows by adjointness that $\mathcal{L}^*$ has the structure of a contravariant (pseudo-) functor,

$$\mathcal{L}^* : \mathbf{S} \to \mathbf{CAT}.$$

13.5. A map of simplicial sets $u : A \to B$ is final iff the functor $\mathcal{R}_!(u)$ preserves terminal objects. A map $u : A \to B$ is fully faithful iff the functor $\mathcal{R}_!(u)$ is fully faithful.

13.6. We say that a map of simplicial sets $u : A \to B$ is *dominant* if the functor $\mathcal{R}^*(u)$ is fully faithful.

13.7. A map $u : A \to B$ is dominant iff the opposite map $u^o : A^o \to B^o$ is dominant iff the map $X^u : X^B \to X^A$ is fully faithful for every quategory X.

13.8. The functor $\tau_1 : \mathbf{S} \to \mathbf{Cat}$ takes a fully faithful map to a fully faithful functor, and a dominant map to a 0-dominant functor.

13.9. If B is a quategory, then a map of simplicial sets $u : A \to B$ is dominant iff the simplicial set $\mathrm{Fact}(f, A)$ defined by the pullback square

$$\begin{array}{ccc} \mathrm{Fact}(f,A) & \longrightarrow & A \\ \downarrow & & \downarrow u \\ \mathrm{Fact}(f,B) & \longrightarrow & B \end{array}$$

is weakly contractible for every arrow $f \in B$, where $\mathrm{Fact}(f, B) = f\backslash(B/b) = (a\backslash B)/f$ is the simplicial set of factorisations of the arrow $f : a \to b$.

13.10. A dominant map is both final and initial. A map of simplicial set $u : A \to B$ is dominant iff its base change any right fibration is final iff its base change any left fibration is initial. The base change of a dominant map along a left or a right fibration is dominant. A (weak) reflection and a (weak) coreflection are dominant. An iterated homotopy localisation is dominant.

13.11. The functor

$$\mathcal{R}_! : \mathbf{S} \to \mathbf{CAT}$$

has the structure of a 2-functor covariant on 2-cells. In particular, we have a functor

$$\mathcal{R}_!(-) : \tau_1(A, B) \to \mathbf{CAT}(\mathcal{R}(A) \to \mathcal{R}(B)).$$

for every pair of simplicial sets A and B. It associates a natural transformation

$$\mathcal{R}_!(\alpha) : \mathcal{R}_!(u) \to \mathcal{R}_!(v) : \mathcal{R}(A) \to \mathcal{R}(B)$$

to every morphism $\alpha : u \to v : A \to B$ in the category $\tau_1(A, B)$. Let us describe $\mathcal{R}_!(\alpha)$ in the case where α is the mprphism $[h] : i_0 \to i_1$ defined by the canonical homotopy $h : i_0 \to i_1 : A \to A \times I$. For any $X \in \mathbf{S}/A$ we have $(i_0)_!(X) = X \times \{0\}$ and $(i_1)_!(X) = X \times \{1\}$. The inclusion $X \times \{1\} \subseteq X \times I$ is a contravariant equivalence in $\mathbf{S}/(A \times I)$, since it is right anodyne; it is thus invertible in the category $\mathcal{R}(A \times I)$. The morphism $\mathcal{R}_!([h]) : \mathcal{R}_!(i_0)(X) \to \mathcal{R}_!(i_1)(X)$ is obtained by composing the inclusion $X \times \{0\} \subseteq X \times I$ with the inverse morphism $X \times I \to X \times \{1\}$.

13.12. It follows from the above that the (pseudo-) functor $\mathcal{R}^*$ has the structure of a contravariant (pseudo-) 2-functor,

$$\mathcal{R}^* : \mathbf{S} \to \mathbf{CAT},$$

contravariant on 2-cells.

13.13. If (α, β) is an adjunction between two maps $u : A \leftrightarrow B : v$ in the 2-category $\mathbf{S}^{\tau_1}$, then the pair $(\mathcal{R}_!(\alpha), \mathcal{R}_!(\beta))$ is an adjunction $\mathcal{R}_!(u) \vdash \mathcal{R}_!(v)$ in the 2-category $\mathbf{CAT}$ and the pair $(\mathcal{R}^*(\beta), \mathcal{R}^*(\alpha))$ is an adjunction $\mathcal{R}^*(u) \vdash \mathcal{R}^*(v)$. We thus have a canonical isomorphism $\mathcal{R}_!(v) \simeq \mathcal{R}^*(u)$,

$$\mathcal{R}_!(u) \vdash \mathcal{R}_!(v) \simeq \mathcal{R}^*(u) \vdash \mathcal{R}^*(v).$$

13.14. Dually, the functor

$$\mathcal{L}_! : \mathbf{S} \to \mathbf{CAT}$$

has the structure of a covariant 2-functor, contravariant on 2-cells. The (pseudo-) functor

$$\mathcal{L}^* : \mathbf{S}^{\tau_1} \to \mathbf{CAT}$$

has the structure of a contravariant (pseudo-) 2-functor, covariant on 2-cells. If (α, β) is an adjunction between two maps $u : A \leftrightarrow B : v$ in the 2-category $\mathbf{S}^{\tau_1}$, then the pair $(\mathcal{L}_!(\beta), \mathcal{L}_!(\alpha))$ is an adjunction $\mathcal{L}_!(v) \vdash \mathcal{L}_!(u)$ in the 2-category $\mathbf{CAT}$, and the pair $(\mathcal{L}^*(\alpha), \mathcal{L}^*(\beta))$ is an adjunction $\mathcal{L}^*(v) \vdash \mathcal{L}^*(u)$. We thus have a canonical isomorphism $\mathcal{L}_!u) \simeq \mathcal{L}^*(v)$,

$$\mathcal{L}_!(v) \vdash \mathcal{L}_!!(u) \simeq \mathcal{L}^*_!(v) \vdash \mathcal{L}^*_!(u).$$

13.15. The (pseudo-) 2-functor $\mathcal{L}^*$ induces a functor

$$\mathcal{L}^* : \tau_1(A, B) \to \mathbf{CAT}\big(\mathcal{L}(B), \mathcal{L}(A)\big)$$

for each pair of simplicial sets A and B. In particular, it induces a functor

$$\mathcal{L}^* : \tau_1(B) \to \mathbf{CAT}\big(\mathcal{L}(B), \mathcal{L}(1)\big)$$

for each simplicial set B. If $X \in \mathcal{L}(B)$ and $b \in B_0$, let us put $D(X)(b) = \mathcal{L}^*(b)(X)$. This defines a functor

$$D(X) : \tau_1(B) \to \mathcal{L}(1) = Ho(\mathbf{S}, \mathrm{Who}).$$

We shall say that $D(X)$ is the *homotopy diagram* of X. This extends the notion introduced in 8.15. Dually, every object $X \in \mathcal{R}(B)$ has a contravariant homotopy diagram

$$D(X) : \tau_1(B)^o \to \mathcal{R}(1) = Ho(\mathbf{S}, \mathrm{Who}).$$

14. Cylinders, correspondances, distributors and spans

In this section we introduce the notions of cylinder, correspondance, distributor and span between simplicial sets. To each notion is associated a Quillen model structure and the three model structures are Quillen equivalent. The homotopy bicategory of spans is symmetric monoidal and compact closed. There is an equivalent symmetric monoidal compact closed structure on the homotopy bicategory of distributors.

14.1. Recall that if C is a category, then a set S of objects of C is said to be a *sieve* if the implication

$$\mathrm{target}(f) \in S \Rightarrow \mathrm{source}(f) \in S$$

is true for every arrow $f \in C$. We shall often identify a sieve S with the full subcategory of C spanned by the object of C. A *cosieve* in C is defined dually. The opposite of a sieve $S \subseteq C$ is a cosieve $S^o \subseteq C^o$. For any sieve $S \subseteq C$ (resp. cosieve), there exists a unique functor $p : A \to I$ such that $S = p^{-1}(0)$ (resp. $S = p^{-1}(1)$). We shall say that the sieve $p^{-1}(0)$ and the cosieve $p^{-1}(1)$ are *complementary*. Complementation is a bijection between the sieves and the cosieves of C.

14.2. We shall say that an object of the category $\mathbf{Cat}/I$ is a *0-cylinder*, or just a *cylinder* if the context is clear. The *cobase* of a cylinder $p : C \to I$ is the sieve $C(0) = p^{-1}(0)$ and its *base* is the cosieve $C(1) = p^{-1}(1)$. The category $\mathbf{Cat}/I$ is cartesian closed. If i denotes the inclusion $\partial I \subset I$, then the functor

$$i^* : \mathbf{Cat}/I \to \mathbf{Cat} \times \mathbf{Cat}$$

is a Grothendieck bifibration; its fiber at (A, B) is the category $\mathrm{Cyl}_0(A, B)$ of 0-cylinders with cobase A and base B. The functor i^* has a left adjoint $i_!$ and a right adjoint $i_!$. The cylinder $i_!(A, B) = A \sqcup B$ is the initial object of the category $\mathrm{Cyl}_0(A, B)$ and the cylinder $i_*(A, B) = A \star B$ is the terminal object.

14.3. The model structure $(\mathbf{Cat}, Eq)$ induces a cartesian closed model structure on the category $\mathbf{Cat}/I$.

14.4. If A and B are small categories, we shall say that a functor $F : A^o \times B \to \mathbf{Set}$ is a *0-distributor* (or just a *distributor* if the context is clear), and we shall write $F : A \Rightarrow B$. The distributors $A \Rightarrow B$ are the objects of a category $\mathrm{Dist}_0(A, B) = [A^o \times B, \mathbf{Set}]$. To every cylinder $C \in \mathrm{Cyl}_0(A, B)$ we can associate a distributor $D(C) \in \mathrm{Dist}_0(A, B)$ by putting $D(C)(a, b) = C(a, b)$ for every pair of objects $a \in A$ and $b \in B$. The resulting functor

$$D : \mathrm{Cyl}_0(A, B) \to \mathrm{Dist}_0(B, A)$$

is an equivalence of categories. The inverse equivalence associate to a distributor $F : A^o \times B$ the *collage cylinder* $C = col(F) = A \star_F B$ constructed as follows: $Ob(C) = Ob(A) \sqcup Ob(B)$ and for every $x, y \in Ob(A) \sqcup Ob(B)$,

$$C(x,y) = \begin{cases} A(x,y) & \text{if } x \in A \text{ and } y \in A \\ B(x,y) & \text{if } x \in B \text{ and } y \in B \\ F(x,y) & \text{if } x \in A \text{ and } y \in B \\ \emptyset & \text{if } x \in B \text{ and } y \in A. \end{cases}$$

Composition of arrows is obvious. The obvious functor $p : C \to I$ gives the category C the structure of a cylinder with base B and cobase A. The collage of the distributor $hom : A^o \times A \to \mathbf{Set}$ is the cylinder $A \times I$; the collage of the terminal distributor $1 : A^o \times B \to \mathbf{Set}$ is the join $A \star B$; the collage of the empty distributor $\emptyset : A^o \times A \to \mathbf{Set}$ is the coproduct $A \sqcup A$.

14.5. We shall say that a full simplicial subset $S \subseteq X$ of a simplicial set X is a *sieve* if the implication $\mathrm{target}(f) \in S \Rightarrow \mathrm{source}(f) \in S$ is true for every arrow $f \in X$. If $h : X \to \tau_1 X$ is the canonical map, then the map $S \mapsto h^{-1}(S)$ induces a bijection between the sieves in the category $\tau_1 X$ and the sieves in X. For any sieve $S \subseteq X$ there exists a unique map $g : X \to I$ such that $S = g^{-1}(0)$. This defines a bijection between the sieves in X and the maps $X \to I$. Dually, we shall say that a full simplicial subset $S \subseteq X$ is a *cosieve* if the implication $\mathrm{source}(f) \in S \Rightarrow \mathrm{target}(f) \in S$ is true for every arrow $f \in X$. A simplicial subset $S \subseteq X$ is a cosieve iff the opposite subset $S^o \subseteq X^o$ is a sieve. For any cosieve $S \subseteq X$ there exists a unique map $g : X \to I$ such that $S = g^{-1}(1)$. The cosieve $g^{-1}(1)$ and the sieve $g^{-1}(0)$ are said to be *complementary*. Complementation is a bijection between the sieves and the cosieves of X.

14.6. We shall say that an object $p : C \to I$ of the category $\mathbf{S}/I$ is a *(simplicial) cylinder*. The *base* of the cylinder is the cosieve $C(1) = p^{-1}(1)$ and its *cobase* is the sieve $C(0) = p^{-1}(0)$. If $C(0) = 1$ we say that C is a *projective cone*, and if $C(1) = 1$ we say that it is an *inductive cone*. If $C(0) = C(1) = 1$, we say that C is a *spindle*. If i denotes the inclusion $\partial I \subset I$, then the functor

$$i^* : \mathbf{S}/I \to \mathbf{S} \times \mathbf{S}$$

has left adjoint $i_!$ and a right adjoint i_*. The functor i^* is a Grothendieck bifibration and its fiber at (A, B) is the category $\mathrm{Cyl}(A, B)$ of cylinders with cobase A and base B. The initial object of this category is the cylinder $A \sqcup B$ and its terminal object is the cylinder $A \star B$. An object $q : X \to A \star B$ of the category $\mathbf{S}/A \star B$ belongs to $\mathrm{Cyl}(A, B)$ iff the map $q^{-1}(A \sqcup B) \to A \sqcup B$ induces by q is an isomorphism. It follows that following forgetful functors

$$\mathrm{Cyl}(A,B) \to \mathbf{S}/A \star B, \quad \mathrm{Cyl}(A,B) \to A \sqcup B \backslash \mathbf{S}, \quad \mathrm{Cyl}(A,B) \to A \sqcup B \backslash \mathbf{S}/A \star B$$

are fully faithful. To each pair of maps of simplicial sets $u : A \to A'$ and $v : B \to B'$ is associated a pair of adjoint functors

$$(u, v)_! : \mathrm{Span}(A, B) \leftrightarrow \mathrm{Span}(A', B') : (u, v)^*,$$

since i^* is a bifibration. If $X \in \mathrm{Cyl}(A, B)$, then $(u, v)_!(X)$ is calculated by the following pushout square of simplicial sets,

$$\begin{array}{ccc} A \sqcup B & \xrightarrow{u \sqcup v} & A' \sqcup B' \\ \downarrow & & \downarrow \\ X & \longrightarrow & (u, v)_!(X). \end{array}$$

If $Y \in \mathrm{Cyl}(A', B')$, then $(u, v)^*(Y)$ is calculated by the following pullback square of simplicial sets,

$$\begin{array}{ccc} (u, v)^*(Y) & \longrightarrow & Y \\ \downarrow & & \downarrow \\ A \star B & \xrightarrow{u \star v} & A' \star B'. \end{array}$$

The model category $\mathbf{S}, \mathrm{Wcat})$ induces a model structure on the category $\mathrm{Cyl}(A, B)$. By definition, a map in $\mathrm{Cyl}(A, B)$ is a cofibration (resp. a weak equivalence, resp. a fibration) iff the underlying map in $\mathbf{S}$ is a cofibration (resp. a weak equivalence, resp. a fibration) in $(\mathbf{S}, \mathrm{Wcat})$. We shall denote this model structure by $(\mathrm{Cyl}(A, B), Wcat)$. We conjecture that a cylinder $X \in \mathrm{Cyl}(A, B)$ is fibrant iff the map $X \to A \star B$ is a mid fibration, and that a map between fibrant cylinders is a fibration iff it is a mid fibration. The model structure $(\mathbf{S}, Wcat)$ induces a cartesian closed model structure $(\mathbf{S}/I, Wcat)$ on the category $\mathbf{S}/I$. The pair of adjoint functors

$$(u, v)_! : \mathrm{Span}(A, B) \leftrightarrow \mathrm{Span}(A', B') : (u, v)^*.$$

is a Quillen adjunction for every pair of maps $u : A \to A'$ and $v : B \to B'$, and it is a Quillen equivalence if u and v are weak categorical equivalences. Hence the model category $(\mathbf{S}/I, Wcat)$ is bifibered by the functor i^* over the model category

$$(\mathbf{S}, Wcat) \times (\mathbf{S}, Wcat) = (\mathbf{S} \times \mathbf{S}, Wcat \times Wcat).$$

It induces the model structure $(\mathrm{Cyl}(A, B), \mathrm{Wcat})$ on each fiber $\mathrm{Cyl}(A, B)$.

14.7. The *opposite* of a cylinder $C \in \mathrm{Cyl}(A, B)$ is a cylinder $C^o \in \mathrm{Cyl}(B^o, A^o)$. The functor

$$(-)^o : \mathrm{Cyl}(A, B) \to \mathrm{Cyl}(B^o, A^o)$$

is isomorphism between the model categories $(\mathrm{Cyl}(A, B), Wcat)$ and $(\mathrm{Cyl}(B^o, A^o), Wcat)$.

14.8. The *inductive mapping cone* of a map of simplicial sets $u : A \to B$ is the simplicial set $C(u)$ defined by the following pushout square,

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ \downarrow & & \downarrow \\ A \diamond 1 & \longrightarrow & C(u). \end{array}$$

The simplicial set $C(u)$ is equipped with a map $C(u) \to I$ obtained from the canonical map $A \diamond 1 \to I$. The resulting functor

$$C : \mathbf{S}/B \to \mathrm{Cyl}(B, 1)$$

is the left adjoint in a Quillen equivalence between the model categories $(\mathbf{S}/B, \mathrm{Wcont})$ and $(\mathrm{Cyl}(B, 1), \mathrm{Wcat})$. Dually, the *projective mapping cone* of a map of simplicial sets $u : A \to B$ is the simplicial set $C^o(u) = C(u^o)^o$ constructed by the folllowing pushout square,

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ \downarrow & & \downarrow \\ 1 \diamond A & \longrightarrow & C^o(u). \end{array}$$

The resulting functor

$$C^o : \mathbf{S}/B \to \mathrm{Cyl}(1, B)$$

is the left adjoint in a Quillen equivalence between the model categories $(\mathbf{S}/B, \mathrm{Wcov})$ and $(\mathrm{Cyl}(1, B), \mathrm{Wcat})$. The (unreduced) *suspension* of a simplicial set A is the simplicial set $\Sigma_u(A)$ defined by the following pushout square,

$$\begin{array}{ccc} A \sqcup A & \longrightarrow & 1 \sqcup 1 \\ \downarrow & & \downarrow \\ A \times I & \longrightarrow & \Sigma_u(A). \end{array}$$

The simplicial set $\Sigma_u(A)$ is equipped with a map $\Sigma^u(A) \to I$ obtained from the projection $A \times I \to I$. The resulting functor

$$\Sigma_u : \mathbf{S} \to \mathrm{Cyl}(1, 1)$$

is the left adjoint in a Quillen equivalence between the model category $(\mathbf{S}, \mathrm{Who})$ and the model category $(\mathrm{Cyl}(1, 1), \mathrm{Wcat})$.

14.9. Let $\mathbf{S}^{(2)} = [\Delta^o \times \Delta^o, \mathbf{Set}]$ be the category of bisimplicial sets. If $A, B \in \mathbf{S}$, let us put

$$(A \Box B)_{mn} = A_m \times B_n$$

for $m, n \geq 0$. If X is a bisimplicial set, a map $X \to A \Box 1$ is called a *column augmentation* of X and a map $X \to 1 \Box B$ is called a *row augmentation*. We shall say that a map $X \to A \Box B$ is a *biaugmentation* of X or that it is a *(simplicial) correspondence* $X : A \Rightarrow B$. The correspondences $A \to B$ form a category

$$\mathrm{Cor}(A, B) = \mathbf{S}^{(2)}/A \Box B.$$

The simplicial set $\Delta[m] \star \Delta[n]$ has the structure of a cylinder for every $m, n \geq 0$. To every cylinder $C \in \mathbf{S}/I$ we can associate a correspondance $cor(C) \to C(0) \Box C(1)$ by by putting

$$cor(C)_{mn} = Hom(\Delta[m] \star \Delta[n], C)$$

for every $m, n \geq 0$. The structure map $cor(C) \to C(0) \Box C(1)$ is defined from the inclusions $\Delta[m] \sqcup \Delta[n] \subseteq \Delta[m] \star \Delta[n]$. The induced functor

$$cor : \mathrm{Cyl}(A, B) \to \mathrm{Cor}(A, B).$$

is an equivalence of categories. See[Gon]. A *map* between a correspondance $X \in \mathrm{Cor}(A,B)$ and a correspondance $Y \in \mathrm{Cor}(A',B')$ is defined to be a triple of maps $u : A \to A'$, $v : B \to B'$ and $f : X \to Y$ fitting in a commutative square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ A\Box B & \xrightarrow{u\Box v} & A'\Box B'. \end{array}$$

The correspondances form a category Cor with these maps. The obvious projection functor

$$p : \mathrm{Cor} \to \mathbf{S} \times \mathbf{S}$$

is a Grothendieck bifibration whose fiber at (A,B) is the category $\mathrm{Cor}(A,B)$. The equivalence $cor : \mathrm{Cyl}(A,B) \to \mathrm{Cor}(A,B)$ can be extended as an equivalence of bifibered categories

$$cor : \mathrm{Cyl} \to \mathrm{Cor}.$$

The category Cor has then a model structure (Cor, Wcor) obtained by transporting the model structure (Cyl, Wcat) along this equivalence. The model structure (Cor, Wcor) is bifibered by the projection functor p over the model category

$$(\mathbf{S}, Wcat) \times (\mathbf{S}, Wcat) = (\mathbf{S} \times \mathbf{S}, Wcat \times Wcat).$$

It induces a model structure $(\mathrm{Cor}(A,B), \mathrm{Wcor})$ on each fiber $\mathrm{Cor}(A,B)$ and the functor $cor : \mathrm{Cyl}(A,B) \to \mathrm{Cor}(A,B)$ is an equivalence of model categories.

14.10. A *distributor* $X : A \Rightarrow B$ between two simplicial sets A and B is defined to be a pair of maps

$$\begin{array}{ccc} & X & \\ {}^{s}\swarrow & & \searrow^{t} \\ A^o & & B. \end{array}$$

Equivalently, a distributor $A \Rightarrow B$ is an object of the category

$$\mathrm{Dist}(A,B) = \mathbf{S}/(A^o \times B).$$

We give the category $\mathrm{Dist}(A,B)$ the model structure $(\mathbf{S}/(A^o \times B), \mathrm{Wcov})$ and we shall denote it shortly by $(\mathrm{Dist}(A,B), \mathrm{Wbiv})$. A distributor $X \in \mathrm{Dist}(A,B)$ is fibrant for this model structure iff the map $X \to A^o \times B$ is a left fibration. We shall put

$$hDist(A,B) = Ho(\mathrm{Dist}(A,B), \mathrm{Wcat}).$$

A *map* between a distributor $X \in \mathrm{Dist}(A,B)$ and a distributor $Y \in \mathrm{Dist}(A',B')$ is defined to be a triple of maps $u : A \to A'$, $v : B \to B'$ and $f : X \to Y$ fitting in a commutative square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ A^o \times B & \xrightarrow{u^o \times v} & A'^o \times B'. \end{array}$$

The distributors form a category Dist with these maps. The obvious projection functor

$$p : \mathrm{Dist} \to \mathbf{S} \times \mathbf{S}$$

is a Grothendieck bifibration whose fiber at (A, B) is the category $\mathrm{Dist}(A, B)$. If $u : A \to A'$ and $v : B \to B'$ is a pair of maps of simplicial sets, then the pair of adjoint functor

$$(u, v)_! : \mathrm{Dist}(A, B) \leftrightarrow \mathrm{Dist}(A', B') : (u, v)^*$$

is a Quillen adjunction, and it is a Quillen equivalence if u and v are weak categorical equivalences. We shall say that a map of distributors $(f, u, v) : X \to Y$ as above is a *distributor equivalence* if u and v are weak categorical equivalences and the map $(u, v)_!(X) \to Y$ induces by f is a covariant equivalence in $\mathrm{Dist}(A', B')$. The category Dist admits a model structure in which a cofibration is a monomorphism and a weak equivalence is a distributor equivalence. We shall denote the resulting model category by $(\mathrm{Dist}, \mathrm{Wdist})$, where Wdist denotes the class of distributor equivalences. The model category $(\mathrm{Dist}, \mathrm{Wdist})$ is left proper and cartesian closed. It is bifibered by the projection functor $\mathrm{Dist} \to \mathbf{S} \times \mathbf{S}$ over the model category $(\mathbf{S}, \mathrm{Wcat}) \times (\mathbf{S}, \mathrm{Wcat})$. It induces the model structure $(\mathrm{Dist}(A, B), \mathrm{Wcov})$ on each fiber $\mathrm{Dist}(A, B)$. The *tensor product* of a distributor $X \in \mathrm{Dist}(A, B)$ with a distributor $Y \in \mathrm{Dist}(C, D)$ is defined to be the distributor $X \otimes Y = X \times Y \in \mathrm{Dist}(A \times C, B \times D)$. The tensor product functor

$$\otimes : \mathrm{Dist}(A, B) \times \mathrm{Dist}(C, D) \to \mathrm{Dist}(A \times C, B \times D)$$

is a left Quillen functor of two variables.

14.11. The *transpose* ${}^tX$ of a distributor $(s, t) : X \to A^o \times B$ is the distributor $(t, s) : X \to B \times A^o$. The transposition functor induces an isomorphism of model categories

$${}^t(-) : (\mathrm{Dist}(A, B), \mathrm{Wcov}) \to (\mathrm{Dist}(B^o, A^o), \mathrm{Wcov}).$$

There are canonical isomorphisms of model categories

$$\begin{array}{rcl} (\mathrm{Dist}(1, B), \mathrm{Wcov}) & = & (\mathbf{S}/B, \mathrm{Wcov}) \\ (\mathrm{Dist}(A, 1), \mathrm{Wcov}) & = & (\mathbf{S}/A^o, \mathrm{Wcov}) \simeq (\mathbf{S}/A, \mathrm{Wcont}) \end{array}$$

where the last isomorphism is induced by the functor $X \mapsto X^o$. The model category $(\mathrm{Dist}(1, 1), \mathrm{Wcov})$ is isomorphic to the model category $(\mathbf{S}, \mathrm{Who})$.

14.12. A *span* $S : A \Rightarrow B$ between two simplicial sets is a pair of maps

$$\begin{array}{ccc} & S & \\ {}^{s}\swarrow & & \searrow^{t} \\ A & & B. \end{array}$$

Equivalently, a span $A \Rightarrow B$ is an object of the category

$$\mathrm{Span}(A, B) = \mathbf{S}/(A \times B).$$

The terminal object of this category is the span $A \times_s B$ defined by the pair of projections

$$\begin{array}{ccc} & A \times B & \\ {}^{p_A}\swarrow & & \searrow^{p_B} \\ A & & B. \end{array}$$

We shall say that a map $u : S \to T$ in $\mathrm{Span}(A, B)$ is a *bivariant equivalence* if the map

$$X \times_A u \times_B Y : X \times_A S \times_B Y \to X \times_A T \times_B Y$$

is a weak homotopy equivalence for every $X \in \mathbf{L}(A)$ and $Y \in \mathbf{R}(B)$. For each vertex $a \in A$, let us choose a factorisation $1 \to La \to A$ of the map $a : 1 \to A$ as a left anodyne map $1 \to La$ followed by a left fibration $La \to A$. Dually, for each vertex $b \in B$, let us choose a factorisation $1 \to Rb \to B$ of the map $b : 1 \to B$ as a right anodyne map $1 \to Rb$ followed by a right fibration $Rb \to B$. Then a map $u : S \to T$ in $\mathrm{Span}(A, B)$ is a bivariant equivalence iff the map

$$La \times_A u \times_B Rb : La \times_A S \times_B Rb \to La \times_A T \times_B Rb$$

is a weak homotopy equivalence for every pair of vertices $(a, b) \in A \times B$. If A and B are quategories, we can take $La = a\backslash A$ and $Rb = B/b$. In this case, a map $u : S \to T$ in $\mathrm{Span}(A, B)$ is a bivariant equivalence iff the map

$$a\backslash u/b : a\backslash S/b \to a\backslash T/b$$

is a weak homotopy equivalence for every pair of objects $(a, b) \in A \times B$, where the simplicial set $a\backslash S/b$ is defined by the pullback square

$$\begin{array}{ccc} a\backslash S/b & \longrightarrow & S \\ \downarrow & & \downarrow \\ a\backslash A \times B/b & \longrightarrow & A \times B. \end{array}$$

The category $\mathrm{Span}(A, B)$ admits a model structure in which a weak equivalence is a bivariant equivalence and a cofibration is a monomorphism. We shall say that a fibrant object $S \in \mathrm{Span}(A, B)$ is *bifibrant*. We shall denote this model category shortly by $\mathrm{Span}(A, B), \mathrm{Wbiv})$ and put

$$hSpan(A, B) = Ho(\mathrm{Span}(A, B), \mathrm{Wbiv}).$$

A span can be defined to be a simplicial presheaf $X : P^o \to \mathbf{S}$ on the poset P of non-empty subsets of $\{0, 1\}$,

$$\begin{array}{ccccc} & & X(01) & & \\ & \swarrow^{s} & & \searrow^{t} & \\ X(0) & & & & X(1). \end{array}$$

A map between a span $X \in \mathrm{Span}(A, B)$ and a span $Y \in \mathrm{Span}(A', B')$ is a triple of maps $u : A \to A'$, $v : B \to B'$ and $f : X \to Y$ fitting in a commutative square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ A \times B & \xrightarrow{u \times v} & A' \times B'. \end{array}$$

We shall denote the category of spans $[P^o, \mathbf{S}]$ by Span. The obvious projection

$$\mathrm{Span} \to \mathbf{S} \times \mathbf{S}$$

is a Grothendieck bifibration whose fiber at (A, B) is the category $\mathrm{Span}(A, B)$. For any pair of maps of simplicial sets $u : A \to A'$ and $v : B \to B'$, the pair of adjoint functors

$$(u, v)_! : \mathrm{Span}(A, B) \leftrightarrow \mathrm{Span}(A', B') : (u, v)^*$$

is a Quillen adjunction, and it is a Quillen equivalence if u and v are weak categorical equivalences. We shall say that a map $(f, u, v) : X \to Y$ in the category Span is a *span equivalence* if u and v are weak categorical equivalences and the map $(u, v)_!(X) \to Y$ induced by f is a bivariant equivalence in $\mathrm{Span}(A', B')$. The category Span admits a model structure in which a cofibration is a monomorphism and a weak equivalence is a span equivalence. We shall denote the resulting model category by (Span, Wspan), where Wspan denotes the class of span equivalences. The model category is left proper and cartesian closed. It is bifibered by the projection functor $\mathrm{Span} \to \mathbf{S} \times \mathbf{S}$ over the model category $(\mathbf{S}, \mathrm{Wcat}) \times (\mathbf{S}, \mathrm{Wcat})$. It induces the model structure $(\mathrm{Span}(A, B), \mathrm{Wbiv})$ on each fiber $\mathrm{Span}(A, B)$.

14.13. For any span $(s, t) : S \to A \times B$, the composite

$$\mathcal{L}(A) \xrightarrow{\mathcal{L}^*(s)} \mathcal{L}(S) \xrightarrow{\mathcal{L}_!(t)} \mathcal{L}(B)$$

is a functor $\mathcal{L}\langle S\rangle : \mathcal{L}(A) \to \mathcal{L}(B)$. If $u : S \to T$ is a map of spans,

$$\begin{array}{ccccc} & & S & & \\ & \swarrow^{s} & & \searrow^{t} & \\ A & & \downarrow u & & B \\ & \nwarrow_{l} & & \nearrow_{r} & \\ & & T & & \end{array},$$

then form the counit $\mathcal{L}_!(u)\mathcal{L}^*(u) \to id$, we obtain a natural transformation

$$\mathcal{L}\langle u\rangle : \mathcal{L}\langle S\rangle = \mathcal{L}_!(t)\mathcal{L}^*(s) = \mathcal{L}_!(r)\mathcal{L}_!(u)\mathcal{L}^*(u)\mathcal{L}^*(l) \to \mathcal{L}_!(r)\mathcal{L}^*(l) = \mathcal{L}\langle T\rangle.$$

This defines a functor

$$\mathcal{L}\langle -\rangle : \mathrm{Span}(A, B) \to \mathbf{CAT}(\mathcal{L}(A), \mathcal{L}(B)).$$

A map $u : S \to T$ in $\mathrm{Span}(A, B)$ is a bivariant equivalence iff the natural transformation

$$\mathcal{L}\langle u\rangle : \mathcal{L}\langle S\rangle \to \mathcal{L}\langle T\rangle$$

is invertible.

14.14. Dually, for any span $(s, t) : S \to A \times B$, the composite

$$\mathcal{R}(B) \xrightarrow{\mathcal{R}^*(t)} \mathcal{R}(S) \xrightarrow{\mathcal{R}_!(s)} \mathcal{R}(A)$$

is a functor $\mathcal{R}\langle S\rangle : \mathcal{R}(B) \to \mathcal{R}(A)$. To every map $u : S \to T$ in $\mathrm{Span}(A, B)$ we can associate a natural transformation

$$\mathcal{R}\langle u\rangle : \mathcal{R}\langle S\rangle \to \mathcal{R}\langle T\rangle.$$

We obtain a functor

$$\mathcal{R}\langle -\rangle : \mathrm{Span}(A, B) \to \mathbf{CAT}(\mathcal{R}(B), \mathcal{R}(A)).$$

A map $u : S \to T$ in $\mathrm{Span}(A, B)$ is a bivariant equivalence iff the natural transformation $\mathcal{R}\langle u\rangle$ is invertible.

14.15. If A and B are quategories, then a span $(s,t): S \to A \times B$ is bifibrant iff the following conditions are satisfied:

- the source map $s: S \to A$ is a Grothendieck fibration;
- the target map $t: S \to B$ is a Grothendieck opfibration;
- an arrow $f \in S$ is inverted by t iff f is cartesian with respect to s;
- an arrow $f \in S$ is inverted by s iff f is cocartesian with respect to t.

The last two conditions are equivalent in the presence of the first two. Let us denote by $S(a,b)$ the fiber of the map $(s,t)S \to A \times B$ at $(a,b) \in A_0 \times B_0$. The simplicial set $S(a,b)$ is a Kan complex if S is bifibrant. A map between bifibrant spans $u: S \to T$ in $\mathrm{Span}(A,B)$ a bivariant equivalence iff the map

$$S(a,b) \to T(a,b)$$

induced by u is a homotopy equivalence for every pair $(a,b) \in A_0 \times B_0$.

14.16. The *conjugate* $S^\dagger$ of a span $(s,t): S \to A \times B$ is defined to be the span $(t^o, s^o): S^o \to B^o \times A^o$. The conjugation functor induces an isomorphism of model categories

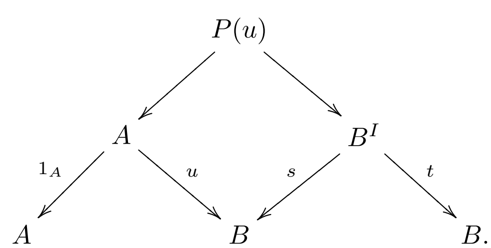
$$(-)^\dagger : \mathrm{Span}(A,B) \to \mathrm{Span}(B^o, A^o).$$

There are canonical isomorphisms of model categories

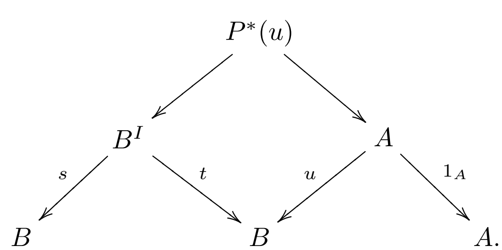
$$(\mathrm{Span}(1,B), \mathrm{Wbiv}) = (\mathbf{S}/B, \mathrm{Wcov}) \quad \text{and} \quad (\mathrm{Span}(A,1), \mathrm{Wbiv}) = (\mathbf{S}/A, \mathrm{Wcont}).$$

The model category $(\mathrm{Span}(1,1), \mathrm{Wbiv})$ is isomorphic to the model category $(\mathbf{S}, \mathrm{Who})$.

14.17. If A is a quategory, then a bifibrant replacement of the span $(1_A, 1_A): A \to A \times A$ is the span $\delta_A = (s,t): A^I \to A \times A$. If $u: A \to B$ is a map between quategories then a fibrant replacement of the span $(1_A, u): A \to A \times B$ is the span $P(u) \to A \times B$ defined by the pullback diagram,



Dually, a bifibrant replacement of the span $(u, 1_A): A \to B \times A$ is the span $P^*(u) \to B \times A$ defined in the pullback diagram,



14.18. For every $n \geq 0$, the simplicial set $\rho[n] = \Delta[n]^o \star \Delta[n]$ has the structure of a cylinder, The *twisted core* of a cylinder C is the simplicial set $\rho^*(C)$ defined by putting

$$\rho^*(C)_n = Hom_I(\rho[n], C)$$

for every $n \geq 0$. The simplicial set $\rho^*(C)$ has the structure of a distributor, $(s,t) : \rho^*(C) \to C(0)^o \times C(1)$, where s is defined from the inclusion $\Delta[n]^o \subset \Delta[n]^o \star \Delta[n]$ and t from the inclusion $\Delta[n] \subset \Delta[n]^o \star \Delta[n]$. The resulting functor $\rho^* : \mathbf{S}/I \to \mathrm{Dist}$ has a right adjoint ρ_* and the pair of adjoint functors

$$\rho^* : \mathbf{S}/I \leftrightarrow \mathrm{Dist} : \rho_*$$

is a Quillen equivalence between the model categories $(\mathrm{Dist}, \mathrm{Wdist})$ and $(\mathbf{S}/I, \mathrm{Wcat})$. The functor ρ^* is cartesian with respect to the fibered structure on these categories. The induced pair of adjoint functors

$$\rho^* : \mathrm{Cyl}(A,B) \leftrightarrow \mathrm{Dist}(A,B) : \rho_*$$

is a Quillen equivalence between the model category $\mathrm{Dist}(A,B), \mathrm{Wcov})$ and the model category $\mathrm{Cyl}(A,B), \mathrm{Wcat})$ for any pair (A,B).

14.19. The *path space* of a cylinder $C \to I$ is defined to be simplicial set $[I, C]$ of global sections of the map $C \to I$. The simplicial set $[I, C]$ has the structure of a span

$$\begin{array}{ccc} & [I,C] & \\ {}^{s}\swarrow & & \searrow^{t} \\ C(0) & & C(1), \end{array}$$

where s is defined from the inclusion $\{0\} \subset I$ and t from the inclusion : $\{1\} \subset I$. This defines a functor $[I, -] : \mathbf{S}/I \to \mathrm{Span}$. The *realisation* of a span $S \in \mathrm{Span}(A,B)$ is the simplicial set defined by the following pushout square,

$$\begin{array}{ccc} S \sqcup S & \xrightarrow{s \sqcup t} & A \sqcup B \\ \downarrow & & \downarrow \\ I \times S & \longrightarrow & R(S). \end{array}$$

The simplicial set $R(S)$ has the structure of a cylinder. The resulting functor $R : \mathrm{Span} \to \mathbf{S}/I$ is left adjoint to the functor $[I, -]$. Moreover, rhe pair of adjoint functors

$$R : \mathrm{Span} \leftrightarrow \mathbf{S}/I : [I, -]$$

is a Quillen equivalence between the model category $(\mathrm{Span}, \mathrm{Wspan})$ and the model category $\mathbf{S}/I, \mathrm{Wcat})$. The adjoint pair is compatible with the fibered model structure on these categories. It thus induces a Quillen equivalence

$$R : \mathrm{Span}(A,B) \leftrightarrow \mathrm{Cyl}(A,B) : [I, -]$$

between the model category $(\mathrm{Span}(A,B), \mathrm{Wbiv})$ and the model category $\mathrm{Cyl}(A,B), \mathrm{Wcat})$ for any pair of simplicial sets (A,B).

14.20. The simplicial set $\delta[n] = \Delta[n] \star \Delta[n]$ has the structure of a cylinder for every $n \geq 0$. The *core* of a cylinder C is the simplicial set $\delta^*(C)$ obtained by putting

$$\delta^*(C)_n = Hom(\delta[n], C)$$

for every $n \geq 0$. The simplicial set $\delta^*(C)$ has the structure of a span

$$\begin{array}{ccc} & \delta^*(C) & \\ {}^{s}\swarrow & & \searrow^{t} \\ C(0) & & C(1), \end{array}$$

where s and t are defined from the canonical inclusions $\Delta[n] \subset \Delta[n] \star \Delta[n]$. The resulting functor $\delta^* : \mathbf{S}/I \to \mathrm{Span}$ has a right adjoint δ_* and the pair

$$\delta^* : \mathrm{Cyl} \to \mathrm{Span} : \delta_*$$

is a Quillen equivalence between the model category $(\mathbf{S}/I, \mathrm{Wcat})$ and the model category $(\mathrm{Span}, \mathrm{Wbiv})$. The functor δ^* is cartesian with respect to the fibered structure on these categories. The induced adjoint pair

$$\delta^* : \mathrm{Cyl}(A, B) \leftrightarrow \mathrm{Span}(A, B) : \delta_*$$

is also a Quillen equivalence between the model category $\mathrm{Cyl}(A, B), \mathrm{Wcat})$ and the model category $(\mathrm{Span}(A, B), \mathrm{Wbiv})$ for any pair (A, B).

14.21. For any pair of simplicial sets A and B we have $\delta^*(A \star B = A \times_s B$ and $R(A \times_s B) = A \diamond B$. Hence the map $\theta_{AB} : A \diamond B \to A \star B$ of 9.18 is a map $\theta_{AB} : R\delta^*(A \star B) \to A \star B$. There is a unique natural transformation

$$\theta_C : R\delta^*(C) \to C$$

which extends the maps θ_{AB} to every cylinder $C \to I$. The maps θ_C is a weak categorical equivalence for every $C \in \mathbf{S}/I$. This shows that the left derived functors $L(R)$ and $L(\delta^*)$ are mutually inverse (up to a natural isomorphism).

14.22. Recall that the *composite* of a span $S : A \to B$ with a span $T : B \Rightarrow C$ is the span $T \circ S = S \times_B T : A \Rightarrow C$, defined by the pullback diagram,

$$\begin{array}{ccccc} & & S \times_B T & & \\ & \swarrow & & \searrow & \\ & S & & T & \\ {}^{s}\swarrow & & {}^{t}\searrow \quad {}^{s}\swarrow & & \searrow^{t} \\ A & & B & & C. \end{array}$$

This defines a functor

$$- \circ - : \mathrm{Span}(B, C) \times \mathrm{Span}(A, B) \to \mathrm{Span}(A, C).$$

For any three spans $S : A \Rightarrow B$, $T : B \Rightarrow C$ and $U : C \Rightarrow D$, the canonical isomorphism

$$(U \circ T) \circ S = S \times_B (T \times_C U) \simeq (S \times_B T) \times_C U = U \circ (T \circ S)$$

satisfies the coherence condition of MacLane. The span $(1_A, 1_A) : A \to A \times A$ is a unit $A \to A$ for this composition law. This defines the *bicategory of spans* Span.

The cartesian product of a span $X \in \mathrm{Span}(A, B)$ with a span $Y \in \mathrm{Span}(C, D)$ is a span $X \times Y \in \mathrm{Span}(A \times C, B \times D)$. The product functor

$$\mathrm{Span}(A, B) \times \mathrm{Span}(C, D) \to \mathrm{Span}(A \times C, B \times D)$$

define a a symmetric monoidal structure on the bicategory Span. If the span $S \in \mathrm{Span}(A, B)$ is bifibrant, then the functor

$$(-) \circ S : \mathrm{Span}(B, C) \to \mathrm{Span}(A, C)$$

is a left Quillen functor. Dually, if the span $T \in \mathrm{Span}(A, B)$ is bifibrant, then the functor

$$T \circ (-) : \mathrm{Span}(A, B) \to \mathrm{Span}(A, C)$$

is a left Quillen functor. Let us denote by $\mathrm{Span}(A, B)_f$ the full subcategory of $\mathrm{Span}(A, B)$ spanned by the bifibrant spans. Then the composition functor

$$- \circ - : \mathrm{Span}(B, C)_f \times \mathrm{Span}(A, B)_f \to \mathrm{Span}(A, C),$$

induces a *derived composition*

$$- \circ - : \mathrm{hSpan}(B, C) \times \mathrm{hSpan}(A, B) \to \mathrm{hSpan}(A, C).$$

The derived composition is coherently associative. A unit $I_A \in \mathrm{hSpan}(A, A)$ for this composition is a fibrant replacement of the span $(1_A, 1_A) : A \to A \times A$. We thus obtain a bicategory hSpan called the *homotopy bicategory of spans*. The product functor

$$\mathrm{Span}(A, B) \times \mathrm{Span}(C, D) \to \mathrm{Span}(A \times C, B \times D)$$

is a left Quillen functor of two variables with respect to the bivariant model structures on these categories. The corresponding derived functor

$$\otimes : \mathrm{hSpan}(A, B) \times \mathrm{hSpan}(C, D) \to \mathrm{hSpan}(A \times C, B \times D)$$

defines a symmetric monoidal structure on the bicategory hSpan. . .

14.23. The *twisted diagonal* A^δ of a simplicial set A is defined to be the twisted core of the cylinder $A \times I$. By definition, we have

$$(A^\delta)_n = \mathbf{S}(\Delta[n]^o \star \Delta[n], A)$$

for every $n \geq 0$. For example, the twisted diagonal of a category C is the category of elements of the hom functor $C^o \times C \to \mathbf{Set}$. The functor $(-)^\delta : \mathbf{S} \to \mathrm{Dist}$ has a left adjoint $(-)_\delta$, and the adjoint pair

$$(-)_\delta : \mathrm{Dist} \leftrightarrow \mathbf{S} : (-)^\delta$$

is a Quillen adjunction between the model categories $(\mathrm{Dist}, \mathrm{Wdist})$ and $(\mathbf{S}, \mathrm{Wcat})$. Hence the canonical map $A^\delta \to A^o \times A$ is a left fibration when A is a quategory.

14.24. The symmetric monoidal bicategory hSpan is compact closed. The dual of a simplicial set A is the opposite simplicial set A^o. The duality is defined by a pair of spans

$$\eta_A : 1 \Rightarrow A^o \times A \quad \text{and} \quad \epsilon_A : A \times A^o \Rightarrow 1$$

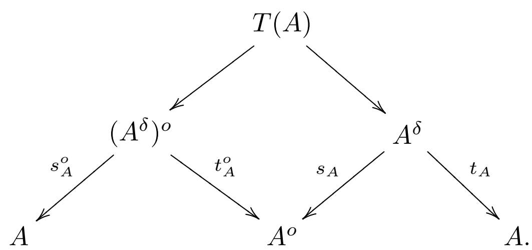
together with a pair of isomorphisms,

$$\alpha_A : I_A \simeq (\epsilon_A \otimes A) \circ (A \otimes \eta_A) \quad \text{and} \quad \beta_A : I_{A^o} \simeq (A^o \otimes \epsilon_A) \circ (\eta_A \otimes A^o).$$

The unit η_A is defined by the left fibration $p : A^\delta \to 1 \times (A^o \times A)$ of 14.23 and the counit ϵ_A by the opposite right fibration $p^o : (A^\delta)^o \to (A \times A^o) \times 1$. The isomorphism α_A can be described as follows, It is easy to see that the span

$$T(A) = (\epsilon_A \otimes A) \circ (A \otimes \eta_A).$$

can be constructed by the following pullback diagram,



A simplex $\Delta[n] \to T(A)$ is a pair of simplices $x : \Delta[n] \star \Delta[n]^o \to A$ and $y : \Delta[n]^o \star \Delta[n] \to A$ such that $x \mid \Delta[n]^o = y \mid \Delta[n]^o$. The isomorphism α_A of is obtained by composing in $\mathrm{hSpan}(A, A)$ a chain of bivariant equivalences

$$A^I \xleftarrow{q_A} U(A) \xrightarrow{p_A} T(A)$$

in $\mathrm{Span}(A, A)$. The simplicial set $U(A)$ is defined by putting

$$U(A)_n = \mathbf{S}(\Delta[n] \star \Delta[n]^o \star \Delta[n], A)$$

for every $n \geq 0$ and the structure map $U(A) \to A \times A$ is obtained from the obvious inclusion $i_n : \Delta[n] \sqcup \Delta[n] \subset \Delta[n] \star \Delta[n]^o \star \Delta[n]$. Let us describe the map $p_A : U(A) \to T(A)$. If $z : \Delta[n] \star \Delta[n]^o \star \Delta[n] \to A$ is a simplex of $U(A)$, then we have $p_A(z) = (x, y)$, where $x = z \mid \Delta[n] \star \Delta[n]^o$ and $y = z \mid \Delta[n]^o \star \Delta[n]$. Let us describe the map $q_A : U(A) \to A^I$. We have $q_A(x) = xj_n$ for every $x \in U(A)_n$, where

$$j_n : \Delta[n] \times I \to \Delta[n] \star \Delta[n]^o \star \Delta[n]$$

denotes the a unique extension of i_n along the inclusion $\Delta[n] \sqcup \Delta[n] = \Delta[n] \times \{0, 1\} \subset \Delta[n] \times I$, The isomorphism β_A has a similar description.

14.25. The conjugate $S^\dagger$ of a span $S \in \mathrm{hSpan}(A, B)$ is naturally isomorphic to the span

$$(A^o \otimes \epsilon_B) \circ (A^o \otimes S \otimes B^o) \circ (\eta_A \otimes B^o)$$

The *scalar product* of a span $S \in \mathrm{hSpan}(A, B)$ with a span $T \in \mathrm{hSpan}(B, A)$ is the object $\langle S|T\rangle \in \mathrm{hSpan}(1, 1) = Ho(\mathbf{S}, \mathrm{Who})$ defined by putting

$$\langle S|T\rangle = \epsilon_B \circ (S \otimes T^\dagger) \circ \eta_{A^o},$$

where $T^\dagger \in \mathrm{hSpan}(A^o, B^o)$ is the conjugate of T. A map $u : S \to S'$ in $\mathrm{hSpan}(A, B)$ is invertible iff the map

$$\langle u|T\rangle : \langle S|T\rangle \to \langle S'|T\rangle$$

is invertible for every $T \in \mathrm{hSpan}(B, A)$. The *trace* $Tr(X)$ of a span $X \in \mathrm{Span}(A, A)$ is defined by putting

$$Tr(X) = \langle X|I_A\rangle$$

where $I_A \in \mathrm{Span}(A, A)$ is a unit span. There is a natural isomorphism

$$Tr(X) \simeq \epsilon_A \circ (X \otimes A^o) \circ \eta_{A^o}$$

in the category $Ho(\mathbf{S}, \text{Who}) = \text{hSpan}(1, 1)$. The scalar product of $S \in \text{hSpan}(A, B)$ and $T \in \text{hSpan}(B, A)$ is naturally isomorphic to the trace of $T \circ S$ and to the trace of $S \circ T$. Hence there is a natural isomorphism,

$$Tr(T \circ S) \simeq Tr(S \circ T).$$

The C-*trace* of a span $X \in \text{hSpan}(A \times C, B \times C)$ is the span $Tr_C(X) \in \text{hSpan}(A, B)$ defined by putting

$$Tr_C(X) = (B \otimes \epsilon_C) \circ (X \otimes C^o) \circ (A \otimes \eta_{C^o}).$$

The composite of a span $S \in \text{hSpan}(A, B)$ with a span $T \in \text{hSpan}(B, C)$ is canonically isomorphic to the B-trace of the span $S \otimes T \in \text{hSpan}(A \times B, B \times C)$.

14.26. If we compose the equivalence between spans and cylinders

$$R : \text{Span}(A, B) \leftrightarrow \text{Cyl}(A, B) : [I, -]$$

of 14.19 with the equivalence between cylinders and distributors

$$\rho^* : \text{Cyl}(A, B) \leftrightarrow \text{Dist}(A, B) : \rho_*$$

of 14.18, we obtain an equivalence between spans and distributors

$$\text{Span}(A, B) \leftrightarrow \text{Dist}(A, B).$$

The derived equivalence

$$\text{hSpan}(A, B) \leftrightarrow \text{hDist}(A, B)$$

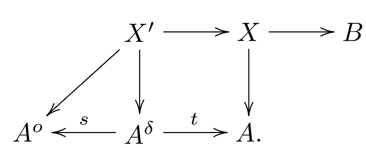
can be obtained more simply by using the isomorphism

$$\text{hSpan}(1, A^o \times B) = \mathcal{L}(A^o \times B) = \text{hDist}(A, B)$$

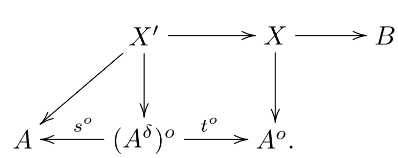
and the duality

$$\text{hSpan}(A, B) \to \text{hSpan}(1, A^o \times B).$$

The equivalence associates to a bifibrant span $S \to A \times B$ the distributor $S' \to A^o \times B$ calculated by following diagram with a pullback square,



The inverse equivalence associates to a fibrant distributor $X \to A^o \times B$ the span $X' \to A \times B$ calculated by following diagram with a pullback square,



14.27. The *scalar product* of two distributors $S \in \mathrm{Dist}(A, B)$ and $T \in \mathrm{Dist}(B, A)$ is defined by putting

$$\langle S|T\rangle = p_!(S \times_{A^o \times B} T^\dagger)$$

where $T^\dagger$ denotes and $(t, s) : T \to A \times B^o$ (the adjoint of T), and where p is the map $A^o \times B \to 1$. The functor $S \mapsto \langle S|T\rangle$ is a left Quillen functor

$$(\mathrm{Dist}(A, B), \mathrm{Wcov}) \to (\mathbf{S}, \mathrm{Who})$$

when T is fibrant. Similarly, the functor $T \mapsto \langle S|T\rangle$ is a left Quillen functor

$$(\mathrm{Dist}(B, A), \mathrm{Wcov}) \to (\mathbf{S}, \mathrm{Who})$$

when S is fibrant. There is a resulting derived scalar product

$$\langle -|-\rangle : \mathrm{hDist}(A, B) \times \mathrm{hDist}(B, A) \to \mathrm{hDist}(1, 1).$$

The *trace* $Tr(X)$ of a distributor $X \in \mathrm{Dist}(A, A)$ is defined by putting

$$Tr(X) = \langle X|A^\delta\rangle,$$

where A^δ is the distributor defined in 14.23. The trace functor is a left Quillen functor

$$Tr : \mathrm{Dist}(A, A) \to \mathrm{Dist}(1, 1)$$

when A is a quategory. The B-*trace* of a distributor $X \in \mathrm{Dist}(A \times B, B \times C)$ is the distributor $Tr_B(X) \in \mathrm{Dist}(A, C)$ defined by putting $Tr_B(X) = p_! q^*(X)$, where $q = B^o \times (s, t) \times C$ and p is the projection

$$A^o \times C \xleftarrow{\ p\ } A^o \times B^\delta \times C \xrightarrow{\ q\ } A^o \times B^o \times B \times C.$$

The B-trace functor is a left Quillen functor

$$Tr_B : \mathrm{Dist}(A \times B, B \times C) \to \mathrm{Dist}(A, C)$$

when B is a quategory. It thus induces a functor

$$Tr_B : \mathrm{hDist}(A \times B, B \times C) \to \mathrm{hDist}(A, C).$$

The *composite* of a distributor $S \in \mathrm{hDist}(A, B)$ with a distributor $T \in \mathrm{hDist}(B, C)$ is defined to be the B-trace of their tensor product $S \otimes T \in \mathrm{hDist}(A \times B, B \times C)$. The resulting composition functor

$$\circ : \mathrm{hDist}(B, C) \times \mathrm{hDist}(A, B) \to \mathrm{hDist}(A, C)$$

is coherently associative and the distributor $A^\delta \in \mathrm{hDist}(A, B)$ is a unit for this composition. We thus obtain a bicategory hDist called the *homotopy bicategory of distributors*. The bicategory hDist is symmetric monoidal and compact closed. The equivalence $\mathrm{hSpan}(A, B) \to \mathrm{hDist}(A, B)$ of 14.26 can be extended as an equivalence symmetric monoidal bicategories,

$$\mathrm{hSpan} \simeq \mathrm{hDist}.$$

14.28. There are simplicially enriched versions of the notions of cylinder and distributor. See 51.13 and 51.15. for a comparaison with the notions presented in this section.

15. Yoneda lemmas

Te extension of Yoneda lemma to quategories has many incarnations. We describe five forms of the extension. We use the lemma to strictify a quategory.

15.1. (Yoneda lemma 1) Recall that the category $\mathbf{S}/B$ is enriched over simplicial sets for any simplicial set B. Let us denote by $[X, Y]$ the simplicial set of maps $X \to Y$ between two objects of $\mathbf{S}/B$. A vertex $b \in B$ defines a map $b : 1 \to B$ and for every object $p : X \to B$ of $\mathbf{S}/B$ we have $[b, X] = X(b)$, where $X(b) = p^{-1}(b)$. We shall say an object E of $\mathbf{S}/B$ is *represented* by a vertex $a \in E(b)$ if the resulting morphism $a : b \to E$ is a contravariant equivalence in $\mathbf{S}/B$. Equivalently, E is a represented by $a \in E(b)$ if the evaluation map

$$a^* : [E, X] \to X(b)$$

is an homotopy equivalence for every $X \in \mathbf{R}(B)$. An object $E \in \mathbf{R}(B)$ is represented by a vertex $a \in E$ iff a is a terminal vertex of the simplicial set E. For example, if B is a quategory, then the right fibration $B/b \to B$ is represented by the unit $1_b \in B/b$. Hence the evalutation map

$$1_b^* : [B/b, X] \to X(b)$$

is a homotopy equivalence for every $X \in \mathbf{R}(B)$.

15.2. Dually, we shall say an object E of $\mathbf{S}/B$ is *corepresented* by a vertex $a \in E(b)$ if the resulting morphism $a : b \to E$ is a covariant equivalence in $\mathbf{S}/B$. Equivalently, E is a represented by $a \in E(b)$ if the evaluation map

$$a^* : [E, X] \to X(b)$$

is an homotopy equivalence for every $X \in \mathbf{L}(B)$. An object $E \in \mathbf{L}(B)$ is represented by a vertex $a \in E$ iff a is an initial vertex of the simplicial set E. For example, if B is a quategory, then the left fibration $b\backslash B \to B$ is represented by the unit $1_b \in b\backslash B$. Hence the evalutation map

$$1_b^* : [b\backslash B, X] \to X(b)$$

is a homotopy equivalence for every $X \in \mathbf{L}(B)$.

15.3. If B is a quategory, then the right fibration $B/\!/b \to B$ is represented by the unit $1_b \in B/\!/b$. Hence the evalutation map

$$i_b^* : [B/\!/b, X] \to X(b)$$

is a homotopy equivalence for every $X \in \mathbf{R}(B)$. In particular, the evalutation map

$$[B/\!/b, B/\!/c] \to B(b, c)$$

is a homotopy equivalence for every object $c \in B$. Consider the simplicial category $\overline{B}$ having the same objects as B and defined by putting

$$\overline{B}(a, b) = [B/\!/a, B/\!/b].$$

The category $\overline{B}$ is enriched over Kan complexes and its coherent nerve is equivalent to B.

15.4. Let $f : a \to b$ be an arrow in a quategory X. Then by Yoneda lemma, there is a map $f_! : B/a \to B/b$ in $\mathbf{S}/B$ such that $f_!(1_a) = b$ and f is homotopy unique. We shall say that $f_!$ is the *pushforward map along f*.

15.5. Recall that the quategory $\mathcal{K}$ is the coherent nerve of the category **Kan** of Kan complexes. Let us put $\mathcal{K}' = 1\backslash\mathcal{K}$, where 1 denotes the terminal object of the quategory $\mathcal{K}$. Then the canonical map $p : \mathcal{K}' \to \mathcal{K}$ is a universal left fibration. The universality means that for any left fibration $f : E \to A$ there exists a homotopy pullback square in $(\mathbf{S}, \mathrm{Wcat})$,

$$\begin{array}{ccc} E & \xrightarrow{g'} & \mathcal{K}' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow{g} & \mathcal{K}, \end{array}$$

and moreover that the pair (g, g') is homotopy unique. We shall say that g is the *classifying map* of the left fibration $E \to A$.

15.6. The *simplicial set of elements* $el(f)$ of a map $f : B \to \mathcal{K}$ is defined by the pullback square

$$\begin{array}{ccc} el(f) & \xrightarrow{q} & \mathcal{K}' \\ \downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{f} & \mathcal{K}. \end{array}$$

The map q is a left fibration, since p is a left fibration. Moreover, f is classifying q. The simplicial set $el(f)$ is a quategory if B is a quategory. We shall say that a map $f : B \to \mathcal{K}$ is *represented* by an element $a : 1 \to f(b)$ if the left fibration $el(f) \to B$ is corepresentes by the vertex $a \in el(f)(b)$.

15.7. A *prestack* on a simplicial set A is defined to be a map $B^o \to \mathcal{K}$. The prestacks on B form a quategory

$$\mathcal{P}(B) = \mathcal{K}^{B^o} = [B^o, \mathcal{K}].$$

The *simplicial set of elements* $El(g)$ of a prestack $g : B^o \to \mathcal{K}$ is defined by putting $El(g) = el(g)^o$. The canonical map $El(g) \to B$ is a right fibration. We shall say that a prestack $g : B^o \to \mathcal{K}$ is *represented* by an element $a : 1 \to g(b)$ if the right fibration $El(g) \to B$ is represented by the vertex $a \in El(g)(b)$.

15.8. Recall that the *twisted diagonal* C^δ of a category C is the category of elements of the hom functor $C^o \times C \to \mathbf{Set}$. Similarly, the twisted diagonal of a quategory A is the domain of a left fibration $(s, t) : A^\delta \to A^o \times A$ by 14.23. The *hom map*

$$hom_A : A^o \times A \to \mathcal{K}$$

is defined to be the classifying map of this left fibration, The *Yoneda map*,

$$y_A : A \to \mathcal{P}(A)$$

is obtained by transposing the map hom_A.

15.9. (Yoneda lemma 2) The vertices of the simplicial set B^δ are the arrows of B. In particular, to every vertex $b \in B$ corresponds an arrow $1_b \in B^\delta$ with $s(1_b) = t(1_b) = b$. This defines a morphism $1_b : 1 \to hom_B(b, b)$, hence also a morphism $1_b : 1 \to y(b)(b)$. If B is a quategory, then the evaluation map

$$ev_b : \mathcal{P}(B) \to \mathcal{K}$$

is represented by the element $1_b : 1 \to y(b)(b)$.

15.10. The left fibration $L_A \to A \times \mathcal{K}^A$ defined by the pullback square

$$\begin{array}{ccc} L_A & \longrightarrow & \mathcal{K}' \\ \downarrow & & \downarrow \\ A \times \mathcal{K}^A & \xrightarrow{ev} & \mathcal{K} \end{array}$$

is universal, where *ev* denotes the evaluation map. The universality means that for any simplicial set B and any left fibration $E \to A \times B$, there exists a homotopy pullback square in $(\mathbf{S}, \mathrm{Wcat})$,

$$\begin{array}{ccc} E & \xrightarrow{g'} & L_A \\ \downarrow & & \downarrow \\ A \times B & \xrightarrow{A \times g} & A \times \mathcal{K}^A \end{array}$$

and that the pair (g, g') is homotopy unique.

15.11. Dually, the left fibration M_A defined by the pullback square

$$\begin{array}{ccc} M_A & \longrightarrow & \mathcal{K}' \\ \downarrow & & \downarrow \\ A^o \times \mathcal{P}(A) & \xrightarrow{ev} & \mathcal{K} \end{array}$$

is a universal distributor $A \Rightarrow \mathcal{P}(A)$. More precisely, for any simplicial set B and any fibrant distributor $E : A \Rightarrow B$, there exists a homotopy pullback square in the model category $(\mathbf{S}, \mathrm{Wcat})$,

$$\begin{array}{ccc} E & \xrightarrow{g'} & M_A \\ \downarrow & & \downarrow \\ A^o \times B & \xrightarrow{A^o \times g} & A^o \times \mathcal{P}(A), \end{array}$$

and the pair (g, g') is homotopy unique. We shall say that g *classifies* the distributor $E : A \Rightarrow B$ and that $M_A : A \Rightarrow \mathcal{P}(A)$ is a *Yoneda distributor*. A distributor $E : A \Rightarrow B$ is essentially the same thing as a map $B \to \mathcal{P}(A)$.

15.12. (Yoneda lemma 3) The twisted diagonal $A^\delta \to A^o \times A$ is classified by the Yoneda map $y_A : A \to \mathcal{P}(A)$. We have a diagram of homotopy pullback squares in $(\mathbf{S}, \mathrm{Wcat})$,

$$\begin{array}{ccccc} A^\delta & \longrightarrow & M_A & \longrightarrow & \mathcal{P}(A)^\delta \\ \downarrow & & \downarrow & & \downarrow \\ A^o \times A & \xrightarrow{A^o \times y_A} & A^o \times \mathcal{P}(A) & \xrightarrow{y_A^o \times \mathcal{P}(A)} & \mathcal{P}(A)^o \times \mathcal{P}(A). \end{array}$$

The composite square shows that the map y_A is fully faithful.

15.13. The Quillen equivalence 14.18 between distributors and cylinders implies the existence of a universal cylinder $C_A \in Cyl(A, \mathcal{P}(A))$. The cylinder C_A turns out to be a fibrant replacement of the cylinder $Cl(y_A)$ defined by the pushout square of simplicial sets,

$$\begin{array}{ccc} A & \xrightarrow{y_A} & \mathcal{P}(A) \\ {\scriptstyle i_1}\downarrow & & \downarrow \\ A \times I & \longrightarrow & Cl(y_A) \end{array}$$

The universality of C_A means that for any simplicial set B and any cylinder $E \in Cyl(A, B)$, there exists a homotopy pullback square in the model category $(\mathbf{S}, \mathrm{Wcat})$,

$$\begin{array}{ccc} E & \xrightarrow{g'} & C_A \\ \downarrow & & \downarrow \\ A \star B & \xrightarrow{1_A \star g} & A \star \mathcal{P}(A), \end{array}$$

and the pair (g, g') is homotopy unique. We shall say that g *classifies* the cylinder $E \in Cyl(A, B)$ and that $C_A \in Cyl(A, \mathcal{P}(A))$ is a *Yoneda cylinder*. A cylinder $C : A \Rightarrow B$ is essentially the same thing as a map $B \to \mathcal{P}(A)$.

15.14. (Yoneda lemma 4) The cylinder $A \times I \in Cyl(A, A)$ is classified by the Yoneda map $y_A : A \to \mathcal{P}(A)$. We have a diagram of homotopy pullback squares in $(\mathbf{S}, \mathrm{Wcat})$,

$$\begin{array}{ccccc} A \times I & \longrightarrow & C_A & \longrightarrow & \mathcal{P}(A) \times I \\ \downarrow & & \downarrow & & \downarrow \\ A \star A & \xrightarrow{A \star y_A} & A \star \mathcal{P}(A) & \xrightarrow{y_A \star \mathcal{P}(A)} & \mathcal{P}(A) \star \mathcal{P}(A). \end{array}$$

15.15. The Quillen equivalence **??** between cylinders and spans implies the existence of a universal span $P_A \in Span(A, \mathcal{P}(A))$. The universality of P_A means that for any simplicial set B and any bifibrant span $S : A \Rightarrow B$, there exists a homotopy pullback square in the model category $(\mathbf{S}, \mathrm{Wcat})$,

$$\begin{array}{ccc} S & \xrightarrow{g'} & P_A \\ \downarrow & & \downarrow \\ A \times B & \xrightarrow{A \times g} & A \times \mathcal{P}(A), \end{array}$$

and the pair (g, g') is homotopy unique. We shall say that g *classifies* the span $S : A \Rightarrow B$ and that $P_A : A \Rightarrow \mathcal{P}(A)$ is a *Yoneda span*. A bifibrant span $S : A \Rightarrow B$ is essentially the same thing as a map $B \to \mathcal{P}(A)$.

15.16. (Yoneda lemma 5) The span $A^I \to A \times A$ is classified by the Yoneda map $y_A : A \to \mathcal{P}(A)$. We have a diagram of homotopy pullback squares in $(\mathbf{S}, \mathrm{Wcat})$,

$$\begin{array}{ccccc} A^I & \longrightarrow & P_A & \longrightarrow & \mathcal{P}(A)^I \\ \downarrow & & \downarrow & & \downarrow \\ A \times A & \xrightarrow{A \times y_A} & A \times \mathcal{P}(A) & \xrightarrow{y_A \times \mathcal{P}(A)} & \mathcal{P}(A) \times \mathcal{P}(A). \end{array}$$

The composite square shows that the map y_A is fully faithful.

15.17. If X is a small simplicial category, let us denote by $[X, \mathbf{S}]^f$ the full subcategory of fibrant objects of the model category $[X, \mathbf{S}]^{inj}$. Then the evaluation functor $ev : X \times [X, \mathbf{S}] \to \mathbf{S}$ induces a functor $e : X \times [X, \mathbf{S}]^f \to \mathbf{Kan}$. The coherent nerve of this functor is a map of simplicial sets

$$C^! X \times C^! [X, \mathbf{S}]^f \ \to \mathcal{K}.$$

When X is enriched over Kan complexes, the corresponding map

$$C^! [X, \mathbf{S}]^f \to \mathcal{K}^{C^! X}$$

is an equivalence of quategories. It follows by adjointness that for any simplicial set A we have an equivalence of quategories

$$C^! [C_! A, \mathbf{S}]^f \to \mathcal{K}^A.$$

16. Morita equivalences

In this section, we introduce the notion of Morita equivalence between simplicial sets. The category of simplicial sets admits a model structure in which the weak equivalences are the Morita equivalences and the cofibration are the monomorphisms. The fibrant objects are the Karoubi complete quategories. We construct explicitly the Karoubi envelope of a quategory. The results of the section are taken from [J2].

16.1. Recall that a functor $u : A \to B$ between small categories is said to be a *Morita equivalence* if the base change functor

$$u^* : [B^o, \mathbf{Set}] \to [A^o, \mathbf{Set}]$$

is an equivalence of categories. We shall say that u is u is *Morita surjective* if the functor u^* is conservative. A functor $u : A \to B$ is a Morita surjective iff every object $b \in B$ is a retract of an object in the image of u. A functor $u : A \to B$ is a Morita equivalence iff it is fully faithful and Morita surjective.

16.2. Recall an idempotent $e : b \to b$ in a category is said to *split* if there exists a pair of arrows $s : a \to b$ and $r : b \to a$ such that $e = sr$ and $rs = 1_a$. A category C is said to be *Karoubi complete* if every idempotent in C splits.

16.3. The model structure $(\mathbf{Cat}, Eq)$ admits a Bousfield localisation with respect to Morita equivalences. The local model structure is cartesian closed and left proper. We shall denote it shortly by $(\mathbf{Cat}, \mathrm{Meq})$. A category is fibrant iff it is Karoubi complete. We call a fibration a *Morita fibration.* A *Karoubi envelope* $\mathrm{Kar}(C)$ of a category C is a fibrant replacement of C in the model structure $(\mathbf{Cat}, \mathrm{Meq})$. The category $\mathrm{Kar}(C)$ is well defined up to an equivalence of categories. The envelope is well defined up to an equivalence of quategories.

16.4. We shall denote by $i : C \to \kappa(C)$ the following explicit construction of the Karoubi envelope of a category C. An *object* of the category $\kappa(C)$ is a pair (c, e), where c is an object of C and $e \in C(c, c)$ is an idempotent. An *arrow* $f : (c, e) \to (c', e')$ of $\kappa(C)$ is a morphism $f \in C(c, c')$ such that $fe = f = e'f$. The composite of $f : (c, e) \to (c', e')$ and $g : (c', e') \to (c", e")$ is the arrow $gf : (c, e) \to (c", e")$. The arrow $e : (c, e) \to (c, e)$ is the unit of (c, e). The functor $i : C \to \kappa(C)$ takes an object $c \in C$ to the object $(c, 1_c) \in \kappa(C)$.

16.5. Let *Split* be the category freely generated by two arrows $s : 0 \to 1$ and $r : 1 \to 0$ such that $rs = 1_0$. The monoid $E = \mathit{Split}(1, 1)$ is freely generated by one idempotent $e = sr$ and we have $\kappa(E) = \mathit{Split}$. A functor is a Morita fibration iff it has the right lifting property with respect to the inclusion $E \subset \mathit{Split}$.

16.6. We shall say that a map of simplicial sets $u : A \to B$ is a *Morita equivalence* if the base change functor

$$\mathcal{R}^*(u) : \mathcal{R}(B) \leftrightarrow \mathcal{R}(A)$$

is an equivalence of categories. We shall say that a map of simplicial sets $u : A \to B$ is *Morita surjective* if the base change functor

$$\mathcal{R}^*(u) : \mathcal{R}(B) \leftrightarrow \mathcal{R}(A)$$

is conservative. A map $u : A \to B$ is Morita surjective iff the functor $\tau_1(u) : \tau_1(A) \to \tau_1(B)$ is Morita surjective. A map $u : A \to B$ is a Morita equivalence iff it is fully faithful and Morita surjective. Hence a map $u : A \to B$ is a Morita equivalence iff the opposite map $u^o : A^o \to B^o$ is a Morita equivalence. A weak categorical equivalence is a Morita equivalence.

16.7. An *idempotent* in a quategory X is defined to be a map $e : E \to X$, where E is the monoid freely generated by one idempotent. We shall say that an idempotent $e : E \to X$ *split* if it can be extended to a map $\mathit{Split} \to X$. We shall say that a quategory X is *Karoubi complete* if every idempotent in X splits. If X is Karoubi complete, then so are the quategories X/b and $b\backslash X$ for every object $b \in X$.

16.8. The model category $(\mathbf{S}, \mathrm{Wcat})$ admits a Bousfield localisation with respect to Morita equivalences. The local model structure is cartesian closed and left proper. We shall denote it shortly by $(\mathbf{S}, \mathrm{Meq})$. A fibration is called a *Morita fibration.* A quategory is fibrant iff it is Karoubi complete. The *Karoubi envelope* $\mathrm{Kar}(X)$ of a quategory X is defined to be a fibrant replacement of X in the model structure $(\mathbf{S}, \mathrm{Meq})$. The envelope is well defined up to an equivalence of quategories.

16.9. The pair of adjoint functors

$$\tau_1 : \mathbf{S} \leftrightarrow \mathbf{Cat} : N$$

is a Quillen adjunction between the model categories $(\mathbf{S}, \mathrm{Meq})$ and $(\mathbf{Cat}, \mathrm{Meq})$. A functor $u : A \to B$ in **Cat** is a Morita equivalence (resp. a Morita fibration) iff the map $Nu : NA \to NB$ is a Morita equivalence (resp. a Morita fibration).

16.10. A mid fibration between quategories is a Morita fibration iff it has the right lifting property with respect to the inclusion $E \subset Split$. The base change of a Morita equivalence along a left or a right fibration is a Morita equivalence. Every right (resp. left) fibration is a Morita fibration.

16.11. The canonical map $X \to hoX$ is a Morita fibration for any quategory X. It follows that an idempotent $u : E \to X$ splits iff its image $hu : E \to hoX$ splits in hoX. Hence a quategory X is Karoubi complete iff every idempotent $u : E \to X$ which splits in hoX splits in X.

16.12. Let E be the monoid freely generated by one idempotent. Then a quategory X is Karoubi complete iff the projection $X^{Split} \to X^E$ defined by the inclusion $E \subset Split$ is a trivial fibration.

16.13. The Karoubi envelope of a quategory X has functorial construction $X \to \kappa(X)$. Observe that the functor $\kappa : Cat \to Cat$ has the structure of a monad, with a left adjoint comonad L. To see this, we need the notion of semi-category. By definition, a *semi-category* B is a category without units. More precisely, it is a graph $(s,t) : B_1 \to B_0 \times B_0$ equipped with a composition law $B_1 \times_{s,t} B_1 \to B_1$ which is associative. There is an obvious notion of semi-functor between semi-categories. Let us denote by **sCat** the category of small semi-categories and semi-functors. The forgetful functor $U : \mathbf{Cat} \to \mathbf{sCat}$ has a left adjoint F and a right adjoint G. The existence of F is clear by a general result of algebra. If B is a semi-category, then the category $G(B)$ has the following description. An *object* of $G(B)$ is a pair (b,e), where $b \in B_0$ and $e : b \to b$ is an idempotent; an *arrow* $f : (b,e) \to (b',e')$ of $G(B)$ is a morphism $f \in B(b,b')$ such that $fe = f = e'f$. Composition of arrows is obvious. The unit of (b,e) is the morphism $e : (b,e) \to (b,e)$. It is easy to verify that we have $U \vdash G$. By construction, we have $\kappa(C) = GU(C)$ for any category C. It follows that the functor κ has the structure of a monad. Moreover, we have $L \vdash \kappa$, where $L = FU$. The functor L has the structure of a comonad by adjointness. The category $L[n]$ has the following presentation for each $n \geq 0$. It is generated by a chain of arrows

$$0 \xrightarrow{f_1} 1 \xrightarrow{f_2} 2 \longrightarrow \cdots \xrightarrow{f_n} n,$$

and a sequence of idempotents $e_i : i \to i$ $(0 \leq i \leq n)$. In addition to the relation $e_i e_i = e_i$ for each $0 \leq i \leq n$, we have the relation $f_i e_{i-1} = f_i = e_i f_i$ for each $0 < i \leq n$. If A is a simplicial set, let us put

$$\kappa(A)_n = \mathbf{S}(L[n], A)$$

for every $n \geq 0$. This defines a continuous functor $\kappa : \mathbf{S} \to \mathbf{S}$ having the structure of a monad. If X is a quategory, then the unit $X \to \kappa(X)$ is a Karoubi envelope of X. A map between quategories $f : X \to Y$ is a Morita equivalence iff the map $\kappa(f) : \kappa(X) \to \kappa(Y)$ is an equivalence of quategories.

16.14. The model category $(\mathbf{S}, \mathrm{Wcat})$ admits a uniform homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of Morita equivalences. A map $p : X \to Y$ belongs to $\mathcal{B}$ iff it admits a factorisation $p'w : X \to X' \to Y$ with p' a Morita fibration and w a weak categorical equivalence.

17. Adjoint maps

We introduce the notion of adjoint maps between quategories and formulate a necessary an sufficient condition for the existence of adjoints. We also introduce a weaker form of the notion of adjoint for maps between simplicial sets.

17.1. Recall from 1.11 that the category $\mathbf{S}$ has the structure of a 2-category $\mathbf{S}^{\tau_1}$. If $u : A \to B$ and $v : B \to A$ are maps of simplicial sets, an *adjunction* $(\alpha, \beta) : u \dashv v$ between u and v

$$u : A \leftrightarrow B : v$$

is a pair of natural transformations $\alpha : 1_A \to vu$ and $\beta : uv \to 1_B$ satisfying the *adjunction identities*:

$$(\beta \circ u)(u \circ \alpha) = 1_u \quad \text{and} \quad (v \circ \beta)(\alpha \circ v) = 1_v.$$

The map u is the *left adjoint* and the map v the *right adjoint*. The natural transformation α is the *unit* of the adjunction and the natural transformation β is the *counit*. We shall say that a homotopy $\alpha : 1_A \to vu$ is an *adjunction unit* if the natural transformation $[\alpha] : 1_A \to vu$ is the unit of an adjunction $u \dashv v$. Dually, we say that a homotopy $\beta : uv \to 1_B$ is an *adjunction counit* if the natural transformation $[\beta] : uv \to 1_B$ is the counit of an adjunction $u \dashv v$.

17.2. The functor $\tau_1 : \mathbf{S} \to \mathbf{Cat}$ takes an adjunction to an adjunction. A composite of left adjoints $A \to B \to C$ is left adjoint to the composite of the right adjoints $C \to B \to A$.

17.3. An object a in a quategory X is initial iff the map $a : 1 \to X$ is left adjoint to the map $X \to 1$.

17.4. A map between quategories $g : Y \to X$ is a right adjoint iff the quategory $a\backslash Y$ defined by the pullback square

$$\begin{array}{ccc} a\backslash Y & \longrightarrow & Y \\ \downarrow & & \downarrow g \\ a\backslash X & \longrightarrow & X \end{array}$$

admits an initial object for every object $a \in X$. An object of the quategory $a\backslash Y$ is a pair (b, u), where $b \in Y_0$ and $u : a \to f(b)$ is an arrow in X. We shall say that the arrow u is *universal* if the object (b, u) is initial in $a\backslash Y$. If f is a map $X \to Y$, then a homotopy $\alpha : 1_X \to gf$ is an adjunction unit iff the arrow $\alpha(a) : a \to gf(a)$ is universal for every object $a \in X$. Dually, a map between quategories $f : X \to Y$ is a left adjoint iff the quategory X/b defined by the pullback square

$$\begin{array}{ccc} X/b & \longrightarrow & X \\ \downarrow & & \downarrow f \\ Y/b & \longrightarrow & Y \end{array}$$

admits a terminal object for every object $b \in Y$. An object of the quategory X/b is a pair (a, v), where $a \in X_0$ and $v : f(a) \to b$ is an arrow in Y; we shall say that the arrow v is *couniversal* if the object (a, v) is terminal in X/b. If g is a map $Y \to X$, then a homotopy $\beta : fg \to 1_Y$ is an adjunction counit iff the arrow $\beta(b) : fg(b) \to b$ is couniversal for every object $b \in Y$.

17.5. The base change of left adjoint between quategories along a right fibration is a left adjoint.

17.6. If $f : X \leftrightarrow Y : g$ is a pair of adjoint maps between quategories, then the right adjoint g is fully faithful iff the counit of the adjunction $\beta : fg \to 1_Y$ is invertible, in which case the left adjoint f is said to be a *reflection* and the map g to be *reflective*. Dually, the left adjoint f is fully faithful iff the unit of the adjunction $\alpha : 1_X \to gf$ is invertible, in which case the right adjoint g is said to be a *coreflection* and the map f to be *coreflective*.

17.7. The base change of a reflective map along a left fibration is reflective. Dually, the base change of a coreflective map along a right fibration is coreflective.

17.8. We shall say that a map of simplicial sets $u : A \to B$ is a *weak left adjoint* if the functor

$$\tau_1(u, X) : \tau_1(B, X) \to \tau_1(A, X)$$

is a right adjoint for every quategory X. Dually, we shall say that $u : A \to B$ is a *weak right adjoint* if the functor $\tau_1(u, X)$ is a left adjoint for every quategory X. A map of simplicial sets $u : A \to B$ is a weak left adjoint iff the opposite map $u^o : A^o \to B^o$ is a weak right adjoint.

17.9. A map between quategories is a weak left adjoint iff it is a left adjoint. The notion of weak left adjoint is invariant under weak categorical equivalences. The functor $\tau_1 : \mathbf{S} \to \mathbf{Cat}$ takes a weak left adjoint to a left adjoint.

17.10. Weak left adjoints are closed under composition. The base change of weak left adjoint along a right fibration is a weak left adjoint. A weak left adjoint is an initial map. A vertex $a \in A$ in simplicial set A is initial iff the map $a : 1 \to A$ is a weak left adjoint.

17.11. Let B a simplicial set. For each vertex $b \in B$, let us choose a factorisation $1 \to Rb \to B$ of the map $b : 1 \to B$ as a right anodyne map $1 \to Rb$ followed by a right fibration $Rb \to B$. Then a map of simplicial sets $u : A \to B$ is a weak left adjoint iff the simplicial set $Rb \times_B A$ admits a terminal vertex for each vertex $b \in B$.

17.12. We say that a map $v : B \to A$ is a *weak reflection* if the functor

$$\tau_1(v, X) : \tau_1(A, X) \to \tau_1(B, X)$$

is coreflective for every quategory X. We say that a map of simplicial sets $u : A \to B$ is *weakly reflective* if the functor

$$\tau_1(u, X) : \tau_1(B, X) \to \tau_1(A, X)$$

is a coreflection for every quategory X. There are dual notions of weak coreflection and of weakly coreflective maps.

17.13. If a map of simplicial sets is both a weak left adjoint and a weak right adjoint, then it is a weak reflection iff it is weak coreflection.

17.14. A weak left adjoint is a weak reflection iff it is dominant iff it is a localisation. Dually, a weak right adjoint is a weak coreflection iff it is dominant iff it is a localisation.

18. Quasi-localisations

The notion of simplicial localisation was introduced by Dwyer and Kan. The corresponding notion for quategories is called quasi-localisation. We formalise the theory of quasi-localisation with the theory of homotopy factorisation systems.

18.1. We say that a map of simplicial sets $u : A \to B$ *inverts* an arrow $f \in A$ if $u(f)$ is invertible in $\tau_1(B)$. We say that u *inverts* a set of arrows $S \subseteq A$ if it inverts every arrow in S. We shall say that u is a *quasi-localisation with respect to S* if it inverts S universally. The universality means that for any quategory X, the map $X^u : X^B \to X^A$ induces an equivalence between X^B and the full simplicial subset of X^A spanned by the maps $A \to X$ which invert S. In general, we shall say that a map of simplicial sets $u : A \to B$ is a *quasi-localisation* if it is a quasi-localisation with respect to the set $\Sigma(u)$ of arrows which are inverted by u.

18.2. Let $S \to A_1$ be a family of arrows in a simplicial set A. If J is the groupoid generated by one arrow $0 \to 1$, then the map $A \to A[S^{-1}]$ in the pushout square

$$\begin{array}{ccc} S \times I & \longrightarrow & A \\ \downarrow & & \downarrow \\ S \times J & \longrightarrow & A[S^{-1}] \end{array}$$

is a quasi-localisation with respect to S.

18.3. For any set S of arrows in a category A, there is a functor $l_S : A \to S^{-1}A$ which inverts S universally. Such a functor is said to be a *strict localisation* in 47.4. There is also a notion of *iterated strict localisation.* Recall that that the category **Cat** admits a factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{B}$ is the class of conservative functors and $\mathcal{A}$ is the class of iterated strict localisations. A functor $u : A \to B$ is said to be a *localisation* (resp *iterated localisation*) in 11.14 if it is equivalent to a strict localisation (resp. an *iterated strict localisation*). The model category $(\mathbf{Cat}, Eq)$ admits a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{B}$ is the class of conservative functors and $\mathcal{A}$ is the class of iterated localisations.

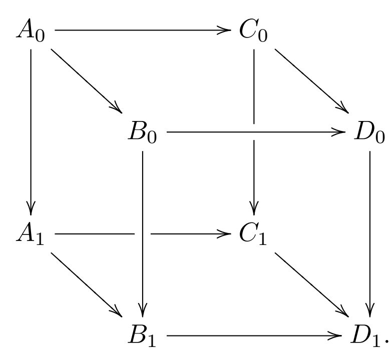
18.4. We shall say that a functor $u : C \to D$ *inverts* an iterated localisation $l : C \to L$ if there exits a functor $v : L \to D$ together with an isomorphism $vl \simeq u$. We shall say that a map of simplicial sets $u : A \to B$ *inverts* an iterated localisation $l : \tau_1 A \to L$ if the functor $\tau_1(u) : \tau_1(A) \to \tau_1(B)$ inverts L. We shall say that $u : A \to B$ is an *iterated quasi-localisation* with respect to l if it inverts l universally. More precisely, this means that for any quategory X, the map $X^u : X^B \to X^A$ induces an equivalence between X^B and the full simplicial subset of X^A spanned by the maps $A \to X$ which coinvert l. In general, we shall say that a map of simplicial sets $u : A \to B$ is an *iterated quasi-localisation* if the functor $\tau_1(u)$ is an iterated localisation and the map u is an iterated quasi-localisation with respect to $\tau_1(u)$. The model category $(\mathbf{S}, \mathrm{Wcat})$ admits a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{B}$ is the class of conservative maps and $\mathcal{A}$ is the class of iterated quasi-localisations. A monomorphism of simplicial sets is an iterated quasi-localisation iff it has the left lifting property with respect to every conservative pseudo-fibration between quategories.

18.5. The functor $\tau_1 : \mathbf{S} \to \mathbf{Cat}$ takes an iterated quasi-localisation to an iterated localisation. An iterated quasi-localisation $u : A \to B$ is a quasi-localisation iff the functor $\tau_1(u) : \tau_1 A \to \tau_1 B$ is a localisation.

18.6. An iterated quasi-localisation is dominant and essentially surjective. A weak reflection (resp. coreflection) is a quasi-localisation. The base change of a quasi-localisation along a left or a right fibration is a quasi-localisation. Similarly for the base change of an iterated quasi-localisation.

18.7. Recall from 47.5 that if C is a category, then the full subcategory of $C\backslash\mathbf{Cat}$ spanned by the iterated strict localisations $C \to L$ is equivalent to a complete lattice $Loc(C)$. The canonical functor $C\backslash\mathbf{Cat} \to Ho(C\backslash\mathbf{Cat}, \mathrm{Eq})$ induces an equivalence between $Loc(C)$ and the full subcategory of $Ho(C\backslash\mathbf{Cat}, \mathrm{Eq})$ spanned by the iterated localisations $C \to L$. If A is a simplicial set, then the functor τ_1 induces an equivalence between the full subcategory of the homotopy category $Ho(A\backslash\mathbf{S}, \mathrm{Wcat})$ spanned by the iterated quasi-localisations $A \to L$ and the lattice $Loc(\tau_1 A)$.

18.8. Suppose that we have a commutative cube of simplicial sets



in which the top and the bottom faces are homotopy cocartesian. If the maps $A_0 \to A_1$, $B_0 \to B_1$ and $C_0 \to C_1$ are quasi-localisations, then so is the maps $D_0 \to D_1$. Similarly for iterated quasi-localisations.

18.9. Every simplicial set X is the quasi-localisation

$$t_X : \Delta/X \to X$$

of its category of elements Δ/X. The map t_X was introduced by Illusie in [Illu]. Let us first describe t_X in the case where X is (the nerve of) a category C. The functor $t_C : \Delta/CtoC$ is defined by putting $t_C(x) = x(n)$ for a functor $x : [n] \to C$. The family of maps $t_C : C/\Delta \to C$, for $C \in \mathbf{Cat}$, can be extended uniquely as a natural transformation $t_X : \Delta/X \to X$ for $X \in \mathbf{S}$. Let us now show that t_X is a quasi-localisation. Let Δ' be the subcategory of Δ whose morphisms are the maps $u : [m] \to [n]$ with $u(m) = n$. The map $t_X : X/\Delta \to X$ takes every arrow in

$$\Delta'/X$$

to a unit in X. It thus induces a canonical map

$$w_X : \Delta/X[(\Delta'/X)^{-1}] \to X.$$

The result will be proved if we show that w_X a weak categorical equivalence. We only sketch of the proof. The domain $F(X)$ of w_X is a cocontinuous functor of X. Moreover, the functor F takes a monomorphism to a monomorphism. The result is easy to verify in the case where $X = \Delta[n]$. The result then follows from a formal argument using the the skeleton filtration of X and the cube lemma.

18.10. If $u : A \to B$ is quasi-localisation, then the base change functor $\mathcal{R}^*(u) : \mathcal{R}(B) \to \mathcal{R}(A)$ is fully faithful, since a quasi-localisation is dominant. An object $X \in \mathcal{R}(A)$ belongs to the essential image of the functor $\mathcal{R}^*(u)$ iff its (contravariant) homotopy diagram $D(X) : \tau_1(A)^o \to Ho(\mathbf{S}, Who)$ inverts the localisation $\tau_1(A) \to \tau_1(B)$.

18.11. If $f : a \to b$ is an arrow in a simplicial set A, then the inclusion $i_0 : \{0\} \to I$ induces a map $f' : a \to f$ between the objects $a : 1 \to A$ and $f : I \to A$ of the category $\mathbf{S}/A$. If S is a set of arrows in A, we shall denote by $(\mathbf{S}/A, S \cup Wcont)$. the Bousfield localisation of the model structure $(\mathbf{S}/A, Wcont)$ with respect to the set of maps $\{f' : f \in S\}$. An object $X \in \mathbf{R}(A)$ is fibrant in the localised structure iff the map $f^* : X(b) \to X(a)$ of the contravariant homotopy diagram of X is a weak homotopy equivalence for every arrow $f : a \to b$ in S. If $p : A \to A[S^{-1}]$ is the canonical map, then the pair of adjoint functors

$$p_! : \mathbf{S}/A \leftrightarrow \mathbf{S}/A[S^{-1}] : p^*$$

is a Quillen equivalence between the model category $(\mathbf{S}/A, \Sigma \cup Wcont)$ and the model category $(\mathbf{S}/A[S^{-1}], Wcont)$.

18.12. It follows from 18.11 that a right fibration $X \to B$ is a Kan fibration iff the map $f^* : X(b) \to X(a)$ of the contravariant homotopy diagram of X is a weak homotopy equivalence for every arrow $f : a \to b$ in B.

19. Limits and colimits

In this section we study the notions of limit and colimit in a quategory. We define the notions of cartesian product, of fiber product, of coproduct and of pushout. The notion of limit in a quategory subsume the notion of homotopy limits. For example. the loop space of a pointed object is a pullback and its suspension a pushout. We consider various notions of complete and cocomplete quategories. Many results of this section are taken from [J1] and [J2].

19.1. If X is a quategory and A is a simplicial set, we say that a map $d : A \to X$ is a *diagram* indexed by A in X The quategory X can be large. The *cardinality* of a diagram $d : A \to X$ is the cardinality of A. A diagram $d : A \to X$ is *small* (resp. *finite*) if A is small (resp. finite). .

19.2. Recall that a *projective cone* with base $d : A \to X$ in a quategory X is a map $c : 1 \star A \to X$ which extends d along the inclusion $A \subset 1 \star A$. The projective cones with base d are the vertices of a quategory X/d by 9.7. We say that a projective cone $c : 1 \star A \to X$ with base d is a *limit cone* if it is a terminal object of the quategory X/d; in this case, the vertex $l = c(1) \in X$ is said to be the (*homotopy*) *limit* of d and we can write

$$l = \lim_{a \in A} d(a) = \lim_{A} d.$$

19.3. If $d : A \to X$ is a diagram in a quategory X, then the full simplicial subset of X/d spanned by the limit cones with base d is a contractible Kan complex when non-empty. It follows that the limit of a diagram is homotopy unique when it exists.

19.4. The notion of limit can also be defined by using fat projective cones $1 \diamond A \to X$ instead of projective cones $1 \star A \to X$. But the canonical map $X/d \to X /\!\!/ d$ obtained from the canonical map $1 \diamond A \to 1 \star A$ is an equivalence of quategories by 9.18. It thus induces an equivalence between the Kan complex spanned by the terminal vertices of X/d and the Kan complex spanned by the terminal vertices of $X /\!\!/ d$.

19.5. The colimit of a diagram with values in a quategory X is defined dually. We recall that an *inductive cone* with *cobase* $d : A \to X$ in a quategory X is a map $c : A \star 1 \to X$ which extends d along the inclusion $A \subset A \star 1$. The inductive cones with a fixed cobase d are the objects of a quategory $d \backslash X$. We say that an inductive cone $c : 1 \star A \to X$ with cobase d is a *colimit cone* if it is an initial object of the quategory $d \backslash X$; in this case the vertex $l = c(1) \in X$ is said to be the (*homotopy*) *colimit* of d and we can write

$$l = colim_{a \in A} d(a) = colim_A d.$$

The notion of colimit can also be defined by using fat inductive cones $A \diamond 1 \to X$, but the two notions are equivalent.

19.6. If X is a quategory and A is a simplicial set, then the diagonal map $X \to X^A$ has a right (resp. left) adjoint iff every diagram $A \to X$ has a limit (resp. colimit).

19.7. We shall say that a (large) quategory X is *complete* if every (small) diagram $A \to X$ has a limit. There is a dual notion of a *cocomplete* quategory. We shall say that a large quategory is *bicomplete* if it is complete and cocomplete.

19.8. We say that a quategory X is *finitely complete* or *cartesian* if every finite diagram $A \to X$ has a limit. There is dual notion of a *finitely cocomplete* or *cocartesian* quategory. We shall say that a quategory X is *finitely bicomplete* or *bicartesian* if it is finitely complete and cocomplete.

19.9. The homotopy localisation $L(\mathcal{E})$ of a model category $\mathcal{E}$ is finitely bicomplete, and it is (bi)complete when the category $\mathcal{E}$ is (bi)complete.

19.10. The coherent nerve of the category of Kan complexes is a bicomplete quategory $\mathcal{K} = \mathcal{Q}_0$. Similarly for the coherent nerve of the category of small quategories is a bicomplete quategory $\mathcal{Q}_1$

19.11. We shall say that a map between quategories $f : X \to Y$ *preserves* the limit of a diagram $d : A \to X$ if it takes a limit cone $c : 1 \star A \to X$ with base d to a limit cone $fc : 1 \star A \to Y$. Dually, we shall say f *preserves* the the colimit of a diagram $d : A \to X$ if it takes a colimit cone $c : A \star 1 \to X$ with cobase d to a colimit cone. We shall say that a map $f : X \to Y$ is *continuous* if it takes every small limit cone in X to a limit cone. Dually, we shall say that f is *cocontinuous* if it takes every (small) colimit cone in X to a colimit cone. We shall say that f is *bicontinuous* if it is both continuous and cocontinuous. We shall say that a map between cartesian quategories is *finitely continuous* or *left exact* if it preserves finite limits. Dually, we shall say that a map between cocartesian quategories is *finitely cocontinuous* or *right exact* if it preserves finite colimits.

19.12. In a pair of adjoint maps $f : X \leftrightarrow Y : g$, the left adjoint f is cocontinuous and the right adjoint g is continuous.

19.13. If X is a quategory and S is a discrete simplicial set (ie a set), then a projective cone $c : 1 \star S \to X$ is the same thing as a family of morphisms $(p_i : y \to x_i \mid i \in S)$ with domain $y = c(1)$. When c is a limit cone, the object y is said to be the *product* of the family $(x_i : i \in S)$, the morphism $p_i : y \to x_i$ to be a *projection* and we write

$$y = \prod_{i \in S} x_i.$$

Dually, if S is a discrete simplicial set, then an inductive cone $c : S \star 1 \to X$ is the same thing as a family of morphisms $(u_i : x_i \to y \mid i \in S)$ with codomain $y = c(1)$. When c is a colimit cone, the object y is said to be the *coproduct* of the family $(x_i : i \in S)$, the arrow $u_i : x_i \to y$ to be an *inclusion* and we write

$$y = \coprod_{i \in S} x_i.$$

19.14. The canonical map $X \to hoX$ preserves products and coproducts.

19.15. We say that a quategory X *has finite products* if every finite family of objects of X has a product. A quategory with a terminal object and binary products has finite products. We say that a large quategory X *has products* if every small family of objects of X has a product. There are dual notions of a quategory *with finite coproducts* and of large quategory *with coproducts*

19.16. If X is a quategory and $b \in X_0$, then an object of the quategory X/b is an arrow $a \to b$ in X. The *fiber product* of two arrows $f : a \to b$ and $g : c \to b$ in X is defined to be their product as objects of the quategory X/b,

$$\begin{array}{ccc} a \times_b c & \longrightarrow & c \\ \downarrow & & \downarrow{\scriptstyle g} \\ a & \xrightarrow{f} & b. \end{array}$$

The square $I \times I$ is a projective cone $1 \star \Lambda^2[2]$. We shall say that a commutative square $I \times I \to X$ is *cartesian*, or that it is a *pullback* if the projective cone $1 \star \Lambda^2[2]$ is a limit cone. A diagram $d : \Lambda^2[2] \to X$ is the same thing as a pair of arrows $f : a \to b$ and $g : c \to b$ in X; the limit of d is the fiber product of f and g. Dually, an object of the quategory $a\backslash X$ is an arrow $a \to b$ in X. The *amalgameted sum* of two arrows $u : a \to b$ and $v : a \to c$ in X is defined to be their coproduct as objects of the quategory $a\backslash X$,

$$\begin{array}{ccc} a & \xrightarrow{v} & c \\ {\scriptstyle u}\downarrow & & \downarrow \\ b & \longrightarrow & b \sqcup_a c. \end{array}$$

The square $I \times I$ is an inductive cone $\Lambda^0[2] \star 1$. We shall say that a commutative square $I \times I \to X$ is *cocartesian*, or that it is a *pushout* if the inductive cone $d : \Lambda^0[2] \star 1 \to X$ is a colimit cone. A diagram $d : \Lambda^0[2] \to X$ is the same thing as a pair of arrows $u : a \to b$ and $v : a \to c$ in X; the colimit of d is the amalgamated sum of u and v.

19.17. We shall say that a quategory X *has pullbacks* if every diagram $\Lambda^2[2] \to X$ has a limit. A quategory X has pullbacks iff the quategory X/b has finite products for every object $b \in X$. Dually, we say that a quategory X has *pushouts*) if every diagram $\Lambda^0[2] \to X$ has a colimit. A quategory X has pushouts iff the quategory $a\backslash X$ has finite coproducts for every object $a \in X$.

19.18. A quategory with terminal objects and pullbacks is cartesian. A map between cartesian quategories is finitely continuous iff it preserves terminal objects and pullbacks. Dually, a quategory with initial objects and pushouts is cocartesian. A map between cocartesian quategories is finitely cocontinuous iff it preserves initial objects and pushouts.

19.19. A quategory with (arbitrary) products and pullbacks is complete. A map between complete quategories is continuous iff it preserves products and pullbacks. Dually, a quategory with (arbitrary) coproducts and pushouts is cocomplete. A map between cocomplete quategories is cocontinuous iff it preserves coproducts and pushouts.

19.20. We say that a quategory X is *cartesian closed* if it has finite products and the product map $a \times (-) : X \to X$ has a right adjoint $[a, -] : X \to X$, called the *exponential*, for every object $a \in X$. We say that a quategory X is *locally cartesian closed* if the slice quategory X/a is cartesian closed for every object $a \in X$.

19.21. The quategory $\mathcal{K}$ is locally cartesian closed. The quategories $\mathcal{Q}_1$ and $\mathcal{Q}_1/I$ are cartesian closed, where $I = \Delta[1]$.

19.22. The *base change* of a morphism $f : a \to b$ in a quategory along another morphism $u : a' \to a$ is the morphism f' in a pullback square,

$$\begin{array}{ccc} a' & \longrightarrow & a \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ b' & \xrightarrow{u} & b. \end{array}$$

19.23. To every arrow $f : a \to b$ in a quategory X we can associate a pushforward map $f_! : X/a \to X/b$ by 15.4. The map $f_!$ is unique up to a unique invertible 2-cell in the 2-category **QCat**. The quategory X has pullbacks iff the map $f_! : X/a \to X/b$ has a right adjoint

$$f^* : X/b \to X/a$$

for every arrow $f : a \to b$. We shall say that f^* is the *base change map* along f. A cartesian quategory X is locally cartesian closed iff the base change map

$$f^* : X/b \to X/a$$

has a right adjoint f_* for every arrow $f : a \to b$.

19.24. Let $d : B \to X$ a diagram with values in a quategory X and let $u : A \to B$ a map of simplicial sets. If the colimit of the diagrams d and du exist, then there is a canonical morphism

$$\mathrm{colim}_A du \to \mathrm{colim}_B d$$

in the category hoX. Let us suppose that the map $u : A \to B$ is final. Then the map $d\backslash X \to du\backslash X$ induced by u is an equivalence of quategories by 9.15. It follows that the colimit of d exists iff the colimit of du exists, in which cases the canonical morphism above is invertible and the two colimits are isomorphic.

19.25. Let $d : B \to X$ a diagram with values in a quategory X. If $u : (M, p) \to (N, q)$ is a contravariant equivalence in the category $\mathbf{S}/B$, then the map $dq\backslash X \to dp\backslash X$ induced by u is an equivalence of quategories. It follows that the colimit of dp exists iff the colimit of dq exists, in which case the two colimits are naturally isomorphic in the category hoX.

19.26. Let $(A_i \mid i \in S)$ be a family of simplicial sets and let us put

$$A = \bigsqcup_{i \in S} A_i.$$

If X is a quategory, then a diagram $d : A \to X$ is the same thing as a family of diagrams $d_i : A_i \to X$ for $i \in S$. If each diagram d_i has a colimit x_i, then the diagram d has a colimit iff the coproduct of the family $(x_i : i \in I)$ exists, in which case we have

$$\mathrm{colim}_A d = \coprod_{i \in S} \mathrm{colim}_{A_i} d.$$

19.27. Suppose we have a pushout square of simplicial sets

$$\begin{array}{ccc} A & \xrightarrow{u} & C \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ B & \xrightarrow{v} & T. \end{array}$$

with i monic. Let $d : T \to X$ be a diagram with values in a quategory X and suppose that each diagram dv, dvi and dj has a colimit. Then the diagram d has a colimit iff the pushout square

$$\begin{array}{ccc} \mathrm{colim}_A dvi & \longrightarrow & \mathrm{colim}_C dj \\ \downarrow & & \downarrow \\ \mathrm{colim}_B dv & \longrightarrow & Z \end{array}$$

exists, in which case $\mathrm{colim}_T d = Z$.

19.28. In a quategory with finite colimit X, the coproduct of n objects can be computed inductively by taking pushouts starting from the initial object. More generally, the colimit of any finite diagram $d : A \to X$ can be computed inductively by taking pushouts and the initial object. To see this, let us put

$$l_n = \mathrm{colim}_{Sk^n A} d \mid Sk^n A$$

for each $n \geq 0$. The object l_0 is the coproduct of the family $d \mid A_0$. If $n > 0$, the object l_n can be constructed from l_{n-1} by taking pushouts. To see this, let us denote by $C_n(A)$ the set of non-degenerate n-simplices of A. We then have a pushout square

$$\begin{array}{ccc} C_n(A) \times \partial\Delta[n] & \longrightarrow & Sk^{n-1}A \\ \downarrow & & \downarrow \\ C_n(A) \times \Delta[n] & \longrightarrow & Sk^n A \end{array}$$

for each $n \geq 1$. The colimit of a simplex $x : \Delta[n] \to X$ is equal to $x(n)$, since n is a terminal object of $\Delta[n]$. Let us denote by $\delta(x)$ the colimit of the simplicial sphere $x \mid \partial\Delta[n]$. There is then a canonical morphism $\delta(x) \to x(n)$, since $\partial\Delta[n] \subset \Delta[n]$. It then follows from 19.27 that we have a pushout square,

$$\begin{array}{ccc} \coprod_{x \in C_n(A)} \delta(x) & \longrightarrow & l_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{x \in C_n(A)} x(n) & \longrightarrow & l_n. \end{array}$$

The construction shows that a quategory with initial object and pushouts is finitely cocomplete.

19.29. Recall from 16.7 than an *idempotent* in a quategory X is defined to be a map $e : E \to X$, where E is the monoid freely generated by one idempotent. An idempotent $e : E \to X$ splits iff the diagram $e : E \to X$ has a limit iff it has a colimit. A complete quategory is Karoubi complete Beware that the simplicial set E is not quasi-finite. Hence a cartesian quat is not necessarly Karoubi complete.

19.30. The Karoubi envelope of a cartesian quategory is cartesian. The Karoubi envelope of a quategory with finite products has finite products.

19.31. Every cocartesian quategory X admits a natural action $\mathbf{S}_f \times X \to X$ by the category of finite simplicial sets. The action associates to a pair (A, x) the colimit $A \cdot x$ of the constant diagram $A \to X$ with value x. The map $x \mapsto A \cdot x$ can be obtained by composing the diagonal $X \to X^A$ with its left adjoint $X^A \to X$. There is also a canonical homotopy equivalence

$$X(A \cdot x, y) \simeq X(x, y)^A$$

for every $y \in X$. For a fixed object $x \in X$, the map $A \mapsto A \wedge x$ takes a weak homotopy equivalence to an isomorphism and a homotopy pushout square to a pushout square in X. Dually, every cartesian quategory X admits a natural coaction $X \times \mathbf{S}_f^o \to X$ by the category of finite simplicial sets. The coaction associates to a pair (x, A) the limit x^A of the constant diagram $A \to X$ with value x. The map

$x \mapsto x^A$ can be obtained by composing the diagonal $X \to X^A$ with its right adjoint $X^A \to X$. There is then a a canonical homotopy equivalence

$$X(y, x^A) \simeq X(y, x)^A$$

for every object $y \in X$. The coaction by A on X is dual to the action of A^o on X^o, since $(x^A)^o = A^o \cdot x^o$. Moreover, when X is bicartesian, the map $A \cdot (-) : X \to X$ is left adjoint to the map $(-)^A : X \to X$.

19.32. If X is a null-pointed quategory. then the projection $X^I \to X \times X$ admits a section which associates to a pair of objects $x, y \in X$ a null morphism $0 : x \to y$ by 10.8. If X is cocartesian, then there is a natural action $1\backslash\mathbf{S}_f \times X \to X$ by the category of finite pointed simplicial sets. The action associates to a pair (A, x) the *smash product* $A \wedge x \in X$ defined by the pushout square,

$$\begin{array}{ccc} 1 \cdot x & \longrightarrow & 1 \cdot 0 \\ {\scriptstyle a\cdot x}\downarrow & & \downarrow \\ A \cdot x & \longrightarrow & A \wedge x, \end{array}$$

where $a : 1 \to A$ is the base point. For example, $S^1 \wedge x$ is the *suspension* $\Sigma(x)$ of an object $x \in X$. More generally, $S^n \wedge x$ is the *n-fold suspension* $\Sigma^n(x)$ of x. There is also a canonical homotopy equivalence

$$X(Ax, y) \simeq [A, X(x, y)]$$

for every $y \in X$, where $[A, X(x, y)]$ is the simplicial set of pointed maps $A \to X(x, y)$. For a fixed object $x \in X$, the map $A \mapsto A \wedge x$ takes a weak homotopy equivalence to an isomorphism and a homotopy pushout square to a pushout square in X. Dually, a null-pointed cartesian quategory X admits a natural coaction by finite pointed simplicial sets. The coaction associates to a pair (x, A) the *cotensor* $[A, x] \in X$ defined by the pullback square,

$$\begin{array}{ccc} [A, x] & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ x^A & \xrightarrow{x^a} & x^1, \end{array}$$

where $a : 1 \to A$ is the base point. For examp;le, $[S^1, x]$ is the *loop space* $\Omega(x)$ of an object $x \in X$. More generally, $[S^n, x]$ is the *n-fold loop space* $\Omega^n(x)$ of x. The coaction by A on X is dual to the action by A^o on X^o, since $[A, x]^o \simeq A^o \wedge x^o$. Moreover, when X is bicartesian, the map $[A, -] : X \to X$ is right adjoint to the map $A \wedge (-) : X \to X$

19.33. Unless exception, we only consider small ordinals and cardinals. Recall that an ordinal α is said to be a *cardinal* if it is smallest among the ordinals with the same cardinality. Recall that a cardinal α is said to be *regular* if the sum of a family of cardinals $< \alpha$, indexed by a set of cardinality $< \alpha$, is $< \alpha$.

19.34. Let α be a regular cardinal. We say that a diagram $A \to X$ in a quategory X is α-*small* if A is a simplicial set of cardinality $< \alpha$. We shall say that the limit of an α-small diagram is an α-*limit*. An α-*product* is the product of a family of objects indexed by a set of cardinality $< \alpha$. We say that a quategory X is α-*complete* if every α-small diagram $A \to X$ has a limit. We say that map $X \to Y$ between α-complete quategories is α-*continuous* if it preserves the limit every α-small diagram $K \to X$. There are dual notions of α-cocomplete quategory, and of α-*cocontinuous* map.

19.35. For any simplicial set A, the map

$$t_A : \Delta/A \to A$$

defined in 19.35 is initial, since a localisation is dominant and a dominant map is initial. Hence the limit of a diagram $d : A \to X$ in a quategory X is isomorphic to the limit of the composite $dt_A : \Delta/A \to X$. Observe that the projection $q : \Delta/A \to \Delta$ is a discrete fibration. If $d : A \to X$ is a diagram in a quategory with products X, then the map $dt_A : \Delta/A \to X$ admits a right Kan extension $\Pi_A(d) = \Pi_q(dt_A) : \Delta \to X$ along the projection q. See section 22 for Kan extensions. Moreover, we have

$$\Pi_A(d)(n) = \prod_{a \in A_n} d(a(n))$$

for every $n \geq 0$. The diagram d has a limit iff the diagram $\Pi_A(d)$ has a limit, in which case we have

$$\lim_A d = \lim_\Delta \Pi_A(d).$$

It follows that a quategory with products and Δ-indexed limits is complete.

19.36. Dually, for any simplicial set A, the opposite

$$s_A =: \Delta^o/A \to A.$$

of the map $t_{A^o} : \Delta/A^o \to A^o$ is final. Observe that the canonical projection $p : \Delta^o/A \to \Delta^o$ is a discrete opfibration. If $d : A \to X$ is a diagram in a quategory with coproducts X, then the map $ds^o_A : \Delta/A^o \to X$ admits a left Kan extension $\Sigma_A(d) = \Sigma_p : \Delta^o \to X$ along the projection p. We have

$$\Sigma_A(d)_n = \coprod_{a \in A_n} d(a(0))$$

for every $n \geq 0$. The diagram d has a colimit iff the diagram $\Sigma_A(d)$ has a colimit, in which case we have

$$\mathrm{colim}_A d = \mathrm{colim}_\Delta \Sigma_A(d).$$

19.37. A quategory is cocomplete iff it has coproducts and Δ^o-indexed colimits. A map between cocomplete quategories is cocontinuous iff it preserves coproducts and Δ^o-indexed colimits.

20. Grothendieck fibrations

20.1. We first recall the notion of Grothendieck fibration between categories. A morphism $f : a \to b$ in a category E is said to be *cartesian* with respect a functor $p : E \to B$ if for every morphism $g : c \to b$ in E and every factorisation $p(g) = p(f)u : p(c) \to p(a) \to p(b)$ in B, there is a unique morphism $v : c \to a$ in E such that $g = fv$ and $p(v) = u$. A morphism $f : a \to b$ is cartesian with respect to the functor p iff the square of categories

$$\begin{array}{ccc} E/a & \longrightarrow & E/b \\ \downarrow & & \downarrow \\ B/p(a) & \longrightarrow & B/p(b) \end{array}$$

is cartesian, where the functor $E/a \to E/b$ (resp. $B/p(a) \to B/p(b)$) is obtained by composing with f (resp. $p(f)$). A functor $p : E \to B$ is called a *Grothendieck fibration over* B if for every object $b \in E$ and every morphism $g \in B$ with target $p(b)$ there exists a cartesian morphism $f \in E$ with target b such that $p(f) = g$. There are dual notions of *cocartesian morphism* and of Grothendieck *opfibration*. A functor $p : E \to B$ is a Grothendieck opfibration iff the opposite functor $p^o : E^o \to B^o$ is a Grothendieck fibration. We shall say that a functor $p : E \to B$ is a *Grothendieck bifibration* if it is both a fibration and an opfibration.

20.2. If X and Y are two Grothendieck fibrations over B, then a functor $X \to Y$ in $\mathbf{Cat}/B$ is said to be *cartesian* if its takes every cartesian morphism in X to a cartesian morphism in Y. There is a dual notion of cocartesian functor between Grothendieck opfibrations over B and a notion of bicartesian functor between Grothendieck bifibrations.

20.3. Observe that a morphism $f : a \to b$ in a category E is cartesian with respect a functor $p : E \to B$ iff every commutative square

$$\begin{array}{ccc} \Lambda^2[2] & \xrightarrow{x} & E \\ \downarrow & & \downarrow p \\ \Delta[2] & \longrightarrow & B \end{array}$$

with $x(1, 2) = f$ has a unique diagonal filler.

20.4. Let $p : E \to B$ be a mid fibration between simplicial sets. We shall say that an arrow $f \in E$ is *cartesian* if every commutative square

$$\begin{array}{ccc} \Lambda^n[n] & \xrightarrow{x} & E \\ \downarrow & & \downarrow p \\ \Delta[n] & \longrightarrow & B \end{array}$$

with $n > 1$ and $x(n-1, n) = f$ has a diagonal filler. Equivalently, an arrow $f \in E$ with target $b \in E$ is cartesian with respect to p if the map $E/f \to B/pf \times_{B/pb} E/b$

obtained from the commutative square

$$\begin{array}{ccc} E/f & \longrightarrow & E/b \\ \downarrow & & \downarrow \\ B/pf & \longrightarrow & B/pb \end{array}$$

is a trivial fibration. Every isomorphism in E is cartesian when B is a quategory by 9.12. We call a map of simplicial sets $p : E \to B$ a *Grothendieck fibration* if it is a mid fibration and for every vertex $b \in E$ and every arrow $g \in B$ with target $p(b)$ there exists a cartesian arrow $f \in E$ with target b such that $p(f) = g$.

20.5. A map $X \to 1$ is a Grothendieck fibration iff X is a quategory. A right fibration is a Grothendieck fibration whose fibers are Kan complexes. Every Grothendieck fibration is a pseudo-fibration.

20.6. The class of Grothendieck fibrations is closed under composition and base changes. The base change of a weak left adjoint along a Grothendieck fibration is a weak left adjoint [Malt2].

20.7. If X is a quategory, then the source map $s : X^I \to X$ a Grothendieck fibration. More generally, if a monomorphism of simplicial sets $u : A \to B$ is (weakly) coreflective, then the map $X^u : X^B \to X^A$ is a Grothendieck fibration.

20.8. If X is a quategory with pullbacks, then the target map $t : X^I \to X$ a Grothendieck fibration. More generally, if a monomorphism of simplicial sets $u : A \to B$ is (weakly) reflective, then the map $X^u : X^B \to X^A$ is a Grothendieck fibration.

20.9. If $p : X \to T$ is a Grothendieck fibration, then so is the map

$$\langle u, p \rangle : X^B \to Y^B \times_{Y^A} X^A$$

obtained from the square

$$\begin{array}{ccc} X^B & \longrightarrow & X^A \\ \downarrow & & \downarrow \\ Y^B & \longrightarrow & Y^A, \end{array}$$

for any monomorphism of simplicial sets $A \to B$. Moreover, the map $\langle u, f \rangle$ is a trivial fibration if u is mid anodyne.

20.10. Recall from 12.1 that the category $\mathbf{S}/B$ is enriched over $\mathbf{S}$ for any simplicial set B. Let us denote by $[X, Y]$ the simplicial set of maps $X \to Y$ between two objects of $\mathbf{S}/B$. If E is an object of $\mathbf{S}/B$ and $b : 1 \to B$, then the simplicial set $[b, E]$ is the fiber $E(b)$ of the structure map $E \to B$ at $b \in B$. If $f : a \to b$ is an arrow in B, consider the projections $p_0 : [f, E] \to E(a)$ and $p_1 : [f, E] \to E(b)$ respectively defined by the inclusions $\{0\} \subset I$ and $\{1\} \subset I$. If the structure map $E \to B$ is a Grothendieck fibration, then the projection $p_1 : [f, E] \to E(b)$ has a right adjoint $i_1 : E(b) \to [f, E]$ and the composite

$$f^* = p_0 i_1 : E(b) \to E(a)$$

is well defined up to a unique invertible 2-cell. We shall say that f^* is the *base change* along f, or the *pullback* along f If $t : \Delta[2] \to B$ is a simplex with boundary $\partial t = (g, h, f)$,

$$\begin{array}{ccccc} & & b & & \\ & {}^{f}\nearrow & & \searrow^{g} & \\ a & & \xrightarrow[h]{} & & c, \end{array}$$

then we can define a canonical invertible 2-cell

$$h^* \simeq f^* g^* : E(c) \to E(b) \to E(a).$$

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20.11. We shall say that a map $g : X \to Y$ between two Grothendieck fibrations in $\mathbf{S}/B$ is *cartesian* if it takes every cartesian arrow in X to a cartesian arrow in Y. A cartesian map $g : X \to Y$ respects base changes. More precisely, for any arrow $f : a \to b$ in B, the following square commutes up to a canonical invertible 2-cell,

$$\begin{array}{ccc} X(b) & \longrightarrow & Y(b) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ X(a) & \longrightarrow & Y(a), \end{array}$$

where the horizontal maps are induced by g.

20.12. If X and Y are quategories, then every map $u : X \to Y$ admits a factorisation

$$u = gi : X \to P(u) \to Y$$

with g a Grothendieck fibration and i a fully faithful right adjoint [Malt2]. The simplicial set $P(u)$ is constructed by the pullback square

$$\begin{array}{ccc} P(u) & \xrightarrow{q} & Y^I \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle t} \\ X & \xrightarrow{u} & Y, \end{array}$$

where t is the target map. If $s : Y^I \to Y$ is the source map, then the composite $g = sq : P(u) \to Y$ is a Grothendieck fibration. There is a unique map $i : X \to P(u)$ such that $pi = 1_X$ and $qi = \delta u$, where $\delta : Y \to Y^I$ is the diagonal. We have $g \vdash i$ and the counit of this adjunction is the identity of $gi = 1_X$. Thus, i is fully faithful. If $p : Z \to Y$ is a Grothendieck fibration, then for every map $f : X \to Z$ in $\mathbf{S}/Y$ there exists a cartesian map $c : P(u) \to Z$ such that $f = ci$. Moreover, c is unique up to a unique invertible 2-cell.

20.13. If $p : E \to B$ and $q : F \to B$ are two Grothendieck fibrations. We shall say that a commutative square

$$\begin{array}{ccc} E & \xrightarrow{g} & F \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ B & \xrightarrow{f} & C \end{array}$$

is a *morphism* $p \to q$ if the induced map $E \to B \times_C F$ is cartesian. We shall denote by **Cart** the category whose objects are the Grothendieck fibrations between quategories and whose arrows are the morphisms so defined. The category **Cart** is enriched over **S**. We shall denote by $\mathrm{Mor}(p,q)$ the simplicial set of morphisms $p \to q$ between two Grothendieck fibrations. By definition, a simplex $\Delta[n] \to Hom(p,q)$ is a morphism $\Delta[n] \times p \to q$. The simplicial set $\mathrm{Mor}(p,q)$ is a quategory for any pair of objects $p, q \in$ **Cart**. Hence the category **Cart** is actually enriched over **QCat**. It is thus enriched over Kan complexes if we put $\mathrm{Hom}(p.q) = J\mathrm{Mor}(p,q)$.

20.14. There are dual notions of *cocartesian* arrow and of *Grothendieck opfibration.* A map $p : E \to B$ is a Grothendieck opfibration iff the opposite map $p^o : E \to B$ is a Grothendieck fibration. We shall say that a map is a *Grothendieck bifibration* if it is both a Grothendieck fibration and a Grothendieck opfibration.

20.15. A Kan fibration is a Grothendieck bifibration whose fibers are Kan complexes.

20.16. If X is a bicomplete quategory and $u : A \to B$ is a fully faithful monomorphism of simplicial sets, then the map $X^u : X^B \to X^A$ is a Grothendieck bifibration.

20.17. If $p : E \to B$ is a Grothendieck opfibration, then the projection $p_0 : [f, E] \to E(a)$ has a left adjoint $i_0 : E(a) \to [f, E]$ and the composite

$$f_! = p_1 i_0 : E(a) \to E(b)$$

is well defined up to a unique invertible 2-cell. We shall say that $f_!$ is the *cobase change* along f, or the *pushforward* along f. The map $f_!$ is well defined of to a unique invertible 2-cell. If $p : E \to B$ is a Grothendieck bifibration, the map $f_!$ is left adjoint to the map f^*.

20.18. The quategory $\mathcal{Q}_1$ is the target of a *universal opfibration* $p : \mathcal{Q}_1' \to \mathcal{Q}_1$. The universality means that for any opfibration $f : E \to A$ there exists a homotopy pullback square in $(\mathbf{S}, \mathrm{Wcat})$,

$$\begin{array}{ccc} E & \xrightarrow{g'} & \mathcal{Q}_1' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow{g} & \mathcal{Q}_1, \end{array}$$

and the pair (g, g') is homotopy unique. We shall say that the map g *classifies* the opfibration $E \to A$.

20.19. The *Grothendieck construction* associates to a map of simplicial sets $g : A \to \mathcal{Q}_1$ its *simplicial set of elements* $el(g)$ defined by the pullback square,

$$\begin{array}{ccc} el(g) & \longrightarrow & \mathcal{Q}_1' \\ \downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow{g} & \mathcal{Q}_1. \end{array}$$

The map $el(g) \to A$ is an opfibration. The simplicial set $el(g)$ is a quategory when A is a quategory. The Grothendieck construction also associates to a map of simplicial sets $g : A^o \to \mathcal{Q}_1$ its *simplicial set of elements* $El(g) = el(g)^o$. The map $El(g) \to A$ is a Grothendieck fibration.

21. Proper and smooth maps

The notions of proper and of smooth functors were introduced by Grothendieck in 51. We extend these notions to maps of simplicial sets. The results of the section are taken from [J2].

21.1. We shall say that a map of simplicial sets $u : A \to B$ is *proper* if the pullback functor $u^* : \mathbf{S}/B \to \mathbf{S}/A$ takes a right anodyne map to a right anodyne map. A map of simplicial sets $u : A \to B$ is proper iff the inclusion $u^{-1}(b(n)) \subseteq b^*(E)$ is right anodyne for every simplex $b : \Delta[n] \to B$.

21.2. A Grothendieck opfibration is proper. In particular, a left fibration is proper. The class of proper maps is closed under composition and base changes. A projection $A \times B \to B$ is proper.

21.3. The pullback functor $u^* : \mathbf{S}/B \to \mathbf{S}/A$ has a right adjoint u_* for any map of simplicial sets $u : A \to B$. When u is proper, the pair of adjoint functors

$$u^* : \mathbf{S}/B \leftrightarrow \mathbf{S}/A : u_*.$$

is a Quillen pair with respect to the contravariant model structures on these categories. The functor u^* takes a contravariant equivalence to a contravariant equivalence and we obtain an adjoint pair of derived functors

$$\mathcal{R}^*(u) : \mathcal{R}(B) \leftrightarrow \mathcal{R}(A) : \mathcal{R}_*(u).$$

21.4. Dually, we shall say that a map of simplicial sets $p : E \to B$ is *smooth* if the functor $p^* : \mathbf{S}/B \to \mathbf{S}/E$ takes a left anodyne map to a left anodyne map. A map p is smooth iff the opposite map $p^o : E^o \to B^o$ is proper.

21.5. The functor $\mathcal{R}^*(u)$ admits a right adjoint $\mathcal{R}_*(u)$ for any map of simplicial sets $u : A \to B$. In order to see this, it suffices by Morita equivalence to consider the case where A and B are quategories. But in this case we have a factorisation $u = pi : A \to C \to B$, with i a left adjoint and p a Grothendieck opfibration by 22.10. Hence it suffices to prove that each functor $\mathcal{R}^*(p)$ and $\mathcal{R}^*(i)$ admit a right adjoint. But the functor $\mathcal{R}^*(p)$ admits a right adjoint $\mathcal{R}_*(p)$ by 21.3, since p is a Grothendieck opfibration and a Grothendieck opfibration is proper by 21.2. Let $v : C \to A$ be a right adjoint to i. Then the functor $\mathcal{R}^*(v)$ is right adjoint to $\mathcal{R}^*(i)$ by 13.13. The composite $\mathcal{R}_*(p)\mathcal{R}^*(v)$ is right adjoint to the composite $\mathcal{R}^*(u) = \mathcal{R}^*(i)\mathcal{R}^*(p)$.

21.6. Suppose that we have a commutative square of simplicial sets

$$\begin{array}{ccc} F & \xrightarrow{v} & E \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow{u} & B \end{array}$$

Then the following square commutes,

$$\begin{array}{ccc} \mathcal{R}(F) & \xrightarrow{\mathcal{R}_!(v)} & \mathcal{R}(E) \\ {\scriptstyle \mathcal{R}_!(q)}\downarrow & & \downarrow{\scriptstyle \mathcal{R}_!(p)} \\ \mathcal{R}(A) & \xrightarrow{\mathcal{R}_!(u)} & \mathcal{R}(B). \end{array}$$

From the adjunctions $\mathcal{R}_!(p) \vdash \mathcal{R}^*(p)$ and $\mathcal{R}_!(q) \vdash \mathcal{R}^*(q)$ we can define a canonical natural transformation

$$\alpha : \mathcal{R}_!(v)\mathcal{R}^*(q) \to \mathcal{R}^*(p)\mathcal{R}_!(u).$$

We shall say that the *Beck-Chevalley law holds* if α is invertible. This means that the following square commutes up to a canonical isomorphism,

$$\begin{array}{ccc} \mathcal{R}(F) & \xrightarrow{\mathcal{R}_!(v)} & \mathcal{R}(E) \\ {\scriptstyle \mathcal{R}^*(q)}\uparrow & & \uparrow{\scriptstyle \mathcal{R}^*(p)} \\ \mathcal{R}(A) & \xrightarrow{\mathcal{R}_!(u)} & \mathcal{R}(B). \end{array}$$

Equivalently, this means that the following square of right adjoints commutes up to a canonical isomorphism,

$$\begin{array}{ccc} \mathcal{R}(F) & \xleftarrow{\mathcal{R}^*(v)} & \mathcal{R}(E) \\ {\scriptstyle \mathcal{R}_*(q)}\downarrow & & \downarrow{\scriptstyle \mathcal{R}_*(p)} \\ \mathcal{R}(A) & \xleftarrow{\mathcal{R}^*(u)} & \mathcal{R}(B). \end{array}$$

21.7. (Proper or smooth base change) [J2] Suppose that we have a cartesian square of simplicial sets,

$$\begin{array}{ccc} F & \xrightarrow{v} & E \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow{u} & B. \end{array}$$

Then the Beck-Chevalley law holds if p is proper *or* if u is smooth.

22. Kan extensions

We introduce the notion of Kan extension for maps between quategories . The results of the section are taken from [J2].

22.1. Let $\mathcal{C}$ be a 2-category. We shall call a 1-cell of $\mathcal{C}$ a *map*. The *left Kan extension* of a map $f : A \to X$ along a map $u : A \to B$ is a pair (g, α), where $g : B \to X$ is a map and $\alpha : f \to gu$ is a 2-cell, which reflects the map f along the functor

$$\mathcal{C}(u, X) : \mathcal{C}(B, X) \to \mathcal{C}(A, X).$$

This means that for any map $g' : B \to X$ and any 2-cell $\alpha' : f \to g'u$, there is a unique 2-cell $\beta : g \to g'$ such that $(\beta \circ u)\alpha = \alpha'$. The pair (g, α) is unique up to a unique invertible 2-cell when it exists, in which case we shall put $g = \Sigma_u(f)$. Dually, the *right Kan extension* of a map $f : A \to X$ along a map $u : A \to B$ is a pair (g, β), where $g : B \to X$ is a map and $\beta : gu \to f$ is a 2-cell, which coreflects the map f along the functor $\mathcal{C}(X, u)$. This means that for any map $g' : B \to X$ and any 2-cell $\alpha' : g'u \to f$, there is a unique 2-cell $\beta : g' \to g$ such that $\alpha(\beta \circ u) = \alpha'$. The pair (g, β) is unique up to a unique invertible 2-cell when it exists, in which case we shall put $g = \Pi_u(f)$.

22.2. If $u : A \leftrightarrow B : v$ is an adjoint pair in a 2-category $\mathcal{C}$, then we have $\mathcal{C}(X, v) \vdash \mathcal{C}(X, u)$ for any object X. Hence we have $fv = \Sigma_u(f)$ for every $f : A \to X$ and we have $gu = \Pi_v(g)$ for every map $g : B \to X$.

22.3. The category $\mathbf{S}$ has the structure of a 2-category ($= \mathbf{S}^{\tau_1}$). Hence there is a notion of Kan extension for maps of simplicial sets. We will only consider Kan extension of maps with values in a quategory . If X is a quategory, we shall denote by $\Sigma_u(f)$ the *left Kan extension* of a map $f : A \to X$ along a map of simplicial sets $u : A \to B$. Dually, we shall denote by $\Pi_u(f)$ the *right Kan extension* of a map $f : A \to X$ along $u : A \to B$. By duality we have

$$\Pi_u(f)^o = \Sigma_{u^o}(f^o).$$

22.4. If X is a cocomplete quategory and $u : A \to B$ is a map between (small) simplicial sets, then every map $f : A \to X$ has a left Kan extension $\Sigma_u(f) : B \to X$ and the map $X^u : X^B \to X^A$ has a left adjoint

$$\Sigma_u : X^A \to X^B.$$

Dually, if X is a complete quategory, then every map $f : A \to X$ has a right Kan extension $\Pi_u(f) : B \to X$ and the map X^u has a right adjoint

$$\Pi_u : X^A \to X^B.$$

22.5. If $u : A \to B$ is a map of simplicial sets, then the colimit of a diagram $d : A \to X$ is isomorphic to the colimit of its left Kan extension $\Sigma_u(d) : B \to X$, when they exist. Dually, the limit of a diagram $d : A \to X$ is isomorphic to the limit of its right Kan extension $\Pi_u(d) : B \to X$, when they exist.

22.6. If $u : A \to B$ and $v : B \to C$ are maps of simplicial sets, then we have a canonical isomorphism

$$\Sigma_v \circ \Sigma_u = \Sigma_{vu} : X^A \to X^C$$

for any cocomplete quategory X. Dually, we have a canonical isomorphism

$$\Pi_v \circ \Pi_u = \Pi_{vu} : X^A \to X^C$$

for any complete quategory X.

22.7. Let X be a bicomplete quategory. If $u : A \leftrightarrow B : v$ is an adjunction between two maps of simplicial sets, then we have three adjunctions and two isomorphisms,

$$\Sigma_v \vdash \Sigma_u = X^v \vdash X^u = \Pi_v \vdash \Pi_u.$$

22.8. Every map between quategories $u : A \to B$ admits a factorisation

$$u = qi : A \to P \to B$$

with q a Grothendieck opfibration and i a fully faithful left adjoint (a coreflection) by 22.10. If $p : P \to A$ is the righ adjoint of i, then we have $X^p = \Sigma_i$ for any cocomplete quategory X, since we have $X^p \vdash X^i$. Thus

$$\Sigma_u = \Sigma_q \circ \Sigma_i = \Sigma_q \circ X^p.$$

22.9. If $u : A \to B$ is a map between (small) simplicial sets, we shall denote the map $\mathcal{K}^{u^o}$ by u^*, the map Σ_{u^o} by $u_!$ and the map Π_{u^o} by u_*. We have $u_! \vdash u^* \vdash u_*$,

$$u_! : \mathcal{P}(A) \leftrightarrow \mathcal{P}(B) : u^* : \mathcal{P}(B) \leftrightarrow \mathcal{P}(A) : u_*.$$

Notice the equality $(vu)^* = u^* v^*$ and the isomorphisms $(vu)_! \simeq v_! u_!$ and $(vu)_* \simeq v_* u_*$. for a pair of maps $u : A \to B$ and $v : B \to C$. More generally, if X is a complete quategory anhd $u : A \to B$ is a map between (small) simplicial sets, we may denote the map X^{u^o} by u^*, the map Σ_{u^o} by $u_!$ and the map Π_{u^o} by u_*.

22.10. If $u : A \leftrightarrow B : v$ is an adjunction between two maps of simplicial sets, then we have three adjunctions and two isomorphisms,

$$u_! \vdash v_! = u^* \vdash v^* = u_* \vdash v_*.$$

22.11. Suppose that we have commutative square of simplicial sets,

$$\begin{array}{ccc} F & \xrightarrow{v} & E \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow{u} & B. \end{array}$$

If X is a cocomplete quategory. then from the commutative square

$$\begin{array}{ccc} X^F & \xleftarrow{X^v} & X^E \\ {\scriptstyle X^q}\uparrow & & \uparrow{\scriptstyle X^p} \\ X^A & \xleftarrow{X^u} & X^B. \end{array}$$

then from the adjunctions $\Sigma_u \vdash X^u$ and $\Sigma_v \vdash X^v$, we can define a natural transformation

$$\alpha : \Sigma_v X^q \to X^p \Sigma_u.$$

We shall say that the *Beck-Chevalley law holds* if α is invertible. Dually, if X is complete, then from the adjunctions $X^p \vdash \Pi_p$ and $X^q \vdash \Pi_q$ we obtain natural transformation

$$\beta : X^u \Pi_p \to \Pi_q X^v.$$

We shall say that the *Beck-Chevalley law holds* if β is invertible. When X is bicomplete, the transformation β is the right transpose of α. Thus, β is invertible iff α is invertible. Hence the Beck-Chevalley law holds in the first sense iff it holds in the second sense. The Beck-Chevalley law holds in the first sense if the square $pv = uq$ is cartesian and u is a smooth map. The Beck-Chevalley law holds in the second sense if the square $pv = uq$ is cartesian and p is a proper map.

22.12. Suppose that we have commutative square of simplicial sets,

$$\begin{array}{ccc} F & \xrightarrow{v} & E \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow{u} & B. \end{array}$$

If X is a complete quategory. then from the commutative square

$$\begin{array}{ccc} X^{F^o} & \xleftarrow{v^*} & X^{E^o} \\ {\scriptstyle q^*}\uparrow & & \uparrow{\scriptstyle p^*} \\ X^{A^o} & \xleftarrow{u^*} & X^{B^o}. \end{array}$$

and the adjunctions $p^* \vdash p_*$ and $q^* \vdash q_*$, we obtain natural transformation

$$\alpha : u^* p_* \to q_* v^*.$$

We shall say that the *Beck-Chevalley law holds* if α is invertible. The Beck-Chevalley law holds if the square $pv = uq$ is cartesian and p is a proper map. Dually, if X is a cocomplete quategory, then from the adjunctions $u_! \vdash u^*$ and $v_! \vdash v^*$, we obtain natural transformation

$$\beta : v_! q^* \to p^* u_!.$$

We shall say that the *Beck-Chevalley law holds* if β is invertible. The Beck-Chevalley law holds if the square $pv = uq$ is cartesian and u is a smooth map. When X is bicomplete, the transformation β is the left transpose of α. Thus, β is invertible iff α is invertible. Hence the Beck-Chevalley law holds in the first sense iff it holds in the second sense.

22.13. If $p : E \to B$ is a proper map and $E(b)$ is the fiber of p at $b \in B_0$, then the Beck-Chevalley law holds for the square

$$\begin{array}{ccc} E(b) & \xrightarrow{v} & E \\ \downarrow & & \downarrow{\scriptstyle p} \\ 1 & \xrightarrow{b} & B. \end{array}$$

This means that if X is a complete quategory, then we have

$$p_*(f)(b) = \varprojlim_{x \in E(b)} f(x)$$

for any map $f : E^o \to X$. Dually, If $p : E \to B$ is a smooth map and X is a cocomplete quategory, then we have

$$p_!(f)(b) = \varinjlim_{x \in E(b)} f(x)$$

for any map $f : E^o \to X$.

22.14. It follows from 22.13 that if $p : E \to B$ is a smooth map and X is a complete quategory, then we have

$$\Pi_p(f)(b) = \varprojlim_{x \in E(b)} f(x),$$

for every map $f : E \to X$ and every $b \in B_0$. Dually, if $p : E \to B$ is a proper map and X is a cocomplete quategory, then we have

$$\Sigma_p(f)(b) = \varinjlim_{x \in E(b)} f(x)$$

for every map $f : E \to X$ and every $b \in B_0$.

22.15. If X is a cocomplete quategory and B is a quategory, let us compute the left Kan extension of a map $f : A \to X$ along a map $u : A \to B$. We shall apply the Beck-Chevalley law to the pullback square

$$\begin{CD} A/b @>>> B/b \\ @VVV @VV{p}V \\ A @>{u}>> B \end{CD}$$

The value of $\Sigma_u(f)$ at $b : 1 \to B$ is obtained by composing the maps

$$X^A \xrightarrow{\Sigma_u} X^B \xrightarrow{X^b} X.$$

If $t : 1 \to B/b$ is the terminal vertex, then we have $X^b = X^t X^p$, since we haved $b = pt$. The map $t : 1 \to B/b$ is right adjoint to the map $r : B/b \to 1$. It follows that $X^t = \Sigma_r$. Thus,

$$X^b \Sigma_u = X^t X^p \Sigma_u = \Sigma_r X^p \Sigma_u.$$

The projection p is smooth since a right fibration is smooth. Hence the following square commutes up to a natural isomorphism by 22.12,

$$\begin{CD} X^{A/b} @<{X^q}<< X^A \\ @V{\Sigma_v}VV @VV{\Sigma_u}V \\ X^{B/b} @<{X^p}<< X^B. \end{CD}$$

Thus,

$$\Sigma_r X^p \Sigma_u \simeq \Sigma_r \Sigma_v X^q \simeq \Sigma_{rv} X^q.$$

But Σ_{rv} is the colimit map

$$\varinjlim : X^{A/b} \to X,$$

since rv is the map $A/b \to 1$. Hence the square

$$\begin{CD} X^{A/b} @<{X^q}<< X^A \\ @V{\varinjlim}VV @VV{\Sigma_u}V \\ X @<{X^b}<< X^B \end{CD}$$

commutes up to a canonical isomorphism. This yields Kan's formula

$$\Sigma_u(f)(b) = \varinjlim_{u(a)\to b} f(a).$$

22.16. Dually, if X is a complete quategory and B is a quategory, then the right Kan extension of a map $f : A \to X$ along a map $u : A \to B$ is computed by Kan's formula

$$\Pi_u(f)(b) = \varprojlim_{b\to u(a)} f(a),$$

where the limit is taken over the simplicial set $b\backslash A$ defined by the pullback square

$$\begin{CD} b\backslash A @>>> b\backslash B \\ @VVV @VV{p}V \\ A @>{u}>> B. \end{CD}$$

22.17. A map of simplicial sets $u : A \to B$ is fully faithful iff the map $\Sigma_u : X^A \to X^B$ is fully faithful for every cocomplete quategory X.

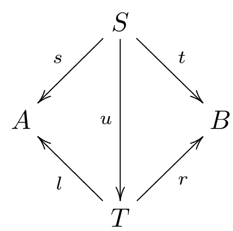
22.18. Let X be a cocomplete quategory. For any span $(s, t) : S \to A \times B$, the composite

$$X^A \xrightarrow{X^s} X^S \xrightarrow{\Sigma_t} X^B$$

is a cocontinuous map

$$X\langle S\rangle : X^A \to X^B.$$

If $u : S \to T$ is a map in $\mathrm{Span}(A, B)$, then from the commutative diagram



and the counit $\Sigma_u \circ X^u \to id$, we can define a 2-cell,

$$X\langle u\rangle : X\langle S\rangle = \Sigma_t \circ X^s = \Sigma_r \circ \Sigma_u \circ X^u \circ X^l \to \Sigma_r \circ X^l = X\langle T\rangle.$$

This defines a functor

$$X\langle -\rangle : \mathrm{Span}(A, B) \to \tau_1(X^A, X^B).$$

A map $u : S \to T$ in $\mathrm{Span}(A, B)$ is a bivariant equivalence if the 2-cell

$$X\langle u\rangle : X\langle S\rangle \to X\langle T\rangle.$$

is invertible for any cocomplete quategory X iff the 2-cell

$$\mathcal{K}\langle u\rangle : \mathcal{K}\langle S\rangle \to \mathcal{K}\langle T\rangle.$$

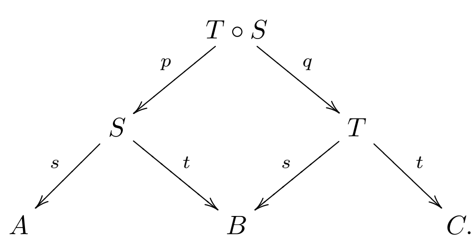
is invertible. We thus obtain a functor

$$X\langle -\rangle : \mathrm{hSpan}(A, B) \to \tau_1(X^A, X^B).$$

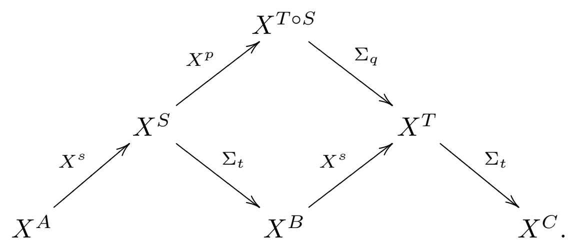
22.19. If $S \in \mathrm{Span}(A, B)$ and $T \in \mathrm{Span}(B, C)$ are bifibrant spans, then we have a canonical isomorphism

$$X\langle T \circ S\rangle \simeq X\langle T\rangle \circ X\langle S\rangle$$

for any cocomplete quategory X. To see this, it suffices to consider the case where A, B and C are quategories . We have a pullback diagram,



The map $s : T \to B$ is smooth, since it is Grothendieck fibration by 14.15. It then follows from 22.12 that the Beck-Chevalley law holds for the square in the following diagram,



Thus,

$$X\langle T \circ S\rangle = \Sigma_r \Sigma_q X^p X^s \simeq \Sigma_t X^p \Sigma_t X^s = X\langle T\rangle \circ X\langle S\rangle.$$

We have defined a (pseudo) functor

$$X\langle -\rangle : \mathrm{hSpan} \to \mathbf{CC},$$

where **CC** the category of cocomplete quategories and cocontinuous maps.

22.20. If X and Y are cocomplete quategories, let us denote by $\mathrm{CC}(X, Y)$ the full simplicial subset of Y^X spanned by the cocontinuous maps $X \to Y$. If A is a (small) simplicial set, then we have a natural equivalence of categories

$$\mathrm{CC}(X^{A^o}, Y) \simeq \mathrm{CC}(X, Y^A).$$

More precisely, then the endo-functor $X \mapsto X^{A^o}$ is left adjoint to the endo-functor $Y \mapsto Y^A$. The unit of the adjunction is the map $X\langle\eta_A\rangle : X \to X^{A^o\times A}$ and the counit is the map $X\langle\epsilon_A\rangle : X^{A\times A^o} \to X$. It folllows from this adjunction that the quategory X^{A^o} can be regarded as the *tensor product* $A \otimes X$ of X by A. More precisely, the map

$$c_A : A \times X \to X^{A^o}$$

which corresponds to the map $X\langle\eta_A\rangle : X \to X^{A^o\times A}$ by the exponential adjointness is cocontinuous in the second variable and universal with respect to that property. This means that for any cocomplete quategory Y and any map $f : A \times X \to Y$ cocontinuous in the second variable, there exists a cocontinuous map $g : X^{A^o} \to Y$ together with an isomorphism $\alpha : f \simeq gc_A$ and moreover that the pair (f, α) is unique up to unique isomorphism. Notice that we have $c_A(a, x)(b^o) = Hom_A(b, a)\cdot x$ for every $a, b \in A$ and $x \in X$. The 2-category **CC** becomes tensored over the category hSpan^{rev} if we put $A \otimes X = X^{A^o}$ and

$$\langle S\rangle \otimes X = X\langle S^o\rangle : A \otimes X \to B \otimes X$$

for $S \in \mathrm{Span}(B, A)$. In particular, we have $A \otimes \mathcal{K} = \mathcal{P}(A)$.

22.21. The counit of the adjunction $(-)^{A^o} \vdash (-)^A$ described above is the *trace map*

$$Tr_A = X\langle\epsilon_A\rangle : X^{A\times A^o} \to X.$$

In category theory, the trace of a functor $f : A \times A^o \to Y$ is called the *coend*

$$coend_A(f) = \int^{a\in A} f(a, a).$$

We shall use the same notation for the trace of a map $f : A \times A^o \to Y$. Notice that

$$Tr_A(f) = Tr_{A^o}({}^t f),$$

where ${}^t f : A^o \times A \to Y$ is the transpose of f. The inverse of the equivalence

$$\mathrm{CC}(X^{A^o}, Y) \simeq \mathrm{CC}(X, Y^A)$$

associates to a map cocontinuous in the first variable $f : X \times A \to Y$ the map $g : X^{A^o} \to Y$ obtained by putting

$$g(z) = \int^{a \in A} f(z(a), a).$$

for every $z \in X^{A^o}$.

22.22. If X is a complete quategory, the *cotrace map*

$$Tr^o_A : X^{A^o \times A} \to X$$

is defined to be the opposite of the trace map $Tr_A : (X^o)^{A \times A^o} \to X^o$. In category theory, the cotrace of a functor $f : A^o \times A \to X$ is the *end*

$$end_A(f) = \int_{a \in A} f(a, a),$$

and we shall use the same notation. Notice that

$$Tr^o_A(f) = Tr^o_{A^o}({}^t f),$$

where ${}^t f : A \times A^o \to X$ is the transpose of f.

22.23. If X is a quategory, then the contravariant functor $A \mapsto ho(A, X) = ho(X^A)$ is a kind of cohomology theory with values in **Cat**. When X is bicomplete, the map $ho(u, X) : ho(B, X) \to ho(A, X)$ has a left adjoint $ho(\Sigma_u)$ and a right adjoint $ho(\Pi_u)$ for any map $u : A \to B$. If we restrict the functor $A \mapsto ho(A, X)$ to the subcategory $\mathbf{Cat} \subset \mathbf{S}$, we obtain a *homotopy theory* in the sense of Heller, also called a *derivateur* by Grothendieck [Malt1] Most derivateurs occuring naturally in mathematics can be represented by bicomplete quategories .

23. The quategory $\mathcal{K}$

The quategory $\mathcal{K}$ is cocomplete and freely generated by its terminal object. A prestack on a simplicial set A is defined to be a map $A^o \to \mathcal{K}$. The simplicial set of prestacks on A is cocomplete and freely generated by A. A cocomplete quategory is equivalent to a quategory of prestacks iff it is generated by a small set of atoms.

23.1. Recall that the quategory $\mathcal{K} = \mathcal{Q}_0$ is defined to be the coherent nerve of the category of Kan complexes. The quategory $\mathcal{K}$ is bicomplete and freely generated by the object $1 \in \mathcal{K}$ as a cocomplete quategory. More precisely, the evaluation map

$$ev : \mathrm{CC}(\mathcal{K}, X) \to X$$

defined by putting $ev(f) = f(1)$ is an equivalence for any cocomplete quategory X. The map ev is actually a trivial fibration. If s is a section of ev, then the map

$$\cdot : \mathcal{K} \times X \to X$$

defined by putting $k \cdot x = s(x)(k)$ is cocontinuous in each variable and we have $1 \cdot x = x$ for every $x \in X$.

23.2. The Yoneda map $y_A : A \to \mathcal{P}(A)$ exibits the quategory $\mathcal{P}(A)$ as the free completion of A under colimits. More precisely, for any cocomplete quategory X the map

$$y_A^* : \mathrm{CC}(\mathcal{P}(A), X) \to X^A$$

induced by y_A is an equivalence. The inverse equivalence associates to a map $f : A \to X$ its left Kan extension $f_! : \mathcal{P}(A) \to X$ along y_A. The value of $f_!$ on a prestack $k \in \mathcal{P}(A)$ is the colimit of the composite $fp : El(k) \to A \to X$, where $p : El(k) \to A$ is the quategory of lements of k In other words, we have

$$f_!(k) = \varinjlim_{El(k)} f.$$

Compare with Dugger [Du].

23.3. The left Kan extension of the Yoneda map $y_A : A \to \mathcal{P}(A)$ along itself is the identity of $\mathcal{P}(A)$. It follows that we have

$$k = \varinjlim_{El(k)} y_A$$

for every object $k \in \mathcal{P}(A)$.

23.4. A map $f : A \to B$ between small quategories induces a map

$$f^* : \mathcal{P}(B) \to \mathcal{P}(A).$$

If $y_B : B \to \mathcal{P}(B)$ denotes the Yoneda map, The composite $f^! = f^* y_B : B \to \mathcal{P}(A)$ is the *probe map* associated to f. By definition, we have $f(b)(a) = hom_B(fa, b)$ for every $a \in A$ and $b \in B$. The probe map $f^!$ can be defined under the weaker assumption that B is locally small. If B is locally small and cocomplete, then $f^!$ is right adjoint to the left Kan extension

$$f_! : \mathcal{P}(A) \to B$$

of f along y_A.

23.5. For example, if f is the map $\Delta \to \mathcal{Q}_1$ obtained by applying the coherent nerve functor to the inclusion $\Delta \to \mathbf{QCat}$, then the probe map

$$f^! : \mathcal{Q}_1 \to \mathcal{P}(\Delta)$$

associates to an object $C \in \mathcal{Q}_1$ its *nerve* $N(C) : \Delta^o \to \mathcal{K}$. By construction, we have

$$N(C)_n = J(C^{\Delta[n]})$$

for every $n \geq 0$.

23.6. For any simplicial set A, the quategory $\mathcal{P}(A)$ is the homotopy localisation of the model category $(\mathbf{S}/A, \mathrm{Wcont})$. More precisely, we saw in 19.35 that the map $\lambda_A : \Delta/A \to A$ is a homotopy localisation. The left Kan extension of the composite

$$y_A \lambda_A : \Delta/A \to \mathcal{P}(A)$$

along the inclusion $\Delta/A \to \mathbf{S}/A$ induces an equivalence of quategories

$$L(\mathbf{S}/A, \mathrm{Wcont}) \to \mathcal{P}(A).$$

The inverse equivalence associates to a prestack $f : A \to \mathcal{K}$ the right fibration $El(f) \to A$.

23.7. If $f : A \to B$ is a map between small quategories, then the map $f^* = \mathcal{K}^{f^o}$ is the probe of the composite $y_B f : A \to B \to \mathcal{P}(B)$ and we have

$$f_! : \mathcal{P}(A) \leftrightarrow \mathcal{P}(B) : f^*.$$

23.8. It follows from Yoneda lemma that the quategory of elements $El(g)$ of a prestack $g \in \mathcal{P}(A)$, is equivalent to the quategory A/g defined by the pullback square

$$\begin{array}{ccc} A/g & \xrightarrow{q} & \mathcal{P}(A)/g \\ \downarrow & & \downarrow \\ A & \xrightarrow{y_A} & \mathcal{P}(A), \end{array}$$

The adjoint pair

$$q_! : \mathcal{P}(A/g) \leftrightarrow \mathcal{P}(A)/g : q^!$$

obtained from the map q is an equivalence of quategories.

23.9. Let X be a locally small quategory. If A is a small simplicial set, we shall say that a map $f : A \to X$ is *dense* if the probe map $f^! : X \to \mathcal{P}(A)$ is fully faithful. We shall say that a small full simplicial subset $A \subseteq X$ is *dense* if the inclusion $i : A \subseteq X$ is dense; we shall say that a set of objects $S \subseteq X$ is *dense* if the full simplicial subset spanned by S is dense.

23.10. For example, the Yoneda map $y_A : A \to \mathcal{P}(A)$ is dense, since the map $(y_A)_!$ is the identity. In particular, the map $1 : 1 \to \mathcal{K}$ is dense. The map $f : \Delta \to \mathcal{Q}_1$ defined in **??** is dense; this means that the nerve map

$$N : \mathcal{Q}_1 \to \mathcal{P}(\Delta)$$

is fully faithful.

23.11. A map of simplicial sets $u : A \to B$ is dominant iff the map $y_B u : A \to \mathcal{P}(B)$ is dense.

23.12. Let X be a locally small quategory. If A is a simplicial set, then a map $f : A \to X$ is dense iff the counit of the adjunction

$$f_! : \mathcal{P}(A) \leftrightarrow X : f^!$$

is invertible. The value of this counit at $x \in X$ is the canonical morphism

$$\varinjlim_{A/x} f \to x$$

where the diagram $A/x \to A$ is defined by the pullback square

$$\begin{array}{ccc} A/x & \longrightarrow & A \\ \downarrow & & \downarrow f \\ X/x & \longrightarrow & X. \end{array}$$

23.13. Let X be a locally small quategory. We shall say that a map $f : A \to X$ with a small domain A is *separating* if the probe map $f^! : X \to \mathcal{P}(A)$ is conservative. A dense map is separating. We shall say that a set of objects $S \subseteq X$ is *separating* if the inclusion $S \subseteq X$ is separating.

23.14. If X is a quategory, we shall say that a simplicial subset $S \subseteq X$ is *replete* if every object which is isomorphic to an object in S belongs to S. If X is cocomplete, we shall say that a set objects $S \subseteq X$ *generates* X if X is the smallest full replete simplicial subset of X which contains S and which is closed under colimits. If X cocomplete and locally small, then every generating subset $S \subseteq X$ separates X.

23.15. Let X be a (locally small) cocomplete quategory. We shall say that an object $x \in X$ is *atomic* if the map $hom_X(x, -) : X \to \mathcal{K}$ is cocontinuous.

23.16. An object $K \in \mathcal{K}$ is atomic iff K is contractible.

23.17. If A is a simplicial set, then a prestack $f \in \mathcal{P}(A)$ is atomic iff it is a retract of a representable. The Yoneda map $y_A : A \to \mathcal{P}(A)$ induces an equivalence between the Karoubi envelope of A and the full simplicial subset of $\mathcal{P}(A)$ spanned by the atomic objects.

23.18. An arrow $f \to g$ in $\mathcal{P}(A)$ is atomic as an object of the quategory $\mathcal{P}(A)/g$ iff the object f is atomic in $\mathcal{P}(A)$.

23.19. Let X be a cocomplete quategory and $A \subseteq X$ be a small full sub-quategory of atomic objects. Then the left Kan extension

$$i_! : \mathcal{P}(A) \to X$$

of the inclusion $i : A \subseteq X$ along $Y_A : A \to \mathcal{P}(A)$ is fully faithful Moreover, $i_!$ is an equivalence if A generates or separates X.

23.20. A cocomplete quategory X is equivalent to a quategory of prestacks iff it is generated by a small set of atoms.

23.21. If A is a simplicial set, we shall say that a prestack $g \in \mathcal{P}(A)$ is *finitely presentable*, or that it is of *finite type*, if it is the colimit of a finite diagram of representable prestacks. Let us denote by $\mathcal{P}_f(A)$ the full simplicial subset of $\mathcal{P}(A)$ spanned by the prestacks of finite types. Then the map $y : A \to \mathcal{P}_f(A)$ induced by the Yoneda map $A \to \mathcal{P}(A)$ exibits the quategory $\mathcal{P}_f(A)$ as the free cocompletion of A under finite colimits. More precisely, for any quategory with finite colimits X the map

$$y^* : \mathrm{fCC}(\mathcal{P}_f(A), X) \to X^A$$

induced by y is an equivalence, where the domain of y^* is the quategory of maps $\mathcal{P}_f(A) \to X$ which preserve finite colimits. The inverse equivalence associates to a map $g : A \to X$ its left Kan extension $g_! : \mathcal{P}_f(A) \to X$ along y. A quategory A has finite colimits iff the canonical map $y : A \to \mathcal{P}_f(A)$ has a left adjoint.

23.22. An object in $\mathcal{P}(1) = \mathcal{K}$ is of finite type iff it has a finite homotopy type. We conjecture that a prestack $a \in \mathcal{P}(A)$ is representable iff it is atomic and of finite type. A map of simplicial sets $u : A \to B$ can be extended as a map preserving finite colimits. $u_! : \mathcal{P}_f(A) \to \mathcal{P}_f(B)$. The map $u_!$ is fully faithful iff u is fully faithful. Moreover, $u_!$ is an equivalence when u is an equivalence; we conjecture that the converse is true.

23.23. We conjecture that a simplicial set A is essentially finite iff the map $hom :$ $A^o \times A \to \mathcal{K}$ (regarded as a prestack) is finitely presentable (the necessity is obvious).

23.24. If A is a simplicial set, we shall say that a prestack $g \in \mathcal{P}(A)$ is *free* (resp. *finitely free*) if it is a coproduct (resp. finite coproduct) of representable prestacks. Let us denote by $\mathrm{fCup}(A)$ the full simplicial subset of $\mathcal{P}(A)$ spanned by the finitely free prestacks. Then the map $y : A \to \mathrm{fCup}(A)$ induced by the Yoneda map $A \to \mathcal{P}(A)$ exibits the quategory $\mathrm{fCup}(A)$ as the free cocompletion of A under finite coproducts. More precisely, for any quategory with finite coproducts X the map

$$y^* : \mathrm{fCoprod}(\mathrm{fCup}(A), X) \to X^A$$

induced by y is an equivalence, where the domain of y^* is the quategory of maps $\mathcal{P}_f(A) \to X$ which preserve finite coproducts . The inverse equivalence associates to a map $g : A \to X$ its left Kan extension $g_! : \mathrm{fCup}(A) \to X$ along y. The quategory $\mathrm{fCup}(A)$ is (equivalent to) a category when A is a category. For example, $\mathrm{fCup}(1)$ can be taken to be the category $\underline{N}$, whose objects are the natural numbers and whose arrows are the maps $\underline{m} \to \underline{n}$, where $\underline{n} = \{1, \cdots, n\}$. A quategory A has finite coproducts iff the map $u : A \to \mathrm{fCup}(A)$ has a left adjoint.

23.25. Let α be a regular cardinal (recall that 0 and 1 are the finite regular cardinals). If A is a simplicial set, we shall say that a prestack $g \in \mathcal{P}(A)$ is *α-presentable* if it is the colimit of an α-small diagram of representable prestacks. Let us denote by $\mathcal{P}_\alpha(A)$ the full simplicial subset of $\mathcal{P}(A)$ spanned by α-presentable prestacks. Then the map $y : A \to \mathcal{P}_\alpha(A)$ induced by the Yoneda map $A \to \mathcal{P}(A)$ exibits the quategory $\mathcal{P}_\alpha(A)$ as the free cocompletion of A under α-colimits. More precisely, for any α-cocomplete quategory X the map

$$y^* : \mathrm{C}_\alpha(\mathcal{P}_\alpha(A), X) \to X^A$$

induced by y is an equivalence, where the domain of y^* is the quategory of maps $\mathcal{P}_\alpha(A) \to X$ which preserve α-colimits. The inverse equivalence associates to a map $g : A \to X$ its left Kan extension $g_! : \mathcal{P}_\alpha(A) \to X$ along y. A quategory A is α-cocomplete iff the map $u : A \to \mathcal{P}_\alpha(A)$ has a left adjoint.

24. Factorisation systems in quategories

In this section, we introduce the notion of factorisation system in a quategory. It is closely related to the notion of homotopy factorisation system in a model category introduced in section 11.

24.1. We first define the orthogonality relation $u \perp f$ between the arrows of a quategory X. If $u : a \to b$ and $f : x \to y$ are two arrows in X, then an arrow $s \in X^I(u, f)$ in the quategory X^I is a a commutative square $s : I \times I \to X$,

$$\begin{array}{ccc} a & \longrightarrow & x \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle f} \\ b & \longrightarrow & y, \end{array}$$

such that $s|\{0\} \times I = u$ and $s|\{1\} \times I = f$. A *diagonal filler* for s is a map $I \star I \to X$ which extends s along the inclusion $I \times I \subset I \star I$. The projection $q : X^{I \star I} \to X^{I \times I}$ defined by the inclusion $I \times I \subset I \star I$ is a Kan fibration. We shall say that u is *left orthogonal* to f, or that f is *right orthogonal* to u, and we shall write $u \perp f$, if the fiber of q at s is contractible for every commutative square $s \in X^I(u, f)$. An arrow $f \in X$ is invertible iff we have $f \perp f$.

24.2. When X has a terminal object 1, then an arrow $x \to 1$ is right orthogonal to an arrow $u : a \to b$ iff the map

$$X(u, x) : X(b, x) \to X(a, x)$$

induced by u is a homotopy equivalence. In this case we shall say that x is *right orthogonal* to the arrow u, or that x *local* with respect to u, and we shall write $u \perp x$.

24.3. If $h : X \to hoX$ is the canonical map, then the relation $u \perp f$ between the arrows of X implies the relation $h(u) \pitchfork h(f)$ in hoX. However, if $h(u) = h(u')$ and $h(f) = h(f')$, then the relations $u \perp f$ and $u' \perp f'$ are equivalent. Hence the relation $u \perp f$ only depends on the homotopy classes of u and f. If A and B are two sets of arrows in X, we shall write $A \perp B$ to indicate the we have $u \perp f$ for every $u \in A$ and $f \in B$. We shall put

$$A^{\perp} = \{f \in X_1 : \forall u \in A,\ \ u \perp f\}, \qquad {}^{\perp}A = \{u \in X_1 : \forall f \in A,\ \ u \perp f\}.$$

The set $A^{\perp}$ contains the isomorphisms, it is closed under composition and it has the left cancellation property. It is closed under retracts in the quategory X^I. And it is closed under base changes when they exist. This means that the implication $f \in A^{\perp} \Rightarrow f' \in A^{\perp}$ is true for any pullback square

$$\begin{array}{ccc} x' & \longrightarrow & x \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ y' & \longrightarrow & y \end{array}$$

in X.

24.4. Let X be a (large or small) quategory. We shall say that a pair (A, B) of class of arrows in X is a *factorisation system* if the following two conditions are satisfied:

- $A^{\perp} = B$ and $A = {}^{\perp}B$;
- every arrow $f \in X$ admits a factorisation $f = pu$ (in hoX) with $u \in A$ and $p \in B$.

We say that A is the *left class* and that B is the *right class* of the factorisation system.

24.5. If X is a quategory, then the image by the canonical map $h : X \to hoX$ of a factorisation system (A, B) is a weak factorisation system $(h(A), h(B))$ on the category hoX. Moreover, we have $A = h^{-1}h(A)$ and $B = h^{-1}h(B)$. Conversely, if (C, D) is a weak factorisation system on the category $ho(X)$, then the pair $(h^{-1}(C), h^{-1}(D))$ is a factorisation system on X iff we have $h^{-1}(C) \perp h^{-1}(D)$.

24.6. The left class A of a factorisation system (A, B) in a quategory has the right cancellation property and the right class B the left cancellation property. Each class is closed under composition and retracts. The class A is closed under cobase changes when they exist. and the class B under base changes when they exist.

24.7. The intersection $A \cap B$ of the classes of a factorisation system (A, B) on a quategory X is the class of isomorphisms in X. Let us denote by A' the 1-full simplicial subset of X spanned by A. The simplicial set A' is a quategory by **??**, since we have $A = h^{-1}h(A)$ and $h(A)$ is a subcategory of hoX. We shall say that it is the *sub-quategory spanned* by A. If B' is the sub-quategory spanned by B, then we have $A' \cap B' = J(X)$, where $J(X)$ is the largest sub Kan complex of X.

24.8. Let (A, B) be a factorisation system in a quategory X. Then the full sub-quategory of X^I spanned by the elements in B is reflective; it is thus closed under limits. Dually, the full sub-quategory of X^I spanned by the elements in A is coreflective; it is thus closed under colimits.

24.9. Let (A, B) be a factorisation system in a quategory X. If $p : E \to X$ is a left or a right fibration, then the pair $(p^{-1}(A), p^{-1}(B))$ is a factorisation system in E; we shall say that the system $(p^{-1}(A), p^{-1}(B))$ is obtained by *lifting* the system (A, B) to E along p. In particular, every factorisation system on X can lifted to X/b (resp. $b\backslash X$) for any vertex $b \in X$.

24.10. A factorisation system (A, B) on a quategory X induces a factorisation system (A_S, B_S) on the quategory X^S for any simplicial set S. By definition, a natural transformation $\alpha : f \to g : S \to X$ belongs to A_S (resp. B_S) iff the arrow $\alpha(s) : f(s) \to g(s)$ belongs to A (resp. B) for every vertex $s \in S$. We shall say that the system (A_S, B_S) is *induced* by the system (A, B).

24.11. Let $p : \mathcal{E} \to L(\mathcal{E})$ be the homotopy localisation of a model category. If (A, B) is a factorisation system in $L(\mathcal{E})$, then the pair $(p^{-1}(A), p^{-1}(B)$ is a homotopy factorisation system in $\mathcal{E}$, and this defines a bijection between the factorisation systems in $L(\mathcal{E})$ and the homotopy factorisation systems in $\mathcal{E}$.

24.12. If A is the class of essentially surjective maps in the quategory $\mathcal{Q}_1$ and B is the class of fully faithful maps, then the pair (A, B) is a factorisation system. If A is the class of final maps in $\mathcal{Q}_1$ and B is the class of right fibrations then the pair (A, B) is a factorisation system. If B is the class of conservative maps in $\mathcal{Q}_1$ and A is the class of iterated homotopy localisations, then the pair (A, B) is a factorisation system. If A is the class of weak homotopy equivalences in $\mathcal{Q}_1$ and B is the class of Kan fibrations then the pair (A, B) is a factorisation system.

24.13. Let $p : X \to Y$ be a Grothendieck fibration between quategories. If $A \subseteq X$ is the set of arrows inverted by p and $B \subseteq X$ is the set of cartesian arrows, then the pair (A, B) is a factorisation system on X.

24.14. If X is a quategory with pullbacks then the target functor $t : X^I \to X$ is a Grothendieck fibration. It thus admits a factorisation system (A, B) in which B is the class of pullback squares. An arrow $u : a \to b$ in X^I belongs to A iff the arrow u_1 in the square

$$\begin{array}{ccc} a_0 & \xrightarrow{u_0} & b_0 \\ \downarrow & & \downarrow \\ a_1 & \xrightarrow{u_1} & b_1 \end{array}$$

is invertible.

24.15. We say that a factorisation system (A, B) in a quategory with finite products X is *stable under finite products* if the class A is closed under products in the category X^I. When X has pullbacks, we say that a factorisation system (A, B) is *stable under base changes* if the class A is closed under base changes. This means that the implication $f \in A \Rightarrow f' \in A$ is true for any pullback square

$$\begin{array}{ccc} x' & \longrightarrow & x \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ y' & \longrightarrow & y. \end{array}$$

24.16. Every factorisation system in the quategory $\mathcal{K}$ is stable under finite products.

24.17. We shall say that an arrow $u : a \to b$ in a quategory X is a *monomorphism* or that it is *monic* if the commutative square

$$\begin{array}{ccc} a & \xrightarrow{1_a} & a \\ {\scriptstyle 1_a}\downarrow & & \downarrow{\scriptstyle u} \\ a & \xrightarrow{u} & b \end{array}$$

is cartesian. Every monomorphism in X is monic in the category hoX but the converse is not necessarly true. A map between Kan complexes $u : A \to B$ is monic in $\mathcal{K}$ iff it is homotopy monic.

24.18. We shall say that an arrow in a cartesian quategory X is *surjective*, or that is a *surjection*, if it is left orthogonal to every monomorphism of X. We shall say that a cartesian quategory X admits *surjection-mono factorisations* if every arrow $f \in X$ admits a factorisation $f = up$, with u a monomorphism and p a surjection. In this case X admits a factorisation system (A, B), with A the set of surjections and B the set of monomorphisms. If a quategory X admits surjection-mono factorisations, then so do the quategories $b\backslash X$ and X/b for every vertex $b \in X$, and the quategory X^S for every simplicial set S.

24.19. If a quategory X admits surjection-mono factorisations, then so does the category hoX.

24.20. Recall that a simplicial set A is said to be a *0-object* if the canonical map $A \to \pi_0(A)$ is a weak homotopy equivalence, If X is a quategory, we shall say that an object $a \in X$ is *discrete* or that it is a 0-*object* if the simplicial set $X(x, a)$ is a 0-object for every object $x \in X$. When the product $a \times a$ exists, the object $a \in X$ is a 0-object iff the diagonal $a \to a \times a$ is monic. When the exponential a^{S^1} exists, the object $a \in X$ is a 0-object iff the projection $a^{S^1} \to a$ is invertible. We shall say that an arrow $u : a \to b$ in X is a *0-cover* if it is a 0-object of the slice quategory X/b. An arrow $u : a \to b$ is a 0-cover iff the map $X(x, u) : X(x, a) \to X(x, b)$ is a 0-cover for every node $x \in X$. We shall say that an arrow $u : a \to b$ in X is *0-connected* if it is left orthogonal to every 0-cover in X. We shall say that a quategory X admits *0-factorisations* if every arrow $f \in X$ admits a factorisation $f = pu$ with u a 0-connected arrow and p a 0-cover. In this case X admits a factorisation system (A, B) with A the set of 0-connected maps and B the set of 0-covers. If a quategory

X admits 0-factorisations, then so do the quategories $b\backslash X$ and X/b for every vertex $b \in X$, and the quategory X^S for every simplicial set S.

24.21. There is a notion of n-cover and of n-connected arrow in every quategory for every $n \geq -1$. If X is a quategory, we shall say that a vertex $a \in X$ is a *n-object* if the simplicial set $X(x,a)$ is a n-object for every vertex $x \in X$. If $n = -1$, this means that $X(x,a)$ is contractible or empty. When the exponential $a^{S^{n+1}}$ exists, then a is a n-object iff the projection $a^{S^{n+1}} \to a$ is invertible. We shall say that an arrow $u : a \to b$ is a *n-cover* if it is a n-object of the slice quategory X/b. If $n \geq 0$ and the product $a \times a$ exists, the vertex a is a n-object iff the diagonal $a \to a \times a$ is a $(n-1)$-cover. We shall say that an arrow in a quategory X is *n-connected* if it is left orthogonal to every n-cover. We shall say that a quategory X admits *n-factorisations* if every arrow $f \in X$ admits a factorisation $f = pu$ with u a n-connected map and p a n-cover. In this case X admits a factorisation system (A, B) with A the set of n-connected morphism and B the class of n-covers. If $n = -1$, this is the surjection-mono factorisation system. If X admits k-factorisations for every $-1 \leq k \leq n$, then we have a sequence of inclusions

$$A_{-1} \supseteq A_0 \supseteq A_1 \supseteq A_2 \cdots \supseteq A_n$$
$$B_{-1} \subseteq B_0 \subseteq B_1 \subseteq B_2 \cdots \subseteq B_n,$$

where (A_k, B_k) denotes the k-factorisation system in X.

24.22. The quategory $\mathcal{K}$ admits n-factorisations for every $n \geq -1$ and the system is stable under base change.

24.23. If a quategory X admits n-factorisations, then so do the quategories $b\backslash X$ and X/b for every vertex $b \in X$, and the quategory X^S for every simplicial set S.

24.24. Suppose that X admits k-factorisations for every $0 \leq k \leq n$. If $k > 0$, we shall say that a k-cover $f : x \to y$ in X is an *Eilenberg-MacLane k-gerb* and f is $(k-1)$-connected. A *Postnikov tower* (of height n) for an arrow $f : a \to b$ is a factorisation of length $n+1$ of f

$$a \xleftarrow{p_0} x_0 \xleftarrow{p_1} x_1 \xleftarrow{p_2} \cdots \xleftarrow{p_n} x_n \xleftarrow{q_n} b,$$

where p_0 is a 0-cover, where p_k is an EM k-gerb for every $1 \leq k \leq n$ and where q_n is n-connected. The tower can be augmented by further factoring p_0 as a surjection followed by a monomorphism. Every arrow in X admits a Postnikov tower of height n and the tower is unique up to a homotopy unique isomorphism in the quategory $X^{\Delta[n+1]}$.

24.25. We shall say that a factorisation system (A, B) in a quategory X is *generated* by a set Σ of arrows in X if we have $B = \Sigma^{\perp}$. Let X be a cartesian closed quategory. We shall say that a factorisation system (A, B) in X is *multiplicatively generated* by a set of arrows Σ if it is generated by the set

$$\Sigma' = \bigcup_{a \in X_0} a \times \Sigma.$$

A multiplicatively generated system is stable under products. For example, in the quategory $\mathcal{K}$, the n-factorisations system is multiplicatively generated by the map $S^{n+1} \to 1$. In the quategory $\mathcal{Q}_1$, the system of essentially surjective maps and fully faithful maps is multiplicatively generated by the inclusion $\partial I \subset I$. The system of final maps and right fibrations is multiplicatively generated by the inclusion $\{1\} \subset I$. The dual system of initial maps and left fibrations is multiplicatively generated by the inclusion $\{0\} \subset I$. The system of iterated homotopy localisations and conservative maps is multiplicatively generated by the map $I \to 1$ (or by the inclusion $I \subset J$, where J is the groupoid generated by one isomorphism $0 \to 1$). The system of weak homotopy equivalences and Kan fibrations is multiplicatively generated by the pair of inclusions $\{0\} \subset I$ and $\{1\} \subset I$.

25. n-objects

25.1. Recall that a simplicial set X is said to be a *n-object*, where $n \geq 0$, if we have $\pi_i(X, x) = 1$ for every $i > n$ and $x \in X$. A Kan complex X is a n-object iff every sphere $\partial\Delta[m] \to X$ of dimension $m - 1 > n$ can be filled. We shall say that a map of simplicial sets $u : A \to B$ is a *weak homotopy n-equivalence* if the map $\pi_0(u) : \pi_0(A) \to \pi_0(B)$ is bijective as well as the maps $\pi_i(u, a) : \pi_i(A, a) \to \pi_i(B, u(a))$ for every $1 \leq i \leq n$ and $a \in A$. The model category $(\mathbf{S}, \mathrm{Who})$ admits a Bousfield localisation with respect to the class of weak homotopy n-equivalences. We shall denote the local model structure shortly by $(\mathbf{S}, \mathrm{Who}\langle n\rangle)$, where $\mathrm{Who}\langle n\rangle$ denotes the class of weak homotopy n-equivalences. Its fibrant objects are the Kan n-objects.

25.2. Recal that a simplicial set X is said to be a (-1)-*object* if it is contractible or empty (ie if the map $X \to \exists X$ is a weak homotopy equivalence, where $\exists X \subseteq 1$ denotes the image of the map $X \to 1$). A Kan complex X is a (-1)-object iff every sphere $\partial\Delta[m] \to X$ with $m > 0$ can be filled. We shall say that a map of simplicial sets $u : A \to B$ is a (-1)-*equivalence* if it induces a bijection $\exists A \to \exists B$. The model category $(\mathbf{S}, \mathrm{Who})$ admits a Bousfield localisation with respect to the class of weak homotopy (-1)-equivalences. We shall denote the local model structure shortly by $(\mathbf{S}, \mathrm{Who}[-1])$, where $\mathrm{Who}[-1]$ denotes the class of weak homotopy (-1)-equivalences. Its fibrant objects are the Kan (-1)-objects.

25.3. Recall that a simplicial set X is said to be a (-2)-*object* if it is contractible. Every map of simplicial sets is by definition a (-2)-*equivalence*. The model category $(\mathbf{S}, \mathrm{Who})$ admits a Bousfield localisation with respect to the class of (-2)-equivalences (ie of all maps). The local model can be denoted by by $(\mathbf{S}, \mathrm{Who}[-2])$, where $\mathrm{Who}[-2]$ denotes the class of all maps. Its fibrant objects are the contractible Kan complexes.

25.4. The *homotopy n-type* of a simplicial set A is defined to be a fibrant replacement of $A \to \pi_{\langle n\rangle}(A)$ of A in the model category $(\mathbf{S}, \mathrm{Who}\langle n\rangle)$.

25.5. If $n \geq -2$, we shall denote by $\mathcal{K}\langle n\rangle$ the coherent nerve of the category of Kan n-objects. It is the full simplicial subset of $\mathcal{K}$ spanned by these objects. We have an infinite sequence of quategories,

$$\mathcal{K}\langle -2\rangle \longrightarrow \mathcal{K}\langle -1\rangle \longrightarrow \mathcal{K}\langle 0\rangle \longrightarrow \mathcal{K}\langle 1\rangle \longrightarrow \mathcal{K}\langle 2\rangle \longrightarrow \cdots .$$

The quategory $\mathcal{K}\langle -2\rangle$ is equivalent to the terminal quategory 1. The quategory $\mathcal{K}\langle -1\rangle$ is equivalent to the poset $\{0, 1\}$ and the quategory $\mathcal{K}\langle 0\rangle$ to the category of sets. The quategory $\mathcal{K}\langle 1\rangle$ is equivalent to the coherent nerve of the category of groupoids. Each quategory $\mathcal{K}\langle n\rangle$ is bicomplete and locally cartesian closed. The inclusion $\mathcal{K}\langle n\rangle \to \mathcal{K}$ is reflective and its left adjoint is the map

$$\pi_{\langle n\rangle} : \mathcal{K} \to \mathcal{K}\langle n\rangle$$

which associates to a Kan complex its homotopy n-type. The map $\pi_{\langle n\rangle}$ preserves finite products.

25.6. We shall say that a map $f : X \to Y$ in $\mathbf{S}/B$ is a *fibrewise homotopy n-equivalence* if the map $X(b) \to Y(b)$ induced by f between the homotopy fibers of X and Y is a weak homotopy n-equivalence for every vertex $b \in B$. The model category $(\mathbf{S}/B, \mathrm{Who})$ admits a Bousfield localisation with respect to the fibrewise homotopy n-equivalences. We shall denotes the local model structure shortly by $(\mathbf{S}/B, \mathrm{Who}_B\langle n\rangle)$, where $\mathrm{Who}_B\langle n\rangle$ denotes the class of fibrewise homotopy n-equivalences in $\mathbf{S}/B$. Its fibrant objects are the Kan n-covers $X \to B$.

25.7. If $u : A \to B$ is a map of simplicial sets, then the pair of adjoint functors

$$u_! : \mathbf{S}/A \to \mathbf{S}/B : u^*$$

is a Quillen adjunction between the model category $(\mathbf{S}/A, \mathrm{Who}_A\langle n\rangle)$ and the model category $(\mathbf{S}/B, \mathrm{Who}_B\langle n\rangle)$. Moreover, it is a Quillen equivalence when u is a weak homotopy $(n+1)$-equivalence. This is true in particular when u is the canonical map $A \to \pi_{\langle n+1\rangle}A$.

26. Truncated quategories

26.1. We shall say that a quategory X is *1-truncated* if the canonical map $X \to \tau_1 X$ is a weak categorical equivalence. A quategory X is 1-truncated iff the following equivalent conditions are satisfied:

- the simplicial set $X(a, b)$ is a 0-object for every pair $a, b \in X_0$.
- every simplicial sphere $\partial\Delta[m] \to X$ with $m > 2$ can be filled.

A Kan complex is 1-truncated iff it is a 1-object.

26.2. A category C is equivalent to a poset iff the set $C(a, b)$ has at most one element for every pair of objects $a, b \in C$. We say that a quategory X is *0-truncated* if it is 1-truncated and the category $\tau_1 X$ is equivalent to a poset. A quategory X is 0-truncated iff the following equivalent conditions are satisfied:

- the simplicial set $X(a, b)$ is empty or contractible for every pair $a, b \in X_0$;
- every simplicial sphere $\partial\Delta[m] \to X$ with $m > 1$ can be filled.

A Kan complex is 0-truncated iff it is a 0-object.

26.3. For any $n \geq 2$, we say that a quategory X is *n-truncated* if the simplicial set $X(a, b)$ is a $(n-1)$-object for every pair $a, b \in X_0$. A quategory X is n-truncated iff every simplicial sphere $\partial\Delta[m] \to X$ with $m > n+1$ can be filled. A Kan complex is n-truncated iff it is a n-object.

26.4. The quategory $\mathcal{K}\langle n\rangle$ is $(n+1)$ truncated for every $n \geq -1$.

26.5. We shall say that a map of simplicial sets $u : A \to B$ is a *weak categorical n-equivalence* if the map

$$\tau_0(u, X) : \tau_0(B, X) \to \tau_0(A, X)$$

is bijective for every n-truncated quategory X. The model structure $(\mathbf{S}, \mathrm{Wcat})$ admits a Bousfield localisation with respect to the class $\mathrm{Wcat}\langle n \rangle$ of weak categorical n-equivalences. The fibrant objects are the n-truncated quategories. The localised model structure is cartesian closed and left proper. We shall denote it shortly by $(\mathbf{S}, \mathrm{Wcat}\langle n \rangle)$.

26.6. If $n \geq 0$, then a map between quategories $f : X \to Y$ is a categorical n-equivalence iff it is essentially surjective and the map $X(a, b) \to Y(fa, fb)$ induced by f is a homotopy $(n-1)$-equivalence for every pair of objects $a, b \in X$. A map of simplicial sets $uj : A \to B$ is a weak categorical 1-equivalence iff the functor $\tau_1(u) : \tau_1 \to \tau_1 B$ is an equivalence of categories. A map of simplicial sets $u : A \to B$ is a weak categorical 0-equivalence iff it induces an isomorphism between the poset reflections of A and B.

26.7. The *categorical n-truncation* of a simplicial set A is defined to be a fibrant replacement of $A \to \tau_{\langle n \rangle}(A)$ of A in the model category $(\mathbf{S}, \mathrm{Wcat}\langle n \rangle)$. The fundamental category $\tau_1 A$ is a categorical 1-truncation of A. The poset reflection of A is a categorical 0-truncation of A.

26.8. If $n \geq 0$, we shall denote by $\mathcal{Q}_1\langle n \rangle$ the coherent nerve of the (simplicial) category of n-truncated quategories. It is the full simplicial subset of $\mathcal{Q}_1$ spanned by the n-truncated quategories. We have an infinite sequence of quategories,

$$\mathcal{Q}_1\langle 0 \rangle \longrightarrow \mathcal{Q}_1\langle 1 \rangle \longrightarrow \mathcal{Q}_1\langle 2 \rangle \longrightarrow \mathcal{Q}_1\langle 3 \rangle \longrightarrow \cdots$$

The quategory $\mathcal{Q}_1\langle 0 \rangle$ is equivalent to the category of posets and the quategory $\mathcal{Q}_1\langle 1 \rangle$ to the coherent nerve of **Cat**. We have $\mathcal{Q}\langle n \rangle = \mathcal{Q} \cap \mathcal{Q}_1\langle n \rangle$ for every $n \geq 0$. The inclusion $\mathcal{Q}_1\langle n \rangle \to \mathcal{Q}_1$ is reflective and its left adjoint is the map

$$\tau_{\langle n \rangle} : \mathcal{Q}_1 \to \mathcal{Q}_1\langle n \rangle$$

which associates to a quategory its categorical n-truncation. The map $\tau_{\langle n \rangle}$ preserves finite products. The quategory $\mathcal{Q}_1\langle n \rangle$ is is cartesian closed and $(n+1)$-truncated.

26.9. We right fibration $X \to B$ is said to be *n-truncated* if its fibers are n-objects. The model category $(\mathbf{S}/B, \mathit{Wcont})$ admits a Bousfield localisation in which the fibrant objects are the right n-fibrations $X \to B$. The weak equivalences of the localised structure are called *contravariant n-equivalences*. The localised model structure is simplicial. We shall denotes it by $(\mathbf{S}/B, \mathit{Wcont}\langle n \rangle)$,

26.10. A map $u : M \to N$ in $\mathbf{S}/B$ is a contravariant n-equivalence if the map

$$\pi_0[u, X] : \pi_0[M, X] \to \pi_0[N, X]$$

is bijective for every right n-fibration $X \to B$.

26.11. For each vertex $b \in B$, let us choose a factorisation $1 \to Lb \to B$ of the map $b : 1 \to B$ as a left anodyne map $1 \to Lb$ followed by a left fibration $Lb \to B$. Then a map $u : M \to N$ in $\mathbf{S}/B$ is a contravariant n-equivalence iff the map $Lb \times_B u : Lb \times_B M \to Lb \times_B N$ is a homotopy n-equivalence for every vertex $b \in B$. When B is a quategory, we can take $Lb = b\backslash B$. In this case a map $u : M \to N$ in $\mathbf{S}/B$ is a contravariant n-equivalence iff the map $b\backslash u = b\backslash M \to b\backslash N$ is a homotopy n-equivalence for every object $b \in B$.

26.12. If $u : A \to B$ is a map of simplicial sets, then the pair of adjoint functors

$$u_! : \mathbf{S}/A \to \mathbf{S}/B : u^*$$

is a Quillen adjunction between the model category $(\mathbf{S}/A, Wcont\langle n\rangle)$ and the model category $(\mathbf{S}/B, Wcont\langle n\rangle)$. Moreover, it is a Quillen equivalence when u is a categorical $(n+1)$-equivalence. This is true in particular when u is the canonical map $A \to \tau_{\langle n+1\rangle} A$.

26.13. Dually, we say that a map $u : M \to N$ in $\mathbf{S}/B$ is a *covariant n-equivalence* if the map $u^o : M^o \to N^o$ is a contravariant n-equivalence in $\mathbf{S}/B^o$. The model category $(\mathbf{S}/B, Wcov)$ admits a Bousfield localisation with respect to the class of covariant n-equivalences for any $n \geq 0$. A fibrant object of this model category is a left n-fibration $X \to B$. The localised model structure is simplicial. We shall denote it by $(\mathbf{S}/B, Wcov\langle n\rangle)$,

27. Accessible quategories and directed colimits

27.1. Recall that a quategory is said to be *cartesian* if it has finite limits. Recall that a map between cartesian quategories is said to be *left exact* iff it preserves finite limits. More generally, let α be a regular cardinal (recall that 0 and 1 are the finite regular cardinals). We shall say that a quategory X is *α-cartesian* if it has α-limits. We shall say that a map between α-cartesian quategories is *α-continuous* if it preserves α-limits.

27.2. We shall say that a (small) simplicial set A is *directed* if the colimit map

$$\varinjlim_A : \mathcal{K}^A \to \mathcal{K}$$

is left exact. We shall say that A is *filtered* if the opposite simplicial set A^o is directed. More generally, if α is a regular cardinal, we shall say that a simplicial set A is *α-directed* if the colimit map

$$\varinjlim_A : \mathcal{K}^A \to \mathcal{K}$$

is α-continuous. We shall say that A is *α-filtered* if A^o is α-directed.

27.3. Every quategory is 0-cartesian and every map is 0-continuous. A quategory is 1-cartesian iff it has a terminal object and a map is 1-continuous iff it preserves terminal objects.

27.4. The notion of directed simplicial set is invariant under Morita equivalence: this means that if two simplicial sets A and B are Morita equivalent, then A is directed iff B is directed. A category is directed iff its nerve is directed. A quategory with a terminal object is directed. A quategory with finite colimits is directed. A monoid generated by one idempotent is directed. If a quategory A is directed then the canonical map $d\backslash A \to A$ is final for any finite diagram $d : K \to A$. More generally, let α be a regular cardinal. The notion of α-directed simplicial set is invariant under Morita equivalence. A quategory with a terminal object is α-directed. A quategory with α-colimits is directed. A monoid generated by one idempotent is α-directed. Every simplicial set is 0-directed. A simplicial set A is 1-directed iff it is weakly contratible (ie iff the map $A \to 1$ is a weak homotopy equivalence). If a quategory A is α-directed then the canonical map $d\backslash A \to A$ is final for any diagram $d : K \to A$ of cardinality $< \alpha$.

27.5. We say that a diagram $d : K \to A$ in a quategory A is *bounded above* if it admits an extension $K \star 1 \to A$. Dually, we say that d is *bounded below* if it admits an extension $1 \star K \to A$.

27.6. A quategory A is directed iff every finite diagram $K \to A$ is bounded above. More generally, if α is a regular cardinal $\geq \omega$, then a quategory A is α-directed iff every diagram $K \to A$ of cardinality $< \alpha$ is bounded above.

27.7. A quategory A is directed iff every simplicial sphere $\partial\Delta[n] \to A$ is bounded above.

27.8. Recall that the *barycentric subdivision* $Sd[n]$ of $\Delta[n]$ is defined to be the nerve of the poset of non-empty subsets of $[n]$ ordered by the inclusion. A map $f : [m] \to [n]$ induces a map $Sd(f) : Sd[m] \to Sd[n]$ by putting $Sd(f)(S) = f(S)$ for every $S \in Sd[m]$. This defines a functor $Sd : \Delta \to \mathbf{S}$. Recall that the *barycentric expansion* of a simplicial set A is the simplicial set $Ex(A)$ defined by putting

$$Ex(A)_n = \mathbf{S}(Sd[n], A)$$

for every $n \geq 0$. A quategory A is directed iff the simplicial set $Ex(A)$ is a contractible Kan complex.

27.9. The notion of α-directed simplicial set is invariant under Morita equivalence: this means that if two simplicial sets A and B are Morita equivalent, then A is α-directed iff B is α-directed. A category is α-directed iff its nerve is α-directed. A monoid generated by one idempotent is α-directed. A quasi-category with a terminal object is α-directed. A quategory with α-colimits is α-directed.

27.10. A quategory A is α-directed iff there exists an α-directed category C together with a final map $C \to A$; moreover C can be chosen to be a poset.

27.11. A simplicial set A is α-directed iff the canonical map $u : A \to \mathcal{P}_\alpha(A)$ is final, where $\mathcal{P}_\alpha(A)$ is the free cocompletion of A under α-colimits in 23.25.

27.12. We shall say that a diagram $d : A \to X$ in a quategory X is *directed* if the indexing simplicial set A is directed, in which case we shall say that the colimit of d is *directed* if it exists. We shall say that a quategory X *has* directed colimits if every (small) directed diagram $A \to X$ has a colimit. We shall say that a map between two quategories is *finitary* if it preserves directed colimits. More generally, if α is a regular cardinal, we shall say that a diagram $d : A \to X$ is α*-directed* if A is α-directed, in which case we shall say that the colimit of d is α*-directed* if it exists. We shall say that a quategory X *has* α-directed colimits if every (small) α-directed diagram $A \to X$ has a colimit. We shall say that a map between two quategories is α*-finitary* if it preserves α-directed colimits. A map is ω-finitary iff it is finitary.

27.13. A quategory with directed colimits is Karoubi complete. A quategory with directed colimits and finite colimits is cocomplete. A finitary map between cocomplete quategories is cocontinuous iff it preserves finite colimits. More generally, let α be a regular cardinal. A quategory with α-directed colimits is Karoubi complete. A quategory with α-directed colimits and α-colimits is cocomplete. A map between cocomplete quategories is cocontinuous iff it preserves α-directed colimits and α-colimits.

27.14. A quategory has 0-directed colimits iff it is cocomplete. A diagram $d : K \to X$ is 1-directed iff K is weakly contractible, in which case we shall say that d is *weakly contractible*. A quategory X has 1-directed colimits iff every weakly contractible diagram $d : K \to X$ has a colimit. A Kan complex has α-directed colimits for every regular cardinal $\alpha \geq 1$.

27.15. If A is a simplicial set, we shall say that a prestack $g \in \mathcal{P}(A)$ is *inductive* if the simplicial set A/g (or $El(g)$) is directed. We shall denote by $\mathrm{Ind}(A)$ the full sub-quategory of $\mathcal{P}(A)$ spanned by the inductive objects and by $y : A \to \mathrm{Ind}(A)$ the map induced by the Yoneda map $A \to \mathcal{P}(A)$. The quategory $\mathrm{Ind}(A)$ is closed under directed colimits and the map $y : A \to \mathrm{Ind}(A)$ exibits the quategory $\mathrm{Ind}(A)$ as the free cocompletion of A under directed colimits. More precisely, let us denote by $\mathrm{Fin}(X, Y)$ the quategory of finitary maps $X \to Y$ between two quategories. Then the map

$$y^* : \mathrm{Dir}(\mathrm{Ind}(A), X) \to X^A$$

induced by y is an equivalence of quategories for any quategory with directed colimits X. The inverse equivalence associates to a map $g : A \to X$ its left Kan extension $g_! : \mathrm{Ind}(A) \to X$ along y. More generally, if α is a regular cardinal, we shall say that a prestack $g \in \mathcal{P}(A)$ is α*-inductive* if the simplicial set A/g (or $El(g)$) is α-directed. We shall denote by $\mathrm{Ind}_\alpha(A)$ the full sub-quategory of $\mathcal{P}(A)$ spanned by the α-inductive objects and by $y : A \to \mathrm{Ind}_\alpha(A)$ the map induced by the Yoneda map $A \to \mathcal{P}(A)$. The quategory $\mathrm{Ind}_\alpha(A)$ is closed under α-directed colimits and the map $y : A \to \mathrm{Ind}_\alpha(A)$ exibits the quategory $\mathrm{Ind}_\alpha(A)$ as the free cocompletion of A under α-directed colimits.

27.16. By definition, we have decreasing sequence of inclusions,

$$\mathcal{P}(A) = \mathrm{Ind}_0(A) \supseteq \mathrm{Ind}_1(A) \supseteq \mathrm{Ind}_\omega(A) \supseteq \mathrm{Ind}_{\omega_1}(A) \supseteq \cdots$$

where $\mathrm{Ind}_\omega(A) = \mathrm{Ind}(A)$. A quategory A has α-directed colimits iff the canonical map $y : A \to \mathrm{Ind}_\alpha(A)$ has a left adjoint.

27.17. We shall say that a quategory is *accessible* if it is equivalent to a quategory $\mathrm{Ind}_\alpha(A)$ for for some regular cardinal α and and some small quategory A. More precisely, we shall say that a quategory is an *α-accessible* if it is equivalent to a quategory $\mathrm{Ind}_\alpha(A)$. We shall say that a quategory is a *finitary accessible* if it is ω-accessible, that is, if if it is equivalent to a quategory $\mathrm{Ind}(A)$ for a small quategory A.

27.18. We shall say that a map of between accessible quategories $f : X \to Y$ is *α-accessible* if X and Y are α-accessible and f is α-finitary. We shall say that f is *accessible* if it is α-accessible for some regular cardinal α. We shall say that f is *finitary accessible* if it is ω-accessible.

27.19. If $\alpha < \beta$ are two regular cardinals, we shall write $\alpha \triangleleft \beta$ to indicate that every α-accessible quategory is β-accessible. For any set S of regular cardinals, there is a regular cardinal β such that $\alpha \triangleleft \beta$ for all $\alpha \in S$. See [MP].

27.20. A quategory is 0-accessible iff it is equivalent to a prestack quategory $\mathcal{P}(A)$. Let $\alpha > 0$ be a regular cardinal. If a quategory X is α-accessible (resp. accessible) then so are the slice quategories $a\backslash X$ and X/a for any object $a \in X$, and the quategory X^A for any simplicial set A.

27.21. If A is a quategory and K is a simplicial set of cardinality $< \alpha$, then the canonical map $\mathrm{Ind}_\alpha(A^K) \to \mathrm{Ind}_\alpha(A)^K$ is an equivalence. If $(A_i | i \in S)$ is a family of quategories and $\mathrm{Card}(S) < \alpha$, then the canonical map

$$\mathrm{Ind}_\alpha(\prod_{i \in S} A_i) \to \prod_{i \in S} \mathrm{Ind}_\alpha(A_i)$$

is an equivalence.

27.22. If If A is a small quategory with finite colimits, then the quategory $\mathrm{Ind}(A)$ is cocomplete and the map $y : A \to \mathrm{Ind}(A)$ preserves finite colimits; a prestack $f : A^o \to \mathcal{K}$ is inductive iff it preserves finite limits. Moreover, the map $y : A \to \mathrm{Ind}(A)$ exibits the quategory $\mathrm{Ind}(A)$ as the free cocompletion of A. More precisely, let us denote the quategory of maps preserving finite colimits between two quategories by $\mathrm{fCC}(X, Y)$. Then the map

$$y^* : \mathrm{CC}(\mathrm{Ind}(A), X) \to \mathrm{fCC}(A, X)$$

induced by y is an equivalence of quategories for any cocomplete quategory X. The inverse equivalence associates to a map which preserves finite colimits $f : A \to X$ its left Kan extension $f_! : \mathrm{Ind}(A) \to X$ along y. More generally, if A is a small quategory with α-colimits, then the quategory $\mathrm{Ind}_\alpha(A)$ is cocomplete and the map $y : A \to \mathrm{Ind}_\alpha(A)$ preserves α-colimits; a prestack $f : A^o \to \mathcal{K}$ is α-inductive iff it preserves α-limits. Moreover, the map $y : A \to \mathrm{Ind}_\alpha(A)$ exibits the quategory $\mathrm{Ind}_\alpha(A)$ as the free cocompletion of A.

27.23. The left Kan extension of the inclusion $i : \mathcal{P}_\alpha(A) \subseteq \mathcal{P}(A)$ is an equivalence of quategories,

$$\mathrm{Ind}_\alpha(\mathcal{P}_\alpha(A)) \to \mathcal{P}(A).$$

27.24. Let X be a (locally small) quategory with directed colimits. We shall say that an object $a \in X$ is *compact* if the map

$$hom_X(a,-) : X \to \mathcal{K}$$

is finitary. More generally, let α be a regular cardinal and X be a quategory with α-directed colimits. We shall say that an object $a \in X$ is *α-compact* if the map $hom_X(a,-)$ is α-finitary.

27.25. An object of a cocomplete quategory is 0-compact iff it is atomic. An inital object is 1-compact.

27.26. The class of compact objects is closed under finite colimits and retracts. An object $x \in \mathcal{K}$ is compact iff it is a retract of a finite homotopy type. Not every compact object of $\mathcal{K}$ has finite hmotopy type. If A is a simplicial set, then a prestack $g \in \mathcal{P}(A)$ is compact iff it is a retract of a finitely presented prestack. More generally, let α be a regular cardinal. Then the class of α-compact objects is closed under α-colimits and retracts. If A is a simplicial set, then a prestack $g \in \mathcal{P}(A)$ is α-compact iff it is a retract of a prestack in $\mathcal{P}_\alpha(A)$. If β is a regular cardinal $\geq \alpha$, then an object $g \in \mathrm{Ind}_\alpha(A)$ is β-compact iff it is β-compact in $\mathcal{P}(A)$. Hence the sub-quategory of β-compact objects of an α-accessible quategory is essentially small.

27.27. Let X be a quategory with directed colimits. Then X is finitary accessible iff its subcategory of compact objects is essentially small and every object in X is a directed colimit of a diagram of compact objects. More precisely, if $K \subseteq X$ is a small full sub-quategory of compact objects, then the left Kan extension

$$i_! : \mathrm{Ind}(K) \to X$$

of the inclusion $i : K \subseteq X$ is fully faithful. Moreover, $i_!$ is an equivalence if every object of X is a directed colimit of a diagram of objects of K. More generally, let α be a regular cardinal and X be a quategory with α-directed colimits. Then X is α-accessible iff its subcategory of α-compact objects is essentially small and every object in X is an α-directed colimit of a diagram of α-compact objects. More precisely, if $K \subseteq X$ is a small full sub-quategory of α-compact objects, then the left Kan extension

$$i_! : \mathrm{Ind}_\alpha(K) \to X$$

of the inclusion $i : K \subseteq X$ is fully faithful. Moreover, $i_!$ is an equivalence if every object of X is an α-directed colimit of a diagram of objects of K.

27.28. A cocomplete (locally small) quategory X is finitary accessible iff it is generated by a set of compact objects. More precisely, let $K \subseteq X$ be a small full sub-quategory of compact objects. If K is closed under finite colimits, then the quategory $\mathrm{Ind}(K)$ is cocomplete and the left Kan extension

$$i_! : \mathrm{Ind}(K) \to X$$

of the inclusion $i : K \to X$ along $y : K \to\to \mathrm{Ind}(K)$ is fully faithful and cocontinuous. Moreover, $i_!$ is an equivalence if K generates or separates X. More generally, if α is a regular cardinal, then a cocomplete quategory X is α-accessible iff it is generated by a small set of α-compact objects. More precisely, let $K \subseteq X$ be a small full sub-quategory of α-compact objects. If K is closed under α-colimits, then the quategory $\mathrm{Ind}_\alpha(K)$ is cocomplete and the left Kan extension

$$i_! : \mathrm{Ind}_\alpha(K) \to X$$

of the inclusion $i : K \to X$ along $y : K \to\to \mathrm{Ind}_\alpha(K)$ is fully faithful and cocontinuous. Moreover, $i_!$ is an equivalence if K generates or separates X.

27.29. Let $f : X \leftrightarrow Y; g$ be a pair of adjoint maps between cocomplete quategories. If the map g is finitary, then the map f preserves compact objects, and the converse is true if X is finitary accessible. More generally, if the map g is α-finitary, then the map f preserves α-compact objects, and the converse is true if X is α-accessible.

27.30. The category **ACC** of accessible quategories and accessible maps is closed under (homotopy) limits and the forgetful functor $\mathbf{ACC} \to \mathbf{QCAT}$ is continuous. See **??**.

28. Limit sketches and arenas

In this section we extend the theory of limit sketches to quategories. The quategory of models of a limit sketch is called an *arena*. A quategory is an arena iff it is generated by a set of compact objects.

28.1. Recall that a *projective cone* in a simplicial set A is a map of simplicial sets $c : 1 \star K \to A$. A *limit sketch* is a pair (A, P), where A is a simplicial set and P is a set of projective cones in A. If X is a quategory, we shall say that a map $f : A \to X$ is a *model* of limit sketch (A, P) if it takes every cone $c : 1 \star K \to A$ in P to an exact cone $fc : 1 \star K \to X$. We shall write $f : A/P \to X$ to indicate that a map $f : A \to X$ is a model of (A, P). We shall denote by $\mathrm{Model}(A/P, X)$ the full simplicial subset of X^A spanned by the models $A/P \to X$ and we shall put

$$\mathrm{Model}(A/P) = \mathrm{Model}(A/P, \mathcal{K}).$$

We shall say that a structure is *essentially algebraic* if it is a model of a limit sketch.

28.2. The *cardinality* of a cone $c : 1 \star K \to A$ is defined to be the cardinality of K (ie the cardinality of the set of non-degenerate simplices of K). We shall say that a limit sketch (A, P) is *finitary* if every cone in P is finite. More generally, if α is a regular cardinal, we shall say that (A, P) is *α-bounded* if every cone in P has cardinality $< \alpha$. A *stack* on a fixed topological space X is a model of a certain limit sketch associated to the space. The sketch is not finitary in general.

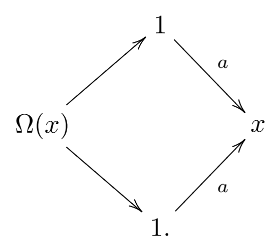
28.3. Remark. If $\alpha = 0$, a limit sketch (A, P) is α-bounded iff $P = \emptyset$. Hence we have $\mathrm{Model}(A/P) = \mathcal{K}^A$ in this case. If $\alpha = 1$, then a cone $c : 1 \star K \to A$ of cardinality $< \alpha$ is just a vertex $c(1) \in A$, since $K = \emptyset$ in this case. Hence a 1-bounded limit sketch is the same thing as a pair (A, S) where S is a set of nodes in A. A map $f : A \to \mathcal{K}$ is a model of (A, S) iff we have $f(s) \simeq 1$ for every $s \in S$. For example, if $A = I$ and $S = \{0\}$, then a model $f : A/S \to \mathcal{K}$ is a morphism $f : 1 \to x$ in the quategory $\mathcal{K}$. Hence the quategory $\mathrm{Model}(I/\{0\})$ is equivalent to the quategory $1\backslash\mathcal{K}$ of pointed Kan complexes.

28.4. If (A, P) is a limit sketch then the inclusion $\mathrm{Model}(A/P) \subseteq \mathcal{K}^A$ has a left adjoint and the quategory $\mathrm{Model}(A/P)$ is bicomplete. We shall say that a quategory is *locally presentable* or that it is an *arena* it is equivalent to a quategory $\mathrm{Model}(A/P)$ of a limit sketch (A, P). More generally, we shall say that an arena is *α-presentable* if it is equivalent to a quategory $\mathrm{Model}_\alpha(A/P)$ for an α-bounded limit sketch (A, P). We shall say that an arena is *finitary presentable* if it is ω-presentable.

28.5. Remark. If an arena is α-presentable then it is β-presentable for any regular cardinal $\beta \geq \alpha$. An arena is 0-presentable iff it is equivalent to a quategory $\mathcal{P}(A)$ for a simplicial set A. The quategory $1\backslash\mathcal{K}$ of pointed Kan complexes is 1-presentable. The quategory $\mathcal{K}^o$ is not an arena.

28.6. If X is an arena then so are the quategories $a\backslash X$ and X/a for any object $a \in X$ and the quategory $\mathrm{Model}(A/P, X)$ for any limit sketch (A, P). More precisely, let α be a regular cardinal ≥ 1. If an arena X is α-presentable, then so are the arenas $a\backslash X$ and X/a for any object $a \in X$ and the arena $\mathrm{Model}(A/P, X)$ for any α-bounded limit sketch (A, P). If an arena X is 0-presentable, then so is the arena X/a for any object $a \in X$ and the arena X^A for any simplicial set A.

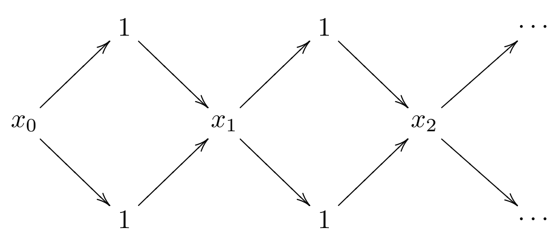
28.7. (Example) The loop space of a pointed object $a : 1 \to x$ in a cartesian quategory X is defined by a cartesian square



The object $\Omega(x)$ is naturally pointed A *pre-spectrum* in X is defined to be an infinite sequence of pointed objects $1 \to x_n$ together with an infinite sequence of pointed morphisms

$$u_n : x_n \to \Omega(x_{n+1}).$$

It can thus be defined by an infinite sequence of commutative squares



in the quategory $1\backslash X$. It follows that the notion of pre-spectrum is defined by a 1-bounded limit sketch. Hence the quategory of pre-spectra is 1-presentable. We shall say that a pre-spectrum (u_n) in X is a *spectrum* or a *stable object* if the morphism u_n is invertible for every $n \geq 0$. Equivalently, a pre-spectrum (u_n) is a spectrum iff every square of the sequence above is cartesian. It follows that the notion of spectrum is defined by a finitary limit sketch (A, P) and that the *quategory of spectra* $\mathcal{S}p = \mathrm{Mod}(A, P)$ is finitary presentable.

28.8. (Example) A morphism $a \to b$ in a cartesian quategory X is monic iff the square

$$\begin{array}{ccc} a & \xrightarrow{1_a} & a \\ {\scriptstyle 1_a}\downarrow & & \downarrow{\scriptstyle u} \\ a & \xrightarrow{u} & b \end{array}$$

is cartesian. Hence the notion of monomorphism can be described by a finitary limit sketch. An object $a \in X$ is discrete iff the diagonal $a \to a \times a$ is monic. This condition is expressed by two exact cones,

$$\begin{array}{ccccc} & & b & & \\ & {\scriptstyle p_1}\swarrow & & \searrow{\scriptstyle p_2} & \\ a & & & & a, \end{array} \qquad \begin{array}{ccc} a & \xrightarrow{1_a} & a \\ {\scriptstyle 1_a}\downarrow & & \downarrow{\scriptstyle d} \\ a & \xrightarrow{d} & b. \end{array}$$

and two relations $pd = qd = 1_a$. Hence the notion of discrete object can be described by a finitary limit sketch. Hence the quategory $\mathcal{K}\langle 0\rangle$ of discrete objects in $\mathcal{K}$ is finitary presentable. It folllows that the category of sets **Set** is finitary presentable. An arrow $a \to b$ in X is a 0-cover iff the diagonal $a \to a \times_b a$ is monic. Hence the notion of 0-cover can be described by a finitary limit sketch. An object $a \in X$ is a 1-object iff the diagonal $a \to a \times a$ is a 0-cover. Hence the notion of 1-object can be described by a finitary limit sketch. It is easy to see by induction on $n \geq 0$ that the notions of n-object and of n-cover can be described by a finitary limit sketch. Hence the quategory $\mathcal{K}\langle n\rangle$ of n-objects in $\mathcal{K}$ is finitary presentable for every $n \geq 0$.

28.9. (Example) The notion of category object in a cartesian quategory X is defined by a finitary limit sketch. More precisely, a simplicial object $C : \Delta^o \to X$ is said to be a *category object* if the map C takes every square of the form

$$\begin{array}{ccc} [0] & \xrightarrow{0} & [n] \\ {\scriptstyle m}\downarrow & & \downarrow \\ [m] & \longrightarrow & [m+n], \end{array}$$

to a pullback square in X. In other words, a simplicial object C is a category object if it satisfies the *Segal condition*. If $C : \Delta^o \to X$ is a category object, we shall say that $C_0 \in X$ is the *object of objects* of C and that C_1 is the *object of arrows*. The morphism $\partial_1 : C_1 \to C_0$ is the *source morphism* the morphism $\partial_0 : C_1 \to C_0$ is the *target morphism*, and the morphism $\sigma_0 : C_0 \to C_1$ is the *unit morphism*. The morphism $\partial_1 : C_2 \to C_1$ is the *multiplication*.

28.10. (Example) The notion of groupoid object in a cartesian quategory X is defined by a finitary limit sketch. We shall say that a category object $C : \Delta^o \to X$ is a *groupoid* if C takes the squares

$$\begin{array}{ccc} [0] & \xrightarrow{d_0} & [1] \\ {\scriptstyle d_0}\downarrow & & \downarrow{\scriptstyle d_0} \\ [1] & \xrightarrow{d_1} & [2], \end{array} \qquad \begin{array}{ccc} [0] & \xrightarrow{d_1} & [1] \\ {\scriptstyle d_1}\downarrow & & \downarrow{\scriptstyle d_2} \\ [1] & \xrightarrow{d_1} & [2] \end{array}$$

to pullback squares,

$$\begin{array}{ccc} C_2 & \xrightarrow{m} & C_1 \\ {\scriptstyle \partial_0}\downarrow & & \downarrow{\scriptstyle t} \\ C_1 & \xrightarrow{t} & C_0, \end{array} \qquad \begin{array}{ccc} C_2 & \xrightarrow{m} & C_1 \\ {\scriptstyle \partial_2}\downarrow & & \downarrow{\scriptstyle s} \\ C_1 & \xrightarrow{s} & C_0. \end{array}$$

(one is enough).

28.11. (Example) A *monoid* in a cartesian quategory X is is a category object $C : \Delta^o \to X$ such that $C_0 \simeq 1$. A *group* is a groupoid object $C : \Delta^o \to X$ such that $C_0 \simeq 1$.

28.12. Remark. The notions of groupoid and of group can be defined alternatively by using symmetric simplicial objects as in **??**. Let us denote by $\Sigma\Delta$ the category having the same objects as Δ but where every map of sets $[m] \to [n]$ is a morphism. A *symmetric simplicial object* in a quategory X is defined to be a map $(\Sigma\Delta)^o \to X$. A *groupoid object* in X can be defined to be a symmetric simplicial object $G : (\Sigma\Delta)^o \to X$. which takes every pushout square

$$\begin{array}{ccc} A & \longrightarrow & A' \\ {\scriptstyle i}\downarrow & & \downarrow \\ B & \longrightarrow & B', \end{array}$$

in which i is monic to a pullback square in X. A groupoid object G is a *group* if $G_0 \simeq 1$.

28.13. (Example) The notion of E_∞-space in a quategory X can be defined by using a sketch introduced by Segal in [S2]. Let us denote by Γ the category of finite pointed sets and pointed maps. Every object of Γ is isomorphic to a set $[n]$ pointed by $0 \in [n]$. For each $1 \le k \le n$, let δ_k be the pointed map $[n] \to [1]$ defined by putting

$$\delta_k(x) = \begin{cases} 1 & \text{if } x = k \\ 0 & \text{if } x \neq k. \end{cases}$$

A Γ-*object* in a quategory X is defined to be a map $E : \Gamma \to X$. If X has finite products, then from the morphisms $E(\delta_k) : E_n \to E_1$ we obtain a morphism

$$p_n : E_n \to \prod_{k=1}^{n} E_1.$$

We shall say that E is an E_∞-*space* if p_n is invertible for every $n \ge 0$. The notion of E_∞-space is defined by a finitary limit sketch (Γ, P) and the quategory of E_∞-spaces $\mathcal{E}_\infty = \mathrm{Mod}(\Gamma, P)$ is finitary presentable. Consider the functor $i : \Delta^o \to \Gamma$ obtained by putting $i[n] = Hom(\Delta[n], S^1)$ for every $n \ge 0$, where $S^1 = \Delta[1]/\partial\Delta[1]$ is the pointed circle. If X is a cartesian quategory, then the map $X^i : X^\Gamma \to X^{\Delta^o}$ takes an E_∞-space $E \in \mathrm{Model}(\Gamma/P, X)$ to a monoid $i^*(E) : \Delta^o \to X$ (the monoid underlying E). We shall that E is an *infinite loop space* if the monoid $i^*(E)$ is a group. The notion of infinite loop space is described by a finitary limit sketch (Γ, P') and the quategory of infnite loop spaces $\mathcal{L}_\infty = \mathrm{Mod}(\Gamma, P')$ is finitary presentable.

28.14. (Example) We shall say that a category object $C : \Delta^o \to X$ in a cartesian quategory X is a *preorder* (on C_0) if the vertex map $C_1 \to C_0 \times C_0$ is monic. A *preorder* $C : \Delta^o \to X$ is an *equivalence relation* if the quategory C is a groupoid. Theses notions have the following classical descriptions. A *binary relation* on an object $a \in X_0$ can be defined to be a monomorphism $r \to a \times a$. More generally, if $n \geq 0$, a *n-ary relation* on a can be defined to be a monomorphism $r \to a^n$ is The notion of n-ary relation is essentially algebraic and finitary, as well as the following notions. A binary relation $r \to a \times a$ is *reflexive* if the diagonal $a \to a \times a$ can be factored through the morphism $r \to a \times a$. A binary relation $r \to a \times a$ is *transitive* if the morphism $p_{12}^*(r) \cap p_{23}^*(r) \to a \times a \times a$ can be factored through the morphism $p_{13}^*(r) \to a \times a \times a$. A binary relation $r \to a \times a$ is *symmetric* if it can be factored through its transpose ${}^t r \to a \times a$. A binary relation $r \to a \times a$ is a *preorder* if it is reflexive and transitive. A binary relation $r \to a \times a$ is an *equivalence* if it is reflexive, symmetric and transitive.

28.15. Recall that an *inductive cone* in a simplicial set A is a map of simplicial sets $K \star 1 \to A$. The opposite of an inductive cone $c : K \star 1 \to A$ is a projective cone $c^o : 1 \star K^o \to A^o$. A *colimit sketch* is defined to be a pair (A, Q), where A is a simplicial set and Q is a set of inductive cones in A. The opposite of a colimit sketch (A, Q) is a limit sketch (A^o, Q^o), where $Q^o = \{c^o : c \in Q\}$. Dually, the opposite of a limit sketch (A, P) is a colimit sketch (A^o, P^o). We shall say that a colimit sketch (A, Q) is α-bounded if the opposite sketch is α-bounded. A *comodel* of colimit sketch (A, Q) with values in a cocomplete quategory X is a map $f : A \to X$ which takes every cone $c : K \star 1 \to A$ in Q to a coexact cone $fc : K \star 1 \to X$ in X. We shall write $f : Q\backslash A \to X$ to indicate that the map $f : A \to X$ is a comodel of (A, Q). The comodels of (A, Q) with values in X form a quategory $\mathrm{CoModel}(Q\backslash A, X)$. By definition, it is the full simplicial subset of X^A spanned by the comodels $Q\backslash A \to X$. Every colimit sketch (A, Q) has a universal comodel $u : Q\backslash A \to U$ with values in a cocomplete quategory U. More precisely, let us denote by $\mathrm{CC}(X, Y)$ the quategory of cocontinuous maps between two cocomplete quategories. Then the map

$$u^* : \mathrm{CC}(U, X) \to \mathrm{CoMod}(Q\backslash A, X)$$

induced by u is an equivalence of quategories for any cocomplete quategory X. We shall say that the comodel $u : Q\backslash A \to U$ is a *presentation* of the U by (A, Q). A cocomplete quategory X is an arena iff it admits a presentation $u : Q\backslash A \to X$ by a colimit sketch (A, Q). More precisely, if (A, P) is a limit sketch, then the inclusion $\mathrm{Model}(A/P) \subseteq \mathcal{K}^A$ has a left adjoint $r : \mathcal{K}^A \to \mathrm{Model}(A/P)$. The map

$$ry : A^o \to \mathrm{Model}(A/P)$$

obtained by composing r with the Yoneda map $y : A^o \to \mathcal{K}^A$ is a universal comodel of the colimit sketch (A^o, P^o). More generally, a cocomplete quategory X is α-presentable iff it admits a presentation $u : Q\backslash A \to X$ by an α-bounded colimit sketch (A, Q).

28.16. The α-directed colimits commute with the α-limits in any α-presentable quategory (and this is true in any quategory if $\alpha = 0, 1$).

28.17. A cocomplete quategory X is an arena iff it is accesssible. An arena is α-presentable iff it is generated by α-compact objects. The full sub-quategory of α-compact objects of an arena is essentially small.

28.18. If X is an arena, then every continuous map $X^o \to \mathcal{K}$ is representable. Moreover, every cocontinuous map $f : X \to Y$ with values in a cocomplete locally small quategory has a right adjoint $g : X \to Y$.

28.19. Let $f : X \leftrightarrow Y : g$ be a pair of adjoint maps between cocomplete quategories. If the right adjoint $g : Y \to X$ is α-finitary, then f preserves α-compact objects, and the converse is true if X is generated by α-compact objects.

28.20. If X, Y and Z are arenas, then a map $f : X \times Y \to Z$ cocontinuous in each variable can be divided on both sides. See**??**. More precisely, for every object $x \in X$ the map $f(x, -) : Y \to Z$ has a right adjoint $x\rfloor - : Z \to Y$ is called the *left division* by x. The map $l : X^o \times Z \to Y$ defined by putting $l(x^o, z) = x\rfloor z$ is continuous in each variable. Dually, for every object $y \in Y$ the map $f(-, y) : X \to Z$ has a right adjoint $-\lfloor y : Z \to Y$ called the *right division* by y. Moreover, the map $r : Z \times Y^o \to X$ defined by putting $r(z, y^o) = z\lfloor y$ is continuous in each variable.

28.21. We denote by **AR** the category of arenas and cocontinuous maps. If X and Y are arenas, then so is the quategory $\mathrm{CC}(X, Y)$ of cocontinuous maps $X \to Y$. The (simplicial) category **AR** is symmetric monoidal closed. The *tensor product* $X \otimes Y$ of two arenas is the target of a map $\phi : X \times Y \to X \otimes Y$ cocontinuous in each variable and universal with respect to that property. More precisely, for any cocomplete quategory Z, let us denote by $\mathrm{CC}(X, Y; Z)$ the full simplicial subset of $Z^{X\times Y}$ spanned by the maps $X \times Y \to Z$ cocontinuous in each variable. Then the map

$$\phi^* : \mathrm{CC}(X \otimes Y, X) \to \mathrm{CC}(X, Y; Z)$$

induced by ϕ is an equivalence of quategories. By combining this equivalence with the natural isomorphism

$$\mathrm{CC}(X, Y; Z) = \mathrm{CC}(X, \mathrm{CC}(Y, Z)$$

we obtain an equivalence of quategories

$$\mathrm{CC}(X \otimes Y, Z) \simeq \mathrm{CC}(X, \mathrm{CC}(Y, Z)).$$

The unit object for the tensor product is the quategory $\mathcal{K}$. The equivalence

$$\mathcal{K} \otimes X \simeq X$$

is induced by a product map $(A, x) \mapsto A \cdot x$ described in 23.1 in the case of a finite simplicial set A. The quategory $\mathcal{K}$ is cartesian closed and every arena X is enriched and cocomplete over $\mathcal{K}$. The enrichement *hom* $: X^o \times X \to \mathcal{K}$ can be obtained by dividing the canonical (right) action $X \times \mathcal{K} \to X$. on the right.

28.22. The terminal object 1 of the category **AR**. is also the initial, since the quategory $\mathrm{CC}(1, X)$ is equivalent to the quategory 1 for every $X \in$ **AR**. More generally, the product $X = \prod_{i\in I} X_i$ of a (small) family of arenas is also their coproduct. More precisely, if 0_i denotes the initial object of X_i, then the map $u_i : X_i \to X$ defined by putting

$$u_i(x)_k = \begin{cases} x & \text{if } k = i \\ 0_k & \text{if } k \neq i, \end{cases}$$

is cocontinuous (it is left adjoint to the projection $p_i : X \to X_i$). The family of maps (u_i) turns the object X into the coproduct of the family of objects (X_i) since the map

$$(u_i^*) : \mathrm{CC}(\prod_{i \in I} X_i, Y) \to \prod_{i \in I} \mathrm{CC}(X_i, Y)$$

induced by the family (u_i) is an equivalence of quategories for every $Y \in \mathbf{AR}$.

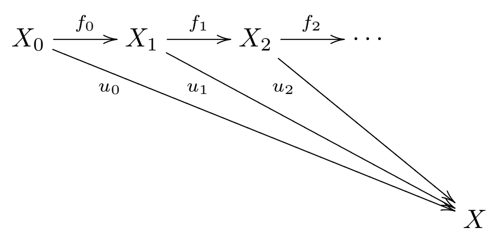
28.23. (A duality). The (simplicial) category $\mathbf{AR}$ is closed under (homotopy) limits and the forgetful functor $\mathbf{AR} \to \mathbf{QCAT}$ is continuous. The right adjoint of a map $X \to Y$ in $\mathbf{AR}$ is continuous and accessible. Conversely, every accessible continuous map $Y \to X$ has a left adjoint $X \to Y$. .Let us denote by $\mathbf{AR}^*$ the (simplicial) category having the same objects as $\mathbf{AR}$ but whose morphisms are the accessible continuous maps. The (simplicial) category $\mathbf{AR}^*$ is closed under (homotopy) limits and the forgetful functor $\mathbf{AR}^* \to \mathbf{QCAT}$ is continuous. The (simplicial) category $\mathbf{AR}^*$ and $\mathbf{AR}$ are mutually opposite. Hence the homotopy colimit of a diagram in $\mathbf{AR}$ can be constructed as the homotopy limit of a dual diagram in $\mathbf{AR}^*$. For example, the homotopy colimit of an infinite sequence of maps in $\mathbf{AR}$,

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots$$

can be constructed as the homotopy limit X in $\mathbf{QCAT}$ of the corresponding sequence of right adjoints

$$X_0 \xleftarrow{g_0} X_1 \xleftarrow{g_1} X_2 \xleftarrow{g_2} \cdots .$$

An object of X is a pair (x, k), where $x = (x_n)$ is a sequence of objects $x_n \in X_n$ and $k = (k_n)$ is a sequence of isomorphisms $k_n : x_n \simeq g_n(x_{n+1})$. Each projection $p_n : X \to X_n$ has a left adjoint $u_n : X_n \to X$ and each isomorphism $k_n : p_n \simeq g_n p_{n+1}$ has a left transpose $i_n : u_{n+1} f_n \simeq u_n$. The dual diagram



is a (homotopy) colimit diagram in $\mathbf{AR}$.

28.24. Let $\alpha : f \to g : X \to Y$ be a natural transformation between two maps in $\mathbf{AR}$. Let us say that a map $q : Y \to Z$ in $\mathbf{AR}$ *inverts* α if the morphism $q \circ \alpha : qf \to qg$ is invertible. There is then a map $p : Y \to Y^{[\alpha]}$ which inverts α universally. More precisely, if Z is an arena, let us denote by $\mathrm{CC}^{[\alpha]}(Y, Z)$ the full simplicial subset of $\mathrm{CC}(Y, Z)$ spanned by the maps $Y \to Z$ which invert α. Then the map

$$p^* : \mathrm{CC}(Y^{[\alpha]}, Z) \to \mathrm{CC}^{[\alpha]}(Y, Z).$$

induced by p is an equivalence of quategories. The map $f : X \to Y$ has a right adjoint $f_* : Y \to X$ in $\mathbf{AR}^*$. Let $\alpha_* : g_* \to f_* : Y \to X$ be the right transpose of the natural transformation α. We shall say that an object $y \in Y$ *coinverts* α_* if the morphism $\alpha_*(y) : g_*(y) \to f_*(y)$ is invertible in X. Let us denote by $Coinv(\alpha_*)$ the full simplicial subset of Y spannned by the objects which coinvert α_*. Then the inclusion $Coinv(\alpha_*) \subseteq Y$ has a left adjoint $p : Y \to Coinv(\alpha_*)$ and we have $Y^{[\alpha]} = Coinv(\alpha_*)$.

28.25. If X is a quategory then the map $hom_X : X^o \times X \to \mathcal{K}$ is continuous in each variable; hence the opposite map $hom^o_X : X \times X^o \to \mathcal{K}^o$ is cocontinuous in each variable. If X is an arena, then the resulting map

$$X^o \to \mathrm{CC}(X, \mathcal{K}^o)$$

is an equivalence of quategories. More generally, if X and Y are arenas, then we have two equivalences of quategories,

$$\begin{aligned} (X \otimes Y)^o &\simeq \mathrm{CC}(X, Y^o) \\ &\simeq \mathrm{CC}(Y, X^o). \end{aligned}$$

28.26. The *tensor product* of two limit sketches (A, P) and (B, Q) is defined to be the limit sketch

$$(A \times B, P \times' Q) = (A \times B, P \times B_0 \sqcup A_0 \times Q),$$

where

$$P \times B_0 = \{c \times b : c \in P, b \in B_0\} \quad \text{and} \quad A_0 \times Q = \{a \times c : a \in A_0, c \in Q\}.$$

If X is a complete quategory, then a map $f : A \times B \to X$ is a model of the sketch $(A \times B, P \times' Q)$ iff the map $f(-, b) : A \to X$ is a model of (A, P) for every vertex $b \in B_0$ and the map $f(a, -) : B \to X$ is a model of (B, Q) for every vertex $a \in A_0$. By definition, we have two equivalences of quategories:

$$\begin{aligned} \mathrm{Model}(A \times B/P \times' Q, X) &\simeq \mathrm{Model}(A/P, \mathrm{Model}(B/Q, X)) \\ &\simeq \mathrm{Model}(B/Q, \mathrm{Model}(A/P, X)). \end{aligned}$$

The *external tensor product* of a model $f \in \mathrm{Model}(A/P)$ with a model $g \in \mathrm{Model}(B/Q)$ is the model $f \otimes g \in \mathrm{Model}(A \times B/P \times' Q)$ is obtained by applying the left adjoint to the inclusion $\mathrm{Model}(A \times B/P \times' Q) \subseteq \mathcal{K}^{A \times B}$ to the map $(a, b) \mapsto f(a) \cdot g(b)$. The map $(f, g) \mapsto f \otimes g$ is cocontinuous in each variable and the induced map

$$\mathrm{Model}(A/P) \otimes \mathrm{Model}(A/Q) \to \mathrm{Model}(A \times B/P \times' Q)$$

is an equivalence of quategories.

28.27. Recall that the *smash product* of two pointed simplicial sets $A = (A, a)$ and $B = (B, b)$ is a pointed simplicial set $A \wedge B$ defined by the pushout square

$$\begin{array}{ccc} (A \times b) \cup (a \times B) & \longrightarrow & A \times B \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & A \wedge B. \end{array}$$

A pointed simplicial set (A, a) can be regarded as a limit sketch (A, P), where P contains only the cone $a : 1 \star \emptyset \to A$. The sketch is 1-bounded and a model of (A, P) is a pointed map $f : A \to 1\backslash\mathcal{K}$ If $B = (B, b)$ is another pointed simplicial set, then the external tensor product of a model $f \in \mathrm{Model}(A, a)$ with a model $g \in \mathrm{Model}(B, b)$ is their smash product $f \wedge g : A \wedge B \to 1\backslash\mathcal{K}$, where $(f \wedge g)(x \wedge y) = f(x) \wedge g(y)$.

28.28. Let (A, P) is a limit sketch and X be an arena. The *external tensor product* of a model $f \in \mathrm{Model}(A/P)$ with an object $x \in X$ is defined the map $f \otimes x : A/P \to X$ obtained by applying the left adjoint to the inclusion $\mathrm{Model}(A/P, X) \subseteq X^A$ to the map $a \mapsto f(a) \cdot x$. The map $(f, x) \mapsto f \otimes x$ is cocontinuous in each variable and the induced map

$$\mathrm{Model}(A/P) \otimes X \simeq \mathrm{Model}(A/P, X)$$

is an equivalence of quategories.

28.29. We shall say that a pair $(\mathcal{A}, \mathcal{B})$ of classes of maps in **AR** is a *homotopy factorisation system* if the following conditions are satisfied:

- the classes $\mathcal{A}$ and $\mathcal{B}$ are invariant under categorical equivalences;
- the pair $(\mathcal{A} \cap \mathcal{C}, \mathcal{B} \cap \mathcal{F})$ is a weak factorisation system in **AR**, where $\mathcal{C}$ is the class of monomorphisms and $\mathcal{F}$ is the class of pseudo-fibrations;
- the class $\mathcal{A}$ has the right cancellation property;
- the class $\mathcal{B}$ has the left cancellation property.

The last two conditions are equivalent in the presence of the others. We shall say that $\mathcal{A}$ is the *left class* of the system and that $\mathcal{B}$ is the *right class*.

28.30. The category **AR** admits a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{B}$ is the class of fully faithful maps. A map $f : X \to Y$ belongs to $\mathcal{A}$ iff its right adjoint $Y \to X$ is conservative iff $f(X_0)$ generates Y.

28.31. The category **AR** admits a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{B}$ is the class of conservative maps and $\mathcal{A}$ is the class of reflections. We shall say that a map in $\mathcal{A}$ is a *Bousfield localisation*. Every Bousfield localisation $l : X \to Y$ is equivalent to a reflection $r : X \to X^{[\Sigma]}$ where Σ is a (small) set of arrows in X.

28.32. Let Σ be a (small) set of arrows in an arena X. Then the pair $({}^{\perp}(\Sigma^{\perp}), \Sigma^{\perp})$ is a factorisation system. We say that an object $a \in X$ is Σ-*local* if it is right orthogonal to every arrow in Σ (see 24.2). Let us denote by $X^{[\Sigma]}$ the full simplicial subset of X spanned by the Σ-local objects. Then the quategory $X^{[\Sigma]}$ is an arena and the inclusion $i : X^{[\Sigma]} \subseteq X$ has a left adjoint $r : X \to X^{[\Sigma]}$. Hence the quategory $X^{[\Sigma]}$ is a Bousfield localisation of X. Conversely, every Bousfield localisation of X is equivalent to to a sub-quategory $X^{[\Sigma]}$ for a set Σ of arrows in X. If Y is an arena, let us denote by $\mathrm{CC}^{[\Sigma]}(X, Y)$ the full simplicial subset of $\mathrm{CC}(X, Y)$ spanned by the maps $X \to Y$ which invert every arrows in Σ. Then the map

$$r^* : \mathrm{CC}(X^{[\Sigma]}, Y) \to \mathrm{CC}^{[\Sigma]}(X, Y)$$

induced by r is an equivalence of quategories. Every arena is equivalent to a quategory $\mathcal{P}(A)^{[\Sigma]}$ for a small category A and a a (small) set Σ of arrows in $\mathcal{P}(A)$.

28.33. We shall say that a map in **AR** is *coterminal* if it preserves terminal objects. Every map $f : X \to Y$ in **AR** admits a factorisation $f = pf' = X \to Y/f(1) \to Y$, where f' is a coterminal map and where p is the projection $Y/f(1) \to Y$. The category **AR** admits a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of coterminal maps. A map belongs to $\mathcal{B}$ iff it is equivalent to a right fibration.

28.34. We recall that if X is a quategory with finite coproducts and a is an object of X then the projection $p_a : a\backslash X \to X$ has a left adjoint $i_a : X \to a\backslash X$ called the *cobase change* along a. We shall say that a map in **AR** is *coinitial* if its right adjoint preserves initial objects. The category **AR** admits a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of cobase changes and $\mathcal{B}$ is the class of coinitial maps. Let us see that every map $f : X \to Y$ in **AR** admits a factorisation $f = f' i_a : X \to a\backslash X \to Y$ where i_a is a cobase change and f' a coinitial map. For this it suffices to construct the dual factorisation of the right adjoints $g = p_a g' = Y \to a\backslash X \to X$. By construction $a = g(0)$ and g' is induced by g.

28.35. Let (A, P) be a limit sketch. For every cone $c : 1 \star K \to A$ in P, let us denote by $i(c)$ the inclusion $K \subset 1 \star K$ regarded as a morphism of the category $\mathbf{S}/A$. Then the model category $(\mathbf{S}/A, \mathrm{Wcov})$ admits a Bousfield localisation with respect to the set of morphisms $i(P) = \{i(c) | c \in P\}$. We shall say that fibrant local object is a *vertical model* of (A, P). A left fibration $p : E \to A$ is a vertical model iff the map

$$[i(c), E] : [1 \star K, E] \to [K, E]$$

is a trivial fibration for every cone $c : 1 \star K \to A$ in P. A map $f : A \to \mathcal{K}$ is a model of A iff the left fibration $el(f) \to A$ is a vertical model of (A, P). The coherent nerve of the simplicial category of vertical models of (A, P) is equivalent to the quategory $\mathrm{Model}(A/P)$.

29. Duality for prestacks and null-pointed prestacks

29.1. If X and Y are two arenas, we shall say that a map $\epsilon : X \otimes Y \to \mathcal{K}$ is a *pairing* between X and Y. We shall say that the pairing is *exact* if the map

$$\epsilon^\sharp : \mathrm{CC}(U, V \otimes X) \to \mathrm{CC}(U \otimes Y, V)$$

defined by putting $\epsilon^\sharp(f) = (V \otimes \epsilon)(f \otimes Y)$ is an equivalence of quategories for any arenas U and V. A pairing ϵ is exact iff it is the counit of an adjunction $X \vdash Y$. The unit is a map $\eta : \mathcal{K} \to Y \otimes X$ together with a pair of isomorphisms,

$$I_X \simeq (\epsilon \otimes X) \circ (X \otimes \eta) \quad \text{and} \quad I_Y \simeq (Y \otimes \epsilon) \circ (\eta \otimes Y).$$

When the pairing $\epsilon : X \otimes Y \to \mathcal{K}$ is exact, the map

$$Y \to \mathrm{CC}(X, \mathcal{K})$$

induced by ϵ is an equivalence of quategories. We shall say that Y is the *dual* of X and put $Y = X^*$. An arena X is *dualisable* iff the canonical pairing

$$X \otimes \mathrm{CC}(X, \mathcal{K}) \to \mathcal{K}$$

is exact.

29.2. An arena is dualisable iff it is a retract of an arena $\mathcal{P}(A)$ for some simplicial set A. The *external cartesian product* of a prestack $f \in \mathcal{P}(A)$ with a prestack $g \in \mathcal{P}(B)$ is the prestack $f \hat{\times} g \in \mathcal{P}(A \times B)$ obtained by putting

$$(f \hat{\times} g)(a, b) = f(a) \times g(b)$$

for every pair of objects $(a, b) \in A \times B$. The map $(f, g) \mapsto f \hat{\times} g$ is cocontinuous in each variable and the induced map

$$\mathcal{P}(A) \otimes \mathcal{P}(B) \to \mathcal{P}(A \times B).$$

is an equivalence of quategories. The cocontinuous extension of the map $hom_A : A^o \times A \to \mathcal{K}$ is an exact pairing

$$\mathcal{P}(A^o) \otimes \mathcal{P}(A) \to \mathcal{K}.$$

Hence the arena $\mathcal{P}(A^o)$ is dual to the arena $\mathcal{P}(A)$. The unit map $\eta : \mathcal{K} \to \mathcal{P}(A) \otimes \mathcal{P}(A^o)$ is determined by $\eta(1) = Hom_A : (A \otimes A^o)^o \to \mathcal{K}$. It follows from the duality that for any arena X we have an equivalence of quategories,

$$\mathcal{P}(A) \otimes X \simeq X^{A^o}.$$

Hence the functor $X \mapsto X^{A^o}$ is left adjoint to the functor $X \mapsto X^A$ as in 22.20.

29.3. A map $A^o \times B \to \mathcal{K}$ is essentially the same thing as a distributor $A \Rightarrow B$ by 15.5. The *quategory of distributors* $A \Rightarrow B$ is defined to be the quategory

$$\mathcal{D}(A, B) = \mathcal{K}^{A^o \times B} = \mathcal{P}(A \times B^o) \simeq \mathrm{CC}(\mathcal{P}(B), \mathcal{P}(A)).$$

The composition law for distributors

$$\mathcal{D}(B, C) \times \mathcal{D}(A, B) \to \mathcal{D}(A, C)$$

is equivalent to the composition law of cocontinuous maps

$$\mathrm{CC}(\mathcal{P}(B), \mathcal{P}(A)) \times \mathrm{CC}(\mathcal{P}(C), \mathcal{P}(B)) \to \mathrm{CC}(\mathcal{P}(C), \mathcal{P}(A)).$$

The distributors form a bicategory $\mathcal{D}$ enriched over the (simplicial) monoidal category **AR**.

29.4. The *trace map*

$$Tr_A : \mathcal{P}(A^o \times A) \to \mathcal{K}$$

defined in 22.21 is a cocontinuous extension of the map $hom_A : A^o \times A \to \mathcal{K}$. It is equivalent to the counit of the duality

$$\epsilon : \mathcal{P}(A^o) \otimes \mathcal{P}(A) \to \mathcal{K}.$$

The *scalar product* of $f \in \mathcal{P}(A)$ and $g \in \mathcal{P}(A^o)$ is defined by putting

$$\langle f|g\rangle = Tr_A(g\Box f).$$

The map $\langle f|-\rangle : \mathcal{P}(A^o) \to \mathcal{K}$ is a cocontinuous extension of the map $f : A^o \to \mathcal{K}$ and the map $\langle -|g\rangle : \mathcal{P}(A) \to \mathcal{K}$ a cocontinuous extension of the map $g : A \to \mathcal{K}$.

29.5. Recall that a quategory is *null-pointed* if it admits a null object, and that a map between null-pointed quategories is *pointed* if it preserves null objects. The quategory of pointed Kan complexes $1\backslash\mathcal{K}$ is null-pointed and symmetric monoidal closed, where the tensor product is taken to be the smash product. Moreover, every null-pointed arena is enriched over $1\backslash\mathcal{K}$ and bicomplete as an enriched quategory. More precisely, If X and Y are two arenas and if X or Y is null-pointed then the quategories $X \otimes Y$ and $\mathrm{CC}(X, Y)$ are null-pointed. We shall denote by $\mathbf{AR}(1\backslash\mathcal{K})$ the full sub-category of **AR** spanned by the null-pointed arenas. The inclusion $\mathbf{AR}(1\backslash\mathcal{K}) \subset \mathbf{AR}$ has both a left and a right adjoint. The left adjoint is the functor $X \mapsto \top\backslash X$, where $\top$ denotes the terminal object of X, and the right adjoint is the functor $X \mapsto X/\bot$, where $\bot$ denotes the initial object of X. The (simplicial) category $\mathbf{AR}(1\backslash\mathcal{K})$ is symmetric monoidal closed if the unit object is taken to be the quategory $1\backslash\mathcal{K}$. If X is a null-pointed arena, then the equivalence $1\backslash\mathcal{K}\otimes X \simeq X$ is induced by the *smash product*

$$\wedge : 1\backslash\mathcal{K} \times X \to X.$$

29.6. If X and Y are null-pointed arenas, we shall say that a map $\epsilon : X \otimes Y \to 1\backslash\mathcal{K}$ is a *(pointed) pairing* between X and Y. We shall say that the pairing is *exact* if the map

$$\epsilon^\sharp : \mathrm{CC}(U, V \otimes X) \to \mathrm{CC}(U \otimes Y, V)$$

defined by putting $\epsilon^\sharp(f) = (V \otimes \epsilon)(f \otimes Y)$ is an equivalence of quategories for any null-pointed arenas U and V. A pairing ϵ is exact iff it is the counit of an adjunction $X \vdash Y$ in the monoidal category $\mathbf{AR}_\bullet$. When the pairing $\epsilon : X \otimes Y \to 1\backslash\mathcal{K}$ is exact, the map

$$Y \to \mathrm{CC}(X, 1\backslash\mathcal{K}) \simeq \mathrm{CC}(X, \mathcal{K})$$

induced by ϵ is an equivalence of quategories; we shall say that Y is the *pointed dual* of X and put $Y = X^*$.

29.7. If A is a simplicial set with null object $0 \in A$, we shall say that a prestack $f : A^o \to \mathcal{K}$ is *pointed* if $f(0) \simeq 1$. We shall denote by $\mathcal{P}_0(A)$ the full sub-quategory of $\mathcal{P}(A)$ spanned by the pointed prestacks $A^o \to \mathbf{K}$. If B is a simplicial set with null object $0 \in B$, the *external smash product* of a null-pointed prestack $f \in \mathcal{P}_0(A)$ with a null-pointed prestack $g \in \mathcal{P}_0(B)$ is the null-pointed prestack $f \bar{\wedge} g \in \mathcal{P}_0(A \wedge B)$ obtained by putting

$$(f \bar{\wedge} g)(a \wedge b) = f(a) \wedge g(b)$$

for every pair of objects $(a, b) \in A \times B$. The map $(f, g) \mapsto f \bar{\wedge} g$ is cocontinuous in each variable and the induced map

$$\mathcal{P}_0(A) \otimes \mathcal{P}_0(B) \to \mathcal{P}_0(A \times B).$$

is an equivalence of quategories. If A is a null-pointed quategory, the cocontinuous extension of the map $hom_A : A^o \times A \to 1\backslash\mathcal{K}$ is an exact pairing

$$\mathcal{P}_0(A^o) \otimes \mathcal{P}_0(A) \to 1\backslash\mathcal{K}.$$

Hence the arena $\mathcal{P}_0(A^o)$ is the pointed dual to the arena $\mathcal{P}_0(A)$. It follows from the duality that for any null-pointed arena X we have an equivalence of quategories,

$$\mathcal{P}(A) \otimes X \simeq [A^o, X]$$

where $[A^o, X]$ denotes the quategory of pointed maps $A^o \to X$. Hence the functor $X \mapsto [A^o, X]$ is left adjoint to the functor $X \mapsto [A, X]$.

30. Cartesian theories

A cartesian theory is a small quategory with finite limits. We show that the (simplicial) category of cartesian theories is symmetric monoidal closed. We introduce the notion of α-cartesian theory for any regular cardinal $\alpha \geq 0$.

30.1. A *cartesian theory* is a small cartesian quategory T. If X is a cartesian quategory (possibly large), we shall say that a left exact map $T \to X$ is a *model* or an *interpretation* of T in X. We shall denote by $\mathrm{Model}(T, X)$, or by $T(X)$, the full simplicial subset of X^T spanned by the models $T \to X$. We shall say that a model $T \to \mathcal{K}$ is a *homotopy model* and we shall write

$$\mathrm{Model}(T) = \mathrm{Model}(T, \mathcal{K}).$$

A *morphism* $S \to T$ of cartesian theories is a model $S \to T$. The identity morphism $T \to T$ is the *generic* or *tautological* model of T. We shall denote by $\mathbf{CT}$ the category of cartesian theories and morphisms. More generally, if α is a regular cardinal, we shall say that a small α-cartesian quategory T is an *α-cartesian theory.* If X is an α-cartesian quategory (possibly large), we shall say that an α-continuous map $T \to X$ is a *model* or an *interpretation* of T in X. We shall denote by $\mathrm{Model}_\alpha(T, X)$, or by $T(X)$, the full simplicial subset of X^T spanned by the models $T \to X$. We shall say that a model $T \to \mathcal{K}$ is a *homotopy model* and we shall write

$$\mathrm{Model}_\alpha(T) = \mathrm{Model}_\alpha(T, \mathcal{K}).$$

A *morphism* $S \to T$ of α-cartesian theories is a model $S \to T$. The identity morphism $T \to T$ is the *generic* or *tautological* model of T. We shall denote by $\mathbf{CT}_\alpha$ the category of α-cartesian theories and morphisms.

30.2. Remark A 0-cartesian theory is just a small quategory A; and a model of A in a quategory X is just a map $A \to X$. Thus,

$$\mathrm{Model}_0(A, X) = X^A.$$

A 1-cartesian theory is just a small quategory with terminal object A; and a model of A in a quategory with terminal object X is a map $A \to X$ which preserves terminal objects.

30.3. If T is a cartesian theory, then the inclusion $\mathrm{Model}(T) \subseteq \mathcal{K}^T$ has a left adjoint and the quategory $\mathrm{Model}(T)$ is a finitary presentable arena. If $u : S \to T$ is a morphism of cartesian theories, then the map

$$u^* : \mathrm{Model}(T) \to \mathrm{Model}(S)$$

induced by u has a left adjoint $u_!$. More generally, if α is a regular cardinal and T is an α-cartesian theory, then the inclusion $\mathrm{Model}_\alpha(T) \subseteq \mathcal{K}^T$ has a left adjoint and the quategory $\mathrm{Model}_\alpha(T)$ is an α-presentable arena. If $u : S \to T$ is a morphism of α-algebraic theories, then the map

$$u^* : \mathrm{Model}_\alpha(T) \to \mathrm{Model}_\alpha(S)$$

induced by u has a left adjoint $u_!$.

30.4. If T is a cartesian theory, then the map $y(a) = hom_T(a, -) : T \to \mathcal{K}$ is model for every object $a \in T$. We shall say that a model $f \in \mathrm{Model}(T)$ is *representable* if it is isomorphic to a model $y(a)$ for some object $a \in T$. The map

$$y : T^o \to \mathrm{Model}(T)$$

induced by the Yoneda map $T^o \to \mathcal{K}^T$ is fully faithful and it induces an equivalence between T^o and the full sub-quategory of $\mathrm{Model}(T)$ spanned by the representable models. A model of T is a retract of a representable iff it is compact. The full sub-quategory of compact models of T is equivalent to the Karoubi envelope $Kar(T^o) = Kar(T)^o$. The quategory $\mathrm{Kar}(T)$ is cartesian and the map

$$i^* : \mathrm{Model}(Kar(T)) \to: \mathrm{Model}(T)$$

induced by the inclusion $i : T \to \mathrm{Kar}(T)$ is an equivalence of quategories. More generally, a morphism of cartesian theories $u : S \to T$ is a Morita equivalence iff the map $u^* : \mathrm{Model}(T) \to \mathrm{Model}(S)$ induced by u is an equivalence of quategories. More generally, if α is a regular cardinal and T is an α-cartesian theory, then the map $y(a) = hom_T(a, -) : T \to \mathcal{K}$ is model for every object $a \in T$. We shall say that a model $f \in \mathrm{Model}_\alpha(T)$ is *representable* if it is isomorphic to a model $y(a)$ for some object $a \in T$. The map

$$y : T^o \to \mathrm{Model}_\alpha(T)$$

induced by the Yoneda map $T^o \to \mathcal{K}^T$ is fully faithful and it induces an equivalence between T^o and the full sub-quategory of $\mathrm{Model}_\alpha(T)$ spanned by the representable models. A model of T is a retract of a representable iff it is is α-compact. It follows that the full sub-quategory of α-compact models of T is equivalent to $Kar(T^o) = Kar(T)^o$. Notice that we have $\mathrm{Kar}(T) = T$ if $\alpha > \omega$. The quategory $\mathrm{Kar}(T)$ is α-cartesian and the map

$$i^* : \mathrm{Model}_\alpha(Kar(T)) \to: \mathrm{Model}_\alpha(T)$$

induced by the inclusion $i : T \to \mathrm{Kar}(T)$ is an equivalence of quategories. More generally, a morphism of α-cartesian theories $u : S \to T$ is a Morita equivalence iff the map $u^* : \mathrm{Model}_\alpha(T) \to \mathrm{Model}_\alpha(S)$ induced by u is an equivalence of quategories.

30.5. It T is a cartesian theory, then the Yoneda map $y : T^o \to \mathrm{Model}(T)$ preserves finite colimits and it exibits the quategory $\mathrm{Model}(T)$ as the free cocompletion of T^o. More precisely, let us denote by $\mathrm{fCC}(X, Y)$ the quategory of maps preserving finite colimits between two quategories X and Y. Then the map

$$y^* : \mathrm{CC}(\mathrm{Model}(T), X) \to \mathrm{fCC}(T^o, X)$$

induced by y is an equivalence of quategories for any cocomplete quategory X. The inverse equivalence associates to a finitely cocontinuous map $f : T^o \to X$ its left Kan extension $f_! : \mathrm{Model}(T) \to X$ along y. More generally, let α be a regular cardinal and T be an α-cartesian theory. Then the Yoneda map $y : T^o \to \mathrm{Model}_\alpha(T)$ preserves α-colimits and it exibits the quategory $\mathrm{Model}_\alpha(T)$ as the free cocompletion of T^o.

30.6. Remark. : It T is a (finitary) cartesian theory, then we have

$$\mathrm{Model}(T) = \mathrm{Ind}(T^o)$$

since a map $f : T \to \mathcal{K}$ preserves finite limits iff its quategory of elements is directed. More generally, if α is a regular cardinal, then we have $\mathrm{Model}_\alpha(T) = \mathrm{Ind}_\alpha(T^o)$ for any α-cartesian theory T.

30.7. The forgetful functor $\mathbf{CT} \to \mathbf{S}$ admits a left adjoint which associates to a simplicial set A a cartesian theory $\mathcal{C}[A]$ equipped with a map $u : A \to \mathcal{C}[A]$. By definition, for every cartesian quategory X, the map $u^* : \mathrm{Model}(\mathcal{C}[A], X) \to X^A$ induced by u is an equivalence of quategories. The quategory $\mathcal{C}[A]$ is the opposite of the quategory $\mathcal{P}_f(A^o)$ described in 23.25. The cartesian theory $\mathcal{C} = \mathcal{C}[1]$ is freely generated by one object $u \in \mathcal{C}$. The quategory $\mathcal{C}$ is equivalent to the opposite of the quategory $\mathcal{K}_f$ of finite homotopy types. The equivalence $\mathcal{K}_f^o \to \mathcal{C}$ is induced by the map $x \mapsto u^x$. More generally, if α is a regular cardinal, then the forgetful functor $\mathbf{CT}_\alpha \to \mathbf{S}$ admits a left adjoint which associates to a simplicial set A an α cartesian theory $\mathcal{C}_\alpha[A]$ equipped with a map $u : A \to \mathcal{C}_\alpha[A]$. The quategory $\mathcal{C}_\alpha[A]$ is the opposite of the quategory $\mathcal{P}_\alpha(A^o)$ described in 23.25. The cartesian theory $\mathcal{C}_\alpha = \mathcal{C}_\alpha[1]$ is freely generated by one object $u \in \mathcal{C}_\alpha$.

30.8. Notice that $\mathcal{C}_0[A] = A$ and $\mathcal{C}_1[A] = A \star 1$ for any small quategory A. In particular, $\mathcal{C}_0 = 1$ and $\mathcal{C}_1 = I$.

30.9. Every finitary limit sketch (A, P) has a universal model $u : A/P \to \mathcal{C}[A/P]$ with values in a cartesian theory called the *envelopping theory* of (A, P). The universality means that the map

$$u^* : \text{Model}(T, X) \to \text{Model}(A/P, X)$$

induced by u is an equivalence for any cartesian quategory X. In particular, the map

$$u^* : \text{Model}(\mathcal{C}[A/P]) \to \text{Model}(A/P)$$

induced by u is an equivalence of quategories. More generally, if α is a regular cardinal, then every α-bounded limit sketch (A, P) has a universal model $u : A/P \to \mathcal{C}_\alpha[A/P]$ with values in an α-cartesian theory called the *envelopping theory* of (A, P).

30.10. If (A, P) is a limit sketch, then by composing the Yoneda map $y : A^o \to \mathcal{K}^A$ with the left adjoint r to the inclusion $\text{Model}(A/P) \subseteq \mathcal{K}^A$ we obtain a map

$$ry : A^o \to \text{Model}(A/P).$$

We shall say that a model $f \in \text{Model}(A/P)$ is *representable* if it belongs to the essential image of ry. If the sketch (A, P) is finitary, we shall say that f is *finitely presentable* if it is the colimit of a finite diagram of representable models. We shall denote by $\text{Model}(A/P)_f$ the full sub-quategory of $\text{Model}(A/P)$ spanned by the finitely presentable models. Then the quategory $\mathcal{C}[A/P]$ is the opposite of the quategory $\text{Model}(A/P)_f$ and the canonical map $u : A \to \mathcal{C}[A/P]$ is the opposite of the map $A^o \to \text{Model}(A/P)_f$ induced by ry. More generally, if α is a regular cardinal and (A, P) is an α-bounded limit sketch, we shall say that a model $f \in \text{Model}(A/P)$ is *α-presentable* if it is the colimit of a diagram of cardinality $< \alpha$ of representable models. We shall denote by $\text{Model}(A/P)^\alpha$ the full sub-quategory of $\text{Model}(A/P)$ spanned by the α-presentable models. The quategory $\mathcal{C}_\alpha[A/P]$ is the opposite of the quategory $\text{Model}(A/P)^\alpha$ and the map $u : A \to \mathcal{C}_\alpha[A/P]$ is the opposite of the map $A^o \to \text{Model}(A/P)^\alpha$ induced by ry.

30.11. (Example) We saw in 28.8 that the notion of n-object is essentially algebraic and finitary for any $n \geq 0$. We shall denote the *cartesian theory of n-objects* by $\mathcal{C}\langle n\rangle$. By definition, it is freely generated by a n-object $u \in \mathcal{C}\langle n\rangle$. Hence the map

$$u^* : \text{Model}(\mathcal{C}\langle n\rangle) = \mathcal{K}\langle n\rangle$$

defined by putting $u^*(f) = f(u)$ is an equivalence, where $\mathcal{K}\langle n\rangle$ is the quategory of n-objects in $\mathcal{K}$. Let us say that an object of $\mathcal{K}\langle n\rangle$ is *truncated finite* if it is the n-truncation of a finite homotopy type. It then follows from **??** that the quategory $\mathcal{C}\langle n\rangle$ is the oppposite of the quategory $\mathcal{K}\langle n\rangle_f$ of truncated finite n-objects. In particular, the quategory $\mathcal{C}\langle 0\rangle$ is equivalent to the opposite of the category of finite sets, and the quategory $\mathcal{C}\langle 1\rangle$ to the opposite of the quategory of finitely presentable groupoids.

30.12. If S and T are two cartesian theories, then so is the quategory $\text{Model}(S, T)$ of models $S \to T$. The category of cartesian theories **CT** is simplicial and symmetric monoidal closed. The *tensor product* of two cartesian theories S and T is defined to be the target of a map $S \times T \to S \odot T$ left exact in each variable and universal with respect to that property. More precisely, if X is a cartesian quategory, let us denote by $\mathrm{Model}(S,T;X)$ the full simplicial subset of $X^{S\times T}$ spanned by the maps $S \times T \to X$ left exact in each variable. Then the map

$$\phi^* : \mathrm{Model}(S \odot T, X) \to \mathrm{Model}(S,T;X)$$

induced by ϕ is an equivalence for any cartesian quategory X. It follows that we have two canonical equivalences of quategories

$$\mathrm{Model}(S \odot T, X) \simeq \mathrm{Model}(S, \mathrm{Model}(T,X)) \simeq \mathrm{Model}(T, \mathrm{Model}(S,X)).$$

In particular, we have two canonical equivalences

$$\mathrm{Model}(S \odot T) \simeq \mathrm{Model}(S, \mathrm{Model}(T)) \simeq \mathrm{Model}(T, \mathrm{Model}(S)).$$

The unit for the tensor product is the cartesian theory $\mathcal{C}$ described in 30.7. More generally, if α is a regular cardinal and S and T are two α-cartesian theories, then so is the quategory $\mathrm{Model}_\alpha(S,T)$ of models $S \to T$. The category $\mathbf{CT}_\alpha$ is simplicial and symmetric monoidal closed. The *tensor product* of two α-cartesian theories S and T is defined to be the target of a map $S \times T \to S \odot_\alpha T$ α-continuous in each variable and universal with respect to that property. The unit for the tensor product is the theory $\mathcal{C}_\alpha$ described in 30.7.

30.13. Recall that a 1-cartesian theory T is a quasi-category with terminal object 1. The tensor product $S \odot_1 T$ of two 1-cartesian theories S and T is a equivalent to the smash product $S \wedge T$ of the pointed simplicial sets $(S,1)$ and $(T,1)$.

30.14. If A and B are simplicial sets, then the morphism

$$\phi : \mathcal{C}[A] \odot \mathcal{C}[B] \to \mathcal{C}[A \times B]$$

defined by putting $\phi(a \odot b) = (a,b)$ for every pair of objects $(a,b) \in A \times B$ is an equivalence of quategories. Hence the functor

$$\mathcal{C}[-] : \mathbf{S} \to \mathbf{CT}$$

preserves tensor products (where the tensor product on $\mathbf{S}$ is the cartesian product). If T is a cartesian theory, we shall put

$$T[A] = \mathcal{C}[A] \odot T$$

Then for any cartesian quategory X we have two equivalences of quategories

$$\mathrm{Model}(T[A], X) \simeq \mathrm{Model}(T, X^A) \simeq \mathrm{Model}(T,X)^A.$$

This shows that $T[A]$ is the cartesian theory of A-diagrams of models of T. In particular, $T[I]$ is the cartesian theory of maps between two models of T. More generally, if α is a regular cardinal, then the functor

$$\mathcal{C}_\alpha[-] : \mathbf{S} \to \mathbf{CT}_\alpha$$

preserves tensor products. If T is an α-cartesian theory, we shall put

$$T[A] = \mathcal{C}_\alpha[A] \odot_\alpha T$$

30.15. The cartesian product of a (small) family of cartesian quategories is cartesian. Hence the category **CT** has cartesian products. The cartesian product of a finite family of cartesian theories is also their coproduct. This means that the (simplicial) category **CT** is semi-additive. For example, the terminal quategory 1 is also the initial object of **CT**, since the quategory Model$(1, T)$ is equivalent to the quategory 1 for every T. Moreover, if S and T are cartesian theories, consider the maps

$$i_S : S \to S \times T \quad \text{and} \quad i_T : S \to S \times T$$

defined by putting $i_S(x) = (x, 1)$ and $i_T(y) = (1, y)$ for every $x \in S$ and $y \in T$. Then the map

$$(i_S^*, i_T^*) : \mathrm{Model}(S \times T, X) \to \mathrm{Model}(S, X) \times \mathrm{Model}(T, X)$$

induced by the pair (i_S, i_T) is an equivalence for any cartesian quategory X. More generally, the category $\mathbf{CT}_\alpha$ has cartesian products for any regular cardinal α. The cartesian product of a family of α-cartesian theories indexed by a set of cardinality $< \alpha$ is also their coproduct.

30.16. If T is a cartesian theory and X is an arena, then the *external tensor product* of a model $f \in \mathrm{Model}(T)$ with an object $x \in X$ is defined to be the map $f \otimes x : T \to X$ obtained by applying the left adjoint to the inclusion $\mathrm{Model}(T, X) \subseteq X^T$ to the map $a \mapsto f(a) \cdot x$. See for the action of $\mathcal{K}$ on X. This defines a map $(f, x) \mapsto f \otimes x$ cocontinuous in each variable and the induced map

$$\mathrm{Model}(T) \otimes X \simeq \mathrm{Model}(T, X)$$

is an equivalence of quategories. In particular, the *external tensor product* of a model $f \in \mathrm{Model}(S)$ with a model $g \in \mathrm{Model}(T)$ is the model $f \otimes g \in \mathrm{Model}(S \odot T)$ defined in 28.28. The map $(f, g) \mapsto f \otimes g$ is cocontinuous in each variable and the induced map

$$\mathrm{Model}(S) \otimes \mathrm{Model}(T) \simeq \mathrm{Model}(S \odot T)$$

is an equivalence of quategories. More generally, if α be a regular cardinal, then the external tensor induces an equivalence of quategories

$$\mathrm{Model}_\alpha(T) \otimes X \simeq \mathrm{Model}_\alpha(T, X)$$

for any α-cartesian theory T and any arena X. In particular it induces the equivalence of quategories

$$\mathrm{Model}_\alpha(S) \otimes \mathrm{Model}_\alpha(T) \simeq \mathrm{Model}_\alpha(S \odot_\alpha T)$$

of 28.28.

30.17. (Example) Recall that a *stable object* or of *spectrum* in a cartesian quategory X is a model of a finitary limit sketch (A, P) by 28.7. Hence the notion of stable object is essentially algebraic and finitary. We shall denote the *cartesian theory of spectra* by *Spec* and the quategory of spectra in X by $Spec(X)$. The quategory $Spec(X)$ is the (homotopy) projective limit of the infinite sequence of quategories

$$1\backslash X \xleftarrow{\Omega} 1\backslash X \xleftarrow{\Omega} 1\backslash X \xleftarrow{\Omega} \cdots .$$

The quategory Model(*Spec*) is the *quategory of spectra* $\mathcal{S}p$. If $\mathcal{C}'$ denotes the cartesian theory of pointed objects, consider the interpretation $i : \mathcal{C}' \to Spec$ defined by the pointed object x_0 of the generic spectrum (x_n). Then the adjoint adjoint pair

$$i_! : \mathrm{Model}(\mathcal{C}') \leftrightarrow \mathrm{Model}(Spec) : i^*$$

is the classical adjoint pair

$$\Sigma^\infty : 1\backslash\mathcal{K} \leftrightarrow \mathcal{S}p : \Omega^\infty$$

between pointed spaces and spectra.

30.18. (Example) Recall that a *category object* in a cartesian quategory X is a model of a finitary limit sketch by 28.9. Hence the notion of category object is essentially algebraic and finitary. We shall denote the *cartesian theory of categories* by Cat and the quategory of category objects in X by $Cat(X)$. We shall say that a morphism $f : C \to D$ in $Cat(X)$ is a *functor*.

30.19. (Example) Recall that a *groupoid object* in a cartesian quategory X is a model of a finitary limit sketch by 28.10. Hence the notion of groupoid object is essentially algebraic and finitary. We shall denote the *cartesian theory of groupoids* by Gpd and the quategory of groupoid objects in X by $Gpd(X)$. If i denotes the canonical morphism $i : Cat \to Gpd$, then the map

$$i_! : \mathrm{Model}(Cat) \leftrightarrow \mathrm{Model}(Gpd) : i^*$$

associates to a category C the groupoid freely generated by it.

30.20. (Example) If X is a cartesian quategory, then the forgetful map $Ob : Gpd(X) \to X$ has both a left and a right adjoint. The left adjoint $Sk^0 : X \to Gpd(X)$ associate to an object $b \in X$ the constant simplicial object $Sk^0(b) : \Delta^o \to X$ with value b. The right adjoint $Cosk^0 : X \to Gpd(X)$ associates to b a simplicial object obtained by putting $Cosk^0(b)_n = b^{[n]}$ for each $n \geq 0$. We say that $Cosk^0(b)$ is the *Cech groupoid* of b. More generally, the *Cech groupoid* $Cech(f)$ of an arrow $f : a \to b$ in X is defined to be the image by the canonical map $X/b \to X$ of the the Cech groupoid of the object $f \in X/b$. Let $\mathcal{C}[I]$ be the cartesian theory of maps and consider the interpretation $j : Gpd \to \mathcal{C}[I]$ defined by the Cech groupoid of the generic map. Then the map

$$j_! : \mathrm{Model}(Gpd) \to \mathrm{Model}(\mathcal{C}[I]) = \mathcal{K}^I$$

takes a groupoid C to its classifying space BC equipped with the canonical map $C_0 \to BC$. It induces an equivalence between Model(Gpd) and the full subquategory of $\mathcal{K}^I$ spanned by the surjections. It follows that the map $j : Gpd \to \mathcal{C}[I]$ is fully faithful since $j_!$ is fully faithful

30.21. (Example) If X is a cartesian quategory, then the inclusion $Gpd(X) \subseteq Cat(X)$ has a right adjoint which associates to a category $C \in Cat(X)$ its groupoid of isomorphisms $J(C)$. We have $J(C) = i^*(C)$, where $i : Gpd \to Cat$ is the interpretation defined by the groupoid of isomorphisms of the generic category object in Cat. If A is a simplicial set, we call a map $f : A \to X$ *essentially constant* if it belongs to the essential image of the diagonal $X \to X^A$. A category object $C : \Delta^o \to X$ is essentially constant iff the unit map $C_0 \to C_1$ is invertible. We shall say that a quategory C satisfies the *Rezk condition*, or that it is *reduced*, if the groupoid $J(C)$ is essentially constant. The notion of a reduced category object is essentially algebraic and finitary. We shall denote the *cartesian theory of reduced categories* by $RCat$ and the quategory of reduced category objects in X by $RCat(X)$.

30.22. We observe that the nerve functor $N : \mathcal{Q}_1 \to [\Delta^o, \mathcal{K}]$ defined in 23.5. induces an equivalence of quategories

$$N : \mathcal{Q}_1 \simeq \text{Model}(RCat).$$

30.23. (Example) We say that a category object C in a cartesian quategory X is *n-truncated* if the map $C_1 \to C_0 \times C_0$ is a $(n-1)$-cover. If $n = 0$, this means that C is a preorder. The notion of n-truncated category is essentially algebraic and finitary. We shall denote the *cartesian theory of n-truncated categories* by $Cat\langle n\rangle$. If a reduced category object $C \in Cat(X)$ is n-truncated, then C_k is a n-object for every $k \geq 0$. The notion of n-truncated reduced category is essentially algebraic. We shall denote the *cartesian theory of n-truncated reduced categories* by $RCat\langle n\rangle$.

30.24. (Example) A *double category object* in cartesian quategory X is a double simplicial object $C : \Delta^o \times \Delta^o \to X$ which is a category object in each variable. If Cat denotes the cartesian theory of categories then $Cat^2 = Cat \odot Cat$ is the cartesian theory of double categories. If $Cat^2(X)$ denotes the quategory of double category objects in X, then we have

$$Cat^2(X) = Cat(Cat(X)).$$

More generally, a *n-fold category object* in X is a n-fold simplicial object $C : (\Delta^n)^o \to X$ which is a category object in each variable. We shall denote the *cartesian theory of n-fold categories* by Cat^n and the quategory of n-fold category objects in X by $Cat^n(X)$. We shall say that a n-fold category $C : (\Delta^n)^o \to X$ is *reduced* if it is reduced in each variable. If $RCat$ denotes the cartesian theory of reduced categories, then $RCat^n = RCat^{\odot n}$ is the *cartesian theory of reduced n-fold categories*.

30.25. (Exemple) A *2-category object* in cartesian quategory X is a double category $C : \Delta^o \to Cat(X)$ such that the map $C_0 : \Delta^o \to X$ is essentially constant. The notion of 2-category object is essentially algebraic and finitary. We shall denote the cartesian theory of 2-categories by Cat_2 and the quategory of 2-category objects in X by $Cat_2(X)$. A morphism $f : C \to D$ in $Cat_2(X)$ is a *2-functor*. More generally, the quategory $Cat_n(X)$ of n-category objects in X is defined by induction on $n \geq 2$: a category object $C : \Delta^o \to Cat_{n-1}(X)$ is a *n-category* if C_0 is essentially constant. The notion of n-category object is essentially algebraic and finitary. We shall denote the *cartesian theory of n-categories* by Cat_n and the quategory of n-category objects in X by $Cat_n(X)$. We say that a n-category $C \in Cat_n(X)$ is *reduced* if it is reduced as a n-fold category. A n-category $C : \Delta^o \to Cat_{n-1}(X)$ is reduced iff the quategory $C : \Delta^o \to Cat_{n-1}(X)$ is reduced and the $(n-1)$-category C_1 is reduced. We denote the cartesian theory of reduced n-categories by $RCat_n$ and the quategory of reduced n-category objects in X by $RCat_n(X)$.

30.26. A quategory X is equivalent to a quategory of models of a cartesian theory iff it is finitary presentable arena iff it is generated by a small set of compact objects. More precisely, let $K \subseteq X$ be a small full sub-quategory of compact objects. If K is closed under finite colimits, then the left Kan extension

$$i_! : \text{Model}(K^o) \to X$$

of the inclusion $i : K \to X$ along $y : K \to\to \text{Model}(K^o)$ is fully faithful and cocontinuous. Moreover, $i_!$ is an equivalence if K generates or separates X. More generally, if α is a regular cardinal, then a quategory X is equivalent to the quategory of models of an α-cartesian theory iff X is an α-presentable arena iff it is generated by a small set of α-compact objects. More precisely, let $K \subseteq X$ be a small full sub-quategory of α-compact objects. If K is closed under α-colimits, then the left Kan extension

$$i_! : \mathrm{Model}_\alpha(K^o) \to X$$

of the inclusion $i : K \to X$ along $y : K \to\to \mathrm{Model}_\alpha(K^o)$ is fully faithful and cocontinuous. Moreover, $i_!$ is an equivalence if K generates or separates X.

30.27. Recall from 32.23 that a cocontinuous map betwen finitary presentable arenas $f : X \to Y$ preserves compact objects iff its right adjoint $g : Y \to X$ is finitary accessible. Let us denote by $\mathbf{AR}_\omega$ the category whose objects are the arenas and whose morphisms are the cocontinuous maps preserving compact objects. If $u : S \to T$ is a morphism of cartesian theories, then the map

$$u_! : \mathrm{Model}(S) \to \mathrm{Model}(T)$$

preserves compact objects. The resulting functor

$$\mathrm{Model} : \mathbf{CT} \to \mathbf{AR}_\omega.$$

has a right adjoint k^o which associates to X the opposite of its sub-quategory $k(X)$ of compact objects (or a small quategory equivalent to it). The quategory $k(X)$ is Karoubi complete and the counit of the adjunction $\epsilon_X : \mathrm{Model}(k^o(X)) \to X$ is fully faithful; and it is an equivalence iff X is finitary presentable. The unit of the adjunction $\eta_T : T \to k^o(\mathrm{Model}(T))$ is a Morita equivalence; and it is an equivalence iff T is Karoubi complete. Hence the adjoint pair $\mathrm{Model} \vdash k^o$ induces an equivalence between the full subcategory of $\mathbf{CT}$ spanned by the Karoubi complete theories and the full sub category of $\mathbf{AR}_\omega$ spanned by the finitary presentable quategories. More generally, if α is a regular cardinal, then a cocontinuous map betwen α-presentable quategories $f : X \to Y$ preserves α-compact objects iff its right adjoint $g : Y \to X$ is α-accessible. Let us denote by $\mathbf{AR}_\alpha$ the category whose objects are the arenas and whose morphisms are the cocontinuous maps preserving α-compact objects. If $u : S \to T$ is a morphism of α-cartesian theories, then the map

$$u_! : \mathrm{Model}_\alpha(S) \to \mathrm{Model}_\alpha(T)$$

preserves α-compact objects. The resulting functor

$$\mathrm{Model}_\alpha : \mathbf{CT}_\alpha \to \mathbf{AR}_\alpha$$

has a right adjoint k^o_α which associates to X the opposite of its sub-quategory $k_\alpha(X)$ of α-compact objects (or a small quategory equivalent to it). The quategory $k_\alpha(X)$ is Karoubi complete and the counit of the adjunction $\epsilon_X : \mathrm{Model}_\alpha(k^o_\alpha(X)) \to X$ is fully faithful; and it is an equivalence iff X is α-presentable. The unit of the adjunction $\eta_T : T \to k^o_\alpha(\mathrm{Model}_\alpha(T))$ is a Morita equivalence; and it is an equivalence iff T is Karoubi complete. Hence the adjoint pair $\mathrm{Model}_\alpha \vdash k^o_\alpha$ induces an equivalence between the full subcategory of $\mathbf{CT}_\alpha$ spanned by the Karoubi complete theories and the full sub category of $\mathbf{AR}_\alpha$ spanned by the α-presentable quategories.

30.28. The notion of homotopy factorisation system in the category **CT** is defined as in 28.29. The category **CT** admits a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of essentially surjective morphisms and $\mathcal{B}$ the class of fully faithful morphisms. More generally, if α is a regular cardinal, then the category $\mathbf{CT}_\alpha$ admits a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of essentially surjective morphisms and $\mathcal{B}$ the class of fully faithful morphisms. A morphism $u : S \to T$ in $\mathbf{CT}_\alpha$ is fully faithful, iff the map $u_! : \mathrm{Model}_\alpha(S) \to \mathrm{Model}_\alpha(T)$ is fully faithful. The map $u^* : \mathrm{Model}_\alpha(T) \to \mathrm{Model}_\alpha(S)$ is conservative iff u is Morita surjective.

30.29. Let Σ be a set of morphisms in a cartesian theory T. If Σ is closed under base changes then the quategory $L(T, \Sigma)$ is cartesian and the canonical map $T \to L(T, \Sigma)$ is left exact. We shall say that a morphism of cartesian theories is a *quasi-localisation* (resp. *iterated quasi-localisation*) if it is a quasi-localisation (resp. an iterated quasi-localisation) as a map of quategories. The category **CT** admits a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{B}$ is the class of conservative morphisms and $\mathcal{A}$ is the class of iterated quasi-localisations. This is true also of the category $\mathbf{CT}_\alpha$.

30.30. The cartesian theories Gpd and $RCat$ are cartesian localisations of Cat. The cartesian theory $RCat_n$ is a a cartesian of Cat_n for every $n \geq 0$. The cartesian theory $\mathcal{C}\langle n \rangle$ is a cartesian localisation of $\mathcal{C}$ for every $n \geq 0$.

30.31. The initial model of a cartesian theory T is representable by its terminal object $1 \in T$. We shall say that a morphism of cartesian theories $u : S \to T$ is *coinitial* if the map

$$u^* : \mathrm{Model}(T) \to \mathrm{Model}(S)$$

preserves initial models, that is, if $u^*(\bot) = \bot$. A morphism $u : S \to T$ is coinitial iff the map $S(1, x) \to T(1, ux)$ induced by u is a homotopy equivalence for every object $x \in S$. The category **CT** admits a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{B}$ is the class of coinitial morphisms. We shall say that a morphism in the class $\mathcal{A}$ is *elementary*. For every model f of a cartesian theory T there is an elementary morphism $i : T \to T[f]$ with an isomorphism $i^*(\bot) = f$. We shall say that $T[f]$ is the *envelopping theory* of the model f. The map

$$\tilde{i}^* : \mathrm{Model}(T[f]) \to f \backslash \mathrm{Model}(T)$$

induced by the map $i^* : \mathrm{Model}(T[f]) \to \mathrm{Model}(T)$ is an equivalence of quategories. When f is representable by an object $a \in T$, we have$T[f] = T/a$ and $i : T \to T/a$ is the base change map. More generally, if α is a regular cardinal > 0, then the category $\mathbf{CT}_\alpha$ admits a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{B}$ is the class of coinitial morphisms. A morphism in the class $\mathcal{A}$ is said to be *elementary*. For any model f of an α-cartesian theory T there is an elementary morphism $i : T \to T[f]$ such that $i^*(\bot) = f$. The map

$$\tilde{i}^* : \mathrm{Model}_\alpha(T[f]) \to f \backslash \mathrm{Model}_\alpha(T)$$

induced by i^* is an equivalence of quategories. We shall say that $T[f]$ is the *envelopping theory* of the model f.

30.32. We say that a morphism of cartesian theories $u : S \to T$ is *coterminal* if the map

$$u_! : \mathrm{Model}(S) \to \mathrm{Model}(T)$$

takes a terminal model $\top_S$ of S to a terminal model of T. The category $\mathbf{CT}$ admits a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of coterminal morphisms. A morphism $u : S \to T$ belongs to $\mathcal{B}$ iff it is equivalent to a left fibration $E \to T$. For any model $f \in \mathrm{Model}(T)$, the left fibration $p : el(f) \to T$ belongs to $\mathcal{B}$. Moreover, we have $p_!(\top) \simeq f$ and the map $\tilde{p}_! : \mathrm{Model}(el(f)) \to \mathrm{Model}(T)/f$ induced by the map $p_! : \mathrm{Model}(el(f)) \to \mathrm{Model}(T)$ is an equivalence of quategories. More generally, if α is a regular cardinal, we say that a morphism of α-cartesian theories $u : S \to T$ is *coterminal* if the map

$$u_! : \mathrm{Model}_\alpha(S) \to \mathrm{Model}_\alpha(T)$$

preserves terminal objects. The category $\mathbf{CT}_\alpha$ admits a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of coterminal morphisms. A morphism $u : S \to T$ belongs to $\mathcal{B}$ iff it is equivalent to a left fibration $E \to T$. For any model $f \in \mathrm{Model}_\alpha(T)$, the left fibration $p : el(f) \to T$ belongs to $\mathcal{B}$. Moreover, we have $p_!(\top) \simeq f$ and the map

$$\tilde{p}_! : \mathrm{Model}_\alpha(el(f)) \to \mathrm{Model}_\alpha(T)/f$$

induced by $p_!$ is an equivalence of quategories.

30.33. A morphism of cartesian theories $u : S \to T$ is coterminal iff the map $u_! : \mathrm{Model}(S) \to: \mathrm{Model}(T)$ is coterminal in $\mathbf{AR}$. A morphism $u : S \to T$ is equivalent to a left fibration iff the map $u_!$ is equivalent to a right fibration. There is a similar result for a morphism of α-cartesian theories $u : S \to T$.

30.34. A map of simplicial sets $u : A \to B$ is initial iff the morphism of cartesian theories $\mathcal{C}[u] : \mathcal{C}[A] \to \mathcal{C}[B]$ is coterminal. A map $u : A \to B$ is equivalent to a left fibration iff the morphism $\mathcal{C}[u] : \mathcal{C}[A] \to \mathcal{C}[B]$ is equivalent to a left fibration. There is a similar result for the morphism $\mathcal{C}_\alpha[u]$.

30.35. Let us denote by $\mathbf{FP}$ the (simplicial) category whose objects are the finitary presentable quategories and whose morphisms are the finitary maps. The cartesian product of two finitary presentable quategories X and Y is finitary presentable and we have $k(X \times Y) \simeq k(X) \times k(Y)$, where $k(X)$ denotes the quategory of α-compact objects of X. The quategory $\mathbf{FP}$ is cartesian closed. More precisely, the map

$$i^* : \mathrm{Fin}(X, Y) \to Y^{k(X)}$$

induced by the inclusion $i : k(X) \subseteq X$ is an equivalence, where the domain of i^* is the simplicial set of finitary maps $X \to Y$.

30.36. Let us say that a quategory A is *finite* if it is finitely presented and the simplicial set $Hom_A(a, b)$ is homotpy finite for every pair of objects $a, b \in A$. In this case the cartesian theories $C_\omega[A]$ and $C_\omega[A^o]$ are mutually dual in the symmetric monoidal category $\mathbf{CT}$. The counit of this duality is the map

$$\epsilon : \mathcal{C}[A] \odot \mathcal{C}[A^o] \to \mathcal{C}$$

induced by the opposite of the map $Hom_A : A^o \times A \to \mathcal{K}_f = \mathcal{C}^o$. Moreover, we have an equivalence of quategories $T[A^o] \simeq T^A$ for any cartesian theory T. It follows that we have an equivalence of quategories,

$$\mathrm{Model}(S^{A^o}, T) \simeq \mathrm{Model}(S, T^A)$$

for any $S, T \in \mathbf{CT}$.

30.37. If T is a cartesian theory, we shall say that a left fibration $E \to T$ is a *vertical model* of T if the quategory E is cartesian and the map $E \to T$ is left exact. A map $f : T \to \mathcal{K}$ is a model of T iff the left fibration $el(f) \to T$ is a vertical model. The model category $(\mathbf{S}/T, \mathrm{Wcov})$ admits a Bousfield localisation whose fibrant objects are the vertical models over T. The coherent nerve of the simplicial category of vertical models of T is equivalent to the quategory $\mathrm{Model}(T)$. More generally, if α is a regular cardinal and T is an α-cartesian theory, we shall say that a left fibration $E \to T$ is a *vertical model* of T if the quategory E is α-cartesian and the map $E \to T$ is α-continuous. A map $f : T \to \mathcal{K}$ is a model of T iff the left fibration $el(f) \to T$ is a vertical model. The model category $(\mathbf{S}/T, \mathrm{Wcov})$ admits a Bousfield localisation whose fibrant objects are the vertical models of T. The coherent nerve of the simplicial category of vertical models of T is equivalent to the quategory $\mathrm{Model}_\alpha(T)$.

31. Sifted colimits

31.1. This notion of sifted category was introduced by C. Lair in [La] under the name of categorie *tamisante*. We shall say that a simplicial set A is *weakly directed* or *sifted* if the colimit map

$$\varinjlim : \mathcal{K}^A \to \mathcal{K}$$

preserves finite products. More generally, if α is a regular cardinal, we shall say that a simplicial set A is is *weakly α-directed* is *α-sifted* if the colimit map

$$\varinjlim : \mathcal{K}^A \to \mathcal{K}$$

preserves α-products.

31.2. The notion of sifted simplicial set is invariant under Morita equivalence. A directed simplicial set is sifted. A quategory with finite coproducts is sifted. A sifted simplicial set is weakly contractible. A non-empty simplicial set A is sifted iff the diagonal $A \to A \times A$ is final. A non-empty quategory A is sifted iff the simplicial set $a\backslash A \times_A b\backslash A$ defined by the pullback square

$$\begin{array}{ccc} a\backslash A \times_A b\backslash A & \longrightarrow & b\backslash A \\ \downarrow & & \downarrow \\ a\backslash A & \longrightarrow & A \end{array}$$

is weakly contractible for any pair of objects $a, b \in A$. The category Δ^o is sifted. More generally, let α be a regular cardinal. The notion of α-sifted simplicial set is invariant under Morita equivalence. An α-directed simplicial set is α-sifted. A quategory with α-coproducts is α-sifted. A simplicial set A is α-sifted iff the diagonal $A \to A^S$ is final for every set S of cardinality $< \alpha$. A quategory A is sifted iff for every set S of cardinality $< \alpha$ and every family of objects $(a_i) \in A^S$

the (fiber) product over A of the left fibration $a_i\backslash A \to A$ has a total space which is contractible.

31.3. A simplicial set A is sifted iff the canonical map $A \to \mathrm{Cup}(A)$ is final, where $\mathrm{Cup}(A)$ is the free cocompletion of A under finite coproducts 23.24. More generally, if α is a regular cardinal, then a simplicial set A is α-sifted iff the canonical map $A \to \mathrm{Cup}_\alpha(A)$ is final, where $\mathrm{Cup}_\alpha(A)$ is the free cocompletion of A under finite coproducts 23.24.

31.4. We shall say that a diagram $d : A \to X$ in a quategory X is *sifted* if the indexing simplicial set A is sifted, in which case we shall say that the colimit of d is *sifted* if it exists. We shall say that a quategory X *has* sifted colimits if every (small) sifted diagram $A \to X$ has a colimit. We shall say that a map between two quategories is *fair* if it preserves sifted colimits. More generally, if α is a regular cardinal, we shall say that a diagram $d : A \to X$ in a quategory X is α*-sifted* if A is α-sifted, in which case we shall say that the colimit of d is α*-sifted* if it exists. We shall say that a quategory X *has* α-sifted colimits if every (small) α-sifted diagram $A \to X$ has a colimit. We shall say that a map between two quategories is α*-fair* if it preserves α-sifted colimits.

31.5. A quategory with sifted colimits and finite coproducts is cocomplete. See 19.37. A fair map between cocomplete quategories is cocontinuous iff it preserves finite coproducts. More generally, let α be a regular cardinal. A quategory with α-directed colimits and α-coproducts is cocomplete. An α-fair map between cocomplete quategories is cocontinuous iff it preserves α-coproducts.

31.6. A finitary map between cocomplete quategories is fair iff it preserves Δ^o-indexed colimits. See [] for a proof.

31.7. We say that a prestack $g \in \mathcal{P}(A)$ is *weakly inductive*, if the simplicial set A/g (or $El(g)$) is sifted. We shall denote by $\mathrm{Wind}(A)$ the full sub-quategory of $\mathcal{P}(A)$ spanned by the weakly inductive objects and by $y : A \to \mathrm{Wind}(A)$ the map induced by the Yoneda map $A \to \mathcal{P}(A)$. The quategory $\mathrm{Wind}(A)$ is closed under sifted colimits and the map $y : A \to \mathrm{Wind}(A)$ exibits the quategory $\mathrm{Wind}(A)$ as the free cocompletion of A under sifted colimits. More precisely, let us denote by $\mathrm{Fair}(X, Y)$ the quategory of fair maps $X \to Y$ between two quategories. Then the map

$$y^* : \mathrm{Fair}(\mathrm{Wind}(A), X) \to X^A$$

induced by y is an equivalence of quategories for any quategory with sifted colimits X. The inverse equivalence associates to a map $g : A \to X$ its left Kan extension $g_! : \mathrm{Wind}(A) \to X$ along u. More generally, if α is a regular cardinal, we shall say that a prestack $g \in \mathcal{P}(A)$ is *weakly* α*-inductive* if the simplicial set A/g (or $El(g)$) is α-sifted. We shall denote by $\mathrm{Wind}_\alpha(A)$ the full sub-quategory of $\mathcal{P}(A)$ spanned by the weakly α-inductive objects and by $y : A \to \mathrm{Wind}_\alpha(A)$ the map induced by the Yoneda map $A \to \mathcal{P}(A)$. The quategory $\mathrm{Wind}_\alpha(A)$ is closed under α-sifted colimits and the map $y : A \to \mathrm{Wind}_\alpha(A)$ exibits the quategory $\mathrm{Wind}_\alpha(A)$ as the free cocompletion of A under α-sifted colimits.

31.8. A quategory A has α-sifted colimits iff the map $y : A \to \mathrm{Wind}_\alpha(A)$ has a left adjoint.

31.9. We shall say that a quategory is *weakly accessible* if it is equivalent to a quategory $\mathrm{Wind}_\alpha(A)$ for for some regular cardinal α and and some small quategory A. More precisely, we shall say that a quategory is an *weakly α-accessible* if it is equivalent to $\mathrm{Wind}_\alpha(A)$. We shall say that a quategory is a *weakly finitary accessible* if it is weakly ω-accessible, that is, if if it is equivalent to a quategory $\mathrm{Wind}(A)$ for a small quategory A.

31.10. If A is a small quategory with finite coproducts, then the quategory $\mathrm{Wind}(A)$ is cocomplete and the map $y : A \to \mathrm{Wind}(A)$ preserves finite coproducts; a prestack $f : A^o \to \mathcal{K}$ is weakly inductive iff it preserves finite products. Moreover, the map $y : A \to \mathrm{Wind}(A)$ exibits the quategory $\mathrm{Wind}(A)$ as the free cocompletion of A. More precisely, let us denote the quategory of maps preserving finite coproducts between two quategories by $\mathrm{fCprod}(X, Y)$. Then the map

$$y^* : \mathrm{CC}(\mathrm{Wind}(A), X) \to \mathrm{fCprod}(A, X)$$

induced by y is an equivalence of quategories for any quategory with finite coproducts X. The inverse equivalence associates to a map which preserves finite coproducts $f : A \to X$ its left Kan extension $f_! : \mathrm{Wind}(A) \to X$ along y. More generally, if A is a small quategory with α-coproducts, then the quategory $\mathrm{Wind}_\alpha(A)$ is cocomplete and the map $y : A \to \mathrm{Wind}_\alpha(A)$ preserves α-coproducts; a prestack $f : A^o \to \mathcal{K}$ is weakly α-inductive iff it preserves α-products. Moreover, the map $y : A \to \mathrm{Wind}_\alpha(A)$ exibits the quategory $\mathrm{Wind}_\alpha(A)$ as the free cocompletion of A.

31.11. If A is a small simplicial set, let us denote by $\mathrm{Cup}(A)$ the free cocompletion of A under finite coproducts. Then the left Kan extension of the inclusion $i : \mathrm{Cup}(A) \subseteq \mathcal{P}(A)$ is an equivalence of quategories,

$$\mathrm{Wind}(\mathrm{Cup}(A)) \to \mathcal{P}(A).$$

More generally, if α is a regular cardinal, let us denote by $\mathrm{Cup}_\alpha(A)$ the free cocompletion of A under α-coproducts. The left Kan extension of the inclusion $i : \mathrm{Cup}_\alpha(A) \subseteq \mathcal{P}(A)$ is an equivalence of quategories,

$$\mathrm{Wind}_\alpha(\mathrm{Cup}_\alpha(A)) \to \mathcal{P}(A).$$

31.12. Let X be a (locally small) quategory with sifted colimits. We shall say that an object $a \in X$ is *perfect* if the map

$$hom(a, -) : X \to \mathcal{K}$$

is fair. More generally, let α be a regular cardinal and X be a quategory with α-siftted colimits. We shall say that an object $a \in X$ is *α-perfect* if the map $hom_X(a, -)$ is α-fair.

31.13. An object is 0-perfect iff it is 0-compact iff it is atomic. An object is 1-perfect iff it is 1-compact.

31.14. A perfect object is compact. The class of perfect objects is closed under finite coproducts and retracts. An object $x \in \mathcal{K}$ is perfect iff it is discrete and finite. If A is a simplicial set, then a prestack $g \in \mathcal{P}(A)$ is perfect iff it is a finite coproduct of atomic prestacks. A prestack $g \in \mathrm{Wind}(A)$ is perfect iff it is atomic. More generally, let α be a regular cardinal. The class of α-perfect objects is closed under α-coproducts and retracts. An object $x \in \mathcal{K}$ is 0-perfect iff it is contractibe, and it is 1-perfect iff it is contractible or empty. If A is a simplicial set, then a prestack $g \in \mathcal{P}(A)$ is perfect iff it is an α-coproduct of atomic prestacks. A prestack $g \in \mathrm{Wind}_\alpha(A)$ is perfect iff it is atomic. Hence the quategory α-perfect objects of a weakly α-accessible quategory is essentially small.

31.15. A compact object of a cocomplete quategory X is perfect iff the map

$$hom(a, -) : X \to \mathcal{K}$$

preserves Δ^o-indexed colimits.

31.16. Let X be a (locally small) quategory with sifted colimits. Then X is weakly finitary accessible iff its subcategory of perfect objects is essentially small and every object in X is a sifted colimit of a diagram of perfect objects. More precisely, if $K \subseteq X$ is a small full sub-quategory of perfect objects, then the left Kan extension

$$i_! : \mathrm{Wind}(K) \to X$$

of the inclusion $i : K \subseteq X$ is fully faithful. Moreover, $i_!$ is an equivalence if every object of X is a sifted colimit of a diagram of objects of K. More generally, let α be a regular cardinal and X be a quategory with α-directed colimits. Then X is weakly α-accessible iff its subcategory of α-perfect objects is essentially small and every object in X is an α-sifted colimit of a diagram of α-perfect objects. More precisely, if $K \subseteq X$ is a small full sub-quategory of α-perfect objects, then the left Kan extension

$$i_! : \mathrm{Wind}_\alpha(K) \to X$$

of the inclusion $i : K \subseteq X$ is fully faithful. Moreover, $i_!$ is an equivalence if every object of X is an α-sifted colimit of a diagram of objects of K.

31.17. A cocomplete (locally small) quategory X is weakly finitary accessible iff it is generated by a set of perfect objects. More precisely, let K be a small full sub-quategory of perfect objects of X. If K is closed under finite coproducts, then the left Kan extension

$$i_! : \mathrm{Wind}(K) \to X$$

of the inclusion $i : K \subseteq X$ along $y : K \to: \mathrm{Wind}(K)$ is fully faithful and cocontinuous. Moreover, $i_!$ is an equivalence if in addition K generates or separates X. More generally, if α is a regular cardinal, then a cocomplete quategory X is weakly α-accessible iff it is generated by a small set of α-perfect objects. More precisely, let $K \subseteq X$ be a small full sub-quategory of α-perfect objects. If K is closed under α-colimits, then the quategory $\mathrm{Ind}_\alpha(K)$ is cocomplete and the left Kan extension

$$i_! : \mathrm{Wind}_\alpha(K) \to X$$

of the inclusion $i : K \to X$ along $y : K \to\to \mathrm{Wind}_\alpha(K)$ is fully faithful and cocontinuous. Moreover, $i_!$ is an equivalence if K generates or separates X.

31.18. Let $f : X \leftrightarrow Y ; g$ be a pair of adjoint maps between cocomplete quategories. If g is fair, then f preserves perfect objects, and the converse is true if X is weakly finitary accessible.. More generally, if g is α-fair, then f preserves α-perfect objects, and the converse is true if X is weakly α-accessible.

32. Algebraic theories and theaters

Universal algebra was formulated in categorical terms by Lawvere [La]. It was extended to homotopy invariant algebraic structures by Boardman and Vogt [BV] and more recently by Badzioch [**?**] and Bergner [B4]. Here we further extends this theory to quategories. The quategory of models of an algebraic theory is called a *theater*. A quategory is a theater iff it is generated by a set of perfect objects.

32.1. There is a classical theory of simplicialy enriched algebraic theories and their models. See [**?**] See 51.16 and 51.17 for a comparaison with the theory presented in this section.

32.2. We shall say that a small quategory with finite products T is a (finitary) *algebraic theory*. A *model* of T in a quategory X (possibly large) is a map $T \to X$ which preserves finite products. We also say that a model $T \to X$ is an *algebra* of T in X. We shall denote by $\mathrm{Alg}(T, X)$, or by $T(X)$, the full simplicial subset of X^T spanned by the algebras $T \to X$ and we shall write

$$\mathrm{Alg}(T) = \mathrm{Alg}(T, \mathcal{K}).$$

A *morphism* of algebraic theories $S \to T$ is a model $S \to T$. The identity morphism $T \to T$ is the *generic* or *tautological* model of T. We shall denote by **ALG** the category of algebraic theories and morphisms. More generally, if α is a regular cardinal, we shall say that a small quategory T is α-*algebraic* if it has α-products. A *model* of T in a quategory X (possibly large) is a map $T \to X$ which preserves α-products. We shall say that a model $T \to X$ is an *algebra* of T in X. We shall denote by $\mathrm{Alg}_\alpha(T, X)$, or by $T(X)$, the full simplicial subset of X^T spanned by the models $T \to X$ and we shall write

$$\mathrm{Alg}_\alpha(T) = \mathrm{Alg}_\alpha(T, \mathcal{K}).$$

A *morphism* of α-algebraic theories $S \to T$ is a model $S \to T$. We shall denote by $\mathbf{ALG}_\alpha$ the category of α-algebraic theories and morphisms.

32.3. We shall say that a quategory X is a *theater* if it is equivalent to a quategory $\mathrm{Alg}_\alpha(T)$ for some α-algebraic theory T (and some regular cardinal α). More precisely, we shall say that X is an α-*theater* if it is equivalent to a quategory $\mathrm{Alg}_\alpha(T)$. We shall say that X is a *finitary theater* if it is an ω-theater.

32.4. Every small quategory is a 0-algebraic theory. A small quategory with terminal object is a 1-algebraic theory. For example, the category *Split* generated by two objects a and b and by two morphisms $i : a \to b$ and $r : b \to a$ such that $ri = 1_a$ has a terminal object a. A model of $Split \to X$ is a pointed object in X. Thus $\mathrm{Alg}_1(Split, X) \simeq 1\backslash X$. An α-theater is also a β-theater for every regular cardinal $\beta \geq \alpha$. A quategory X is a 0-theater iff it is equivalent to a quategory $\mathcal{P}(A)$ for a small simplicial set A. The quategory $1\backslash\mathcal{K}$ of pointed Kan complexes is a 1-theater.

32.5. If T is an α-algebraic theory, then so is the category hoT and the map $T \to hoT$ preserves α-products. We shall say that a model $f : T \to \mathcal{K}$ is *discrete* if the object $f(x)$ is discrete for every object $x \in T$. The quategory $\mathrm{Alg}_\alpha(T, \mathcal{K}\langle 0 \langle)$ of discrete models of T is equivalent to the category $\mathrm{Alg}_\alpha(hoT, \mathbf{Set})$ of set valued models of hoT. We shall say that T is *discrete* if the map $T \to hoT$ is an equivalence of quategories. In other words, T is discrete iff the quategory T is 1-truncated.

32.6. If T is a finitary algebraic theory , then the inclusion $\mathrm{Alg}(T) \subseteq \mathcal{K}^T$ has a left adjoint and the quategory $\mathrm{Alg}(T)$ is finitary presentable. If $u : S \to T$ is a morphism of algebraic theories, then the map

$$u^* : \mathrm{Alg}(T) \to \mathrm{Alg}(S)$$

induced by u has a left adjoint $u_!$. More generally, if α is a regular cardinal and T is an α-algebraic theory, then the inclusion $\mathrm{Alg}_\alpha(T) \subseteq \mathcal{K}^T$ has a left adjoint and the quategory $\mathrm{Alg}_\alpha(T)$ is α-presentable. If $u : S \to T$ is a morphism of α-algebraic theories, then the map

$$u^* : \mathrm{Alg}_\alpha(T) \to \mathrm{Alg}_\alpha(S)$$

induced by u has a left adjoint $u_!$.

32.7. If T is an algebraic theory, then the map $y(a) = hom_T(a, -) : T \to \mathbf{K}$ is model for every object $a \in T$. We shall say that an algebra $f \in \mathrm{Alg}(T)$ is *representable* if it is isomorphic to a model $y(a)$ for some object $a \in T$. The map

$$y : T^o \to \mathrm{Alg}(T)$$

induced by the Yoneda map $T^o \to \mathcal{K}^T$ is fully faithful and it induces an equivalence between T^o and the full sub-quategory of $\mathrm{Alg}(T)$ spanned by the representable algebras. A model of T is a retract of a representable iff it is perfect. The full sub-quategory of perfect models of T is equivalent to the Karoubi envelope $Kar(T^o) = Kar(T)^o$. The quategory $\mathrm{Kar}(T)$ has finite products and the map

$$i^* : \mathrm{Alg}(Kar(T)) \to: \mathrm{Alg}(T)$$

induced by the inclusion $i : T \to \mathrm{Kar}(T)$ is an equivalence of quategories More generally, a morphism of algebraic theories $u : S \to T$ is a Morita equivalence iff the map $u^* : \mathrm{Alg}(T) \to \mathrm{Alg}(S)$ induced by u is an equivalence of quategories. More generally, if α is a regular cardinal and T is an α-algebraic theory, then the map $y(a) = hom_T(a, -) : T \to \mathbf{K}$ is model for every object $a \in T$. We shall say that an algebra $f \in \mathrm{Alg}_\alpha(T)$ is *representable* if it is isomorphic to a model $y(a)$ for some object $a \in T$. The map

$$y : T^o \to \mathrm{Alg}_\alpha(T)$$

induced by the Yoneda map $T^o \to \mathcal{K}^T$ is fully faithful and it induces an equivalence between T^o and the full sub-quategory of $\mathrm{Alg}_\alpha(T)$ spanned by the representable algebras. A model of T is a retract of a representable iff it is α-perfect. The full sub-quategory of $\mathrm{Alg}(T)$ spanned by the α-perfect models is equivalent to $Kar(T^o) = Kar(T)^o$. The quategory $\mathrm{Kar}(T)$ has α-products and the map

$$i^* : \mathrm{Alg}_\alpha(Kar(T)) \to: \mathrm{Alg}_\alpha(T)$$

induced by the inclusion $i : T \to \mathrm{Kar}(T)$ is an equivalence of quategories. More generally, a morphism of algebraic theories $u : S \to T$ is a Morita equivalence iff the map $u^* : \mathrm{Alg}(T) \to \mathrm{Alg}(S)$ induced by u is an equivalence of quategories.

32.8. If T is a finitary algebraic theory, then the Yoneda map $y : T^o \to \mathrm{Alg}(T)$ preserves finite coproducts and it exibits the quategory $\mathrm{Alg}(T)$ as the free cocompletion of T^o. More precisely, let us denote by $\mathrm{fCprod}(X, Y)$ the quategory of maps preserving finite coproducts between two quategories X and Y. Then the map

$$y^* : \mathrm{CC}(\mathrm{Alg}(T), X) \to \mathrm{fCprod}(T^o, X)$$

induced by y is an equivalence for any cocomplete quategory X. The inverse equivalence associates to a map preserving finite coproducts $f : T^o \to X$ its left Kan extension $f_! : \mathrm{Mod}(T) \to X$ along y. More generally, let α be a regular cardinal and T be an α-algebraic theory. Then the Yoneda map $y : T^o \to \mathrm{Alg}_\alpha(T)$ preserves α-coproducts and it exibits the quategory $\mathrm{Alg}_\alpha(T)$ as the free cocompletion of T^o.

32.9. Remark. : It T is a finitary algebraic theory, then we have

$$\mathrm{Alg}(T) = \mathrm{Wind}(T^o)$$

since a map $f : T \to \mathcal{K}$ preserves finite products iff its quategory of elements is sifted. More generally, if α is a regular cardinal, then we have $\mathrm{Alg}_\alpha(T) = \mathrm{Wind}_\alpha(T^o)$ for any α-algebraic theory T.

32.10. The forgetful functor $\mathbf{ALG} \to \mathbf{S}$ admits a left adjoint which associates to a simplicial set A a finitary algebraic theory $\mathcal{O}[A]$ equipped with a map $u : A \to \mathcal{O}[A]$. By definition, for any quategory with finite products X, the map $u^* : \mathrm{Alg}(\mathcal{O}[A], X) \to X^A$ induced by u is an equivalence. This means that $\mathcal{O}[A]$ is the algebraic theory of A-diagrams. In particular, $\mathcal{O}[I]$ is the algebraic theory of maps. The algebraic theory $\mathcal{O} = \mathcal{O}[1]$ is freely generated by one object $u \in \mathcal{O}$. The quategory $\mathcal{O}[A]$ is the opposite of the quategory $\mathrm{Cup}(A^o)$ described in 23.24. The quategory $\mathcal{O}$ is equivalent to the opposite of the category of finite cardinals $\underline{N}$. The equivalence $\underline{N}^o \to \mathcal{O}$ takes a natural number $n \geq 0$ to the object $u^n \in \mathcal{O}$. More generally, if α is a regular cardinal, then the forgetful functor $\mathbf{ALG}_\alpha \to \mathbf{S}$ admits a left adjoint which associates to a simplicial set A an α-algebraic theory $\mathcal{O}_\alpha[A]$ equipped with a map $u : A \to \mathcal{O}_\alpha[A]$. The algebraic theory $\mathcal{O}_\alpha = \mathcal{O}_\alpha[1]$ is freely generated by one object $u \in \mathcal{O}_\alpha$.

32.11. Notice that $\mathcal{O}_0[A] = \mathcal{C}_0[A] = A$ and that $\mathcal{O}_1[A] = \mathcal{C}_1[A] = A \star 1$ for any small quategory A. In particular, $\mathcal{O}_0 = 1$ and $\mathcal{O}_1 = I$.

32.12. We shall say that an object a of a finitary algebraic theory T is a *power generator* if every object of T is isomorphic to a power a^n for some $n \geq 0$. Then the *forgetful map*

$$a^* : \mathrm{Alg}(T) \to \mathcal{K}$$

defined by putting $a^*(f) = f(a)$ is conservative. We shall say that a finitary algebraic theory is *unisorted* if it is equipped with a a power generator. For any object a of a finitary algebraic theory T there is a morphism $f : \mathcal{O} \to T$ such that $f(u) = a$; the object a is a power generator iff the map f is essentially surjective. If S is a set, we shall say that a finitary algebraic theory T is *S-multisorted* if it is equipped with an essentially surjective morphism $s : \mathcal{O}[S] \to T$. in which case the corresponding *forgetful map*

$$s^* : \mathrm{Alg}(T) \to \mathcal{K}^S$$

is conservative. More generally, if α is a regular cardinal, we shall say that an object a of an α-algebraic theory T is a *power generator* if every object of T is isomorphic to a power a^E for some set E of cardinality $< \alpha$. We shall say that an α-algebraic theory is *uni-sorted* if it is equipped with a a power generator. More generally, if S is a set, we shall say that an α-algebraic theory T is *S-multisorted* if it is equipped with an essentially surjective morphism $s : \mathcal{O}_\alpha[S] \to T$.

32.13. We shall say that a projective cone $c : 1 \star K \to A$ in a simplicial set A is *discrete* if K is a discrete simplicial set. In this case c is the same thing as a family of arrows $(c(1) \to c(k)|k \in A)$ in A. We shall say that a discrete cone $c : 1 \star K \to X$ with values in a quategory X is a *product cone* if it is exact. A *product sketch* is a limit sketch (A, P) in which P is a set of discrete projective cones. A product sketch (A, P) is *finitary* if every cone in P is finite. Every finitary product sketch (A, P) has a universal model $u : A \to \mathcal{O}[A/P]$ with values in a finitary algebraic theory called the *envelopping theory* of (A, P). The universality means that the map

$$u^* : \mathrm{Alg}(T, X) \to \mathrm{Model}(A/P, X)$$

induced by u is an equivalence for any quategory with finite products X. In particular, the map

$$u^* : \mathrm{Alg}(\mathcal{O}[A/P]) \to \mathrm{Model}(A/P)$$

induced by u is an equivalence of quategories. More generally, if α is a regular cardinal, then every α-bounded product sketch (A, P) has a universal model $u : A/P \to \mathcal{O}_\alpha[A/P]$ with values in an α-cartesian theory called the *envelopping theory* of (A, P).

32.14. If (A, P) is a product sketch, then by composing the Yoneda map $y : A^o \to \mathcal{K}^A$ with the left adjoint r to the inclusion $\mathrm{Model}(A/P) \subseteq \mathcal{K}^A$ we obtain a map

$$ry : A^o \to \mathrm{Model}(A/P).$$

We shall say that a model of (A, P) is *representable* if it belongs to the essential image of ry. We shall say that f is *free* if it is a coproduct of representables. If (A, P) is finitary we shall say that f is *finitely free* if it is a finite coproduct of representables. In this case we shall denote by $\mathrm{Model}(A/P)_{(f)}$ the full sub-quategory of $\mathrm{Model}(A/P)$ spanned by the finitely free models; More generally, if (A, P) is α-bounded we shall say that f is *α-free* if it is an α-coproduct of representables and we shall denote by $\mathrm{Model}(A/P)^{(\alpha)}$ the full sub-quategory of $\mathrm{Model}(A/P)$ spanned by the α-free models. The quategory $\mathcal{O}_\alpha[A/P]$ is the opposite of the quategory $\mathrm{Model}(A/P)^{(\alpha)}$ and the map $u : A \to \mathcal{O}_\alpha[A/P]$ is the opposite of the map $A^o \to \mathrm{Model}(A/P)^{(\alpha)}$ induced by ry.

32.15. If S and T are two finitary algebraic theories then so is the quategory $\mathrm{Alg}(S, T)$ of morphisms $S \to T$. The category of finitary algebraic theories **ALG** is simplicial and symmetric monoidal closed. The *tensor product* of two finitary algebraic theories S and T is defined to be the target of a map $\phi : S \times T \to S \odot T$ which preserves finite products in each variable and which is universal with respect to that property [BV]. More precisely, if X is a quategory with finite products, let us denote by $\mathrm{Alg}(S, T; X)$ the full simplicial subset of $X^{S \times T}$ spanned by the maps $S \times T \to X$ which preserves finite products in each variable. Then the map

$$\phi^* : \mathrm{Alg}(S \odot T, X) \to \mathrm{Alg}(S, T; X)$$

induced by ϕ is an equivalence for every quategory with finite products X. It follows that we have two canonical equivalence of quategories

$$\mathrm{Alg}(S \odot T, X) \simeq \mathrm{Alg}(S, \mathrm{Alg}(T, X)) \simeq \mathrm{Alg}(T, \mathrm{Alg}(S, X)).$$

In particular, we have two equivalences of quategories ,

$$\mathrm{Alg}(S \odot T) \simeq \mathrm{Alg}(S, \mathrm{Alg}(T)) \simeq \mathrm{Alg}(T, \mathrm{Alg}(S)).$$

The unit for the tensor product is the algebraic theory $\mathcal{O}$ described above. More generally, if α is a regular cardinal and S and T are two α-algebraic theories, then so is the quategory $\mathrm{Alg}_\alpha(S,T)$ of models $S \to T$. The category $\mathbf{ALG}_\alpha$ is simplicial and symmetric monoidal closed. The *tensor product* of two α-algebraic theories S and T is defined to be the target of a map $S \times T \to S \odot_\alpha T$ preserving α-products in each variable and universal with respect to that property. The unit for the tensor product is the theory $\mathcal{O}_\alpha$ described in 30.7.

32.16. If A and B are simplicial sets, then the morphism

$$\phi : \mathcal{O}[A] \odot \mathcal{O}[B] \to \mathcal{O}[A \times B]$$

defined by putting $\phi(a \odot b) = (a,b)$ for every pair of objects $(a,b) \in A \times B$ is an equivalence of quategories. Hence the functor

$$\mathcal{O}[-] : \mathbf{S} \to \mathcal{A}LG$$

preserves tensor products (where the tensor product on $\mathbf{S}$ is the cartesian product). If T is a finitary algebraic theory, we shall put

$$T[A] = \mathcal{O}[A] \odot T$$

Then for any quategory with finite product X we have two equivalences of quategories

$$\mathrm{Alg}(T[A], X) \simeq \mathrm{Alg}(T, X^A) \simeq \mathrm{Alg}(T, X)^A.$$

This shows that $T[A]$ is the algebraic theory of A-diagrams of models of T. In particular, $T[I]$ is the algebraic theory of maps between two models of T. More generally, if α is a regular cardinal, then the functor

$$\mathcal{O}_\alpha[-] : \mathbf{S} \to \mathbf{CT}_\alpha$$

preserves tensor products. If T is an α-cartesian theory, we shall put

$$T[A] = \mathcal{O}_\alpha[A] \odot_\alpha T$$

32.17. The cartesian product of a (small) family of finitary algebraic theories is a finitary algebraic theory. Hence the category $\mathbf{ALG}$ has cartesian products. The terminal quategory 1 is also the initial object of $\mathbf{ALG}$, since the quategory $\mathrm{Alg}(1,T)$ is equivalent to the quategory 1 for every algebraic theory T. Moreover, the cartesian product $S \times T$ of two algebraic theories is also their coproduct. More precisely, consider the maps

$$i_S : S \to S \times T \quad \text{and} \quad i_T : S \to S \times T$$

defined by putting $i_S(x) = (x,1)$ and $i_T(y) = (1,y)$ for every $x \in S$ and $y \in T$. Then the map

$$(i_S^*, i_T^*) : \mathrm{Alg}(S \times T, X) \to \mathrm{Alg}(S,X) \times \mathrm{Alg}(T,X)$$

induced by the pair (i_S, i_T) is an equivalence for any quategory with finite products X. This shows that the (simplicial) category $\mathbf{ALG}$ is semi-additive. More generally, the category $\mathbf{ALG}_\alpha$ has cartesian products for any regular cardinal α. The cartesian product of a family of α-algebraic theories indexed by a set of cardinality $< \alpha$ is also their coproduct.

32.18. Let T be a cartesian theory and X be a locally presentable quategory. Then the *external tensor product* of an algebra $f \in \mathrm{Alg}(T)$ with an object $x \in X$ is defined to be the model $f \otimes x : T \to X$ obtained by applying the left adjoint to the inclusion $\mathrm{Alg}(T, X) \subseteq X^T$ to the map $a \mapsto f(a) \cdot x$. The map $(f, x) \mapsto f \otimes x$ is cocontinuous in each variable and the induced map

$$\mathrm{Alg}(T) \otimes X \simeq \mathrm{Alg}(T, X)$$

is an equivalence of quategories. If S and T are two finitary algebraic theories, then the *external tensor product* of an algebra $f \in \mathrm{Alg}(S)$ with an algebra $g \in \mathrm{Alg}(T)$ is the algebra $f \otimes g \in \mathrm{Alg}(S \odot T)$ defined in 28.28. The map $(f, g) \mapsto f \otimes g$ is cocontinuous in each variable and the induced map

$$\mathrm{Alg}(S) \otimes \mathrm{Alg}(T) \simeq \mathrm{Alg}(S \odot T)$$

is an equivalence of quategories. More generally, if α be a regular cardinal, then the external tensor induces an equivalence of quategories

$$\mathrm{Alg}_\alpha(T) \otimes X \simeq \mathrm{Alg}_\alpha(T, X).$$

for any α-algebraic theory T and α-cartesian quategory X. In particular it induces the equivalence of quategories

$$\mathrm{Alg}_\alpha(S) \otimes \mathrm{Alg}_\alpha(T) \simeq \mathrm{Alg}_\alpha(S \odot_\alpha T)$$

of **??**.

32.19. The forgetful functor $\mathbf{CT} \to \mathbf{ALG}$ has a left adjoint which associates to an algebraic theory T the cartesian theory cT freely generated by T. The freeness means that the map

$$u^* : \mathrm{Model}(cT, X) \to \mathrm{Alg}(T, X)$$

induced by the canonical morphism $u; T \to cT$ is an equivalence for any cartesian quategory X. The map $u; T \to cT$ is the opposite of the canonical map $T^o \to \mathrm{Alg}(T)_f$, where $\mathrm{Alg}(T)_f$ is the quategory of finitely presented models of T. The functor $c(-)$ preserves the tensor product $\odot$. This means that the canonical map $cS \odot cT \to c(S \odot T)$ is an equivalence of quategories for any pair of finitary algebraic theories S and T. More generally, if α is a regular cardinal, then the forgetful functor $\mathbf{CT}_\alpha \to \mathbf{ALG}_\alpha$ has a left adjoint which associates to an α-algebraic theory T the α-cartesian theory cT freely generated by T. The functor $c(-)$ preserves the tensor product $\odot_\alpha$.

32.20. If a quategory X is a theater then so are the slice quategories $a\backslash X$ and X/a for any object $a \in X$, and the quategory $\mathrm{Alg}(T, X)$ for any finitary algebraic theory T. More precisely let α be a regular cardinal ≥ 1. If a quategory X is an α-theater then so are the quategories $a\backslash X$ and X/a for any object $a \in X$ and the quategory $\mathrm{Alg}_\alpha(T, X)$ for any α-cartesian theory T. If a quategory X is 0-theater then so is the quategory X/a for any object $a \in X$ and the quategory X^A for any simplicial set A.

32.21. Finite coproducts and sifted colimits commute in any finitary theater. More generally, α-coproducts and α-sifted colimits commute in any α-theater.

32.22. A cocomplete quategory X is a finitary theater iff it is generated by a small set of perfect objects. More precisely, let $K \subseteq X$ be a small full sub-quategory of perfect objects of X. If K is closed under finite coproducts, then the left Kan extension

$$i_! : \mathrm{Alg}(K^o) \to X$$

of the inclusion $i : K \to X$ along $y : K \to \mathrm{Alg}(K^o)$ is fully faithful and cocontinuous. Moreover, $i_!$ is an equivalence if K generates or separates X, More generally, if α is a regular cardinal, then a locally presentable quategory X is an α-theater iff it is generated by a small set of α-perfect objects. More precisely, let $K \subseteq X$ be a small full sub-quategory of α-perfect objects. If K is closed under α-coproducts then the left Kan extension

$$i_! : \mathrm{Alg}_\alpha(K^o) \to X$$

of the inclusion $i : K \to X$ along $y : K \to\to \mathrm{Alg}_\alpha(K^o)$ is fully faithful and cocontinuous. Moreover, $i_!$ is an equivalence if K generates or separates X.

32.23. A cocontinuous map between finitary theaters $f : X \to Y$ preserve perfect objects iff its right adjoint $Y \to X$ is fair. Let us denote by $\mathbf{AR}_{[\omega]}$ the category whose objects are locally presentable quategories and whose morphisms are cocontinuous maps preserving perfect objects. If $u : S \to T$ is a morphism of finitary algebraic theories, then the map

$$u_! : \mathrm{Alg}(S) \to \mathrm{Alg}(T)$$

preserve perfect objects (ie takes a perfect objects to a perfect object). The resulting functor

$$\mathrm{Alg} : \mathbf{ALG} \to \mathbf{AR}_{[\omega]}$$

has a right adjoint pf^o which associates to X the opposite of its sub quategory of perfect objects $pf(X)$ (or a small quategory equivalent to it). The quategory $pf(X)$ is Karoubi complete and the counit of the adjunction $\epsilon_X : \mathrm{Alg}(pf^o(X)) \to X$ is fully faithful; and it is an equivalence iff X is a finitary theater. The unit of the adjunction $\eta_T : T \to pf^o(\mathrm{Alg}(T))$ is a Morita equivalence; and it is an equivalence iff T is Karoubi complete. Hence the adjoint pair Model $\vdash pf^o$ induces an equivalence between the full subcategory of $\mathbf{ALG}$ spanned by the Karoubi complete theories and the full sub category of $\mathbf{AR}_{[\omega]}$ spanned by the finitary theaters. More generally, if α is a regular cardinal, then a cocontinuous map betwen α-theaters $f : X \to Y$ preserves α-perfect objects iff its right adjoint $g : Y \to X$ is α-fair. Let us denote by $\mathbf{AR}_{[\alpha]}$ the category whose objects are the locally presentable quategories and whose morphisms are the cocontinuous maps preserving α-perfect objects. If $u : S \to T$ is a morphism of α-algebraic theories, then the map

$$u_! : \mathrm{Alg}_\alpha(S) \to \mathrm{Alg}_\alpha(T)$$

preserves α-perfect objects. The resulting functor

$$\mathrm{Alg}_\alpha : \mathbf{ALG}_\alpha \to \mathbf{AR}_{[\alpha]}$$

has a right adjoint pf^o_α which associates to X the opposite of its sub-quategory $pf^o_\alpha(X)$ of α-perfect objects (or a small quategory equivalent to it). The quategory $pf_\alpha(X)$ is Karoubi complete and the counit of the adjunction $\epsilon_X : \mathrm{Alg}_\alpha(pf^o_\alpha(X)) \to X$ is fully faithful; and it is an equivalence iff X is an α-theater. The unit of the adjunction $\eta_T : T \to pf^o_\alpha(\mathrm{Alg}_\alpha(T))$ is a Morita equivalence; and it is an equivalence iff T is Karoubi complete. Hence the adjoint pair $\mathrm{Model}_\alpha \vdash pf^o_\alpha$ induces an equivalence between the full subcategory of $\mathbf{ALG}_\alpha$ spanned by the Karoubi complete theories and the full sub category of $\mathbf{AR}_{[\alpha]}$ spanned by the α-theaters.

32.24. A *monoid* in a quategory with finite products X is defined to be a simplicial object $M : \Delta^o \to X$ which satisfies the following Segal condition:

- $M_0 \simeq 1$;
- the "edge morphism" $M_n \to M_1^n$ defined from the inclusions $(i-1, i) \subseteq [n]$ $(1 \leq i \leq n)$ is invertible for every $n \geq 2$.

The notion of monoid is algebraic and finitary. We shall denote the algebraic theory of monoids by Mon. The theory Mon is discrete and the opposite quategory Mon^o is equivalent to the category of finitely generated free monoids in **Set**. A *n-fold monoid* is defined to be a model of the tensor power $Mon^n = Mon^{\odot n}$, where $n \geq 1$, In topology, a n-fold monoid is called an *E_n-space*. See [BFV]. A 2-fold monoid is called a *braided monoid*. The theory Mon is unisorted and from the canonical morphism $u : O \to Mon$ we obtain a morphism $u_n = u \odot M^n : Mon^n \to Mon^{n+1}$ for every $n \geq 0$. The (homotopy) colimit of the infinite sequence ot theories,

$$O \xrightarrow{u_0} Mon \xrightarrow{u_1} Mon^2 \xrightarrow{u_2} Mon^3 \xrightarrow{u_3} \cdots,$$

is the theory of *coherently commutative monoid* $CMon$. A model of this theory is called an *E_∞-space*. We shall denote the quategory $\mathrm{Alg}(CMon)$ of E_∞-spaces by $\mathcal{E}_\infty$.

32.25. The theory of coherently commutative monoids $CMon$ is 2-truncated. This is because the free models of $CMon$ are $K(\pi, 1)$-spaces. In other words, the quategory $CMon$ is equivalent to a category enriched over groupoids. More precisely, let us denote by $\underline{N}$ the category of finite cardinals and maps. If $a \in CMon$ denotes the generating object, then $CMon(a^m, a^n)$ is the groupoid of isomorphisms of the category of functors $\underline{N}^m \to \underline{N}^n$ which preserve finite coproducts. Notice the equivalence of categories $\underline{N}^n \simeq \underline{N}/\underline{n}$ where $\underline{n} = \{1, \cdot, n\}$. The groupoid $CMon(a^m, a^n)$ is equivalent to the groupoid of isomorphisms of the category $\underline{N}/\underline{m} \times \underline{n}$. The equivalence associates to a span

$$\underline{m} \xleftarrow{\;s\;} S \xrightarrow{\;t\;} \underline{n}$$

the functor $S_! : \underline{N}/\underline{m} \to \underline{N}/\underline{n}$ obtained by putting $S_!(X) = X \times_m S$.

32.26. The notion of coherently commutative monoid can be defined by a product sketch (Γ, C) introduced by Segal in [S2], where Γ denotes the category of finite pointed sets and basepoint preserving maps. For every $n \geq 0$, let us put $\underline{n} = \{1, \cdot, n\}$ and $n_+ = \underline{n} \sqcup \{\star\}$. The set n_+ is pointed with base point $\star$. For each $k \in \underline{n}$ let δ_k be the map $: n_+ \to 1_+$ which takes the value 1 at k and $\star$ elsewere. The family of maps $(\delta_k : k \in n)$ defines a discrete cone $c_n : 1 \star n \to \Gamma$. The sketch (Γ, C) is then defined by putting $C = \{c_n : n \geq 0\}$. Consider the functor $i : \Delta^o \to \Gamma$ obtained by putting $i[n] = Hom(\Delta[n], S^1)$ for every $n \geq 0$, where $S^1 = \Delta[1]/\partial\Delta[1]$ is the pointed circle. If X is a quategory with finite products, then the map $X^i :: X^\Gamma \to X^{\Delta^o}$ takes a model of (Γ, C) to a monoid $i^*(E) : \Delta^o \to X$ (the monoid underlying E).

More generally, for every $n \geq 1$, consider the functor $i_n : (\Delta^n)^o \to \Gamma$ obtained by composiing

$$(\Delta^n)^o \xrightarrow{i^n} \Gamma^n \xrightarrow{\wedge^n} \Gamma,$$

where $\wedge^n$ denotes the n-fold smash product functor. Then the map X^{i_n} takes a model of (Γ, C) to a n-fold monoid $i_n^*(E) : (\Delta^n)^o \to X$. We saw above in that the quategory $CMon$ is equivalent to a 2-category enriched over groupoids M. Hence the sketch (Γ, C) admits a universal model $u : \Gamma/C \to M$. The pseudo-functor u associates to a pointed map $f : m_+ \to n_+$, the functor $f_! : \underline{N}^m \to \underline{N}^n$ defined by putting

$$f_!(X)(j) = \bigsqcup_{f(i)=j} X(i)$$

for every $j \in \underline{n}$.

32.27. A *monoidal quategory* is a simplicial object $M : \Delta^o \to \mathbf{QCat}$ satisfying the following *Segal condition*:

- the canonical map $M_0 \to 1$ is a categorical equivalence;
- the edge map $M_n \to M_1^n$ is a categorical equivalence for every $n \geq 2$.

The category $\mathbf{S}^{\Delta^o}$ admits a Quillen model structure in which the fibrant objects are the Reedy fibrant monoidal quategory (where the Reedy model structure is defined from the model structure $(\mathbf{S}, \text{Wcat})$. The coherent nerve of the category of fibrant objects is equivalent to the quategory $Mon(\mathcal{Q}_1)$.

32.28. A *braided monoidal quategory* is a bisimplicial object $M : (\Delta \times \Delta)^o \to \mathbf{QCat}$ satisfying the Segal condition in each variable:

- the edge maps $M_{nm} \to M_{1m}^n$ and $M_{nm} \to M_{n1}^m$ are categorical equivalences for every $m, n \geq 1$.
- the canonical maps $M_{0n} \to 1$ and $M_{n0} \to 1$ are categorical equivalences for every $n \geq 0$.

More generally, a *n-fold monoidal quategory* is a n-fold simplicial object $M : (\Delta^n)^o \to \mathbf{QCat}$ satisfying the Segal condition in each variable.

32.29. A *symmetric monoidal quategory* is a functor $M : \Gamma \to \mathbf{QCat}$ satisfying the following *Segal condition*:

- the canonical map $M(n_+) \to M(1_+)^n$ is a categorical equivalence for every $n \geq 2$;
- the canonical map $M(0_+) \simeq 1$ is a categorical equivalence.

32.30. The tensor product of an n-fold monoid $N \in \text{Alg}(Mon^n)$ with an m-fold monoid $M \in \text{Alg}(Mon^m)$ is an $(n+m)$-fold monoid $N \otimes M \in \text{Alg}(Mon^{n+m})$. The tensor product is symmetric and it gives the disjoint union

$$\bigsqcup_{n \geq 0} \text{Alg}(Mon^n)$$

the structure of a symmetric monoidal quategory.

32.31. We shall say that a monoid $M : \Delta^o \to X$ is a *group* iff the morphism

$$(\sigma_1, \partial_0) : M_2 \to M_1 \times M_1$$

is invertible. The notion of group is algebraic and finitary. We shall denote the algebraic theory of groups by *Grp*. The theory *Grp* is discrete and the opposite category Grp^o is equivalent to the category of finitely free groups (in **Set**). The theory *Grp* is a multiplicative localisation of the theory of monoids Mon. We say that a n-fold monoid $M : Mon^n \to X$ is a *n-fold group* if its underlying monoid is a group. In topology, a n-fold group is called an *n-fold loop space*. The notion of n-fold group is algebraic and finitary. The algebraic theory of n-fold groups is the tensor power $Grp^n = Grp^{\odot n}$. The theory Grp^n is a multiplicative localisation of the theory Mon^n. A *coherently commutative group* or *coherently abelian group* is a coherently commutative monoid whose underlying monoid is a group. The notion of coherently abelian group is algebraic and finitary. W e shall denote the algebraic theory of coherently abelian groups by *CGrp*. The algebraic theory *CGrp* is a multiplicative localisation of the algebraic theory $CMon$. In topology, a model of *CGrp* is called an *infinite loop space*. We shall denote the quategory of infinite loop spaces by $\mathcal{L}_\infty$. Recall that the theory of groups *Grp* is unisorted; from the canonical morphism $u : O \to Grp$ we can define a morphism $u_n : Grp^n \to Grp^{n+1}$ for every $n \geq 0$. The algebraic theory *CGrp* is the (homotopy) colimit of the infinite sequence of theories,

$$O \xrightarrow{u_0} Grp \xrightarrow{u_1} Grp^2 \xrightarrow{u_2} Grp^3 \xrightarrow{u_3} \cdots .$$

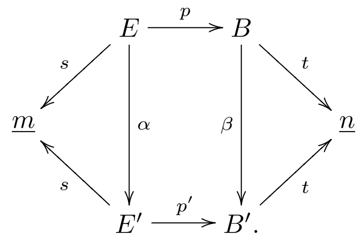
32.32. Recall that a *rig* is a ring without negative inverse (ie in which the additive structure is a commutative monoid). We now describe the algebraic theory $CRig$ of *coherently commutative rig*. In topology, a model of $CRig$ is an *E_∞-rig space*. The quategory $CRig$ is 2-truncated and equivalent to a 2-category enriched over groupoids. More precisely, let us denote by $\underline{N}$ the category of finite cardinals and maps. If $m, n \geq 0$, we shall say that a functor $f : \underline{N}^m \to \underline{N}^n$ is *polynomial* if we have

$$f(X)(j) = \coprod_{i \in B(j)} \prod_{k \in E(i)} X(l(k))$$

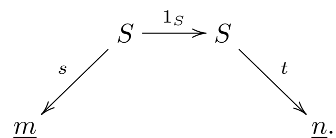
, where $B(j)$ and $E(i)$ are respectively the fibers of the maps r and p in a diagram of finite sets

$$\begin{array}{ccccc} & E & \xrightarrow{p} & B & \\ \swarrow^{l} & & & & \searrow^{r} \\ \underline{m} & & & & \underline{n} \end{array} .$$

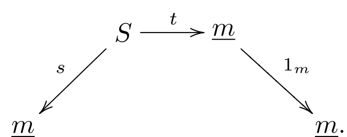
Notice that $f = r_! p_* l^*$, where l^* is the pullback functor along l, where p_* is the right adjoint to p^* and $r_!$ is the left adjoint to r^*. If $a \in CRig$ denotes the generating object, then $CRig(a^m, a^n)$ is the groupoid of isomorphisms of the category of polynomial functors $\underline{N}^m \to \underline{N}^n$. Let us denote by $Pol(m, n)$ the groupoid whose objects are the diagrams (l, E, p, B, r) as above and whose arrows $(l, E, p, B, r) \to (l', E', p', B', r')$ are the pair of bijections (α, β) in a commutative diagram



Then the functor $Pol(m,n) \to CRig(a^m, a^n)$ which associates to a diagram (l, E, p, B, r) the polynomial functor $f = r_! p_* l^*$ is an equivalence of groupoids. The algebraic theory $CMon$ admits two interpretations in the theory $CRig$. The *additive interpretation* $u : CMon \to CRig$ is induced by the functor which takes a span $(s,t) : S \to \underline{m} \times \underline{n}$ to the polynomial



The resulting map $u^* : \mathrm{Alg}(CRig) \to \mathrm{Alg}(CMon)$ takes a coherently commutative rig to its underlying additive structure. The *multiplicative interpretation* $v : CMon \to CRig$ is induced by the functor which takes a span $(s,t) : S \to \underline{m} \times \underline{n}$ to the polynomial



The resulting map $v^* : \mathrm{Alg}(CRig) \to \mathrm{Alg}(CMon)$ takes a coherently commutative rig to its underlying mutiplicative structure. A (coherently commutative) *ring* is defined to be is a coherently commutative rig whose underlying additive structure is a group. We shall denote by $CRing$ the algebraic theory of coherently commutative rings.

32.33. The notion of homotopy factorisation system in the category **ALG** is defined as in 28.29. The category **ALG** admits a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of essentially surjective morphisms and $\mathcal{B}$ the class of fully faithful morphisms. More generally, if α is a regular cardinal, then the category $\mathbf{ALG}_\alpha$ admits a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of essentially surjective morphisms and $\mathcal{B}$ the class of fully faithful morphisms. A morphism $u : S \to T$ in $\mathbf{ALG}_\alpha$ is fully faithful, iff the map $u_! : \mathrm{Alg}_\alpha(S) \to \mathrm{Alg}_\alpha(T)$ is fully faithful. The map $u^* : {}_\alpha(T) \to \mathrm{Alg}_\alpha(S)$ is conservative iff u is Morita surjective.

32.34. Let Σ be a set of morphisms in a finitary algebraic theory T. We shall say that Σ is *multiplicatively closed* if it is closed under finite products. In this case the quategory $L(T, \Sigma)$ has finite products and the canonical map $X \to L(T, \Sigma)$ preserves finite products. We shall say that a morphism of algebraic theories is a *quasi-localisation* (resp. *iterated quasi-localisation*) if it is a quasi-localisation (resp. an iterated quasi-localisation) as a map of quategories. The category **ALG** admits a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{B}$ is the class of conservative morphisms and $\mathcal{A}$ is the class of iterated quasi-localisations. This is true also of the category $\mathbf{ALG}_\alpha$.

32.35. The algebraic theory *Grp* is a quasi-localisation of Mon. The algebraic theory *CGrp* is a quasi-localisation of $CMon$. The algebraic theory *CRing* is a quasi-localisation of $CRig$.

32.36. Every algebraic theory T is the quasi-localisation of a free theory $\mathcal{O}[C]$, where C is a category.

32.37. The initial model of an algebraic theory T is representable by its terminal object $1 \in T$. We shall say that a morphism of (finitary) algebraic theories $u : S \to T$ is *coinitial* if the map

$$u^* : \mathrm{Alg}(T) \to \mathrm{Alg}(S)$$

preserves initial algebras, that is, if $u^*(\bot) = \bot$. A morphism $u : S \to T$ is coinitial iff the map $S(1, x) \to T(1, ux)$ induced by u is a homotopy equivalence for every object $x \in S$. The category $\mathbf{ALG}$ admits a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{B}$ is the class of coinitial morphisms. We shall say that a morphism in the class $\mathcal{A}$ is *elementary.* For any model f of a finitary algebraic theory T there is an elementary morphism $i : T \to T[f]$ with an isomorphism $i^*(\bot) = f$. The map

$$\tilde{i}^* : \mathrm{Alg}(T[f]) \to f\backslash \mathrm{Alg}(T)$$

induced by the map $i^* : \mathrm{Alg}(T[f]) \to \mathrm{Alg}(T)$ is an equivalence of quategories. We shall say that $T[f]$ is the *envelopping theory* of the model f. More generally, if α is a regular cardinal > 0, then the category $\mathbf{ALG}_\alpha$ admits a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{B}$ is the class of coinitial morphisms. A morphism in the class $\mathcal{A}$ is said to be *elementary.* For any model f of an α-cartesian theory T there is an elementary morphism $i : T \to T[f]$ such that $i^*(\bot) = f$. The map

$$\tilde{i}^* : \mathrm{Alg}_\alpha(T[f]) \to f\backslash \mathrm{Alg}_\alpha(T)$$

induced by i^* is an equivalence of quategories. We shall say that $T[f]$ is the *envelopping theory* of the model f.

32.38. We say that a morphism of finitary algebraic theories $u : S \to T$ is *coterminal* if the map

$$u_! : \mathrm{Alg}(S) \to \mathrm{Alg}(T)$$

preserves terminal algebras, that is, if $u_!(\top) = \top$. The category $\mathbf{ALG}$ admits a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of coterminal morphisms. A morphism $u : S \to T$ belongs to $\mathcal{B}$ iff it is equivalent to a left fibration $E \to T$. For any algebra $f \in \mathrm{Alg}(T)$, the left fibration $p : el(f) \to T$ belongs to $\mathcal{B}$. Moreover, we have $p_!(\top) \simeq f$ and the map $\tilde{p}_! : \mathrm{Alg}(el(f)) \to \mathrm{Alg}(T)/f$ induced by the map $p_! : \mathrm{Alg}(el(f)) \to \mathrm{Alg}(T)$ is an equivalence of quategories. See [BJP]. More generally, if α is a regular cardinal, we say that a morphism of α-algebraic theories $u : S \to T$ is *coterminal* if the map

$$u_! : \mathrm{Alg}_\alpha(S) \to \mathrm{Alg}_\alpha(T)$$

preserves terminal objects. The category $\mathbf{ALG}_\alpha$ admits a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of coterminal morphisms. A morphism $u : S \to T$ belongs to $\mathcal{B}$ iff it is equivalent to a left fibration $E \to T$. For any algebra $f \in \mathrm{Alg}_\alpha(T)$, the left fibration $p : el(f) \to T$ belongs to $\mathcal{B}$. Moreover, we have $p_!(\top) \simeq f$

and the map $\tilde{p}_! : \mathrm{Alg}_\alpha(elf(f)) \to \mathrm{Alg}_\alpha(T)/f$ induced by $p_!$ is an equivalence of quategories.

32.39. A morphism of finitary algebraic theories $u : S \to T$ is coterminal iff the map $u_! : \mathrm{Alg}(S) \to: \mathrm{Alg}(T)$ is coterminal in **AR**. A morphism $u : S \to T$ is equivalent to a left fibration iff the map $u_!$ is equivalent to a right fibration. There is a similar result for a morphism of α-algebraic theories $u : S \to T$.

32.40. A map of simplicial sets $u : A \to B$ is initial iff the morphism $\mathcal{O}[u] : \mathcal{O}[A] \to \mathcal{O}[B]$ is coterminal. A map $u : A \to B$ is equivalent to a left fibration iff the morphism $\mathcal{O}[u] : \mathcal{O}[A] \to \mathcal{O}[B]$ is equivalent to a left fibration. There is a similar result for the morphism $\mathcal{O}_\alpha[u]$.

32.41. Some algebraic theories can be defined semantically. If X is a quategory with finite products, then the *algebraic theory of operations* on an object $z \in X$ is defined to be the full sub-quategory $Op(z)$ of X spanned by the objects z^n for $n \geq 0$. The theory $Op(z)$ is unisorted and generated by the object $z \in Op(z)$. If Y is a quategory with finite products, then so is the quategory Y^A for any simplicial set A. There is thus a (finitary) algebraic theory $Op(z)$ for any map $z : A \to Y$. The quategory $Op(z)$ can be small even when A and Y are large simplicial sets. This is true for example, when A is a locally small quategory and z is representable by an object $a \in A$. in this case the quategory $Op(z)$ is equivalent to the opposite of the full sub-quategory of A spanned by the objects $n \cdot a = \sqcup_n a$ for $(n \geq 0)$.

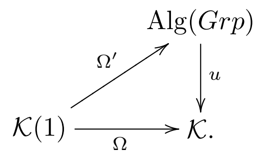
32.42. Let X be a locally small quategory and z be a map $X \to \mathcal{K}$. Let us assume that the quategory of operations $Op(z)$ is small. Then the map $X \to \mathcal{K}^{Op(z)}$ induced by the inclusion $Op(z) \to \mathcal{K}^X$ factors through the inclusion $\mathrm{Alg}(Op(z)) \subseteq \mathcal{K}^{Op(z)}$; it induces a map z' which fits in a commutative diagram

$$\begin{array}{ccc} & & \mathrm{Alg}(Op(z)) \\ & \overset{z'}{\nearrow} & \downarrow u \\ X & \xrightarrow[z]{} & \mathcal{K}, \end{array}$$

where u is the forgetful map. We shall say that the map z is *monadic* if z' is an equivalence of quategories. In this case z admits a left adjoint, since u admits a left adjoint; moreover, z is representable, since u is representable.

32.43. The quategory of pointed Kan complexes is equivalent to the quategory $1\backslash\mathcal{K}$. Let us compute the operations on the loop space map $\Omega : 1\backslash\mathcal{K} \to \mathcal{K}$. The map Ω is representable by the pointed circle $s^1\ in 1\backslash\mathcal{K}$. Hence the space of n-ary operations $\Omega^n \to \Omega$ is homotopy equjivalent to the space of pointed maps $s^1 \to \vee_n s^1$. The fundamental group of $\vee_n s^1$ is the free group on n-generators $F(n)$ by Van Kampen theorem. A wedge of circles is a $K(\pi, 1)$-space by a classical theorem. Hence the space of pointed maps $s^1 \to \vee_n s^1$ is homotopy equivalent to the set of group homomorphisms $F(1) \to F(n)$ equipped with the discrete topology. It follows that the algebraic theory $Op(\Omega)$ is discrete and equivalent to the usual theory of groups *Grp*.

32.44. Let $\mathcal{K}(1)$ be the quategory of pointed connected Kan complexes. The algebraic theory of operations on the loop space map $\Omega : \mathcal{K}(1) \to \mathcal{K}$ is equivalent to the theory of groups Grp. The map Ω is monadic by a theorem of Jon Beck. Hence the map Ω' in the following diagram is an equivalence of quategories,



It follows that the quategory $\mathcal{K}(1)$ is a finitary theater.

subsection If $n > 0$, let us denote by $\mathcal{K}(n)$ the quategory of pointed $(n-1)$-connected objects in $\mathcal{K}$. The n-fold loop space functor Ω^n induces an equivalence of quategories

$$\Omega^n : \mathcal{K}(n) \simeq \mathrm{Alg}(Grp^n).$$

It follows that the quategory $\mathcal{K}(n)$ is a finitary theater. The tensor product of an m-fold group $G \in \mathrm{Alg}(Grp^n)$ with an m-fold group $H \in \mathrm{Alg}(Grp^m)$ is an $(n+m)$-fold group $G \otimes H \in \mathrm{Alg}(Grp^{n+m})$. The tensor product is symmetric and it gives the disjoint union

$$\bigsqcup_{n \geq 0} \mathrm{Alg}(Grp^n)$$

the structure of a symmetric monoidal quategory. The smash product of an object $x \in \mathcal{K}(n)$ with an object $y \in \mathcal{K}(m)$ is an object $x \wedge y \in \mathcal{K}(n+m)$ and the canonical map $\Omega^n(x) \times \Omega^m(y) \to \Omega^{n+m}(x \wedge y)$ induces an isomorphism The smash product functor $\wedge : \mathcal{K}(n) \times \mathcal{K}(m) \to \mathcal{K}(n+m)$ is cocontinuous in each variable and it induces an equivalence of quategories,

$$\mathcal{K}(n) \otimes \mathcal{K}(m) \simeq \mathcal{K}(n+m).$$

It follows that the n-fold smash product functor induces an equivalence of quategories,

$$\mathcal{K}(1)^{\otimes n} \simeq \mathcal{K}(n)$$

for every $n > 0$.

32.45. The quategory of pointed quategories is equivalent to the quategory $1\backslash\mathcal{Q}_1$. If (X, x_0) is a pointed quategory, let us put $End(X, x_0) = X(x_0, x_0)$. This defines a functor

$$End : 1\backslash\mathcal{Q}_1 \to \mathcal{K}.$$

Let us compute the operations on the functor End. By construction we have $End(X) = Map(S^1, X)$ for any pointed quategory $X = (X, x_0)$, where S^1 is the (pointed) circle $\Delta[1]/\partial\Delta[1]$ and where Map is the simplicial set of pointed maps between pointed simplicial sets. This shows that the functor End is representable by a fibrant replacement of S^1 in the model category for pointed quategories. This fibrant replacement is the free monoid on one generator by 2.12. More generally, the free monoid on n-generators $M(n)$ is a fibrant replacement of $\vee_n S^1$. Hence the space of operations $End^n \to End$ is homotopy equivalent to the set of homomorphisms $M(1) \to M(n)$ equipped with the discrete topology. It follows that the algebraic theory $Op(End)$ is discrete and equivalent to the usual theory of monoids Mon.

32.46. Let us say that a quategory X is *strongly connected* if $\tau_0 X = 1$. A quategory is strongly connected iff it is equivalent to a quasi-monoid. Let us denote by $\mathcal{Q}_1(1)$ the quategory of pointed strongly connected quategories. It is easy to see, by using the same argument as above, that the algebraic theory of operations on the functor $End : \mathcal{Q}_1(1) \to \mathcal{K}$ is the theory of monoids Mon. The functor End is monadic and the functor End' in the folllowing diagram is an equivalence of quategories,

$$\begin{array}{ccc} & & \mathrm{Alg}(Mon) \\ & \overset{End'}{\nearrow} & \downarrow u \\ \mathcal{Q}_1(1) & \xrightarrow[End]{} & \mathcal{K}. \end{array}$$

32.47. Let us say that a finite category is *absolutely finite* if it is finitely generated as a quategory. For example, a finite poset is absolutely finite. When a category C is absolutely finite, the theories $\mathcal{O}[C]$ and $\mathcal{O}[C^o]$ are mutually dual in the symmetric monoidal category **ALG**. The counit of this duality is the map

$$\epsilon : \mathcal{O}[C] \odot \mathcal{O}[C^o] \to \mathcal{O}$$

induced by the opposite of the map $Hom_C : C^o \times C \to \underline{N} = \mathcal{O}^o$. Moreover, we have an equivalence of quategories $T[C^o] \simeq T^C$ for any algebraic theory T. It follows that we have an equivalence of quategories,

$$\mathrm{Alg}(S^{C^o}, T) \simeq \mathrm{Alg}(S, T^C)$$

for any $S, T \in$ **ALG**.

32.48. If X and Y are finitary theaters then the map

$$i^* : \mathrm{Fair}(X, Y) \to Y^{pf(X)}$$

induced by the inclusion $i : pf(X) \subseteq X$ is an equivalence. It follows that the simplicial sets $\mathrm{Fair}(X, Y)$ is a finitary theater. Let us denote by **FT** the category whose objects are the finitary theaters and whose morphisms are the fair maps. If $X, Y \in$ **FT** then $X \times Y \in$ **FT** and $pf(X \times Y) = pf(X) \times pf(Y)$. Thus, If $X, Y, Z \in$ **FT** then

$$\mathrm{Fair}(X \times Y, Z) \simeq Y^{pf(X \times Y)} \simeq Y^{pf(X) \times pf(Y))} \simeq \mathrm{Fair}(X, \mathrm{Fair}(Y, Z))$$

and this shows that the category **FT** is cartesian closed.

32.49. If T is a finitary algebraic theory, we shall say that a left fibration $E \to T$ is a *vertical algebra* over T if the quategory E has finite products and the map $E \to T$ preserves finite products. A map $f : T \to \mathcal{K}$ is a model of T iff the left fibration $el(f) \to T$ is a vertical algebra. The model category $(\mathbf{S}/T, \mathrm{Wcov})$ admits a Bousfield localisation whose fibrant objects are the vertical algebras over T. The coherent nerve of the simplicial category of vertical algebras over T is equivalent to the quategory $\mathrm{Alg}(T)$. More generally, if α is a regular cardinal, and T is an α-algebraic theory, we shall say that a left fibration $E \to T$ is a *vertical algebra* over T if the quategory E is has α-products and the map $E \to T$ preserves α-products. A map $f : T \to \mathcal{K}$ is a model of T iff the left fibration $el(f) \to T$ is a vertical algebra. The model category $(\mathbf{S}/T, \mathrm{Wcov})$ admits a Bousfield localisation whose fibrant objects are the vertical algebras over T. The coherent nerve of the simplicial category of vertical algebras over T is equivalent to the quategory $\mathrm{Alg}_\alpha(T)$.

32.50. See 51.16 and 51.17 for some aspects of the theory of simplicialy enriched algebraic theories.

33. Fiber sequences

33.1. Let X be a null-pointed quategory. Recall from 10.8 that X admits a *null flow* which associates to a pair of objects $x, y \in X$ a null morphism $m(x \star y) = 0 : x \to y$. We shall say that a 2-simplex $t \in X_2$

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ & {\scriptstyle 0}\searrow & \downarrow{\scriptstyle g} \\ & & c. \end{array}$$

with boundary $\partial t = (g, 0, f)$ is a *null sequence* and we shall write $\partial t = (g, 0, f) : a \to b \to c$. Let us denote by $Nul(X/g)$ the full simplicial subset of X/g spanned by the null sequences $t \in X$ with $\partial_0 t = g$. We shall say that a null sequence $t \in Nul(X/g)$ is a *fiber sequence* if it is a terminal object of the quategory X/g. We shall say that the arrow f of a fiber sequence $\partial t = (g, 0, f) : a \to b \to c$ is the *fiber* of the arrow g and we shall put $a = fib(g)$,

$$\begin{array}{ccc} fib(g) & \xrightarrow{f} & b \\ & {\scriptstyle 0}\searrow & \downarrow{\scriptstyle g} \\ & & c \end{array}$$

The *loop space* $\Omega(x)$ of an object $x \in X$ is defined to be the fiber of the morphism $0 \to x$,

$$\begin{array}{ccc} \Omega(x) & \xrightarrow{0} & 0 \\ & {\scriptstyle 0}\searrow & \downarrow{\scriptstyle 0} \\ & & x. \end{array}$$

Dually, let us denote by $Nul(f\backslash X)$ the full simplicial subset of $f\backslash X$ spanned by the null sequences $t \in X$ with $\partial_2 t = f$. We shall say that a null sequence $t \in Nul(f\backslash X)$ is a *cofiber sequence* if it is an initial object of the quategory X/g. We shall say that the arrow g of a cofiber sequence $\partial t = (g, 0, f) : a \to b \to c$ is the *cofiber* of the arrow g and we shall write $c = cofib(f)$,

$$\begin{array}{ccc} a & & \\ {\scriptstyle f}\downarrow & \searrow^{\scriptstyle 0} & \\ b & \xrightarrow{q} & cofib(f). \end{array}$$

The *suspension* $\Sigma(x)$ of an object $x \in X$ is defined to be the cofiber of the morphism $x \to 0$,

$$\begin{array}{ccc} x & \xrightarrow{0} & 0 \\ & {\scriptstyle 0}\searrow & \downarrow{\scriptstyle 0} \\ & & \Sigma(x). \end{array}$$

33.2. Let X be a cartesian quategory. We shall say that a functor $p : E \to C$ in $Cat(X)$ is a *left fibration* if the naturality square

$$\begin{array}{ccc} E_1 & \xrightarrow{s} & E_0 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_0} \\ C_1 & \xrightarrow{s} & C_0 \end{array}$$

is cartesian, where s is the source map. The notion of right fibration is defined dually with the target map. The two notions are equivalent when the category C is a groupoid. We shall denote by X^C the full simplicial subset of $Cat(X)/C$ spanned by the left fibrations $E \to C$. The pullback of a left fibration $E \to C$ along a functor $u : D \to C$ is a left fibration $u^*(E) \to D$.

33.3. Let X be a cartesian quategory. Recall from 37.1 that the Cech groupoid $Cech(u)$ of an arrow $u : a \to b$ in X is the image by the canonical map $X/b \to X$ of the the Cech groupoid of the object $u \in X/b$. The map $u : a \to b$ induces a functor $\tilde{u} : Cech(u) \to Sk^0(b)$. The lifted base change map

$$\tilde{u}^* : X/b \to X^{Cech(u)}.$$

associates to an arrow $e \to b$ the arrow $a \times_b e \to a$

$$\begin{array}{ccc} a \times_b e & \longrightarrow & e \\ \downarrow & & \downarrow \\ a & \xrightarrow{u} & b \end{array}$$

equipped with a natural action of the groupoid $Cech(u)$.

33.4. Let X be a cartesian quategory. The *loop group* $\Omega(b) = \Omega_u(b)$ of a pointed object $u : 1 \to b$ in X is the Cech groupoid of the arrow $u : 1 \to b$. The lifted base change map

$$\tilde{u}^* : X/b \to X^{\Omega_u(b)}$$

associates to an arrow $e \to b$ its fiber $e(u) = u^*(e)$ equipped with the natural action (say on the right) of the group $\Omega_u(b)$. In the special case where $p = u : 1 \to b$, this gives the natural right action of $\Omega_u(b)$ on itself. If $l : e' \to e$ is an arrow in X/b, then the arrow $u^*(l) : u^*(e') \to u^*(e)$ respects the right action by $\Omega_u(b)$. Suppose that we have a base point $v : 1 \to e$ over the base point $u : 1 \to b$. Then the arrow $\partial = u^*(v) : \Omega_u(b) \to e(u)$ respects the right action by $\Omega_u(b)$. The top square of the following commutative diagram is cartesian, since the bottom square and the boundary rectangle are cartesians,

$$\begin{array}{ccc} \Omega_u(b) & \longrightarrow & 1 \\ {\scriptstyle \partial}\downarrow & & \downarrow{\scriptstyle v} \\ u^*(e) & \xrightarrow{i} & e \\ \downarrow & & \downarrow{\scriptstyle p} \\ 1 & \xrightarrow{u} & b. \end{array}$$

Hence the arrow $\partial : \Omega_u(b) \to e(u)$ is the fiber at v of the arrow $u^*(e) \to e$. The base point $v : 1 \to e$ lifts naturally as a base point $w : 1 \to u^*(e)$. Let us show that the arrow $\Omega(p) : \Omega_v(e) \to \Omega_u(b)$ is the fiber at w of the arrow ∂. For this, it suffices to show that we have a cartesian square

$$\begin{array}{ccc} \Omega_v(e) & \xrightarrow{\Omega(p)} & \Omega_u(b) \\ \downarrow & & \downarrow{\scriptstyle \partial} \\ 1 & \xrightarrow{w} & u^*(e) \end{array}$$

By working in the quategory $Y = X/b$, we can suppose that $b = 1$, since the canonical map $X/b \to X$ preserves pullbacks. For clarity, we shall use a magnifying glass by denoting the objects of $Y = X/b$ by capital letters. The base point $u : 1 \to b$ defines an object $T \in Y$ and the arrow $p : e \to b$ an object $E \in Y$. The base point $v : 1 \to e$ defines a morphism $v : T \to E$. Observe that the image of the projection $p_2 : T \times E \to E$ by the canonical map $Y \to X$ is the arrow $i : u^*(u) \to e$. Similarly, the image of the canonical morphism $j : T \times_E T \to T \times T$ by the map $Y \to X$ is the arrow $\Omega(p) : \Omega(e) \to \Omega(b)$. The square in the NE corner of the following commutative diagram is cartesian,

$$\begin{array}{ccccc} T \times_E T & \xrightarrow{j} & T \times T & \xrightarrow{p_2} & T \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle T\times v} & & \downarrow{\scriptstyle v} \\ T & \xrightarrow{(1_T, v)} & T \times E & \xrightarrow{p_2} & E \\ & & {\scriptstyle p_1}\downarrow & & \downarrow \\ & & T & \longrightarrow & 1. \end{array}$$

It follows that the square in the NW corner is cartesian, since the composite of the top squares is cartesian. This shows that the square above is cartesian and hence that the arrow $\Omega(p) : \Omega_v(e) \to \Omega_u(b)$ is the fiber at w of the arrow ∂. We thus obtain a fiber sequence of length four,

$$\Omega(e) \xrightarrow{\Omega(p)} \Omega(b) \xrightarrow{\partial} f \xrightarrow{i} e \xrightarrow{p} b\ .$$

By iterating, we obtain the *long fiber sequence*

$$\cdots \longrightarrow \Omega^2(e) \xrightarrow{\partial} \Omega(f) \xrightarrow{\Omega(i)} \Omega(e) \xrightarrow{\Omega(p)} \Omega(b) \xrightarrow{\partial} f \xrightarrow{i} e \xrightarrow{p} b\ .$$

33.5. The considerations above can be dualised. Let X be a pointed cocartesian quategory with nul object $0 \in X$. The *cofiber* of an arrow $u : x \to y$ is the arrow $v : x \to y$ defined by a pushout square

$$\begin{array}{ccc} x & \longrightarrow & 0 \\ {\scriptstyle u}\downarrow & & \downarrow \\ y & \xrightarrow{v} & z. \end{array}$$

The suspension $\Sigma(x)$ is the cofiber of the nul arrow $x \to 0$. It follows from the duality that $\Sigma(x)$ has the structure of a cogroup object in X. We obtain the *Puppe* *cofiber sequence*

$$x \xrightarrow{u} y \xrightarrow{v} z \xrightarrow{\partial} \Sigma(x) \xrightarrow{\Sigma(u)} \Sigma(y) \xrightarrow{\Sigma(v)} \Sigma(z) \xrightarrow{\partial} \Sigma^2(z) \longrightarrow \cdots .$$

34. Additive quategories

We extend the theory of additive categories to quategories.

34.1. If X is a pointed quategory, we shall say that an object $c \in X$ equipped with four morphisms

$$\begin{array}{ccccc} a & & & & a \\ & \searrow^{i_1} & & \nearrow^{p_1} & \\ & & c & & \\ & \nearrow_{i_2} & & \searrow_{p_2} & \\ b & & & & b \end{array}$$

is the *direct sum* of the objects $a, b \in X$ if the following 3 conditions are satisfied:

- $p_1 i_1 = 1_a$, $p_2 i_2 = 1_b$, $p_2 i_1 = 0$ and $p_1 i_2 = 0$ in hoX;
- the pair (p_1, p_2) is a product diagram,

$$\begin{array}{ccc} & & a \\ & \nearrow^{p_1} & \\ a \oplus b & & \\ & \searrow_{p_2} & \\ & & b \end{array}$$

- the pair (i_1, i_2) is a coproduct diagram,

$$\begin{array}{ccc} a & & \\ & \searrow^{i_1} & \\ & & a \oplus b, \\ & \nearrow_{i_2} & \\ b & & \end{array}$$

We shall write $c = a \oplus b$ to indicate that c is the direct sum of a and b. The cartesian product of two objects $a, b \in X$ is a direct sum iff the pair of morphisms

$$\begin{array}{ccc} a & & \\ & \searrow^{(1_a, 0)} & \\ & & a \times b \\ & \nearrow_{(0, 1_b)} & \\ b & & \end{array}$$

is a coproduct diagram. Dually, the coproduct of two objects $a, b \in X$ is a direct sum iff the pair of morphisms

$$\begin{array}{ccc} & & a \\ & \overset{(1_a,0)}{\nearrow} & \\ a \sqcup b & & \\ & \underset{(0,1_b)}{\searrow} & \\ & & b \end{array}$$

is a product diagram. We shall say that a pointed quategory X is *semi-additive* if every pair of objects in X has a direct sum. The opposite of a semi-additive quategory is semi-additive. The homotopy category of a semi-additive quategory is semi-additive. In a semi-additive quategory, the coproduct of a finite family of objects is also their product; the coproduct of an arbitrary family of objects $(a_i : i \in I)$ is denoted as a direct sum

$$\bigoplus_{i \in I} a_i = \bigsqcup_{i \in I} a_i.$$

Similarly, the coproduct of an arbitrary family of morphisms $f_i : a_i \to b_i$ is denoted as a direct sum

$$\bigoplus_{i \in I} f_i : \bigoplus_{i \in I} a_i \to \bigoplus_{i \in I} b_i.$$

We shall say that a map $X \to Y$ between semi-additive quategories is *additive* if it preserves finite direct sums. A map $f : X \to Y$ is additive iff the opposite map $f^o : X^o \to Y^o$ is additive. The canonical map $X \to hoX$ is additive for any semi-additive quategory X. We shall denote by $\mathrm{Add}(X, Y)$ the full simplicial subset of Y^X spanned by the additive maps $X \to Y$ between two semi-additive quategories. We shall say that a semi-additive quategory X is *additive* if the category hoX is additive. We shall say that an algebraic theory T is *additive* (resp. *semi-additive*) if the quategory T is additive (resp. semi-additive). A *morphism* of *additive* (resp. *semi-additive*) theories is an additive map.

34.2. An algebraic theory T is additive (resp. semi-additive) iff the quategory $\mathrm{Alg}(T)$ is additive. We shall say that a quategory is an *additive theater* (resp. *semi-additive theater*) if it is equivalent to a quategory $\mathrm{Alg}(T)$ for some additive (resp. semi-additive) theory T.

34.3. If T is a semi-additive theory, we shall say that a power generator $a \in T$ is an *additive generator*. An object $a \in T$ is an additive generator iff every object of T is isomorphic to a sum $\oplus_n a$ for some $n \geq 0$. An object $a \in T$ is an additive generator iff the opposite object $a^o \in T$ is an additive generator. An additive map $f : CMon \to T$ is essentially surjective iff the object $f(1) \in T$ is an additive generator. A ring (resp. rig) is essentially the same thing as a unisorted additive theory (resp. semi-additive theory).

34.4. An algebraic theory T is additive (resp. semi-additive), iff the quategory $\mathrm{Alg}(T)$ is additive (resp. semi-additive), The theory of E_∞-spaces $CMon$ is semi-additive hence also the quategory of E_∞-spaces $\mathcal{C}M = \mathrm{Alg}(CMon)$. The theory $CMon$ is freely generated by the object $1 \in CMon$. More precisely, the evaluation map $e : \mathrm{Add}(CMon, X) \to X$ defined by putting $e(f) = f(1)$ is an equivalence for any semi-additive quategory X. Similarly, the theory of infinite loop-spaces $CGrp$ is additive hence also the quategory of infinite loop spaces $\mathcal{C}G = \mathrm{Alg}(CGrp)$.

and freely generated by the object $1 \in CGrp$. Hence the evalutation map $e : \mathrm{Add}(CGrp, X) \to X$ defined by putting $e(f) = f(1)$ is an equivalence for any additive quategory X.

The theory of E_∞-spaces $CMon$ is semi-additive and the theory of infinite loop-spaces $CGrp$ is additive. Hence the quategory $\mathcal{L}_\infty = \mathrm{Alg}(CGrp)$ is additive. and the quategory $\mathcal{E}_\infty = \mathrm{Alg}(CMon)$ is semi-additive.

34.5. The theory of E_∞-spaces $CMon$ is semi-additive and freely generated by the object $1 \in CMon$. More precisely, the evalutation map $e : \mathrm{Add}(CMon, X) \to X$ defined by putting $e(f) = f(1)$ is an equivalence for any semi-additive quategory X. Similarly, the theory of infinite loop-spaces $CGrp$ is additive and freely generated by the object $1 \in CGrp$.

34.6. A quategory with finite product X is semi-additive iff one (and then both) of the following forgetful maps

$$CMon(X) \to X \quad \text{and} \quad Mon(X) \to X$$

is an equivalence. Thus, if X is semi-additive then every object $a \in X$ has the structure of a monoid, where the multiplication and unit are given by

$$(1_a, 1_a) : a \oplus a \to a \qquad 0 \to a.$$

Similarly, a quategory with finite product X is additive iff one (and then both) of the following forgetful maps

$$CGrp(X) \to X \quad \text{and} \quad Grp(X) \to X$$

is an equivalence. Thus, if X is additive, then every object $a \in X$ has the structure of a group; the inverse operation $a \to a$ is -1_a, the additive inverse of 1_a in the group $hoX(a, a)$.

34.7. The quategory of E_∞-spaces $\mathcal{E}_\infty = \mathrm{Alg}(CMon)$ is semi-additive and symmetric monoidal closed. Moreover every semi-additive arena is enriched over $\mathcal{E}_\infty$ and bicomplete as an enriched quategory. More precisely, if X and Y are two arenas and if X or Y is semi-additive, then the arenas $X \otimes Y$ and $\mathrm{CC}(X, Y)$ are semi-additive. Let us denote by $\mathbf{AR}(\mathcal{E}_\infty)$ the full sub-category of $\mathbf{AR}$ spanned by the semi-additive arenas. Then the inclusion functor $\mathbf{AR}(\mathcal{E}_\infty) \subset \mathbf{AR}$ has both a left and a right adjoint. The left adjoint is the functor $X \mapsto \mathcal{E}_\infty \otimes X = CMon(X)$ and the right adjoint is the functor $X \mapsto \mathrm{CC}(\mathcal{E}_\infty, X) = CMon(X^o)^o$. The (simplicial) category $\mathbf{AR}(\mathcal{E}_\infty)$ is symmetric monoidal closed if the unit object is taken to be the quategory $\mathcal{E}_\infty$. If $X \in \mathbf{AR}(\mathcal{E}_\infty)$, then the equivalence $\mathcal{E}_\infty \otimes X \simeq X$ is induced by a map

$$\otimes : \mathcal{E}_\infty \times X \to X$$

called the *tensor product*. Similarly, the quategory $\mathcal{L}_\infty = \mathrm{Alg}(CGrp)$ is additive and symmetric monoidal closed. And every additive arena is enriched over $\mathcal{L}_\infty$ and bicomplete as an enriched quategory. More precisely, if X and Y are two arenas and if X or Y is additive, then the arenas $X \otimes Y$ and $\mathrm{CC}(X, Y)$ are additive. Let us denote by $\mathbf{AR}(\mathcal{L}_\infty)$ the full sub-category of $\mathbf{AR}$ spanned by the additive arenas. Then the inclusion functor $\mathbf{AR}(\mathcal{L}_\infty) \subset \mathbf{AR}$ has both a left and a right adjoint. The left adjoint is the functor $X \mapsto \mathcal{L}_\infty \otimes X = CGrp(X)$ and the right adjoint is the functor $X \mapsto \mathrm{CC}(\mathcal{L}_\infty, X) = CGrp(X^o)^o$. The (simplicial) category $\mathbf{AR}(\mathcal{L}_\infty)$ is symmetric monoidal closed if the unit object is taken to be the quategory $\mathcal{L}_\infty$. If $X \in \mathbf{AR}(\mathcal{L}_\infty)$, then the equivalence $\mathcal{L}_\infty \otimes X \simeq X$ is induced by a map

$$\otimes : \mathcal{L}_\infty \times X \to X$$

called the *tensor product*.

34.8. If X is an additive arena, then the opposite of the map $Hom_X : X^o \times X \to \mathcal{L}_\infty$ is cocontinuous in each variable and the resulting map

$$X^o \to \mathrm{CC}(X, \mathcal{L}_\infty^o)$$

is an equivalence of quategories as in 28.25. Similarly, if X is a semi-additive arena then the opposite of the map $Hom_X : X^o \times X \to \mathcal{E}_\infty$ is cocontinuous in each variable and the resulting map

$$X^o \to \mathrm{CC}(X, \mathcal{E}_\infty^o)$$

is an equivalence of quategories.

34.9. Recall that the (simplicial) category of algebraic theories is denoted by **ALG**. We shall denote by **ADD** the full sub-category of **ALG** spanned by the additive theories. If $S, T \in$ **ALG** and one of the theories S or T is additive, then the quategories $S \odot T$ and $\mathrm{Alg}(S,T)$ is additive. When S and T are both additive, we shall put

$$S \otimes T := S \odot T \quad \text{and} \quad \mathrm{Add}(S,T) := \mathrm{Alg}(S,T).$$

The (simplicial) category **ADD** is symmetric monoidal closed if the unit object is taken to be the theory $CGrp$. The opposite of an additive theory is additive and the functor $T \mapsto T^o$ respects the symmetric monoidal structure. In particular the quategory $CGrp$ is equivalent to its opposite. The inclusion

$$\mathbf{ADD} \subset \mathbf{ALG}$$

has both a left and a right adjoint. The left adjoint is the functor $T \mapsto CGrp \odot T$ and its right adjoint is the functor $T \mapsto CGrp(T)$. An algebraic theory T is additive iff the quategory $\mathrm{Alg}(T)$ is additive iff one (and then all) of the following canonical maps

$$T \to T \odot CGrp \qquad CGrp(T) \to T$$
$$T \to T \odot Grp \qquad Grp(T) \to T$$

is an equivalence of quategories. Similarly, we shall denote by **SAD** the full sub-category of **ALG** spanned by the semi-additive theories. If $S, T \in$ **ALG** and one of the theories S or T is semi-additive, then the quategories $S \odot T$ and $\mathrm{Alg}(S,T)$ are semi-additive. When S and T are both semi-additive, we shall put $S \otimes T := S \odot T$ and $\mathrm{Add}(S,T) = \mathrm{Alg}(S,T)$. The (simplicial) category **SAD** is symmetric monoidal closed if the unit object is taken to be the theory $CMon$. The opposite of a semi-additive theory is semi-additive, and the functor $T \mapsto T^o$ respects the symmetric monoidal structure. In particular the quategory $CMon$ is equivalent to its opposite. The inclusion

$$\mathbf{SAD} \subset \mathbf{ALG}$$

has both a left and a right adjoint. The left adjoint is the functor $T \mapsto CMon \odot T$ and its right adjoint is the functor $T \mapsto CMon(T)$. An algebraic theory T is semi-additive iff the quategory Alg(T) is semi-additive iff one (and then all) of the following canonical maps

$$T \to T \odot CMon \qquad CMon(T) \to T$$

$$T \to T \odot Mon \qquad Mon(T) \to T$$

is an equivalence of quategories.

34.10. If $T \in \mathbf{ADD}$ and $X \in \mathbf{AR}$ then the map

$$\mathrm{Add}(T, CGrp(X)) = \mathrm{Alg}(T, CGrp(X)) \to \mathrm{Alg}(T, X)$$

induced by the forgetful map $CGrp(X) \to X$ is an equivalence of quategories. In particular, the map

$$\mathrm{Add}(T, \mathcal{L}_\infty) = \mathrm{Alg}(T, \mathcal{L}_\infty) \to \mathrm{Alg}(T, \mathcal{K})$$

induced by the forgetful map $\mathcal{L}_\infty \to \mathcal{K}$ is an equivalence of quategories. We shall say that an additive map $f : T \to \mathcal{L}_\infty$ is a *left T-module* and put

$$\mathrm{Mod}(T) = \mathrm{Add}(T, \mathcal{L}_\infty).$$

Dually, we shall say that an additive map $f : T^o \to \mathcal{L}_\infty$ is a *right T-module.* If S and T are additive theories, we shall say that an additive map $f : S^o \otimes T \to \mathcal{L}_\infty$ is a (T, S)*-bimodule* and put

$$\mathrm{Mod}(S, T) = \mathrm{Mod}(S^o \otimes T).$$

34.11. If $u : S \to T$ is a morphism of additive theories, then the map

$$u^* : \mathrm{Mod}(T) \to \mathrm{Mod}(S)$$

induced by u has a left adjoint $u_!$ and a right adjoint u_*.

34.12. If T is an additive theory, then the map $hom : T^o \times T \to \mathcal{K}$ preserves finite direct sum in each variable. It thus induces an additive map $Hom_T : T^o \otimes T \to \mathcal{L}_\infty$ by 38.7. The resulting *Yoneda map*

$$y : T^o \to \mathrm{Mod}(T)$$

is fully faithful and additive. We say that a left module $T \to \mathcal{L}_\infty$ is *representable* if it is isomorphic to a module $y(a)$ for some object $a \in T$. Then the map y induces an equivalence between T^o and the full sub-quategory of Mod(T) spanned by the representable left modules. There is a dual *Yoneda map*

$$y : T \to \mathrm{Mod}(T^o)$$

and a notion of representable right module.

34.13. To every commutative square in an additive quategory

$$\begin{array}{ccc} a & \xrightarrow{v} & c \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle f} \\ b & \xrightarrow{g} & d \end{array}$$

we can associate a null sequence,

$$\begin{array}{ccc} a & \xrightarrow{(u,v)} & b \oplus c \\ & {\scriptstyle 0}\searrow & \downarrow{\scriptstyle (-g,f)} \\ & & d. \end{array}$$

The square is a pullback iff the null sequence is a fiber sequence. Dually, the square is a pushout iff the null sequence is a cofiber sequence. An additive quategory is cartesian iff every morphism has a fiber. An additive map between additive quategories is left exact iff it preserves fibers. An additive quategory is cocomplete iff it has arbitrary coproducts and cofibers. An additive map between cocomplete additive quategories is cocontinuous iff it preserves coproducts.

34.14. If an additive quategory X is cartesian, then to each arrow $f : x \to y$ with fiber $i : z \to x$ in X we can associate a *long fiber sequence*

$$\cdots \longrightarrow \Omega^2(y) \xrightarrow{\partial} \Omega(z) \xrightarrow{\Omega(i)} \Omega(x) \xrightarrow{\Omega(f)} \Omega(y) \xrightarrow{\partial} z \xrightarrow{i} x \xrightarrow{f} y\ .$$

Dually, if X is cocartesian, then to each arrow $f : x \to y$ with cofiber $p : y \to z$ in X we can associate a *long cofiber sequence*

$$x \xrightarrow{u} y \xrightarrow{p} z \xrightarrow{\partial} \Sigma(x) \xrightarrow{\Sigma(u)} \Sigma(y) \xrightarrow{\Sigma(p)} \Sigma(z) \xrightarrow{\partial} \Sigma^2(z) \longrightarrow \cdots .$$

34.15. To every idempotent $e : x \to x$ of an additive quategory we can associate a *complementary idempotent* $1-e : x \to x$. The idempotent e splits iff the morphism $1-e$ has a fiber. When the idempotents e and $1-e$ split, we obtain a decomposition

$$x \simeq Fib(e) \oplus Fib(1-e).$$

The Karoubi envelope of an additive theory T is additive. A cartesian additive quategory is Karoubi complete.

34.16. If T is an additive theory, then the Yoneda map $y : T \to \mathrm{Mod}(T^o)$ exibits the quategory $\mathrm{Mod}(T^o)$ as the free cocompletion of T under colimits. More precisely, if X is a cocomplete additive quategory, then the map

$$y^* : \mathrm{CC}(\mathrm{Mod}(T^o), X) \to \mathrm{Add}(T, X)$$

induced by y is an equivalence of quategories. The inverse equivalence associates to an additive map $f : T \to X$ its left Kan extension $f_! : \mathrm{Mod}(T^o) \to X$ along y. In particular, if S and T are additive theories, then the map

$$\mathrm{CC}(\mathrm{Mod}(T^o), \mathrm{Mod}(S^o)) \to \mathrm{Add}(T, \mathrm{Mod}(S^o))$$

induced by the Yoneda map $y : T \to \mathrm{Mod}(T^o)$ is an equivalence of quategories,

$$\mathrm{CC}(\mathrm{Mod}(T^o), \mathrm{Mod}(S^o)) \to \mathrm{Mod}(S^o \otimes T) = \mathrm{Mod}(S, T).$$

The inverse equivalence associates to a bimodule $f \in \mathrm{Mod}(S,T)$ a cocontinuous map $R(f) : \mathrm{Mod}(T^o) \to \mathrm{Mod}(S^o)$ such that $R(f)(y(b))(a) = f(a^o \otimes b)$ for every pair of objects $a \in S$ and $b \in T$. There is also an equivalence

$$L : \mathrm{Mod}(S,T) \to \mathrm{CC}(\mathrm{Mod}(S), \mathrm{Mod}(T))$$

such that $L(f)(y(a))(b) = f(a^o \otimes b)$ for every pair of objects $a \in S$ and $b \in T$. The *composite* of a bimodule $f \in \mathrm{Mod}(S^o \otimes T)$ with a bimodule $g \in \mathrm{Mod}(T^o \otimes U)$ is defined to be the bimodule $g \circ_T f \in \mathrm{Mod}(S^o \otimes T)$ such that

$$L(g \circ_T f) \simeq L(g)L(f).$$

34.17. If X is a cocomplete additive quategory, then a map $X \to \mathbf{L}_\infty$ is cocontinuous iff the composite $X \to \mathbf{L}_\infty \to \mathcal{K}$ preserves sifted colimits. An object $a \in X$ is perfect iff the map $Hom(a, -) : X \to \mathbf{L}_\infty$ is cocontinuous.

34.18. If T is an additive theory, then a (left or right) T-module is perfect iff it is a retract of a representable. The full sub-quategory of $\mathrm{Mod}(T^o)$ spanned by the perfect modules is equivalent to the Karoubi envelope $Kar(T)$. A morphism of additive theories $u : S \to T$ is a Morita equivalence iff the map $u^* : \mathrm{Mod}(T) \to \mathrm{Mod}(S)$ induced by u is an equivalence of quategories.

34.19. An additive quategory X is a theater iff it is generated by a set of perfect objects. More precisely, let $K \subseteq X$ be a (small) full sub-quategory of perfect objects of X. If K is closed under finite direct sum then the left Kan extension

$$i_! : \mathrm{Mod}(K^o) \to X$$

of the inclusion $i : K \to X$ along $y : K \to \mathrm{Mod}(K^o)$ is fully faithful and cocontinuous. Moreover, $i_!$ is an equivalence if K generates or separates X,

34.20. We shall denote by **MOD** the full subcategory of $\mathbf{AR}(\mathcal{L}_\infty)$ spanned by the additive varieties. If S and T are additive theories, then the *exterior tensor product* of a module $f \in \mathrm{Mod}(S)$ with a module $g \in \mathrm{Mod}(T)$ is the module $f \otimes g \in \mathrm{Mod}(S \otimes T)$ defined by putting $(f \otimes g)(a \otimes b) = f(a) \otimes g(b)$ for every pair of objects $a \in S$ and $b \in T$. The map $(f, g) \mapsto f \otimes g$ is cocontinuous in each variable and the induced map

$$\mathrm{Mod}(S) \otimes \mathrm{Mod}(T) \to \mathrm{Mod}(S \otimes T)$$

is an equivalence of quategories. Hence the (simplicial) category **MOD** is symmetric monoidal and the functor

$$\mathrm{Mod} : \mathbf{ADD} \to \mathbf{MOD}$$

preserves tensor products. If S and T are additive theories, then the map

$$\mathrm{CC}(\mathrm{Mod}(S), \mathrm{Mod}(T)) \longrightarrow \mathrm{Mod}(S^o \otimes T)$$

induced by the Yoneda map $y : S^o \to \mathrm{Mod}(S)$ is an equivalence of quategories by 34.16. It follows that the monoidal category **MOD** is compact closed. The dual of a quategory $X = \mathrm{Mod}(T)$ is the quategory $X^* = \mathrm{Mod}(T^o)$ and the counit $\epsilon_X : X \otimes X^* \to \mathcal{L}_\infty$ is the cocontinuous extension of the map

$$Hom_T : T^o \otimes T \to \mathcal{L}_\infty.$$

The value of the unit $\eta_X : \mathcal{L}_\infty \to X^* \otimes X$ at the unit object $1 \in \mathcal{L}_\infty$ is the module $Hom_T : T^o \otimes T \to \mathcal{L}_\infty$. It follows from the duality that if $X \in \mathbf{MOD}$ and Y is an additive arena, then the map

$$i : X^* \otimes Y \to \mathrm{CC}(X, Y)$$

defined by putting $i(x^* \otimes y)(x) = x^*(x) \otimes y$ is an equivalence of quategories. More precisely, the *external tensor product* of a module $f \in \mathrm{Mod}(T)$ with an object $y \in Y$ is the additive map $f \otimes y : T \to Y$ defined by putting $(f \otimes y)(a) = f(a) \otimes y$ for every object $a \in T$. The map $(f, y) \mapsto f \otimes y$ is cocontinuous in each variable and the induced map

$$\mathrm{Mod}(T) \otimes Y \to \mathrm{Add}(T, Y) \simeq \mathrm{CC}(\mathrm{Mod}(T^o), Y)$$

is an equivalence of quategories. Hence the functor $Y \mapsto \mathrm{Add}(T, Y)$ is left adjoint to the functor $Y \mapsto \mathrm{Add}(T^o, Y)$ as in 29.2. We thus have an equivalence

$$\mathrm{CC}(\mathrm{Add}(T, X), Y) \simeq \mathrm{CC}(X, \mathrm{Add}(T^o, Y))$$

for any pair of additive arenas X and Y.

34.21. The *trace* of an endomorphism $f : X \to X$ in $\mathbf{MOD}$ is the object of $Tr(f) \in \mathcal{L}_\infty$ obtained by putting

$$Tr(f) = \epsilon_X \circ (f \otimes X^*) \circ \eta_{X^*}.$$

If $f : X \to Y$ and $g : Y \to X$ are two maps in $\mathbf{MOD}$, then there is a canonical isomorphism $Tr(fg) \simeq Tr(gf)$. The *scalar product* of f and g is defined by putting

$$\langle f \mid g \rangle = Tr(fg).$$

For example the scalar product of an object $x^* \in X^*$ with an object $x \in X$ is the object $\langle x^* \mid x \rangle = x^*(x)$. For any endomorphism $f : X \to X$ we have $Tr(f) \simeq \langle f | 1_X \rangle$. The *transpose* of a map $f : X \to Y$ in $\mathbf{MOD}$ is the map ${}^t f : Y^* \to X^*$ obtained by putting

$${}^t f = (X^* \otimes \epsilon_Y) \circ (X^* \otimes f \otimes Y^*) \circ (\eta_{X^*} \otimes Y^*).$$

There is then a canonical isomorphism,

$$\langle y^* | f(x) \rangle \simeq \langle {}^t f(y^*) | x \rangle$$

for $x \in X$ and $y^* \in Y^*$. If $f : X \to Y$ and $g : Y \to X$, then there is then a canonical isomorphism,

$$\langle {}^t f \mid {}^t g \rangle \simeq \langle f \mid g \rangle.$$

In particular, if $f : X \to X$, then $Tr(f^*) \simeq Tr(f)$. If $Z \in \mathbf{MOD}$, then the Z*-trace* of a map $f : X \otimes Z \to Y \otimes Z$ in $\mathbf{AR}(\mathcal{L}_\infty)$ is defined to be the map $Tr_Z(f) : X \to Y$ obtained by putting

$$Tr_Z(f) = (Y \otimes \epsilon_Z) \circ (f \otimes Z^*) \circ (X \otimes \eta_{Z^*}).$$

The composite of a map $f \in \mathrm{CC}(X, Z)$ with a map $g \in \mathrm{CC}(Z, Y)$ is the Z-trace of the map $f \otimes g \in \mathrm{CC}(X \otimes Z, Z \otimes Y)$.

34.22. The considerations above can be formulated for bimodules instead of maps. The *transpose* of a bimodule $f \in \mathrm{Mod}(S^o \otimes T) = \mathrm{Mod}(S,T)$ is a bimodule ${}^t f \in \mathrm{Mod}(T^o \otimes S) = \mathrm{Mod}(T^o, S^o)$. The *scalar product* $\langle f, g\rangle$ of a module $f \in \mathrm{Mod}(T^o)$ with a module $g \in \mathrm{Mod}(T)$ is isomorphic to $f_!(g)$, where $f_! : \mathrm{Mod}(T) \to \mathcal{L}_\infty$ is the cocontinuous extension of f along the Yoneda map, and also isomorphic to $g_!(f)$, where $g_! : \mathrm{Mod}(T^o) \to \mathcal{L}_\infty$ is the cocontinuous extension of g along the Yoneda map. The *scalar product* $\langle f, g\rangle$ of a bimodule $f \in \mathrm{Mod}(S,T)$ with a bimodule $g \in \mathrm{Mod}(T,S) \simeq \mathrm{Mod}(S^o, T^o)$ is defined similarly. This defines a map

$$\mathrm{Mod}(S,T) \otimes \mathrm{Mod}(T,S) \to \mathcal{L}_\infty.$$

The *trace* of a bimodule $f \in \mathrm{Mod}(T,T)$ can be defined by putting $Tr(f) = \langle f|h_T\rangle$ where h_T is the bimodule $Hom_T : T^o \otimes T \to \mathcal{L}_\infty$. This defines a map

$$Tr : \mathrm{Mod}(T,T) \to \mathcal{L}_\infty.$$

More generally, the T-*trace* of a bimodule $f \in \mathrm{Mod}(S \otimes T, T \otimes U)$ is a bimodule $Tr_T(f) \in \mathrm{Mod}(S,U)$. The composite of a bimodule $f \in \mathrm{Mod}(S,T)$ with a bimodule $g \in \mathrm{Mod}(T,U)$ is the T-trace of bimodule $f \otimes g \in \mathrm{Mod}(S \otimes T, T \otimes U)$

34.23. To every morphism of additive theories $u : S \to T$ we associate two bimodules $\gamma_!(u) \in \mathrm{Mod}(S,T)$ and $\gamma^*(u) \in \mathrm{Mod}(T,S)$ by putting

$$\gamma_!(u)(a,b) = Hom_T(u(a), b) \quad \text{and} \quad \gamma^*(u)(b,a) = Hom_T(b, u(a))$$

for every pair of objects $a \in S$ and $b \in T$. Then we have

$$L(\gamma_!(u)) = u_! : \mathrm{Mod}(S) \to \mathrm{Mod}(T) : u^* = L(\gamma^*(u)).$$

It shows that the map u^* is cocontinuous and that it has a right adjoint u_*.

34.24. The notion of homotopy factorisation system in the category **ADD** is defined as in 28.29. The category **ADD** admits a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of essentially surjective maps and $\mathcal{B}$ the class of fully faithful maps. There is a similar homotopy factorisation system on **SAD**.

34.25. If $T \to L$ is a quasi-localisation of algebraic theories and the theory T is additive then L is additive. The same result is true for an iterated quasi-licalisation $T \to L$. It follows that the category **ADD** admits a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{B}$ is the class of conservative maps and $\mathcal{A}$ is the class of iterated quasi-localisations. There is a similar homotopy factorisation system on the category **SAD**.

34.26. The category $\mathbf{AR}(\mathcal{L}_\infty)$ admits a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{B}$ is the class of fully faithful maps. A map $f : X \to Y$ belongs to $\mathcal{A}$ iff its right adjoint $Y \to X$ is conservative iff $f(X_0)$ generates Y.

34.27. A Bousfield localisation an additive arena is additive. The category $\mathbf{AR}(\mathcal{L}_\infty)$ admits a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{B}$ is the class of conservative maps and $\mathcal{A}$ is the class of Bousfield localisations.

35. Dold-Kan correspondance and finite differences calculus

35.1. Let Ch be the pointed category whose objects are the natural numbers and whose arrows are given by

$$Ch(m,n) = \begin{cases} \{\partial, 0\} & \text{if } m = n+1 \\ \{id, 0\} & \text{if } m = n. \\ \{0\} & \text{otherwise} \end{cases}$$

By definition, $\partial\partial = 0$. A *chain complex* in an additive quategory X is defined to be a pointed map $Ch \to X$. We shall denote by $Ch(X)$ the full simplicial subset of X^{Ch} spanned by the chain complexes in X. The quasi-category $Ch(X)$ is additive.

35.2. Let X be an additive quasi-category which is Karoubi complete. Then the quategory $[\Delta^o, X]$ of simplicial object in X is equivalent to the quategory $[\Delta^o_+, X]$ of augmented simplicial objects in X. The equivalence is given by the *first difference functor*

$$\delta_+ : [\Delta^o, X] \to [\Delta^o_+, X]$$

defined as follow. We first need the *successor functor* $s : \Delta \to \Delta$ defined by putting $s([n]) = [n] \star [0] = [n+1]$ for every $n \geq 0$. The *shift* EC of simplicial object $C : \Delta^o \to X$ is then defined by putting $EC = s^*(C)$. The maps $d_{n+1} : [n] \to [n+1]$ define a natural transformation $Id \to s$, hence also a morphism $\epsilon(C) : EC \to C$ for every $C : \Delta^o \to X$. The morphism $\epsilon(C)_n : EC_n \to C_n$ admits a section $C(s_n)$ for every $n \geq 0$. We thus obtain a decomposition $EC \simeq C \oplus \delta C$, where δC is the fiber of $\epsilon(C)$. The simplicial object δC is naturally augmented, with an augmentation $\partial : \delta C_0 \to C_0$ obtained by composing the canonical morphism $\delta C_0 \to C_1$ with $\partial_0 : C_1 \to C_0$ The resulting augmented simplicial object is denoted $\delta_+ C$, and the resulting map

$$\delta_+ : [\Delta^o, X] \to [\Delta^o_+, X]$$

is an equivalence of quasi-categories. The inverse equivalence associates to an augmented simplicial object $D : \Delta^o_+ \to X$ a simplicial object ΣD and we have

$$(\Sigma D)_n = \bigoplus_{i=0}^{n} D(i)$$

for every $n \geq 0$. Let us describe ΣD more explicitly. If u denotes the inclusion $\Delta \subset \Delta_+$, then we have $u^*(D)_n = D(n+1)$. Let $\sigma : \Delta_+ \to \Delta$ be the functor defined by putting $\sigma(n) = n+1 = [n]$. If $\sigma_!(D)$ denotes the left Kan extension of D along σ, then we have

$$\sigma_!(D)_n = \bigoplus_{i=0}^{n+1} D(i).$$

The simplicial object ΣD is the cofiber of a canonical morphism $\alpha_D : u^*(D) \to \sigma_!(D)$ which can be described as follows. From the obvious natural transformation $Id \to \sigma u$, we obtain a natural transformation $\beta : Id \to \sigma_! u_!$. If $\epsilon : u_! u^* \to Id$ is the counit of the adjunction $u_! \vdash u^*$, then

$$\alpha_D = \sigma_!(\epsilon_D)\beta_{u^*(D)} : u^*(D) \to \sigma_! u_! u^*(D) \to \sigma_!(D).$$

35.3. The first difference $C \mapsto \delta C$ can be iterated. If $C : \Delta^o \to X$, then the simplicial object $\delta(C)$ is equipped with an augmentation $\partial : \delta C_0 \to C_0$. The second difference $\delta^2 C = \delta(\delta(C))$ is equipped with an augmentation $\partial : \delta^2 C_0 \to \delta C_0$ and we have $\partial\partial = 0$. By iterating, we obtain a chain complex $\delta^* C_0$,

$$C_0 \xleftarrow{\partial} \delta C_0 \xleftarrow{\partial} \delta^2 C_0 \xleftarrow{\partial} \cdots .$$

It follows from the construction that $\delta^n C_0$ is the fiber of the map

$$(\partial_1, \ldots, \partial_n) : C_n \to \bigoplus_{i=1}^{n} C_{n-1}$$

for every $n \geq 1$. The boundary morphism $\partial : \delta^{n+1} C_0 \to \delta^n C_0$ is induced by the morphism $\partial_0 : C_{n+1} \to C_n$ and $d = (0, \partial_0 p_2, \cdots, \partial_0 p_{n+1})$ in the following diagram,

$$\begin{array}{ccc} \delta^{n+1} C_0 & \xrightarrow{\partial} & \delta^n C_0 \\ \downarrow & & \downarrow \\ C_{n+1} & \xrightarrow{\partial_0} & C_n \\ \downarrow & & \downarrow \\ \bigoplus_{i=1}^{n+1} C_{n+1} & \xrightarrow{d} & \bigoplus_{i=1}^{n} C_n . \end{array} ,$$

The *Dold-Kan correspondance* is the map

$$ch : [\Delta^o, X] \to Ch(X)$$

obtained by putting $ch(C) = \delta^* C_0$ for every simplicial object $C : \Delta^o \to X$, where $Ch(X)$ denotes the quategory of chain complexes in X, The correspondance is an equivalence of quategories. The inverse equivalence associates to a chain complex $D \in Ch(X)$ a simplicial object $S(D)$ obtained by putting

$$S(D)_n = \bigoplus_{k=0}^{n} \binom{n}{k} D_k$$

for every $n \geq 0$. Let us describe the simplicial object $S(D)$ more explictly. Observe that the binomial coefficient $\binom{n}{k}$ is the number of surjections $[n] \to [k]$. Let $\Delta_{mono} \subset \Delta$ be the subcategory of monomorphisms. Consider the functor $G : \Delta_{mono} \to Ch$ which takes a monomorphism $f : [m] \to [n]$ to the morphism $G(f) : m \to n$ defined by putting

$$G(f) = \begin{cases} \partial & \text{if } n = m+1 \quad \text{and} \quad f = d_0 \\ id & \qquad \text{if } m = n. \\ 0 & \qquad \text{otherwise} \end{cases}$$

Then the map $S(D) : \Delta^o \to X$ is the left Kan extension of the composite $D \circ G : \Delta^o_{mono} \to X$ along the inclusion $\Delta_{mono} \subset \Delta$. The equivalence

$$C_n \simeq S(\delta^* C_0)_n = \bigoplus_{k=0}^{n} \binom{n}{k} \delta^k C_0$$

is *Newton's formula* of finite differences calculus.

35.4. Let X be an additive cartesian quategory. Then a simplicial object $C : \Delta^o \to X$ is a groupoid iff we have $\delta^n C = 0$ for every $n > 1$. The Dold-Kan correspondance associates to a groupoid $C : \Delta^o \to X$ the morphism $\partial(C) = \partial : \delta C_0 \to C_0$. It induces an equivalence

$$\partial : Gpd(X) \simeq X^I$$

between the quasi-category $Gpd(X)$ of groupoids in X and the quategory X^I of morphisms in X. If C is the equivalence groupoid of an arrow $u : x \to y$, then the morphism $\partial(C)$ is the fiber $Ker(u) \to x$. A functor $p : E \to C$ is a left (or right) fibration iff the morphism $\delta p_0 : \delta E_0 \to \delta C_0$ is invertible. Hence the Dold-Kan correspondance induces an equivalence between the quategory X^C and the quategory $Fact(\partial(C), X)$ of factorisations of the arrow $\partial(C)$.

36. Stabilisation

36.1. Consider the category $End(\mathbf{Q})$ whose objects are the pairs (X, f), where $f : X \to X$ is an endomorphism of a quasi-category, and whose morphisms $(X, f) \to (Y, g)$ are the pairs (u, α), where $u : X \to Y$ is a map and α is an invertible 2-cell $\alpha : uf \to gu$ in the square

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{u} & Y. \end{array}$$

The category $End(\mathbf{Q})$ has the structure of a 2-category induced by that of **QCat**. We shall say that an endomorphism $f : X \to X$ is a *self-equivalence* if f is an equivalence of quasi-categories. Let us denote by $SEq(\mathbf{Q})$ the full sub-2-category of $End(\mathbf{Q})$ whose objects are the self equivalences. Then the inclusion functor

$$SEq(\mathbf{Q}) \subset End(\mathbf{Q})$$

has a left adjoint which associates to an object $(X, f) \in End(\mathbf{Q})$ an object $(X', f') \in SEq(\mathbf{Q})$ equipped with a morphism $(u, \alpha) : (X, f) \to (X', f')$ The quategory X' is the (homotopy) colimit of the infinite sequence of quategories

$$X \xrightarrow{f} X \xrightarrow{f} X \xrightarrow{f} \cdots .$$

By construction, we have an infinite sequence of maps $u_n : X \to X'$ together with an infinite sequence of isomorphisms $i_n : u_{n+1} f \simeq u_n$,

$$\begin{array}{c} X \xrightarrow{f} X \xrightarrow{f} X \xrightarrow{f} \cdots \\ u_0, u_1, u_2 \searrow \\ X' \end{array}$$

The canonical map $u : X \to X'$ is the map $u_0 : X \to X'$. The self-equivalence $f' : X' \to X'$ is the (homotopy) colimit of the sequence of maps

$$\begin{array}{ccccccc} X & \xrightarrow{f} & X & \xrightarrow{f} & X & \xrightarrow{f} & \cdots \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \\ X & \xrightarrow{f} & X & \xrightarrow{f} & X & \xrightarrow{f} & \cdots \end{array}$$

and its (pseudo) inverse g' is the (homotopy) colimit of the positive shift

$$\begin{array}{ccccccc} X & \xrightarrow{f} & X & \xrightarrow{f} & X & \xrightarrow{f} & \cdots \\ & \searrow^{1_X} & & \searrow^{1_X} & & \searrow^{1_X} & \\ X & \xrightarrow{f} & X & \xrightarrow{f} & X & \xrightarrow{f} & X \cdots . \end{array}$$

By construction, there is a canonical isomorphism $\alpha_n : u_n f \simeq f' u_n$ for every $n \geq 0$. We have $(u, \alpha) = (u_0, \alpha_0) : (X, f) \to (X', f')$. Notice that we have $u_n(x) \simeq (g')^n u(x)$.

36.2. Consider the category $End(\mathbf{AR})$ whose objects are the pairs (X, f), where X is an arena and $f : X \to X$ is a cocontinuous map, and whose morphisms $(X, f) \to (Y, g)$ are the pairs (u, α), where $u : X \to Y$ is a map in $\mathbf{AR}$ and α is an invertible 2-cell $\alpha : uf \simeq gu$ in the square

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{u} & Y. \end{array}$$

The category $End(\mathbf{AR})$ has the structure of a 2-category induced by that of $\mathbf{AR}$. Let us denote by $SEq(\mathbf{AR})$ the full sub-2-category of $End(\mathbf{AR})$ whose objects are the self equivalences. Then the inclusion functor

$$SEq(\mathbf{AR}) \subset End(\mathbf{AR})$$

has a left adjoint which associates to an object $(X, f) \in End(\mathbf{AR})$ its *stabilisation* $(X', f') \in SEq(\mathbf{AR})$ equipped with a morphism $(u, \alpha) : (X, f) \to (X', f')$. By construction, X' is the (homotopy) colimit in $\mathbf{AR}$ of the infinite sequence

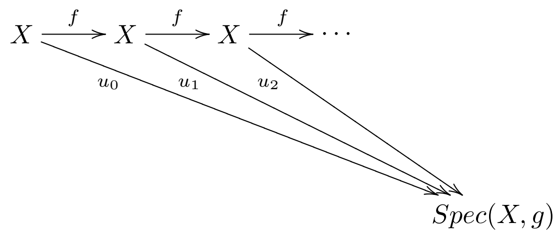
$$X \xrightarrow{f} X \xrightarrow{f} X \xrightarrow{f} \cdots .$$

We thus have an infinite sequence of maps $u_n : X \to X'$ and an infinite sequence of isomorphisms $i_n : u_{n+1} f \simeq u_n$. The quategory X' can be computed by using the duality between $\mathbf{AR}$ and $\mathbf{AR}^*$ as in **??** (A duality). If $g : X \to X$ is right adjoint to $f : X \to X$, then the quategory $Spec(X, g)$ of *g-spectra* in X is defined to be the (homotopy) limit of the sequence of quategories

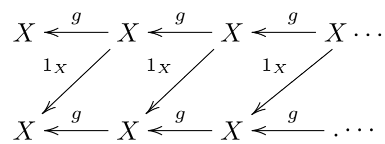
$$X \xleftarrow{g} X \xleftarrow{g} X \xleftarrow{g} \cdots .$$

Hence a g-spectrum is a pair (x, k), where $x = (x_n)$ is a sequence of objects $x_n \in X$ and $k = (k_n)$ is a sequence of isomorphisms $k_n : x_n \simeq g(x_{n+1})$. We have $X' = Spec(X, g)$, the canonical map $u_n : X \to Spec(X, g)$ is left adjoint to the projection

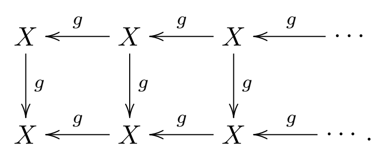
$p_n : X' \to X$ and the canonical isomorphism $i_n : u_{n+1} f \simeq u_n$ is the left transpose of the isomorphism $k_n : p_n \simeq g p_{n+1}$,



The map $f' : Spec(X, g) \to Spec(X, g)$ is the homotopy limit of the negative shift map



Thus, $f'(x, k)_n = (x_{n+1}, k_{n+1})$ for every $n \geq 0$. The inverse $g' : Spec(X, g) \to Spec(X, g)$ of f' is the homotopy limit $g' : X' \to X'$ of the sequence of maps



Thus, $g'(x, k)_n = (g(x_n), g(k_n))$ for every $n \geq 0$.

36.3. Let X be an arena and $f : X \to X$ be a cocontinuous map with right adjoint $g : X \to X$. A *prefixed object* for g is a pair (x, a), where $x \in X$ and $a : x \to g(x)$ is a morphism; a prefixed object (x, a) is *fixed* if a is invertible. Let us denote the quategory of prefixed objects for g by $\mathrm{PFix}(X, g)$ and the quategory of fixed objects for g by $\mathrm{Fix}(X, g)$. Then the inclusion $\mathrm{Fix}(X, g) \subset \mathrm{PFix}(X, g)$ has a left adjoint

$$\rho : \mathrm{PFix}(X, g) \to \mathrm{Fix}(X, g).$$

Let us now suppose that the map g is finitary. Then we have $\rho(x, a) = (y, i)$, where y is the colimit the sequence

$$x \xrightarrow{a} g(x) \xrightarrow{g(a)} g^2(x) \xrightarrow{g^2(a)} \cdots .$$

and i is the canonical isomorphism $y \simeq g(y)$. Moreover, the map ρ is finitary in this case. Let us suppose that directed colimits and finite limits commute in X, and that g is left exact and finitary. Then the map ρ is left exact and finitary.

36.4. Let X be an arena and $f : X \to X$ be a cocontinuous map with right adjoint $g : X \to X$. A *pre-g-spectrum* in X is defined to be a pair (x, a), where $x = (x_n)$ is a sequence of objects of X and $a = (a_n)$ is a sequence of morphisms $a_n : x_n \to g(x_{n+1})$. A pre-g-spectrum (x, a) is a g-spectrum iff the morphism a_n is invertible for every $n \geq 0$. Let us denote the quategory of pre-g-spectra in X by $PSpec(X, g)$ and the quategory of g-spectra in X by $Spec(X, g)$. Then the inclusion $Spec(X, g) \subseteq PSpec(X, g)$ has a left adjoint $\rho : PSpec(X, g) \to Spec(X, g)$. Let us put $Y = X^N$ and let $\omega : Y \to Y$ be the map obtained by putting $\omega(x)_n = g(x_{n+1})$

for every $x = (x_n) \in Y$. The map $\omega : Y \to Y$ has a left adjoint $\sigma : Y \to Y$ obtained by putting $\sigma(x)_0 = \perp$ and $\sigma(x)_n = f(x_{n-1})$ for $n > 0$. By definition, we have $PSpec(X, g) = \mathrm{PFix}(Y, \omega)$ and $Spec(X, g) = \mathrm{Fix}(Y, \omega)$. Let us now suppose that g is finitary. It then follows from 36.3 that we have $\rho(x, a) = (y, i)$ where y is the colimit in Y of the sequence

$$x \xrightarrow{a} \omega(x) \xrightarrow{\omega(a)} \omega^2(x) \xrightarrow{\omega^2(a)} \cdots .$$

and i is the canonical isomorphism $y \simeq \omega(y)$. It folllows that the reflection $\rho : PSpec(X, g) \to Spec(X, g)$ is finitary. Let us now suppose that directed colimits and finite limits commute in X. In this case ρ is left exact and finitary when g is finitary.

36.5. Let X be a null-pointed arena. Then the suspension $\Sigma : X \to X$ has a right adjoint $\Omega : X \to X$. A *spectrum* in X is an infinite sequence of objects $x_n \in X$ together with an infinite sequence of isomorphisms $i_n : x_n \simeq \Omega(x_{n+1})$. It follows from 36.4 that $Spec(X) = Spec(X, \Omega)$. Hence the quategory $Spec(X)$ is the (homotopy) colimit in **AR** of the infinite sequence of quategories,

$$X \xrightarrow{\Sigma} X \xrightarrow{\Sigma} X \xrightarrow{\Sigma} \cdots .$$

The suspension map $\Sigma : Spec(X) \to Spec(X)$ is invertible. A *pre-spectrum* in X is an infinite sequence of objects $x_n \in X$ together with an infinite sequence of morphisms $a_n : \Sigma(x_n) \to x_{n+1}$. The quategory $PSpec(X)$ of prespectra in X is equivalent to the quategory $PSpec(X, \Omega)$. If directed colimits and finite limits commmute in X, then the reflection

$$\rho : PSpec(X) \to Spec(X)$$

is left exact and finitary.

36.6. Let $f : X \to X$ be an endomorphism in **AR** and $\alpha : f \to 1_X$ be a natural transformation. Then the map $p : X \to X^{[\alpha]}$ which inverts α universally can be constructed as follows by 36.6. Let $g : X \to X$ be the right adjoint of f and $\beta : 1 \to g : X \to X$ be the right transpose of α. An object $x \in X$ is said to *coinvert* β if the morphism $\beta(x) : x \to g(x)$ is invertible. Let us denote by $Coinv(\beta)$ the full simplicial subset of X spanned by the objects which coinvert β. Then the inclusion $Coinv(\beta) \subseteq X$ has a left adjoint $p : Y \to Coinv(\beta)$ and we have $Y^{[\alpha]} = Coinv(\beta)$. Let us now suppose that the natural transformations $f \circ \alpha, \alpha \circ f : f^2 \to f$ are homotopic (ie that $f \circ [\alpha] = [\alpha] \circ f$ in the 2-category **S**). If g is finitary then p is finitary and $p(x)$ is the colimit of the sequence

$$x \xrightarrow{\beta} g(x) \xrightarrow{\beta} g^2(x) \xrightarrow{\beta} \cdots ,$$

for every object $x \in X$, where $\beta : g^n(x) \to g^{n+1}(x)$ is $\beta(g^n(x))$. The morphism $\beta : p(x) \to g(p(x))$ is the colimit of the sequence of morphisms

$$\begin{array}{ccccccc}
x & \xrightarrow{\beta} & g(x) & \xrightarrow{\beta} & g^2(x) & \xrightarrow{\beta} & \cdots \\
\downarrow{\beta} & & \downarrow{\beta} & & \downarrow{\beta} & & \\
g(x) & \xrightarrow{\beta} & g^2(x) & \xrightarrow{\beta} & g^3(x) & \xrightarrow{\beta} & \cdots
\end{array}$$

since $g \circ \beta \simeq \beta \circ g$. Its inverse $g(p(x)) \to p(x)$ is the colimit of the positive shift

$$\begin{array}{ccccccccc} g(x) & \xrightarrow{\beta} & g^2(x) & \xrightarrow{\beta} & g^3(x) & \xrightarrow{\beta} & \cdots & & \\ & \searrow^{id} & & \searrow^{id} & & \searrow^{id} & & & \\ x & \xrightarrow{\beta} & g(x) & \xrightarrow{\beta} & g^2(x) & \xrightarrow{\beta} & g^3(x) & \to & \cdots . \end{array}$$

Moreover, p is left exact if directed colimits and finite limits commute in X
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36.7. Let $PSpec$ be the cartesian theory of parametrised spectra described in **??**. If X is a para-variety (resp. an ∞-topos), let us show that $PSpec(X)$ is a para-variety (resp. an ∞-topos). Let $PPreSpec$ be the cartesian theory of parametrized pre-spectra. An object of $PPreSpec(X)$ is a pre-spectrum in X/b for some object $b \in X$. A pointed object of X/b is an arrow $p : x \to b$ equipped with a section $s : b \to x$. A *pre-spectrum* in X/b is an infinite sequence of pointed objects (x_n, p_n, s_n) together with an infinite sequence of commutative squares

$$\begin{array}{ccc} x_n & \xrightarrow{p_n} & b \\ {\scriptstyle p_n}\downarrow & & \downarrow{\scriptstyle s_{n+1}} \\ b & \xrightarrow{s_{n+1}} & x_{n+1}. \end{array}$$

Clearly, a parametrised pre-spectrum in X is a map $B \to X$, where B is a certain simplicial set. Hence the quasi-category $PPreSpec(X)$ of parametrized pre-spectra in X is of the form X^B for some simplicial set B. It is thus a para-variety (resp. an ∞-topos), since X is a para-variety (resp. an ∞-topos). But the quasi-category $PSpec(X)$ is a left exact reflection of $PPreSpec(X)$ by 36.2, since directed colimits commute with finite limits in X by **??**. It is thus a para-variety (resp. an ∞-topos).

37. Perfect quategories and descent

We introduce the notions of regular and of perfect quategories. The general notion of descent diagram is due to Charles Rezk.

37.1. The map $Eq : \mathcal{K}^I \to Gpd(\mathcal{K})$ which associates to an arrow $f : a \to b$ its equivalence groupoid $Eq(f)$ has a left adjoint B which associates to a groupoid G its *classifying space* BG equipped with the canonical morphism $G_0 \to BG$. If $Surj(\mathcal{K})$ denotes the full sub-quategory of $\mathcal{K}^I$ spanned by the surjections, then the adjoint pair $B \vdash Eq$ induces an equivalence of quategories

$$B : Gpd(\mathcal{K}) \leftrightarrow Surj(\mathcal{K}) : Eq.$$

Similarly, the map $\Omega : 1\backslash\mathcal{K} \to Grp(\mathcal{K})$ which associates to a pointed object $1 \to b$ its loop group $\Omega(b)$ has a left adjoint B which associates to a group object G to its pointed classifying space BG. If $\mathcal{K}(1)$ denotes the full sub-quategory of connected objects of $1\backslash\mathcal{K}$, then the adjoint pair $B \vdash Eq$ induces an equivalence of quategories

$$B : Grp(\mathcal{K}) \leftrightarrow \mathcal{K}(0) : \Omega.$$

37.2. Let X be a cartesian quategory. Consider the map $Eq : X^I \to Gpd(X)$ which associates to an arrow $u : a \to b$ its equivalence groupoid $Eq(u)$. We shall say that a groupoid object $C : \Delta^o \to X$ is *effective* if it has a colimit $p : C_0 \to BC$ and the canonical functor $C \to Eq(p)$ is invertible.

37.3. We say that a cartesian quategory X is *regular* if it admits surjection-mono factorisations and the following conditions are satisfied:

- the base change of a surjection is a surjection;
- every surjection is the colimit of its equivalence groupoid.

We say that a regular quategory X is *perfect* if every groupoid is effective.

37.4. The quategory $\mathcal{K}$ is perfect. An α-variety is perfect for any regular ordinal α. If X is a perfect quategory, then so are the quategories $b\backslash X$ and X/b for any object $b \in X$, the quategory X^A for any simplicial set A and the quategory $Alg(T, X)$ for any (finitary) algebraic theory T. A variety of modules is perfect. A left exact reflection of a perfect quategory is perfect.

37.5. We say that a left exact map between regular quategories is *exact* if it preserves surjections. If X is a regular quategory, we say that an object $a \in X$ is *projective* if the map $hom(a, -) : X \to \mathcal{K}$ is exact.

37.6. If X is a regular quategory, then the base change map $u^* : X/b \to X/a$ is exact for any morphism $u : a \to b$. An object $a \in X$ is projective iff every surjection $b \to a$ has a section.

37.7. In a variety of homotopy algebras X, an object $a \in X$ is perfect iff it is compact and projective.

37.8. Let X be a perfect quategory. Then the map $Eq : X^I \to Gpd(X)$ which associates to an arrow $f : a \to b$ its equivalence groupoid $Eq(f)$ has left adjoint B which associates to a groupoid C its *classifying space* BC equipped with the canonical morphism $C_0 \to BC$. If $Surj(X)$ denotes the full sub-quategory of X^I spanned by the surjections, then the adjoint pair $B \vdash Eq$ induces an equivalence of quategories

$$B : Gpd(X) \leftrightarrow Surj(X) : Eq.$$

The canonical map $Ob : Gpd(X) \to X$ is a Grothendieck fibration and its fiber at $a \in X$ is a quategory $Gpd(X, a)$. By definition, an object of $Gpd(X, a)$ is a groupoid $C \in Gpd(X)$ with $C_0 = a$. If $Surj(a, X)$ denotes the full simplicial subset of $a\backslash X$ spanned by the surjection $a \to x$, then the adjoint pair $B \vdash Eq$ induces an equivalence of quategories

$$B : Gpd(X, a) \leftrightarrow Surj(a, X) : Eq.$$

37.9. A perfect quategory X admits n-factorisations for every $n \geq -1$. A morphism $a \to b$ in X is -1-connected iff it is surjective. If $n \geq 0$, a morphism $a \to b$ is n-connected iff it is surjective and the diagonal $a \to a \times_b a$ is $(n-1)$-connected. If $a \to e \to b$ is the n-factorisation of a morphism $a \to b$, then $a \to a \times_e a \to a \times_b a$ is the $(n-1)$-factorisation of the diagonal $a \to a \times_b a$. An exact map $f : X \to Y$ between perfect quategories preserves the n-factorisations for every $n \geq 0$.

37.10. Let X be an perfect pointed quategory. Then the map $\Omega : X \to Grp(X)$ which associates to a pointed object $0 : 1 \to b$ its loop group $\Omega(b)$ has a left adjoint B which associates to a group object G to its (pointed) classifying space BG. If $n \geq 0$, then an object $x \in X$ is n-connected iff the morphism $0 \to x$ is $(n-1)$-connected. Let us denote by $X(n)$ the full sub-quategory of X spanned by the pointed $(n-1)$-connected object of X. Then the adjoint pair $B \vdash Eq$ induces an equivalence of quategories

$$B : Grp(X) \leftrightarrow X(1) : \Omega.$$

Hence the quategory $X(1)$ is perfect, since the quasi-category $Grp(X)$ is perfect. A morphism in $X(1)$ is n-connected iff it is $(n+1)$ connected in X. Similarly, a morphism in $X(1)$ is a n-cover iff it is a $(n+1)$ cover in X. By ierating the equivalence above we obtain an equivalence of quategories

$$B^n : Grp^n(X) \leftrightarrow X(n) : \Omega^n$$

for every $n \geq 1$.

37.11. An additive quategory X is perfect iff the following five conditions are satisfied:

- X admits surjection-mono factorisations;
- the base change of a surjection is a surjection;
- every morphism has a fiber and a cofiber;
- every morphism is the fiber of its cofiber;
- every surjection is the cofiber of its fiber.

37.12. If a quategory X is additive and perfect, then so is the quategory X^A for any simplicial set A and the quategory $Mod^{\times}(T, X)$ for any algebraic theory T.

37.13. Let X be a perfect additive quategory. If $f : x \to y$ is a surjection, then a null sequence $0 = fi : z \to x \to y$ is a fiber sequence iff it is a cofiber sequence.

37.14. Let X be a perfect additive quategory. An object $a \in X$ is discrete iff $\Omega(a) = 0$. A morphism $u : a \to b$ in X is a 0-cover iff its fiber $Ker(u)$ is discrete. An object $a \in X$ is connected iff the morphism $0 \to a$ is surjective. A morphism $a \to b$ is 0-connected iff it is surjective and the fiber $Ker(u)$ is connected. The suspension $\Sigma : X \to X$ induces an equivalence between X and the full sub-quategory of connected objects of X.

37.15. Let X be a perfect additive quategory. Then an object $a \in X$ is a n-object iff $\Omega^n(a) = 0$. An arrow $u : a \to b$ in X is a n-cover iff its fiber $Ker(u)$ is a n-object. An object $a \in X$ is n-connected iff it is connected and $\Omega(a)$ is $(n-1)$-connected. A morphism $a \to b$ is n-connected iff it is surjective and its fiber $Ker(u)$ is n-connected.

37.16. Let X be a cartesian quategory. We say that a a functor $f : C \to D$ in $Cat(X)$ is *fully faithful* if the commutative square

$$\begin{array}{ccc} C_1 & \xrightarrow{f_1} & D_1 \\ {\scriptstyle (s,t)}\downarrow & & \downarrow{\scriptstyle (s,t)} \\ C_0 \times C_0 & \xrightarrow{f_0 \times f_0} & D_0 \times D_0 \end{array}$$

is cartesian. For example, the canonical functor $Eq(u) \to Sk^0(b)$ is fully faithful for every arrow $u : a \to b$ in X. If X is regular, we shall say that a functor $f : C \to D$ is *essentially surjective* if the morphism $q : P \to D_0$ defined in the following pullback square

$$\begin{array}{ccc} P & \xrightarrow{f_1} & J(D)_1 \\ {\scriptstyle (p,q)}\downarrow & & \downarrow{\scriptstyle (s,t)} \\ C_0 \times D_0 & \xrightarrow{f_0 \times D_0} & D_0 \times D_0 \end{array}$$

is surjective, where $J(D)_1$ is the object of isomorphisms of D; we say that $f : C \to D$ is a *weak equivalence* if it is fully faithful and essentially surjective. For example, the canonical functor $Eq(u) \to Sk^0(b)$ is a weak equivalence for every surjection $u : a \to b$ in X.

37.17. Let X be a cartesian quategory. We shall say that a functor $f : C \to D$ in $Cat(X)$ is a *Morita equivalence* if the base change map $f^* : X^D \to X^C$ is an equivalence of quategories. We shall say that an arrow $f : a \to b$ in X is a *descent morphism* if the canonical functor $Eq(f) \to Sk^0(b)$ is a Morita equivalence. An arrow $u : a \to b$ is a descent morphism iff the lifted base change map $\tilde{u}^* : X/b \to X^{Eq(u)}$ is an equivalence of quategories. If X is perfect, then every weak equivalence $f : C \to D$ is Morita equivalence. In particular, every surjection $u : a \to b$ is a descent morphism.

37.18. Let X be a perfect quategory. If $RCat(X)$ denotes the quategory of reduced category objects in X, then the inclusion $RCat(X) \subseteq Cat(X)$ has a left adjoint which associates to a category $C \in Cat(X)$ a reduced category RC. The canonical functor $C \to RC$ is a weak equivalence for every $C \in Cat(X)$.

37.19. Let X be a cartesian quasi-category. If A is a simplicial set, recall that the *appex* of a projective cone $c : A \star 1 \to X$ is defined to be the object $c(1) \in X$. The *appex map* $a : X^{A\star 1} \to X$ defined by putting $a(c) = c(1)$ is a Grothendieck fibration. The base change of a cone $c : A \star 1 \to X$ along an arrow $u : e \to c(1)$ is a cone $u^*(c) : A \star 1 \to X$ obtained by putting

$$u^*(c)(a) = e \times_{c(1)} c(a)$$

for every $a \in A$. We shall say that a colimit cone c and its colimit $c(1)$ are *stable* if the cone $u^*(c)$ is a colimit cone for every arrow $u : e \to c(1)$ in X.

37.20. Let X be a cartesian quasi-category and A be a simplicial set. We say that a natural transformation $\alpha : f \to g : A \to X$ is *cartesian* if the naturality square

$$\begin{array}{ccc} f(a) & \xrightarrow{f(u)} & f(b) \\ \downarrow & & \downarrow \\ g(a) & \xrightarrow{g(u)} & g(b) \end{array}$$

is cartesian for every arrow $u : a \to b$ in A. This notion only depends on the homotopy class of of the natural transformation α. It is thus a property of the 2-cell $[\alpha] : f \to g$ defined by α. The set of cartesian natural transformations is invariant under isomorphism in X^A. Moreover, it is closed under composition, base changes and it has the left cancellation property. We shall say that a cartesian natural transformation $\alpha : f \to g$ a *gluing datum* over $g : A \to X$. We shall denote by $Glue(g)$ the full simplicial subset of X^A/g spanned by the gluing data over g. When the category $\tau_1 A$ is a groupoid, every natural transformation $\alpha : f \to g : A \to X$ is cartesian. Hence we have $Glue(g) = X^A/g$ in this case.

37.21. A functor $f : C \to D$ in $Cat(X)$ is a natural transformation $f : C \to D : \Delta^o \to X$. The natural transformation is cartesian iff the funhctor f is both a left and a right fibration.

37.22. Let X be a cartesian quasi-category. If $u : A \to B$ is a map of simplicial sets, then the map $X^u : X^B \to X^A$ takes a cartesian natural transformation to a cartesian natural transformation. It thus induces a map

$$u^* : Glue(g) \to Glue(gu)$$

for any diagram $g : B \to X$. We shall say that u^* is the *restriction* along u. The restriction along a final map is an equivalence of quategories. In particular, for any cone $c : A \star 1 \to X$, the restriction along the inclusion $1 \subseteq A \star 1$ is an equivalence of quategories, $Glue(c) \to X/c(1)$. The *spread map*:

$$X/c(1) \to Glue(ci).$$

is obtained by composing the inverse equivalence with the restriction

$$i^* : Glue(c) \to Glue(ci)$$

along the inclusion $i : A \subseteq A \star 1$. We say that a diagram $d : A \to X$ is a *descent diagram* if it has a colimit b and the spread map

$$\sigma : X/b \to Glue(d)$$

associated to the colimit cone $c : A \star 1 \to X$ is an equivalence of quategories. In which case, the inverse equivalence associates to a cartesian morphism $f \to g$ its colimit

$$\varinjlim_{a \in A} f(a) \to \varinjlim_{a \in A} g(a) = b.$$

The colimit of a descent diagram is stable under base change.

37.23. (Rezk) In the quategory $\mathcal{K}$, every diagram is a descent diagram (and every colimit is stable under base change). Let us sketch a proof using the correspondance $f \mapsto El(f)$ of 15.5. If B is a simplicial set, let us denote by $\mathcal{K}(B)$ the full subcategory of $\mathbf{S}/B$ whose objects are the Kan fibrations $X \to B$. The category $\mathcal{K}(B)$ is enriched over Kan complexes. Moreover, if $i : A \to B$ is a weak homotopy equivalence, then the map $i^* : \mathcal{K}(B) \to \mathcal{K}(A)$ is a Dwyer-Kan equivalence. If $g : A \to \mathcal{K}$ is a diagram and $G = El(g)$, let us choose a weak homotopy equivalence $i : G \subseteq Y$ with Y a Kan complex. Then the object $Y \in \mathcal{K}$ is the colimit of g. A natural transformation $\alpha : f \to g : A \to \mathcal{K}$ is cartesian iff the map $El(\alpha) : El(f) \to El(g) = G$ is a *homotopy covering* in the sense of 11.22. It follows that the quasi-category $Glue(g)$ is equivalent to the coherent nerve of the simplicial category $\mathcal{K}(G)$. Moreover, the spread map $\mathcal{K}/Y \to Glue(g)$ is induced by the functor $i^* : \mathcal{K}(Y) \to \mathcal{K}(G)$. It is thus an equivalence of quategories, since i is a weak homotopy equivalence.

37.24. In a finitary presentable quategory, every directed diagram is a descent diagram (and every directed colimit is stable under base changes). In a variety of homotopy algebras, every sifted diagram is a descent diagram (and every sifted colimit is stable under base changes).

38. Stable quategories

38.1. Let X be a null-pointed quategory. Recall that the loop space $\Omega(x)$ of an object $x \in X$ is defined to be the fiber of the arrow $0 \to x$. By definition, we have a pullback square

$$\begin{array}{ccc} \Omega(x) & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & x. \end{array}$$

We shall say that X is *stable* if if every object $x \in X$ has a loop space, and the map $\Omega : X \to X$ is an equivalence of quategories. We shall say that a pointed map between stable quategories $f : X \to Y$ is *stable* if the canonical morphism $f \circ \Omega \to \Omega \circ f$ is invertible. A stable quategory with finite products is additive.

38.2. If X is a stable quategory, then the inverse of the map $\Omega : X \to X$ is the suspension $\Sigma : X \to X$. We shall put

$$\Omega^{-n} = \Sigma^n \quad \text{and} \quad \Sigma^{-n} = \Omega^n$$

for every $n \geq 0$. A null sequence

$$\begin{array}{ccc} x & \xrightarrow{0} & 0 \\ & {\scriptstyle 0}\searrow & \downarrow{\scriptstyle 0} \\ & & y \end{array}$$

is a fiber sequence iff it is a cofiber sequence. The opposite of a stable quategory is stable and we have $\Omega(x^o) = \Sigma(x)^o$ for every object $x \in X$. The opposite of stable map $f : X \to Y$ between stable quategories is stable.

38.3. In a stable cartesian quategory, a null sequence

$$\begin{array}{ccc} x & \longrightarrow & y \\ & \searrow & \downarrow \\ & & z \end{array}$$

is a fiber sequence iff it is a cofiber sequence; a commutative square

$$\begin{array}{ccc} x & \longrightarrow & y \\ \downarrow & & \downarrow \\ u & \longrightarrow & z \end{array}$$

is cartesian iff it is cocartesian. If a stable quategory is cartesian iff it is cocartesian. A map between stable cartesian quategories is finitely continuous iff it is finitely cocontinuous.

38.4. A locally presentable quategory X is stable iff it is null-pointed and the suspension $\Sigma : X \to X$ is an equivalence of quategories. If $X, Y \in \mathbf{LP}$ and one of the quategories X or Y is stable then the quategories $X \otimes Y$ and $\mathrm{CC}(X, Y)$ are stable. We shall denote by **SLP** the full sub-category of **LP** spanned by the stable quategories. The inclusion $\mathbf{SLP} \subset \mathbf{LP}$ has both a left and a right adjoint; the left adjoint is the functor $X \mapsto \mathcal{S}_\infty \otimes X = Spec(X)$ and the right adjoint is the functor $X \mapsto \mathrm{CC}(\mathcal{S}_\infty, X) = Spec(X^o)^o$. The (simplicial) category **SLP** is symmetric monoidal closed if the unit object is taken to be the quategory $\mathcal{S}_\infty = \mathrm{Mod}Spec$. If $X \in \mathbf{SLP}$, then the equivalence $\mathcal{S}_\infty \otimes X \simeq X$ is induced by a map

$$\otimes : \mathcal{S}_\infty \times X \to X$$

called the *tensor product*. The tensor product is the basic ingredient of a symmetric monoidal closed structure on the quategory $\mathcal{S}_\infty$. Every quategory $X \in \mathbf{SLP}$ is enriched and cocomplete over the monoidal quategory $\mathcal{S}_\infty$.

38.5. If X is a locally presentable stable quategory, then the opposite of the map $Hom_X : X^o \times X \to \mathcal{S}_\infty$ is cocontinuous in each variable and the resulting map

$$X^o \to \mathrm{CC}(X, \mathcal{S}_\infty^o)$$

is an equivalence of quategories as in 28.25.

38.6. Recall that the category of cartesian theories and left exact maps is denoted by **CT**. We shall denote by **SCT** the full subcategory of **CT** spanned by the stable cartesian theories. If $S, T \in \mathbf{SCT}$ and one of the theories S or T is stable then so are the quategories $S \odot T$ and $\mathrm{Model}(S, T)$. When S and T are both additive, we shall put

$$S \otimes T := S \odot T.$$

The (simplicial) category **SCT** is symmetric monoidal closed if the unit object is taken to be the theory *Spec*. The opposite of a stable theory is a stable theory and the functor $T \mapsto T^o$ respects the symmetric monoidal structure. In particular the quategory *Spec* is equivalent to its opposite. The inclusion functor

$$\mathbf{SCT} \to \mathbf{CT}$$

admits both a left and a right adjoint. The left adjoint is the functor $T \mapsto Spec \odot T$ and its right adjoint is the functor $T \mapsto Spec(T)$. If T is null-pointed, then the quategory $Spec \odot T$ is the (homotopy) colimit of the sequence of quategories

$$T \xrightarrow{\Omega} T \xrightarrow{\Omega} T \xrightarrow{\Omega} \cdots .$$

38.7. If $T \in SCT$ and $X \in \mathbf{LP}$ then the map

$$\mathrm{Model}(T, Spec(X)) \to \mathrm{Model}(T, X)$$

induced by the forgetful map $Spec(X) \to X$ is an equivalence of quategories. In particular, the map

$$\mathrm{Model}(T, \mathcal{L}_\infty) \to \mathrm{Model}(T, \mathcal{K})$$

induced by the forgetful map $\mathcal{S}_\infty \to \mathcal{K}$ is an equivalence of quategories. We shall say that a model $f : T \to \mathcal{L}_\infty$ is a *stable left T-module* and put

$$\mathrm{SMod}(T) = \mathrm{Model}(T, \mathcal{S}_\infty).$$

Dually, we shall say that model $f : T^o \to \mathcal{S}_\infty$ is a *stable right T-module*. If S and T are stable theories, we say that a model $f : S^o \otimes T \to \mathcal{S}_\infty$ is a *stable (T, S)-bimodule* and put

$$\mathrm{SMod}(S, T) = \mathrm{SMod}(S^o \otimes T).$$

38.8. We shall say that a quategory is a *stable variety* if it is equivalent to a quategory $\mathrm{SMod}(T)$ for a stable cartesian theory T. For example, the quategory $\mathcal{S}_\infty = \mathrm{SMod}(Spec)$ is a stable variety.

38.9. If $u : S \to T$ is a morphism of stable cartesian theories, then the map

$$u^* : \mathrm{SMod}(T) \to \mathrm{SMod}(S)$$

induced by u has a left adjoint $u_!$ and a right adjoint u_*.

38.10. If T is a stable cartesian theory, then the map $hom : T^o \times T \to \mathcal{K}$ is left exact in each variable. It thus induces a left exact map $Hom_T : T^o \otimes T \to \mathcal{E}_\infty$ by 38.7. The resulting *Yoneda map*

$$y : T^o \to \mathrm{SMod}(T)$$

is fully faithful and left exact. We say that a stable module $T \to \mathcal{L}_\infty$ is *representable* if it is isomorphic to a module $y(a)$ for some object $a \in T$. Then the map y induces an equivalence between T^o and the full sub-quategory of $\mathrm{SMod}(T)$ spanned by the representable sable left modules. There is a dual *Yoneda map*

$$y : T \to \mathrm{SMod}(T^o)$$

and a notion of representable stable right module.

38.11. We say that a quategory is a *para-variety of stable modules* if it is equivalent to a quategory $\mathrm{SMod}(T)$ for some stable additive theory T.

38.12. If $u : S \to T$ is a morphism of stable theories, then the map

$$u^* : \mathrm{SMod}(T) \to \mathrm{SMod}(S)$$

induced by u has a left adjoint $u_!$ and a right adjoint u_*.

38.13. If T is stable cartesian theory, then the map $hom : T^o \times T \to \mathcal{K}$ is finitely bicontinuous in each variable. It thus induces a cartesian map $Hom_T : T^o \otimes_s T \to \mathcal{S}_\infty$. The resulting *Yoneda map*

$$y : T^o \to \mathrm{SMod}(T)$$

is fully faithful and finitely bicontinuous.

We say that a stable left module $T \to \mathcal{S}_\infty$ is *representable* if it is isomorphic to a module $y(a)$. Then the map y induces an equivalence between T^o and the full sub-quategory of $\mathrm{SMod}(T)$ spanned by the representable stable left modules. There is a dual *Yoneda map*

$$y : T \to \mathrm{SMod}(T^o)$$

and a notion of representable stable right module.

38.14. If X is a cocomplete stable additive quategory, we shall say that an object $a \in X$ is *perfect* iff the map $Hom(a, -) : X \to \mathbf{L}_\infty$ is cocontinuous. If X is a stable

38.15. We denote by $SAdd$ the stable algebraic theory freely generated by one object $u \in SAdd$. Every object $f \in SAdd$ is a finite direct sum

$$f = \bigoplus_{i \in F} \Sigma^{n_i}(u)$$

where n_i is an integer. We say that a stable algebraic theory T is *unisorted* if it is equipped with an essentially surjective map $SAdd \to T$. A "ring spectrum" is essentially the same thing as a unisorted stable theory. In other words, a stable algebraic theory T is a "ring spectrum with many objects". A stable model $f : T \to \mathbf{Spec}$ is a left T-module.

38.16. The opposite of a stable algebraic theory is a stable algebraic theory. The stable theory $SAdd^o$ is freely generated by the object $u^o \in SAdd$. Hence the stable morphism $SAdd \to SAdd^o$ which takes u to u^o is an equivalence. The duality takes the object $\Sigma^n(u)$ to the object $\Sigma^{-n}(u)$ for every integer n.

38.17. Every stable algebraic theory T generates freely a cartesian theory $u : T \to T_c$. By definition, T_c is a pointed cartesian theory and $u : T \to T_c$ is a stable morphism which induces an equivalence of quasi-categories

$$Mod(T_c, X) \simeq SProd(T, X)$$

for any pointed cartesian quasi-category X. The cartesian theory T_c is stable. For example, we have $SAdd_c = Spec$.

38.18. The quasi-category of spectra $\mathbf{Spec}$ is exact. More generally, if T is a stable algebraic theory, then the quasi-category $SProd(T)$ is stable and exact.

38.19. Let us sketch a proof of 38.18. The quasi-category $\mathbf{Spec}$ is a para-variety by 39.7. It is thus exact by **??**. Let us show that the quasi-category $SProd(T, \mathbf{Spec})$ is stable and exact. It is easy to see that it is stable. Let us show that it is a para-variety. The quasi-category $Prod(T, \mathbf{Spec})$ is a para-variety by 39.4. Hence it suffices to show that the quasi-category $SProd(T, \mathbf{Spec})$ is a left exact reflection of the quasi-category $Prod(T, \mathbf{Spec})$. A model $f : T \to \mathbf{Spec}$ is stable iff the the canonical natural transformation $\alpha : f \to \Omega f \Sigma$ is invertible. By iterating, we obtain an infinite sequence

$$f \xrightarrow{\alpha} \Omega f \Sigma \xrightarrow{\Omega\alpha\Sigma} \Omega^2 f \Sigma^2 \longrightarrow \cdots$$

The colimit $R(f)$ of this sequence is a stable map $T \to \mathbf{Spec}$. This defines a left exact reflection

$$R : Prod(T, \mathbf{Spec}) \to SProd(T, \mathbf{Spec}).$$

Thus, $SProd(T, \mathbf{Spec})$ is a para-variety. Hence it is exact by **??**.

38.20. An additive quasi-category X is stable and exact iff the following two conditions are satisfied:

- Every morphism has a fiber and a cofiber;
- A null sequence $z \to x \to y$ is a fiber sequence iff it is a cofiber sequence.

38.21. Let us sketch a proof of 37.11. (⇒) Every morphism in X has a fiber and a cofiber by 37.11, since X is exact and additive. Let us show that every arrow is surjective. For this it suffices to show that every monomorphism is invertible, since every arrow is right orthogonal to every quasi-isomorphism. If $u : a \to b$ is a monomorphism, then we have a fiber sequence

$$\Omega(a) \xrightarrow{\Omega(u)} \Omega(b) \longrightarrow 0 \longrightarrow a \xrightarrow{u} b\ .$$

by 34.14. Thus, $\Omega(u)$ invertible, since it is the fiber of a nul morphism, It follows that u is invertible, since the map $\Omega : X \to X$ is an equivalence. We have proved that every arrow is surjective. It then follows from 37.13 that a nul sequence $z \to x \to y$ is a fiber sequence iff it is a cofiber sequence. (⇐) Let us show that X is stable. If $x \in X$, then we have $\Sigma\Omega(x) \simeq x$, since the fiber sequence $\Omega(x) \to 0 \to x$ is a cofiber sequence. Moreover, we have $x \simeq \Omega\Sigma(x)$, since the cofiber sequence $x \to 0 \to \Sigma(x)$ is a fiber sequence. This shows that X is stable. It remains to show that X is exact. For this, it suffices to show that the conditions of 37.11 are satisfied. Let us first show that X admits surjection-mono factorisations. For this, it suffices to show that every monomorphism is invertible. If $x \to y$ is monic, then the sequence $0 \to x \to y$ is a cofiber sequence, since it is a fiber sequence. It follows that the arrow $x \to y$ is invertible. This proves that every monomorphism is invertible. Thus, every morphism is surjective. Hence the base change of a surjection is a surjection.

38.22. The opposite of an exact stable quasi-category is exact and stable.

38.23. An additive map $X \to Y$ between two exact stable quasi-categories is exact iff it is left exact iff it is right exact.

38.24. Let X be an exact stable quasi-category. Then to each arrow $f : x \to y$ in X we can associate by 34.14 a *two-sided long fiber sequence*,

$$\cdots \Omega(x) \xrightarrow{\Omega(f)} \Omega(y) \xrightarrow{\partial} z \xrightarrow{i} x \xrightarrow{f} y \xrightarrow{\partial} \Sigma(z) \xrightarrow{\Sigma(i)} \Sigma(x) \cdots .$$

where $i : z \to x$ is the fiber of f. The sequence is entirely described by a triangle

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ {\scriptstyle i}\nwarrow & & \swarrow{\scriptstyle \partial} \\ & z & \end{array}\ .$$

where ∂ is now regarded as a morphism of degree -1 (ie as a morphism $y \to \Sigma(x)$).

38.25. If A and B are two stable algebraic theories then so is the quasi-category $SProd(A, B)$ of stable models $A \to B$. The 2-category **SAT** is symmetric monoidal closed. The *tensor product* $A \odot_S B$ of two stable algebraic theories is the target of a map $A \times B \to A \odot_S B$ which is a stable morphism in each variable (and which is universal with respect to that property). There is a canonical equivalence of quasi-categories

$$SProd(A \odot_S B, X) \simeq SProd(A, SProd(B, X))$$

for any cartesian quasi-category X. In particular, we have two equivalences of quasi-categories,

$$SProd(A \odot_S B) \simeq SProd(A, SProd(B)) \simeq SProd(B, Prod(A)).$$

The unit for the tensor product is the theory $SAdd$ described in **??**. The opposite of the canonical map $S \times T \to S \odot T$ can be extended along the Yoneda maps as a map cocontinuous in each variable.

$$SProd(A) \times SProd(B) \to SProd(A \odot_S B).$$

38.26. Let **CT** be the (2-)category of cartesian theories. Then the full sub(2-)category **SCT** of **CT** spanned by the stable cartesian theories is (pseudo) reflective and coreflective. The left adjoint to the inclusion **SCT** $\subset$ **CT** is the functor $T \mapsto T \odot_c Spec$ and its right adjoint is the functor $T \mapsto Mod(Spec, T) \simeq SProd(SAdd, T)$.

38.27. Let **LP** be the (2-)category of locally representable quasi-categories. Then the full sub(2-)category **SLP** of **LP** spanned by the stable locally presentable quasi-categories is (pseudo) reflective and coreflective. The left adjoint to the inclusion **SLP** $\subset$ **LP** is the functor $X \mapsto Mod(Spec, X) \simeq SProd(SAdd, X) \simeq X \otimes \mathbf{Spec}$ and its right adjoint is the functor $X \mapsto Map(\mathbf{Spec}, X)$.

38.28. If A is a stable quasi-category, then the map $hom_A : A^o \times A \to \mathbf{U}$ admits a factorisation

$$\begin{array}{ccc} & & \mathbf{Spec} \\ & \overset{hom'_A}{\nearrow} & \big\downarrow U \\ A^o \times A & \xrightarrow{\ hom_A\ } & \mathbf{U}, \end{array}$$

where the map hom'_A is stable in each variable, and where U is the forgetful map. The factorisation is unique up to a unique invertible 2-cell. This defines an "enrichement" of the quasi-category A over the quasi-category of spectra **Spec**. The *Yoneda map*

$$y : A^o \to \mathbf{Spec}^A$$

is obtained from hom'_A by exponential adjointness.

38.29. If T is a stable algebraic theory, then the Yoneda map $y : T^o \to \mathbf{Spec}^T$ induces a map $y : T^o \to SProd(T)$. We say that a model $f : T \to \mathbf{Spec}$ is *representable* if it belongs to the essential image of the Yoneda map.

39. Para-varieties

39.1. Recall that a map between two quategories $r : Y \to X$ is said to be a *reflection* if it has a fully faithful right adjoint. A reflection $r : Y \to X$ is *left exact* if it preserves finite limits. If X and Y are locally presentables, a reflection $r : Y \to X$ is called a *Bousfield localisation.* We shall say that a locally presentable quategory X is a *para-variety* if it is a left exact Bousfield localisation $Y \to X$ of a a variety of homotopy algebras Y.

39.2. A locally presentable quategory X is a left exact Bousfield localisation of a finitary presentable quategory iff directed colimits and finite limits commute in X.

39.3. Sifted colimits and finite products commute in any para-variety. A locally presentable quategory X is a para-variety iff the following conditions are satisfied:

- X is exact;
- directed colimits and finite limits commute in X;
- Δ^o-indexed colimits commute with finite products in X.

39.4. A left exact Bousfield localisation of a para-variety is a para-variety. If X is a para-variety, then so are the slice quategories $a\backslash X$ and X/a for any object $a \in X$ and the quategory X^A for any simplicial set A. More generally, the quategory $Alg(T, X)$ is a para-variety for any algebraic theory T.

39.5. Let $X \subseteq Y$ be a left exact reflection of a cartesian quategory Y. Then a diagram $g : A \to X$ which is a descent diagram in Y is also a descent diagram in X. Let us sketch a proof. Let i be the inclusion $X \subseteq Y$. The composite $ig : A \to Y$ is a descent diagram by assumption. If b is the colimit of the ig, then $r(b)$ is the colimit of g in X. Consider the diagram

$$\begin{array}{ccccc} X/r(b) & & \xrightarrow{\sigma'} & & Glue(g) \\ \downarrow{\scriptstyle i_0} & & & & \downarrow{\scriptstyle i_1} \\ Y/r(b) & \xrightarrow{p^*} & Y/b & \xrightarrow{\sigma} & Glue(ig), \end{array}$$

where i_0 and i_1 are induced by $i : X \subseteq Y$, where σ and σ' are the spread maps, and where p^* is base change along the canonical arrow $p : b \to r(b)$. It is easy to see that the diagram commutes up to a canonical isomorphism. The map $q : Y/b \to X/r(b)$ induced by r is left adjoint to the composite $p^*i_0 : X/r(b) \to Y/r(b) \to Y/b$. Moreover, the counit of the adjunction $q \vdash p^*i_0$ is invertible by the left exactness of r. Thus. p^*i_0 is fully faithful. It follows that $i_1\sigma' = \sigma p^* i_0$ is fully faithful, since σ is an equivalence by assumption. Thus, σ' is fully faithful, since i_1 is fully faithful. It remains to show that σ' is essentially surjective. Let $\alpha : f \to g$ be an object of $Glue(g)$ and $u : a \to b$ be the colimit of α in Y. Then the canonical square

$$\begin{array}{ccc} f(a) & \longrightarrow & a \\ \downarrow{\scriptstyle \alpha(a)} & & \downarrow{\scriptstyle u} \\ g(a) & \longrightarrow & b \end{array}$$

is a pullback for every $a \in A$, since g is a descent diagram in Y. Hence the square

$$\begin{array}{ccc} f(a) & \longrightarrow & r(a) \\ \downarrow{\scriptstyle \alpha(a)} & & \downarrow{\scriptstyle r(u)} \\ g(a) & \longrightarrow & r(b), \end{array}$$

is also a pullback in X, since r is left exact. This proves that σ' is essentially surjective.

39.6. In a para-variety, every sifted diagram is a descent diagram (and every sifted colimit is stable under base changes).

39.7. If X is a para-variety, then so is the quasi-category $Spec(X)$ of spectra in X. Let us sketch a proof. We can suppose that X is pointed. We then have $Spec(X) = S(X, \Sigma)$, where $\Sigma : X \to X$ is the suspension map. Let us show that $S(X, \Sigma)$ is a para-variety if X is a para-variety A *pre-spectrum* in X is an infinite sequence of pointed objects (x_n) together with an infinite sequence of commutative squares

$$\begin{array}{ccc} x_n & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & x_{n+1}, \end{array}$$

The notion of pre-spectrum is essentially algebraic and finitary. Let us denote by PS the algebraic theory of pre-spectra. The quasi-category $PS(X) = \mathrm{Alg}(PS, X)$ is a para-variety by 39.4, since X is a para-variety. But the quasi-category $Spec(X)$ is a left exact reflection of $PS(X)$ by 36.1, since directed colimits commute with finite limits in X by **??**. It is thus a para-variety.

40. Homotopoi (∞-topoi)

The notion of homotopos (∞-topos) presented here is due to Carlos Simpson and Charles Rezk.
EEE

40.1. Every diagram in an ∞-topos is a descent diagram (and every colimit is stable under base changes).
EEE

40.2. Recall that a category $\mathcal{E}$ is said to be a *Grothendieck topos*, but we shall say a *1-topos*, if it is a left exact reflection of a presheaf category $[C^o, \mathbf{Set}]$. This means that $\mathcal{E}$ is equivalent to a reflective category of $[C^o, \mathbf{Set}]$, with a reflection functor $[C^o, \mathbf{Set}] \to \mathcal{E}$ which is left exact.

40.3. We call a locally presentable quasi-category X an ∞*-topos* if it is a left exact reflection of a quasi-category of pre-stacks $\mathbf{P}(A)$ for some simplicial set A. If $n \geq 0$ we call a locally presentable quasi-category X a n*-topos* if it is a left exact reflection of a quasi-category of n-pre-stacks $\mathbf{P}(A)(n)$ for some simplicial set A.
EEE

40.4. Recall from 40.2 that a category $\mathcal{E}$ is said to be a *Grothendieck topos* if it is a left exact reflection of a presheaf category $[C^o, \mathbf{Set}]$. A *homomorphism* $\mathcal{E} \to \mathcal{F}$ between Grothendieck topoi is a cocontinuous functor $f : \mathcal{E} \to \mathcal{F}$ which preserves finite limits. The 2-category of Grothendieck topoi and homomorphism is has the structure of a 2-category, where a 2-cell is a natural transformation. Every homomorphism has a right adjoint. A *geometric morphism* $\mathcal{E} \to \mathcal{F}$ is an adjoint pair

$$g^* : \mathcal{F} \leftrightarrow \mathcal{E} : g_*$$

with g^* a homomorphism. The map g^* is called the *inverse image part of* g and the map g_* its *direct image part.* . We shall denote by **Gtop** the category of Grothendieck topoi and geometric morphisms. The category **Gtop** has the structure of a 2-category, where a 2-cell $\alpha : f \to g$ is a natural transformation $\alpha : g^* \to f^*$. The 2-category **Gtop** is equivalent to the opposite of the 2-category of Grothendieck topoi and homomophism.

40.5. Recall from 40.3 that a locally presentable quasi-category X is said to be a *homotopos*, or an ∞-*topos*, if it is a left exact reflection of a quasi-category of prestacks. $\mathbf{P}(A)$ for some simplicial set A. The quasi-category of homotopy types **Tp** is the archtype of a homotopos. If X is a homotopos, then so is the quasi-category X/a for any object $a \in X$ and the quasi-category X^A for any simplicial set A.

40.6. Recall that a cartesian quasi-category X is said to be *locally cartesian closed* if the quasi-category X/a is cartesian closed for every object $a \in X$. A cartesian quasi-category X is locally cartesian closed iff the base change map $f^* : X/b \to X/a$ has a right adjoint $f_* : X/a \to X/b$ for any morphism $f : a \to b$ in X.

40.7. A locally presentable quasi-category X is locally cartesian closed iff the base change map $f^* : X/b \to X/a$ is cocontinuous for any morphism $f : a \to b$ in X.

40.8. (Giraud's theorem)[Lu1] A locally presentable quasi-category X is a homotopos iff the following conditions are satisfied:

- X is locally cartesian closed;
- X is exact;
- the canonical map

$$X/\sqcup a_i \to \prod_i X/a_i$$

is an equivalence for any family of objects $(a_i : i \in I)$ in X.

40.9. A *homomorphism* $X \to Y$ between utopoi is a cocontinuous map $f : X \to Y$ which preserves finite limits. Every homomorphism has a right adjoint. A *geometric morphism* $X \to Y$ between utopoi is an adjoint pair

$$g^* : Y \leftrightarrow X : g_*$$

with g^* a homomorphism. The map g^* is called the *inverse image part of* g and the map g_* the *direct image part.* . We shall denote by **Utop** the category of utopoi and geometric morphisms. The category **Utop** has the structure of a 2-category, where a 2-cell $\alpha : f \to g$ between geometric morphisms is a natural transformation $\alpha : g^* \to f^*$. The opposite 2-category $\mathbf{Utop}^o$ is equivalent to the sub (2-)category of **LP** whose objects are utopoi, whose morphisms (1-cells) are the homomorphisms, and whose 2-cells are the natural transformations.

40.10. If $u : A \to B$ is a map of simplicial sets, then the pair of adjoint maps $u^* : \mathbf{P}(B) \to \mathbf{P}(A) : u_*$ is a geometric morphism $\mathbf{P}(A) \to \mathbf{P}(B)$. If X is a homotopos, then the adjoint pair $f^* : X/b \to X/a : f_*$ is a geometric morphism $X/a \to X/b$ for any arrow $f : a \to b$ in X.

40.11. Recall that if X is a bicomplete quasi-category and A is a simplicial set, then every map $f : A \to X$ has a left Kan extension $f_! : \mathbf{P}(A) \to X$. A locally presentable quasi-category X is a homotopos iff the map $f_! : \mathbf{P}(T) \to X$ is left exact for any cartesian theory T and any cartesian map $f : T \to X$.

40.12. If X is a homotopos, we shall say that a reflexive sub quasi-category $S \subseteq X$ is a *sub-homotopos* if it is locally presentable and the reflection functor $r : X \to S$ preserves finite limits. If $i : S \subseteq X$ is a sub-homotopos and $r : X \to S$ is the reflection, then the pair (r, i) is a geometric morphism $S \to X$. In general, we say that a geometric morphism $g : X \to Y$ is an *embedding* if the map $g_* : X \to Y$ is fully faithful. We say that a geometric morphism $g : X \to Y$ is *surjective* if the map $g^* : Y \to X$ is conservative. The (2-) category **Utop** admits a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class if surjections and $\mathcal{B}$ the class of embeddings.

40.13. If X is a homotopos, then the quasi-category $Dis(X)$ spanned by the 0-objects of X is (equivalent to) a Grothendieck topos. The inverse image part of a geometric morphism $X \to Y$ induces a homorphism $Dis(Y) \to Dis(X)$, hence also a geometric morphism $Dis(X) \to Dis(Y)$. The 2-functor

$$Dis : \mathbf{Utop} \to \mathbf{Gtop}$$

has a right adjoint constructed as follows. If $\mathcal{E}$ is a Grothendieck topos, then the category $[\Delta^o, \mathcal{E}]$ of simplicial sheaves on $\mathcal{E}$ has a simplicial model structure. The coherent nerve of the category of fibrant objects of $[\Delta^o, \mathcal{E}]$ is a homotopos $\hat{\mathcal{E}}$ and there is a canonical equivalence of categories $Dis(\hat{\mathcal{E}}) \simeq \mathcal{E}$. The 2-functor

$$(\hat{-}) : \mathbf{Gtop}(1) \to \mathbf{Utop}$$

is fully faithful and left adjoint to the functor Dis. Hence the (2-)-category **Gtop** is a reflective sub-(2)-category of **Utop**.

40.14. A set Σ of arrows in a homotopos X is called a *Grothendieck topology* if the quasi-category of Σ-local objects $X^\Sigma \subseteq X$ is a sub-homotopos. Every sub-homotopos of X is of the form X^Σ for a Grothendieck topology Σ. In particular, if A is a simplicial set, every sub-homotopos of $\mathbf{P}(A)$ is of the form $\mathbf{P}(A)^\Sigma$ for a Grothendieck topology Σ on A. The pair (A, Σ) is called a *site* and a Σ-local object $f \in \mathbf{P}(A)$ is called a *stack*.

40.15. For every set Σ of arrows in a homotopos X, the sub-quasi-category X^Σ contains a largest sub-homotopos $L(X^\Sigma)$. We shall say that a Grothendieck topology Σ' is *generated* by Σ if we have $X^{\Sigma'} = L(X^\Sigma)$.

is contained in a Grothendieck topology Σ' with the property that a subtopos

then we have $f_*(X) \subseteq Y^\Sigma$ iff f^* take every arrow in Σ to a quasi-isomorphism in X.

40.16. If Σ is Grothendieck topology on Y, then we have $f_*(X) \subseteq Y^\Sigma$ iff f^* take every arrow in Σ to a quasi-isomorphism in X.

40.17. Every simplicial set A generates freely a cartesian quasi-category $A \to C(A)$. Similarly, every simplicial set A generates freely an homotopos $i : A \to UT(A)$. The universality means that every map $f : A \to X$ with values in a homotopos has an homomorphic extension $f' : UT(A) \to X$ which is unique up to a unique invertible 2-cell. By construction, $UT(A) = \mathbf{P}(C(A))$. The map $i : A \to UT(A)$ is obtained by composing the canonical map $A \to C(A)$ with the Yoneda map $C(A) \to \mathbf{P}(C(A))$.

40.18. A *geometric sketch* is a pair (A, Σ), where Σ is a set of arrows in $UT(A)$. A *geometric model* of (A, Σ) with values in a homotopos X is a map $f : A \to X$ whose homomorphic extension $f' : UT(A) \to X$ takes every arrow in Σ to an equimorphism in X. We shall denote by $Mod(A/\Sigma, X)$ the full simplicial subset of X^A spanned by the models $A \to X$.

40.19. Every geometric sktech has a *universal geometric model* $u : A \to UT(A/\Sigma)$. The universality means that for every homotopos X and every geometric model $f : A \to X$ there exists a homomorphism $f' : UT(A/\Sigma) \to X$ such that $f'u = f$, and moreover that f' is unique up to a unique invertible 2-cell. We shall say that $UT(A/\Sigma)$ is the *classifying homotopos* of (A, Σ). The homotopos $UT(A/\Sigma)$ is a sub-homotopos of the homotopos $UT(A)$. We have $UT(A/\Sigma) = UT(A)^{\Sigma'}$, where $\Sigma' \subset UT(A)$ is the Grothendieck topology generated by Σ.

41. Meta-stable quasi-categories

41.1. We say that an exact quasi-category X is *meta-stable* if every object in X is ∞-connected. A cartesian quasi-category X is meta-stable iff if it satisfies the following two conditions:

- Every morphism is a descent morphism;
- Every groupoid is effective.

41.2. The sub-quasi-category of ∞-connected objects in an exact quasi-category is meta-stable. We shall see in 51 that the quasi-category of spectra is meta-stable. In a meta-stable quasi-category, every monomorphism is invertible and every morphism is surjective.

41.3. If a quasi-category X is meta-stable then so are the quasi-categories $b\backslash X$ and X/b for any vertex $b \in X$, the quasi-category X^A for any simplicial set A, and the quasi-category $Prod(T, X)$ for any algebraic theory T. A left exact reflection of a meta-stable quasi-category is meta-stable.

41.4. Let $u : a \to b$ be an arrow in a meta-stable quasi-category X. Then the lifted base change map

$$\tilde{u}^* : X/b \to X^{Eq(u)}$$

of **??** is an equivalence of quasi-categories. In particular, if $u : 1 \to b$ is a pointed object, then the map

$$\tilde{u}^* : X/b \to X^{\Omega_u(b)}$$

defined in 33.4 is an equivalence of quasi-categories.

41.5. Let X be a meta-stable quasi-category. Then the map $Eq : X^I \to Gpd(X)$ which associates to an arrow $u : a \to b$ the equivalence groupoid $Eq(u)$ is invertible. We thus have an equivalence of quasi-categories

$$B : Gpd(X) \leftrightarrow X^I : Eq.$$

The equivalence can be iterated as in **??**. It yields an equivalence of quasi-categories

$$B^n : Gpd^n(X) \leftrightarrow X^{I^n} : Eq^n$$

for each $n \geq 1$.

41.6. Let X be a meta-stable quasi-category. Then the equivalence

$$B : Gpd(X) \leftrightarrow X^I : Eq.$$

induces an equivalence

$$B : Gpd(X,a) \leftrightarrow a\backslash X : Eq$$

for each object $a \in A$, where $Gpd(X,a)$ is the quasi-category of groupoids $C \in Gpd(X)$ with $C_0 = a$. In particular, it induces an equivalence

$$B : Grp(X) \leftrightarrow 1\backslash X : \Omega,$$

where $Grp(X)$ is the quasi-category of groups in X. By iterating, we obtain an equivalence

$$B^n : Grp^n(X) \leftrightarrow 1\backslash X : \Omega^n,$$

for each $n \geq 1$.

41.7. Let **Ex** be the category of exact categories and exact maps. If **MEx** is the full sub-quasi-category of **Ex** spanned by the meta stable quasi-categories, then the inclusion **MEx** $\subset$ **Ex** has a right adjoint which associates to an exact quasi-category X its full sub-quasi-category of meta-stable objects.

42. Higher categories

We introduce the notions of n-fold category object and of n-category object in a quasi-category. We finally introduced the notion of truncated n-category object.

42.1. Let X be a quasi-category. If A is a simplicial set, we say that a map $f : A \to X$ is *essentially constant* if it belongs to the essential image of the diagonal $X \to X^A$. If A is weakly contractible, then a map $f : A \to X$ is essentially constant iff it takes every arrow in A to an isomorphism in X. A simplicial object $C : \Delta^o \to X$ in a quasi-category X is essentially constant iff the canonical morphism $sk^0(C_0) \to C$ is invertible. A category object $C : \Delta^o \to X$ is essentially constant iff it inverts the arrow $[1] \to [0]$. A n-fold category $C : (\Delta^n)^o \to X$ is essentially constant iff C inverts the arrow $[\epsilon] \to [0^n]$ for every $\epsilon = (\epsilon_1, \cdots, \epsilon_n) \in \{0,1\}^n$, where $[0^n] = [0, \ldots, 0]$.

42.2. Let X be a cartesian quasi-category. We call a double category $C : \Delta^o \to Cat(X)$ a *2-category* if the simplicial object $C_0 : \Delta^o \to X$ is essentially constant. A double category $C \in Cat^2(X)$ is a 2-category iff it inverts every arrow in $[0] \times \Delta$. Let us denote by Id the set of identity arrows in Δ. Then the set of arrows

$$\Sigma_n = \bigsqcup_{i+1+j=n} Id^i \times [0] \times \Delta^j$$

is a subcategory of Δ^n. We say that a n-fold category object $C \in Cat^n(X)$ is a *n-category* if it inverts every arrow in Σ_n. The notion of n-category object in X can be defined by induction on $n \geq 0$. A category object $C : \Delta^o \to Cat_{n-1}(X)$ is a n-category iff the $(n-1)$-category C_0 is essentially constant. We denote by Cat_n the cartesian theory of n-categories and by $Cat_n(X)$ the quasi-category of n-category objects in X.

42.3. The object of *k-cells* $C(k)$ of a n-category $C : (\Delta^o)^n \to X$ is the image by C of the object $[1^k 0^{n-k}]$. The *source map* $s : C(k) \to C(k-1)$ is the image of the map $[1^{k-1}] \times d_1 \times [0^{n-k}]$ and the *target map* $t : C(k) \to C(k-1)$ is the image of the map $[1^{k-1}] \times d_0 \times [0^{n-k}]$. From the pair of arrows $(s,t) : C(k) \to C(k-1) \times C(k-1)$ we obtain an arrow $\partial : C(k) \to C(\partial k)$, where $C(\partial k)$ is defined by the following pullback square

$$\begin{array}{ccc} C(\partial k) & \longrightarrow & C(k-1) \\ \downarrow & & \downarrow{\scriptstyle (s,t)} \\ C(k-1) & \xrightarrow{(s,t)} & C(k-2) \times C(k-2). \end{array}$$

If $n = 1$, $\partial = (s,t) : C(1) \to C(0) \times C(0)$.

42.4. There is a notion of n-fold reduced category for every $n \geq 0$. If $RCat$ denotes the cartesian theory of reduced categories, then $RCat^n$ is the theory of n-fold reduced categories. If X is a cartesian quasi-category, then we have

$$RCat^{n+1}(X) = RCat(RCat^n(X))$$

for every $n \geq 0$.

42.5. We say that a n-category $C \in Cat_n(X)$ is *reduced* if it is reduced as a n-fold category. We denote by $RCat_n$ the cartesian theory of reduced n-categories. A n-category $C : \Delta^o \to Cat_{n-1}(X)$ is reduced iff it is reduced as a category object and the $(n-1)$-category C_1 is reduced. If X is an exact quasi-category, then the inclusion $RCat_n(X) \subseteq Cat_n(X)$ has a left adjoint

$$R : Cat_n(X) \to RCat_n(X)$$

which associates to a n-category $C \in Cat_n(X)$ its *reduction* $R(C)$. We call a map $f : C \to D$ in $Cat_n(X)$ an *equivalence* if the map $R(f) : R(C) \to R(D)$ is invertible in $RCat_n(X)$. The quasi-category

$$\mathbf{Typ}_n = Mod(RCat_n)$$

is cartesian closed.

42.6. The object $[0]$ is terminal in Δ. Hence the functor $[0] : 1 \to \Delta$ is right adjoint to the functor $\Delta \to 1$. It follows that the inclusion $i_n : \Delta^n = \Delta^n \times [0] \subseteq \Delta^{n+1}$ is right adjoint to the projection $p_n : \Delta^{n+1} = \Delta^n \times \Delta \to \Delta^n$. For any cartesian quasi-category X, the pair of adjoint maps

$$p_n^* : [(\Delta^o)^n, X] \leftrightarrow [(\Delta^o)^{n+1}, X] : i_n^*$$

induces a pair of adjoint maps

$$inc : Cat_n(X) \leftrightarrow Cat_{n+1}(X) : res.$$

The "inclusion" *inc* is fully faithful and we can regard it as an inclusion by adopting the same notation for $C \in Cat_n(X)$ and $inc(C) \in Cat_{n+1}(X)$. The map *res* associates to $C \in Cat_{n+1}(X)$ its *restriction* $res(C) \in Cat_n(X)$. The adjoint pair $p_n \vdash i_n^*$ also induces an adjoint pair

$$inc : RCat_n(X) \leftrightarrow RCat_{n+1}(X) : res.$$

In particular, it induces an adjoint pair

$$inc : \mathbf{Typ}_n \leftrightarrow \mathbf{Typ}_{n+1} : res.$$

When $n = 0$, the map *inc* is induced by the inclusion $\mathbf{Kan} \subset \mathbf{QCat}$ and the map *res* by the functor $J : \mathbf{QCat} \to \mathbf{Kan}$. The inclusion $\mathbf{Typ}_n \subset \mathbf{Typ}_{n+1}$ has also a left adjoint which associates to a reduced $(n+1)$-category C the reduced n-category obtained by inverting the $(n+1)$-cells of C.

42.7. Recall from **??** that a quasi-category X is said to be *n-truncated* if the simplicial set $X(a, b)$ is a $(n-1)$-object for every pair $a, b \in X_0$. A quasi-category X has a nerve $NX : \Delta^o \to \mathbf{Typ}$ which is a (reduced) category object in $\mathbf{Typ}$ by 30.22. By construction we have $(NX)_p = J(X^{\Delta[p]})$ for every $p \geq 0$. A quasi-category X is n-truncated iff the morphism $(NX)_1 \to (NX)_0 \times (NX)_0$ is a $(n-1)$-cover.

42.8. Let X be a cartesian quasi-category. We say that a category object C in X is *n-truncated* if the morphism $C_1 \to C_0 \times C_0$ is a $(n-1)$-cover. If C is n-truncated and reduced, then C_k is a n-object for every $k \geq 0$.

42.9. The notion of n-truncated category is essentially algebraic and finitary. We denotes the cartesian theory of n-truncated categories by $Cat[n]$. The notion of n-truncated reduced category is also essentially algebraic. We denotes the cartesian theory of n-truncated reduced categories by $RCat[n]$. The equivalence $N : \mathbf{Typ}_1 \simeq Mod(RCat)$ of 30.22 induces an equivalence

$$\mathbf{Typ}_1[n] \simeq Mod(RCat[n])$$

for every $n \geq 0$. In particular, an ordinary category is essentially the same thing as a 1-truncated reduced category in $\mathbf{Typ}$. Recall from **??** that if X is an exact quasi-category, then the inclusion $RCat(X) \subseteq Cat(X)$ has a left adjoint $R : Cat(X) \to RCat(X)$ which associates to a category $C \in Cat(X)$ its *reduction* $R(C)$. If $C \in Cat[n](X)$, then $R(C) \in RCat[n](X)$.

42.10. Let C be a n-category object in a cartesian quasi-category X. If $1 \leq k \leq n$ and $C(k)$ is the object of k-cells of C, then from the pair of arrows $(s, t) : C(k) \to C(k-1) \times C(k-1)$ we obtain an arrow $\partial : C(k) \to C(\partial k)$ by 42.3. If $m \geq n$, we say that C is *m-truncated* if the map $C(n) \to C(\partial n)$ is a $(m-n)$-cover. If $n = 1$, this means that the category C is m-truncated in the sense of 30.23. We shall denote by $Cat_n[m]$ the cartesian theory of m-truncated n-categories. We shall denote by $RCat_n[m]$ the cartesian theory of m-truncated reduced n-categories. If X is an exact quasi-category, then a n-category $C \in Cat_n(X)$ is m-truncated iff its reduction $R(C) \in RCat_n(X)$ is m-truncated. Hence the notion of m-truncated n-category in X is invariant under equivalence of n-categories. If $C \in Cat_n[m](X)$ and $n < m$, then $inc(C) \in Cat_{n+1}[m](X)$. Moreover, if $C \in RCat_n[m](X)$, then $res(C) \in RCat_{n-1}[m](X)$ and C_p is a m-object for every $p \in \Delta^n$. Hence the canonical morphism

$$RCat_n[m] \to RCat_n[m] \odot_c OB(m)$$

is an equivalence of quasi-categories for every $m \geq n$.

43. Higher monoidal categories

The stabilisation hypothesis of Breen-Baez-Dolan was proved by Simpson in [Si2]. We show that it is equivalent to a result of classical homotopy theory 43.1.
EEE

43.1. The suspension theorem of Freudenthal implies that a pointed n-connected space with vanishing homotopy groups in dimension $> 2n$ is naturally a loop space [May2]. The $(n+1)$-fold loop space functor induces an equivalence between the homotopy category of pointed n-connected spaces and the homotopy category of $(n+1)$-fold loop spaces by a classical result [**?**]. The $(n+1)$-fold loop space of a $2n$-object is a $(n-1)$ object. It then follows from Freudenthal theorem that a $(n+1)$-fold loop space with vanishing homotopy groups in dimension $> n-1$ is naturally a $(n+2)$-fold loop space. This means that the forgetful map

$$Grp^{n+2}(\mathbf{U}[n-1]) \to Grp^{n+1}(\mathbf{U}[n-1])$$

is an equivalence of quategories for every $n \geq 1$. It follows that the quategory

$$Grp^{n+1}(\mathbf{U}[n-1]) = \mathrm{Mod}(OB[n-1] \odot Grp^{n+1})$$

is additive for every $n \geq 1$, hence also the cartesian theory $OB[n-1] \odot Grp^{n+1}$. Equivalently, the cartesian theory

$$OB[n] \odot Grp^{n+2}$$

is additive for every $n \geq 0$.

43.2. (Generalised Suspension Theorem) The cartesian theory

$$OB[n] \odot Mon^{n+2}$$

is semi-additive for every $n \geq 0$.
EEE

43.3. If Mon denotes the theory of monoids, then Mon^k is the theory of k-monoids and $Mon^k \odot Cat_n$ the theory of k-monoidal n-categories. For any cartesian quasi-category X we have

$$Mod(Mon^k \odot Cat_n, X) = Cat_n(Mon^k(X)).$$

If X is an exact quasi-category, then inclusion $RCat_n(Mon^k(X)) \subseteq Cat_n(Mon^k(X))$ has a left adjoint

$$R : Cat_n(Mon^k(X)) \to RCat_n(Mon^k(X)),$$

since the quasi-category $Mon^k(X)$ is exact. We call a map $f : C \to D$ between k-monoidal n-categories in X an *equivalence* if the map $R(f) : R(C) \to R(D)$ is invertible in $RCat_n(Mon^k(X))$.

43.4. An object of the quasi-category $Mod^k(Cat_n[n](X))$ is a k-fold monoidal n-truncated n-category. The stabilisation hypothesis of Baez and Dolan in [BD] can be formulated by saying that the forgetful map

$$Mon^{k+1}(Cat_n[n](X)) \to Mon^k(Cat_n[n](X))$$

is an equivalence if $k \geq n+2$ and $X = \mathbf{Typ}$. But this formulation cannot be totally correct, since it it does use the correct notion of equivalence between n-categories. In order to take this notion into account, it suffices to replace $Cat_n(X)$ by $RCat_n(X)$. If correctly formulated, the hypothesis asserts the forgetful map

$$Mon^{k+1}(RCat_n[n](X)) \to Mon^k(RCat_n[n](X))$$

is an equivalence of quasi-categories if $k \geq n+2$ and $X = \mathbf{Typ}$. A stronger statement is that it is an equivalence for any X. In other words, that the canonical map

$$Mon^k \odot_c RCat_n[n] \to Mon^{k+1} \odot_c RCat_n[n]$$

is an equivalence if $k \geq n+2$.

43.5. Let us show that the stabilisation hypothesis of Breen-Baez-Dolan is equivalent to the Generalised Suspension Conjecture in 43.1. We first prove the the implication GSC⇒BBD. For this it suffices to show by **??** that the cartesian theory $Mon^k \odot_c RCat_n[n]$ is semi-additive for $k \geq n+2$. But for this, it suffices to show that the cartesian theory $Mon^{n+2} \odot_c RCat_n[n]$ is semi-additive. Let us show more generally that the the cartesian theory $RCat_n[m] \odot Mon^{m+2}$ is semi-additive for every $m \geq n$. But we have an equivalence $RCat_n[m] \simeq RCat_n[m] \odot_c OB(m)$ by 42.10. Hence it suffices to show that the cartesian theory

$$RCat_n[m] \odot_c OB(m) \odot Mon^{m+2}$$

is semi-additive. But this is true of the cartesian theory $OB(m) \odot Mon^{m+2}$ by the GSC in 43.1. Hence the canonical map

$$RCat_n[m] \odot_c OB(m) \odot Mon^{m+2} \to RCat_n[m] \odot_c OB(m) \odot Mon^{m+3}$$

is an equivalence. The implication GSC⇒BBD is proved. Conversely, let us prove the implication BBD⇒GSC. The cartesian theory $RCat_0[m] \odot Mon^{m+2}$ is semi-additive if we put $n = 0$. But we have $RCat_0[m] = OB(m)$. Hence the cartesian theory $OB(m) \odot Mon^{m+2}$ is semi-additive.

44. Disks and duality

44.1. We begin by recalling the duality between the category Δ and the category of intervals. An *interval* I is a linearly ordered set with a first and last elements respectively denoted $\bot$ and $\top$ or 0 and 1. If $0 = 1$ the interval is *degenerate*, otherwise we say that is *strict*. A *morphism* $I \to J$ between two intervals is defined to be an order preserving map $f : I \to J$ such that $f(0) = 0$ and $f(1) = 1$. We shall denote by $\mathcal{D}(1)$ the category of finite strict intervals (it is the category of finite 1-*disk*). The category $\mathcal{D}(1)$ is the opposite of the category Δ. The duality functor $(-)^* : \Delta^o \to \mathcal{D}(1)$ associates to $[n]$ the set $[n]^* = \Delta([n],[1]) = [n+1]$ equipped with the pointwise ordering. The inverse functor $\mathcal{D}(1)^o \to \Delta$ associates to an interval $I \in \mathcal{D}(1)$ the set $I^* = \mathcal{D}(1)(I,[1])$ equipped with the pointwise ordering. A morphism $f : I \to J$ in $\mathcal{D}(1)$ is surjective (resp. injective) iff the dual morphism $f^* : J^* \to I$" is injective (resp. surjective). A simplicial set is usually defined to be a contravarint functors $\Delta^o \to \mathbf{Set}$; it can be defined to be a covariant functor $\mathcal{D}(1) \to \mathbf{Set}$.

44.2. If I is a strict interval, we shall put $\partial I = \{0,1\}$ and $int(I) = I \setminus \partial I$. We say that a morphism of strict intervals $f : I \to J$ is *proper* if $f(\partial I) \subseteq \partial J$. We shall say that $f : I \to J$ is a *contraction* if it induces a bijection $f^{-1}(int(J)) \to int(J)$. A morphism $f : I \to J$ is a contraction iff it has a unique section. If $\mathcal{A}$ is the class of contractions and $\mathcal{B}$ is the class of proper morphisms then the pair $(\mathcal{A},\mathcal{B})$ is a factorisation system in $\mathcal{D}(1)$.

44.3. The *euclidian-ball* of dimension $n \geq 0$ $B^n = \{x \in \mathbf{R}^n :\| x \| \leq 1\}$ is the main geometric example of an n-disk. The boundary of the ball is a sphere ∂B^n of dimension $n-1$. The sphere ∂B^n is the union of two disks, the lower an upper hemispheres. In order to describe this structrure, it is convenient to use the projection $q : B^n \to B^{n-1}$ which forget the last coordinate. Each fiber $q^{-1}(x)$ is a strict interval except when $x \in \partial B^{n-1}$ in which case it is reduced to a point. There are two canonical sections $s_0, s_1 : B^{n-1} \to B^n$ obtained by selecting the bottom and the top elements in each fiber. The image of s_0 is the lower hemisphere of ∂B^n and the image of s_1 the upper hemisphere; observe that $s_0(x) = s_1(x)$ iff $x \in \partial B^{n-1}$.

44.4. A *bundle of intervals* over a set B is an interval object in the category $\mathbf{Set}/B$. More explicitly, it is a map $p : E \to B$ whose fibers $E(b) = p^{-1}(b)$ have the structure on an interval. The map p has two canonical sections $s_0, s_1 : B \to E$ obtained by selecting the bottom and the top elements in each fiber. The interval $E(b)$ is degenerated iff $s_0(b) = s_1(b)$. If $s_0(b) = s_1(b)$, we shall say that b is in the *singular set* indexAsingular set—textbf. The projection $q : B^n \to B^{n-1}$ is an example of bundle of intervals. Its singular set is the boundary ∂B^{n-1}. If we order the coordinates in R^n we obtain a sequence of bundles of intervals:

$$1 \leftarrow B^1 \leftarrow B^2 \leftarrow \cdots B^{n-1} \leftarrow B^n.$$

44.5. A *n-disk* D is defined to be a sequence of length n of bundles of intervals

$$1 = D_0 \leftarrow D_1 \leftarrow D_2 \leftarrow \cdots D_{n-1} \leftarrow D_n$$

such that the singular set of the projection $p : D_{k+1} \to D_k$ is equal to the *boundary* $\partial D_k := s_0(D_{k-1}) \cup s_1(D_{k-1})$ for every $0 \leq k < n$. By convention $\partial D_0 = \emptyset$. If $k = 0$, the condition means that the interval D_1 is strict. It follows from the definition that we have $s_0 s_0 = s_1 s_0$ and $s_0 s_1 = s_1 s_1$. The *interior* of D_k is defined to be $int(D_k) = D_k \backslash \partial D_k$. There is then a decomposition

$$\partial D_n \simeq \bigsqcup_{k=0}^{n-1} 2 \cdot int(D_k).$$

We shall denote by $\mathcal{B}^n$ the n-disks defined by the sequence of projections

$$1 \leftarrow B^1 \leftarrow B^2 \leftarrow \cdots B^{n-1} \leftarrow B^n.$$

44.6. A *morphism* between two bundles of intervals $E \to B$ and $E' \to B'$ is a pair of maps (f, g) in a commutative square

$$\begin{array}{ccc} B & \longleftarrow & E \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ B' & \longleftarrow & E' \end{array}$$

such that the map $E(b) \to E'(f(b))$ induced by g is a morphism of intervals for every $b \in B$. A *morphism* $f : D \to D'$ between n-disks is defined to be a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longleftarrow & D_1 & \longleftarrow & D_2 & \longleftarrow \cdots & D_{n-1} & \longleftarrow & D_n \\ \downarrow & & {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_{n-1}}\downarrow & & {\scriptstyle f_n}\downarrow \\ 1 & \longleftarrow & D'_1 & \longleftarrow & D'_2 & \longleftarrow \cdots & D'_{n-1} & \longleftarrow & D'_n \end{array}$$

and which the squares are morphisms of bundles of intervals. Every morphism $f : D \to D'$ can be factored as a surjection $D \to f(D)$ followed by an inclusion $f(D) \subseteq D'$.

44.7. A *planar tree* T of *height* $\leq n$, or a *n-tree*, is defined to be a sequence of maps

$$1 = T_0 \leftarrow T_1 \leftarrow T_2 \leftarrow \cdots \leftarrow T_{n-1} \leftarrow T_n$$

with linearly ordered fibers. If D is a n-disk, then we have $p(int(D_k)) \subseteq int(D_{k-1})$ for every $1 \leq k \leq n$, where p is the projection $D_k \to D_{k-1}$. The sequence of maps

$$1 \leftarrow int(D_1) \leftarrow int(D_2) \leftarrow \cdots int(D_{n-1}) \leftarrow int(D_n)$$

has the structure of a planar tree called the *interior* of D and denoted $int(D)$. Every n-tree T is the interior of a n-disk $\bar{T}$. By construction, we have $\bar{T}_k = T_k \sqcup \partial \bar{T}_k$ for every $1 \leq k \leq n$, where

$$\partial \bar{T}_k = \bigsqcup_{i=0}^{k-1} 2 \cdot T_i.$$

We shall say that $\bar{T}$ is the *closure* of T. We have $\overline{int(D)} = D$ for every disk D. A morphism of disks $f : D \to D'$ is completely determined by its values on the sub-tree $int(D) \subseteq D$. More precisely, a *morphism of trees* $g : S \to T$ is defined to be a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longleftarrow & S_1 & \longleftarrow & S_2 & \longleftarrow \cdots & S_{n-1} & \longleftarrow & S_n \\
\downarrow & & \downarrow g_1 & & \downarrow g_2 & & \downarrow g_{n-1} & & \downarrow g_n \\
1 & \longleftarrow & T_1 & \longleftarrow & T_2 & \longleftarrow \cdots & T_{n-1} & \longleftarrow & T_n
\end{array}$$

in which f_k preserves the linear order on the fibers of the projections for each $1 \leq k \leq n$. If $Disk(n)$ denotes the category of n-disks and $Tree(n)$ the category of n-trees, then the forgetful functor $Disk(n) \to Tree(n)$ has a left adjoint $T \mapsto \bar{T}$. If $D \in Disk(n)$, then a morphism of trees $T \to D$ can be extended uniquely to a morphism of disks $\bar{T} \to D$. It follows that there a bijection between the morphisms of disks $D \to D'$ and the morphisms of trees $int(D) \to D'$.

44.8. We shall say that a morphism of disks $f : D \to D'$ is *proper* if we have $f(int(D_k)) \subseteq int(D'_k)$ for every $1 \leq k \leq n$. An proper morphism $f : D \to D'$ induces a morphism of trees $int(f) : int(D) \to int(D')$. The functor $T \mapsto \bar{T}$ induces an equivalence between the category $Tree(n)$ and the sub-category of proper morphisms of $Disk(n)$. We shall say that a morphism of disks $f : D \to D'$ is a *contraction* if it induces a bijection $f^{-1}(int(D)) \to int(D')$. Every contraction $f : D \to D'$ has a section and this section is unique. If $\mathcal{A}$ is the class of contractions and $\mathcal{B}$ is the class of proper morphisms then the pair $(\mathcal{A}, \mathcal{B})$ is a factorisation system in $\mathcal{D}(n)$. Every surjection $f : D \to D'$ admits a factorisation $f = up$ with p a contraction and u a proper surjection and this factorisation is essentially unique.

44.9. A *sub-tree* of a n-tree T is a sequence of subsets $S_k \subseteq T_k$ closed under the projection $T_k \to T_{k-1}$ for $1 \leq k \leq n$ and with $S_0 = 1$. If $T = int(D)$ then the map $C \mapsto C \cap T$ induces a bijection between the sub-disks of D and the sub-trees of T. The set of sub-disks of D is closed under non-empty unions and arbitrary intersections.

44.10. We shall say that a n-disk D is *finite* if D_n is a finite set. The *degree* $\mid D \mid$ of a finite disk D, is defined to be the number of edges of the tree $int(D)$. By definition,

$$\mid D \mid = \sum_{k=1}^{n} \mathrm{Card}(int(D_k)).$$

We have

$$2(1+ \mid D \mid) = \mathrm{Card}(D_n) + \mathrm{Card}(int(D_n)).$$

The set

$$D^{\vee} = hom(D, \mathcal{B}^n)$$

has the structure of a topological ball of dimension $\mid D \mid$. The space $D^{\vee}$ has the following description. Let us transport the order relation on the fibers of the planar tree $T = int(D)$ to its edges. Then $D^{\vee}$ is homeomorphic to the space of maps $f : edges(T) \to [-1, 1]$ which satisfy the following conditions

- $f(e) \leq f(e')$ for any two edges $e \leq e'$ with the same target;
- $\sum_{e \in C} f(e)^2 \leq 1$ for every maximal chain C connecting the root to a leaf.

We can associate to f a map of n-disks $f' : D \to \mathcal{B}^n$ by putting

$$f'(x) = (f(e_1), \cdots, f(e_k))$$

where $(e_1, \cdots, e_k)$ is the chain of edges which connects the root to the vertex $x \in T_k$. The map $f' : D \to \mathcal{B}^n$ is monic iff f belongs to the interior of the ball $D^{\vee}$. Every finite n-disk D admits an embedding $D \to \mathcal{B}^n$.

44.11. We shall denote by $\Theta(n)$ the category opposite to $\mathcal{D}(n)$. We call an object of $\Theta(n)$ a *cell* of height $\leq n$. To every disk $D \in \mathcal{D}(n)$ corresponds a *dual cell* $D^* \in \Theta(n)$ and to every cell $C \in \Theta(n)$ corresponds a *dual disk* $C^* \in \mathcal{D}(n)$. The *dimension* of C is the degree of C^*. A $\Theta(n)$-*set* is defined to be a functor

$$X : \Theta(n)^o \to \mathbf{Set},$$

or equivalently a functor $X : \mathcal{D}(n) \to \mathbf{Set}$. We shall denote by $\hat{\Theta}(n)$ the category of $\Theta(n)$-sets. If t is a finite n-tree we shall denote by $[t]$ the cell dual to the disk $\bar{t}$. The dimension of $[t]$ is the number of edges of t. We shall denote by $\Theta[t]$ the image of $[t]$ by the Yoneda functor $\Theta(n) \to \hat{\Theta}(n)$. The *realisation* of a cell C is defined to be the topological ball $R(C) = (C^*)^{\vee}$, This defines a functor $R : \Theta(n) \to \mathbf{Top}$, where **Top** denotes the category of compactly generated spaces. Its left Kan extension

$$R_! : \hat{\Theta}(n) \to \mathbf{Top}$$

preserves finite limits. We call $R_!(X)$ the *geometric realisation* of X.

44.12. We shall say that a map $f : C \to E$ in $\Theta(n)$ is *surjective* (resp. *injective*) if the dual map $f^* : E^* \to C^*$ is injective (resp. surjective). Every surjection admits a section and every injection admits a retraction. If $\mathcal{A}$ is the class of surjections and $\mathcal{B}$ is the class of injections, then the pair $(\mathcal{A}, \mathcal{B})$ is a factorisation system in $\Theta(n)$. If D' and D" are sub-disks of a disk $D \in \mathcal{D}(n)$, then the intersection diagram

$$\begin{array}{ccc} D' \cap D" & \longrightarrow & D" \\ \downarrow & & \downarrow \\ D' & \longrightarrow & D \end{array}$$

is *absolute*, ie it is preserved by any functor with codomain $\mathcal{D}(n)$. Dually, for every pair of surjections $f : C \to C'$ and $g : C \to C$" in the category $\Theta(n)$, we have an absolute pushout square (Eilenberg-Zilber lemma). square

$$\begin{array}{ccc} C & \xrightarrow{g} & C" \\ {\scriptstyle f}\downarrow & & \downarrow \\ C' & \longrightarrow & C'". \end{array}$$

If X is a $\Theta(n)$-set, we shall say that a cell $x : \Theta[t] \to X$ of dimension $n > 0$ is *degenerate* if it admits a factorisation $\Theta[t] \to \Theta[s] \to X$ via a cell of dimension $< n$, otherwise we shall say that x is *non-degenerate*. Every cell $x : \Theta[t] \to X$ admits a unique factorisation $x = yp\Theta[t] \to \Theta[s] \to X$ with p a surjection and y a non-degenerate cell.

44.13. For each $0 \le k \le n$, let put $b^k = \Theta[t^k]$, where t^k is the tree which consists of a unique chain of k-edges. There is a unique surjection $b^k \to b^{k-1}$ and the sequence of surjections

$$1 = b^0 \leftarrow b^1 \leftarrow b^2 \leftarrow \cdots b^n$$

has the structure of a n-disk β^n in the topos $\hat{\Theta}^n$. It is the *generic* n-disk in the sense of classifying topos. The geometric realisation of β^n is the euclidian n-disk $\mathcal{B}^n$.

44.14. We shall say that a map $f : C \to E$ in $\Theta(n)$ is *open* (resp. *is an inflation*) if the dual map $f^* : E^* \to C^*$ is proper (resp. is a contraction). Every inflation admits a unique retraction. If $\mathcal{A}$ is the class of open maps in $\Theta(n)$ and $\mathcal{B}$ is the class of inflations then the pair $(\mathcal{A}, \mathcal{B})$ is a factorisation system. Every monomorphism of cells $i : D \to D'$ admits a factorisation $i = qu$ with u an open monomorphism and q an inflation.

44.15. Recall that a *globular set* X is defined to be a sequence of pairs of maps $s_n, t_n : X_{n+1} \to X_n$ $(n \ge 0)$ such that we have

$$s_n s_{n+1} = s_n t_{n+1} \quad \text{and} \quad t_n s_{n+1} = t_n t_{n+1}$$

for every $n \ge 0$. An element $x \in X_n$ is called an *n-cell*; if $n > 0$ the element $s_{n-1}(x)$ is said to be the *source* and the element $t_{n-1}(x)$ to be the *target* of x. A globular set X can be defined to be a presheaf $X : \mathcal{G}^o \to \mathbf{Set}$ on a category $\mathcal{G}$ of *globes* which can be defined by generators and relations. By definition $Ob\mathcal{G} = \{G_0, G_1, \ldots\}$; there are two generating maps $i_0^n, i_1^n : G_n \to G_{n+1}$ for each $n \ge 0$; the relations

$$i_0^{n+1} i_0^n = i_1^{n+1} i_0^n \quad \text{and} \quad i_0^{n+1} i_1^n = i_1^{n+1} i_1^n.$$

is a presentation. The relations imply that there is exactly two maps $i_0, i_1 : G_m \to G_n$ for each $m < n$. A globular set X is thus equipped with two maps $s, t : X_n \to X_m$ for each $m < n$. A *reflexive* globular set is defined to be a globular set X equipped with a sequence of maps $u_n : X_n \to X_{n+1}$ such that $s_n u_n = t_n u_n = id$. By composing we obtain a map $u : X_m \to X_n$ for each $m < n$. There is also a notion of globular set of *height* $\le n$ for each $n \ge 0$. It can be defined to be a presheaf $\mathcal{G}_n^o \to \mathbf{Set}$, where $\mathcal{G}_n$ is the full sub-category of $\mathcal{G}$ spanned by the globes G_k with $k \le n$. Notice that a globular set of of height ≤ 0 is the same thing as a set and that globular set of of height ≤ 1 is a graph.

44.16. Recall that a (strict) *category* is a graph $s, t : X_1 \to X_0$, equipped with an associative composition operation

$$\circ : X_1 \times_{s,t} X_1 \to X_1$$

and a unit map $u : X_0 \to X_1$. A *functor* between two categories is a map of graphs $f : X \to Y$ which preserves composition and units. A (strict) ω-*category* is defined to be a reflexive globular set X equipped with a category structure

$$\circ_k : X_n \times_k X_n \to X_n$$

for each $0 \leq k < n$, where $X_n \times_k X_n$ is defined by the pullback square

$$\begin{array}{ccc} X_n \times_k X_n & \longrightarrow & X_n \\ \downarrow & & \downarrow{\scriptstyle t} \\ X_n & \xrightarrow{\ s\ } & X_k. \end{array}$$

The unit map $u : X_k \to X_n$ is given by the reflexive graph structure. The operations should obey the interchange law

$$(x \circ_k y) \circ_m (u \circ_k v) = (x \circ_m u) \circ_k (y \circ_m v)$$

for each $k < m < n$. A *functor* $f : X \to Y$ between ω-categories is a map of globular sets which preserves composition and units. We shall denote by Cat_ω the category of ω-categories. The notion of (strict) n-*category* is defined similarly but by using a globular set of height $\leq n$. We shall denote by Cat_n the category of n-categories.

44.17. We saw in 44.12 that the sequence of cells

$$1 = b^0 \leftarrow b^1 \leftarrow b^2 \leftarrow \cdots b^n$$

has the structure of a n-disk in the category Θ^n. The lower and upper sections $s_0, s_1 : b^k \to b^(k+1)$ give the sequence the structure of a co-globular set of height $\leq n$. This defines a functor $b : \mathcal{G}_n \to \Theta^n$ from which we obtain a functor

$$b^! : \Theta^n \to \hat{\mathcal{G}}_n.$$

Let us see that the functor $b^!$ can be lifted to Cat_n,

$$\begin{array}{ccc} & & Cat_n \\ & \nearrow^{\tilde{b}^!} & \downarrow{\scriptstyle U} \\ \Theta^n & \xrightarrow{\ b^!\ } & \hat{\mathcal{G}}_n \end{array}$$

EEEE

We shall denote by $\Theta(n)$ the category opposite to $\mathcal{D}(n)$ and by $\Theta(\infty)$ the category opposite to $\mathcal{D}(\infty)$. We call an object of $\Theta(\infty)$ a *cell.* To every disk $D \in \mathcal{D}(\infty)$ corresponds a *dual cell* $D^* \in \Theta(\infty)$ and to every cell $C \in \Theta(\infty)$ corresponds a *dual disk* $C^* \in \mathcal{D}(\infty)$. The *dimension* of C is defined to be the degree of C^* and the *height* of C to be the height of C^*. If t is a finite planar tree, we shall denote by $[t]$ the cell opposite to the disk $\bar{t}$. The dimension of $[t]$ is the number of edges of t and the height of $[t]$ is the height of t.

44.18. The *height* of a n-tree T is defined to be the largest integer $k \geq 0$ such that $T_k \neq \emptyset$. The *height* of a n-disk D is defined to be the height of its interior $int(D)$. If $m < n$, the obvious restriction functor $Disk(n) \to Disk(m)$ has a left adjoint $Ex^n : Disk(m) \to Disk(n)$. The extension functor Ex^n is fully faithful and its essential image is the full subcategory of $Disk(n)$ spanned by the disks of height $\leq n$. We shall identify the category $Disk(m)$ with a full subcategory of $Disk(n)$ by adoptiong the same notation for a disk $D \in Disk(m)$ and its extension $Ex^n(D) \in Disk(n)$. We thus obtain an increasing sequence of coreflexive subcategories,

$$Disk(1) \subset Disk(2) \subset \cdots \subset Disk(n).$$

Hence also an increasing sequence of coreflexive subcategories,

$$\mathcal{D}(1) \subset \mathcal{D}(2) \subset \cdots \subset \mathcal{D}(n).$$

The coreflection functor $\rho^k : \mathcal{D}(n) \to \mathcal{D}(k)$ takes a disk $\overline{T}$ to the sub-disk $\overline{T^k} \subset \overline{T}$, where T^k is the k-*truncation* of T. We shall denote by $\mathcal{D}(\infty)$ the union of the categories $\mathcal{D}(n)$,

$$\mathcal{D}(\infty) = \bigcup_n \mathcal{D}(n)$$

An object of $\mathcal{D}(\infty)$ is an infinite sequence of bundles of finite intervals

$$1 = D_0 \leftarrow D_1 \leftarrow D_2 \leftarrow$$

such that

- the singular set of the projection $D_{n+1} \to D_n$ is the set $\partial D_n := s_0(D_{n-1}) \cup s_1(D_{n-1})$ for every $n \geq 0$;
- the projection $D_{n+1} \to D_n$ is bijective for n large enough.

We have increasing sequence of reflexive subcategories,

$$\Theta(1) \subset \Theta(2) \subset \cdots \subset \Theta(\infty),$$

where $\Theta(k)$ is the full subcategory of $\Theta(\infty)$ spanned by the cells of height $\leq k$. By 44.1, we have $\Theta^1 = \Delta$ A cell $[t]$ belongs to Δ iff the height of t is ≤ 1. If $n \geq 0$ we shall denote by n the unique planar tree height ≤ 1 with n edges. A cell $[t]$ belongs to Δ iff we have $t = n$ for some $n \geq 0$. The reflection functor $\rho^k : \Theta(\infty) \to \Theta(k)$ takes a cell $[t]$ to the cell $[t^k]$, where t^k is the k-*truncation* of t.

44.19. A Θ-*set* of height $\leq n$ is defined to be a functor

$$X : \Theta(n)^o \to \mathbf{Set},$$

or equivalently a functor $X : \mathcal{D}(n) \to \mathbf{Set}$. We shall denote by $\hat{\Theta}(n)$ the category of Θ-sets of height $\leq n$. If t is a finite tree of height $\leq n$, we shall denote by $\Theta[t]$ the image of $[t]$ by the Yoneda functor $\Theta(n) \to \hat{\Theta}(n)$. Consider the functor $R : \Theta(n) \to \mathbf{Top}$ defined by putting $R(C) = (C^*)^\vee = Hom(C^*, \mathcal{B}^n)$, where $\mathbf{Top}$ denotes the category of compactly generated spaces. Its left Kan extension $R : \hat{\Theta}(n) \to \mathbf{Top}$ preserves finite limits. We call $R(X)$ the *geometric realisation* of the Θ-set X.

The left Kan extension of the inclusion

$$\Theta^1 \subset \Theta^m$$

44.20. For each $0 \leq k \leq n$, let us denote by E^k the n-disk whose interior is a chain of k edges. The geometric realisation of dual cell $b^k = (E^k)^*$ is the euclidian k-ball B^k. There is a unique open map of disks $E^{k-1} \to E^k$, hence a map of cells $b^k \to b^{k-1}$. The sequence

$$1 = b^0 \leftarrow b^1 \leftarrow b^2 \leftarrow \cdots b^n$$

has the structure of a n-disk β^n in the topos $\hat{\Theta}^n$. It is the generic n-disk in the sense of classifying topos.

44.21. Recall that a *globular set* X is defined a sequence of sets $(X_n : n \geq 0)$ equipped with a sequence of pair of maps $s_n, t_n : X_{n+1} \to X_n$ such that we have

$$s_n s_{n+1} = s_n t_{n+1} \quad \text{and} \quad t_n s_{n+1} = t_n t_{n+1}$$

for every $n \geq 0$. An element $x \in X_n$ is called an *n-cell*; if $n > 0$ the element $s_{n-1}(x)$ is said to be the *source* and the element $t_{n-1}(x)$ to be the *target* of x. A globular set X can be defined to be a presheaf $X : \mathcal{G}^o \to \mathbf{Set}$ on a category $\mathcal{G}$ of *globes* which can be defined by generators and relations. By definition $Ob\mathcal{G} = \{G_0, G_1, \ldots\}$; there are two generating maps $i_0^n, i_1^n : G_n \to G_{n+1}$ for each $n \geq 0$; the relations

$$i_0^{n+1} i_0^n = i_1^{n+1} i_0^n \quad \text{and} \quad i_0^{n+1} i_1^n = i_1^{n+1} i_1^n.$$

is a presentation. The relations imply that there is exactly two maps $i_0, i_1 : G_m \to G_n$ for each $m < n$. A globular set X is thus equipped with two maps $s, t : X_n \to X_m$ for each $m < n$. A *reflexive* globular set is defined to be a globular set X equipped with a sequence of maps $u_n : X_n \to X_{n+1}$ such that $s_n u_n = t_n u_n = id$. By composing we obtain a map $u : X_m \to X_n$ for each $m < n$. There is also a notion of globular set of *height* $\leq n$ for each $n \geq 0$. It can be defined to be a presheaf $\mathcal{G}_n^o \to \mathbf{Set}$, where $\mathcal{G}_n$ is the full sub-category of $\mathcal{G}$ spanned by the globes G_k with $k \leq n$. Notice that a globular set of of height ≤ 0 is the same thing as a set and that globular set of of height ≤ 1 is a graph.

44.22. Recall that a (strict) *category* is a graph $s, t : X_1 \to X_0$, equipped with an associative composition operation

$$\circ : X_1 \times_{s,t} X_1 \to X_1$$

and a unit map $u : X_0 \to X_1$. A *functor* between two categories is a map of graphs $f : X \to Y$ which preserves composition and units. A (strict) ω-*category* is defined to be a reflexive globular set X equipped with a category structure

$$\circ_k : X_n \times_k X_n \to X_n$$

for each $0 \leq k < n$, where $X_n \times_k X_n$ is defined by the pullback square

$$\begin{array}{ccc} X_n \times_k X_n & \longrightarrow & X_n \\ \downarrow & & \downarrow{\scriptstyle t} \\ X_n & \xrightarrow{s} & X_k. \end{array}$$

The unit map $u : X_k \to X_n$ is given by the reflexive graph structure. The operations should obey the interchange law

$$(x \circ_k y) \circ_m (u \circ_k v) = (x \circ_m u) \circ_k (y \circ_m v)$$

for each $k < m < n$. A *functor* $f : X \to Y$ between ω-categories is a map of globular sets which preserves composition and units. We shall denote by Cat_ω the category of ω-categories. The notion of (strict) *n-category* is defined similarly but by using a globular set of height $\leq n$. We shall denote by Cat_n the category of n-categories.

44.23. We saw in 44.20 that the sequence of cells

$$1 = b^0 \leftarrow b^1 \leftarrow b^2 \leftarrow \cdots b^n$$

has the structure of a n-disk in the category Θ^n. The lower and upper sections $s_0, s_1 : b^k \to b^{(}k+1)$ give the sequence the structure of a co-globular set of height $\leq n$. This defines a functor $b : \mathcal{G}_n \to \Theta^n$ from which we obtain a functor

$$b^! : \Theta^n \to \hat{\mathcal{G}}_n.$$

Let us see that the functor $b^!$ can be lifted to Cat_n,

$$\begin{array}{ccc} & & Cat_n \\ & \overset{\tilde{b}^!}{\nearrow} & \downarrow U \\ \Theta^n & \xrightarrow{b^!} & \hat{\mathcal{G}}_n \end{array}$$

if $0 \leq k \leq n$, let us denote by E^k the n-disk whose interior is a chain of k edges. There is a unique element $e^k \in int(E^k)_k$. The interval over e^k has exactly two points. There are two map of disks $p_0, p_1 : E^k \to E^{k-1}$. The first takes $e^k \in E^k$ to the top element of the interval over $e^{k-1} \in E^{k-1}$, and the second to the top element of the interval over $e^{k-1} \in E^{k-1}$.

There is a unique map of disks $e_{k-1} \to e_k$ and two maps of disks

let us denote by e_k the n-disk whose interior is a chain of k edges. The geometric realisation of the cell $b^k = {}^*e_k$ is the euclidian n-ball. There is a unique map of disks $e_{k-1} \to e_k$, hence also a unique map of cells $b^k \to b^{k-1}$. The sequence

$$1 = b^0 \leftarrow b^1 \leftarrow b^2 \leftarrow \cdots b^n$$

has the structure of a n-disk b in the topos $\hat{\Theta}^n$. It is the generic n-disk in the sense of classifying topos.

44.24. The *composite* $D \circ E$ of a n-disk D with a m-disk E is the $m+n$ disk

$$1 = D_0 \leftarrow D_1 \leftarrow \cdots \leftarrow D_n \leftarrow (D_n, \partial D_n) \times E_1 \leftarrow \cdots \leftarrow (D_n, \partial D_n) \times E_m,$$

where $(D_n, \partial D_n) \times E_k$ is defined by the pushout square

$$\begin{array}{ccc} \partial D_n \times E_k & \longrightarrow & D_n \times E_k \\ \downarrow & & \downarrow \\ E_k & \longrightarrow & (D_n, \partial D_n) \times E_k. \end{array}$$

This composition operation is associative.

44.25. The category $\mathbf{S}^{(n)} = [(\Delta^n)^o, \mathbf{Set}]$, contains n intervals

$$I_k = 1\Box 1\Box \cdots 1\Box I\Box 1 \cdots 1\Box 1,$$

one for each $0 \leq k \leq n$. It thus contain a n-disk $I^{(n)} : I_1 \circ I_2 \circ \cdots \circ I_n$. Hence there is a geometric morphism

$$(\rho^*, \rho_*) : \mathbf{S}^{(n)} \to \hat{\Theta},$$

such that $\rho^*(b) = I^{(n)}$. We shall say that a map of Θ^n-sets $f : X \to Y$ is a *weak categorical equivalence* if the map $\rho^*(f) : \rho^*(X) \to \rho^*(Y)$ is a weak equivalence in the model structure for reduced Segal n-spaces. The category $\hat{\Theta}_n$ admits a model structure in which the weak equivalences are the weak categorical equivalences and the cofibrations are the monomorphisms. We shall say that a fibrant object is a Θ^n*-category*. The model structure is cartesian closed and left proper. We call it the *model structure for* Θ^n*-categories*. We denote by $\Theta^n\mathbf{Cat}$ the category of Θ^n-categories. The pair of adjoint functors

$$\rho^* : \hat{\Theta}_n \to \mathbf{S}^{(n)} : \rho_*$$

is a Quillen equivalence between the model structure for Θ^n-categories and the model structure for reduced Segal n-spaces.

45. Higher quasi-categories

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A n*-quasi-category* can be defined to be a fibrant object with respect to a certain model structure structure on the category of presheaves on certain category Θ_n. The category Θ_n was introduced for this purpose by the author in 1998. It was first defined as the opposite of the category of finite n-disks. It was later conjectured (jointly by Batanin, Street and the author) to be isomorphic to a category T_n^* introduced by Batanin in his theory of higher operads [**?**]. The category T_n^* is a full subcategory of the category of strict n-categories. The conjecture was proved by Makkai and Zawadowski in [MZ] and by Berger in [Ber]. The model structure for n-quasi-categories can be described in various ways. In principle, the model structure for n-quasi-categories can be described by specifying the fibrant objects, since the cofibrations are supposed to be the monomorphisms. But a complete list of the filling conditions defining the n-quasi-categories is still missing (a partial list was proposed by the author in 1998). An alternative approach is find a way of specifying the class $Wcat_n$ of weak equivalences (the *weak categorical* n*-equivalences*). Let us observe that the class $Wcat$ in $\mathbf{S}$ can be extracted from the canonical map $i : \Delta \to \mathbf{U}_1$, since a map of simplicial sets $u : A \to B$ is a weak categorical equivalence if the arrow $i_!(u) : i_!A \to i_!B$ is invertible in $\mathbf{U}_1$, where $i_! : \hat{\Delta} \to \mathbf{U}_1$ denotes the left Kan extension of i along the Yoneda functor. In general, it should suffices to exibit a map $i : \Theta_n \to \mathbf{U}_n$ with values in a cocomplete quasi-category chosen appropriately. The quasi-category $\mathbf{U}_1$ is equivalent to the quasi-category of reduced category object in $\mathbf{U}$. It seems reasonable to suppose that $\mathbf{U}_n$ is the quasi-category of reduced n-category object in $\mathbf{U}$. A n*-category object* in $\mathbf{U}$ is defined to be a map $C : \Theta_n^o \to \mathbf{U}$ satisfying a certain Segal condition. A n-category C is *reduced* if every invertible cell of C is a unit. The notion of reduced n-category object is essentially algebraic. Hence the quasi-category $\mathbf{U}_n$ is cocomplete, since it is locally presentable. The canonical map $i : \Theta_n \to \mathbf{U}_n$ is obtained from the inclusion of Θ_n in the category of reduced strict n-categories. A map $u : A \to B$ in $\hat{\Theta}_n$ is then defined to be a *weak categorical* n*-equivalence* if the arrow $i_!(u) : i_!A \to i_!B$ is invertible in $\mathbf{U}_n$, where $i_! : \hat{\Theta}_n \to \mathbf{U}_n$ denotes the left Kan extension of i along the Yoneda functor. The model category $(\hat{\Theta}_n, Wcat_n)$ is cartesian closed and its full subcategory of fibrant objects $\mathbf{QCat_n}$ has the structure of a simplicial category enriched over Kan complexes. We conjecture that the coherent nerve of $\mathbf{QCat_n}$ is equivalent to $\mathbf{U}_n$. There is another description of $Wcat_n$ which is conjectured by Cisinski and the author. It is easy to show that the localizer $Wcat$ is generated by inclusions $I[n] \subseteq \Delta[n]$ $(n \geq 0)$, where $I[n]$ is the union of the edges $(i-1, i)$ for

$1 \leq i \leq n$. The simplicial set $I[n]$ is said to be the *spine* of $\Delta[n]$. The objects of Θ_n are indexed by finite planar trees of height $\leq n$. For each tree t, let us denote by $\Theta[t]$ the representable presheaf generated by the object $[t]$ of Θ_n. The *spine* $S[t] \subseteq \Theta[t]$ is the union of the generators of the n-category $[t]$ (it is the globular diagram associted to t by Batanin). It is conjectured that $Wcat_n$ is the localizer generated by the inclusions $S[t] \subseteq \Theta[t]$.

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45.1. There is a notion of *n-fold Segal space* for every $n \geq 1$. Recall that the category $[(\Delta^o)^n, \mathbf{S}] = \mathbf{S}^{(n)}\mathbf{S}$ of n-fold simplicial spaces admits a Reedy model structure in which the weak equivalences are the level wise weak homotopy equivalences and the cofibrations are the monomorphisms. A *n-fold Segal space* is defined to be a Reedy fibrant n-fold simplicial space $C : (\Delta^o)^n \to \mathbf{S}$ which satisfies the Segal condition **??** in each variable. The Reedy model structure admits a Bousfield localisation in which the fibrant objects are the n-fold Segal spaces. The model structure is simplicial. It is the *model structure for n-fold Segal spaces*. The coherent nerve of the simplicial category of n-fold Segal spaces is equivalent to the quasi-category $Cat^n(\mathbf{Typ})$.

45.2. There is a notion of *n-fold Rezk space* for every $n \geq 1$. It is a n-fold Segal space which satisfies the Rezk condition **??** in each variable. The Reedy model structure admits a Bousfield localisation in which the fibrant objects are the n-fold Rezk spaces. The model structure is simplicial. It is the *model structure for n-fold Rezk spaces*. The coherent nerve of the simplicial category of n-fold Rezk spaces is equivalent to the quasi-category $RCat^n(\mathbf{Typ})$.

45.3. There is a notion of *Segal n-space* for every $n \geq 1$. It is defined by induction on $n \geq 1$. If $n = 1$, it is a Segal space $C : \Delta^o \to \mathbf{S}$. If $n > 1$, it is a n-fold Segal space $C : \Delta^o \to \mathbf{S}^{(n-1)}\mathbf{S}$ such that

- C_k is a Segal n-space for every $k \geq 0$,
- $C_0 : (\Delta^o)^{n-1} \to \mathbf{S}$ is homotopically constant.

The model structure for n-fold Segal spaces admits a Bousfield localisation in which the fibrant objects are the Segal n-spaces. The model structure is simplicial. It is the *model structure for Segal n-spaces*. The coherent nerve of the simplicial category of Segal n-spaces is equivalent to the quasi-category $Cat_n(\mathbf{Typ})$.

45.4. There is a notion of *Rezk n-space* for every $n \geq 1$. By definition, it is a Segal n-space which satisfies the Rezk condition **??** in each variable. The model structure for Segal n-spaces admits a Bousfield localisation for which the fibrant objects are the Rezk n-spaces. It is the *model structure for Rezk n-spaces*. The coherent nerve of the simplicial category of Rezk n-spaces is equivalent to the quasi-category $RCat_n(\mathbf{Typ})$.

45.5. There is a notion of *n-fold quasi-category* for every $n \geq 1$. If $n = 1$, this is a quasi-category. The projection $p : \Delta^n \times \Delta \to \Delta^n$ is left adjoint to the functor $i : \Delta^n \to \Delta^n \times \Delta$ defined by putting $i(a) = (0, [0])$ for every $n \geq 0$. We thus obtain a pair of adjoint functors

$$p^* : \mathbf{S}^{(n)} \leftrightarrow \mathbf{S}^{(n+1)} : i^*.$$

Let us say that a map $f : X \to Y$ in $\mathbf{S}^{(n)}$ is a *weak equivalence* if the map $p^*(f)$ is a weak equivalence in the model structure for n-fold Rezk spaces. Then the category

$\mathbf{S}^{(n)}$ admits a unique Cisinski model structure with these weak equivalences. We call it the *model structure for n-fold quasi-categories*, A fibrant object for this model structure is a *n-fold quasi-category.* The pair of adjoint functors (p^*, i^*) is a Quillen equivalence between the model structure for n-fold quasi-categories and the model structure for n-fold Rezk spaces.

45.6. There is box product functor

$$\Box : \mathbf{S}^{(m)} \times \mathbf{S}^{(n)} \to \mathbf{S}^{(m+n)}$$

for every $m, n \geq 0$. The functor is a left Quillen functor of two variables with respect to the model structures for p-fold quasi-categories, where $p \in \{m, n, m+n\}$.

45.7. There is a notion of *quasi-n-category* for every $n \geq 1$. Let $p^* : \mathbf{S}^{(n)} \leftrightarrow \mathbf{S}^{(n+1)} : i^*$ be the pair of adjoint functors of 45.5. Let us say that a map $f : X \to Y$ in $\mathbf{S}^{(n)}$ is a *weak equivalence* if the map $p^*(f)$ is a weak equivalence in the model structure for Rezk n-spaces. Then the category $\mathbf{S}^{(n)}$ admits a unique Cisinski model structure with these weak equivalences. We call it the *model structure for quasi-n-categories.* A fibrant object for this model structure is a *quasi-n-category.* The pair of adjoint functors (p^*, i^*) is a Quillen equivalence between the model structure for quasi-n-categories and the model structure for Rezk n-spaces.

45.8. The *composite* $D \circ E$ of a n-disk D with a m-disk E is the $m+n$ disk

$$1 = D_0 \leftarrow D_1 \leftarrow \cdots \leftarrow D_n \leftarrow (D_n, \partial D_n) \times E_1 \leftarrow \cdots \leftarrow (D_n, \partial D_n) \times E_m,$$

where $(D_n, \partial D_n) \times E_k$ is defined by the pushout square

$$\begin{array}{ccc} \partial D_n \times E_k & \longrightarrow & D_n \times E_k \\ \downarrow & & \downarrow \\ E_k & \longrightarrow & (D_n, \partial D_n) \times E_k. \end{array}$$

This composition operation is associative.

45.9. The category $\mathbf{S}^{(n)} = [(\Delta^n)^o, \mathbf{Set}]$, contains n intervals

$$I_k = 1 \Box 1 \Box \cdots 1 \Box I \Box 1 \cdots 1 \Box 1,$$

one for each $0 \leq k \leq n$. It thus contain a n-disk $I^{(n)} : I_1 \circ I_2 \circ \cdots \circ I_n$. Hence there is a geometric morphism

$$(\rho^*, \rho_*) : \mathbf{S}^{(n)} \to \hat{\Theta},$$

such that $\rho^*(b) = I^{(n)}$. We shall say that a map of n-cellular sets $f : X \to Y$ is a *weak categorical equivalence* if the map $\rho^*(f) : \rho^*(X) \to \rho^*(Y)$ is a weak equivalence in the model structure for quasi-n-categories. The category $\hat{\Theta}_n$ admits a model structure in which the weak equivalences are the weak categorical equivalences and the cofibrations are the monomorphisms. We say that a fibrant object is a *n-quasi-category* The model structure is cartesian closed and left proper. We call it the *model structure for n-quasi-categories.* We denote the category of n-quasi-categories by $\mathbf{QCat}_n$. The pair of adjoint functors

$$\rho^* : \hat{\Theta}_n \to \mathbf{S}^{(n)} : \rho_*$$

is a Quillen equivalence between the model structure for n-quasi-categories and the model structure for quasi-n-categories.

46. Appendix on category theory

46.1. We fix three arbitrary Grothendieck universes $\mathbf{U}_1 \in \mathbf{U}_2 \in \mathbf{U}_3$. Sets in $\mathbf{U}_1$ are said to be *small*, sets in $\mathbf{U}_2$ are said to be *large* and sets in $\mathbf{U}_3$ are said to be *extra-large*. Beware that a small set is large and that a large set is extra-large. We denote by **Set** the category of small sets and by **SET** the category of large sets. A category is said to be *small* (resp. *large*, *extra-large*) if its set of arrows belongs to $\mathbf{U}_1$ (resp. $\mathbf{U}_2$, $\mathbf{U}_3$). The category **Set** is large and the category **SET** extra-large. We denote by **Cat** the category of small categories and by **CAT** the category of large categories. The category **Cat** is large and the category **CAT** is extra-large. A large category is *locally small* if its hom sets are small. We shall often denote small categories by ordinary capital letters and large categories by curly capital letters.

46.2. We shall denote by A^o the opposite of a category A. It can be useful to distinguish between the objects of A and A^o by writing $a^o \in A^o$ for each object $a \in A$, with the convention that $a^{oo} = a$. If $f : a \to b$ is a morphism in A, then $f^o : b^o \to a^o$ is a morphism in A^o. Beware that the *opposite* of a functor $F : A \to B$ is a functor $F^o : A^o \to B^o$. A *contravariant functor* $F : A \to B$ between two categories is defined to be a (covariant) functor $F : A^o \to B$; but we shall often denote the value of F at $a \in A$ by $F(a)$ instead of $F(a^o)$. .

46.3. We shall say that a functor $u : A \to B$ is *biunivoque* if the map $Ob(u) : ObA \to ObB$ is bijective. Every functor $u : A \to B$ admits a factorisation $u = pq$ with q a biunivoque functor and p a fully faithful functor. The factorisation is unique up to unique isomorphism. It is called the *Gabriel factorisation* of the functor u.

46.4. The categories **Cat** and **CAT** are cartesian closed. We shall denote the category of functors $A \to B$ between two categories by B^A or $[A, B]$ If $\mathcal{E}$ is a locally small category, then so is the category $\mathcal{E}^A = [A, \mathcal{E}]$ for any small category A. Recall that a *presheaf* on a small category A is defined to be a functor $X : A^o \to \mathbf{Set}$. A *map* of presheaves $X \to Y$ is a natural transformation. The presheaves on A form a locally small category

$$\hat{A} = \mathbf{Set}^{A^o} = [A^o, \mathbf{Set}].$$

The category $\hat{A}$ is cartesian closed; if $X, Y \in \hat{A}$ we shall denote the presheaf of maps $X \to Y$ by Y^X.

46.5. If A is a small category, then the *Yoneda functor* $y_A : A \to \hat{A}$ associates to an object $a \in A$ the presheaf $A(-, a)$. The *Yoneda lemma* asserts that for any object $a \in A$ and any presheaf $X \in \hat{A}$, the evaluation $x \mapsto x(1_a)$ induces a bijection between the set of natural transformation $A(-, a) \to X$ and the set $X(a)$. The lemma implies that the Yoneda functor is fully faithful. We shall often regard the functor as an inclusion $A \subset \hat{A}$ by adopting the same notation for an object $a \in A$ and the presheaf $A(-, a)$. Moreover, we shall identify a natural transformation $x : a \to X$ with the element $x(1_a) \in X(a)$. If $u : a \to b$ is a morphism in A, then the image of an element $x \in X(b)$ by the map $X(u) : X(b) \to X(a)$ is denoted as the composite of $x : b \to X$ by $u : a \to b$. We say that a presheaf X is *represented* by an element $x \in X(a)$ if the natural transformation $x : a \to X$ is invertible. A presheaf X is *representable* if it can be represented by a pair (a, x). Recall that the *category of elements* $El(X)$ of a presheaf $X : A \to \mathbf{Set}$ is the category whose objects are the pairs (a, x), where $a \in A$ and $x \in X(a)$, and whose arrows $(a, x) \to (b, y)$ are the morphism $f : a \to b$ in A such that $X(f)(y) = x$. It follows from Yoneda lemma that we have $El(X) = A/X$, where A/X is the full subcategory of $\hat{A}/X$ whose objects are the maps $a \to X$ with $a \in A$. A presheaf X is represented by an element $x \in X(a)$ iff the object (a, x) of $El(X)$ is terminal. Thus, a presheaf X representable iff its category of elements $El(X)$ has a terminal object.

46.6. The *dual Yoneda functor* $y^o_A : A^o \to [A, \mathbf{Set}]$ associates to an object $a \in A$ the set valued functor $A(a, -)$. The *Yoneda lemma* asserts that for any object $a \in A$ and any functor $F : A \to \mathbf{Set}$, the evaluation $x \mapsto x(1_a)$ induces a bijection between the set of natural transformations $x : A(a, -) \to F$ and $F(a)$. We shall identify these two sets by adopting the same notation for a natural transformation $x : A(a, -) \to F$ and the element $x(1_a) \in F(a)$. The dual Yoneda functor is fully faithful. and we shall often regard it as an inclusion $A^o \subset [A, \mathbf{Set}]$ by adopting the same notation for an object $a^o \in A^o$ and the presheaf $A(a, -)$. We say that a functor $F : A \to \mathbf{Set}$ is *represented* by an element $x \in F(a)$ if the corresponding natural transformation $x : a^o \to X$ is invertible. The functor F is said to be *representable* if it can be represented by an element (a, x). The *category of elements* of a (covariant) functor $F : A \to \mathbf{Set}$ is the category $el(F)$ whose objects are the pairs (a, x), where $a \in A$ and $x \in F(a)$, and whose arrows $(a, x) \to (b, y)$ are the morphisms $f : a \to b$ in A such that $F(f)(x) = y$. The functor X is represented by an element $x \in F(a)$ iff (a, x) is an initial object of the category $el(X)$. Thus, F representable iff the category $el(F)$ has an initial object.

46.7. Recall that a *2-category* is a category enriched over $\mathbf{Cat}$. An object of a 2-category $\mathcal{E}$ is often called a 0-*cell*. If A and B are 0-cells, an object of the category $\mathcal{E}(A, B)$ is called a 1-*cell* and an arrow is called a 2-*cell*. We shall often write $\alpha : f \to g : A \to B$ to indicate that α is a 2-cell with source the 1-cell $f : A \to B$ and target the 1-cell $g : A \to B$. The composition law in the category $\mathcal{E}(A, B)$ is said to be *vertical* and the composition law

$$\mathcal{E}(B, C) \times \mathcal{E}(A, B) \to \mathcal{E}(A, C)$$

horizontal. The vertical composition of a 2-cell $\alpha : f \to g$ with a 2-cell $\beta : g \to h$ is a 2-cell denoted by $\beta\alpha : f \to h$. The horizontal composition of a 2-cell $\alpha : f \to g : A \to B$ with a 2-cell and $\beta : u \to v : B \to C$ is a 2-cell denoted by $\beta \circ \alpha : uf \to vg : A \to C$.

46.8. There is a notion of adjoint in any 2-category. If $u : A \to B$ and $v : B \to A$ are 1-cells in a 2-category, an *adjunction* $(\alpha, \beta) : u \dashv v$ is a pair of 2-cells $\alpha : 1_A \to vu$ and $\beta : uv \to 1_B$ for which the *adjunction identities* hold:

$$(\beta \circ u)(u \circ \alpha) = 1_u \quad \text{and} \quad (v \circ \beta)(\alpha \circ v) = 1_v.$$

The 1-cell u is the *left adjoint* and the 1-cell v the *right adjoint*. The 2-cell α is the *unit* of the adjunction and the 2-cell β the *counit*. Each of the 2-cells α and β determines the other.

46.9. In any 2-category, there is a notion of left (and right) Kan extension of 1-cell $f : A \to X$ along a 1-cell $u : A \to B$. More precisely, the *left Kan extension* of f along u is a pair (g, α) where $g : B \to X$ and $\alpha : f \to gu$ is a 2-cell which reflects the object $f \in Hom(A, X)$ along the functor $Hom(u, X) : Hom(B, X) \to Hom(A, X)$. The *right Kan extension* of f along u is a pair (g, β) where $g : B \to X$ and $\beta : gu \to f$ is a 2-cell which coreflects the object $f \in Hom(A, X)$ along the functor $Hom(u, X) : Hom(B, X) \to Hom(A, X)$.

46.10. Recall that a full subcategory $A \subseteq B$ is said to be *reflective* if the inclusion functor $A \subseteq B$ has a left adjoint called a *reflection*. In general, the right adjoint v of an adjunction $u : A \leftrightarrow B : v$ is fully faithful iff the counit of the adjunction $\beta : uv \to 1_B$ is invertible, in which case u is said to be a *reflection* and v to be *reflective*. These notions can be defined in any 2-category. If the counit $\beta : uv \to 1_B$ of an adjunction $u : A \leftrightarrow B : v$ is invertible, the left adjoint is said to be a *reflection* and v to be *reflective*. Dually, a full subcategory $A \subseteq B$ is said to be *coreflective* if the inclusion functor $A \subseteq B$ has a right adjoint called a *coreflection*. These notions can be defined in any 2-category: if the counit $\beta : uv \to 1_B$ of an adjunction $u : A \leftrightarrow B : v$ is invertible, then the right adjoint v is said to be a *coreflection* and u to be *coreflective*.

46.11. The notion of *0-distributor* (called *distributor* if the context is clear) between two categories was defined in 14.4. The *composite* of two distributors $F : A \Rightarrow B$ and $G : B \Rightarrow C$) is the distributor $G \circ F = F \otimes_B G : A \Rightarrow C$ defined by putting

$$(F \otimes_B G)(a, c) = \int^{b \in B} F(a, b) \times G(b, c).$$

The composition of distributors

$$\circ : \mathrm{Dist}_0(B, C)\mathrm{Dist}_0(A, B) \to \mathrm{Dist}_0(A, C)$$

is coherently associative, and the distributor $hom : A^o \times A \to \mathbf{S}$ is a unit. This defines a bicategory Dist_0 whose objects are the small categories. The bicategory Dist_0 is biclosed. This means that the composition functor $\circ$ is divisible on each side. See 50.25 for this notion. For every $H \in \mathrm{Dist}_0(A, C)$, $F \in \mathrm{Dist}_0(A, B)$ and $G \in \mathrm{Dist}_0(B, C)$ we have

$$G\backslash H = Hom_C(G, H) \quad \text{and} \quad H/F = Hom_A(F, H).$$

Notice that $\mathrm{Dist}_0(1, A) = [A, \mathbf{Set}]$ and that $\mathrm{Dist}_0(A, 1) = [A^o, \mathbf{Set}] = \hat{A}$. To every distributor $F : A \Rightarrow B$ we can associate a cocontinuous functor $- \circ F : \hat{B} \to \hat{A}$. This defines an equivalence between the category of distributors $A \Rightarrow B$ and the category of cocontinuous functors $\hat{B} \to \hat{A}$. Dually, to every distributor $F \in \mathrm{Dist}_0(A, B)$ we can associate a cocontinuous functor $F \circ - : [A, \mathbf{Set}] \to [A, \mathbf{Set}]$. This defines an equivalence between the category of distributors $A \Rightarrow B$ and the category of cocontinuous functors $[A, \mathbf{Set}] \to [B, \mathbf{Set}]$. Notice that we have a natural isomorphism

$$G \circ (F \circ X) \simeq (G \circ F) \circ X$$

for every $X : A \to \mathbf{Set}]$, $F : A \Rightarrow B$ and $G : B \Rightarrow C$

46.12. The bicategory Dist_0 is symmetric monoidal. The *tensor product* of $F : A \Rightarrow B$ and $G : C \Rightarrow D$ is the distributor $F \otimes G : A \times C \Rightarrow B \times D)$ defined by putting

$$(F \times G)((a, c), (b, d)) = F(a, b) \times G(c, d)$$

for every quadruple of objects $(a, b, c, d) \in A \times B \times C \times D$.

46.13. The symmetric monoidal bicategory Dist_0 is compact closed. The *dual* of a category A is the category A^o and the *adjoint* of a distributor $F : A \Rightarrow B$ is the distributor $F^* : B^o \Rightarrow A^o$ obtained by putting $F^*(b^o, a^o) = F(a, b)$. The unit of the adjunction $A \vdash A^o$ is a distributor $\eta_A \in 1 \Rightarrow A^o \times A)$ and the counit a distributor $\epsilon_A : A \times A^o \Rightarrow 1$. We have $\eta_A = \epsilon_A = Hom_A : A^o \times A \to \mathbf{Set}$. The adjunction $A \vdash A^o$ is defined by a pair of invertible 2-cells,

$$\alpha_A : I_A \simeq (\epsilon_A \otimes A) \circ (A \otimes \eta_A) \quad \text{and} \quad \beta_A : I_{A^o} \simeq (A^o \otimes \epsilon_A) \circ (\eta_A \otimes A^o).$$

each of which is defined by using fthe canonical isomorphism

$$\int_{b \in A} \int_{c \in A} A(a, b) \times A(b, c) \times A(c, d) \simeq A(a, d).$$

46.14. The *trace* of a distributor $F : A \Rightarrow A$ defined by putting

$$Tr_A(F) = \epsilon_A \circ (F \otimes A^o) \circ \eta_{A^o}$$

is isomorphic to the *coend*

$$coend_A(F) = \int^{a \in A} F(a, a).$$

of the functor $F : A^o \times A \to \mathbf{Set}$.

46.15. To every functor $u : A \to B$ in **Cat** is associated a pair of adjoint functor

$$u_! : [A^o, \mathbf{Set}] \leftrightarrow [B^o, \mathbf{Set}] : u^*.$$

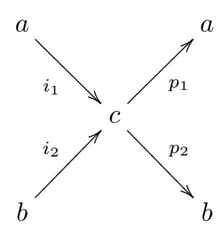
We have $u^*(Y) = \Gamma(u) \otimes_B Y = Y \circ \Gamma(u)$ for every $Y \in [B^o, \mathbf{Set}]$, where the distributor $\Gamma(u) \in \mathrm{Dist}_0(A, B)$ obtained by putting $\Gamma(u)(a, b) = B(ua, b)$ for every pair of objects $a \in A$ and $b \in B$. We have $u_!(X) = \Gamma^*(u) \otimes_A X = X \circ \Gamma^*(u)$ for every $X \in [A^o, \mathbf{Set}]$, where the distributor $\Gamma(u) \in \mathrm{Dist}_0(B, A)$ is defined by putting $\Gamma^*(u)(b, a) = B(b, ua)$. Notice that the functor u^* has a right adjoint u_* and that we have $u_*(X) = X/\Gamma(u)$ for every $X \in [A^o, \mathbf{Set}]$.

46.16. The functor $1 \star 1 \to 1$ gives the category 1 the structure of a monoid in the monoidal category $(\mathbf{Cat}, \star)$. If C is a category with terminal object $t \in C$ then there is unique functor $r : C \star 1 \to C$ which extends the identity $1_C : C \to C$ along the inclusion $C \subset C \star 1$ and such that $r(1) = t$. This defines a right action of the monoid 1 on C, and every right action of 1 on C is of this form. Dually, If C is a category with initial object $i \in C$ then there is unique functor $l : 1 \star C \to C$ which extends the identity $1_C : C \to C$ along the inclusion $C \subset 1 \star C$ and such that $l(1) = i$. This defines a left action of the monoid 1 on C, and every left action of 1 on C is of this form. We shall say that an object of a category is *null* if it is both initial and terminal. We shall say that a category C is *nullpointed* if it admits a null object $0 \in C$. A functor between nullpointed categories is *pointed* if it takes a null object to a null object. If C is nullpointed, then there is a unique functor $m : C \star C \to C$ which extends the codiagonal $C \sqcup C \to C$ and such that $m(1 \star 1) = 1_0$. The image by m of the unique arrow in $C \star C$ between $a \in C \star \emptyset$ and

$b \in \emptyset \star C$ the *null morphism* $0 = 0_{ba} : a \to b$ obtained by composing the morphism $a \to 0 \to b$. The functor m gives the category C the structure of a monoid object in the monoidal category $(\mathbf{Cat}, \star)$ and the map $0 : 1 \to C$ is an homomorphism.

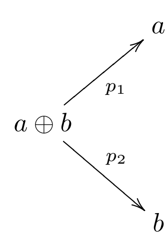
46.17. nullpointed category. We shall say that an object of a category is *null* if it is both initial and terminal. We shall say that a category C is *nullpointed* if it admits a null object $0 \in C$ (we could say more generally that C is *nullpointed* if its Karoubi envelope admits a null object). The *null morphism* $0 = 0_{ba} : a \to b$ between two objects of C is obtained by composing the morphism $a \to 0 \to b$. A functor between nullpointed categories is *pointed* if it takes a null object to a nul object.

46.18. If C is a nullpointed category, then the *direct sum* of two objects $a, b \in C$ is defined to be an object $c = a \oplus b$ equipped with four morphisms

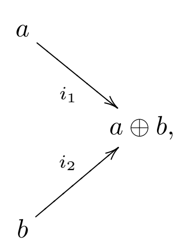


satisfying the following conditions:

- $p_1 i_1 = 1_a$, $p_2 i_2 = 1_b$, $p_2 i_1 = 0$ and $p_1 i_2 = 0$ in hoX;
- the pair (p_1, p_2) is a product diagram,



- the pair (i_1, i_2) is a coproduct diagram,



The cartesian product $a \times b$ of two objects $a, b \in C$ is a direct sum iff the pair of morphisms

$$a \xrightarrow{(1_a,0)} a \times b \xleftarrow{(0,1_b)} b$$

is a coproduct diagram. Dually, the coproduct $a \sqcup b$ of two objects $a, b \in C$ is a direct sum iff the pair of morphisms

$$a \xleftarrow{(1_a,0)} a \sqcup b \xrightarrow{(0,1_b)} b$$

is a product diagram. We shall say that a nullpointed category is *semi-additive* if it has binary direct sums. In a semi-additive category, the coproduct of an arbitrary family of objects $(a_i : i \in I)$ is denoted as a direct sum

$$\bigoplus_{i \in I} a_i = \bigsqcup_{i \in I} a_i.$$

The direct sum is also a product when I is finite. Similarly, the coproduct of an arbitrary family of morphisms $f_i : a_i \to b_i$ is denoted as a direct sum

$$\bigoplus_{i \in I} f_i : \bigoplus_{i \in I} a_i \to \bigoplus_{i \in I} b_i.$$

The opposite of a semi-additive category is semi-additive. We shall say that a functor between semi-additive categories is *finitely additive* if it preserves finite direct sums. A functor between semi-additive categories is finitely additive iff it preserves finite products. A functor $f : C \to D$ between semi-additive categories is finitely additive iff the opposite functor $f^o : C^o \to D^o$ is finitely additive. The *sum* $f + g : a \to b$ of two morphisms $f, g : a \to b$ of an additive category is defined to be the composite,

$$a \xrightarrow{(1_a,1_a)} a \oplus a \xrightarrow{f \oplus g} b \oplus b \xrightarrow{(1_b,1_b)} b.$$

This gives the set $C(a, b)$ the structure of a commutative monoid, with the null morphism $0 : a \to b$ for the neutral element. The composition

$$C(b, c) \times C(a, b) \to C(a, c)$$

is distributive with respect to the addition of morphisms for every triple of objects $a, b, c \in C$. A semi-additive category C is said to be *additive* if the monoid $C(a, b)$ is a group for every pair of objects $a, b \in C$. The opposite of an additive category is additive.

47. Appendix on factorisation systems

In this appendix we study the notion of factorisation system. We give a few examples of factorisation systems in **Cat**.

Definition 47.1. *If $\mathcal{E}$ is a category, we shall say that a pair $(\mathcal{A}, \mathcal{B})$ of classes of maps in $\mathcal{E}$ is a* (strict) factorisation system *if the following conditions are satisfied:*

- *each class $\mathcal{A}$ and $\mathcal{B}$ is closed under composition and contains the isomorphisms;*
- *every map $f : A \to B$ admits a factorisation $f = pu : A \to E \to B$ with $u \in \mathcal{A}$ and $p \in \mathcal{B}$, and the factorisation is unique up to unique isomorphism.*

We say that $\mathcal{A}$ is the left *class and $\mathcal{B}$ the* right *class of the weak factorisation system.*

In this definition, the uniqueness of the factorisation $f = pu : A \to E \to B$ means that for any other factorisation $f = p'u' : A \to E' \to B$ with $u' \in \mathcal{A}$ and $p' \in \mathcal{B}$, there exists a unique isomorphism $i : E \to F$ such that $iu = u'$ and $p'i = p$,

$$\begin{array}{ccc} A & \xrightarrow{u'} & E' \\ {\scriptstyle u}\downarrow & \overset{i}{\nearrow} & \downarrow{\scriptstyle p'} \\ E & \xrightarrow[p]{} & B. \end{array}$$

Recall that a class of maps $\mathcal{M}$ in a category $\mathcal{E}$ is said to be *invariant under isomorphisms* if for every commutative square

$$\begin{array}{ccc} A & \longrightarrow & A' \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u'} \\ B & \longrightarrow & B' \end{array}$$

in which the horizontal maps are isomorphisms we have $u \in \mathcal{M} \Leftrightarrow u' \in \mathcal{M}$. It is obvious from the definition that each class of a factorisation system is invariant under isomorphism.

Definition 47.2. *We shall say that a class of maps $\mathcal{M}$ in a category $\mathcal{E}$ has the* right cancellation property *if the implication*

$$vu \in \mathcal{M} \text{ and } u \in \mathcal{M} \ \Rightarrow \ v \in \mathcal{M}$$

is true for any pair of maps $u : A \to B$ and $v : B \to C$. Dually, we shall say that $\mathcal{M}$ has the left cancellation property *if the implication*

$$vu \in \mathcal{M} \text{ and } v \in \mathcal{M} \ \Rightarrow \ u \in \mathcal{M}$$

is true.

Proposition 47.3. *The intersection of the classes of a factorisation system $(\mathcal{A}, \mathcal{B})$ is the class of isomorphisms. Moreover,*

- *the class $\mathcal{A}$ has the right cancellation property;*
- *the class $\mathcal{B}$ has the left cancellation property.*

Proof: If a map $f : A \to B$ belongs to $\mathcal{A} \cap \mathcal{B}$, consider the factorisations $f = f1_A$ and $f = 1_B f$. We have $1_A \in \mathcal{A}$ and $f \in \mathcal{B}$ in the first, and we have $f \in \mathcal{A}$ and $1_B \in \mathcal{B}$ in the second. Hence there exists an isomorphism $i : B \to A$ such that $if = 1_A$ and $fi = 1_B$. This shows that f is invertible. If $u \in \mathcal{A}$ and $vu \in \mathcal{A}$, let us show that $v \in \mathcal{A}$. For this, let us choose a factorisation $v = ps : B \to E \to C$, with $s \in \mathcal{A}$ and $p \in \mathcal{B}$, and put $w = vu$. Then w admits the factorisation $w = p(su)$ with $su \in \mathcal{A}$ and $p \in \mathcal{B}$ and the factorisation $w = 1_C(vu)$ with $vu \in \mathcal{A}$ and $1_C \in \mathcal{B}$. Hence there exists an isomorphism $i : E \to C$ such that $i(su) = vu$ and $1_C i = p$. Thus, $p \in \mathcal{A}$ since $p = i$ and every isomorphism belongs to $\mathcal{A}$. It follows that $v = ps \in \mathcal{A}$, since $\mathcal{A}$ is closed under composition. ∎

Definition 47.4. *We say that a map $u : A \to B$ in a category $\mathcal{E}$ is* left orthogonal *to a map $f : X \to Y$, or that f is* right orthogonal *to u, if every commutative square*

$$\begin{array}{ccc} A & \xrightarrow{x} & X \\ {\scriptstyle u}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\ B & \xrightarrow[y]{} & Y \end{array}$$

has a unique diagonal filler $d : B \to X$ (that is, $du = x$ and $fd = y$). We shall denote this relation by $u \perp f$.

Notice that the condition $u \perp f$ means that the square

$$\begin{array}{ccc} Hom(B,X) & \xrightarrow{Hom(u,X)} & Hom(A,X) \\ {\scriptstyle Hom(B,f)}\downarrow & & \downarrow{\scriptstyle Hom(A,f)} \\ Hom(B,Y) & \xrightarrow[Hom(u,Y)]{} & Hom(A,Y) \end{array}$$

is cartesian. If $\mathcal{A}$ and $\mathcal{B}$ are two classes of maps in $\mathcal{E}$, we shall write $\mathcal{A} \perp \mathcal{B}$ to indicate that we have $a \perp b$ for every $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

If $\mathcal{M}$ is a class of maps in a category $\mathcal{E}$, we shall denote by ${}^\perp\mathcal{M}$ (resp. $\mathcal{M}^\perp$) the class of maps which are left (resp. right) orthogonal to every map in $\mathcal{M}$. Each class ${}^\perp\mathcal{M}$ and $\mathcal{M}^\perp$ is closed under composition and contains the isomorphisms. The class ${}^\perp\mathcal{M}$ has the right cancellation property and the class $\mathcal{M}^\perp$ the left cancellation property. If $\mathcal{A}$ and $\mathcal{B}$ are two classes of maps in $\mathcal{E}$, then

$$\mathcal{A} \subseteq {}^\perp\mathcal{B} \ \Leftrightarrow\ \mathcal{A} \perp \mathcal{B} \ \Leftrightarrow\ \mathcal{A}^\perp \supseteq \mathcal{B}.$$

Proposition 47.5. *If $(\mathcal{A}, \mathcal{B})$ is a factorisation system then*

$$\mathcal{A} = {}^\perp\mathcal{B} \quad \text{and} \quad \mathcal{B} = \mathcal{A}^\perp.$$

Proof Let us first show that we have $\mathcal{A} \perp \mathcal{B}$. If $a : A \to A'$ is a map in $\mathcal{A}$ and $b : B \to B'$ is a map in $\mathcal{B}$, let us show that every commutative square

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ A' & \xrightarrow[u']{} & B' \end{array}$$

has a unique diagonal filler. Let us choose a factorisation $u = ps : A \to E \to B$ with $s \in \mathcal{A}$ and $p \in \mathcal{B}$ and a factorisation $u' = p's' : A' \to E' \to B'$ with $s' \in \mathcal{A}$ and $p' \in \mathcal{B}$. From the commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{s} & E & \xrightarrow{p} & B \\ {\scriptstyle a}\downarrow & & & & \downarrow{\scriptstyle b} \\ A' & \xrightarrow[s']{} & E' & \xrightarrow[p']{} & B', \end{array}$$

we can construct a square

$$\begin{array}{ccc} A & \xrightarrow{s} & E \\ {\scriptstyle s'a}\downarrow & & \downarrow{\scriptstyle bp} \\ E' & \xrightarrow[p']{} & B'. \end{array}$$

Observe that $s \in \mathcal{A}$ and $bp \in \mathcal{B}$ and also that $s'a \in \mathcal{A}$ and $p' \in \mathcal{B}$. By the uniqueness of the factorisation of a map, there is a unique isomorphism $i : E' \to E$ such that $is'a = s$ and $bpi = p'$:

$$\begin{array}{ccccc} A & \xrightarrow{s} & E & \xrightarrow{p} & B \\ {\scriptstyle a}\downarrow & & \uparrow{\scriptstyle i} & & \downarrow{\scriptstyle b} \\ A' & \xrightarrow[s']{} & E' & \xrightarrow[p']{} & B'. \end{array}$$

The composite $d = pis'$ is then a diagonal filler of the first square

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ {\scriptstyle a}\downarrow & \overset{d}{\nearrow} & \downarrow{\scriptstyle b} \\ A' & \xrightarrow[u']{} & B'. \end{array}$$

It remains to prove the uniqueness of d. Let d' be an arrow $A' \to B$ such that $d'a = u$ and $bd' = u'$. Let us choose a factorisation $d' = qt : A' \to F \to B$ with $t \in \mathcal{A}$ and $q \in \mathcal{B}$. From the commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{s} & E & \xrightarrow{p} & B \\ & & & \overset{q}{\nearrow} & \\ {\scriptstyle a}\downarrow & & F & & \downarrow{\scriptstyle b} \\ & \overset{t}{\nearrow} & & & \\ A' & \xrightarrow[s']{} & E' & \xrightarrow[p']{} & B'. \end{array}$$

we can construct two commutative squares

$$\begin{array}{ccc} A & \xrightarrow{s} & E \\ {\scriptstyle ta}\downarrow & & \downarrow{\scriptstyle p} \\ F & \xrightarrow[q]{} & B, \end{array} \qquad \begin{array}{ccc} A' & \xrightarrow{t} & F \\ {\scriptstyle s'}\downarrow & & \downarrow{\scriptstyle bq} \\ E' & \xrightarrow[p']{} & B'. \end{array}$$

Observe that we have $ta \in \mathcal{A}$ and $q \in \mathcal{B}$. Hence there exists a unique isomorphism $j : F \to E$ such that $jta = s$ and $pj = q$. Similarly, there exists a unique isomorphism $j' : E' \to F$ such that $j's' = t$ and $bqj' = p'$. The maps fits in the following commutative diagram,

$$\begin{array}{ccccc} A & \xrightarrow{s} & E & \xrightarrow{p} & B \\ & & \uparrow{\scriptstyle j} & \nearrow{\scriptstyle q} & \\ \downarrow{\scriptstyle a} & & F & & \downarrow{\scriptstyle b} \\ & \nearrow{\scriptstyle t} & \uparrow{\scriptstyle j'} & & \\ A' & \xrightarrow[s']{} & E' & \xrightarrow[p']{} & B'. \end{array}$$

Hence the diagram

$$\begin{array}{ccc} A & \xrightarrow{s} & E \\ {\scriptstyle s'a}\downarrow & \nearrow{\scriptstyle jj'} & \downarrow{\scriptstyle bp} \\ E' & \xrightarrow[p']{} & B'. \end{array}$$

commutes. It follows that we have $jj' = i$ by the uniqueness of the isomorphism between two factorisations. Thus, $d' = qt = (pj)(j's') = pis' = d$. The relation $\mathcal{A} \perp \mathcal{B}$ is proved. This shows that $\mathcal{A} \subseteq {}^{\perp}\mathcal{B}$. Let us show that ${}^{\perp}\mathcal{B} \subseteq \mathcal{A}$. If a map $f : A \to B$ is in ${}^{\perp}\mathcal{B}$. let us choose a factorisation $f = pu : A \to C \to B$ with $u \in \mathcal{A}$ and $p \in \mathcal{B}$. Then the square

$$\begin{array}{ccc} A & \xrightarrow{u} & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow[1_B]{} & B \end{array}$$

has a diagonal filler $s : B \to C$, since $f \in {}^{\perp}\mathcal{B}$. We have $ps = 1_B$. Let us show that $sp = 1_C$. Observe that the maps sp and 1_C are both diagonal fillers of the square

$$\begin{array}{ccc} A & \xrightarrow{u} & C \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle p} \\ C & \xrightarrow[p]{} & B. \end{array}$$

This proves that $sp = 1_C$ by the uniqueness of a diagonal filler. Thus, $p \in \mathcal{A}$, since every isomorphism is in $\mathcal{A}$. Thus, $f = pu \in \mathcal{A}$. ∎

Corollary 47.6. *Each class of a factorisation system determines the other.*

47.1. We shall say that a class of maps $\mathcal{M}$ in a category $\mathcal{E}$ is *closed under limits* if the full subcategory of $\mathcal{E}^I$ spanned by the maps in $\mathcal{M}$ is closed under limits. There is a dual notion of a class of maps closed under colimits.

Proposition 47.7. *The class $\mathcal{M}^{\perp}$ is closed under limits for any class of maps $\mathcal{M}$ in a category $\mathcal{E}$. Hence the right class of a factorisation system is closed under limits.*

Proof: For any pair of morphisms $u : A \to B$ and $f : X \to Y$ in $\mathcal{E}$, we have a commutative square $Sq(u, f)$:

$$\begin{array}{ccc} \mathcal{E}(B,X) & \xrightarrow{\mathcal{E}(u,X)} & \mathcal{E}(A,X) \\ {\scriptstyle \mathcal{E}(B,f)}\downarrow & & \downarrow{\scriptstyle \mathcal{E}(A,f)} \\ \mathcal{E}(B,Y) & \xrightarrow[\mathcal{E}(u,Y)]{} & \mathcal{E}(A,Y). \end{array}$$

The resulting functor

$$Sq : (\mathcal{E}^o)^I \times \mathcal{E}^I \to \mathbf{Set}^{I\times I}$$

continuous in each variable. An arrow $f \in \mathcal{E}$ belongs to $\mathcal{M}^\perp$ iff the square $Sq(u, f)$ is cartesian for every arrow $u \in \mathcal{M}$. This proves the result, since the full subcategory of $\mathbf{Set}^{I\times I}$ spanned by the cartesian squares is closed under limits. QED

Recall that a map $u : A \to B$ in a category $\mathcal{E}$ is said to be a *retract* of another map $v : C \to D$, if u is a retract of v in the category of arrows $\mathcal{E}^I$. A class of maps $\mathcal{M}$ in a category $\mathcal{E}$ is said to be *closed under retracts* if the retract of a map in $\mathcal{M}$ belongs to $\mathcal{M}$.

Corollary 47.8. *The class $\mathcal{M}^\perp$ is closed under retracts for any class of maps $\mathcal{M}$ in a category $\mathcal{E}$. Each class of a factorisation system is closed under retracts.*

47.2. Let $(\mathcal{A}, \mathcal{B})$ be a factorisation system in a category $\mathcal{E}$. Then the full subcategory of $\mathcal{E}^I$ spanned by the elements of $\mathcal{B}$ is reflective. Dually, the full subcategory of $\mathcal{E}^I$ spanned by the elements of $\mathcal{A}$ is coreflective.
Proof: Let us denote by $\mathcal{B}'$ the full subcategory of $\mathcal{E}^I$ whose objects are the arrows in $\mathcal{B}$. Every map $u : A \to B$ admits a factorisation $u = pi : A \to E \to B$ with $i \in \mathcal{A}$ and $p \in \mathcal{B}$. The pair $(i, 1_B)$ defines an arrow $u \to p$ in $\mathcal{E}^I$. Let us show that the arrow reflects u in the subcategory $\mathcal{B}'$. For this, it suffices to show that for every arrow $f : X \to Y$ in $\mathcal{B}$ and every commutative square

$$\begin{array}{ccc} A & \xrightarrow{x} & X \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[y]{} & Y, \end{array}$$

there exists a unique arrow $z : E \to X$ such that $fz = yp$ and $zi = x$. But this is clear, since the square

$$\begin{array}{ccc} A & \xrightarrow{x} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ E & \xrightarrow[yp]{} & Y. \end{array}$$

has a unique diagonal filler by 47.5. ∎

Recall that the projection $A \times_B E \to A$ in a pullback square

$$\begin{array}{ccc} A \times_B E & \longrightarrow & E \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array}$$

is said to be the *base change* of the map $E \to B$ along the map $A \to B$. A class of maps $\mathcal{B}$ in a category $\mathcal{E}$ is said to be *closed under base changes* if the base change of a map in $\mathcal{B}$ along any map in $\mathcal{E}$ belongs to $\mathcal{B}$ when this base change exists. The class $\mathcal{M}^{\perp}$ is closed under base changes for any class of maps $\mathcal{M} \subseteq \mathcal{E}$. In particular, the right class of a factorisation system is closed under base change. Recall that the map $B \to E \sqcup_A B$ in a pushout square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ E & \longrightarrow & E \sqcup_A B \end{array}$$

is said to be the *cobase change* of the map $A \to E$ along the map $A \to B$. A class of maps $\mathcal{A}$ in category $\mathcal{E}$ is said to be *closed under cobase changes* if the cobase change of a map in $\mathcal{A}$ along any map in $\mathcal{E}$ belongs to $\mathcal{A}$ when this cobase change exists. The class ${}^{\perp}\mathcal{M}$ is closed under cobase changes for any class of maps $\mathcal{M} \subseteq \mathcal{E}$. In particular, the left class of a factorisation system is closed under cobase changes.

47.3. Let us say that an arrow $f : X \to Y$ in a category with finite limits is *surjective* if it is left orthogonal to every monomorphism. The class of surjections is closed under cobase change, under colimits and it has the right cancellation property. Every surjection is an epimorphism, but the converse is not necessarly true.

We now give some examples of factorisation systems.

Proposition 47.9. *Let $p : \mathcal{E} \to \mathcal{C}$ be a Grothendieck fibration. Then the category $\mathcal{E}$ admits a factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{B}$ is the class of cartesian morphisms. An arrow $u \in \mathcal{E}$ belongs to $\mathcal{A}$ iff the arrow $p(u)$ is invertible.*

Dually, if $p : \mathcal{E} \to \mathcal{C}$ is a Grothendieck opfibration, then the category $\mathcal{E}$ admits a factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of cocartesian morphisms. A morphism $u \in \mathcal{E}$ belongs to $\mathcal{B}$ iff the morphism $p(u)$ is invertible.

If $\mathcal{E}$ is a category with pullbacks, then the target functor $t : \mathcal{E}^I \to \mathcal{E}$ is a Grothendieck fibration. A morphism $f : X \to Y$ of the category $\mathcal{E}^I$ is a commutative square in $\mathcal{E}$,

$$\begin{array}{ccc} X_0 & \xrightarrow{f_0} & Y_0 \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle y} \\ X_1 & \xrightarrow{f_1} & Y_1. \end{array}$$

The morphism f is cartesian iff the square is a pullback (also called a *cartesian square*). Hence the category $\mathcal{E}^I$ admits a factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{B}$ is the class of cartesian squares. A square $f : X \to Y$ belongs to $\mathcal{A}$ iff the morphism $f_1 : X_1 \to Y_1$ is invertible.

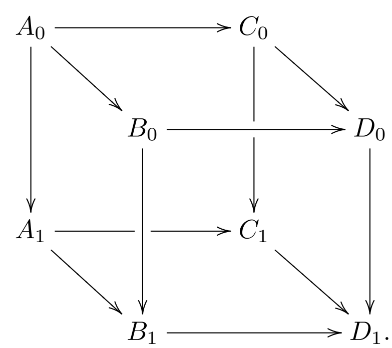
Corollary 47.10. *Suppose that we have a commutative diagram*

$$\begin{array}{ccccc} A_0 & \longrightarrow & B_0 & \longrightarrow & C_0 \\ \downarrow & & \downarrow & & \downarrow \\ A_1 & \longrightarrow & B_1 & \longrightarrow & C_1 \end{array}$$

in which the right hand square is cartesian. Then the left hand square is cartesian iff the composite square is cartesian.

Proof: This follows from the left cancellation property of the right class of a factorisation system. ■

Corollary 47.11. *Suppose that we have a commutative cube*



in which the left face, the right face and front face are cartesian. Then the back face is cartesian.

We now give a few examples of factorisation systems in the category **Cat**.

Recall that a functor $p : E \to B$ is said to be a *discrete fibration* if for every object $e \in E$ and every arrow $g \in B$ with target $p(e)$, there exists a unique arrow $f \in E$ with target e such that $p(f) = e$. There is a dual notion of *discrete opfibration*. Recall that a functor between small categories $u : A \to B$ is said to be *final* (but we shall say *0-final*) if the category $b\backslash A = (b\backslash B) \times_B A$ defined by the pullback square

$$\begin{array}{ccc} b\backslash A & \xrightarrow{h} & A \\ \downarrow & & \downarrow u \\ b\backslash B & \longrightarrow & B. \end{array}$$

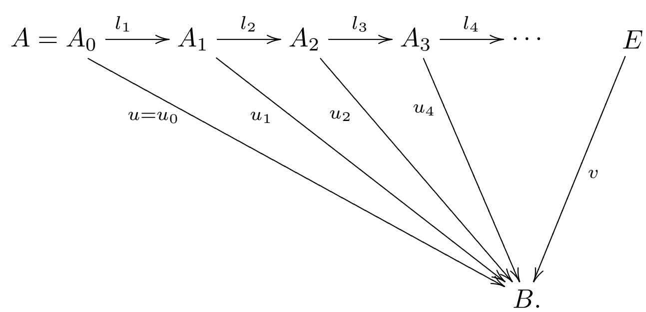
is connected for every object $b \in B$. There is a dual notion of *initial functor* (but we shall say *0-initial*).

Theorem 47.12. [Street] *The category* **Cat** *admits a factorisation system* $(\mathcal{A}, \mathcal{B})$ *in which* $\mathcal{B}$ *is the class of discrete fibrations and* $\mathcal{A}$ *the class of 0-final functors. Dually, category* **Cat** *admits a factorisation system* $(\mathcal{A}', \mathcal{B}')$ *in which* $\mathcal{B}'$ *is the class of discrete opfibrations and* $\mathcal{A}'$ *is the class of 0-initial functors.*

47.4. Recall that a functor $p : C \to D$ is said to be *conservative* if the implication

$$p(f) \text{ invertible } \Rightarrow f \text{ invertible}$$

is true for every arrow $f \in C$. The model category $(\mathbf{Cat}, Eq)$ admits a factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{B}$ is the class of conservative functors. A functor in the class $\mathcal{A}$ is an *iterated strict localisation* Let us describe the strict localisations explicitly. We say that a functor $g : A \to B$ *inverts* a set S of arrows in A if every arrow in $g(S)$ is invertible. When the category A is small. there is a functor $l_S : A \to S^{-1}A$ which inverts S universally. The universality means that for any functor $g : A \to B$ which inverts S there exists a unique functor $h : S^{-1}A \to B$ such that $hl_S = g$. The functor l_S is a strict localisation. Every functor $u : A \to B$ admits a factorisation $u = u_1 l_1 : A \to S_0^{-1}A \to B$, where S_0 is the set of arrows inverted by u and where $l_1 = l_{S_0}$. Let us put $A_1 = S_0^{-1}A$. The functor u_1 is not necessarily conservative but it admits a factorisation $u_1 = u_2 l_2 : A_1 \to S_1^{-1}A_1 \to B$, where S_1 is the set of arrows inverted by u_1. Let us put $A_2 = S_1^{-1}A_1$. By iterating this process, we obtain an infinite sequence of categories and functors,



If the category E is the colimit of the sequence, then the functor $v : E \to B$ is conservative. and the canonical functor $l : A \to E$ is an iterated strict localisation.

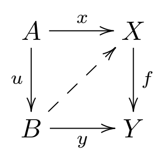
47.5. For any category C, the full subcategory of $C\backslash\mathbf{Cat}$ spanned by the iterated strict localisations $C \to L$ is equivalent to a complete lattice $Loc(C)$. Its maximum element is defined by the localisation $C \to \pi_1 C$ which inverts every arrow in C. A functor $u : C \to D$ induces a pair of adjoint maps

$$u_! : Loc(C) \to Loc(D) : u^*,$$

where $u_!$ is defined by cobase change along u.

48. Appendix on weak factorisation systems

48.1. Recall that an arrow $u : A \to B$ in a category $\mathcal{E}$ is said to have the *left lifting property* with respect to another arrow $f : X \to Y$, or that f has the *right lifting property* with respect to u, if every commutative square



has a diagonal filler $d : B \to X$ (that is, $du = x$ and $fd = y$). We shall denote this relation by $u \pitchfork f$. If the diagonal filler is unique we shall write $u \perp f$ and say that

u is *left orthogonal* to f, ot that f is *right orthogonal* to u. For any class of maps $\mathcal{M} \subseteq \mathcal{E}$, we shall denote by ${}^{\pitchfork}\mathcal{M}$ (resp. $\mathcal{M}^{\pitchfork}$) the class of maps having the left lifting property (resp. right lifting property) with respect to every map in $\mathcal{M}$. Each class ${}^{\pitchfork}\mathcal{M}$ and $\mathcal{M}^{\pitchfork}$ contains the isomorphisms and is closed under composition. If $\mathcal{A}$ and $\mathcal{B}$ are two classes of maps in $\mathcal{E}$, we shall write $\mathcal{A} \pitchfork \mathcal{B}$ to indicate that we have $u \pitchfork f$ for every $u \in \mathcal{A}$ and $f \in \mathcal{B}$. Then

$$\mathcal{A} \subseteq {}^{\pitchfork}\mathcal{B} \iff \mathcal{A} \pitchfork \mathcal{B} \iff \mathcal{B} \subseteq \mathcal{A}^{\pitchfork}.$$

48.2. We say that a pair $(\mathcal{A}, \mathcal{B})$ of classes of maps in a category $\mathcal{E}$ is a *weak factorisation system* if the following two conditions are satisfied:

- every map $f \in \mathcal{E}$ admits a factorisation $f = pu$ with $u \in \mathcal{A}$ and $p \in \mathcal{B}$;
- $\mathcal{A} = {}^{\pitchfork}\mathcal{B}$ and $\mathcal{A}^{\pitchfork} = \mathcal{B}$.

We say that $\mathcal{A}$ is the *left class* and $\mathcal{B}$ the *right class* of the weak factorisation system.

48.3. Every factorisation system is a weak factorisation system.

48.4. We say that a map in a topos is a trivial fibration if it has the right lifting property with respect to every monomorphism. This terminology is non-standard but useful. The trivial fibrations often coincide with the acyclic fibrations (which can be defined in any model category). An object X in a topos is said to be *injective* if the map $X \to 1$ is a trivial fibration. If $\mathcal{B}$ is the class of trivial fibrations in a topos and $\mathcal{A}$ is the class monomorphisms, then the pair $(\mathcal{A}, \mathcal{B})$ is a weak factorisation system. A map of simplicial sets is a trivial fibration iff it has the right lifting property with respect to the inclusion $\delta_n : \partial\Delta[n] \subset \Delta[n]$ for every $n \geq 0$.

48.5. We say that a Grothendieck fibration $E \to B$ is a *1-fibration* if its fibers $E(b)$ are groupoids. We say that a category C is *1-connected* if the functor $\pi_1 C \to 1$ is an equivalence. We say that functor $u : A \to B$ is is *1-final*) if the category $b\backslash A = (b\backslash B) \times_B A$ is 1-connected for every object $b \in B$. The category **Cat** admits a weak factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{B}$ is the class of 1-fibrations and $\mathcal{A}$ the class of 1-final functors.

48.6. Let $\mathcal{E}$ be a cocomplete category. If $\alpha = \{i : i < \alpha\}$ is a non-zero ordinal, we shall say that a functor $C : \alpha \to \mathcal{E}$ is an *α-chain* if the canonical map

$$\varinjlim_{i<j} C(i) \to C(j)$$

is an isomorphism for every non-zero limit ordinal $j < \alpha$. The *composite* of C is the canonical map

$$C(0) \to \varinjlim_{i<\alpha} C(i).$$

We shall say that a subcategory $\mathcal{A} \subseteq \mathcal{E}$ is closed under *transfinite composition* if the composite of any α-chain $C : \alpha \to \mathcal{E}$ with values in $\mathcal{A}$ belongs to $\mathcal{A}$.

48.7. Let $\mathcal{E}$ be a cocomplete category. We shall say that a class of maps $\mathcal{A} \subseteq \mathcal{E}$ is *saturated* if it satisfies the following conditions:

- $\mathcal{A}$ contains the isomorphisms and is closed under composition ;
- $\mathcal{A}$ is closed under transfinite composition;
- $\mathcal{A}$ is closed under cobase change and retract;

Every class of maps $\Sigma \subseteq \mathcal{E}$ is contained in a smallest saturated class $\overline{\Sigma}$ called the saturated class *generated* by Σ. We shall say that a saturated class $\mathcal{A}$ is *accessible* if it is generated by a set of maps $\Sigma \subseteq \mathcal{A}$.

48.8. [Ci1] The class $\mathcal{M}$ of monomorphisms in a Grothendieck topos is accessible.

48.9. If Σ is a set of maps in a locally presentable category, then the pair $(\overline{\Sigma}, \Sigma^{\pitchfork})$ is a weak factorisation system, where $\overline{\Sigma}$ denotes the saturated class generated by Σ.

49. Appendix on simplicial sets

We fix some notations about simplicial sets. The category of finite non-empty ordinals and order preserving maps is denoted Δ. It is standard to denote the ordinal $n+1 = \{0, \ldots, n\}$ by $[n]$. A map $u : [m] \to [n]$ in Δ can be specified by listing its values $(u(0), \ldots, u(m))$. The order preserving injection $[n-1] \to [n]$ which omits $i \in [n]$ is denoted d_i, and the order preserving surjection $[n] \to [n-1]$ which repeats $i \in [n-1]$ is denoted s_i.

49.1. Recall that a (small) *simplicial set* is a presheaf on the category Δ. We shall denote the category of simplicial sets by $\mathbf{S}$. If X is a simplicial set and $n \geq 0$, it is standard to denote the set $X([n])$ by X_n. An element of X_n is said to be a *simplex of dimension* n, or a n*-simplex* of X; a 0-simplex is called a *vertex* and a 1-simplex an *arrow*. If $n > 0$ and $i \in [n]$, the map $X(d_i) : X_n \to X_{n-1}$ is denoted by ∂_i, and if $i \in [n-1]$, the map $X(s_i) : X_{n-1} \to X_n$ is denoted by σ_i. The simplex $\partial_i(x) \in X_{n-1}$ is the *i-th face* of a simplex $x \in X_n$. The *source* of an arrow $f \in X_1$ is defined to be the vertex $\partial_1(f)$ and its *target* to be the vertex $\partial_0(f)$; we shall write $f : a \to b$ to indicate that $a = \partial_1(f)$ and $b = \partial_0(f)$.

49.2. The *combinatorial simplex* $\Delta[n]$ of dimension n is defined to be the simplicial set $\Delta(-, [n])$; the simplex $\Delta[1]$ is the *combinatorial interval* and we shall denote it by I; the simplex $\Delta[0]$ is the terminal object of the category $\mathbf{S}$ and we shall denote it by 1. We shall often identify a morphism $u : [m] \to [n]$ in Δ with its image $y_\Delta(u) : \Delta[m] \to \Delta[n]$ by the Yoneda functor $y_\Delta : \Delta \to \mathbf{S}$. By the Yoneda lemma, for every $X \in \mathbf{S}$ and $n \geq 0$ the evaluation map $x \mapsto x(1_{[n]})$ is a bijection between the maps $\Delta[n] \to X$ and the elements of X_n; we shall identify these two sets by adopting the same notation for a map $\Delta[n] \to X$ and the corresponding simplex in X_n. If $u : [m] \to [n]$ and $x \in X_n$, we shall denote the simplex $X(u)(x) \in X_m$ as a composite $xu : \Delta[m] \to X$. For example, $\partial_i(x) = xd_i : \Delta[n-1] \to X$ for every $x \in X_n$ and $\sigma_i(x) = xs_i$ for every $x \in X_{n-1}$. A simplex $x \in X_n$ is said to be *degenerate* if it belongs to the image of $\sigma_i : X_{n-1} \to X_n$ for some $i \in [n-1]$. To every vertex $a \in X_0$ is associated a degenerate arrow $\sigma_0(a) : a \to a$ that we shall denote as a *unit* $1_a : a \to a$.

49.3. The *cardinality* of a simplicial set X is defined to be the cardinality of the set of non-degenerate simplices of X. A simplicial set is *finite* if it has a finite number of non-degenerate simplices. A simplicial set can be large. A *large simplicial set* is defined to be a functor $\Delta^o \to \mathbf{SET}$, where $\mathbf{SET}$ is the category of large sets. We say that a large simplicial set X is *locally small* if the vertex map $X_n \to X_0^{n+1}$ has small fibers for every $n \geq 0$. If X is locally small, then so is the simplicial set X^A for any small simplicial set A.

49.4. Let $\tau : \Delta \to \Delta$ be the automorphism of the category Δ which reverses the order of each ordinal. If $u : [m] \to [n]$ is a map in Δ, then $\tau(u)$ is the map $u^o : [m] \to [n]$ obtained by putting $u^o(i) = n - f(m-i)$. The *opposite* X^o of a simplicial set X is obtained by composing the (contravariant) functor $X : \Delta \to \mathbf{Set}$ with the functor τ. We distinguish between the simplices of X and X^o by writing $x^o \in X^o$ for each $x \in X$, with the convention that $x^{oo} = x$. If $f : a \to b$ is an arrow in X, then $f^o : b^o \to a^o$ is an arrow in X^o. Beware that the *opposite* of a map of simplicial sets $u : A \to B$ is a map $u^o : A^o \to B^o$. A *contravariant map* $p : A \to B$ between two simplicial sets is defined to be a map $q : A^o \to B$; we shall often write $p(a)$ instead of $q(a^o)$ for $a \in A$. .

49.5. If X is a simplicial set, we say that a subfunctor $A \subseteq X$ is a *simplicial subset* of X. If $n > 0$ and $i \in [n]$ the image of the map $d_i : \Delta[n-1] \to \Delta[n]$ is denoted $\partial_i\Delta[n] \subset \Delta[n]$. The *simplicial sphere* $\partial\Delta[n] \subset \Delta[n]$ is the union the faces $\partial_i\Delta[n]$ for $i \in [n]$; by convention $\partial\Delta[0] = \emptyset$. If $n > 0$, a map $x : \partial\Delta[n] \to X$ is said to be a *simplicial sphere of dimension* $n-1$ *in* X; it is determined by the sequence of its faces $(x_0, \ldots, x_n) = (xd_0, \ldots, xd_n)$. A simplicial sphere $\partial\Delta[2] \to X$ is called a *triangle*. Every n-simplex $y : \Delta[n] \to X$ has a *boundary* $\partial y = (\partial_0 y, \ldots, \partial_n y) = (yd_0, \ldots, yd_n)$ obtained by restricting y to $\partial\Delta[n]$. A simplex y is said to *fill* a simplicial sphere x if we have $\partial y = x$. A simplicial sphere $x : \partial\Delta[n] \to X$ *commutes* if it can be filled.

49.6. If $n > 0$ and $k \in [n]$, the *horn* $\Lambda^k[n] \subset \Delta[n]$ is defined to be the union of the faces $\partial_i\Delta[n]$ with $i \neq k$. A map $x : \Lambda^k[n] \to X$ is called a *horn in* X; it is determined by a lacunary sequence of faces $(x_0, \ldots, x_{k-1}, *, x_{k+1}, \ldots, x_n)$. A *filler* for x is a simplex $\Delta[n] \to X$ which extends x. Recall that a simplicial set X is said to be a *Kan complex* if every horn $\Lambda^k[n] \to X$ $(n > 0, k \in [n])$ has a filler $\Delta[n] \to X$,

$$\begin{array}{ccc} \Lambda^k[n] & \xrightarrow{\forall} & X \\ \downarrow & \nearrow_{\exists} & \\ \Delta[n]. & & \end{array}$$

49.7. Let us denote by $\Delta(n)$ the full subcategory of Δ spanned by the objects $[k]$ for $0 \leq k \leq n$. We say that a presheaf on $\Delta(n)$ is a *n-truncated simplicial set* and we put $\mathbf{S}(n) = [\Delta(n)^o, \mathbf{Set}]$. If i_n denotes the inclusion $\Delta(n) \subset \Delta$, then the restriction functor $i_n^* : \mathbf{S} \to \mathbf{S}(n)$ has a left adjoint $(i_n)_!$ and a right adjoint $(i_n)_*$. The functor $Sk^n = (i_n)_!(i_n)^* : \mathbf{S} \to \mathbf{S}$ associates to a simplicial set X its *n-skeleton* $Sk^n X \subseteq X$; it is the simplicial subset of X generated by the simplices $x \in X_k$ of dimension $k \leq n$. The functor $Cosk^n = (i_n)_*(i_n)^* : \mathbf{S} \to \mathbf{S}$ associates to a simplicial set X its *n-coskeleton* $Cosk^n X$. A simplex $\Delta[k] \to Cosk^n X$ is the same thing as a simplex $Sk^n \Delta[k] \to X$.

49.8. We say that a map of simplicial sets $f : X \to Y$ is *biunivoque* if the map $f_0 : X_0 \to Y_0$ is bijective. We say that a map of simplicial sets $f : X \to Y$ is *n-full* if the ollowing square of canonical maps is a pullback,

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ Cosk^n(X) & \xrightarrow{Cosk^n(f)} & Cosk^n(Y). \end{array}$$

The n-full maps are closed under composition and base change. Every map $f : X \to Y$ admits a factorisation $f = pq : X \to Z \to Y$ with p a 0-full map and q biunivoque. The factorisation is unique up to unique isomorphism. It is the *Gabriel factorisation* of the map. A 0-full map between quasi-categories is fully faithful. We say that a simplicial subset S of a simplicial set X is *n-full* if the inclusion of the subset $S \subseteq X$ is n-full. The inclusion of a subcategory in a category is always 1-full.

49.9. Let **Top** be the category of (small) topological spaces. Consider the functor $r : \Delta[n] \to \mathbf{Top}$ which associates to $[n]$ the geometric simplex

$$\Delta^n = \{(x_1, \ldots, x_n) : 0 \le x_1 \le \cdots \le x_n \le 1\}.$$

The *singular complex* of a topological space Y is the simplicial set $r^! Y$ defined by putting

$$(r^! Y)_n = \mathbf{Top}(\Delta^n, Y)$$

for every $n \ge 0$. The simplicial set $r^! Y$ is a Kan complex. The singular complex functor $r^! : \mathbf{Top} \to \mathbf{S}$ has a left adjoint $r_!$ which associates to a simplicial set X its *geometric realisation* $r_! X$. A map of simplicial sets $u : A \to B$ is said to be a *weak homotopy equivalence* if the map $r_!(u) : r_! A \to r_! B$ is a homotopy equivalence of topological spaces.

50. Appendix on model categories

50.1. We shall say that a class $\mathcal{W}$ of maps in a category $\mathcal{E}$ has the *"three for two"* property if the following condition is satisfied:

- If two of three maps $u : A \to B$, $v : B \to C$ and $vu : A \to C$ belong to $\mathcal{W}$, then so does the third.

50.2. Let $\mathcal{E}$ be a finitely bicomplete category. We shall say that a triple $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ of classes of maps in $\mathcal{E}$ is a *model structure* if the following conditions are satisfied:

- $\mathcal{W}$ has the "three for two" property;
- the pairs $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ are weak factorisation systems.

A map in $\mathcal{W}$ is said to be *acyclic* or to be a *weak equivalence*. A map in $\mathcal{C}$ is called a *cofibration* and a map in $\mathcal{F}$ a *fibration* . An object $X \in \mathcal{E}$ is said to be *fibrant* if the map $X \to \top$ is a fibration, where $\top$ is the terminal object of $\mathcal{E}$. Dually, an object $A \in \mathcal{E}$ is said to be *cofibrant* if the map $\bot \to A$ is a cofibration, where $\bot$ is the initial object of $\mathcal{E}$. A *Quillen model category* is a category $\mathcal{E}$ equipped with a model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$.

50.3. We shall say that a model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ in a cocomplete category $\mathcal{E}$ is *accessible* or *cofibrantly generated* if the saturated classes $\mathcal{C}$ and $\mathcal{C}\cap\mathcal{W}$ are accessible.

50.4. A model structure is said to be *left proper* if the cobase change of a weak equivalence along a cofibration is a weak equivalence. Dually, a model structure is said to be *right proper* if the base change of a weak equivalence along a fibration is a weak equivalence. A model structure is *proper* if it is both left and right proper.

50.5. If $\mathcal{E}$ is a model category, then so is the slice category $\mathcal{E}/B$ for each object $B \in \mathcal{E}$. By definition, a map in $\mathcal{E}/B$ is a weak equivalence (resp. a cofibration , resp. a fibration) iff the underlying map in $\mathcal{E}$ is a weak equivalence (resp. a cofibration , resp. a fibration). Dually, each category $B\backslash\mathcal{E}$ is a model category.

50.6. Let $\mathcal{E}$ be a finitely bicomplete category equipped a class of maps $\mathcal{W}$ having the "three-for-two" property and two factorisation systems $(\mathcal{C}_W, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F}_W)$. Suppose that the following two conditions are satisfied:

- $\mathcal{C}_W \subseteq \mathcal{C}\cap\mathcal{W}$ and $\mathcal{F}_W \subseteq \mathcal{F}\cap\mathcal{W}$;
- $\mathcal{C}\cap\mathcal{W} \subseteq \mathcal{C}_W$ or $\mathcal{F}\cap\mathcal{W} \subseteq \mathcal{F}_W$.

Then we have $\mathcal{C}_W = \mathcal{C}\cap\mathcal{W}$, $\mathcal{F}_W = \mathcal{F}\cap\mathcal{W}$ and $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ is a model structure.

50.7. The *homotopy category* of a model category $\mathcal{E}$ is defined to be the category of fractions $Ho(\mathcal{E}) = \mathcal{W}^{-1}\mathcal{E}$. We shall denote by $[u]$ the image of a map $u \in \mathcal{E}$ by the canonical functor $\mathcal{E} \to Ho(\mathcal{E})$. A map $u : A \to B$ is a weak equivalence iff $[u]$ invertible in $Ho(\mathcal{E})$ by [Q].

50.8. We shall denote by $\mathcal{E}_f$ (resp. $\mathcal{E}_c$) the full sub-category of fibrant (resp. cofibrant) objects of a model category $\mathcal{E}$. We shall put $\mathcal{E}_{fc} = \mathcal{E}_f \cap \mathcal{E}_c$. A *fibrant replacement* of an object $X \in \mathcal{E}$ is a weak equivalence $X \to RX$ with codomain a fibrant object. Dually, a *cofibrant replacement* of X is a weak equivalence $LX \to X$ with domain a cofibrant object. Let us put $Ho(\mathcal{E}_f) = \mathcal{W}_f^{-1}\mathcal{E}_f$ where $\mathcal{W}_f = \mathcal{W}\cap\mathcal{E}_f$ and similarly for $Ho(\mathcal{E}_c)$ and $Ho(\mathcal{E}_{fc})$. Then the diagram of inclusions

$$\begin{array}{ccc} \mathcal{E}_{fc} & \longrightarrow & \mathcal{E}_f \\ \downarrow & & \downarrow \\ \mathcal{E}_c & \longrightarrow & \mathcal{E} \end{array}$$

induces a diagram of equivalences of categories

$$\begin{array}{ccc} Ho(\mathcal{E}_{fc}) & \longrightarrow & Ho(\mathcal{E}_f) \\ \downarrow & & \downarrow \\ Ho(\mathcal{E}_c) & \longrightarrow & Ho(\mathcal{E}). \end{array}$$

50.9. A *path object* for an object X in a model category is obtained by factoring the diagonal map $X \to X \times X$ as weak equivalence $\delta : X \to PX$ followed by a fibration $(p_0, p_1) : PX \to X \times X$. A *right homotopy* $h : f \sim_r g$ between two maps $u, v : A \to X$ is a map $h : A \to PX$ such that $u = p_0 h$ and $v = p_1 h$. Two maps $u, v : A \to X$ are *right homotopic* if there exists a right homotopy $h : f \sim_r g$ with codomain a path object for X. The right homotopy relation on the set of maps $A \to X$ is an equivalence if X is fibrant. There is a dual notion of *cylinder object* for A obtained by factoring the codiagonal $A \sqcup A \to A$ as a cofibration

$(i_0, i_1) : A \sqcup A \to IA$ followed by a weak equivalence $p : IA \to A$. A *left homotopy* $h : u \sim_l v$ between two maps $u, v : A \to X$ is a map $h : IA \to X$ such that $u = hi_0$ and $v = hi_1$. Two maps $u, v : A \to X$ are *left homotopic* if there exists a left homotopy $h : u \sim_l v$ with domain some cylinder object for A. The left homotopy relation on the set of maps $A \to X$ is an equivalence if A is cofibrant. The left homotopy relation coincides with the right homotopy relation if A is cofibrant and X is fibrant; in which case two maps $u, v : A \to X$ are said to be homotopic if they are left (or right) homotopic; we shall denote this relation by $u \sim v$.

Proposition 50.1. [Q]. *If A is cofibrant and X is fibrant, let us denote by $\mathcal{E}(A, X)^\sim$ the quotient of the set $\mathcal{E}(A, X)$ by the homotopy relation $\sim$. Then the canonical map $u \mapsto [u]$ induces a bijection*

$$\mathcal{E}(A, X)^\sim \simeq Ho(\mathcal{E})(A, X).$$

A map $X \to Y$ in $\mathcal{E}_{cf}$ is a homotopy equivalence iff it is a weak equivalence.

50.10. A model structure $M = (\mathcal{C}, \mathcal{W}, \mathcal{F})$ in a category $\mathcal{E}$ is determined by its class $\mathcal{C}$ of cofibrations together with its class of fibrant objects $Fob(M)$. If $M' = (\mathcal{C}, \mathcal{W}', \mathcal{F}')$ is another model structure with the same cofibrations, then the relation $\mathcal{W} \subseteq \mathcal{W}'$ is equivalent to the relation $Fob(M') \subseteq Fob(M)$.
Proof: Let us prove the first statement. It suffices to show that the class $\mathcal{W}$ is determined by $\mathcal{C}$ and $Fob(M)$. The class $\mathcal{F} \cap \mathcal{W}$ is determined by $\mathcal{C}$, since the pair $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ is a weak factorisation system. For any map $u : A \to B$, there exists a commutative square

$$\begin{array}{ccc} A' & \longrightarrow & A \\ {\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle u} \\ B' & \longrightarrow & B \end{array}$$

in which the horizontal maps are acyclic fibrations and the objects A' and B' are cofibrants. The map u is acyclic iff the map u' is acyclic. Hence it suffices to show that the class $\mathcal{W} \cap \mathcal{E}_c$ is is determined by $\mathcal{C}$ and $Fob(M)$. If A and B are two objects of $\mathcal{E}$, let us denote by $h(A, B)$ the set of maps $A \to B$ between in the category $Ho(\mathcal{E})$. A map $u : A \to B$ in $\mathcal{E}$ is invertible in $Ho(\mathcal{E})$ iff the map $h(u, X) : h(B, X) \to h(A, X)$ is bijective for every object $X \in \mathcal{E}$ by Yoneda lemma. Hence a map $u : A \to B$ in $\mathcal{E}$ belongs to $\mathcal{W}$ iff the map $h(u, X) : h(B, X) \to h(A, X)$ is bijective for every object $X \in Fob(M)$, since every object in $Ho(\mathcal{E})$ is isomorphic to a fibrant object. If A is cofibrant and X is fibrant, let us denote by $\mathcal{E}(A, X)^\sim$ the quotient of the set $\mathcal{E}(A, X)$ by the homotopy relation. It follows from 50.1 that a map $u : A \to B$ in $\mathcal{E}_c$ belongs to $\mathcal{W}$ iff the map $\mathcal{E}(B, X)^\sim \to \mathcal{E}(A, X)^\sim$ induced by the map $\mathcal{E}(u, X)$ is bijective for every object $X \in Fob(M)$. Hence the result will be proved if we show that the homotopy relation $\sim$ on the set $\mathcal{E}(A, X)$ only depends on the class $\mathcal{C}$ if A is cofibrant and X is fibrant. But two maps $A \to X$ are homotopic iff they are left homotopic, since A is cofibrant and X is fibrant. A cylinder for A can be constructed by factoring the codiagonal $A \sqcup A \to A$ as a cofibration $(i_0, i_1) : A \sqcup A \to I(A)$ followed by an acyclic fibration $I(A) \to A$. Two maps $f, g : A \to X$ are left homotopic iff there exists a map $h : I(A) \to X$ such that $hi_0 = f$ and $hi_1 = g$. The construction of $I(A)$ only depends on $\mathcal{C}$, since it only depends on the factorisation system $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$. Hence the left homotopy relation on the set $\mathcal{E}(A, X)$ only depends on $\mathcal{C}$. The first statement of the proposition follows. The proof of the second statement is left to the reader. ∎

50.11. Recall from [Ho] that a cocontinuous functor $F : \mathcal{U} \to \mathcal{V}$ between two model categories is said to be a *left Quillen functor* if it takes a cofibration to a cofibration and an acyclic cofibration to an acyclic cofibration. A left Quillen functor takes a weak equivalence between cofibrant objects to a weak equivalence. Dually, a continuous functor $G : \mathcal{V} \to \mathcal{U}$ between two model categories is said to be a *right Quillen functor* if it takes a fibration to a fibration and an acyclic fibration to an acyclic fibration. A right Quillen functor takes a weak equivalence between fibrant objects to a weak equivalence.

50.12. A left Quillen functor $F : \mathcal{U} \to \mathcal{V}$ induces a functor $F_c : \mathcal{U}_c \to \mathcal{V}_c$ hence also a functor $Ho(F_c) : Ho(\mathcal{U}_c) \to Ho(\mathcal{V}_c)$. Its *left derived functor* is a functor

$$F^L : Ho(\mathcal{U}) \to Ho(\mathcal{V})$$

for which the following diagram of functors commutes up to isomorphism,

$$\begin{array}{ccc} Ho(\mathcal{U}_c) & \xrightarrow{Ho(F_c)} & Ho(\mathcal{V}_c) \\ \downarrow & & \downarrow \\ Ho(\mathcal{U}) & \xrightarrow{F^L} & Ho(\mathcal{V}), \end{array}$$

The functor F^L is unique up to a canonical isomorphism. It can be computed as follows. For each object $A \in \mathcal{U}$, we can choose a cofibrant replacement $\lambda_A : LA \to A$, with λ_A an acyclic fibration. We can then choose for each arrow $u : A \to B$ an arrow $L(u) : LA \to LB$ such that $u\lambda_A = \lambda_B L(u)$,

$$\begin{array}{ccc} LA & \xrightarrow{\lambda_A} & A \\ {\scriptstyle L(u)}\downarrow & & \downarrow{\scriptstyle u} \\ LB & \xrightarrow{\lambda_B} & B. \end{array}$$

Then

$$F^L([u]) = [F(L(u))] : FLA \to FLB.$$

50.13. Dually, a right Quillen functor $G : \mathcal{V} \to \mathcal{U}$ induces a functor $G_f : \mathcal{V}_f \to \mathcal{U}_f$ hence also a functor $Ho(G_f) : Ho(\mathcal{V}_f) \to Ho(\mathcal{U}_f)$. Its *right derived functor* is a functor

$$G^R : Ho(\mathcal{V}) \to Ho(\mathcal{U})$$

for which the following diagram of functors commutes up to a canonical isomorphism,

$$\begin{array}{ccc} Ho(\mathcal{V}_f) & \xrightarrow{Ho(G_f)} & Ho(\mathcal{U}_f) \\ \downarrow & & \downarrow \\ Ho(\mathcal{V}) & \xrightarrow{G^R} & Ho(\mathcal{U}). \end{array}$$

The functor G^R is unique up to a canonical isomorphism. It can be computed as follows. For each object $X \in \mathcal{V}$ let us choose a fibrant replacement $\rho_X : X \to RX$, with ρ_X an acyclic cofibration. We can then choose for each arrow $u : X \to Y$ an arrow $R(u) : RX \to RY$ such that $R(u)\rho_X = \rho_Y u$,

$$\begin{array}{ccc} X & \xrightarrow{\rho_X} & RX \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle R(u)} \\ Y & \xrightarrow{\rho_Y} & RY. \end{array}$$

Then

$$G^R([u]) = [G(R(u))] : GRX \to GRY.$$

50.14. Let $F : \mathcal{U} \leftrightarrow \mathcal{V} : G$ be an adjoint pair of functors between two model categories. Then the following two conditions are equivalent:

- F is a left Quillen functor;
- G is a right Quillen functor.

When these conditions are satisfied, the pair (F, G) is said to be a *Quillen pair*. In this case, we obtain an adjoint pair of functors

$$F^L : Ho(\mathcal{U}) \leftrightarrow Ho(\mathcal{V}) : G^R.$$

If $A \in \mathcal{U}$ is cofibrant, the adjunction unit $A \to G^R F^L(A)$ is obtained by composing the maps $A \to GFA \to GRFA$, where $FA \to RFA$ is a fibrant replacement of FA. If $X \in \mathcal{V}$ is fibrant, the adjunction counit $F^L G^R(X) \to X$ is obtained by composing the maps $FLGX \to FGX \to X$, where $LGX \to GX$ is a cofibrant replacement of GX.

50.15. We shall say that a Quillen pair $F : \mathcal{U} \leftrightarrow \mathcal{V} : G$ a *homotopy reflection* of $\mathcal{U}$ into $\mathcal{V}$ if the right derived functor G^R is fully faithful. Dually, we shall say that (F, G) is a *homotopy coreflection* of $\mathcal{V}$ into $\mathcal{U}$ if the left derived functor F^L is fully faithful. We shall say that (F, G) is called a *Quillen equivalence* if the adjoint pair (F^L, G^R) is an equivalence of categories.

50.16. A Quillen pair $F : \mathcal{U} \leftrightarrow \mathcal{V} : G$ is a homotopy reflection iff the map $FLGX \to X$ is a weak equivalence for every fibrant object $X \in \mathcal{V}$, where $LGX \to GX$ denotes a cofibrant replacement of GX. A homotopy reflection $F : \mathcal{U} \leftrightarrow \mathcal{V} : G$ is a Quillen equivalence iff the functor F reflects weak equivalences between cofibrant objects.

50.17. Let $F : \mathcal{U} \leftrightarrow \mathcal{V} : G$ be a homotopy reflection beween two model categories. We shall say that an object $X \in \mathcal{U}$ is *local* (with respect to the the pair (F, G)) if it belongs to the essential image of the right derived functor $G^R : Ho(\mathcal{V}) \to Ho(\mathcal{U})$.

50.18. Let $\mathcal{M}_i = (\mathcal{C}_i, \mathcal{W}_i, \mathcal{F}_i)$ $(i = 1, 2)$ be two model structures on a category $\mathcal{E}$. If $\mathcal{C}_1 \subseteq \mathcal{C}_2$ and $\mathcal{W}_1 \subseteq \mathcal{W}_2$, then the identity functor $\mathcal{E} \to \mathcal{E}$ is a homotopy reflection $\mathcal{M}_1 \to \mathcal{M}_2$. The following conditions on an object A are equivalent:

- A is local;
- there exists a $\mathcal{M}_1$-equivalence $A \to A'$ with codomain a $\mathcal{M}_2$-fibrant object A';
- (every $\mathcal{M}_2$-fibrant replacement $A \to A'$ is a $\mathcal{M}_1$-fibrant replacement.

In particular, every $\mathcal{M}_2$-fibrant object is local. A map between local objects is a $\mathcal{M}_1$-equivalence iff it is a $\mathcal{M}_2$-equivalence.

50.19. Let $\mathcal{M}_i = (\mathcal{C}_i, \mathcal{W}_i, \mathcal{F}_i)$ $(i = 1, 2)$ be two model structures on a category $\mathcal{E}$. If $\mathcal{C}_1 = \mathcal{C}_2$ and $\mathcal{W}_1 \subseteq \mathcal{W}_2$, we shall say that $\mathcal{M}_2$ is a *Bousfield localisation* of $\mathcal{M}_1$. We shall say that $\mathcal{M}_1$ is the *localised model structure* and $\mathcal{M}_2$ is the *local model structure*.

50.20. Let $\mathcal{M}_2 = (\mathcal{C}_2, \mathcal{W}_2, \mathcal{F}_2)$ be a Bousfield localisation of a model structure $\mathcal{M}_1 = (\mathcal{C}_1, \mathcal{W}_1, \mathcal{F}_1)$ on a category $\mathcal{E}$. A local object is $\mathcal{M}_1$-fibrant iff it is $\mathcal{M}_2$-fibrant. An object A is local iff every $\mathcal{M}_1$-fibrant replacement $i : A \to A'$ is a $\mathcal{M}_2$-fibrant replacement. A map between $\mathcal{M}_2$-fibrant objects is a $\mathcal{M}_2$-fibration iff it is a $\mathcal{M}_1$-fibration.

50.21. Let $\odot : \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3$ be a functor of two variables with values in a finitely cocomplete category $\mathcal{E}_3$. If $u : A \to B$ is map in $\mathcal{E}_1$ and $v : S \to T$ is a map in $\mathcal{E}_2$, we shall denote by $u \odot' v$ the map

$$A \odot T \sqcup_{A \odot S} B \odot S \longrightarrow B \odot T$$

obtained from the commutative square

$$\begin{array}{ccc} A \odot S & \longrightarrow & B \odot S \\ \downarrow & & \downarrow \\ A \odot T & \longrightarrow & B \odot T. \end{array}$$

This defines a functor of two variables

$$\odot' : \mathcal{E}_1^I \times \mathcal{E}_2^I \to \mathcal{E}_3^I,$$

where $\mathcal{E}^I$ denotes the category of arrows of a category $\mathcal{E}$.

50.22. [Ho] We shall say that a functor of two variables $\odot : \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3$ between three model categories is a *left Quillen functor* it is concontinuous in each variable and the following conditions are satisfied:

- $u \odot' v$ is a cofibration if $u \in \mathcal{E}_1$ and $v \in \mathcal{E}_2$ are cofibrations;
- $u \odot' v$ is an acyclic cofibration if $u \in \mathcal{E}_1$ and $v \in \mathcal{E}_2$ are cofibrations and one of the maps u or v is acyclic.

Dually, we shall say that the functor of two variables $\odot$ is a *right Quillen functor* if the opposite functor $\odot^o : \mathcal{E}_1^o \times \mathcal{E}_2^o \to \mathcal{E}_3^o$ is a left Quillen functor.

50.23. [Ho] A model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ on monoidal closed category $\mathcal{E} = (\mathcal{E}, \otimes)$ is said to be *monoidal* if the tensor product $\otimes : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ is a left Quillen functor of two variables and if the unit object of the tensor product is cofibrant.

50.24. A model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ on a category $\mathcal{E}$ is said to be *cartesian* if the cartesian product $\times : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ is a left Quillen functor of two variables and if the terminal object 1 is cofibrant.

50.25. We say that a functor of two variables $\odot : \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3$ is *divisible on the left* if the functor $A \odot (-) : \mathcal{E}_2 \to \mathcal{E}_3$ admits a right adjoint $A\backslash(-) : \mathcal{E}_3 \to \mathcal{E}_2$ for every object $A \in \mathcal{E}_1$. In this case we obtain a functor of two variables $(A, X) \mapsto A\backslash X$,

$$\mathcal{E}_1^o \times \mathcal{E}_3 \to \mathcal{E}_2,$$

called the *left division functor*. Dually, we say that $\odot$ is *divisible on the right* if the functor $(-) \odot B : \mathcal{E}_1 \to \mathcal{E}_3$ admits a right adjoint $(-)/B : \mathcal{E}_3 \to \mathcal{E}_1$ for every object $B \in \mathcal{E}_2$. In this case we obtain a functor of two variables $(X, B) \mapsto X/B$,

$$\mathcal{E}_3 \times \mathcal{E}_2^o \to \mathcal{E}_1,$$

called the *right division functor*.

50.26. If a functor of two variables $\odot : \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3$ is divisible on both sides, then so is the left division functor $\mathcal{E}_1^o \times \mathcal{E}_3 \to \mathcal{E}_2$ and the right division functor $\mathcal{E}_3 \times \mathcal{E}_2^o \to \mathcal{E}_1$. This is called a *tensor-hom-cotensor* situation by Gray [**?**]. There is then a bijection between the following three kinds of maps

$$A \odot B \to X, \qquad B \to A\backslash X, \qquad A \to X/B.$$

The contravariant functors $A \mapsto A\backslash X$ and $B \mapsto B\backslash X$ are mutually right adjoint for any object $X \in \mathcal{E}_3$.

50.27. Suppose the category $\mathcal{E}_2$ is finitely complete and that the functor $\odot : \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3$ is divisible on the left. If $u : A \to B$ is map in $\mathcal{E}_1$ and $f : X \to Y$ is a map in $\mathcal{E}_3$, we denote by $\langle u\backslash\ f\rangle$ the map

$$B\backslash X \to B\backslash Y \times_{A\backslash Y} A\backslash X$$

obtained from the commutative square

$$\begin{array}{ccc} B\backslash X & \longrightarrow & A\backslash X \\ \downarrow & & \downarrow \\ B\backslash Y & \longrightarrow & A\backslash Y. \end{array}$$

The functor $f \mapsto \langle u\backslash f\rangle$ is right adjoint to the functor $v \mapsto u \odot' v$ for every map $u \in \mathcal{E}_1$. Dually, suppose that the category $\mathcal{E}_1$ is finitely complete and that the functor $\odot$ is divisible on the right. If $v : S \to T$ is map in $\mathcal{E}_2$ and $f : X \to Y$ is a map in $\mathcal{E}_3$, we denote by $\langle f/v\rangle$ the map

$$X/T \to Y/T \times_{Y/S} X/S$$

obtained from the commutative square

$$\begin{array}{ccc} X/T & \longrightarrow & X/S \\ \downarrow & & \downarrow \\ Y/T & \longrightarrow & Y/S. \end{array}$$

the functor $f \mapsto \langle f/v\rangle$ is right adjoint to the functor $u \mapsto u \odot' v$ for every map $v \in \mathcal{E}_2$.

50.28. Let $\odot : \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3$ be a functor of two variables divisible on both sides, where $\mathcal{E}_i$ is a finitely bicomplete category for $i = 1, 2, 3$. If $u \in \mathcal{E}_1$, $v \in \mathcal{E}_2$ and $f \in \mathcal{E}_3$, then

$$(u \odot' v) \pitchfork f \quad \Longleftrightarrow \quad u \pitchfork \langle f/v \rangle \quad \Longleftrightarrow \quad v \pitchfork \langle u \backslash f \rangle.$$

50.29. Let $\odot : \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3$ be a functor of two variables divisible on each side between three model categories. Then the functor $\odot$ is a left Quillen functor iff the corresponding left division functor $\mathcal{E}_1^o \times \mathcal{E}_3 \to \mathcal{E}_2$ is a right Quillen functor iff the the corresponding right division functor $\mathcal{E}_1^o \times \mathcal{E}_3 \to \mathcal{E}_2$ is a right Quillen functor.

50.30. Let $\mathcal{E}$ be a *symmetric* monoidal closed category. Then the objects X/A and $A\backslash X$ are canonicaly isomorphic; we can identify them by adopting a common notation, for example $[A, X]$. Similarly, the maps $\langle f/u \rangle$ and $\langle u \backslash f \rangle$ are canonicaly isomorphic; we shall identify them by adopting a common notation, for example $\langle u, f \rangle$. A model structure on $\mathcal{E}$ is monoidal iff the following two conditions are satisfied:

- if u is a cofibration and f is a fibration, then $\langle u, f \rangle$ is a fibration which is acyclic if in addition u or f is acyclic;
- the unit object is cofibrant.

50.31. Recall that a functor $P : \mathcal{E} \to \mathcal{K}$ is said to be a *bifibration* if it is both a Grothendieck fibration and a Grothendieck opfibration. If P is a bifibration, then every arrow $f : A \to B$ in $\mathcal{E}$ admits a factorisation $f = c^f u^f$ with c^f a cartesian arrow and u^f a unit arrow (ie $P(u^f) = 1_{P(A)})$), together with a factorisation $f = u_f c_f$ with c^f a cocartesian arrow and u_f a unit. Let us denote by $\mathcal{E}(S)$ the fiber of the functor P at an object $S \in \mathcal{K}$. Then for every arrow $g : S \to T$ in $\mathcal{K}$ we can choose pair of adjoint functors

$$g_! : \mathcal{E}(S) \to \mathcal{E}(T) : g^*.$$

The *pullback functor* g^* is obtained by choosing for each object $B \in \mathcal{E}(T)$ a cartesian lift $g^*(B) \to B$ of the arrow g. The *pushforward functor* $g_!$ is obtained by choosing for each object $A \in \mathcal{E}(S)$ a cocartesian lift $A \to g_!(A)$ of the arrow g.

50.32. Let $P : \mathcal{E} \to \mathcal{K}$ be a Grothendieck bifibration where $\mathcal{K}$ is a model category. We shall say that a model structure $\mathcal{M} = (\mathcal{C}, \mathcal{W}, \mathcal{F})$ on $\mathcal{E}$ is *bifibered* by the functor P if the following conditions are satisfied:

- The intersection $\mathcal{M}(S) = (\mathcal{C} \cap \mathcal{E}(S), \mathcal{W} \cap \mathcal{E}(S), \mathcal{F} \cap \mathcal{E}(S))$ is a model structure on $\mathcal{E}(S)$ for each object $S \in \mathcal{K}$;
- The pair of adjoint functors
$$g_! : \mathcal{E}(S) \to \mathcal{E}(T) : g^*$$
is a Quillen pair for each arrow $g : S \to T$ in $\mathcal{K}$ and it is a Quillen equivalence if g is a weak equivalence;
- An arrow $f : A \to B$ in $\mathcal{E}$ is a cofibration iff the arrows $u_f \in \mathcal{E}(B)$ and $P(f) \in \mathcal{K}$ are cofibrations;
- An arrow $f : A \to B$ in $\mathcal{E}$ is a fibration iff the arrows $u^f \in \mathcal{E}(A)$ and $P(f) \in \mathcal{K}$ are fibrations.

It follows from these conditions that the functor P takes a fibration to a fibration, a cofibration to a cofibration and a weak equivalence to a weak equivalence. For another notion of bifibered model category, see [Ro].

50.33. Let $P : \mathcal{E} \to \mathcal{K}$ be a bifibered model category over a model category $\mathcal{K}$. Then the model structure on $\mathcal{E}$ is determined by the model structure on $\mathcal{K}$ together with the model structure on $\mathcal{E}(S)$ for each object $S \in \mathcal{K}$.

- An arrow $f : A \to B$ in $\mathcal{E}$ is an acyclic cofibration iff the arrows $u_f \in \mathcal{E}(B)$ and $P(f) \in \mathcal{K}$ are acyclic cofibrations;
- An arrow $f : A \to B$ in $\mathcal{E}$ is an acyclic fibration iff the arrows $u^f \in \mathcal{E}(A)$ and $P(f) \in \mathcal{K}$ are acyclic fibrations.

51. Appendix on simplicial categories

51.1. Recall that a *simplicial category* is a category enriched over simplicial sets and that a *simplicial functor* is a functor enriched over simplicial sets. We shall denote by **SCat** the category of small simplicial categories and simplicial functors. The *opposite* of a simplicial category A is a simplicial category A^o if we put $A^o(a^o, b^o) = A(a, b)^o$ for every pair of objects $a, b \in A$. Beware that the *opposite* of a simplicial functor $F : A \to B$ is a simplicial functor $F^o : A^o \to B^o$. A *contravariant simplicial functor* $F : A \to B$ between two simplicial categories A and B is defined to be a simplicial functor $F : A^o \to B$; but we shall often denote the value of F at $a \in A$ by $F(a)$ instead of $F(a^o)$. .

51.2. The category **SCat** is cartesian closed. If A and B are small simplicial category, we shall denote the simplicial category of simplicial functors $A \to B$ by B^A or by $[A, B]$. If A is a small simplicial category, we shall denote the large simplicial category of simplicial functors $A \to \mathbf{S}$ by $[A, \mathbf{S}]$. A *simplicial presheaf* on A is defined to be a contravariant simplicial functor $A \to \mathbf{S}$. The simplicial presheaves on A form a locally small simplicial category

$$\mathbf{S}^{A^o} = [A^o, \mathbf{S}].$$

We shall denote by $[X, Y]$ the simplicial sets of maps $X \to Y$ between two simplicial presheaves. If $u : A \to B$ is a simplicial functor between small simplicial categories, then the simplicial functor

$$u^* = [u^o, \mathbf{S}] : [B^o, \mathbf{S}] \to [A^o, \mathbf{S}]$$

induced by u has a left adjoint $u_!$ and a right adjoint u_*.

51.3. If A is a small simplicial category, then the *Yoneda functor* $y_A : A \to [A^o, \mathbf{S}]$ associates to an object $a \in A$ the simplicial presheaf $A(-, a)$. The *Yoneda lemma* asserts that for any object $a \in A$ and any simplicial presheaf X on A the the evaluation $x \mapsto x(1_a)$ induces an isomorphism of simplicial sets

$$[A(-, a), X] \simeq X(a).$$

The Yoneda lemma implies that the Yoneda functor is fully faithful; we shall often regard it as an inclusion $A \subset [A^o, \mathbf{S}]$ by adopting the same notation for an object $a \in A$ and the simplicial presheaf $A(-, a)$. Moreover, we shall identify a natural transformation $x : a \to X$ with the vertex $x(1_a) \in X(a)$. We shall say that a presheaf X is *(strictly) representable* if there exists an object $a \in A$ together with a vertex $x \in X(a)$ such that the corresponding natural transformation $x : a \to X$ is invertible.

51.4. An ordinary category can be viewed a simplicial category with discrete hom. We shall say that a simplicial functor $f : A \to B$ is *homotopy fully faithful* if the map $A(a, b) \to B(fa, fb)$ induced by f is a weak homotopy equivalence for every pair of objects $a, b \in A$. The inclusion functor $\mathbf{Cat} \subset \mathbf{SCat}$ has a left adjoint

$$ho : \mathbf{SCat} \to \mathbf{Cat}$$

which associates to a simplicial category A its *homotopy category* hoA. By construction, we have $(hoA)(a, b) = \pi_0 A(a, b)$ for every pair of objects $a, b \in A$. We shall say that a simplicial functor $f : A \to B$ is *homotopy essentially surjective* if the functor $ho(f) : hoA \to hoB$ is essentially surjective. We shall say that a simplicial functor is a *Dwyer-Kan equivalence* if it is homotopy fully faithful and homotopy essentially surjective. We shall say that a simplicial functor $f : A \to B$ is a *Dwyer-Kan fibration* if the map $A(a, b) \to B(fa, fb)$ is a Kan fibration for every pair of objects $a, b \in A$ and the functor $ho(f) : hoA \to hoB$ is an isofibration. The category $\mathbf{SCat}$ admits a Quillen model structure in which the weak equivalences are the Dwyer-Kan equivalences and the fibrations are the Dwyer-Kan fibrations [B1]. A functor $f : A \to B$ is an acyclic fibration iff the map $Ob(f) : ObA \to ObA$ is surjective and the map $A(a, b) \to B(fa, fb)$ is a trivial fibration for every pair of objects $a, b \in A$. The model structure is left proper and the fibrant objects are the categories enriched over Kan complexes. We say that it is the *Bergner model structure* or the *model structure for simplicial categories.* We shall denote it by $(\mathbf{SCat}, DK)$, where DK denotes the class of Dwyer-Kan equivalences.

51.5. We shall say the a simplicial functor $f : A \to B$ is a *Dwyer-Kan-Morita equivalence* if it is homotopy fully faithful and the functor $ho(f) : hoA \to hoB$ is a Morita equivalence in $\mathbf{Cat}$. Then the Bergner model structure $(\mathbf{SCat}, DK)$ admits a Bousfield localisation with respect to the class of DKM-equivalences.

51.6. Recall that the category $[A, \mathbf{S}]$ of simplicial presheaves on simplicial category A admits a model structure, called the *projective model structure*, in which a weak equivalence is a term-wise weak homotopy equivalence and a fibration is a term-wise Kan fibrations [Hi]. We shall denote this model structure by $[A, \mathbf{S}]^{proj}$. If $u : A \to B$ is a simplicial functor, then the pair

$$u_! : [A, \mathbf{S}] \to [B, \mathbf{S}] : u^*$$

is a Quillen adjunction with respect to the projective model structures on these categories. The pair is a Quillen equivalence iff u is a Dwyer-Kan-Morita equivalence [Hi].

51.7. Recall that the category $[A, \mathbf{S}]$ of simplicial presheaves on simplicial category A admits a cartesian close model structure, called the *injective model structure*, in which a weak equivalence is a term-wise weak homotopy equivalence and a cofibration is a monomorphism [Hi]. We shall denote this model structure by $[A, \mathbf{S}]^{inj}$. The identity functor

$$[A, \mathbf{S}]^{proj} \to [A, \mathbf{S}]^{inj}.$$

is the left adjoint in a Quillen equivalence between the projective and the injective model structures. If $u : A \to B$ is a simplicial functor, then the pair

$$u^* : [B, \mathbf{S}] \to [A, \mathbf{S}] : u_*$$

is a Quillen adjunction with respect to the injective model structures on these categories. The pair is a Quillen equivalence iff u is a Dwyer-Kan-Morita equivalence [Hi].

51.8. We shall say that a simplicial presheaf X on a small simplicial category A is is *(homotopy) representable* if there exists an object $a \in A$ together with a vertex $x \in X(a)$ such that the corresponding natural transformation $x : a \to X$ is a term-wise homotopy equivalence.

51.9. We shall say that object t in a simplicial category A is *(homotopy) terminal* if the simplicial set $X(x,t)$ is contractible for every object $x \in X$. The *(homotopy) cartesian product* of two objects $a, b \in X$ is an object $a \times b$ equipped with a pair of morphisms $p_1 : a \times b \to a$ and $p_2 : a \times b \to c$ such that the induced map

$$X(x, a \times b) \to X(x,a) \times X(x,b)$$

is a weak homotopy equivalence for every object $x \in X$.

51.10. We shall say that a simplicial functor $f : A \to B$ has a *(homotopy) right adjoint* $g : B \to A$ if for every object $b \in B$, the simplicial presheaf $x \mapsto B(f(x), b)$ is (homotopy) representable by an object $g(b) \in B$ with a morphism $f(g(b)) \to b$. The (homotopy) right adjoint is not a (strict) simplicial functor in general. But it is when A is cofibrant and B is fibrant in the Bergner model structure. Dually, we shall say that a simplicial functor $g : B \to A$ has a *(homotopy) left adjoint* $f : A \to B$ if for every object $a \in A$, the simplicial functor $x \mapsto A(a, g(x))$ is (homotopy) (co)representable by an object $f(a) \in B$ with a morphism $a \to g(f(a))$. The (homotopy) left adjoint is not a (strict) simplicial functor in general. But it is when A is cofibrant and B is fibrant in the Bergner model structure.

51.11. ?? If A and B are small simplicial categories, we shall say that a simplicial functor $F : A^o \times B \to \mathbf{S}$ is a *S-distributor*, or an *S-distributor* $F : A \Rightarrow B$. The S-distributors $A \Rightarrow B$ are the objects of a simplicial category $\mathrm{SDist}(A,B) = [A^o \times B, \mathbf{S}]$.

51.12. A *S-cylinder*, is defined to be an object $p : C \to I$ of the category $\mathbf{SCat}/I$, where the category $I = [n]$ is regarded as a simplicial category. The *base* of $p : C \to I$ is the cosieve $C(1) = p^{-1}(1)$ and its *cobase* is the sieve $C(0) = p^{-1}(0)$. If i denotes the inclusion $\{0,1\} \subset I$, then the pullback functor

$$i^* : \mathbf{SCat}/I \to \mathbf{SCat} \times \mathbf{SCat}$$

has left adjoint $i_!$ and a right adjoint i_*. The functor i^* is a Grothendieck bifibration and its fiber at (A,B) is the category $\mathrm{SCyl}(A,B)$ of simplicial S-cylinders with cobase A and base B. To every S-cylinder $C \in \mathrm{SCyl}(A,B)$ we can associate a S-distributor $D(C) \in \mathrm{SDist}(A,B)$ by putting $D(C)(a,b) = C(a,b)$ for every pair of objects $a \in A$ and $b \in B$. The resulting functor

$$D : \mathrm{SCyl}(A,B) \to \mathrm{SDist}(A,B)$$

is an equivalence of categories. The inverse equivalence associates to a S-distributor $F : A \Rightarrow B$ its *collage cylinder* $col(F) = A \star_F B$.

51.13. The Quillen equivalence

$$C_! : \mathbf{S} \leftrightarrow \mathbf{SCat} : C^!$$

of 3.5 induces a Quillen equivalence

$$C_! : \mathbf{S}/I \leftrightarrow \mathbf{SCat}/I : C^!,$$

since we have $C^!(I) = I$ and $C_!(I) = I$. The pair $(C_!, C^!)$ also induces a Quillen equivalence

$$C_! : \mathrm{Cyl}(A, B) \leftrightarrow \mathrm{SCyl}(C_!A, C_!B) : C^!$$

for any pair pair of simplicial sets A and B. By composing the equivalence with the Quillen equivalence

$$\rho_! : \mathbf{S}/A^o \times B \leftrightarrow \mathrm{Cyl}(A, B) : \rho^*$$

of 14.18 and the equivalence of categories

$$D : \mathrm{SCyl}(C_!A, C_!B) \to \mathrm{SDist}(C_!A, C_!B) : col$$

of 51.12, we obtain a a Quillen equivalence

$$\mathbf{S}/A^o \times B \leftrightarrow \mathrm{SDist}(C_!A, C_!B)$$

between the model category $(\mathbf{S}/A^o \times B, \mathrm{Wcov})$ and the projective model category $\mathrm{SDist}(C_!A, C_!B)$. In particular, this yields a Quillen equivalence

$$\mathbf{S}/B \leftrightarrow [C_!B, \mathbf{S}]$$

between the model category $(\mathbf{S}/B, \mathrm{Wcov})$ and the projective model category $[C_!B, \mathbf{S}]$.

51.14. Dually, the pair $(C_!, C^!)$ induces a Quillen equivalence

$$C_! : \mathrm{Cyl}(C^!X, C^!Y) \leftrightarrow \mathrm{SCyl}(X, Y) : C^!$$

for any pair of fibrant simplicial categories X and Y. By composing the equivalence with the Quillen equivalence

$$\rho_! : \mathbf{S}/C^!X^o \times C^!Y \leftrightarrow \mathrm{Cyl}(C^!X, C^!Y) : \rho^*$$

of 14.18 and the equivalence of categories

$$D : \mathrm{SCyl}(X, Y) \to \mathrm{SDist}(X, Y) : col$$

of 51.12, we obtain a a Quillen equivalence

$$\mathbf{S}/C^!X^o \times C^!Y \leftrightarrow \mathrm{SDist}(X, Y)$$

between the model category $(\mathbf{S}/C^!X^o \times C^!Y, \mathrm{Wcov})$ and the projective model category $\mathrm{SDist}(X, Y)$. In particular, this yields a Quillen equivalence

$$\mathbf{S}/C^!Y \leftrightarrow [Y, \mathbf{S}]$$

between the model category $(\mathbf{S}/C^!Y, \mathrm{Wcov})$ and the projective model category $[Y, \mathbf{S}]$.

51.15. If Y is a small simplicial category, let us denote by $[Y, \mathbf{S}]^f$ the category of fibrant objects of the injective model category $[Y, \mathbf{S}]^{inj}$. If Y is enriched over Kan complexes, then the functor

$$[Y, \mathbf{S}] \to \mathbf{S}/C^!Y]$$

defined in induces a Dwyer-Kan equivalence of simplicial categories

$$[Y, \mathbf{S}]^f \to \mathbf{L}(C^!Y).$$

51.16. We shall say that a small simplicial category with finite products T is a *strict simplicial algebraic theory.* A *strict model* of T is a simplicial functor $T \to \mathbf{S}$ which preserves finite products strictly. We shall denote by $\mathrm{SAlg}(T)$ the full subcategory of $[T, \mathbf{S}]$ spanned by the strict models of T The category $\mathrm{SAlg}(T)$ admits a simplicial model structure, called *the projective model structure*, in which the weak equivalences and the fibrations are the term-wise weak homotopy equivalences and the term-wise Kan fibrations [Q][B4]. We shall say that a simplicial functor $T \to \mathbf{S}$ is a *Segal model* if it preserves finite products up to weak homotopy equivalence. The projective model structure $[T, \mathbf{S}]^{proj}$ admits a Bousfield localisation $[T, \mathbf{S}]^{bad}$ in which the (fibrant) local objects are the fibrant Segal models. The inclusion functor $\mathrm{SAlg}(T)^{proj} \to [T, \mathbf{S}]^{bad}$ is the right adjoint in a Quiilen equivalence of model categories by a result of Badzioch [Bad1] and Bergner [B4].

51.17. We shall say that a small simplicial category with finite homotopy products T is a *simplicial algebraic theory.* Its coherent nerve is an algebraic theory $C^!T$ when T is T is DK-fibrant. A *homotopy model* of T is a simplicial functor $F : T \to \mathcal{S}$ which preserves finite homotopy products. The projective model structure $[T, \mathbf{S}]^{proj}$ admits a Bousfield localisation $[T, \mathbf{S}]^{bad}$ in which the (fibrant) local objects are the fibrant homotopy models. Let us denote by $\mathrm{Alg}_{fc}(T)$ the full subcategory of fibrant-cofibrant objects of the localised model structure. Then the coherent nerve of the functor $T \times \mathrm{Alg}_{fc}(T) \to \mathbf{Kan}$. induced by the evaluation functor $T \times [T, \mathbf{S}] \to \mathbf{S}$ is a map of simplicial sets

$$C^!T \times C^!\mathrm{Alg}_{fc}(T) \to \mathbf{U}.$$

The corresponding map $C^!\mathrm{Alg}_{fc}(T) \to \mathbf{U}^{C^!T}$ induces an equivalence of quategories ,

$$C^!\mathrm{Alg}_{fc}(T) \to \mathrm{Alg}(C^!T)$$

when T is DK-fibrant. Dually, if $T \in \mathbf{QCat}$ is an algebraic theory, then $C_!T$ is a simplicial algebraic theory and we have an equivalence of quategories

$$C^!\mathrm{Alg}_{fc}(C_!T) \to \mathrm{Alg}(T).$$

52. Appendix on Cisinski theory

We briefly describe Cisinki's theory of model structures on a Grothendieck topos. It can be used to generate the model structure for n-quasi-category for every $n \geq 1$.

52.1. We shall say that a combinatorial model structure on a Grothendieck topos $\mathcal{E}$ is a *Cisinski structure* if its cofibrations are the monomorphisms.

52.2. The classical model structure $(\mathbf{S}, \mathit{Who})$ is a Cisinski model structure. Also the model structure for quasi-categories. The model structure for Segal categories is a Cisinski model structure on **PCat**. The model structure for Segal spaces is a Cisinski structure on $\mathbf{S}^{(2)}$, and also the model structure for Rezk categories.

52.3. Let $\mathcal{E}$ be a finitely bicomplete category and $(\mathcal{C}, \mathcal{T})$ be a weak factorisation system in $\mathcal{E}$ ($\mathcal{C}$=the cofibrations and $\mathcal{T}$=the trivial fibrations) We shall say that a class of maps $\mathcal{W} \subseteq \mathcal{E}$ is a *localizer* (with respect to $\mathcal{C}$) if the following conditions are satisfied:

- $\mathcal{W}$ has the"three for two" property;
- $\mathcal{T} \subseteq \mathcal{W}$;
- $\mathcal{C} \cap \mathcal{W}$ is the left class of a weak factoriszation system.

A class $\mathcal{W}$ is a localizer iff the triple $M(\mathcal{W}) = (\mathcal{C}, \mathcal{W}, (\mathcal{C}\cap\mathcal{W})^{\pitchfork})$ is a model structure. The map $\mathcal{W} \mapsto M(\mathcal{W})$ induces a bijection between the localizers with respect to $\mathcal{C}$ and the model structures on $\mathcal{E}$ having $\mathcal{C}$ for class of cofibrations. If $\mathcal{W}$ and $\mathcal{W}'$ are two localizers with respect to $\mathcal{C}$, then the model structure $M(\mathcal{W}')$ is a Bousfield localisation of the model structure $M(\mathcal{W})$ iff we have $\mathcal{W} \subseteq \mathcal{W}'$. This defines a partial order relation on the class of model structures having $\mathcal{C}$ for class of cofibrations.

52.4. [Ci1] We say that a class $\mathcal{W}$ of maps in a Grothendieck topos $\mathcal{E}$ is a *localizer* if it is a localizer with respect to the class $\mathcal{C}$ of monomorphisms. We shall say that a localizer $\mathcal{W}$ is *accessible* if the saturated class $\mathcal{C}\cap\mathcal{W}$ is accessible (ie generated by a set of maps). A localizer $\mathcal{W} \subseteq \mathcal{E}$ is accessible iff the triple $M(\mathcal{W}) = (\mathcal{C}, \mathcal{W}, \mathcal{C}\cap\mathcal{W})^{\pitchfork})$ is a Cisinski model structure. The map $\mathcal{W} \mapsto M(\mathcal{W})$ induces a bijection between the accessible localizers and the Cisinski model structures.

52.5. (Cisinski) In the category $\mathbf{S}$, the localizer *Who* is generated by the maps $\Delta[n] \to 1$ for $n \geq 0$. The localizer *Wcat* is generated by the inclusions $I[n] \subseteq \Delta[n]$ for $n \geq 0$.

52.6. Let us sketch a proof that the localizer Wcat is generated by the spine inclusions $I[n] \subseteq \Delta[n]$ for $n \geq 0$. We shall first prove that if a localiser $W \subseteq \mathbf{S}$ contains the inclusions $I[n] \subseteq \Delta[n]$ for every $n \geq 0$, then it contains the mid anodyne maps. If $\mathcal{C} \subset \mathbf{S}$ is the class of monomorphisms, then the intersection $W \cap \mathcal{C}$ is saturated. Moreover, the class $W \cap \mathcal{C}$ has the right cancellation property, since W satisfies "3 for 2". It follows that every mid anodyne map belongs to $W \cap \mathcal{C}$ by [JT2]. Thus, every fibrant object of the model structure defined by W is a quategory. The result then follows from 50.10.

52.7. [Ci1] If $\mathcal{E}$ is a Grothendieck topos, then every set of maps $S \subseteq \mathcal{E}$ is contained in a smallest accessible localizer $\mathcal{W}(S)$ called the localizer *generated* by S In particular, there is a smallest localizer $\mathcal{W}_0 = \mathcal{W}(\emptyset)$. We say that the model structure $M(\mathcal{W}_0)$ is *minimal*. The minimal Cisinski model structure $M(\mathcal{W}_0)$ is cartesian closed and proper. Every Cisinski model structure is a Bousfield localisation of $M(\mathcal{W}_0)$. .

52.8. [Ci2] Let L be the Lawvere object in a topos $\mathcal{E}$ and let $t_0, t_1 : 1 \to L$ be the canonical elements (the first is classifying the subobject $\emptyset \subseteq 1$ and the second the subobject $1 \subseteq 1$). Then an object $X \in \hat{C}$ is fibrant with respect to minimal Cisinski model structure $(\mathcal{C}, \mathcal{W}_0, \mathcal{F}_0)$ iff the projection $X^{t_i} : X^L \to X$ is a trivial fibration for $i = 0, 1$. A monomorphism $A \to B$ is acyclic iff the map $X^B \to X^A$ is a trivial fibration for every fibrant object X.

References

[ADR] J. Adámek, F.W. Lawvere and J. Rosický, *On the duality between varieties and algebraic theories* Algebra Universalis 49 (2003), 35-49.

[AR1] J. Adámek and J. Rosický, *Locally Presentable and Acessible Categories* London Mathematical Society Lecture Note Series No. 189, 1994.

[AR2] J. Adámek and J. Rosický, *On sifted colimits and generalised varieties* Theory Appl. Categ. 8 (2001), 33-53.

[Bad1] B. Badzioch, *Algebraic theories in homotopy theory*, Annals of Math. 155 (2002), 895-913.

[Bad2] B. Badzioch, *From Γ-spaces to algebraic theories*, Trans. Amer. Math. Soc. 357 (2005), 1779-1799.

[Bat1] M. Batanin, *Categorical strong shape theory*, Cahier de Topologie et de Gomtrie Diffrentielle Catgorique 38 (1997) no1, 3-66.

[Bat2] M. Batanin, *Homotopy coherent category theory and A_∞-structures in monoidal categories*,

[Bat3] M. Batanin, *Monoidal globular categories as a natural environment for the theory of weak n-categories*, Advances in Mathematics 136 (1998), 39103.

[BD] J. C. Baez, J. Dolan, *Higher-dimensional Algebra and Topological Quantum Fields Theory* J. Math. Phys. 36 (1995) 6073-6105.

[BS] M. Batanin, R. Street, *The universal property of the multitude of trees* JPAA 154(2000) 3-13.

[BJP] H. J. Baues, M. Jibladze andT. Pirashvili, *Strengthening track theories* arXiv:math/0907185v1, 2003.

[BJP2] H. J. Baues, M. Jibladze andT. Pirashvili, *Third MacLane Cohomology* arXiv:math/0907185v1, 2003.

[Bec] J. Beck, *On H-spaces and infinite loop spaces*, Lect. Notes Math. 99, 139-153, Springer-Verlag 1969.

[Bek] T. Beke, *Sheafable homotopy model categories*, Math. Proc. Camb. Phil. Soc. 129 (2000), no.3, 447-475

[Ber] C. Berger, *A cellular nerve for higher categories*, Advances in Mathematics 169 (2002), 118175.

[B1] J. E. Bergner, *A model category structure on the category of simplicial categories*, Trans. Math. Soc., 359 (2007) 2043-2058.

[B2] J. E. Bergner, *Three models for the homotopy theory of homotopy theories*, Topology. Vol. 46, Issue 4, September 2007, 397-436.

[B3] J. E. Bergner, *A characterisation of fibrant Segal categories*, preprint available at arXiv:math.AT/0603400.

[B4] J. E. Bergner, *Rigidification of algebras over multisorted theories*, arXiv:math.AT/0508152

[B5] J. E. Bergner, *A survey of $(\infty,1)$-categories*, preprint. July 14, 2006.

[B6] J. E. Bergner, *Adding inverses to diagrams encoding algebraic structures*, preprint. October 9, 2006.

[BJ] T. Bisson, A. Joyal *The Dyer-Lashof algebra in bordism*, C.R. Math. Rep. Acad. Sci. Vol. 17 no.4 (1995) pp 135-140.

[BV] M. Boardman and R. Vogt, *Homotopy invariant algebraic structures on topological spaces*, Lectures Notes in Math. vol. 347, Springer Verlag, 1973.

[Bo] F. Borceux, *Handbook of Categorical Algebra, I, II, III*, Encyclopedia of Mathematics and its Applications. vol 50, 51, 52. Cambridge University Press (1994).

[BC] D. Bourn and J.M. Cordier, *A general formulation of homotopy limits*, Jour. Pure Appl. Alg. 29(1983), 129-141.

[BK] A. K. Bousfield and D. M. Kan, *Homotopy limits, completions and localisations*, Lectures Notes in Math. vol. 304, Springer Verlag, 1972.

[Br] M. Brun, *Witt vectors and Tambara functors*, Adv. Math.193, no.2, 233-256 (2005).

[BFV] M. Brun, Z. Fiedorowicz and R.M. Vogt *On the multiplicative structure of Topological Hochschild Homology*, arXiv:math.AT/0410367

[CC] C. Casacuberta and B. Chorny, *The orthogonal subcategory problem in homotopy theory*, Proceedings of the Arolla conference on Algebraic Topology 2004.

[Ci1] D.C. Cisinski, *Théories homotopiques dans les topos*, Journal of Pure and Applied Algebra 174 (2002) 43-82.

[Ci2] D.C. Cisinski,*Les prfaisceaux comme modles des types d'homotopie*, Astrisque 308, Soc. Math. France (2006), xxiv+392 pages.

[Ci3] D.C. Cisinski,*Batanin higher groupoids and homotopy types*, in Categories in Algebra, Geometry and Physics, Street Festschrift, A. Davydov et al. eds., Contemp. Math. 431, Amer. Math. Soc., pp. 171-186 (2007).

[C] J. M. Cordier, *Sur la notion de diagramme homotopiquement coherent*, Proc. 3ieme Colloque sur les Catégories, Amiens (1980).

[CP1] J. M. Cordier and T. Porter, *Vogt's theorem on categories of homotopy coherent diagrams*, Math. Proc. Camb. Phil. Soc. (1986), 100, 65-90.

[CP2] J. M. Cordier and T. Porter, *Homotopy coherent category theory*, Trans. Amer. Math. Soc. 349(1997), 1-54.

[Da] B. Day,

[Du] D. Dugger, *Universal homotopy theories*, Adv. in Math. 164, (2001), no. 1, 144-176.

[DHKS] W.G. Dwyer, P.S. Hirschhorn, D.M. Kan and J.H. Smith, *Homotopy limit functors on model categories and homotopical categories*, Mathematical Surveys and Monographs 113, American Mathematical Society, Providence, RI, 1999.

[DK1] W.G. Dwyer and D. Kan, *Functions complexes in homotopical algebra* , Topology 19 (1980), 427-440.

[DK2] W.G. Dwyer and D. Kan, *Simplicial localisations of categories*, J. of Pure and Appl. Algebra. vol 17 (1980), 427-440.

[DS] W.G. Dwyer and J. Spalinski, *Homotopy theories and model categories*, Handbook of Algebraic Topology, Elsevier, 1995.

[EM] S. Eilenberg and S. MacLane, *General theory of natural equivalences*, Trans. Amer. Math. Soc., 58, (1945)

[Eh] C. Ehresmann, *Espece de structures locales*, Séminaire de topologie et de géomtrie différentielle, Paris (1961)

[GU] P. Gabriel and F. Ulmer, *Lokal prsentierbare Kategorien*, Springer Lecture Notes in Math. 221 (1971).

[GZ] P. Gabriel and M. Zisman, *Calculus of fractions and homotopy theory*, Springer-Verlag, New York, 1963.

[Ga] N. Gambino, *Homotopy limits in 2-categories*, to appear in the Mathematical Proceedings of the Cambridge Philosophical Society.

[GJ] P. G. Goerss and J.F. Jardine, *Simplicial homotopy theory*, Progress in Mathematics, vol 174, Birkhauser, 1999.

[Gon] Beatriz Rodriguez Gonzales, *Simplicial Descent Categories*, Ph Thesis, Universidad de Sevilla, 2008. arXiv:0804.2154.

[Gra] J.Gray, *Closed categories, lax limits and homotopy limits*, J. Pure and Applied Algebra 19 (1980) 127-158

[Gro] A.Grothendieck, *Pursuing Stacks*, Manuscript 1983, To appear in Documents Mathématiques, edited by Georges Maltsiniotis.

[He] A. Heller, *Homotopy theories*, Memoir of the American Math. Soc. 383, 1988.

[Hi] P. S. Hirschhorn, *Model Categories and Their Localisations*, Mathematical Surveys and Monographs, 99, American Mathematical Society, Providence, RI, 2003.

[HS] A. Hirschowitz and C. Simpson, *Descente pour les n-champs*, arXiv:math. AG/9807049.

[Ho] M. Hovey, *Model Categories*, Mathematical Surveys and Monographs 63, American Mathematical Society, Providence, RI, 1999.

[Illu] L. Illlusie, *Complexes cotangents et dformations II*, Lectures Notes in Math. vol. 283, Springer Verlag, 1972.

[J1] A. Joyal, *Quasi-categories and Kan complexes*, J. of Pure and Appl. Algebra. vol 175 (2002), 207-222.

[J2] A. Joyal, *The theory of quasi-categories I.* in preparation.

[J3] A. Joyal, *The theory of quasi-categories II.* in preparation.

[J4] A. Joyal, *Quasi-categories vs Simplicial categories*, in preparation.

[J6] A. Joyal *A course on quasi-categories and applications.* CRM Lecture Notes, Barcelona.

[JT1] A. Joyal and M. Tierney, *Strong stacks and classifying spaces*, Lectures Notes in Math. vol. 1588, Springer Verlag, 1991.

[JT2] A. Joyal and M. Tierney, *Quasi-categories vs Segal spaces*, Categories in Algebra, Geometry and Mathematical Physics. AMS Contemporary Mathematics 431. A. Davydov. M. Batanin, M. Johnson, S.Lack, A. Neeman editors.

[JT3] A. Joyal and M. Tierney, *Elements of Simplicial Homotopy Theory.* in preparation.

[Ka] D. Kan, *Adjoint functors*, Trans. Amer. Math. Soc. 87; 330-346(1958)

[Ke] M. Kelly, *Basic concepts of enriched category theory*, LMS Lecture note series. 64, Cambridge university Press, 1982.

[KP] K. H. Kamps and T. Porterf, *Abstract homotopy and simple homotopy theory*, World Scientific, 1997.

[KJBM] J. Kock, A.Joyal, M.Batanin, J. Mascari, *Polynomial functors and opetopes*, arXiv:0706.1033v1

[La] C. Lair, *Sur le genre d'esquissibilté des catégories modélables (accessibles) possédant les produits de deux*, Diagrammes **35** (1996), 25-52.

[Law] F. W Lawvere, *Equality in hyperdoctrines and comprehension scheme as an adjoint functor*, Proc. Symp. Pure Math. Vol XVII, AMS (1970).

[Law] W. Lawvere, *Functorial semantics of algebraic theories*, Proc. Nat. Acad, Sc. 50(1963), 869-872.

[Le] T. Leinster, *Higher Operads, Higher Categories*, arXiv:math.CT/0305049v1, 2003.

[Lu1] J. Lurie, *Higher Topos Theory*, arXiv:math.CT/0608040, 2007.

[Lu2] J. Lurie, *Derived algebraic Geometry I: Stable ∞-Categories*, arXiv:math.CT/0608228v4, 2007.

[Lu3] J. Lurie, *Derived Algebraic Geometry II: Non-commutative Algebra*, arXiv:math.CT/0702299, 2007.

[Lu4] J. Lurie, *Derived Algebraic Geometry III: Commutative Algebra*, arXiv:math.CT/0703204, 2007.

[Lu5] J. Lurie, *Derived Algebraic Geometry IV: Deformation Theory*, arXiv:math.CT/0709.3091, 2007.

[Mac] S. MacLane, *Categories for the Working Mathematicians*, Graduate Texts in Mathematics 5, Springer-Verlag(1971).

[Malt1] G. Maltsiniotis, *Introduction à la théorie des dérivateurs d'après Grothendieck*, unpublished manuscript available at http://www.math.jussieu.fr/ maltsin/.

[Malt2] *Une mthode complique pour dfinir un groupe, ou une catgorie additive*, unpublished manuscript available at http://www.math.jussieu.fr/ maltsin/.

[Malt2] G. Maltsiniotis, *Structure d'asphricit, foncteurs lisses, et fibrations,* Ann. Math. Blaise Pascal, 12, pp. 1-39 (2005)

[Malt3] G. Maltsiniotis, *La théorie de l'"homotopie de Grothendieck*, Asterisque 301, 2005.

[Malt4] G. Maltsiniotis, *Infini groupoides d'aprs Grothendieck*, Preprint. http://www.institut.math.jussieu.fr/ maltsin/ps/infgrart.pdf

[Malt5] G. Maltsiniotis, *Infini catgories non strictes, une nouvelle définition*, Preprint. http://www.institut.math.jussieu.fr/ maltsin/ps/infctart.pdf.

[Malt6] *Une mthode compliquée pour définir un groupe, ou une catégorie additive*, unpublished manuscript available at http://www.math.jussieu.fr/ maltsin/.

[MP] M. Makkai, R. Par *Accessible Categories, The Foundation of Categorical Model Theory* AMS Contemporary Mathematics, No.104, 1989.

[MZ] M. Makkai, M. Zawadowski *duality for simple ω-categories and disks*, Theory and Applications pf Categories, Vol 8, No. 5, pp. 114-243 (2001).

[May1] P. May, *The Geometry of Iterated loop spaces,* Lect. Notes in Math. 271, Springer Verlag, 1972.

[May2] P. May, *A concise Course in Algebraic Topology* Chicago Lect. Notes in Math., The University of Chigago Press. 1999.

[MW1] I. Moerdijk, I. Weiss *Dendroidal Sets*, arXiv:math,AT/0701293, 2007.

[MW2] I. Moerdijk, I. Weiss *On inner Kan complexes in the category of dendroidal sets* arXiv:math,AT/0701295, 2007.

[Ni] J.P. Nichols-Barrer, *On quasi-categories as a foundation for Higher Algebraic Stacks* PhD Thesis, MIT, June 2001.

[P] R. Pellissier. *Catégories enrichies faibles*. Thèse, Université de Nice-Sophia Antipolis. June 2002, arXiv:math/0308246, 2003.

[Q] D. Quillen, *Homotopical algebra*, Lectures Notes in Math. vol. 43, Springer Verlag, 1967.

[Ree] C.L. Reedy, *Homotopy theories of model categories*, unpublished manuscript.

[Rez] C. Rezk, *Toposes and homotopy toposes*, unpublished manuscript. Dec 2005.

[Rez] C. Rezk, *A model for the homotopy theory of homotopy theory*, Trans. Amer. Math Soc., 353(2001), no. 3, 973-1007.

[Ro] A. Roig, *Model category structures in bifibered categories*, JPAA 95 (1994) 203-223.

[Ros] J. Rosicky, *On homotopy varieties*, Advances in Mathematics, Vol 214, N.2, 1 Oct 2007.

[ScVo] R. Schwnzl and R. Vogt *Homotopy homomorphisms and the hammock localisation*,

[Sch] S. Schwede, *Spectra in model categories and applications to the algebraic cotangent complex*, JPAA 120 (1997) 77-104.

[Sch] S. Schwede, *Stable homotopy of algebraic theories*, Topology 40 (2001) 1-41

[S1] G.B. Segal, *Classifying spaces and spectral sequences*, IHES Publ. Math. 34, 105-112(1968).

[S2] G. Segal, *Categories and cohomology theories*, Topology 13 (1974) 293-312.

[Si4] C. Simpson, *Some properties of the theory of n-categories* arXiv:math.CT/0110273, 2001.

[Si3] C. Simpson, *A Giraud-type characterisation of the simplicial category associated to closed model categories as ∞-pretopoi* arXiv:math.CT/990316,1999.

[Si2] C. Simpson, *On the Breen-Baez-Dolan stabilisation hypothesis for Tamsamani weak n-categories* arXiv:math.CT/9810058, 1998.

[Si1] C. Simpson, *A closed model structure for n-categories, internal Hom, n-stacks and generalised Seifert-Van Kampen* arXiv:math.AG/9704006v2, 2000.

[SGA] A. Grothendieck, J.L. Verdier *Topos theory and etale cohomology, SGA4*,

[SW] R. Street and R. Walters, *The comprehensive factorisation of a functor*, Bulletin American Math. Soc. **79** (1973) 936-941.

[Sta] J,D, Stasheff, *Homotopy associativity of H-spaces I, II,* Trans. Amer. Math, Soc. 108 (1963), 275-312.

[Tamb] D. Tambara *On multiplicative transfer* Commun. Algebra 21, No 4, 1393-1420 (1993)

[Ta] Z. Tamsamani *On non-strict notions of n-category and n-groupoid via multisimplicial sets*, arXiv:math.AG/9512006, 1995.

[T1] B. Töen *Homotopical and Higher Categorical Structures in Algebraic Geometry*, Habilitation thesis, May 2003. Unpublished manuscript, preprint available at arXiv:math.AG/0312262.

[TV1] B. Toen and G. Vezzosi, *Algebraic Geometry over model categories (a general approach to derived algebraic geometry* arXiv:math.AG/0110109

[TV2] B. Toen and G. Vezzosi, *Homotopical algebraic geometry I: Topos theory*, Adv. Math. 193 (2005), 257-372, arXiv:math.AG/0207028

[TV3] B. Toen and G. Vezzosi, *Segal topoi and stacks over Segal categories*, arXiv:math.AG/0212330

[Vi] E. Vitale *Localizations of algebraic categories II*, Journal of Pure and Applied Algebra 133 (1998) 317-326.

[V] R.M. Vogt, *Homotopy limits and colimits*, Math.Z. 134. 11-52 (1973)

[W] M. Weber, *Generic Morphisms, parametric representations and weakly cartesian monads*, Theory and Applications of Categories, Vol. 13, No. 14, 2004, pp. 191234.

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