Volume II

The Theory of Quasi-Categories and its Applications

André Joyal

Contents

Introduction

The notion of quasi-category was introduced by Boardman and Vogt in their work on homotopy invariant algebraic structures [BV]. A Kan complex and the nerve of a category are examples. The goal of our work is to extend category theory to quasi-categories and to develop applications to homotopy theory, higher category theory and (higher) topos theory.

Quasi-category are examples of $(\infty, 1)$-categories in the sense of Baez and Dolan. Other examples are simplicial categories, Segal categories and complete Segal spaces (here called Rezk categories). To each example is associated a model category and the model categories are connected by a network of Quillen equivalences. Simplicial categories were introduced by Dwyer and Kan in their work on simplicial localisation. Segal categories were introduced by Hirschowitz and Simpson in their work on higher stacks in algebraic geometry. Many aspects of category theory were extended to Segal categories. A notion of Segal topos was introduced by Toen and Vezzosi, and a notion of stable Segal category by Hirschowitz, Simpson and Toen. A notion of higher Segal category was studied by Tamsamani, and a notion of enriched Segal category by Pellisier. The theory of Segal categories is a source of inspiration for the theory of quasi-categories.

Jacob Lurie has recently formulated his work on higher topoi in the language of quasi-categories [Lu1]. In doing so, he has extended a considerable amount of category theory to quasi-categories. He also developed a theory of stable quasi-categories [Lu2] and applications to geometry in [Lu3], [Lu4] and [Lu5]. Our lectures may serve as an introduction to his work.

The present notes were prepared for a course on quasi-categories given at the CRM in Barcelona in February 2008. The material is taken from two manuscripts under preparation. The first is a book in two volumes called the "Theory of Quasi-categories" which I hope to finish before I leave this world if God permits. The second is a paper called "Notes on Quasi-categories" to appear in the Proceedings of an IMA Conference in Minneapolis in 2004. The two manuscripts have somewhat different goals. The aim of the book is to teach the subject at a technical level by giving all the relevant details while the aim of the paper is to brush the subject in perspective. Our goal in the course is to bring the participants at the cutting edge of the subject. The perspective presented in the course is very sketchy and a more complete one will be found in the IMA Conference Proceedings. To the eight lectures that were originally planned for the course we added four complementary lectures for a total of twelve. We included support material organised in eight appendices. The last appendix called "Open boxes and prismes" was originally a chapter of the book. But it is so technical that we putted it as an appendix.

The results presented here are the fruits of a long term research project which began around thirty years ago. We suspect that some of our results could be given a simpler proofs. The extension by Cisinski [Ci2] of the homotopy theory of Grothendieck [Mal2] appears to be the natural framework for future developements. We briefly describe this theory in the perspective and we use some of the results.

The fact that category theory can be extended to quasi-categories is not obvious a priori but it can discovered by working on the subject. The theory of quasi-categories depends strongly on homotopical algebra. Quasi-categories are the fibrant objects of a Quillen model structure on the category of simplicial sets. Many results of homotopical algebra become more conceptual and simpler when reformulated in the language of quasi-categories. We hope that this reformulation will help to shorten the proofs. In mathematics, many details of a proof are omitted because they are considered obvious. But what is "obvious" in a given subject evolves through times. It is the result of an implicit agreement between the reseachers based on their knowledge and experience. A mathematical theory is a social construction. The theory of quasi-categories is presently in its infancy.

The theory of quasi-categories can analyse phenomena which belong properly to homotopy theory. The notion of stable quasi-category is an example. The notion of meta-stable quasi-category introduced in the notes is another. We give a proof that the quasi-category of parametrized spectra is an utopos (joint work with Georg Biedermann). All the machinery of universal algebra can be transfered to homotopy theory. We introduce the notion of para-variety (after a suggestion by Mathieu Anel).

In the last chapters we venture a few steps in the theory of (∞, n)-categories. We introduce a notion of n-disk and of n-cellular sets. If $n = 1$, a n-disk is an interval and a n-cellular set is a simplicial set. A n-quasi-category is defined to be a fibrant n-cellular set for a certain model structure on n-cellular sets. In the course, we shall formulate a conjecture of Cisinski about this model structure.

A few words on terminology. A quasi-category is sometime called a *weak Kan complex* in the literature [KP]. The name *Boardman complex* was recently proposed by Vogt. The purpose of our terminology is to stress the analogy with categories. The theory of quasi-categories is very closely *apparented* to category theory. We are calling *utopos* (upper topos) a "higher topos"; alternatives are "homotopy topos" or “homotopos”. We are calling *pseudo-fibration* a fibration in the model structure for quasi-categories; alternatives are “iso-fibration”, “categorical fibration” and “quasi-fibration”. We are calling *isomorphism* a morphism which is invertible in a quasi-category; alternatives are “quasi-isomorphism”,“equimorphism” and “equivalence”.

I warmly thank Carles Casacuberta and Joachim Kock for the organisation of the Advanced Course and their support. The 2007-2008 CRM research program on Homotopy Theory and Higher Categories is the fruit of their initiative. I thank the director of the CRM Joaquim Bruna and the former director Manuel Castellet for their support. The CRM is a great place for mathematical research and Barcelona a marvelous cultural center. Long live to Catalunya!

Perspective

The notion of quasi-category

Recall that a simplicial set X is called a *Kan complex* if it satisfies the *Kan condition*: every horn $\Lambda^k[n] \to X$ can be filled by a simplex $\Delta[n] \to X$,

$$\begin{array}{ccc} \Lambda^k[n] & \xrightarrow{\forall} & X \\ \downarrow & \nearrow_{\exists} & \\ \Delta[n]. & & \end{array}$$

The notion of quasi-category is a slight modification of this notion. A simplicial set X is called a *quasi-category* if it satisfies the *Boardman condition*: every horn $\Lambda^k[n] \to X$ *with* $0 < k < n$ can be filled by a simplex $\Delta[n] \to X$. A quasi-category is sometime called a *weak Kan complex* in the literature [KP]. The name *Boardman complex* was recently proposed by Vogt. A Kan complex and the nerve of a category are examples of quasi-categories. The purpose of our terminology is to stress the analogy with categories. The theory of quasi-categories is very closely *apparented* to category theory. We often say that a vertex of a quasi-category is an *object* of this quasi-category, and that an arrow is a *morphism*. A *map* of quasi-categories $f : X \to Y$ is a map of simplicial sets. We denote the category of (small) categories by **Cat** and the category of (small) quasi-categories by **QCat**. If X is a quasi-category, then so is the simplicial set X^A for any simplicial sets A. Hence the category **QCat** is cartesian closed.

The notion of quasi-category has many equivalent descriptions. For $n > 0$, the *n-chain* $I[n] \subseteq \Delta[n]$ is defined to be the union of the edges $(i-1, i) \subseteq \Delta[n]$ for $1 \le i \le n$. We shall put $I[0] = 1$. A simplicial set X is a quasi-category iff the projection $X^{\Delta[2]} \to X^{I[2]}$ defined by the inclusion $I[2] \subset \Delta[2]$ is a trivial fibration.

The nerve functor

The nerve functor $N : \mathbf{Cat} \to \mathbf{S}$, from the category of small categories to the category of simplicial sets is fully faithful. It can be regarded as an inclusion by adopting the same notation for a small category and its nerve. If I denotes the category generated by one arrow $0 \to 1$ and J the groupoid generated by one isomorphism $0 \to 1$, then we have $\Delta[1] = I \subset J$. The nerve functor has a left adjoint $\tau_1 : \mathbf{S} \to \mathbf{Cat}$ which associates to a simplicial set X its *fundamental category* $\tau_1 X$. The fundamental category of a quasi-category X is isomorphic to the *homotopy category* hoX constructed by Boardman and Vogt. If X is a simplicial and $a, b \in X_0$, let us denote by $X(a, b)$ the fiber at (a, b) of the projection

$$(s,t) : X^I \to X^{\{0,1\}} = X \times X$$

defined by the inclusion $\{0, 1\} \subset I$. We have $(hoX)(a, b) = \pi_0 X(a, b)$. The composition law

$$(hoX)(b,c) \times (hoX)(a,b) \;\; \to \;\; (hoX)(a,c)$$

is defined by filling horns $\Lambda^1[2] \to X$.

Quasi-categories and Kan complexes

We say that an arrow in a quasi-category X is *invertible*, or that it is an *isomorphism*, if the arrow is invertible in the category hoX. An arrow $f \in X$ is invertible iff the map $f : I \to X$ can be extended along the inclusion $I \subset J$. If **Kan** denotes the category of Kan complexes, then the inclusion functor $\mathbf{Kan} \subset \mathbf{QCat}$ has a right adjoint $J : \mathbf{QCat} \to \mathbf{Kan}$ which associates to a quasi-category X its *simplicial set of isomorphisms* $J(X)$: a simplex $x : \Delta[n] \to X$ belongs to $J(X)$ iff the arrow $x(i, j) : x(i) \to x(j)$ is invertible for every $i < j$. There is an analogy between Kan complexes and groupoids. A quasi-category X is a Kan complex iff its homotopy category hoX is a groupoid.

The 2-category of simplicial sets

The functor τ_1 preserves finite products by a result of Gabriel and Zisman. If we apply it on the composition map $C^B \times B^A \to C^A$, we obtain the composition law

$$\tau_1(B,C) \times \tau_1(A,B) \to \tau_1(A,C)$$

of a 2-category $\mathbf{S}^{\tau_1}$, where we put $\mathbf{S}^{\tau_1}(A, B) = \tau_1(A, B)$. A 1-cell of this 2-category is a map of simplicial sets. Hence the category $\mathbf{S}$ has the structure of a 2-category $\mathbf{S}^{\tau_1}$.. We call a map of simplicial sets $X \to Y$ a *categorical equivalence* if it is an equivalence in this 2-category. If X and Y are quasi-categories, a categorical equivalence $X \to Y$ is called an *equivalence of quasi-categories*, or just an *equivalence* if the context is clear. An *adjunction* between two maps of simplicial sets $f : X \leftrightarrow Y : g$ is defined to be an adjunction in the 2-category $\mathbf{S}^{\tau_1}$. We remark here that in any 2-category, there is a notion of left (and right) Kan extension of a map $A \to X$ along a map $A \to B$.

A map between quasi-categories is an equivalence iff it is fully faithful and essentially surjective. Let us define these notions. A map between quasi-categories $f : X \to Y$ is said to be *fully faithful* if the map $X(a, b) \to Y(fa, fb)$ induced by f is a weak homotopy equivalence for every pair of objects $a, b \in X_0$. A map of simplicial sets $u : A \to B$ is said to be *essentially surjective* if the functor $\tau_1(u) : \tau_1(A) \to \tau_1(B)$ is essentially surjective.

Limits and colimits in a quasi-category

There is a notion of limit (and colimit) for a diagram with values in any quasi-category. A *diagram* in a quasi-category X is defined to be a map $A \to X$, where A is an arbitrary simplicial set. The notion of limit depends on the notions of terminal object and of exact projective cone. An object a in a quasi-category X is said to be *terminal* if every simplical sphere $x : \partial\Delta[n] \to X$ with target $x(n) = a$ can be filled. An object $a \in X$ is terminal iff the simplicial set $X(x, a)$ is contractible for evry object $x \in X$ iff the map $a : 1 \to X$ is right adjoint to the map $X \to 1$. The notion of projective cone is defined by using the join $A \star B$ of two simplicial sets A and B. A *projective cone* with base $d : A \to X$ in X is a map $c : 1 \star A \to X$ which extends the map d; the object $c(1) \in X$ is the *apex* of the cone. There a quasi-category X/d of projective cones with base d in X. A simplex $\Delta[n \to X/d$ is a map $\Delta[n \star A \to X$ which extends d. A projective cone $c \in X/d$ is said to be *exact* if it is a terminal object of X/d. The *limit*

$$l = \varprojlim_{a \in A} d(a).$$

is defined to be the apex $l = c(1) \in X$ of an exact cone $c : 1 \star A \to X$. The full simplicial subset of X/d spanned by the exact projective cones is a contractible Kan complex when non-empty. Hence the limit of a diagram is homotopy unique if it exists. The colimit of a diagram is defined dually with the notions of initial object and of coexact inductive cone.

A simplicial set A is said to be *finite* if it has a finite number on non-degenerate cell. A diagram $A \to X$ is said to be *finite* if A is finite. A quasi-category X is said to be *finitely complete* or *cartesian* if every finite diagram $d : A \to X$ has a limit. A quasi-category is cartesian iff it has pullbacks and a terminal object iff the diagonal $X \to X^A$ has a right adjoint for any finite simplicial set A. A quasi-category X is said to be *finitely cocomplete* or *cocartesian* if its opposite X^o

is cartesian. A (large) quasi-category is cocomplete iff it has pushouts and coproducts iff the diagonal $X \to X^A$ has a left adjoint for any (small) simplicial set A. If X is a cocomplete quasi-category and $u : A \to B$ is a map of simplicial sets, then the map $X^u : X^B \to X^A$ has a left adjoint $u_!$. The map $u_!(f) : B \to A$ is the left Kan extension of a map $f : A \to X$ along u.

The *loop space* $\Omega_u(x)$ of a pointed object $u : 1 \to x$ in a cartesian quasi-category is defined by a pullback square

$$\begin{array}{ccc} \Omega_u(x) & \longrightarrow & 1 \\ \downarrow & & \downarrow{\scriptstyle u} \\ 1 & \xrightarrow{u} & x. \end{array}$$

A *null object* in a quasi-category X is an object $0 \in X$ which is both initial and terminal. The *suspension* $\Sigma(x)$ of an object x in a cocartesian quasi-category with null object 0 is defined by a pushout square

$$\begin{array}{ccc} x & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \Sigma(x) \end{array}$$

A quasi-category A is said to be *cartesian closed* if it admits finite products and the map $a \times - : A \to A$ has a right adjoint for any object $a \in A$. A quasi-category A is said to be *locally cartesian closed* if the quasi-category A/a is cartesian for any object $a \in A$.

Cisinski theory

We briefly describe Cisinki's theory of model structures on a Grothendieck topos. It can be used to generate the model structure for quasi-categories. It can also used to generate the model structure for higher quasi-categories.

We say that a map in a topos $\mathcal{E}$ is a *trivial fibration* if it has the right lifting property with respect to the monomorphisms. This terminology is non-standard but useful. If $\mathcal{A}$ is the class of monomorphisms in a topos $\mathcal{E}$ and $\mathcal{B}$ is the class of trivial fibrations, then the pair $(\mathcal{A}, \mathcal{B})$ is a weak factorisation system D.1.12. A map of simplicial sets is a trivial fibration iff it has the right lifting property with respect to the inclusion $\delta_n : \partial\Delta[n] \subset \Delta[n]$ for every $n \geq 0$. See B.0.9. An object I in a topos is said to be *injective* if the map $I \to 1$ is a trivial fibration. For example, the Lawvere object L of a topos is injective. An injective object I equipped with a monomorphism $(i_0, i_1) : \{0, 1\} \to I$ is called an *injective interval.* For example, the Lawvere object L is an injective interval, where $i_1 : 1 \to L$ is the map which classifies the subobject $1 \subseteq 1$ and $i_0 : 1 \to L$ is the map which classifies the subobject $\emptyset \subseteq 1$. The (nerve of the) groupoid J is an injective interval in the topos of simplicial sets $\mathbf{S}$.

We say that a cofibrantly generated model structure on a Grothendieck topos $\mathcal{E}$ is a *Cisinski model structure* if the cofibrations are the monomorphisms. The acylic fibrations of a Cisinski model structure are the trivial fibrations. The Bousfield localisation of a Cisinski model structure with respect to a (small) set of maps $\Sigma \subseteq \mathcal{E}$ is a Cisinski model structure.

A model structure on a category is determined by its cofibrations and its fibrant objects by E.1.10. Hence a Cisinski model structure on a topos is determined by its class of fibrant objects.

The classical model structure on the category of simplicial sets $\mathbf{S}$ is a Cisinski model structure whose fibrant objects are the Kan complexes. The weak equivalences are the *weak homotopy equivalences* and the fibrations are the *Kan fibrations*. We say that it is the *Kan* model structure on $\mathbf{S}$ and denote it shortly by $(\mathbf{S}, \mathbf{Kan})$ or by $(\mathbf{S}, \mathit{Who})$, where *Who* denotes the class of weak homotopy equivalences.

We say that a model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ on a category $\mathcal{E}$ is *cartesian* if the cartesian product $\times : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ is a left Quillen functor of two variables and the terminal object 1 is cofibrant (definition E.3.8). A Cisinki model structure is cartesian iff the cartesian product of two weak equivalences is a weak equivalence. The classical model structure on $\mathbf{S}$ is cartesian. If X is a fibrant object in a cartesian Cisinski model $\mathcal{E}$, then so is the object X^A for any object $A \in \mathcal{E}$. Hence the (full) subcategory $\mathcal{E}_f$ of fibrant object of $\mathcal{E}$ is cartesian closed.

[Ci1] Let $\mathcal{C}$ be the class of monomorphisms in a Grothendieck topos $\mathcal{E}$. A class of maps $\mathcal{W} \subseteq \mathcal{E}$ is called an *(accessible) localizer* if the following conditions are satisfied:

- $\mathcal{W}$ has the"three for two" property;
- the class $\mathcal{C} \cap \mathcal{W}$ is saturated and accessible;
- $\mathcal{W}$ contains the trivial fibrations.

If $\mathcal{W} \subseteq \mathcal{E}$ is a localizer and $\mathcal{F} = (\mathcal{C} \cap \mathcal{W})^{\pitchfork}$, then the triple $M(\mathcal{W}) = (\mathcal{C}, \mathcal{W}, \mathcal{F})$ is a Cisinski model structure. The map $\mathcal{W} \mapsto M(\mathcal{W})$ induces a bijection between the class of localizers in $\mathcal{E}$ and the class of Cisinski model structures on $\mathcal{E}$. The partially ordered class of localizers in $\mathcal{E}$ is closed under (small) intersection. Its maximum element is the class $\mathcal{W} = \mathcal{E}$. Every set of maps $S \subset \mathcal{E}$ is contained in a smallest localizer $\mathcal{W}(S)$ called the localiser *generated* by S . In particular, there is a smallest localizer $\mathcal{W}_0 = \mathcal{W}(\emptyset)$. We shall say that $M(\mathcal{W}_0)$ is the *minimal Cisinski structure*. The model structure $M(\mathcal{W}_0)$ is is cartesian closed. Every Cisinski model structure on $\mathcal{E}$ is a Bousfield localisation of $M(\mathcal{W}_0)$.

[Ci2] Let $I = (I, i_0, i_1)$ be an injective interval in a topos $\mathcal{E}$. Then an object $X \in \mathcal{E}$ is fibrant with respect to minimal Cisinski model structure iff the projection $X^{i_k} : X^I \to X$ is a trivial fibration for $k = 0, 1$. A monomorphism $A \to B$ is acyclic iff the map $X^B \to X^A$ is a trivial fibration for every fibrant object X.

The model structure for quasi-categories

We say that a functor $p : A \to B$ (in **Cat**) is a *pseudo-fibration* if for every object $a \in A$ and every isomorphism $g \in B$ with source $p(a)$, there exists an isomorphism $f \in A$ with source a such that $p(f) = g$. Equivalently, a functor is a pseudo-fibration iff it has the right lifting property with respect to the inclusion $\{0\} \subset J$. We say that a functor $u : A \to B$ is *monic on objects* if the induced map $Ob(A) \to Ob(B)$ is monic. The category **Cat** admits a Quillen model structure in which a cofibration is a functor monic on objects, a weak equivalence is an *equivalence of categories* and a fibration is a pseudo-fibration. Every object is fibrant and cofibrant. The model structure is cartesian and proper. We call it the *natural* model structure on **Cat**. We shall denote it by $(\mathbf{Cat}, Eq)$, where Eq is the class of equivalences of categories. It induces a (natural) model structure on **Grp**.

The category of simplicial sets **S** admits a Cisinski model structure in which the fibrant objects the quasi-categories. We say that it is the *model structure for quasi-categories* and we denote it shortly by $(\mathbf{S}, \mathbf{QCat})$. A weak equivalence is called a *weak categorical equivalence* and a fibration a *pseudo-fibration*. The model structure is cartesian. We call it the *model structure for quasi-categories*. We denote it shortly by $(\mathbf{S}, \mathbf{QCat})$ or by $(\mathbf{S}, Wcat)$, where $Wcat$ denotes the class of weak categorical equivalences.

We shall say that the n-chain $I[n] \subseteq \Delta[n]$ is the *spine* of $\Delta[n]$. The localizer *Wcat* is generated by the spine inclusions $I[n] \subset \Delta[n]$.

A map between quasi-categories is a weak categorical equivalence iff it is a categorical equivalence. We call a map of simplicial set a *mid fibration* if it has the right lifting property with respect to the inclusions $\Lambda^k[n] \subset \Delta[n]$ with $0 < k < n$. A map between quasi-categories $f : X \to Y$ is a pseudo-fibration iff it is a mid fibration and the functor $ho(f) : hoX \to hoY$ is a pseudo-fibration.

The pair of adjoint functors $\tau_1 : \mathbf{S} \leftrightarrow \mathbf{Cat} : N$ is a Quillen adjunction between the model structure for quasi-categories and the natural model structure on **Cat**. A functor $u : A \to B$ in **Cat** is a pseudo-fibration iff the map $Nu : NA \to NB$ is a pseudo-fibration in **S**.

The Kan model structure on : **S** is a Bousfield localisation of the model structure for quasi-categories. Hence a weak categorical equivalence is a weak homotopy equivalence and the converse is true for a map between Kan complexes by E.2.18. A Kan fibration is a pseudo-fibration and that the converse is true for a map between Kan complexes.

The Kan model structure on **S** is a Bousfield localisation of the model structure for quasi-categories. Many results on Kan complexes can be extended to quasi-categories. For example, every quasi-category has a skeletal (or minimal) model which is unique up to isomorphism.

Equivalence with simplicial categories

The theory of simplicial categories was developped by Dwyer and Kan in their work on simplicial localisation.

Recall that a category enriched over simplicial sets is called a *simplicial category*. An enriched functor between simplicial categories is said to be *simplicial*. We denote by **SCat** the category of simplicial categories and simplicial functors. An ordinary category is a simplicially enriched category with discrete hom. The inclusion functor $\mathbf{Cat} \subset \mathbf{SCat}$ has a left adjoint

$$ho : \mathbf{SCat} \to \mathbf{Cat}$$

which associates to a simplicial category X its *homotopy category* hoX. By construction, we have $(hoX)(a, b) = \pi_0 X(a, b)$ for every pair of objects $a, b \in X$. A simplicial functor $f : X \to Y$ is said to be *homotopy fully faithful* if the map $X(a, b) \to Y(fa, fb)$ is a weak homotopy equivalence for every pair of objects $a, b \in X$. A simplicial functor $f : X \to Y$ is said to be *homotopy essentially surjective* if the functor $ho(f) : hoX \to hoY$ is essentially surjective. A simplicial functor $f : X \to Y$ is called a *Dwyer-Kan equivalence* if it is homotopy fully faithful and homotopy essentially surjective. A simplicial functor $f : X \to Y$ is called a *Dwyer-Kan fibration* if the map $X(a, b) \to Y(fa, fb)$ is a Kan fibration for every pair of objects $a, b \in X$, and the functor $ho(f)$ is a pseudo-fibration. The category **SCat** admits a Quillen model structure in which the weak equivalences are the Dwyer-Kan equivalences and the fibrant objects the Dwyer-Kan fibrations. We call it the *Bergner model structure* on **SCat**. The fibrant objects are the categories enriched over Kan complexes.

Recall that a *reflexive graph* is a 1-truncated simplicial set. Let **Grph** be the category of reflexive graphs. The obvious forgetful functor $U : \mathbf{Cat} \to \mathbf{Grph}$ has a left adjoint F. The composite $C = FU$ has the structure of a comonad on **Cat**. Hence the sequence of categories $C_n A = C^{n+1}(A)$ $(n \geq 0)$ has the structure of a simplicial object $C_*(A)$ in **Cat** for any small category A. The simplicial set $n \mapsto Ob(C_n A)$ is constant with value $Ob(A)$. It follows that $C_*(A)$ can be viewed as a simplicial category instead of a simplicial object in **Cat**. This defines a functor

$$C_* : \mathbf{Cat} \to \mathbf{SCat}.$$

If A is a category then the augmentation $C_*(A) \to A$ is a cofibrant replacement of A in the model category **SCat**. If X is a simplicial category, then a simplicial functor $C_*(A) \to X$ is said to be a *homotopy coherent diagram* $A \to X$ [V].

The simplicial category $C_\star[n]$ has the following simple description. The objects of $C_\star[n]$ are the elements of $[n]$. If $i, j \in [n]$ and $i \leq j$, then the category $C_\star[n](i,j)$ is the poset of subsets $S \subseteq [i.j]$ such that $\{i,j\} \subseteq S$. If $i > j$, then $C_\star[n](i,j) = \emptyset$. If $i \leq j \leq k$, then the composition operation

$$C_\star[n](j,k) \times C_\star[n](i,j) \to C_\star[n](i,k)$$

is the union $(T,S) \mapsto T \cup S$.

The *coherent nerve* of a simplicial category X is the simplicial set $C^! X$ defined by putting

$$(C^! X)_n = \mathbf{SCat}(C_\star[n], X)$$

for every $n \geq 0$. A *homotopy coherent diagram* $A \to X$ indexed by a category A is a map of simplicial sets $A \to C^! X$. The functor $C^! : \mathbf{SCat} \to \mathbf{S}$ has a left adjoint $C_!$ and we have $C_! A = C_\star A$ when A is a category [J4]. The pair of adjoint functors

$$C_! : \mathbf{S} \leftrightarrow \mathbf{SCat} : C^!$$

is a Quillen equivalence between the model category for quasi-categories and the Bergner model structure[J4][Lu1]. The simplicial set $C^!(X)$ is a quasi-category when the simplicial category X is enriched over Kan complexes [CP].

A quasi-category can be large. The (large) *quasi-category of homotopy types* **U** is defined to be the coherent nerve of the (large) simplicial category of Kan complexes **Kan**. The quasi-category **U** is bicomplete and locally cartesian closed. It is the archetype of an *utopos*.

The category **QCat** becomes enriched over Kan complexes if we put

$$Hom(X,Y) = J(Y^X)$$

for $X, Y \in \mathbf{QCat}$. The (large) *quasi-category of (small) quasi-categories* $\mathbf{U}_1$ is defined to be the coherent nerve of **QCat**. The quasi-category $\mathbf{U}_1$ is bicomplete and cartesian closed.

Equivalence with Segal categories

The notion of Segal categories was introduced by Hirschowitz and Simpson in their work on higher stacks in algebraic geometry.

A *bisimplicial set* is a contravariant functor $\Delta \times \Delta \to \mathbf{Set}$. We denote the category of bisimplicial sets by $\mathbf{S}^{(2)}$. A *simplicial space* is a contravariant functor $\Delta \to \mathbf{S}$. We can regard a simplicial space X as a bisimplicial set by putting

$X_{mn} = (X_m)_n$ for every $m, n \geq 0$. Conversely, we can regard a bisimplicial set X as a simplicial space by putting $X_m = X_{m\star}$ for every $m \geq 0$. The *box product* of two simplicial sets A and B is the bisimplicial set $A \Box B$ obtained by putting

$$(A \Box B)_{mn} = A_m \times B_n$$

for every $m, n \geq 0$. This defines a functor of two variables $\Box : \mathbf{S} \times \mathbf{S} \to \mathbf{S}^{(2)}$.

A simplicial space $X : \Delta^o \to \mathbf{S}$ is called a *precategory* if the simplicial set X_0 is discrete. We shall denote by **PCat** the full subcategory of $\mathbf{S}^{(2)}$ spanned by the precategories. A bisimplicial set $X : (\Delta^o)^2 \to \mathbf{Set}$ is a precategory iff it takes every map in $[0] \times \Delta$ to a bijection. Let us put

$$\Delta^{|2} = ([0] \times \Delta)^{-1}(\Delta \times \Delta)$$

and let π be the canonical functor $\Delta^2 \to \Delta^{|2}$. We can regard the functor π^* as an inclusion by adopting the same notation for a contravariant functor $X : \Delta^{|2} \to \mathbf{Set}$ and the precategory $\pi^*(X)$. The functor $\pi^* : \mathbf{PCat} \subset \mathbf{S}^{(2)}$ has a left adjoint $\pi_!$ and a right adjoint π_*.

If X is a precategory and $n \geq 1$, then the vertex map $v_n : X_n \to X_0^{n+1}$ takes its values in a discrete simplicial set. We thus have a decomposition

$$X_n = \bigsqcup_{a \in X_0^{[n]_0}} X(a),$$

where $X(a) = X(a_0, a_1, \ldots, a_n)$ denotes the fiber of v_n at $a = (a_0, a_1, \cdots, a_n)$. If $u : [m] \to [n]$ is a map in Δ, then the map $X(u) : X_n \to X_m$ induces a map

$$X(a_0, a_1, \ldots, a_n) \to X(a_{u(0)}, a_{u(1)}, \ldots, a_{u(m)})$$

for every $a \in X_0^{[n]_0}$. A precategory X is called a *Segal category* if the canonical map

$$X(a_0, a_1, \ldots, a_n) \to X(a_0, a_1) \times \cdots \times X(a_{n-1}, a_n)$$

is a weak homotopy equivalence for every $a \in X_0^{[n]_0}$ and $n \geq 2$. This condition is called the *Segal condition.*

If C is a small category, then the bisimplicial set $N(C) = C \Box 1$ is a Segal category. The functor $N : \mathbf{Cat} \to \mathbf{PCat}$ has a left adjoint

$$\tau_1 : \mathbf{PCat} \to \mathbf{Cat}.$$

We say that $\tau_1 X$ is the *fundamental category* of a precategory X. A map of precategories $f : X \to Y$ is said to be *essentially surjective* if the functor $\tau_1(f) : \tau_1 X \to \tau_1 Y$ is essentially surjective. A map of precategories $f : X \to Y$ is said to be *fully faithful* if the map

$$X(a, b) \to Y(fa, fb)$$

is a weak homotopy equivalence for every pair $a, b \in X_0$. We say that $f : X \to Y$ is an *equivalence* if it is fully faithful and essentially surjective.

Hirschowitz and Simpson construct a completion functor

$$S : \mathbf{PCat} \to \mathbf{PCat}$$

which associates to a precategory X a Segal category $S(X)$ "generated" by X. A map of precategories $f : X \to Y$ is called a *weak categorical equivalence* if the map $S(f) : S(X) \to S(Y)$ is an equivalence of Segal categories. The category **PCat** admits a model structure in which a a weak equivalence is a weak categorical equivalence and a cofibration is a monomorphism. The model structure is left proper and it is cartesian closed by a result of Pellisier in [P]. We say that it is the *model structure for Segal categories.*

We recall that the category of simplicial spaces $[\Delta^o, \mathbf{S}]$ admits a Reedy model structure in which the weak equivalences are the level wise weak homotopy equivalences and the cofibrations are the monomorphisms. A Segal category is fibrant iff it is Reedy fibrant as a simplicial space by a result of Bergner [B3]. Hence the Hirschowitz-Simpson model structure is the Cisinki model structure on **PCat** for which the fibrant objects are the Reedy fibrant Segal category.

The functor $i_i : \Delta \to \Delta \times \Delta$ defined by putting $i_1([n]) = ([n], 0)$ is right adjoint to the projection $p_1 : \Delta \times \Delta \to \Delta$. The projection p_1 inverts every arrow in $[0] \times \Delta$. Hence there is a unique functor $q : \Delta^{|2} \to \Delta$ such that $q\pi = p_1$. The composite functor $j = \pi i_1 : \Delta \to \Delta^{|2}$ is right adjoint to the functor q. Hence the functor $j^* : \mathbf{PCat} \to \mathbf{S}$ is right adjoint to the functor q^*. If X is a precategory, then $j^*(X)$ is the first row of X. If $A \in \mathbf{S}$, then $q^*(A) = A\Box 1$. It was conjectured in [T1] and proved in [JT2] that the adjoint pair of functors

$$q^* : \mathbf{S} \leftrightarrow \mathbf{PCat} : j^*$$

is a Quillen equivalence between the model category for quasi-categories and the model category for Segal categories.

Let us put $d = \pi\delta : \Delta \to \Delta^{|2}$, where δ is the diagonal functor $\Delta \to \Delta \times \Delta$. The simplicial set $d^*(X)$ is the diagonal of a precategory X. The functor

$$d^* : \mathbf{PCat} \to \mathbf{S}$$

admits a left adjoint $d_!$ and a right adjoint d_*. It was proved in [JT2] that the adjoint pair of functors

$$d^* : \mathbf{PCat} \leftrightarrow \mathbf{S} : d_*$$

is a Quillen equivalence between the model category for Segal categories and the model category for quasi-categories.

Equivalence with Rezk categories

Rezk categories were introduced by Charles Rezk under the name of complete Segal spaces. We describe the equivalence between Rezk categories and quasi-categories.

The box product funtor $\Box : \mathbf{S} \times \mathbf{S} \to \mathbf{S}^{(2)}$ is divisible on both sides. This means that the functor $A\Box(-) : \mathbf{S} \to \mathbf{S}^{(2)}$ admits a right adjoint $A\backslash(-) : \mathbf{S}^{(2)} \to \mathbf{S}$ for every simplicial set A, and that the functor $(-)\Box B : \mathbf{S} \to \mathbf{S}^{(2)}$ admits a right adjoint $(-)/B : \mathbf{S}^{(2)} \to \mathbf{S}$ for every simplicial set B. Let $I_n \subseteq \Delta[n]$ be the n-chain. For any simplicial space X we have a canonical bijection

$$I_n\backslash X = X_1 \times_{\partial_0,\partial_1} X_1 \times \cdots \times_{\partial_0,\partial_1} X_1,$$

where the successive fiber products are calculated by using the face maps $\partial_0, \partial_1 : X_1 \to X_0$. We shall say that a simplicial space X satisfies the *Giraud condition* if the map

$$i_n\backslash X : \Delta[n]\backslash X \longrightarrow I_n\backslash X$$

obtained from the inclusion $i_n : I_n \subseteq \Delta[n]$ is an isomorphism for every $n \geq 2$ (the condition is trivially satisfied if $n < 2$). We say that a simplicial space X satisfies the *Segal condition* if the same map is a weak homotopy equivalence for every $n \geq 2$.

We recall that the category $[\Delta^o, \mathbf{S}]$ of simplicial spaces admits a Reedy model structure in which the weak equivalences are the level wise weak homotopy equivalences and the cofibrations are the monomorphisms. We say that a simplicial space $X : \Delta^o \to \mathbf{S}$ is a *Segal space* if it is Reedy fibrant and satisfies the Segal condition. The Reedy model structure admits a Bousfield localisation in which the fibrant objects are the Segal spaces by a theorem of Rezk in [Rezk1]. We call the localised model structure the *model structure for Segal spaces.*

Let J be the groupoid generated by one isomorphism $0 \to 1$. We regard J as a simplicial set via the nerve functor. Wel say that a Segal space X satisfies the *Rezk condition* if the map

$$1\backslash X \longrightarrow J\backslash X$$

obtained from the map $J \to 1$ is a weak homotopy equivalence. We say that a Segal space which satisfies the Rezk condition is *complete*, or that it is a *Rezk category.* The model structure for Segal spaces admits a Bousfield localisation in which the fibrant objects are the Rezk categories by a theorem of Rezk in [Rezk1]. It is the *model structure for Rezk categories.*

[JT2] The first projection $p_1 : \Delta \times \Delta \to \Delta$ is left adjoint to the functor $i_1 : \Delta \to \Delta \times \Delta$ defined by putting $i_1([n]) = ([n], [0])$ for every $n \geq 0$. The simplicial set $i_1^*(X)$ is the first row of a bisimplicial set X. Notice that we have $p_1^*(A) = A\Box 1$ for every simplicial set A. The pair of adjoint functors

$$p_1^* : \mathbf{S} \leftrightarrow \mathbf{S}^{(2)} : i_1^*$$

is a Quillen equivalence between the model category for quasi-categories and the model category for Rezk categories.

[JT2] Recall that the inclusion functor $\pi^* : \mathbf{PCat} \subset \mathbf{S}^{(2)}$ has a left adjoint $\pi_!$ and a right adjoint π_*. It was proved by Bergner in [B2] that the pair of adjoint functors

$$\pi^* : \mathbf{PCat} \leftrightarrow \mathbf{S}^{(2)} : \pi_*$$

is a Quillen equivalence between the model structure for Segal categories and the model structure for Rezk categories. The functor π^* preserves and reflects weak equivalences.

Homotopy localisations

The theory of simplicial localisation of Dwyer and Kan can be formulated in the language of quasi-categories. The *homotopy localisation* of a quasi-category X with respect to a set Σ of arrows in X is the quasi-category $L(X, \Sigma)$ defined by a homotopy pushout square

$$\begin{array}{ccc} \Sigma \times I & \longrightarrow & X \\ \downarrow & & \downarrow \\ \Sigma \times J & \longrightarrow & L(X, \Sigma), \end{array}$$

where the vertical map on the left side is induced by the inclusion $I \subset J$. The functor τ_1 preserves homotopy pushouts and it follows that there is an equivalence of categories,

$$hoL(X, \Sigma) \simeq \Sigma^{-1} hoX.$$

If C is a category, then $L(C, \Sigma)$ is equivalent to the coherent nerve of the Dwyer-Kan localisation of C with respect to the set Σ. If X is a quasi-category, then every map $C \to X$ which inverts every arrow in Σ admits an extension $L(C, \Sigma) \to X$ which is unique up to a unique 2-cell. A pair (C, Σ) is a *homotopical category* in the sense of Dwyer, Hirschhorn, Kan and J.H. Smith [DHKS]. If X is a quasi-category, we call a map $C \to X$ a *representation* of X by (C, Σ) if its extension $L(C, \Sigma) \to X$ is an equivalence of quasi-categories. Every quasi-category admits a representation by a homotopical category (C, Σ). The *homotopy localisation* of a model category $\mathcal{E}$ is defined to be the quasi-category $L(\mathcal{E}) = L(\mathcal{E}, \mathcal{W})$, where $\mathcal{W}$ is the class of weak equivalences. Notice the equivalence of categories

$$hoL(\mathcal{E})) \simeq \mathcal{W}^{-1}\mathcal{E} = Ho(\mathcal{E}).$$

It follows from a result of Simpson [Si3] and of [Du] that the quasi-category $L(\mathcal{E})$ is locally presentable when the model category $\mathcal{E}$ is combinatorial [Hi]. Conversely, every locally presentable quasi-category is the homotopy localisation of a combinatorial model category. We conjecture that the quasi-category $L(\mathcal{E})$ is finitely bicomplete for any model category $\mathcal{E}$ and conversely, that every (small) finitely bicomplete quasi-category is the homotopy localisation of a (small) model category.

Homotopy factorisation systems

Factorisation systems arise in category theory and homotopical algebra. They play an important role in the theory of quasi-categories. We consider four kinds of factorisation systems: strict, weak, Bousfield and homotopy. Strict factorisation systems occur in category theory and weak factorisation systems in homotopical algebra. Homotopy factorisation systems were introduced in homotopy theory by Bousfield as a side product of his localisation theory. We introduce a general notion to formalise natural examples from category theory, homotopy theory and the theory of quasi-categories. Each class of a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ is homotopy replete and closed under composition. The left class $\mathcal{A}$ has the right cancellation property and the right class $\mathcal{B}$ has the left cancellation property; the intersection $\mathcal{A} \cap \mathcal{B}$ is the class of weak equivalences. If $\mathcal{A}$ is the class of essentially surjective functors and $\mathcal{B}$ is the class of fully faithful functors, then the pair $(\mathcal{A}, \mathcal{B})$ is a homotopy factorisation system in the category **Cat** (equipped with the natural model structure). Each class determines of a homotopy factorisation system determines the other. The category **Cat** admits a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{B}$ is the class of conservative functors. A functor $u : A \to B$ belongs to the class $\mathcal{A}$ of this system iff it admits a factorisation $u = eu' : A \to B' \to B$ with e an equivalence and u' an iterated localisation. Each of these systems is related to a corresponding homotopy factorisation system in the model category $(\mathbf{S}, \mathbf{QCat})$. Let us say that map of simplicial sets $u : A \to B$ is *essentially surjective* if the functor $\tau_1(u)$ is essentially surjective. The model category $(\mathbf{S}, \mathbf{QCat})$ admits a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of essentially surjective maps; a map in $\mathcal{B}$ is said to be *fully faithful*. Let us say that a map of simplicial sets $f : X \to Y$ is *conservative* if the functor $\tau_1(f)$ is conservative. The model category $(\mathbf{S}, \mathbf{QCat})$ admits a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{B}$ is the class of conservative maps; a map in $\mathcal{A}$ is an *iterated homotopy localisation*. We say that a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ is *strong* if the pair $(\mathcal{A}', \mathcal{B}') = (\mathcal{A} \cap \mathcal{C}, \mathcal{B} \cap \mathcal{F})$ is a weak factorisation system; the pair $(\mathcal{A}', \mathcal{B}')$ is what we call a *Bousfield factorisation system*. There is a bijection between the strong homotopy factorisation systems and the Bousfield factorisation systems. In the model category **Cat**, every homotopy factorisation system in the model category **Cat** is strong but this is false in the model category $(\mathbf{S}, \mathbf{QCat})$.

Recall that a functor $u : A \to B$ induces a pair of adjoint functors between the presheaf categories

$$u_! : [A^o, \mathbf{Set}] \leftrightarrow [B^o, \mathbf{Set}] : u^*.$$

A functor u is said to be *final*, but we shall say *0-final*, if the functor $u_!$ takes a terminal object to a terminal object. A functor $u : A \to B$ is final iff the category $b\backslash A$ defined by the pullback square

$$\begin{array}{ccc} b\backslash A & \longrightarrow & A \\ \downarrow & & \downarrow{\scriptstyle u} \\ b\backslash B & \longrightarrow & B \end{array}$$

is connected for every object $b \in B$. The model category **Cat** admits a (strict) factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of 0-final functors and $\mathcal{B}$ is the class of discrete fibrations. The system is not a homotopy factorisation system, since the class $\mathcal{B}$ is not invariant under equivalences. There is however an associated homotopy factorisation system $(\mathcal{A}, \mathcal{B}')$, where a functor $f : X \to Y$ belongs to $\mathcal{B}'$ iff it admits a factorisation $f = f'e : X \to X' \to Y$ with e an equivalence and f' a discrete fibration. The notion of 0-final functor $u : A \to B$ can be strengthtened. A functor $u : A \to B$ is said to be *1-final* if the category $b\backslash A$ is 1-connected for every object $b \in B$. The model category **Cat** admits a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of 1-final functors. A functor $f : X \to Y$ belongs to $\mathcal{B}$ iff it admits a factorisation $f = f'e : X \to X' \to Y$ with e an equivalence and f' a 1-fibration. A *1-fibration* is a Grothendieck fibration whose fibers are groupoids. There is an obvious notion of 2-final functor but the model category **Cat** does not admit a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of 2-final functors. But such a system exists if we replace the model category **Cat** by the model category $(\mathbf{S}, \mathbf{QCat})$. There is a notion of *n-final* map of simplicial sets for every $n \geq 0$, and the model category $(\mathbf{S}, \mathbf{QCat})$ admits a homotopy factorisation system $(\mathcal{A}_n, \mathcal{B}_n)$ in which $\mathcal{A}_n$ is the class of n-final maps. There is also a notion of ∞-*final* map of simplicial sets and the model category $(\mathbf{S}, \mathbf{QCat})$ admits a homotopy factorisation system $(\mathcal{A}_\infty, \mathcal{B}_\infty)$ in which $\mathcal{A}_\infty$ is the class of ∞-final maps. For simplicity, a map in $\mathcal{A}_\infty$ is said to be *final*. A map $f : X \to Y$ belongs to $\mathcal{B}_\infty$ iff it admits a factorisation $f = f'e : X \to X' \to Y$ with e a weak categorical equivalence and f' a *right* fibration.

Left and right fibrations

We call a map of simplicial sets a *left fibration*, or a *covariant fibration*, if it has the right lifting property with respect to the inclusions $\Lambda^k[n] \subset \Delta[n]$ with $0 \leq k < n$. A map $f : X \to Y$ is a left fibration iff the map $X^I \to Y^I \times_Y X$ obtained from the square

$$\begin{array}{ccc} X^I & \xrightarrow{X^{i_1}} & X \\ \downarrow & & \downarrow \\ Y^I & \xrightarrow{Y^{i_1}} & Y \end{array}$$

is a trivial fibration, where i_0 denotes the inclusion $\{0\} \subset I = \Delta[1]$. We say that a map is *left anodyne* if it belongs to the saturated class generated by the inclusions $\Lambda^k[n] \subset \Delta[n]$ with $0 \leq k < n$. The category $\mathbf{S}$ admits a factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of left anodyne maps and $\mathcal{B}$ is the class of left fibrations. The system $(\mathcal{A}, \mathcal{B})$ is a Bousfield system with respect to the model structure for quasi-categories. There is an associated homotopy factorisation system $(\mathcal{A}', \mathcal{B}')$ where $\mathcal{A}'$ is the class of initial maps, where a map $u : A \to B$ is *initial* iff it admits a factorisation $u = eu' : A \to B' \to B$ with u' a left anodyne map and e a weak categorical equivalence. A functor $f : X \to Y$ belongs to $\mathcal{B}'$ iff it admits a factorisation $f = f'e : X \to X' \to Y$ with e a weak categorical equivalence and f' a left fibration. Dually, we call a map of simplicial sets a *right fibration*, or a *contravariant fibration*, if it has the right lifting property with respect to the inclusions $\Lambda^k[n] \subset \Delta[n]$ with $0 < k \leq n$. We say that a map is *right anodyne* if it belongs to the saturated class generated by the inclusions $\Lambda^k[n] \subset \Delta[n]$ with $0 < k \leq n$. If $\mathcal{A}$ is the class of right anodyne maps and $\mathcal{B}$ is the class of right fibrations, then the pair $(\mathcal{A}, \mathcal{B})$ is a Bousfield factorisation system in the model category $(\mathbf{S}, \mathbf{QCat})$. We say that a map of simplicial sets $u : A \to B$ is *terminal* if it admits a factorisation $u = eu' : A \to B' \to B$ with u' a right anodyne map and e a weak categorical equivalence.

If $u : A \to B$ is a final map, then the colimit of a diagram $d : B \to X$ with values in a quasi-category X exists iff the colimit of the composite diagram $du : A \to X$ exists, in which case the two colimits are naturally isomorphic. Dually, if a map $u : A \to B$ is initial, then the limit of a diagram $d : B \to X$ exists iff the limit of $du : A \to X$ exists, in which case the two limits are naturally isomorphic.

Contravariant and covariant model structures

The category $\mathbf{S}/B$ is enriched over the category $\mathbf{S}$ for any simplicial set B. We denote by $[X, Y]_B$, or more simply by $[X, Y]$, the simplicial set of maps $X \to Y$ between two objects of $\mathbf{S}/B$. If we apply the functor π_0 to the composition map

$$[Y, Z] \times [X, Y] \to [X, Z]$$

of a triple $X, Y, Z \in \mathbf{S}/B$, we obtain a composition law

$$\pi_0[Y, Z] \times \pi_0[X, Y] \to \pi_0[X, Z]$$

for a category $(\mathbf{S}/B)^{\pi_0}$, where we put $(\mathbf{S}/B)^{\pi_0}(X,Y) = \pi_0[X,Y]$. We say that a map $X \to Y$ in $\mathbf{S}/B$ is a *fibrewise homotopy equivalence* if the map is invertible in the category $(\mathbf{S}/B)^{\pi_0}$. If $X \in \mathbf{S}/B$, let us denote by $X(b)$ the fiber of the structure map $X \to B$ over a vertex $b \in B$. If a map $f : X \to Y$ in $\mathbf{S}/B$ is a fibrewise homotopy equivalence, then the map $f_b : X(b) \to Y(b)$ induced by f is a homotopy equivalence for each vertex $b \in B$. Let $\mathbf{R}(B)$ (resp. $\mathbf{L}(B)$) be the full subcategory of $\mathbf{S}/B$ spanned by the right (resp. left) fibrations with target B. Then a map $f : X \to Y$ in $\mathbf{R}(B)$ (resp. in $\mathbf{L}(B)$) is a fibrewise homotopy equivalence iff the map $f_b : X(b) \to Y(b)$ induced by f is a homotopy equivalence for every vertex $b \in B$.

We call a map $u : M \to N$ in $\mathbf{S}/B$ a *contravariant equivalence* if the map

$$\pi_0[u,X] : \pi_0[M,X] \to \pi_0[N,X]$$

is bijective for every $X \in \mathbf{R}(B)$. A fibrewise homotopy equivalence is a contravariant equivalence and the converse is true for a map in $\mathbf{R}(B)$. A final map is a contravariant equivalence and the converse is true for a map with codomain in $\mathbf{R}(B)$. The category $\mathbf{S}/B$ admits a simplicial Cisinski model structure called the *contravariant* model structure, in which the weak equivalences are the contravariant equivalences. A fibration is called a *dexter fibration* and a fibrant object is an object of $\mathbf{R}(B)$. We denote the model structure by $(\mathbf{S}/B, \mathbf{R}(B))$. A dexter fibration is a right fibration and the converse is true for a map in $\mathbf{R}(B)$. Dually, we say that $u : M \to N$ in $\mathbf{S}/B$ is a *covariant equivalence* if the map $\pi_0[u,X]$ is bijective for every $X \in \mathbf{L}(B)$. The category $\mathbf{S}/B$ admits a simplicial Cisinski model structure, called the *covariant* model structure, in which the weak equivalences are the covariant equivalences. A fibration is called a *sinister fibration* and a fibrant object is an object of $\mathbf{L}(B)$. We denote the model structure by $(\mathbf{S}/B, \mathbf{L}(B))$.

If C is a small category, then the category $[C^o, \mathbf{S}]$. of simplicial presheaves $C^o \to, \mathbf{S}$ admits two model structures respectively called the *projective* and the *injective* model structures [GJ]. The weak equivalences are the pointwise weak homotopy equivalences in both model structures. A fibration is a pointwise Kan fibration in the projective structure and a cofibration is a pointwise cofibration in the injective structure. Consider the functor $\Gamma : \mathbf{S}/C \to [C^o, \mathbf{S}]$ which associates to an object $E \in \mathbf{S}/C$ the simplicial presheaf $c \mapsto Hom_C(C/c, E)$. The functor Γ is the right adjoint in a Quillen equivalence between the dexter model category $(\mathbf{S}/C, \mathbf{R}(C))$ and the projective model category $[C^o, \mathbf{S}]$.

Morita Equivalences

For any simplicial set A, let us put $\mathcal{P}(A) = Ho(\mathbf{S}/A, \mathbf{R}(A)$. A map of simplicial sets $u : A \to B$ induces a pair of adjoint functors

$$u_! : \mathbf{S}/A \leftrightarrow \mathbf{S}/B : u^*.$$

The pair is a Quillen adjunction with respect to the contravariant model structure on these categories. We thus obtain a derived adjunction

$$\mathcal{P}_!(u) : \mathcal{P}(A) \leftrightarrow \mathcal{P}(B) : \mathcal{P}^*(u),$$

where $\mathcal{P}_!(u) = Lu_!$ is the left derived of $u_!$ and $\mathcal{P}^*(u) = Ru^*$ is the right derived of u^*. Many properties of the map $u : A \to B$ can be related to properties of the functrors $\mathcal{P}_!(u)$ and $\mathcal{P}_!(u)$. For example, the map u is final iff the functor $\mathcal{P}_!(u)$ takes a terminal object to a terminal object. The map u is fully faithful iff the functor $\mathcal{P}_!(u)$ is fully faithful. A map u is said to be *dominant* if the functor $\mathcal{P}^*(u)$ is fully faithful. It is not obvious (but true) that the notion of dominant map is self dual: a map $u : A \to B$ is dominant iff the the opposite map $u^o : A^o \to B^o$ is dominant. A homotopy localisation is dominant. A map u is called a *Morita equivalence* if the adjunction $\mathcal{P}_!(u) \vdash \mathcal{P}^*(u)$ is an equivalence. A map u is a Morita equivalence iff it is fully faithful and every object of $\tau_1 B$ is a retract of an object in the image of u.

Karoubi envelopes

Recall that a category A is said to be *Karoubi complete* if every idemptent in A splits. Every category A has a *Karoubi envelope* $\kappa(A)$ obtained by splitting freely the idempotents in A. A functor $f : A \to B$ is a Morita equivalence iff the functor $\kappa(f) : \kappa(A) \to \kappa(B)$ is an equivalence of categories. Let E be the monoid freely generated by one idempotent $e \in E$. Its Karoubi envelope is the category E' freely generated by two arrows $s : 0 \to 1$ and $r : 1 \to 0$ such that $rs = 1_0$. A category A is Karoubi complete iff every functor $E \to A$ can be extended along the inclusion $E \subset E'$. The category **Cat** admits a model structure in which a weak equivalence is a Morita equivalence and a cofibration is a functor monic on objects. A category is fibrant iff it is Karoubi complete.

A quasi-category X is said to be *Karoubi complete* if every map $u : E \to X$ can be extended along the inclusion $E \subset E'$. The category **S** admits a Cisinki model structure in which the fibrant objects are the Karoubi complete quasi-categories. A weak equivalence is a Morita equivalence. The fibrant replacement of a quasi-category A is its Karoubi envelope $\kappa(A)$.

Grothendieck fibrations

There is a notion of Grothendieck fibration for maps between simplicial sets. If $p : E \to B$ is a mid fibration between simplicial sets, we say that an arrow $f : a \to b$ in E is *cartesian* if the map $E/f \to B/pf \times_{B/pb} E/b$ obtained from the commutative square

$$\begin{array}{ccc} E/f & \longrightarrow & E/b \\ \downarrow & & \downarrow \\ B/pf & \longrightarrow & B/pb \end{array}$$

is a trivial fibration. We say that a mid fibration $p : E \to B$ is a *Grothendieck fibration* if for every vertex $b \in E$ and every arrow $g \in B$ with target $p(b)$, there exists a cartesian arrow $f \in E$ with target b such that $p(f) = g$. Every right fibration is a Grothendieck fibration and every Grothendieck fibration is pseudo-fibration.

There is a dual notion of *cocartesian arrow* and a dual notion of *Grothendieck opfibration.*

If X is a quasi-category, then the source map $s : X^I \to X$ is a Grothendieck fibration, and it is an opfibration when X admits pushouts. Dually, the target map $t : X^I \to X$ a Grothendieck opfibration, and it is a fibration when X admits pullbacks.

Every map between quasi-categories $u : X \to Y$ admits a factorisation

$$u = qi : X \to P \to Y$$

with q a Grothendieck fibration and i a fully faithful right adjoint. The quasi-category P can be constructed by the pullback square

$$\begin{array}{ccc} P & \xrightarrow{h} & Y^I \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle t} \\ X & \xrightarrow{u} & Y, \end{array}$$

where t is the target map. If $s : Y^I \to Y$ is the source map, then the composite $q = sh : P \to Y$ is a Grothendieck fibration. There is a unique map $i : X \to P$ such that $pi = 1_X$ and $hi = \delta u$, where $\delta : Y \to Y^I$ is the diagonal. We have $p \vdash i$ and the counit of the adjunction is the identity of $pi = 1_X$. Thus, i is fully faithful.

Proper maps

There is a notion of proper (resp. smooth) map between quasi-categories and more generally beween simplicial sets. We say that a map of simplicial sets $u : A \to B$ is *proper* if the pullback functor $u^* : \mathbf{S}/B \to \mathbf{S}/A$ takes a right anodyne map to a right anodyne map. A map of simplicial sets $u : A \to B$ is proper iff the inclusion $u^{-1}(b(n)) \subseteq b^*(E)$ is right anodyne for every simplex $b : \Delta[n] \to B$. A

Grothendieck opfibration is proper. A map of simplicial sets $u : A \to B$ induces a pair of adjoint functors

$$u^* : \mathbf{S}/B \leftrightarrow \mathbf{S}/A : u_*.$$

When u is proper, it is a Quillen adjunction with respect to the contravariant model structure on these categories. We thus obtain a derived adjunction

$$\mathcal{P}^*(u) : \mathcal{P}(B) \leftrightarrow \mathcal{P}(A) : \mathcal{P}_*(u),$$

where $\mathcal{P}^*(u)$ is both the left and the right derived functor of the functor u^*, and where $\mathcal{P}_*(u)$ is the right derived functor of u_*. The base change of a proper map $u : A \to B$ along any map $v : B' \to B$ is a proper map $u : A' \to B'$,

$$\begin{array}{ccc} A' & \xrightarrow{v'} & A \\ {\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle u} \\ B' & \xrightarrow{v} & B. \end{array}$$

Moreover, the *Beck-Chevalley law holds*. This means that the following square of functors commutes up to a canonical isomorphism,

$$\begin{array}{ccc} \mathcal{P}(A') & \xleftarrow{\mathcal{P}^*(v')} & \mathcal{P}(A) \\ {\scriptstyle \mathcal{P}_*(u')}\downarrow & & \downarrow{\scriptstyle \mathcal{P}_*(u)} \\ \mathcal{P}(B') & \xleftarrow{\mathcal{P}^*(v)} & \mathcal{P}(B). \end{array}$$

Dually, a map of simplicial sets $u : A \to B$ is said to be *smooth* if the opposite map u^o is proper. A Grothendieck fibration is smooth.

The right derived functor

$$\mathcal{P}^*(u) : \mathcal{P}(B) \to \mathcal{P}(A)$$

has a right adjoint for any map of simplicial sets $u : A \to B$. To see this, it suffices by Morita equivalence to consider the case where u is a map between quasi-categories. The result is obvious in the cases where u is proper and where u has a right adjoint. The general case follows by factoring u as a left adjoint followed by a Grothendieck opfibration.

The quasi-category U

Recall that the *quasi-category of homotopy types* $\mathbf{U}$ is defined to be the coherent nerve of the category **Kan**. An object of the quasi-category $\mathbf{U}' = 1\backslash\mathbf{U}$ is a *pointed homotopy type*. The canonical map $q : \mathbf{U}' \to \mathbf{U}$ is a universal left fibration. he universality means that the functor $\mathbf{S}^o \to \mathbf{CAT}$ which associate to a simplicial set B the homotopy category $Q(B)$ of left fibrations with target B is "representable" by the object $\mathbf{U}' \in Q(\mathbf{U})$ (it is not truly representable, since the quasi-category $\mathbf{U}$ fails to be small). Concretely, the universality means that for any left fibration $E \to B$ there exists a homotopy pullback square

$$\begin{array}{ccc} E & \xrightarrow{f_0} & \mathbf{U}' \\ \downarrow & & \downarrow \\ B & \xrightarrow{f_1} & \mathbf{U} \end{array}$$

and moreover that the pair (f_0, f_1) is unique up to a unique invertible 2-cell in a certain 2-category of maps. The *simplicial set of elements* $E(f)$ of a map $f : A \to \mathbf{U}$ is defined by putting $E(f) = f^*(\mathbf{U}')$. If e_A is the evaluation map $A \times \mathbf{U}^A \to \mathbf{U}$, then the left fibration L_A defined by the pullback square

$$\begin{array}{ccc} L_A & \longrightarrow & \mathbf{U}' \\ \downarrow & & \downarrow \\ A \times \mathbf{U}^A & \xrightarrow{e_A} & \mathbf{U} \end{array}$$

"represents" the functor $\mathbf{S}^o \to \mathbf{CAT}$ which associates to a simplicial set B the homotopy category $Q(A \times B)$ of left fibrations with target $A \times B$. When $B = 1$, this gives an equivalence of categories

$$ho\ \mathbf{U}^A \simeq \mathcal{Q}(A).$$

A *prestack* on a simplicial set A is defined to be a map $A^o \to \mathbf{U}$. The prestacks form a cartesian closed quasi-category

$$\mathbf{P}(A) = \mathbf{U}^{A^o} = [A^o, \mathbf{U}].$$

The simplicial set of elements $E^o(f)$ of a prestack $f : A^o \to \mathbf{U}$ is defined by putting $E^o(f) = E(f)^o$. The canonical map $E^o(f) \to A$ is a right fibration. The left fibration $L_{A^o} \to A^o \times \mathbf{P}(A)$ defined above "represents" the functor $\mathbf{S}^o \to \mathbf{CAT}$ which associates to a simplicial set B the homotopy category $Q(A^o \times B)$ of left fibrations with target $A^o \times B$. When $B = 1$, this gives an equivalence of categories

$$ho\ \mathbf{P}(A) \simeq \mathcal{P}(A).$$

Yoneda Lemma

The *twisted category of arrows* $\theta(C)$ of a category C is the category of elements of the hom functor $C^o \times C \to \mathbf{Set}$. The *twisted quasi-category of arrows* of a quasi-category A is defined by putting $\theta(A) = a^*(A)$, where $a : \Delta \to \Delta$ is the functor obtained by putting $a([n]) = [n]^o \star [n]$ for every $n \geq 0$. The canonical map $(s,t) : \theta(A) \to A^o \times A$ is a left fibration; it is thus classified by a map

$$hom_A : A^o \times A \to \mathbf{U}.$$

This defines the *Yoneda map*

$$y_A : A \to \mathbf{P}(A)$$

by adjointness. We say that a prestack on A is *representable* if it belongs to the essential image of y_A. The map $hom_X : X^o \times X \to \mathbf{U}$ can be defined for any locally small quasi-category X; if A is a simplicial set and $f : A \to X$, then by composing the maps

$$A^o \times X \xrightarrow{f^o \times X} X^o \times X \xrightarrow{hom_X} \mathbf{U}$$

we obtain a map $A^o \times X \to \mathbf{U}$, hence also a map $f^! : X \to \mathbf{P}(A)$ by adjointness. One form of the *Yoneda lemma* says that the map $f^!$ is the identity of $\mathbf{P}(A)$ when f is the Yoneda map $y_A : A \to \mathbf{P}(A)$. It implies that for any $f \in \mathbf{P}(A)$ we have a homotopy pullback square

$$\begin{array}{ccc} E^o(f) & \longrightarrow & \mathbf{P}(A)/f \\ \downarrow & & \downarrow \\ A & \xrightarrow{y_A} & \mathbf{P}(A). \end{array}$$

If X is a locally small quasi-category and A is a simplicial set, we say that a map $f : A \to X$ is *dense* if the map $f^! : X \to \mathbf{P}(A)$ is fully faithful. For example, let $i : \Delta \to \mathbf{U}_1$ be the map obtained by applying the coherent nerve functor to the inclusion $\Delta \to \mathbf{QCat}$. It can be proved that the map

$$i^! : \mathbf{U}_1 \to \mathbf{P}(\Delta)$$

is fully faithful. This means that the map i is dense.

The quasi-category $\mathbf{U}$ is cocomplete, and it is freely generated by the object $1 \in \mathbf{U}$ as a cocomplete quasi-category. More generally, if A is a simplicial set, then the quasi-category $\mathbf{P}(A)$ is cocomplete and freely generated by the Yoneda map $y_A : A \to \mathbf{P}(A)$. If X is a cocomplete locally small quasi-category, then the left Kan extension $f_! : \mathbf{P}(A) \to X$ of a map $f : A \to X$ is left adjoint to the map $f^! : X \to \mathbf{P}(A)$.

Factorisation systems in a quasi-category

There is a notion of factorisation system in a quasi-category. Let us define the orthogonality relation $u \perp f$ between the arrows of a quasi-category X. If $u : a \to b$ and $f : x \to y$ is a pair of arrows in X, then a commutative square

$$\begin{array}{ccc} a & \longrightarrow & x \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle f} \\ b & \longrightarrow & y \end{array}$$

is a map $s : I \times I \to X$ such that $s|\{0\} \times I = u$ and $s|\{1\} \times I = f$. A *diagonal filler* for s is a map $I \star I \to X$ which extends s along the inclusion $I \times I \subset I \star I$. Let us denote by $Fill(s)$ the fiber at s of the projection $q : X^{I \star I} \to X^{I \times I}$ defined by the inclusion $I \times I \subset I \star I$. The simplicial set $Fill(s)$ is a Kan complex, since q is a Kan fibration. We shall say that the arrow u is *left orthogonal* to f, or that f is *right orthogonal* to u, and we shall write $u \perp f$, if the simplicial set $Fill(s)$ is contractible for every commuative square s such that $s|\{0\} \times I = u$ and $s|\{1\} \times I = f$.

We say that an object x in a quasi-category X is *local* with respect to an arrow $u : a \to b$, and we write $u \perp x$, if the map

$$hom_X(u, x) : hom_X(b, x) \to hom_X(a, x)$$

is invertible. When$X$ has a terminal object 1, an object x is local with respect to an arrow $u : a \to b$ iff the arrow $x \to 1$ is right orthogonal to u.

If $h : X \to hoX$ is the canonical map, then the relation $u \perp f$ between the arrows of X implies the relation $h(u) \perp h(f)$ in hoX, but *the converse is not necessarly true*. However, the relation $u \perp f$ only depends on the homotopy classes of u and f. If A and B are two sets of arrows in X, we shall write $A \perp B$ to indicate the we have $u \perp f$ for every $u \in A$ and $f \in B$. We shall put

$$A^{\perp} = \{f \in X_1 : \forall u \in A,\ \ i \perp f\}, \qquad {}^{\perp}A = \{u \in X_1 : \forall f \in A,\ \ u \perp f\}.$$

The set $A^{\perp}$ contains the isomorphisms, has the left cancellation property, and it is closed under composition and retracts. It is also closed under the base changes which exists.

Let X be a (large or small) quasi-category. We say that a pair (A, B) of class of arrows in X is a *factorisation system* if the following two conditions are satisfied:

- $A^{\perp} = B$ and $A = {}^{\perp}B$;
- every arrow $f \in X$ admits a factorisation $f = pu$ (in hoX) with $u \in A$ and $p \in B$.

We say that A is the *left class* and that B is the *right class* of the factorisation system.

If X is a quasi-category, then the image by the canonical map $h : X \to hoX$ of a factorisation system (A, B) is a *weak* factorisation system $(h(A), h(B))$ on the category hoX. Moreover, we have $A = h^{-1}h(A)$ and $B = h^{-1}h(B)$. Conversely, if (C, D) is a weak factorisation system on the category $ho(X)$, then the pair $(h^{-1}(C), h^{-1}(D))$ is a factorisation system in X iff we have $h^{-1}(C) \perp h^{-1}(D)$.

The intersection $A \cap B$ of the classes of a factorisation system (A, B) on a quasi-category X is the class of isomorphisms in X. The class A of a factorisation system (A, B) has the right cancellation property and the class B the left cancellation property. Each class is closed under composition and retracts. The class A is closed under the cobase changes which exist. and the class B under the base changes which exist.

Let (A, B) be a factorisation system in a quasi-category X. Then the full sub-quasi-category of X^I spanned by the elements in B is reflective. Hence this sub-quasi-category is closed under limits. Dually, the full sub=quasi-category of X^I spanned by the elements in A is coreflective.

If $p : X \to Y$ is a left or a right fibration between quasi-categories and (A, B) is a factorisation system on Y, then the pair $(p^{-1}(A), p^{-1}(B))$ is a factorisation system on X. We say that it is obtained by *lifting* the system (A, B) along p. In particular, every factorisation system on X can lifted to X/b (resp. $b\backslash X$) for any vertex $b \in X$.

Let $p : \mathcal{E} \to L(\mathcal{E})$ be the localisation of a model category with respect the class of weak equivalences. If (A, B) is a factorisation system in $L(\mathcal{E})$, then the pair $(p^{-1}(A), p^{-1}(B)$ is a homotopy factorisation system in $\mathcal{E}$, and this defines a bijection between the factorisation systems in $L(\mathcal{E})$ and the homotopy factorisation systems in $\mathcal{E}$.

We say that a factorisation system (A, B) in a quasi-category with products X is *closed under products* if the class A is closed under products (as a class of objects in X^I). The notion of a factorisation *closed under finite products* in a quasi-category with finite products X is defined similarly. When X has pullbacks, we say that a factorisation system (A, B) is *stable under base changes* if the class A is closed under base changes, in other words, if the implication $f \in A \Rightarrow f' \in A$ is true for any pullback square

$$\begin{array}{ccc} x' & \longrightarrow & x \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ y' & \longrightarrow & y. \end{array}$$

We say that an arrow $u : a \to b$ in a quasi-category X is a *monomorphism* or that it is *monic* if the commutative square

$$\begin{array}{ccc} a & \xrightarrow{1_a} & a \\ {\scriptstyle 1_a}\downarrow & & \downarrow{\scriptstyle u} \\ a & \xrightarrow{u} & b \end{array}$$

is cartesian. A monomorphism in X is monic in hoX but the converse is not necessarily true. A map between Kan complexes $u : A \to B$ is monic in $\mathbf{U}$ iff it is homotopy monic.

We say that an arrow in a cartesian quasi-category X is *surjective*, or that is a *surjection* , if it is left orthogonal to every monomorphism of X. Wel say that a cartesian quasi-category X admits *surjection-mono factorisations* if every arrow $f \in X$ admits a factorisation $f = up$, with u a monomorphism and p a surjection. In this case X admits a factorisation system (A, B), with A the set of surjections and B the set of monomorphisms. If quasi-category X admits surjection-mono factorisations, then so do the quasi-categories $b\backslash X$ and X/b for every vertex $b \in X$, and the quasi-category X^S for every simplicial set S.

We say that a cartesian quasi-category X is *regular* if it admits surjection-mono factorisations stable under base changes. The quasi-category $\mathbf{U}$ is regular. If a quasi-category X is regular then so are the quasi-categories $b\backslash X$ and X/b for any vertex $b \in X$ and the quasi-category X^A for any simplicial set A.

Recall that a simplicial set A is said to be a *0-object* if the canonical map $A \to \pi_0(A)$ is a weak homotopy equivalence, If X is a quasi-category, we shall say that an object $a \in X$ is *discrete* or that it is a 0-*object* if the simplicial set $X(x, a)$ is a 0-object for every node $x \in X$. When the square $a \times a$ exists, an object $a \in X$ is a 0-object iff the diagonal $a \to a \times a$ is monic. When the exponential a^{S^1} exists, an object $a \in X$ is a 0-object iff the projection $a^{S^1} \to a$ is quasi-invertible. We shall say that an arrow $u : a \to b$ is a *0-cover* if it is a 0-object of the slice quasi-category X/b. An arrow $u : a \to b$ is a 0-cover iff the map $X(x, u) : X(x, a) \to X(x, b)$ is a 0-cover for every node $x \in X$. We shall say that an arrow $u : a \to b$ in X is *0-connected* if it is left orthogonal to every 0-cover in X. We shall say that a quasi-category X admits *0-factorisations* if every arrow $f \in X$ admits a factorisation $f = pu$ with u a 0-connected arrow and p a 0-cover. In this case X admits a factorisation system (A, B) with A the set of 0-connected maps and B the set of 0-covers. The quasi-category $\mathbf{U}$ admits 0-factorisations and they are stable under base changes. If a quasi-category X admits 0-factorisations, then so do the quasi-categories $b\backslash X$ and X/b for every vertex $b \in X$, and the quasi-category X^S for every simplicial set S.

There is a notion of n-cover and of n-connected arrow in every quasi-category for every $n \geq -1$. If X is a quasi-category, we shall say that a vertex $a \in X$ is a *n-object* if the simplicial set $X(x,a)$ is a n-object for every vertex $x \in X$. See [Bi2] for the homotopy theory of n-objects. If $n = -1$, this means that $X(x,a)$ is contractible or empty. When the exponential $a^{S^{n+1}}$ exists, the vertex a is a n-object iff the projection $a^{S^{n+1}} \to a$ is quasi-invertible. We shall say that an arrow $u : a \to b$ is a *n-cover* if it is a n-object of the slice quasi-category X/b. If $n \geq 0$ and the product $a \times a$ exists, the vertex a is a n-object iff the diagonal $a \to a \times a$ is a $(n-1)$-cover. We shall say that an arrow in a quasi-category X is *n-connected* if it is left orthogonal to every n-cover. We shall say that a quasi-category X admits *n-factorisations* if every arrow $f \in X$ admits a factorisation $f = pu$ with u a n-connected map and p a n-cover. In this case X admits a factorisation system (A, B) with A the set of n-connected maps and B the class of n-covers. If $n = -1$, (A, B) is the surjection-mono factorisation system. The quasi-category $\mathbf{U}$ admits n-factorisations and they are stable under base changes. If a quasi-category X admits n-factorisations, then so do the quasi-categories $b\backslash X$ and X/b for every vertex $b \in X$, and the quasi-category X^S for every simplicial set S.

Suppose that a quasi-category X admits k-factorisations for every $-1 \leq k \leq n$. Then we have a sequence of inclusions

$$A_{-1} \supseteq A_0 \supseteq A_1 \supseteq A_2 \supseteq A_3 \cdots \supseteq A_n$$

$$B_{-1} \subseteq B_0 \subseteq B_1 \subseteq B_2 \subseteq B_3 \cdots \subseteq B_n,$$

where (A_k, B_k) denotes the k-factorisation system in X. If $n > 0$, we shall say that a n-cover $f : x \to y$ is an *Eilenberg-MacLane n-gerb* if f is $(n-1)$-connected. A *Postnikov tower* (of height n) for an arrow $f : a \to b$ is defined to be a factorisation of length $n+1$ of f

$$a \xleftarrow{q_0} x_0 \xleftarrow{p_1} x_1 \xleftarrow{p_2} \cdots \xleftarrow{p_n} x_n \xleftarrow{q_n} b,$$

where q_0 is a 0-cover, where p_k is an EM k-gerb for $1 \leq k \leq n$ and where q_n is n-connected. The tower could be augmented by further factoring q_0 as a surjection $p_0 : x_0 \to x_{-1}$ followed by a monomorphism $x_{-1} \to a$. Every arrow in X admits a Postnikov tower of height n and this tower is unique up to a unique isomorphism in the homotopy category of the quasi-category of towers.

We say that a factorisation system (A, B) in a quasi-category X is *generated* by a set Σ of arrows in X if we have $B = \Sigma^{\perp}$. Let X be a cartesian closed quasi-category. We shall say that a factorisation system (A, B) in X is *multiplicatively generated* by a set of arrows Σ if it is generated by the set

$$\Sigma' = \bigcup_{a \in X_0} a \times \Sigma.$$

Most factorisation systems of interest are multiplicatively generated. For example, in the quasi-category $\mathbf{U}$, the surjection-mono factorisation system is multiplicatively generated by the map $S^0 \to 1$. More generally, the n-factorisation system is multiplicatively generated by the map $S^{n+1} \to 1$. In the quasi-category $\mathbf{U}_2$, the system of essentially surjective maps and fully faithful maps is multiplicatively generated by the inclusion $\partial I \subset I$. The system of final maps and right fibrations described is multiplicatively generated by the inclusion $\{1\} \subset I$. The dual system of initial maps and left fibrations is multiplicatively generated by the inclusion $\{0\} \subset I$. The system of localisations and conservative maps is multiplicatively generated by the map $I \to 1$ (or by the inclusion $I \subset J$, where J is the groupoid generated by one isomorphism $0 \to 1$). The system of weak homotopy equivalences and Kan fibrations is multiplicatively generated by the pair of inclusions $\{0\} \subset I$ and $\{1\} \subset I$.

Distributors, cylinders and spans

The purpose of the theory of distributors is to give a representations of the cocontinuous maps $\mathbf{P}(A) \to \mathbf{P}(B)$, where A and B are simplicial sets. A cocontinuous map $\mathbf{P}(A) \to \mathbf{P}(B)$ is determined by its composite with the Yoneda map $y_A : A \to \mathbf{P}(A)$, since $\mathbf{P}(A)$ is freely generated by y_A as a cocomplete quasi-category. But a map $A \to \mathbf{P}(B) = \mathbf{U}^{B^o}$ is the same thing as a map $B^o \times A \to \mathbf{U}$. Hence the quasi-category of cocontinuous maps $\mathbf{P}(A) \to \mathbf{P}(B)$ is equivalent to the quasi-category $\mathbf{U}^{B^o \times A}$. The quasi-category $\mathbf{U}^{B^o \times A}$ is the homotopy localisation of the model category $(\mathbf{S}/(B^o \times A, \mathbf{L}(B^o \times A))$ A *distributor* $B \Rightarrow A$ is defined to be an object D of the category $\mathbf{S}/B^o \times A$; the distributor is *fibrant* if its structure map $D \to B^o \times A$ is a left fibration. Every cocontinuous map $\mathbf{P}(A) \to \mathbf{P}(B)$ can be represented by a fibrant distributor $X \to B^o \times A$.

A *cylinder* is defined to be a simplicial set C equipped with a map $p : C \to I$. The *base* of a cylinder $p : C \to I$ is the simplicial set $C(1) = p^{-1}(1)$ and its *cobase* is the simplicial set $C(0) = p^{-1}(0)$. For example the join $A \star B$ of two simplicial sets has the structure of a cylinder with base B and cobase A. Every cylinder C with base B and cobase A is equipped with a pair of maps $A \sqcup B \to C \to A \star B$ which factors the inclusion $A \sqcup B \subseteq A \star B$. The category $\mathbf{C}(A, B)$ of cylinders with base B and cobase A is a full subcategory of $\mathbf{S}/A \star B$. The model structure for quasi-categories induces a model structure on $\mathbf{C}(A, B)$ for any pair of simplcial sets A and B. A cylinder $X \in \mathbf{C}(A, B)$ is fibrant for this model structure iff the canonical map $X \to A \star B$ is a mid fibration. The simplicial set $\Delta[n]^o \star \Delta[n]$ has the structure of a cylinder for every $n \geq 0$. The *anti-diagonal* of a cylinder C is the simplicial set $a^*(C)$ obtained by putting

$$a^*(C)_n = Hom_I(\Delta[n]^o \star \Delta[n], C)$$

for every $n \geq 0$. The simplicial set $a^*(C)$ has the structure of a distributor $C(0) \Rightarrow C(1)$. The resulting functor $a^* : \mathbf{C}(A, B) \to \mathbf{S}/A^o \times B$ has a left adjoint $a_!$ and the pair $(a_!, a^*)$ is a Quillen equivalence between the model category $\mathbf{C}(A, B)$ and the model category $(\mathbf{S}/(B^o \times A, \mathbf{L}(B^o \times A))$. The cocontinuous map $\mathbf{P}(A) \to \mathbf{P}(B)$

associated to a cylinder $C \in \mathbf{C}(B,A)$ is the map $i_A^*(i_B)_! : \mathbf{P}(B) \to \mathbf{P}(A)$, where $(i_A, i_B) : B \sqcup A \to C$ is the inclusion.

A *span* $A \Rightarrow B$ between two simplicial sets is defined to be a map $(s,t) : S \to A \times B$. The spans $A \Rightarrow B$ form a category $Span(A,B) = \mathbf{S}/A \times B$. The *realisation* of a span $(s,t) : S \in Span(A,B)$ is the simplicial set $\sigma^*(S)$ defined by the pushout square of canonical maps,

$$\begin{array}{ccc} S \sqcup S & \xrightarrow{s \sqcup t} & A \sqcup B \\ \downarrow & & \downarrow \\ I \times S & \longrightarrow & \sigma^*(S). \end{array}$$

The simplicial set $\sigma^*(S)$ has the structure of a cylinder in $\mathbf{C}(A,B)$. The resulting functor

$$\sigma^* : Span(A,B) \to \mathbf{C}(A,B)$$

has a a right adjoint σ_*. We call a map $u : S \to T$ in $Span(A,B)$ a *bivalence* if the map $\sigma^*(u) : \sigma^*(S) \to \sigma^*(T)$ is a weak categorical equivalence. The category $Span(A,B)$ admits a Cisinski model structure in which a weak equivalence is a bivalence. The pair of adjoint functors (σ^*, σ_*) is a Quillen equivalence between the model categories $Span(A,B)$ and $\mathbf{C}(A,B)$. The simplicial set $\Delta[n] \star \Delta[n]$ has the structure of a cylinder for every $n \geq 0$. The *diagonal* of a cylinder C is the simplicial set $\delta^*(C)$ obtained by putting

$$\delta^*(C)_n = Hom_I(\Delta[n] \star \Delta[n], C)$$

for every $n \geq 0$. The simplicial set $\delta^*(C)$ has the structure of a span $C(0) \Rightarrow C(1)$. The resulting functor $\delta^* : \mathbf{C}(A,B) \to Span(A,B)$ has a right adjoint δ_* and the pair (δ^*, δ_*) is a Quillen equivalence between model categories. The cocontinuous map $\mathbf{P}(A) \to \mathbf{P}(B)$ associated to a span $S \in Span(B,A)$ is the map $(p_B)_! p_A^* : \mathbf{P}(A) \to \mathbf{P}(B)$, where $(p_B, p_A) : S \to B \times A$ is the structure map.

Limit sketches

The notion of limit sketch was introduced by Ehresmann [Eh]. A structure which can be defined by a limit sketch is said to be *essentially algebraic* by Gabriel and Ulmer [GU]. Recall that a *projective cone* in a simplicial set A is a map of simplicial sets $1 \star K \to A$. A *limit sketch* is a pair (A, P), where A is a simplicial set and P is a set of projective cones in A. The sketch is *finitary* if every cone in P is finite. A *model* of the sketch with values in a quasi-category X is a map $f : A \to X$ which takes every cone $c : 1 \star K \to A$ in P to an exact cone $fc : 1 \star K \to X$. We write $f : A/P \to X$ to indicate that a map $f : A \to X$ is a model of (A,P). A model $A/P \to \mathbf{U}$ is called a *homotopy model*, or just a *model* if the context is clear. A model $A/P \to \mathbf{Set}$ is said to be *discrete*. We say that an essentially algebraic structure is *finitary* if it can be defined by a finitary limit sketch. The notion of *stack* on a fixed topological space is essentially algebraic, but it is not finitary in general. The models of (A, P) with values in a quasi-category X form a quasi-category $Mod(A/P, X)$; by definition, it is the full simplicial subset of X^A spanned by the models $A/P \to X$. We shall write

$$Mod(A/P) = Mod(A/P, \mathbf{U}).$$

The quasi-category $Mod(A/P)$ is bicomplete and the inclusion $Mod(A/P) \subseteq \mathbf{U}^A$ has a left adjoint.

Recall that a quasi-category with finite limits is said to be *cartesian*. A *cartesian theory* is defined to be a small cartesian quasi-category T. A *model* of T with values in a quasi-category X is a map $f : T \to X$ which preserves finite limits (also called a left exact map). We also say that a model $T \to X$ with values in a quasi-category X is an *interpretation* of T into X. The identity morphism $T \to T$ is the *generic model* of T. The models of $T \to X$ form a quasi-category $Mod(T, X)$, also denoted $T(X)$. By definition, it is the full simplicial subset of X^T spanned by the models $T \to X$. We say that a model $T \to \mathbf{U}$ is a *homotopy model*, or just a *model* if the context is clear. We say that a model $T \to \mathbf{Set}$ is *discrete*. We shall write

$$Mod(T) = Mod(T, \mathbf{U}).$$

The quasi-category $Mod(T)$ is bicomplete and the inclusion $Mod(T) \subseteq \mathbf{U}^T$ has a left adjoint.

Every finitary limit sketch (A, P) has a universal model $u : A \to T(A/P)$ with values in a cartesian theory $T(A/P)$. The universality means that the map

$$u^* : Mod(T(A/P), X) \to Mod(A/P, X)$$

induced by u is an equivalence for any cartesian quasi-category X. We say that $T(A/P$ is the cartesian theory *generated* by the sketch (A, P).

A *morphism* $S \to T$ of cartesian theories is a left exact map. (ie a model $S \to T$). We shall denote by $\mathbf{CT}$ the category of cartesian theories and morphisms. The category $\mathbf{CT}$ has the structure of a 2-category induced by the 2-category structure of the category of simplicial sets. If $u : S \to T$ is a morphism of theories, then the map

$$u^* : Mod(T) \to Mod(S)$$

induced by u has a left adjoint $u_!$. The adjoint pair $(u_!, u^*)$ an equivalence iff the map $u : S \to T$ is a Morita equivalence.

If S and T are two cartesian theories then so is the quasi-category $Mod(S, T)$ of models $S \to T$. The (2-)category $\mathbf{CT}$ is symmetric monoidal closed. The *tensor product* $S \odot T$ of S and T is the target of a map $S \times T \to S \odot T$ left exact in each variable and universal with respect to that property. The unit object for the tensor product is the cartesian category freely generated by one object. For any cartesian quasi-category X, large or small, we have an equivalence of quasi-categories,

$$Mod(S \odot T, X) \simeq Mod(S, Mod(T, X))$$

In particular, we have two equivalences,

$$Mod(S \odot T) \simeq Mod(S, Mod(T)) \simeq Mod(T, Mod(S)).$$

The notion of *spectrum* or *stable object* is essentially algebraic and finitary. By definition, a stable object in a cartesian quasi-category X is an infinite sequence of pointed objects (x_n) together with an infinite sequence of isomorphisms

$$u_n : x_n \to \Omega(x_{n+1}).$$

This shows that the notion of stable objects is defined by a finitary limit sketch (A, P). The theory $T(A/P)$ is the (cartesian) *theory of spectra Spec* We denote by *Spec*(X) the quasi-category of stable objects in a cartesian quasi-category X.

The notion of monomorphism between two objects of a quasi-category is essentially algebraic (and finitary): an arrow $a \to b$ is monic iff the square

$$\begin{array}{ccc} a & \xrightarrow{1_a} & a \\ {\scriptstyle 1_a}\downarrow & & \downarrow{\scriptstyle u} \\ a & \xrightarrow{u} & b \end{array}$$

is cartesian. The notion of (homotopy) discrete object is essentially algebraic: an object a is discrete iff the diagonal $a \to a \times a$ is monic. The condition is expressed by two exact cones,

$$\begin{array}{ccc} & b & \\ {\scriptstyle p_1}\swarrow & & \searrow{\scriptstyle p_2} \\ a & & a, \end{array} \qquad \begin{array}{ccc} a & \xrightarrow{1_a} & a \\ {\scriptstyle 1_a}\downarrow & & \downarrow{\scriptstyle d} \\ a & \xrightarrow{d} & b. \end{array}$$

and two relations $pd = qd = 1_a$. The notion of 0-cover is also essentially algebraic, since an arrow $a \to b$ is a 0-cover iff the diagonal $a \to a \times_b a$ is monic. It follows that the notion of 1-object is essentially algebraic, since an object a is a 1-object iff its diagonal $a \to a \times a$ is a 0-cover. It is easy to see by induction on n that the notions of n-object and of n-cover are essentially algebraic for every $n \geq 0$. We denote by $OB(n)$. the cartesian theory of n-objects. We have

$$Mod(OB(n)) = \mathbf{U}[n],$$

where $\mathbf{U}[n]$ is the quasi-category of n-objects in $\mathbf{U}$. In particular, the quasi-category $Mod(OB(0))$ is equivalent to the category **Set**. If T is a cartesian theory, then a model of the theory $T \odot OB(n)$ is a model of T in $\mathbf{U}[n]$. In particular, $T \odot OB(0)$ is the theory of discrete models of T.

The notion of category object is essentially algebraic and finitary. If X is a cartesian category, a *simplicial object* $C : \Delta^o \to X$ is said to be a *category* if it satisfies the *Segal condition*. This condition can be expressed in many ways, for example by demanding that C takes every pushout square of the form

$$\begin{array}{ccc} [0] & \xrightarrow{\;0\;} & [n] \\ {\scriptstyle m}\downarrow & & \downarrow \\ [m] & \longrightarrow & [m+n], \end{array}$$

to a pullback square in X. If $C : \Delta^o \to X$ is a category object, we say that $C_0 \in X$ is the *object of objects* of C and that C_1 is the *object of arrows*. The *source morphism* $s : C_1 \to C_0$ is the image of the arrow $d_1 : [0] \to [1]$, the *target morphism* $t : C_1 \to C_0$ is the image of $d_0 : [0] \to [1]$, the *unit morphism* $u : C_0 \to C_1$ is the image of $s_0 : [1] \to [0]$. and the *multiplication* $C_2 \to C_1$ is image of $d_1 : [1] \to [2]$. If Q is the set of pushout squares which express the Segal condition, then the pair (Δ^o, Q^o) is a finitary limit sketch. The (cartesian) *theory of categories* Cat is defined to be the cartesian theory $T((\Delta^o/Q^o)$ We denote the quasi-category of category objects in a cartesian quasi-category X by $Cat(X)$.

The notion of groupoid object is essentially algebraic and finitary. By definition, a category object $C : \Delta^o \to X$ is said to be a *groupoid* if it takes the squares (but one is enough)

$$\begin{array}{ccc} [0] & \xrightarrow{d_0} & [1] \\ {\scriptstyle d_0}\downarrow & & \downarrow{\scriptstyle d_0} \\ [1] & \xrightarrow{d_1} & [2], \end{array} \qquad \begin{array}{ccc} [0] & \xrightarrow{d_1} & [1] \\ {\scriptstyle d_1}\downarrow & & \downarrow{\scriptstyle d_2} \\ [1] & \xrightarrow{d_1} & [2] \end{array}$$

to pullback squares,

$$\begin{array}{ccc} C_2 & \xrightarrow{m} & C_1 \\ {\scriptstyle \partial_0}\downarrow & & \downarrow{\scriptstyle t} \\ C_1 & \xrightarrow{t} & C_0, \end{array} \qquad \begin{array}{ccc} C_2 & \xrightarrow{m} & C_1 \\ {\scriptstyle \partial_2}\downarrow & & \downarrow{\scriptstyle s} \\ C_1 & \xrightarrow{s} & C_0. \end{array}$$

We denote the (cartesian) theory of groupoids by *Gpd* and the quasi-category of groupoid objects in a cartesian quasi-category X by $Gpd(X)$. The inclusion $Gpd(X) \subseteq Cat(X)$ has a right adjoint which associates to a category $C \in Cat(X)$ its groupoid of isomorphisms $J(C)$.

If X is a quasi-category, we say that a diagram $d : A \to X$ is *essentially constant* if it belongs to the essential image of the diagonal $X \to X^A$. A simplicial object $C : \Delta^o \to X$ is essentially constant iff the map C inverts every arrow. If the quasi-category X is cartesian, then a category object $C : \Delta^o \to X$ is essentially constant iff the unit morphism $C_0 \to C_1$ is invertible. We say that a category C satisfies the *Rezk condition*, or that it is *reduced*, if the groupoid $J(C)$ is essentially constant. The notion of a reduced category is essentially algebraic and finitary. We denote the cartesian theory of reduced categories by $RCat$ and the quasi-category of reduced category objects in a cartesian quasi-category X by $RCat(X)$.

Let $\mathbf{U}_1$ be the quasi-categories of small quasi-categories and let $i : \Delta \to \mathbf{U}_1$ be the map obtained by applying the coherent nerve functor to the inclusion $\Delta \to \mathbf{QCat}$. It follows from [JT2] that the map $i^! : \mathbf{U}_1 \to \mathbf{P}(\Delta)$ is fully faithful and that its essential image is the subcategory $Mod(RCat) \subset \mathbf{P}(\Delta)$. It thus induces an equivalence of quasi-categories

$$\mathbf{U}_1 \simeq Mod(RCat).$$

This means that quasi-category is a reduced category.

If T is a cartesian theory and b is an object of a cartesian quasi-category X, we say that a model $T \to X/b$ is a *parametrized model* or a *based model* of T in X; the object b is the *parameter space* or the *base* of the model. For any algebraic theory T, there is another algebraic theory T' whose models are the parametrized models of T. A model $T' \to \mathbf{U}$ is essentially the same thing as a model $T \to \mathbf{U}/K$ or a model $\mathbf{U}^K$ for some Kan complex K. It is a *Kan diagram* of models of T.

Locally presentable quasi-categories

The theory of locally presentable categories of Gabriel and Ulmer [GU] can be extended to quasi-categories. See [Lu1] for a different approach and a more complete treatment.

Recall that an *inductive cone* in a simplicial set A is a map of simplicial sets $K \star 1 \to A$. A *colimit sketch* is a pair (A, Q), where A is a simplicial set and Q is a set of inductive cones in A. A *model* of the sketch with values in a quasi-category X is a map $f : A \to X$ which takes every cone $c : K \star 1 \to A$ in Q to a coexact cone $fc : K \star 1 \to X$ in X. We shall write $f : Q\backslash A \to X$ to indicate that a map $f : A \to X$ is a model of (A, Q). The models of (A, Q) with values in a quasi-category X form a quasi-category $Mod(Q\backslash A, X)$. By definition, it is the full simplicial subset of X^A spanned by the models $Q\backslash A \to X$. Every colimit sketch (A, Q) has a universal model $u : A \to U(Q\backslash A)$ with values in a (locally small) cocomplete quasi-category $U(Q\backslash A)$. We say that a quasi-category X is *locally presentable* is if it is equivalent to a quasi-category $U(Q\backslash A)$ for some colimit sketch (A, Q). The universal model $Q\backslash A \to X$ is a *presentation* of X

by (A,Q). Every locally presentable quasi-category is bicomplete. If X is locally presentable, then so are the slice quasi-categories $a\backslash X$ and X/a for any object $a \in X$ and the quasi-category X^A for any simplicial set A. More generally, the quasi-category $Mod(A/P,X)$ is locally presentable for any limit sketch (A,P). The quasi-category $\mathbf{U}$ is locally presentable but its opposite $\mathbf{U}^o$ is not.

A colimit sketch (A,Q) is said to be *finitary* if every cone in Q is finite. We say that a quasi-category X is *finitary presentable* if it is equivalent to a quasi-category $U(Q\backslash A)$ for some finitary colimit sketch (A,Q). If X is finitary presentable, then so are the slice quasi-categories $a\backslash X$ and X/a for any object $a \in X$ and the quasi-category X^A for any simplicial set A. More generally, the quasi-category $Mod(A/P,X)$ is finitary presentable for any finitary limit sketch (A,P).

The opposite of an inductive cone $c : K \star 1 \to A$ is a projective cone $c^o : 1 \star K^o \to A^o$. The opposite of a colimit sketch (A,Q) is a limit sketch (A^o,Q^o), where $Q^o = \{c^o : c \in Q\}$. If $u : A \to U(Q\backslash A)$ is the canonical map, then the map

$$\rho : U(Q\backslash A) \simeq Mod(A^o/Q^o)$$

defined by putting $\rho(x) = hom(u(-),x) : A^o \to \mathbf{U}$ for every $x \in A$ is an equivalence of quasi-categories. Hence the quasi-category $U(Q\backslash A)$ is equivalent to the quasi-category of models of the limit sketch (A^o,Q^o). Conversely, the opposite of a limit sketch (A,P) is a colimit sketch (A^o,P^o). The quasi-category $Mod(A/P)$ is equivalent to the quasi-category $U(P^o\backslash A^o)$. Hence a quasi-category is locally presentable iff it is equivalent to the quasi-category of models of a limit sketch. A quasi-category is finitary presentable if it is equivalent to a quasi-category of models of a finitary limit sketch.

If X is a locally presentable quasi-category, then every cocontinuous map $X \to Y$ with codomain a locally small cocomplete quasi-category has a right adjoint. In particular, every continuous map $X^o \to \mathbf{U}$ is representable.

If X and Y are locally presentable quasi-categories, then so is the quasi-category $Map(X,Y)$ of cocontinuous maps $X \to Y$. The 2-category $\mathbf{LP}$ of locally presentable quasi-categories and cocontinuous maps is symmetric monoidal closed. The *tensor product* $X \otimes Y$ of two locally presentable quasi-categories is the target of a map $X \times Y \to X \otimes Y$ cocontinuous in each variable and universal with respect to that property. This means that there is an equivalence of quasi-categories

$$Map(X \otimes Y, Z) \simeq Map(X, Map(Y,Z))$$

for any cocomplete quasi-category Z, The unit object for the tensor product is the quasi-category $\mathbf{U}$. There is thus a natural action $\cdot : \mathbf{U} \times X \to X$ of the quasi-category $\mathbf{U}$ on any quasi-category $X \in \mathbf{LP}$. The action associates to a pair $(A,x) \in \mathbf{U} \times X$ the colimit $A \cdot x$ of the constant diagram $A \to X$ with value x.

If A is a (small) simplicial set, then the quasi-category $\mathbf{P}(A)$ is locally presentable and freely generated by the Yoneda map $y_A : A \to \mathbf{P}(A)$. It follows that the map

$$y_A^* : Map(\mathbf{P}(A), X) \to X^A$$

is an equivalence of quasi-categories for any $X \in \mathbf{LP}$. If we compose the maps

$$A^o \times A \times X \xrightarrow{hom_A \times X} \mathbf{U} \times X \xrightarrow{\quad \cdot \quad} X$$

we obtain a map $A^o \times A \times X \to X$, hence also a map $A \times X \to X^{A^o}$; it can be extended as a map $\mathbf{P}(A) \times X \to X^{A^o}$ cocontinuous in each variable. The resulting map

$$\mathbf{P}(A) \otimes X \to X^{A^o}$$

is an equivalence of quasi-categories. It follows that the functor $X \mapsto X^{A^o}$ is left adjoint to the functor $X \mapsto Map(\mathbf{P}(A), X) = X^A$. We thus obtain an equivalence of quasi-categories

$$Map(X^{A^o}, Y) \simeq Map(X, Y^A)$$

for $X.Y \in \mathbf{LP}$.

The *external product* of a pre-stack $f \in \mathbf{P}(A)$ with a pre-stack $g \in \mathbf{P}(B)$ is defined to be the prestack $f \square g \in \mathbf{P}(A \times B)$ obtained by putting

$$(f \square g)(a, b) = f(a) \times g(b)$$

for every pair of objects $(a, b) \in A \times B$. The map $(f, g) \mapsto f \square g$ is cocontinuous in each variable and the induced map

$$\mathbf{P}(A) \otimes \mathbf{P}(B) \to \mathbf{P}(A \times B)$$

is an equivalence of quasi-categories. The *trace map*

$$Tr_A : \mathbf{P}(A^o \times A) \to \mathbf{U}$$

is defined to be the cocontinuous extension of the map $hom_A : A^o \times A \to \mathbf{U}$. The quasi-categories $\mathbf{P}(A)$ and $\mathbf{P}(A^o)$ are mutually dual as objects of the monoidal category $\mathbf{LP}$. The pairing $\mathbf{P}(A^o) \otimes \mathbf{P}(A) \to \mathbf{U}$ which defines the duality is obtained by composing the equivalence $\mathbf{P}(A^o) \otimes \mathbf{P}(A) \simeq \mathbf{P}(A^o \times A)$ with the map Tr_A.

We say that a (small) simplicial set A is *directed* if the colimit map

$$\varinjlim_A : \mathbf{U}^A \to \mathbf{U}$$

is preserves finite limits. This extends the classicial notion of a directed category. A non-empty quasi-category A is directed iff the simplicial set $d \backslash A$ is (weakly) contractible for any diagram $d : \Lambda^0[2] \to A$.

We say that a diagram $d : A \to X$ in a quasi-category X is *directed* if A is directed, in which case we shall say that the colimit of d is *directed* when it exists. We say that an object a in a quasi-category X with directed colimits is *compact* if the map

$$hom_X(a, -) : X \to \mathbf{U}$$

preserves directed colimits.

A model of a cartesian theory T is compact iff it is a retract of a representable model. The map $y : T^o \to Mod(T)$ induces an equivalence between the Karoubi envelope of T^o and the full simplicial subset of $Mod(T)$ spanned by the compact models.

A locally small cocomplete quasi-category X is finitary presentable iff it is generated by a small set of compact objects (ie every object of X is a colimit of a diagram of compact objects).

Universal algebra

Recall that an algebraic theory in the sense of Lawvere is a small category with finite products [Law1]. We can extend this notion by declaring that an *algebraic theory* is a small quasi-category T with finite products. A theory T is *discrete* if it is equivalent to a category. A *model* of a theory T with values in a quasi-category with finite products X (possibly large) is a map $f : T \to X$ which preserves finite products. We also say that a model $T \to X$ is an *interpretation* of T into X. The identity map $T \to T$ is the *generic model* of T. The models $T \to X$ form a quasi-category $Mod^{\times}(T, X)$, also denoted $T(X)$. By definition, it is the full simplicial subset of X^T spanned by the models $T \to X$. We call a model $T \to \mathbf{U}$ a *homotopy algebra* and a model $T \to \mathbf{Set}$ a *discrete algebra*. We shall put

$$Mod^{\times}(T) = Mod^{\times}(T, \mathbf{U}).$$

The quasi-category $Mod^{\times}(T)$ is bicomplete and the inclusion $Mod^{\times}(T) \subseteq \mathbf{U}^T$ has a left adjoint.

A *morphism* $S \to T$ between two algebraic theories is a map which preserves finite products (ie a model $S \to T$). We shall denote by $\mathbf{AT}$ the category of algebraic theories and morphisms. The category $\mathbf{AT}$ has the structure of a 2-category induced by the 2-category structure of the category of simplicial sets. If $u : S \to T$ is a morphism of theories, then the map

$$u^* : Mod^{\times}(T) \to Mod^{\times}(S)$$

induced by u has a left adjoint $u_!$. The adjoint pair $(u_!, u^*)$ an equivalence iff the map $u : S \to T$ is a Morita equivalence.

If T is an algebraic theory, then the Yoneda map $T^o \to \mathbf{U}^T$ induces a map $y : T^o \to Mod^\times(T)$ which preserves finite coproducts. We say that a model of T is *representable* or *finitely generated free* if it belongs to the essential image of y. The Yoneda map induces an equivalence between the opposite quasi-category T^o and the full sub-quasi-category of $Mod^\times(T)$ spanned by the finitely generated free models. We say that a model of T is *finitely presented* if it is a finite colimit of representables.

The quasi-category $Mod^\times(S, T)$ of morphisms $S \to T$ between two algebraic theories is an algebraic theory since it has finite products. This defines the internal hom of a symmetric monoidal closed structure on the 2-category **AT**. The *tensor product* $S \odot T$ of two algebraic theories is defined to be the target of a map $S \times T \to S \odot T$ which preserves finite products in each variable and which is universal with respect to that property. There is then a canonical equivalence of quasi-categories

$$Mod^\times(S \odot T, X) \simeq Mod^\times(S, Mod^\times(T, X))$$

for any quasi-category with finite product X. In particular, we have two equivalences of quasi-categories,

$$Mod^\times(S \odot T) \simeq Mod^\times(S, Mod^\times(T)) \simeq Mod^\times(T, Mod^\times(S)).$$

The unit for the tensor product is the algebraic theory O generated by one object. The opposite of the canonical map $S \times T \to S \odot T$ can be extended along the Yoneda maps as a map $Mod^\times(S) \times Mod^\times(T) \to Mod^\times(S \odot T)$ cocontinuous in each variable. The resulting cocontinuous map

$$Mod^\times(S) \otimes Mod^\times(T) \to Mod^\times(S \odot T)$$

is an equivalence of quasi-categories.

We denote by Mon the algebraic theory of monoids. By definition, Mon^o is the category of finitely generated free monoids. The theory Mon is unisorted. If $u : OB \to Mon$ is the canonical morphism, we conjecture that the two morphisms

$$u \odot Mon : Mon \to Mon \odot Mon \quad \text{and} \quad Mon \odot u : Mon \to Mon \odot Mon$$

are canonically isomorphic in two ways. We conjecture that $Mon^2 = Mon \odot Mon$ is the algebraic theory of braided monoids. For example, if $\tilde{\mathbf{Cat}}$ denotes the coherent nerve of **Cat** (viewed as a category enriched over groupoids), then the quasi-category $Mon^2(\tilde{\mathbf{Cat}})$ is equivalent to the coherent nerve of the category of braided monoidal categories. More generally, we conjecture that $Mon^n = Mon^{\odot n}$ is the algebraic theory of E_n-monoids for every $n \geq 1$. algebraic theory of E_n-monoids—textbf This means that the quasi-category $Mod^\times(Mon^n)$ is equivalent to the coherent nerve of the simplicial category of E_n-spaces. Let us put

$$u_n = u \odot Mon^n : Mon^n \to Mon^{n+1}$$

for every $n \geq 0$. We conjecture that the (homotopy) colimit of the infinite sequence ot theories

$$OB \xrightarrow{u_0} Mon \xrightarrow{u_1} Mon^2 \xrightarrow{u_2} Mon^3 \xrightarrow{u_3} \cdots$$

is the theory of E_∞-spaces.

We denote by Grp the algebraic theory of groups. By definition, Grp^o is the category of finitely generated free groups. The conjecture above implies that $Grp^n = Grp^{\odot n}$ is the algebraic theory of n-fold loop spaces for every $n \geq 1$. The theory Grp is unisorted. By tensoring with the canonical morphism $u : O \to Grp$, we obtain a morphism $u_n : Grp^n \to Grp^{n+1}$ for every $n \geq 0$. The (homotopy) colimit of the infinite sequence

$$OB \xrightarrow{u_0} Grp \xrightarrow{u_1} Grp^2 \xrightarrow{u_2} Grp^3 \xrightarrow{u_3} \cdots$$

is the algebraic theory of infinite loop spaces [BD].

Varieties of homotopy algebras

We call a quasi-category a *variety of homotopy algebras* if it is equivalent to a quasi-category $Mod^\times(T)$ for some (finitary) algebraic theory T. If X is a variety of homotopy algebras then so are the slice quasi-categories $a\backslash X$ and X/a for any object $a \in X$ and the quasi-category X^A for any simplicial set A. More generally, the quasi-category $Mod^\times(T, X)$ is a variety for any finitary algebraic theory T.

Recall that a category C is said to be *sifted*, but we shall say *0-sifted*, if the colimit functor

$$\varinjlim : \mathbf{Set}^C \to \mathbf{Set}$$

preserves finite products. This notion was introduced by C. Lair in [Lair] under the name of *categorie tamisante*. We say that a simplicial set A is *(homotopy) sifted* if the colimit map

$$\varinjlim : \mathbf{U}^A \to \mathbf{U}$$

preserves finite products. The notion of homotopy sifted category was introduced by Rosicky [Ros]. A non-empty quasi-category A is sifted iff the simplicial set $a\backslash A \times_A b\backslash A$ defined by the pullback square

$$\begin{array}{ccc} a\backslash A \times_A b\backslash A & \longrightarrow & b\backslash A \\ \downarrow & & \downarrow \\ a\backslash A & \longrightarrow & A \end{array}$$

is (weakly) contractible for any pair of objects $a, b \in A$. A category is homotopy sifted iff it is a test category in the sense of Grothendieck [Gro] (Rosicky). The category Δ^o is sifted.

If X is a quasi-category, we say that a diagram $d : A \to X$ is *sifted* if the simplicial set A is sifted, in which case the colimit of d is said to be *sifted* if it exists. A quasi-category with sifted colimits and finite coproducts is cocomplete. A map between cocomplete quasi-category is cocontinuous iff it preserves finite coproducts and sifted colimits iff it preserves directed colimits and Δ^o-indexed colimits.

Let X be a (locally small) quasi-category with sifted colimits. We say that an object a in a cocomplete (locally small) quasi-category X is *bicompact* if the map

$$hom_X(a, -) : X \to \mathbf{U}$$

preserves sifted colimits; we say that a is *adequate* if the same map preserves Δ^o-indexed colimits. An object is bicompact iff it is compact and adequate (this is a theorem). A (locally small) cocomplete quasi-category X is a homotopy variety iff it is generated by a small set of bicompact objects.

Para-varieties and descent

We call a locally presentable quasi-category X a *para-variety* if it is a left exact reflection of a variety of homotopy algebras. If X is a para-variety, then so are the slice quasi-categories $a\backslash X$ and X/a for any object $a \in X$ and the quasi-category X^A for any simplicial set A. More generally, the quasi-category $Prod(T, X)$ is a para-variety for any (finitary) algebraic theory T. If X is a para-variety, then the colimit map

$$\varinjlim_A : X^A \to X$$

preserves finite products for any sifted simplicial set A. This is true in particular if $A = \Delta^o$.

A para-variety admits surjection-mono factorisations and the factorisations are stable under base changes. More generally, it admits n-factorisations stable under base changes for every $n \geq -1$.

Let X be a cartesian quasi-category. Then the map $Ob : Gpd(X) \to X$ has a left adjoint $Sk^0 : X \to Gpd(X)$ and a right adjoint $Cosk^0 : X \to Gpd(X)$. The left adjoint associate to $b \in X$ the constant simplicial object $Sk^0(b) : \Delta^o \to X$ with value b. The right adjoint associates to b the simplicial object $Cosk^0(b)$ obtained by putting $Cosk^0(b)_n = b^{[n]}$ for each $n \geq 0$. We say that $Cosk^0(b)$ is the *coarse groupoid* of b. More generally, the *equivalence groupoid* $Eq(f)$ of an arrow $f : a \to b$ in X is defined to be the coarse groupoid of the object $f \in X/b$ (or rather its image by the canonical map $X/b \to X$). The *loop group* $\Omega(b)$ of a pointed object $1 \to b$ is the equivalence groupoid of the arrow $1 \to b$.

If $C : \Delta^o \to X$ is a category object in a cartesian quasi-category X, we call a functor $p : E \to C$ in $Cat(X)$ a *left fibration* if the naturality square

$$\begin{array}{ccc} E_1 & \xrightarrow{s} & E_0 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_0} \\ C_1 & \xrightarrow{s} & C_0 \end{array}$$

is cartesian, where s is the source map. We denote by X^C the full simplicial subset of $Cat(X)/C$ spanned by the left fibrations $E \to C$. The pullback of a left fibration $E \to D$ along a functor in $Cat(X)$ is a left fibration $f^*(E) \to C$. This defines the *base change map*

$$f^* : X^D \to X^C.$$

Let X be a cartesian quasi-category. The equivalence groupoid of an arrow $u : a \to b$ is equipped with a map $Eq(u) \to b$. and the base change map $u^* : X/b \to X/a$ admits a lifting $\tilde{u}^*$,

$$\begin{array}{ccc} & & X^{Eq(u)} \\ & \overset{\tilde{u}^*}{\nearrow} & \downarrow{\scriptstyle p} \\ X/b & \xrightarrow[u^*]{} & X/a, \end{array}$$

where p is the forgeful map. We call $\tilde{u}^*$ the *lifted base change map*; it associates to an arrow $e \to b$ the arrow $a \times_b e \to a$

$$\begin{array}{ccc} a \times_b e & \longrightarrow & e \\ \downarrow & & \downarrow \\ a & \xrightarrow{u} & b \end{array}$$

equipped with a natural action of the groupoid $Eq(u)$. We say that an arrow $u : a \to b$ is a *descent morphism* if the lifted base change map $\tilde{u}^*$ is an equivalence of quasi-categories. If $u : 1 \to b$ is a pointed object in a cartesian quasi-category X, then the groupoid $Eq(u)$ is the loop group $\Omega_u(b)$. In this case, the lifted base change map

$$\tilde{u}^* : X/b \to X^{\Omega_u(b)}$$

associates to an arrow $e \to b$ its fiber $e(u) = u^*(e)$ equipped with the natural action (say on the right) of the group $\Omega_u(b)$.

In the quasi-category **U**, every surjection is a descent morphism. This is true more generally of any surjection in a para-variety.

Exact quasi-categories

In a cartesian quasi-category X, we say that a groupoid $C \in Gpd(X)$ is *effective* if it has a colimit $p : C_0 \to BC$ and the canonical functor $C \to Eq(p)$ is invertible.

Recall that a cartesian quasi-category X is said to be *regular* if it admits surjection-mono factorisations stable under base changes. We say that a regular quasi-category X is *exact* if it satisfies the following two conditions:

- Every surjection is a descent morphism;
- Every groupoid is effective.

The quasi-category $\mathbf{U}$ is exact. If X is an exact quasi-category, then so are the quasi-categories $b\backslash X$ and X/b for any vertex $b \in X$, the quasi-category X^A for any simplicial set A and the quasi-category $Mod^{\times}(T, X)$ for any algebraic theory T. A variety of homotopy algebras is exact. A left exact reflection of an exact quasi-category is exact. A para-variety is exact.

We say that a map $X \to Y$ between regular quasi-categories is *exact* if it is left exact and preserves surjections. If $u : a \to b$ is an arrow in an exact quasi-category X, then the base change map $u^* : X/b \to X/a$ is exact. Moreover, u^* is conservative if u is surjective. Moreover, the lifted base change map

$$\tilde{u}^* : X/b \to X^{Eq(u)}$$

is an equivalence of quasi-categories. A pointed object $u : 1 \to b$ is connected iff u is a surjective map. In this case the map

$$\tilde{u}^* : X/b \to X^{\Omega_u(b)}$$

is an equivalence of quasi-categories.

An exact quasi-category X admits n-factorisations for every $n \geq 0$. An object a is connected iff the arrows $a \to 1$ and $a \to a \times a$ are surjective. An arrow $a \to b$ is 0-connected iff it is surjective and the diagonal $a \to a \times_b a$ is surjective. If $n > 0$, an arrow $a \to b$ is n-connected iff it is surjective and the diagonal $a \to a \times_b a$ is $(n-1)$-connected. If $a \to e \to b$ is the n-factorisation of an arrow $a \to b$, then $a \to a \times_e a \to a \times_b a$ is the $(n-1)$-factorisation of the arrow $a \to a \times_b a$. An exact map $f : X \to Y$ between exact quasi-categories preserves the n-factorisations for every $n \geq 0$.

Let X be a cartesian quasi-category. We say that a functor $f : C \to D$ in $Cat(X)$ is a *Morita equivalence* if the induced map $f^* : X^D \to X^C$ is an equivalence of quasi-categories. If X is regular, we say that a functor $f : C \to D$

in $Gpd(X)$ is *essentially surjective* if the morphism tp_1 in the square

$$\begin{array}{ccccc} & & D_1 \times_{D_0} C_0 & \xrightarrow{p_2} & C_0 \\ & \swarrow^{tp_1} & \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle f_0} \\ D_0 & \xleftarrow[t]{} & D_1 & \xrightarrow{s} & D_0 \end{array}$$

is surjective. Let $J : Cat(X) \to Gpd(X)$ be the right adjoint to the inclusion $Gpd(X) \subseteq Cat(X)$. We say that a functor $f : C \to D$ in $Cat(X)$ is *essentially surjective* if the functor $J(f) : J(C) \to J(D)$ is essentially surjective. We say that f is a *weak equivalence* if it is fully faithful and essentially surjective. For example, if $u : a \to b$ is a surjection in X, then the canonical functor $Eq(u) \to b$ is a weak equivalence. If X is an exact quasi-category, then every weak equivalence $f : C \to D$ is a Morita equivalence, and the converse is true if C and D are groupoids.

Let X be an exact quasi-category. Then the map $Eq : X^I \to Gpd(X)$ which associates to an arrow $u : a \to b$ its equivalence groupoid $Eq(u)$ has left adjoint $B : Gpd(X) \to X^I$ which associates to a groupoid C its "quotient" or "classifying space" BC equipped with the canonical map $C_0 \to BC$. Let us denote by $Surj(X)$ the full simplicial subset of X^I spanned by the surjections. The map B is fully faithful and its essential image is equal to $Surj(X)$. Hence the adjoint pair $B \vdash Eq$ induces an equivalence of quasi-categories

$$B : Gpd(X) \leftrightarrow Surj(X) : Eq.$$

Let X be a pointed exact quasi-category. Then an object $x \in X$ is connected iff the morphism $0 \to x$ is surjective. More generally, an object $x \in X$ is n-connected iff the morphism $0 \to x$ is $(n-1)$-connected. If $CO(X)$ denotes the quasi-category of connected objects in X, then we have an equivalence of quasi-categories

$$B : Grp(X) \leftrightarrow CO(X) : \Omega.$$

Hence the quasi-category $CO(X)$ is exact, since the quasi-category $Grp(X)$ is exact. A morphism in $CO(X)$ is n-connected iff it is $(n+1)$ connected in X. Similarly, a morphism in $CO(X)$ is a n-cover iff it is a $(n+1)$ cover in X. Let us put $CO^{n+1}(X) = CO(CO^n(X))$ for every $n \geq 1$. This defines a decreasing chain

$$X \supseteq CO(X) \supseteq CO^2(X) \supseteq \cdots.$$

An object $x \in X$ belongs to $CO^n(X)$ iff x is $(n-1)$-connected. Let $Grp^n(X)$ be the quasi-category of n-fold groups in X. By iterating the equivalence above we obtain an equivalence

$$B^n : Grp^n(X) \leftrightarrow CO^n(X) : \Omega^n$$

for every $n \geq 0$.

Let X be an exact quasi-category. We say that an arrow in X is ∞-*connected* if it is n-connected for every $n \geq 0$. An arrow $f \in X$ is ∞-connected iff it is a n-equivalence for every $n \geq 0$. We say that X is *t-complete* if every ∞-connected arrow is invertible.

Let X be an exact quasi-category. If $RCat(X)$ is the quasi-category of reduced categories in X, then the inclusion $RCat(X) \subseteq Cat(X)$ has a left adjoint

$$R : Cat(X) \to RCat(X)$$

which associates to a category $C \in Cat(X)$ its *reduction* $R(C)$. When C is a groupoid, we have $R(C) = B(C)$. In general, we have a a pushout square in $Cat(X)$,

$$\begin{array}{ccc} J(C) & \longrightarrow & C \\ \downarrow & & \downarrow \\ B(J(C)) & \longrightarrow & R(C), \end{array}$$

where $J(C)$ is the groupoid of isomorphisms of a category C. The simplicial object $R(C)$ can be constructed by putting $(RC)_n = B(J(C^{[n]}))$ for every $n \geq 0$, where $C^{[n]}$ is the (internal) category of functor $[n] \to C$. The canonical map $C \to R(C)$ is an equivalence of categories, hence it is also a Morita equivalence. A functor $f : C \to D$ in $Cat(X)$ is an equivalence iff the functor $R(f) : R(C) \to R(D)$ is a isomorphism in $RCat(X)$. If $W \subseteq Cat(X)$ is the set of equivalences, then the induced map

$$L(Cat(X), W) \to RCat(X)$$

is an equivalence of quasi-categories.

Additive quasi-categories

We say that a quasi-category X is *pointed* if the natural projection $X^I \to X \times X$ admits a section $X \times X \to X^I$. The section is homotopy unique when it exists; it then associates to a pair of objects $a, b \in X$ a *null arrow* $0 : a \to b$. The homotopy category of a pointed quasi-category X is pointed. In a pointed quasi-category, every initial object is terminal. A *null object* in a quasi-category X is an object $0 \in X$ which is both initial and terminal. A quasi-category X with a null object is pointed; the null arrow $0 = a \to b$ between two objects of X is obtained by composing the arrows $a \to 0 \to b$. The quasi-category $a\backslash X/a$ has a null object for any object a of a quasi-category X.

The product of two objects $x \times y$ in a pointed quasi-category X is called a *direct sum* $x \oplus y$ if the pair of arrows

$$x \xrightarrow{(1_x,0)} x \times y \xleftarrow{(0,1_y)} y$$

is a coproduct diagram. A pointed quasi-category with finite products X is said to be *semi-additive* if the product $x \times y$ of any two objects is a direct sum $x \oplus y$. The opposite of a semi-additive quasi-category X is semi-additive. The homotopy category of a semi-additive quasi-category is semi-addditive. The set of arrows between two object of a semi-additive category has the structure of a commutative monoid. A map $f : X \to Y$ between semi-additive quasi-categories preserves finite products iff it preserves finite coproducts iff it preserves finite direct sums. Such a map is said to be *additive*. The canonical map $X \to hoX$ is additive for any semi-additive quasi-category X.

A semi-additive category C is said to be *additive* if the monoid $C(x, y)$ is a group for any pair of objects $x, y \in C$. A semi-additive quasi-category X is said to be *additive* if the category hoX is additive. The opposite of an additive quasi-category is additive. If a quasi-category X is semi-additive (resp. additive), then so is the quasi-category X^A for any simplicial set A and the quasi-category $Mod^\times(T, X)$ for any algebraic theory T.

The *fiber* $a \to x$ of an arrow $x \to y$ in a quasi-category with null object 0 is defined by a pullback square

$$\begin{array}{ccc} a & \longrightarrow & x \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & y. \end{array}$$

The cofiber of an arrow is defined dually. An additive quasi-category is cartesian iff every arrow has a fiber. Let X be a cartesian additive quasi-category. Then to each arrow $f : x \to y$ in X we can associate a *long fiber sequence*,

$$\cdots \longrightarrow \Omega^2(y) \xrightarrow{\partial} \Omega(z) \xrightarrow{\Omega(i)} \Omega(x) \xrightarrow{\Omega(f)} \Omega(y) \xrightarrow{\partial} z \xrightarrow{i} x \xrightarrow{f} y\ .$$

where $i : z \to x$ is the fiber of f. An additive map between cartesian additive quasi-categories is left exact iff it preserves fibers.

An additive quasi-category X is exact iff the following five conditions are satisfied:

- X admits surjection-mono factorisations;
- The base change of a surjection is a surjection;
- Every morphism in has a fiber and a cofiber;
- Every morphism is the fiber of its cofiber;
- Every surjection is the cofiber of its fiber.

Let X be an exact additive quasi-category. If a morphism $f : x \to y$ is surjective, then a null sequence $0 = fi : z \to x \to y$ is a fiber sequence iff it is a cofiber sequence.

Stable quasi-categories

See Lurie [Lu2] for another approach and a more complete treatment.

Let X be a quasi-category with null object $0 \in X$. Recall that the *loop space* $\Omega(x)$ of an object $x \in X$ is defined to be the fiber of the arrow $0 \to x$. We say that X is *stable* if every object $x \in X$ has a loop space, and the loop space map $\Omega : X \to X$ is an equivalence of quasi-categories. If X is a stable quasi-category, then the inverse of the map Ω is the *suspension* $\Sigma : X \to X$. The opposite of a stable quasi-category X is stable. The loop space map $\Omega : X^o \to X^o$ is obtained by putting $\Omega(x^o) = \Sigma(x)^o$ for every object $x \in X$. A stable quasi-category with finite products is additive.

Let *Spec* be the cartesian theory of spectra. If X is a cartesian quasi-category, then the quasi-category $Spec(X)$ of stable objects in X is stable. In particular, the quasi-category of spectra $\mathbf{Spec} = Mod(Spec)$ is stable. The quasi-category of spectra **Spec** is exact.

An additive quasi-category X is stable and exact iff the following two conditions are satisfied:

- Every morphism has a fiber and a cofiber;
- A null sequence $z \to x \to y$ is a fiber sequence iff it is a cofiber sequence.

The opposite of an exact stable quasi-category is exact and stable.

Utopoi

The notion of utopos (higher, upper topos) presented here is attributed to Charles Rezk. See Lurie [Lu1] for a more complete treatment.

Recall that a category $\mathcal{E}$ is said to be a *Grothendieck topos*, but we shall say a *1-topos* if it is a left exact reflection of a presheaf category $[C^o, \mathbf{Set}]$. A *homomorphism* $\mathcal{E} \to \mathcal{F}$ between Grothendieck topoi is a cocontinuous functor $f : \mathcal{E} \to \mathcal{F}$ which preserves finite limits. Every homomorphism has a right adjoint. A *geometric morphism* $\mathcal{E} \to \mathcal{G}$ between Grothendieck topoi is an adjoint pair

$$g^* : \mathcal{F} \leftrightarrow \mathcal{E} : g_*$$

with g^* a homomorphism. The map g^* is called the *inverse image part of g* and the map g_* its *direct image part.* . We shall denote by **Gtop** the category of Grothendieck topoi and geometric morphisms. The category **Gtop** has the structure of a 2-category, where a 2-cell $\alpha : f \to g$ between geometric morphisms is a natural transformation $\alpha : g^* \to f^*$.

We say that a locally presentable quasi-category X is an *upper topos* or an *utopos* if it is a left exact reflection of a quasi-category of pre-stacks $\mathbf{P}(A)$ for some simplicial set A. The quasi-category $\mathbf{U}$ is the archetype an utopos. If X is an utopos, then so is the quasi-category X/a for any object $a \in X$ and the quasi-category X^A for any simplicial set A.

Recall that a cartesian quasi-category X is said to be *locally cartesian closed* if the quasi-category X/a is cartesian closed for every object $a \in X$. A cartesian quasi-category X is locally cartesian closed iff the base change map $f^* : X/b \to X/a$ has a right adjoint $f_* : X/a \to X/b$ for any morphism $f : a \to b$ in X. A locally presentable quasi-category X is locally cartesian closed iff the base change map $f^* : X/b \to X/a$ is cocontinuous for any morphism $f : a \to b$ in X.

(Giraud's theorem)[Toen, Vezzozi] A locally presentable quasi-category X is an utopos iff the following conditions are satisfied:

- X is locally cartesian closed and exact;
- the canonical map
$$X/\sqcup a_i \to \prod_i X/a_i$$
is an equivalence for any family of objects $(a_i : i \in I)$ in X.

Recall that if X is a bicomplete quasi-category and A is a simplicial set, then every map $f : A \to X$ has a left Kan extension $f_! : \mathbf{P}(A) \to X$. A locally presentable quasi-category X is an utopos iff the map $f_! : \mathbf{P}(A) \to X$ is left exact for any cartesian category A and any left exact map $f : A \to X$.

A *homomorphism* $X \to Y$ between utopoi is a cocontinuous map $f : X \to Y$ which preserves finite limits. Every homomorphism has a right adjoint. A *geometric morphism* $X \to Y$ is an adjoint pair

$$g^* : Y \leftrightarrow X : g_*$$

with g^* a homomorphism. The map g^* is called the *inverse image part of* g and the map g_* the *direct image part.* . We shall denote by **UTop** the category of utopoi and geometric morphisms. The category **UTop** has the structure of a 2-category, where a 2-cell $\alpha : f \to g$ between geometric morphisms is a natural transformation $\alpha : g^* \to f^*$. The opposite 2-category $\mathbf{UTop}^o$ is equivalent to the sub (2-)category of **LP** whose objects are utopoi, whose morphisms (1-cells) are the homomorphisms, and whose 2-cells are the natural transformations.

If $u : A \to B$ is a map of simplicial sets, then the pair of adjoint maps $u^* : \mathbf{P}(B) \to \mathbf{P}(A) : u_*$ is a geometric morphism $\mathbf{P}(A) \to \mathbf{P}(B)$. If X is an utopos, then the adjoint pair $f^* : X/b \to X/a : f_*$ is a geometric morphism $X/a \to X/b$ for any arrow $f : a \to b$ in X.

If X is an utopos, we shall say that a reflexive sub quasi-category $S \subseteq X$ is a *sub-utopos* if it is locally presentable and the reflection functor $r : X \to S$ preserves finite limits. If $i : S \subseteq X$ is a sub-utopos and $r : X \to S$ is the reflection, then the pair (r, i) is a geometric morphism $S \to X$. In general, we say that a geometric morphism $g : X \to Y$ is an *embedding* if the map $g_* : X \to Y$ is fully faithful. We say that a geometric morphism $g : X \to Y$ is *surjective* if the map $g^* : Y \to X$ is conservative. The (2-) category **UTop** admits a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class if surjections and $\mathcal{B}$ the class of embeddings.

Stabilisation

The homotopy colimit of an infinite sequence of maps

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots$$

in the category **LP** can be computed as the homotopy limit in the category **QCAT** of the corresponding sequence of right adjoints

$$X_0 \xleftarrow{g_0} X_1 \xleftarrow{g_1} X \xleftarrow{g_2} \cdots .$$

An object of this limit L is a pair (x, a), where $x = (x_n)$ is a sequence of objects $x_n \in X_n$ and $a = (a_n)$ is a sequence of isomorphisms $a_n : x_n \simeq g_n(x_{n+1})$. The canonical map $u_0 : X_0 \to L$ has no simple description, but its right adjoint $L \to X$ is the projection $(x, a) \mapsto x_0$. The quasi-category L can also be obtained by localising another locally presentable quasi-category of L' constructed as follows. An object of L' is pair (x, b), where $x = (x_n)$ is a sequence of objects $x_n \in X_n$ and $b = (b_n)$ is a sequence of morphisms $b_n : f_n(x_n) \to x_{n+1}$. The object (x, b) can also be described as a pair $y = (x, a)$, where $x = (x_n)$ is a sequence of objects $x_n \in X_n$ and $a = (a_n)$ is a sequence of morphisms $a_n : x_n) \to g_n(x_{n+1})$. The obvious inclusion $L \subseteq L'$ has a left adjoint $q : L' \to L$ which can be described explicitly by a colimit process using transfinite iteration. If $y = (x, a) \in L'$ let us put $\rho(y) = \rho(x, a) = \rho(x), \rho(a))$, where $\rho(x)_n = g_n(x_{n+1})$ and $g(a)_n = g_n(a_{n+1})$. This defines a map $\rho : L' \to L'$ and the sequence $a_n : x_n \to g_n(x_{n+1})$ defines a morphism $\theta(y) : y \to \rho(y)$ in L' which is natural in y. It is easy to see that we have $\theta \circ \rho = \rho \circ \theta : \rho \to \rho^2$. By iterating ρ transfinitly, we obtain a cocontinuous chain

$$Id \xrightarrow{\theta} \rho \xrightarrow{\theta} \rho^2 \xrightarrow{\theta} \rho^3 \xrightarrow{\theta} \cdots$$

where

$$\rho^\alpha(y) = \varinjlim_{i<\alpha} \rho^i(y).$$

for a limit ordinal α. The chain stabilises enventually and we have

$$q(x) = \varinjlim_{\alpha} \rho^\alpha(x).$$

If directed colimits commute with finite limits in each X_n, then the reflection $q : L' \to L$ is left exact. We conjecture that the quasi-category L' is a para-variety (resp. an utopos) if each quasi-category X_n is a para-variety (resp. an utopos) It follows that the quasi-category L is a para-variety (resp. an utopos) in this case. If $\phi : X \to X$ is a cocontinuous endomorphism of a locally presentable quasi-category X, we shall denote by $S(X, \phi)$ the (homotopy) colimit in **LP** of the sequence of quasi-categories

$$X \xrightarrow{\phi} X \xrightarrow{\phi} X \xrightarrow{\phi} \cdots$$

If $\omega : X \to X$ is right adjoint to $\phi : X \to X$, then $S(X, \phi)$ is the (homotopy) limit of the sequence of quasi-categories

$$X \xleftarrow{\omega} X \xleftarrow{\omega} X \xleftarrow{\omega} \cdots$$

An object of X' is an ω-*spectrum*.

(Joint work with Georg Biedermann) If X is a para-variety (resp. an utopos), then so is the quasi-category of parametrised spectra in X. Let us sketch the proof. Let us denote by *Spec* be the cartesian theory of spectra and by *Spec*$'$ the cartesian theory of parametrized spectra. Similarly, let us denote by *PSpec* be the cartesian theory of pre-spectra and by *PSpec*$'$ be the cartesian theory of parametrized pre-spectra. The quasi-category *Spec*(X) is a left exact reflection of the quasi-category *PSpec*(X). Similarly, the quasi-category *Spec*$'(X)$ is a left exact reflection of the quasi-category *PSpec*$'(X)$. An object of *PSpec*$'(X)$ is a pre-spectrum in X/b for some object $b \in X$. A pointed object of X/b is an arrow $p : x \to b$ equipped with a section $s : b \to x$. A pre-spectrum in X/b is an infinite sequence of pointed objects (x_n, p_n, s_n) together with an infinite sequence of commutative squares

$$\begin{array}{ccc} x_n & \xrightarrow{p_n} & b \\ {\scriptstyle p_n}\downarrow & & \downarrow{\scriptstyle s_{n+1}} \\ b & \xrightarrow{s_{n+1}} & x_{n+1}. \end{array}$$

Clearly, a parametrised pre-spectrum in X is a map $B \to X$, where B is a certain category. Hence the quasi-category *PSpec*$'(X)$ of parametrized pre-spectra in X is of the form X^B. It is thus a para-variety (resp. an utopos), since X is a para-variety (resp. an utopos). But the quasi-category *Spec*$'(X)$ is a left exact reflection of *PSpec*$'(X)$. ∎

Meta-stable quasi-categories

We say that an exact quasi-category X is *meta-stable* if every object in X is ∞-connected. A cartesian quasi-category X is meta-stable iff if it satisfies the following two conditions:

- Every morphism is a descent morphism;
- Every groupoid is effective.

The sub-quasi-category of ∞-connected objects in an exact quasi-category is meta-stable. The quasi-category of spectra is meta-stable. In a meta-stable quasi-category, every monomorphism is invertible and every morphism is surjective.

If a quasi-category X is meta-stable then so are the quasi-categories $b\backslash X$ and X/b for any vertex $b \in X$, the quasi-category X^A for any simplicial set A, and the quasi-category $Prod(T, X)$ for any algebraic theory T. A left exact reflection of a meta-stable quasi-category is meta-stable.

Let $u : a \to b$ be an arrow in a meta-stable quasi-category X. Then the lifted base change map

$$\tilde{u}^* : X/b \to X^{Eq(u)}$$

is an equivalence of quasi-categories. In particular, if $u : 1 \to b$ is a pointed object, then the map

$$\tilde{u}^* : X/b \to X^{\Omega_u(b)}$$

is an equivalence of quasi-categories.

Let X be a meta-stable quasi-category. Then the map $Eq : X^I \to Gpd(X)$ which associates to an arrow $u : a \to b$ the equivalence groupoid $Eq(u)$ is invertible. We thus have an equivalence of quasi-categories

$$B : Gpd(X) \leftrightarrow X^I : Eq.$$

The equivalence can be iterated and it yields an equivalence of quasi-categories

$$B^n : Gpd^n(X) \leftrightarrow X^{I^n} : Eq^n$$

for each $n \geq 1$.

Let X be a meta-stable quasi-category. Then the equivalence

$$B : Gpd(X) \leftrightarrow X^I : Eq.$$

induces an equivalence

$$B : Gpd(X, a) \leftrightarrow a\backslash X : Eq$$

for each object $a \in A$, where $Gpd(X, a)$ is the quasi-category of groupoids $C \in Gpd(X)$ with $C_0 = a$. In particular, it induces an equivalence

$$B : Grp(X) \leftrightarrow 1\backslash X : \Omega,$$

where $Grp(X)$ is the quasi-category of groups in X. By iterating, we obtain an equivalence

$$B^n : Grp^n(X) \leftrightarrow 1\backslash X : \Omega^n,$$

for each $n \geq 1$.

Higher Categories

We introduce the notions of n-fold category object and of n-category object in a quasi-category. We also introduce the notions of reduced n-category and of equivalence of n-categories.

If Cat denotes the cartesian theory of categories then $Cat^2 = Cat \odot_c Cat$ is the theory of double categories. If X is a cartesian quasi-category, then an object of

$$Cat^2(X) = Cat(Cat(X))$$

is a *double category* in X. By definition, a double simplicial object $\Delta^o \times \Delta^o \to X$ is a double category iff it is a category object in each variable. We shall denote by $Cat^n(X)$ the quasi-category of n-fold categories in X and by Cat^n the cartesian theory of n-fold categories.

Let X be a quasi-category. If A is a simplicial set, we say that a map $f : A \to X$ is *essentially constant* if it belongs to the essential image of the diagonal $X \to X^A$. If A is weakly contractible, then a map $f : A \to X$ is essentially constant iff it takes every arrow in A to an isomorphism in X. A simplicial object $C : \Delta^o \to X$ in a quasi-category X is essentially constant iff the canonical morphism $sk^0(C_0) \to C$ is invertible. A category object $C : \Delta^o \to X$ is essentially constant iff it inverts the arrow $[1] \to [0]$. A n-fold category $C : (\Delta^n)^o \to X$ is essentially constant iff C inverts the arrow $[\epsilon] \to [0^n]$ for every $\epsilon = (\epsilon_1, \cdots, \epsilon_n) \in \{0,1\}^n$, where $[0^n] = [0, \ldots, 0]$.

Let X be a cartesian quasi-category. We call a double category $C : \Delta^o \to Cat(X)$ a *2-category* if the simplicial object $C_0 : \Delta^o \to X$ is essentially constant. A double category $C \in Cat^2(X)$ is a 2-category iff it inverts every arrow in $[0] \times \Delta$. Let us denote by Id the set of identity arrows in Δ. Then the set of arrows

$$\Sigma_n = \bigsqcup_{i+1+j=n} Id^i \times [0] \times \Delta^j$$

is a subcategory of Δ^n. We say that a n-fold category object $C \in Cat^n(X)$ is a *n-category* if it inverts every arrow in Σ_n. The notion of n-category object in X can be defined by induction on $n \geq 0$. A category object $C : \Delta^o \to Cat_{n-1}(X)$ is a n-category iff the $(n-1)$-category C_0 is essentially constant. We denote by Cat_n the cartesian theory of n-categories and by $Cat_n(X)$ the quasi-category of n-category objects in X.

The object of *k-cells* $C(k)$ of a n-category $C : (\Delta^o)^n \to X$ is the image by C of the object $[1^k 0^{n-k}]$. The *source map* $s : C(k) \to C(k-1)$ is the image of the map $[1^{k-1}] \times d_1 \times [0^{n-k}]$ and the *target map* $t : C(k) \to C(k-1)$ is the image of the map $[1^{k-1}] \times d_0 \times [0^{n-k}]$. From the pair of arrows $(s,t) : C(k) \to C(k-1) \times C(k-1)$ we obtain an arrow $\partial : C(k) \to C(\partial k)$, where $C(\partial k)$ is defined by the following pullback square

$$\begin{array}{ccc} C(\partial k) & \longrightarrow & C(k-1) \\ \downarrow & & \downarrow{\scriptstyle (s,t)} \\ C(k-1) & \xrightarrow{(s,t)} & C(k-2)\times C(k-2). \end{array}$$

If $n = 1$, $\partial = (s,t) : C(1) \to C(0) \times C(0)$.

There is a notion of n-fold reduced category for every $n \geq 0$. If $RCat$ denotes the cartesian theory of reduced categories, then $RCat^n$ is the theory of n-fold reduced categories. If X is a cartesian quasi-category, then we have

$$RCat^{n+1}(X) = RCat(RCat^n(X))$$

for every $n \geq 0$.

We say that a n-category $C \in Cat_n(X)$ is *reduced* if it is reduced as a n-fold category. We denote by $RCat_n$ the cartesian theory of reduced n-categories. A n-category $C : \Delta^o \to Cat_{n-1}(X)$ is reduced iff it is reduced as a category object and the $(n-1)$-category C_1 is reduced. If X is an exact quasi-category, then the inclusion $RCat_n(X) \subseteq Cat_n(X)$ has a left adjoint

$$R : Cat_n(X) \to RCat_n(X)$$

which associates to a n-category $C \in Cat_n(X)$ its *reduction* $R(C)$. We call a map $f : C \to D$ in $Cat_n(X)$ an *equivalence* if the map $R(f) : R(C) \to R(D)$ is invertible in $RCat_n(X)$. The quasi-category

$$\mathbf{U}_n = Mod(RCat_n)$$

is cartesian closed.

The object $[0]$ is terminal in Δ. Hence the functor $[0] : 1 \to \Delta$ is right adjoint to the functor $\Delta \to 1$. It follows that the inclusion $i_n : \Delta^n = \Delta^n \times [0] \subseteq \Delta^{n+1}$ is right adjoint to the projection $p_n : \Delta^{n+1} = \Delta^n \times \Delta \to \Delta^n$. For any cartesian quasi-category X, the pair of adjoint maps

$$p_n^* : [(\Delta^o)^n, X] \leftrightarrow [(\Delta^o)^{n+1}, X] : i_n^*$$

induces a pair of adjoint maps

$$inc : Cat_n(X) \leftrightarrow Cat_{n+1}(X) : res.$$

The "inclusion" inc is fully faithful and we can regard it as an inclusion by adopting the same notation for $C \in Cat_n(X)$ and $inc(C) \in Cat_{n+1}(X)$. The map res associates to $C \in Cat_{n+1}(X)$ its *restriction* $res(C) \in Cat_n(X)$. The adjoint pair $p_n \vdash i_n^*$ also induces an adjoint pair

$$inc : RCat_n(X) \leftrightarrow RCat_{n+1}(X) : res.$$

In particular, it induces an adjoint pair

$$inc : \mathbf{U}_n \leftrightarrow \mathbf{U}_{n+1} : res.$$

When $n = 0$, the map inc is induced by the inclusion $\mathbf{Kan} \subset \mathbf{QCat}$ and the map res by the functor $J : \mathbf{QCat} \to \mathbf{Kan}$. The inclusion $\mathbf{U}_n \subset \mathbf{U}_{n+1}$ has also a left adjoint which associates to a reduced $(n+1)$-category C the reduced n-category obtained by inverting the $(n+1)$-cells of C.

Part I

Twelve lectures

Chapter 1

Elementary aspects

In this second lecture we introduce the notion of quasi-category, the fundamental category of a simplicial set, the homotopy category of a quasi-category, the notion of weak categorical equivalence and describe the model structure for quasi-categories. We also introduce a notion of adjoint maps between simplicial sets.

We denote by **Cat** the category of small categories and by **S** the category of simplicial sets. Recall that the *nerve* of a category C is the simplicial set NC defined by putting

$$(NC)_n = \mathbf{Cat}([n], C)$$

for every $n \geq 0$. A simplex $x \in (NC)_n$ can be represented as a chain of length n of arrows in C,

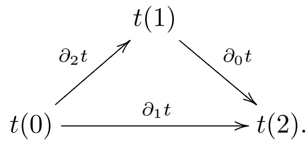
$$x(0) \xrightarrow{x_1} x(1) \xrightarrow{x_2} x(2) \longrightarrow \cdots \xrightarrow{x_n} x(n).$$

The nerve functor $N : \mathbf{Cat} \to \mathbf{S}$ is fully faithful. We shall regard it is an inclusion by adopting the same notation for a small category and its nerve. The nerve functor has a left adjoint $\tau_1 : \mathbf{S} \to \mathbf{Cat}$, which associates to a simplicial set X its *fundamental category* $\tau_1 X$. The *fundamental groupoid* $\pi_1 X$ is obtained by freely inverting all the arrows of $\tau_1 X$. If **Gpd** denotes the category of small groupoids, then the functor $\pi_1 : \mathbf{S} \to \mathbf{Gpd}$ is left adjoint to the inclusion $\mathbf{Gpd} \subset \mathbf{S}$ defined by the nerve.

The category $\tau_1 X$ can be described by generators and relations. Let GX be the graph of non-degenerate arrows of X, and let FX be the category freely generated by GX. By construction, a morphism $a \to b$ in FX is a path in GX,

$$a = a_0 \xrightarrow{f_1} a_1 \xrightarrow{f_2} a_2 \cdots \qquad \xrightarrow{f_n} a_n = b.$$

There is a natural injection $X_1 \to F(X)$ which takes a non-degenerate arrow in to a path of length one and a degenerate arrow to a unit. And let $\equiv$ the congruence relation on FX generated by the relations $(td_0)(td_2) \equiv td_1$, one for each non-degenerate simplex $t \in X$ with boundary $\partial t = (\partial_0 t, \partial_1 t, \partial_2 t)$:



The proof of the following result is left as an exercice.

Proposition 1.1. [GZ] *Let FX be the category freely generated by the graph of non-degenerate arrows of a simplicial set X. Then we have*

$$\tau_1 X = FX/\equiv,$$

where $\equiv$ is the congruence described above. Moreover, the functor $\tau_1 Sk^2 X \to \tau_1 X$ induced by the inclusion $Sk^2 X \subseteq X$ is an isomorphism of categories.

Corollary 1.2. *The nerve of a category is 2-coskeletal.*

Proof: For any category C there is a natural bijection between the maps $Sk^2\Delta[n] \to NC$ and the functors $\tau_1 Sk^2\Delta[n] \to C$. We have $\tau_1 Sk^2\Delta[n] = \tau_1\Delta[n]$ by 1.1. It follows that every map $Sk^2\Delta[n] \to NC$ has a unique extension $\Delta[n] \to NC$. ■

Proposition 1.3. [GZ] *The functor $\tau_1 : \mathbf{S} \to \mathbf{Cat}$ preserves finite products.*

See B.0.15 for a proof.

We denote by $\tau_0 A$ the set of isomorphism classes of objects of the category $\tau_1 A$. This defines a functor

$$\tau_0 : \mathbf{S} \to \mathbf{Set}.$$

It follows from 1.3 that

Corollary 1.4. *The functor τ_0 preserves finite products.*

Recall that a simplicial set X is said to be *n-coskeletal* if the canonical map $X \to Cosk^n X$ is an isomorphism. A simplicial set X is n-coskeletal iff every simplicial sphere $\partial\Delta[m] \to X$ with $m > n$ has a unique filler.

We say that a horn $\Lambda^k[n]$ is *inner* if we have $0 < k < n$.

Definition 1.5. [BV] *We call a simplicial set X a* quasi-category *if every inner horn $\Lambda^k[n] \to X$ has a filler $\Delta[n] \to X$.*

The notion of quasi-category was introduced by Boardman and Vogt in their work on homotopy invariant algebraic structures [BV]. A Kan complex and the nerve of a category are examples. A quasi-category is sometime called a *weak Kan complex* in the literature [KP]. The purpose of our terminology is to stress the analogy with categories.

A *map* of quasi-categories is a map of simplicial sets. We shall denote the category of (small) quasi-categories by **QCat**. Notice the diagram of inclusions

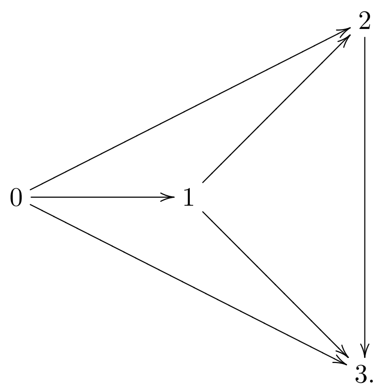
$$\begin{array}{ccc} \mathbf{Gpd} & \longrightarrow & \mathbf{Cat} \\ \downarrow & & \downarrow \\ \mathbf{Kan} & \longrightarrow & \mathbf{QCat}. \end{array}$$

where the vertical inclusions are induced by the nerve functor. We have $\mathbf{Kan} \cap \mathbf{Cat} = \mathbf{Gpd}$, since the nerve of a category is a Kan complex iff the category is a groupoid. A Kan complex may be called a *quasi-groupoid*. A quasi-category with a single vertex may be called a *quasi-monoid*. The *opposite* of a quasi-category is a quasi-category. We shall prove in 4.16 that a quasi-category X is a Kan complex *iff* the category $\tau_1 X$ is a groupoid.

The following lemma is useful for proving that the nerve of a category is a quasi-category. From the inclusion $h_n^k : \Lambda^k[n] \subset \Delta[n]$, we obtain a functor $\tau_1(h_n^k) : \Lambda^k[n] \to \tau_1\Delta[n]$.

Lemma 1.6. *If $0 < k < n$, then the functor $\tau_1(h_n^k) : \Lambda^k[n] \to \tau_1\Delta[n]$ is an isomorphism of categories.*

Proof: If $n > 3$, we have $Sk^2\Lambda^k[n] = Sk^2\Delta[n]$; hence we have $\tau_1\Lambda^k[n] = \tau_1\Delta[n]$ in this case by 1.1. If $n = 2$ then $k = 1$, since $0 < k < 2$. The simplicial set $\Lambda^1[2]$ has two non-degenerate arrows $0 \to 1 \to 2$ and no higher dimensional simplicies. It follows from this description and from 1.1 that $\tau_1\Lambda^1[2] = [2]$. If $n = 3$, then $k = 1$ or $k = 2$, since $0 < k < 3$. It is enough to consider the case $k = 1$ by symmetry. The 1-skeleton of $\Lambda^1[3]$ has six (non-degenerate) arrows $f_{ji} : i \to j$, one for each $0 \leq i < j \leq 3$,



The category $\tau_1 Sk^1\Lambda^1[3]$ is freely generated by this graph. The simplicial set $\Lambda^1[3]$ has three non-degenerate 2-simplices (012), (013) and (123). It follows by 1.1 that the category $\tau_1\Lambda^1[3]$ is the quotient of the category $\tau_1 Sk^1\Lambda^1[3]$ by the congruence $\equiv$ generated by the conditions:

$$f_{21}f_{10} \equiv f_{20}, \quad f_{31}f_{10} \equiv f_{30}, \quad f_{32}f_{21} \equiv f_{31}.$$

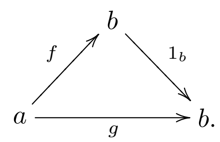
But these conditions imply that we have $f_{32}f_{20} \equiv f_{30}$. It follows that we have $\tau_1\Lambda^1[3] = [3]$. ∎

Proposition 1.7. [BV] *If C is a category, then every inner horn $\Lambda^k[n] \to C$ has a unique filler $\Delta[n] \to C$. Hence the nerve of a category is a quasi-category.*

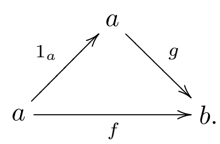
Proof: If $0 < k < n$, let us show that every map $\Lambda^k[n] \to NC$ can be extended uniquely to a map $\Delta[n] \to NC$. It is equivalent to showing that every functor $\tau_1\Lambda^k[n] \to C$ can be extended uniquely to a functor $\tau_1\Delta[n] \to C$, since we have $\tau_1 \vdash N$. But the functor $\tau_1\Lambda^k[n] \to \tau_1\Delta[n]$ induced by the inclusion $\Lambda^k[n] \subset \Delta[n]$ is an isomorphism by 1.6. The result follows. ∎

The fundamental category of a simplicial set X has a nice description when X is a quasi-category: it is the *homotopy category* hoX introduced by Boardman and Vogt in [BV].

Definition 1.8. *If X is a quasi-category, $a, b \in X_0$ and $f, g : a \to b$ a* left homotopy *$u : f \Rightarrow_L g$ is a 2-simplex $u : \Delta[2] \to X$ with boundary $\partial u = (1_b, g, f)$,*



Dually, a right homotopy *$v : f \Rightarrow_R g$ is a 2-simplex $v : \Delta[2] \to X$ with boundary $\partial v = (g, f, 1_a)$,*



Two arrows $f, g : a \to b$ in a quasi-category X are said to be *left homotopic* (resp *right homotopic*) exists a left homotopy $f \Rightarrow_L g$ (resp. right homotopy $f \Rightarrow_R g$).

If X is a simplicial and $a, b \in X_0$, let us denote by $X(a, b)$ the fiber at (a, b) of the projection

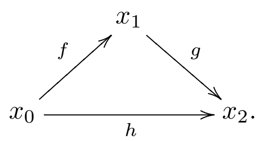
$$(s, t) : X^I \to X^{\{0,1\}} = X \times X$$

defined by the inclusion $\{0,1\} \subset I$. A vertex of the simplicial set $X(a,b)$ is an arrow $a \to b$ in X. The proof of the following lemma is left as an exercise to the reader.

Recall that two vertices u and v of a simplicial set Z are said to be *homotopic* if there exists an arrow $u \to v$ in Z_1.

Lemma 1.9. [BV] *If X is a quasi-category and $a, b \in X_0$ then the homotopy relation on the simplicial set $X(a,b)$ is an equivalence relation. Moreover, two arrows $f, g : a \to b$ are homotopic iff they are left homotopic iff they are right homotopic.*

We shall denote the homotopy relation between the arrows $a \to b$ by $f\tilde{g}$ and the homotopy class of an arrow f by $[f]$. A map $x = (g,h,f) : \partial\Delta[2] \to X$ is a triangle of arrows in X,



We say that the triangle *commutes* if it can be extended to a simplex $\Delta[2] \to X$. Let us write $gf \sim h$ to indicate that a triangle (g,h,f) commutes.

We leave the proof of the following theorem as an exercise to the reader.

Theorem 1.10. [BV] *Let X be a quasi-category. If $f \in X_1(a,b)$, $g \in X_1(b,c)$ and $h \in X_1(a,c)$, then the relation $gf \sim h$ depends only on the homotopy classes of f, g and h. It induces a composition law*

$$hoX(b,c) \times hoX(a,b) \quad \to \quad hoX(a,c)$$

for a category hoX. We have $[g][f] = [h]$ in hoX iff the triangle $(g,h,f) : \partial\Delta[2] \to X$ commutes.

Let X be a quasi-category. If $x : \Delta[n] \to X$, then $ho(x) : [n] \to hoX$, since $ho\Delta[n] = [n]$. This defines a map $h : X \to hoX$ if we put $h(x) = ho(x)$. Hence also a functor

$$i : \tau_1 X \to hoX$$

by the universality of $\tau_1 X$.

Proposition 1.11. *Let X be a quasi-category. Then the canonical functor $i : \tau_1 X \to hoX$ is an isomorphism of categories.*

Proof: It suffices to show that the map $h : X \to hoX$ reflects the simplicial set in the subcategory $\mathbf{Cat} \subseteq \mathbf{S}$. That is, we have to show that for any category C and any map $u : X \to C$ there is a unique functor $v : hoX \to C$ such that $vh = u$. The existence of v is clear, since we can take $v = ho(u) : hoX \to hoC = C$. The uniqueness is left to the reader. ∎

Definition 1.12. *We shall say that an arrow in a simplicial set A is* invertible *or that it is an* isomorphism *if the arrow is invertible in the category $\tau_1 A$.*

Proposition 1.13. *If X is a quasi-category, then two objects $a, b \in X$ are isomorphic in $ho(X) = \tau_1 X$ iff there exists an isomorphism $a \to b$ in X.*

Proof: We have $\tau_1 X = hoX$ by 1.11. Thus, a and b are isomorphic in $\tau_1 X$ iff they are isomorphic in hoX. ∎

Proposition 1.14. *Let X be a quasi-category. Then an arrow $f : a \to b$ in X is invertible iff there exists an arrow $g : b \to a$ such that $gf \sim 1_a$ and $fg \sim 1_b$.*

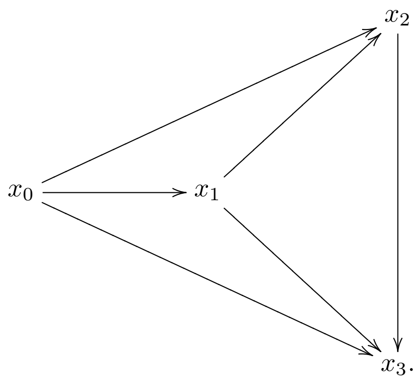
Proof: We have $\tau_1 X = hoX$ by 1.11. Thus, the arrow f is invertible iff the morphism $[f] \in hoX$ is invertible. But the morphism $[f]$ is invertible in hoX iff there exists an arrow $g : b \to a$ in X such that $[g][f] = 1_a$ and $[f][g] = 1_b$. This proves the result, since the relation $[g][f] = 1_a$ is equivalent to the relation $gf \sim 1_a$ and the relation $[f][g] = 1_b$ equivalent to the relation $fg \sim 1_b$ by 1.10. ∎

1.1 Exercises

Exercise 1.15. *Prove Proposition 1.1.*

Exercise 1.16. *Show that the fundamental category of a Kan complex is a groupoid.*

If X is a simplicial set, a map $Sk^1\Delta[3] \to X$ is a diagram of six arrows in X,



Each face of the diagram is a triangle $\partial_k x : \partial\Delta[2] \to X$.

Exercise 1.17. [BV] *Let X be a quasi-category and $x : Sk^1\Delta[3] \to X$ be a diagram as above. Suppose that the triangles $\partial_0 x$ and $\partial_3 x$ commute. Show that the triangle $\partial_1 x$ commutes iff the triangle $\partial_2 x$ commutes.*

Exercise 1.18. *Prove Lemma 1.9.*

Exercise 1.19. *Prove Theorem 1.10.*

1.2 Equivalences

We recall the construction of the homotopy category of simplicial sets $\mathbf{S}^{\pi_0}$ by Gabriel and Zisman [GZ]. The category $\mathbf{S}$ is cartesian closed and the functor $\pi_0 : \mathbf{S} \to \mathbf{Set}$ preserves finite products. If $A, B \in \mathbf{S}$ let us put

$$\pi_0(A, B) = \pi_0(B^A).$$

If we apply the functor π_0 to the composition map $C^B \times B^A \to C^A$ we obtain a composition law

$$\pi_0(B, C) \times \pi_0(A, B) \to \pi_0(A, C)$$

for a cartesian closed category $\mathbf{S}^{\pi_0}$, where we put $\mathbf{S}^{\pi_0}(A, B) = \pi_0(A, B)$. We call a map of simplicial sets a *combinatorial homotopy equivalence* if the map is invertible in the homotopy category $\mathbf{S}^{\pi_0}$. A map of simplicial set $u : A \to B$ is a *weak homotopy equivalence* iff the map

$$\pi_0(u, X) : \pi_0(B, X) \to \pi_0(A, X)$$

is bijective for every Kan complex X.

The functor $\tau_0 : \mathbf{S} \to \mathbf{Set}$ preserves finite products by 1.4. If $A, B \in \mathbf{S}$ let us put

$$\tau_0(A, B) = \tau_0(B^A).$$

If we apply the functor τ_0 to the composition map $C^B \times B^A \to C^A$ we obtain a composition law

$$\tau_0(B, C) \times \tau_0(A, B) \to \tau_0(A, C)$$

for a cartesian closed category $\mathbf{S}^{\tau_0}$, if we put $\mathbf{S}^{\tau_0}(A, B) = \tau_0(A, B)$.

Definition 1.20. *We call a map of simplicial sets a* categorical equivalence *if the map is invertible in the category* $\mathbf{S}^{\tau_0}$. *If X and Y are quasi-categories, then a categorical equivalence $X \to Y$ is called an* equivalence of quasi-categories . *We call a map $u : A \to B$ a* weak categorical equivalence *if the map*

$$\tau_0(u, X) : \tau_0(B, X) \to \tau_0(A, X)$$

is bijective for every quasi-category X.

Proposition 1.21. *A categorical equivalence is a weak categorical equivalence. The converse is true for a map between quasi-categories.*

Proof: Let us prove the second statement. Let us denote by $\mathbf{QCat}^{\tau_0}$ the full subcategory of $\mathbf{S}^{\tau_0}$ spanned by the quasi-categories. A map between quasi-categories is an equivalence iff it is invertible in the category $\mathbf{QCat}^{\tau_0}$. Let $u : X \to Y$ be a weak categorical equivalence between quasi-categories. Then the map $\tau_0(u, X) : \tau_0(Y, Z) \to \tau_0(X, Z)$ is bijective for every object $Z \in \mathbf{QCat}^{\tau_0}$. It follows by Yoneda Lemma that u is invertible in $\mathbf{QCat}^{\tau_0}$. ∎

Recall that map of simplicial sets is called a *trivial fibration* if has the right lifting property with respect to every monomorphism. A trivial fibration $f : X \to Y$ has a section $s : Y \to X$, since the square

$$\begin{array}{ccc} \emptyset & \longrightarrow & X \\ \big\downarrow & & \big\downarrow{\scriptstyle f} \\ Y & \xrightarrow{1_Y} & Y \end{array}$$

has a diagonal filler.

Proposition 1.22. *A trivial fibration $f : X \to Y$ in $\mathbf{S}$ is a categorical equivalence.*

Proof: A trivial fibration $f : X \to Y$ has a section $s : Y \to X$ by the above. We have $fs = 1_Y$, by definition of a section. Let us show that we have $1_X = sf$ in $\tau_0(X, X) = \tau_0(X^X)$. For this it suffices to show that the maps 1_X and sf are isomorphic in $\tau_1(X^X)$ Let J be the groupoid generated by one isomorphism $\sigma : 0 \to 1$ and let $j = (j_0, j_1)$ be the inclusion $\{0, 1\} \subset J$. Let us put $u = (1_X, sf) : X \times \{0, 1\} \to X$ and let p_1 be the projection $X \times J \to X$. The following square commutes, since we have $f = fsf$ and $fp_1(X \times j) = f(1_X, 1_X) = (f, f)$,

$$\begin{array}{ccc} X \times \{0,1\} & \xrightarrow{u} & X \\ {\scriptstyle X\times j}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ X \times J & \xrightarrow{fp_1} & Y. \end{array}$$

Hence the square has a diagonal filler $d : X \times J \to X$, since f is a trivial fibration and $X \times j$ is monic. From the map d we obtain a map $k : J \to X^X$ such that $kj_0 = 1_X \in X^X$ and $kj_0 = sf \in X^X$. Thus, $k(\sigma) : 1_X \to sf$, since $\sigma : 0 \to 1$. The arrow $k(\sigma)$ is invertible in $\tau_1(X^X)$, since σ is invertible in J. This proves that the maps 1_X and sf are isomorphic in $\tau_1(X^X)$. ■

Proposition 1.23. *The functor $\tau_1 : \mathbf{S} \to \mathbf{Cat}$ takes a weak categorical equivalence to an equivalence of categories.*

Proof: If $X \in \mathbf{S}$ and $C \in \mathbf{Cat}$, then we have $C^X = C^{\tau_1 X}$ by B.0.16. Hence we have $\tau_0(X, C) = \tau_0(\tau_1 X, C)$. It follows that we have $\tau_0(u, C) = \tau_0(\tau_1 u, C)$ for any map of simplicial sets $u : X \to Y$ and any category $C \in \mathbf{Cat}$. If $u : X \to Y$ is a weak categorical equivalence, then the map $\tau_0(u, C)$ is bijective for any category C (since a category is a quasi-category by 1.7). Hence also the map $\tau_0(\tau_1 u, C)$. It follows by Yoneda Lemma that $\tau_1 u$ is invertible in the category $\mathbf{Cat}^{\tau_0}$. Thus, $\tau_1 u$ is an equivalence of categories. ■

1.3 The 2-category of simplicial sets

Recall that a 2-*category* is a category enriched over **Cat**. An object of a 2-category $\mathcal{E}$ is often called a 0-*cell*. If A and B are 0-cells, an object of the category $\mathcal{E}(A, B)$ is called a 1-*cell* and an arrow is called a 2-*cell*. We shall often write $\alpha : f \to g : A \to B$ to indicate that α is a 2-cell with source the 1-cell $f : A \to B$ and target the 1-cell $g : A \to B$. The composition law in a fixed hom category $\mathcal{E}(A, B)$ is said to be *vertical*. We shall denote the vertical composition $\alpha : f \to g$ and $\beta : g \to h$ in $\mathcal{E}(A, B)$ by $\beta\alpha : f \to h$. The composition law

$$\mathcal{E}(B, C) \times \mathcal{E}(A, B) \to \mathcal{E}(A, C)$$

is said to be *horizontal*. We shall denote the horizontal composition $\alpha : f \to g : A \to B$ and $\beta : u \to v : B \to C$ by $\beta \circ \alpha : uf \to vg : A \to C$.

If A and B are simplicial sets, let us put

$$\tau_1(A, B) = \tau_1(B^A).$$

The functor $\tau_1 : \mathbf{S} \to \mathbf{Cat}$ preserves finite products by Proposition 1.3. If we apply the functor τ_1 to the composition map $C^B \times B^A \to C^A$, we obtain the composition law

$$\tau_1(B, C) \times \tau_1(A, B) \to \tau_1(A, C)$$

of a 2-category $\mathbf{S}^{\tau_1}$ if we put $\mathbf{S}^{\tau_1}(A, B) = \tau_1(A, B)$. A 1-cell of this 2-category is a map of simplicial sets. A 2-cell $f \to g : A \to B$ is a morphism of the category $\tau_1(B^A)$. We should be careful to distinguish between a *homotopy* $\alpha : f \to g$ between two maps $f, g : A \to B$ and the corresponding 2-cell $[\alpha] : f \to g$ in $\tau_1(A, B)$. The former is an arrow in the simplicial set B^A and the latter is an arrow in the category $\tau_1(B^A)$.

There is a notion of equivalence in any 2-category. Recall that 1-cell $u : A \to B$ is said to be an *equivalence* if there exists a 1-cell $v : B \to A$ together with a pair of invertible 2-cells $1_A \to vu$ and $1_B \to uv$. A map of simplicial sets is an equivalence in the 2-category $\mathbf{S}^{\tau_1}$ iff it is a categorical equivalence.

1.4 Exercises

Exercise 1.24. *Show that a map of simplicial set $u : A \to B$ is a weak homotopy equivalence iff the map*

$$\pi_0(u, X) : \pi_0(B, X) \to \pi_0(A, X)$$

is bijective for every Kan complex X.

Exercise 1.25. *Show that the class of weak categorical equivalences has the "3 for 2" property.*

1.5 Adjoint maps

There is a notion of adjunction in any 2-category. Recall that if $u : A \to B$ and $v : B \to A$ are 1-cells, then a pair of of 2-cells $\alpha : 1_A \to vu$ and $\beta : uv \to 1_B$ is called an *adjunction* if the following *adjunction identities* hold:

$$(\beta \circ u)(u \circ \alpha) = 1_u \quad \text{and} \quad (v \circ \beta)(\alpha \circ v) = 1_v.$$

The 2-cell α is said to be the *unit* of the adjunction and the 2-cell β to be the *counit*. Each of these 2-cells determines the others. Hence it suffices to specify the unit (resp. counit) of an adjunction to specify the adjunction. The 1-cell u is said to be the *left adjoint* and the 1-cell v to be the *right adjoint*. We shall write $(\alpha, \beta) : u \vdash v$ to indicate that the pair (α, β) is an adjunction between u and v. We shall write $u \vdash v$ to indicate that u is the left adjoint and v the right adjoint of an adjunction $(\alpha, \beta) : u \vdash v$.

Definition 1.26. *We shall say that a map of simplicial sets is a* left adjoint *(resp.* right adjoint*) if it is a left (resp. right) adjoint in the 2-category* $\mathbf{S}^{\tau_1}$*. We shall say that a homotopy* $\alpha : 1_A \to vu$ *is an* adjunction unit *if the 2-cell* $[\alpha] : 1_A \to vu$ *is the unit of an adjunction in the 2-category* $\mathbf{S}^{\tau_1}$*. Dually, we shall say that a homotopy* $\beta : uv \to 1_B$ *is an* adjunction counit *if the 2-cell* $[\beta] : uv \to 1_B$ *is the counit of an adjunction.*

Proposition 1.27. *The functor* $\tau_1 : \mathbf{S}^{\tau_1} \to \mathbf{Cat}$ *is a 2-functor. Hence it takes a categorical equivalence to an equivalence of categories, an adjunction to an adjunction, a left adjoint to a left adjoint and a right adjoint to a right adjoint.*

Proof: We have $\tau_1(A) = \tau_1(1, A) = \mathbf{S}^{\tau_1}(1, A)$ for any simplicial set A. Hence the functor τ_1 is representable by the object $1 \in \mathbf{S}^{\tau_1}$. It is thus a 2- functor. ∎

The notion of adjoint map in $\mathbf{S}^{\tau_1}$ can be weakened. Observe that 1-cell $u : A \to B$ in a 2-category $\mathcal{E}$ is a left adjoint iff the functor

$$\mathcal{E}(u, X) : \mathcal{E}(B, X) \to \mathcal{E}(A, X)$$

is a right adjoint for every object $X \in \mathcal{E}$. This motivates the following definition:

Definition 1.28. *We say that a map of simplicial sets* $u : A \to B$ *is a* weak left adjoint *if the functor*

$$\tau_1(u, X) : \tau_1(B, X) \to \tau_1(A, X)$$

is a right adjoint for every quasi-category X*. Dually, we say that* u *is a* weak right adjoint *if the functor* $\tau_1(u, X)$ *is a left adjoint for every quasi-category* X*.*

A map of simplicial sets $u : A \to B$ is a weak left adjoint iff the opposite map $u^o : A^o \to B^o$ is a weak right adjoint. The composite of two weak left adjoints is a weak left adjoint. A map between quasi-categories $X \to Y$ is a left adjoint iff it is a weak left adjoint. The notion of weak left adjoint is invariant under weak categorical equivalence. This means that given a commmutative square of simplicial sets

$$\begin{array}{ccc} A & \longrightarrow & A' \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u'} \\ B & \longrightarrow & B' \end{array}$$

in which the horizontal maps are weak categorical equivalences, then u is a weak left adjoint iff u' is a weak left adjoint.

1.6 Exercises

Exercise 1.29. *Show that a map of simplicial sets is an equivalence in the 2-category* $\mathbf{S}^{\tau_1}$ *iff it is a categorical equivalence.*

Exercise 1.30. *Show that each of the 2-cells of an adjunction* $(\alpha, \beta) : u \vdash v$ *determines the other.*

Exercise 1.31. *Show that a map between quasi-categories* $X \to Y$ *is a left adjoint iff it is a weak left adjoint.*

Exercise 1.32. *Show that the notion of weak left adjoint is invariant under weak categorical equivalence.*

Exercise 1.33. *Show that the functor* $\tau_1 : \mathbf{S} \to \mathbf{Cat}$ *takes a weak categorical equivalence to an equivalence of categories, and a weak left (resp. right) adjoint to a left (resp. right) adjoint.*

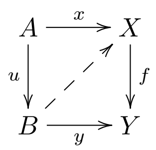
Chapter 2

Three classes of fibrations

In this chapter we introduce three classes of fibrations respectively called *left*, *mid* and *right* fibrations. Each class of fibrations forms a factorisation system with a class of anodyne maps respectively called *left*, *mid* and *right* anodyne maps. We also study the behavior of fibrations under exponentiation and apply the results to weak categorical equivalences

2.1 Left, right and mid fibrations

Recall from D.1.1 that an arrow $u : A \to B$ in a category is said to have the left lifting property (LLP) with respect to an arrow $f : X \to Y$, or that f is said to have the right lifting property (RLP) with respect to u, if every commutative square



has a diagonal filler $d : B \to X$ (that is, $du = x$ and $fd = y$). We shall denote this relation by $u \pitchfork f$.

Recall that a map of simplicial sets $p : X \to Y$ is said to be a *Kan fibration* if it has the RLP with respect to the inclusion $h^k_n : \Lambda^k[n] \subset \Delta[n]$ for every $n > 0$ and $0 \le k \le n$. This suggests the following definition.

Definition 2.1. *[J1] We shall say that a map of simplicial sets is a*

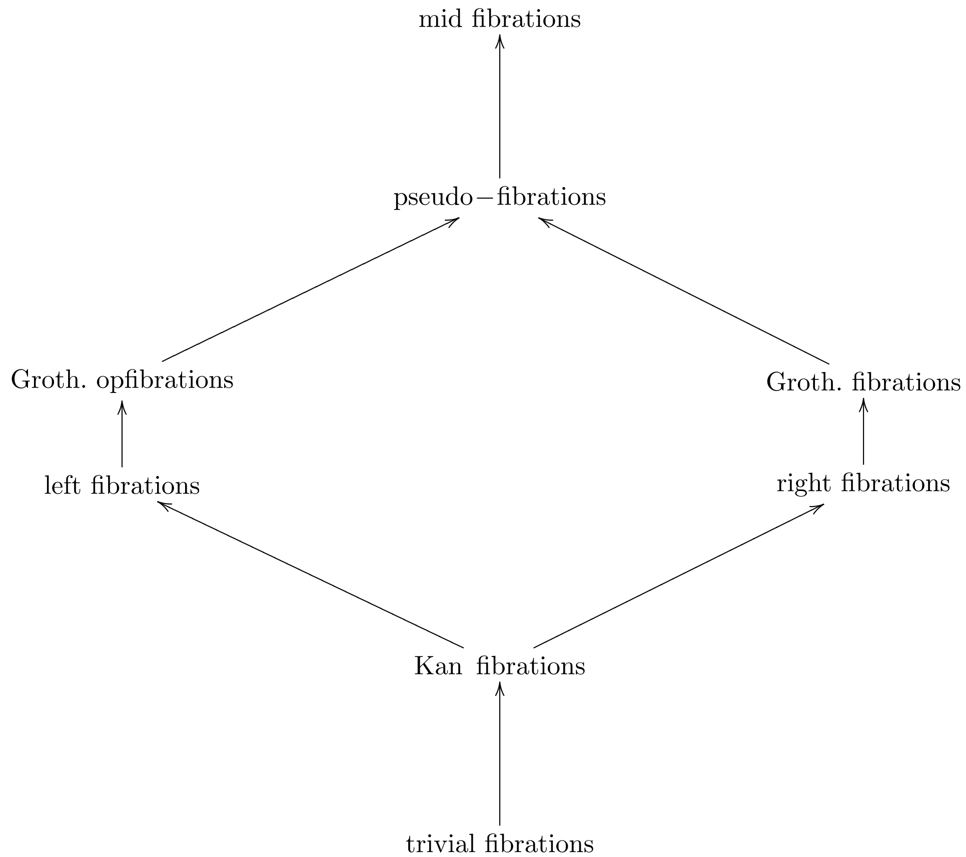
- left fibration *or a* covariant fibration *if it has the RLP with respect to* h^k_n *for every* $0 \le k < n$*;*
- mid fibration *if it has the RLP with respect to* h^k_n *for every* $0 < k < n$*;*

- right fibration *or a* contravariant fibration *if it has the RLP with respect to* h_n^k *for every* $0 < k \leq n$.

A map $p : X \to Y$ is a left fibration iff the opposite map $p^o : X^o \to Y^o$ is a right fibration. A map is a Kan fibration iff it is both a left and a right fibration. Each class of fibrations is closed under composition and base change. A simplicial set X is a quasi-category iff the map $X \to 1$ is a mid fibration.

We remark that if $p : X \to Y$ is a right fibration, then for every vertex $b \in X$ and every arrow $g \in Y$ with codomain $p(b)$, then there exists an arrow $f \in X$ with codomain b such that $p(f) = g$. There is a dual property for left fibration.

There are other many classes of fibrations than the three classes introduced in this chapter. We recall from B.0.9 that a map of of simplicial sets is a *trivial fibration* iff it has the right lifting property with respect to the inclusion $\delta_n : \partial\Delta[n] \subset \Delta[n]$ for every $n \geq 0$. A notion of *pseudo-fibration* will be introduced later in these notes6.3. The pseudo-fibrations are the fibrations of the model structure for quasi-categories 6.12. There is also a notion of Grothendieck fibration between simplicial sets and a dual notion of Grothendieck opfibration In all, we have the following diagram of inclusions between eight classes of fibrations.



Proposition 2.2. *If X is a quasi-category and C is a category, then every map $X \to C$ is a mid fibration. In particular, every functor in* **Cat** *is a mid fibration.*

Proof: We have to show that if $0 < k < n$, then every commutative square

$$\begin{array}{ccc} \Lambda^k[n] & \xrightarrow{x} & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ \Delta[n] & \xrightarrow{y} & C \end{array}$$

has a diagonal filler. The horn $x : \Lambda^k[n] \to X$ has a filler $z : \Delta[n] \to X$, since X is a quasi-category. Let us show that $pz = y$. But the maps $pz : \Delta[n] \to C$ and $y : \Delta[n] \to C$ both fill the same horn $px : \Lambda^k[n] \to C$. Thus, $pz = y$ by 1.7, since C is a category. This shows that z is a diagonal filler of the square. ∎

2.1.1 Supplement

Recall that a Grothendieck fibrations $p : E \to B$ in **Cat** is called a *1-fibration* if its fibers are groupoids.

Proposition 2.3. *A functor $p : E \to B$ in* **Cat** *is a right fibration in* **S** *iff it is a 1-fibration.*

Proof: ($\Leftarrow$) Let $p : E \to B$ be a 1-fibration in **Cat**. If $0 < k \le n$, let us show that every commutative square

$$\begin{array}{ccc} \Lambda^k[n] & \xrightarrow{x} & E \\ \downarrow & & \downarrow p \\ \Delta[n] & \xrightarrow{y} & B \end{array}$$

has a diagonal filler. This is clear if $n = 1$ and $k = 1$, since p is a Grothendieck fibration. This is clear if $0 < k < n$, since p is a mid fibration by 2.2. It remains to consider the case where $k = n > 1$. If $n = 2$, we have $px(02) = y(02)$ and $px(12) = y(12)$, since the square commutes. Hence there is a unique arrow $u : x(0) \to x(1)$ such that $p(u) = y(01)$ and $x(02) = x(12)u$, since every arrow in E is cartesian. The chain of arrows

$$x(0) \xrightarrow{u} x(1) \xrightarrow{x(12)} x(2)$$

determines a 2-simplex $\Delta[2] \to E$ which is a diagonal filler of the square. If $n > 2$ we have $(i, i+1) \in \Lambda^n[n]$ for every $0 \le i < n$. The chain of arrows

$$x(0) \xrightarrow{x(01)} x(1) \xrightarrow{x(12)} x(2) \longrightarrow \cdots \longrightarrow x(n-1) \xrightarrow{x(n-1,n)} x(n)$$

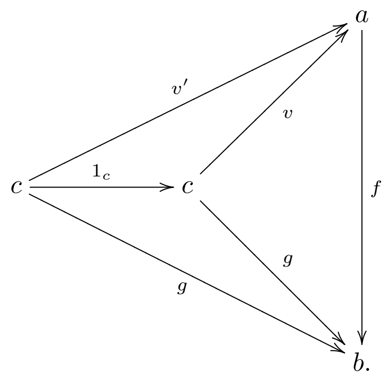
defines a simplex $\Delta[n] \to E$ which is a diagonal filler of the square. ($\Rightarrow$) Let us first show that every arrow $f : a \to b$ in E is cartesian with respect to p. If $g : c \to b$ is an arrow in E and $p(g) = p(f)u : p(c) \to p(a) \to p(b)$ is a factorisation in B, let us show that there is a unique arrow $v : c \to a$ such that $g = fv$ and $p(v) = u$. Consider the commutative square

$$\begin{array}{ccc} \Lambda^2[2] & \xrightarrow{x} & E \\ \downarrow & & \downarrow p \\ \Delta[2] & \xrightarrow{y} & B, \end{array}$$

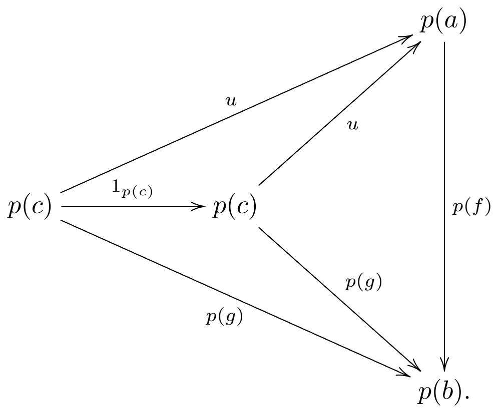
where where x is the horn $(f, g, \star)$ and where y is the 2-simplex defined by the chain of arrows

$$p(a) \xrightarrow{u} p(b) \xrightarrow{p(f)} p(c).$$

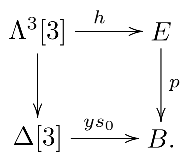
The square has a diagonal filler $z : \Delta[2] \to E$, since p is a right fibration. If $v = zd_2$, then $v : c \to a$, $g = fv$ and $p(v) = u$. This proves the existence of v. It remains to prove its uniqueness. Let $v' : c \to a$ be another arrow such that $g = fv'$ and $p(v') = u$. Let $s : Sk^1\Delta[3] \to E$ be the map defined by the following diagram of six arrows in E:



The faces $\partial_0 s$, $\partial_1 s$ and $\partial_2 s$ of this diagram commutes. Hence we have $\partial_0 s = \partial h_0$, $\partial_1 s = \partial h_1$ and $\partial_2 s = \partial h_2$, where $h_i : \Delta[2] \to E$. This defines a horn $h = (h_0, h_1, h_2, \star) : \Lambda^3[3] \to E$. The image by p of the diagram of arrows above is the following diagram,



The diagram commutes and it defines a degenerate simplex $ys_0 : \Delta[3] \to E$. This shows that the following square commutes,



The square has a diagonal filler $e : \Delta[3] \to E$, since p is a right fibration. If $e_0 = ed_3$, then $\partial e_0 = (v, v', 1_c)$. Thus, $v = v'$. We have proved that every arrow in E is cartesian. Let us now show that p is a Grothendieck fibration. For every object $a \in E$ and every arrow $g \in B$ with target $p(a)$, there exists an arrow $f \in E$ with target b such that $p(f) = g$, since the square

$$\begin{array}{ccc} \Lambda^1[1] & \xrightarrow{a} & E \\ \downarrow & & \downarrow{\scriptstyle p} \\ \Delta[1] & \xrightarrow{g} & B \end{array}$$

has a diagonal filler. The arrow f is cartesian, since every arrow is cartesian. This proves that p is a Grothendieck fibration. ■

2.1.2 Exercises

Exercise 2.4. (Descent property of fibrations) *If the base change of a map $p : X \to B$ along a surjection $u : A \to B$ is a Kan fibration, then p is a Kan fibration. A similar result is true for right fibrations, left fibrations, mid fibrations and trivial fibrations.*

Proposition 2.5. *A directed colimit of Kan fibrations is a Kan fibration. A similar result is true for right fibrations, left fibrations, mid fibrations and trivial fibrations.*

Proof: If $u : A \to B$ and $f : X \to Y$ are two maps of simplicial sets let us denote by $F(u, f)$ the map

$$\mathbf{S}(B, X) \to \mathbf{S}(B, Y) \times_{\mathbf{S}(A,Y)} \mathbf{S}(A, X)$$

obtained from the square

$$\begin{array}{ccc} \mathbf{S}(B, X) & \xrightarrow{\mathbf{S}(v,X)} & \mathbf{S}(A, X) \\ {\scriptstyle \mathbf{S}(B,f)}\downarrow & & \downarrow{\scriptstyle \mathbf{S}(A,f)} \\ \mathbf{S}(B, Y) & \xrightarrow{\mathbf{S}(v,Y)} & \mathbf{S}(A, Y). \end{array}$$

If we fix u, this defines a functor $F(u, -) : \mathbf{S}^I \to \mathbf{Set}^I$. It is easy to verify that the functor $F(u, -)$ preserves directed colimits when u is a map between finite simplicial sets (finite simplicial set=finitely generated=finitely presented). In particular, the functor $F(h_n^k, -)$ preserves directed colimits, since h_n^k is a map between finite simplicial sets. But a map $f : X \to Y$ is Kan fibration iff the map $F(h_n^k, f)$ is surjective for every horn h_n^k. This proves the result, since a directed colimit of surjections is a surjection. ■

Recall from Definition B.0.11 that a map of simplicial sets $X \to Y$ is said to be 0-*full* if the naturality square

$$\begin{array}{ccc} X & \longrightarrow & Cosk^0 X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Cosk^0 Y \end{array}$$

is cartesian. We shall say that a simplicial subset $S \subseteq X$ is *full* f the map $S \to X$ defined by the inclusion is 0-full.

Exercise 2.6. *A 0-full map is a mid fibration. In particular, a full simpliical subset of a quasi-category is a quasi-category.*

2.2 Left, right and mid anodyne maps

Recall from D.2.2 that a class of maps $\mathcal{A}$ in a cocomplete category is said to be *saturated* if it satisfies the following conditions:

- $\mathcal{A}$ contains the isomorphisms and is closed under composition ;
- $\mathcal{A}$ is closed under cobase change and retract;
- $\mathcal{A}$ is closed under transfinite composition.

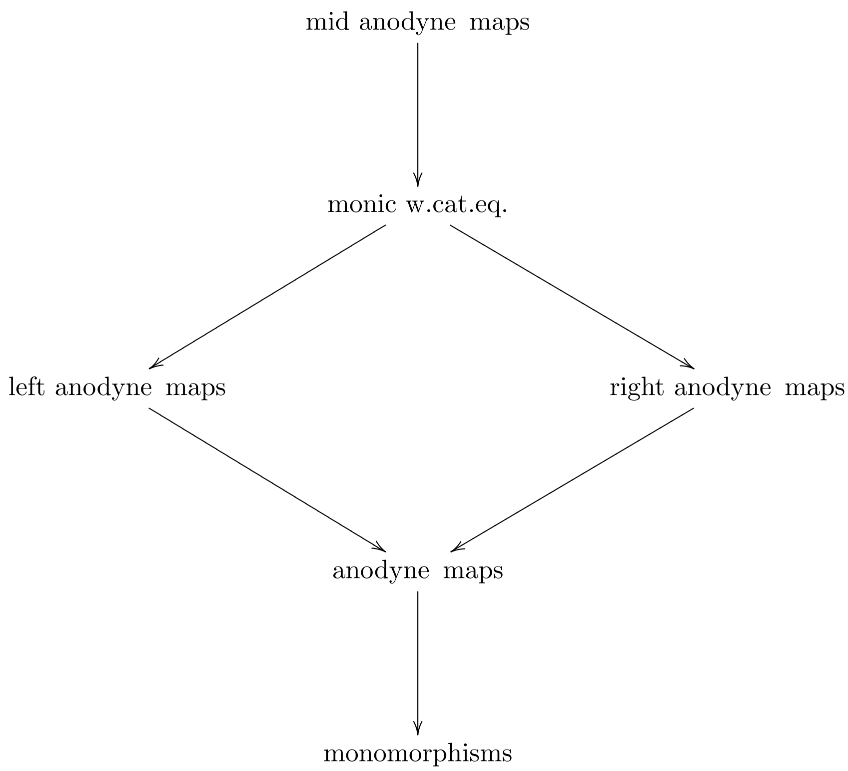
In a cocomplete category, every class of maps Σ is contained in a smallest saturated class $\overline{\Sigma}$ called the saturated class *generated* by Σ. For example, in the category $\mathbf{S}$, the saturated class of monomorphisms is generated by the set of inclusions $\delta_n : \partial\Delta[n] \subset \Delta[n]$ for $n \geq 0$. See Proposition B.0.8.

Recall from [GZ] that a map of simplicial sets is said to be *anodyne* if it belongs to the saturated class generated by the inclusions $\Lambda^k[n] \subset \Delta[n]$ with $0 \leq k \leq n > 0$. Every anodyne map is monic.

Definition 2.7. *We say that a map of simplicial sets is*

- left anodyne *if it belongs to the saturated class generated by the inclusions* $\Lambda^k[n] \subset \Delta[n]$ *with* $0 \leq k < n$*;*
- mid anodyne *if it belongs to the saturated class generated by the inclusions* $\Lambda^k[n] \subset \Delta[n]$ *with* $0 < k < n$*;*
- right anodyne *if it belongs to the saturated class generated by the inclusions* $\Lambda^k[n] \subset \Delta[n]$ *with* $0 < k \leq n$*.*

These classes of maps fit in the following diagram of inclusions.



A map $u : A \to B$ is left anodyne iff the opposite map $u^o : A^o \to B^o$ is right anodyne. We shall see in A that a monic weak categorical equivalence is a both left and right anodyne.

Theorem 2.8. *Each of the following pairs* $(\mathcal{A}, \mathcal{B})$ *of classes of maps in* $\mathbf{S}$ *is a weak factorisation system:*

- $\mathcal{A}$ *is the class of monomorphisms and* $\mathcal{B}$ *the class of trivial fibrations;*
- $\mathcal{A}$ *is the class of anodyne maps and* $\mathcal{B}$ *the class of Kan fibrations;*
- $\mathcal{A}$ *is the class of mid anodyne maps and* $\mathcal{B}$ *the class of mid fibrations;*
- $\mathcal{A}$ *is the class of left anodyne maps and* $\mathcal{B}$ *the class of left fibrations;*
- $\mathcal{A}$ *is the class of right anodyne maps and* $\mathcal{B}$ *the class of right fibrations.*

Proof: The first two results are classical [GZ]. See D.1.11 for a proof of the first. The others follow from Theorem D.2.11. ∎

Definition 2.9. *We say that a map of simplicial sets $u : A \to B$ is biunivoque indexmap!biunivoque—textbf if the map $u_0 : A_0 \to B_0$ is bijective.*

A functor $u : A \to B$ in **Cat** is biunivoque iff the map $Ob(A) \to Ob(B)$ induced by u is bijective. indexfunctor!biunivoque—textbf

Proposition 2.10. *A mid anodyne map is biunivoque. The functor $\tau_1 : \mathbf{S} \to \mathbf{Cat}$ takes a mid anodyne map to an isomorphism of categories.*

Proof: It is easy to see that the class of biunivoque maps is saturated. Every inner horn $h_n^k : \Lambda^k[n] \subset \Delta[n]$ is biunivoque, since h_n^k is biunivoque when $n > 1$. It follows that every mid anodyne map is biunivoque. The first statement is proved. Let us prove the second statement. Let $\mathcal{A} \subset \mathbf{S}$ be the class of maps $u : A \to B$, such that $\tau_1(u)$ is an isomorphism of categories. The class $\mathcal{A}$ is saturated by Proposition D.2.5, since the functor τ_1 is cocontinuous. Every inner horn $h_n^k : \Lambda^k[n] \subset \Delta[n]$ belongs to $\mathcal{A}$ by Lemma 1.6. It follows that every mid anodyne map belongs to $\mathcal{A}$. ■

We say that a functor $u : A \to B$ is said to be *1-final* if the category $b\backslash A = (b\backslash B) \times_B A$ defined by the pullback square

$$\begin{array}{ccc} b\backslash A & \xrightarrow{h} & A \\ \downarrow & & \downarrow u \\ b\backslash B & \longrightarrow & B. \end{array}$$

is simply connected for every object $b \in B$.

Proposition 2.11. *The functor $\tau_1 : \mathbf{S} \to \mathbf{Cat}$ takes a right anodyne map to a 1-final functor (hence also to a 0-final functor).*

Proof: If $\mathcal{B} \subset \mathbf{S}$ is the class of right fibrations and $\mathcal{A}$ is the class of right anodyne maps, then the pair $(\mathcal{A}, \mathcal{B})$ is a weak factorisation system in **S** by 2.8. If $\mathcal{B}' \subset \mathbf{Cat}$ is the class of 1-fibrations and $\mathcal{A}'$ is the class of 1-final cofibrations then the pair $(\mathcal{A}', \mathcal{B}')$ is a weak factorisation system in **Cat** by D.1.8. We have, $N(\mathcal{B}') \subseteq \mathcal{B}$ by 2.3. It follows by adjointness D.1.14 that we have $\tau_1(\mathcal{A}) \subseteq \mathcal{A}'$. ■

2.2.1 Supplement

If A is a subset of $[n]$, the *generalised horn* $\Lambda^A[n]$ is the simplicial subset of $\Delta[n]$ defined by putting

$$\Lambda^A[n] = \bigcup_{i \notin A} \partial_i \Delta[n].$$

Notice that $\Lambda^{\{k\}}[n] = \Lambda^k[n]$. If $A \subseteq B \subseteq [n]$ then $\Lambda^B[n] \subseteq \Lambda^A[n]$. If $a, b \in [n]$ and $a \leq b$, let us put $[a, b] = \{x \in [n] : a \leq x \leq b\}$. We shall say that $[a, b]$ is an *interval.*

Proposition 2.12. *Let $A \subseteq [n]$ be a non-empty subset of $[n]$.*

- *(i) if A is a proper subset, then the inclusion $i_A : \Lambda^A[n] \subset \Delta[n]$ is anodyne;*
- *(ii) if $A \subseteq [0, n-1]$, then i_A is left anodyne;*
- *(iii) if $A \subseteq [1, n]$, then i_A is right anodyne;*
- *(iv) if the complement of A is not an interval, then i_A is mid anodyne.*

Proof: Let us prove (i). We shall argue by induction on $r = \mathrm{Card}(A)$. If $r = 1$, then we have $\Lambda^A[n] = \Lambda^k[n]$ for some $k \in [n]$. The result is obvious in this case. Let us now suppose $r > 1$. In this case, let us choose an element $a \in A$ and put $B = A \backslash \{a\}$. The inclusion $\Lambda^B[n] \subset \Delta[n]$ is anodyne by the induction hypothesis, since B is a proper non-empty subset of $[n]$ and $\mathrm{Card}(B) < r$. Hence the result will be proved if we show that the inclusion $\Lambda^A[n] \subset \Lambda^B[n]$ is anodyne. The square

$$\begin{array}{ccc} \partial_a\Delta[n] \cap \Lambda^A[n] & \longrightarrow & \Lambda^A[n] \\ \downarrow & & \downarrow \\ \partial_a\Delta[n] & \longrightarrow & \Lambda^B[n] \end{array}$$

is a pushout, since $\Lambda^B[n] = \partial_a\Delta[n] \cup \Lambda^A[n]$. Hence it suffices to show that the inclusion $\partial_a\Delta[n] \cap \Lambda^A[n] \subset \partial_a\Delta[n]$ is anodyne. Let $S \subseteq [n-1]$ be the inverse image of the subset A by the map $d_a : [n-1] \to [n]$. The map $d_a : \Delta[n-1] \to \Delta[n]$ induces an isomorphism between $\Lambda^S[n-1]$ and $\partial_a\Delta[n] \cap \Lambda^A[n]$. It follows that the inclusion $\partial_a\Delta[n] \cap \Lambda^A[n] \subset \partial_a\Delta[n]$ is isomorphic to the inclusion $\Lambda^S[n-1] \subset \Delta[n-1]$. Hence it suffices to show that the inclusion $\Lambda^S[n-1] \subset \Delta[n-1]$ is anodyne. The map d_a induces a bijection between S and B. Thus, $S \neq \emptyset$, since $B \neq \emptyset$. Moreover, $\mathrm{Card}(S) = \mathrm{Card}(B) = r - 1 \leq n - 1$, since $r = \mathrm{Card}(A) \leq n$. This shows that S is a proper non-empty subset of $[n-1]$. Hence the inclusion $\Lambda^S[n-1] \subset \Delta[n-1]$ is anodyne by the induction hypothesis, since $\mathrm{Card}(S) < r$. The result is proved.

Let us prove (ii) and (iii). By symmetry, it is enough to prove (ii). Let us suppose that $A \subseteq [0, n-1]$ and show that i_A is left anodyne. We shall argue by induction on $r = \mathrm{Card}(A)$. If $r = 1$, we have $\Lambda^A[n] = \Lambda^k[n]$ for some $0 \leq k < n$. The result is obvious in this case. Let us suppose that $r > 1$. In this case, let us choose an element $a \in A$ and put $B = A \backslash \{a\}$. The inclusion $\Lambda^B[n] \subset \Delta[n]$ is left anodyne by the induction hypothesis, since $\mathrm{Card}(B) < r$. Hence the result will be proved if we show that the inclusion $\Lambda^A[n] \subset \Lambda^B[n]$ is left anodyne. As above, let $S \subseteq [n-1]$ be the inverse image of A by the map $d_a : [n-1] \to [n]$. It suffices to show that the inclusion $\Lambda^S[n-1] \subset \Delta[n-1]$ is left anodyne. As above, S is a non-empty subset of $[n-1]$. We have $a \neq n$, since $n \notin A$ and $a \in A$. Thus, $d_a(n-1) = n$. It follows that $S \subset [0, n-1]$. We have $\mathrm{Card}(S) = \mathrm{Card}(A) - 1 < r$, since $a \in A$. Hence the inclusion $\Lambda^S[n-1] \subset \Delta[n-1]$ is a left anodyne by the induction hypothesis. The result is proved.

Let us prove (iv). If $A' = [n] \setminus A$ is not an interval, let us show that i_A is mid anodyne. We shall prove the result by induction on $r = \mathrm{Card}(A)$. If $r = 1$, then we have $A = \{k\}$ for some $0 < k < n$. The result is obvious in this case. We can suppose that $r > 1$. There are elements $s < b < t$ with $s, t \in A'$ and $b \in A$, since A' is not an interval. There is an element $a \in A \setminus \{b\}$, since $r > 1$. Let us put $B = A \backslash \{a\}$. We have $B \neq \emptyset$, since $r > 1$. The complement $B' = [n] \setminus B$ is not an interval, since $s, t \in B'$ and $b \in B$. Hence the inclusion $\Lambda^B[n] \subset \Delta[n]$ is mid anodyne by the induction hypothesis, since $\mathrm{Card}(B) < r$. Hence the result will be proved if we show that the inclusion $\Lambda^A[n] \subset \Lambda^B[n]$ is mid anodyne. As above, let $S \subseteq [n-1]$ be the inverse image of A by the map $d_a : [n-1] \to [n]$. It suffices to show that the inclusion $\Lambda^S[n-1] \subset \Delta[n-1]$ is mid anodyne. As above, S is a non-empty subset of $[n-1]$. The elements s, t and b belongs to the image of d_a, since $a \notin \{s, t, b\}$. It follows that S is not an interval. Hence the inclusion $\Lambda^S[n-1] \subset \Delta[n-1]$ is mid anodyne by the induction hypothesis, since $\mathrm{Card}(S) = \mathrm{Card}(A) - 1 < r$. It follows that the inclusion $\Lambda^A[n] \subset \Lambda^B[n]$ is mid anodyne. ∎

For $n > 0$, the *n-chain* $I[n] \subseteq \Delta[n]$ is defined to be the union of the edges $(i-1, i) \subseteq \Delta[n]$ for $1 \leq i \leq n$. We shall put $I[0] = 1$.

Proposition 2.13. *The inclusion $I[n] \subseteq \Delta[n]$ is mid anodyne for every $n \geq 0$.*

Proof: Let us first show that the inclusion $\partial_0\Delta[n] \cup I[n] \subseteq \Delta[n]$ is mid anodyne by induction on $n > 0$. This is clear if $n = 1$. Let us suppose $n > 1$. The square

$$\begin{array}{ccc} I[n-1] \cup \partial_0\Delta[n-1] & \longrightarrow & \partial_0\Delta[n] \cup I[n] \\ \downarrow & & \downarrow \\ \partial_n\Delta[n] & \longrightarrow & \partial_0\Delta[n] \cup \partial_n\Delta[n] \end{array}$$

is a pushout, since

$$\partial_n\Delta[n] \cap (\partial_0\Delta[n] \cup I[n]) = I[n-1] \cup \partial_0\Delta[n-1]$$

and

$$\partial_n\Delta[n] \cup (\partial_0\Delta[n] \cup I[n]) = \partial_0\Delta[n] \cup \partial_n\Delta[n].$$

The inclusion $I[n-1] \cup \partial_0\Delta[n-1] \subseteq \partial_n\Delta[n]$ is mid anodyne by the induction hypothesis. It follows that the inclusion $\partial_0\Delta[n] \cup I[n] \subseteq \partial_0\Delta[n] \cup \partial_n\Delta[n]$ is mid anodyne. But the inclusion $\partial_0\Delta[n] \cup \partial_n\Delta[n] \subseteq \Delta[n]$ is mid anodyne by 2.12. It follows by composing that the inclusion $\partial_0\Delta[n] \cup I[n] \subseteq \Delta[n]$ is mid anodyne. Hence also the inclusion $I[n] \cup \partial_n\Delta[n] \subseteq \Delta[n]$ by symmetry. We can now prove by induction on n that the inclusion $I[n] \subseteq \Delta[n]$ is mid anodyne. We can suppose

$n > 1$. The square

$$\begin{array}{ccc} I[n-1] & \longrightarrow & I[n] \\ \downarrow & & \downarrow \\ \partial_n\Delta[n] & \longrightarrow & I[n]\cup\partial_n\Delta[n] \end{array}$$

is a pushout, since $I[n-1] = \partial_n\Delta[n] \cap I[n]$. The inclusion $I[n-1] \subseteq \partial_n\Delta[n]$ is mid anodyne by the induction hypothesis. It follows that the inclusion $I[n] \subseteq I[n]\cup\partial_n\Delta[n]$ is mid anodyne. But we saw that the inclusion $I[n]\cup\partial_n\Delta[n] \subseteq \Delta[n]$ is mid anodyne. It follows by composing that the inclusion $I[n] \subseteq \Delta[n]$ is mid anodyne. ∎

2.3 Function spaces

The proofs of this section depend heavily on the Appendix on boxes and prisms.

If $u : A \to B$ and $f : X \to Y$ are maps of simplicial sets, we shall denote by $\langle u, f\rangle$ the map

$$X^B \to Y^B \times_{Y^A} X^A.$$

obtained from the square

$$\begin{array}{ccc} X^B & \longrightarrow & X^A \\ \downarrow & & \downarrow \\ Y^B & \longrightarrow & Y^A, \end{array}$$

The main result of the chapter is the following theorem. The first statement is classical.

Theorem *If $f : X \to Y$ is a Kan fibration (resp. mid fibration, left fibration, right fibration) then so is the map*

$$\langle u, f\rangle : X^B \to Y^B \times_{Y^A} X^A$$

for any monomorphism $u : A \to B$. Moreover, $\langle u, f\rangle$ is a trivial fibration if in addition u is anodyne (resp. mid anodyne, left anodyne, right anodyne).

The theorem is proved in 2.18. It follows from this theorem that if X is a quasi-category, then so is the simplicial set X^A for any simplicial set A.

We shall denote by $\mathbf{S}^I$ the category of arrows in $\mathbf{S}$. A morphism $u \to f$ in $\mathbf{S}^I$ is a commutative square in $\mathbf{S}$:

$$\begin{array}{ccc} A & \xrightarrow{x} & X \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{y} & Y. \end{array}$$

If $u : A \to B$ and $v : S \to T$ are two maps in $\mathbf{S}$, we shall denote by $u \times' v$ the map

$$(A \times T) \sqcup_{A\times S} (B \times S) \to B \times T$$

obtained from the commutative square

$$\begin{array}{ccc} A \times S & \xrightarrow{u\times S} & B \times S \\ {\scriptstyle A\times v}\downarrow & & \downarrow{\scriptstyle B\times v} \\ A \times T & \xrightarrow{u\times T} & B \times T. \end{array}$$

The operation $(u, v) \mapsto u \times' v$ is functorial in $u, v \in \mathbf{S}^I$. The resulting functor

$$\times' : \mathbf{S}^I \times \mathbf{S}^I \to \mathbf{S}^I$$

defines a symmetric monoidal structure on $\mathbf{S}^I$. The unit object given by the arrow $\emptyset \to 1$.

Lemma 2.14. *The monoidal category* $(\mathbf{S}^I, \times')$ *is closed. The right adjoint to the functor* $v \mapsto u \times' v$ *is the functor* $f \mapsto \langle u, f\rangle$. *We have*

$$u \pitchfork \langle v, f\rangle \iff (u \times' v) \pitchfork f \iff v \pitchfork \langle u, f\rangle.$$

Proof: This follows from D.1.18 ∎

Proposition 2.15. [GZ] *If* $u : A \to B$ *and* $v : S \to T$ *are monic, then* $u \times' v$ *is monic.*

Proof: We can suppose that u is an inclusion $A \subseteq B$ and that v is an inclusion $S \subseteq T$. In this case we have

$$(A \times T) \cap (B \times S) = A \times S,$$

where the intersection is taken in $B \times T$. It follows that the square of inclusions

$$\begin{array}{ccc} A \times S & \longrightarrow & B \times S \\ \downarrow & & \downarrow \\ A \times T & \longrightarrow & (A \times T) \cup (B \times S) \end{array}$$

is a pushout, where the union is taken in $B \times T$. Thus, $u \times' v$ is the inclusion

$$(A \times T) \cup (B \times S) \subseteq B \times T.$$

∎

The following result is classical.

Proposition 2.16. [GZ] *If $f : X \to Y$ is a trivial fibration then so is the map*

$$\langle u, f\rangle : X^B \to X^A \times_{Y^A} Y^B$$

for any monomorphism $u : A \to B$.

Proof: Let us show that $\langle u, f\rangle$ has the right lifting property with respect to every monomorphism $v : S \to T$. But the condition $v \pitchfork \langle u, f\rangle$ is equivalent to the condition $(u \times' v) \pitchfork f$ by 2.14. But we have $(u \times' v) \pitchfork f$, since $u \times' v$ is monic by 2.15. ∎

Theorem 2.17. *If a monomorphism of simplicial sets $u : A \subseteq B$ is anodyne (resp. mid anodyne, left anodyne, right anodyne), then so is the monomorphism $u \times' v$ for any monomorphism $v : S \to T$.*

Proof: Let us prove the second statement. We shall use D.2.6. Let us denote by $\overline{\mathcal{M}}$ the saturated class generated by a class $\mathcal{M}$ of maps in **S**. If $\mathcal{M}$ and $\mathcal{N}$ are two classes of maps we denote by $\mathcal{M} \times' \mathcal{N}$ the class of maps $u \times' v$, for $u \in \mathcal{M}$ and $v \in \mathcal{N}$. Let us show that we have

$$\mathcal{A} \times' \mathcal{C} \subseteq \mathcal{A},$$

where $\mathcal{A}$ is the class of mid anodyne maps and $\mathcal{C}$ is the class of monomorphisms. The class $\mathcal{C}$ is saturated and generated by the set Σ_1 of inclusions $\delta_n : \partial\Delta[n] \subset \Delta[n]$ for $n \geq 0$ by B.0.8. The class $\mathcal{A}$ is saturated and generated by the set Σ_2 of inclusions $h^k_m : \Lambda^k[m] \subset \Delta[m]$ with $0 < k < m$. We have $\Sigma_1 \times' \Sigma_2 \subseteq \mathcal{A}$, since we have $h^k_m \times' \delta_n \in \mathcal{A}$ for every $0 < k < m$ and $n \geq 0$ by H.0.20. Hence we have $\overline{\Sigma}_1 \times' \overline{\Sigma}_2 \subseteq \mathcal{A}$ by D.2.6. The inclusion $\mathcal{A} \times' \mathcal{C} \subseteq \mathcal{A}$ is proved. The other statements have a similar proof. ∎

We can now prove the main theorem of the section:

Theorem 2.18. *If $f : X \to Y$ is a Kan fibration (resp. mid fibration, left fibration, right fibration) then so is the map*

$$\langle u, f\rangle : X^B \to Y^B \times_{Y^A} X^A$$

for any monomorphism $u : A \to B$. Moreover, $\langle u, f\rangle$ is a trivial fibration if in addition u is anodyne (resp. mid anodyne, left anodyne, right anodyne).

Proof: Let us prove the second statement. If $\mathcal{M}$ and $\mathcal{N}$ are two classes of maps in **S**, let us denote by $\mathcal{M} \times' \mathcal{N}$ the class of maps $u \times' v$, for $u \in \mathcal{M}$ and $v \in \mathcal{N}$, and by $\langle \mathcal{M}, \mathcal{N}\rangle$ the class of maps $\langle u, v\rangle$, for $u \in \mathcal{M}$ and $v \in \mathcal{N}$. We have $\mathcal{A} \times' \mathcal{C} \subseteq \mathcal{A}$ by 2.17. It follows that we have

$$\langle \mathcal{C}, \mathcal{A}\rangle \subseteq \mathcal{A}$$

by D.1.20. The other statements have a similar proof. ∎

Corollary 2.19. *If X is a quasi-category, then so is the simplicial set X^A for any simplicial set A. In particular, the category* **QCat** *is cartesian closed.*

Proof: It sufices to apply theorem 2.18 to the maps $\emptyset \subseteq A$ and $X \to 1$. ∎

Corollary 2.20. *If X is a quasi-category, then the map $X^u : X^B \to X^A$ is a mid fibration for any monomorphism of simplicial sets $u : A \to B$.*

Proof: It sufices to apply theorem 2.18 to the maps $i : A \to B$ and $X \to 1$.

Corollary 2.21. *If $f : X \to Y$ is a map between quasi-categories then the simplicial set $Y^B \times_{Y^A} X^A$ is a quasi-category for any monomorphism of simplicial sets $u : A \to B$.*

Proof: The map $Y^u : Y^B \to Y^A$ is a mid fibration by 2.20. Hence also the projection p_2 in the cartesian square

$$\begin{array}{ccc} Y^B \times_{Y^A} X^A & \xrightarrow{p_2} & X^A \\ \downarrow & & \downarrow \\ Y^B & \longrightarrow & Y^A. \end{array}$$

It follows that the simplicial set $Y^B \times_{Y^A} X^A$ is a quasi-category, since X^A is a quasi-category by 2.19. ∎

A homotopy $\alpha : f \to g$ between two maps $f, g : B \to X$ can be represented as a map $\alpha : B \times I \to X$, or as a map $\lambda^I \alpha : B \to X^I$ or as an arrow $\lambda^B \alpha : I \to X^B$. If $p : X \to Y$, we shall denote the homotopy $p\alpha : B \times I \to Y$ by $p \circ \alpha : pf \to pg$; if $u : A \to B$, we shall denote the homotopy $\alpha(u \times I) : A \times I \to X$ by $\alpha \circ u : fu \to gu$.

Proposition 2.22. (Covering homotopy extension property for left fibrations) Suppose that we have a commutative square

$$\begin{array}{ccc} A & \xrightarrow{a} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{b} & Y \end{array}$$

in which p is a left fibration and i is monic. Suppose also that we have a map $c : B \to X$ and two homotopies $\alpha : ci \to a$ and $\beta : pc \to b$ such that $p \circ \alpha = \beta \circ i$. Then there exists a pair (d, σ), where $d : B \to X$ is a diagonal filler of the square and $\sigma : c \to d$ is a homotopy such that $\sigma \circ i = \alpha$ and $p \circ \sigma = \beta$.

Proof: We can suppose that i is an inclusion $A \subseteq B$. We have $\alpha(x,0) = c(x)$ for every $x \in A$. Hence there is a map

$$u : (A \times I) \cup (B \times \{0\}) \to X$$

such that $u(x,t) = \alpha(x,t)$ for $(x,t) \in A \times I$, and such that $u(x,0) = c(x)$ for $x \in B$. The following square commutes

$$\begin{array}{ccc} (A \times I) \cup (B \times \{0\}) & \xrightarrow{u} & X \\ \downarrow & & \downarrow p \\ B \times I & \xrightarrow{\beta} & Y, \end{array}$$

since we have $p\alpha(x,t) = \beta(i(x),t)$ for $(x,t) \in A \times I$ and we have $pc(x) = \beta(x,0)$ for $x \in B$. The inclusion $(A \times I) \cup (B \times \{0\}) \subseteq B \times I$ is left anodyne by Theorem 2.18, since the inclusion $\{0\} \subset I$ is left anodyne. Hence the square has a diagonal filler $\sigma : B \times I \to X$, since p is a left fibration. We then have $\sigma(x,t) = u(x,t) = \alpha(x,t)$ for $(x,t) \in A \times I$, and $f\sigma(x,t) = \beta(x,t)$ for $(x,t) \in B \times I$. Let us put $d(x) = \sigma(x,1)$ for $x \in B$. The map $d : B \to X$ is a diagonal filler of the original square, since we have $d(i(x)) = \sigma(i(x),1) = \alpha(x,1) = a(x)$ for $x \in A$, and we have $fd(x) = f\sigma(x,1) = \beta(x,1) = b(x)$ for $x \in B$. We have $\sigma : c \to d$, since $\sigma(x,0) = u(x,0) = c(x)$. ■

2.3.1 Supplement

Let us denote by h_n^k the inclusion $\Lambda^k[n] \subset \Delta[n]$.

Lemma 2.23. *If $0 < k < n$, then the inclusion h_n^k is a retract of the inclusion*

$$h_n^k \times' h_2^1 : \big(\Lambda^k[n] \times \Delta[2]\big) \cup \big(\Delta[n] \times \Lambda^1[2]\big) \subset \Delta[n] \times \Delta[2].$$

Proof: Let $s : [n] \to [n] \times [2]$ be the map defined by

$$s(x) = \begin{cases} (x,0) & \text{if } x < k \\ (x,1) & \text{if } x = k \\ (x,2) & \text{if } x > k, \end{cases}$$

and $r : [n] \times [2] \to [n]$ the map defined by

$$r(x,t) = \begin{cases} x \wedge k & \text{if } t = 0 \\ k & \text{if } t = 1 \\ x \vee k & \text{if } t = 2. \end{cases}$$

We have $rs(x) = x$ for every $x \in [n]$. The result will be proved if we show that s induces a map

$$s' : \Lambda^k[n] \to \big(\Lambda^k[n] \times \Delta[2]\big) \cup \big(\Delta[n] \times \Lambda^1[2]\big)$$

and that r induces a map

$$r' : \big(\Lambda^k[n] \times \Delta[2]\big) \cup \big(\Delta[n] \times \Lambda^1[2]\big) \to \Lambda^k[n].$$

If $i \in [n]$, then we have $p_1 s d_i(x) = d_i(x) \neq i$ for every $x \in [n-1]$. Hence we have

$$s(\partial_i \Delta[n]) \subseteq \partial_i \Delta[n] \times \Delta[2]$$

for every $i \in [n]$. This shows that s induces a map s'. If $i \neq k$, then we have $r(d_i(x), t) \neq i$ for every $x \in [n-1]$. Thus,

$$r(\partial_i \Delta[n] \times \Delta[2]) \subseteq \partial_i \Delta[n]$$

in this case. If $k > 0$ and $t > 0$, then we have $r(x,t) \geq k > 0$ for every $x \in [n]$. Thus,

$$r(\Delta[n] \times \partial_0 \Delta[2]) \subseteq \partial_0 \Delta[n]$$

in this case. If $k < n$ and $t < 2$, then we have $r(x,t) \leq k < n$ for every $x \in [n]$. Thus,

$$r(\Delta[n] \times \partial_2 \Delta[2]) \subseteq \partial_2 \Delta[n]$$

in this case. Altogether, this shows that r induces a map r'. ■

If X is a simplicial set, consider the projection $p_n : X^{\Delta[n]} \to X^{I[n]}$ defined from the inclusion $I[n] \subseteq \Delta[n]$.

Proposition 2.24. *The following conditions on a simplicial set X are equivalent:*

- *(i) X is a quasi-category;*
- *(ii) the projection $p_2 : X^{\Delta[2]} \to X^{\Lambda^1[2]}$ is a trivial fibration;*
- *(iii) the projection $p_n : X^{\Delta[n]} \to X^{I[n]}$ is a trivial fibration for every $n \geq 0$.*

Proof: Let us prove the implication (i)⇒(iii). The projection p_n is a trivial fibration by 2.18, since the inclusion $I[n] \subseteq \Delta[n]$ is mid anodyne by 2.13. The implication (i)⇒(iii) is proved. The implication (iii)⇒(ii) is trivial, since $\Lambda^1[2] = I_2$. Let us prove the implication (ii)⇒(i). We have $p_2 = \langle h_2^1, p_X \rangle$ where p_X is the map $X \to 1$. If p_2 is a trivial fibration, then we have $h_n^k \pitchfork \langle h_2^1, p_X \rangle$ for every h_n^k. Hence we have $(h_n^k \times' h_2^1) \pitchfork p_X$ for every h_n^k by 2.14. But the map h_n^k is a retract of the map $h_n^k \times' h_2^1$ if $0 < k < n$ by 2.23. It follows that we have $h_n^k \pitchfork p_X$ for every $0 < k < n$. This shows that X is a quasi-category. ■

Lemma 2.25. *If $0 < k \leq n$, the inclusion h_n^k is a retract of the inclusion*

$$h_n^k \times' h_1^1 : \big(\Lambda^k[n] \times I\big) \cup \big(\Delta[n] \times \{1\}\big) \subset \Delta[n] \times I.$$

Proof: Consider the map $s : [n] \to [n] \times [1]$ given by $s(x) = (x, 0)$ and the map $r : [n] \times [1] \to [n]$ given by

$$r(x,t) = \begin{cases} x & \text{if } t = 0 \\ x \vee k & \text{if } t = 1. \end{cases}$$

We have $rs(x) = x$ for every $x \in [n]$. The result will be proved if we show that s induces a map

$$s' : \Lambda^k[n] \to \big(\Lambda^k[n] \times I\big) \cup \big(\Delta[n] \times \{1\}\big)$$

and that r induces a map

$$r' : \big(\Lambda^k[n] \times I\big) \cup \big(\Delta[n] \times \{1\}\big) \to \Lambda^k[n].$$

We have $s(\Lambda^k[n]) = \Lambda^k[n] \times \{0\} \subseteq \Lambda^k[n] \times I$. This shows that s induces a map s'. If $i \neq k$, we have $r(d_i(x), t) \neq i$ for every $x \in [n-1]$ and $t \in [1]$. It follows that we have $r(\partial_i \Delta[n] \times \Delta[2]) \subseteq \partial_i \Delta[n]$ for every $i \neq k$. Moreover, we have $r(x, 1) = x \vee k \geq k > 0$ for every $x \in [n]$. Thus, $r(\Delta[n] \times \{1\}) \subseteq \partial_0 \Delta[n]$. This shows that r induces a map r'. ∎

Proposition 2.26. *A map of simplicial sets $f : X \to Y$ is a right fibration iff the map*

$$\langle i_1, f \rangle : X^I \to X \times_Y Y^I$$

defined from the inclusion $i_1 : \{1\} \subset I$ is a trivial fibration.

Proof: ($\Rightarrow$) The projection $\langle i_1, f \rangle$ is a trivial fibration by 2.16, since i_1 is right anodyne. ($\Leftarrow$) Let us suppose that $\langle i_1, f \rangle$ is a trivial fibration. Then we have $h_n^k \pitchfork \langle i_1, f \rangle$ for every horn $h_n^k : \Lambda^k[n] \subset \Delta[n]$. It follows by 2.14 that we have $(h_n^k \times' i_1) \pitchfork f$. If $0 < k \leq n$, the map h_n^k is a retract of the map $h_n^k \times' h_1^1 = h_n^k \times' i_1$ by 2.25. It follows that we have $h_n^k \pitchfork f$ for every $0 < k \leq n$. This shows that f is a right fibration. ∎

2.4 Applications to weak categorical equivalences

Recall from definition 1.20 that a map of simplicial sets $u : A \to B$ is said to be a weak categorical equivalence if the map

$$\tau_0(u, X) : \tau_0(B, X) \to \tau_0(A, X)$$

is bijective for every quasi-category X.

Proposition 2.27. *A map $u : A \to B$ is a weak categorical equivalence iff the map*

$$X^u : X^B \to X^A$$

is an equivalence of quasi-categories for every quasi-category X.

Proof: ($\Rightarrow$) Let u be a weak categorical equivalence. If X is a quasi-category then so is the simplicial set X^S for any simplicial set S by 2.19. Hence the map

$$\tau_0(u, X^S) : \tau_0(B, X^S) \to \tau_0(A, X^S)$$

is bijective, since u is a weak categorical equivalence. But the map $\tau_0(u, X^S)$ is isomorphic to the map

$$\tau_0(S, X^u) : \tau_0(S, X^B) \to \tau_0(S, X^A),$$

since the category $\mathbf{S}^{\tau_0}$ is cartesian closed. Hence the map $\tau_0(S, X^u)$ is bijective for any simplicial set S. It follows by Yoneda Lemma that X^u is a categorical equivalence. ($\Leftarrow$) If X is a quasi-category, then the map

$$\tau_0(1, X^u) : \tau_0(1, X^B) \to \tau_0(1, X^A)$$

is bijective, since the map X^u is a categorical equivalence. But the map $\tau_0(1, X^u)$ is isomorphic to the map

$$\tau_0(u, X) : \tau_0(B, X) \to \tau_0(A, X).$$

This shows u is a weak categorical equivalence. ∎

Proposition 2.28. *The cartesian product of two weak categorical equivalences is a weak categorical equivalence.*

Proof: Let $u : A \to B$ and $v : S \to T$ be two weak categorical equivalences. Consider the decomposition $u \times v = (B \times v)(u \times S)$. It suffices to show that $u \times S$ is a weak categorical equivalence. The map $\tau_0(u \times S, X)$ is isomorphic to the map $\tau_0(u, X^S)$ for any simplicial set X, since the category $\mathbf{S}^{\tau_0}$ is cartesian closed. Thus, it suffices to show that the map $\tau_0(u, X^S)$ is bijective for any quasi-category X. But this is clear, since X^S is a quasi-category by 2.19, and since u is a weak categorical equivalence by hypothesis. ∎

Corollary 2.29. *Every mid anodyne map is a weak categorical equivalence.*

Proof: If $u : A \to B$ is mid anodyne then the map $X^u : X^B \to X^A$ is a trivial fibration for any quasi-category X by 2.18 applied to the maps $u : A \to B$ and $X \to 1$. It is thus a categorical equivalence by 1.22. The result then follows from 2.27. ∎

It follows from D.2.8 that there exists a functor $Q : \mathbf{S} \to \mathbf{S}$ together with a natural transformation $\eta : Id \to Q$ having the following properties:

- the simplicial set $Q(A)$ is a quasi-category for every $A \in \mathbf{S}$;
- the map $\eta_A : A \to Q(A)$ is mid anodyne for every $A \in \mathbf{S}$;

Corollary 2.30. *A map of simplicial set $u : A \to B$ is a weak categorical equivalence iff the map $Q(u) : Q(A) \to Q(B)$ is an equivalence of quasi-categories.*

Proof: Consider the naturality square

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & Q(A) \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle Q(u)} \\ B & \xrightarrow{\eta_B} & Q(B). \end{array}$$

The horizontal maps of the square are weak categorical equivalences by 2.29, since they are mid anodyne. It follows by three-for-two that u is a weak categorical equivalence iff $Q(u)$ is a weak categorical equivalence. But $Q(u)$ is a weak categorical equivalence iff it is an equivalence of quasi-categories by 1.21. ∎

Chapter 3

Join and slices

In this chapter we introduce and study the *join* $A \star B$ of two simplicial sets as well as the *upper slice* $a\backslash X$ and the *lower slice* X/a of a simplicial set X by a map $a : A \to X$. The chapter has two sections.

3.1 Join and slice for categories

Before defining the join of two simplicial sets, we describe this operation for categories. The *join* of two categories A and B is the category $C = A \star B$ obtained as follows: $Ob(C) = Ob(A) \sqcup Ob(B)$ and for any pair of objects $x, y \in Ob(A) \sqcup Ob(B)$ we have

$$C(x,y) = \begin{cases} A(x,y) & \text{if } x \in A \text{ and } y \in A \\ B(x,y) & \text{if } x \in B \text{ and } y \in B \\ 1 & \text{if } x \in A \text{ and } y \in B \\ \emptyset & \text{if } x \in B \text{ and } y \in A. \end{cases}$$

Composition of arrows is obvious. Notice that the category $A \star B$ is a poset if A and B are posets: it is the *ordinal sum* of the posets A and B. The operation $(A, B) \mapsto A \star B$ is functorial and coherently associative. It defines a monoidal structure on **Cat**, with the empty category as the unit object. The monoidal category $(\mathbf{Cat}\star)$ is not symmetric but there is a natural isomorphism

$$(A \star B)^o = B^o \star A^o.$$

The category $1 \star A$ is called the *projective cone* with *base* A and the category $A \star 1$ the *inductive cone* with *base* A. The object 1 is terminal in $A \star 1$ and initial in $1 \star A$. The category $A \star B$ can be equipped with the functor $A \star B \to I = 1 \star 1$ obtained by joining the functors $A \to 1$ and $B \to 1$. The resulting functor $\star : \mathbf{Cat} \times \mathbf{Cat} \to \mathbf{Cat}/I$ is right adjoint to the functor

$$i^* : \mathbf{Cat}/I \to \mathbf{Cat} \times \mathbf{Cat},$$

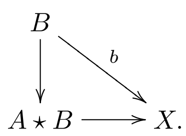
where i denotes the inclusion $\{0,1\} = \partial I \subset I$,. This gives another description of the join operation.

3.1.1

The monoidal category $(\mathbf{Cat}, \star)$ is not closed. But for any category $B \in \mathbf{Cat}$, the functor

$$(-) \star B : \mathbf{Cat} \to B\backslash\mathbf{Cat}$$

which associates to $A \in \mathbf{Cat}$ the inclusion $B \subseteq A \star B$ has a right adjoint. The right adjoint takes a functor $b : B \to X$ to a category that we shall denote by X/b. We shall say that X/b is the *lower slice* of X by b. For any category A, there is a bijection between the functors $A \to X/b$ and the functors $A \star B \to X$ which extend b along the inclusion $B \subseteq A \star B$,



In particular, an object $1 \to X/b$ is a functor $c : 1 \star B \to X$ which extends b; it is a *projective cone* with *base* b.

3.1.2

Dually, the functor $A \star (-) : \mathbf{Cat} \to A\backslash\mathbf{Cat}$ has a right adjoint which takes a functor $a : A \to X$ to a category that we shall denote $a\backslash X$. We shall say that $a\backslash X$ is the *upper slice* of X by a. An object $1 \to a\backslash X$ is a functor $c : A \star 1 \to C$ which extends a; it is an *inductive cone* with *base* a.

3.2 Join for simplicial sets

We use augmented simplicial sets for defining the join operations for simplicial sets. Let $\Delta^+ \supset \Delta$ be the category of finite ordinals and order preserving maps. The empty ordinal $0 = \emptyset$ is the only object of Δ^+ which is not in Δ. We shall denote the ordinal n by n, so that we have $n = [n-1]$ for $n \geq 1$. We may occasionally denote the ordinal 0 by $[-1]$. The ordinal sum $(m,n) \mapsto m+n$ is functorial with respect to order preserving maps. This defines a monoidal structure on Δ^+,

$$+ : \Delta^+ \times \Delta^+ \to \Delta^+,$$

with 0 as the unit object.

An *augmented simplicial set* is defined to be a contravariant functor $\Delta^+ \to$ **Set**. We shall denote by $\mathbf{S}^+$ the category $[(\Delta^+)^o, \mathbf{Set}]$ of augmented simplicial sets. By a general procedure due to Brian Day [Da], the monoidal structure $+ : \Delta^+ \times \Delta^+ \to \Delta^+$ can be extended to $\mathbf{S}^+$ as a closed monoidal structure

$$\star : \mathbf{S}^+ \times \mathbf{S}^+ \to \mathbf{S}^+.$$

We call $X \star Y$ the *join* of X and Y. By construction, the contravariant functor $X \star Y : \Delta^+ \to \mathbf{Set}$ is the left Kan extension of the contravariant functor $(p,q) \mapsto X(p) \times Y(q)$ along the functor $\Delta^+ \times \Delta^+ \to \Delta^+$. The unit object for this operation is the augmented simplicial set $0 = \Delta^+(-,0)$.

Proposition 3.1. *If $X, Y \in \mathbf{S}^+$, then we have*

$$(X \star Y)(n) = \bigsqcup_{i+j=n} X(i) \times Y(j)$$

for every $n \geq 0$.

Proof: By construction, we have

$$(X \star Y)(n) = \int^{p \in \Delta^+} \int^{q \in \Delta^+} \Delta^+(n, p+q) \times X(p) \times Y(q).$$

It follows that we have

$$(X \star Y)(n) = \varinjlim_{E_n} X(p) \times Y(q),$$

where the colimit is taken over the category E_n of elements of the functor $(p,q) \mapsto \Delta^+(n, p+q)$. But every map $f : n \to p+q$ in Δ^+ is of the form $f = u + v : i + j \to p + q$ for a unique pair of maps $u : i \to p$ and $v : j \to q$. Hence the set of decompositions $n = i + j$ is initial in the category E_n. The result follows. ∎

From the inclusion $i : \Delta \subset \Delta^+$ we obtain a pair of adjoint functors

$$i^* : \mathbf{S}^+ \leftrightarrow \mathbf{S} : i_*.$$

The functor i^* delete the augmentation of an augmented simplicial set, and the functor i_* augment a simplicial set A with the trivial augmentation $A_0 \to 1$. More precisely, we have $i_*(A)(0) = 1$ and $i_*(A)(n) = X_{n-1}$ for every $n \geq 1$. Clearly, the functor i_* is full and faithful. An augmented simplicial set X belongs the the essential image of i_* iff $X(0) = 1$. If $X(0) = 1$ and $Y(0) = 1$, then $(X \star Y)(0) = X(0) \times Y(0) = 1$ 3.1. Hence the operation $\star : \mathbf{S}^+ \times \mathbf{S}^+ \to \mathbf{S}^+$ induces a monoidal structure on $\mathbf{S}$,

$$\star : \mathbf{S} \times \mathbf{S} \to \mathbf{S}.$$

By definition, we have

$$i_*(A \star B) = i_*(A) \star i_*(B)$$

for any pair $A, B \in \mathbf{S}$. We call $A \star B$ the *join* of the simplicial sets A and B. Notice that we have

$$A \star \emptyset = A = \emptyset \star A$$

for any simplicial set A, since the augmented simplicial set $i_*(\emptyset) = 0$ is the unit object for the operation $\star$ on $\mathbf{S}^+$. Hence the empty simplicial set is the unit object for the operation $\star$ on $\mathbf{S}$. The monoidal structure $\star$ is not a symmetric but there is a natural isomorphism

$$(A \star B)^o = B^o \star A^o.$$

For every pair $m, n \geq 0$ we have

$$\Delta[m] \star \Delta[n] = \Delta[m+1+n],$$

since we have $[m] + [n] = [m+n+1]$. In particular,

$$1 \star 1 = \Delta[0] \star \Delta[0] = \Delta[1] = I.$$

We call the simplicial set $1 \star A$ the *projective cone* with *base* A and the simplicial set $A \star 1$ the *inductive cone*. The join of the maps $A \to 1$ and $B \to 1$ is a canonical map $A \star B \to I$.

Proposition 3.2. *If $A, B \in \mathbf{S}$, then we have*

$$(A \star B)_n = A_n \sqcup B_n \sqcup \bigsqcup_{i+1+j=n} A_i \times B_j$$

for every $n \geq 0$.

Proof: This follows from 3.2. ∎

The formula shows that we have $A \sqcup B \subseteq A \star B$. It shows also that if a simplex of $x : \Delta[n] \to A \star B$ does not belongs to $A \sqcup B$, then it admits a unique decomposition $x = u \star v : [i] \star [j] \to A \star B$, for a unique pair of simplices $u : [i] \to A$ and $v : [j] \to B$.

Corollary 3.3. *The nerve functor $N : \mathbf{Cat} \subset \mathbf{S}$ preserves the join operation.*

Proof: If $A, B \in \mathbf{Cat}$, let us first define a natural map

$$\theta : N(A) \star N(B) \to N(A \star B)$$

By the formula in 3.2 we have

$$(N(A) \star N(B))_n = N(A)_n \sqcup N(B)_n \sqcup \bigsqcup_{i+1+j=n} N(A)_i \times N(B)_j$$

for every $n \geq 0$. The map $N(A) \sqcup N(B) \to N(A \star B)$ induced by θ is defined by the inclusion $A \sqcup B \subseteq A \star B$. The map $N(A)_i \times N(B)_j \to N(A \star B)_n$ induced by θ takes a pair $u : [i] \to A$ and $v : [j] \to B$ to the simplex $u \star v : [i] \star [j] \to A \star B$. It is easy to verify that θ is bijective. ∎

If $X, Y \in \mathbf{S}$, then we have $X \sqcup Y \subseteq X \star Y$ by 3.2. If we join the maps $X \to 1$ and $Y \to 1$, we obtain a canonical map $X \star Y \to I$, since $1 \star 1 = I$. This defines a functor

$$\star : \mathbf{S} \times \mathbf{S} \to \mathbf{S}/I.$$

Lemma 3.4. *The following square of canonical maps is cartesian,*

$$\begin{array}{ccc} X \sqcup Y & \longrightarrow & X \star Y \\ \downarrow & & \downarrow \\ \partial I & \longrightarrow & I, \end{array}$$

where $\partial I = 1 \sqcup 1 = \{0, 1\} \subset I$.

Proof: We have $I = 1 \star 1$. It then follows from 3.2 that we have

$$I_n = (1 \star 1)_n = 1 \sqcup 1 \sqcup \bigsqcup_{i+1+j=n} 1 \times 1$$

for every $n \geq 0$. The canonical map $X \star Y \to I$ takes $X_n \sqcup Y_n$ to $1 \sqcup 1$ and $X_i \times Y_j$ to 1×1. This proves the result, since $(\partial I)_n = 1 \sqcup 1$. ∎

We have $\mathbf{S}/\partial I = \mathbf{S} \times \mathbf{S}$, since $\partial I = \{0, 1\} = 1 \sqcup 1$. If i denotes the inclusion $\partial I \subset I$, then the pullback functor

$$i^* : \mathbf{S}/I \to \mathbf{S}/\partial I = \mathbf{S} \times \mathbf{S},$$

has a right adjoint $i_* : \mathbf{S} \times \mathbf{S} \to \mathbf{S}/I$. If $X, Y \in \mathbf{S}$, then we have $i^*(X \star Y) = X \sqcup Y = (X, Y)$ by 3.4. We thus obtain a canonical map $c : X \star Y \to i_*(X, Y)$ by the adjointness $i^* \vdash i_*$.

Proposition 3.5. *The canonical map* $X \star Y \to i_*(X, Y)$ *is an isomorphism.*

Proof: Let p be the canonical map $i_*(X, Y) \to I$. By definition, a simplex $x : \Delta[n] \to i_*(X, Y)$ is a map $y : i^*(\Delta[n], f) \to X \sqcup Y$, where $f = px$. It is easy to see that

$$\bigsqcup_{f:\Delta[n]\to I} i^*(\Delta[n], f) = \Delta[n] \sqcup \Delta[n] \sqcup \bigsqcup_{i+1+j=n} \Delta[i] \sqcup \Delta[j].$$

The result follows from this formula and from 3.2. ∎

Proposition 3.6. *If* A, B, S *and* T *are simplicial sets, then*

$$(A \star B) \times_I (S \times T) = (A \times S) \star (B \times Y).$$

Proof: The functor $i_* : \mathbf{S} \times \mathbf{S} \to \mathbf{S}/I$ preserves cartesian products, since it is a right adjoint. But we have

$$(A, B) \times (S, T) = (A \times S, B \times T)$$

in the category $\mathbf{S} \times \mathbf{S}$. It follows that that we have

$$i_*(A, B) \times_I i_*(S, T) = i_*(A \times S, B \times Y).$$

This proves the formula by Proposition 3.5. ∎

Corollary 3.7. *If A and B are simplicial sets, then*

$$A \star B = (A \star 1) \times_I (1 \star B)$$

Lemma 3.8. *If $A \subseteq X$ and $B \subseteq Y$, then $A \star B \subseteq X \star Y$. Moreover, if $A, A' \subseteq X$ and $B, B' \subseteq Y$, then*

$$\begin{array}{rcl} (A \star B) \cap (A' \star B') & = & (A \cap A') \star (B \cap B'). \\ (A \cup A') \star Y & = & (A \star Y) \cup (A' \star Y), \\ X \star (B \cup B') & = & (X \star B) \cup (X \star B') \end{array}$$

Proof: The functor $i_* :: \mathbf{S} \times \mathbf{S} \to \mathbf{S}/I$ preserves monomorphisms, since it is a right adjoint. Thus, if $A \subseteq X$ and $B \subseteq Y$, then $i_*(A, B) \subseteq i_*(X, Y)$. Moreover, if $A, A' \subseteq X$ and $B, B' \subseteq Y$ then

$$i_*(A \cap A'S, B \cap B') = i_*(A, B) \cap i_*(B, B').$$

The other formulas can be proved by using Proposition 3.5. ∎

If $u : A \to B$ and $v : S \to T$ are two maps in $\mathbf{S}$, we shall denote by $u \star' v$ the map

$$(A \star T) \sqcup_{A \star S} (B \star S) \to B \star T$$

obtained from the commutative square

$$\begin{array}{ccc} A \star S & \xrightarrow{u \star S} & B \star S \\ {\scriptstyle A \star v} \downarrow & & \downarrow {\scriptstyle B \star v} \\ A \star T & \xrightarrow{u \star T} & B \star T. \end{array}$$

This defines a functorial operation

$$\star' : \mathbf{S}^I \times \mathbf{S}^I \to \mathbf{S}^I,$$

where $\mathbf{S}^I$ is the category of maps in $\mathbf{S}$. The operation is coherently associative but it has no unit object.

Lemma 3.9. *If u is the inclusion $A \subseteq B$ and v the inclusion $S \subseteq T$, then the map $u \star' v$ is the inclusion*

$$(A \star T) \cup (B \star S) \subseteq B \star T.$$

Proof: It follows from 3.8 that the square of inclusions

$$\begin{array}{ccc} A \star S & \longrightarrow & B \star S \\ \downarrow & & \downarrow \\ A \star T & \longrightarrow & B \star T \end{array}$$

is a pullback. Hence the square

$$\begin{array}{ccc} A \star S & \longrightarrow & B \star S \\ \downarrow & & \downarrow \\ A \star T & \longrightarrow & (A \star T) \cup (B \star S) \end{array}$$

is a pushout, where the union is taken in $B \star T$. ■

Lemma 3.10. • *(i) If $n \geq 0$, then*

$$\partial\Delta[0] \star \Delta[n] = \partial_0\Delta[n+1] \quad \textit{and} \quad \Delta[n] \star \partial\Delta[0] = \partial_{n+1}\Delta[n+1];$$

- *(ii) if $m > 0$ and $i \in [m]$, then*

$$\partial_i\Delta[m] \star \Delta[n] = \partial_i\Delta[m+n+1];$$

- *(iii) if $n > 0$ and $j \in [n]$, then*

$$\Delta[m] \star \partial_j\Delta[n] = \partial_{m+j+1}\Delta[m+n+1];$$

- *(iv) If $n \geq 0$, then*

$$\partial\Delta[n] \star 1 = \Lambda^{n+1}[n+1] \quad \textit{and} \quad 1 \star \partial\Delta[n] = \Lambda^0[n+1].$$

Proof: Let us prove (i). If i denotes the inclusion $\emptyset \subset \Delta[0]$, then $i \star 1_{[n]} = d_0 : \Delta[n] \to \Delta[n+1]$. This proves the first formula. The second formula is proved similarly. Let us prove (ii). If $d_i : [m-1] \to [m]$, then $d_i \star 1_{[n]} = d_i : [m+n] \to [m+n+1]$. This proves formula (ii). Formula (iii) is proved similarly. Let us prove the first formula in (iv). If $n = 0$, then

$$\partial\Delta[0] \star 1 = \emptyset \star 1 = \partial_0\Delta[1] = \Lambda^1[1].$$

If $n > 0$ and $i \in [n]$, then $\partial_i\Delta[n] \star 1 = \partial_i\Delta[n+1]$. It follows that by 3.8 that

$$\begin{aligned} \partial\Delta[n] \star 1 &= \Big(\bigcup_{i\in[n]} \partial_i\Delta[n]\Big) \star 1 \\ &= \bigcup_{i\in[n]} \partial_i\Delta[n] \star 1 \\ &= \bigcup_{i\in[n]} \partial_i\Delta[n+1] \\ &= \Lambda^{n+1}[n+1]. \end{aligned}$$

The first formula in (iv) is proved. The second formula proved similarly. ■

Lemma 3.11.

- *(i) If $m, n \geq 0$, then*

$$\big(\partial\Delta[m] \star \Delta[n]\big) \cup \big(\Delta[m] \star \partial\Delta[n]\big) = \partial\Delta[m+1+n];$$

- *(ii) if $m > 0$, $k \in [m]$ and $n \geq 0$, then*

$$\big(\Lambda^k[m] \star \Delta[n]\big) \cup \big(\Delta[m] \star \partial\Delta[n]\big) = \Lambda^k[m+1+n];$$

- *(iii) if $m \geq 0$, $n > 0$ and $k \in [n]$, then*

$$\big(\partial\Delta[m] \star \Delta[n]\big) \cup \big(\Delta[m] \star \Lambda^k[n]\big) = \Lambda^{m+1+k}[m+1+n].$$

Proof We shall use lemmas 3.8 and 3.10. Let us prove (i). If $m = 0$ and $n = 0$, then $(\emptyset \star 1) \cup (1 \star \emptyset) = \partial\Delta[1]$. The formula (i) is proved in this case. If $m > 0$ and $n = 0$, then

$$\begin{aligned}(\partial\Delta[m] \star \Delta[0]) \cup (\Delta[m] \star \partial\Delta[0]) &= \Lambda^{m+1}[m+1] \cup \partial_{m+1}\Delta[m+1] \\ &= \partial\Delta[m+1].\end{aligned}$$

And similarly in the case $m = 0$ and $n > 0$. If $m, n > 0$, then

$$\begin{aligned}\partial\Delta[m] \star \Delta[n] &= \Big(\bigcup_{i\in[m]} \partial_i\Delta[m]\Big) \star \Delta[n] \\ &= \bigcup_{i\in[m]} \partial_i\Delta[m] \star \Delta[n] \\ &= \bigcup_{i\in[m]} \partial_i\Delta[m+n+1].\end{aligned}$$

Also

$$\begin{aligned}\Delta[m] \star \partial\Delta[n] &= \Delta[m] \star \Big(\bigcup_{j\in[n]} \partial_j\Delta[n]\Big) \\ &= \bigcup_{j\in[n]} \Delta[m] \star \partial_j\Delta[n] \\ &= \bigcup_{j\in[n]} \partial_{j+m+1}\Delta[m+n+1].\end{aligned}$$

By taking union we then obtain that

$$(\partial\Delta[m] \star \Delta[n]) \cup (\Delta[m] \star \partial\Delta[n]) = \partial\Delta[m+n+1].$$

This proves (i). The formulas (ii) and (iii) are proved similarly. ■

3.3 Slice for simplicial sets

The functor $(-)\star B : \mathbf{S} \to \mathbf{S}$ does not preserve the initial object when $B \neq \emptyset$, since we have $\emptyset \star B \neq \emptyset$ in this case. Hence the functor $(-) \star B$ does not have a right adjoint if $B \neq \emptyset$. Hence the monoidal category $(\mathbf{S}, \star)$ is not closed. But consider the functor

$$(-) \star B : \mathbf{S} \to B\backslash\mathbf{S}$$

which associates to $X \in \mathbf{S}$ the inclusion $B \subseteq X \star B$.

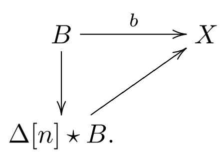
Proposition 3.12. *The functor* $(-) \star B : \mathbf{S} \to B\backslash\mathbf{S}$ *has a right adjoint.*

Proof: It suffice to show that the functor $(-) \star B : \mathbf{S} \to B\backslash\mathbf{S}$ is cocontinuous, since its domain is a presheaf category. We first make a few observations. The evaluation functor $U_n : \mathbf{S} \to \mathbf{Set}$ defined by putting $U_n(X) = X_n$ is cocontinuous for every $n \geq 0$. Hence also the functor $V_n : B\backslash\mathbf{S} \to B_n\backslash\mathbf{Set}$ which takes a map $B \to X$ to the map $B_n \to X_n$. Moreover, the functors $(U_n : n \geq 0)$ are collectively conservative (this means that a map of simplicial set f is invertible iff the map $U_n(f)$ is invertible for every $n \geq 0$). Hence also the functors $(V_n : n \geq 0)$. It follows from these observations that a functor $F : \mathbf{S} \to B\backslash\mathbf{S}$ is cocontinuous iff the functor $V_nF : \mathbf{S} \to B_n\backslash\mathbf{Set}$ is cocontinuous for every $n \geq 0$. Let us now show that the functor $F(-) = (-) \star B : \mathbf{S} \to B\backslash\mathbf{S}$ is cocontinuous. If $X \in \mathbf{S}$, let us put

$$G_n(X) = X_n \sqcup \bigsqcup_{i+1+j=n} X_i \times B_j.$$

Then we have $V_nF(X) = G_n(X) \sqcup B_n$ by 3.2. The functor $G_n : \mathbf{S} \to \mathbf{Set}$ is cocontinuous, since it is a coproduct of cocontinuous functors. The functor $(-) \sqcup B_n : \mathbf{Set} \to B_n\backslash\mathbf{Set}$ is also cocontinuous, since it is a left adjoint. This shows that the functor $X \mapsto V_nF(X) = G_n(X) \sqcup B_n$ is cocontinuous. We have proved that F is cocontinous. It follows that it has a right adjoint ∎

The right adjoint to the functor $(-)\star B : \mathbf{S} \to B\backslash\mathbf{S}$ takes a map of simplicial sets $b : B \to X$ to a simplicial set that we shall denote by X/b. We shall say that X/b is the *lower slice* of X by b. It follows by the adjointness that for any simplicial set A, there is a bijection between the maps $A \to X/b$ and the maps $A\star B \to X$ which extends b along the inclusion $B \subseteq A\star B$. In particular, a simplex $\Delta[n] \to X/b$ is a map $\Delta[n] \star B \to X$ which extends b,



A vertex $1 \to X/b$ is a map $c : 1 \star B \to X$, it is a *projective cone* with *base* b; the *apex* of c is the vertex $c(1) \in X_0$.

Dually, the right adjoint to the functor $A \star (-) : \mathbf{S} \to A\backslash\mathbf{S}$ takes a map of simplicial sets $a : A \to X$ to a simplicial set that we shall denote $a\backslash X$. By duality, we have $(a\backslash X)^o = (X^o/a^o)$. We shall say that the simplicial set $a\backslash X$ is the upper slice of X by a. t follows by the adjointness that for any simplicial set B, there is a bijection between the maps $A \to a\backslash X$ and the maps $A \star B \to X$ which extends the map a along the inclusion $A \subseteq A \star B$. A simplex $\Delta[n] \to a\backslash X$ is a map $A \star \Delta[n] \to X$ which extends a. A vertex $1 \to a\backslash X$ is an *inductive cone* $A \star 1 \to X$ with *base* a.

The functor
$$A \star (-) \star B : \mathbf{S} \to A \star B\backslash\mathbf{S}$$
which associates to a simplicial set X the inclusion $A \star B \subseteq A \star X \star B$ has a right adjoint which associates to a map $f : A \star B \to X$ a simplicial set that we denote by $Fact(f, X)$. A vertex of $Fact(f, X)$ is a map $g : A \star 1 \star B \to X$ which extends the map $f : A \star B \to X$ along the inclusion $A \star B \subseteq A \star 1 \star B$. When $A = B = 1$, the map f is the same thing as an arrow $f : a \to b$ in X and $Fact(f, X)$ is the simplicial set of *factorisations* of the arrow f By restricting the map $f : A \star B \to X$ to the simplicial subsets $A \subseteq A \star B$ and $B \subseteq A \star B$ we obtain a pair of maps $f_A : A \to X$ and $f_B : B \to X$, hence also a pair of maps $f'_B : B \to f_A\backslash X$ and $f'_A : A \to X/f_B$ by adjointness. It is easy to see that we have
$$Fact(f, X) = (f_A\backslash X)/f'_B) = f'_A\backslash(X/f_B).$$

Proposition 3.13. *The nerve functor $N : \mathbf{Cat} \subset \mathbf{S}$ preserves the slice operations.*

Proof: Let $b : B \to X$ be a functor in **Cat**. There is natural bijection between the simplices $\Delta[n] \to N(X/b)$, the functors $[n] \to X/b$, the functors $[n] \star B \to X$ which extend the functor b, the maps $\Delta[n] \star N(B) \to N(X)$ which extend the map $N(b)$ (since the nerve functor preserves the join operation by 3.3) and the simplices $\Delta[n] \to N(X)/N(b)$. It follows by Yoneda Lemma that the simplicial sets $N(X/b)$ and $N(X)/N(b)$ are canonically isomorphic. ∎

The simplicial set X/b depends functorially on the map $b : B \to X$. More precisely, to every commutative diagram
$$\begin{array}{ccc} B & \xleftarrow{u} & A \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle a} \\ X & \xrightarrow{f} & Y \end{array}$$
we can associated a map
$$f/u : X/b \to Y/a.$$
By definition. the image of a simplex $x : \Delta[n] \to X/b$ by the map f/u is defined to be the composite the maps
$$\Delta[n] \star A \xrightarrow{\Delta[n]\star u} \Delta[n] \star B \xrightarrow{x} X \xrightarrow{f} Y.$$

For any chain of three maps

$$S \xrightarrow{s} T \xrightarrow{t} X \xrightarrow{f} Y$$

we shall denote by $\langle s, t, f\rangle$ the map

$$X/t \to Y/ft \times_{Y/fts} X/ts$$

obtained from the commutative square

$$\begin{array}{ccc} X/t & \longrightarrow & X/ts \\ \downarrow & & \downarrow \\ Y/ft & \longrightarrow & Y/fts, \end{array}$$

Our next goal is to prove the following theorem:

Theorem *Let $s : S \to T$, $t : T \to X$, $f : X \to Y$ be a chain of three maps of simplicial sets, where s is monic.*

- (i) *if f is a mid fibration then $\langle s, t, f\rangle$ is a right fibration;*
- (ii) *if f is a left fibration then $\langle s, t, f\rangle$ is a Kan fibration ;*

Moreover, $\langle s, t, f\rangle$ is a trivial fibration in each of the following cases:

- (iii) *f is a trivial fibration;*
- (iv) *f is a right fibration and s is anodyne:*
- (v) *f is a mid fibration and s is left anodyne.*

The proof is given in 3.19. We need to establish a few intermediate results.

Let $u : A \to B$ and $v : S \to T$ then the map

$$u \star' v : (A \star T) \sqcup_{A \star S} (B \star S) \to B \star T$$

can be viewed as a map in the category $T\backslash\mathbf{S}$, since $T \subseteq A \star T$.

Lemma 3.14. *Let $s : S \to T$, $t : T \to X$ and $f : X \to Y$ be a chain of three maps of simplicial sets. Then for any map $u : A \to B$, there is a natural bijection between the following commutative squares*

$$\begin{array}{ccc} A & \longrightarrow & X/t \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle \langle s,t,f\rangle} \\ B & \longrightarrow & Y/ft \times_{Y/fts} X/ts \end{array}$$

and

$$\begin{array}{ccc} (A\star T)\sqcup_{A\star S}(B\star S) & \longrightarrow & X \\ {\scriptstyle u\star' s}\downarrow & & \downarrow{\scriptstyle f} \\ B\star T & \longrightarrow & Y, \end{array}$$

where the top map of the second square is a map in the category $T\backslash\mathbf{S}$. Moreover, if one of the square has a diagonal filler then so has the other.

Proof: Let us prove the first statement. We shall use lemma D.1.15. We first define two functors $G_0, G_1 : T\backslash\mathbf{S} \to \mathbf{S}$ and a natural transformation $\beta : G_1 \to G_0$. By definition, $G_1(X,t) = X/t$ and $G_0(X,t) = X/ts$. The map $\beta(X,t) : G_1(X,t) \to G_0(X,t)$ is the map $X/t \to X/ts$ obtained by composing with $s : S \to T$. If $f : X \to Y$ is a map in $T\backslash\mathbf{S}$. then the map

$$\beta^\bullet(f) : G_1X \to G_1Y \times_{G_0Y} G_0X$$

obtained from the naturality square

$$\begin{array}{ccc} G_1X & \xrightarrow{\beta_X} & G_0X \\ {\scriptstyle G_1f}\downarrow & & \downarrow{\scriptstyle G_0f} \\ G_1Y & \xrightarrow{\beta_Y} & G_0Y. \end{array}$$

is equal to the map $\langle s,t,f\rangle$. The functor G_1 has a left adjoint F_1 which takes a simplicial set X to the canonical map $T \to T\star X$. From the commutative square

$$\begin{array}{ccc} S & \longrightarrow & X\star S \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle X\star v} \\ T & \longrightarrow & X\star T, \end{array}$$

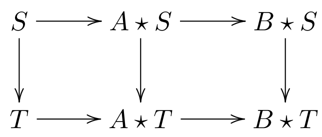
we obtain a map $\alpha_X : T\sqcup_S (X\star S) \to X\star T$. The functor G_0 has a left adjoint F_0 which takes a simplicial set X to the canonical map $T \to T\sqcup_S (X\star S)$. The natural transformation $\alpha : F_0 \to F_1$ defined by the map α_X is the left transpose of of the natural transformation $\beta : G_1 \to G_0$. If $u : A \to B$ is a map of simplicial sets, let us compute the map

$$\alpha_\bullet(u) : F_0B \sqcup_{F_0A} F_1A \to F_1B$$

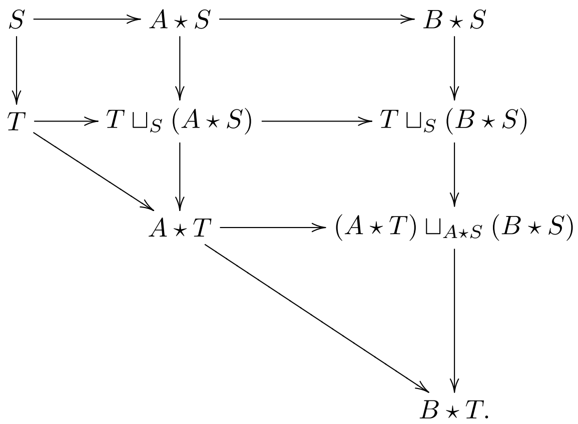
obtained from the naturality square

$$\begin{array}{ccc} F_0A & \xrightarrow{\alpha_A} & F_1A \\ {\scriptstyle F_0u}\downarrow & & \downarrow{\scriptstyle F_1u} \\ F_0B & \xrightarrow{\alpha_B} & F_1B. \end{array}$$

From the commutative diagram



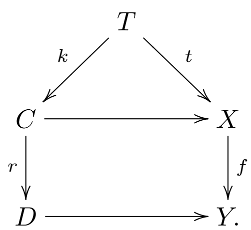
we can construct the following diagram in which every square is a pushout:



The diagram shows that $\alpha_\bullet(u) = u \star' s$. It then follows from lemma D.1.15, that there is a bijection between the maps $u \to \beta^\bullet(f)$ in the category $\mathbf{S}^I$ and the maps $\alpha_\bullet(u) \to f$ in the category of $(T\backslash\mathbf{S})^I$. The first statement of the lemma is proved. Moreover, there is a bijection between the diagonal fillers of a square $u \to \beta^\bullet(f)$ and the diagonal fillers of the corresponding square $\alpha_\bullet(u) \to f$. The second statement of the lemma is proved. It follows that $\alpha_\bullet(u) \pitchfork f \Leftrightarrow u \pitchfork \beta^\bullet(f)$. ■

Lemma 3.15. *Let $u : A \to B$, $s : S \to T$ and $f : X \to Y$. Then we have $u \pitchfork \langle s, t, f\rangle$ for every $t : T \to X$ iff we have $(u \star' s) \pitchfork f$.*

Proof: If $k : T \to C$ and $r : C \to D$ are two maps in $\mathbf{S}$, let us denote by (r, k) the map $(C, k) \to (D, rk)$ defined by r in the category $T\backslash\mathbf{S}$. It is easy to verify that we have $r \pitchfork f$ iff for every $t : T \to X$ we have $(r, k) \pitchfork (f, t)$,



Let k be be the canonical map $T \to (A \star T) \cup (B \star S)$ and let us put $r = u \star' s$. Then the condition $(r, k) \pitchfork (f, t)$ is equivalent to the condition $u \pitchfork \langle s, t, f \rangle$ by Lemma 3.14. The result follows. ■

In the following lemma, $\overline{\Sigma}$ denotes the saturated class generated by a class of maps $\Sigma \subseteq \mathbf{S}$.

Lemma 3.16. *Let $(\mathcal{A}, \mathcal{B})$ be a weak factorisation system in the category $\mathbf{S}$. If $\Sigma_1 \subseteq \mathbf{S}$ and $\Sigma_2 \subseteq \mathbf{S}$ are two classes of maps then,*

$$\Sigma_1 \star' \Sigma_2 \subseteq \mathcal{A} \quad \Longrightarrow \quad \overline{\Sigma}_1 \star' \overline{\Sigma}_2 \subseteq \mathcal{A}.$$

Proof: It suffices to prove the implication $\Sigma_1 \star' \Sigma_2 \subseteq \mathcal{A} \Rightarrow \overline{\Sigma}_1 \star' \Sigma_2 \subseteq \mathcal{A}$. For this, it suffices to prove the implication $(\Sigma_1 \star' \Sigma_2) \pitchfork \mathcal{B} \Rightarrow (\overline{\Sigma}_1 \star' \Sigma_2) \pitchfork \mathcal{B}$, since $\mathcal{A} = {}^{\pitchfork}\mathcal{B}$. For this, it suffices to prove the implication $(\Sigma_1 \star' v) \pitchfork f \Rightarrow (\overline{\Sigma}_1 \star' v) \pitchfork f$ for any pair of maps $v : S \to T$ and $f : X \to Y$. Let $\mathcal{C}$ be the class of maps $i : A \to B$ such that $(i \star' v) \pitchfork f$. It follows from 3.15 that a map $i : A \to B$ belongs to $\mathcal{C}$ iff we have $i \pitchfork \langle j, t, f \rangle$ for every map $t : T \to X$. This description shows that the class $\mathcal{C}$ is saturated. Thus, $\Sigma_1 \subseteq \mathcal{C} \Rightarrow \overline{\Sigma}_1 \subseteq \mathcal{C}$. ■

Theorem 3.17. *If $u : A \to B$ and $v : S \to T$ are monomorphisms of simplicial sets, then the map $u \star' v : A \star T \sqcup_{A \star S} B \star S \subseteq B \star T$ is*

- *(i) mid anodyne, if u is right anodyne or v is left anodyne;*
- *(ii) left anodyne, if u is anodyne;*
- *(iii) right anodyne, if v is anodyne.*

Proof: Let us denote the class of monomorphisms by $\mathcal{C}$, the class of mid anodyne maps by $\mathcal{C}_m$, the class of right anodyne maps by $\mathcal{C}_r$, the class of left anodyne maps by $\mathcal{C}_l$ and the class of anodyne maps by $\mathcal{C}_a$. By B.0.8, the class $\mathcal{C}$ is generated as a saturated class by the set of inclusions $\delta_n : \partial\Delta[n] \subset \Delta[n]$ for $n \geq 0$. By 2.7, the saturated class $\mathcal{C}_r$ is generated is generated by the set of inclusions h^k_m with $0 < k \leq m$. We have $h^k_m \star' \delta_n = h^k_{n+m+1}$ by lemma 3.11. But we have $0 < k \leq m \Rightarrow 0 < k < m + n + 1$ for every $n \geq 0$. Thus, $h^k_m \star' \delta_n \in \mathcal{C}_m$ for every $0 < k \leq m$ and every $n \geq 0$. It follows by 3.16 that we have $\mathcal{C}_r \star' \mathcal{C} \subseteq \mathcal{C}_m$. This shows that $u \star' v$ is mid anodyne if u is right anodyne. It follows by symmetry that $u \star' v$ is mid anodyne if v is left anodyne. We have $0 \leq k \leq m \Rightarrow 0 \leq k < m+n+1$ for every $n \geq 0$. Thus, $h^k_m \star' \delta_n \in \mathcal{C}_l$ for every $0 \leq k \leq m$ and every $n \geq 0$. Hence we have $\mathcal{C}_a \star' \mathcal{C} \subseteq \mathcal{C}_l$ by 3.16. The statement (ii) is proved. The statement (iii) follows from (ii) by duality. ■

Corollary 3.18. *If $u : A \to B$ and $v : S \to T$ are mid anodyne (resp. anodyne), then so is the map $u \star v : A \star S \to B \star T$*

Proof: Let us first show that the map $u \star T$ is mid anodyne. There is no loss of generality in supposing that u is an inclusion $A \subseteq B$. The inclusion u is right anodyne, since a mid anodyne map is right anodyne. Hence the inclusion $(A \star T) \cup (B \star \emptyset) \subseteq B \star T$ is mid anodyne by theorem 3.17. The square of inclusions

$$\begin{array}{ccc} A & \longrightarrow & A \star T \\ \downarrow & & \downarrow \\ B & \longrightarrow & (A \star T) \cup (B \star \emptyset) \end{array}$$

is a pushout, since $(A \star T) \cap (B \star \emptyset) = A \star \emptyset$ by 3.8. Hence the inclusion $A \star T \subseteq (A \star T) \cup (B \star \emptyset)$ is mid anodyne, since the inclusion $A \subseteq B$ is mid anodyne by hypothesis. It follows that the composite

$$u \star T : A \star T \subseteq (A \star T) \cup (B \star \emptyset) \subseteq B \star T$$

is mid anodyne. Similarly, the map $A \star v$ is mid anodyne by symmetry. Hence also the composite $u \star v = (u \star T)(A \star v)$. The first statement is proved. The second statement is proved similarly. ∎

Theorem 3.19. *Let $s : S \to T$, $t : T \to X$, $f : X \to Y$ be a chain of three maps of simplicial sets where s is monic. Consider the map*

$$\langle s, t, f \rangle : X/t \to Y/ft \times_{Y/fts} X/ts.$$

- (i) *if f is a mid fibration then $\langle s, t, f \rangle$ is a right fibratio;*
- (ii) *if f is a left fibration then $\langle s, t, f \rangle$ is a Kan fibration;*

Moreover, $\langle s, t, f \rangle$ is a trivial fibration in each of the following cases:

- (iii) *f is a trivial fibration;*
- (iv) *f is a right fibration and s is anodyne;*
- (v) *f is a mid fibration and s is left anodyne:*

Proof: Let us put $p = \langle s, t, f \rangle$. Let us start with the easiest case (iii). In order to prove that p is a trivial fibration, it suffices to show that we have $u \pitchfork \langle s, t, f \rangle$ for every monomorphism u. But the map $u \star' s$ is monic by 3.9, since u and s are monic. Hence we have $(u \star' s) \pitchfork f$, since f is a trivial fibration. It follows that we have $u \pitchfork p$ by 3.15. Let us prove (i). For this it suffices to show that we have $u \pitchfork \langle s, t, f \rangle$ for every right anodyne map u. But we have $(u \star' s) \pitchfork f$, since f is a mid fibration by assumption, and since $u \star' s$ is mid anodyne by 3.17. This shows that we have $u \pitchfork p$ by 3.15. Let us prove (ii). For this it suffices to show that we have $u \pitchfork \langle s, t, f\rangle$ for every anodyne map u. But we have $(u \star' s) \pitchfork f$, since f is a left fibration by assumption, and since $u \star' s$ is left anodyne by 3.17. Hence we have $u \pitchfork p$ by 3.15. Let us prove (iv). For this it suffices to show that we have $u \pitchfork \langle s, t, f\rangle$ for every monomorphism u. But we have $(u \star' s) \pitchfork f$, since f is a right fibration by assumption, and since $u \star' s$ is right anodyne by 3.17. Hence we have $u \pitchfork p$ by 3.15. Let us prove (v). For this it suffices to show that we have $u \pitchfork \langle s, t, f\rangle$ for every monomorphism u. But we have $(u \star' s) \pitchfork f$, since f is a mid fibration by assumption, and since $u \star' s$ is mid anodyne by 3.17. Hence we have $u \pitchfork p$ by 3.15. ∎

Corollary 3.20. *If X is a quasi-category then so is the simplicial set X/t for any map $t : T \to X$. Moreover, the projection $X/t \to X/ts$ is a right fibration for any monomorphism $s : S \to T$.*

Proof: The projection $X/t \to X/ts$ is a right fibration by theorem 3.19 applied to the monomorphism $s : S \to T$ and to the map $X \to 1$. The second statement is proved. Hence the projection $X/t \to X$ is a right fibration by the same result in the case $S = \emptyset$. But a right fibration is a mid fibration. This shows that X/t is a quasi-category, since X is a quasi-category. ∎

If B is an object of a category $\mathcal{E}$, and $\mathcal{M}$ is a class of maps in $\mathcal{E}$, we shall say that a morphism $f : (X, p) \to (Y, q)$ in the category $\mathcal{E}/B$ *belongs* to $\mathcal{M}$ if this is true of the map $f : X \to Y$.

Lemma 3.21. *If i_0 denotes the inclusion $\{0\} \subset I$, then the functor*

$$i_0^* : \mathbf{S}/I \to \mathbf{S}$$

preserves mid (resp. left) anodyne maps.

Proof: Let $\mathcal{A}$ be the class of mid anodyne maps in $\mathbf{S}$ and let $\mathcal{A}'$ be the class of mid anodyne maps in $\mathbf{S}/I$. By definition, we have $\mathcal{A}' = U^{-1}(\mathcal{A})$, where U is the forgetful functor $\mathbf{S}/I \to \mathbf{S}$. Let us show that we have $i_0^*(\mathcal{A}') \subseteq \mathcal{A}$. The class $\mathcal{A}'$ is saturated, since the class $\mathcal{A}$ is saturated. The class $\mathcal{A}$ is generated by the set Σ of inclusions $h_n^k : \Lambda^k[n] \subset \Delta[n]$ with $0 < k < n$. If follows by Proposition D.2.7 that the class $\mathcal{A}'$ is generated by the set $\Sigma' = U^{-1}(\Sigma)$. Let us show that $i_0^*(\Sigma') \subseteq \mathcal{A}$. For this, we have to show that the inclusion $u^{-1}(0) \cap \Lambda^k[n] \subseteq u^{-1}(0)$ belongs to $\mathcal{A}$ for any map $u : \Delta[n] \to I$ and any $0 < k < n$. This is clear if $u^{-1}(0) = \emptyset$. This is also clear if $u^{-1}(0) = \Delta[n]$. Otherwise we have $u^{-1}(0) = \Delta[r]$ for some $0 \le r < n$. Observe that we have $\Delta[r] \subseteq \partial_n\Delta[n] \subseteq \Lambda^k[n]$, since $r < n$ and $k < n$. It follows that $u^{-1}(0) \cap \Lambda^k[n] = u^{-1}(0)$. Hence the inclusion $u^{-1}(0) \cap \Lambda^k[n] \subseteq u^{-1}(0)$ belongs to $\mathcal{A}$. Thus, $i_0^*(\Sigma') \subseteq \mathcal{A}$. It follows that we have $i_0^*(\mathcal{A}') \subseteq \mathcal{A}$, since the functor i_0^* is cocontinuous, and since $\mathcal{A}'$ is generated by Σ'. The first statement is proved. The second statement is proved similarly. ∎

Proposition 3.22. *If $f : S \to T$ and $g : X \to Y$ are mid fibrations, then so is the map $f \star g : S \star X \to T \star Y$.*

Proof: We shall use 3.5. It suffices to show that the functor $i_* : \mathbf{S} \times \mathbf{S} \to \mathbf{S}/I$ takes a mid fibration to a mid fibration. For this, it suffices to show that the functor $i^* : \mathbf{S}/I \to \mathbf{S} \times \mathbf{S}$ preserves mid anodyne maps by Lemma D.1.14. But we have $i^* = (i_0^*, i_1^*)$, where i_0 denotes the inclusion $\{0\} \subset I$ and i_1 the inclusion $\{1\} \subset I$. Hence it suffices to shows that each functor $i_0^*, i_1^* : \mathbf{S}/I \to \mathbf{S}$ preserves mid anodyne maps. The result follows from Lemma 3.21. ∎

Corollary 3.23. *If X and Y are quasi-categories then so is the simplicial set $X \star Y$.*

Proof: If the maps $X \to 1$ and $Y \to 1$ are mid fibrations, then the map $X \star Y \to 1 \star 1 = I$ is a mid fibration by Proposition 3.22. Hence also the map $X \star Y \to 1$, since I is a quasi-category. ∎

Proposition 3.24. *A mid fibration $p : E \to B$ is a right fibration iff the map $p/a : E/a \to B/pa$ is a trivial fibration for every vertex $a \in E$.*

Proof: A mid fibration $p : E \to B$ is a right fibration iff it has the right lifting property with respect to the inclusion $h_n^n : \Lambda^n[n] \subset \Delta[n]$ for every $n > 0$. If σ_n denotes the inclusion $\partial\Delta[n] \subset \Delta[n]$ then we have $h_{n+1}^{n+1} = \sigma_n \star 1$ by 3.11. The functor from $(-) \star 1 : \mathbf{S} \to 1\backslash\mathbf{S}$ is left adjoint to the functor $(X, a) \mapsto X/a$. For each vertex $a \in E$ the map $p : E \to B$ induces a map $(p, a) : (E, a) \to (B, pa)$ in the category $1\backslash\mathbf{S}$. It is easy to verify the equivalence

$$(\sigma_n \star 1) \pitchfork p \quad \Leftrightarrow \quad \forall a \in E \quad (\sigma_n \star 1, 1) \pitchfork (p, a).$$

But the condition $(\sigma_n \star 1, 1) \pitchfork (p, a)$ is equivalent to the condition $\sigma_n \pitchfork (p/a)$ by 3.14. Thus, p is a right fibration iff the map p/a is a trivial fibration for every $a \in E$. ∎

Corollary 3.25. *The map $i \star 1 : A \star 1 \to B \star 1$ is right anodyne for any monomorphism $i : A \to B$.*

Proof: It suffices to show that we have $(i \star 1) \pitchfork f$ for every right fibration $f : X \to Y$. If $a = x(1)$, then the square

$$\begin{array}{ccc} S & \longrightarrow & X/a \\ \downarrow & & \downarrow{\scriptstyle f/a} \\ T & \longrightarrow & Y/fa \end{array}$$

has a diagonal filler, since f/a is a trivial fibration by 3.24. Hence also the square

$$\begin{array}{ccc} A \star 1 & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ B \star 1 & \longrightarrow & Y \end{array}$$

by 3.14. ■

Corollary 3.26. *The inclusion* $1 \subseteq B \star 1$ *is right anodyne for any simplicial set* B*.*

Proof: The inclusion $\emptyset \star 1 \subseteq B \star 1$ is right anodyne by Corollary 3.25. ■

Chapter 4

Quasi-categories and Kan complexes

In this chapter we introduce the notion of pseudo-fibration between quasi-categories. We show that a quasi-category is a Kan complex iff its fundamental category is a groupoid. We show that every quasi-category contains a largest sub Kan complex. The chapter has three sections.

4.1 Pseudo-fibrations between quasi-categories

Definition 4.1. *We say that a functor is a* pseudo-fibration *if for every object $x \in E$ and every isomorphism $g \in B$ with source $p(x)$, there exists an isomorphism $f \in E$ with source x such that $p(f) = g$.*

A functor $p : E \to B$ is a pseudo-fibration iff the opposite functor $p^o : E^o \to B^o$ is a pseudo-fibration. Hence a functor $p : E \to B$ is a pseudo-fibration iff for every object $y \in E$ and every isomorphism $g \in B$ with target $p(y)$, there exists an isomorphism $f \in E$ with target x such that $p(f) = g$. We shall see in 6.2 that the category **Cat** admits a model structure in which a a weak equivalence is an equivalence of categories and a fibration is a pseudo-fibration.

Recall from Definition 1.12 that an arrow in a quasi-category X is said to an *isomorphism* if the arrow is invertible in the category hoX.

Definition 4.2. *We call a map between quasi-categories $p : X \to Y$ a* pseudo-fibration *if it is a mid fibration and for every object $x \in X$ and every isomorphism $g \in Y$ with source $p(x)$, there exists an isomorphism $f \in X$ with source x such that $p(f) = g$.*

The general notion of pseudo-fibration for maps between simplicial sets will be defined in 6.3.

The composite of two pseudo-fibrations is a pseudo-fibration. The canonical map $X \to 1$ is a pseudo-fibration for any quasi-category X.

Proposition 4.3. *A functor $p : E \to B$ in* **Cat** *is a pseudo-fibration iff the map $Np : NE \to NB$ is a pseudo-fibration in* **QCat**.

Proof: A functor in **Cat** is a mid fibration by 2.2. ∎

Proposition 4.4. *A trivial fibration between quasi-categories is a pseudo-fibration. The canonical map $X \to hoX$ is a pseudo-fibration for any quasi-category X.*

Proof: Let $p : X \to Y$ a trivial fibration between quasi-category. A trivial fibration is a mid fibration. Let $a \in X_0$ and let $g \in Y$ be an isomorphism with source $p(a)$. Then there exists an arrow $f \in X$ with source a such that $p(f) = g$, since p has the right lifting property with respect to the inclusion $\{0\} \subset I$. The map p is a categorical equivalence by 1.22. Hence the functor $ho(p) : hoX \to hoY$ is an equivalence of categories by 1.27, since $ho(p) = \tau_1(p)$. Hence the arrow f is invertible in hoX, since the arrow $p(f)$ is invertible in hoY. This proves that p is a pseudo-fibration. Let us prove the second statement. The canonical map $X \to hoX$ is a mid fibration by 2.2, since hoX is a category. If $a \in X_0$, then every arrow with source a in hoX is of the form $[f]$ for an arrow $f \in X$ with source a. Moreover f is invertible, since $[f]$ is invertible. The second statement is proved. ∎

Proposition 4.5. *A mid fibration between quasi-categories $p : X \to Y$ is a pseudo-fibration iff the functor $ho(p) : hoX \to hoY$ is a pseudo-fibration.*

Proof: ($\Rightarrow$) Let $a \in X_0$ and let $g' \in hoY$ be an isomorphism with source $p(a)$. The construction of hoY shows that we have $g' = [g]$ for an arrow $g \in Y$ with target $p(a)$. The arrow g is invertible, since g' is invertible. Hence there exists an isomorphism $f \in X$ with target a such that $p(f) = g$, since p is a pseudo-fibration by assumption. The arrow $f' = [f] \in hoX$ is invertible and we have $ho(p)(f') = g'$, since we have $p(f) = g$. This shows that the functor $ho(p)$ is a pseudo-fibration. ($\Leftarrow$) Let $a \in X_0$ and $g \in Y_1$ be an isomorphism with target $p(a)$. The arrow $g' = [g]$ is invertible, since g is invertible. Hence there exists an arrow $f' \in hoX$ with target a such that $ho(p)(f') = g'$, since $ho(p)$ is a pseudo-fibration by assumption. The construction of hoX shows that we have $f' = [f]$ for an arrow $f \in X$ with target a. The arrow f is invertible, since f' is invertible. We have $[p(f)] = [g]$, since $ho(p)(f') = g'$. Thus, $p(f)$ is homotopic to g. Hence there exists a 2-simplex $t \in Y$ with boundary $(1_{pa}, g, p(f))$. Consider the commutative square

$$\begin{array}{ccc} \Lambda^0[2] & \xrightarrow{h} & X \\ \downarrow & & \downarrow p \\ \Delta[2] & \xrightarrow{t} & Y, \end{array}$$

where h is the horn $(1_a, \star, f)$. The square has a diagonal filler $u : \Delta[2] \to Y$, since p is a mid fibration by assumption. Let us put $k = ud_1$. The target of k is equal to a, since $kd_0 = ud_1d_0 = ud_0d_0 = 1_ad_0 = a$. Moreover, we have $pk = pud_1 = td_1 = g$. The arrows k and f are homotopic, since $\partial u = (1_a, k, f)$. This shows that k is invertible in hoX, since f is invertible in hoX. Thus, k is an isomorphism. The implication ($\Leftarrow$) is proved. ∎

Corollary 4.6. *A map between quasi-categories $p : X \to Y$ is a pseudo-fibration iff the opposite map $p^o : X^o \to Y^o$ is a pseudo-fibration.*

Proof: The map p is a mid fibration iff the opposite map p^o is a mid fibration. The functor $ho(p)$ is pseudo-fibration iff the opposite functor $ho(p)^o = ho(p^o)$ is pseudo-fibration. The result then follows from 4.5. ∎

Recall that a functor $u : A \to B$ is said to be *conservative*indexconservative!functor—textbf if the implication

$$u(f) \quad \text{invertible} \quad \Rightarrow \quad f \quad \text{invertible}$$

is true for every arrow $f \in A$.

Definition 4.7. *We shall say that a map of simplicial sets $u : A \to B$ is* conservative *indexconservative!map of simplicial sets—textbf if the functor $\tau_1(u) : \tau_1 A \to \tau_1 B$ is conservative.*

Proposition 4.8. *A map between quasi-categories $p : X \to Y$ is conservative iff the implication*

$$p(f) \quad \text{invertible} \quad \Rightarrow \quad f \quad \text{invertible}$$

is true for every arrow $f \in X$.

Proof: The functor $\tau_1(p) : \tau_1 X \to \tau_1 Y$ is isomorphic to the functor $ho(p) : hoX \to hoY$ by 1.11. Moreover, every arrow in hoX is the homotopy class of an arrow in X by 1.10. The result follows. ∎

Proposition 4.9. *Every left (resp. right) fibration between quasi-categories is conservative.*

Proof: Let $p : X \to Y$ be a left fibration between quasi-categories. Let $f : a \to b$ be an arrow in X and suppose that $p(f) : pa \to pb$ is invertible. Then there exists an arrow $g : pb \to pa$ such that $[g][p(f)] = 1_{pa}$. Hence there exists a 2-simplex $t \in Y$ with boundary $(g, 1_{pa}, p(f))$ by theorem 1.10. We then have commutative square

$$\begin{array}{ccc} \Lambda^0[2] & \xrightarrow{z} & X \\ \big\downarrow & & \big\downarrow p \\ \Delta[2] & \xrightarrow{t} & Y, \end{array}$$

where z is the horn $(\star, 1_a, f)$. The square has a diagonal filler $u : \Delta[2] \to X$, since p is a left fibration. Let us put $f' = ud_0$. Then we have $p(f') = pud_0 = td_0 = g$. Moreover, we have $[f'][f] = 1_a$ in hoX, since $\partial u = (f', 1_a, f)$. This shows that $[f']$ is a left inverse of $[f]$ in hoX. We shall prove that $[f]$ is invertible by showing that $[f']$ has also a left inverse. The arrow $[g]$ is invertible in hoY, since we have $[g][p(f)] = 1_{pa}$, and since $[p(f)]$ is invertible. Hence the arrow $p(f') = g$ is invertible in Y. If we repeat the argument above with f' instead of f, we obtain that $[f']$ has a left inverse in hoX. This shows that $[f]$ is invertible in hoX. Thus, p is a conservative map. ∎

Proposition 4.10. *If Y is a quasi-category, then every left (resp. right) fibration $p : X \to Y$ is a pseudo-fibration.*

Proof: The map p is a mid fibration, since a left fibration is a mid fibration. Thus, X is a quasi-category, since Y is a quasi-category. Let $a \in X_0$ and let $g \in Y_1$ be an isomorphism with source $p(a)$. The square

$$\begin{array}{ccc} \Lambda^0[1] & \xrightarrow{a} & Y \\ \big\downarrow & & \big\downarrow p \\ \Delta[1] & \xrightarrow{g} & X \end{array}$$

has a diagonal filler $f : \Delta[1] \to Y$, since p is a left fibration. The arrow f is invertible, since $p(f)$ is invertible and p is conservative by 4.9. This proves that p is a pseudo-fibration. ∎

4.2 Quasi-categories and Kan complexes

Lemma 4.11. *Let $f : X \to Y$ be a map between quasi-categories, T a simplicial set, $S \subseteq T$ a simplicial subset and $t : T \to X$ a map. Then the simplicial set*

$Y/T \times_{Y/S} X/S$ defined by the pullback square

$$\begin{array}{ccc} Y/T \times_{Y/S} X/S & \xrightarrow{pr_2} & X/S \\ \downarrow & & \downarrow \\ Y/T & \longrightarrow & Y/S, \end{array}$$

is a quasi-category.

Proof: The projection $Y/T \to Y/S$ obtained from the inclusion $S \subseteq T$ is a right fibration by 3.20. Hence also the projection pr_2 in the pullback square. It follows that pr_2 is a mid fibration, since every right fibration is a mid dibration. The simplicial set X/S is a quasi-category by 3.20. This shows that $Y/T \times_{Y/S} X/S$ is a quasi-category. ∎

If $S \subseteq T$ are simplicial sets, consider the inclusion

$$(\{0\} \star T) \cup (I \star S) \subseteq I \star T$$

where $I = \Delta[1]$. Observe that $I \subseteq (\{0\} \star T) \cup (I \star S)$.

Lemma 4.12. *Suppose that we have a commutative square*

$$\begin{array}{ccc} (\{0\} \star T) \cup (I \star S) & \xrightarrow{u} & X \\ \downarrow & & \downarrow p \\ I \star T & \xrightarrow{v} & Y, \end{array}$$

where p is a mid fibration between quasi-categories. If the arrow $u(I) \in X$ is invertible, then the square has a diagonal filler.

Proof: Let i be the inclusion $\{0\} \subset I$ and let us put $t = u|T$. We shall use lemma 3.14 applied to the triple of maps $i : \{0\} \subset I$, $j : S \subseteq T$, $t : T \to X$ and $p : X \to Y$. The lemma shows that the square has a diagonal filler iff the following square has a diagonal filler,

$$\begin{array}{ccc} \{0\} & \xrightarrow{u'_1} & X/T \\ \downarrow & & \downarrow q \\ I & \xrightarrow{(v', u'_2)} & Y/T \times_{Y/S} X/S, \end{array}$$

where $q = \langle j, t, p \rangle$, where $v' : I \to Y/T$ corresponds by adjointness to $v : I \star T \to Y$, where $u'_1 : \{0\} \to X/T$ corresponds to $u|\{0\} \star T \to X$ and where $u'_2 : I \to X/S$

correspnds to $u|I \star S \to X$. A diagonal filler of the second square is an arrow $f \in X/T$ with domain $u_1' \in X/T$ such that $qf = g$. We shall prove the existence of f by showing that q is a pseudo-fibration between quasi-categories and that g is invertible. Let us first show that q is a pseudo-fibration between quasi-categories. By 4.10, it suffices to show that q is a right fibration between quasi-categories. But q is a right fibration by 3.19, since p is a mid-fibration. The codomain of q is a quasi-category by 4.11. Thus, q is a pseudo-fibration between quasi-categories. Let us now show that the arrow g is invertible. The canonical projection $k : X/S \to X$ is a right fibration by 3.20. Hence also the composite kp_2. Thus, kp_2 is conservative by 4.10. But we have $kp_2 g = ku_2' = u|I$. This shows that the arrow g is invertible, since the arrow $u|I = u(I)$ is invertible by assumption. Hence there exists an arrow $f \in X/T$ with domain $u_1' \in X/T$ such that $qf = g$, since q is a pseudo-fibration. We have proved that the square has a diagonal filler. ∎

The horn $\Lambda^0[n] \subset \Delta[n]$ contains the edge $(0,1) \subset [n]$ if $n > 1$.

Theorem 4.13. *Suppose that we have a commutative square*

$$\begin{array}{ccc} \Lambda^0[n] & \xrightarrow{x} & X \\ \downarrow & & \downarrow p \\ \Delta[n] & \longrightarrow & Y, \end{array}$$

in which p is a mid fibration between quasi-categories. If $n > 1$ and the arrow $x(0,1) \in X$ is invertible, then the square has a diagonal filler.

Proof: Observe that $\Delta[n] = I \star \Delta[n-2]$ and that

$$\Lambda^0[n] = (\{0\} \star \Delta[n-2]) \cup (I \star \partial\Delta[n-2]).$$

The result then follows from 4.12. ∎

Theorem 4.14. *A quasi-category X is a Kan complex iff the category hoX is a groupoid.*

Proof: The implication $(\Rightarrow)$ follows from 1.16. $(\Leftarrow)$ Let us show that every horn $x : \Lambda^k[n] \to X$ can be filled. This is true if $n = 1$, since each inclusion $\{0\} \subset I$ and $\{1\} \subset I$ admits a retraction. This is true if $0 < k < n$, since X is a quasi-category. This is also true if $n > 1$ and $k = 0$ by 4.13 applied to the map $X \to 1$, since every arrow in X is invertible. This is true if $n > 1$ and $k = n$ by duality, since $hoX^o = (hoX)^o$ is a groupoid. ∎

Corollary 4.15. *Let $p : X \to Y$ be a conservative map between quasi-categories. If Y is a Kan complex, then so is X.*

Proof: The functor $ho(p) : ho(X) \to ho(Y)$ is conservative, since p is conservative. But the category $ho(Y)$ is a groupoid, since Y is a Kan complex. Hence also the category $ho(X)$, since $ho(p)$ is conservative. It then follows by 4.14 that Y is a Kan complex. ∎

Corollary 4.16. *A simplicial set X is a Kan complex iff the map $X \to 1$ is a right fibration.*

Proof: The implication ($\Rightarrow$) is clear, since a Kan fibration is a right fibration. Conversely, if the map $X \to 1$ is a right fibration, let us show that X is a Kan complex. The simplicial set X is a quasi-category, since a right fibration is a mid fibration. But the map $X \to 1$ is conservative, since a right fibration between quasi-categories is conservative by 4.10. It then follows from 4.15, that X is a Kan complex. ∎

Corollary 4.17. *The fibers of a right fibration are Kan complexes.*

Proof: Let $p : E \to B$ be a right fibration. If $b \in B_0$ then the map $p^{-1}(b) \to 1$ is a right fibration, since it is a base change of the map $p : E \to B$. It is thus a Kan complex by 4.16. ∎

Let **Grpd** be the category of (small) groupoids. The inclusion functor **Grpd** $\subset$ **Cat** admits a right adjoint,

$$J : \mathbf{Cat} \to \mathbf{Grpd},$$

where $J(C)$ is the groupoid of isomorphisms of a category C. If X is a quasi-category, let us denote by $J(X)$ the simplicial subset of X defined by the pullback square

$$\begin{array}{ccc} J(X) & \longrightarrow & J(hoX) \\ \downarrow & & \downarrow \\ X & \xrightarrow{\ h\ } & hoX, \end{array}$$

where h is the canonical map.

Lemma 4.18. *The inclusion $J(X) \subseteq X$ is 1-full (see Definition B.0.11). A simplex $x : \Delta[n] \to X$ belongs to $J(X)$ iff the arrow $x(i, j)$ is invertible for every $0 \leq i < j \leq n$.*

Proof: A subcategory of a category is 1-full. Hence the inclusion $J(hoX) \subseteq hoX$ is 1-full. Hence also the inclusion $J(X) \subseteq X$ by base change. Hence a simplex $x : \Delta[n] \to X$ belongs to $J(X)$ iff the arrow $x(i, j)$ is invertible for every $0 \leq i < j \leq n$. ∎

Theorem 4.19. [J1] *The simplicial set $J(X)$ is a Kan complex for any quasi-category X and it is the largest sub-Kan complex of X. The functor*

$$J : \mathbf{QCat} \to \mathbf{Kan}$$

is right adjoint to the inclusion functor $\mathbf{Kan} \subset \mathbf{QCat}$.

Proof: Let us first verify that $J(X)$ is a quasi-category. The inclusion $J(hoX) \subseteq hoX$ is a mid fibration by 2.2, since hoX is a category. Hence the inclusion $J(X) \subseteq X$ is a mid fibration by base change. Thus, $J(X)$ is a quasi-category, since X is a quasi-category. Let us now show that $J(X)$ is a Kan complex. By 4.14, it suffices to show that every arrow $f : a \to b$ in $J(X)$ has an inverse in $J(X)$. But f has an inverse in X, by definition of $J(X)$. If $g : b \to a$ is an inverse of f, then there is a 2-simplex $s \in X$ with boundary $(g, 1_a, f)$ and a 2-simplex $t \in X$ with boundary $(f, 1_b, g)$ by 1.10. We have $s, t \in J(X)$, since the arrows f, g and 1_a belongs to $J(X)$, and since the inclusion $J(X) \subseteq X$ is 1-full. This shows that g is an inverse of f in $J(X)$. Thus, $J(X)$ is a Kan complex. Let us now prove that the functor $J : \mathbf{QCat} \to \mathbf{Kan}$ is right adjoint to the inclusion functor $\mathbf{Kan} \subset \mathbf{QCat}$. For this, it suffices to show that if K is a Kan complex, then every map $u : K \to X$ factors through the inclusion $J(X) \subseteq X$. But the functor $ho(u) : hoK \to hoX$ factors through the inclusion $J(hoX) \subseteq hoX$, since hoK is a groupoid by 4.14. This proves the result, since $J(X) = h^{-1}J(hoX)$. This proves also that $J(X)$ is the largest sub Kan complex of X. ■

The canonical map $X \to hoX$ induces a map $J(X) \to J(hoX)$ hence also a functor $hoJ(X) \to J(hoX)$.

Proposition 4.20. *The canonical functor $hoJ(X) \to J(hoX)$ is an isomorphism for any quasi-category X.*

Proof: The functor is obviously bijective on objects. Let us show that it is fully faithful. The canonical map $h : X \to hoX$ is surjective on arrows, hence also the induced map $J(X) \to J(hoX)$, since $J(X) = h^{-1}J(hoX)$. It follows that the functor $hoJ(X) \to J(hoX)$ is surjective on arrows. It remains to show that the induced map $hoJ(X)(a, b) \to J(hoX)(a, b)$ is injective for every pair $a, b \in X_0$. For this, it suffices to show that if two arrows $f, g : a \to b$ in $J(X)$ are homotopic in X, then they are homotopic in $J(X)$. But if $f \simeq g$, there is a 2-simplex $t \in X_2$ such that $\partial t = (f, g, 1_a)$. We have $t \in J(X)$, since the inclusion $J(X) \subseteq X$ is 1-full by 4.18. This shows that f and g are homotopic in $J(X)$. This completes the proof that the canonical functor $hoJ(X) \to J(hoX)$ is an isomorphism. ■

Proposition 4.21. *Let A be a simplicial set and X be a quasi-category. If $\tau_1 A$ is a groupoid, then every map $u : A \to X$ factors through the inclusion $J(X) \subseteq X$ by*

Proof: The inclusion $J(X) \subseteq X$ is 1-full by Lemma 4.18. Hence it suffices to show that we have $u(A_1) \subseteq J(X)$. But every arrow in A_1 is invertible, since $\tau_1 A$ is a groupoid. Hence also every arrow in $u(A_1)$. This shows that $u(A_1) \subseteq J(X)$. ■

Let J be the groupoid generated by one isomorphism $0 \to 1$.

Proposition 4.22. *An arrow f in a quasi-category X is invertible iff the corresponding map $f : I \to X$ can be extended along the inclusion $I \subset J$.*

Proof: ($\Rightarrow$) If $f \in X_1$ is invertible, then the map $f : I \to X$ can be factored through the inclusion $J(X) \subseteq X$, since $f \in J(X)$. The simplicial set $J(X)$ is a Kan complex by 4.19. The inclusion $I \subset J$ is anodyne by a classical result [GZ], since it is a weak homotopy equivalence. Hence the map $f : I \to J(X)$ can be extended along the inclusion $I \subset J$. ($\Leftarrow$) Let $g : J \to X$ be an extension of the map $f : I \to X$. The arrow $i = (0, 1)$ is invertible in J, since J is a groupoid. It follows that $f = g(i)$ is invertible in X. ∎

Corollary 4.23. *Two objects a and b of a quasi-category X are isomorphic iff there exist a map $u : J \to X$ such that $u(0) = a$ and $u(1) = b$.*

Proof: ($\Rightarrow$) There exist an isomorphism $f : a \to b$ in X by 1.13. The corresponding map $f : I \to X$ can be extended as a map $u : J \to X$ by 4.22, since f is invertible. ($\Leftarrow$) Let $u : J \to X$ be a map such that $u(0) = a$ and $u(1) = b$. The arrow $u(0, 1) : a \to b$ is invertible in X, since the arrow $(0, 1)$ is invertible in J. ∎

An homotopy $\alpha : A \times I \to B$ between two maps $f, g : A \to B$ defines an arrow $[\alpha] : f \to g$ in the category $\tau_1(A, B)$.

Definition 4.24. *We say that a homotopy $\alpha : A \times I \to B$ between two maps $f, g : A \to B$ is* invertible *f the arrow $[\alpha] : f \to g$ is invertible in $\tau_1(A, B)$.*

Let j_0 be the inclusion $\{0\} \subset J$ and j_1 the inclusion $\{1\} \subset J$.

Corollary 4.25. *If X is a quasi-category, then a homotopy $\alpha : A \times I \to X$ is invertible iff it can extended along the inclusion $A \times I \subseteq A \times J$. Two maps $f, g : A \to X$ are isomorphic in $\tau_1(A, X)$ iff there exist a map $h : A \times J \to X$ such that $h(1_A \times j_0) = f$ and $h(1_A \times j_1) = g$.*

Proof: This follows from 4.22, since the simplicial set X^A is a quasi-category by 2.19. ∎

Recal that two maps $f, g : X \to Y$ are said to be *isomorphic* if they are isomorphic in the category $\tau_1(X, Y) = \tau_1(Y^X)$.

Proposition 4.26. *Let X and Y be quasi-categories. If two maps $f, g : X \to Y$ are isomorphic, then so are the maps $J(f), J(g) : J(X) \to J(Y)$. If a map $f : X \to Y$ is a categorical equivalence, then so is the map $J(f) : J(X) \to J(Y)$.*

Proof: By 4.25, an isomorphism $f \to g$ can be represented by a map $h : X \times J \to Y$ such that $h(1_X \times j_0) = f$ and $h(1_X \times j_1) = g$. The functor $J : \mathbf{QCat} \to \mathbf{Kan}$ preserves products, since it is a right adjoint by 4.19. We have $J(J) = J$, since the simplicial set J is a Kan complex. Hence the map $J(h) : J(X) \times J \to J(Y)$ represents an isomorphism $J(f) \to J(g)$. The first statement is proved. The second statement follows. ∎

4.3 Pseudo-fibrations and Kan fibrations

Proposition 4.27. *Let X and Y be two quasi-categories. Then a mid fibration $p : X \to Y$ is a pseudo-fibration iff the map $J(p) : J(X) \to J(Y)$ is a Kan fibration.*

Proof: ($\Rightarrow$) If p is a pseudo-fibration, let us show that every square

$$\begin{array}{ccc} \Lambda^k[n] & \xrightarrow{x} & J(X) \\ \downarrow & & \downarrow{\scriptstyle J(p)} \\ \Delta[n] & \xrightarrow{y} & J(Y) \end{array}$$

has a diagonal filler. This is true if $n = 1$ and $k = 0$, since p is a pseudo-fibration. This is true also if $n = 1$ and $k = 1$, since p^o is a pseudo-fibration by 4.6. We can thus suppose $n > 1$ for the rest of the proof. Let us first show that the following square has a diagonal filler,

$$\begin{array}{ccc} \Lambda^k[n] & \xrightarrow{i_X x} & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ \Delta[n] & \xrightarrow{i_Y y} & Y, \end{array}$$

where i_X denotes the inclusion $J(X) \subseteq X$ and i_Y the inclusion $J(Y) \subseteq Y$. This is true if $0 < k < n$, since p is a mid fibration. This is true if $k = 0$ by 4.13, since the arrow $x(0,1) \in J(X)$ is invertible. This is also true if $k = n$ by symmetry. We have completed the proof that the second square has a diagonal filler. Let us now prove hat the first square has a diagonal filler. For this it suffice to show that every diagonal filler $d : \Delta[n] \to X$ of the second square factors through the inclusion i_X. The inclusion i_X is 1-full by 4.18. Hence it suffices to show that that d takes every arrow of $\Delta[n]$ to an isomorphism in X. If $n > 2$, every arrow of $\Delta[n]$ is in $\Lambda^k[n]$. The result follows in this case, since the restriction $d|\Lambda^k[2]$ factors through i_X. It remains to consider the case $n = 2$. We have $d(0,2) = d(1,2)d(0,1)$ in hoX. Thus, if two of the arrows $d(0,1)$, $d(1,2)$ and $d(0,2)$ is an isomorphism then so is the third. But two at least of these arrows is invertible, since the restriction $d|\Lambda^k[2]$ factors through i_X. This proves that d factor through the inclusion i_X. Let us prove the implication ($\Leftarrow$). Let $a \in X_0$ and let $g \in Y$ be an isomorphism with source $p(a)$. Then there exists an arrow $f \in J(X)$ with source a such that $p(f) = g$, since $J(p)$ is a Kan fibration. The arrow f is invertible in X, since $f \in J(X)$. This shows that p is a pseudo-fibration. ∎

Corollary 4.28. *Let X and Y are Kan complexes, then every pseudo-fibration $p : X \to Y$ is a Kan fibration.*

Proof: We have $J(p) = p$, since X and Y are Kan complexes. But $J(p)$ is a Kan fibration by 4.27, since p is a pseudo-fibration. ∎

Lemma 4.29. *A map between quasi-categories $p : X \to Y$ is conservative iff the square*

$$\begin{array}{ccc} J(X) & \longrightarrow & X \\ {\scriptstyle J(p)}\downarrow & & \downarrow{\scriptstyle p} \\ J(Y) & \longrightarrow & Y \end{array}$$

is cartesian.

Proof: ($\Rightarrow$) We have $J(X) \subseteq p^{-1}(J(Y))$, since the square commutes. Let us show that we have $p^{-1}(J(Y)) \subseteq J(X)$. If a simplex $x : \Delta[n] \to X$ belongs to $p^{-1}(J(Y))$ then the simplex $px : \Delta[n] \to Y$ belongs to $J(Y)$. Hence the arrow $px(i,j)$ is invertible in Y for every $i < j$. It follows that the arrow $x(i,j)$ is invertible in X for every $i < j$, since p is conservative. Hence the simplex $x : \Delta[n] \to X$ belong to $J(X)$ by 4.18. The implication ($\Rightarrow$) is proved. The converse is obvious. ∎

Proposition 4.30. *Let $p : X \to Y$ be a conservative pseudo-fibration between quasi-categories. If Y is a Kan complex then p is a Kan fibration.*

Proof: The square

$$\begin{array}{ccc} J(X) & \longrightarrow & X \\ {\scriptstyle J(p)}\downarrow & & \downarrow{\scriptstyle p} \\ J(Y) & \longrightarrow & Y \end{array}$$

is a pullback by lemma 4.29, since p is conservative. If Y is a Kan complex, then $J(Y) = Y$. Hence we have $J(p) = p$, since the square is a pullback. Thus, p is a Kan fibration, since $J(p)$ is a Kan fibration by 4.27. ∎

Corollary 4.31. *If $p : X \to Y$ is a right fibration and Y is a Kan complex then p is a Kan fibration.*

Proof: The simplicial set X is a quasi-category, since Y is a quasi-category and a right fibration is a mid fibration. Moreover, p is a conservative pseudo-fibration by 4.9. It follows that p is a Kan fibration by 4.30. ∎

Proposition 4.32. *A mid fibration between quasi-categories $p : X \to Y$ is a pseudo-fibration iff it has the right lifting property with respect to the inclusion $j_0 : \{0\} \subset J$.*

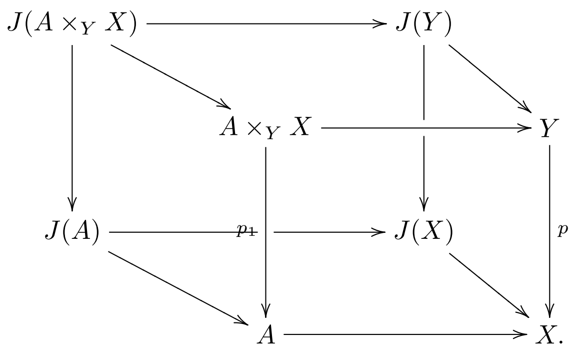
Proof: ($\Rightarrow$) The functor $J : \mathbf{QCat} \to \mathbf{Kan}$ is right adjoint to the inclusion functor $\mathbf{Kan} \subset \mathbf{QCat}$ by 4.19. Hence the condition $j_0 \pitchfork J(p)$ is equivalent to the condition $j_0 \pitchfork p$ by D.1.14. But we have $j_0 \pitchfork J(p)$, since $J(p)$ is a Kan fibration by 4.27. It follows that we have $j_0 \pitchfork p$. ($\Leftarrow$) Let $a \in X_0$ and let $g \in Y_1$ be an isomorphism with source $p(a)$. We shall prove that there exists an arrow $f \in X$ with source a such that $p(f) = g$. The map $g : I \to Y$ can be extended to a map $g' : J \to Y$ by 4.22. The square

$$\begin{array}{ccc} 1 & \xrightarrow{a} & X \\ {\scriptstyle j_0}\downarrow & & \downarrow{\scriptstyle p} \\ J & \xrightarrow{g'} & Y \end{array}$$

has then a diagonal filler $f' : J \to X$, since we have $j_0 \pitchfork p$ by assumption. The arrow $f = f'(0,1)$ is invertible in X, since the arrow $(0,1)$ is invertible in J. We have $pf = g$, since $pf' = g'$. Moreover, $f(0) = f'(0) = a$. ∎

Corollary 4.33. *The class of pseudo-fibrations in* **QCat** *is closed under base change and retracts. Similarly for the class of conservative pseudo-fibrations in* **QCat**.

Proof: Let us prove the first statement. Let $\mathcal{B}$ be the class of mid fibrations in $\mathbf{S}$. The base change of a mid fibration between quasi-categories along a map of quasi-categories is a mid fibration between quasi-categories. Hence the class $\mathcal{B} \cap \mathbf{QCat}$ is closed under base change. It is also closed under retracts. If j_0 denotes the inclusion $\{0\} \subset J$, then the class $\{j_0\}^{\pitchfork} \subset \mathbf{S}$ is closed under base change and retracts, since this is true of every class of the form $\{u\}^{\pitchfork}$ for any map $u \in \mathbf{S}$. It follows that the intersection $\{j_0\}^{\pitchfork} \cap \mathcal{B} \cap \mathbf{QCat}$ is closed under base change and retracts. This proves the result by 4.32. Let us prove the second statement. Let $p : X \to Y$ be a conservative pseudo-fibration between quasi-categories and let $A \to X$ be a map between quasi-categories. Let us show that the map $p_1 : A \times_Y X \to A$ is conservative. Consider the cube



The front face is a pullback square of quasi-categories. Hence the back face is a pullback square in the category of Kan complexes, since the functor $J : \mathbf{QCat} \to \mathbf{Kan}$ is a right adjoint by 4.19. But $J(p)$ is a Kan fibration by 4.27. Hence the back face is actually a pullback square in the category of simplicial sets. The right hand face is a pullback by 4.29. It then follows by the cube lemma C.0.29, that the left hand face is a pullback. This proves that p_1 is conservative. ∎

Corollary 4.34. *Every fiber of a conservative pseudo-fibration between quasi-categories is a Kan complex.*

Proof: Let $p : X \to Y$ be a conservative pseudo-fibration between quasi-categories. If $y \in Y$, then the map $p^{-1}(y) \to 1$ is a conservative pseudo-fibration by 4.33. It is thus a Kan fibration by 4.30. This shows that $p^{-1}(y)$ is a Kan complex. ∎

Chapter 5

Pseudo-fibrations and function spaces

The main results of the chapter are the following.

Theorem A *If $f : X \to Y$ is a pseudo-fibration between quasi-categories, then so is the map*

$$\langle u, f\rangle : X^B \to Y^B \times_{Y^A} X^A$$

for any monomorphism of simplicial sets $u : A \to B$.

The theorem will be proved in 5.13.

Let J be the groupoid generated by one isomorphism $0 \to 1$.

Theorem B *A map between quasi-categories $f : X \to Y$ is a pseudo-fibration iff the projection*

$$X^J \to Y^J \times_Y X$$

defined from the inclusion $\{0\} \subset J$ is a trivial fibration.

The theorem will be proved in 5.22 .

If A and X are simplicial sets, consider the projection $X^A \to X^{A_0}$ defined from the inclusion $A_0 \subseteq A$.

Theorem C. *If X is a quasi-category, then the projection*

$$X^A \to X^{A_0}$$

is conservative.

The theorem will be proved in 5.14. It says that a homotopy $\alpha : f \to g$ between two maps $A \to X$ is invertible in X^A iff the arrow $\alpha(a) : f(a) \to g(a)$ is invertible in X for each $a \in A_0$.

Theorem D *A pseudo-fibration between quasi-categories is a trivial fibration iff it is an equivalence.*

The theorem will be proved in 5.15.

Our first goal is to prove theorem C in 5.14. Recall by lemma 4.29 that a map between quasi-categories $X \to Y$ is conservative iff the square

$$\begin{array}{ccc} J(X) & \longrightarrow & X \\ \downarrow & & \downarrow \\ J(Y) & \longrightarrow & Y \end{array}$$

is cartesian. Thus, theorem C will be proved if we can show that the following square is cartesian,

$$\begin{array}{ccc} J(X^A) & \longrightarrow & X^A \\ {\scriptstyle J(p)}\downarrow & & \downarrow{\scriptstyle p} \\ J(X^{A_0}) & \longrightarrow & X^{A_0}, \end{array}$$

where p is the projection defined from the inclusion $A_0 \subseteq A$. Let us put

$$J(A, X) = p^{-1}(J(X^{A_0})).$$

By definition, the following square is cartesian,

$$\begin{array}{ccc} J(A, X) & \longrightarrow & X^A \\ \downarrow & & \downarrow{\scriptstyle p} \\ J(X^{A_0}) & \longrightarrow & X^{A_0}. \end{array}$$

Let us record the following elementary fact:

Proposition 5.1. *We have $J(X^A) \subseteq J(A, X)$.*

We shall prove that $J(X^A) = J(A, X)$ in A. The simplicial set $J(X^A)$ is the largest sub-Kan complex of X^A by 4.19. Hence we can prove the equality $J(X^A) = J(A, X)$ by showing that $J(A, X)$ is a Kan complex. Observe that the simplicial set $J(A, X)$ depends contravariantly on A. The contravariant functor

$A \mapsto J(A, X)$ is a subfunctor of the contravariant functor $A \mapsto X^A$. We shall see that it has a right adjoint.

A 0-full simplicial subset $U \subseteq X$ of a simplicial set X is obviously determined by the subset $U_0 \subseteq X_0$; we shall say that X is *spanned* by U_0. A 0-full simplicial subset of a quasi-category is a quasi-category. If X is a quasi-category and S is a simplicial set, let us denote by $X^{(S)}$ the 0-full simplicial subset of X^S which is spanned by the set of maps $S \to X$ which can be factored through the inclusion $J(X) \subseteq X$. The simplicial set $X^{(S)}$ depends contravariantly on S. The contravariant functor $S \mapsto X^{(S)}$ is a subfunctor of the contravariant functor $S \mapsto X^S$.

For any simplicial sets A, S and X, there are natural bijections between the following three kinds of maps

$$S \to X^A, \quad A \times S \to X, \quad A \to X^S.$$

The bijections associate to a map $u : A \times S \to X$ a map $\lambda^A u : S \to X^A$ and a map $\lambda^S u : A \to X^S$. It shows that the contravariant functor $A \mapsto X^A$ is right adjoint to itself.

Proposition 5.2. *Let X be a quasi-category. Then the contravariant functors*

$$A \mapsto J(A, X) \quad \text{and} \quad S \mapsto X^{(S)}$$

are mutually right adjoint. If $u : A \times S \to X$, then the map $\lambda^A u : S \to X^A$ (resp. $\lambda^S u : A \to X^S$) can be factored through the inclusion $J(A, X) \subseteq X^A$ (resp $X^{(S)} \subseteq X^S$) iff the arrow $u(1_a, f)$ is quasi-invertible in X for every vertex $a \in A_0$ and every arrow $f \in S_1$.

Proof: Let us denote by $F(S, A; X)$ the set of maps $u : S \times A \to X$ such that the arrow $u(1_a, f)$ is quasi-invertible in X for every vertex $a \in A_0$ and every $f \in S_1$. This defines a subfunctor $(A, S) \mapsto F(A, S; X)$ of the contravariant set-valued functor $(A, S) \mapsto \mathbf{S}(A \times S, X)$. Let us show that the map $u \mapsto \lambda^A u$ induces a bijection $F(A, S; X) \simeq \mathbf{S}(S, J(A, X))$ and that the map $u \mapsto \lambda^S u$ induces a bijection $F(A, S; X) \simeq \mathbf{S}(A, X^{(S)})$. Let $p_A : X^A \to X^{A_0}$ be the projection defined by the inclusion $i_A : A_0 \subseteq A$. By definition, we have $J(A, X) = p_A^{-1}(J(X^{A_0}))$. Hence the map $\lambda^A u$ can be factored through the inclusion $J(A, X) \subseteq X^A$ iff the map $p_A \lambda^A u = \lambda^{A_0}(u(S \times i_A))$ can be factored through the inclusion $J(X^{A_0}) \subseteq X^{A_0}$. But we have $J(X^{A_0}) = J(X)^{A_0}$, since the functor $J : \mathbf{QCat} \to \mathbf{Kan}$ is a right adjoint, and since a right adjoint preserves products. Hence the map $\lambda^{A_0}(u(S \times i_A))$ can be factored through the inclusion $J(X^{A_0}) \subseteq X^{A_0}$ iff the map $u(-, a) : S \to X$ can be factored through the inclusion $J(X) \subseteq X$ for each vertex $a \in A_0$. By 4.18, the map $u(-, a) : S \to X$ can be factored through the inclusion $J(X) \subseteq X$ iff the arrow $u(f, a)$ is quasi-invertible in X for every arrow $f \in S_1$. This proves that the map $u \mapsto \lambda^A u$ induces a bijection $F(A, S; X) \simeq \mathbf{S}(S, J(A, X))$.

Similarly, the map $\lambda^S u$ can be factored through the inclusion $X^{(S)} \subseteq X^S$ iff we have $(\lambda^S u)(A_0) \subseteq X^{(S)}$, since the inclusion $X^{(S)} \subseteq X^S$ is 0-full. For every $a \in A_0$ we have $(\lambda^S u)(a) = u(-,a) : S \to X$. Hence the condition $(\lambda^S u)(A_0) \subseteq X^{(S)}$ means that the map $u(-,a) : S \to X$ can be factored through the inclusion $J(X) \subseteq X$ for every $a \in A_0$. By 4.18, the map $u(-,a) : S \to X$ can be factored through the inclusion $J(X) \subseteq X$ iff the arrow $u(f,a)$ is quasi-invertible in X for every arrow $f \in S_1$. This proves that the map $u \mapsto \lambda^S u$ induces a bijection $F(A,S;X) \simeq \mathbf{S}(A, X^{(S)})$. By composing the two bijections, we obtain is a natural bijection

$$\mathbf{S}(S, J(A,X)) \simeq \mathbf{S}(A, X^{(S)}).$$

■

Proposition 5.3. *If X is a quasi-category and S is a simplicial set, then we have $J(X^{(S)}) = J(X)^S$. Moreover, $X^{(S)} = X^S$ and $J(X)^S = J(X^S) = J(S,X)$ if in addition $\tau_1 S$ is a groupoid.*

Proof: We have $J(X)^S \subseteq X^{(S)}$, since $X^{(S)}$ is a 0-full simplicial subset of X^S and every map $S \to J(X)$ belongs to $X^{(S)}$. The simplicial set $J(X)^S$ is a Kan complex, since $J(X)$ is a Kan complex. Thus, $J(X)^S \subseteq J(X^{(S)})$, since $J(X^{(S)})$ is the largest Kan subcomplex of $X^{(S)}$. Let us show that $J(X^{(S)}) \subseteq J(X)^S$. We shall prove this by showing that if A is a Kan complex, then every map $u : A \to X^S$ which can be factored through the inclusion $X^{(S)} \subseteq X^S$ can also be factored through the inclusion $J(X)^S \subseteq X^S$. Let us put $v(s,x) = u(x)(s)$. This defines a map $v : S \times A \to X$ and we have $\lambda^S v = u$. We wish to prove that the map $v : S \times A \to X$ can be factored through the inclusion $J(X) \subseteq X$. For this, it suffices to show that the image by v of every pair of arrows $(g,f) \in S_1 \times A_1$ is quasi-invertible in hoX, since the inclusion $J(X) \subseteq X$ is 1-full by 4.18. If $g : c \to d$ and $f : a \to b$ then we have a decomposition $v(g,f) = v(g,1_a)v(1_d,f)$ in the category hoX. Hence it suffices to show that each arrow $v(g,1_a)$ and $v(1_d,f)$ is invertible in hoX. The arrow $v(g,1_a)$ is invertible in hoX by 5.2, since the map u can be factored through the inclusion $X^{(S)} \subseteq X^S$ by the hypothesis. The arrow $u(1_d,f)$ is invertible in hoX, since f is invertible in hoA. The equality $J(X^{(S)}) = J(X)^S$ is proved. Let us now suppose that $\tau_1(S)$ is a groupoid. In this case let us show that $X^{(S)} = X^S$. For this, it suffices to show that every map $u : S \to X$ can be factored through the inclusion $J(X) \subseteq X$. But the image by u of every arrow in S is invertible in $\tau_1 X$, since every arrow in S is invertible in $\tau_1 S$. It follows that u can be factored through the inclusion $J(X) \subseteq X$. The equality $X^{(S)} = X^S$ is proved. Under the same hypothesis on $\tau_1 S$, let us now show that $J(X)^S = J(X^S) = J(S,X)$. We have $J(X)^S = J(X^{(S)})$ by the first part of the proof. We have $J(X^{(S)}) = J(X^S)$, since we have $X^{(S)} = X^S$ by what we just proved. We have $J(X^S) \subseteq J(S,X)$ by 5.1. Thus,

$$J(X)^S = J(X^S) \subseteq J(S,X) \subseteq X^S.$$

Let us prove that we have $J(S,X) \subseteq J(X)^S$. We shall prove this by showing that if A is a simplicial set, then every map $u : A \to J(S,X)$ can be factored through the inclusion $J(X)^S \subseteq X^S$. Let us put $v(s,x) = u(x)(s)$. This defines a map $v : S \times A \to X$ and we have $\lambda^S v = u$. The arrow $u(1_d, f)$ is quasi-invertible in X for every vertex $d \in S_0$ and every arrow $f \in A_1$ by 5.2, since the map $u = \lambda^S v$ can be factored through the inclusion $J(S,X) \subseteq X^S$. The image by v of every arrow of the form $(g, 1_a)$ is quasi-invertible in X, since g is quasi-invertible in S. It follows that the image by v of every pair $(g,f) \in S_1 \times A_1$ is quasi-invertible in X. Thus, v can be factored through the inclusion $J(X) \subseteq X$. It follows that $u = \lambda^S v$ can be factored through the inclusion $J(X)^S \subseteq X^S$. ∎

Let $f : X \to Y$ be a map between quasi-categories. If $v : S \to T$ is a map of simplicial sets, we shall denote by $\langle (v), f \rangle$ the map

$$X^{(T)} \to X^{(S)} \times_{Y^{(S)}} Y^{(T)}$$

obtained from the commutative square

$$\begin{array}{ccc} X^{(T)} & \xrightarrow{X^{(v)}} & X^{(T)} \\ {\scriptstyle f^{(T)}}\downarrow & & \downarrow{\scriptstyle f^{(S)}} \\ Y^{(T)} & \xrightarrow{Y^{(v)}} & Y^{(S)}. \end{array}$$

Similarly, if $u : A \to B$ is a map of simplicial sets, we shall denote by $J'(u,f)$ the map

$$J(B,X) \to J(B,Y) \times_{J(A,Y)} J(A,X)$$

obtained from the commutative square

$$\begin{array}{ccc} J(B,X) & \xrightarrow{J(u,X)} & J(A,X) \\ {\scriptstyle J(B,f)}\downarrow & & \downarrow{\scriptstyle J(A,f)} \\ J(B,Y) & \xrightarrow{J(u,Y)} & J(A,Y). \end{array}$$

Lemma 5.4. *Let $f : X \to Y$ be a map between quasi-categories. If $u : A \to B$ and $v : S \to T$ are maps of simplicial sets, then we have*

$$u \pitchfork \langle (v), f \rangle \quad \Longleftrightarrow \quad v \pitchfork J'(u,f).$$

Proof: We shall use 5.2 and D.1.15. Let $f : X \to Y$ be a fixed map between quasi-categories. For every simplicial set A, let us put $F_0(A) = J(A,X)$, $F_1(A) = J(A,Y)$ and $\alpha_A = J(A,f)$. This defines two contravariant functors $F_0, F_1 : \mathbf{S} \to \mathbf{S}$ and a natural transformation $\alpha : F_0 \to F_1$. For every simplicial set S, let us put $G_0(S) = Y^{(S)}$, $G_1(S) = Y^{(S)}$ and $\beta_S = f^{(S)}$. This defines two contravariant functors $G_0, G_1 : \mathbf{S} \to \mathbf{S}$ and a natural transformation $\beta_S : G_0 \to G_1$. The functor G_0 is right adjoint to the functor F_0 by 5.2 and the functor G_1 is right adjoint to the functor F_1. Moreover, the natural transformation β is the right transpose of the natural transformation α. Hence the condition $u \pitchfork \beta^\bullet(v)$ is equivalent to the condition $v \pitchfork \alpha^\bullet(u)$ by D.1.15. This proves the result, since $\beta^\bullet(v) = \langle (v), f \rangle$ and $\alpha^\bullet(u) = J'(u, f)$. ∎

Lemma 5.5. *Suppose that we have a commutative square of simplicial sets*

$$\begin{array}{ccc} A & \xrightarrow{i} & A' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ B & \xrightarrow{j} & B', \end{array}$$

in which j is monic and i is 0-full. If f' is a mid fibration, then so is f.

Proof: The map f is a base change of the map jf along j, since j is monic. Hence it suffices to to show that jf is a mid fibration. But we have $jf = f'i$, since the square commutes. The map i is a mid fibration by 2.6, since it is 0-full. Hence also the composite $f'i$, since f' is a mid fibration by assumption. ∎

Lemma 5.6. *If $f : X \to Y$ is a map between quasi-categories and $v : S \to T$ is a monomorphism of simplicial sets, then $\langle (v), f \rangle$ is a map between quasi-categories and we have $J\langle (v), f \rangle = \langle v, J(f) \rangle$.*

Proof: The simplicial set $X^{(T)}$ is a quasi-category, since the inclusion $X^{(T)} \subseteq X^T$ is 0-full and the simplicial set X^T is a quasi-category by 2.19. Let us show that the codomain of $\langle (v), f \rangle$ is a quasi-category. Consider the pullback square

$$\begin{array}{ccc} X^{(S)} \times_{Y^{(S)}} Y^{(T)} & \xrightarrow{p_2} & Y^{(T)} \\ \downarrow & & \downarrow \\ Y^{(T)} & \xrightarrow{Y^{(v)}} & Y^{(S)}. \end{array}$$

It suffices to show that the projection p_2 is a mid fibration, since $Y^{(T)}$ is a quasi-category. But for this it suffices to show that $Y^{(v)}$ is a mid fibration. Consider the square

$$\begin{array}{ccc} Y^{(T)} & \xrightarrow{i} & Y^T \\ {\scriptstyle Y^{(v)}}\downarrow & & \downarrow{\scriptstyle Y^v} \\ X^{(S)} & \xrightarrow{j} & Y^S, \end{array}$$

where the horizontal maps are inclusions. The map Y^v is a mid fibration by 2.20 The inclusion i is 0-full. It follows by lemma 5.5 that $Y^{(v)}$ is a mid fibration. It follows from 5.3 that the image by the functor J of the square

$$\begin{array}{ccc} X^{(T)} & \xrightarrow{X^{(v)}} & X^{(T)} \\ {\scriptstyle f^{(T)}}\downarrow & & \downarrow{\scriptstyle f^{(S)}} \\ Y^{(T)} & \xrightarrow{Y^{(v)}} & Y^{(S)} \end{array}$$

is the square

$$\begin{array}{ccc} J(X)^T & \xrightarrow{J(X)^v} & J(Y)^T \\ {\scriptstyle J(f)^T}\downarrow & & \downarrow{\scriptstyle J(f)^S} \\ J(Y)^T & \xrightarrow{J(Y)^v} & J(Y)^S. \end{array}$$

But the functor $J : \mathbf{QCat} \to \mathbf{Kan}$ preserves pullbacks, since it is a right adjoint. Thus, $J\langle(v), f\rangle = \langle v, J(f)\rangle$. ∎

Theorem 5.7. *If $f : X \to Y$ is a pseudo-fibration between quasi-categories, then so is the map*

$$\langle(v), f\rangle : X^{(T)} \to Y^{(T)} \times_{Y^{(S)}} X^{(S)}$$

for any monomorphism of simplicial sets $v : S \to T$.

Proof: We saw in 5.6 that $\langle(v), f\rangle$ is a map between quasi-categories. Let us show that it is a mid fibration. Consider the commutative square

$$\begin{array}{ccc} X^{(T)} & \xrightarrow{\quad i \quad} & X^T \\ {\scriptstyle \langle(v),f\rangle}\downarrow & & \downarrow{\scriptstyle \langle v,f\rangle} \\ Y^{(T)} \times_{Y^{(S)}} X^{(S)} & \xrightarrow{j} & Y^T \times_{Y^S} X^S, \end{array}$$

where i and j are the inclusions. The map $\langle v, f\rangle$ is a mid fibration by theorem 2.18, since f is a mid fibration. The inclusion i is 0-full. It follows by lemma 5.5 that $\langle(v), f\rangle$ is a mid fibration. Let us show that $\langle(v), f\rangle$ is a pseudo-fibration. By 4.27 it suffices to show that the map $J\langle(v), f\rangle$ is a Kan fibration, since it is a mid fibration between quasi-categories. But we have $J\langle(v), f\rangle = \langle v, J(f)\rangle$ by 5.6. The map $J(f)$ is a Kan fibration by 4.27, since f is a pseudo-fibration by the hypothesis. It follows that $\langle v, J(f)\rangle$ is a Kan fibration. This complete the proof that $\langle(v), f\rangle$ is a pseudo-fibration. ∎

If $m > 0$, the open box

$$\Lambda^m[m,n] = \big(\Lambda^m[m] \times \Delta[n]\big) \cup \big(\Delta[m] \times \partial\Delta[n]\big)$$

contains the edge $(m-1,n) \to (m,n)$ of $\Delta[m,n] = \Delta[m] \times \Delta[n]$.

Lemma 5.8. *Suppose that we have a commutative square*

$$\begin{array}{ccc} \Lambda^m[m,n] & \xrightarrow{x} & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ \Delta[m,n] & \longrightarrow & Y, \end{array}$$

in which p is a mid fibration between quasi-categories. If $m > 0$, $m+n > 1$ and the image the arrow $(m-1,n) \to (m,n)$ by x is quasi-invertible, then the square has a diagonal filler.

Proof: We shall use the notation of lemma H.0.21. The lemma shows that the following square

$$\begin{array}{ccccc} \Lambda^m[m,n] & & \xrightarrow{\qquad x \qquad} & & X \\ \downarrow & & \text{(a)} & & \downarrow{\scriptstyle p} \\ C(P') \cup \Lambda^m[m,n] & \longrightarrow & \Delta[m,n] & \xrightarrow{y} & Y \end{array}$$

has a diagonal filler $d : C(P') \cup \Lambda^m[m,n] \to X$, since p is a mid fibration. Hence it suffices to show that the following square (b) has a diagonal filler:

$$\begin{array}{ccc} C(P') \cup \Lambda^m[m,n] & \xrightarrow{d} & X \\ \downarrow & \text{(b)} & \downarrow{\scriptstyle p} \\ \Delta[m] \times \Delta[n] & \xrightarrow{y} & Y. \end{array}$$

We shall use the notation of lemmas H.0.18 and H.0.19. The poset $P = [m] \times [n]$ is the shadow of the following maximal chain ω,

$$(0,0) < (0,1) < \cdots < (0,n-1) < (0,n) < (1,n) < \cdots < (m,n).$$

We have $\dot{C}(P) = C(P')$ by lemma H.0.18, since $(0,n)$ is the only upper corner of ω. It follows that we have $C(P) = C(P') \cup \Delta[\omega]$ by the same lemma. The set of lower corners of ω is empty and (m,n) is the lowest element of ω on the vertical line $\{m\} \times [n]$. It follows from H.0.19 that we have

$$\Delta[\omega] \cap (C(P') \cup \Lambda^m[m,n]) = \Lambda^{m+n}[\omega].$$

Therefore, the following square is a pushout,

$$\begin{array}{ccc}
\Lambda^{m+n}[\omega] & \longrightarrow & C(P') \cup \Lambda^m[m,n] \\
\downarrow & \text{(c)} & \downarrow \\
\Delta[\omega] & \longrightarrow & \Delta[m,n].
\end{array}$$

Hence the square (b) has a diagonal filler iff the following square (b+c)

$$\begin{array}{ccc}
\Lambda^{n+m}[\omega] & \xrightarrow{d'} & X \\
\downarrow & \text{(b+c)} & \downarrow{\scriptstyle p} \\
\Delta[\omega] & \xrightarrow{y'} & Y
\end{array}$$

has a diagonal filler, But the image of the arrow $m+n-1 \to m+n$ by d' is equal to the image of the arrow $(n-1,m) \to (n,m)$ by x. This image is quasi-invertible in X by hypothesis. Hence the square (b+c) has a diagonal filler by 4.13, since $m+n > 1$. ∎

Lemma 5.9. *Let $f : X \to Y$ be a map between quasi-categories and $u : A \to B$ be a monomorphism of simplicial sets. If the map $u_0 : A_0 \to B_0$ is bijective, then the square*

$$\begin{array}{ccc}
J(B,X) & \longrightarrow & X^B \\
{\scriptstyle J'(u,f)}\downarrow & & \downarrow{\scriptstyle \langle u,f\rangle} \\
J(B,Y) \times_{J(A,Y)} J(A,X) & \longrightarrow & Y^B \times_{Y^A} X^A
\end{array}$$

is cartesian.

Proof: We can suppose that u is an inclusion $A \subseteq B$ and that $A_0 = B_0$. Consider the commutative diagram

$$\begin{array}{ccc}
J(B,X) & \longrightarrow & X^B \\
{\scriptstyle J'(u,f)}\downarrow & & \downarrow{\scriptstyle \langle u,f\rangle} \\
J(B,Y) \times_{J(A,Y)} J(A,X) & \longrightarrow & Y^B \times_{Y^A} X^A \\
{\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle p_2} \\
J(A,X) & \longrightarrow & X^A \\
\downarrow & & \downarrow \\
J(X)^{A_0} & \longrightarrow & X^{A_0},
\end{array}$$

in which the horizontal maps are inclusions. It suffices to show that the boundary square is cartesian. But the boundary square coincide with the square

$$\begin{array}{ccc} J(B,X) & \longrightarrow & X^B \\ \downarrow & & \downarrow \\ J(X)^{B_0} & \longrightarrow & X^{B_0}, \end{array}$$

since we have $A_0 = B_0$ by the hypothesis. This last square is cartesian by definition of $J(B,X)$. ∎

Theorem 5.10. *Let $f : X \to Y$ be pseudo-fibration between quasi-categories. Then the map*

$$J'(u,f) : J(B,X) \to J(B,Y) \times_{J(A,Y)} J(A,X)$$

is a Kan fibration for any monomorphism $u : A \to B$ and the map

$$\langle (v), f \rangle : X^{(T)} \to Y^{(T)} \times_{Y^{(S)}} X^{(S)}$$

is a trivial fibration for any anodyne map $v : S \to T$.

Proof: Let us first verify that the map $J'(u,f)$ is a Kan fibration in the case where u is the inclusion $\sigma_n : \partial\Delta[n] \subset \Delta[n]$. If $n = 0$ we we have $J'(\delta_n, f) = J(f)$. The result follows from 4.27 in the case. We can thus suppose $n > 0$. In this case we shall prove that every commutative square (a)

$$\begin{array}{ccc} \Lambda^k[m] & \xrightarrow{\quad x \quad} & J(\Delta[n],X) \\ \downarrow & \text{(a)} & \downarrow J'(\delta_n,f) \\ \Delta[m] & \xrightarrow{(y,z)} & J(\Delta[n],Y) \times_{J(\partial\Delta[n],Y)} J(\partial\Delta[n],X) \end{array}$$

has a diagonal filler. The following square (b)

$$\begin{array}{ccc} J(\Delta[n],X) & \longrightarrow & X^{\Delta[n]} \\ J'(\delta_n,f) \downarrow & \text{(a)} & \downarrow \langle \delta_n, f \rangle \\ J(\Delta[n],Y) \times_{J(\partial\Delta[n],Y)} J(\partial\Delta[n],X) & \longrightarrow & Y^{\Delta[n]} \times_{Y^{\partial\Delta[n]}} X^{\partial\Delta[n]} \end{array}$$

is cartesian by Lemma 5.9, since inclusion $\partial\Delta[n] \subset \Delta[n]$ is bijective on 0-cells when $n > 0$. Hence the square (a) has a diagonal filler iff the following composite square (a+b) has a diagonal filler,

$$\begin{array}{ccc} \Lambda^k[m] & \xrightarrow{\quad x' \quad} & X^{\Delta[n]} \\ \downarrow & \text{(a+b)} & \downarrow \langle \delta_n, f \rangle \\ \Delta[m] & \xrightarrow{(y',z')} & Y^{\Delta[n]} \times_{Y^{\partial\Delta[n]}} X^{\partial\Delta[n]}. \end{array}$$

But it follows from lemma 2.14 that the square (a+b) has a diagonal filler iff the following square (c) has a diagonal filler,

$$\begin{array}{ccc} \big(\Lambda^k[m]\times\Delta[n]\big)\cup\big(\Delta[m]\times\partial\Delta[n]\big) & \xrightarrow{\quad \overline{x} \quad} & X \\ \Big\downarrow & \text{(c)} & \Big\downarrow f \\ \Delta[m]\times\Delta[n] & \xrightarrow{\quad \overline{y} \quad} & Y, \end{array}$$

where $\overline{x}$ is defined by x' and z', and where $\overline{y}$ is defined by y'. The square (c) has a diagonal filler if $0 < k < m$ by H.0.20, since f is a mid fibration. We can thus suppose $k = m$ (the case $k = 0$ is similar). The image by $\overline{x}$ of the arrow $(m-1,n) \to (m,n)$ is quasi-invertible in X by 5.2, since x' factors through the inclusion $J(\Delta[n],X) \subseteq X^{\Delta[n]}$ by definition of x. It then follows from lemma 5.8 that the square (c) has a diagonal filler. We have proved that $J'(\delta_n, f)$ is a Kan fibration. Let us now show that $\langle (v), f\rangle$ is a trivial fibration when v is anodyne. For this, it suffices to show that we have $\delta_n \pitchfork \langle (v), f\rangle$ for every $n \geq 0$. But the condition $\delta_n \pitchfork \langle (v), f\rangle$ is equivalent to the condition $v \pitchfork J'(\delta_n, f)$ by 5.4. And we have $v \pitchfork J'(\delta_n, f)$, since $J'(\delta_n, f)$ is a Kan fibration by what we just proved, and since v is anodyne. This completes the proof that $\langle (v), f\rangle$ is a trivial fibration when v is anodyne. We can now prove that $J'(u,f)$ is a Kan fibration for any monomorphism u. For this, it suffices to show that we have $v \pitchfork J'(u,f)$ for every anodyne map v. But the condition $v \pitchfork J'(u,f)$ is equivalent to the condition $u \pitchfork \langle (v), f\rangle$ by 5.4. The result follows, since the map $\langle (v), f\rangle$ is a trivial fibration when v is anodyne. ■

Corollary 5.11. *If X is a quasi-category, then we have $J(A,X) = J(X^A)$ for any simplicial set A. The contravariant functors $A \mapsto J(X^A)$ and $S \mapsto X^{(S)}$ are mutually right adjoint. If $f : X \to Y$ is a map between quasi-categories, then we have*

$$J'(u,f) = J\langle u, f\rangle$$

for any monomorphism of simplicial sets $u : A \to B$.

Proof: Let us prove the first statement. We have $J(X^A) \subseteq J(A,X)$ by definition of $J(A,X)$. The map $J(A,X) \to 1$ is a Kan fibration by 5.10 applied to the map $X \to 1$ and to the inclusion $\emptyset \subseteq A$. This shows that $J(A,X)$ is a Kan complex. Thus, $J(A,X) = J(X^A)$, since $J(X^A)$ is the largest sub Kan complex of X^A. The first statement is proved. The second statement follows from 5.2, since we have $J(A,X) = J(X^A)$. Let us prove the third statement. The following square

$$\begin{array}{ccc} Y^B \times_{Y^A} X^A & \longrightarrow & X^A \\ \Big\downarrow & & \Big\downarrow \\ Y^B & \longrightarrow & Y^A, \end{array}$$

is a cartesian square of quasi-categories by 2.21. The functor $J : \mathbf{QCat} \to \mathbf{Kan}$ preserves pullbacks, since it is a right adjoint. Hence we have

$$\begin{aligned} J(Y^B \times_{Y^A} X^A) &= J(Y^B) \times_{J(Y^A)} J(X^A) \\ &= J(B, Y) \times_{J(A,Y)} J(A, X). \end{aligned}$$

It follows that we have $J\langle u, f\rangle = J'(u, f)$. ∎

Corollary 5.12. *If X is a quasi-category and A is a simplicial set, then a homotopy $\alpha : A \times I \to X$ is quasi-invertible in X^A iff the corresponding map $A \to X^I$ can be factored through the inclusion $X^{(I)} \subseteq X^I$.*

Proof: The map $\lambda^I \alpha : A \to X^I$ can be factored through the inclusion $X^{(I)} \subseteq X^I$ iff the map $\lambda^A \alpha : I \to X^A$ can be factored through the inclusion $J(A, X) \subseteq X^A$ by 5.2. But we have $J(A, X) = J(X^A)$ by 5.11. But the map $\lambda^A \alpha : I \to X^A$ can be factored through the inclusion $J(X^A) \subseteq X^A$ iff the homotopy α is quasi-invertible in X^A. ∎

Theorem 5.13. *If $f : X \to Y$ is a pseudo-fibration between quasi-categories, then so is the map*

$$\langle u, f\rangle : X^B \to Y^B \times_{Y^A} X^A$$

for any monomorphism of simplicial sets $u : A \to B$.

Proof: The map $\langle u, f\rangle$ is a mid fibration by 2.18. It is a map between quasi-categories by 2.19 and 2.21. Hence it suffices to show that the map $J\langle u, f\rangle$ is a Kan fibration by 4.27. We have $J\langle u, f\rangle = J'(u, f)$ by 5.11. But $J'(u, f)$ is a Kan fibration by 5.10. This proves that $\langle u, f\rangle$ is a pseudo-fibration. ∎

If X is a quasi-category and A is a simplicial set, consider the projection $X^A \to X^{A_0}$ defined from the inclusion $A_0 \subseteq A$.

Theorem 5.14. *If X is a quasi-category and A is a simplicial set, then the projection $X^A \to X^{A_0}$ is conservative.*

Proof: The following square is cartesian by definition of $J(A, X)$,

$$\begin{array}{ccc} J(A, X) & \longrightarrow & X^A \\ \downarrow & & \downarrow{\scriptstyle p} \\ J(X^{A_0}) & \longrightarrow & X^{A_0}. \end{array}$$

Hence also the square

$$\begin{array}{ccc} J(X^A) & \longrightarrow & X^A \\ {\scriptstyle J(p)}\downarrow & & \downarrow{\scriptstyle p} \\ J(X^{A_0}) & \longrightarrow & X^{A_0}, \end{array}$$

since $J(A, X) = J(X^A)$ by 5.11. This proves that p is conservative by 4.29. ■

Theorem 5.15. *A pseudo-fibrations between quasi-categories is a trivial fibration iff it is a categorical equivalence.*

Proof: The implication ($\Rightarrow$) was proved in 4.4. ($\Leftarrow$) Let $f : X \to Y$ be a pseudo-fibration between quasi-categories. If f is a categorical equivalence, let us show that it is a trivial fibration. For this, we shall prove that f has the right lifting property with respect to every monomorphism of simplicial sets $u : A \to B$. For this it suffices to show that the map $\langle u, f\rangle$ is surjective on 0-cells. Equivalently, it suffices to show that the map $J\langle u, f\rangle$ is surjective on 0-cells. We shall prove that $J\langle u, f\rangle$ is a trivial fibration. The map $J\langle u, f\rangle$ is a Kan fibration by Theorem 5.10, since we have $J\langle u, f\rangle = J'(u, f)$ by 5.11. Thus, it suffices to show that $J\langle u, f\rangle$ is a weak homotopy equivalence. Consider the square

$$\begin{array}{ccc} X^B & \longrightarrow & Y^A \\ \downarrow & & \downarrow \\ Y^B & \longrightarrow & X^A. \end{array}$$

The vertical maps of the square are categorical equivalences, since f is a categorical equivalence. Moreover, the horizontal maps are pseudo-fibrations by 5.13. Its image by the functor $J : \mathbf{QCat} \to \mathbf{Kan}$ is a square

$$\begin{array}{ccc} J(X^B) & \longrightarrow & J(X^A) \\ \downarrow & & \downarrow \\ J(Y^B) & \longrightarrow & J(Y^A). \end{array}$$

The vertical maps of this square are homotopy equivalences by 4.26. The horizontal maps are Kan fibrations by 4.27. Hence the map

$$J(X^B) \to J(Y^B) \times_{J(Y^A)} J(X^A)$$

is a weak homotopy equivalence. We have proved that $J\langle u, f\rangle$ is a trivial fibration. This completes the proof that f is a trivial fibration. ■

Let X be a quasi-category. From the inclusions $i_0 : \{0\} \subset I$ and $i_1 : \{1\} \subset I$ we obtain two projections

$$p_0 : X^{(I)} \to X \quad \text{and} \quad p_1 : X^{(I)} \to X.$$

From the map $I \to 1$, we obtain a a diagonal map $\delta_X : X \to X^{(I)}$. We have $p_0\delta_X = 1_X = p_1\delta_X$.

Proposition 5.16. (Path space 1) *If X is a quasi-category, then the map*

$$(p_0, p_1) : X^{(I)} \to X \times X$$

is a pseudo-fibration and each projection $p_0, p_1 : X^{(I)} \to X$ is a trivial fibration. Moreover, the diagonal $\delta_X : X \to X^{(I)}$ is an equivalence of quasi-categories.

Proof: We have $(p_0, p_1) = X^{(i)}$, where i denotes the inclusion $\{0,1\} \subset I$. Thus, (p_0, p_1) is a pseudo-fibration by 5.7 applied to the map $X \to 1$ and to the inclusion $\{0,1\} \subset I$. Moreover, the projection $p_0 = X^{(i_0)}$ is trivial fibration 5.10 applied to the map $X \to 1$, since i_0 is anodyne. Similarly for p_1. It follows that p_0 is an equivalence of quasi-categories by 1.22, since it is a trivial fibration. It then follows that δ_X i is an equivalence of quasi-categories by the "three-for-two" property of equivalences, since we have $p_0\delta_X = 1_X$. ■

If $f : X \to Y$ is a map between quasi-categories we define the *mapping path space* $P(f)$ by the pullback square

$$\begin{array}{ccc} P(f) & \xrightarrow{pr_2} & X \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle f} \\ Y^{(I)} & \xrightarrow{p_0} & Y. \end{array}$$

There is a unique map $i_X : X \to P(f)$ such that $pr_1 i_X = \delta_Y f$ and $pr_2 i_X = 1_X$. Let us put $p_X = pr_2$ and $p_Y = p_1 pr_1 : P(f) \to Y^{(I)} \to Y$.

Proposition 5.17. (Mapping path space factorisation 1). *Let $f : X \to Y$ be a map between quasi-categories. Then the simplicial set $P(f)$ is a quasi-category and we have a factorisation*

$$f = p_Y i_X : X \to P(f) \to Y,$$

where i_X an equivalence of quasi-categories and p_Y a pseudo-fibration. Moreover, we have $p_X i_X = 1_X$ where $p_X : P(f) \to X$ a trivial fibration.

Proof: We have $p_Y i_X = p_1 pr_1 i_X = p_1 \delta_Y f = 1_Y f = f$. Let us show that p_Y is a pseudo-fibration. We shall first prove that the joint map $(p_X, p_Y) : P(f) \to X \times Y$ is a pseudo-fibration. Consider the commutative diagram

$$\begin{array}{ccccc} P(f) & \xrightarrow{(p_X,p_Y)} & X \times Y & \xrightarrow{pr_1} & X \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle f\times Y} & & \downarrow{\scriptstyle f} \\ Y^{(I)} & \xrightarrow{(p_0,p_1)} & Y \times Y & \xrightarrow{pr_1} & Y. \end{array}$$

The boundary square is cartesian by definition of $P(f)$. The square on the right is obviously cartesian. It follows that the square on the left is cartesian. Hence the map (p_X, p_Y) is a base change of the map (p_0, p_1). But (p_0, p_1) is a pseudo-fibration by 5.16. The class of pseudo-fibrations in **QCat** is closed under base change by 4.33. This proves that the map (p_X, p_Y) is a pseudo-fibration. The projection $pr_2 : X \times Y \to Y$ is a pseudo-fibration by base change, since the map $X \to 1$ is a pseudo-fibration. Thus, $p_Y = pr_2(p_X, p_Y)$ is a mid fibration. We have $p_X i_X = 1_X$, since $p_X = pr_2$ and $pr_2 i_X = 1_X$. Let us show that i_X is an equivalence of quasi-categories. For this, it suffices to show that p_X is an equivalence of quasi-categories, since we have $p_X i_X = 1_X$. But for this, it suffices to show that p_X is a trivial fibration by 1.22. But p_X is a base change of the projection $p_0 : X^{(I)} \to X$. It is thus a trivial fibration, since p_0 is a trivial fibration by 5.16. ∎

Let $\alpha : B \times I \to X$ be a homotopy between two maps $f, g : B \to X$. If $u : A \to B$ we shall denote the homotopy $\alpha(u \times I) : A \times I \to X$ by $\alpha \circ u : fu \to gu$. If $p : X \to Y$, we shall denote the homotopy $p\alpha : B \times I \to Y$ by $p \circ \alpha : pf \to pg$.

Proposition 5.18. (Covering homotopy extension property 1) *Suppose that we have a commutative square*

$$\begin{array}{ccc} A & \xrightarrow{a} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{b} & Y, \end{array}$$

in which p is a pseudo-fibration between quasi-categories and i is monic. Suppose also that we have a map $c : B \to X$ together with two quasi-invertible homotopies $\alpha : ci \to a$ and $\beta : pc \to b$ such that $p \circ \alpha = \beta \circ i$. Then the square has a diagonal filler $d : B \to X$ and there exists a quasi-invertible homotopy $\sigma : c \to d$ such that $\sigma \circ i = \alpha$ and $p \circ \sigma = \beta$.

Proof: Let i_0 be the inclusion $\{0\} \subset I$ and i_1 be the inclusion $\{1\} \subset I$. We have $\alpha(A \times i_0) = ci$ and $\alpha(A \times i_1) = a$, since $\alpha : ci \to a$. Similarly, we have $\beta(B \times i_0) = pc$ and $\beta(B \times i_1) = b$, since $\beta : pc \to b$. The map $\lambda^I \alpha : A \to X^I$ factors through the inclusion $X^{(I)} \subseteq X^I$ by 5.12, since α is quasi-invertible. It thus defines a map $\alpha' : A \to X^{(I)}$. Similarly, the map $\lambda^I \beta : B \to Y^I$ factors through the inclusion

$Y^{(I)} \subseteq Y^I$ and it defines a map $\beta' : B \to Y^{(I)}$. The condition $p \circ \alpha = \beta \circ i$ implies that the following square commutes,

$$\begin{array}{ccc} A & \xrightarrow{\alpha'} & X^{(I)} \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle q} \\ B & \xrightarrow{(\beta',c)} & Y^{(I)} \times_Y X, \end{array}$$

where $q = (p^{(I)}, p_0)$. By definition, we have $q = \langle (i_0), p \rangle$. Thus, q is a trivial fibration by 5.10, since i_0 is anodyne. Hence the square has a diagonal filler $\sigma' : B \to X^{(I)}$, since i is monic. We have $\sigma' i = \alpha'$, $p^{(I)}\sigma' = \beta'$ and $p_0\sigma' = c$, since σ' is a diagonal filler of the square. Consider the homotopy $\sigma : B \times I \to X$ defined by putting $\sigma(x,t) = \sigma'(x)(t)$. The homotopy σ is is quasi-invertible by 5.12. We have $\sigma \circ i = \alpha$, since we have $\sigma' i = \alpha'$. We have $p \circ \sigma = \beta$, since we have $p^{(I)}\sigma' = \beta'$. We have $\sigma(B \times i_0) = c$, since we have $p_0\sigma' = c$. If $d = \sigma(B \times i_1) : B \to X$, then $\sigma : c \to d$. The relation $\sigma \circ i = \alpha$ implies that $di = a$. The relation $p \circ \sigma = \beta$ implies that $pd = b$. This shows that d is a diagonal filler of the square. ∎

Let J be the groupoid generated by one isomorphism $0 \to 1$. For any quasi-category X we have $X^{(J)} = X^J$ by 5.3, since J is a groupoid. Hence the projection $X^{(J)} \to X^{(I)}$ defined from the inclusion $I \subset J$ is a map $X^J \to X^{(I)}$.

Proposition 5.19. *The canonical map $X^J \to X^{(I)}$ is a trivial fibration for any quasi-category X. A homotopy $\alpha : A \times I \to X$ is quasi-invertible in X^A iff it can be extended to $A \times J$.*

Proof: The inclusion $I \subset J$ is anodyne, since it is a weak homotopy equivalence. Hence the map $X^{(i)} : X^{(J)} \to X^{(I)}$ is a trivial fibration by 5.10. This proves the first statement, since $X^{(J)} = X^J$. The second statement follows (it also follows from 4.25). ∎

Let X be a simplicial set. From the inclusions $j_0 : \{0\} \subset J$ and $j_1 : \{1\} \subset J$ we obtain two projections

$$q_0 : X^J \to X \quad \text{and} \quad q_1 : X^J \to X.$$

From the map $J \to 1$, we obtain a a diagonal map $\Delta_X : X \to X^J$. We have $q_0 \Delta_X = 1_X = q_1 \Delta_X$.

Proposition 5.20. (Path space 2) *If X is a quasi-category, then the map*

$$(q_0, q_1) : X^J \to X \times X$$

is a pseudo-fibration and each projection $q_0, q_1 : X^J \to X$ is a trivial fibration. Moreover, the diagonal $\Delta_X : X \to X^J$ is an equivalence of quasi-categories.

Proof: We have $(q_0, q_1) = X^j$, where j denotes the inclusion $\{0, 1\} \subset J$. Thus, (q_0, q_1) is a pseudo-fibration by 5.13. We have $q_0 = X^{j_0}$, where j_0 denotes the inclusion $\{0\} \subset J$. But we have $X^{j_0} = X^{(j_0)}$ by 5.3, since j_0 is a map between groupoids. The inclusion $\{0\} \subset J$ is anodyne by a classical result [GZ], since it is a weak homotopy equivalence. Hence the map $X^{(j_0)}$ is a trivial fibration by 5.10. The rest of the proof is similar to the proof of 5.16. ∎

If $f : X \to Y$ is a map of simplicial sets, we can define a mapping path space $Q(f)$ by the pullback square

$$\begin{array}{ccc} Q(f) & \xrightarrow{pr_2} & X \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle f} \\ Y^J & \xrightarrow{q_0} & Y. \end{array}$$

There is a unique map $j_X : X \to Q(f)$ such that $pr_1 j_X = \Delta_Y f$ and $pr_2 j_X = 1_X$. Let us put $q_X = pr_2$ and $q_Y = q_1 pr_1 : Q(f) \to Y^J \to Y$.

Proposition 5.21. (Mapping path space factorisation 2). *Let $f : X \to Y$ be a map between quasi-categories. Then the simplicial set $Q(f)$ is a quasi-category and we have a factorisation*

$$f = q_Y j_X : X \to Q(f) \to Y,$$

where j_X an equivalence of quasi-categories and q_Y a pseudo-fibration. Moreover, we have $q_X j_X = 1_X$ and $q_X : Q(f) \to X$ is a trivial fibration.

Proof: Similar to the proof of 5.17. ∎

Let i_0 be the inclusion $\{0\} \subset I$ and j_0 be the inclusion $\{0\} \subset J$.

Theorem 5.22. *Let $f : X \to Y$ be a map between quasi-categories. Then the folllowing conditions are equivalent:*

- (i) *f is a pseudo-fibration;*
- (ii) *f has the RLP with respect to every monic weak categorical equivalence;*
- (iii) *the map $\langle (i_0), f \rangle : X^{(I)} \to Y^{(I)} \times_Y X$ is a trivial fibration;*
- (iv) *the map $\langle j_0, f \rangle : X^J \to Y^J \times_Y X$ is a trivial fibration;*
- (v) *the map $\langle u, f \rangle : X^B \to Y^B \times_{Y^A} X^A$ is a trivial fibration for any monic weak categorical equivalence $u : A \to B$.*

Proof: Let us prove the implication (i)⇒(iii). If f is a pseudo-fibration, then the map $\langle (i_0), f\rangle : X^{(I)} \to Y^{(I)} \times_Y X$ is a trivial fibration by 5.10, since the inclusion i_0 is anodyne. Let us prove the converse (iii)⇒(i). The following diagram commutes

$$\begin{array}{ccccc} X^{(I)} & \xrightarrow{\langle (i_0),f\rangle} & Y^{(I)} \times_Y X & \xrightarrow{pr_1} & Y^{(I)} \\ {\scriptstyle p_1}\downarrow & & & & \downarrow{\scriptstyle p_1} \\ X & & \xrightarrow{\quad f \quad} & & Y \end{array}$$

The composite $p_Y = p_1 pr_1$ is a pseudo-fibration by 5.17. The map $\langle (i_0), f\rangle$ is a pseudo-fibration by 4.4, since it is a trivial fibration by hypothesis. Hence the composite $p_Y \langle (i_0), f\rangle = f p_1$ is a pseudo-fibration. The following diagram commutes, since $(fp_1)\delta_X = f(p_1\delta_X) = f1_X = f$,

$$\begin{array}{ccccc} X & \xrightarrow{\delta_X} & X^{(I)} & \xrightarrow{p_1} & X \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle fp_1} & \swarrow{\scriptstyle f} & \\ & & Y. & & \end{array}$$

It shows that f is a retract of fp_1. Thus, f is a pseudo-fibration by 4.33, since fp_1 is a pseudo-fibration. The equivalence (i)⇔(iii) is proved. The equivalence (i)⇔(iv) is proved similarly. Let us prove the implication (i)⇒(v). If $f : X \to Y$ is a pseudo-fibration then the map $\langle u, f\rangle$ is a pseudo-fibration between quasi-categories by 5.13. Let us show that it is an equivalence. We need the commutative diagram

$$\begin{array}{ccccc} X^B & \xrightarrow{\langle u,f\rangle} & Y^B \times_{Y^A} X^A & \xrightarrow{pr_2} & X^A \\ & \searrow & \downarrow{\scriptstyle pr_1} & & \downarrow{\scriptstyle f^A} \\ & & Y^B & \xrightarrow{Y^u} & Y^A, \end{array}$$

where $pr_2\langle u, f\rangle = X^u$. The map X^u is an equivalence of quasi-categories by 2.27, since u is a weak categorical equivalence. Similarly for the map Y^u. But Y^u is a pseudo-fibration by 5.13. It is thus a trivial fibration by 5.15. Thus, pr_2 is a trivial fibration by base change. It is thus a categorical equivalence by 1.22. Therefore, $\langle u, f\rangle$ is an equivalence of quasi-categories by three-for-two. This proves that $\langle u, f\rangle$ is a trivial fibration by 5.15. Let us prove the implication (v)⇒ (ii). If $u : A \to B$ is a monic weak categorical equivalence, then the map $\langle u, f\rangle$ is surjective on 0-cells, since it is a trivial fibration by (v). This proves that $u \pitchfork f$. Let us prove the implication (ii)⇒ (i). Suppose that $f : X \to Y$ has the RLP with respect every monic weak categorical equivalence. Then f is a mid fibration, since every mid anodyne map is a weak categorical equivalence by 2.29. The inclusion $j_0 : \{0\} \subset J$ is a weak categorical equivalence, since it is an equivalence of categories. Hence we have $j_0 \pitchfork f$ by the hypothesis on f. Therefore, f is a pseudo-fibration by 4.32.

∎

Chapter 6

The model structure for quasi-categories

6.1 Introduction

In this chapter we show that the category of simplicial sets admits a Quillen model structure in which the fibrant objects are the quasi-categories. It is *the model structure for quasi-categories*. The cofibrations are the monomorphisms, the weak equivalences are the weak categorical equivalences and the fibrations are the pseudo-fibrations. The classical model structure on the category of simplicial sets is both a homotopy reflection and coreflection of the model structure for quasi-categories. We compare the model structure for quasi-categories with the natural model structure on **Cat**.

See E.1.2 for the notion of model structure. Recall that a map of simplicial sets $u : A \to B$ is a *weak homotopy equivalence* if and only if the map

$$\pi_0(u, X) : \pi_0(B, X) \to \pi_0(A, X)$$

is bijective for every Kan complex X. The following theorem describes the *classical* model structure on **S** also called the *Kan* model structure.

Theorem 6.1. [Q] *The category of simplicial sets* **S** *admits a model structure in which a cofibration is a monomorphism, a weak equivalence is a weak homotopy equivalence and a fibration is a Kan fibration. The fibrant objects are Kan complexes. The acyclic fibrations are the trivial fibrations. The model structure is cartesian and proper.*

See [JT2] for a purely combinatorial proof. The Kan model structure is a Cisinski model structure. It is thus determined by its class of fibrant objects. We shall denote it shortly by $(\mathbf{S}, \mathbf{Kan})$ or by $(\mathbf{S}, Who)$, where *Who* denotes the class of weak homotopy equivalences.

Recall from Definition 4.1 that a functor $p : E \to B$ (in **Cat**) is said to be a *pseudo-fibration* if for every object $a \in E$ and every isomorphism $g \in B$ with target $p(a)$, there exists an isomorphism $f \in E$ with target a such that $p(f) = g$. Recall also that a functor $u : A \to B$ is said to be *monic on objects* if the induced map $Ob(A) \to Ob(B)$ is monic. The following theorem describes the *natural model structure on* **Cat**.

Theorem 6.2. [JT1][Rez] *The category* **Cat** *admits a model structure in which a cofibration is a functor monic on objects, a weak equivalence is an equivalence of categories and a fibration is a pseudo-fibration. The model structure is cartesian and proper. Every object is fibrant and cofibrant. A functor is an acyclic fibration iff it is an equivalence surjective on objects.*

We shall denote this model category/structure shortly by $(\mathbf{Cat}, Eq)$, where Eq is the class of equivalences between small categories.

Recall that a map of simplicial sets $u : A \to B$ is said to be a *weak categorical equivalence* if the map

$$\tau_0(u, X) : \tau_0(B, X) \to \tau_0(A, X)$$

is bijective for every quasi-category X (Definition 1.20). Let *Wcat* be the class of weak categorical equivalences and $\mathcal{C}$ be the class of monomorphisms.

Definition 6.3. *We call a map of simplicial sets* $f : X \to Y$ *a (general)* pseudo-fibration *if it has the right lifting property with respect to the maps in* $\mathcal{C} \cap Wcat$.

It follows from Theorem 5.22 that this notion extends the notion of pseudo-fibration between quasi-categories introduced in 4.2. The main theorem of the chapter is the following:

The following theorem describes the *model structure for quasi-categories.*

Theorem *The category of simplicial sets* **S** *admits a Cisinki model structure in which a fibrant object is a quasi-category. A weak equivalence is a weak categorical equivalence and a fibration is a pseudo-fibration. The model structure is cartesian.*

The theorem will be proved in 6.12.

6.2 General pseudo-fibrations

The main theorem of the section is the following:

Theorem *Let Wcat be the class of weak categorical equivalences, $\mathcal{C}$ be the class of monomorphisms and $\mathcal{F}$ is the class of (general) pseudo-fibrations. Then the pair $(\mathcal{C} \cap Wcat, \mathcal{F})$ is a weak factorisation system.*

The theorem will proved in 6.11. We first extend Theorem 5.15.

Theorem 6.4. *A (general) pseudo-fibration is a weak categorical equivalence iff it is a trivial fibration.*

Proof: A trivial fibration is a (weak) categorical equivalence by 1.22. Conversely, let $f : X \to Y$ be a (general) pseudo-fibration. If f is a weak categorical equivalence, let us show that it is a trivial fibration. By D.1.12, there exists a factorisation $f = qi : X \to P \to Y$ with $i \in \mathcal{C}$ and q a trivial fibration. The map i is a weak categorical equivalence by three-for-two, since q is a weak categorical equivalence by 1.22. It follows that the square

$$\begin{array}{ccc} X & \xrightarrow{1_X} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ P & \xrightarrow{q} & Y \end{array}$$

has a diagonal filler $r : P \to X$. The relations $ri = 1_X$, $fr = q$ and $qi = f$ show that the map f is a retract of the map q. Therefore f is a trivial fibration, since q is a trivial fibration. ∎

Let $\mathcal{F}_Q$ be the class of pseudo-fibrations between quasi-categories.

Lemma 6.5. *We have $\mathcal{C} \cap Wcat = {}^{\pitchfork}\mathcal{F}_Q$. Hence the class $\mathcal{C} \cap Wcat$ is saturated.*

Proof: We have $\mathcal{C} \cap Wcat \subseteq {}^{\pitchfork}\mathcal{F}_Q$, since we have $(\mathcal{C} \cap Wcat) \pitchfork \mathcal{F}_Q$ by 5.22. Let us show that we have ${}^{\pitchfork}\mathcal{F}_Q \subseteq \mathcal{C} \cap Wcat$. Let $u : A \to B$ be a map in ${}^{\pitchfork}\mathcal{F}_Q$. Let us first verify that $u \in \mathcal{C}$. For this, let us choose a mid anodyne map $i : A \to X$ with values in a quasi-categorie X (this can be done by factoring the map $A \to 1$ as a mid anodyne map $i : A \to X$ followed by a mid fibration). The square

$$\begin{array}{ccc} A & \xrightarrow{i} & X \\ {\scriptstyle u}\downarrow & & \downarrow \\ B & \longrightarrow & 1 \end{array}$$

has a diagonal filler $d : B \to X$, since the map $X \to 1$ belongs to $\mathcal{F}_Q$. Thus, u is monic, since $du = i$ is monic. Let us now show that $u \in Wcat$. For this, let us first show that the map $\langle u, f \rangle$ is a trivial fibration for any map $f \in \mathcal{F}_Q$. For this, it suffices to show that we have $v \pitchfork \langle u, f \rangle$ for every monomorphism $v : S \to T$. But the condition $v \pitchfork \langle u, f \rangle$ is equivalent to the condition $u \pitchfork \langle v, f \rangle$ by 2.14. We have $\langle v, f \rangle \in \mathcal{F}_Q$ by 5.13. Hence we have $u \pitchfork \langle v, f \rangle$, since we have $u \in {}^{\pitchfork}\mathcal{F}_Q$ by hypothesis. Thus, $v \pitchfork \langle u, f \rangle$. This proves that $\langle u, f \rangle$ is a trivial fibration. We can now show that u is a weak categorical equivalence. If X is a quasi-category, then the map $X^u : X^B \to X^A$ is a trivial fibration by what we have proved applied to the map $X \to 1$. Thus, X^u is a categorical equivalence by 1.22. Hence the functor $\tau_1(X^u)$ is an equivalence of categories by 1.27. It follows that the map $\tau_0(u, X) = \tau_0(X^u)$ is bijective. Therefore, $u \in Wcat$. ∎

We can now extend theorem 5.13.

Theorem 6.6. *If $f : X \to Y$ is a (general) pseudo-fibration then so is the map*

$$\langle u, f \rangle : X^B \to Y^B \times_{Y^A} X^A$$

for any monomorphism $u : A \to B$. Moreover, $\langle u, f \rangle$ is a trivial fibration if in addition u is a weak categorical equivalence.

Proof: Let us show that $\langle u, f \rangle$ is a pseudo-fibration. By definition, we have to show that we have $v \pitchfork \langle u, f \rangle$ for every map $v \in \mathcal{C} \cap Wcat$. But the condition $v \pitchfork \langle u, f \rangle$ is equivalent to the condition $(u \times' v) \pitchfork f$ by 2.14. Hence it suffices to show that we have $u \times' v \in \mathcal{C} \cap Wcat$. For this, it suffices to show that we have $(u \times' v) \pitchfork g$ for every $g \in \mathcal{F}_Q$ by 6.5. But the condition $(u \times' v) \pitchfork g$ is equivalent to the condition $v \pitchfork \langle u, g \rangle$ by 2.14. But the map $\langle u, g \rangle$ is a pseudo-fibration between quasi-categories by 5.13, since g is a pseudo-fibration between quasi-categories. Therefore, we have $v \pitchfork \langle u, g \rangle$ by 5.22. This completes the proof that $\langle u, f \rangle$ is a pseudo-fibration. The first statement is proved. The second statement follows from the first and the equivalence $v \pitchfork \langle u, f \rangle \Leftrightarrow u \pitchfork \langle v, f \rangle$. ∎

Every map $f : X \to Y$ in **QCat** admits a mapping path space factorisation

$$f = p_Y i_X : X \to P(f) \to Y$$

by 5.17. The factorisation depends functorially on the map f. Let us put $MP(f) = p_Y : P(f) \to Y$. This defines a functor

$$MP : \mathbf{QCat}^I \to \mathbf{QCat}^I.$$

Notice that a directed colimit of quasi-categories is a quasi-category.

Lemma 6.7. *The functor $MP : \mathbf{QCat}^I \to \mathbf{QCat}^I$ preserves directed colimits.*

Proof: The simplicial set $P(f)$ is defined by the pullback square

$$\begin{array}{ccc} P(f) & \xrightarrow{pr_2} & X \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle f} \\ Y^{(I)} & \xrightarrow{p_0} & Y. \end{array}$$

Hence it suffices to show that that the functor $X \mapsto X^{(I)}$ preserves directed colimits. But we have a pull-back square

$$\begin{array}{ccc} X^{(I)} & \longrightarrow & X^I \\ \downarrow & & \downarrow \\ Cosk^0(X^{(I)}) & \longrightarrow & Cosk^0(X^I), \end{array}$$

since the inclusion $X^{(I)} \subseteq X^I$ is 0-full. It thus suffices to show that each functor $X \mapsto X^I$, $X \mapsto Cosk^0(X^I)$ and $X \mapsto Cosk^0(X^{(I)})$ preserves directed colimits. We have $Cosk^0(X^I) = Cosk^0(X_1)$ and $Cosk^0(X^I) = Cosk^0(X'_1)$, where $X'_1 \subseteq X_1$ denotes the set of quasi-invertible arrows in X. Hence it suffices to show that each functor $X \mapsto Cosk^0(X_1)$ and $X \mapsto Cosk^0(X'_1)$ preserves directed colimits. This is clear for the first, since the functor $Cosk^0 : \mathbf{S} \to \mathbf{S}$ preserve directed colimits. Hence it remains to show that the functor $X \mapsto X'_1$ preserves directed colimits. But we have a pull-back square

$$\begin{array}{ccc} X'_1 & \longrightarrow & Iso(\tau_1(X)) \\ \downarrow & & \downarrow \\ X_1 & \longrightarrow & Ar(\tau_1(X)), \end{array}$$

where $Ar(C)$ (resp. $Iso(C)$) denotes the set of arrows of a category C. The functor $\tau_1 : \mathbf{S} \to \mathbf{Cat}$ is cocontinuous. Hence it suffices to show that the functors Ar and $Iso : \mathbf{Cat} \to \mathbf{Set}$ preserve directed colimits. We have $Ar(C) = \mathbf{Cat}(I, C)$ and $Ar(C) = \mathbf{Cat}(J, C)$, where $I = [1]$ and J is the groupoid generated by one isomorphism $0 \to 1$. But the functors $\mathbf{Cat}(I, -)$ and $\mathbf{Cat}(J, -)$ preserves directed colimits, since the categories I and J are finitely presentable. ∎

A functor $R : \mathcal{E}^I \to \mathcal{E}^I$ together with a natural transformation $\rho : Id \to R$ associates to a map $u : A \to B$ a commutative square of simplicial sets

$$\begin{array}{ccc} A & \xrightarrow{\rho_1(u)} & R_1(u) \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle R(u)} \\ B & \xrightarrow{\rho_0(u)} & R_0(u). \end{array}$$

Lemma 6.8. *There exists a functor $R : \mathbf{S}^I \to \mathbf{S}^I$ together with a natural transformation $\rho : Id \to R$ such that:*

- *R preserves directed colimits;*
- *the map $R(u)$ is a pseudo-fibration between quasi-categories;*
- *the maps $\rho_0(u)$ and $\rho_1(u)$ are weak categorical equivalences.*

Proof: Let Σ be the set of inner horns $h_n^k : \Lambda^k[n] \subset \Delta[n]$. It follows from D.2.10 that there exists a functor $F : \mathbf{S} \to \mathbf{S}$ together with a natural transformation $\phi : Id \to F$ having the following properties:

- the map $\phi_X : X \to F(X)$ is mid anodyne for every X;
- the simplicial set $F(X)$ is a quasi-categories for every X.

Moreover, the functor F preserves directed colimits, since every $\Lambda^k[n]$ is a finitely presented simplicial set. If $u : A \to B$ is a map of simplicial sets, consider the diagram

$$\begin{array}{ccccc} A & \xrightarrow{\phi_A} & F(A) & \xrightarrow{i_{F(A)}} & P(F(u)), \\ {\scriptstyle u}\downarrow & & {\scriptstyle F(u)}\downarrow & \swarrow{\scriptstyle p_{F(B)}} & \\ B & \xrightarrow{\phi_B} & F(B) & & , \end{array}$$

where $F(u) = p_{F(B)} i_{F(A)}$ is the mapping path factorisation of map $F(u)$. Let us put $R(u) = p_{F(B)}$, $\rho_0(u) = i_{F(A)}\phi_A$ and $\rho_1(u) = \phi_B$. This defines a functor $R : \mathbf{S}^I \to \mathbf{S}^I$ together with a natural transformation $\rho : Id \to R$. The map ϕ_B is a weak categorical equivalence, since a mid anodyne map is a weak categorical equivalence by 2.29. Similarly, the map ϕ_A is a weak categorical equivalence. Hence the composite $i_{F(A)}\phi_A$ is a weak categorical equivalence, since $i_{F(A)}$ is an equivalence of quasi-categories by 5.17. Hence the maps $\rho_0(u)$ and $\rho_1(u)$ are weak categorical equivalences. The map $R(u)$ is a pseudo-fibration between quasi-categories by 5.17. By definition, $R(u) = MP(F(u))$, where MP is the mapping path functor in 6.7. Hence the functor R preserves directed colimits, since the functor F preserves directed colimits and the functor MP preserves directed colimits by 6.7. ∎

Lemma 6.9. *A map of simplicial set $u : A \to B$ is a weak categorical equivalence iff the map $R(u)$ is a trivial fibration.*

Proof: The horizontal maps in the following square are weak categorical equivalences,

$$\begin{array}{ccc} A & \xrightarrow{\rho_1(u)} & R_1(u) \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle R(u)} \\ B & \xrightarrow{\rho_0(u)} & R_0(u). \end{array}$$

It follows by three-for-two that u is a weak categorical equivalence iff $R(u)$ is a weak categorical equivalence. But $R(u)$ is a weak categorical equivalence iff it is a trivial fibration by 6.4, since it is a pseudo-fibration. ∎

Corollary 6.10. *A directed colimit of weak categorical equivalences $Wcat \subset \mathbf{S}$ is a weak categorical equivalence.*

Proof: The functor $R : \mathbf{S}^I \to \mathbf{S}^I$ preserves directed colimits. A directed colimits of trivial fibrations in the category $\mathbf{S}$ is a trivial fibration by 2.5. The result follows. ∎

Theorem 6.11. *If $\mathcal{F}$ is the class of pseudo-fibrations, then the pair $(\mathcal{C} \cap Wcat, \mathcal{F})$ is a weak factorisation system*

Proof: The class $\mathcal{C} \cap Wcat$ is saturated by 6.5. Let us show that it is generated by a set of maps. We shall use Theorem D.2.16. It suffices to show that the class $\mathcal{C} \cap Wcat$ can be defined by an accessible equation. Let us first show that the class $Wcat$ can be defined by an accessible equation. We shall use the functor R of Lemma 6.8. A map $u : A \to B$ is a weak categorical equivalence iff the map $R(u)$ is a trivial fibration by 6.9. The functor R is accessible, since it preserves directed colimits by 6.8 But the class of trivial fibrations can be defined by an accessible equations by D.2.14. It follows by composing that the class of weak categorical equivalences can be defined by an accessible equation. The class of monomorphisms $\mathcal{C}$ can be defined by an accessible equation by D.1. It follows that the intersection $\mathcal{C} \cap Wcat$ can be defined by an accessible equation by D.2.13. It follows by D.2.16 that the saturated class $\mathcal{C} \cap Wcat \subset \mathbf{S}$ is generated by a set of maps Σ. Then we have $\Sigma^{\pitchfork} = \mathcal{F}$, since we have $(\mathcal{C} \cap Wcat)^{\pitchfork} = \mathcal{F}$ by definition of $\mathcal{F}$. But the pair $(\overline{\Sigma}, \Sigma^{\pitchfork})$ is a weak factorisation system by D.2.11. This shows that the pair $(\mathcal{C} \cap Wcat, \mathcal{F})$ is a weak factorisation system. ∎

6.3 The model structure

We can now establish the *model structure for quasi-categories*:

Theorem 6.12. *The category of simplicial sets $\mathbf{S}$ admits a Cisinki model structure in which a fibrant object is a quasi-category. A weak equivalence is a weak categorical equivalence and a fibration is a pseudo-fibration. The model structure is cartesian.*

Proof: Let us denote by $Wcat$ the class of weak categorical equivalences, by $\mathcal{C}$ the class of monomorphisms, and by $\mathcal{F}$ the class of pseudo-fibrations. The class $Wcat$ has the "three-for-two" property by 1.25. The pair $(\mathcal{C} \cap Wcat, \mathcal{F})$ is a weak factorisation system by 6.11. The intersection $\mathcal{F} \cap Wcat$ is the class of trivial fibrations by 6.4. Hence the pair $(\mathcal{C}, \mathcal{F} \cap Wcat)$ is a weak factorisation system by D.1.12. This proves that the triple $(\mathcal{C}, Wcat, \mathcal{F})$ is a model structure. The model structure is cartesian, since the product of two weak categorical equivalences is a weak categorical equivalence by 2.28. Let us now show that the fibrant objects are the quasi-categories. If X is a quasi-category, then the map $X \to 1$ is a pseudo-fibration by 4.4. The converse is obvious since a pseudo-fibration is a mid fibration. This shows that the fibrant objects are the quasi-categories. ∎

The model structure for quasi-categories is a Cisinski model structure. It is thus determined by its class of fibrant objects. We shall denote the model structure for quasi-categories shortly by $(\mathbf{S}, \mathbf{QCat})$ or by $(\mathbf{S}, Wcat)$.

Remark 6.13. *The model structure* $(\mathbf{S}, \mathbf{QCat})$ *is not right proper. However, it can be shown that the base change of a weak categorical equivalence along a left (resp. right) fibration is a weak categorical equivalence.*

Recall also form Definition E.2.15 that a Quillen pair (F, G) is called a *homotopy reflection* if the right derived functor G^R is fully faithful. Dually, the pair is called a *homotopy coreflection* if the left derived functor F^L is fully faithful.

Proposition 6.14. *The pair of adjoint functors*

$$\tau_1 : \mathbf{S} \leftrightarrow \mathbf{Cat} : N$$

is a homotopy reflection between the model categories $(\mathbf{S}, \mathbf{Kan})$ *and* $(\mathbf{Cat}, Eq)$.

Proof: The functor τ_1 takes a monomorphism of simplicial sets to a functor monic on objects, since we have $Ob\tau_1 A = A_0$ for every simplicial set A. It takes a weak categorical equivalence to an equivalence of categories by Proposition 1.23. This shows that the pair (τ_1, N) is a Quillen adjunction. Let us show that it is a homotopy reflection. By Proposition E.2.17, it suffices to show that the map $\tau_1 LNC \to C$ is an equivalence for every $C \in \mathbf{Cat}$, where $LNC \to NC$ denotes a cofibrant replacement of NC. But this is clear, since NC is cofibrant and $\tau_1 NC = C$. ∎

Recall from Definition E.2.22 that a model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ on a category $\mathcal{E}$. is said to be a *Bousfield localisation* of another model structure $(\mathcal{C}', \mathcal{W}', \mathcal{F}')$ on the same category if $\mathcal{C} = \mathcal{C}'$ and $\mathcal{W}' \subseteq \mathcal{W}$; in which case the first is also a homotopy reflection of the second.

Proposition 6.15. *The classical model structure* $(\mathbf{S}, \mathbf{Kan})$ *is a Bousfield localisation of the model structure* $(\mathbf{S}, \mathbf{QCat})$. *It is thus a homotopy reflection of the model structure for quasi-categories.*

Proof: The cofibrations are the monomorphisms in both model structures. Every weak categorical equivalence is a weak homotopy equivalence by 1.21. ∎

Corollary 6.16. *A weak categorical equivalence is a weak homotopy equivalence and a Kan fibration is a pseudo-fibration. The converse is true for a map between Kan complexes.*

Proof: This follows from Proposition E.2.21 and Proposition E.2.23. ■

Corollary 6.17. *The following conditions on a simplicial set A are equivalent:*

- *(i) $\tau_1(A)$ is a groupoid*
- *(ii) there exists a weak categorical equivalence $A \to A'$ with codomain a Kan complex A';*
- *(iii) every weak homotopy equivalence $A \to A'$ with codomain a Kan complex A' is a weak categorical equivalence.*

Proof: Let $i : A \to A'$ be a weak categorical equivalence with codomain a quasi-category A'. The functor $\tau_1(i) : \tau_1(A) \to \tau_1(A')$ is an equivalence of categories by Proposition 6.14. Hence the category $\tau_1(A)$ is a groupoid iff the category $\tau_1(A')$ is a groupoid. But $\tau_1(A')$ is a groupoid iff A' is a Kan complex by Theorem 4.14, since A' is a quasi-category. The equivalence (i)⇒(ii) is proved. The equivalence (ii)⇒(iii) then follows from Proposition E.2.21. ■

Proposition 6.18. *The groupoid J is a fibrant interval of the model category $(\mathbf{S}, \mathbf{QCat})$. If A is a simplicial set, then the simplicial set $A \times J$ is a cylinder object for A. If X is a quasi-category, then the quasi-categories X^J and $X^{(I)}$ are both path objects for X.*

Proof: The inclusion $\{0, 1\} \subset J$ is a cofibration since it is monic. The map $J \to 1$ is an acyclic fibration in $(\mathbf{S}, \mathbf{QCat})$ since it is an acyclic fibration in $(\mathbf{Cat}, Eq)$ and the nerve functor $N : \mathbf{Cat} \to \mathbf{S}$ is a right Quillen functor by 6.14. The first statement is proved. The second statement follows from the first, since the model structure is cartesian. The second statement follows from 5.16 and 5.20. ■

Definition 6.19. *We shall say that two maps of simplicial sets $f, g : A \to B$ are* quasi-isomorphic *f they define the same morphism in the homotopy category $Ho(\mathbf{S}, \mathbf{QCat})$.*

Proposition 6.20. *If B is a quasi-category, then two maps $f, g : A \to B$ are quasi-isomorphic iff they are isomorphic in the category $\tau_1(A, B) = \tau_1(B^A)$.*

Proof: If B is a quasi-category, then the simplicial set B^J is a path object for B by 6.18, where J is the groupoid generated by one isomorphism $0 \to 1$. Let $q_0, q_1 : B^J \to B$ be the projections. By A, two maps $f, g : A \to B$ are equal in the homotopy category iff there exists a map $h : A \to B^J$ such that $q_0 h = f$ and

$q_1 h = g$, since A is cofibrant and B fibrant. But the homotopy h is the same thing as a map $k : J \to B^A$ such that $k(0) = f$ and $k(1) = g$. But the simplicial set B^A is a quasi-category, since B is a quasi-category. Thus, the existence of k is equivalent to the existence of an isomorphism $f \to g$ in the quasi-category B^A by 4.22. This proves the result, since f and g are isomorphic in the quasi-category B^A iff they are isomorphic in the category $\tau_1(B^A)$ by 1.13. ■

Proposition 6.21. *Let $\alpha : f \to g : A \to B$ be a homotopy between two maps of simplicial sets. If the arrow $\alpha(a) : f(a) \to g(a)$ is invertible in the category $\tau_1 B$ for every vertex $a \in A$, then the maps $f, g : A \to B$ are quasi-isomorphic.*

Proof: Let us choose a weak categorical equivalence $i : B \to X$ with values in a quasi-category X. The arrow $i\alpha(a) : if(a) \to ig(a)$ is invertible in the category $\tau_1 X$ for every vertex $a \in A$, since the arrow $\alpha(a)$ is invertible in the category $\tau_1 B$ by assumption. It follows by 5.14 that the homotopy $i \circ \alpha$ is invertible in X^A by 5.14. It then follows by 6.20 that $if = ig$ in the homotopy category $Ho(\mathbf{S}, \mathbf{QCat})$. Thus, $f = g$ in the homotopy category, since i is invertible in this category. ■

Consider the functor $k : \Delta \to \mathbf{S}$ defined by putting $k[n] = \Delta'[n]$ for every $n \geq 0$, where $\Delta'[n]$ denotes the (nerve of the) groupoid freely generated by the category $[n]$. If $X \in \mathbf{S}$, let us put

$$k^!(X)_n = \mathbf{S}(\Delta'[n], X)$$

for every $n \geq 0$. This defines a functor $k^! : \mathbf{S} \to \mathbf{S}$. From the inclusion $\Delta[n] \subseteq \Delta'[n]$, we obtain a map $k^!(X)_n \to X_n$ for each $n \geq 0$ and hence a map of simplicial sets $\beta_X : k^!(X) \to X$. This defines a natural transformation $\beta : k^! \to Id$. The functor $k^!$ has a left adjoint $k_!$ which is the left Kan extension of the functor $k : \Delta \to \mathbf{S}$ along the Yoneda functor. The natural transformation $\beta : k^! \to Id$ has a left adjoint $\alpha : Id \to k_!$

Theorem 6.22. *The pair of adjoint functors*

$$k_! : \mathbf{S} \leftrightarrow \mathbf{S} : k^!$$

is a Quillen adjunction between the model categories $(\mathbf{S}, \mathbf{Kan})$ and $(\mathbf{S}, \mathbf{QCat})$. Moreover, the map $\alpha_X : X \to k_!(X)$ is a weak homotopy equivalence for every X.

Proof: The functor $k_!$ takes the inclusion $\partial\Delta[1] \to \Delta[1]$ to the inclusion $\partial\Delta[1] \to \Delta'[1]$. It follows by B.0.17 that the functor $k_!$ preserves monomorphisms. If $X = \Delta[n]$, then the map $\alpha_X : X \to k_! X$ coincides with the natural inclusion $\Delta[n] \subseteq \Delta'[n]$. It is thus a weak homotopy equivalence for every $n \geq 0$. It follows from B.0.18 that α_X is a weak homotopy equivalence for every X. Hence the horizontal maps of the following commutative square are acyclic,

$$\begin{array}{ccc} X & \xrightarrow{\alpha_X} & k_!(X) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle k_!(f)} \\ Y & \xrightarrow{\alpha_Y} & k_!(Y). \end{array}$$

It follows by three-for-two that the functor $k_!$ takes an acyclic map $f : X \to Y$ to an acyclic map $k_!(f) : k_!(X) \to k_!(Y)$. We saw that the functor $k_!$ takes a monomorphism to a monomorphism. This shows that $k_!$ is a left Quillen functor. ∎

Proposition 6.23. *For every $X \in \mathbf{S}$, we have $\tau_1 k_! X = \pi_1 X$.*

Proof: The functors $\tau_1 k_!$ and $\pi_1 X$ are cocontinuous. Hence it suffices to prove the equality $\tau_1 k_! X = \pi_1 X$ in the case where $X = \Delta[n]$. But in this case where have

$$\tau_1 k_! \Delta[n] = \tau_1 \Delta'[n] = \pi_1 \Delta[n].$$

∎

For any map of simplicial sets $u : A \to B$, let us denote by $\alpha_\bullet(u)$ the map

$$B \sqcup_A k_!(A) \to k_!(B)$$

obtained from the square

$$\begin{array}{ccc} A & \xrightarrow{\alpha_A} & k_!(A) \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle k_!(u)} \\ B & \xrightarrow{\alpha_B} & k_!(B). \end{array}$$

Dually, for any map $f : X \to Y$ let us denote by $\beta^\bullet(f)$ the map

$$k^!(X) \to k^!(Y) \times_Y X$$

obtained from the square

$$\begin{array}{ccc} k^!(X) & \xrightarrow{\beta_X} & X \\ {\scriptstyle k^!(f)}\downarrow & & \downarrow{\scriptstyle f} \\ k^!(Y) & \xrightarrow{\beta_Y} & Y. \end{array}$$

Lemma 6.24. *The map $\alpha_\bullet(u)$ is monic if u is monic.*

Proof: Let us denote by $\mathcal{A}$ the class of maps $u : A \to B$ such that $\alpha_\bullet(u)$ is monic. Let us show that $\mathcal{A}$ is saturated. By D.1.12, the map $\alpha_\bullet(u)$ is monic iff we have $\alpha_\bullet(u) \pitchfork f$ for every trivial fibration f. But the condition $\alpha_\bullet(u) \pitchfork f$ is equivalent to the condition $u \pitchfork \beta^\bullet(f)$ by D.1.15. Thus, u belongs to $\mathcal{A}$ iff we have $u \pitchfork \beta^\bullet(f)$ for every trivial fibration f. This shows that the class $\mathcal{A}$ is saturated. Let us now prove that every monomorphism belongs to $\mathcal{A}$. By B.0.8, it suffices to show that the inclusion $\delta_n : \partial\Delta[n] \subset \Delta[n]$ belongs to $\mathcal{A}$ for every $n \geq 0$. This is clear if $n = 0$ since the map $\alpha_0 : \Delta[0] \to \Delta'[0]$ is an isomorphism. Let us now suppose $n > 0$. It is easy to verify that the square of monomomorphisms

$$\begin{array}{ccc} \Delta[n-1] & \hookrightarrow & \Delta'[n-1] \\ {\scriptstyle d_i}\downarrow & & \downarrow{\scriptstyle d'_i} \\ \Delta[n] & \xrightarrow{\alpha_n} & \Delta'[n] \end{array}$$

is cartesian for every $i \in [n]$. Let us denote by $\partial_i\Delta'[n]$ the image of the map $d'_i : \Delta'[n-1] \to \Delta'[n]$. Then we have $\alpha_n^{-1}(\partial_i\Delta'[n]) = \partial_i\Delta[n]$ since the square obove is cartesian. The functor $k_!$ preserves monomorphisms by 6.22. It follows that it preserves union of sub-objects, since it is cocontinuous. Thus,

$$k_!(\partial\Delta[n]) = \bigcup_{i\in[n]} \partial_i\Delta'[n].$$

It follows that we have

$$\alpha_n^{-1}(k_!(\partial\Delta[n])) = \bigcup_{i\in[n]} \partial_i\Delta[n] = \partial\Delta[n].$$

Hence the square

$$\begin{array}{ccc} \partial\Delta[n] & \longrightarrow & k_!(\partial\Delta[n]) \\ {\scriptstyle \delta_n}\downarrow & & \downarrow{\scriptstyle k_!(\delta_n)} \\ \Delta[n] & \xrightarrow{\alpha_n} & \Delta'[n] \end{array}$$

is cartesian. This shows that $\alpha_\bullet(\delta_n)$ is monic. Thus, $\delta_n \in \mathcal{A}$ for every $n \geq 0$. It follows that every monomorphism belongs to $\mathcal{A}$. ■

Proposition 6.25. *The map $\alpha_A : A \to k_!(A)$ is monic for any simplicial set A.*

Proof: We have $\alpha_A = \alpha_\bullet(i_A)$, where i_A denotes the inclusion $\emptyset \to A$. ■

If X is a quasi-category, then the simplicial set $k^!(X)$ is a Kan complex, since the functor $k^!$ is a right Quillen functor Hence the map $\beta_X : k^!(X) \to X$ can be factored through inclusion $J(X) \subseteq X$ by 4.19.

Proposition 6.26. *The map $k^!(X) \to J(X)$ induced by β_X is a trivial fibration for every quasi-category X.*

Proof: Every map $\Delta'[n] \to X$ factors through the inclusion $J(X) \subseteq X$ since the simplicial set $\Delta'[n]$ is a Kan complex. Thus, $k^!(X) = k^!(J(X))$. Hence it suffices to show that the map $\beta_X : k^!(X) \to X$ is a trivial fibration when X, is a Kan complex. We shall first prove that $\beta^\bullet(f)$ is a trivial fibration if f is a Kan fibration. For this, it suffices to show that we have $u \pitchfork \beta^\bullet(f)$ for every monomorphism u by D.1.12, But the condition $u \pitchfork \beta^\bullet(f)$ is equivalent to the condition $\alpha_\bullet(u) \pitchfork f$ by D.1.15. Hence it suffices to show that $\alpha_\bullet(u)$ is a monic weak homotopy equivalence. The map $\alpha_\bullet(u)$ is monic by 6.24. Consider the square.

$$\begin{array}{ccc} A & \xrightarrow{\alpha_A} & k_!(A) \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle k_!(u)} \\ B & \xrightarrow{\alpha_B} & k_!(B). \end{array}$$

The horizontal maps are monic weak homotopy equivalence by 6.25 and 6.22. This shows that $\alpha_\bullet(u)$ is a weak homotopy equivalence. We have proved that $\beta^\bullet(f)$ is trivial fibration if f is a Kan fibration. We have $\beta^\bullet(f) = \beta_X$ if f is the map $X \to 1$. Thus, β_X is a trivial fibration if X is a Kan complex. ∎

Proposition 6.27. *The Quillen adjunction*

$$k_! : (\mathbf{S}, \mathbf{Kan}) \leftrightarrow (\mathbf{S}, \mathbf{QCat}) : k^!$$

is a homotopy coreflection.

Proof: Let $\eta : Id \to k^! k_!$ be the unit of the adjunction $k_! \vdash k^!$. We shall use the criterion of E.2.17. For this we need to show that the composite

$$k_!(i_X)\eta_X : X \to k^! k_! X \to k^! X'$$

is a weak homotopy equivalence for any simplicial set X, where $i_X : k_! X \to X'$ denotes a fibrant replacement of $k_! X$ in the model category $(\mathbf{S}, \mathbf{QCat})$. We shall use the commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{\eta_X} & k^! k_! X & \xrightarrow{k^!(i_X)} & k^! X' \\ & {\scriptstyle \alpha_X}\searrow & \downarrow{\scriptstyle \beta_{k_! X}} & & \downarrow{\scriptstyle \beta_{X'}} \\ & & k_! X & \xrightarrow{i_X} & X'. \end{array}$$

Let us first show that $\beta_{X'}$ is a weak homotopy equivalence. But the functor $\tau_1(i_X) : \tau_1 k_! X \to \tau_1 X'$ is an equivalence of categories by 6.14, since i_X is a weak categorical equivalence. Hence the category $\tau_1 X'$ is a groupoid, since the category $\tau_1 k_! X$ is a groupoid by 6.23. Thus, X' is a Kan complex by 4.14. The map $k^!(X') \to J(X')$ induced by $\beta_{X'}$ is a trivial fibration by 6.26. But we have $J(X') = X'$, since X' is a Kan complex. Thus, $\beta_{X'}$ is a trivial fibration. This shows that $\beta_{X'}$ is a weak homotopy equivalence. Let us now prove that the composite $k_!(i_X)\eta_X$ is a weak homotopy equivalence. For this, it suffices to show that the composite $\beta_{X'} k_!(i_X)\eta_X$ is a weak homotopy equivalence by three-for-two, since $\beta_{X'}$ is a weak homotopy equivalence. But we have $\beta_{X'} k_!(i_X)\eta_X = i_X \alpha_X$, since the diagram above commutes. The map i_X is a weak homotopy equivalence by 6.15, since it is a weak categorical equivalence. The map α_X is a also a weak homotopy equivalence by 6.22. Hence the composite $i_X \alpha_X$ is a weak homotopy equivalence. ■

Corollary 6.28. *A map of simplicial set $A \to B$ is a weak homotopy equivalence iff the map $k_!(u) : k_!(A) \to k_!(B)$ is a weak categorical equivalence.*

Proof: This follows from 6.25 if we use E.2.18. ■

It follows from Proposition 3.12 that the functor $(-) \star B : \mathbf{S} \to B\backslash\mathbf{S}$ has a right adjoint for any simplicial set B. The right adjoint takes a map of simplicial sets $b : B \to X$ to a simplicial set that we shall denote by X/b or more simply by X/B if the map b is clear from the context.

Let us denote by $(\mathbf{S}/B, \mathbf{QCat})$ the model category/structure on the category $\mathbf{S}/B$ which is induced the model category $(\mathbf{S}, \mathbf{QCat})$

Proposition 6.29. *The pair of adjoint functors*

$$(-) \star B : \mathbf{S} \leftrightarrow B\backslash\mathbf{S} : (-)/B$$

is a Quillen pair between the model categories $(\mathbf{S}, \mathbf{QCat})$ and $(\mathbf{S}/B, \mathbf{QCat})$.

Proof The functor $(-)\star B$ takes a monomorphism to a monomorphism by 3.8. Let us show that $u \star B$ is a weak categorical equivalence if the map $u : S \to T$ is a monic weak categorical equivalence. Consider the commutative diagram

$$\begin{array}{ccccc} S & \longrightarrow & S \star B & & \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle i_2} & \searrow^{u\star B} & \\ T & \xrightarrow{i_1} & (T \star \emptyset) \cup (S \star B) & \xrightarrow[u\star' i_B]{} & T \star B, \end{array}$$

where i_B is the inclusion $\emptyset \subseteq B$. The map i_2 is a cobase change of the map u. Thus, i_2 is a (monic) weak categorical equivalence, since u is a (monic) weak categorical equivalence. Hence the result will be proved by three-for-two if we show that the inclusion $u \star' i_B$ is a weak categorical equivalence. By Lemma E.2.13, it suffices to show that we have $(u \star' i_B) \pitchfork f$ for every pseudo-fibration between quasi-categories $f : X \to Y$. But by 3.15, we have $(u \star' i_B) \pitchfork f$ iff we have $u \pitchfork \langle i_B, t, f \rangle$ for every map $t : T \to X$. The map $p = \langle i_B, t, f \rangle$ is a right fibration by 3.19 since f is a mid fibration. The codomain of p is a quasi-category by 4.11. It follows that p is a pseudo-fibration by 4.10. This shows that we have $u \pitchfork \langle i_B, t, f \rangle$, since u is a monic weak categorical equivalence. We have proved that $u \star' i_B$ is a weak categorical equivalence. It follows that $u \star B$ is a weak categorical equivalence. ∎

Chapter 7

The model structure for cylinders

7.1 Categorical cylinders and sieves

Recall that a full subcategory S of a category A is said to be a *sieve* if the implication $\mathrm{target}(f) \in S \Rightarrow \mathrm{source}(f) \in S$ is true for every arrow $f \in A$. Dually, a full subcategory $S \subseteq A$ is said to be a *cosieve* if the implication $\mathrm{source}(f) \in S \Rightarrow \mathrm{target}(f) \in S$ is true for every arrow $f \in A$. If $S \subseteq A$ is a sieve (resp. cosieve), then there exists a unique functor $p : A \to I$ such that $S = p^{-1}(0)$ (resp. $S = p^{-1}(1)$); we say that the sieve $p^{-1}(0)$ and the cosieve $p^{-1}(1)$ are *complementary*. Complementation defines a bijection between the sieves and the cosieves of A.

We shall say that an object of the category $\mathbf{Cat}/I$ is a *categorical cylinder*. The *base* of a cylinder $p : C \to I$ is the category $C(1) = p^{-1}(1)$ and its *cobase* is the category $C(0) = p^{-1}(0)$. The base of a cylinder (C, p) is a cosieve in C and its cobase is a sieve.

Recall that if A and B are small categories, a *distributor* $R : A \Rightarrow B$ is defined to be a functor $R : A^o \times B \to \mathbf{Set}$. The distributors $A \Rightarrow B$ form a category

$$D(A, B) = [A^o \times B, \mathbf{Set}].$$

To every distributor $R : A \Rightarrow B$ is associated a *collage* category $C = A \star_R B$ constructed as follows: $Ob(C) = Ob(A) \sqcup Ob(B)$ and for $x, y \in Ob(C)$, we put

$$C(x,y) = \begin{cases} A(x,y) & \text{if } x \in A \text{ and } y \in A \\ B(x,y) & \text{if } x \in B \text{ and } y \in B \\ R(x,y) & \text{if } x \in A \text{ and } y \in B \\ \emptyset & \text{if } x \in B \text{ and } y \in A. \end{cases}$$

Composition of arrows is obvious. Notice that there is a canonical pair of fully faithful functors,

$$A \xrightarrow{s} A \star_R B \xleftarrow{t} B.$$

Notice also that $A \star_\emptyset B = A \sqcup B$ and that $A \star_1 B = A \star B$, where 1 is the terminal distributor $A \Rightarrow B$.

The obvious canonical functor $c : A \star_R B \to I$ shows that $A \star_K B$ has the structure of a cylinder. An *external map* from a distributor $K : A \Rightarrow B$ to a distributor $R : C \Rightarrow D$ is defined to be a pair of functors $f : A \to C$ and $g : B \to D$ together with a natural transformation $K \to (f \times g)^*(R)$. This defines the morphisms of a (fibered) category that we shall denote by **Dist**. The collage functor induces an equivalence of categories

$$\mathbf{Dist} \simeq \mathbf{Cat}/I.$$

Theorem 7.1.1. *The category* $\mathbf{Cat}/I$ *is cartesian closed. The model category* $(\mathbf{Cat}, Eq)$ *induces a cartesian closed model structure on the category* $\mathbf{Cat}/I$*. Every object is fibrant and cofibrant.*

7.2 Simplicial cylinders and sieves

If X is a simplicial set, we say that a full simplicial subset $S \subseteq X$ is a *sieve* if the implication $\mathrm{target}(f) \in S \Rightarrow \mathrm{source}(f) \in S$ is true for every arrow $f \in X$. Dually, we say that S is a *cosieve* if the implication $\mathrm{source}(f) \in S \Rightarrow \mathrm{target}(f) \in S$ is true for every arrow $f \in X$. If $h : X \to \tau_1 X$ is the canonical map, then the map $S \mapsto h^{-1}(S)$ induces a bijection between the sieves of X and the sieve of the category $\tau_1 X$, and similarly for the cosieves. If $S \subseteq X$ is a sieve (resp. cosieve) there exists a unique map $f : X \to I$ such that $S = f^{-1}(0)$ (resp. $S = f^{-1}(1)$). There is thus a bijection between the sieves and the cosieves of X. We shall say that the sieve $f^{-1}(0)$ and the cosieve $f^{-1}(1)$ are *complementary.*

We call an object of the category $\mathbf{S}/I$ is a (simplicial) *cylinder*. The *base* of a cylinder $p : C \to I$ is the simplicial set $C(1) = p^{-1}(1)$ and its *cobase* is the simplicial set $C(0) = p^{-1}(0)$. The cobase of a cylinder (C, p) is a sieve in C and its cobase is a cosieve. If $C(1) = 1$ (resp. $C(0) = 1$) we say that C is an *inductive cone* (resp *projective cone*). If $C(0) = C(1) = 1$ we say that C is a *spindle.*

Lemma 7.1. *If* X *is a quasi-category, then every map* $X \to I$ *is a pseudo-fibration.*

Proof: Every map $X \to I$ is a mid fibration by 2.2, since I is a category. Every map $X \to I$ is thus a pseudo-fibration, since every isomorphism of I is a unit. ∎

The model structure for quasi-categories $(\mathbf{S}, \mathbf{QCat})$ induces a model structure on the category $\mathbf{S}/B$ for any simplicial set B. In particular, it induces a model structure on the category $\mathbf{S}/I$. It follows from the lemma that its category of fibrant objects is $\mathbf{QCat}/I$. We shall denote the model structure by $(\mathbf{S}/I, \mathbf{QCat}/I)$. The following theorem is the main result of the chapter:

Theorem *The model category* $(\mathbf{S}/I, \mathbf{QCat}/I)$ *is cartesian closed. Hence the fiber product over* I *of two weak categorical equivalences is a weak categorical equivalence.*

The theorem will be proved in 7.9. For this we need to establish a few intermediate results.

If i denotes the inclusion $\partial I \subset I$ then the pullback functor $i^* : \mathbf{S}/I \to \mathbf{S}/\partial I = \mathbf{S} \times \mathbf{S}$ associates to a cylinder X the pair of simplicial sets $(X(0), X(1))$. The functor i^* has a left adjoint $i_!$ and a right adjoint i_*. We have $i_!(A, B) = A \sqcup B$ for any pair of simplicial sets A and B. The structure map of $A \sqcup B$ is the map $A \sqcup B \to I$ which takes the value 0 on A and the value 1 on B. Moreover, we have $i_*(A, B) = A \star B$ by Proposition 3.5. The structure map of $A \star B$ is obtainned by joining the maps $A \to 1$ and $B \to 1$. Every cylinder X is equipped with two canonical maps

$$X(0) \sqcup X(1) \to X \to X(0) \star X(1).$$

Recall that a functor is said to be a *Grothendieck bifibration* if it is both a Grothendieck fibration and a Grothendieck opfibration. Recall that a functor is said to be a *bireflection* if it is both a reflection and a coreflection. If a pseudo-fibration is bireflection, then it is a Grothendieck bifibration.

Proposition 7.2. *The functor* $i^* : \mathbf{S}/I \to \mathbf{S} \times \mathbf{S}$ *is a Grothendieck bifibration.*

Proof: The functor $i_!$ is fully faithful, since the map $i : \partial I \to I$ is monic. Hence the functor i^* is a bireflection, since it has a right adjoint. It is easy to verify that i^* is a pseudo-fibration. It is thus a Grothendieck bifibration. ∎

A map of cylinders $f : X \to Y$ induces a pair of maps of simplicial sets $f_0 : X(0) \to Y(0)$ and $f_1 : X(1) \to Y(1)$. The map is cartesian iff the following square is a pullback,

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ \downarrow & & \downarrow \\ X(0) \star X(1) & \xrightarrow{f_0 \star f_1} & Y(0) \star Y(1). \end{array}$$

The map is cocartesian iff the following square is a pushout,

$$\begin{array}{ccc} X(0)\sqcup X(1) & \xrightarrow{f_0\sqcup f_1} & Y(0)\sqcup Y(1) \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y. \end{array}$$

We shall denote by $\mathbf{C}(A,B)$ the fiber at (A,B) of the functor $i^*:\mathbf{S}/I\to\mathbf{S}\times\mathbf{S}$. It is the *category of cylinders with with cobase A and base B*. The initial object of the category $\mathbf{C}(A,B)$ is the cylinder $A\sqcup B$, and its terminal object is the cylinder $A\star B$.

Let K be a small category and let us put $\mathcal{E}=\hat{K}=[K^o,\mathbf{Set}]$. If $B\in\mathcal{E}$, we shall denote by $el(B)$, or by K/B, the category of elements of B. The Yoneda functor $y:K\to\mathcal{E}$ induces a functor $y/B:K/B\to\mathcal{E}/B$. The following result is classical:

Lemma 7.2.1. *The "singular" functor*

$$(y/B)^!:\mathcal{E}/B\to[(K/B)^o,\mathbf{Set}]$$

is an equivalence of categories.

We shall say that an object $(X,p)\in\mathcal{E}/B$ is *trivial* over a sub presheaf $A\subseteq B$ if the induced map $p^{-1}(A)\to A$ is an isomorphism. Let us denote by $\mathcal{E}/(B,A)$ the full subcategory of $\mathcal{E}/B$ spanned by the objects which are trivial over A. The category $el(A)$ is a sieve in the category $el(B)$. Let us denote the category of elements of the complementary cosieve by $el(B,A)$.

Lemma 7.2.2. *If i denotes the inclusion $el(B,A)\subseteq el(B)$, then the composite*

$$\mathcal{E}/(B,A)\longrightarrow\mathcal{E}/B\xrightarrow{(y/B)^!}[el(B)^o,\mathbf{Set}]\xrightarrow{i^*}[el(B,A)^o,\mathbf{Set}]$$

is an equivalence of categories.

The join of two simplices $x:\Delta[m]\to A$ and $y:\Delta[n]\to B$ is a simplex

$$x\star y:\Delta[m+1+n]\to A\star B.$$

This defines a functor $\star:el(A)\times el(B)\to el(A\star B)$.

Lemma 7.3. *The functor $\star:el(A)\times el(B)\to el(A\star B)$ induces an equivalence of categories*

$$el(A)\times el(B)\simeq el(A\star B,A\sqcup B).$$

Proof: If $x : \Delta[m] \to A$ and $y : \Delta[n] \to B$, we have a commutative diagram of canonical maps

$$\begin{array}{ccc}
\Delta[m] \sqcup \Delta[n] & \longrightarrow & \Delta[m] \star \Delta[n] \\
{\scriptstyle x \sqcup y}\downarrow & & \downarrow{\scriptstyle x \star y} \\
A \sqcup B & \longrightarrow & A \star B \\
\downarrow & & \downarrow{\scriptstyle p} \\
\partial I & \longrightarrow & I,
\end{array}$$

where $\partial I = 1 \sqcup 1 = \{0, 1\} \subset I$. The bottom square is a pullback by Lemma 3.4. And also the composite square by the same lemma. It follows that the top square is a pullback. This show that $x \star y$ cannot be factored through the inclusion $A \sqcup B \subset A \star B$. Thus, $x \star y \in el(A \star B, A \sqcup B)$. Conversely, if a simplex $f : \Delta[p] \to A \star B$ belongs to $el(A \star B, A \sqcup B)$, let us show that we have $f = x \star y$ for a unique pair of simplices $x : \Delta[m] \to A$ and $y : \Delta[n] \to B$. The assumption implies that the map $pf : \Delta[p] \to I$ cannot be factored through the inclusion $\partial I \subset I$, since the bottom square in the diagram above is a pullback. It follows that $pf : \Delta[p] \to I$ is the join of two maps $\Delta[m] \to 1$ and $\Delta[n] \to 1$, where $\Delta[m] = p^{-1}(0)$ and $n = p - m - 1$. The simplex $x : \Delta[m] \to A$ is then obtained by composing f with the inclusion $\Delta[m] \subset \Delta[m] \star \Delta[n]$ and the simplex $y : \Delta[n] \to A$ by composing f with the inclusion $\Delta[n] \subset \Delta[m] \star \Delta[n]$. ∎

Let $\mathbf{S}^{(2)} = [\Delta^o \times \Delta^o, \mathbf{Set}]$ be the category of bisimplicial sets. If $A, B \in \mathbf{S}$, let us put

$$(A \Box B)_{mn} = A_m \times B_n$$

for $m, n \geq 0$. An object of the category $\mathbf{S}^{(2)}/A \Box B$ is a bisimplicial set X equipped with two augmentations, a row augmentation $X \to A$ and a column augmentation $X \to B$. The functor

$$\Box : el(A) \times el(B) \to el(A \Box B)$$

is obviously an equivalence of categories. By Proposition 7.2.1, we have an equivalence of categories

$$\mathbf{S}^{(2)}/A \Box B = [el(A \Box B), \mathbf{Set}]$$

By combining these equivalences with the equivalence of Lemma 7.3, we obtain an equivalence of categories

$$D : \mathbf{C}(A, B) \simeq \mathbf{S}^{(2)}/A \Box B.$$

The equivalence associates to a cylinder $X \to I$ the bisimplicial set $D(X)$ defined by putting

$$D(X)_{mn} = (\mathbf{S}/I)(\Delta[m] \star \Delta[n], X)$$

for every $m, n \geq 0$. If $X \in \mathbf{C}(A, B)$, the bisimplicial set $D(X)$ is augmented by a map $(\epsilon_1, \epsilon_2) : D(X) \to A \Box B$. The image of $x : \Delta[m] \star \Delta[n] \to X$ by $\epsilon_1 : D(X) \to A$ is obtained by restricting it to $\Delta[m] \subset \Delta[m] \star \Delta[n]$ and its image by $\epsilon_2 : D(X) \to B$ is obtained by restricting it to $\Delta[n] \subset \Delta[m] \star \Delta[n]$.

We shall denote by $\mathbf{S}^{(2)}/\mathbf{S}^2$ the category defined by the pullback square

$$\begin{array}{ccc} \mathbf{S}^{(2)}/\mathbf{S}^2 & \longrightarrow & (\mathbf{S}^{(2)})^I \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle t} \\ \mathbf{S} \times \mathbf{S} & \xrightarrow{\Box} & \mathbf{S}^{(2)}, \end{array}$$

where $(\mathbf{S}^{(2)})^I$ is the arrow category of $\mathbf{S}^{(2)}$ and where t is the target functor. The functor p is a Grothendieck bifibration, since the functor t is a Grothendieck bifibration. An object of the category $\mathbf{S}^{(2)}/\mathbf{S}^2$ is a quadruple (X, p, A, B), where X is a bisimplicial set, where (A, B) is pair of simplicial sets and where q is a map of bisimplicial sets $X \to A \Box B$. A map $(X, q, A, B) \to (X', q', A', B')$ is a triple (f, u, v), where $f : X \to X'$, $u : A \to A'$ and $v : B \to B'$ are maps fitting in the commutative square

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q'} \\ A \Box B & \xrightarrow{u \Box v} & A' \Box B'. \end{array}$$

The square is a pullback iff the map (f, u, v) is cartesian. Recall from Proposition 7.2 that the functor $i^* : \mathbf{S}/I \to \mathbf{S} \times \mathbf{S}$ is a Grothendieck bifibration.

Proposition 7.4. *The functor D induces an equivalence of fibered categories,*

$$D : \mathbf{S}/I \to \mathbf{S}^{(2)}/\mathbf{S}^2.$$

Proof: It suffices to show that the functor D takes a cartesian morphism to a cartesian morphism, since it induces an equivalence between the fibers. Let $u : A' \to A$ and $v : B' \to B$ be two maps of simplicial sets, and let $C \in \mathbf{C}(A, B)$, Then the cylinder $C' = (u, v)^*(C) \in \mathbf{C}(A', B')$ is defined by a pullback square of simplicial sets

$$\begin{array}{ccc} C' & \longrightarrow & C \\ \downarrow & & \downarrow \\ A' \star B' & \xrightarrow{u \star v} & A \star B, \end{array}$$

where the vertical maps are canonical. We then have to show that the following square is a pullback,

$$\begin{array}{ccc} D(C') & \longrightarrow & D(C) \\ \downarrow & & \downarrow \\ A'\Box B' & \xrightarrow{u\Box v} & A\Box B. \end{array}$$

But this follows directly form the description of D given above. ∎

Let us describe the functor D^{-1}, (quasi-) inverse to the functor D. Let $\sigma : \Delta \times \Delta \to \Delta$ be the ordinal sum functor. The functor $\sigma^* : \mathbf{S} \to \mathbf{S}^{(2)}$ has a left adjoint $\sigma_!$ and a right adjoint σ_*. We have $\sigma_!(\Delta[m]\Box\Delta[n]) = \Delta[m] \star \Delta[n]$ for every $m, n \geq 0$, by definition of $\sigma_!$. In particular, $\sigma_!(1) = \sigma_!(\Delta[0]\Box\Delta[0]) = \Delta[1] = I$. It follows that $\sigma_!$ has a natural lift

$$\sigma_! : \mathbf{S}^{(2)} \to \mathbf{S}/I.$$

If i_0 denotes the inclusion $\{0\} \subset I$, then the functor $i_0^*\sigma_! : \mathbf{S}^{(2)} \to \mathbf{S}$ is cocontinuous and we have

$$i_0^*\sigma_!(\Delta[m]\Box\Delta[n]) = \Delta[m]$$

for every $m, n \geq 0$. It follows that we have

$$\sigma_!(X)(0)_m = i_0^*\sigma_!(X)_m = \pi_0 X_{m\star}$$

for every $X \in \mathbf{S}^{(2)}$ and every $m \geq 0$. Similarly, if i_1 denotes the inclusion $\{1\} \subset I$, then we have

$$\sigma_!(X)(1)_n = i_1^*\sigma_!(X)_n = \pi_0 X_{\star n}$$

for every $n \geq 0$. If $X \to A\Box B$ is the structure map of an object in $\mathbf{S}^{(2)}/\mathbf{S}^2$, then from the augmentation $X \to A$ we obtain a map $\sigma_!(X)(0) \to A$ and from the augmentation $X \to B$ a map $\sigma_!(X)(1) \to A$. The cylinder $D^{-1}(X)$ is then constructed by the following pushout square

$$\begin{array}{ccc} \sigma_!(X)(0) \sqcup \sigma_!(X)(1) & \longrightarrow & \sigma_!(X) \\ \downarrow & & \downarrow \\ A \sqcup B & \longrightarrow & D^{-1}(X), \end{array}$$

If A and B are simplicial sets, let us denote by $A \boxdot B$ the object of $\mathbf{S}^{(2)}/A\Box B$ defined by the identity map $A\Box B \to A\Box B$.

Lemma 7.5. *We have $D(A \star B) = A \boxdot B$.*

Proof: We have $A \star B = i_*(A, B)$. Hence the functor $\star : \mathbf{S} \times \mathbf{S} \to \mathbf{S}/I$ is right adjoint to the functor $i^* : \mathbf{S}/I \to \mathbf{S} \times \mathbf{S}$. But we habe $i^* = pD$ by Proposition 7.4, where p is the canonical functor $\mathbf{S}^{(2)}/\mathbf{S}^2 \to \mathbf{S}^2$. This proves the result, since the functor $(A, B) \mapsto A \boxdot B$ is right adjoint to the functor p. ∎

If $u : A \to B$ and $v : S \to T$ are two maps in $\mathbf{S}/I$, we shall denote by $u \times_I' v$ the map

$$(A \times_I T) \sqcup_{A \times_I S} (B \times_I S) \to B \times_I T$$

obtained from the commutative square

$$\begin{array}{ccc} A \times_I S & \xrightarrow{u \times_I S} & B \times_I S \\ {\scriptstyle A \times_I v} \downarrow & & \downarrow {\scriptstyle B \times_I v} \\ A \times_I T & \xrightarrow{u \times_I T} & B \times_I T. \end{array}$$

We have $D(u \times_I' v) = D(u) \times' D(v)$, since the functor D is an equivalence of categories by Proposition 7.4.

Lemma 7.6. *We have a canonical isomorphism between the maps*

$$(u_1 \star' u_2) \times_I' (v_1 \star' v_2) \simeq (u_1 \times' v_1) \star' (u_2 \times' v_2),$$

for any quadruple of maps of simplicial sets $u_1 : A_1 \to B_1$, $u_2 : A_2 \to B_2$, $v_1 : S_1 \to T_1$ and $v_2 : S_2 \to T_2$.

Proof: We shall use the equivalence of categories $D : \mathbf{S}/I \to \mathbf{S}^{(2)}/\mathbf{S}^2$ of Proposition 7.4. If A is a simplicial set, let us put $\theta_1(A) = A \boxdot 1$ and $\theta_2(A) = 1 \boxdot A$. We have a canonical isomorphism of maps

$$\theta_1(u_1) \times' \theta_2(u_2) \times' \theta_1(v_1) \times' \theta_2(v_2) \simeq \theta_1(u_1) \times' \theta_1(v_1) \times' \theta_2(u_2) \times' \theta_2(v_2),$$

since the operation $\times'$ is coherently associative and symmetric. Notice that we have

$$\theta_1(A) \times \theta_2(B) = (A \boxdot 1) \times (1 \boxdot B) = A \boxdot B$$

for any pair of simplicial sets A and B, since we have $(A \Box 1) \times (1 \Box B) = A \Box B$. Thus, $D(A \star B) = \theta_1(A) \times \theta_2(B)$ by Lemma 7.5. It follows that we have $D(u \star' v) = \theta_1(u) \times' \theta_2(v)$ for any pair of maps $u : A \to B$ and $v : S \to T$. Thus,

$$\begin{aligned} \theta_1(u_1) \times' \theta_2(u_2) \times' \theta_1(v_1) \times' \theta_2(v_2) &= D(u_1 \star' u_2) \times' D(v_1 \star' v_2) \\ &= D((u_1 \star' u_2) \times_I' (v_1 \star' v_2)). \end{aligned}$$

The functor θ_1 preserves pullbacks, since the functor $A \mapsto A \Box 1$ preserves pullbacks. It also preserves pushout, since the functor $A \mapsto A \star 1$ preserves pushout by Proposition 3.12, since D is an equivalence of categories and since we have $D(A \boxdot 1) = A \star 1$ by Lemma 7.5. It follows that we have $\theta_1(u \times' v) = \theta_1(u) \times' \theta_1(v)$ for any pair of maps of simplicial sets $u : A \to B$ and $v : S \to T$. Similarly, we have $\theta_2(u \times' v) = \theta_2(u) \times' \theta_2(v)$. Thus,

$$\begin{aligned} \theta_1(u_1) \times' \theta_1(v_1) \times' \theta_2(u_2) \times' \theta_2(v_2) &= \theta_1(u_1 \times' v_1) \times' \theta_2(u_2 \times' v_2) \\ &= D((u_1 \times' v_1) \star' (u_2 \times' v_2)). \end{aligned}$$

We have constructed a canonical isomorphism between the maps

$$D((u_1 \star' u_2) \times'_I (v_1 \star' v_2)) \simeq D((u_1 \times' v_1) \star' (u_2 \times' v_2)),$$

This proves the result, since the functor D is an equivalence of categories. ∎

Let us denote the inclusion $\partial\Delta[n] \subset \Delta[n]$ by δ_n and the inclusion $\Lambda^k[n] \subset \Delta[n]$ by h_n^k.

Lemma 7.7. *The saturated class of monomorphisms in* $\mathbf{S}/I$ *is generated by the following maps*

- *the map* $\delta_m \star \emptyset$ *for* $m \geq 0$*;*
- *the map* $\delta_m \star' \delta_n$ *for* $m, n \geq 0$*;*
- *the map* $\emptyset \star \delta_n$ *for* $n \geq 0$.

The saturated class of mid anodyne maps in $\mathbf{S}/I$ *is generated by the following maps*

- *the map* $h_m^k \star \emptyset$ *for* $0 < k < m$
- *the map* $h_m^k \star' \delta_n$ *for* $0 < k \leq m$ *and* $n \geq 0$*;*
- *the map* $\delta_m \star' h_n^k$ *for* $m \geq 0$ *and* $0 \leq k < n$*;*
- *the map* $\emptyset \star h_n^k$ *for* $0 < k < n$

Proof Let us denote the restriction of a simplex $u : \Delta[n] \to I$ to $\partial\Delta[n]$ by ∂u. The saturated class of monomorphisms in $\mathbf{S}/I$ is generated by the set of inclusions $(\partial\Delta[n], \partial u) \subset (\Delta[n], u)$, where u runs in the simplices of I. But the simplices of I are of the following three kinds:

- a simplex $\Delta[m] \star \emptyset \to I$ for $m \geq 0$;
- a simplex $\Delta[m] \star \Delta[n] \to I$ for $m, n \geq 0$;
- a simplex $\emptyset \star \Delta[n] \to I$ for $n \geq 0$.

Obviuously, we have

- $\partial(\Delta[m] \star \emptyset) = \partial\Delta[m] \star \emptyset$ for every $m \geq 0$;
- $\partial(\emptyset \star \Delta[n]) = \emptyset \star \partial\Delta[n]$ for every $n \geq 0$;

Moreover, we have

$$\partial(\Delta[m] \star \Delta[n]) = \big(\partial\Delta[m] \star \Delta[n]\big) \cup \big(\Delta[m] \star \partial\Delta[n]\big)$$

for every $m, n \geq 0$ by Lemma 3.11. The first statement is proved. The second statement is proved similarly by using Lemma 3.11. ∎

Theorem 7.8. *If a map $u : A \subseteq B$ in $\mathbf{S}/I$ is mid anodyne then so is the map $u \times_I' v$ for any monomorphism $v : S \to T$ in $\mathbf{S}/I$.*

Proof: The class of mid anodyne maps in $\mathbf{S}/I$ is generated by a set of maps Σ_1 described in Lemma 7.7. The class of monomorphism in $\mathbf{S}/I$ is generated by a set of maps Σ_2 described in the same lemma. By Proposition D.2.6, it suffices to show if $u \in \Sigma_1$ and $v \in \Sigma_2$, then the map $u \times_I' v$ is mid anodyne. There are twelve cases to consider, most of which are trivial. We consider the non-trivial cases first. Suppose that $v = \delta_p \star' \delta_q$, where $p, q \geq 0$. We first consider the case where $u = h_m^k \star' \delta_n$ with $0 < k \leq m$ and $n \geq 0$. By Lemma 7.6, we have

$$u \times_I' v = (h_m^k \times' \delta_p) \star' (\delta_n \times' \delta_q).$$

But the map $h_m^k \times' \delta_p$ is right anodyne by Theorem 2.17 since the map h_m^k is right anodyne when $0 < k \leq m$. Hence the map $(h_m^k \times' \delta_p) \star' (\delta_n \times' \delta_q)$ is mid anodyne by Theorem 3.17, since the map $\delta_n \times' \delta_q$ is monic by Lemma 2.15. This proves that the map $u \times_I' v$ is mid anodyne in this case. Let us now consider the case where $u = \delta_m \star' h_n^k$, with $m \geq 0$ and $0 \leq k < n$. By Lemma 7.6, we have

$$u \times_I' v = (\delta_m \times' \delta_p) \star' (h_n^k \times' \delta_q).$$

But the map $h_m^k \times' \delta_p$ is left anodyne by Theorem 2.17, since the map h_n^k is left anodyne when $0 \leq k < n$. Hence the map $(\delta_m \times' \delta_p) \star' (h_n^k \times' \delta_q)$ is mid anodyne by Theorem 3.17, since the map $(\delta_m \times' \delta_p)$ is monic. We have proved that the map $u \times_I' v$ is mid anodyne in the non trivial cases. Let us now suppose that $u = h_m^k \star \emptyset$. Observe that we have

$$(A \star \emptyset) \times_I (S_1 \star S_2) = (A \times S_1) \star \emptyset$$

for any triple of simplicial sets A, S_1 and S_2. It follows from this observation that if $v_1 : S_1 \to T_1$ and $v_2 : S_2 \to T_2$ are two maps of simplicial sets, then the image of the square

$$\begin{array}{ccc} S_1 \star T_1 & \longrightarrow & S_2 \star T_1 \\ \downarrow & & \downarrow \\ S_1 \star T_2 & \longrightarrow & S_2 \star T_2 \end{array}$$

by the functor $(A \star \emptyset) \times_I (-)$ is equal to the square

$$\begin{array}{ccc} (A \times S_1) \star \emptyset & \longrightarrow & (A \times S_2) \star \emptyset \\ \downarrow & & \downarrow \\ (A \times S_1) \star \emptyset & \longrightarrow & (A \times S_2) \star \emptyset, \end{array}$$

which is trivially a pushout. Hence the map $(A \star \emptyset) \times_I (v_1 \star' v_2)$ is an isomorphism. It follows that the map $(u \star \emptyset) \times_I' (v_1 \star' v_2)$ is an isomorphism for any map $u : A \to B$.

In particular, the map $(h^k_m \star \emptyset)) \times'_I (\delta_p \star' \delta_q)$ is an isomorphism. We have proved that the map $u \times'_I v$ is mid anodyne in this cases. The other cases are left to the reader. ∎

Theorem 7.9. *The model category* $(\mathbf{S}/I, \mathbf{QCat}/I)$ *is cartesian closed. Hence the product over* I *of two weak categorical equivalences is a weak categorical equivalence.*

Proof: We have to show that the cartesian product functor

$$\times_I : \mathbf{S}/I \times \mathbf{S}/I \to \mathbf{S}/I$$

is a left Quillen functor of two variables. For this, we shall use Proposition E.3.4. If u and v are two monomorphisms in $\mathbf{S}/I$, then the map $u \times_I v$ is a monomorphism, since this property is true in any topos. If $C = (C, r) \in \mathbf{S}/I$, let us show that the functor $C \times_I (-) = r^*(-) : \mathbf{S}/I \to \mathbf{S}/C$ takes an acyclic cofibration to an acyclic cofibration. If a map $u : A \to B$ in $\mathbf{S}/I$ is mid anodyne, then so is the map $C \times_I u$ by Theorem 7.8. If $u : A \to B$ is an acyclic cofibration in $\mathbf{S}/I$, let us show that the map $r^*(u)$ is an acyclic cofibration. For this, let us choose a factorisation of the structure map $B \to I$ as a mid anodyne map $y : B \to Y$ followed by a mid fibration $p : Y \to I$, together with a factorisation of the composite $yu : A \to Y$ as a mid anodyne map $x : A \to X$ followed by a mid fibration $g : X \to Y$. The horizontal maps in the commutative square

$$\begin{array}{ccc} A & \xrightarrow{x} & X \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xrightarrow{y} & Y \end{array}$$

are weak categorical equivalences, since a mid anodyne map is a weak categorical equivalence by Corollary 2.29. Hence the map g is a weak categorical equivalence by three-for-two. The horizontal maps in the following commutative square

$$\begin{array}{ccc} r^*(A) & \xrightarrow{r^*(x)} & r^*(X) \\ {\scriptstyle r^*(u)}\downarrow & & \downarrow{\scriptstyle r^*(g)} \\ r^*(C) & \xrightarrow{r^*(y)} & r^*(Y) \end{array}$$

are mid anodynes, since the functor r^* takes a mid anodyne map to a mid anodyne map. Hence they are are weak categorical equivalences. Let us show that $r^*(g)$ is a weak categorical equivalence. But g is a map between fibrant objects of the model category $(\mathbf{S}/I, \mathbf{QCat}/I)$. The functor $r^* : \mathbf{S}/I \to \mathbf{S}/C$ is a right Quillen functor by Proposition E.2.4. It follows by Ken Brown's Lemma E.2.6 that $r^*(g)$ is a weak equivalence. It then follows by three-for-two that $r^*(u)$ is a weak categorical equivalence. Hence the conditions of Proposition E.3.4 are satisfied. This shows that the product functor $\times_I$ is a left Quillen functor of two variables. ∎

If $u : A \to B$ is a map of simplicial sets, then the pullback functor $u^* : \mathbf{S}/B \to \mathbf{S}/A$ has a right adjoint u_*. Let us denote by $(\mathbf{S}/A, Wcat)$ the model structure on $\mathbf{S}/A$ induces by the model structures for quasi-categories on $\mathbf{S}$.

Corollary 7.10. *If $u : A \to I$, then the pair of adjoint functors*

$$u^* : \mathbf{S}/I \leftrightarrow \mathbf{S}/A : u_*$$

is a Quillen pair between the model categories $(\mathbf{S}/I, \mathbf{QCat}/I)$ and $(\mathbf{S}/A, Wcat)$.

Proof: The functor $u^* = A \times_I (-)$ is a left Quillen functor by Theorem 7.9. ■

If i denotes the inclusion $\{0, 1\} = \partial I \subset I$, then the functor $i^* : \mathbf{S}/I \to \mathbf{S} \times \mathbf{S}$ is given by $i_*(A, B) = A \star B$.

Corollary 7.11. *The pair of adjoint functors*

$$i^* : \mathbf{S}/I \leftrightarrow \mathbf{S} \times \mathbf{S} : \star,$$

is a Quillen pair between the model category $(\mathbf{S}/I, \mathbf{QCat}/I)$ and the model category $(\mathbf{S}, \mathbf{QCat}) \times (\mathbf{S}, \mathbf{QCat})$.

Proof: We have $(\mathbf{S}/\partial I, \mathbf{QCat}/\partial I) = (\mathbf{S}, \mathbf{QCat}) \times (\mathbf{S}, \mathbf{QCat})$. ■

The functor $\tau_1 : \mathbf{S} \to \mathbf{Cat}$ induces a functor $\tau_1 : \mathbf{S}/I \to \mathbf{Cat}/I$ since $\tau_1 I = I$. From the inclusion $N : \mathbf{Cat} \subset \mathbf{S}$ we obtain an inclusion $N : \mathbf{Cat}/I \subset \mathbf{S}/I$.

Proposition 7.12. *The induced functor $\tau_1 : \mathbf{S}/I \to \mathbf{Cat}/I$ preserves finite products. The resulting pair of adjoint functors*

$$\tau_1 : \mathbf{S}/I \to \mathbf{Cat}/I : N$$

is a Quillen pair between the model categories $(\mathbf{S}/I, \mathbf{QCat}/I)$ and $(\mathbf{Cat}/I, Eq)$.

Proof: Obviously, we have $\tau_1 I = I$. Hence the functor $\tau_1 : \mathbf{S}/I \to \mathbf{Cat}/I$ preserves terminal objects. Let us show that the canonical map

$$i_{XY} : \tau_1(X \times_I Y) \to \tau_1 X \times_I \tau_1 Y$$

is an isomorphism for every $X, Y \in \mathbf{S}/I$. The functor τ_1 is cocontinuous since it is a left adjoint. Hence the functor $(X, Y) \mapsto \tau_1(X \times_I Y)$ is cocontinuous in each variable, since the category $\mathbf{S}/I$ is cartesian closed. Similarly, the functor $(X, Y) \mapsto \tau_1 X \times_I \tau_1 Y$ is cocontinuous in each variable, since the category $\mathbf{Cat}/I$ is cartesian closed by Theorem 7.1.1. Every object of $\mathbf{S}/I$ is a colimit of a diagram of simplices $u : \Delta[n] \to I$. Hence it suffices to prove that the natural transformation i_{XY} is invertible in the case where $X = (\Delta[m], u)$ and $Y = (\Delta[n], v)$. We have $(\Delta[m], u) = N([m], u)$ and $(\Delta[n], v) = N([n], v)$. The functor N preserves fiber products since it is a right adjoint. We have $\tau_1 NC = C$ for every category C. It follows that we have

$$\begin{aligned}\tau_1(X \times_I Y) &= \tau_1(N([m],u) \times_I N([n],v)) = \tau_1 N(([m],u) \times_I ([n],v)) \\ &= ([m],u) \times_I ([n],v) = \tau_1(\Delta[m],u) \times_I \tau_1(\Delta[n],v) \\ &= \tau_1 X \times_I \tau_1 Y.\end{aligned}$$

The first statement is proved. The second statement.is a direct consequence of Proposition 6.14. ∎

Chapter 8

The contravariant model structure

In this chapter we introduce the *contravariant model structure* on the category $\mathbf{S}/B$ whose fibrant objects are the contravariant (ie right) fibrations $X \to B$; the weak equivalences are called *contravariant* equivalences and the fibrations *dexter* fibrations. We also introduce the dual *covariant* model structure whose fibrant objects are the covariant (ie left) fibrations $X \to B$; the weak equivalences are called *covariant* equivalences and the fibrations *sinister* fibrations.

8.1 Introduction

The category $\mathbf{S}/B$ is enriched over the category $\mathbf{S}$ for any simplicial set B. We shall denote by $[X,Y]_B$, or more simply by $[X,Y]$, the simplicial set of maps $X \to Y$ between two objects of $\mathbf{S}/B$. By definition, a simplex $\Delta[n] \to [X,Y]$ is a map $\Delta[n] \times X \to Y$ in $\mathbf{S}/B$, where $\Delta[n] \times (X,p) = (\Delta[n] \times X, pp_2)$ and where p_2 is the projection $\Delta[n] \times X \to X$.

The enriched category $\mathbf{S}/B$ admits tensor and cotensor products. The *tensor product* of an object $X = (X,p)$ by a simplicial set A is the object $A \times X = (A \times X, pp_2)$. The *cotensor product* of $X = (X,p)$ by A is an object denoted $X^{[A]}$. If $q : X^{[A]} \to B$ is the structure map, then a simplex $x : \Delta[n] \to X^{[A]}$ over a simplex $y = qx : \Delta[n] \to B$ is a map $A \times (\Delta[n], y) \to (X,p)$. The object $(X^{[A]}, q)$ can be constructed by a pullback square

$$\begin{array}{ccc} X^{[A]} & \longrightarrow & X^A \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p^A} \\ B & \longrightarrow & B^A, \end{array}$$

where the bottom map is the diagonal. There are canonical isomorphisms

$$[A \times X, Y] = [X, Y]^A = [X, Y^{[A]}]$$

for any pair $X, Y \in \mathbf{S}/B$ and any simplicial set A.

We say that two maps $f, g : X \to Y$ in $\mathbf{S}/B$ are *fibrewise homotopic* if they belong the same connected component of the simplicial set $[X, Y]$. If we apply the functor π_0 to the composition map

$$[Y, Z] \times [X, Y] \to [X, Z]$$

of a triple $X, Y, Z \in \mathbf{S}/B$, we obtain a composition law

$$\pi_0[Y, Z] \times \pi_0[X, Y] \to \pi_0[X, Z]$$

for a category $(\mathbf{S}/B)^{\pi_0}$, where we put $(\mathbf{S}/B)^{\pi_0}(X, Y) = \pi_0[X, Y]$. There is an obvious canonical functor $\mathbf{S}/B \to (\mathbf{S}/B)^{\pi_0}$.

Definition 8.1. *We say that a map $X \to Y$ in $\mathbf{S}/B$ is a* fibrewise homotopy equivalence *if the map is invertible in the homotopy category $(\mathbf{S}/B)^{\pi_0}$.*

If $X \in \mathbf{S}/B$, let us denote by $X(b)$ the fiber of the structure map $X \to B$ over a vertex $b \in B$. We call a map $f : X \to Y$ in $\mathbf{S}/B$ a *pointwise homotopy equivalence* if the map $f_b : X(b) \to Y(b)$ induced by f is a homotopy equivalence for each vertex $b \in B$. A fibrewise homotopy equivalence is a pointwise homotopy equivalence but the converse is not necessary true. Let $\mathbf{R}(B)$ (resp. $\mathbf{L}(B)$) be the full subcategory of $\mathbf{S}/B$ spanned by the right (resp. left) fibrations with target B.

Theorem *A map $f : X \to Y$ in $\mathbf{R}(B)$ (resp. in $\mathbf{L}(B)$) is a fibrewise homotopy equivalence iff the map $f_b : X(b) \to Y(b)$ induced by f is a homotopy equivalence for every vertex $b \in B$.*

The Theorem is proved in 8.28.

Definition 8.2. *We say that a map $u : M \to N$ in $\mathbf{S}/B$ is a* contravariant equivalence *if the map*

$$\pi_0[u, X] : \pi_0[M, X] \to \pi_0[N, X]$$

is bijective for every $X \in \mathbf{R}(B)$. Dually, we say that $u : M \to N$ is a covariant equivalence if the map $\pi_0[u, X]$ is bijective for every $X \in \mathbf{L}(B)$.

A map $u : M \to N$ in $\mathbf{S}/B$ is a contravariant equivalence iff the opposite map $u^o : M^o \to N^o$ is a covariant equivalence in $\mathbf{S}/B^o$

We shall denote by $WR(B)$ the class of contravariant equivalences in $\mathbf{S}/B$ and by $WL(B)$ the class of covariant equivalences.

The class of fibrewise homotopy equivalences in $\mathbf{S}/B$ has the three-for-two property. This is true also of the class of contravariant equivalences and of the class of covariant equivalences The following stronger result is also obvious but useful.

Proposition 8.3. *The class of fibrewise homotopy equivalences in $\mathbf{S}/B$ has the six-for-two property. This is true also of the classes of dexter and covariant equivalences.*

Proposition 8.4. *Every fibrewise homotopy equivalence is a contravariant equivalence and the converse is true for a map in $\mathbf{R}(B)$. Dually, every fibrewise homotopy equivalence is a covariant equivalence and the converse is true for a map in $\mathbf{L}(B)$.*

Proof: Let $u : X \to Y$ be a contravariant equivalence in $\mathbf{R}(B)$. Let us show that u is a fibrewise homotopy equivalence. For this, let us denote by $\mathbf{R}(B)^{\pi_0}$ the full subcategory of $(\mathbf{S}/B)^{\pi_0}$ spanned by the objects of $\mathbf{R}(B)$. The map

$$\mathbf{R}(B)^{\pi_0}(u, Z) : \mathbf{R}(B)^{\pi_0}(Y, Z) \to \mathbf{R}(B)^{\pi_0}(X, Z)$$

is bijective for every object $Z \in \mathbf{R}(B)^{\pi_0}$ by the assumption on u. It follows by Yoneda lemma that u is invertible in the category $\mathbf{R}(B)^{\pi_0}$. It is thus invertible in the category $(\mathbf{S}/B)^{\pi_0}$. This shows that u is a fibrewise homotopy equivalence. ∎

Definition 8.5. *We say that a map in $\mathbf{S}/B$ is a* dexter fibration *if it has the right lifting property with respect to every monic contravariant equivalence. Dually, we say that a map in $\mathbf{S}/B$ is a sinister fibration if it has the right lifting property with respect to every monic covariant equivalence.*

Theorem *The category $\mathbf{S}/B$ admits a simplicial Cisinski structure in which a fibrant object is a right fibration with target B. A weak equivalence is a contravariant equivalence and a fibration is a dexter fibration. Every fibrewise homotopy equivalence is a contravariant equivalence and the converse is true for a map in $\mathbf{R}(B)$. Every dexter fibration is a right fibration and the converse is true for a map in $\mathbf{R}(B)$.*

We denote this model structure by $(\mathbf{S}/B, \mathbf{R}(B))$ and we say that it is the *model structure for right fibrations with target B* or the *contravariant* model structure on $\mathbf{S}/B$.

The theorem is proved in 8.20. Dually,

Theorem *The category $\mathbf{S}/B$ admits a simplicial Cisinski structure in which a fibrant object is a left fibration with target B. A weak equivalence is a covariant equivalence and a fibration is a sinister fibration. Every fibrewise homotopy equivalence is a covariant equivalence and the converse is true for a map in $\mathbf{R}(B)$.*

Every sinister fibration is a left fibration and the converse is true for a map in $\mathbf{L}(B)$.

We denote this model structure by $(\mathbf{S}/B, \mathbf{L}(B))$ and we say that it is the *model structure for left fibrations with target* B or the *covariant model structure* on $\mathbf{S}/B$.

8.2 The contravariant model structure

Proposition 8.6. *If* $X \in \mathbf{R}(B)$, *then* $X^{[A]} \in \mathbf{R}(B)$ *for any simplicial set* A.

Proof: By construction, we have a pullback square

$$\begin{array}{ccc} X^{[A]} & \longrightarrow & X^A \\ \downarrow & & \downarrow{\scriptstyle p^A} \\ B & \xrightarrow{q} & B^A, \end{array}$$

where p is the structure map $X \to B$. But the map p^A is a right fibration by Theorem 2.18, since p is a right fibration. Hence also the map $X^{[A]} \to B$ by base change. ■

Proposition 8.7. *If a map* $v : M \to N$ *in* $\mathbf{S}/B$ *is a dexter equivalence, then the map*

$$[v, X] : [M, X] \to [N, X]$$

is a homotopy equivalence for every $X \in \mathbf{R}(B)$.

Proof: For any simplicial set A we have $X^{[A]} \in \mathbf{R}(B)$ by 8.6. Hence the map $\pi_0[v, X^{[A]}]$ is bijective, since v is a contravariant equivalence by assumption. But the map $[v, X^{[A]}]$ is isomorphic to the map $[v, X]^A$ by the properties of the cotensor product. Hence the map $\pi_0(A, [v, X]) = \pi_0[v, X]^A$ is bijective for every simplicial set A. It follows by Yoneda Lemma that the map $[v, X]$ is invertible in the category $\mathbf{S}^{\pi_0}$. It is thus a homotopy equivalence. ■

Lemma 8.8. *A trivial fibration in* $\mathbf{S}/B$ *is a fibrewise homotopy equivalence. Hence it is a contravariant equivalence.*

Proof: Similar to the proof of 1.22 .

Proposition 8.9. *If a dexter fibration in* $\mathbf{S}/B$ *is a contravariant equivalence, then it is trivial fibration.*

Proof: Let $f : X \to Y$ be a dexter fibration which is a a contravariant equivalence. Let us show that f is a trivial fibration. By Theorem D.1.12 in the appendix, there exists a factorisation $f = qi : X \to P \to Y$ with i a monomorphism and q a trivial fibration. The map i is a dexter equivalence by three-for-two, since q is a contravariant equivalence by Lemma 8.8. It follows that the square

$$\begin{array}{ccc} X & \xrightarrow{1_X} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ P & \xrightarrow{q} & Y \end{array}$$

has a diagonal filler $r : P \to X$. The relations $ri = 1_X$, $fr = q$ and $qi = f$ show that the map f is a retract of the map q. Therefore f is a trivial fibration, since q is a trivial fibration. ∎

If $u : S \to T$ is a map in $\mathbf{S}$ and $v : M \to N$ is a map in $\mathbf{S}/B$, we shall denote by $u \times' v$ the map

$$(S \times N) \sqcup_{S\times M} (T \times M) \to T \times N$$

in $\mathbf{S}/B$ obtained from the commutative square

$$\begin{array}{ccc} S \times M & \longrightarrow & T \times M \\ \downarrow & & \downarrow \\ S \times N & \longrightarrow & T \times N. \end{array}$$

If $v : M \to N$ and $f : X \to Y$ is a pair of maps in $\mathbf{S}/B$, we shall denote by $[f/v]$ the map

$$[N, X] \to [N, Y] \times_{[M,Y]} [M, X]$$

in $\mathbf{S}/B$ obtained from the commutative square

$$\begin{array}{ccc} [N, X] & \longrightarrow & [M, X] \\ \downarrow & & \downarrow \\ [N, Y] & \longrightarrow & [M, Y]. \end{array}$$

Finally, if $u : S \to T$ is a map in $\mathbf{S}$ and $f : X \to Y$ is a map in $\mathbf{S}/B$, we shall denote by $[u\backslash f]$ the map

$$X^{[T]} \to Y^{[T]} \times_{Y^{[S]}} X^{[S]}$$

in $\mathbf{S}$ obtained from the commutative square

$$\begin{array}{ccc} X^{[T]} & \longrightarrow & X^{[S]} \\ \downarrow & & \downarrow \\ Y^{[T]} & \longrightarrow & Y^{[S]}. \end{array}$$

Lemma 8.10. *If $u : S \to T$ is a map in* $\mathbf{S}$ *and if $v : M \to N$ and $f : X \to Y$ are maps in* $\mathbf{S}/B$*, then*

$$u \pitchfork [f/v] \iff (u \times' v) \pitchfork f \iff v \pitchfork [u\backslash f].$$

Proof: This follows from Proposition D.1.18 in the appendix. ■

We shall say that a map $f : X \to Y$ in $\mathbf{S}/B$ *belongs* to a class of maps in $\mathbf{S}$ if this is true of the map underlying f.

Theorem 8.11. *Let $v : M \to N$ and $f : X \to Y$ be two maps in* $\mathbf{S}/B$ *and let $u : S \to T$ be a map in* $\mathbf{S}$*. Let us suppose that u and v are monic. Then*

- *if f is a trivial fibration, then so are the maps $[u\backslash f]$ and $[f/v]$;*
- *if f is a right fibration, then so are the map $[u\backslash f]$ and $[f/v]$;*
- *if f is a right fibration and u is right anodyne, then $[u\backslash f]$ a trivial fibration;*
- *if f is a right fibration and v is right anodyne, then $[f/v]$ a trivial fibration.*

Proof: If u and v are monic, then the map $u \times' v$ is monic by Proposition 2.15. Moreover, $u \times' v$ is right anodyne if in addition u or v is right anodyne by Theorem 2.17. The result then follows by using Lemma 8.10. See Corollary D.1.20. ■

Corollary 8.12. *A right anodyne map in* $\mathbf{S}/B$ *is a contravariant equivalence.*

Proof: Let $v : M \to N$ be a right anodyne map in $\mathbf{S}/B$. If $X \in \mathbf{R}(B)$ then the map

$$[v, X] : [N, X] \to [M, X]$$

is a trivial fibration by Theorem 8.11 applied the the right fibration $X \to B$. Hence the map $\pi_0[v, X]$ is bijective, since a trivial fibration is a homotopy equivalence. This proves that v is a contravariant equivalence. ■

Proposition 8.13. *If $f : X \to Y$ is a right fibration in* $\mathbf{R}(B)$*, then the map*

$$[f/v] : [N, X] \to [N, Y] \times_{[M,Y]} [M, X]$$

is a Kan fibration between Kan complexes for any monomorphism $v : M \to N$ in $\mathbf{S}/B$*.*

Proof: The map $[f/v]$ is a right fibration by Theorem 8.11. The result will be proved by Corollary 4.31 if we show that the codomain of $[f/v]$ is a Kan complex. Let us first show that the simplicial set $[M, X]$ is a Kan complex. The map $[M, X] \to 1$ is a right fibration by Theorem 8.11 applied to the map $Y \to B$ and to the inclusion $\emptyset \subseteq N$. This shows that $[M, X]$ is a Kan complex by Corollary 4.31. Consider the the pullback square

$$\begin{array}{ccc} [N,Y] \times_{[M,Y]} [M,X] & \xrightarrow{pr_2} & [M,X] \\ \downarrow & & \downarrow \\ [N,Y] & \xrightarrow{[u,Y]} & [M,Y]. \end{array}$$

The map $[u, Y]$ is a right fibration by Theorem 8.11 applied to the map $Y \to B$ and to the monomorphism u. Hence the projection pr_2 is also a right fibration by base change. It follows that the domain of pr_2 is a Kan complex by Corollary 4.31, since $[M, X]$ is a Kan complex. This shows that the codomain of $[f/v]$ is a Kan complex. ■

Proposition 8.14. *Every dexter fibration in* $\mathbf{S}/B$ *is a right fibration and the converse is true for a map in* $\mathbf{R}(B)$.

Proof: Every right anodyne map is a monic contravariant equivalence by Corollary 8.12. It follows that every dexter fibration is a right fibration. Conversely, let us show that a right fibration $f : X \to Y$ in $\mathbf{R}(B)$ is a dexter fibration. We have to show that if $v : M \to N$ is a monic contravariant equivalence in $\mathbf{S}/B$, then we have $v \pitchfork f$. For this it suffices to show that the map

$$[f/v] : [N, X] \to [N, Y] \times_{[M,Y]} [M, X]$$

is a trivial fibration, since a trivial fibration is surjective on 0-cells. But the map $[f/v]$ is a Kan fibration between Kan complexes by Proposition 8.13. Hence it suffices to show that $[f/v]$ is a weak homotopy equivalence. But the horizontal maps of the square

$$\begin{array}{ccc} [N,X] & \longrightarrow & [M,X] \\ \downarrow & & \downarrow \\ [N,Y] & \longrightarrow & [M,Y] \end{array}$$

are Kan fibrations by Proposition 8.13. They are also weak homotopy equivalences by Proposition 8.7, since v is a contravariant equivalence. It follows that $[f/v]$ is a weak homotopy equivalence. It is thus a trivial fibration. This shows that we have $v \pitchfork f$. ■

Corollary 8.15. *Let* $u : A \to B$ *and* $v : B \to C$ *be two monomorphisms of simplicial sets. If* u *and* vu *are right anodyne, then so is* v.

Proof: The maps u and vu are contravariant equivalences in $\mathbf{S}/C$ since a right anodyne map is a contravariant equivalence by Proposition 8.12. Thus, v is a contravariant equivalence in $\mathbf{S}/C$ by three-for-two. Let us choose a factorisation $v = ip : B \to E \to C$ with $i : B \to E$ a right anodyne map and $p : E \to C$ a right fibration. The map i is a contravariant equivalence in $\mathbf{S}/C$, since it is right anodyne. Thus, p is a contravariant equivalence in $\mathbf{S}/C$ by three-for-two. But p is a dexter fibration in $\mathbf{S}/C$ by Proposition 8.14, since it is a right fibration in $\mathbf{R}(C)$. It is thus a trivial fibration by Proposition 8.9. Hence the square

$$\begin{array}{ccc} B & \xrightarrow{i} & E \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle p} \\ C & \xrightarrow{1_C} & C. \end{array}$$

has a diagonal filler $s : C \to E$, since v is monic. This shows that v is a (codomain) retract if i. Thus, v is right anodyne, since i is right anodyne. ■

Let $WR(B)$ be the class of contravariant equivalences in $\mathbf{S}/B$.

Lemma 8.16. *If $\mathcal{C}$ is the class of monomorphisms in $\mathbf{S}/B$ and $RF(B)$ is the class of right fibrations in $\mathbf{R}(B)$, then we have*

$$WR(B) \cap \mathcal{C} = {}^{\pitchfork}\mathcal{F}_0.$$

Hence the class $WR(B) \cap \mathcal{C}$ is saturated.

Proof: It follows from Proposition 8.14 that we have $WR(B) \cap \mathcal{C} \subseteq {}^{\pitchfork}RF(B)$. Conversely, if a map $v : M \to N$ in $\mathbf{S}/B$ has the left lifting property with respect to the maps in $RF(B)$, let us show that it is a monic contravariant equivalence. Let us first choose a factorisation $N \to Y \to B$ of the structure map $N \to B$ as a right anodyne map $j : N \to Y$ followed by a right fibration $Y \to B$. And then choose a factorisation $fi : M \to X \to Y$ of the composite $jv : M \to Y$ as a right anodyne map $i : M \to X$ followed by a right fibration $f : X \to Y$. Then the square

$$\begin{array}{ccc} M & \xrightarrow{i} & X \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle f} \\ N & \xrightarrow{j} & Y \end{array}$$

has a diagonal filler $k : N \to X$ by the assumption on v, since $f \in RF(B)$. Thus v is monic, since $kv = j$ is monic. The maps i and j are contravariant equivalences by Corollary 8.12, since they are mid anodyne. It follows by six-for-two in Proposition 8.3 that v is a contravariant equivalence. ■

If $v : M \to N$ is a monomorphism in $\mathbf{S}/B$ and $u : S \to T$ is a monomorphism in $\mathbf{S}$, then the map $u \times' v$ is monic by Proposition 2.15.

Theorem 8.17. *If $v : M \to N$ is a monic contravariant equivalence in $\mathbf{S}/B$* or *if $u : S \to T$ is monic weak homotopy equivalence, then $u \times' v$ is a monic contravariant equivalence.*

Proof: Let us suppose that v is a monic contravariant equivalence and that u is monic. In this case, let us show that $u \times' v$ is a contravariant equivalence. By Lemma 8.16, it suffices to show that we have $(u \times' v) \pitchfork f$ for every right fibration $f : X \to Y$ in $\mathbf{R}(B)$. But the condition $(u \times' v) \pitchfork f$ is equivalent to the condition $v \pitchfork [u\backslash f]$ by Lemma 8.10. The map $[u\backslash f]$ is a right fibration by Proposition 8.11, Let us show that it is a map in $\mathbf{R}(B)$. For this, it suffices to show that its codomain belongs to $\mathbf{R}(B)$, since it is a right fibration. Consider the pullback square

$$\begin{array}{ccc} X^{[T]} \times_{Y^{[S]}} X^{[S]} & \longrightarrow & X^{[S]} \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle f^{[S]}} \\ Y^{[T]} & \longrightarrow & Y^{[S]}. \end{array}$$

The object $Y^{[T]}$ belongs to $\mathbf{R}(B)$ by Proposition 8.11 applied to the map $Y \to B$ and to the inclusion $\emptyset \subseteq T$. The map $f^{[S]}$ is a right fibration by the same proposition applied to the map $f : X \to Y$ and to the inclusion $\emptyset \subseteq T$. Hence the projection p_1 is a right fibration by base change. It follows that the domain of p_1 belongs to $\mathbf{R}(B)$, since its codomain belongs to $\mathbf{R}(B)$. We have proved that the map $[u\backslash f]$ is a right fibration in $\mathbf{R}(B)$. It is thus a dexter fibration by Proposition 8.14. Hence we have $v \pitchfork [u\backslash g]$, since v is a monic contravariant equivalence by assumption. We have proved that $u \times' v$ is a contravariant equivalence. Let us now suppose that u is a monic weak homotopy equivalence and that v is monic. In this case, let us show that $u \times' v$ is a contravariant equivalence. By Lemma 8.16, it suffices to show that we have $(u \times' v) \pitchfork f$ for every right fibration $f : X \to Y$ in $\mathbf{R}(B)$. But the condition $(u \times' v) \pitchfork f$ is equivalent to the condition $u \pitchfork [f/v]$ by Lemma 8.10. The map $[f/v]$ is a Kan fibration by Proposition 8.13. Hence we have $u \pitchfork [f/v]$, since u is a monic weak homotopy equivalence by assumption. We have proved that $u \times' v$ is a contravariant equivalence. ∎

Lemma 8.18. *There exists a functor $R : (\mathbf{S}/B)^I \to (\mathbf{S}/B)^I$ together with a natural transformation $\rho : Id \to R$ such that:*

- *R preserves directed colimits;*
- *the map $R(u)$ is a right fibration in $\mathbf{R}(B)$ for every map u;*
- *the maps $\rho_0(u)$ and $\rho_1(u)$ are right anodyne for every map u.*

Proof: To every simplex $x : \Delta[n] \to B$ and every horn $h_n^k : \Lambda^k[n] \subset \Delta[n]$ we can associate a map

$$(h_n^k, x) : (\Lambda^k[n], x h_n^k) \to (\Delta[n], x)$$

in $\mathbf{S}/B$. Let us denote by Σ the set of maps (h_n^k, x) with $0 < k \leq n$. The result follows from Corollary D.2.9 in the appendix applied to the category $\mathcal{E} = \mathbf{S}/B$ and to the set Σ. ∎

Proposition 8.19. *If $\mathcal{A}$ is the class of monic contravariant equivalences in $\mathbf{S}/B$ and $\mathcal{B}$ is the class of dexter fibrations, then the pair $(\mathcal{A}, \mathcal{B})$ is a weak factorisation system.*

Proof: We have $\mathcal{A} = WR(B) \cap \mathcal{C}$, where $WR(B)$ is the class of contravariant equivalences in $\mathbf{S}/B$ and $\mathcal{C}$ is the class of monomorphisms. The class $\mathcal{A}$ is saturated by Lemma 8.16. Let us show that it is generated by a set of maps. It suffices to show that the class $\mathcal{A}$ can be defined by an accessible equation by Theorem D.2.16. Let us first show that the class $WR(B)$ can be defined by an acessible equation. We shall use Lemma 8.18. Let $v : M \to N$ be a map in $\mathbf{S}/B$. The horizontal maps in the following square are cofinal equivalences, since a right anodyne map in $\mathbf{S}/B$ is a contravariant equivalence by Corollary 8.12,

$$\begin{array}{ccc} M & \xrightarrow{\rho_1(v)} & R_1(v) \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle R(v)} \\ N & \xrightarrow{\rho_0(v)} & R_0(v). \end{array}$$

It follows by three-for-two that v is a contravariant equivalence iff $R(v)$ is a dexterequivalence. But $R(v)$ is a dexter fibration by Proposition 8.14, since it is a right fibration by Lemma 8.18. Thus, $R(v)$ a contravariant equivalence iff it is a trivial fibration by Proposition 8.9 and Proposition 8.8. But the class of trivial fibrations can be defined by an accessible equations by Proposition D.2.14 in the appendix. The functor R is accessible, since it preserves directed colimits. It follows by composing that the class of contravariant equivalences can be defined by an accessible equation. The class of monomorphisms $\mathcal{C}$ can be defined by an accessible equation by Lemma D.1. Hence also the intersection $WR(B) \cap \mathcal{C}$ by Proposition D.2.13. This proves by Theorem D.2.16 that the saturated class $WR(B) \cap \mathcal{C}$ is generated by a set of maps Σ. Then we have $\Sigma^{\pitchfork} = \mathcal{B}$ since we have $\mathcal{A}^{\pitchfork} = \mathcal{B}$ by definition of $\mathcal{B}$. But the pair $(\overline{\Sigma}, \Sigma^{\pitchfork})$ is a weak factorisation system by Theorem D.2.11. This shows that the pair $(\mathcal{A}, \mathcal{B})$ is a weak factorisation system. ∎

We can now establish the *contravariant model structure* in $\mathbf{S}/B$:

Theorem 8.20. *The category $\mathbf{S}/B$ admits a simplicial Cisinski structure in which a fibrant object is a right fibration with target B. A weak equivalence is a contravariant equivalence and a fibration is a dexter fibration. Every dexter fibration is a right fibration and the converse is true for a map in $\mathbf{R}(B)$.*

Proof: For simplicity, let us denote by $\mathcal{W}$ the class of contravariant equivalences in $\mathbf{S}/B$, by $\mathcal{C}$ the class of monomorphisms and by $\mathcal{F}$ the class of dexter fibrations. Let us show that the triple $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ is a model structure in $\mathbf{S}/B$. The intersection $\mathcal{F} \cap \mathcal{W}$ is the class of trivial fibrations by Proposition 8.8 and Proposition 8.9. This shows that the pair $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ is a weak factorisation system by Theorem D.1.12. The pair $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ is a weak factorisation system by Proposition 8.19. We have proved that the triple $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ is a model structure. The fibrant objects are the right fibrations with target B by Proposition 8.14. Moreover, a map between fibrant objects is a dexter fibration iff it is right fibration by the same lemma. The (tensor) product

$$\times : (\mathbf{S}, \mathbf{Kan}) \times (\mathbf{S}/B, \mathcal{W}) \to (\mathbf{S}/B, \mathcal{W})$$

is a left Quillen functor of two variables by 8.17. Hence the model structure is simplicial. ∎

The classical model structure $(\mathbf{S}, \mathbf{Kan})$ induces a model structure on the category $\mathbf{S}/B$. We denote the induced model structure by $(\mathbf{S}/B, Who)$.

Proposition 8.21. *The model structure* $(\mathbf{S}/B, Who)$ *is a Bousfield localisation of the contravariant model structure* $(\mathbf{S}/B, \mathbf{R}(B))$.

Proof: The two model structrures have the same cofibrations. Let us show that every contravariant equivalence is a weak homotopy equivalence. Let us show that a contravariant equivalence $v : M \to N$ in $\mathbf{S}/B$ is a weak homotopy equivalence. For this, let us choose a factorisation of the structure map $N \to B$ as a right anodyne map $j : N \to Y$ followed by a right fibration $q : Y \to B$, together with a factorisation of the composite $jv : M \to Y$ as a right anodyne map $i : M \to X$ followed by a right fibration $g : X \to Y$. The following square commutes by construction,

$$\begin{array}{ccc} M & \xrightarrow{i} & X \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle g} \\ N & \xrightarrow{j} & Y. \end{array}$$

The horizontal maps of the square are contravariant equivalences by the first part of the proof. It follows by three-for-two that g is a contravariant equivalence. But we have $Y \in \mathbf{R}(B)$, since q is a right fibration, and we have $X \in \mathbf{R}(B)$, since qg is a right fibration. Thus, g is a fibrewise homotopy equivalence by Proposition 8.4. It is thus a homotopy equivalence. But the horizontal maps of the square are weak homotopy equivalences since a right anodyne map is anodyne. It follows by three-for-two that v is a weak homotopy equivalence. ∎

Proposition 8.22. *A map is a contravariant equivalence in* $\mathbf{S}/1 = \mathbf{S}$ *iff it is a weak homotopy equivalence. The model structures* $(\mathbf{S}/1, \mathbf{R}(1))$ *and* $(\mathbf{S}, \mathbf{Kan})$ *coincide.*

Proof: The cofibrations are the same in both model structures. Let us show that the weak equivalences are the same. A simplicial set X is a Kan complex iff the map $X \to 1$ is a right fibration by 4.17. It follows from the definitions, that a map is a weak homotopy equivalence iff it is a contravariant equivalence in $\mathbf{S}/1$. ∎

8.3 Pointwise homotopy equivalences

Recall that a map $f : X \to Y$ in $\mathbf{S}/B$ is called a *pointwise homotopy equivalence* if the map $f_b : X(b) \to Y(b)$ induced by f is a homotopy equivalence for each vertex $b \in B$. We shall prove in 8.28 that a map $f : X \to Y$ in $\mathbf{R}(B)$ (resp. in $\mathbf{L}(B)$) is a fibrewise homotopy equivalence iff it is a pointwise homotopy equivalence.

Recall that a simplicial set X is said to be *contractible* if the map $X \to 1$ is homotopy equivalence. Recall also that X is said to be *weakly contractible* if the map $X \to 1$ is a weak homotopy equivalence. A Kan complex is contractible iff it is weakly contractible. The following result is classical:

Proposition 8.23. *A Kan fibration is a trivial fibration iff its fibers are contractible.*

Proof: If $p : X \to B$ is a trivial fibration then so is the map $X(b) \to 1$ for every vertex $b \in B$ by base change. Hence the map $X(b) \to 1$ is a homotopy equivalence by 1.22. Conversely, if the fibers of a Kan fibration $p : X \to B$ are contractible, let us show that p is a trivial fibration. Let us first consider the case where $B = \Delta[n]$. The map $0 : 1 \to \Delta[n]$ is a weak homotopy equivalence, since $\Delta[n]$ is contractible. Hence the inclusion $X(0) \subseteq X$ is a weak homotopy equivalence, since the base change of a weak homotopy equivalence along a Kan fibration is a weak homotopy equivalence by Theorem 6.1. This shows that X is contractible, since $X(0)$ is contractible by assumption. Thus, p is a weak homotopy equivalence. It is thus a trivial fibration. Let us now consider the general case. Let us first show that the base change of $p : X \to B$ along any simplex $b : \Delta[n] \to B$ is a trivial fibration $b^*(X) \to \Delta[n]$. Every fiber of the map $b^*(X) \to \Delta[n]$ is a fiber of the map $X \to B$ by transitivity of base change. Thus, every fiber of the map $b^*(X) \to \Delta[n]$. is contractible. Hence the map $b^*(X) \to \Delta[n]$ is a trivial fibration, since it is a Kan fibration. This shows by the descent property of trivial fibrations in 2.4 that the map $p : X \to B$ is a trivial fibration. ■

If $X \in \mathbf{S}/B$ and $b \in B_0$, then we have

$$X(b) = [b, X],$$

where b is the map $b : 1 \to B$ with value the vertex $b \in B$. If $f : a \to b$ is an arrow in B, let us put

$$X(f) = [f, X],$$

where the map $f : I \to X$ is representing the arrow f. From the inclusions $i_0 : \{0\} \subset I$ and $i_1 : \{1\} \subset I$, we obtain two projections

$$q_0 = [i_0, X] : X(f) \to X(a) \quad \text{and} \quad q_1 = [i_1, X] : X(f) \to X(b).$$

Lemma 8.24. *If $p : X \to B$ is a right fibration, then the projection $q_1 : X(f) \to X(b)$ is a trivial fibration for every arrow $f : a \to b$ in B.*

Proof: This follows from Theorem 8.11 applied to the structure map $X \to B$ and to the inclusion $i_1 : \{1\} \subset I$, since i_1 is right anodyne. ■

The following result is classical:

Lemma 8.25. *The fibers of a Kan fibration over a connected base are homotopically equivalent.*

Proof: Let $p : X \to B$ be a Kan fibration over a connected simplicial set B. If $f : a \to b$ is an arrow in B, then the two projections

$$X(a) \xleftarrow{q_0} X(f) \xrightarrow{q_1} X(b)$$

are trivial fibrations by Lemma 8.24, since p is both a left and a right fibration. Thus, $X(a)$ is homotopically equivalent to $X(b)$, since a trivial fibration is a homotopy equivalence. The result follows, since B is connected. ∎

Lemma 8.26. *Let $p : X \to B$ be a right fibration with connected fibers. If B is connected then X is connected.*

Proof: Let us first show that if $f : a \to b$ is an arrow in B, then for every pair of vertices $(x, y) \in X(a) \times Y(b)$ there is a path $x \to y$ in X. Here a path is defined to be a sequence of arrows in either directions. To see this, observe that there exist an arrow $g \in X$ with target y such that $p(g) = f$, since p is a right fibration. If x' is the source of g, then there is a path $\gamma : x \to x'$ in $X(a)$, since $X(a)$ is connected. By concatenating γ with g, we obtain a path $x \to y$. Let us now show that X is connected. For every pair of vertices $x, y \in X$, let us construct a path $x \to y$. There is a path $\beta : p(x) \to p(y)$, since B is connected. Let $(b_0, b_1, \cdots, b_n)$ be the sequence of nodes of β. The map $p_0 : X_0 \to B_0$ is surjective since the fibers of p are non-empty. For each $0 \leq i \leq n$, let us choose a vertex $x_i \in X$ such that $p(x_i) = b_i$. There is then a path $\gamma_i : x_i \to x_{i+1}$ in X by what we have proved, since the vertices b_i and b_{i+1} are connected by an arrow in either direction. By concatenating the paths γ_i we obtain a path $x \to y$. ∎

Proposition 8.27. *A left fibration is a trivial fibration iff its fibers are contractible.*

Proof: The necessity is clear. Conversely, if $p : X \to B$ be a left fibation with contractible fibers, let us show that it is a trivial fibration. The fibers of p are Kan complexes by Corollary 4.17, since p is a left fibration. Hence the map $X(b) \to 1$ is a trivial fibration for every vertex $b \in B$, since it is a (weak) homotopy equivalence by assumption. Let us show that every commutative square

$$\begin{array}{ccc} \partial\Delta[n] & \xrightarrow{u} & X \\ \downarrow & \text{(i)} & \downarrow p \\ \Delta[n] & \xrightarrow{v} & B \end{array}$$

has a diagonal filler. This is true if $n = 0$, since the fibers of p are non-empty by the assumption. Let us suppose $n > 0$. By pulling back p over $\Delta[n]$ we can suppose that $Y = \Delta[n]$ and that v the identity map. We shall use Lemma 3.14 where the maps $u : A \to B$, $s : S \to T$, $t : T \to X$ and $f : X \to Y$ are respectively the maps $\partial\Delta[n-1] \subset \Delta[n-1]$, $\emptyset \to 1$, $u(n) : 1 \to X$ and $p : X \to \Delta[n]$. We have $B \star T = \Delta[n-1] \star 1 = \Delta[n]$ and

$$(A \star T) \sqcup_{A \star S} (B \star S) = \big(\partial\Delta[n-1] \star 1\big) \cup \big(\Delta[n-1] \star \emptyset\big) = \partial\Delta[n].$$

Moreover, $X/t = X/u(n)$ and

$$Y/ft \times_{Y/fts} X/ts = \Delta[n]/n \times_{\Delta[n]} X = X.$$

It follows from Lemma 3.14 that the square (i) has a diagonal filler iff the following square

$$\begin{array}{ccc} \partial\Delta[n-1] & \longrightarrow & X/u(n) \\ \downarrow & \text{(ii)} & \downarrow{\scriptstyle q} \\ \Delta[n-1] & \longrightarrow & X. \end{array}$$

has a diagonal filler, where q is the projection. Let us show that q is a trivial fibration. It is a Kan fibration by Theorem 3.19, since p is a left fibration. Let us show that q has contractible fibers. The simplicial set X is connected by Lemma 8.26, since $\Delta[n]$ is connected and p has connected fibers . But the fibers of a Kan fibration with a connected base are homotopically equivalent by Lemma 8.25. We can thus proves that q has contractible fibers by showing that the fiber $F = q^{-1}(u(n))$ is contractible. We have $u(n) \in X(n)$, where $X(n)$ is the fiber at n of $p : X \to \Delta[n]$. Let us show that F is the fiber at $u(n)$ of the projection $X(n)/u(n) \to X(n)$. The pullback square

$$\begin{array}{ccc} X(n) & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ 1 & \xrightarrow{\ n\ } & \Delta[n]. \end{array}$$

is a pullback square in the category $1\backslash\mathbf{S}$, if the simplicial set $X(n)$ is pointed by $u(n) \in X(n)$. The functor $(-)/1 : 1\backslash\mathbf{S} \to \mathbf{S}$ preserves pullbacks, since it is a right adjoint. If we apply the functor to the pullback square above, we obtain a pullback square

$$\begin{array}{ccc} X(n)/u(n) & \longrightarrow & X/u(n) \\ \downarrow & & \downarrow{\scriptstyle q} \\ 1 & \xrightarrow{\ n\ } & \Delta[n], \end{array}$$

since $1/1 = 1$ and $\Delta[n]/n = \Delta[n]$. Consider the diagram

$$\begin{array}{ccccc}
F & \longrightarrow & X(n)/u(n) & \longrightarrow & X/u(n) \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle q} \\
1 & \xrightarrow{u(n)} & X(n) & \longrightarrow & X \\
 & & \downarrow & & \downarrow{\scriptstyle p} \\
 & & 1 & \xrightarrow{n} & \Delta[n],
\end{array}$$

The top square on the right is cartesian by Corollary C.0.28 in the appendix, since its composite with the bottom square is cartesian. It follows that the top square on the left is cartesian by the same lemma. This shows that F is the fiber at $u(n)$ of the projection $X(n)/u(n) \to X(n)$. But this projection is a trivial fibration by Theorem 3.19, since the map $X(n) \to 1$ is a trivial fibration. This shows that F is contractible and hence that the Kan fibration $q : X/u(n) \to X$ has contractible fibers. It is thus a trivial fibration by Proposition 8.23. This shows that the square (ii) has a diagonal filler, and hence that the square (i) has a diagonal filler, ∎

Theorem 8.28. *A map $f : X \to Y$ in $\mathbf{L}(B)$ (resp. in $\mathbf{R}(B)$) is a fibrewise homotopy equivalence iff it is a pointwise homotopy equivalence.*

Proof: The necessity is clear. Conversely, suppose that the map $f_b : X(b) \to Y(b)$ is an homotopy equivalence for every vertex $b \in B$. Let us show that f is a fibrewise homotopy equivalence. Let us first consider the case where f is a left fibration. In this case the map $f_b : X(b) \to Y(b)$ is a left fibration for every vertex $b \in B$ by base change. The simplicial set $Y(b)$ is a Kan complex, since the fibers of a right fibration are Kan complexes by Corollary 4.17. Therefore, f_b is a Kan fibration, since a right fibration whose codomain is a Kan complex is a Kan fibration by Corollary 4.31. It is thus a trivial fibration by Theorem 6.1, since it is a homotopy equivalence by assumption. It follows that f_b has contractible fibers. But every fiber of f is a fiber of a map f_b for some $b \in B_0$. Thus, f has contractible fibers. This shows that f is a trivial fibration by Proposition 8.27. In the general case. let us choose a factorisation $f = pi : X \to E \to Y$ with i a left anodyne map and p a right fibration. The map $i : X \to E$ is a covariant equivalence by Corollary 8.12, since it is left anodyne. It is thus a fibrewise homotopy equivalence, since a covariant equivalence in $\mathbf{L}(B)$ is a fibrewise homotopy equivalence by Proposition 8.4. It this thus a pointwise homotopy equivalence. It follows by three-for-two that p is a pointwise homotopy equivalence, since $f = pi$ is a pointwise homotopy equivalence by assumption. Thus, p is a fibrewise homotopy equivalence by the first part of the proof. This shows that $f = pi$ is a fibrewise homotopy equivalence, since i is a fibrewise homotopy equivalence. ∎

Chapter 9

Minimal fibrations

In this chapter we show that every left fibration over a base has a minimal model which is unique up to isomorphism.

Recall that the category $\mathbf{S}/B$ is enriched over simplicial sets for any simplicial set B. If $X, Y \in \mathbf{S}/B$, we denote by $[X, Y]$ the simplicial object of maps $X \to Y$. We denote by $\mathbf{L}(B)$ the full subcategory of $\mathbf{S}/B$ whose objects are the left fibrations $X \to B$.

Definition 9.1. *If $X = (X, p) \in \mathbf{L}(B)$, we shall say that a simplicial subset $S \subseteq X$ is a* model of X *if the induced map $S \subseteq X \to B$ is a left fibration and the inclusion $S \subseteq X$ is a fibrewise homotopy equivalence. We shall say that X (or p) is* minimal *if it has no proper model.*

The main result of the chapter is the following theorem.

Theorem *Every object $X \in \mathbf{L}(B)$ has a minimal model. Any two minimal models of X are isomorphic.*

The theorem is proved in 9.11 and in 9.13.

If $p : X \to B$, we shall denote by $X(b)$ the fiber of p at a vertex $b \in B$. We have $X(b) = [b, X]$, where b is the object of $\mathbf{S}/B$ defined by the map $b : 1 \to B$. More generally, if $u \in B_n$, we shall often by $\Delta[u]$ the object of $\mathbf{S}/B$ defined by the map $u : \Delta[n] \to B$. We shall denote by $\partial u]$ the map $\partial\Delta[n] \to B$ obtained by restricting the map u to $\partial\Delta[n]$ and by $\partial\Delta[u]$ the object of $\mathbf{S}/B$ defined by the map $\partial u : \partial\Delta[n] \to B$. If $X \in \mathbf{S}/B$ we shall put put

$$X(u) = [\Delta[u], X] \quad \text{and} \quad X(\partial u) = [\partial\Delta[u], X].$$

A vertex $x \in X(\partial u)$ is a map $x : \partial\Delta[n] \to X$ which fits in a commutative square

$$\begin{array}{ccc} \partial\Delta[n] & \xrightarrow{x} & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ \Delta[n] & \xrightarrow{u} & B, \end{array}$$

where p is the structure map of X. From the inclusion $\partial\Delta u \subset \Delta[u]$ we obtain a projection

$$\partial : X(u) \to X(\partial u).$$

We shall denote by $X(x/u)$ the fiber of the map ∂ at a vertex $x \in X(\partial u)$. A vertex of $X(x/u)$ is a diagonal filler of the square above.

Let us consider the case $n = 1$. In this case, u is is an arrow $f : a \to b$ in B. We have $\partial f = (a, b)$ and $X(\partial f) = X(a) \times X(b)$. A vertex $x \in X(\partial f)$ is a pair of vertices $(x_0, x_1) \in X(a) \times X(b)$. A vertex $g \in X(x/f)$ is an arrow $g : x_0 \to x_1$ in X such that $p(g) = f$.

Let us consider the case $n = 0$. In this case, u is a vertex $b \in B$. We have $\partial b = \emptyset$ and $X(\partial b) = 1$. If $\emptyset$ denote the (empty) map $\partial b \to X$, then $X(\emptyset/b) = X(b)$.

Proposition 9.2. *If $X \in \mathbf{L}(B)$, then he projection*

$$\partial : X(u) \to X(\partial u)$$

is a Kan fibration between Kan complexes for any simplex $u : \Delta[n] \to B$. Moreover, the simplicial set $X(x/u)$ is a Kan complex for any map $x : \partial u \to X$.

Proof: The map $\partial : X(u) \to X(\partial u)$ is equal to the map

$$\partial = [i, X] : [\Delta[u], X] \to [\partial\Delta[u], X],$$

where i denotes the inclusion $\partial u \subset u$. But the map $[i, X]$ is a Kan fibration between Kan complexes by Proposition 8.13. Hence its fibers are Kan complexes. ∎

Definition 9.3. *Let $(X, p) \in \mathbf{L}(B)$. We shall say that two simplicies $a, b \in X_n$ are* fibrewise homotopic with fixed boundary *if $pa = pb$, $\partial a = \partial b$ and a is homotopic to b in the simplicial set $X(\partial a/pa) = X(\partial b/b)$.*

We shall write $a \cong b$ to indicate that that two simplicies $a, b \in X_n$ are fibrewise homotopic with fixed boundary.

If $X \in \mathbf{S}/B$, $S \subseteq X$ and $x \in X_n$ we shall often write $\partial x \in S$ to indicate that the map $\partial x : \partial\Delta[n] \to X$ can be factored through the inclusion $S \subseteq X$.

Theorem *A simplicial subset $S \subseteq X$ is a model of an object $X \in \mathbf{L}(B)$ iff for every simplex $a \in X$ such that $\partial a \in S$, there exist a simplex $b \in S$ such that $b \cong a$.*

The theorem will be proved in 9.9.

From a map $f : X \to Y$ in $\mathcal{S}/B$ and a simplex $u : \Delta[n] \to B$ we obtain a commutative square

$$\begin{array}{ccc} X(u) & \longrightarrow & Y(u) \\ {\scriptstyle \partial}\downarrow & & \downarrow{\scriptstyle \partial} \\ X(\partial u) & \longrightarrow & Y(\partial u), \end{array}$$

where the horizontal maps are induced by f. The top map of the square induces a map between the fibers of the vertical maps. If $x : \partial u \to X$, this defines a map

$$f(x/u) : X(x/u) \to Y(fx/u).$$

If $n = 0$ and $u = b \in B_0$, the map $f(\emptyset/b)$ is equal to the map $f_b : X(b) \to Y(fu)$ induced by f between the fibers at b.

If $v : M \to N$ and $f : X \to Y$ is a pair of maps in $\mathbf{S}/B$, we shall denote by $[f/v]$ the map

$$[N, X] \to [N, Y] \times_{[M,Y]} [M, X]$$

in $\mathbf{S}/B$ obtained from the commutative square

$$\begin{array}{ccc} [N, X] & \longrightarrow & [M, X] \\ \downarrow & & \downarrow \\ [N, Y] & \longrightarrow & [M, Y]. \end{array}$$

Proposition 9.4. *Let $f : X \to Y$ be a map in $\mathbf{L}(B)$. If $u : \Delta[n] \to B$ and $x : \partial\Delta[u] \to X$, then the map $f(x/u) : X(x/u) \to Y(fx/u)$*

- *is a Kan fibration if f is a left fibration;*
- *is a homotopy equivalence if f is a fibrewise homotopy equivalence.*

Proof: Let us prove the first statement. Consider the following commutative diagram

$$\begin{array}{ccccc} X(x/u) & \longrightarrow & X(u) & & \\ {\scriptstyle f(x/u)}\downarrow & \text{(a)} & \downarrow{\scriptstyle q} & & \\ Y(fx/u) & \longrightarrow & X(\partial u) \times_{Y(\partial u)} Y(u) & \xrightarrow{p_2} & Y(u) \\ \downarrow & \text{(b)} & {\scriptstyle p_1}\downarrow & \text{(c)} & \downarrow{\scriptstyle \partial} \\ 1 & \xrightarrow{x} & X(\partial u) & \longrightarrow & Y(\partial u), \end{array}$$

where $p_1 q = \partial$ and $p_2 q$ is the map $X(u) \to Y(u)$ induced by f. The square (c) is a pullback by construction. The square $(b{+}c)$ is a pullback by definition of $Y(fx/u)$. Hence the square (b) is a pullback by the concellation property of pullback squares in Corollary C.0.28. The square $(a{+}b)$ is a pullback by definition of $X(x/u)$. Hence the square (a) is a pullback by the concellation property of pullback squares. This shows that $f(x/u)$ is a base change of q. But q is isomorphic to the map

$$[f/i] : [\Delta[u], X] \to [\Delta[u], Y] \times_{[\partial\Delta[u], Y]} [\partial\Delta[u], X],$$

where i denotes the inclusion $\partial\Delta[u] \subset \Delta[u]$. Thus, q is a Kan fibration by Proposition 8.13. This shows that $f(x/u)$ is a Kan fibration. The first statement is proved. Let us prove the second statement. The vertical maps of the following square are Kan fibrations between Kan complexes by 9.2.

$$\begin{array}{ccc} X(u) & \longrightarrow & Y(u) \\ \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} \\ X(\partial u) & \longrightarrow & Y(\partial u) \end{array}$$

The horizontal maps are homotopy equivalences, since f is a fibrewise homotopy equivalence and the covariant model structure is simplicial. It then follows from the Cube lemma F.4.6 that the top map induces a homotopy equivalence between the fibers of the vertical maps. This shows that $f(x/u)$ is a homotopy equivalence. ∎

Definition 9.5. *We shall say that a map $f : X \to Y$ in $\mathbf{L}(B)$ satisfies* condition C *if the map*

$$\pi_0 f(x/u) : \pi_0 X(x/u) \to \pi_0 Y(fx/u)$$

is surjective for every $u : \Delta[n] \to B$ and $x : \partial u \to X$.

Lemma 9.6. *A fibrewise homotopy equivalence in $\mathbf{L}(B)$ satisfies condition C. Let $f : X \to Y$ and $g : Y \to Z$ be two maps in $\mathbf{L}(B)$.*

- *If f and g satisfy condition C then so is gf;*
- *If g is a fibrewise homotopy equivalence and gf satisfies condition C, then f satisfies condition C.*

Proof: Let $f : X \to Y$ be a fibrewise homotopy equivalence in $\mathbf{L}(B)$. Then the map $f(x/u) : X(x/u) \to Y(fx/u)$ is a homotopy equivalence for every $u : \Delta[n] \to B$ and $x : \partial\Delta[u] \to X$ by Proposition 9.4. Hence the map $\pi_0 f(x/u)$ is bijective. This shows that f satisfies condition C. Let us prove the second statement. Let $f : X \to Y$ and $g : Y \to Z$ be two maps in $\mathbf{L}(B)$. If $u : \Delta[n] \to B$ and $x : \partial u \to X$, then $(gf)(x/u) = g(fx/u) \circ f(x/u)$. Thus, $\pi_0(gf)(x/u) = \pi_0 g(fx/u) \circ \pi_0 f(x/u)$. Thus, if f and g satisfy condition C then so does gf. Let us now suppose that gf satisfies condition C. If g is a fibrewise homotopy equivalence, then the map $\pi_0 g(fx/u)$ is bijective. Thus, $\pi_0 f(x/u)$ is surjective since the composite $\pi_0(gf)(x/u) = \pi_0 g(fx/u) \circ \pi_0 f(x/u)$ is surjective by assumption. This shows that f satisfies condition C. ∎

Recall that a fibrewise homotopy between two maps $f, g : X \to Y$ in $\mathbf{S}/B$ is a map $h : I \times X \to Y$ in $\mathbf{S}/B$ such that $h(i_0 \times X) = f$ and $h(i_1 \times X) = g$, where i_0 and i_1 denote respectively the inclusions $\{0\} \subset I$ and $\{1\} \subset I$. Equivalently, it is a map $k : X \to Y^{[I]}$ such that $p_0 k = f$ and $p_1 k = g$, where p_0 and p_1 are the canonical projections $Y^{[I]} \to Y$. If $Y \in \mathbf{L}(B)$, then $Y^{[I]}$ is a path object for Y by Theorem 8.20.

Lemma 9.7. *If two maps in $\mathbf{L}(B)$ are fibrewise homotopic and one of the maps satisfies condition C, then so does the other. If the composite of two maps $f : X \to Y$ and $g : Y \to Z$ in $\mathbf{L}(B)$ satisfies condition C and f is a fibrewise homotopy equivalence, then g satisfies condition C.*

Proof Let us prove the first statement. Let $k : X \to Y^{[I]}$ be a homotopy between two maps $f, g : X \to Y$ in $\mathbf{L}(B)$. If f satisfies condition C, let us show that g satisfies condition C. The projection $p_0 : Y^{[I]} \to Y$ is a fibrewise homotopy equivalence, since $Y^{[I]}$ is a path object for Y. Thus, p_0 satisfies condition C by Lemma 9.6. It follows that h satisfies condition C by the same lemma, since $p_0 h = f$ satisfies condition C by assumption. Hence the composite $p_1 h = g$ satisfies condition C by the same lemma, since the projection p_1 also satisfies condition C. Let us prove the second statement. Suppose that the composite of two maps $f : X \to Y$ and $g : Y \to Z$ in $\mathbf{L}(B)$ satisfies condition C and that f is a fibrewise homotopy equivalence. Let $e : Y \to X$ be a fibrewise homotopy equivalence quasi-inverse to the map $f : X \to Y$. The map e satisfies condition C by Lemma 9.6, since it is a fibrewise homotopy equivalence. Thus, the composite gfe satisfies condition C by the same lemma, since gf satisfies condition C by assumption. But gfe is fibrewise homotopic to g, since fe is fibrewise homotopic to 1_X. Therefore, g is satisfies condition C by the first part. ∎

Theorem 9.8. *A map in $\mathbf{L}(B)$ is a fibrewise homotopy equivalence iff it satisfies condition C.*

Proof: The necessity was proved in 9.6. Conversely, if a map $f : X \to Y$ in $\mathbf{L}(B)$ satisfies condition C, let us show that it is a fibrewise homotopy equivalence. Let us first consider the case where f is a left fibration. In this case, we shall prove that f is a trivial fibration. For this, it suffices to show that every commutative square

$$\begin{array}{ccc} \partial\Delta[n] & \xrightarrow{x} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ \Delta[n] & \xrightarrow{b} & Y \end{array}$$

has a diagonal filler. If q is the structure map $Y \to B$ and $u = qb$, it is equivalent to showing that the map $f(x/u) : X\langle x/u\rangle \to Y\langle fx/u\rangle$ is surjective on 0-cells. The map $\pi_0 f(x/u)$ is surjective by assumption. It follows that $f(x/u)$ is surjective on 0-cells, since it is a Kan fibration by Proposition 9.4. This proves that f is a trivial fibration. It is thus a fibrewise homotopy equivalence by Theorem 8.20. The result is proved in the case where f is a left fibration. In the general case, let us factor f as a left anodyne map $i : X \to P$ followed by a left fibration $q : P \to Y$. The map i is a covariant equivalence by Corollary 8.12. It is thus a fibrewise homotopy equivalence by Proposition 8.12, since it is a map in $\mathbf{L}(B)$. It follows by Lemma 9.7 that q satisfies condition C, since $qi = f$ satisfies condition C by assumption. Thus, q is a fibrewise homotopy equivalence by the first part of the proof. This shows that $f = qi$ is a fibrewise homotopy equivalence. ■

Theorem 9.9. *A simplicial subset $S \subseteq X$ is a model of an object $X \in \mathbf{L}(B)$ iff for every simplex $x \in X$ such that $\partial x \in S$, there exist a simplex $x' \in S$ such that $x' \cong x$.*

Proof: Let $p : X \to B$ is the structure map. ($\Rightarrow$). Let $x \in X$ be a simplex such that $\partial x \in S$. Let us put $u = px$. The simplicial set $X(\partial x/u)$ is a Kan complex by Proposition 9.2. The map

$$\pi_0 S(\partial x/u) \to \pi_0 X(\partial x/u)$$

induced by the inclusion $S \subseteq X$ is surjective by Theorem 9.8, since the inclusion is a fibrewise homotopy equivalence in $\mathbf{L}(B)$ by the assumption on S. Hence there exists an element $x' \in S(\partial x/u)$ homotopic to $x \in X(\partial x/u)$. The implication ($\Rightarrow$) is proved. Let us prove the implication ($\Leftarrow$). Let us first show that the map $pi : S \to B$ is a left fibration, where i is the inclusion $S \subseteq X$. For this, we have to show that if $0 \leq k < n$, then every commutative square

$$\begin{array}{ccc} \Lambda^k[n] & \xrightarrow{a} & S \\ \downarrow & (1) & \downarrow{\scriptstyle pi} \\ \Delta[n] & \xrightarrow{u} & B \end{array}$$

has a diagonal filler. Let us first examine the case $n = 1$, in which case we have $k = 0$. The square

$$\begin{array}{ccc} \Lambda^0[1] & \xrightarrow{ia} & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ \Delta[1] & \xrightarrow{u} & B \end{array}$$

has a diagonal filler $v : \Delta[1] \to X$, since p is a left fibration. We have $v : a \to c$ for some $c \in X$. There exists a vertex $c' \in S$ such that $c \cong c'$ by the assumption on

S. If $p(c) = b$, then there exists an arrow $w : c \to c'$ in the fiber $X(b)$, since $c \cong c'$. The following square commutes,

$$\begin{array}{ccc} \Lambda^1[2] & \xrightarrow{h} & X \\ \downarrow & & \downarrow p \\ \Delta[2] & \xrightarrow{us_1} & B, \end{array}$$

where $h : \Lambda^1[2] \to X$ is the horn $(w, \star, v)$. The square has a diagonal filler $t : \Delta[2] \to X$, since p is a left fibration. Let us put $z = td_1$. Then $z : a \to c'$ and $p(z) = u$,

$$\begin{array}{ccc} a & \xrightarrow{v} & c \\ & {\scriptstyle z}\searrow & \downarrow w \\ & & c'. \end{array}$$

We have $\partial z \in S$, since $a \in S$ and $c' \in S$. There is then an element $d \in S$ such that $d \cong z$ by the assumption on S. We have $d : a \to c'$ and $p(d) = p(z) = u$. Hence the map $d : \Delta[1] \to S$ is a diagonal filler of the square (1). Let us now consider the case $n > 1$. The following square

$$\begin{array}{ccc} \Lambda^k[n] & \xrightarrow{ia} & X \\ \downarrow & & \downarrow p \\ \Delta[n] & \xrightarrow{u} & B \end{array}$$

has a diagonal filler $v : \Delta[n] \to X$, since p is a left fibration. Let us put $c = vd^k$ and $b = ud_k$. We have $\partial c \in S$, since we have $a \in S$ and since ∂c is equal to the restriction of a to the boundary of $\partial_k \Delta[n]$. Hence there exists a simplex $c' \in S$ such that $c' \cong c$ by the assumption on S. We have $pc = pc'$ and $\partial c = \partial c'$, since $c' \cong c$. There is then a unique map $x : \partial\Delta[n] \to X$ such that $x \mid \partial_k\Delta[n] = c'$ and $x \mid \Lambda^k[n] = a$, since c' and a coincide on the intersection $\partial_k\Delta[n] \cap \Lambda^k[n]$. We have $px = \partial u$, since

$$px \mid \Lambda^k[n] = pa = u \mid \Lambda^k[n] = \partial u \mid \Lambda^k[n]$$

and

$$px \mid \partial_k\Delta[n] = pc' = pvd^k = ud^k = \partial u \mid \partial_k\Delta[n].$$

Let us show that the square

$$\begin{array}{ccc} \partial\Delta[n] & \xrightarrow{x} & X \\ \downarrow & (2) & \downarrow p \\ \Delta[n] & \xrightarrow{u} & B \end{array}$$

has a diagonal filler. For this, it suffices to show that the simplicial set $X\langle x/u\rangle$ is non-empty. By definition, it is the fiber of the projection $\partial : X(u) \to X(\partial u)$ at $x \in X(\partial u)$. The projection is a Kan fibration by 9.4. Its fiber at ∂v is non-empty since it contains v. Hence it suffices to show that x is homotopic to ∂v in $X(\partial u)$. The following square of inclusions is a pushout in the category $\mathbf{S}/B$,

$$\begin{array}{ccccccc} \partial\partial_k\Delta[n] & \hookrightarrow & \partial_k\Delta[n] & & & & \\ \downarrow & & \downarrow & & & & \\ \Lambda^k[n] & \longrightarrow & \partial\Delta[n] & \longrightarrow & \Delta[n] & \xrightarrow{u} & B, \end{array}$$

Therefore, the corresponding square of projections is a pullback,

$$\begin{array}{ccc} X(\partial u) & \longrightarrow & X(b) \\ {\scriptstyle q}\downarrow & (3) & \downarrow{\scriptstyle \partial} \\ X(ui) & \longrightarrow & X(\partial b), \end{array}$$

where $b = ud_k$ and where we put $X(ui) = [ui, X]$. We have

$$q(x) = x \mid \Lambda^k[n] = a = \partial v \mid \Lambda^k[n] = q(\partial v).$$

Thus, x and ∂v belongs to the fiber at $a \in X(ui)$ of the map q. But this fiber is isomorphic to the fiber $X(\partial c/b)$ of the map $\partial : X(b) \to X(\partial b)$ at $\partial c \in X(\partial b)$, since the square (3) is a pullback and since $x \mid \partial\partial_k\Delta[n] = \partial c$. Hence it suffices to show that the elements $x \mid \partial_k\Delta[n]$ and $\partial v \mid \partial_k\Delta[n]$ are homotopic in $X(\partial c/b)$. But we have $x \mid \partial_k\Delta[n] = c'$ and $\partial v \mid \partial_k\Delta[n] = c$. The elements c and c' are homotopic in $X(\partial c/b)$ since we have $c' \simeq c$ by assumption. Therefore, the simplicial set $X(x/u)$ is non-empty. We have proved that the square (2) has a diagonal filler $z : \Delta[n] \to X$. Notice that $\partial z = x \in S$, since $c' \in S$ and $a \in S$. Hence there exists an element $d \in S$ such that $d \cong z$ by the assumption on S. We have $pd = pz = u$, since $d \cong z$. We have $di = a$, since $\partial d = \partial z = x$ and $x \mid \Lambda^k[n] = a$. Therefore, the map $d : \Delta[n] \to S$ is a diagonal filler of the square (1). We have proved that pi is a left fibration. It remains to prove that the inclusion $S \subseteq X$ is a fibrewise homotopy equivalence. But this follows from Theorem 9.8. ∎

Lemma 9.10. *Let x and y be two degenerate n-simplicies of a simplicial set X. If $\partial x = \partial y$ then $x = y$.*

Proof: We have $xd_k = yd_k$ for every $k \in [n]$, since we have $\partial x = \partial y$ by assumption. But we have $x = xd_is_i$ for some $i \in [n]$, since x is degenerate. Similarly, we have $y = yd_js_j$ for some $j \in [n]$. If $i = j$ then $x = y$. Otherwise, we can suppose that $i < j$. Then

$$x = xd_is_i = yd_is_i = yd_js_jd_is_i = yd_jd_is_{j-1}s_i = yd_jd_is_is_j.$$

Thus, $x = zs_j$, where $z = yd_jd_is_i$. Hence $xd_j = zs_jd_j = z$ and it follows that $x = xd_js_j = yd_js_j = y$. ∎

Theorem 9.11. *An object $X \in \mathbf{L}(B)$ is minimal iff the implication*

$$a \cong b \Longrightarrow a = b$$

is true for every pair of simplices $a, b \in X$. Every object $X \in \mathbf{L}(B)$ contains a minimal model.

Proof:($\Leftarrow$) If $S \subseteq X$ is a model, let us show that $S = X$. For this, we shall prove by induction on n that we have $S_n = X_n$. If $a \in X_0$, then we have $a \cong b$ for some element $b \in S_0$ by Theorem 9.9. But we have $a = b$ by the assumption on X. Thus, $a \in S_0$. If $n > 0$ and $a \in X_n$ then we have $\partial a \in S$ since $X_{n-1} = S_{n-1}$ by the induction hypothesis. Hence we have $a \cong b$ for some element $b \in S_n$ by 9.9. But we have $a = b$ by the assumption on X. Thus, $a \in S_n$. We have proved that $S_n = X_n$. Thus, $S = X$ and this shows that X is minimal. Let us now show that every object $X \in \mathbf{L}(B)$ contains a minimal model. Let $\mathcal{P}$ be the set of simplicial subsets $A \subseteq X$ such that A contains and at most one representative of each equivalence class of the relation $\cong$ on X. It is obvious that $\mathcal{P}$ is closed under directed union. It thus contains a maximal element S by Zorn lemma. We claim that if $a \in X_n$ and $\partial a \in S$ then there exist $b \in S_n$ such that $b \cong a$. We can suppose that $a \notin S$, since otherwise we can take $b = a$. In this case, let us show that a is non-degenerate. If a is degenerate then we have $a = ad_is_i$ for some $i \in [n]$. But we have $ad_i \in S$, since $\partial a \in S$ by the assumption on a. Thus, $a \in S$ since $a = (ad_i)s_i$ and S is closed under the degeneracy operators. This is a contradiction. Thus, a is non-degenerate. Let S' be the simplicial subset of X generated by S and a. The simplicies of S' not in S are of the form as, for some surjection $s : [m] \to [n]$, since $\partial a \in S$ by assumption. We have $S' \notin \mathcal{P}$, since $S' \neq S$ and S is maximal. Thus, S' contains two simplices $u \neq v$ such that $u \simeq v$. One of these simplices must belongs to $S' \setminus S$, since $S \in \mathcal{P}$. They cannot both belong to S', since $as \cong at$ implies $as = at$ by Lemma 9.10. Hence, there exists a surjection $s : [m] \to [n]$ and an element $b \in S$ such that $as \cong b$. Let $i : [n] \to [m]$ a map such that $si = Id$. If $m > n$ the relation $\partial b = \partial(as)$ implies that we have $bi = asi = a$. This is a contradiction, since $a \notin S$ and $bi \in S$. Thus, $m = n$ and we have $b \simeq a$. This proves the claim made above that if $a \in X_n$ and $\partial a \in S$, then there exist $b \in S_n$ such that $b \simeq a$. It follows that S is a model of X by Theorem 9.9. Let us show that S is minimal. If $a, b \in S$ and $a \cong b$ in S then we have $a \cong b$ in X and hence $a = b$ by definition of S. This shows that S is minimal by the first part of the proof. We have proved that X contains a minimal model $S \subseteq X$. We can now prove the implication ($\Rightarrow$). If X is minimal, then $S = X$. Thus, $a \cong b \Rightarrow a = b$. ∎

Proposition 9.12. *Let $f : X \to Y$ be a fibrewise homotopy equivalence in $\mathbf{L}(B)$. If X is minimal then f is monic. If Y is minimal, then f is a trivial fibration. If X and Y are minimal then f is an isomorphism.*

Proof: If $p : X \to Y$ and $q : Y \to B$ are the structure maps, then $qf = p$. Let us suppose that X is minimal. We shall prove by induction on n that the map $f_n : X_n \to Y_n$ is monic. Let us consider the case $n = 0$. Let $a, b \in X_0$ be two vertices such that $fa = fb$. Then $pa = pb$ since $p = qf$. Let us put $u = pa = pb$. Then $a, b \in X(u)$, the fiber of p at $u \in B_0$. The map $f_u : X(u) \to Y(u)$ is a homotopy equivalence, since f is a fibrewise homotopy equivalence by assumption. Thus, a and b are homotopic in $X(u)$, since $f_u(a) = f_u(b)$. It follows that we have $a \cong b$. Hence we have $a = b$ by Theorem 9.11, since X is minimal by assumption. Let us now suppose $n > 0$. Let $a, b \in X_n$ be two simplicies such that $fa = fb$. We have $f\partial a = \partial fa = \partial fb = f\partial b$. Thus, $\partial a = \partial b$ since the map $Sk^{n-1}f : Sk^{n-1}X \to Sk^{n-1}Y$ is monic by the induction hypothesis. We have $pa = pb$, since $p = qf$. Let us put $u = pa = pb$. Hence we have $a, b \in X\langle x/u\rangle$, where $x = \partial a = \partial b$. The map $f(x/u) : X(x/u) \to Y(fx/u)$ is a homotopy equivalence by 9.4, since f is a fibrewise homotopy equivalence by assumption. Thus a and b are homotopic in $X\langle x/u\rangle$ since we have $fa = fb$. Hence we have $a = b$ by 9.11, since X is minimal by assumption. The first statement is proved. Let us show that f is a trivial fibration if Y is minimal. For this we shall prove that every commutative square

$$\begin{array}{ccc} \partial\Delta[n] & \xrightarrow{x} & X \\ \downarrow & & \downarrow f \\ \Delta[n] & \xrightarrow{y} & Y \end{array}$$

has a diagonal filler. Let us put $u = q(y)$. The map $X\langle x/u\rangle \to Y(fx/u)$ is a homotopy equivalence by 9.4, since f is a fibrewise homotopy equivalence. We have $y \in Y(fx/u)$ and hence there exist $z \in X(x/u)$ such that $f(z)$ is homotopic to y in $Y(fx/u)$. Thus $f(z) \cong y$ and it follows that $f(z) = y$ since Y is minimal. This shows that the map $z : \Delta[n] \to X$ is a diagonal filler of the square. The third statement follows from the first two since a trivial fibration is surjective. ∎

Proposition 9.13. *Let $X \in \mathbf{L}(B)$. If $S \subseteq X$ is a minimal model, then S is a domain retract of X and every fibrewise retraction $X \to S$ is a trivial fibration. Two minimal models $S \subseteq X$ and $T \subseteq X$ are isomorphic.*

Proof: Let $i : S \subseteq X$ be a minimal model of X. The square

$$\begin{array}{ccc} S & \xrightarrow{1_S} & S \\ i\downarrow & & \downarrow pi \\ X & \xrightarrow{p} & B \end{array}$$

has a diagonal filler $r : X \to S$ by theorem 8.20 since pi is a left fibration and i is a monic fibrewise equivalence. Thus, S is a domain retract of X since $pri = p$. Every fibrewise retraction $X \to S$ is a fibrwise equivalence by three-for-two. It is thus a trivial fibration by 9.12 since it is surjective. The first statement of the proposition is proved. Let us prove the second statement. The inclusion $T \subseteq X$ has a fibrewise retraction $r : T \to X$ by the first part of the proof. The retraction is a a fibrewise homotopy equivalence, since it is a trivial fibration. Hence the composite $ri : S \subseteq X \to T$ is a fibrewise homotopy equivalence. It is thus an isomorphism by 9.12 since S and T are minimal. ∎

Proposition 9.14. *Every left fibration $f : X \to B$ admits a factorisation $f = f'p : X \to X' \to B$, whith $p : X \to X'$ a trivial fibration and $f' : X' \to B$ is a minimal left fibration.*

Proof: Let $i : X' \subseteq X$ be a minimal model of the object $(X, f) \in \mathbf{L}(B)$. Then the map $f' = fi : X' \to X$ is a minimal left fibration. There exists a fibrewise retraction $p : X \to X'$ by 9.13 and it is a trivial fibration. We have $f'p = fip = f$ since the retraction is fibrewise. ∎

A map of simplicial sets $f : A \to B$ induces a pair of adjoint functors

$$f_! : \mathbf{S}/A \longleftrightarrow \mathbf{S}/B : f^*.$$

It is easy to verify that the functors $f_!$ and f^* are simplicial and that adjunction $f_! \dashv f^*$ is strong. This means that we have a natural isomorphism of simplicial sets,

$$\theta : [X, f^*Y] \to [f_!X, Y]$$

for $X \in \mathbf{S}/A$ and $Y \in \mathbf{S}/B$. If $q : Y \to B$ is the structure map, then we have pullback square

$$\begin{array}{ccc} f^*(Y) & \xrightarrow{f'} & Y \\ {\scriptstyle q'}\downarrow & & \downarrow{\scriptstyle q} \\ A & \xrightarrow{f} & B \end{array}$$

If $g : X \to f^*Y$ is a map in $\mathbf{S}/A$, then $\theta(g) = f'g : f_!X \to Y$.

Lemma 9.15. *If $Y \in \mathbf{L}(B)$, then we have a canonical isomorphism*

$$\theta : f^*(Y)(x/u) \to Y(\theta(x)/fu)$$

for $u : \Delta[n] \to A$ and $x : \partial\Delta[u] \to f^*(Y)$,

Proof: If $u : \Delta[n] \to A$, then $f_!(\Delta[u]) = \Delta[fu]$ and $f_!(\partial\Delta[u]) = \partial\Delta[fu]$. If $x : \Delta[u] \to f^*Y$, then $\theta(x) = f'x : \Delta[fu] \to Y$. It follows from the naturality of θ

that we have a commutative square of simplicial sets

$$\begin{array}{ccc} [\Delta[u], f^*(Y)] & \xrightarrow{\theta} & [\Delta[fu], Y] \\ \downarrow & & \downarrow \\ [\partial\Delta[u], f^*(Y)] & \xrightarrow{\theta} & [\partial\Delta[fu], Y] \end{array}$$

where the vertical maps are defined from the inclusions $\partial\Delta[u] \subset \Delta[u]$ and $\partial\Delta[fu] \subset \Delta[fu]$. The square can be written as a square

$$\begin{array}{ccc} f^*(Y)(u) & \xrightarrow{\theta} & Y(fu) \\ \partial\downarrow & & \downarrow\partial \\ f^*(Y)(\partial u) & \xrightarrow{\theta} & Y(\partial fu). \end{array}$$

The horizontal maps of the square are isomorphisms. Hence, they induce an isomorphism between the fibers of the vertical maps. The result follows. ■

Lemma 9.16. *Suppose that* $Y \in \mathbf{L}(B)$*. If* $u : \Delta[n] \to A$ *and* $a, b : \Delta[u] \to f^*Y$*, then*

$$a \cong b \quad \Longleftrightarrow \quad \theta(a) \cong \theta(b).$$

Proof: We have $\partial a = \partial b \Leftrightarrow \theta(\partial a) = \theta(\partial b)$ since the adjunction θ induces an isomorphism $f^*(Y)(\partial u) \simeq Y(\partial fu)$. If $\partial a = \partial b = x$, then a is homotopic to b in $f^*(Y)(x/u)$ iff $\theta(a)$ is homotopic to $\theta(b)$ in $X(\theta(x)/fu)$. since θ induces an isomorphism $f^*(Y)(x/u) \simeq Y(\theta(x)/fu)$ by Lemma 9.15. ■

Proposition 9.17. *The base change of a minimal left fibration is minimal. If the base change of a map* $q : Y \to B$ *along a surjection* $A \to B$ *is a minimal left fibration, then* q *is a minimal left fibration.*

Proof: Let $p : X \to A$ be the base change of a map $q : Y \to B$ along a map $f : A \to B$. If q is a minimal left fibration, let us show that p is a minimal left fibration. The map p is a left fibration, since the base change of a left fibration is a left fibration. If $a, b \in X$ and $a \cong b$, then we have $pa = pb$. If $u = pa = pb$, then $u : \Delta[n] \to A$ and $a, b : \Delta[u] \to X$. We have $\theta(a) \cong \theta(b)$ by lemma 9.16 since we have $a \cong b$. Thus, $\theta(a) = \theta(b)$ since $q : Y \to B$ is a minimal left fibration by assumption. Thus, $a = b$ since θ is bijective. This proves that the map $p : X \to A$ is a minimal left fibration by 9.11. Let us prove the second statement. Suppose that $f : A \to B$ is surjective and that p is a minimal left fibration. We shall prove that q is a minimal left fibration. The map q is a left fibration by the descent property of left fibrations 2.4. Let us show that it is minimal. By Proposition 9.14, there exists a factorisation $q = q'w : Y \to Y' \to B$, with $w : Y \to Y'$ a trivial fibration and $q' : Y' \to B$ a minimal fibration. By pulling back this factorisation along f we obtain a factorisation $p = p'v : X \to X' \to A$, with v a trivial fibration and p' a minimal left fibration by the first part of the proof. Hence the map v is an isomorphism by 9.12, since a trivial fibration is a fibrewise homotopy equivalence and p is a minimal left fibration by assumption. The pullback functor $f^* : \mathbf{S}/B \to \mathbf{S}/A$ is conservative, since f is surjective. Hence the map $w : Y \to Y'$ is an isomorphism, since the map $v : X \to X'$ is an isomorphism. This proves that the left fibration p is minimal, since the left fibration p' is minimal. ∎

Chapter 10

Base changes

10.1 Functoriality

If A is a simplicial set, we shall put

$$\mathcal{P}(A) = Ho(\mathbf{S}/A, \mathbf{R}(A)).$$

The category $\mathcal{P}(1)$ is the classical homotopy category $Ho(\mathbf{S}, \mathbf{Kan})$. Dually, for any simplicial set A we shall put

$$\mathcal{Q}(A) = Ho(\mathbf{S}/A, \mathbf{L}(A)).$$

The functor $X \mapsto X^o$ induces an isomorphism of model categories,

$$(\mathbf{S}/A, \mathbf{R}(A)) \simeq (\mathbf{S}/A^o, \mathbf{L}(A^o)),$$

hence also of homotopy categories,

$$\mathcal{P}(A) \simeq \mathcal{Q}(A^o).$$

A map of simplicial sets $u : A \to B$ induces a pair of adjoint functors

$$u_! : \mathbf{S}/B \leftrightarrow \mathbf{S}/A : u^*,$$

where $u_!$ is the composition functor $(X, p) \mapsto (X, up)$ and u^* is the base change functor. Recall that the category $\mathbf{S}/A$ is enriched over $\mathbf{S}$ for any simplicial set A. The functors $u_!$ and u^* are simplicial and the adjunction $u_! \dashv u^*$ is strong. The proof of the following proposition is left to the reader.

Proposition 10.1. *The adjuntion $u_! \dashv u^*$ induces an adjunction*

$$u_! : (\mathbf{S}/A)^{\pi_0} \leftrightarrow (\mathbf{S}/B)^{\pi_0} : u^*$$

Theorem 10.2. *If $u : A \to B$ is a map of simplicial sets, then the pair of adjoint functors*

$$u_! : \mathbf{S}/A \leftrightarrow \mathbf{S}/B : u^*$$

is a Quillen pair with respect to the contravariant model structures on these categories. Moreover, the functor $u_!$ takes a dexter equivalence to a dexter equivalence.

Proof: By Proposition E.2.14 it suffices to show that the functor $u_!$ takes a cofibration to a cofibration and that the functor u^* takes a fibration between fibrant objects to a fibration. It is obvious that $u_!$ preserves cofibrations, since the cofibrations are the monomorphisms. Let us show that the functor u^* takes a fibration between fibrant objects to a fibration. The category of fibrant objects of the model category $(\mathbf{S}/B, \mathbf{R}(B))$ is the category $\mathbf{R}(B)$. A map in $\mathbf{R}(B)$ is a fibration iff it is a right fibration by Theorem 8.20. But the base change of a right fibration is a right fibration. This shows that u^* takes a fibration between fibrant objects to a fibration. The first statement is proved. Let us prove the second statement. Every object of the model category $(\mathbf{S}/B, \mathbf{R}(B))$ is cofibrant. It follows that the functor $u_!$ takes a weak equivalence to a weak equivalence by Lemma E.2.6. ∎

For any map $u : A \to B$, the functor $u_! : \mathbf{S}/A \to \mathbf{S}/B$ induces a functor,

$$\mathcal{P}_!(u) : \mathcal{P}(A) \to \mathcal{P}(B)$$

since it preserves weak equivalences. If $v : B \to C$, then we have $v_! u_! = (vu)_!$. It follows that we have

$$\mathcal{P}_!(vu) = \mathcal{P}_!(v)\mathcal{P}_!(u).$$

We thus obtain a functor

$$\mathcal{P}_! : \mathbf{S} \to \mathbf{CAT}.$$

We shall prove in A that the functor $\mathcal{P}_!$ has the structure of a 2-functor with respect to the 2-category structure on $\mathbf{S}$.

For any map $u : A \to B$, the functor $u_! : \mathbf{S}/A \to \mathbf{S}/B$ is a left Quillen functor with respect to the contravariant model structures by 10.2. The functor $\mathcal{P}_!(u)$ induced by $u_!$ is the left derived functor $Lu_!$. It follows that the functor $\mathcal{P}_!(u)$ has a right adjoint Ru^* which is the right derived functor of the functor u^*. We thus have a pair of adjoint functors

$$\mathcal{P}_!(u) : \mathcal{P}(A) \leftrightarrow \mathcal{P}(B) : \mathcal{P}^*(u),$$

where we put $\mathcal{P}^*(u) = Ru^*$. If $v : B \to C$, then we have a canonical isomorphism

$$\mathcal{P}^*(vu) \simeq \mathcal{P}^*(u)\mathcal{P}^*(v)$$

by uniqueness of adjoints. We thus obtain a pseudo-functor,

$$\mathcal{P}^* : \mathbf{S}^o \to \mathbf{CAT}.$$

Remark: We shall see later that the functor $\mathcal{P}^*(u)$ has a right adjoint $\mathcal{P}_*(u)$ for any map $u : A \to B$.

10.2 2-Functoriality

If A is a simplicial set, then the projection $p : A \times I \to A$ has two canonical sections $i_0, i_1 : A \to A \times I$. If $X \in \mathbf{S}/A$, then $p^*(X) = X \times I$, $i_{0!}(X) = X \times \{0\}$ and $i_{1!}(X) = X \times \{1\}$. From the inclusions $X \times \{0\} \subseteq X \times I$ and $X \times \{1\} \subseteq X \times I$, we obtain two natural maps

$$i_{0!}(X) \to p^*(X) \quad \text{and} \quad i_{1!}(X) \to p^*(X).$$

This defines two natural transformations $i_{0!} \to p^*$ and $i_{1!} \to p^*$.

Lemma 10.3. *The natural map $i_{1!}(X) \to p^*(X)$ is right anodyne for every object $X \in \mathbf{S}/A$.*

Proof: We have to show that the inclusion $X \times \{1\} \subseteq X \times I$ is right anodyne. This follows from Theorem 2.17, since the inclusion $\{1\} \subset I$ is right anodyne. ∎

The Lemma shows that the second map in the following diagram

$$i_{0!}(X) \to p^*(X) \leftarrow i_{1!}(X).$$

is invertible in the homotopy category $\mathcal{P}(A \times I)$ by Corollary 8.12. By composing the first map with the inverse of the second we obtain a natural map

$$\sigma_X : i_{0!}(X) \to i_{1!}(X)$$

in the category $\mathcal{P}(A \times I)$. This defines a natural transformation

$$\sigma : \mathcal{P}_!(i_0) \to \mathcal{P}_!(i_1) : \mathcal{P}(A) \to \mathcal{P}(A \times I).$$

A homotopy $\alpha : f \to g$ between two maps $A \to B$ is a map $h : A \times I \to B$. By composing the derived functor

$$\mathcal{P}_!(h) : \mathcal{P}(A \times I) \to \mathcal{P}(B)$$

with the natural transformation σ above we obtain a natural transformation

$$\mathcal{P}_!(\alpha) = \mathcal{P}_!(h) \circ \sigma : \mathcal{P}_!(f) \to \mathcal{P}_!(g) : \mathcal{P}(A) \to \mathcal{P}(B).$$

Lemma 10.4. *If $u : U \to A$ and $v : B \to V$, then*

$$\mathcal{P}_!(v \circ \alpha) = \mathcal{P}_!(v) \circ \mathcal{P}_!(\alpha) \quad \text{and} \quad \mathcal{P}_!(\alpha \circ u) = \mathcal{P}_!(\alpha) \circ \mathcal{P}_!(u).$$

Proof: The homotopy $v \circ \alpha : vf \to vg$ is defined by the map $vh : A \times I \to V$, since the homotopy $\alpha : f \to g$ is defined by the map $h : A \times I \to B$. Thus,

$$\begin{aligned} \mathcal{P}_!(v \circ \alpha) &= \mathcal{P}_!(vh) \circ \sigma \\ &= \big(\mathcal{P}_!(v)\mathcal{P}_!(h)\big) \circ \sigma \\ &= \mathcal{P}_!(v) \circ \big(\mathcal{P}_!(h) \circ \sigma\big) \\ &= \mathcal{P}_!(v) \circ \mathcal{P}_!(\alpha). \end{aligned}$$

The first formula is proved. Let us prove the second formula. The homotopy $\alpha \circ u : fu \to gu$ is defined by the map $h(u \times I) : U \times I \to B$. Let q be the projection $U \times I \to U$ and let j_0 and j_1 be the canonical sections $U \to U \times I$. For every $X \in \mathbf{S}/U$ we have a commutative diagram of canonical maps

$$\begin{array}{ccccc} i_{0!}(u_!X) & \longrightarrow & p^*(u_!X) & \longleftarrow & i_{1!}(u_!X) \\ \downarrow & & \downarrow & & \downarrow \\ (u \times I)_! j_{0!}(X) & \longrightarrow & (u \times I)_! q^*(X) & \longleftarrow & (u \times I)_! j_{1!}(X) \end{array}$$

in which the vertical maps are isomorphisms. Notice the vertical maps on the extremities are identity maps since $(u \times I)j_0 = i_0 u$ and $(u \times I)j_1 = i_1 u$. It follows that we have a commutative square of maps in $\mathcal{P}(A \times I)$,

$$\begin{array}{ccc} i_{0!}(u_!X) & \xrightarrow{\sigma_{u_!X}} & i_{1!}(u_!X) \\ \downarrow & & \downarrow \\ (u \times I)_! j_{0!}(X) & \xrightarrow{(u \times I)_!(\sigma_X)} & (u \times I)_! j_{1!}(X) \end{array}$$

in which the vertical natural transformations are identities. This shows that we have

$$\mathcal{P}_!(u \times I) \circ \sigma = \sigma \circ \mathcal{P}_!(u).$$

Thus,

$$\begin{aligned} \mathcal{P}_!(\alpha \circ u) &= \mathcal{P}_!(h(u \times I)) \circ \sigma \\ &= \big(\mathcal{P}_!(h)\mathcal{P}_!(u \times I)\big) \circ \sigma \\ &= \mathcal{P}_!(h) \circ \big(\mathcal{P}_!(u \times I) \circ \sigma\big) \\ &= \mathcal{P}_!(h) \circ \big(\sigma \circ \mathcal{P}_!(u)\big) \\ &= \big(\mathcal{P}_!(h) \circ \sigma\big) \circ \mathcal{P}_!(u) \\ &= \mathcal{P}_!(\alpha) \circ \mathcal{P}_!(u). \end{aligned}$$

■

We say that a simplicial subset $i : A \subseteq B$ is *reflexive* if there exists a retraction $r : B \to A$ together with a homotopy $\alpha : 1_B \to ir$ such that $\alpha \circ i = 1_i$; we say that the homotopy α is *reflecting* B into A. Dually, we say that $i : A \subseteq B$ is *coreflexive* if there exists a retraction $r : B \to A$ together with a homotopy $\alpha : ir \to 1_B$ such that $\alpha \circ i = 1_i$; we say that the homotopy α is *coreflecting* B into A.

Lemma 10.5. *If a simplicial subset $A \subseteq B$ is coreflexive, then the inclusion $i : A \subseteq B$ is left anodyne.*

Proof: Let $r : B \to A$ be a retraction and $\alpha : ir \to 1_B$ be a coreflecting homotopy. We then have a commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{i_1} & (A \times I) \cup (B \times 0) & \longrightarrow & A \\ {\scriptstyle i}\downarrow & & \downarrow & & \downarrow{\scriptstyle i} \\ B & \xrightarrow{i_1} & B \times I & \xrightarrow{\alpha} & B \end{array}$$

where $i_1(x) = (x, 1)$. The diagram shows that the map i is a retract of the middle map of the diagram. But the middle map is left anodyne by Theorem 2.17, since the inclusion $\{0\} \subset I$ is left anodyne. This proves that i is left anodyne. ■

The barycentric subdivision of the poset $[n]$ is defined to be the poset $B[n]$ of non-empty subsets of $[n]$. If $S \in B[n]$, let us denote by $\Delta[S]$ the full simplicial subset of $\Delta[n]$ spanned by the elements of S. Consider the map $\mu : B[n] \to [n]$ which associates to $S \in B[n]$ its maximum element $\mu(S)$.

Lemma 10.6. *If $S \in B[n]$ and $\mu(S) = n$, then the inclusion $\Delta[S] \subseteq \Delta[n]$ is right anodyne.*

Proof The inclusion $S \subseteq [n]$ admits a left adjoint $r : [n] \to S$ since $n \in S$. It then follows from Lemma 10.5 that the inclusion $\Delta[S] \subseteq \Delta[n]$ is right anodyne. ■

If A is a simplicial set, we define a functor

$$q_S : \mathbf{S}/A \to \mathbf{S}/(A \times \Delta[n])$$

for each $S \in B[n]$ by putting $q_S(X) = X \times \Delta[S]$ for every $X \in \mathbf{S}/A$. If $S \subseteq T$, then $X \times \Delta[S] \subseteq X \times \Delta[T]$. This defines a natural transformation $q_S \to q_T$.

Lemma 10.7. *If $S, T \in B[n]$, $S \subseteq T$ and $\mu(S) = \mu(T)$, then the map $q_S(X) \to q_S(X)$ is right anodyne for every $X \in \mathbf{S}/A$.*

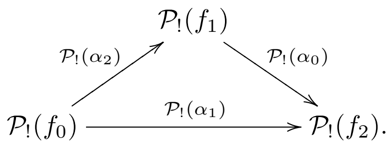
Proof: The inclusion $\Delta[S] \subseteq \Delta[T]$ is right anodyne by Lemma 10.6. Hence also the inclusion $X \times \Delta[S] \subseteq X \times \Delta[T]$ by Theorem 2.17. ■

A 2-simplex in a simplicial set B^A is a map $z : A \times \Delta[2] \to B$. Let us put $f_k = z(A \times k) : A \to B$ and $\alpha_k = z(A \times d_k) : A \times I \to B$ for every $k \in [2]$. This defines a triangle of homotopies in the simplicial set B^A.

$$\begin{array}{ccccc} & & f_1 & & \\ & {\scriptstyle \alpha_2}\nearrow & & \searrow{\scriptstyle \alpha_0} & \\ f_0 & & \xrightarrow{\alpha_1} & & f_2 \end{array}$$

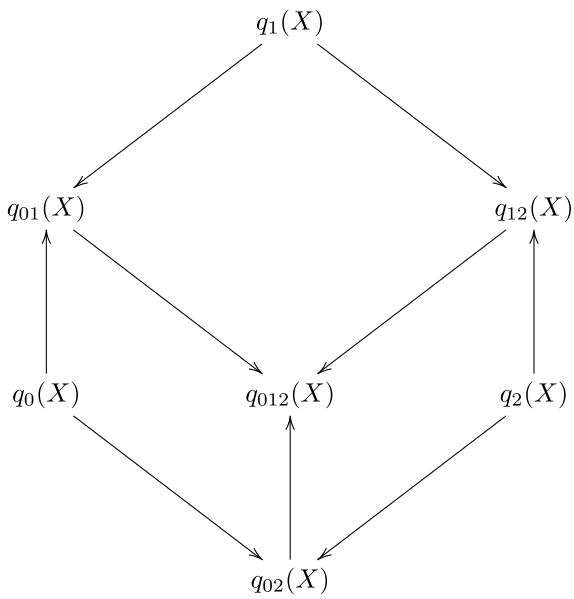
The triangle commutes in the category $\tau_1(B^A)$.

Lemma 10.8. *The triangle of natural transformations*

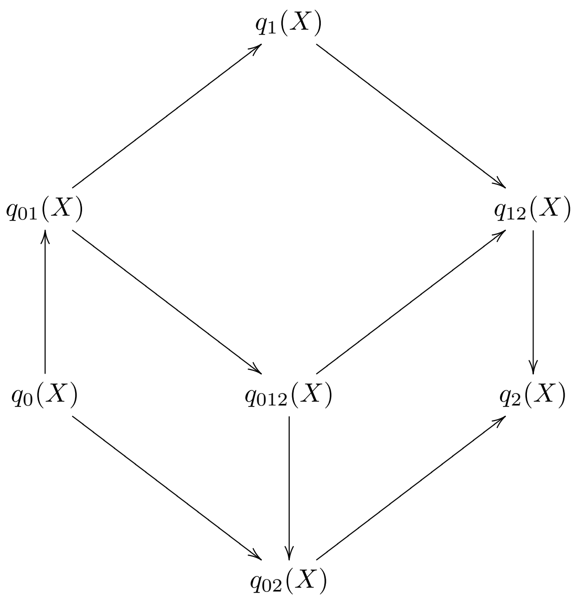


commutes

Proof: Notice that we have a natural isomorphism $(i_k)_!(X) \simeq q_k(X)$ for every $k \in [2]$ where i_k is the canonical sections $A \times k : A \to A \times \Delta[2]$. For every $X \in \mathbf{S}/A$, we have a diagram of inclusions.



If $S \subseteq T$ and $\mu(S) = \mu(T)$, then the map $q_S(X) \to q_S(Y)$ invertible in the category $\mathcal{P}(A \times \Delta[2])$ by Lemma 10.7. By inverting these maps we obtain the following commutative diagram in the category $\mathcal{P}(A \times \Delta[2])$,



We thus obtain a commutative triangle in the category $\mathcal{P}(A \times \Delta[2])$,

$$\begin{array}{ccc} & q_1(X) & \\ \nearrow & & \searrow \\ q_0(X) & \longrightarrow & q_2(X). \end{array}$$

It shows that the following triangle commutes in the category $\mathcal{P}(A \times \Delta[2])$,

$$\begin{array}{ccc} & i_{1!}(X) & \\ {\scriptstyle \sigma_2}\nearrow & & \searrow{\scriptstyle \sigma_0} \\ i_{0!}(X) & \xrightarrow{\sigma_1} & i_{2!}(X) \end{array}$$

if we use the natural isomorphism $(i_k)_!(X) \simeq q_k(X)$. Hence the following triangle of natural transformations commutes

$$\begin{array}{ccc} & \mathcal{P}_!(i_1) & \\ {\scriptstyle \sigma_2}\nearrow & & \searrow{\scriptstyle \sigma_0} \\ \mathcal{P}_!(i_0) & \xrightarrow{\sigma_1} & \mathcal{P}_!(i_2). \end{array}$$

By composing it with the functor $\mathcal{P}_!(z) : \mathcal{P}(A \times \Delta[2]) \to \mathcal{P}(A)$, we obtain the following triangle

$$\begin{array}{ccc} & \mathcal{P}_!(f_1) & \\ {}^{\mathcal{P}_!(\alpha_2)}\nearrow & & \searrow^{\mathcal{P}_!(\alpha_0)} \\ \mathcal{P}_!(f_0) & \xrightarrow{\mathcal{P}_!(\alpha_1)} & \mathcal{P}_!(f_2). \end{array}$$

The lemma is proved. ■

Recall from Chapter 2 that the category $\mathbf{S}$ has the structure of a 2-category $\mathbf{S}^{\tau_1}$ if we put $\mathbf{S}^{\tau_1}(A, B) = \tau_1(A, B) = \tau_1(B^A)$.

Theorem 10.9. *The assignement $\alpha \mapsto \mathcal{P}_!(\alpha)$ gives the functor*

$$\mathcal{P}_! : \mathbf{S} \to \mathbf{CAT}$$

the structure of a 2-functor with respect to the 2-category structure on the category $\mathbf{S}$.

Proof: If A and B are simplicial sets, let us denote by $F(A, B)$ the category freely generated by the 1-skeleton of the simplicial set B^A. By Proposition B.0.14 the category $\tau_1(A, B)$ is the quotient of the category $F(A, B)$ by the congruence relation generated by the relations $(zd_0)(zd_2) \equiv zd_1$, one for each 2-simplex $z \in B^A$. It then follows from Lemma 10.8 that the assignement $\alpha \mapsto \mathcal{P}_!(\alpha)$ induces a functor

$$\mathcal{P}_! : \tau_1(A, B) \to \mathbf{CAT}(\mathcal{P}(A), \mathcal{P}(B)).$$

It remains to show that if $u : U \to A$ and $v : B \to V$, then we have

$$\mathcal{P}_!(v \circ \alpha) = \partial^c v_! \circ \mathcal{P}_!(\alpha) \quad \text{and} \quad \mathcal{P}_!(\alpha \circ u) = \mathcal{P}_!\alpha) \circ \partial^c u_!$$

for every arrow $\alpha : f \to g$ in $\tau_1(A, B)$. Let us prove the first equality. Each side of the equality is the value of a functor defined on the category $\tau_1(A, B)$. Hence the equality can be proved by showing that it holds for a generating set of arrows of this category. But this follows from Lemma 10.4. ■

Corollary 10.10. *The functor $\mathcal{P}_! : \mathbf{S} \to \mathbf{CAT}$ takes a categorical equivalence to an equivalence of categories.*

Corollary 10.11. *The contravariant pseudo-functor $\mathcal{P}^* : \mathbf{S} \to \mathbf{CAT}$ has the structure of a pseudo 2-functor.*

Corollary 10.12. *If $u : A \to B : v$ is a pair of adjoint maps between simplicial sets, then we have*

$$\mathcal{P}_!(u) \dashv \mathcal{P}_!(v) \simeq \mathcal{P}^*(u) \dashv \mathcal{P}^*(v).$$

Proof: We have $\mathcal{P}_!(u) \dashv \mathcal{P}_!(v)$, since a 2-functor takes an adjoint pair to an adjoint pair. Hence we have $\mathcal{P}^*(u) \dashv \mathcal{P}^*(v)$ by adjointness. It follows that we have $\mathcal{P}_!(v) \simeq \mathcal{P}^*(u)$ by uniqueness of adjoints. ■

Corollary 10.13. *The functor*

$${}^o\mathcal{P}_! : \mathbf{S} \to \mathbf{CAT}$$

has the structure of a 2-functor contravariant on 2-cells.

Proof: The functor $(-)^o : \mathbf{S} \to \mathbf{S}$ is reversing the direction of the homotopies in the category $\mathbf{S}$. The corresponding 2-functor $(-)^o : \mathbf{S}^{\tau_1} \to \mathbf{S}^{\tau_1}$ is reversing the direction of the 2-cells. The 2-functor ${}^o\mathcal{P}_!$ is isomorphic to the composite of the 2-functor $(-)^o : \mathbf{S}^{\tau_1} \to \mathbf{S}^{\tau_1}$ followed by the 2-functor $\mathcal{P}_!$. ∎

Proposition 10.14. *If $u : A \to B$ is a left fibration then a map $f : M \to N$ in $\mathbf{S}/A$ is a covariant equivalence iff the map $u_!(f) : u_!M \to u_!N$ is a covariant equivalence in $\mathbf{S}/B$.*

Proof: The implication ($\Rightarrow$) is clear, since the functor $u_!$ takes a covariant equivalence to a covariant equivalence by Proposition 10.2. Conversely, if $u_!(f)$ is a covariant equivalence in $\mathbf{S}/B$, let us show that it is a covariant equivalence in $\mathbf{S}/A$. Let us choose a factorisation of the structure map $N \to A$ as a left anodyne map $j : N \to Y$ followed by a left fibration $Y \to A$ together with a factorisation of the composite $jf : M \to Y$ as a left anodyne map $i : M \to X$ followed by a left fibration $g : X \to Y$. The horizontal maps of the square

$$\begin{array}{ccc} M & \xrightarrow{i} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ N & \xrightarrow{j} & Y \end{array}$$

are covariant equivalences in $\mathbf{S}/A$ by Corollary 8.12. Let us show that g is a covariant equivalence in $\mathbf{S}/A$. If we compose the structure map $Y \to A$ with u, the square becomes a square in $\mathbf{S}/B$. The horizontal maps of the square are covariant equivalences in $\mathbf{S}/B$ by Corollary 8.12. Hence also the map g by three-for-two, since f is a covariant equivalence in $\mathbf{S}/B$ by assumption. But g is a left fibration in $\mathbf{L}(A)$, hence also a left fibration in $\mathbf{L}(B)$, since u is a left fibration. Thus, g is a covariant fibration in $\mathbf{L}(B)$ by Theorem 8.20. It is thus a trivial fibration. Hence it is also a covariant equivalence in $\mathbf{S}/A$ by Theorem 8.20. It follows by three-for-two that f is a covariant equivalence in $\mathbf{S}/A$. ∎

Corollary 10.15. *If $u : A \to B$ is a right fibration then the functor*

$$\mathcal{P}_!(u) : \mathcal{P}(A) \to \mathcal{P}(B)$$

is conservative.

proof: This follows from Proposition 10.14 since a map in a model category is invertible in the homotopy category iff it is a weak equivalence by Proposition E.1.4. ∎

If $X \in \mathbf{S}/B$ and $b : 1 \to B$ then $b^*(X)$ is the fiber $X(b)$ of the structure map $X \to B$ at the vertex b. The category $\mathcal{P}(1)$ is equivalent to the homotopy category of Kan complexes by Proposition 8.22. If $X \in \mathcal{P}(B)$ and $b : 1 \to B$, then $\mathcal{P}^*(b)(X)$ is the *contravariant homotopy fiber* of X at the vertex b. It is the fiber a fibrant replacement of X in the contravariant model structure of $\mathbf{S}/B$.

Theorem*A map $f : X \to Y$ in $\mathcal{P}(B)$ is invertible iff the map*

$$\mathcal{P}^*(b)(f) : \mathcal{P}^*(b)(X) \to \mathcal{P}^*(b)(Y)$$

is invertible in the category $\mathcal{P}(1)$ for every vertex $b \in B$.

The theorem is proved in 10.16

Theorem 10.16. *A map $f : X \to Y$ in $\mathcal{P}(B)$ is invertible iff the map*

$$\mathcal{P}^*(b)(f) : \mathcal{P}^*(b)(X) \to \mathcal{P}^*(b)(Y)$$

is invertible in the category $\mathcal{P}(1)$ for every vertex $b \in B$.

Proof: The necessity is clear. Let us prove the converse. We can suppose that the objects X and Y belongs to $\mathbf{R}(B)$, since $\mathbf{R}(B)$ is the category of fibrant objects of the model category $(\mathbf{S}/B, \mathcal{W}^c)$. In this case the map $\mathcal{P}^*(b)(f)$ is represented by the map $f_b : X(b) \to Y(b)$. The result then follows from Theorem 8.28 since a map in a model category is invertible in the homotopy category iff it is a weak equivalence by Proposition E.1.4. ■

We saw in Corollary 10.11 that the contravariant pseudo functor

$$\mathcal{P}^* : \mathbf{S} \to \mathbf{CAT}$$

has the structure of a pseudo 2-functor. It thus defines a contravariant functor

$$\mathcal{P}^* : \tau_1(A, B) \to \mathbf{CAT}(\mathcal{P}(B), \mathcal{P}(A))$$

for any pair of simplicial sets A and B. If $A = 1$, it defines a contravariant functor

$$\mathcal{P}^* : \tau_1 B \to \mathbf{CAT}(\mathcal{P}(B), \mathcal{P}(1)).$$

By adjointeness, this defines a functor

$$\Phi_B : \mathcal{P}(B) \to [(\tau_1 B)^o, \mathcal{P}(1)].$$

By definition, if $X \in \mathcal{P}(B)$ and $b : 1 \to B$, then $\Phi_B(X)(b) = \mathcal{P}^*(b)(X)$. The contravariant functor

$$\Phi_B(X) : \tau_1 B \to \mathcal{P}(1)$$

is called the *homotopy diagram* of the object X.

Corollary 10.17. *The functor*

$$\Phi_B : \mathcal{P}(B) \to [(\tau_1 B)^o, \mathcal{P}(1)].$$

is conservative

Chapter 11

Proper and smooth maps

If $u : A \to B$ is a map of simplicial sets, then the pullback functor

$$u^* : \mathbf{S}/B \to \mathbf{S}/A$$

has a right adjoint u_*. We introduce the notions of proper and smooth maps. When u is proper, the pair of adjoint functors (u^*, u_*) is a Quillen pair for the contravariant model structures on these categories. It induces a pair of derived functors between the homotopy categories

$$\mathcal{P}^*(u) : \mathcal{P}(B) \leftrightarrow \mathcal{P}(A) : \mathcal{P}_*(u).$$

Dually, when u is smooth, the pair (u^*, u_*) is a Quillen pair for the covariant model structures It induces a pair of derived functors between the homotopy categories

$$\mathcal{Q}^*(u) : \mathcal{Q}(B) \leftrightarrow \mathcal{Q}(A) : \mathcal{Q}_*(u).$$

We show that a left fibration is proper and that a right fibration is smooth.

Definition 11.1. *We say that a map of simplicial sets $u : A \to B$ is* proper *if the functor u^* takes a right anodyne map to a right anodyne map. Dually, we say that u is* smooth *if the functor u^* takes a left anodyne map to a left anodyne map.*

Theorem 11.2. *If $u : A \to B$ is smooth, then the pair of adjoint functors*

$$u^* : \mathbf{S}/B \leftrightarrow \mathbf{S}/A : u_*$$

is a Quillen pair for the covariant model structures on these categories and the functor u^* takes a sinister equivalence to a sinister equivalence. Dually, if u is proper, then the pair (u^*, u_*) is a Quillen pair with respect to the contravariant model structures and the functor u^* takes a dexter equivalence to a dexter equivalence.

Proof: Let us show that the functor u^* takes a sinister equivalence to a sinister equivalence. If $f : M \to N$ is a sinister equivalence in $\mathbf{S}/B$, let us choose a factorisation of the structure map $N \to B$ as a left anodyne map $j : N \to Y$ followed by a left fibration $Y \to B$ together with a factorisation of the composite $jf : M \to Y$ as a left anodyne map $i : M \to X$ followed by a left fibration $g : X \to Y$. The horizontal maps in the square

$$\begin{array}{ccc} M & \xrightarrow{i} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ N & \xrightarrow{j} & Y \end{array}$$

are sinister equivalences by Corollary 8.12. Hence also the map g by three-for-two, since f is a sinister equivalence by assumption. Thus, g is a trivial fibration by Theorem 8.20, since it is a left fibration in $\mathbf{L}(B)$. Thus, $u^*(f)$ is a trivial fibration by base change. It is thus a sinister equivalence by Theorem 8.20. The horizontal maps of the square

$$\begin{array}{ccc} u^*(M) & \xrightarrow{u^*(i)} & u^*(X) \\ {\scriptstyle u^*(f)}\downarrow & & \downarrow{\scriptstyle u^*(g)} \\ u^*(N) & \xrightarrow{u^*(j)} & u^*(Y) \end{array}$$

are left anodyne, since u is smooth by assumption. Hence they are sinister equivalences by Corollary 8.12. It follows by three-for-two that $u^*(f)$ is a sinister equivalence. We have proved that u^* takes a sinister equivalence to a sinister equivalence. It follows that u^* is a left Quillen functor, since u^* preserves monomorphisms.

■

When $u : A \to B$ is proper, the functor $u^* : \mathbf{S}/B \leftrightarrow \mathbf{S}/A$ preserves the dexter equivalences. It thus induces a functor between the homotopy categories

$$Ho(u^*) : Ho(\mathbf{S}/B, \mathcal{W}^c) \to Ho(\mathbf{S}/A, \mathcal{W}^c)$$

It follows that we have $u^{*R} = Ho(u^*) = u^{*L}$. This justifies the following notation:

Notation 11.3. *When $u : A \to B$ is proper, we shall denote by*

$$\mathcal{P}^*(u) : \mathcal{P}(B) \leftrightarrow \mathcal{P}(A) : \mathcal{P}_*(u)$$

the pair of derived functors (u^{*L}, u_*^R) in the contravariant case. When $u : A \to B$ is smooth, we shall denote by

$$\mathcal{Q}^*(u) : \mathcal{Q}(B) \leftrightarrow \mathcal{Q}(A) : \mathcal{Q}_*(u)$$

the pair of derived functors (u^{*L}, u_*^R) in the covariant case.

Remark: We shall see in A that the functor $\mathcal{P}^*(u)$ has a right adjoint $\mathcal{P}_*(u)$ for any map $u : A \to B$.

Proposition 11.4. *The class of smooth (resp. proper) maps is closed under composition and base change.*

Proof: Obviously, a composite of smooth maps is smooth. Let $q : Z \times_Y X \to Z$ be the base change of a smooth map $p : X \to Y$ along a map $Z \to Y$. Let us show that q is smooth. For simplicity, let us put $W = Z \times_Y X$. If a map $u : A \to B$ in $\mathbf{S}/Z$ is left anodyne, let us show that the map $u \times_Z W : A \times_Z W \to B \times_Z W$ is left anodyne. The three squares of the following diagram are cartesian by C.0.28.

$$\begin{array}{ccccccc}
A \times_Z W & \longrightarrow & B \times_Z W & \longrightarrow & W & \longrightarrow & X \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\
A & \xrightarrow{\quad u \quad} & B & \longrightarrow & Z & \longrightarrow & Y.
\end{array}$$

It follows that $u \times_Z W = u \times_Y X$. Hence the map $u \times_Z W$ is the base change of u along p. It is thus left anodyne, since p is smooth by assumption. ■

Corollary 11.5. *A projection $A \times B \to B$ is both proper and smooth.*

Proof: By Proposition 11.4, it suffices to show that the map $p : A \to 1$ is smooth. But if a map $v : S \to T$ is left anodyne, then so is the map $p^*(v) = A \times v : A \times S \to A \times T$ by Theorem 2.17. ■

Suppose that we have a cartesian square of simplicial sets

$$\begin{array}{ccc}
F & \xrightarrow{v} & E \\
{\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\
A & \xrightarrow{u} & B
\end{array}$$

Then the following squares of functors commutes,

$$\begin{array}{ccc}
\mathcal{P}(F) & \xrightarrow{\quad \mathcal{P}_!(v) \quad} & \mathcal{P}(E) \\
{\scriptstyle \mathcal{P}_!(q)}\downarrow & & \downarrow{\scriptstyle \mathcal{P}_!(p)} \\
\mathcal{P}(A) & \xrightarrow{\quad \mathcal{P}_!(u) \quad} & \mathcal{P}(B).
\end{array}$$

From the equality $\mathcal{P}_!(p)\mathcal{P}_!(v) = \mathcal{P}_!(u)\mathcal{P}_!(q)$ we deduce the equality

$$\mathcal{P}^*(p)\mathcal{P}_!(p)\mathcal{P}_!(v)\mathcal{P}^*(q) = \mathcal{P}^*(p)\mathcal{P}_!(u)\mathcal{P}_!(q)\mathcal{P}^*(q).$$

By composing with the unit $id \to \mathcal{P}^*(p)\mathcal{P}_!(p)$ and the counit $\mathcal{P}_!(q)\mathcal{P}^*(q) \to id$, we obtain a canonical natural transformation

$$\mathcal{P}_!(v)\mathcal{P}^*(q) \to \mathcal{P}^*(p)\mathcal{P}_!(u)$$

called the *Beck-Chevalley transformation.* We shall say that the *Beck-Chevalley law holds* if the Beck-Chevalley transformation is invertible.

Proposition 11.6. (Proper base change) *If a map $p : E \to B$ is proper, then the Beck-Chevalley law holds for every cartesian square of simplicial sets,*

$$\begin{array}{ccc} F & \xrightarrow{v} & E \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow{u} & B. \end{array}$$

Hence the following squares of functors commute up to a Beck-Chevalley isomorphism,

$$\begin{array}{ccc} \mathcal{P}(F) & \xrightarrow{\mathcal{P}_!(v)} & \mathcal{P}(E) \\ {\scriptstyle \mathcal{P}^*(q)}\uparrow & & \uparrow{\scriptstyle \mathcal{P}^*(p)} \\ \mathcal{P}(A) & \xrightarrow{\mathcal{P}_!(u)} & \mathcal{P}(B), \end{array} \qquad \begin{array}{ccc} \mathcal{P}(F) & \xleftarrow{\mathcal{P}^*(v)} & \mathcal{P}(E) \\ {\scriptstyle \mathcal{P}_*(q)}\downarrow & & \downarrow{\scriptstyle \mathcal{P}_*(p)} \\ \mathcal{P}(A) & \xleftarrow{\mathcal{P}^*(u)} & \mathcal{P}(B). \end{array}$$

Proof: If $(X, f) \in \mathbf{S}/A$, then we have a diagram of cartesian squares,

$$\begin{array}{ccccc} q^*(X) & \longrightarrow & F & \xrightarrow{v} & E \\ \downarrow & & {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & A & \xrightarrow{u} & B. \end{array}$$

The composite square is cartesian by Corollary C.0.28. Hence the Beck-Chevally map $v_!(q^*(X)) \to p^*(u_!(X))$ is invertible. Recall that we have $\mathcal{P}_!(u) = Ho(u_!)$ and $\mathcal{P}_!(v) = Ho(v_!)$, since functors $u_!$ and $v_!$ preserve dexter equivalences by Theorem 10.2. The map q is proper by Proposition 11.4, since p is proper by assumption. Hence the functors p^* and q^* preserve dexter equivalences by Proposition 11.4. It follows that we have $\mathcal{P}^*(p) = Ho(p^*)$ and $\mathcal{P}^*(q) = Ho(q^*)$. From the isomorphism $v_!q^* \simeq p^*u_!$, we obtain an isomorphism $Ho(v_!)Ho(q^*) \simeq Ho(p^*)Ho(u_!)$. This proves that the Beck-Chevalley map

$$\mathcal{P}_!(v)\mathcal{P}^*(q) \to \mathcal{P}^*(p)\mathcal{P}_!(u)$$

is invertible. It then follows by adjointness that its right transpose

$$\mathcal{P}^*(u)\mathcal{P}_*(p) \to \mathcal{P}_*(q)\mathcal{P}^*(v)$$

is invertible. ∎

Recall that a simplicial subset $i : A \subseteq B$ is said to be *coreflexive* if there exists a retraction $r : B \to A$ together with a homotopy $\alpha : ir \to 1_B$ such that $\alpha \circ i = 1_i$; we say that α is coreflecting B into A.

Lemma 11.7. *The inverse image of a coreflexive simplicial subset $A \subseteq B$ by a right fibration $p : E \to B$ is coreflexive.*

Proof: Let $r : B \to A$ be a retraction of the inclusion $i : A \subseteq B$ and let $\alpha : ir \to 1_B$ be a coreflecting homotopy. If j denotes the inclusion $p^{-1}(A) \subseteq E$, we shall construct a retraction $\rho : E \to p^{-1}(A)$ together with a coreflecting homotopy $\beta : j\rho \to 1_E$. Consider the square

$$\begin{array}{ccc} (p^{-1}(A) \times I) \cup (E \times 1) & \xrightarrow{\quad q \quad} & E \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle p} \\ E \times I & \xrightarrow{\quad \alpha \circ p \quad} & B, \end{array}$$

where u is the inclusion, where q is induced by the projection $pr_1 : E \times I \to E$ and where $\alpha \circ p = \alpha(p \times I)$. The square commutes since the arrow $\alpha(p(x), -) : irp(x) \to p(x)$ is a unit for $x \in p^{-1}(A)$ and since $\alpha(p(x), 1) = p(x)$ for every $x \in E$. The inclusion u is right anodyne by theorem 2.17 since the inclusion $\{1\} \subset I$ is right anodyne. Hence the square has a diagonal filler $\beta : E \times I \to E$, since p is a right fibration by hypothesis. Let us put $k(x) = \beta(x, 0)$ for every $x \in E$. Then we have $pk(x) = \alpha(p(x), 0) = irp(x)$ for every $x \in E$. Thus, $k(E) \subseteq p^{-1}(A)$. If $x \in p^{-1}(A)$, then $k(x) = \beta(x, 0) = q(x, 0) = x$. Hence the map $\rho : E \to p^{-1}(A)$ induced by k is a retraction of the inclusion $j : p^{-1}(A) \subseteq E$. We have $\beta : k = j\rho \to 1_E$, since $\beta(x, 1) = q(x, 1) = x$ for every $x \in E$. Moreover, if $x \in p^{-1}(A)$ and $t \in I$, then $\beta(x, t) = q(x, t) = x$. This shows that $\beta \circ j = 1_j$. Thus, $\beta : j\rho \to 1_X$ is a coreflecting homotopy ∎

For every $0 \leq k \leq n$, we have a natural inclusion $\Delta[k] \subseteq \Delta[n]$. If $0 \leq k < n$, then $\Delta[k] \subseteq \Lambda^k[n]$, since $\Delta[n-1] = \partial_n \Delta[n] \subseteq \Lambda^k[n]$.

Lemma 11.8. *The inclusions*

$$\Delta[k] \subset \Delta[n] \quad \textit{and} \quad \Delta[k] \subseteq \Lambda^k[n]$$

are coreflexive for every $0 \leq k < n$.

Proof: The inclusion $i : [k, n] \subseteq [n]$ has a right adjoint $r : [n] \to [0, k]$ given by $r(x) = x \wedge k$. This defines a retraction $r : \Delta[n] \to D[k, n]$. We have $ir(x) \leq x$ for every $x \in [n]$. This defines a coreflecting homotopy $\alpha : ir \to 1_{\Delta[n]}$. Hence the inclusion $\Delta[k] \subset \Delta[n]$ is coreflexive. In order to show that the inclusion $\Delta[k] subseteq \Lambda^k[n]$ is coeflexive, it suffices to show that α induces a homotopy $\Lambda^k[n] \times I \to \Lambda^k[n]$, or equivalently that we have $\alpha(\Lambda^k[n] \times I) \subseteq \Lambda^k[n]$. For this, we must show that we have $\alpha(\partial_i \Delta[n] \times I) \subseteq \partial_i \Delta[n]$ for every $i \neq k$. But if $x \in [n]$ and $x \neq i$ then then $\alpha(x, 1) = x \neq i$ and $\alpha(x, 0) = x \wedge k \neq i$ since $k \neq i$. The inclusion $\alpha(\partial_i \Delta[n] \times I) \subseteq \partial_i \Delta[n]$ is proved. ∎

Theorem 11.9. *A right fibration is smooth and a left fibration is proper.*

Proof Let $\mathcal{A}$ be the class of left anodyne map whose base change along any right fibration is left anodyne. The class $\mathcal{A}$ is saturated by Lemma D.2.17 in the appendix. We shall prove that every left anodyne map belongs to $\mathcal{A}$. For this it suffices to show that the inclusion $\Lambda^k[n] \subset \Delta[n]$ belongs to $\mathcal{A}$ for every $0 \leq k < n$, since the class $\mathcal{A}$ is saturated. But it follows from Lemma 11.8, Lemma 11.7 and Lemma 10.5 that the inclusions $\Delta[k] \subset \Delta[n]$ and $\Delta[k] \subseteq \Lambda^k[n]$ belong to $\mathcal{A}$ for every $0 \leq k < n$. It then follows from Lemma 8.15 that the inclusion $\Lambda^k[n] \subset \Delta[n]$ belongs to $\mathcal{A}$ for every $0 \leq k < n$. ∎

Remark 11.10. *We shall see later that if $u : A \to B$ is a left (resp right) fibration, then the pair of adjoint functors*

$$u^* : \mathbf{S}/B \leftrightarrow \mathbf{S}/A : u_*$$

is a Quillen pair between the model categories $(\mathbf{S}/B, \mathbf{QCat})$ and $(\mathbf{S}/A, \mathbf{QCat})$.

Chapter 12

Higher quasi-categories

The goal of this chapter is to introduce the notion of n-cellular sets. We shall introduce the model structure for n-quasi-categories at the course.

12.1

We begin by recalling the duality between the category Δ and the category of intervals. An *interval* I is a linearly ordered set with a first and last elements respectively denoted $\perp$ and $\top$, or 0 and 1. If $0 \neq 1$, the interval is *strict*, otherwise it is *degenerate*. A *morphism of intervals* $I \to J$ is an order preserving map $f : I \to J$ such that $f(0) = 0$ and $f(1) = 1$. If I is a strict interval, we shall put $\partial I = \{0, 1\}$ and $int(I) = I \setminus \partial I$. We shall say that a morphism of strict intervals $f : I \to J$ is *proper* if $f(int(I)) \subseteq int(J)$. We shall say that f is a *collapse* if the map $f^{-1}(int(J)) \to int(J)$ induced by f is a bijection. A morphism of strict intervals $f : I \to J$ is proper (resp. a collapse) iff the fiber $f^{-1}(x)$ has cardinality 1 for every $x \in \partial I$ (resp.$x \in int(I)$). Every collapse $f : I \to J$ has a unique section. The category of intervals admits a factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of collapses and $\mathcal{B}$ is the class of proper morphisms.

12.2

We shall denote by $\mathcal{D}^1$ the category of finite strict intervals (it is the category of finite 1-*disks*). The categories $\mathcal{D}^1$ and Δ are mutually dual. The duality functor $(-)^* : \Delta^o \to \mathcal{D}^1$ associates to $[n]$ the set $[n]^* = \Delta([n], [1]) = [n+1]$ equipped with the pointwise ordering. The inverse functor $(\mathcal{D}^1)^o \to \Delta$ associates to $I \in \mathcal{D}^1$ the set $I^* = \mathcal{D}^1(I, [1])$ equipped with the pointwise ordering. A morphism $f : I \to J$ in $\mathcal{D}^1$ is surjective (resp. injective) iff the dual morphism $f^* : J^* \to I$" is injective (resp. surjective). A simplicial set can be defined to be a covariant functor $\mathcal{D}^1 \to \mathbf{Set}$.

12.3

We say that a morphism $u : [m] \to [n]$ in Δ is *free* if it is monic and $u([m]) = [u(0), u(m)]$. A morphism $u : [m] \to [n]$ is free iff the dual morphism in $\mathcal{D}^1$ is a collapse. We shall say that a morphism $u : [m] \to [n]$ in Δ is *boundary preserving* if $u(0) = 0$ and $u(m) = n$. A morphism $u : [m] \to [n]$ is boundary preserving iff the dual morphism in $\mathcal{D}^1$ is proper. The category Δ admits a factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of boundary preserving maps and $\mathcal{B}$ is the class of free morphisms.

12.4

The boundary of the *euclidian n-ball*

$$B^n = \{x \in \mathbf{R}^n :|| x ||\leq 1\}$$

is a sphere ∂B^n of dimension $n-1$ which is the union the *lower* and *upper hemispheres*. In order to describe this structrure, it is convenient to use the map $q : B^n \to B^{n-1}$ which projects B^n on the equatorial $(n-1)$-plane by forgetting the last coordinate of $\mathbf{R}^n$. The fiber $q^{-1}(x)$ is a strict interval for every $x \in B^{n-1}$, except when $x \in \partial B^{n-1}$, in which case it is reduced to a point. The projection q has two sections $s_0, s_1 : B^{n-1} \to B^n$ obtained by selecting the bottom and the top elements in each fiber. The image of s_0 is the lower hemisphere of ∂B^n and the image of s_1 the upper hemisphere; observe that $s_0(x) = s_1(x)$ iff $x \in \partial B^{n-1}$.

12.5

A *bundle of intervals* over a set B is an interval object in the category $\mathbf{Set}/B$. More explicitly, a map $p : E \to B$ is a bundle of intervals if each fiber $E(b) = p^{-1}(b)$ is equipped with an interval structure. By selecting the bottom and the top elements in each fiber we obtain two canonical sections $s_0, s_1 : B \to E$. The interval $E(b)$ is degenerated iff $s_0(b) = s_1(b)$; in which case we shall say that b is in the *singular subset* of the bundle. The projection $q : B^n \to B^{n-1}$ is an example of bundle of intervals. Its singular set is the boundary ∂B^{n-1}. We have a sequence of bundles of intervals:

$$1 \leftarrow B^1 \leftarrow B^2 \leftarrow \cdots B^{n-1} \leftarrow B^n.$$

12.6

A *n-disk* D is defined to be a sequence of length n of bundles of intervals

$$1 = D_0 \leftarrow D_1 \leftarrow D_2 \leftarrow \cdots D_{n-1} \leftarrow D_n$$

such that the singular set of the projection $p : D_{k+1} \to D_k$ is equal to the *boundary* $\partial D_k := s_0(D_{k-1}) \cup s_1(D_{k-1})$ for every $0 \leq k < n$. By convention $\partial D_0 = \emptyset$. If $k = 0$, the condition means that the interval D_1 is strict. It follows from the definition of a n-disk that we have $s_0 s_0 = s_1 s_0$ and $s_0 s_1 = s_1 s_1$. The *interior* of D_k is defined to be $int(D_k) = D_k \backslash \partial D_k$. We have $p(int(D_k)) \subseteq int(D_{k-1})$ for every $1 \leq k \leq n$, where p is the projection $D_k \to D_{k-1}$. The boundary ∂D_n admits a natural decomposition

$$\partial D_n \simeq \bigsqcup_{k=0}^{n-1} 2 \cdot int(D_k).$$

We shall denote by $\mathcal{B}^n$ the n-disk defined by the sequence of projections

$$1 \leftarrow B^1 \leftarrow B^2 \leftarrow \cdots B^{n-1} \leftarrow B^n.$$

12.7

A *morphism* between two bundles of intervals $E \to B$ and $E' \to B'$ is a pair of maps (f, g) in a commutative square

$$\begin{array}{ccc} B & \longleftarrow & E \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ B' & \longleftarrow & E' \end{array}$$

such that the map $E(b) \to E'(f(b))$ induced by g for each $b \in B$. is a morphism of intervals. A *morphism* $f : D \to D'$ between n-disks is defined to be a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longleftarrow & D_1 & \longleftarrow & D_2 & \longleftarrow \cdots & D_{n-1} & \longleftarrow & D_n \\ \downarrow & & {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_{n-1}}\downarrow & & {\scriptstyle f_n}\downarrow \\ 1 & \longleftarrow & D'_1 & \longleftarrow & D'_2 & \longleftarrow \cdots & D'_{n-1} & \longleftarrow & D'_n \end{array}$$

and which each square is a morphisms of bundles of intervals. Every morphism $f : D \to D'$ can be factored as a surjection $D \to f(D)$ followed by an inclusion $f(D) \subseteq D'$.

12.8

A *planar tree* T of *height* $\leq n$, or a *n-tree*, is defined to be a sequence of maps

$$1 = T_0 \leftarrow T_1 \leftarrow T_2 \leftarrow \cdots \leftarrow T_{n-1} \leftarrow T_n$$

with linearly ordered fibers. If D is a n-disk, the projection $D_k \to D_{k-1}$ induces a map $int(D_k) \to int(D_{k-1})$ for each $1 \leq k \leq n$. The sequence of maps

$$1 \leftarrow int(D_1) \leftarrow int(D_2) \leftarrow \cdots int(D_{n-1}) \leftarrow int(D_n)$$

has the structure of a planar tree called the *interior* of D and denoted $int(D)$. Every n-tree T is the interior of a n-disk $\bar{T}$. By construction, we have $\bar{T}_k = T_k \sqcup \partial \bar{T}_k$ for every $1 \leq k \leq n$, where

$$\partial \bar{T}_k = \bigsqcup_{i=0}^{k-1} 2 \cdot T_i.$$

We shall say that $\bar{T}$ is the *closure* of T. Every disk D is the closure of its interior: $\overline{int(D)} = D$. A morphism of disks $f : D \to D'$ is completely determined by its values on the sub-tree $int(D) \subseteq D$. More precisely, a *morphism of planar trees* $g : S \to T$ is defined to be a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longleftarrow & S_1 & \longleftarrow & S_2 & \longleftarrow \cdots & S_{n-1} & \longleftarrow & S_n \\
\downarrow & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle g_{n-1}} & & \downarrow{\scriptstyle g_n} \\
1 & \longleftarrow & T_1 & \longleftarrow & T_2 & \longleftarrow \cdots & T_{n-1} & \longleftarrow & T_n
\end{array}$$

in which f_k preserves the linear order on the fibers of each projections for each $1 \leq k \leq n$. If $Disk(n)$ denotes the category of n-disks and $Tree(n)$ the category of n-trees, then the forgetful functor $Disk(n) \to Tree(n)$ has a left adjoint $T \mapsto \bar{T}$. If $D \in Disk(n)$, then a morphism of trees $T \to D$ can be extended uniquely to a morphism of disks $\bar{T} \to D$. It follows that there a bijection between the morphisms of disks $D \to D'$ and the morphisms of trees $int(D) \to D'$.

12.9

A *sub-tree* of a n-tree T is a sequence of subsets $S_k \subseteq T_k$ closed under the projection $T_k \to T_{k-1}$ for every $1 \leq k \leq n$, where $S_0 = 1$. If D is a n-disk and $T = int(D)$, then the map $C \mapsto C \cap T$ induces a bijection between the sub-disks of D and the sub-trees of T. The set of sub-disks of D is closed under non-empty unions and arbitrary intersections.

12.10

We shall say that a morphism of disks $f : D \to D'$ is *proper* if we have $f(int(D_k)) \subseteq int(D'_k)$ for every $1 \leq k \leq n$. A proper morphism $f : D \to D'$ induces a morphism of trees $int(f) : int(D) \to int(D')$. The functor $T \mapsto \bar{T}$ induces an equivalence between the category of trees $Tree(n)$ and the sub-category of proper morphisms of $Disk(n)$. We shall say that a morphism of disks $f : D \to D'$ is a *collapse* if the map $f^{-1}(int(D')) \to int(D')$ induced by f is an isomorphism. Every collapse $f : D \to D'$ has a unique section. The category $Disk(n)$ admits a factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of collapses and $\mathcal{B}$ is the class of proper morphisms.

12.11

We shall say that a n-disk D is *finite* if the set D_n is finite. The *degree* $\mid D \mid$ of a finite disk D is defined to be the number of edges of the tree $int(D)$. By definition, we have

$$\mid D \mid = \sum_{k=1}^{n} \text{Card}(int(D_k)).$$

It is easy to see that The set

$$D^{\vee} = hom(D, \mathcal{B}^n)$$

has the structure of a compact convex set of dimension $\mid D \mid$. The following neat description of $D^{\vee}$ is due to Clemens Berger. There is an obvious bijection between the elements in the fibers of the planar tree $T = int(D)$ and the edges of T. Let us uses this bijection to transport the order relation on each fiber to the edges. Let $K(T)$ the set of maps $f : edges(T) \to [-1, 1]$ satisfying the following conditions:

- $f(e) \leq f(e')$ for any two edges $e \leq e'$ with the same target;
- $\sum_{e \in C} f(e)^2 \leq 1$ for every maximal chain C connecting the root to a leaf.

It is easy to see that $K(T)$ is a compact convex subset of $[-1, 1]^{|D|}$ of maximal dimension. To every map $f \in K(T)$ we can associate a map of n-disks $f' : D \to \mathcal{B}^n$ by putting

$$f'(x) = (f(e_1), \cdots, f(e_k))$$

for every $x \in int(D)$, where $(e_1, \cdots, e_k)$ is the chain of edges which connects $x \in T_k$ to the root of T. The map $f \mapsto f'$ induces a bijection $K(T) \simeq D^{\vee}$. This shows that $D^{\vee}$ has the structure of a compact convex set of dimension $\mid D \mid$. We observe that the map $f' : D \to \mathcal{B}^n$ is monic iff f belongs to the interior of the ball $D^{\vee}$. It follows that D admits an embedding $D \to \mathcal{B}^n$, since $K(T)$ has a non-empty interior.

12.12

We shall denote by $\mathcal{D}^n$ the category of finite n-disks and by Θ^n the category opposite to $\mathcal{D}^n$. We call an object of Θ^n a *cell* of height $\leq n$. To every disk $D \in \mathcal{D}^n$ corresponds a *dual cell* $D^* \in \Theta^n$ and to every cell $C \in \Theta^n$ corresponds a *dual disk* $C^* \in \mathcal{D}^n$. The *dimension* of cell C is defined to be the degree of C^*. If t is a finite n-tree we shall denote by $[t]$ the cell dual to the disk $\bar{t}$. The dimension of $[t]$ is the number of edges of t. The *realisation* of a cell $[t]$ is defined to be the topological ball

$$R([t]) = K(t) = hom(\bar{t}, \mathcal{B}^n).$$

This defines a functor $R : \Theta^n \to \mathbf{Top}$, where **Top** denotes the category of compactly generated spaces.

12.13

A Θ^n-*set*, or a n-*cellular set* is defined to be a functor

$$X : \mathcal{D}^n \to \mathbf{Set},$$

or equivalently a functor $X : (\Theta^n)^o \to \mathbf{Set}$. We shall denote the category of n-cellular sets by $\hat{\Theta}^n$ If t is a finite n-tree, we shall denote by $\Theta[t]$ the image of the cell $[t]$ by the Yoneda functor $\Theta^n \to \hat{\Theta}^n$. The left Kan extension

$$R_! : \hat{\Theta}^n \to \mathbf{Top}$$

of the realisation functor $R : \Theta^n \to \mathbf{Top}$ preserves finite limits. The topological space $R_!(X)$ is the *geometric realisation* of a cellular set X.

12.14

We say that a map $f : C \to E$ in Θ^n is *surjective* (resp. *injective*) if the dual map $f^* : E^* \to C^*$ is injective (resp. surjective). Every surjection admits a section and every injection admits a retraction. The category Θ^n admits a factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of surjections and $\mathcal{B}$ is the class of injections.

12.15

(Eilenberg-Zilber lemma) If A and B are sub-disks of a disk $D \in \mathcal{D}^n$, then the intersection diagram

$$\begin{array}{ccc} A \cap B & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & D \end{array}$$

is *absolute*, ie it is preserved by any functor with codomain $\mathcal{D}^n$. Dually, for every pair of surjections $f : C \to E$ and $g : C \to F$ in the category Θ^n, there is an absolute pushout square,

$$\begin{array}{ccc} C & \xrightarrow{g} & F \\ {\scriptstyle f}\downarrow & & \downarrow \\ E & \longrightarrow & G, \end{array}$$

where the cell G is dual to the intersection of the disks $A = E^*$ and $B = F^*$. If X is a n-cellular set, we say that a cell $x : \Theta[t] \to X$ of dimension $n > 0$ is *degenerate* if it admits a factorisation $\Theta[t] \to \Theta[s] \to X$ with $\dim([s]) < n$; otherwise x is said to be *non-degenerate*. Every cell $x : \Theta[t] \to X$ admits a unique factorisation $x = yp : \Theta[t] \to \Theta[s] \to X$ with p a surjection and y a non-degenerate cell.

12.16

We shall say that a map $f : C \to E$ in Θ^n is *open* (resp. *is an inflation*) if the dual map $f^* : E^* \to C^*$ is proper (resp. a collapse). Every inflation admits a unique retraction. The category Θ^n admits a factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of open maps and $\mathcal{B}$ is the class of inflations.

12.17

For each $0 \leq k \leq n$, a chain of k edges is a planar n-tree t^k. The inclusion $t^{k-1} \subset t^k$ defines a surjection $[t^k] \to [t^{k-1}]$. The sequence of maps

$$1 \leftarrow \Theta[t^1] \leftarrow \Theta[t^2] \leftarrow \cdots \leftarrow \Theta[t^n]$$

has the structure of a n-disk β^n in the topos $\hat{\Theta}^n$. It is the *generic* n-disk in the sense of classifying topos. Its geometric realisation is the euclidian n-disk $\mathcal{B}^n$.

12.18

The *height* of a n-tree T is defined to be the largest integer $k \geq 0$ such that $T_k \neq \emptyset$. The *height* of a n-disk D is defined to be the height of its interior $int(D)$. If $m < n$, the obvious restriction functor $Disk(n) \to Disk(m)$ has a left adjoint $Ex^n : Disk(m) \to Disk(n)$. The extension functor Ex^n is fully faithful and its essential image is the full subcategory of $Disk(n)$ spanned by the disks of height $\leq n$. We shall identify the category $Disk(m)$ with a full subcategory of $Disk(n)$ by adoptiong the same notation for a disk $D \in Disk(m)$ and its extension $Ex^n(D) \in Disk(n)$. We thus obtain an increasing sequence of coreflexive subcategories,

$$Disk(1) \subset Disk(2) \subset \cdots \subset Disk(n).$$

Hence also an increasing sequence of coreflexive subcategories,

$$\mathcal{D}(1) \subset \mathcal{D}(2) \subset \cdots \subset \mathcal{D}(n).$$

The coreflection functor $\rho^k : \mathcal{D}(n) \to \mathcal{D}(k)$ takes a disk $\overline{T}$ to the sub-disk $\overline{T^k} \subset \overline{T}$, where T^k is the *k-truncation* of T. We shall denote by $\mathcal{D}(\infty)$ the union of the categories $\mathcal{D}(n)$,

$$\mathcal{D}(\infty) = \bigcup_n \mathcal{D}(n)$$

An object of $\mathcal{D}(\infty)$ is an infinite sequence of bundles of finite intervals

$$1 = D_0 \leftarrow D_1 \leftarrow D_2 \leftarrow$$

such that

- the singular set of the projection $D_{n+1} \to D_n$ is the set $\partial D_n := s_0(D_{n-1}) \cup s_1(D_{n-1})$ for every $n \geq 0$;
- the projection $D_{n+1} \to D_n$ is bijective for n large enough.

We have increasing sequence of reflexive subcategories,

$$\Theta(1) \subset \Theta(2) \subset \cdots \subset \Theta(\infty),$$

where $\Theta(k)$ is the full subcategory of $\Theta(\infty)$ spanned by the cells of height $\leq k$. By 12.1, we have $\Theta^1 = \Delta$ A cell $[t]$ belongs to Δ iff the height of t is ≤ 1. If $n \geq 0$ we shall denote by n the unique planar tree height ≤ 1 with n edges. A cell $[t]$ belongs to Δ iff we have $t = n$ for some $n \geq 0$. The reflection functor $\rho^k : \Theta(\infty) \to \Theta(k)$ takes a cell $[t]$ to the cell $[t^k]$, where t^k is the *k-truncation* of t.

Part II

Appendices

Appendix A

Accessible categories

The *cardinality* of a small category is the cardinality of its set of arrows. A *diagram* in a category C is a functor $D : K \to C$, where K is a small category; the *cardinality* of D is the cardinality of K. We denote by $K \star 1$ the category obtained by adjoining a terminal object 1 to K (see Chapter 6). A diagram $K \star 1 \to C$ is called an *inductive cone* in C. A diagram $K \to C$ is said to be *bounded above* if it can be extended to $K \star 1$. A small category C is said to be *directed* if every finite diagram $K \to C$ is bounded above. A small category C is directed iff the colimit functor

$$\varinjlim_{C} : \mathbf{Set}^C \to \mathbf{Set}$$

preserves finites limits by a classical result. A diagram is said to be *directed* if its domain is directed. A colimit in a category is said to be *directed* if it is taken over a directed diagram. We say that a category $\mathcal{E}$ is *closed under directed colimits* if every directed diagram $C \to \mathcal{E}$ has a colimit. If $\mathcal{E}$ is closed under directed colimits, an object $A \in \mathcal{E}$ is said to be *finitely presentable*, if the functor $\mathcal{E}(A, -) : \mathcal{E} \to \mathbf{Set}$ preserves directed colimits. We shall say that $\mathcal{E}$ is *ω-accessible* if its full subcategory of finitely presentable objects is essentially small (ie equivalent to a small category) and every object in $\mathcal{E}$ is a colimit of a directed diagram of finitely presentable objects.

Unless exception, we only consider small ordinals and small cardinals. Recall that an ordinal α is said to be a *cardinal* if it is smallest among the ordinals with the same cardinality. Recall that a cardinal α is said to be *regular* if the sum of every family of cardinals $< \alpha$ indexed by a set of cardinality $< \alpha$ is $< \alpha$. Let α be a regular cardimal, A small category C is said to be *α-directed* if every diagram $K \to C$ of cardinality $< \alpha$ is bounded above. A small category C is α-directed iff the colimit functor

$$\varinjlim_{C} : \mathbf{Set}^C \to \mathbf{Set}$$

preserves the limits of every diagram of cardinality $< \alpha$, by a classical result. A diagram is said to be *α-directed* if its domain is α-directed. A colimit in a category is said to be *α-directed* if it is taken over an α-directed diagram. We say that a category $\mathcal{E}$ is *closed under α-directed colimits* if every α-directed diagram $C \to \mathcal{E}$ has a colimit. If $\mathcal{E}$ is closed under α-directed colimits, we shall say that an object $A \in \mathcal{E}$ is *α-presentable*, if the functor $\mathcal{E}(A, -) : \mathcal{E} \to \mathbf{Set}$ preserves α-directed colimits. We shall say that a category If $\mathcal{E}$ is *α-accessible* if its full subcategory of α-presentable objects is essentially small (ie equivalent to a small category) and every object in $\mathcal{E}$ is the colimit of an α-directed diagram of α-presentable objects.

Notation A.1. *If α and β are two regular cardinals, we shall write $\alpha \lhd \beta$ to indicate that $\alpha < \beta$ and that every α-accessible category is β-accessible.*

Theorem A.0.1. *[MP] For any (small) set S of regular cardinals, there is a regular cardinal β such that $\alpha \lhd \beta$ for all $\alpha \in S$.*

Theorem A.0.2. *[MP] If $F : \mathcal{E} \to \mathcal{E}'$ is an accessible functor, then there exists a regular cardinal α such that*

- *F preserves α-directed colimits*
- *F takes α-presentable objects to α-presentable objects,*

We shall say that a category $\mathcal{E}$ is *closed under ∞-directed colimits* if it closed under α-directed colimits for some regular cardinal α (hence also for every regular cardinal $\beta \geq \alpha$). If $\mathcal{E}$ and $\mathcal{F}$ are closed under ∞-directed colimits we shall say that a functor $F : \mathcal{E} \to \mathcal{F}$ *preserves* ∞-directed colimits if it preserves α-directed colimits for a regular cardinal α large enough. If $\mathcal{E}$ admits ∞-directed colimits, we shall say that an object $A \in \mathcal{E}$ is *small* if the functor $\mathcal{E}(A, -) : \mathcal{E} \to \mathbf{Set}$ preserves ∞-directed colimits. We shall say that a category is *accessible* if it is α-accessible for some regular cardinal α. We shall say that a functor $F : \mathcal{E} \to \mathcal{F}$ is *accessible* if the categories $\mathcal{E}$ and $\mathcal{F}$ are accessible and F preserves ∞-directed colimits.

The following elementary results will be used in the book:

Proposition A.0.3. *[MP] Let $D : C \to \mathcal{E}$ be a diagram of α-presentable objects in a category closed under α-directed colimits. If* $\mathrm{Card}(C) < \alpha$*, then the diagram is α-presentable as an object of the category $\mathcal{E}^C$.*

Proposition A.0.4. *[MP] Let $\mathcal{E}$ be a category closed under α-directed colimits. If an object $A \in \mathcal{E}$ is α-presentable then so is the object (A, f) in the category $\mathcal{E}/B$ for any map $f : A \to B$.*

Appendix B

Simplicial sets

We fix some notations about simplicial sets. We denote the category of finite non-empty ordinals and order preserving maps by Δ and we denote the ordinal $n{+}1 = \{0, \ldots, n\}$ by $[n]$. A map $u : [m] \to [n]$ can be specified by listing its values $(u(0), \ldots, u(m))$. We denote the injection $[n-1] \to [n]$ which omits $i \in [n]$ by d_i and the surjection $[n] \to [n-1]$ which repeats $i \in [n-1]$ by s_i.

A *simplicial set* is a presheaf on the category Δ. We shall denote the category of simplicial sets by $\mathbf{S}$. If X is a simplicial set, it is standard to denote the set $X([n])$ by X_n. We often denote the map $X(d_i) : X_n \to X_{n-1}$ by ∂_i and the map $X(s_i) : X_{n-1} \to X_n$ by σ_i. An element of X_n is called a *n-simplex*; a 0-simplex is called a *vertex* and a 1-simplex an *arrow*. For each $n \geq 0$, the simplicial set $\Delta(-, [n])$ is called the *combinatorial simplex* of dimension n and denoted by $\Delta[n]$. The simplex $\Delta[1]$ is called the *combinatorial interval* and we shall denote it by I.The simplex $\Delta[0]$ is the terminal object of the category $\mathbf{S}$ and we shall denote it by 1. By the Yoneda lemma, for every $X \in \mathbf{S}$ the evaluation map $x \mapsto x(1_{[n]})$ defines a bijection between the maps $\Delta[n] \to X$ and the elements of X_n for each $n \geq 0$; we shall identify these two sets by adopting the same notation for a map $x : \Delta[n] \to X$ and the simplex $x(1_{[n]}) \in X_n$. If $u : [m] \to [n]$ we shall denote the simplex $X(u)(x) \in X_m$ as a composite $xu : \Delta[m] \to X$. If $n > 0$ and $x \in X_n$ the simplex $\partial_i(x) = xd_i : \Delta[n-1] \to X$ is called the *i-th face* of x. If $f \in X_1$ we shall say that the vertex $\partial_1(f) = fd_1$ is the *source* of the arrow f and that the vertex $\partial_0(f) = fd_0$ is its *target*. We shall write $f : a \to b$ to indicate that $a = \partial_1(f)$ and that $b = \partial_0(f)$. If $a \in X_0$, we shall denote the (degenerate) arrow as_0 as a *unit* $1_a : a \to a$.

Let $\tau : \Delta \to \Delta$ be the automorphism of the category Δ which reverses the order of each ordinal. If $u : [m] \to [n]$ is a map in Δ, then $\tau(u)$ is the map $u^o : [m] \to [n]$ given by $u^o(i) = n - f(m-i)$. The *opposite* X^o of a simplicial set X is obtained by composing the (contravariant) functor $X : \Delta \to \mathbf{Set}$ with the functor τ. We distinguish between the simplices of X and X^o by writing $x^o \in X^o$

for each $x \in X$, with the convention that $x^{oo} = x$. If $f : a \to b$ is an arrow in X, then $f^o : b^o \to a^o$ is an arrow in X^o.

A *simplicial subset* of a simplicial set X is a sub-presheaf $A \subseteq X$. If $n > 0$ and $i \in [n]$ the image of the map $d_i : \Delta[n-1] \to \Delta[n]$ is denoted $\partial_i\Delta[n] \subset \Delta[n]$. The *simplicial sphere* $\partial\Delta[n] \subset \Delta[n]$ is the union the faces $\partial_i\Delta[n]$ for $i \in [n]$; by convention $\partial\Delta[0] = \emptyset$. If $n > 0$, we shall say that a map $x : \partial\Delta[n] \to X$ is a *simplicial sphere in* X; such a map is determined by the sequence of its faces $(x_0, \ldots, x_n) = (xd_0, \ldots, xd_n)$. A simplicial sphere $\partial\Delta[2] \to X$ is also called a *triangle*. Every n-simplex $y : \Delta[n] \to X$ has a *boundary* $\partial y = (\partial_0 y, \ldots, \partial_n y) = (yd_0, \ldots, yd_n)$ obtained by restricting y to $\partial\Delta[n]$. If $\partial y = x$ we say that the simplex y is a filler for x. We shall say that a simplicial sphere $x : \partial\Delta[n] \to X$ *commutes* if it can be filled.

If $n > 0$ and $k \in [n]$, the *horn* $\Lambda^k[n] \subset \Delta[n]$ is defined to be the union of the faces $\partial_i\Delta[n]$ with $i \neq k$. A map $x : \Lambda^k[n] \to X$ is called a *horn in* X; it is determined by a lacunary sequence of faces $(x_0, \ldots, x_{k-1}, *, x_{k+1}, \ldots, x_n)$. A *filler* for x is a simplex $\Delta[n] \to X$ which extends x.

Recall that a simplex $x : \Delta[n] \to X$ is said to be *degenerate* if it admits a factorisation $x = yu : \Delta[n] \to \Delta[m] \to X$ with $m < n$. Otherwise, the simplex is said to be *non-degenerate*. By the Eilenberg-Zilber lemma, every simplex $x : \Delta[n] \to X$ admits a unique factorisation $x = yp : \Delta[n] \to \Delta[m] \to X$, with $p : [m] \to [n]$ a surjection and y a non-degenerate simplex. We give a proof below based on the notion of absolute limit. Recall that a projective cone $D : 1 \star C \to \mathcal{E}$ in a category $\mathcal{E}$ is said to be *absolutly exact* if the cone $FD : 1 \star C \to \mathcal{F}$ is exact for any functor $F : \mathcal{E} \to \mathcal{F}$. A limit diagram in a category $\mathcal{E}$ is said to be *absolute* if the exact projective cone which defines the limit is absolutely exact. There is dual notion of *absolute colimit*

Lemma B.0.5. [JT3] *In a Karoubi complete category, suppose that we have four maps $s_1 : A_1 \to B$, $r_1 : B \to A_1$, $s_2 : A_2 \to B$ and $r_2 : B \to A_2$ such that $r_1 s_1 = 1_{A_1}$ and $r_2 s_2 = 1_{A_2}$. Let us put $e_1 = s_1 r_1$ and $e_2 = s_2 r_2$. If there existe an integer $n \geq 1$ such that $(e_1 e_2)^n = e_2(e_1 e_2)^n$, then the pull back $A_1 \times_B A_2$ exists and is absolute.*

Proof: Let us put $e = (e_1 e_2)^n$. Then we have $e_1 e = e$ and $e_2 e = e$. It follows that $ee = e$. Let us choose a splitting $r : B \to E$ and $s : E \to B$ of the idempotent e. By definition, $rs = 1_E$ and $e = sr$. Let us put $i_1 = r_1 s$ and $i_2 = r_2 s$. Then,

$$s_1 i_1 = s_1 r_1 s = e_1 s = e_1 srs = e_1 es = es = srs = s.$$

Similarly, $s_2 i_2 = s$. Hence the following square commutes,

$$\begin{array}{ccc} E & \xrightarrow{i_2} & A_2 \\ {\scriptstyle i_1}\downarrow & & \downarrow{\scriptstyle s_2} \\ A_1 & \xrightarrow{s_1} & B. \end{array}$$

Let us show that it is an absolute pullback. For this, it suffices to show that its image by an arbitrary set valued functor $\mathcal{E} \to \mathbf{Set}$ is a pullback square. Hence we can suppose that $\mathcal{E} = \mathbf{Set}$. The maps i_1 and i_2 are moni, since the maps $s_1 i_1 = s$ and $s_2 i_2 = s$ are monic. If we replace the sets A_1, A_2 and E by their image in B, we can suppose that the maps s_1, s_2, i_1, i_2 and s are subset inclusions. We have $E \subseteq A_1 \cap A_2$, since the square commutes. Conversely, let us show that $A_1 \cap A_2 \subseteq E$. If $x \in A_1 \cap A_2$, then $e_1(x) = x$ and $e_2(x) = x$. Thus, $e(x) = (e_1 e_2)^n(x) = x$ and this proves that $x \in E$. ∎

Lemma B.0.6. [JT3] *Every pushout square of surjections in Δ is absolute. Dually, every pullback square of monomorphisms in Δ is absolute.*

Proof: Observe that in Δ a surjection $p : B \to A$ in Δ has a section $s : A \to B$ which is smallest with respect to the pointwise ordering the set of maps $A \to B$. If $e = sr$, then we have $e(x) \leq x$ for every $x \in B$. Let $p_1 : B \to A_1$ and $p_2 : B \to A_1$ be two surjections with smallest sections $s_1 : A_1 \to B$ and $s_2 : A_2 \to B$ respectively. Let us put $e_1 = s_1 r_1$ and $e_2 = s_2 r_2$. Then we have $e_1(x) \leq x$ and $e_2(x) \leq x$ for every $x \in B$. The following decreasing sequence

$$x \geq e_1(x) \geq e_2 e_1(x) \geq e_1 e_2 e_1(x) \geq \cdots .$$

must be stationary by finiteness. Hence we have $e_1(e_2 e_1)^n = (e_2 e_1)^n$ for $n \geq 1$ large enough. The result then follows from B.0.5. Dually, let $A_1 \times_B A_2$ be the pullback of two monomorphisms $i_1 : A_1 \to B$ and $i_2 : A_2 \to B$ in the category Δ. For simplicity, we shall suppose that i_1 and i_2 are subset inclusions, in which case $A_1 \times_B A_2 = A_1 \cap A_2$. The intersection $A_1 \cap A_2$ is non-empty, since the objects of Δ are non-empty. We can then choose an element $c \in A_1 \cap A_2$. For every $x \in B$, let us put

$$r_i(x) = \begin{cases} \min\{y \in A_i : x \leq y\} & \text{if } x \leq c \\ \max\{y \in A_i : y \leq x\} & \text{if } c \leq x. \end{cases}$$

This defines an preserving maps $r_i : E \to A_1$ for $i = 1, 2$. Let us put $e_1 = i_1 r_1$ and $e_2 = i_2 r_2$. If $x \in B$ and $x \leq c$, then $x \leq e_1(x) \leq c$ and $x \leq e_2(x) \leq c$. Hence we have $(e_1 e_2)^n(x) = e_2(e_1 e_2)(x)$ for $n \geq 1$ large enough in this case. Similarly, if $c \leq x$, then $c \leq e_1(x) \leq x$ and $c \leq e_2(x) \leq x$. Hence we have $e_2(e_1 e_2)^n(x) = (e_1 e_2)^n(x)$ for $n \geq 1$ large enough in this case. This shows that we have $e_2(e_1 e_2)^n = (e_1 e_2)^n$ for $n \geq 1$ large enough. The result then follows from B.0.5. ∎

Lemma B.1. (Eilenberg-Zilber Lemma) *Every simplex $x : \Delta[n] \to X$ of a simplicial set X admits a unique factorisation $x = ys : \Delta[n] \to \Delta[m] \to X$, with $s : [m] \to [n]$ a surjection and y a non-degenerate simplex.*

Proof: Let us choose a factorisation $x = ys : \Delta[n] \to \Delta[m] \to X$, with $s : [m] \to [n]$ a surjection and m minimal. The simplex $y : \Delta[m] \to X$ is non-degenerate, since m is minimal. The existence is proved. It remains to prove the uniqueness. Suppose that we have two factorisations, $x = y_i p_i : \Delta[n] \to \Delta[m_i] \to X$ $(i = 1, 2)$ with $p_i : [n] \to [m_i]$ a surjection and y_i a non-degenerate simplex. The two surjections have a pushout in Δ,

$$\begin{array}{ccc} [n] & \xrightarrow{p_2} & [m_2] \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle q_2} \\ [m_1] & \xrightarrow{q_1} & [m]. \end{array}$$

The pushout is absolute by B.0.6. Its image by the Yoneda functor is thus a pushout square of simplicial sets:

$$\begin{array}{ccc} \Delta[n] & \xrightarrow{p_2} & \Delta[m_2] \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle q_2} \\ \Delta[m_1] & \xrightarrow{q_1} & \Delta[m]. \end{array}$$

Hence there exists a unique simplex $y : \Delta[m] \to Y$ such that $yq_1 = y_1$ and $yq_2 = y_2$, since we have $y_1 p_1 = y_2 p_2$. But q_1 must be the identity, since it is surjective and y_1 is non-degenerate. Similarly for q_2 must be the identity. Thus, $m_1 = m_2$ and $y_1 = y_2$. ∎

A simplicial set X is said to be *finite* if it has only a finite number of non-degenerate simplices. Let $\Delta(n)$ be the full sub-category of Δ which is spanned by the ordinals $[k]$ for $k \leq n$. A *n-truncated simplicial set* is a contravariant functor $\Delta(n) \to \mathbf{Set}$. From the inclusion $i_n : \Delta(n) \subset \Delta$ we obtain a truncation functor $i_n^* : \mathbf{S} \to \mathbf{S}(n)$, where $\mathbf{S}(n)$ is the category of n-truncated simplicial sets. The functor i_n^* has a a left adjoint $(i_n)_!$ and a right adjoint $(i_n)_*$. Both adjoints are fully faithful, since the functor i_n is fully faithful. The *n-skeleton* of a simplicial set X is defined by putting $Sk^n X = (i_n)_! i_n^*(X)$, and the *n-coskeleton* by putting $Cosk^n X = (i_n)_* i_n^*(X)$. This defines a pair of adjoint functors,

$$Sk^n : \mathbf{S} \leftrightarrow \mathbf{S} : Cosk^n.$$

Hence a simplex $\Delta[m] \to Cosk^n X$ is the same thing as a map $Sk^n \Delta[m] \to X$. The image of the canonical map $Sk^n X \to X$ is the simplicial subset of X generated by the simplices of dimension $\leq n$. It follows from the Lemma B.1 that the canonical map $Sk^n X \to X$ is monic; the image of this map is the simplicial subset of X

generated by the simplices of dimension $\leq n$. We identify the simplicial set $Sk^n X$ with this simplicial subset of X. A simplicial set X is said to be of *dimension* $\leq n$ if $Sk^n X = X$. A simplicial set of dimension ≤ 0 is discrete. The simplicial set $Sk^n X$ can be constructed from the simplicial set $Sk^{n-1} X$ by attaching non-degenerate n-cells. The following result can be proved by using the Lemma B.1:

Proposition B.0.7. *If $S_n(X) \subseteq X_n$ is the set of non-degenerated n-simplicies of a simplicial set X, then there is a canonical pushout square*

$$\begin{array}{ccc} \bigsqcup_{S_n(X)} \partial\Delta[n] & \longrightarrow & Sk^{n-1}X \\ \downarrow & & \downarrow \\ \bigsqcup_{S_n(X)} \Delta[n] & \longrightarrow & Sk^n X. \end{array}$$

.

The following proposition uses the notion of a saturated class. See D.2.2 and D.2.4 for the notion.

Proposition B.0.8. [GZ] *The class of monomorphisms in the category* **S** *is generated as a saturated class by the set of inclusions*

$$\delta_n : \partial\Delta[n] \subset \Delta[n], \quad \text{for } n \geq 0.$$

Proof: Let us denote by $\mathcal{C}$ the class of monomorphisms in **S**. The class $\mathcal{C}$ is saturated, since the class of monomorphisms in any topos is saturated. Let us denote by Σ be the set of maps δ_n for $n \geq 0$ and let $\overline{\Sigma}$ be the saturated class generated by Σ. We have $\overline{\Sigma} \subseteq \mathcal{C}$, since we have $\Sigma \subseteq \mathcal{C}$ and $\mathcal{C}$. Conversely, let us show that every monomorphism $u : A \to B$ belongs to $\overline{\Sigma}$. We can suppose that u is defined by an inclusion $A \subseteq B$, since a saturated class contains the isomorphisms. We have $B = \bigcup_n Sk^n B$. It follows that the inclusion $A \subseteq B$ is the composite of the infinite sequence of inclusions

$$A \subseteq A \cup Sk^0 B \subseteq A \cup Sk^1 B \subseteq A \cup Sk^2 B \to \cdots .$$

Hence it suffices to show that each inclusion $A \cup Sk^{n-1} B \subseteq A \cup Sk^n B$ belongs to $\overline{\Sigma}$, where we put $Sk^{-1} B = \emptyset$. But it follows from B.0.7 that we have a pushout square

$$\begin{array}{ccc} \bigsqcup_{S_n(B\backslash A)} \partial\Delta[n] & \longrightarrow & A \cup Sk^{n-1}B \\ \downarrow & & \downarrow \\ \bigsqcup_{S_n(B\backslash A)} \Delta[n] & \longrightarrow & A \cup Sk^n B, \end{array}$$

where $S_n(B)$ is the set of non-degenerated n-simplicies of B, and $S_n(B\backslash A) = S_n(B)\backslash S_n(A)$. The vertical map on the left hand side of the square belongs to $\overline{\Sigma}$, since a saturated class is closed under coproducts by D.2.1. Hence the vertical map on the right hand side belongs to $\overline{\Sigma}$, since a saturated class is closed under cobase change. ∎

Corollary B.0.9. [GZ] *A map of of simplicial sets is a trivial fibration iff it has the right lifting property with respect to the inclusion* $\delta_n : \partial\Delta[n] \subset \Delta[n]$ *for every* $n \geq 0$.

Proof: Let $f : X \to Y$ be a map having the right lifting property with respect to the inclusion $\delta_n : \partial\Delta[n] \subset \Delta[n]$ for every $n \geq 0$. Let us show that f is a trivial fibration. Let us denote by $\mathcal{A}$ the class of maps having the left lifting property with respect to f. The class $\mathcal{A}$ is saturated by D.2.3. We have $\delta_n \in \mathcal{A}$ for every $n \geq 0$ by the assumption on f. Thus, every monomorphism belongs to $\mathcal{A}$ by proposition B.0.8. ■

Definition B.0.10. *We shall say that a simplicial set X is* n-coskeletal *if the canonical map* $X \to Cosk^n X$ *is an isomorphism.*

A simplicial set X is n-coskeletal iff every simplicial sphere $\partial\Delta[m] \to X$ with $m > n$ has a unique filler.

Definition B.0.11. *We shall say that a map of simplicial sets* $f : X \to Y$ *is* n-full *if the naturality square*

$$\begin{array}{ccc} X & \longrightarrow & Cosk^n X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Cosk^n Y \end{array}$$

is cartesian. We shall say that a simplicial subset $S \subseteq X$ *is* n-full *if the map* $S \to X$ *defined by the inclusion is* n*-full.*

Proposition B.0.12. *A map* $f : X \to Y$ *is* 0*-full iff it is right orthogonal to the inclusion* $\partial\Delta[m] \subset \Delta[m]$ *for every* $m > n$.

Proposition B.0.13. [GZ] *The nerve functor* $N : \mathbf{Cat} \to \mathbf{S}$ *is fully faithful. We have* $\tau_1 NC = C$ *for every category* C.

Proof: Let us show that the functor N is full and faithful. If $u : C \to D$ is a functor in **Cat**, then then map $(Nu)_0 : (NC)_0 \to (ND)_0$ coincide with the map $Ob(u) : Ob(C) \to Ob(D)$ and the map $(Nu)_1 : (NC)_1 \to (ND)_1$ with the map $Ar(u) : Ar(C) \to Ar(D)$. This shows that N is faithful. It remains to show that if $v : N(C) \to N(D)$ is a map of simplicial sets, that we have $v = N(u)$ for some functor $u : C \to D$. The map $Ob(u) : Ob(C) \to Ob(D)$ is taken to be the map $v_0 : (NC)_0 \to (ND)_0$ and the map $Ar(u) : Ar(C) \to Ar(D)$ is taken to be the map $v_0 : (NC)_1 \to (ND)_1$. Let us show that the pair (u_0, u_1) defines a functor $u : C \to D$. If $f : a \to b$ is an arrow in C, then we have $u(f) : u(a) \to u(b)$, since we have $v_1(f) : v_0(a) \to v_0(b)$. Similarly, if $a \in Ob(C)$, then we have $u(1_a) = 1_{u(a)}$. Let us show that u preserves composition. If $f : a \to b$ and $g : c \to d$ are two arrows in C, there is a unique simplex $y : \Delta[2] \to C$ such that $\partial y = (g, gf, f)$.

We have $\partial(v(y)) = (v(g), v(gf), v(f))$, since v is a map of simplicial sets. This proves that $u(gf) = u(g)u(f)$. We have defined a functor $u : C \to D$. Let us show that $v = N(u)$. By construction, we have $v_0 = u_0$ and $v_1 = u_1$, where $u_n = N(u)_n$. Let us show that we have $v_n = u_n$ for every $n \geq 2$. But a simplex $y : \Delta[n] \to D$ is determined by the family of arrows $y(i,j) : \Delta[2] \to D$ for $i < j$. If $x : \Delta[n] \to C$, let us show that we have $v_n(x)(i,j) = u_n(x)(i,j)$ for every $i < j$. We have $v_n(x)(i,j) = v_1(x(i,j)))$, since v is a map of simplicial sets. Similarly, we have $u_n(x)(i,j) = u_1(x(i,j)))$. It follows that we have $v_n(x)(i,j) = u_n(x)(i,j)$, since we have $u_1 = v_1$. This shows that $v_n = u_n$. The first statement of the proposition is proved. The second statement follows. ∎

Let us denote by FX the category freely generated by the graph of non-degenerate arrows in a simplicial set X. An arrow $a \to b$ in FX is a path of non-degenerate arrows in X,

$$a = a_0 \xrightarrow{f_1} a_1 \xrightarrow{f_2} a_2 \cdots \qquad a_{n-1} \xrightarrow{f_n} a_n = b.$$

And let $\equiv$ the congruence relation on FX which is generated by the relations $(td_0)(td_2) \equiv td_1$, one for each non-degenerate 2-simplex $t \in X$ with boundary $\partial t = (\partial_0 t, \partial_1 t, \partial_2 t)$:

$$\begin{array}{ccccc} & & t(1) & & \\ & \nearrow^{\partial_2 t} & & \searrow^{\partial_0 t} & \\ t(0) & & \xrightarrow{\partial_1 t} & & t(2). \end{array}$$

The degenerate arrows in X are interpreted as units in FX.

Proposition B.0.14. [GZ] *Let FX be the category freely generated by the graph of non-degenerate arrows in X. Then we have*

$$\tau_1 X = FX/\equiv,$$

where $\equiv$ is the congruence described above. Moreover, the functor $\tau_1 Sk^2 X \to \tau_1 X$ induced by the inclusion $Sk^2 X \subseteq X$ is an isomorphism.

Proof: If we apply the functor τ_1 to the pushout square of simplicial sets

$$\begin{array}{ccc} \bigsqcup_{S_1(X)} \partial\Delta[1] & \longrightarrow & Sk^0 X \\ \downarrow & & \downarrow \\ \bigsqcup_{S_1(X)} \Delta[1] & \longrightarrow & Sk^1 X, \end{array}$$

we obtain a pushout square of categories

$$\begin{array}{ccc} \bigsqcup_{S_1(X)}\{0,1\} & \longrightarrow & X_0 \\ \downarrow & & \downarrow \\ \bigsqcup_{S_1(X)}[1] & \longrightarrow & \tau_1 Sk^1 X. \end{array}$$

This shows that $\tau_1 Sk^1 X = FX$. If we apply the functor τ_1 to the pushout square of simplicial sets

$$\begin{array}{ccc} \bigsqcup_{S_2(X)}\partial\Delta[2] & \longrightarrow & Sk^1 X \\ \downarrow & & \downarrow \\ \bigsqcup_{S_2(X)}\Delta[2] & \longrightarrow & Sk^2 X. \end{array}$$

we then obtain a pushout square of categories

$$\begin{array}{ccc} \bigsqcup_{S_2(X)}\tau_1\partial\Delta[2] & \longrightarrow & FX \\ \downarrow & & \downarrow \\ \bigsqcup_{S_2(X)}[2] & \longrightarrow & \tau_1 Sk^2 X. \end{array}$$

This shows that $\tau_1 Sk^2 X = FX/\equiv$. Hence the proposition will be proved if we show that the canonical functor $i_X : \tau_1 Sk^2 X \to \tau_1 X$ is an isomorphism. Observe that i_X is a natural transformation between two concontinuous functors. Hence it suffices to show that i_X is an isomorphism in the case where $X = \Delta[n]$, since every simplicial set is a colimit of a diagram of simplices. This is obvious if $n \leq 2$, since $Sk^2\Delta[n] = \Delta[n]$ in this case. Let us suppose $n > 2$. The category $F\Delta[n]$ is freely generated by a family of arrows

$$f_{ji} : i \to j,$$

one for each pair $0 \leq i < j \leq n$. The congruence $\equiv$ is generated by the relations

$$f_{kj}f_{ji} \equiv f_{ki},$$

one for each triple $0 \leq i < j < k \leq n$. It is clear from this description that $\tau_1 Sk^2\Delta[n] = [n]$. ■

Proposition B.0.15. [GZ] *The functor* $\tau_1 : \mathbf{S} \to \mathbf{Cat}$ *preserves finite products.*

Proof: Obviously, we have $\tau_1 1 = 1$. Hence the functor $\tau_1 : \mathbf{S} \to \mathbf{Cat}/I$ preserves terminal objects. Let us show that the canonical map

$$i_{XY} : \tau_1(X \times Y) \to \tau_1 X \times \tau_1 Y$$

is an isomorphism for every $X, Y \in \mathbf{S}$. The functor τ_1 is cocontinuous, since it is a left adjoint. Hence the functor $(X, Y) \mapsto \tau_1(X \times Y)$ is cocontinuous in each variable, since the category $\mathbf{S}$ is cartesian closed. Similarly, the functor $(X, Y) \mapsto \tau_1 X \times \tau_1 Y$ is cocontinuous in each variable, since the category **Cat** is cartesian closed. Every simplicial set is a colimit of a diagram of simplices. Hence it suffices to prove that the natural transformation i_{XY} is invertible in the case where $X = \Delta[m]$ and $Y = \Delta[n]$. We have $\Delta[m] = N[m]$ and $\Delta[n] = N[n]$. The functor N preserves products, since it is a right adjoint. We have $\tau_1 NC = C$ for every category C. It follows that we have

$$\begin{aligned} \tau_1(X \times Y) &= \tau_1(N[m] \times N[n]) = \tau_1 N([m] \times [n]) \\ &= [m] \times [n] = \tau_1(\Delta[m]) \times \tau_1(\Delta[n]) \\ &= \tau_1 X \times \tau_1 Y. \end{aligned}$$

∎

Proposition B.0.16. *If A and B are small categories, then the canonical map*

$$N(B^A) \to N(B)^{N(A)}$$

is an isomorphism. Moreover, if $X \in \mathbf{S}$ then the map $N(B)^{N\tau_1 X} \to N(B)^X$ induced by the map $X \to N\tau_1 X$ is an isomorphism.

Proof: Let us show that the canonical map $N(B^A) \to N(B)^{N(A)}$ is an isomorphism. If we fix $A \in$ **Cat** the map is a natural transformation between two functors in $B \in$ **Cat**. The functor $B \mapsto N(B^A)$ is right adjoint to the functor $X \mapsto \tau_1(X) \times A$ and the functor $B \mapsto N(B)^{N(A)}$ right adjoint to the functor $X \mapsto \tau_1(X \times N(A))$. Moreover, the natural transformation $N(B^A) \to N(B)^{N(A)}$ is right adjoint to the natural transformation $\tau_1(X \times N(A)) \to \tau_1(X) \times A$. Hence it suffices to show that canonical map

$$\tau_1(X \times N(A)) \to \tau_1(X) \times A$$

is an isomorphism. But this is clear, since the functor τ_1 preserves products by B.0.15 and we have $\tau_1 N(A) = A$ by B.0.13. The first statement of the lemma is proved. Let us prove the second. Let h be the canonical map $X \to N(\tau_1 X)$. It follows from B.0.15 and B.0.13 that the map

$$\tau_1(h \times Y) : \tau_1(X \times Y) \to \tau_1(N(\tau_1 X) \times Y))$$

is an isomorphism for every $X, Y \in \mathbf{S}$. Arguing as above, it follows by adjointness that the map

$$N(B)^h : N(B)^{N(\tau_1 X)} \to N(B)^X$$

is an isomorphism for every $X \in \mathbf{S}$ and $B \in$ **Cat**. This proves the result, since we have

$$N(B)^{N(\tau_1 X)} \simeq N(B^{\tau_1 X})$$

by the first part of the proof. ∎

Proposition B.0.17. [JT3] *Let $\mathcal{E}$ be a topos and $F : \mathbf{S} \to \mathcal{E}$ be a cocontinuous functor. If F takes the inclusion $\partial I \subset I$ to a monomorphism, then it takes every monomorphism to a monomorphism.*

Proof: We only give the proof in the special case where $\mathcal{E} = \mathbf{Set}$. Let $\mathcal{A}$ be the class of maps u such that $F(u)$ is monic. The class $\mathcal{A}$ is saturated, since the functor F is cocontinuous and the class of monomorphisms is saturated in $\mathbf{Set}$. Let us show that every monomorphism belongs to $\mathcal{A}$. For this it suffices to show that the map $\delta_n : \partial\Delta[n] \subset \Delta[n]$ belongs to $\mathcal{A}$ for every $n \geq 0$ by B.0.8. Hence it suffice to show that the map $F(\delta_n)$ is monic for every $n \geq 0$. We have $F(\emptyset) = \emptyset$, since the functor F is cocontinuous. This proves the result in the case $n = 0$. The result is obvious if $n = 1$ by the hypothesis on F. Hence we can suppose $n \geq 2$. We need to compute the set $F(\partial\Delta[n])$. For any simplicial set X we have

$$F(X) = \varinjlim_{\Delta[n] \to X} F\Delta[n],$$

since the functor F is cocontinuous. The colimit is taken over the category of elements Δ/X. The category $\Delta/\partial\Delta[n]$ is isomorphic to the full subcategory C_n of $\Delta/[n]$ spanned by the non-surjective maps $[k] \to [n]$. But the full sub-category M_n of $\Delta/[n]$ spanned by the monomorphisms $[k] \to [n]$ with $k < n$ is final in C_n. Thus,

$$F(\partial\Delta[n]) = \varinjlim_{\substack{f:[k] \rightarrowtail [n] \\ f \in M_n}} F\Delta[k].$$

To each monomorphism $f : [k] \rightarrowtail [n]$ corresponds a canonical map $u(f) : F\Delta[k] \to F(\partial\Delta[n])$ and we have $F(\delta_n)u(f) = F(f)$. Every monomorphism in Δ has a retraction. Hence the map $F(f) : F\Delta[k] \to F\Delta[n]$ is monic for every monomorphism $f : [k] \rightarrowtail [n]$. Let us show that $F(\delta_n)$ is monic. If $a, b \in F(\partial\Delta[n])$ and $F(\delta_n)(a) = F(\delta_n)(b)$, let us show that $a = b$. The category M_n is isomorphic to the poset of proper non-empty subsets on $[n]$. Every proper subset of $[n]$ is included in a maximal proper subset $[n] \setminus \{i\}$ for some $i \in [n]$. If follows that every element $a \in F(\partial\Delta[n])$ is in the image of the map $u(d_i)$ for some $i \in [n]$. Hence we have $a = u(d_i)(a')$ and $b = u(d_j)j(b')$ for some $i, j \in [n]$ and some $a', b' \in F\Delta[n-1]$. If $i = j$, then

$$\begin{aligned} F(d_i)(a') &= F(\delta_n)u(d_i)(a') = F(\delta_n)(a) \\ &= F(\delta_n)(b) = F(\delta_n)u(d_i)(b') = F(d_i)(b'). \end{aligned}$$

Hence we have $a' = b'$, since $F(d_i)$ is monic. It follows that $a = b$. Let us now suppose that $i < j$. It follows from B.0.6 that the image by F of the pullback square

$$\begin{array}{ccc} [n-2] & \xrightarrow{d_i} & [n-1] \\ {\scriptstyle d_{j-1}}\downarrow & & \downarrow{\scriptstyle d_j} \\ [n-1] & \xrightarrow{d_i} & [n] \end{array}$$

is a pullback square

$$\begin{array}{ccc} F\Delta[n-2] & \xrightarrow{F(d_i)} & F\Delta[n-1] \\ {\scriptstyle F(d_{j-1})}\downarrow & & \downarrow{\scriptstyle F(d_j)} \\ F\Delta[n-1] & \xrightarrow{F(d_i)} & F\Delta[n]. \end{array}$$

Hence there exists an element $c \in F\Delta[n-2]$ such that $a' = F(d_{j-1})(c)$ and $b' = F(d_i)(c)$. But then,

$$\begin{aligned} a &= u(d_i)(a') = u(d_i)F(d_{j-1})(c) = u(d_i d_{j-1})(c) \\ &= u(d_j d_i)(c) = u(d_j)F(d_i)(c) = u(d_j)(b') = b. \end{aligned}$$

We have proved that the map $F(\delta_n)$ is monic. It follows that F takes a monomorphism to a monomorphism. ∎

Proposition B.0.18. [JT3] *Let $\mathcal{E}$ be a bicomplete model category and $\alpha : F \to G : \mathbf{S} \to \mathcal{E}$ be a natural transformation between two cocontinuous functors. Suppose that the functors F and G take a monomorphism to a cofibration and that the map $\alpha(n) = \alpha(\Delta[n])$ is a weak equivalence for every $n \geq 0$. Then the map $\alpha(X) : F(X) \to G(X)$ is a weak equivalence for every simplicial set X.*

Proof: The hypothesis on F and G implies that $F(X)$ and $G(X)$ are cofibrant objects for every simplicial set X. Let us show by induction on $n \geq 0$ that the map $\alpha(Sk^n X)$ is a weak equivalence. This is clear if $n = 0$ by G.0.14, since the map $\alpha(Sk^0 X)$ is a coproduct of X_0 copies of the map $\alpha(0)$ and $\alpha(0)$ is a weak equivalence between cofibrant objects. Let us suppose $n > 0$. The image by α of the pushout square

$$\begin{array}{ccc} S_n(X) \times \partial\Delta[n] & \longrightarrow & Sk^{n-1}X \\ \downarrow & & \downarrow \\ S_n(X) \times \Delta[n] & \longrightarrow & Sk^n X. \end{array}$$

of B.0.7 is a cube,

$$\begin{array}{ccccccc} S_n(X) \times F\partial\Delta[n] & & \longrightarrow & & FSk^{n-1}X & & \\ \downarrow & \searrow & & & \downarrow & \searrow & \\ & & S_n(X) \times G\partial\Delta[n] & & \longrightarrow & & GSk^{n-1}X \\ S_n(X) \times F\Delta[n] & & \longrightarrow & & FSk^n X & & \downarrow \\ & \searrow & \downarrow & & & \searrow & \\ & & S_n(X) \times G\Delta[n] & & \longrightarrow & & GSk^n X. \end{array}$$

The front and the back faces of the cube are homotopy pushout squares of cofibrant objects by the assumption on F and G. The map $\alpha(\partial\Delta[n])$ is a weak equivalence between cofibrant objects by the induction hypothesis, hence also the map $S_n(X) \times \alpha(\partial\Delta[n])$. Similarly for the map $\alpha(Sk^{n-1}X)$. It then follows from the Cube Lemma F.4.6 that the map $\alpha(Sk^n X)$ is a weak equivalence, since $\alpha(n)$ is a weak equivalence. We have proved that $\alpha(Sk^n X)$ is a weak equivalence for every $n \geq 0$. Let us now show that $\alpha(X)$ is a weak equvalence. But $\alpha(X)$ is a colimit over $n \geq 0$ of the map $\alpha(Sk^n(X))$. The poset of natural numbers $\mathbf{N}$ is well-founded. Hence the colimit functor $\mathcal{E}^{\mathbf{N}} \to \mathcal{E}$ is a left Quillen functor with respect to the projective model structure on the category $\mathcal{E}^{\mathbf{N}}$ by G.0.13. The image by F of the inclusion $Sk^{n-1}X \to Sk^n X$ is a cofibration between cofibrant object. It follows that the infinite sequences

$$F(Sk^0X) \to F(Sk^1X) \to F(Sk^2X) \to \cdots$$

is a cofibrant object in the projective model category $\mathcal{E}^{\mathbf{N}}$. Similarly for the infinite sequences

$$G(Sk^0X) \to G(Sk^1X) \to G(Sk^2X) \to \cdots .$$

It then follows by Ken Brown's lemma E.2.6 that the map $\alpha(X)$ is a weak equivalence. ■

Appendix C

Factorisation systems

In this appendix we study the notion of factorisation system. We give a few examples of factorisation systems in **Cat**.

Definition C.0.19. *If $\mathcal{E}$ is a category, we shall say that a pair $(\mathcal{A}, \mathcal{B})$ of classes of maps in $\mathcal{E}$ is a* (strict) factorisation system *if the following conditions are satisfied:*

- *the classes $\mathcal{A}$ and $\mathcal{B}$ are closed under composition and contain the isomorphisms;*
- *every map $f : A \to B$ admits a factorisation $f = pu : A \to E \to B$ with $u \in \mathcal{A}$ and $p \in \mathcal{B}$, and the factorisation is unique up to unique isomorphism.*

We say that $\mathcal{A}$ is the left *class and $\mathcal{B}$ the* right *class of the weak factorisation system.*

In this definition, the uniqueness of the factorisation of a map $f : A \to B$ means that for any other factorisation $f = qv : A \to F \to B$ with $v \in \mathcal{A}$ and $q \in \mathcal{B}$, there exists a unique isomorphism $i : E \to F$ such that $iu = v$ and $qi = p$,

$$\begin{array}{ccc} A & \xrightarrow{v} & F \\ {\scriptstyle u}\downarrow & \nearrow_{i} & \downarrow{\scriptstyle q} \\ E & \xrightarrow[p]{} & B. \end{array}$$

Recall that a class $\mathcal{M}$ of maps in a category $\mathcal{E}$ is said to be *invariant under isomorphisms* if for every commutative square

$$\begin{array}{ccc} A & \longrightarrow & A' \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u'} \\ B & \longrightarrow & B' \end{array}$$

in which the horizontal maps are isomorphisms we have $u \in \mathcal{M} \Leftrightarrow u' \in \mathcal{M}$. It is obvious from the definition that the classes of a factorisation system are invariant under isomorphism.

Definition C.0.20. *We shall say that a class of maps $\mathcal{M}$ in a category $\mathcal{E}$ has the* right cancellation property *if the implication*

$$vu \in \mathcal{M} \text{ and } u \in \mathcal{M} \ \Rightarrow \ v \in \mathcal{M}$$

is true for any pair of maps $u : A \to B$ and $v : B \to C$. Dually, we shall say that $\mathcal{M}$ has the left cancellation property *if the implication*

$$vu \in \mathcal{M} \text{ and } v \in \mathcal{M} \ \Rightarrow \ u \in \mathcal{M}$$

is true.

Proposition C.0.21. *The intersection of the classes of a factorisation system $(\mathcal{A}, \mathcal{B})$ is the class of isomorphisms. Moreover,*

- *the class $\mathcal{A}$ has the right cancellation property;*
- *the class $\mathcal{B}$ has the left cancellation property.*

Proof: If a map $f : A \to B$ belongs to $\mathcal{A} \cap \mathcal{B}$ then we have the factorisation $f = f1_A$ with $1_A \in \mathcal{A}$ and $f \in \mathcal{B}$ and the factorisation $f = 1_B f$ with $f \in \mathcal{A}$ and $1_B \in \mathcal{B}$. Hence there exists an isomorphism $i : B \to A$ such that $if = 1_A$ and $fi = 1_B$. This shows that f is invertible. If $u \in \mathcal{A}$ and $vu \in \mathcal{A}$, let us show that $v \in \mathcal{A}$. For this, let us choose a factorisation $v = ps : B \to E \to C$, with $s \in \mathcal{A}$ and $p \in \mathcal{B}$. Let us put $w = vu$. Then w admits the factorisation $w = (p)(su)$ with $su \in \mathcal{A}$ and $p \in \mathcal{B}$ and the factorisation $w = (1_C)(vu)$ with $vu \in \mathcal{A}$ and $1_C \in \mathcal{B}$. Hence there exists an isomorphism $i : E \to C$ such that $i(su) = vu$ and $1_C i = p$. Thus, $p \in \mathcal{A}$ since $p = i$ and every isomorphism is in $\mathcal{A}$. It follows that $v = ps \in \mathcal{A}$, since $\mathcal{A}$ is closed under composition. ■

Definition C.0.22. *We say that a map $u : A \to B$ in a category $\mathcal{E}$ is* left orthogonal *to a map $f : X \to Y$, or that f is* right orthogonal *to u, if every commutative square*

$$\begin{array}{ccc} A & \xrightarrow{x} & X \\ {\scriptstyle u}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\ B & \xrightarrow[y]{} & Y \end{array}$$

has a unique diagonal filler $d : B \to X$ (that is, $du = x$ and $fd = y$). We shall denote this relation by $u \perp f$.

Notice that the condition $u \perp f$ means that the square

$$\begin{array}{ccc} Hom(B,X) & \xrightarrow{Hom(u,X)} & Hom(A,X) \\ {\scriptstyle Hom(B,f)}\downarrow & & \downarrow{\scriptstyle Hom(A,f)} \\ Hom(B,Y) & \xrightarrow[Hom(u,Y)]{} & Hom(A,X) \end{array}$$

is cartesian. If $\mathcal{A}$ and $\mathcal{B}$ are two classes of maps in $\mathcal{E}$, we shall write $\mathcal{A} \perp \mathcal{B}$ to indicate that we have $a \perp b$ for every $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

If $\mathcal{M}$ is a class of maps in a category $\mathcal{E}$, we shall denote by ${}^\perp\mathcal{M}$ (resp. $\mathcal{M}^\perp$) the class of maps which are left (resp. right) orthogonal to every map in $\mathcal{M}$. Each class ${}^\perp\mathcal{M}$ and $\mathcal{M}^\perp$ is closed under composition and contains the isomorphisms. The class ${}^\perp\mathcal{M}$ has the right cancellation property and the class $\mathcal{M}^\perp$ the left cancellation property. If $\mathcal{A}$ and $\mathcal{B}$ are two classes of maps in $\mathcal{E}$, then

$$\mathcal{A} \subseteq {}^\perp\mathcal{B} \ \Leftrightarrow\ \mathcal{A} \perp \mathcal{B} \ \Leftrightarrow\ \mathcal{A}^\perp \supseteq \mathcal{B}.$$

Proposition C.0.23. *If $(\mathcal{A}, \mathcal{B})$ is a factorisation system then*

$$\mathcal{A} = {}^\perp\mathcal{B} \quad \text{and} \quad \mathcal{B} = \mathcal{A}^\perp.$$

Proof Let us first show that we have $\mathcal{A} \perp \mathcal{B}$. If $a : A \to A'$ is a map in $\mathcal{A}$ and $b : B \to B'$ is a map in $\mathcal{B}$, let us show that every commutative square

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ A' & \xrightarrow[u']{} & B' \end{array}$$

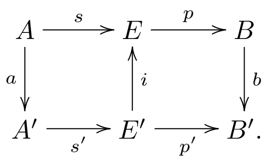
has a unique diagonal filler. Let us choose a factorisation $u = ps : A \to E \to B$ with $s \in \mathcal{A}$ and $p \in \mathcal{B}$ and a factorisation $u' = p's' : A' \to E' \to B'$ with $s' \in \mathcal{A}$ and $p' \in \mathcal{B}$. From the commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{s} & E & \xrightarrow{p} & B \\ {\scriptstyle a}\downarrow & & & & \downarrow{\scriptstyle b} \\ A' & \xrightarrow[s']{} & E' & \xrightarrow[p']{} & B', \end{array}$$

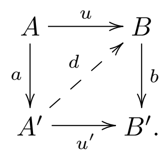
we can construct a square

$$\begin{array}{ccc} A & \xrightarrow{s} & E \\ {\scriptstyle s'a}\downarrow & & \downarrow{\scriptstyle bp} \\ E' & \xrightarrow[p']{} & B'. \end{array}$$

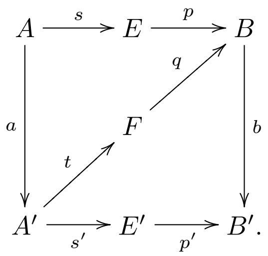
Observe that $s \in \mathcal{A}$ and $bp \in \mathcal{B}$ and also that $s'a \in \mathcal{A}$ and $p' \in \mathcal{B}$. By the uniqueness of the factorisation of a map that there is a unique isomorphism $i : E' \to E$ such that $is'a = s$ and $bpi = p'$:



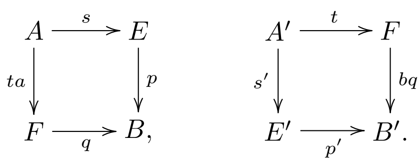
The composite $d = pis'$ is then a diagonal filler of the first square



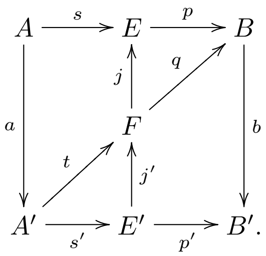
It remains to prove the uniqueness of d. Let d' be an arrow $A' \to B$ such that $d'a = u$ and $bd' = u'$. Let us choose a factorisation $d' = qt : A' \to F \to B$ with $t \in \mathcal{A}$ and $q \in \mathcal{B}$. From the commutative diagram



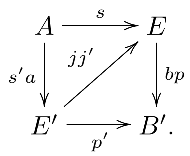
we can construct two commutative squares



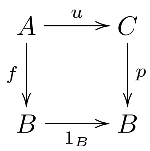
Observe that we have $ta \in \mathcal{A}$ and $q \in \mathcal{B}$. Hence there exists a unique isomorphism $j : F \to E$ such that $jta = s$ and $pj = q$. Similarly, there exists a unique isomorphism $j' : E' \to F$ such that $j's' = t$ and $bqj' = p'$. The maps fits in the following commutative diagram,



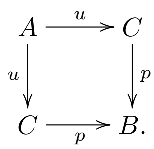
Hence the diagram



commutes. It follows that we have $jj' = i$ by the uniqueness of the isomorphism between two factorisations. Thus, $d' = qt = (pj)(j's') = pis' = d$. The relation $\mathcal{A} \perp \mathcal{B}$ is proved. This shows that $\mathcal{A} \subseteq {}^{\perp}\mathcal{B}$. Let us show that ${}^{\perp}\mathcal{B} \subseteq \mathcal{A}$. If a map $f : A \to B$ is in ${}^{\perp}\mathcal{B}$. let us choose a factorisation $f = pu : A \to C \to B$ with $u \in \mathcal{A}$ and $p \in \mathcal{B}$. Then the square



has a diagonal filler $s : B \to C$, since $f \in {}^{\perp}\mathcal{B}$. We have $ps = 1_B$. Let us show that $sp = 1_C$. Observe that the maps sp and 1_C are both diagonal fillers of the square



This proves that $sp = 1_C$ by the uniqueness of a diagonal filler. Thus, $p \in \mathcal{B}$, since every isomorphism is in $\mathcal{A}$. Thus, $f = pu \in \mathcal{A}$. ■

Corollary C.0.24. *Each class of a factorisation system determines the other.*

Proposition C.0.25. *The right class of a factorisation system is closed under limits.*

Proof: If $(\mathcal{A}, \mathcal{B})$ is a factorisation system, let us denote by $\mathcal{B}'$ be the full subcategory of $\mathcal{E}^I$ whose objects are the arrows in $\mathcal{B}$. The result will be proved if we show that $\mathcal{B}'$ is a reflexive subcategory of $\mathcal{E}^I$. Every map $u : A \to B$ admits a factorisation

$u = pi : A \to E \to B$ with $i \in \mathcal{A}$ and $p \in \mathcal{B}$. The pair $(i, 1_B)$ defines an arrow $u \to p$ in $\mathcal{E}^I$. Let us show that the arrow reflects u in the subcategory $\mathcal{B}'$. For this, it suffices to show that for every arrow $f : X \to Y$ in $\mathcal{B}$ and every commutative square

$$\begin{array}{ccc} A & \xrightarrow{x} & X \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[y]{} & Y, \end{array}$$

there exists a unique arrow $z : E \to X$ such that $fz = py$ and $zi = x$. But this is clear, since the square

$$\begin{array}{ccc} A & \xrightarrow{x} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ E & \xrightarrow[yp]{} & Y. \end{array}$$

has a unique diagonal filler by C.0.23. ∎

Recall that a map $u : A \to B$ in a category $\mathcal{E}$ is said to be a *retract* of another map $v : C \to D$, if u is a retract of v in the category of arrows $\mathcal{E}^I$. A class of maps $\mathcal{M}$ in a category $\mathcal{E}$ is said to be *closed under retracts* if the retract of a map in $\mathcal{M}$ belongs to $\mathcal{M}$.

Corollary C.0.26. *Each class of a factorisation system is closed under retracts.*

Proof: This follolws from Proposition C.0.25

Recall that the *base change* of a map $E \to B$ along a map $A \to B$ is defined to be the projection $A \times_B E \to A$ in a pullback square

$$\begin{array}{ccc} A \times_B E & \longrightarrow & E \\ \downarrow & & \downarrow \\ A & \longrightarrow & B. \end{array}$$

A class of maps $\mathcal{M}$ in a category $\mathcal{E}$ is said to be *closed under base changes* if the base change of a map in $\mathcal{M}$ along any map belongs to $\mathcal{M}$ when it exists. Every class $\mathcal{M}^\perp$ is closed under base change. In particular, the right class of a factorisation system is closed under base change. Recall that the *cobase change* of a map $A \to E$ along a map $u : A \to B$ is the map $B \to E \sqcup_A B$ in a pushout square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ E & \longrightarrow & E \sqcup_A B. \end{array}$$

A class of maps $\mathcal{M}$ in a category $\mathcal{E}$ is said to be *closed under cobase changes* if the cobase change of a map in $\mathcal{M}$ along any map belongs to $\mathcal{M}$ when it exists. Every class ${}^{\perp}\mathcal{M}$ is closed under base change. In particular, the left class of a factorisation system is closed under cobase changes.

Let us give some examples of factorisation systems.

Proposition C.0.27. *Let $p : \mathcal{E} \to \mathcal{C}$ be a Grothendieck fibration. Then the category $\mathcal{E}$ admits a factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{B}$ is the class of cartesian morphisms. An arrow $u \in \mathcal{E}$ belongs to $\mathcal{A}$ iff the arrow $p(u)$ is invertible.*

Dually, if $p : \mathcal{E} \to \mathcal{C}$ is a a Grothendieck opfibration, then the category $\mathcal{E}$ admits a factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of cocartesian morphisms. A morphism $u \in \mathcal{E}$ belongs to $\mathcal{B}$ iff the morphism $p(u)$ is invertible.

If $\mathcal{E}$ is a category with pullbacks, then the target functor $t : \mathcal{E}^I \to \mathcal{E}$ is a Grothendieck fibration. A morphism $f : X \to Y$ of the category $\mathcal{E}^I$ is a commutative square in $\mathcal{E}$,

$$\begin{array}{ccc} X_0 & \xrightarrow{f_0} & Y_0 \\ \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle y} \\ X_1 & \xrightarrow{f_1} & Y_1. \end{array}$$

The morphism f is cartesian iff the square is a pullback (also called a *cartesian square*). Hence the category $\mathcal{E}^I$ admits a factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{B}$ is the class of cartesian square. A square $f : X \to Y$ belongs to $\mathcal{A}$ iff the morphism $f_1 : X_1 \to Y_1$ is invertible.

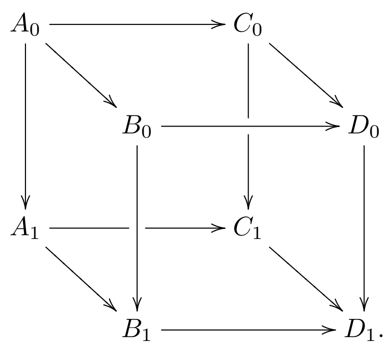
Corollary C.0.28. *Suppose that we have a commutative diagram*

$$\begin{array}{ccccc} A_0 & \longrightarrow & B_0 & \longrightarrow & C_0 \\ \downarrow & & \downarrow & & \downarrow \\ A_1 & \longrightarrow & B_1 & \longrightarrow & C_1 \end{array}$$

in which the right hand square is cartesian. Then the left hand square is cartesian iff the composite square is cartesian.

Proof: This follows from the left cancellation property of the right class of a factorisation system. ∎

Corollary C.0.29. *Suppose that we have a commutative cube*



in which the left face, the right face and front face are cartesian. Then the back face is cartesian.

We now give a few examples of factorisation systems in the category Cat.

Recall that a functor $p : E \to B$ is said to be a *discrete fibration* if for every object $e \in E$ and every arrow $g \in B$ with target $p(e)$, there exists a unique arrow $f \in E$ with target e such that $p(f) = e$. Recall that a functor between small categories $u : A \to B$ is said to be *final* (but we shall say *0-final*) if the category $b\backslash A = (b\backslash B) \times_B A$ defined by the pullback square

$$\begin{array}{ccc} b\backslash A & \xrightarrow{h} & A \\ \downarrow & & \downarrow{\scriptstyle u} \\ b\backslash B & \longrightarrow & B. \end{array}$$

is connected for every object $b \in B$.

Theorem C.0.30. [Street] *The category* **Cat** *admits a factorisation system* $(\mathcal{A}, \mathcal{B})$ *in which* $\mathcal{B}$ *is the class of discrete fibrations and* $\mathcal{A}$ *the class of 0-final functors.*

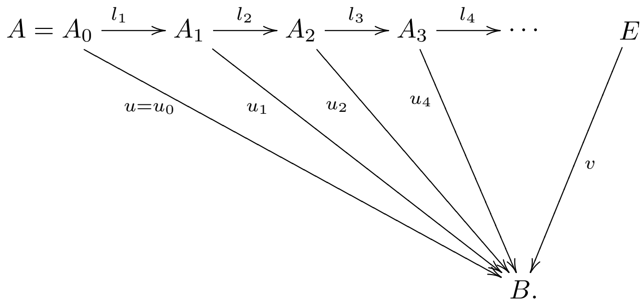
There are a dual notions of *discrete opfibration* and of *0-initial functor*. The category **Cat** admits a factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{A}$ is the class of 0-initial functors and $\mathcal{B}$ is the class of discrete opfibrations.

Recall that a functor $u : A \to B$ is said to be *conservative* if the implication

$$u(f)\text{invertible} \quad \Rightarrow \quad f\text{invertible}$$

is true for every arrow $f \in A$. We say that an arrow f in a category A is *inverted* by a functor $u : A \to B$ if the arrow $u(f)$ has an inverse in the category B. For every set of arrows S in a category A, there is a functor $l_S : A \to S^{-1}A$ which inverts universally every arrow in S. The universality means that if a functor

$u : A \to B$ inverts the arrows in S, then there exists a unique functor $v : S^{-1}A \to B$ such that $vl_S = u$. The functor i_S is called a *localisation*. It is easy to see that a localisation is left orthogonal to every conservative functor. Every functor $u : A \to B$ admits a factorisation $u = u_1 l_1 : A \to S^{-1}A \to B$ where u_1 is a localisation with respect to the set S of arrows inverted by u. Unfortunately, the functor u_1 may not be conservative. Let us put $S_0 = S$ and $A_1 = S^{-1}A$. The functor u_1 admits a factorisation $u_1 = u_2 l_2 : A_1 \to S_1^{-1}A_1 \to B$, where S_1 is the set of arrows inverted by u_1. Let us put $A_2 = S_1^{-1}A_1$. By iterating this process, we obtain an infinite sequence of categories and functors,



The category E is defined to be the colimit of the sequence (A_n) and the functor v to be the extension of the functors u_n. It is easy to verify that the functor v is conservative. The canonical functor $l : A_0 \to E$ is an iterated localisation. Formally, an *iterated localisation* can be defined to be a functor in the class ${}^{\perp}\mathcal{B}$, where $\mathcal{B}$ is the class of conservative functors.

Theorem C.0.31. *The category* **Cat** *admits a factorisation system* $(\mathcal{A}, \mathcal{B})$ *in which* $\mathcal{B}$ *is the class of conservative functors and* $\mathcal{A}$ *the class of iterated localisations.*

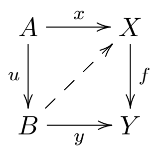
Appendix D

Weak factorisation systems

The theory of weak factorisation systems plays an important role in the theory of quasi-categories and in homotopical algebra, Here we present the basic aspects of the theory. For recent developements, see Casacuberta and al [CF], [CSS] and [CC].

D.1 Basic notions

Definition D.1.1. *A map $u : A \to B$ in a category $\mathcal{E}$ is said to have the* left lifting property (LLP) *with respect to a map $f : X \to Y$, and f is said to have the* right lifting property (RLP) *with respect to u, if every commutative square*



has a diagonal filler $d : B \to X$ (that is, $du = x$ and $fd = y$). We denote this relation by $u \pitchfork f$.

If $\mathcal{A}$ and $\mathcal{B}$ are two classes of maps, we shall write $\mathcal{A} \pitchfork \mathcal{B}$ to indicate that we have $u \pitchfork f$ for every $u \in \mathcal{A}$ and every $f \in \mathcal{B}$.

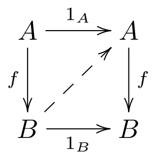
If $\mathcal{M}$ is a class of maps in a category $\mathcal{E}$, we shall denote by ${}^{\pitchfork}\mathcal{M}$ (resp. $\mathcal{M}^{\pitchfork}$) the class of maps in $\mathcal{E}$ having the LLP (resp. RLP) with respect to every map in $\mathcal{M}$. Then

$$\mathcal{A} \subseteq {}^{\pitchfork}\mathcal{B} \iff \mathcal{A} \pitchfork \mathcal{B} \iff \mathcal{B} \subseteq \mathcal{A}^{\pitchfork}.$$

The operations $\mathcal{M} \mapsto \mathcal{M}^{\pitchfork}$ and $\mathcal{M} \mapsto {}^{\pitchfork}\mathcal{M}$ on the classes of maps are contravariant and mutually right adjoint. It follows that each operation $\mathcal{M} \mapsto ({}^{\pitchfork}\mathcal{M})^{\pitchfork}$ and $\mathcal{M} \mapsto {}^{\pitchfork}(\mathcal{M}^{\pitchfork})$ is a closure operator. Each class ${}^{\pitchfork}\mathcal{M}$ and $\mathcal{M}^{\pitchfork}$ contains the isomorphisms and is closed under composition. The intersection ${}^{\pitchfork}\mathcal{M} \cap \mathcal{M}$ (resp. $\mathcal{M} \cap \mathcal{M}^{\pitchfork}$) is a class of isomorphisms by the following lemma.

Lemma D.1.2. *If $f \pitchfork f$, then f is invertible.*

Proof: If $f \pitchfork f$, then the square



has a diagonal filler $g : B \to A$. But then, $gf = 1_A$ and $fg = 1_B$. ■

We say that a class of maps $\mathcal{M}$ in a category with coproducts is *closed under coproducts* if the coproduct

$$\bigsqcup_i u_i : \bigsqcup_i A_i \to \bigsqcup_i B_i$$

of any family of maps $u_i : A_i \to B_i$ in $\mathcal{M}$ belongs to $\mathcal{M}$. The class ${}^{\pitchfork}\mathcal{M}$ is closed under coproducts for any class of maps $\mathcal{M}$ in a category with coproducts. There is a dual notion of a class of maps *closed under products* in a category with products. The class $\mathcal{M}^{\pitchfork}$ is closed under products for any class of maps $\mathcal{M}$ in a category with products.

Proposition D.1.3. *Each class ${}^{\pitchfork}\mathcal{M}$ and $\mathcal{M}^{\pitchfork}$ is closed under composition and retracts. The class $\mathcal{M}^{\pitchfork}$ is closed under base changes and products. Dually, the class ${}^{\pitchfork}\mathcal{M}$ under cobase changes and coproducts.*

We say that a class of maps $\mathcal{M}$ in a category with coproducts is *closed under coproducts* if the coproduct

$$\bigsqcup_i u_i : \bigsqcup_i A_i \to \bigsqcup_i B_i$$

of any family of maps $u_i : A_i \to B_i$ in $\mathcal{M}$ belongs to $\mathcal{M}$. The class ${}^{\pitchfork}\mathcal{M}$ is closed under coproducts for any class of maps $\mathcal{M}$ in a category with coproducts. There is a dual notion of a class of maps *closed under products* in a category with products. The class $\mathcal{M}^{\pitchfork}$ is closed under products for any class of maps $\mathcal{M}$ in a category with products.

Definition D.1.4. *A pair $(\mathcal{A}, \mathcal{B})$ of classes of maps in a category $\mathcal{E}$ is called a* weak factorisation system *if the following two conditions are satisfied:*

- *every map $f \in \mathcal{E}$ admits a factorisation $f = pu$ with $u \in \mathcal{A}$ and $p \in \mathcal{B}$;*
- *$\mathcal{A} = {}^{\pitchfork}\mathcal{B}$ and $\mathcal{A}^{\pitchfork} = \mathcal{B}$.*

We say that $\mathcal{A}$ is the left *class and $\mathcal{B}$ the* right *class of the weak factorisation system.*

Proposition D.1.5. *A factorisation system is a weak factorisation system.*

Proof : Left to the reader.

Each class of a weak factorisation system is closed under composition, retracts, and contains the isomorphisms. The right class is closed under base changes and products. Dually, the left class is closed under cobase changes and coproducts. Each class of a weak factorisation system determines the other.

Proposition D.1.6. *The intersection of the classes of a weak factorisation system is the class of isomorphisms.*

Proof: This follows from D.1.2. ∎

Proposition D.1.7. *Let $(\mathcal{A}, \mathcal{B})$ be a weak factorisation system in a category $\mathcal{E}$. For any object $C \in \mathcal{E}$, let us denote by $\mathcal{A}_C$ (resp. $\mathcal{B}_C$) the class of maps in $\mathcal{E}/C$ with an underlying map in $\mathcal{A}$ (resp. $\mathcal{B}$). Then the pair $(\mathcal{A}_C, \mathcal{B}_C)$ is a weak factorisation system.*

Proof : Left to the reader.

The following conditions on a Grothendieck fibrations $p : E \to B$ are equivalent:

- every arrow in E is cartesian;
- the fibers of p are groupoids.

We call a Grothendieck fibration $p : E \to B$ a *1-fibration* which satisfies these conditions a *1-fibration*.

Recall that the category **Cat** admits a factorisation system in which the right class is the class of discrete fibrations by C.0.30. Let us see that it admits a weak factorisation system in which the right class is the class of 1-fibrations. We say that a category C is *simply connected* if the canonical functor $\pi_1 C \to 1$ is an equivalence, where $\pi_1 C$ is the groupoid freely generated by C. We say that a functor $u : A \to B$ is *1-final* if the category $b\backslash A = (b\backslash B) \times_B A$ defined by the pullback square

$$\begin{array}{ccc} b\backslash A & \xrightarrow{h} & A \\ \downarrow & & \downarrow{\scriptstyle u} \\ b\backslash B & \longrightarrow & B. \end{array}$$

is simply connected for every object $b \in B$. Recall that a functor in **Cat** is a cofibration for the natural model structure on **Cat** iff it is monic on objects.

Theorem D.1.8. *The model category* **Cat** *admits a weak factorisation system* $(\mathcal{A}, \mathcal{B})$ *in which* $\mathcal{B}$ *is the class of 1-fibrations and* $\mathcal{A}$ *is the class of 1-final cofibrations.*

Recall that a map $u : A \to B$ is said to be a *domain retract* of a map $v : C \to B$, if the object (A, u) of the category $\mathcal{E}/B$ is a retract of the object (C, v). A class of maps $\mathcal{M}$ in a category $\mathcal{E}$ is said to be *closed under domain retracts.* if the domain retract of a map in $\mathcal{M}$ belongs to $\mathcal{M}$. There is a dual notion of *codomain retract* and a dual notion of a class of maps *closed under codomain retracts*

Proposition D.1.9. *Let* $(\mathcal{A}, \mathcal{B})$ *be a pair of classes of maps in a category* $\mathcal{E}$*. Suppose that the following conditions are satisfied:*

- *every map* $f \in \mathcal{E}$ *admits a factorisation* $f = pi$ *with* $i \in \mathcal{A}$ *and* $p \in \mathcal{B}$*;*
- $\mathcal{A} \pitchfork \mathcal{B}$*;*
- *the class* $\mathcal{A}$ *is closed under codomain retracts;*
- *the class* $\mathcal{B}$ *is closed under domain retracts.*

Then the pair $(\mathcal{A}, \mathcal{B})$ *is a weak factorisation system.*

Proof: We have $\mathcal{B} \subseteq \mathcal{A}^{\pitchfork}$ since we have $\mathcal{A} \pitchfork \mathcal{B}$. Let us show that $\mathcal{A}^{\pitchfork} \subseteq \mathcal{B}$. If a map $f : X \to Y$ belongs to $\mathcal{A}^{\pitchfork}$, let us choose a factorisation $f = pi : X \to Z \to Y$, with $i \in \mathcal{A}$ and $p \in \mathcal{B}$. The square

$$\begin{array}{ccc} X & \xrightarrow{1_X} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{p} & Y. \end{array}$$

has a diagonal filler $r : Z \to X$ since we have $i \pitchfork f$. This shows that f is a domain retract of p. Thus, $f \in \mathcal{B}$ since $\mathcal{B}$ is closed under domain retracts. ■

Corollary D.1.10. *Let* $(\mathcal{A}, \mathcal{B})$ *be a pair of classes of maps in a category* $\mathcal{E}$*. Suppose that the following conditions are satisfied:*

- *every map* $f \in \mathcal{E}$ *admits a factorisation* $f = pi$ *with* $i \in \mathcal{A}$ *and* $p \in \mathcal{B}$*;*
- $\mathcal{B} = \mathcal{A}^{\pitchfork}$*;*
- *the class* $\mathcal{A}$ *is closed under codomain retracts.*

Then the pair $(\mathcal{A}, \mathcal{B})$ *is a weak factorisation system.*

Proof: We have $\mathcal{A} \pitchfork \mathcal{B}$ since we have $\mathcal{B} = \mathcal{A}^{\pitchfork}$. Moreover, the class $\mathcal{B}$ is closed under codomain retracts for the same reason. This proves the result by D.1.9. ■

If $\mathcal{A}$ be the class of injections in **Set** and $\mathcal{B}$ is the class of surjections, then the pair $(\mathcal{A}, \mathcal{B})$ is a weak factorisation system. We shall see below that the class of monomorphisms in any topos is the left class of a weak factorisation system.

Definition D.1.11. *We shall say that a map in a topos is a* trivial fibration *if it has the right lifting property with respect to every monomorphism.*

This terminology is non-standard but useful. The trivial fibrations often coincide with the acyclic fibrations, which can be defined in a model category. Recall that a *Grothendieck topos* is a category of sheaves with respect to a Grothendieck topology on a small category (we shall only consider toposes of presheaves). We say that an object X in a topos is *injective* if the map $X \to 1$ is a trivial fibration. An object X is injective iff every map $A \to X$ can be extended along every monomorphism $A \to B$.

Theorem D.1.12. *If $\mathcal{A}$ is the class of monomorphisms in a topos, and $\mathcal{B}$ is the class of trivial fibrations, then the pair $(\mathcal{A}, \mathcal{B})$ is a weak factorisation system.*

Proof : Let us denote the topos by $\mathcal{E}$. We shall prove that the conditions of Proposition D.1.10 are satisfied. We have $\mathcal{B} = \mathcal{A}^{\pitchfork}$ by definition. The class $\mathcal{A}$ is obviously closed under (codomain) retracts. It remains to show that every map $f : A \to B$ in $\mathcal{E}$ admits a factorisation $f = pi : A \to Z \to B$ with $i \in \mathcal{A}$ and $p \in \mathcal{B}$. We shall first prove that every object can be embedded into an injective object. Let us first show that the Lawvere object $\Omega \in \mathcal{E}$ is injective. For every object $A \in \mathcal{E}$, let us denote by $\mathcal{P}(A)$ is the set of subobjects of A. The contravariant functor $A \mapsto \mathcal{P}(A)$ is represented by Ω. In order to show that Ω is injective, we have to show that the map $\mathcal{E}(u, \Omega) : \mathcal{E}(B, \Omega) \to \mathcal{E}(A, \Omega)$ is surjective for every monomorphism $u : A \to B$. But the map $\mathcal{E}(u, \Omega)$ is isomorphic to the map $u^* : \mathcal{P}(B) \to \mathcal{P}(A)$, since Ω is representing the functor $\mathcal{P}$. Hence it suffices to show that u^* is surjective. But we have $S = u^*(u(S))$ for every sub-object $S \subseteq A$, since u is monic. This shows that u^* is surjective. We have proved that the object Ω is injective. Let us now show that every object can be embedded into an injective object. It is easy to verify that if Z is an injective object, then so is the object Z^A for any object A. In particular, the object Ω^A is injective for any object A. But the singleton map $A \to \Omega^A$ (which "classifies" the diagonal $A \to A \times A$) is monic by a classical result [Jo]. This show that A can be embedded into an injective object. We can now show that every map $f : A \to B$ in $\mathcal{E}$ admits a factorisation $f = pi : A \to Z \to B$ with $i \in \mathcal{A}$ and $p \in \mathcal{B}$. But the map $p : Z \to B$ is a trivial fibration iff the object (Z, p) of the topos $\mathcal{E}/B$ is injective. Hence the factorisation can be obtained by embedding the object (A, f) of the topos $\mathcal{E}/B$ into an injective object of this topos. The existence of the factorisation is proved. ∎

Proposition D.1.13. *Let $F : \mathcal{D} \leftrightarrow \mathcal{E} : G$ be a pair of adjoint functors. Then for every pair of arrows $u \in \mathcal{D}$ and $f \in \mathcal{E}$ we have*

$$F(u) \pitchfork f \iff u \pitchfork G(f).$$

Proof: The adjunction $\theta : F \dashv G$ induces a bijection between the following commutative squares and their diagonal fillers,

$$\begin{array}{ccc} FA & \xrightarrow{x} & X \\ {\scriptstyle Fu}\downarrow & \nearrow^{d} & \downarrow{\scriptstyle f} \\ FB & \xrightarrow[y]{} & Y, \end{array} \qquad\qquad \begin{array}{ccc} A & \xrightarrow{x'} & GX \\ {\scriptstyle u}\downarrow & \nearrow^{d'} & \downarrow{\scriptstyle Gf} \\ B & \xrightarrow[y']{} & GY, \end{array}$$

where $x' = \theta x$, $y' = \theta y$ and $d' = \theta d$. ■

Lemma D.1.14. *Let $F : \mathcal{D} \leftrightarrow \mathcal{E} : G$ be a pair of adjoint functors. If $(\mathcal{A}, \mathcal{B})$ is a weak factorisation system in $\mathcal{D}$ and $(\mathcal{A}', \mathcal{B}')$ is a weak factorisation system in $\mathcal{E}$ then,*

$$F(\mathcal{A}) \subseteq \mathcal{A}' \quad \Longleftrightarrow \quad G(\mathcal{B}') \subseteq \mathcal{B}.$$

Proof: If $F(\mathcal{A}) \subseteq \mathcal{A}'$, let us show that we have $G(\mathcal{B}') \subseteq \mathcal{B}$. If $g \in \mathcal{B}'$, then we have $F(f) \pitchfork g$ for every $f \in \mathcal{A}$, since $F(\mathcal{A}) \subseteq \mathcal{A}'$ and $\mathcal{A}' \pitchfork \mathcal{B}'$. But the condition $F(f) \pitchfork g$ is equivalent to the condition $f \pitchfork G(g)$ by D.1.13. It follows that we have $f \pitchfork G(g)$ for every $f \in \mathcal{A}$. Thus, $G(g) \in \mathcal{B}$ since $\mathcal{B} = \mathcal{A}^{\pitchfork}$. ■

Let $\mathcal{D}$ and $\mathcal{E}$ be two categories and $\alpha : F_0 \to F_1$ be a natural transformation between two functors $\mathcal{D} \to \mathcal{E}$. Let us suppose that $\mathcal{E}$ admits pushout. If $u : A \to B$ is a map in $\mathcal{D}$, let us denote by $\alpha_\bullet(u)$ the map

$$F_0 B \sqcup_{F_0 A} F_1 A \to F_1 B$$

obtained from the naturality square

$$\begin{array}{ccc} F_0 A & \xrightarrow{\alpha_A} & F_1 A \\ {\scriptstyle F_0 u}\downarrow & & \downarrow{\scriptstyle F_1 u} \\ F_0 B & \xrightarrow{\alpha_B} & F_1 B. \end{array}$$

This defines a functor

$$\alpha_\bullet : \mathcal{D}^I \to \mathcal{E}^I$$

where $\mathcal{D}^I$ (resp. $\mathcal{E}^I$) is the category of arrows of $\mathcal{D}$ (resp. $\mathcal{E}$). Dually, let $\beta : G_1 \to G_0$ be a natural transformation between two functors $\mathcal{E} \to \mathcal{D}$. Let us suppose that $\mathcal{D}$ admits pullbacks. If $f : X \to Y$ is a map in $\mathcal{E}$, let us denote by $\beta^\bullet(f)$ then map

$$G_1 X \to G_1 Y \times_{G_0 Y} G_0 X$$

obtained from the naturality square

$$\begin{array}{ccc} G_1X & \xrightarrow{\beta_X} & G_0X \\ {\scriptstyle G_1f}\downarrow & & \downarrow{\scriptstyle G_0f} \\ G_1Y & \xrightarrow{\beta_Y} & G_0Y. \end{array}$$

This defines a functor

$$\beta^\bullet : \mathcal{E}^I \to \mathcal{D}^I.$$

Suppose now that the functor F_i is left adjoint to the functor G_i,

$$F_i : \mathcal{D} \leftrightarrow \mathcal{E} : G_i$$

for $i = 0, 1$, and that $\alpha : F_0 \to F_1$ is the left transpose of $\beta : G_1 \to G_0$. This means that α is the composite

$$F_0 \xrightarrow{F_0 \circ \mu_1} F_0 \circ \beta \circ F_1 \xrightarrow{\epsilon_0 \circ F_1} F_1,$$

where $\mu_1 : Id \to G_1F_1$ is the unit of the adjunction $F_1 \dashv G_1$ and where $\epsilon_0 : F_0G_0 \to Id$ the counit of the adjunction $F_0 \dashv G_0$. In which case β is the right transpose of α. This means that β is the composite

$$G_1 \xrightarrow{\mu_0 \circ G_1} G_0 \circ \alpha \circ G_1 \xrightarrow{G_1 \circ \epsilon_1} G_0,$$

where $\mu_0 : Id \to G_0F_0$ is the unit of the adjunction $F_0 \dashv G_0$ and where $\epsilon_1 : F_1G_1 \to Id$ is the counit of the adjunction $F_1 \dashv G_1$.

Lemma D.1.15. *With the hypothesis above, we have $\alpha_\bullet \vdash \beta^\bullet$. Thus, for any pair of maps $u : A \to B$ in $\mathcal{D}$ and $f : X \to Y$ in $\mathcal{E}$, there is bijective between the following commutative squares,*

$$\begin{array}{ccc} F_0B \sqcup_{F_0A} F_1A & \longrightarrow & X \\ {\scriptstyle \alpha_\bullet(u)}\downarrow & & \downarrow{\scriptstyle f} \\ F_1B & \longrightarrow & Y, \end{array} \qquad \begin{array}{ccc} A & \longrightarrow & G_0X \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle \beta^\bullet(f)} \\ B & \longrightarrow & G_1Y \times_{G_0Y} G_0X. \end{array}$$

If one of the square has a diagonal filler, so does the other. Thus,

$$\alpha_\bullet(u) \pitchfork f \quad \Longleftrightarrow \quad u \pitchfork \beta^\bullet(f).$$

See D.1.1 for a definition of the relation ψ.

Proof: We only sketch the proof. Let us show that $\alpha_\bullet \vdash \beta^\bullet$. A map $u \to \beta^\bullet(f)$ in $\mathcal{D}^I$ is a commutative square in $\mathcal{D}$:

$$\begin{array}{ccc} A & \xrightarrow{\quad a \quad} & G_1X \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle \beta^\bullet(f)} \\ B & \xrightarrow{\quad (b,c) \quad} & G_1Y \times_{G_0Y} G_0X. \end{array}$$

The square is defined by three maps $a : A \to G_1X$, $b : B \to G_1Y$ and $c : B \to G_0X$ fitting in a commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{a} & G_1X & & \\ {\scriptstyle u}\downarrow & & \big| & \searrow^{\beta_X} & \\ B & \xrightarrow{\qquad\qquad c} & \big| & \longrightarrow & G_0X \\ & \searrow^{b} & \downarrow & & \downarrow{\scriptstyle G_0f} \\ & & G_1Y & \xrightarrow[\beta_X]{} & G_0Y. \end{array}$$

By adjointness, the map $a : A \to G_1X$ corresponds to a map $a' : F_1A \to X$, the map $b : B \to G_1Y$ to a map $b' : F_1B \to Y$, and the map $c : B \to G_0X$ to a map $c' : F_0B \to X$. It is easy to verify that the three maps a', b' and c' fit in the commutative diagram

$$\begin{array}{ccccc} F_0A & \xrightarrow{\alpha_A} & F_1A & & \\ {\scriptstyle F_0u}\downarrow & & \big| & \searrow^{a'} & \\ F_0B & \xrightarrow{\qquad\qquad c'} & \big| & \longrightarrow & X \\ & \searrow^{\alpha_B} & \downarrow & & \downarrow{\scriptstyle f} \\ & & F_1B & \xrightarrow{b'} & Y. \end{array}$$

From the diagram, we obtain a commutative square

$$\begin{array}{ccc} F_0B \sqcup_{F_0A} F_1A & \xrightarrow{\quad (c',a') \quad} & X \\ {\scriptstyle \alpha_\bullet(u)}\downarrow & & \downarrow{\scriptstyle f} \\ F_1B & \xrightarrow{\quad b' \quad} & Y, \end{array}$$

and hence a map $\alpha_\bullet(u) \to f$. This defines the adjunction $\alpha_\bullet \vdash \beta^\bullet$. A diagonal filler of the square $u \to \beta^\bullet(f)$ is given by a map $d : B \to G_1X$ such that $du = a$, $(G_1f)d = b$ and $\beta_X d = c.$. By adjointness, it corresponds to a map $d' : F_1B \to X$ such that $fd' = b'$, $d'(F_1f) = a'$ and $d'\alpha_B = c'$. The map d' is a diagonal filler of the corresponding square $\alpha_\bullet(u) \to f$. This defines a bijection between the diagonal fillers of a square $u \to \beta^\bullet(f)$ and the diagonal fillers of the corresponding square $\alpha_\bullet(u) \to f$. ■

Let $\odot : \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3$ be a functor of two variables with values in a finitely cocomplete category $\mathcal{E}_3$.

Notation D.1.16. *If $u : A \to B$ is map in $\mathcal{E}_1$ and $v : S \to T$ is a map in $\mathcal{E}_2$, we shall denote by $u \odot' v$ the map*

$$A \odot T \sqcup_{A \odot S} B \odot S \longrightarrow B \odot T$$

obtained from the commutative square

$$\begin{array}{ccc} A \odot S & \longrightarrow & B \odot S \\ \downarrow & & \downarrow \\ A \odot T & \longrightarrow & B \odot T. \end{array}$$

This defines a functor of two variables

$$\odot' : \mathcal{E}_1^I \times \mathcal{E}_2^I \to \mathcal{E}_3^I,$$

where $\mathcal{E}^I$ denotes the category of arrows of a category $\mathcal{E}$. Recall that the functor $\odot$ is said to be *divisible on the left* if the functor $A \odot (-) : \mathcal{E}_2 \to \mathcal{E}_3$ admits a right adjoint $A\backslash(-) : \mathcal{E}_3 \to \mathcal{E}_2$ for every object $A \in \mathcal{E}_1$. Dually, the functor $\odot$ is said to be *divisible on the right* if the functor $(-) \odot B : \mathcal{E}_1 \to \mathcal{E}_3$ admits a right adjoint $(-)/B : \mathcal{E}_3 \to \mathcal{E}_1$ for every object $B \in \mathcal{E}_2$.

Notation D.1.17. *Suppose that the functor $\odot : \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3$ is divisible on the left and that the category $\mathcal{E}_2$ is finitely complete. If $u : A \to B$ is map in $\mathcal{E}_1$ and $f : X \to Y$ is a map in $\mathcal{E}_3$, we denote by $\langle u\backslash\, f\rangle$ the map*

$$B\backslash X \to B\backslash Y \times_{A\backslash Y} A\backslash X$$

obtained from the commutative square

$$\begin{array}{ccc} B\backslash X & \longrightarrow & A\backslash X \\ \downarrow & & \downarrow \\ B\backslash Y & \longrightarrow & A\backslash Y. \end{array}$$

Dually, suppose that the category $\mathcal{E}_1$ is finitely complete and that the functor $\odot$ is divisible on the right. If $v : S \to T$ is map in $\mathcal{E}_2$ and $f : X \to Y$ is a map in $\mathcal{E}_3$, we denote by $\langle f/v\rangle$ the map

$$X/T \to Y/T \times_{Y/S} X/S$$

obtained from the commutative square

$$\begin{array}{ccc} X/T & \longrightarrow & X/S \\ \downarrow & & \downarrow \\ Y/T & \longrightarrow & Y/S. \end{array}$$

Proposition D.1.18. *If the functor $\odot : \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3$ is divisible on the left, then the functor $f \mapsto \langle u\backslash f\rangle$ is right adjoint to the functor $v \mapsto u \odot' v$ for every map $u \in \mathcal{E}_1$. Dually, if the functor $\odot$ is divisible on the right, then the functor $f \mapsto \langle f/v\rangle$ is right adjoint to the functor $u \mapsto u \odot' v$ for every map $v \in \mathcal{E}_2$. If $u \in \mathcal{E}_1$, $v \in \mathcal{E}_2$ and $f \in \mathcal{E}_3$ are three maps, then*

$$(u \odot' v) \pitchfork f \quad \Longleftrightarrow \quad u \pitchfork \langle f/v\rangle \quad \Longleftrightarrow \quad v \pitchfork \langle u\backslash f\rangle.$$

Proof We shall use D.1.15. Let $v : S \to T$ be a fixed map in $\mathcal{E}_2$. For every $A \in \mathcal{E}_2$, let us put $F_0(A) = A \otimes S$, $F_1(A) = A \otimes T$ and $\alpha_A = A \otimes v$. This defines a natural transformation $\alpha : F_0 \to F_1$ between two functors $\mathcal{E}_1 \to \mathcal{E}_3$. If $u : A \to B$, then $\alpha_\bullet(u) = u \otimes' v$. The functor F_0 has a right adjoint $X \mapsto X/S = G_0(X)$ and the functor F_1 has a right adjoint $X \mapsto X/T = G_1(X)$. The map $X/v : X/T \to X/S$ defines a natural transformation $\beta : G_1 \to G_0$ which is the right transpose of the natural transformation $\alpha : F_0 \to F_1$. If $f : X \to Y$, then $\beta^\bullet(f) = \langle f/v\rangle$. Hence the functor $f \mapsto \langle f/v\rangle$ is right adjoint to the functor $u \mapsto u \otimes' v$ by D.1.15. Moreover, the condition $(u \otimes' v) \pitchfork f$ is equivalent to the condition $u \pitchfork \langle f/v\rangle$. ■

Notation D.1.19. *Let $\mathcal{E} = (\mathcal{E}, \otimes, \sigma)$ be a symmetric monoidal closed category, with symmetry $\sigma : A \otimes B \simeq B \otimes A$. Then the objects X/A and $A\backslash X$ are canonicaly isomorphic; we can identify them by adopting a common notation, for example $[A, X]$. Similarly, the maps $\langle f/u\rangle$ and $\langle u\backslash f\rangle$ are canonicaly isomorphic. See D.1.17; we can identify them by adopting a common notation, for example $\langle u, f\rangle$.*

In the notation introduced above we have

$$(u \otimes' v) \pitchfork f \quad \Longleftrightarrow \quad u \pitchfork \langle v, f\rangle \quad \Longleftrightarrow \quad v \pitchfork \langle u, f\rangle$$

by D.1.18.

Let $\mathcal{M}_i \subseteq \mathcal{E}_i$ is a class of maps for $i = 1, 2, 3$. We shall denote by $\mathcal{M}_1 \otimes' \mathcal{M}_2$ the class of maps $u_1 \otimes' u_2$ for $u_1 \in \mathcal{M}_1$ and $u_2 \in \mathcal{M}_2$, by $\langle \mathcal{M}_1\backslash\mathcal{M}_3\rangle$ the class of maps $\langle u_1\backslash u_3\rangle$ for $u_1 \in \mathcal{M}_1$ and $u_3 \in \mathcal{M}_3$ and by $\langle \mathcal{M}_3/\mathcal{M}_2\rangle$ the class of maps $\langle u_3/u_2\rangle$ for $u_3 \in \mathcal{M}_3$ and $u_2 \in \mathcal{M}_2$.

Corollary D.1.20. . *Let $\odot : \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3$ be a functor divisible on both sides between finitely biccomplete categories. If $(\mathcal{A}_i, \mathcal{B}_i)$ be a weak factorisation systems in $\mathcal{E}_i$ for $i = 1, 2, 3$, then*

$$\mathcal{A}_1 \odot' \mathcal{A}_2 \subseteq \mathcal{A}_3 \quad \Longleftrightarrow \quad \langle \mathcal{A}_1\backslash\mathcal{B}_3\rangle \subseteq \mathcal{B}_2 \quad \Longleftrightarrow \quad \langle \mathcal{B}_3/\mathcal{A}_2\rangle \subseteq \mathcal{B}_1.$$

Proof: Let us prove the first equivalence. The condition $\mathcal{A}_1 \odot' \mathcal{A}_2 \subseteq \mathcal{A}_3$ is equivalent to the condition $(\mathcal{A}_1 \odot' \mathcal{A}_2) \pitchfork \mathcal{B}_3$, since $\mathcal{A}_3 = {}^{\pitchfork}\mathcal{B}_3$. But the condition $(\mathcal{A}_1 \odot' \mathcal{A}_2) \pitchfork \mathcal{B}_2$ is equivalent to the condition $\mathcal{A}_2 \pitchfork \langle \mathcal{A}_1 \backslash \mathcal{B}_3 \rangle$ by D.1.18. Finally, the condition $\mathcal{A}_2 \pitchfork \langle \mathcal{A}_1 \backslash \mathcal{B}_3 \rangle$ is equivalent to the condition $\langle \mathcal{A}_1 \backslash \mathcal{B}_3 \rangle \subseteq \mathcal{B}_2$, since we have $\mathcal{B}_2 = \mathcal{A}_2^{\pitchfork}$. The first equivalence is proved. The second equivalence follows by symmetry.

■

D.2 Existence of weak factorisation systems

Let $\mathcal{E}$ be a category closed under directed colimits. If $\alpha = \{i : i < \alpha\}$ is a non-zero limit ordinal, we shall say that a functor $C : \alpha \to \mathcal{E}$ is *transfinite chain* if the canonical map

$$\varinjlim_{i<j} C(i) \to C(j)$$

is an isomorphism for every non-zero limit ordinal $j < \alpha$. The *composite* of C is the canonical map

$$C(0) \to \varinjlim_{i<\alpha} C(i).$$

We shall say that a subcategory $\mathcal{A} \subseteq \mathcal{E}$ is closed under *transfinite composition* if the composite of any transfinite chain $C : \alpha \to \mathcal{E}$ with values in $\mathcal{A}$ belongs to $\mathcal{A}$. The class ${}^{\pitchfork}\mathcal{M}$ is closed under transfinite composition for any class of maps $\mathcal{M}$ in $\mathcal{E}$.

Recall that an ordinal α is said to be a *cardinal* if it is smallest among the ordinals with the same cardinality.

Proposition D.2.1. *Let $\mathcal{A}$ be a class of maps in a cocomplete category $\mathcal{E}$. Suppose that $\mathcal{A}$ contains the isomorphisms. If $\mathcal{A}$ is closed under cobase change and transfinite composition, then it is closed under coproducts.*

Proof Let us call an object $A \in \mathcal{E}$ *cofibrant*, if the map $\bot \to A$ belongs to $\mathcal{A}$, where $\bot$ is the initial object of $\mathcal{E}$. Let us first show that the coproduct of any family of cofibrant objects is cofibrant. The identity map $\bot \to \bot$ belongs to $\mathcal{A}$, since $\mathcal{A}$ contains the isomorphisms. Thus, the initial object $\bot$ is cofibrant. This shows that the coproduct of an empty family is cofibrant. since the identity map $\bot \to \bot$ belongs to $\mathcal{A}$. Let us now show that the coproduct of a finite non-empty family of cofibrant objects is cofibrant. For this it suffices to show that the coproduct of two cofibrant objects is cofibrant. If A and B are cofibrant, consider the pushout square

$$\begin{array}{ccc} \bot & \longrightarrow & B \\ \downarrow & & \downarrow{\scriptstyle i_2} \\ A & \xrightarrow{i_1} & A \sqcup B. \end{array}$$

The map i_2 is a cobase change of the map $\bot \to A$. Thus, $i_2 \in \mathcal{A}$, since A is cofibrant and since $\mathcal{A}$ is closed under cobase change. But the map $\bot \to B$ belongs to $\mathcal{A}$, since B is cofibrant. Hence the composite $\bot \to B \to A \sqcup B$ belongs to $\mathcal{A}$, since $\mathcal{A}$ is closed under composition. This shows that the coproduct $A \sqcup B$ is cofibrant. Let us now show by induction on $\alpha = \mathrm{Card}(I)$ that the coproduct

$$A = \bigsqcup_{i \in I} A_i$$

of an infinite family of cofibrant objects is cofibrant. The object $C_j = \bigsqcup_{i<j} A_i$ is cofibrant for every $j < \alpha$ by the induction hypothesis. This defines a chain $C : \alpha \to \mathcal{E}$ since we have

$$\varinjlim_{i<j} C_i = C_j$$

for every $j < \alpha$. Notice that $C_0 = \bot$ and that

$$\varinjlim_{i<\alpha} C_i = \bigsqcup_{i<\alpha} A_i = A.$$

We shall prove that A is cofibrant by showing that the composite of the chain C belongs to $\mathcal{A}$. For every $j \leq k < \alpha$, let us put $C_k^j = \bigsqcup_{j \leq i < k} A_i$. The object C_k^j is cofibrant by the induction hypothesis, since $\mathrm{Card}\{i : j \leq i < k\} < \alpha$. We have a pushout square

$$\begin{array}{ccc} \bot & \longrightarrow & C_j \\ \downarrow & & \downarrow \\ C_k^j & \longrightarrow & C_k. \end{array}$$

Hence the map $C_j \to C_k$ belongs to $\mathcal{A}$, since $\mathcal{A}$ is closed under cobase change. This shows that the composite of the chain C belongs to $\mathcal{A}$, since $\mathcal{A}$ is closed under transfinite composition. This proves that A is cofibrant. Let us now show that the class $\mathcal{A}$ is closed under coproducts. Let $u_i : A_i \to B_i$ ($i \in I$) be a family of maps in $\mathcal{A}$ and let $u : A \to B$ be its coproduct. Let $\mathcal{A}'$ be the class of maps in the category $A\backslash\mathcal{E}$ whose underlying map in $\mathcal{E}$ belongs to the class $\mathcal{A}$. It is easy to verify that the class $\mathcal{A}'$ satisfies the hypothesis of the proposition. Let us put $C_i = B_i \sqcup_{A_i} A$ for each $i \in I$,

$$\begin{array}{ccc} A_i & \longrightarrow & A \\ {\scriptstyle u_i}\downarrow & & \downarrow{\scriptstyle v_i} \\ B_i & \longrightarrow & C_i. \end{array}$$

The map $v_i : A \to C_i$ belongs to $\mathcal{A}$, since $u_i \in \mathcal{A}$ by assumption, and since $\mathcal{A}$ is closed under cobase change. Hence the object (C_i, v_i) of the category $A\backslash\mathcal{E}$ is cofibrant with respect to the class $\mathcal{A}'$. The coproduct of the objects (C_i, v_i) for $i \in I$ is the object (B, u). This shows that the object (B, u) is cofibrant with respect to $\mathcal{A}'$. by the first part of the proof. Thus, $u \in \mathcal{A}$. ∎

Definition D.2.2. *We shall say that a class of maps $\mathcal{A}$ in a cocomplete category is* saturated *if it satisfies the following conditions:*

- *$\mathcal{A}$ contains the isomorphisms and is closed under composition ;*
- *$\mathcal{A}$ is closed under cobase change and retracts;*
- *$\mathcal{A}$ is closed under transfinite composition.*

A saturated class is closed under coproducts by D.2.1. The following result follows from the discussion above.

Proposition D.2.3. *The class ${}^{\pitchfork}\mathcal{M}$ is saturated for any class of maps $\mathcal{M}$ in a cocomplete category.*

In particular, the left class of any weak factorisation system in a cocomplete category is saturated. For example, the class of monomorphisms in any Grothendieck topos is saturated.

Every class of maps $\mathcal{M}$ in a cocomplete category is contained in a smallest saturated class.

Definition D.2.4. *If $\mathcal{M}$ is a class of maps in a cocomplete category $\mathcal{E}$, we denote by $\overline{\mathcal{M}}$ the smallest saturated class which contains $\mathcal{M}$. We shall say that $\overline{\mathcal{M}}$ is the saturated class* generated *by $\mathcal{M}$.*

Proposition D.2.5. *Let $F : \mathcal{U} \to \mathcal{V}$ be a cocontinuous functor between cocomplete categories. If a class of maps $\mathcal{C} \subseteq \mathcal{V}$ is saturated, then so is the class*

$$F^{-1}(\mathcal{C}) = \{f \in \mathcal{U} : F(f) \in (\mathcal{C}\}.$$

Proposition D.2.6. *Let $\odot : \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3$ be a functor divisible on both sides between finitely biccomplete categories. If $\mathcal{A}$ is the left class of a weak factorisation system in $\mathcal{E}_3$ and $\mathcal{M}_i$ is a class of maps in $\mathcal{E}_i$ for $i = 1, 2$, then*

$$\mathcal{M}_1 \odot' \mathcal{M}_2 \subseteq \mathcal{A} \quad \Longrightarrow \quad \overline{\mathcal{M}}_1 \odot' \overline{\mathcal{M}}_2 \subseteq \mathcal{A}.$$

Proof: It suffices to prove the implication $\mathcal{M}_1 \odot' \mathcal{M}_2 \subseteq \mathcal{A} \Rightarrow \overline{\mathcal{M}}_1 \odot' \mathcal{M}_2 \subseteq \mathcal{A}$. If $\mathcal{B} = \mathcal{A}^{\pitchfork}$, then $\mathcal{A} = \mathcal{B}^{\pitchfork}$, since $\mathcal{A}$ is the left class of a weak factorisation system. Hence it suffices to prove the implication $(\mathcal{M}_1 \odot' \mathcal{M}_2) \pitchfork \mathcal{B} \Rightarrow (\overline{\mathcal{M}}_1 \odot' \mathcal{M}_2) \pitchfork \mathcal{B}$. But for this, it suffices to prove the implication $\mathcal{M}_1 \odot' v) \pitchfork f \Rightarrow (\overline{\mathcal{M}}_1 \times' v) \pitchfork f$ for any pair of maps $v : S \to T$ and $f : X \to Y$. Let $\mathcal{C}$ be the class of maps $u : A \to B$ for which we have $(u \odot' v) \pitchfork f$. A map $u : A \to B$ belongs to $\mathcal{C}$ iff we have $u \pitchfork \langle f/v \rangle$ by D.1.18. Hence the class $\mathcal{C}$ is saturated by D.2.3. Thus, $\mathcal{M}_1 \subseteq \mathcal{C} \Rightarrow \overline{\mathcal{M}}_1 \subseteq \mathcal{C}$. This shows that $(\mathcal{M}_1 \odot' v) \pitchfork f \Rightarrow (\overline{\mathcal{M}}_1 \odot' v) \pitchfork f$. The result follows. ∎

Proposition D.2.7. *Let $\mathcal{M}$ be a class of maps in a cocomplete category $\mathcal{E}$. If B is an object of $\mathcal{E}$, let us put $\mathcal{M}_B = U^{-1}(\mathcal{M})$, where U be the forgetful functor $\mathcal{E}/B \to \mathcal{E}$. Then the saturated class generated by $\mathcal{M}_B$ is equal to $U^{-1}(\overline{\mathcal{M}})$, where $\overline{\mathcal{M}}$ is the saturated class generated by $\mathcal{M}$.*

If Σ is a class of maps in a category $\mathcal{E}$, we shall say that an object $X \in \mathcal{E}$ is Σ-*injective* if the map

$$\mathcal{E}(u, X) : \mathcal{E}(B, X) \to \mathcal{E}(A, X)$$

is surjective for every map $u : A \to B$ in Σ. When $\mathcal{E}$ has a terminal object 1, an object $X \in \mathcal{E}$ is Σ-injective iff the map $X \to 1$ belongs to $\Sigma^{\pitchfork}$. Recall that an object A in a cocomplete category $\mathcal{E}$ is said to be *small* if the functor $\mathcal{E}(A, -) : \mathcal{E} \to \mathbf{Set}$ preserves α-directed colimits for α a regular cardinal large enough.

Proposition D.2.8. (Small object argument) *Let Σ be a set of maps in a cocomplete category $\mathcal{E}$ and let $\overline{\Sigma}$ be the saturated class generated by Σ. If the domain of each map in Σ is small, then there exists a functor $R : \mathcal{E} \to \mathcal{E}$ together with a natural transformation $\rho : Id \to R$ such that:*

- *the object $R(X)$ is Σ-injective for every object $X \in \mathcal{E}$;*
- *the map $\rho_X : X \to R(X)$ belongs to $\overline{\Sigma}$ for every $X \in \mathcal{E}$.*

Moreover, the functor R preserves α-directed colimits if the domain of each map in Σ is α-presentable.

Proof: We first consider the case where the domain of each map in Σ is finitely presentable. If S is a set and A is an object in $\mathcal{E}$, we shall denote by $S \otimes A$ the coproduct of S copies of A. For a fixed object A, the functor $S \mapsto S \otimes A$ is left adjoint to the functor $X \mapsto \mathcal{E}(A, X)$. Let $\epsilon_X^A : \mathcal{E}(A, X) \otimes A \to X$ be the counit of the adjunction. For every object $X \in \mathcal{E}$ let us put

$$S(X) = \bigsqcup_{\sigma \in \Sigma} \mathcal{E}(s(\sigma), X) \otimes s(\sigma), \qquad T(X) = \bigsqcup_{\sigma \in \Sigma} \mathcal{E}(s(\sigma), X) \otimes t(\sigma),$$

where $s(\sigma)$ (resp. $t(\sigma)$) is the source (resp. the target) of map σ. The coproduct over $\sigma \in \Sigma$ of the maps

$$\mathcal{E}(s(\sigma), X) \otimes \sigma : \mathcal{E}(s(\sigma), X) \otimes s(\sigma) \to \mathcal{E}(s(\sigma), X) \otimes t(\sigma)$$

is a map $\phi_X : S(X) \to T(X)$. The counits $\epsilon_X^{s(\sigma)} : \mathcal{E}(s(\sigma), X) \otimes s(\sigma) \to X$ induce a map $\epsilon_X : S(X) \to X$. This defines two functors $S, T : \mathcal{E} \to \mathcal{E}$ and two natural transformations $\phi : S \to T$ and $\epsilon : S \to Id$, where Id denotes the identity functor. For every object $X \in \mathcal{E}$, let us denote by $F(X)$ the object defined by the pushout square

$$\begin{array}{ccc} S(X) & \xrightarrow{\epsilon_X} & X \\ {\scriptstyle \phi_X}\downarrow & & \downarrow{\scriptstyle \theta_X} \\ T(X) & \longrightarrow & F(X). \end{array}$$

This defines a functor $F : \mathcal{E} \to \mathcal{E}$ together with a natural transformation $\theta : Id \to F$. Observe that every map $x : A \to X$ admits a canonical factorisation $x = \epsilon^A_X(x' \otimes A) : A \to \mathcal{E}(A, X) \otimes A \to X$, where $x' : 1 \to \mathcal{E}(A, X)$ corresponds to $x : A \to X$. It follows that for every map $\sigma : A \to B$ in Σ and every map $x : A \to X$ we have a commutative diagram of canonical maps,

$$\begin{array}{ccccccc}
A & \xrightarrow{x'\otimes A} & \mathcal{E}(A,X)\otimes A & \longrightarrow & S(X) & \xrightarrow{e_X} & X \\
{\scriptstyle\sigma}\downarrow & & \downarrow & & \downarrow{\scriptstyle\phi_X} & & \downarrow{\scriptstyle\theta_X} \\
B & \xrightarrow{x'\otimes B} & \mathcal{E}(B,X)\otimes B. & \longrightarrow & T(X) & \longrightarrow & F(X),
\end{array}$$

in which the composite of the top maps is equal to x. If $\tilde{x}$ denotes the composite of the bottom maps, then we have $\tilde{x}\sigma = \theta_X x$. Let us denote by $R(X)$ the colimit of the infinite sequence

$$X \xrightarrow{\theta^0} F(X) \xrightarrow{\theta^1} F^2(X) \xrightarrow{\theta^2} F^3(X) \longrightarrow \cdots$$

where $\theta^n = \theta_{F^n(X)}$. This defines a functor $R : \mathcal{E} \to \mathcal{E}$. Let $v_n : F^n(X) \to R(X)$ be the canonical map and let us put $\rho_X = v_0 : X \to R(X)$. This defines a natural transformation $\rho : Id \to R$. Let us show that the object $R(X)$ is Σ-injective for every object X. If $A \in \mathcal{E}$ is a finitely presentable object, then the canonical map

$$\varinjlim_n \mathcal{E}(A, F^n(X)) \to \mathcal{E}(A, R(X))$$

is an isomorphism. Thus, for every $\sigma \in \Sigma$ and every map $x : s(\sigma) \to R(X)$, there exist an integer $n \geq 0$ together with a map $y : s(\sigma) \to F^n(X)$ such that $x = v_n y$, since the codomain of σ is finitely presentable by the assumption. But there is then a map $\tilde{y} : t(\sigma) \to F^{n+1}(X)$ such that $\tilde{y}\sigma = \theta^n y$, by the observation made above,

$$\begin{array}{ccccc}
s(\sigma) & \xrightarrow{y} & F^n(X) & \xrightarrow{v_n} & R(X) \\
{\scriptstyle\sigma}\downarrow & & {\scriptstyle\theta^n}\downarrow & \nearrow{\scriptstyle v_{n+1}} & \\
t(\sigma) & \xrightarrow{\tilde{y}} & F^{n+1}(X). & &
\end{array}$$

If $z = v_{n+1}\tilde{y}$, then $z\sigma = v_{n+1}\tilde{y}\sigma = v_n y = x$. This shows that $R(X)$ is Σ-injective. Let us now show that the map $\rho_X : X \to R(X)$ belongs to $\overline{\Sigma}$. The map ϕ_X belongs to $\overline{\Sigma}$, since a saturated class is closed under coproducts by D.2.1. Hence also the map θ_X by cobase change. This shows that the map θ^n belongs to $\overline{\Sigma}$ for every $n \geq 0$. It follows that $\rho_X : X \to R(X)$ belongs to $\overline{\Sigma}$, since a saturated class is closed under transfinite composition. Let us show that the functor R preserves directed colimits. The functor $\mathcal{E}(A, -)$ preserves directed colimits for any finitely presentable object A. Hence, also the functor $\mathcal{E}(A, -) \otimes B$ for any object B, since the functor $(-) \otimes B$ is cocontinuous. The functor R is by construction a colimit of functors of the form $\mathcal{E}(A,-)\otimes B$, for small objects A. It follows that R preserves directed colimits. This completes the proof of the proposition in the case where the domain of each map in Σ is finitely presentable. Let us now consider the general case. The chain $X \to F(X) \to F^2(X) \to \cdots$ can be extended cocontinuously through all the ordinals by putting $F^{\alpha+1}(X) = F(F^\alpha(X))$ and

$$\theta^\alpha = \theta_{F^\alpha(X)} : F^\alpha(X) \to F^{\alpha+1}(X)$$

for every ordinal α. By construction, we have

$$F^\alpha(X) = \varinjlim_{i<\alpha} F^i(X).$$

if α is a non-zero limit ordinal. If $A \in \mathcal{E}$ is a small object, then there exists a regular cardinal α, such that the canonical map

$$\varinjlim_{i<\alpha} \mathcal{E}(A, F^i(X)) \to \mathcal{E}(A, F^\alpha(X))$$

is an isomorphism for every object X. If α is taken large enough, we can suppose that the map is an isomorphism for the domain A of every map in Σ. Let us then put $R(X) = F^\alpha(X)$ and let $v_i : F^i(X) \to R(X)$ be the canonical map for $i < \alpha$. This defines a functor $R : \mathcal{E} \to \mathcal{E}$ equipped with a natural transformation $\rho_X = v_0 : X \to R(X)$. Let us show that the object $R(X)$ is Σ-injective. For every $\sigma \in \Sigma$ and every map $x : s(\sigma) \to R(X)$, there exist an ordinal $i < \alpha$ and a map $y : s(\sigma) \to F^i(X)$ such that $x = v_i y$, by the hypothesis on α. But there is then a map $\tilde{y} : t(\sigma) \to F^{i+1}(X)$ such that $\tilde{y}\sigma = \theta^i y$, by the observation made above,

$$\begin{array}{ccccc} s(\sigma) & \xrightarrow{\ y\ } & F^i(X) & \xrightarrow{\ v_i\ } & R(X) \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \theta^i} & \nearrow{\scriptstyle v_{i+1}} & \\ t(\sigma) & \xrightarrow{\ \tilde{y}\ } & F^{i+1}(X). & & \end{array}$$

If $z = v_{i+1}\tilde{y}$, then $z\sigma = v_{i+1}\tilde{y}\sigma = v_i y = x$. This shows that $R(X)$ is Σ-injective. We leave to the reader the verification that ρ_X belongs to $\overline{\Sigma}$ and the verification that the functor R preserves α-directed colimits if the domain of each map in Σ is α-presentable. ■

A functor $R : \mathcal{E}^I \to \mathcal{E}^I$ together with a natural transformation $\rho : Id \to R$ associates to a map $f : X \to Y$ a commutative square in $\mathcal{E}$,

$$\begin{array}{ccc} X & \xrightarrow{\ \rho_1(f)\ } & R_1(f) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle R(f)} \\ Y & \xrightarrow{\ \rho_0(f)\ } & R_0(f). \end{array}$$

Corollary D.2.9. *Let Σ be a set of maps in a cocomplete category $\mathcal{E}$ and let $\overline{\Sigma}$ be the saturated class generated by Σ. If the domain of each map in Σ is small, then there exists a functor $R : \mathbf{S}^I \to \mathbf{S}^I$ together with a natural transformation $\rho : Id \to R$ such that:*

- *we have $\rho_0(f) \in \overline{\Sigma}$ and $\rho_1(f) \in \overline{\Sigma}$ for every f;*
- *we have $R(f) \in \Sigma^{\pitchfork}$ for every f;*
- *the objects $R_0(f)$ and $R_1(f)$ are Σ-injective for every f;*

Moreover, the functor R preserves α-directed colimits if every the domain of each map in Σ is α-presentable.

Proof: A map $u : a \to b$ in $\mathcal{E}^I$ is a commutative square in $\mathcal{E}$,

$$\begin{array}{ccc} A_1 & \xrightarrow{u_1} & B_1 \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ A_0 & \xrightarrow{u_0} & B_0. \end{array}$$

Let Σ' be the class of maps $(u_0, u_1) : a \to b$ with $u_0 \in \Sigma$ and $u_1 \in \Sigma$. It is easy to verify that a map $f : X \to Y$ in $\mathcal{E}$ is Σ'-injective as an object of $\mathcal{E}^I$ iff $f \in \Sigma^{\pitchfork}$ and Y is Σ-injective (hence also X). The domain of a map in Σ' is a map between two small objects of $\mathcal{E}$ by the assumption on Σ. It is thus a small object of $\mathcal{E}^I$. It then follows from Proposition D.2.8 that there exists a functor $R : \mathcal{E}^I \to \mathcal{E}^I$ together with a natural transformation $\rho : Id \to R$ such that:

- the object $R(f)$ is Σ'-injective for every arrow $f \in \mathcal{E}$;
- the map $\rho(f) : f \to R(f)$ belongs to $\overline{\Sigma}'$ for every arrow $f \in \mathcal{E}$.

Moreover, the functor R preserves α-directed colimits if the domain of each map in Σ is α-presentable. Let us show that $\rho_0(f) \in \overline{\Sigma}$ and $\rho_1(f) \in \overline{\Sigma}$. Let $\mathcal{A}$ be the class of maps $(u_0, u_1) : a \to b$ in $\mathcal{E}^I$ such that $u_0 \in \overline{\Sigma}$ and $u_1 \in \overline{\Sigma}$. It is easy to verify that $\mathcal{A}$ is saturated. Thus, $\overline{\Sigma}' \subseteq \mathcal{A}$ since $\Sigma' \subseteq \mathcal{A}$. This shows that $\rho_0(f) \in \overline{\Sigma}$ and $\rho_1(f) \in \overline{\Sigma}$. ∎

An object of the category $\mathcal{E}^{[2]}$ is a chain of maps $A_0 \to A_1 \to A_2$ in the category $\mathcal{E}$. Consider the composition functor $\sigma_1 : \mathcal{E}^{[2]} \to \mathcal{E}^{[1]}$ which associates to a chain $A_0 \to A_1 \to A_2$ its composite $A_0 \to A_2$. Let ∂_0 and $\partial_1 : \mathcal{E}^{[1]} \to \mathcal{E}$ be respectively the target and source functors. We shall say that a section of the functor σ_1 is a *functorial factorisation* of the maps in $\mathcal{E}$. A functorial factorisation associates to every map $u : A \to B$ in $\mathcal{E}$ a factorisation

$$A \xrightarrow{\phi_1(u)} F(u) \xrightarrow{\phi_0(u)} B.$$

It is determined by a triple (F, ϕ_0, ϕ_1), where F is a functor $\mathcal{E}^{[1]} \to \mathcal{E}$, where ϕ_0 is a natural transformation $F \to \partial_0$ and where ϕ_1 is a natural transformation $\partial_1 \to F$.

Proposition D.2.10. (Small object argument 2) *Let Σ be a set of maps in a cocomplete category $\mathcal{E}$ and let $\overline{\Sigma}$ be the saturated class generated by Σ. If every map in Σ has a small domain and a small codomain, then there exists a functorial factorisation $(F, \phi_0, \phi_1) : \mathcal{E}^{[1]} \to \mathcal{E}^{[2]}$ such that:*

- *the map $\phi_0(u) : F(u) \to \partial_0(u)$ belongs to $\Sigma^{\pitchfork}$ for every map $u \in \mathcal{E}$;*
- *the map $\phi_1(u) : \partial_1(u) \to F(u)$ belongs to $\overline{\Sigma}$ for every $u \in \mathcal{E}$; .*

Moreover, the functor F preserves α-directed colimits if every map in Σ is α-presentable.

Proof: If $u : A \to B$ is a map in $\mathcal{E}$, let us denote by u' the map $(u, 1_B) : u \to 1_B$ in $\mathcal{E}^I$ defined by the commutative square

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle 1_B} \\ B & \xrightarrow{1_B} & B. \end{array}$$

Let us put $\Sigma' = \{u' : u \in \Sigma\}$. The assumption on Σ implies that the maps in Σ' have a small domain. It is easy to verify that a map $f : X \to Y$ belongs to $\Sigma^{\pitchfork}$ iff f is Σ'-injective as an object of $\mathcal{E}^I$. It follows from D.2.8 that there exists a functor $R : \mathcal{E}^I \to \mathcal{E}^I$ together with a natural transformation $\rho : Id \to R$ such that:

- $R(u)$ is Σ'-injective object of $\mathcal{E}^I$ for every arrow $u \in \mathcal{E}$;
- the map $\rho(u) : X \to R(u)$ belongs to $\overline{\Sigma}'$ for every arrow $u \in \mathcal{E}$;

Moreover, the functor R preserves α-directed colimits if the domain of each map in Σ' is α-presentable. The pair (R, ρ) associates to a map $u : A \to B$ in $\mathcal{E}$ a commutative square

$$\begin{array}{ccc} A & \xrightarrow{\rho_1(u)} & R_1(u) \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle R(u)} \\ B & \xrightarrow{\rho_0(u)} & R_0(u). \end{array}$$

The map $R(u)$ belongs to $\Sigma^{\pitchfork}$ since $R(u)$ Σ'-injective as an object of $\mathcal{E}^I$. Let us show that the map $\rho_0(u)$ is invertible and that the map $\rho_1(u)$ belongs to $\overline{\Sigma}$. Let $\mathcal{A}$ be the class of maps $f = (f_1, f_0)$ in $\mathcal{E}^I$ such that f_0 is invertible and such that $f_1 \in \overline{\Sigma}$. It is easy to verify that $\mathcal{A}$ is saturated. Thus, $\overline{\Sigma}' \subseteq \mathcal{A}$ since $\Sigma' \subseteq \mathcal{A}$. This shows that $\rho_0(u)$ is invertible and that $\rho_1(u) \in \overline{\Sigma}$. Let us put $\phi_0(u) = \rho_0(u)^{-1} R(u)$, $\phi_1(u) = \rho_1(u)$ and $F(u) = R_1(u)$. The map $\phi_0(u) : F(u) \to B$ belongs to $\Sigma^{\pitchfork}$

since $R(u) \in \Sigma^{\pitchfork}$. The map $\phi_1(u) : A \to F(u)$ belongs to $\overline{\Sigma}$ since $\rho_1(u) \in \overline{\Sigma}$. Moreover, the the functor F preserves α-directed colimits if the functor R preserves α-directed colimits. ■

Theorem D.2.11. [] *Let Σ be a set of maps in a cocomplete category $\mathcal{E}$. Suppose that the domain of each map in Σ is small, then the pair $(\overline{\Sigma}, \Sigma^{\pitchfork})$ is a weak factorisation system.*

Proof: Let us show that the pair $(\overline{\Sigma}, \Sigma^{\pitchfork})$ is a weak factorisation system. We shall use Corollary D.1.10. The class $\overline{\Sigma}$ is closed under domain retracts. We have $\Sigma^{\pitchfork} = (\overline{\Sigma})^{\pitchfork}$, since the class ${}^{\pitchfork}(\Sigma^{\pitchfork})$ is saturated. Hence it remains to show that every map $u : A \to B$ admits a factorisation $u = pi : A \to E \to B$ with $i \in \overline{\Sigma}$ and $p \in \Sigma^{\pitchfork}$. This is clear if $B = 1$ by D.2.8 Let us show that the problem can be reduced to this case if we replace the category $\mathcal{E}$ by the category $\mathcal{E}/B$. Let us denote by Σ' the set of maps in $\mathcal{E}/B$ whose underlying map belongs to Σ. The domain of each map in Σ' is small, since the domain of each map in Σ is small. It is easy to verify that an object (E, p) of $\mathcal{E}/B$ is Σ'-injective iff the map $p : E \to B$ belongs to $\Sigma^{\pitchfork}$. It then follows by D.2.8 that for every object (A, u) of $\mathcal{E}/B$ there exists a factorisation $u = pi : A \to E \to B$ with $i \in \overline{\Sigma}'$ and (E, p) a Σ'-injective object. We have $p \in \Sigma^{\pitchfork}$ since (E, p) is Σ'-injective. Let us show that $i \in \overline{\Sigma}$. Let $\mathcal{A}$ be the class of maps in $\mathcal{E}/B$ whose underlying map in $\mathcal{E}$ belongs to $\overline{\Sigma}$. It is easy to verify that the class $\mathcal{A}$ is saturated. Hence we have $\overline{\Sigma}' \subseteq \mathcal{A}$ since we have $\Sigma' \subseteq \mathcal{A}$. This shows that $i \in \overline{\Sigma}$. The existence of the factorisation is proved. ■

Definition D.2.12. *Let $\mathcal{E}$ be a category, T be a functor $\mathcal{E}^I \to \mathbf{Set}$ and $S \subseteq T$ be a subfunctor. We shall say that a class of maps $\mathcal{A} \subseteq \mathcal{E}$ is* defined by the equation *$S = T$ if we have*

$$\mathcal{A} = \{f \in \mathcal{E} : S(f) = T(f)\}.$$

We shall say that the equation is accessible if the category $\mathcal{E}$ is accessible and the functors S and T are accessible.

Let $T_i : \mathbf{S}^I \to \mathbf{Set}$ $(i \in I)$ be a family of functors and $S_i \subseteq T_i$ $(i \in I)$ be a family of sub-functors. For each $i \in I$, let us put

$$\mathcal{A}_i = \{f \in \mathbf{S} : S_i(f) = T_i(f)\}.$$

Then we have

$$\bigcap_i \mathcal{A}_i = \{f \in \mathbf{S} : S(f) = T(f)\},$$

where

$$S(f) = \bigsqcup_{i \in I} S_i(f) \quad \text{and} \quad T(f) = \bigsqcup_{i \in I} T_i(f).$$

If the equation $S_i = T_i$ is accessible for each $i \in I$ and then so is the equation $S = T$. This proves the following result:

Proposition D.2.13. *If the class $\mathcal{A}_i$ is defined by an accessible equation for each $i \in I$, then so is the intersection $\bigcap_i \mathcal{A}_i$.*

Lemma D.1. *The class of monomorphisms in a presheaf category can be defined by an acessible equation. This is true also of the class of epimorphisms and of the class of isomorphisms.*

Proof: Let us prove the first statement. Let $\mathcal{E} = [C^o, \mathbf{Set}]$ be a presheaf category. For every $X \in \mathcal{E}$, let us put

$$\mathrm{Total}(X) = \bigsqcup_{c \in C} X(c).$$

This defines a functor $Total : \mathcal{E} \to \mathbf{Set}$. A map $f : X \to Y$ in $\mathcal{E}$ is an isomorphism iff the map $Total(f) : Total(X) \to Total(Y)$ is bijective. But a map $f : X \to Y$ in $\mathcal{E}$ is monic iff the diagonal $\delta(f) : X \to X \times_Y X$ is an isomorphism. Let us put $T(f) = Total(X \times_Y X)$ and let $S(f) \subseteq T(f)$ be the image of $Total(\delta(f))$. This defines a functor $T : \mathcal{E}^I \to \mathbf{Set}$ together with a subfunctor $S \subseteq T$. A map $f : X \to Y$ in $\mathcal{E}$ is monic iff we have $S = T$. It is easy to see from the construction that the functors S and T are acessible. The first statement is proved. The other statements are proved similarly. ∎

Proposition D.2.14. *Let Σ be a set of maps in a presheaf category $\mathcal{E}$, Then the class $\Sigma^{\pitchfork}$ can be defined by an accessible equations $S = T$. The functor S and T preserve α-directed colimits if Σ is a set of maps between between α-presentable objects.*

Proof: If $u : A \to B$ and $f : X \to Y$ are two maps in $\mathcal{E}$, we shall denote by $\langle u, f \rangle$ the map

$$\mathcal{E}(B, X) \to \mathcal{E}(B, Y) \times_{\mathcal{E}(A,Y)} \mathcal{E}(A, X)$$

obtained from the square

$$\begin{array}{ccc}
\mathcal{E}(B, X) & \xrightarrow{\mathcal{E}(v,X)} & \mathcal{E}(A, X) \\
{\scriptstyle \mathcal{E}(B,f)}\downarrow & & \downarrow{\scriptstyle \mathcal{E}(A,f)} \\
\mathcal{E}(B, Y) & \xrightarrow{\mathcal{E}(v,Y)} & \mathcal{E}(A, Y).
\end{array}$$

A map $f \in \mathcal{E}$ belongs to $\Sigma^{\pitchfork}$ iff the map $\langle u, f \rangle$ is surjective for every map $u \in \Sigma$. Let us denote by $T(u, f)$ the codomain of $\langle u, f \rangle$ and by $S(u, f) \subseteq T(u, f)$ its image. A map $f \in \mathcal{E}$ belongs to $\Sigma^{\pitchfork}$ iff we have $S(u, f) = T(u, f)$ for every map $u \in \Sigma$. Let us put

$$T(f) = \bigsqcup_{u \in \Sigma} T(u, f) \quad \text{and} \quad S(f) = \bigsqcup_{u \in \Sigma} S(u, f).$$

This defines a functor $T : \mathcal{E}^I \to \mathbf{Set}$ together with a subfunctor $S \subseteq T$. A map $f \in \mathcal{E}$ belongs to $\Sigma^{\pitchfork}$ iff we have $S(f) = T(f)$. The construction shows that functor

S and T preserve α-directed colimits if Σ is a set of maps between between α-presentable objects. ∎

Let C be a small category. The *cardinality* of a presheaf X on C is defined to be the cardinality of its category of elements $el(X) = C/X$.

Lemma D.2.15. *Let C be a category of cardinality $\leq \alpha$, where α is an infinite cardinal. Then every presheaf on C is the union of its sub presheaves of cardinality $\leq \alpha$.*

Proof: Let X be a presheaf on C. If $x \in el(X)$, let us denote by $I(x)$ the sub presheaf of X. generated by x. Clearly, X is the union of the presheaves $I(x)$ for $x \in el(X)$. Hence it suffices to show that we have $\mathrm{Card}(I(x)) \leq \alpha$ for every $x \in el(X)$. Let us first show that for every object $c \in C$, the representable presheaf $C(-, c)$ has a cardinality $\leq \alpha$. We have $\mathrm{Card}(C/c) \leq \mathrm{Card}(C)^3$, since an arrow in C/c is a triple of arrows in C. But $\alpha^3 = \alpha$, since α is an infinite cardinal. Thus, $\mathrm{Card}(C/c) \leq \alpha$. If $x \in X(c)$ then by Yoneda lemma, there is a natural transformation $u : C(-, c) \to X$ such that $u(1_c) = x$. The image of u is equal to $I(x)$. There is then a surjection $C(-, c) \to I(x)$. This shows that $\mathrm{Card}(I(x)) \leq \alpha$, since the presheaf $C(-.c)$ has cardinality $\leq \alpha$. ∎

The *cardinality* of a map presheaves $f : X \to Y$ is defined to be the sum

$$\mathrm{Card}(f) = \mathrm{Card}(X) + \mathrm{Card}(Y).$$

Theorem D.2.16. *If a saturated class of monomorphisms in a presheaf category is defined by an accessible equation, then it is generated by a set of maps.*

Proof: Let C be a small category, and let $\mathcal{A} \subset [C^o, \mathbf{Set}]$ be a saturated class of monomorphisms defined by an accessible equation $S = T$, where $T : \mathcal{E}^I \to \mathbf{Set}$ and $S \subseteq T$. By A.0.1 and A.0.2, there exists a regular cardinal α such that

- $\alpha \geq \mathrm{Card}(C)$;
- S and T are α-accessible;
- F takes α-presentable objects to α-presentable sets.

Hence the functor T takes a map of cardinality $\leq \alpha$ to a set of cardinality $\leq \alpha$. Let us denote by Σ the class of maps of cardinality $\leq \alpha$ in the class $\mathcal{A}$. The proposition will be proved if we show that $\overline{\Sigma} = \mathcal{A}$, where $\overline{\Sigma}$ denotes the saturated class generated by Σ. Obviously, $\overline{\Sigma} \subseteq \mathcal{A}$, since the class $\mathcal{A}$ is saturated by hypothesis. Conversely, if a map $i : X \to Y$ belongs to $\mathcal{A}$, let us show that it belongs to $\overline{\Sigma}$. The map i is monic, since every map in $\mathcal{A}$ is monic. There is no loss of generality in supposing that i is an inclusion $X \subseteq Y$, since a saturated class contains the isomorphisms. Let Θ be the set of presheaves $S \subseteq Y$ which contains X and for which the inclusion $X \subseteq S$ belongs to $\overline{\Sigma}$. If $S, T \in \Theta$, let us write $S \preccurlyeq T$ to indicate that we have $S \subseteq T$ and that the inclusion $S \subseteq T$ belongs to $\overline{\Sigma}$. This defines a partial order relation on Θ. The resulting poset is inductive, since a saturated class is closed under transfinite composition. It follows that Θ contains a maximal element M. We shall prove that $M = Y$. Let $\mathcal{P}(Y)$ be the set of presheaves of $S \subseteq Y$ and let $\mathcal{P}_\alpha(Y) \subseteq \mathcal{P}(Y)$ be the set of presheaves $S \subseteq Y$ of cardinality $\leq \alpha$. The presheaf Y is the union of the presheaves in $\mathcal{P}_\alpha(Y)$ by D.2.15. Let us denote by $\mathcal{P}'_\alpha(Y)$ the set of presheaves $S \in \mathcal{P}_\alpha(Y)$ for which the inclusion $S \cap M \subseteq S$ belongs to $\mathcal{A}$. We claim that every $A \in \mathcal{P}_\alpha(Y)$ is contained in a $B \in \mathcal{P}'_\alpha(Y)$. For every $U \in \mathcal{P}(Y)$, let us put $T_0(U) = T(i_U)$ and $S_0(U) = S(i_U)$, where i_U denotes the inclusion $U \cap M \subseteq U$. This defines a functor $T_0 : \mathcal{P}(Y) \to \mathbf{Set}$ and a subfunctor $S_0 \subseteq T_0$. Observe that $i_U \in \mathcal{A}$ iff we have $S_0(U) = T_0(U)$, since the class $\mathcal{A}$ is defined by the equation $S = T$. We shall prove that every $A \in \mathcal{P}_\alpha(Y)$ is contained in a $B \in \mathcal{P}_\alpha(Y)$ for which $S_0(B) = T_0(B)$. Before proving this, we shall prove a weaker property: every $A \in \mathcal{P}_\alpha(Y)$ is contained in a $A' \in \mathcal{P}_\alpha(Y)$ such that the image of the map $T_0(A) \to T_0(A')$ is included in $S_0(A')$,

$$\begin{array}{ccc} S_0(A) & \longrightarrow & S_0(A') \\ \big\downarrow & \nearrow & \big\downarrow \\ T_0(A) & \longrightarrow & T_0(A'). \end{array}$$

The functors S_0 and T_0 preserve α-directed colimits, since the functors S and T preserve α-directed colimits. The presheaf Y is the α-directed colimit of its sub-presheaves $U \in \mathcal{P}_\alpha(Y)$. Hence the set $T_0(Y)$ is the α-directed colimit of the sets $T_0(U)$ with $U \in \mathcal{P}_\alpha(Y)$ by D.2.15. Similarly for the set $S_0(Y)$. But we have $S_0(Y) = T_0(Y)$, since the inclusion $X \subseteq Y$ belongs to $\mathcal{A}$ by hypothesis. It follows that for every element $x \in T_0(A)$ there exists an element $A^x \in \mathcal{P}_\alpha(Y)$ which contains A and such that the image of x by the map $T_0(A) \to T_0(A^x)$ belongs to $S_0(A^x)$,

$$\begin{array}{ccc} 1 & \longrightarrow & S_0(A^x) \\ {\scriptstyle x}\big\downarrow & & \big\downarrow \\ T_0(A) & \longrightarrow & T_0(A^x). \end{array}$$

Let us put

$$A' = \bigcup_{x \in T_0(A)} A^x.$$

Notice that we have $A' \in \mathcal{P}_\alpha(Y)$, since the cardinality of $T_0(A)$ is $\leq \alpha$. Moreover, for every $x \in T_0(A)$, we have a commutative diagram

$$\begin{array}{ccccc} 1 & \longrightarrow & S_0(A^x) & \longrightarrow & S_0(A') \\ {\scriptstyle x}\big\downarrow & & \big\downarrow & & \big\downarrow \\ T_0(A) & \longrightarrow & T_0(A^x) & \longrightarrow & T_0(A'). \end{array}$$

Hence the image of the map $T_0(A) \to T_0(A')$ is included in $S_0(A')$. This proves the weaker property. By using this result, we can construct an increasing sequence of elements of $\mathcal{P}_\alpha(Y)$,

$$A = A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots$$

such that the image of the map $T_0(A_n) \to T_0(A_{n+1})$ is included in $S_0(A_{n+1})$ for each $n \geq 0$,

$$\begin{array}{ccccccccc}
 & & S_0(A_1) & \longrightarrow & S_0(A_2) & \longrightarrow & S_0(A_3) & \longrightarrow & \cdots \\
 & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \\
T_0(A) & \longrightarrow & T_0(A_1) & \longrightarrow & T_0(A_2) & \longrightarrow & T_0(A_3) & \longrightarrow & \cdots .
\end{array}$$

If $\alpha > \omega$, the sequence can be extended as a cocontinuous chain $A_\star : \alpha \to \mathcal{P}_\alpha(Y)$ such that the image of the map $T_0(A_i) \to T_0(A_{i+1})$ is included in $S_0(A_{i+1})$ for each $i < \alpha$. If B is the union of the chain $A_\star$, then we have $S_0(B) = T_0(B)$, since the functors S_0 and T_0 preserve α-directed colimits. Thus, $B \in \mathcal{P}'_\alpha(Y)$ and this proves the stronger property. We can now prove that $M = Y$. For every element $x \in Y$, there exists $A \in \mathcal{P}_\alpha(Y)$ such that $x \in A$. Then there exists a $B \in \mathcal{P}_\alpha(Y)$ which contains A and for which the inclusion $B \cap M \subseteq B$ belongs to $\mathcal{A}$ by what we just proved. The inclusion $B \cap M \subseteq B$ belongs to Σ, since the cardinality of B is $\leq \alpha$. Hence the inclusion $M \subseteq B \cup M$ belongs to $\overline{\Sigma}$, since the square

$$\begin{array}{ccc}
B \cap M & \longrightarrow & M \\
\downarrow & & \downarrow \\
B & \longrightarrow & B \cup M
\end{array}$$

is a pushout. Thus, $B \cup M = M$ by the maximality of M. This proves that $x \in M$, since $x \in B$. This proves that $M = Y$. We have proved that the inclusion $X \subseteq Y$ belongs to $\overline{\Sigma}$. Thus, $\overline{\Sigma} = \mathcal{A}$. ∎

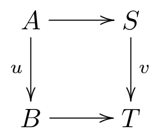
Recall that a class of maps $\mathcal{M}$ in a category $\mathcal{E}$ is said to have the *right cancellation property* if the implication

$$vu \in \mathcal{M} \text{ and } u \in \mathcal{M} \ \Rightarrow\ v \in \mathcal{M}$$

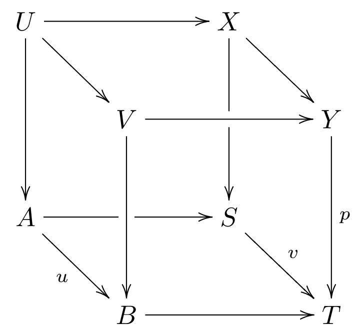
is true for any pair of maps $u : A \to B$ and $v : B \to C$.

Lemma D.2.17. *let $\mathcal{A}$ be a saturated class of maps in a topos $\mathcal{E}$ and let $\mathcal{F} \subseteq \mathcal{E}$ be a class of maps which contains the isomorphisms and which is closed under base change. Then the class $\mathcal{A}'$ of maps $A \to B$ in $\mathcal{A}$ having their base change $A \times_B X \to X$ in $\mathcal{A}$ for every map $X \to B$ in $\mathcal{F}$ is saturated. Moreover, if $\mathcal{A}$ has the right cancellation property, then so does $\mathcal{A}'$.*

Proof: Let us verify that that $\mathcal{A}'$ is closed under cobase change. Consider a pushout square



with $u \in \mathcal{A}'$, and let us prove that $v \in \mathcal{A}'$. For this we have to show that the base change of v along any map $p : Y \to T$ in $\mathcal{F}$ is in $\mathcal{A}$. By pulling back the square along p we obtain a cube



in which the vertical faces are cartesian. The vertical maps belongs to $\mathcal{F}$ since they are base change of p. The map $U \to V$ belongs to $\mathcal{A}$, since it is a base change of u along a map in $\mathcal{F}$. The top face of the cube is a pushout since the functor $p^* : \mathcal{E}/T \to \mathcal{E}/X$ is cocontinuous. It follows that the map $X \to Y$ belongs to $\mathcal{A}$, since $\mathcal{A}$ is closed under cobase change. This proves that the map $S \to T$ belongs to $\mathcal{A}'$. The verification that $\mathcal{A}'$ is closed under retract, composition, and transfinite composition is left to the reader. The last statement about the right cancellation property is also left to the reader. ∎

Appendix E

Model categories

In this appendix we recall some basic notions of homotopical algebra from [Q] and [Ho].

E.1 Model structures

Definition E.1.1. *We shall say that a class $\mathcal{W}$ of maps in a category $\mathcal{E}$ has the* "three-for-two" *property if the following condition is satisfied:*

- *If two of three maps $u : A \to B$, $v : B \to C$ and $vu : A \to C$ belongs to $\mathcal{W}$, then so does the third.*

Definition E.1.2. *A* Quillen model structure *on a finitely bicomplete category $\mathcal{E}$ is a triple $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ of classes of maps in $\mathcal{E}$ such that :*

- *$\mathcal{W}$ has the "three-for-two" property;*
- *Each pair $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ is a weak factorisation system.*

A Quillen model category is a category $\mathcal{E}$ equipped with a model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$.

Each class of a model structure contains the isomorphisms. The class $\mathcal{W}$ is closed under retracts by E.1.3.

A map in the class $\mathcal{W}$ is said to be *acyclic* or to be a *weak equivalence* . A map in $\mathcal{C}$ is called a *cofibration* and a map in $\mathcal{F}$ a *fibration* . A map in $\mathcal{C} \cap \mathcal{W}$ is called an *acyclic cofibration* and a map in $\mathcal{F} \cap \mathcal{W}$ an *acyclic fibration.* An object $X \in \mathcal{E}$ is *fibrant* if the map $X \to \top$ is a fibration, where $\top$ is the terminal object of $\mathcal{E}$. Dually, an object $A \in \mathcal{E}$ is *cofibrant* if the map $\bot \to A$ is a cofibration, where $\bot$ is the initial object of $\mathcal{E}$.

A model structure is said to be *left proper* if the cobase change of a weak equivalence along a cofibration is a weak equivalence. Dually, a model structure is said to be *right proper* if the base change of a weak equivalence along a fibration is a weak equivalence. A model structure is *proper* if it is both left and right proper.

Any two of the classes $\mathcal{C}$, $\mathcal{W}$ and $\mathcal{F}$ of a model structure determines the third. For example, a map $f \in \mathcal{E}$ belongs to $\mathcal{W}$ iff it admits a factorisation $f = pi$ with $i \in {}^{\pitchfork}\mathcal{F}$ and $p \in \mathcal{C}^{\pitchfork}$.

If $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ is a model structure in a category $\mathcal{E}$, then the triple $(\mathcal{F}^o, \mathcal{W}^o, \mathcal{C}^o)$ is a model structure on the opposite category $\mathcal{E}^o$. Hence the opposite of a model category is a model category.

If $\mathcal{E}$ is a model category, then so are the categories $\mathcal{E}/B$ and $B\backslash\mathcal{E}$ for any object $B \in \mathcal{E}$. By definition, a map in $\mathcal{E}/B$ is a weak equivalence (resp. a cofibration , a fibration) iff its underlying map in $\mathcal{E}$ is a weak equivalence (resp. a cofibration , resp. a fibration). And similarly for the model structure on $B\backslash\mathcal{E}$.

Proposition E.1.3. [JT3] *The class $\mathcal{W}$ of a model structure is closed under retracts.*

Proof: Observe first that the class $\mathcal{F} \cap \mathcal{W}$ is closed under retracts since the pair $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ is a weak factorisation Suppose now that a map $f : A \to B$ is a retract of a map $g : X \to Y$ in $\mathcal{W}$. Let us show that $f \in \mathcal{W}$. By definition, we have a commutative diagram,

$$\begin{array}{ccccc} A & \xrightarrow{s} & X & \xrightarrow{t} & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[u]{} & Y & \xrightarrow{v} & B \end{array}$$

where $gf = 1_X$ and $vu = 1_B$. Let us first consider the case where f is a fibration. In this case, let us choose a factorisation $g = qj : X \to Z \to Y$ with $j \in \mathcal{C} \cap \mathcal{W}$ and $q \in \mathcal{F}$. We have $q \in \mathcal{F} \cap \mathcal{W}$ by three-for-two, since $g \in \mathcal{W}$. The square

$$\begin{array}{ccc} X & \xrightarrow{t} & A \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{vq} & B \end{array}$$

has a diagonal filller $d : Z \to A$, since f is a fibration. We then have a commutative diagram,

$$\begin{array}{ccccc} A & \xrightarrow{js} & Z & \xrightarrow{d} & A \\ {\scriptstyle f}\downarrow & & {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{u} & Y & \xrightarrow{v} & B. \end{array}$$

Thus, f is a retract of q, since $d(js) = ts = 1_A$. This shows that $f \in \mathcal{W}$ since $q \in \mathcal{F} \cap \mathcal{W}$. In the general case, let us choose a factorisation $f = pi : A \to E \to B$ with $i \in \mathcal{C} \cap \mathcal{W}$ and $p \in \mathcal{F}$ By taking a pushout, we obtain a commutative diagram

$$\begin{array}{ccccc}
A & \xrightarrow{s} & X & \xrightarrow{t} & A \\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle i} \\
E & \xrightarrow{i_1} & E \sqcup_A X & \xrightarrow{r} & E \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle p} \\
B & \xrightarrow{u} & Y & \xrightarrow{v} & B,
\end{array}$$

where $ki_2 = g$ and $ri_1 = 1_E$. The map i_2 is a cobase change of the map i. Thus, $i_2 \in \mathcal{C} \cap \mathcal{W}$ since $i \in \mathcal{C} \cap \mathcal{W}$. Thus, $k \in \mathcal{W}$ by three-for-two since $g = ki_2 \in \mathcal{W}$ by hypothesis. Thus, $p \in \mathcal{W}$ by the first part since $p \in \mathcal{F}$. Thus $f = pi \in \mathcal{W}$ since $i \in \mathcal{W}$. ∎

The *homotopy category* $Ho(\mathcal{E})$ of a model category $\mathcal{E}$ is the category of fractions $\mathcal{W}^{-1}\mathcal{E}$. We shall denote by $[u]$ the image of a map $u \in \mathcal{E}$ by the canonical functor $\mathcal{E} \to Ho(\mathcal{E})$.

Proposition E.1.4. [Q] *In a model category $\mathcal{E}$, a map $u : A \to B$ is a weak equivalence iff the arrow $[u] : A \to B$ is invertible in the homotopy category $Ho(\mathcal{E})$.*

Definition E.1.5. "Six-for-two" [DHKS] *We shall say that a class $\mathcal{W}$ of maps in a category has the "six-for-two" property, if for any commutative diagram*

$$\begin{array}{ccc}
A & \xrightarrow{x} & X \\
{\scriptstyle u}\downarrow & \overset{d}{\nearrow} & \downarrow{\scriptstyle f} \\
B & \xrightarrow[y]{} & Y
\end{array}$$

with $x \in \mathcal{W}$ and $y \in \mathcal{W}$, the six maps x, y, u, f, d, fdu belongs to $\mathcal{W}$.

It is easy to verify that the six-for-two property implies the three-for-two property.

Corollary E.1.6. *The class $\mathcal{W}$ of weak equivalences of a model structure has the "six-for-two" property. .*

We shall denote by $\mathcal{E}_f$ (resp. $\mathcal{E}_c$) the full sub-category of fibrant (resp. cofibrant) objects of a model category $\mathcal{E}$. We shall put $\mathcal{E}_{fc} = \mathcal{E}_f \cap \mathcal{E}_c$. A *fibrant replacement* of an object $X \in \mathcal{E}$ is a weak equivalence $X \to RX$ with codomain a fibrant object. Dually, a *cofibrant replacement* of X is a weak equivalence $LX \to X$ with domain a cofibrant object.

Let us put $Ho(\mathcal{E}_f) = \mathcal{W}_f^{-1}\mathcal{E}_f$ where $\mathcal{W}_f = \mathcal{W} \cap \mathcal{E}_f$ and similarly for $Ho(\mathcal{E}_c)$ and $Ho(\mathcal{E}_{fc})$. Then the diagram of inclusions

$$\begin{array}{ccc} \mathcal{E}_{fc} & \longrightarrow & \mathcal{E}_f \\ \downarrow & & \downarrow \\ \mathcal{E}_c & \longrightarrow & \mathcal{E} \end{array}$$

induces a diagram of equivalences of categories

$$\begin{array}{ccc} Ho(\mathcal{E}_{fc}) & \longrightarrow & Ho(\mathcal{E}_f) \\ \downarrow & & \downarrow \\ Ho(\mathcal{E}_c) & \longrightarrow & Ho(\mathcal{E}). \end{array}$$

A *path object* for an object X in a model category is obtained by factoring the diagonal map $X \to X \times X$ as weak equivalence $\delta : X \to PX$ followed by a fibration $(p_0, p_1) : PX \to X \times X$. A *right homotopy* $h : f \sim_r g$ between two maps $u, v : A \to X$ is a map $h : A \to PX$ such that $u = p_0 h$ and $v = p_1 h$. Two maps $u, v : A \to X$ are *right homotopic* if there exists a right homotopy $h : f \sim_r g$ with codomain a path object for X. The right homotopy relation on the set of maps $A \to X$ is an equivalence if X is fibrant. There is a dual notion of *cylinder object* for A obtained by factoring the codiagonal $A \sqcup A \to A$ as a cofibration $(i_0, i_1) : A \sqcup A \to IA$ followed by a weak equivalence $p : IA \to A$. A *left homotopy* $h : u \sim_l v$ between two maps $u, v : A \to X$ is a map $h : IA \to X$ such that $u = hi_0$ and $v = hi_1$. Two maps $u, v : A \to X$ are *left homotopic* if there exists a left homotopy $h : u \sim_l v$ with domain some cylinder object for A. The left homotopy relation on the set of maps $A \to X$ is an equivalence if A is cofibrant. The left homotopy relation coincides with the right homotopy relation if A is cofibrant and X is fibrant; in which case two maps $u, v : A \to X$ are said to be homotopic if they are left (or right) homotopic; we shall denote this relation by $u \sim v$.

Proposition E.1.7. [Q] (Covering Homotopy theorem). *Let A be cofibrant with cylinder object $(1_A, 1_A) = p(i_0, i_1) : A \sqcup A \to IA \to A$ and let $f : X \to Y$ be a fibration. If $x : A \to X$ and $h : IA \to Y$ is a left homotopy with $fx = hi_0$, then there exists a left homotopy $H : IA \to X$, with $Hi_0 = x$ and $fH = h$.*

Proposition E.1.8. [Q]. *If A is cofibrant and X is fibrant, then the canonical map $u \mapsto [u]$,*

$$\mathcal{E}(A, X) \to Ho(\mathcal{E})(A, X),$$

is surjective. Moreover, if $u, v : A \to X$, then $[u] = [v] \Leftrightarrow u \sim v$,

A map $X \to Y$ in $\mathcal{E}_{cf}$ is called a *homotopy equivalence* if there exists a map $g : Y \to X$ such that $gf \sim 1_X$ and $fg \simeq 1_Y$.

Corollary E.1.9. [Q]. *A map $X \to Y$ in $\mathcal{E}_{cf}$ is a homotopy equivalence iff it is a weak equivalence.*

Proposition E.1.10. *A model structure $M = (\mathcal{C}, \mathcal{W}, \mathcal{F})$ in a category $\mathcal{E}$ is determined by its class of cofibrations $\mathcal{C}$ together with its class of fibrant objects $F(M)$. If $M' = (\mathcal{C}, \mathcal{W}', \mathcal{F}')$ is another model structure with the same cofibrations, then the relation $\mathcal{W} \subseteq \mathcal{W}'$ is equivalent to the relation $F(M') \subseteq F(M)$.*

Proof: Let us prove the first statement. It suffices to show that the class $\mathcal{W}$ is determined by $\mathcal{C}$ and $F(M)$. The class of acyclic cofibrations is determined by $\mathcal{C}$, since the right class of a weak factorisations system is determined by its left class. For any map $u : A \to B$, there exists a commutative square

$$\begin{array}{ccc} A' & \longrightarrow & A \\ {\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle u} \\ B' & \longrightarrow & B \end{array}$$

in which the horizontal maps are acyclic fibrations and the objects A' and B' are cofibrants. The map u is acyclic iff the map u' is acyclic. Hence it suffices to show that the class $\mathcal{W} \cap \mathcal{E}_c$ is is determined by $\mathcal{C}$ and $F(M)$. If A and B are two objects of $\mathcal{E}$, let us denote by $h(A, B)$ the set of maps $A \to B$ between in the homotopy category $\mathcal{W}^{-1}\mathcal{E}$. A map between two cofibrant objects $u : A \to B$ belongs to $\mathcal{W}$, iff the map $h(u, X) : h(B, X) \to h(A, X)$ is bijective for every object $X \in F(\mathcal{W})$. If $A \in \mathcal{E}_c$ and $X \in F(\mathcal{W})$, then the set $h(A, X)$ is a quotient of the set $\mathcal{E}(A, X)$ by the left homotopy relation. Let us factor the codiagonal $A \sqcup A \to A$ as a cofibration $(i_0, i_1) : A \sqcup A \to IA$ followed by an acyclic fibration $IA \to A$. The construction of the cylinder IA only depends on $\mathcal{C}$. It follows that the left homotopy relation on the set $\mathcal{E}(A, X)$ only depends on $\mathcal{C}$. Hence also the set $h(A, X)$. It follows that $\mathcal{W}$ is determined by $\mathcal{C}$ and $F(M)$. The first statement is proved. The proof of the second statement is left to the reader.

Let $f : X \to Y$ be a map between fibrant objects. If $(p_0, p_1)\delta : Y \to PY \to Y \times Y$ is a path object for Y, the *mapping path space* of f is the object $P(f)$ defined by the pullback square

$$\begin{array}{ccc} P(f) & \xrightarrow{pr_2} & X \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle f} \\ P(Y) & \xrightarrow{p_1} & Y. \end{array}$$

There is a unique map $i_X : X \to P(f)$ such that $pr_1 i_X = \delta_Y f$ and $pr_2 i_X = 1_X$. Let us put $q_X = pr_2$ and $q_Y = p_0 pr_1 : P(f) \to P(Y) \to Y$. The map i_X is acyclic, the map q_Y is a fibration and we have

$$f = q_Y i_X : X \to P(f) \to Y.$$

Moreover, $p_X : P(f) \to A$ an acyclic fibration and we have $p_X i_X = 1_X$. This is called the *mapping path factorisation* of the map f. Dually, let $u : A \to B$ be a map between cofibrant objects. If $p(i_0, i_1) : A \sqcup A \to IA \to A$ is a cylinder for A, followed by a weak equivalence $p : IA \to A$. the *mapping cylinder* of u is the object $C(u)$ defined by the pushout square

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ {\scriptstyle i_1}\downarrow & & \downarrow{\scriptstyle in_2} \\ C(A) & \xrightarrow{in_1} & C(u) \end{array}$$

There is a unique map $q_B : C(u) \to B$ such that $q_B in_1 = up_A$ and $q_B in_2 = 1_B$. Let us put $i_B = in_2$ and $i_A = in_1 i_0 : A \to C(A) \to C(u)$. The map i_A is a cofibration, the map q_B is a weak equivalence and we have

$$u = q_B i_A : A \to C(u) \to B.$$

Moreover, $i_B : B \to C(u)$ an acyclic cofibration and we have $q_B i_B = 1_A$. This is called the *mapping cylinder factorisation* of the map u.

The following proposition is useful for verifying that a triple $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ is a model structure:

Proposition E.1.11. *Let $\mathcal{E}$ be a finitely bicomplete category equipped a class of maps $\mathcal{W}$ having the "three-for-two" property and two factorisation systems $(\mathcal{C}_W, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F}_W)$. Suppose that the following two conditions are satisfied:*

- $\mathcal{C}_W \subseteq \mathcal{C} \cap \mathcal{W}$ and $\mathcal{F}_W \subseteq \mathcal{F} \cap \mathcal{W}$;
- $\mathcal{C} \cap \mathcal{W} \subseteq \mathcal{C}_W$ or $\mathcal{F} \cap \mathcal{W} \subseteq \mathcal{F}_W$.

Then we have $\mathcal{C}_W = \mathcal{C} \cap \mathcal{W}$, $\mathcal{F}_W = \mathcal{F} \cap \mathcal{W}$ and $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ is a model structure.

Proof: We have $\mathcal{C}_W \subseteq \mathcal{C} \cap \mathcal{W}$ and $\mathcal{F}_W \subseteq \mathcal{F} \cap \mathcal{W}$ by hypothesis. If $\mathcal{C} \cap \mathcal{W} \subseteq \mathcal{C}_W$, let us show that $\mathcal{F} \cap \mathcal{W} \subseteq \mathcal{F}_W$. If $f : X \to Y$ belongs to $\mathcal{F} \cap \mathcal{W}$, let us choose a factorisation $f = pi : X \to Z \to Y$ with $i \in \mathcal{C}$ and $p \in \mathcal{F}_W$. We have $p \in \mathcal{W}$ since $\mathcal{F}_W \subseteq \mathcal{W}$. Thus, $i \in \mathcal{W}$ by three-for-two since $f \in \mathcal{W}$. Thus, $i \in \mathcal{C}_W$, since $\mathcal{C} \cap \mathcal{W} \subseteq \mathcal{C}_W$. Hence we have $i \pitchfork f$, since $f \in \mathcal{F}$ and the pair $(\mathcal{C}_W, \mathcal{F})$ is a weak factorisation system. It follows that the square

$$\begin{array}{ccc} X & \xrightarrow{1_X} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{p} & Y. \end{array}$$

has a diagonal filler $r : Z \to X$. This shows that f is a domain retract of p. The class $\mathcal{F}_W$ is closed under domain retracts since the pair $(\mathcal{C}, \mathcal{F}_W)$ is a weak factorisation system. Thus, $f \in \mathcal{F}_W$ since $p \in \mathcal{F}_W$. We have proved that $\mathcal{F} \cap \mathcal{W} \subseteq \mathcal{F}_W$. Thus, $(\mathcal{C}, \mathcal{F} \cap \mathcal{W}) = (\mathcal{C}, \mathcal{F}_W)$. This shows that the triple $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ is a model structure since we have $(\mathcal{C} \cap \mathcal{W}, \mathcal{F}) = (\mathcal{C}_W, \mathcal{F})$ by hypothesis. ∎

Recall that a class of objects $\mathcal{K}$ in a category $\mathcal{E}$ is said to be *replete* or if every object isomorphic to an object of $\mathcal{K}$ belongs to $\mathcal{K}$.

Definition E.1.12. *Let $\mathcal{E}$ be a model category. We say that a class of objects $\mathcal{K} \subseteq \mathcal{E}$ is* homotopy replete *if for every weak equivalence $A \to A'$, we have $A \in \mathcal{K} \Leftrightarrow A' \in \mathcal{K}$. Similarly for a class of objects in one of the subcategories $\mathcal{E}_c$, $\mathcal{E}_f$ and $\mathcal{E}_{cf}$.*

If $\mathcal{K}$ is a class of objects in a model category $\mathcal{E}$, let us put

$$\mathcal{K}_c = \mathcal{K} \cap \mathcal{E}_c, \quad \mathcal{K}_f = \mathcal{K} \cap \mathcal{E}_f \quad \text{and} \quad \mathcal{K}_{cf} = \mathcal{K} \cap \mathcal{E}_{cf}.$$

Lemma E.1.13. *Let $\mathcal{E}$ be a model category and let $\pi : \mathcal{E} \to Ho(\mathcal{E})$ is the canonical functor. Then the map $\mathcal{K} \mapsto \pi^{-1}(\mathcal{K})$ induces a bijection between the replete classes of objects in $Ho(\mathcal{E})$ and the homotopy replete classes of objects in $\mathcal{E}$. Moreover, the map $\mathcal{K} \mapsto \mathcal{K}_c$ induces a bijection between the homotopy replete classes of objects in in $\mathcal{E}$ and in $\mathcal{E}_c$. Similarly for the maps $\mathcal{K} \mapsto \mathcal{K}_f$ and $\mathcal{K} \mapsto \mathcal{K}_{cf}$.*

Proof: Two objects A and X are isomorphic in $Ho(\mathcal{E})$ iff there exists a chain of weak equivalences

$$A \xleftarrow{\ p\ } A' \xrightarrow{\ u\ } X' \xleftarrow{\ i\ } X$$

with A' cofibrant and X' fibrant. The first statement of the lemma follows from this observation. The second statement follows from the fact that every object has a cofibrant replacement together with the fact that two cofibrant objects A and B are isomorphic $Ho(\mathcal{E}_c)$ iff there exists a chain of weak equivalences

$$A \xrightarrow{\ u\ } B' \xleftarrow{\ i\ } B$$

in the subcategory $\mathcal{E}_c$. Similarly for the other statements. ∎

E.2 Quillen functors

Definition E.2.1. *[Ho] We shall say that a cocontinuous functor $F : \mathcal{U} \to \mathcal{V}$ between two model categories is a* left Quillen functor *if it takes a cofibration to a cofibration and an acyclic cofibration to an acyclic cofibration. Dually, we shall say that a continuous functor $G : \mathcal{V} \to \mathcal{U}$ between two model categories is a* right Quillen functor *if it takes a fibration to a fibration and an acyclic fibration to an acyclic fibration.*

Proposition E.2.2. [Q] *Let $F : \mathcal{U} \leftrightarrow \mathcal{V} : G$ be an adjoint pair of functors between two model categories. Then F is a left Quillen functor iff G is a right Quillen functor.*

Definition E.2.3. *[Ho] We shall say that an adjoint pair of functors between two model categories, $F : \mathcal{U} \leftrightarrow \mathcal{V} : G$ is a* Quillen pair *if the conditions of Proposition E.2.2 are satisfied.*

Let $u : A \to B$ be a map in a model category $\mathcal{E}$. The pullback functor $u^* : \mathcal{E}/B \to \mathcal{E}/A$ has a left adjoint $u_!$ obtained by composing a map $X \to A$ with $u : A \to B$. The model structure on $\mathcal{E}$ induces a model structure on each of the category $\mathcal{E}/A$ and $\mathcal{E}/B$,

Proposition E.2.4. *Let $u : A \to B$ be a map in a model category $\mathcal{E}$. Then the pair of adjoint functors*

$$u_! : \mathcal{E}/B \to \mathcal{E}/A : u^*$$

is a Quillen pair.

The following lemma is due to Ken Brown:

Lemma E.2.5. [Ho] *Let $\mathcal{E}$ be a model category and $F : \mathcal{E} \to \mathcal{D}$ be a functor with values in a category equipped with a class of maps $\mathcal{W}$ having the "three-for-two" property. If F takes an acyclic cofibration between cofibrant objects to an element of $\mathcal{W}$, then it takes a weak equivalence between cofibrant objects to an element of $\mathcal{W}$.*

Proof: If $f : A \to B$ is a weak equivalence between cofibrant objects, consider a mapping cylinder factorisation $(f, 1_B) = p(u_A, u_B) : A \sqcup B \to C \to B$. We have $f = pu_A$ and $pu_B = 1_B$. The map u_B is an acyclic cofibration. The cofibration u_A is acyclic by three-for-two since f and p are acyclic. Hence the maps $F(u_A)$ and $F(u_B)$ belong to $\mathcal{W}$. Hence also the map $F(p)$ by three-for-two since $F(p)F(u_B) = 1_{FB}$. It follows that the composite $F(p)F(u_A) = F(f)$ belongs to $\mathcal{W}$. ■

Corollary E.2.6. *A left Quillen functor takes a weak equivalence between cofibrant objects to a weak equivalence.*

The proof of the following result is taken from [JT3].

Proposition E.2.7. *The cobase change along a cofibration of a weak equivalence between cofibrant objects is a weak equivalence.*

Proof: Consider a pushout square

$$\begin{array}{ccc} A & \xrightarrow{k} & A' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ B & \xrightarrow{r} & B', \end{array}$$

where f is a weak equivalence between cofibrant objects and k is a cofibration. Let us show that f' is a weak equivalence. This is clear if f is an acyclic cofibration.

If we factor f with the mapping cylinder factorisation, the problem can reduced to the case where f has a section $s : B \to A$ The cobase change along f is a left Quillen functor $F : A\backslash\mathcal{E} \to B\backslash\mathcal{E}$ by E.2.4. Consider the diagram

$$\begin{array}{ccccc}
B & \xrightarrow{s} & A & \xrightarrow{f} & B \\
{\scriptstyle ks}\downarrow & & \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle ks} \\
A' & \xrightarrow{i_1} & A' \sqcup_B A & \xrightarrow{h} & A' \\
 & {\scriptstyle 1_A}\searrow & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f'} \\
 & & A' & \xrightarrow{f'} & B'
\end{array}$$

where $gi_1 = 1_{A'}$, $gi_2 = k$, $hi_1 = 1_{A'}$ and $hi_2 = ksf$. The top square on the right is pushout since the composite of the top squares is a pushout. It follows that f' is the image of g by the cobase change functor $F : A\backslash\mathcal{E} \to B\backslash\mathcal{E}$. But g is a map between cofibrant objects of the model category $A\backslash\mathcal{E}$. Hence it suffice to show that g is a weak equivalence by Lemma E.2.6. But i_1 is acyclic since s is an acyclic cofibration. It follows by three-for-two that g is a weak equivalence since $gi_1 = 1_{A'}$.

Corollary E.2.8. *If every object of a model category is cofibrant, then the model structure is left proper.*

A left Quillen functor $F : \mathcal{U} \to \mathcal{V}$ induces a functor $F_c : \mathcal{U}_c \to \mathcal{V}_c$ hence also a functor $Ho(F_c) : Ho(\mathcal{U}_c) \to Ho(\mathcal{V}_c)$ by Lemma E.2.6. Its *left derived functor* is a functor

$$F^L : Ho(\mathcal{U}) \to Ho(\mathcal{V})$$

for which the following diagram of functors commutes up to isomorphism,

$$\begin{array}{ccc}
Ho(\mathcal{U}_c) & \xrightarrow{Ho(F_c)} & Ho(\mathcal{V}_c) \\
\downarrow & & \downarrow \\
Ho(\mathcal{U}) & \xrightarrow{F^L} & Ho(\mathcal{V}),
\end{array}$$

It can be computed as follows. For each object $A \in \mathcal{U}$, we can choose a cofibrant replacement $\lambda_A : LA \to A$, with λ_A an acyclic fibration. We can then choose for each arrow $u : A \to B$ an arrow $L(u) : LA \to LB$ such that $u\lambda_A = \lambda_B L(u)$,

$$\begin{array}{ccc}
LA & \xrightarrow{\lambda_A} & A \\
{\scriptstyle L(u)}\downarrow & & \downarrow{\scriptstyle u} \\
LB & \xrightarrow{\lambda_B} & B.
\end{array}$$

Then

$$F^L([u]) = [F(L(u))] : FLA \to FLB.$$

From the map $LA \to A$ we obtain a map $F(LA) \to F(A)$ which is well defined in $Ho(\mathcal{V})$. This defines a a natural transformation $\alpha : F^L P_1 \to P_2 F$, where P_1 and P_2 are the canonical functors.

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{F} & \mathcal{V} \\ {\scriptstyle P_1}\downarrow & & \downarrow{\scriptstyle P_2} \\ Ho(\mathcal{U}) & \xrightarrow{F^L} & Ho(\mathcal{V}). \end{array}$$

Proposition E.2.9. [Q] The natural transformation $\alpha : F^L P_1 \to P_2 F$ exibits the functor F^L as the right Kan extension of the functor $P_2 H$ along the functor P_1.

It follows from this result that the functor F^L depends only on F and on the classes of weak equivalences in the model categories $\mathcal{U}$ and $\mathcal{V}$. We observe that the left Kan extension is *absolute*, ie it remains a left Kan extension when composed with any functor with domain $Ho(\mathcal{U})$.

Dually, a right Quillen functor $G : \mathcal{V} \to \mathcal{U}$ induces a functor $G_f : \mathcal{V}_f \to \mathcal{U}_f$ hence also a functor $Ho(G_f) : Ho(\mathcal{V}_f) \to Ho(\mathcal{U}_f)$ by Lemma E.2.6. Its *right derived functor* is a functor

$$G^R : Ho(\mathcal{V}) \to Ho(\mathcal{U})$$

for which the following diagram of functors commutes up to a canonical isomorphism,

$$\begin{array}{ccc} Ho(\mathcal{V}_f) & \xrightarrow{Ho(G_f)} & Ho(\mathcal{U}_f) \\ \downarrow & & \downarrow \\ Ho(\mathcal{V}) & \xrightarrow{G^R} & Ho(\mathcal{U}). \end{array}$$

The functor G^R is unique up to a canonical isomorphism. It can be computed as follows. For each object $X \in \mathcal{V}$ let us choose a fibrant replacement $\rho_X : X \to RX$, with ρ_X an acyclic cofibration. We can then choose for each arrow $u : X \to Y$ an arrow $R(u) : RX \to RY$ such that $R(u)\rho_X = \rho_Y u$,

$$\begin{array}{ccc} X & \xrightarrow{\rho_X} & RX \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle R(u)} \\ Y & \xrightarrow{\rho_Y} & RY. \end{array}$$

Then

$$G^R([u]) = [G(R(u))] : GRX \to GRY.$$

From the map $X \to RX$ we obtain a map $G(X) \to G(RX)$ which is well defined in $Ho(\mathcal{V})$. This defines a a natural transformation $\alpha : P_1G \to G^RP_2$, where P_1 and P_2 are the canonical functors.

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{G} & \mathcal{U} \\ {\scriptstyle P_2}\downarrow & & \downarrow{\scriptstyle P_1} \\ Ho(\mathcal{V}) & \xrightarrow{G^R} & Ho(\mathcal{U}). \end{array}$$

The natural transformation α exibits the functor G^R as the left Kan extension of the functor P_1G along the functor P_2. We observe that the Kan extension is *absolute*, ie it remains a Kan extension when composed with any functor with domain $Ho(\mathcal{U})$.

Proposition E.2.10. *[Q] A Quillen pair of adjoint functors between two model categories $F : \mathcal{U} \leftrightarrow \mathcal{V} : G$ induces a pair of adjoint derived functors between the homotopy categories:*

$$F^L : Ho(\mathcal{U}) \leftrightarrow Ho(\mathcal{V}) : G^R.$$

If $A \in \mathcal{U}$ is cofibrant, the adjunction unit $A \to G^RF^L(A)$ is obtained by composing the maps $A \to GFA \to GRFA$, where $FA \to RFA$ is a fibrant replacement of FA. If $X \in \mathcal{V}$ is fibrant, the adjunction counit $F^LG^R(X) \to X$ is obtained by composing the maps $FLGX \to FGX \to X$, where $LGX \to GX$ is a cofibrant replacement of GX.

A Quillen pair (F, G) is called a *Quillen equivalence* if the adjoint pair (F^L, G^R) is an equivalence of categories.

The composite of two adjoint pairs

$$F_1 : \mathcal{E}_1 \leftrightarrow \mathcal{E}_2 : G_1 \quad \text{and} \quad F_2 : \mathcal{E}_2 \leftrightarrow \mathcal{E}_3 : G_2$$

is an adjoint pair $F_2F_1 : \mathcal{E}_1 \leftrightarrow \mathcal{E}_3 : G_1G_2$.

Proposition E.2.11. [Ho] *(Three-for-two) The composite of two Quillen pairs (F_1, G_1) and (F_2, G_2) is a Quillen pair (F_2F_1, G_1G_2). Moreover, there are canonical isomorphisms*

$$(F_2F_1)^L \simeq F_2^L F_1^L \quad \text{and} \quad (G_1G_2)^R \simeq G_1^R G_2^R.$$

If two of the pairs (F_1, G_1), (F_2, G_2) and (F_2F_1, G_1G_2) are Quillen equivalences, then so is the third.

Lemma E.2.12. *Let $F_i : \mathcal{U} \leftrightarrow \mathcal{V} : G_i$ $(i = 0, 1)$ be two Quillen pairs of adjoint functors between model categories and let $u : F_0 \to F_1$ and $v : G_1 \to G_0$ be a pair mutually transpose natural transformations. If the map $u_A : F_0(A) \to F_1(A)$ is a weak equivalence for every cofibrant $A \in \mathcal{U}$ then so is the map $v_X : G_1(X) \to G_0(X)$ for every fibrant $X \in \mathcal{V}$.*

Proof: The natural transformation $u : F_0 \to F_1$ induces a natural transformation between left derived functors $u^L : F_0^L \to F_1^L$. The derived natural transformation $v^R : U_0^R \to U_1^R$ is the right transpose of the transformation u^L. Thus, u^L is an isomorphism iff v^R is an isomorphism. The result follows. ■

Lemma E.2.13. *A cofibration is acyclic iff it has the left lifting property with respect to every fibration between fibrant objects.*

Proof: The necessity is clear. Conversely, let us suppose that a cofibration $u : A \to B$ has the left lifting property with respect to every fibration between fibrant objects. We shall prove that u is acyclic. For this, let us choose a fibrant replacement $j : B \to B'$ of the object B together with a factorisation of the composite $ju : A \to B'$ as a weak equivalence $i : A \to A'$ followed by a fibration $p : A' \to B$. The square

$$\begin{array}{ccc} A & \xrightarrow{i} & A' \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{j} & B' \end{array}$$

has a diagonal filler $d : B \to A'$, since p is a fibration between fibrant objects. The arrows i and j are invertible in the homotopy category since they are acyclic. The relations $pd = j$ and $du = i$ then implies that d is invertible in the homotopy category. It thus acyclic by E.1.4. It follows by three-for-two that u is acyclic. ■

Proposition E.2.14. *An adjoint pair of functors $F : \mathcal{U} \leftrightarrow \mathcal{V} : G$ between two model categories is a Quillen pair iff the following two conditions are satisfied:*

- *F takes a cofibration to a cofibration;*
- *G takes a fibration between fibrant objects to a fibration.*

Proof: The necessity is obvious. Let us prove the sufficiency. For this it suffices to show that F is a left Quillen functor by E.2.2. Thus we show that F takes an acyclic cofibration $u : A \to B$ to an acyclic cofibration $F(u) : F(A) \to F(B)$. But $F(u)$ is acyclic iff it has the left lifting property with respect to every fibration between fibrant objects $f : X \to Y$ by lemma E.2.13. But the condition $F(u) \pitchfork f$ is equivalent to the condition $u \pitchfork G(f)$ by the adjointness $F \vdash G$. We have $u \pitchfork G(f)$, since $G(f)$ is a fibration by assumption. This proves that we have $F(u) \pitchfork f$. Thus, $F(u)$ is acyclic. ■

Definition E.2.15. *We call a Quillen pair $F : \mathcal{U} \leftrightarrow \mathcal{V} : G$ a* homotopy reflection *$\mathcal{U} \to \mathcal{V}$ if the right derived functor G^R is full and faithful. Dually, we call the pair a* homotopy reflection *$\mathcal{V} \to \mathcal{U}$ if the left derived functor F^L is full and faithful.*

Proposition E.2.16. *Let*

$$F_1 : \mathcal{T} \leftrightarrow \mathcal{U} : G_1 \quad \text{and} \quad F_2 : \mathcal{U} \leftrightarrow \mathcal{V} : G_2$$

be two Quillen pairs of adjoint functors. If the pair (F_1, G_1) is a homotopy reflection, then the pair (F_2, G_2) is a homotopy reflection iff the pair (F_2F_1, G_1G_2) is a homotopy reflection.

Proposition E.2.17. *The following conditions on a Quillen pair $F : \mathcal{U} \leftrightarrow \mathcal{V} : G$ are equivalent:*

- *The pair (F, G) is a homotopy reflection $\mathcal{U} \to \mathcal{V}$;*
- *The map $FLGX \to X$ is a weak equivalence for every fibrant object $X \in \mathcal{V}$, where $LGX \to GX$ denotes a cofibrant replacement of GX;*
- *The map $FLGX \to X$ is a weak equivalence for every fibrant-cofibrant object $X \in \mathcal{V}$, where $LGX \to GX$ denotes a cofibrant replacement of GX.*

Proof: The functor G^R is full and faithful iff the counit of the adjunction $F^L \dashv G^R$ is an isomorphism. But if $X \in \mathcal{V}$ is fibrant, this counit is obtained by composing the maps $FLGX \to FGX \to X$, where $LGX \to GX$ is a cofibrant replacement of GX. This proves the equivalence (i)⇔(ii). The implication (ii)⇒(iii) is obvious. Let us prove the implication (iii)⇒(ii). For every fibrant objet X, there is a an acyclic fibration $p : Y \to X$ with domain a cofibrant object Y. The map $Gp : GY \to GX$ is an acyclic fibration, since G is a right Quillen functor. Let $q : LGY \to GY$ be a cofibrant replacement of GY. Then the map $FLGY \to FGY \to Y$ is a weak equivalence by assumption, since Y is fibrant-cofibrant. But the composite $G(p)q : LGY \to GY \to GX$ is a cofibrant replacement of GX, since $G(p)$ is a weak equivalence. Moreover, the composite $FLGY \to FGX \to X$ is a weak equivalence, since p is a weak equivalence and the following diagram commutes

$$\begin{array}{ccccc} FLGY & \longrightarrow & FGY & \longrightarrow & Y \\ & \searrow & \downarrow & & \downarrow{\scriptstyle p} \\ & & FGX & \longrightarrow & X. \end{array}$$

This proves that condition (ii) is satisfied for a cofibrant replacement of GX. ■

Proposition E.2.18. *If $F : \mathcal{U} \leftrightarrow \mathcal{V} : G$ is a homotopy reflection, then the right adjoint G preserves and reflects weak equivalences between fibrant objects.*

Proof: The functor G^R is equivalent to the functor $Ho(G_f) : Ho(\mathcal{V}_f) \to Ho(\mathcal{U}_f)$ induced by the functor G. Thus, $Ho(G_f)$ is full and faithful since G^R is full and faithful. This proves the result since a full and faithful functor is conservative. ■

Definition E.2.19. *Let $F : \mathcal{U} \leftrightarrow \mathcal{V} : G$ be a homotopy reflection beween two model categories. We shall say that an object $X \in \mathcal{U}$ is* local *(with respect to the the pair (F, G)) if it belongs to the essential image of the right derived functor $G^R : Ho(\mathcal{V}) \to Ho(\mathcal{U})$.*

Proposition E.2.20. *Let $F : \mathcal{U} \leftrightarrow \mathcal{V} : G$ be a homotopy reflection beween two model categories. If $\eta : I \to GF$ is the unit of the adjunction $F \vdash G$, then a cofibrant object $A \in \mathcal{U}$ is local iff the composite $G(u)\eta_A : A \to GFA \to GRFA$ is a weak equivalence, where $u : FA \to RFA$ denotes a fibrant replacement of FA. The image by G of a fibrant object of $\mathcal{V}$ is local. The class of local objects is invariant under weak equivalences.*

Proof: Let us prove the first statement. An object $A \in \mathcal{U}$ is local iff the unit $A \to G^L F^R A$ of the adjunction $F^L \vdash G^R$ is invertible in $Ho(\mathcal{U})$, since the functor G^R is full and faithful. If A is cofibrant, this unit is represented by the composite $G(u)\eta_A : A \to GFA \to GRFA$, where $u : FA \to RFA$ denotes a fibrant replacement of FA. But a map is invertible in $Ho(\mathcal{U})$ iff it is a weak equivalence by E.1.4. The first statement is proved. If $X \in \mathcal{V}$ is fibrant, then we can take $G^R(X) = G(X)$. Thus, the object $G(X)$ is local, since the object $G^R(X)$ is local. The essential image of the functor G^R is invariant under isomorphisms. It follows from E.1.13 that the class of local objects is invariant under weak equivalences. ■

Proposition E.2.21. *Let $(\mathcal{C}_i, \mathcal{W}_i, \mathcal{F}_i)$ $(i = 1, 2)$ be two model structures on a category $\mathcal{E}$ and let us denote by $\mathcal{M}_i$ the corresponding model category. Suppose that $\mathcal{C}_1 \subseteq \mathcal{C}_2$ and $\mathcal{W}_1 \subseteq \mathcal{W}_2$. Then the identity functor $\mathcal{E} \to \mathcal{E}$ is a homotopy reflection $\mathcal{M}_1 \to \mathcal{M}_2$. The following conditions on an object A are equivalent:*

- *(i) A is local;*
- *(ii) there exists a $\mathcal{M}_1$-equivalence $A \to A'$ with codomain a $\mathcal{M}_2$-fibrant object A';*
- *(iii) every $\mathcal{M}_2$-fibrant replacement $A \to A'$ is a $\mathcal{M}_1$-fibrant replacement.*

In particular, every $\mathcal{M}_2$-fibrant object is local. A map between local objects is a $\mathcal{M}_1$-equivalence iff it is a $\mathcal{M}_2$-equivalence.

Proof: By hypothesis, we have $\mathcal{C}_1 \subseteq \mathcal{C}_2$ and $\mathcal{W}_1 \subseteq \mathcal{W}_2$. It follows that we have $\mathcal{F}_2 \cap \mathcal{W}_2 \subseteq \mathcal{F}_1 \cap \mathcal{W}_1$ and $\mathcal{F}_2 \subseteq \mathcal{F}_1$. Let us prove the first statement. The identity functor $\mathcal{E} \to \mathcal{E}$ is obviously a left Quillen functor $\mathcal{M}_1 \to \mathcal{M}_2$. Moreover, the conditions of proposition E.2.17 are trivially satisfied, since $\mathcal{W}_1 \subseteq \mathcal{W}_2$. Thus, the identity functor is a homotopy reflection $\mathcal{M}_1 \to \mathcal{M}_2$. It follows by Proposition E.2.20 that every $\mathcal{M}_2$-fibrant object is local. Moreover, a $\mathcal{M}_1$-cofibrant object A is local iff the map $i : A \to A'$ is a $\mathcal{M}_1$-equivalence, where $i : A \to A'$ denotes a $\mathcal{M}_2$-fibrant replacement of A. Let us prove the second statement. The implication (ii)⇒(i) is clear, since every $\mathcal{M}_2$-fibrant object is local and the class of local objects is invariant under $\mathcal{M}_1$-equivalences by Proposition E.2.20. The implication (iii)⇒(ii) is obvious. Let us prove the implication (i)⇒(iii). Let $i : A \to A'$ be a $\mathcal{M}_2$-fibrant replacement of A. The object A' is $\mathcal{M}_1$-fibrant, since it is $\mathcal{M}_2$-fibrant. Let us show that $i \in \mathcal{W}_1$. Let us choose a $\mathcal{M}_1$-cofibrant replacement $q : L_1A \to A$. The object L_1A is local, since A is local and $q \in \mathcal{W}_1$. We have $q \in \mathcal{W}_2$, since $\mathcal{W}_1 \subseteq \mathcal{W}_2$. Thus, $iq \in \mathcal{W}_2$, since $i \in \mathcal{W}_2$. Hence there exists a factorisation of $iq = pv : L_1A \to R_2L_1A \to A'$, with $v \in \mathcal{C}_2 \cap \mathcal{W}_2$ and $p \in \mathcal{F}_2 \cap \mathcal{W}_2$,

$$\begin{array}{ccc} L_1A & \xrightarrow{v} & R_2L_1A \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow{i} & A'. \end{array}$$

The object R_2L_1A is $\mathcal{M}_2$-fibrant, since A' is $\mathcal{M}_2$-fibrant and p is a $\mathcal{M}_2$-fibration. Hence the map $v : L_1A \to R_2L_1A$ is a $\mathcal{M}_2$-fibrant replacement of L_1A. Thus, $v \in \mathcal{W}_1$, since L_1A is $\mathcal{M}_1$-cofibrant and local. But we have $p \in \mathcal{W}_1$, since we have $\mathcal{F}_2 \cap \mathcal{W}_2 \subseteq \mathcal{F}_1 \cap \mathcal{W}_1$. Hence we have $i \in \mathcal{W}_1$ by three-for-two. Let us prove the last statement. If $f : X \to Y$ is a map between local objects, let us show that $f \in \mathcal{W}_1 \Leftrightarrow g \in \mathcal{W}_2$. The implication (⇒) is obvious, since $\mathcal{W}_1 \subseteq \mathcal{W}_2$. Conversely, if $f \in \mathcal{W}_2$, let us show that $f \in \mathcal{W}_1$. It is easy to see that there exists a commutative diagram

$$\begin{array}{ccccc} X & \xleftarrow{q_X} & L_1X & \xrightarrow{i_X} & R_2L_1X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ Y & \xleftarrow{q_X} & L_1Y & \xrightarrow{i_Y} & R_2L_1Y \end{array}$$

where q_X and q_Y belongs to $\mathcal{W}_1$, where L_1X and L_1Y are $\mathcal{M}_1$-cofibrant, where i_X and i_Y belongs to $\mathcal{W}_2$ and where R_2L_1X and R_2L_1Y are $\mathcal{M}_2$-fibrant. The horizontal maps of the diagram belongs to $\mathcal{W}_2$ since $\mathcal{W}_1 \subseteq \mathcal{W}_2$. Thus, $h \in \mathcal{W}_2$ by three-for-two since we have $f \in \mathcal{W}_2$ by assumption. Thus, $h \in \mathcal{W}_1$ by E.2.18, since h is a map between $\mathcal{M}_2$-fibrant objects. The object L_1X is local since X is local and $q_X \in \mathcal{W}_1$. Hence the maps i_X and i_Y belongs to $\mathcal{W}_1$ by Proposition E.2.20. Hence the horizontal maps of the diagram belongs to $\mathcal{W}_1$. Thus, $f \in \mathcal{W}_1$ by three-for-two, since $h \in \mathcal{W}_1$. ■

Definition E.2.22. *Let $\mathcal{M}_i = (\mathcal{C}_i, \mathcal{W}_i, \mathcal{F}_i)$ $(i = 1, 2)$ be two model structures on a category $\mathcal{E}$. If $\mathcal{C}_1 = \mathcal{C}_2$ and $\mathcal{W}_1 \subseteq \mathcal{W}_2$, we shall say that the model structure $\mathcal{M}_2$ is a* Bousfield localisation *of the model structure $\mathcal{M}_1$.*

Proposition E.2.23. *Let $\mathcal{M}_2 = (\mathcal{C}_2, \mathcal{W}_2, \mathcal{F}_2)$ be a Bousfield localisation of a model structure $\mathcal{M}_1 = (\mathcal{C}_1, \mathcal{W}_1, \mathcal{F}_1)$ on a category $\mathcal{E}$. Then a map between $\mathcal{M}_2$-fibrant objects is a $\mathcal{M}_2$-fibration iff it is a $\mathcal{M}_1$-fibration. A local object is $\mathcal{M}_1$-fibrant iff it is $\mathcal{M}_2$-fibrant. An object A is local iff every $\mathcal{M}_1$-fibrant replacement $i : A \to A'$ is a $\mathcal{M}_2$-fibrant replacement.*

Proof: By hypothesis, we have $\mathcal{C}_1 = \mathcal{C}_2$ and $\mathcal{W}_1 \subseteq \mathcal{W}_2$. It follows that we have $\mathcal{F}_2 \cap \mathcal{W}_2 = \mathcal{F}_1 \cap \mathcal{W}_1$ and $\mathcal{F}_2 \subseteq \mathcal{F}_1$. Let us prove the first statement. Let us show that a map $f : X \to Y$ between two $\mathcal{M}_2$-fibrant objects is a $\mathcal{M}_2$-fibration iff it is a $\mathcal{M}_1$-fibration. The implication ($\Rightarrow$) is clear, since $\mathcal{F}_2 \subseteq \mathcal{F}_1$. Conversely, if $f \in \mathcal{F}_1$, let us show that $f \in \mathcal{F}_2$. Let us choose a factorisation $f = pi : X \to Z \to Y$ with $i \in \mathcal{C}_2 \cap \mathcal{W}_2$ and $p \in \mathcal{F}_2$. Then $i \in \mathcal{W}_1$ by Lemma E.2.6, since the identity functor is a right Quillen functor $\mathcal{M}_2 \to \mathcal{M}_1$ and since i is a map between $\mathcal{M}_2$-fibrant objects. Thus, $i \in \mathcal{W}_1 \cap \mathcal{C}_1$, since $\mathcal{C}_1 = \mathcal{C}_2$. Hence the square

$$\begin{array}{ccc} X & \xrightarrow{id} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ E & \xrightarrow[p]{} & Y \end{array}$$

has a diagonal filler, making f a retract of p and therefore $f \in \mathcal{F}_2$. Let us prove the second statement. Every $\mathcal{M}_2$-fibrant object is local by E.2.21. Conversely, let X by a M_1-fibrant local object. Let us choose a fibrant replacement $u : X \to R_2X$ with respect to the model structure $\mathcal{M}_2$. We can suppose that $u \in \mathcal{C}_2 \cap \mathcal{W}_2$. We have $u \in \mathcal{W}_1$ by E.2.21 since X is local. Thus, $u \in \mathcal{C}_1 \cap \mathcal{W}_1$, since $\mathcal{C}_1 = \mathcal{C}_2$. Hence the square

$$\begin{array}{ccc} X & \xrightarrow{1_X} & X \\ {\scriptstyle u}\downarrow & & \downarrow \\ R_2X & \longrightarrow & 1 \end{array}$$

has a diagonal filler, since X is $\mathcal{M}_1$-fibrant by assumption. Thus, X a retract of R_2X and therefore X is $\mathcal{M}_2$-fibrant. Let us prove the last statement. ($\Rightarrow$) Let $i : A \to A'$ be a $\mathcal{M}_1$-fibrant replacement of A. The object A' is local, since $i \in \mathcal{W}_1$ and A is local. Thus, A is $\mathcal{M}_2$-fibrant by what we have proved above, since A is $\mathcal{M}_1$-fibrant. We have $i \in \mathcal{W}_2$, since $\mathcal{W}_1 \subseteq \mathcal{W}_2$. This shows that the map $i : A \to A'$ be a $\mathcal{M}_2$-fibrant replacement. ($\Leftarrow$) Let us choose a $\mathcal{M}_1$-fibrant replacement $i : A \to L_1A$. The object L_1A is $\mathcal{M}_2$-fibrant, since the map i is a $\mathcal{M}_2$-fibrant replacement by assumption. This shows that A is local, since $i \in \mathcal{W}_1$. ∎

Proposition E.2.24. *A Quillen pair $F : \mathcal{U} \leftrightarrow \mathcal{V} : G$ is a Quillen equivalence iff the following equivalent conditions are satisfied:*

- *The pair (F, G) is a both a homotopy reflection and coreflection;*
- *The pair (F, G) is a homotopy reflection and the functor F reflects weak equivalences between cofibrant objects;*
- *The pair (F, G) is a homotopy coreflection and the functor G reflects weak equivalences between fibrant objects;*

E.3 Monoidal model categories

Let $\odot : \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3$ be a functor of two variables between three model categories. Recall that if $u : A \to B$ is map in $\mathcal{E}_1$ and $v : S \to T$ is a map in $\mathcal{E}_2$, then $u \odot' v$ denotes the map

$$A \odot T \sqcup_{A \odot S} B \odot S \longrightarrow B \odot T$$

obtained from the commutative square

$$\begin{array}{ccc} A \odot S & \longrightarrow & B \odot S \\ \downarrow & & \downarrow \\ A \odot T & \longrightarrow & B \odot T. \end{array}$$

Definition E.3.1. [Ho] *We shall say that a functor of two variables*

$$\odot : \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3$$

between three model categories is a left Quillen functor *if it is concontinuous in each variable and the following conditions are satisfied:*

- *$u \odot' v$ is a cofibration if u and v are cofibrations;*
- *$u \odot' v$ is an acyclic cofibration if u and v are cofibrations and if u or v is acyclic.*

Dually, we shall say that $\odot$ is a right Quillen functor *if the opposite functor $\odot^o : \mathcal{E}_1^o \times \mathcal{E}_2^o \to \mathcal{E}_3^o$ is a left Quillen functor.*

Proposition E.3.2. *Let $\odot : \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3$ be a left Quillen functor of two variables between three model categories. If $A \in \mathcal{E}_1$ is cofibrant, then the functor $B \mapsto A \odot B$ is a left Quillen functor $\mathcal{E}_2 \to \mathcal{E}_3$.*

Proof: If $A \in \mathcal{E}_1$ is cofibrant, then the map $i_A : \bot \to A$ is a cofibration, where $\bot$ is the initial object. If $v : S \to T$ is a map in $\mathcal{E}_2$, then we have $A \odot v = i_A \odot' v$. Thus, $A \odot v$ is a cofibration if v is a cofibration and $A \odot v$ is acyclic if moreover v is acyclic. ∎

Recall that a functor of two variables

$$\odot : \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3$$

is said to be *divisible on the left* if the functor $A \odot (-) : \mathcal{E}_2 \to \mathcal{E}_3$ admits a right adjoint $A\backslash(-) : \mathcal{E}_3 \to \mathcal{E}_2$ for every object $A \in \mathcal{E}_1$. In this case we obtain a functor of two variables $(A, X) \mapsto A\backslash X$,

$$\mathcal{E}_1^o \times \mathcal{E}_3 \to \mathcal{E}_2,$$

called the left division functor. There is a dual notion of a right division functor. If $u : A \to B$ is map in $\mathcal{E}_1$ and $f : X \to Y$ is a map in $\mathcal{E}_3$, we denote by $\langle u \backslash\, f \rangle$ the map

$$B\backslash X \to B\backslash Y \times_{A\backslash Y} A\backslash X$$

obtained from the commutative square

$$\begin{array}{ccc} B\backslash X & \longrightarrow & A\backslash X \\ \downarrow & & \downarrow \\ B\backslash Y & \longrightarrow & A\backslash Y. \end{array}$$

Proposition E.3.3. *Let $\odot : \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3$ be a functor of two variables between three model categories. If the functor $\odot$ is divisible on the left, then it is a left Quillen functor iff the corresponding left division functor $\mathcal{E}_1^o \times \mathcal{E}_3 \to \mathcal{E}_2$ is a right Quillen functor. Dually, if the functor $\odot$ is divisible on the right, then it is a left Quillen functor iff the corresponding right division functor $\mathcal{E}_3 \times \mathcal{E}_2^o \to \mathcal{E}_1$ is a right Quillen functor.*

Proposition E.3.4. *Let $\odot : \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3$ be a functor of two variables, cocontinuous in each, between three model categories. Suppose that the following three conditions are satisfied:*

- *If $u \in \mathcal{E}_1$ and $v \in \mathcal{E}_2$ are cofibrations, then so $u \otimes' v$;*
- *the functor $(-) \odot B$ preserves acyclic cofibrations for every object $B \in \mathcal{E}_2$;*
- *the functor $A \odot (-)$ preserves acyclic cofibrations for every object $A \in \mathcal{E}_1$.*

Then $\odot$ is a left Quillen functor.

Proof: Let $u : A \to B$ be a cofibration in $\mathcal{E}_1$ and $v : S \to T$ be a cofibration in $\mathcal{E}_2$. Let us show that $u \odot' v$ is acyclic if u or v is acyclic. We only consider the case where v is acyclic. Consider the commutative diagram

$$\begin{array}{ccccc} A \odot S & \xrightarrow{u \odot S} & B \odot S & & \\ {\scriptstyle A\odot v}\downarrow & & {\scriptstyle i_2}\downarrow & \searrow^{B \odot v} & \\ A \odot T & \xrightarrow{i_1} & Z & \xrightarrow{u \odot' v} & B \odot T \end{array}$$

where $Z = A \odot T \sqcup_{A \odot S} B \odot S$ and where $(u \odot' v) i_1 = u \odot T$. The map $A \odot v$ is an acyclic cofibration since v is an acyclic cofibration. Similarly for the map $B \odot v$. It follows that i_2 is an acyclic cofibration by cobase change. Thus, $u \odot' v$ is acyclic by three-for-two since $(u \odot' v) i_2 = B \odot v$ is acyclic.

Definition E.1. [Ho] *A model structure* $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ *on monoidal closed category* $\mathcal{E} = (\mathcal{E}, \otimes)$ *is said to be* monoidal *if the tensor product* $\otimes : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ *is a left Quillen functor of two variables and the unit object of the tensor product is cofibrant.*

Proposition E.3.5. *Let* $\mathcal{E}$ *be a monoidal closed category with unit object* U*. Then a model structure on* $\mathcal{E}$ *for which* U *is cofibrant is monoidal iff the left division functor is a right Quillen functor of two variables iff the right division functor is a right Quillen functor of two variables.*

Definition E.3.6. *If* $\mathcal{V}$ *is a monoidal closed model category we shall say that a model structure on a* $\mathcal{V}$*-category* $\mathcal{E}$ *is* $\mathcal{V}$-enriched *if the hom functor*

$$hom : \mathcal{E}^o \times \mathcal{E} \to \mathcal{V}$$

is a right Quillen functor of two variables.

A $\mathcal{V}$-category equipped with a $\mathcal{V}$-enriched model structure is called a $\mathcal{V}$-*enriched* model category.

Let $\mathcal{E}$ be a *symmetric* monoidal closed category. Then the objects X/A and $A\backslash X$ are canonicaly isomorphic; hence we can use a common notation, for example $[A, X]$. Similarly, the maps $\langle f/u \rangle$ and $\langle u\backslash f \rangle$ are canonicaly isomorphic; hence we can use a common notation, for example $\langle u, f \rangle$.

Recall that a *simplicial category* is a category enriched over $\mathbf{S}$.

Definition E.3.7. [Q] *Let* $\mathcal{E}$ *be a simplicial category. We shall say that a model structure on* $\mathcal{E}$ *is* simplicial *if the it is enriched with respect to the classical model structure* $(\mathbf{S}, \mathit{Who})$.

A simplicial category equipped with a simplicial model structure is called a *simplicial model category*

Definition E.3.8. *We shall say that a model structure* $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ *on a category* $\mathcal{E}$ *is* cartesian *if the cartesian product* $\times : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ *is a left Quillen functor of two variables and if the terminal object 1 is cofibrant.*

If $\mathcal{E}$ is a cartesian closed, then the category $\mathcal{E}/B$ is enriched over $\mathcal{E}$.

Proposition E.3.9. *If* $\mathcal{E}$ *be a cartesian closed model category, then the model category* $\mathcal{E}/B$ *is enriched over* $\mathcal{E}$ *for any object* $B \in \mathcal{E}$*. Similarly, the model category* $A\backslash\mathcal{E}$ *is enriched over* $\mathcal{E}$ *for any object* $A \in \mathcal{E}$*.*

Proof: Let us prove the first statement. Let us denote by $S \otimes X$ the tensor product of an object $X = (X, f)$ in $A\backslash\mathcal{E}$ by an object $S \in \mathcal{E}$. It suffices to show that the tensor product functor $\mathcal{E} \times \mathcal{E}/B \to \mathcal{E}/B$ is a left Quillen functor of two variables. But this is clear since we have $S \otimes X = (S \times X, p(S \times f))$, where p is the projection

$S \times B \to B$. Let us prove the second statement. Let us denote by $X^{[S]}$ the cotensor product of an object $X = (X, u)$ in $A\backslash\mathcal{E}$ by an object $S \in \mathcal{E}$. It suffices to show that the cotensor product functor $\mathcal{E}^o \times A\backslash\mathcal{E} \to A\backslash\mathcal{E}$ is a right Quillen functor of two variables. But this is clear since $X^{[S]} = (X^S, u^S\delta)$, where δ is the diagonal $A \to A^S$. ■

Appendix F

Homotopy factorisation systems

The goal of this appendix is to introduce the notion of homotopy factorisation system and establish their basic properties. Factorisation systems where introduced in homotopy theory by Bousfield [Bous]. The appendix has four sections.

F.1 Homotopy factorisation systems and Bousfield systems

We say that two maps of a category $\mathcal{E}$ are *isomorphic* if they are isomorphic as objects of the category $\mathcal{E}^I$. We say that a class $\mathcal{M}$ of maps in $\mathcal{E}$ is *replete* if every map isomorphic to a map in $\mathcal{M}$ belongs to $\mathcal{M}$.

Definition F.1.1. *Let $\mathcal{E}$ be a model category. We say that a class of morphisms $\mathcal{K} \subseteq \mathcal{E}$ is* homotopy replete *if for every commutative square in $\mathcal{E}'$*

$$\begin{array}{ccc} A & \longrightarrow & A' \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u'} \\ B & \longrightarrow & B' \end{array}$$

for which the horizontal maps are weak equivalences, we have $u \in \mathcal{M} \Leftrightarrow u' \in \mathcal{M}$.

Proposition F.1.2. *Let $\mathcal{E}$ be a model category and let $\pi : \mathcal{E} \to Ho(\mathcal{E})$ is the canonical functor. Then the map $\mathcal{M} \mapsto \pi^{-1}(\mathcal{M})$ induces a bijection between the replete classes of maps in $Ho(\mathcal{E})$ and the homotopy replete classes of maps in $\mathcal{E}$.*

We denote the full subcategory of fibrant (resp. cofibrant) objects of a model category $\mathcal{E}$ by $\mathcal{E}_f$ (resp. $\mathcal{E}_c$). We say that the intersection $\mathcal{E}_{fc} = \mathcal{E}_f \cap \mathcal{E}_c$ is the *core* of $\mathcal{E}$. For any class $\mathcal{M}$ of maps in $\mathcal{E}$, we put

$$\mathcal{M}_c = \mathcal{M} \cap \mathcal{E}_c, \quad \mathcal{M}_f = \mathcal{M} \cap \mathcal{E}_f \quad \text{and} \quad \mathcal{M}_{fc} = \mathcal{M} \cap \mathcal{E}_{fc}.$$

Definition F.1.3. *Let $\mathcal{E}$ be a model category with model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$. We say that a pair $(\mathcal{A}, \mathcal{B})$ of classes of maps in $\mathcal{E}$ is a* homotopy factorisation system *if the following conditions are satisfied:*

- *the classes $\mathcal{A}$ and $\mathcal{B}$ are homotopy replete;*
- *the pair $(\mathcal{A} \cap \mathcal{C}_{fc}, \mathcal{B} \cap \mathcal{F}_{fc})$ is a weak factorisation system in $\mathcal{E}_{fc}$;*
- *the class $\mathcal{A}$ has the right cancellation property;*
- *the class $\mathcal{B}$ has the left cancellation property.*

The class $\mathcal{A}$ is called the left *class of the system, and the class $\mathcal{B}$ the* right *class. The weak factorisation system $(\mathcal{A} \cap \mathcal{C}_{fc}, \mathcal{B} \cap \mathcal{F}_{fc})$ is the* center *of the homotopy factorisation system.*

The pairs $(\mathcal{E}, \mathcal{W})$ and $(\mathcal{W}, \mathcal{E})$ are examples of homotopy factorisation systems.

Let us say that a model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ on a category $\mathcal{E}$ is *discrete* if $\mathcal{W}$ is the class of isomorphisms. In this case we have $\mathcal{C} = \mathcal{E} = \mathcal{F}$. The notions of homotopy factorisation systems and of factorisation systems coincide if the model structure is discrete.

Theorem F.1.4. *Suppose that a pair $(\mathcal{A}, \mathcal{B})$ of classes of maps in a model category $\mathcal{E}$ satisfies the first two conditions of Definition F.2.1. Then the last two conditions are equivalent.*

The Theorem will be proved in F.3.1.

Definition F.1.5. *We say that a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in a model category $\mathcal{E}$ with model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ is* strong *if the pair $(\mathcal{A} \cap \mathcal{C}, \mathcal{B} \cap \mathcal{F})$ is a weak factorisation system in $\mathcal{E}$.*

Definition F.1.6. *Let $\mathcal{E}$ be a model category and let $\mathcal{E}'$ be one of the classes $\mathcal{E}_c$, $\mathcal{E}_f$ or $\mathcal{E}_{fc}$. We say that a class $\mathcal{M}$ of fibrations (resp. cofibrations) in $\mathcal{E}'$ is homotopy replete* in fibrations *(resp.* in cofibrations*) if for every commutative square in $\mathcal{E}'$*

$$\begin{array}{ccc} A & \longrightarrow & A' \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u'} \\ B & \longrightarrow & B' \end{array}$$

in which the horizontal maps are weak equivalences and the vertical maps are fibrations (resp. cofibrations), we have $u \in \mathcal{M} \Leftrightarrow u' \in \mathcal{M}$.

Definition F.1.7. *Let $\mathcal{E}$ be a model category with model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$. We say that a class $\mathcal{M}$ of fibrations has the left cancellation property* in fibrations, *if it has the left cancellation property as a subcategory of $\mathcal{F}$. Dually, we say that a class $\mathcal{M}$ of cofibrations has the right cancellation property* in cofibrations. *if it has the right cancellation property as a subcategory of $\mathcal{C}$.*

Definition F.1.8. *We call a weak factorisation system $(\mathcal{A}, \mathcal{B})$ in a model category $\mathcal{E}$ a* Bousfield factorisation system *if the following conditions are satisfied:*

- *$\mathcal{A}$ is a class of cofibrations homotopy replete in cofibrations;*
- *$\mathcal{B}$ is a class of fibrations homotopy replete in fibrations;*
- *$\mathcal{A}$ has the right cancellation property in cofibrations;*
- *$\mathcal{B}$ has the left cancellation property in fibrations.*

Theorem F.1.9. *Suppose that a pair $(\mathcal{A}, \mathcal{B})$ of classes of maps in a model category $\mathcal{E}$ satisfies the first two conditions of Definition F.1.8. Then the last two conditions are equivalent.*

Theorem F.1.10. *Let $\mathcal{E}$ be a model category with model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$. If $(\mathcal{A}, \mathcal{B})$ is a strong homotopy factorisation system in $\mathcal{E}$, then the pair $(\mathcal{A}', \mathcal{B}') = (\mathcal{A} \cap \mathcal{C}, \mathcal{B} \cap \mathcal{F})$ is a Bousfield factorisation system. This defines a bijection between the strong homotopy factorisations systems in $\mathcal{E}$ and the Bousfield systems. The inverse bijection takes a Bousfield system $(\mathcal{A}', \mathcal{B}')$ to a pair $(\mathcal{A}, \mathcal{B})$, where $\mathcal{A}$ is the class of maps $u : A \to B$ which admits a factorisation $u = eu' : A \to B' \to B$ where e is a weak equivalence and $u' \in \mathcal{A}'$, and where $\mathcal{B}$ is the class of maps $f : X \to Y$ which admits a factorisation $f = f'e : X \to X' \to Y$ where e is a weak equivalence and $f' \in \mathcal{B}'$.*

F.2 Weak homotopy factorisation systems

Many properties of homotopy factorisation systems are also valid for the weaker notion. For example, the intersection of the classes of a weak homotopy factorisation system is the class of weak equivalences.

Definition F.2.1. *Let $\mathcal{E}$ be a model category with model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$. We say that a pair $(\mathcal{A}, \mathcal{B})$ of classes of maps in $\mathcal{E}$ is a* weak homotopy factorisation system *if the following conditions are satisfied:*

- *the classes $\mathcal{A}$ and $\mathcal{B}$ are homotopy replete;*
- *the pair $(\mathcal{A} \cap \mathcal{C}_{fc}, \mathcal{B} \cap \mathcal{F}_{fc})$ is a weak factorisation system in $\mathcal{E}_{fc}$;*

The class $\mathcal{A}$ is the left *class of the system, and the class $\mathcal{B}$ the* right *class. The weak factorisation system $(\mathcal{A} \cap \mathcal{C}_{fc}, \mathcal{B} \cap \mathcal{F}_{fc})$ is the* center *of the system.*

Corollary F.2.2. *Let $\mathcal{M} \subseteq \mathcal{E}$ and $\mathcal{N} \subseteq \mathcal{E}$ be two homotopy replete classes of maps. Then*

$$\mathcal{M} \subseteq \mathcal{N} \iff \mathcal{M}_c \subseteq \mathcal{N}_c \iff \mathcal{M}_f \subseteq \mathcal{N}_f \iff \mathcal{M}_{fc} \subseteq \mathcal{N}_{fc}.$$

Proof: Let us prove the first equivalence. Obviously, $(\mathcal{M} \cap \mathcal{N})_c = \mathcal{M}_c \cap \mathcal{N}_c$. It follows from Lemma F.1.2 that

$$\mathcal{M} = \mathcal{M} \cap \mathcal{N} \iff \mathcal{M}_c = \mathcal{M}_c \cap \mathcal{N}_c.$$

Thus, $\mathcal{M} \subseteq \mathcal{N} \Leftrightarrow \mathcal{M}_c \subseteq \mathcal{N}_c$. This proves the first equivalence. The other equivalences are proved similarly. ∎

Recall from C.0.20 that a class of maps $\mathcal{M}$ in a category $\mathcal{E}$ is said to have the *right cancellation property* if for any pair of maps $u : A \to B$ and $v : B \to C$, the implication

$$vu \in \mathcal{M} \text{ and } u \in \mathcal{M} \;\Rightarrow\; v \in \mathcal{M}$$

is true. Dually, $\mathcal{M}$ is said to have the *left cancellation property* if the implication

$$vu \in \mathcal{M} \text{ and } v \in \mathcal{M} \;\Rightarrow\; u \in \mathcal{M}$$

is true.

Lemma F.2.3. *Let $\mathcal{M}$ be a homotopy replete class of maps in a model category $\mathcal{E}$. Then*

- (i) *If $\mathcal{M}_{fc}$ is closed under composition, then so is $\mathcal{M}$;*
- (ii) *If $\mathcal{M}_{fc}$ has the left (resp. right) cancellation property, then so does $\mathcal{M}$;*
- (iii) *If $\mathcal{M}_{fc}$ contains the isomorphisms in $\mathcal{E}_{fc}$, then $\mathcal{M} \supseteq \mathcal{W}$.*

Proof: Let us prove (i) and (ii). We shall use the category $\mathcal{E}^{I_2}$, where I_2 is the category generated by two arrows $0 \to 1 \to 2$. An object of $\mathcal{E}^{I_2}$ is a pair of maps $a_0 : A_0 \to A_1$ and $a_1 : A_1 \to A_2$ in $\mathcal{E}$. Let us give the category $\mathcal{E}^{I_2}$ the injective model structure of G.0.12. The cofibrations and the weak equivalences of this model structure are defined level-wise. Every fibration is a level-wise fibration (but the converse is not true). For every object $(a_0, a_1) \in \mathcal{E}^{I_2}$, we can choose an acyclic cofibration $i : (a_0, a_1) \to (b_0, b_1)$ with codomain a fibrant object, together with an acyclic fibration $p : (c_0, c_1) \to (b_0, b_1)$ with domain a cofibrant object . This gives a commutative diagram

$$\begin{array}{ccccc}
A_0 & \longrightarrow & B_0 & \longleftarrow & C_0 \\
{\scriptstyle a_0}\downarrow & & {\scriptstyle b_0}\downarrow & & \downarrow{\scriptstyle c_0} \\
A_1 & \longrightarrow & B_1 & \longleftarrow & C_1 \\
{\scriptstyle a_1}\downarrow & & {\scriptstyle b_1}\downarrow & & \downarrow{\scriptstyle c_1} \\
A_2 & \longrightarrow & B_2' & \longleftarrow & C_2
\end{array}$$

in which the horizontal maps are weak equivalences and in which the objects C_0, C_1 and C_2 belongs to $\mathcal{E}_{fc}$. If $\mathcal{M}$ is closed under composition, then

$$a_0, a_1 \in \mathcal{M} \Rightarrow c_0, c_1 \in \mathcal{M}_{fc} \Rightarrow c_1 c_0 \in \mathcal{M}_{fc} \Rightarrow a_1 a_0 \in \mathcal{M},$$

since the class $\mathcal{M}$ is homotopy replete. This shows that the class $\mathcal{M}$ is closed under composition. If $\mathcal{M}$ has the left cancellation property, then

$$a_1 a_0, a_1 \in \mathcal{M} \Rightarrow c_1 c_0, c_1 \in \mathcal{M}_{fc} \Rightarrow c_0 \in \mathcal{M}_{fc} \Rightarrow a_0 \in \mathcal{M},$$

since the class $\mathcal{M}$ is homotopy replete. This shows that the class $\mathcal{M}$ has the left cancellation property. It remains to prove (iii). By F.2.2, it suffices to show that $\mathcal{W}_{fc} \subseteq \mathcal{M}_{fc}$. Let $u : A \to B$ be a map in $\mathcal{W}_{fc}$. We have $1_A \in \mathcal{M}_{fc}$ by the assumption on $\mathcal{M}_{fc}$. It follows that $u \in \mathcal{M}_{fc}$, since the horizontal maps of the square

$$\begin{array}{ccc} A & \xrightarrow{1_A} & A \\ {\scriptstyle 1_A}\downarrow & & \downarrow{\scriptstyle u} \\ A & \xrightarrow{u} & B. \end{array}$$

are weak equivalences and since the class $\mathcal{M}$ is homotopy replete . ■

Proposition F.2.4. *A weak homotopy factorisation system on a model category $\mathcal{E}$ is determined by its center. Each class of a weak homotopy factorisation system determines the other.*

Proof: Let $(\mathcal{A}, \mathcal{B})$ be a weak homotopy factorisation system and let us put $(\mathcal{A}_0, \mathcal{B}_0) = (\mathcal{A} \cap \mathcal{C}_{fc}, \mathcal{B} \cap \mathcal{F}_{fc})$. The class $\mathcal{A}$ is determined by the class $\mathcal{A}_{fc}$ by F.2.2. Let us show that $\mathcal{A}_{fc}$ is determined by the class $\mathcal{A}_{fc} \cap \mathcal{C} = \mathcal{A} \cap \mathcal{C}_{fc} = \mathcal{A}_0$. If $u : A \to B$ is a map in $\mathcal{E}_{fc}$, let us choose a factorisation $u = qi : A \to B' \to B$ with i a cofibration and q an acyclic fibration. We have $B' \in \mathcal{E}_{fc}$, since A is cofibrant and B is fibrant. Thus, $i \in \mathcal{C}_{fc}$. We have $u \in \mathcal{A}_{fc} \Leftrightarrow i \in \mathcal{A}_0$, since the class $\mathcal{A}$ is homotopy replete. This shows that the class $\mathcal{A}_{fc}$ is determined by the class $\mathcal{A}_0$. Hence the class $\mathcal{A}$ is determined by the class $\mathcal{A}_0$. Dually, the class $\mathcal{B}$ is determined by the class $\mathcal{B}_0$. Hence the pair $(\mathcal{A}, \mathcal{B})$ is determined by the pair $(\mathcal{A}_0, \mathcal{B}_0)$. But each class of the pair $(\mathcal{A}_0, \mathcal{B}_0)$ determines the other since the pair $(\mathcal{A}_0, \mathcal{B}_0)$ is a weak factorisation system in $\mathcal{E}_{fc}$. ■

Proposition F.2.5. *Let $(\mathcal{A}, \mathcal{B})$ be a weak homotopy factoorisation system on a model category $\mathcal{E}$. Then we have*

- (i) $\mathcal{W} = \mathcal{A} \cap \mathcal{B}$*;*
- (ii) *the classes $\mathcal{A}$ and $\mathcal{B}$ are closed under composition;*
- (iii) $(\mathcal{A} \cap \mathcal{C}_c) \pitchfork (\mathcal{B} \cap \mathcal{F}_f)$*;*

- (iv) *if* $A \in \mathcal{E}_c$ *and* $X \in \mathcal{E}_f$*, then every map* $f : A \to X$ *admits a factorisation* $f = pi$ *with* $i \in \mathcal{A} \cap \mathcal{C}_c$ *and* $p \in \mathcal{B} \cap \mathcal{F}_f$*.*

Proof: Let us put $(\mathcal{A}_0, \mathcal{B}_0) = (\mathcal{A} \cap \mathcal{C}_{fc}, \mathcal{B} \cap \mathcal{F}_{fc})$. Let us prove (i). Every isomorphism in $\mathcal{E}_{fc}$ belongs to $\mathcal{A}_0$, since $\mathcal{A}_0$ is the left class of a weak factorisation system on $\mathcal{E}_{fc}$. Hence we have $\mathcal{W} \subseteq \mathcal{A}$ by F.2.3. Dually, we have $\mathcal{W} \subseteq \mathcal{B}$. Thus, $\mathcal{W} \subseteq \mathcal{A} \cap \mathcal{B}$. Conversely, let us show that $\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{W}$. By lemma F.2.3, it suffices to show that $\mathcal{A}_{fc} \cap \mathcal{B}_{fc} \subseteq \mathcal{W}_{fc}$. Let $g : X \to Y$ be a map in $\mathcal{A}_{fc} \cap \mathcal{B}_{fc}$. Let us choose a factorisation $g = pu$, with $u : X \to Y'$ a cofibration and $p : Y' \to Y$ an acyclic fibration, together with a factorisation $g = qv$, with $v : X \to X'$ an acyclic cofibration and $q : X' \to Y$ a fibration. We have $u \in \mathcal{A} \cap \mathcal{C}_{fc}$ and $q \in \mathcal{B} \cap \mathcal{F}_{fc}$, since the classes $\mathcal{A}$ and $\mathcal{B}$ are homotopy replete. Hence the square

$$\begin{array}{ccc} X & \xrightarrow{v} & X' \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle q} \\ Y' & \xrightarrow{p} & Y. \end{array}$$

has a diagonal filler $w : Y' \to X'$. The relations $qd = p$ and $dv = u$ implies that d is invertible in the homotopy category $Ho(\mathcal{E}_{fc})$, since the maps p and u are invertible in this category. Thus, d is a weak equivalence by "six-for-two" E.1.5. Hence also v and u by three-for-two. This shows that $g = pv$ is a weak equivalence. Let us prove (ii). By F.2.3, it suffices to show that class $\mathcal{A}_{fc}$ is closed under composition. Let $u : A \to B$ and $v : B \to C$ be two maps in $\mathcal{A}_{fc}$. Let us choose a factorisation $u = pu' : A \to B' \to B$, with u' a cofibration and p an acyclic fibration, together with a factorisation of the composite $vp : B' \to C$ as a cofibration $v' : B' \to C'$ followed by an acyclic fibration $q : C' \to C$,

$$\begin{array}{ccccc} & & B' & \xrightarrow{v'} & C' \\ & {\scriptstyle u'}\nearrow & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ A & \xrightarrow{u} & B & \xrightarrow{v} & C. \end{array}$$

We have $B' \in \mathcal{E}_{fc}$, since $A \in \mathcal{E}_c$ and $B \in \mathcal{E}_f$. Thus, $u' \in \mathcal{C}_{fc}$. Similarly, $v' \in \mathcal{C}_{fc}$. Moreover, we have $u' \in \mathcal{A}$ and $v' \in \mathcal{A}$, since p and q are weak equivalences. Thus, $u' \in \mathcal{A}_0$ and $v' \in \mathcal{A}_0$. It follows that $v'u' \in \mathcal{A}_0$, since the left class of a weak factorisation system is closed under composition. Thus, $vu \in \mathcal{A}_{fc}$, since $q(v'u') = vu$ and q is a weak equivalence. Let us prove (iii). Let $u : A \to B$ be a map in $\mathcal{A} \cap \mathcal{C}_c$ and $f : X \to Y$ be a map in $\mathcal{B} \cap \mathcal{F}_f$. Let us show that every commutative square

$$\begin{array}{ccc} A & \xrightarrow{x} & X \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{y} & Y \end{array}$$

has a diagonal filler. Let us first consider the case where $A \in \mathcal{E}_{fc}$ and $Y \in \mathcal{E}_{fc}$. In this case let us choose a factorisation $x = pi : A \to X' \to X$, with $i : A \to X'$ a cofibration and $p : X' \to X$ an acyclic fibration, together with a factorisation $y = qj$, with $j : B \to B'$ an acyclic cofibration and $q : B' \to Y$ a fibration,

$$\begin{array}{ccccc} A & \xrightarrow{i} & X' & \xrightarrow{p} & X \\ {\scriptstyle u}\downarrow & \searrow & & \searrow & \downarrow{\scriptstyle f} \\ B & \xrightarrow[j]{} & B' & \xrightarrow[q]{} & Y. \end{array}$$

The result will be proved in this case if we show that the middle square of this diagram has a diagonal filler. We have $X' \in \mathcal{E}_{fc}$ and $B' \in \mathcal{E}_{fc}$. Thus, $ju \in \mathcal{C}_{fc}$ and $fp \in \mathcal{F}_{fc}$. Moreover, $ju \in \mathcal{A}$ and $fp \in \mathcal{B}$, since the classes $\mathcal{A}$ and $\mathcal{B}$ are homotopy replete. Thus, $ju \in \mathcal{A} \cap \mathcal{C}_{fc}$ and $fp \in \mathcal{A} \cap \mathcal{F}_{fc}$. It follows that the middle square has a diagonal filler. In the general case, let us choose a factorisation $x = pi : A \to A' \to X$, with $i : A \to A'$ an acyclic cofibration and $p : A' \to X$ a fibration, together with with a factorisation $y = qj$, with $j : B \to Y'$ a cofibration and $q : Y' \to Y$ an acyclic fibration. By taking a pullback and a pushout, we can construct the following commutative diagram

$$\begin{array}{ccccccc} A & \xrightarrow{i} & A' & \longrightarrow & Y' \times_Y X & \xrightarrow{p_2} & X \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle i_2} & & {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{i_1} & B \sqcup_A A' & \longrightarrow & Y' & \xrightarrow{q} & Y. \end{array}$$

It suffices to show that the middle square of the diagram has a diagonal filler. Observe that we have $A' \in \mathcal{E}_{fc}$ and $Y' \in \mathcal{E}_{fc}$. Hence it suffices to show that we have $i_2 \in \mathcal{A} \cap \mathcal{C}$ and $p_1 \in \mathcal{B} \cap \mathcal{F}$ by the first part of the proof. But $i_2 \in \mathcal{C}$ by cobase change since $u \in \mathcal{C}$. We have also $i_1 \in \mathcal{C} \cap \mathcal{W}$ by cobase change since $i \in \mathcal{C} \cap \mathcal{W}$. Hence the maps i and i_1 are acyclic. Thus, $i_2 \in \mathcal{A}$, since $u \in \mathcal{A}$ and the class $\mathcal{A}$ is homotopy replete. This proves that $i_2 \in \mathcal{A} \cap \mathcal{C}$. Dually, $p_1 \in \mathcal{B} \cap \mathcal{F}$. The result is proved. Let us prove (iv). Let us choose an acyclic cofibration $i : A \to A'$ with $A' \in \mathcal{E}_f$ together with an acyclic fibration $p : X' \to X$ with $X' \in \mathcal{E}_c$. Then there exists a map $g : A' \to X'$ fitting in the commutative square,

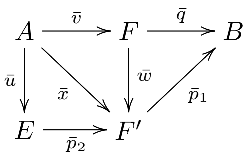
$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & & \uparrow{\scriptstyle p} \\ A' & \xrightarrow{g} & X'. \end{array}$$

We have $A' \in \mathcal{E}_{fc}$ and $X' \in \mathcal{E}_{fc}$ by construction. We can then choose a factorisation $g = qv : A' \to E \to X'$ with $v \in \mathcal{A}_0$ and $q \in \mathcal{B}_0$. This yields a factorisation $f = (pq)(vi)$, with $vi \in \mathcal{C}$ and $pq \in \mathcal{F}$. But we have $vi \in \mathcal{A} \cap \mathcal{C}$, since $v \in \mathcal{A}$ and the class $\mathcal{A}$ is homotopy replete. Similarly, $pq \in \mathcal{B} \cap \mathcal{F}$. ∎

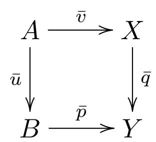
Let $\pi : \mathcal{E} \to Ho(\mathcal{E})$ be the canonical functor. If $(\mathcal{A}, \mathcal{B})$ be a weak homotopy factorisation system in $\mathcal{E}$ then there exists a unique replete class of maps $\mathcal{A}' \subseteq Ho(\mathcal{E})$ such that $\pi^{-1}\mathcal{A}' = \mathcal{A}$ by F.1.2, since the class $\mathcal{A}$ is homotopy replete. Similarly, there exists a unique replete class of maps $\mathcal{B}' \subseteq Ho(\mathcal{E})$ such that $\pi^{-1}\mathcal{B}' = \mathcal{B}$.

Proposition F.2.6. *Let $(\mathcal{A}, \mathcal{B})$ be a weak homotopy factorisation system in a model category $\mathcal{E}$. Then the pair $(\mathcal{A}', \mathcal{B}')$ is a weak factorisation system in the category $Ho(\mathcal{E})$.*

Proof: Let us put $(\mathcal{A}_0, \mathcal{B}_0) = (\mathcal{A} \cap \mathcal{C}_{fc}, \mathcal{B} \cap \mathcal{F}_{fc})$, $\mathcal{A}'' = Ho(\mathcal{E}_{fc}) \cap \mathcal{A}'$ and $\mathcal{B}'' = Ho(\mathcal{E}_{fc}) \cap \mathcal{B}'$. It suffices to show that the pair $(\mathcal{A}'', \mathcal{B}'')$ is a weak factorisation system in $Ho(\mathcal{E}_{fc})$, since the inclusion $Ho(\mathcal{E}_{fc}) \subseteq Ho(\mathcal{E})$ is an equivalence of categories. For this, it suffices to show that the conditions of D.1.9 are satisfied by the pair the $(\mathcal{A}'', \mathcal{B}'')$. We denote by $\bar{u}$ the homotopy class of a map $u : A \to B$ in the category $\mathcal{E}_{fc}$. Every map $A \to B$ in $Ho(\mathcal{E}_{fc})$ is of the form $\pi(u) = \bar{u}$ for a map $u : A \to B$ by E.1.8. Every map $f : X \to Y$ in $\mathcal{E}_{fc}$ admits a factorisation $f = pu : X \to E \to Y$ with $u \in \mathcal{A}_0$ and $p \in \mathcal{B}_0$ since the pair $(\mathcal{A}_0, \mathcal{B}_0)$ is a weak homotopy factorisation system. This shows that the map $\bar{f} : X \to Y$ admits the factorisation $\bar{f} = \bar{p}\bar{u}$, with $\bar{u} \in \mathcal{A}''$ and $\bar{p} \in \mathcal{B}''$. Let us prove that we have $\mathcal{A}'' \pitchfork \mathcal{B}''$. Observe first that every factorisation $\bar{f} = \bar{q}\bar{v} : A \to F \to B$ in $Ho(\mathcal{E}_{fc})$, is *isomorphic* to a factorisation $\bar{f} = \bar{p}\bar{u} : A \to E \to B$ for which we have $f = pu$, where p is a fibration and u is a cofibration. To see this, let us choose a factorisation $q = p_1 w : F \to F' \to B$, with w a weak equivalence and p_1 a fibration. Let us put $x = wv$.



The factorisation $\bar{f} = \bar{q}\bar{v}$ is isomorphic to the factorisation $\bar{f} = \bar{p}_1\bar{x}$, since $\bar{w}$ is invertible. The maps f and $p_1 x$ are homotopic, since $\bar{f} = \bar{p}_1\bar{x}$. It follows by the Covering Homotopy Theorem E.1.7, that there exists a map $y : A \to F'$ such that $\bar{y} = \bar{x}$ and $p_1 y = f$. Let us then choose a factorisation $y = p_2 u : A \to U \to V$, with u a cofibration and p_2 an acyclic fibration. Then we have $f = pu$, where $p = p_2 p_1$ is a fibration. Moreover, the factorisation $\bar{f} = \bar{p}_1\bar{x}$ is isomorphic to the factorisation $\bar{f} = \bar{p}\bar{u}$, since $\bar{p}_2$ is invertible. Hence the factorisation $\bar{f} = \bar{q}\bar{v}$ is isomorphic to the factorisation $\bar{f} = \bar{p}\bar{u}$. Let us now show that any commutative square



with $\bar{u} \in \mathcal{A}''$ and $\bar{q} \in \mathcal{B}''$ has a diagonal filler. We have $\bar{p}\bar{u} = \bar{q}\bar{v}$. By using the procedure above we can replace the square by an isomorphic square for which we have $pu = qv$ and for which $u \in \mathcal{C}$, $p \in \mathcal{F}$, $v \in \mathcal{C}$ and $q \in \mathcal{F}$. In this case we have $B \in \mathcal{E}_{fc}$ and $X \in \mathcal{E}_{fc}$. Moreover, $u \in \mathcal{A}$ and $q \in \mathcal{B}$, since $\bar{u} \in \mathcal{A}''$ and $\bar{q} \in \mathcal{B}''$. Thus, $u \in \mathcal{A}_0$ and $q \in \mathcal{B}_0$, since $u \in \mathcal{C}$ and $q \in \mathcal{F}$. It follows that the square

$$\begin{array}{ccc} A & \xrightarrow{\bar{v}} & X \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle q} \\ B & \xrightarrow{p} & Y \end{array}$$

has a diagonal filler $d : B \to X$, since $\mathcal{A}_0 \pitchfork \mathcal{B}_0$. The map $\bar{d} : B \to X$ is then a diagonal filler of the original square. This completes the proof of the relation $\mathcal{A}'' \pitchfork \mathcal{B}''$. Let us now show that the class $\mathcal{B}''$ is closed under domain retracts. Let $f : X \to Y$ and $g : E \to Y$ be two maps in $\mathcal{E}_{fc}$ with $\bar{f} \in \mathcal{B}''$. If g is a domain retract of f in the category $Ho(\mathcal{E}_{fc})$, then there are maps $s : E \to X$ and $r : X \to E$ such that $\bar{f}\bar{s} = \bar{g}$, $\bar{g}\bar{r} = \bar{f}$, $\bar{r}\bar{s} = \bar{1}_E$. If we factor g as an acyclic cofibration followed by a fibration, we can suppose that g is a fibration (this amount to replace the object (E, g) of the category $Ho(\mathcal{E}_{fc})/Y$ by an isomorphic object). If we factor r as an acyclic cofibration followed by a fibration, we can suppose that r is a fibration and that $f = gr$ (this amount to replace the object (X, f) of the category $Ho(\mathcal{E}_{fc})/Y$ by an isomorphic object). The map f is a fibration since $f = gr$. By the Covering Homotopy Theorem E.1.7, there exists a map $u : E \to X$ homotopic to s such that $ru = 1_E$, since rs is homotopic to 1_E. Then, $fu = gru = g1_E = g$. Hence the map $g : E \to X$ is a domain retract of the map $f : X \to Y$. If $\bar{f} \in \mathcal{B}''$, let us show that $\bar{g} \in \mathcal{B}''$. We have $f \in \mathcal{B}$, since $\bar{f} \in \mathcal{B}''$. Thus, $f \in \mathcal{B}_0$, since $f \in \mathcal{F}_{fc}$. It follows that $g \in \mathcal{B}_0$, since $\mathcal{B}_0$ is closed under domain retracts. Thus, $\bar{g} \in \mathcal{B}''$. We have proved that the class $\mathcal{B}''$ is closed under domain retracts. Dualy, the class $\mathcal{A}''$ is closed under codomain retracts. ∎

Corollary F.2.7. *Each class of a weak homotopy factorisation system is closed under retracts.*

Proof: Let $\pi : \mathcal{E} \to Ho(\mathcal{E})$ be the canonical functor. By Proposition F.2.6 we have $\mathcal{A} = \pi^{-1}\mathcal{A}'$ and $\mathcal{B} = \pi^{-1}\mathcal{B}'$, where $(\mathcal{A}', \mathcal{B}')$ is a weak factorisation system in the homotopy category $Ho(\mathcal{E})$. If a map $u \in \mathcal{E}$ is a retract of a map $v \in \mathcal{A}$, then the map $\pi(u) \in Ho(\mathcal{E})$ is a retract of the map $\pi(v) \in \mathcal{A}'$. Hence we have $\pi(u) \in \mathcal{A}'$, since the class $\mathcal{A}'$ is closed under retracts by Proposition D.1.3. It follows that we have $u \in \mathcal{A}$. ∎

Proposition F.2.8. *Let $\mathcal{E}$ be a model category. Then a weak factorisation system $(\mathcal{A}_0, \mathcal{B}_0)$ on the subcategory $\mathcal{E}_{fc}$ is the center of a weak homotopy factorisation system on $\mathcal{E}$ iff we have $\mathcal{A}_0 \subseteq \mathcal{C}$ and $\mathcal{B}_0 \subseteq \mathcal{F}$.*

Proof: The necessity is obvious. Conversely, let us suppose that we have $\mathcal{A}_0 \subseteq \mathcal{C}$ and $\mathcal{B}_0 \subseteq \mathcal{F}$. Let us show that there is a unique homotopy replete class of maps

$\mathcal{A} \subseteq \mathcal{E}$ such that $\mathcal{A} \cap \mathcal{C}_{fc} = \mathcal{A}_0$. Let $\mathcal{E}^I$ be the category of arrows of the category $\mathcal{E}$. An object of $\mathcal{E}^I$ is a map $u : A \to B$ in $\mathcal{E}$. Let us give the category $\mathcal{E}^I$ the (injective) model structure G.0.12. A map $u : A \to B$ in $\mathcal{E}$ is a cofibrant object of $\mathcal{E}^I$ iff A and B are cofibrant in $\mathcal{E}$. A map $f : X \to Y$ in $\mathcal{E}$ is a fibrant object of $\mathcal{E}^I$ iff f is a fibration between fibrant objects in $\mathcal{E}$. Thus, $f \in (\mathcal{E}^I)_{fc}$ iff $f \in \mathcal{F}_{fc}$. Let us show the subclass $\mathcal{B}_0 \subseteq \mathcal{F}_{fc}$ is homotopy replete in the category $(\mathcal{E}^I)_{fc}$. Consider a commutative square

$$\begin{array}{ccc} X' & \xrightarrow{x} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{y} & Y, \end{array}$$

in which the horizontal maps are weak equivalences and the vertical maps are in $\mathcal{F}_{fc}$. We shall first prove that $f \in \mathcal{B}_0 \Leftrightarrow f' \in \mathcal{B}_0$. Let us first suppose that $f \in \mathcal{B}_0$. Consider the commutative diagram

$$\begin{array}{ccccc} X' & \xrightarrow{(f',x)} & Y' \times_Y X & \xrightarrow{p_2} & X \\ & {\scriptstyle f'}\searrow & \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle f} \\ & & Y' & \xrightarrow[y]{} & Y. \end{array}$$

The projection p_1 is a fibration, since f is a fibration. The projection p_2 is a weak equivalence, since the base change along a fibration of a weak equivalence between fibrant objects is a weak equivalence by E.2.7. Hence the map (f', x) is acyclic by three-for-two, since the composite $p_2(f', x) = x$ is acyclic. Let us choose a factorisation $(f', x) = (g, u)i : X' \to Z \to Y' \times_Y X$, with i an acyclic cofibration and (g, u) an acyclic fibration. The map $u : Z \to X$ is acyclic, since $u = p_2(g, u)$. The map $g : Z \to Y'$ is a fibration, since $g = p_1(g, u)$. Thus, Z is fibrant, since Y' is fibrant. Moreover, Z is cofibrant, since X' is cofibrant and i is a cofibration. Thus, $g \in \mathcal{F}_{fc}$. Let us show that $g \in \mathcal{B}_0$. For this, it suffices to show that g has the right lifting property with respect to every map $v \in \mathcal{A}_0$. But for this, it suffices to show that we have $v \pitchfork (g, u)$ and $v \pitchfork p_1$, since $g = p_1(g, u)$. We have $v \pitchfork (g, u)$, since v is a cofibration and (g, u) is an acyclic fibration. We have $v \pitchfork f$, since $v \in \mathcal{A}_0$ and $f \in \mathcal{B}_0$. Hence we have $v \pitchfork p_1$, since the projection p_1 is a base change of f. This completes the proof that $g \in \mathcal{B}_0$. Let us now show that $f' \in \mathcal{B}_0$. The square

$$\begin{array}{ccc} X' & \xrightarrow{1'_X} & X' \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f'} \\ Z & \xrightarrow{g} & Y' \end{array}$$

has a diagonal filler $r : B' \to P$, since i is an acyclic cofibration and f' is a fibration. This shows that f' is a domain retract of g. Thus, $f' \in \mathcal{B}_0$ since $\mathcal{B}_0$ is closed under codomain retracts. Let us now prove the implication $f' \in \mathcal{B}_0 \Rightarrow f \in \mathcal{B}_0$. The pair (x, y) defines a weak equivalence $f' \to f$ between cofibrant-fibrant objects of the model category $\mathcal{E}^I$. It is thus a homotopy equivalence by E.1.9. Hence there exists a homotopy equivalence in the opposite direction $f \to f'$. Thus, $f \in \mathcal{B}_0$ by what we just proved. We have proved that the subclass $\mathcal{B}_0 \subseteq (\mathcal{E}^I)_{fc}$ is homotopy replete. It then follows from E.1.13 that there is a unique homotopy replete class (of objects) $\mathcal{B} \subseteq \mathcal{E}^I$ such that $\mathcal{B} \cap (\mathcal{E}^I)_{fc} = \mathcal{B}_0$. This defines a homotopy replete class of maps $\mathcal{B} \subseteq \mathcal{E}$ such that $\mathcal{B} \cap \mathcal{C}_{fc} = \mathcal{B}_0$. Dually, there is a unique homotopy replete class of maps $\mathcal{A} \subseteq \mathcal{E}$ such that $\mathcal{A} \cap \mathcal{F}_{fc} = \mathcal{A}_0$. The pair $(\mathcal{A}, \mathcal{B})$ is then a weak homotopy factorisation systeml, since the pair $(\mathcal{A} \cap \mathcal{C}_{fc}, \mathcal{B} \cap \mathcal{F}_{fc}) = (\mathcal{A}_0, \mathcal{B}_0)$ is a weak factorisation system in $\mathcal{E}_{fc}$. ■

Lemma F.2.9. *Let $(\mathcal{A}, \mathcal{B})$ be a weak homotopy factorisation system in a model category $\mathcal{E}$. Let us put $(\mathcal{A}_0, \mathcal{B}_0) = (\mathcal{A} \cap \mathcal{C}_{fc}, \mathcal{B} \cap \mathcal{F}_{fc})$. Then*

- (i) *a map $u \in \mathcal{C}_c$ belongs to $\mathcal{A}$ iff it is has the left lifting property with respect to the maps in $\mathcal{B}_0$;*
- (ii) *a map $f \in \mathcal{F}_f$ belongs to $\mathcal{B}$ iff it is has the right lifting property with respect to the maps in $\mathcal{A}_0$.*

Proof: Let us prove (i). It follows from F.2.5 that we have $\mathcal{A} \cap \mathcal{C}_c \subseteq {}^{\pitchfork}\mathcal{B}_0$. Conversely, if a map $u : A \to B$ in $\mathcal{C}_c$ belongs to ${}^{\pitchfork}\mathcal{B}_0$, let us show that $u \in \mathcal{A}$. Let us first prove the result in the case where $A \in \mathcal{E}_{fc}$. In this case let us choose an acyclic cofibration $i : B \to B'$ with $B' \in \mathcal{E}_f$. The object B' is cofibrant, since B is cofibrant. Thus, $B' \in \mathcal{E}_{fc}$. We have $i \pitchfork \mathcal{B}_0$, since $\mathcal{B}_0 \subseteq \mathcal{F}$ and $i \pitchfork \mathcal{F}$. Thus, $iu \pitchfork \mathcal{B}_0$, since $u \pitchfork \mathcal{B}_0$ by assumption. But $iu \in \mathcal{E}_{fc}$, since $A, B' \in \mathcal{E}_{fc}$. It follows that $iu \in \mathcal{A}_0$, since the pair $(\mathcal{A}_0, \mathcal{B}_0)$ is a factorisation system in $\mathcal{E}_{fc}$. Thus $u \in \mathcal{A}$, since $i \in \mathcal{W}$ and the class $\mathcal{A}$ is homotopy replete. This proves the result in the case where $A \in \mathcal{E}_{fc}$. Let us now consider the general where $A \in \mathcal{E}_c$. Let us choose an acyclic cofibration $i : A \to A'$ with $A' \in \mathcal{E}_f$. Consider the pushout square

$$\begin{array}{ccc} A & \xrightarrow{\ i\ } & A' \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle i_2} \\ B & \xrightarrow{\ i_1\ } & B \sqcup_A A'. \end{array}$$

The maps i_1 is an acyclic cofibration, since i is an acyclic cofibration. The maps i_1 is cofibration, since u is a cofibration. We have $i_2 \in {}^{\pitchfork}\mathcal{B}_0$, since the class ${}^{\pitchfork}\mathcal{B}_0$ is closed under cobase change and we have $u \in {}^{\pitchfork}\mathcal{B}_0$ by assumption. Thus, $i_2 \in \mathcal{A}$ by what we have proved above, since $i_2 \in \mathcal{C}_c$ and $A' \in \mathcal{E}_{fc}$. It follows that $u \in \mathcal{A}$, since the class $\mathcal{A}$ is homotopy replete and the maps i and i_1 are acyclic. This completes the proof of (i). Property (ii) follows by duality. ■

F.3 Homotopy factorisation systems

Recall that the *diagonal* of a map $f : X \to Y$ is the map

$$\delta_f = (1_X, 1_X) : X \to X \times_Y X.$$

Dually, the *codiagonal* of a map $u : A \to B$ is defined to be the map

$$\delta^u = (1_B, 1_B) : B \sqcup_A B \to B.$$

Theorem F.3.1. *. Let $(\mathcal{A}, \mathcal{B})$ be a weak homotopy factorisation system on a model category $\mathcal{E}$, with center $(\mathcal{A}_0, \mathcal{B}_0) = (\mathcal{A} \cap \mathcal{C}_{fc}, \mathcal{B} \cap \mathcal{F}_{fc})$. Then the following conditions are equivalent:*

- (i) *the class $\mathcal{A}$ has the right cancellation property;*
- (ii) *the class $\mathcal{B}$ has the left cancellation property;*
- (iii) *the codiagonal of a map in $\mathcal{A} \cap \mathcal{C}_c$ belongs to $\mathcal{A}$:*
- (iv) *the diagonal of a map in $\mathcal{B} \cap \mathcal{F}_f$ belongs to $\mathcal{B}$;*
- (v) $vu \in \mathcal{A}_0,\ u \in \mathcal{A}_0$ and $v \in \mathcal{C} \ \Rightarrow\ v \in \mathcal{A}_0$;
- (vi) $gf \in \mathcal{B}_0,\ g \in \mathcal{B}_0$ and $f \in \mathcal{F} \ \Rightarrow\ f \in \mathcal{B}_0$.

Proof: (i)⇒(iii). Let $u : A \to B$ be a map in $\mathcal{A} \cap \mathcal{C}_c$. Consider the pushout square

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u_2} \\ B & \xrightarrow{u_1} & B \sqcup_A B \end{array}$$

It follows from F.2.9 that the class $\mathcal{A} \cap \mathcal{C}_c$ is closed under cobase change along a map in $\mathcal{E}_c$. Thus, $u_1 \in \mathcal{A} \cap \mathcal{C}_c$ since $u \in \mathcal{A} \cap \mathcal{C}_c$. The relation $\delta^u u_1 = 1_B$ then implies that $\delta^u \in \mathcal{A}$ by (i), since we have $1_B \in \mathcal{A}$ by F.2.5. The implication (i)⇒(iii) is proved. Let us prove the implication (iii)⇒(vi). Let $f : X \to Y$ and $g : Y \to Z$ be two maps in $\mathcal{F}_{fc}$. If $gf \in \mathcal{B}_0$ and $g \in \mathcal{B}_0$, let us show that $f \in \mathcal{B}_0$. For this, it suffices to show that f has the right lifting property with respect to the maps in $\mathcal{A}_0$. If $u : A \to B$ belongs to $\mathcal{A}_0$, let us show that every commutative square

$$\begin{array}{ccc} A & \xrightarrow{x} & X \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{y} & Y \end{array}$$

has a diagonal filler. But the square

$$\begin{array}{ccc} A & \xrightarrow{x} & X \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle gf} \\ B & \xrightarrow{gy} & Z \end{array}$$

has a diagonal filler $d : B \to X$ since $gf \in \mathcal{B}_0$ by the assumption on gf. We have $fdu = fx = yu$. We thus obtain a map $(fd, y) : B \sqcup_A B \to Y$. Let us choose a factorisation $\delta^u = p(i_0, i_1) : B \sqcup_A B \to C \to B$, with (i_0, i_1) a cofibration and p an acyclic fibration. The map (i_0, i_1) belongs to $\mathcal{A}$, since $\delta^u \in \mathcal{A}$ by assumption and since the class $\mathcal{A}$ is homotopy replete. Thus, $(i_0, i_1) \in \mathcal{A} \cap \mathcal{C}_c$. Hence the square

$$\begin{array}{ccc} B \sqcup_A B & \xrightarrow{(fd,y)} & Y \\ {\scriptstyle (i_0,i_1)}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xrightarrow{gyp} & Z \end{array}$$

has a diagonal filler $h : C \to Y$ by F.2.5, since $(i_0, i_1) \in \mathcal{A} \cap \mathcal{C}_c$ and $g \in \mathcal{B}_0$. The map $i_0 = (i_0, i_1)u_1$ is a cofibration, since (i_0, i_1) and u_1 are cofibrations. Moreover, i_0 is acyclic by three-for-two, since $pi_0 = 1_B$ and p is acyclic. Hence the square

$$\begin{array}{ccc} B & \xrightarrow{d} & X \\ {\scriptstyle i_0}\downarrow & & \downarrow{\scriptstyle f} \\ C & \xrightarrow{h} & Y \end{array}$$

has a diagonal filler $k : C \to Y$, since f is a fibration. The map $t = ki_1$ is then a diagonal filler of the original square, since $ft = fki_1 = hi_1 = y$ and $tu = ki_1u = k(i_0, i_1)u_2u = k(i_0, i_1)u_1u = ki_0u = du = x$. We have proved that $f \in \mathcal{B}_0$. The implication (iii)⇒(vi) is proved. Let us prove the implication (vi)⇒(ii). By F.2.3, it suffices to show that the class $\mathcal{B}_{fc}$ has the left concellation property. Let $f : X \to Y$ and $g : Y \to Z$ be two maps in $\mathcal{E}_{fc}$. If $gf \in \mathcal{B}$ and $g \in \mathcal{B}$, let us show that $f \in \mathcal{B}$. Let us choose a factorisation $g = ki : Y \to Y' \to Z$, with i an acyclic cofibration and k a fibration, together with a factorisation of the composite $if = rj : X \to X' \to Y'$, with j an acyclic cofibration and r a fibration,

$$\begin{array}{ccccc} Y & \xrightarrow{i} & Y' & \xrightarrow{k} & Z \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle r} & & \\ X & \xrightarrow{j} & X'. & & \end{array}$$

The maps k and r belongs to $\mathcal{F}_{fc}$. Moreover, k and kr belongs to $\mathcal{B}_0$, since $ki = f \in \mathcal{B}$, $krj = gf \in \mathcal{B}$ and the class $\mathcal{B}$ is homotopy replete. Thus, $r \in \mathcal{B}_0$ by the assumption on $\mathcal{B}_0$. It follows that $f \in \mathcal{B}$, since $\mathcal{B}$ is homotopy replete. The implications (i)⇒(iii)⇒(vi)⇒(ii) are proved. The implications (ii)⇒(iv)⇒(v)⇒(i) follow by duality. ■

Proposition F.3.2. *Let $\mathcal{E}$ be a model category, A weak factorisation system $(\mathcal{A}_0, \mathcal{B}_0)$ on the subcategory $\mathcal{E}_{fc}$ is the center of a homotopy factorisation system on $\mathcal{E}$ iff we have $\mathcal{A}_0 \subseteq \mathcal{C}$ and $\mathcal{B}_0 \subseteq \mathcal{F}$ and one of the following two conditions are satisfied:*

- $vu \in \mathcal{A}_0,\ u \in \mathcal{A}_0$ and $v \in \mathcal{C} \ \Rightarrow\ v \in \mathcal{A}_0$;
- $gf \in \mathcal{B}_0,\ g \in \mathcal{B}_0$ and $f \in \mathcal{F} \ \Rightarrow\ f \in \mathcal{B}_0$.

Proof: This follows from F.2.4 and Theorem F.3.1. ■

F.4 Homotopy cartesian squares

Recall that a commutative square of fibrant objects in a model category

$$\begin{array}{ccc} A & \longrightarrow & C \\ \downarrow & & \downarrow{\scriptstyle v} \\ B & \xrightarrow{u} & D \end{array}$$

is said to be *homotopy cartesian* if v admits a factorisation $v = v'w : C \to C' \to D$ with w a weak equivalence and v' a fibration, such that the induced map $A \to B \times_D C'$ is a weak equivalence. If the square is cartesian, then the map $A \to B \times_D C'$ is a weak equivalence for *any* factorisation $v = v'w : C \to C' \to D$ as above, and also the map $A \to B' \times_D C$ for *any* factorisation $u = u'w : B \to B' \to B$ with w a weak equivalence and v' a fibration. The class of homotopy cartesian square of fibrant objects is homotopy replete in the subcategory of commutative squares of fibrant objects. It can thus be extended as an homotopy replete class in the category of all commutative squares.

Let $I = [1]$ be the category generated by one arrow $0 \to 1$. An object X of $\mathcal{E}^I$ is a map $x : X_0 \to X_1$ in $\mathcal{E}$. A map $f : X \to Y$ in $\mathcal{E}^I$ is a commutative square in $\mathcal{E}$,

$$\begin{array}{ccc} X_0 & \xrightarrow{f_0} & Y_0 \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle y} \\ X_1 & \xrightarrow{f_1} & Y_1. \end{array}$$

From the square, we obtain map $\langle f \rangle : X_0 \to Y_0 \times_{Y_0} Y_1$. If $\mathcal{E}$ is model category, then the category $\mathcal{E}^I$ admits a model structure called the injective model structure by G.0.12. Recall that the cofibrations and the weak equivalences of this model structure are defined level-wise. A map $f : X \to Y$ is a fibration for the injective model structure G.0.12 on $\mathcal{E}^I$ iff the map $f_1 : X_1 \to Y_1$ and the map $\langle f \rangle$ are fibrations . We shall say that an arrow in $\mathcal{E}^I$ is *homotopy cartesian* if the corresponding square in $\mathcal{E}$ is homotopy cartesian.

Theorem F.4.1. *The model category $\mathcal{E}^I$ (with the injective model structure) admits a homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{B}$ is the class of homotopy cartesian squares. A map $u : A \to B$ in $\mathcal{E}^I$ belongs to $\mathcal{A}$ iff the map $u_1 : A_1 \to B_1$ is a weak equivalence.*

Corollary F.4.2. *Suppose that we have a commutative square in a model category*

$$\begin{array}{ccc} A_0 & \xrightarrow{u_0} & B_0 \\ \downarrow & & \downarrow \\ A_1 & \xrightarrow{u_1} & B_1 \end{array}$$

in which the map u_1 is a weak equivalence. Then the map u_0 is a weak equivalence iff the square is homotopy cartesian.

Proof: We shall use the homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ of Theorem F.4.1. The square is a map $u : A \to B$ in $\mathcal{E}^I$. We have $u \in \mathcal{A}$ by hypothesis. Thus, $u \in \mathcal{A}$ iff $u \in \mathcal{W}$ since we have $\mathcal{W} = \mathcal{A} \cap \mathcal{B}$ by F.2.5. ∎

We say that a commutative square C in a category $\mathcal{E}$ is a *retract* of another commutative square D, if C is a retract of D as an object of the the category $\mathcal{E}^{I \times I}$.

Corollary F.4.3. *A retract of a homotopy cartesian square is homotopy cartesian.*

Proof: This follows from Theorem F.4.1 and Corollary F.2.7. ∎

Proposition F.4.4. *Suppose that we have a commutative diagram in a model category*

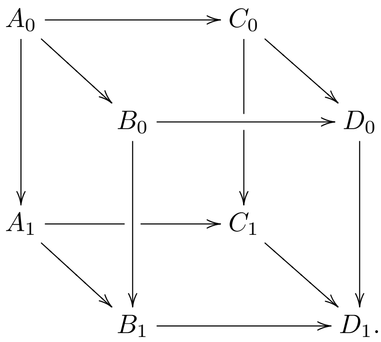
$$\begin{array}{ccccc} A_0 & \longrightarrow & B_0 & \longrightarrow & C_0 \\ \downarrow & & \downarrow & & \downarrow \\ A_1 & \longrightarrow & B_1 & \longrightarrow & C_1 \end{array}$$

in which the right hand square is a homotopy pullback. Then the left hand square is a homotopy pullback iff the composite square is a homotopy pullback.

Proof: Let us denote the model category by $\mathcal{E}$. We shall use the homotopy factorisation system $(\mathcal{A}, \mathcal{B})$ of theorem F.4.1. The diagram is a pair of arrows $u : A \to B$ and $v : B \to C$ in $\mathcal{E}^I$. We have $v \in \mathcal{B}$ by assumption. Thus, $vu \in \mathcal{B} \Leftrightarrow u \in \mathcal{B}$, since the class $\mathcal{B}$ is closed under composition and has the left concellation property.

∎

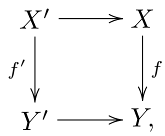
Corollary F.4.5. *Suppose that we have a commutative cube*



in which the left face, the right face and front face are homotopy cartesian. Then the back face is homotopy cartesian.

Lemma F.4.6. (Cube Lemma) *Suppose that we have a commutative cube as above in which the left face and the right face are homotopy cartesian. If the maps $A_1 \to C_1$, $B_1 \to D_1$ and $B_0 \to D_0$ are weak equivalences, then so is the map $A_0 \to C_0$.*

Definition F.4.7. *We shall say that a class of maps $\mathcal{M}$ in a model category $\mathcal{E}$ is closed under* homotopy base change *if for any homotopy cartesian square*



we have $f \in \mathcal{M} \Rightarrow f' \in \mathcal{M}$. There is a dual notion a class of maps closed under homotopy cobase change.

Theorem F.4.8. *The right class of a homotopy factorisation system is closed under homotopy base change. Dually, the left class is closed under homotopy cobase change.*

Appendix G

Reedy theory

We describe Reedy theory of direct categories.

We say that a category C is *artinian* if every infinite decreasing sequence of arrows

$$c_0 \leftarrow c_1 \leftarrow c_2 \leftarrow \cdots$$

is stationnary. There a dual notion of noetherian category. If C is a category and λ is an ordinal, then a functor $d : C \to \lambda$ is called a *linear extension* if it takes a non-identity to a non-identity. A category C is said to be *direct* if it admits a linear extension $d : C \to \lambda$ for some ordinal λ. There is a dual notion of *inverse* category. A category C is direct (resp. inverse) iff it is artinian (resp. noetherian).

Lemma G.0.9. *Let C be a category. For every object $a \in ObC$, the representable presheaf $a = C(-,a)$ contains a maximum proper subobject $\partial a \subset a$.*

.

Proof: The presheaf $F = C(-,a)$ is generated by the element $1_a \in F(a)$. Thus, a sub presheaf $U \subseteq F$ is proper iff $1_A \notin U(a)$. It follows from this observation that the union of all the proper sub presheaves of F is proper. This shows that F contains a maximum proper subobject. ∎

If C is a direct category, then the sub-object $\partial a \subset a$ is generated by the maps $b \to a$ with $b \neq a$.

Let C be a small category and $\mathcal{E}$ be a bicomplete category. The *box product* of a presheaf $A \in \hat{C}$ by an object $B \in \mathcal{E}$ is the functor $(A \Box B) : C^o \to \mathcal{E}$ defined by putting

$$(A \Box B)(c) = A(c) \times B$$

for every object $c \in C$, where $A(c) \times B$ is the coproduct of $A(c)$ copies of the object B. The functor $\Box : \hat{C} \times \mathcal{E} \to [C^o, \mathcal{E}]$ is divisible on both sides. If $X \in [C^o, \mathcal{E}]$ and $A \in \hat{C}$, then

$$A \backslash X = \int_{c \in C} X(c)^{A(c)}.$$

To a map $u : A \to B$ in $\hat{C}$ and a map $v : S \to T$ in $\mathcal{E}$, we can associate the map

$$u\Box' v : A\Box T \sqcup_{A\Box S} B\Box S \longrightarrow B\Box T$$

in $[C^o, \mathcal{E}]$. If $f : X \to Y$ is a map in $[C^o, \mathcal{E}]$ we have a map

$$\langle u\backslash f\rangle : B\backslash X \longrightarrow B\backslash Y \times_{A\backslash Y} A\backslash X$$

in $\mathcal{E}$. In particular, for every object $a \in ObC$ we have a map

$$\langle \delta_a\backslash f\rangle : X(a) \longrightarrow Y(a) \times_{\partial a\backslash Y} \partial a\backslash X,$$

where δ_a denotes the inclusion $\partial a \subset a$.

It follows from D.1.18 that we have

$$(u\Box' v) \pitchfork f \quad \Longleftrightarrow \quad v \pitchfork \langle u\backslash f\rangle.$$

Let $\mathcal{M}$ be a class of maps in a category $\mathcal{E}$. If C is a small category, we shall say that a map $f : X \to Y$ in the category $[C^o, \mathcal{E}]$ is *level-wise* in $\mathcal{M}$ if the map $f(a) : X(a) \to Y(a)$ belongs to $\mathcal{M}$ for every object $a \in C$.

Proposition G.0.10. *Let $\mathcal{E}$ be a bicomplete category equipped with a weak factorisation system $(\mathcal{A}, \mathcal{B})$ and let C be a direct category. Let $\mathcal{A}' \subseteq [C^o, \mathcal{E}]$ be the class of maps level-wise in $\mathcal{A}$ and let $\mathcal{B}' \subseteq [C^o, \mathcal{E}]$ be the class of maps $f : X \to Y$ for which we have $\langle \delta_a\backslash f\rangle \in \mathcal{B}$ for every object $a \in C$. Then the pair $(\mathcal{A}', \mathcal{B}')$ is a weak factorisation system.*

.

Proof: We shall use proposition D.1.9. The classes $\mathcal{A}'$ and $\mathcal{B}'$ are closed under retracts since the classes $\mathcal{A}$ and $\mathcal{B}$ are closed under retracts. Let us show that we have $\mathcal{A}' \pitchfork \mathcal{B}'$ Let S be a commutative square

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle f} \\ B & \longrightarrow & Y \end{array}$$

with $u \in \mathcal{A}'$ and $f \in \mathcal{B}'$. Let us show that it has a diagonal filler. Let $d : C \to \lambda$ be a linear extension. For every $\alpha \le \lambda$, let C_α the full subcategory of C spanned by the objects i with $d(i) < \alpha$. If we restrict the square S to the subcategory C_α, we obtain a square S_α:

$$\begin{array}{ccc} A_\alpha & \longrightarrow & X_\alpha \\ {\scriptstyle u_\alpha}\downarrow & & \downarrow{\scriptstyle f_\alpha} \\ B_\alpha & \longrightarrow & Y_\alpha. \end{array}$$

Let D_α be the set of diagonal fillers of S_α. There is a canonical projection $D_\beta \to D_\alpha$ for every $\alpha \leq \beta$. We shall prove that $D_\lambda \neq \emptyset$ by showing that the projection $D_{\alpha+1} \to D_\alpha$ is surjective for every $\alpha < \lambda$. Let $k \in D_\alpha$. If $d(a) = \alpha$, we have a commutative diagram,

$$\begin{array}{ccccc} A(a) & \longrightarrow & X(a) & \longrightarrow & \partial a\backslash X \\ {\scriptstyle u(a)}\downarrow & \nearrow & \downarrow & & \downarrow {\scriptstyle \partial a\backslash f} \\ B(a) & \longrightarrow & Y(a) & \longrightarrow & \partial a\backslash Y, \end{array}$$

where the diagonal is obtained by composing the map $B(a) \to \partial a\backslash B$ with the map $\partial a\backslash k : \partial a\backslash B \to \partial a\backslash X$ which is induced by $k : B_\alpha \to X_\alpha$ since $\partial a\backslash B = \partial a\backslash B_\alpha$ and $\partial a\backslash X = \partial a\backslash X_\alpha$. From the diagram, we obtain a commutative square

$$\begin{array}{ccc} A(a) & \longrightarrow & X(a) \\ {\scriptstyle u(a)}\downarrow & & \downarrow {\scriptstyle \langle \delta_a\backslash f\rangle} \\ B(a) & \longrightarrow & Y(a) \times_{\partial a\backslash Y} \partial a\backslash X. \end{array}$$

The square which has a diagonal filler $k(a)$ since the map $u(a) \in \mathcal{A}$ and $\langle \delta_a\backslash f\rangle \in \mathcal{B}$. The maps $k(a)$ for $d(a) = \alpha$ are defining an extension of k to $D_{\alpha+1}$. This proves the result. Let us now show that every map $f : X \to Y$ in $[C^o, \mathcal{E}]$ admits a factorisation $f = pu$ with $u \in \mathcal{A}'$ and $p \in \mathcal{B}'$. We can argue by induction on $\alpha < \lambda$. Let $\mathcal{A}'_\alpha$ and $\mathcal{B}'_\alpha$ be the corresponding classes of maps in the category $[C^o_\alpha, \mathcal{E}]$. Let us suppose that we have factorisation of the restricted map $f_\alpha = pu : X_\alpha \to P \to Y_\alpha$ with $u \in \mathcal{A}'_\alpha$ and $p \in \mathcal{B}'_\alpha$. If $d(a) = \alpha$ we have a commutative diagram

$$\begin{array}{ccccc} X(a) & & \longrightarrow & & Y(a) \\ \downarrow & & & & \downarrow \\ \partial a\backslash X & \longrightarrow & \partial a\backslash P & \longrightarrow & \partial a\backslash Y \end{array}$$

and hence a map $X(a) \to \partial a\backslash P \times_{\partial a\backslash Y} Y(a)$. By factoring this map as a map $X(a) \to P(a)$ in $\mathcal{A}$ followed by a map $P(a) \to \partial a\backslash P \times_{\partial a\backslash Y} Y(a)$ in $\mathcal{B}$, we can extend the given factorisation $f_\alpha = pu : X_\alpha \to P \to Y_\alpha$ to a factorisation of $f_{\alpha+1}$. The existence of the factorisation of the map $f : X \to Y$ is proved. The conditions of D.1.9 are thus satisfied. Hence the pair $(\mathcal{A}', \mathcal{B}')$ is a weak factorisation system. ∎

Proposition G.0.11. *Let $\mathcal{E}$ be a bicomplete category equipped with a weak factorisation system $(\mathcal{A}, \mathcal{B})$ and let C be a direct category. Let $f : X \to Y$ be a map in $[C^o, \mathcal{E}]$. If the map $\langle \delta_a\backslash f\rangle$ belongs to $\mathcal{B}$ for every object $a \in C$, then so is the map*

$$\langle u\backslash f\rangle : B\backslash X \longrightarrow B\backslash Y \times_{A\backslash Y} A\backslash X$$

for any monomorphism $u : A \to B$ in $\hat{C}$.

Proof: It suffices to show that we have $v \pitchfork \langle u\backslash f\rangle$ for any map $v : S \to T$ in $\mathcal{A}$. But the condition $v \pitchfork \langle u\backslash f\rangle$ is equivalent to the condition $(u\square' v) \pitchfork f$. Hence it suffices to show that $u\square' v \in \mathcal{A}'$. For this we have to show that the map

$$u(a) \times' v : B(a) \times S \sqcup_{A(a)\times S} A(a) \times T \to B(a) \times T$$

obtained from the square

$$\begin{array}{ccc} A(a) \times S & \longrightarrow & A(a) \times T \\ \downarrow & & \downarrow \\ B(a) \times S & \longrightarrow & B(a) \times T \end{array}$$

belongs to $\mathcal{A}$ for every object $a \in C$. We can suppose that u is an inclusion $A \subseteq B$. Let us denote by $\tilde{A}(a)$ the complement of $A(a)$ in the set $B(a)$. Then we have a decomposition $u(a) = i \cup A(a)$, where i is the inclusion $\emptyset \subset \tilde{A}(a)$. Thus,

$$u(a) \times' v = (i \times' v) \sqcup (A(a) \times' v) = (\tilde{A}(a) \times v) \sqcup (A(a) \times T).$$

Hence the map $u(a) \times' v$ is a base change of the map $\tilde{A}(a) \times v$. The map $\tilde{A}(a) \times v$ belongs to $\mathcal{A}$, since $\mathcal{A}$ is closed under a coproducts. This proves that $u\square' v \in \mathcal{A}'$ and hence that $\langle u\backslash f\rangle \in \mathcal{B}$. ∎

Proposition G.0.12. (Injective model structure) *Let $\mathcal{E}$ be a bicomplete model category and let C be a direct category. Then the category $[C^o, \mathcal{E}]$ admits a model structure in which the cofibrations and the weak equivalences are level-wise. A map $f : X \to Y$ is a fibration iff the map $\langle \delta_a\backslash f\rangle$ is a fibration for every object $a \in C$.*

The model structure of proposition G.0.12 the *injective* model structure on $[C^o, \mathcal{E}]$. In C is an inverse category, there is a dual *projective* model structure on $[C^o, \mathcal{E}]$.

Proposition G.0.13. *Let $\mathcal{E}$ be a bicomplete model category and let C be a direct category. Then the limit functor*

$$\varprojlim_C : [C^o, \mathcal{E}] \to \mathcal{E}.$$

is a right Quillen functor with respect to the injective model structure on $[C^o, \mathcal{E}]$.

Proof The limit functor is right adjoint to the constant diagram functor $c : \mathcal{E} \to [C^o, \mathcal{E}]$. It is obvious that the constant diagram functor is a left Quillen functor.

∎

Corollary G.0.14. *The cartesian product of a family of weak equivalences between fibrant objects is a weak equivalence.*

Appendix H

Open boxes and prisms

The cartesian product of two simplices $\Delta[m]$ and $\Delta[n]$ is a *prism* of dimension $m+n$,

$$\Delta[m,n] := \Delta[m] \times \Delta[n].$$

Its boundary can be calculated by using Leibnitz formula:

$$\partial\Delta[m,n] = \big(\partial\Delta[m] \times \Delta[n]\big) \cup \big(\Delta[m] \times \partial\Delta[n]\big).$$

If we remove the face $\partial_k\Delta[m] \times \Delta[n]$ from this boundary, we obtain the *open box*,

$$\Lambda^k[m,n] := \big(\Lambda^k[m] \times \Delta[n]\big) \cup \big(\Delta[m] \times \partial\Delta[n]\big).$$

And if we remove the face $\Delta[m] \times \partial_k\Delta[n]$ we obtain the open box,

$$\Lambda^{m+1+k}[m,n] := \big(\partial\Delta[m] \times \Delta[n]\big) \cup \big(\Delta[m] \times \Lambda^k[n]\big).$$

The main result of the appendix is the following theorem:

Theorem *If $0 \leq k \leq m$ (resp. $0 < k < m$, $0 \leq k < m$, $0 < k \leq m$), then the inclusion*

$$\Lambda^k[m,n] \subset \Delta[m,n]$$

is anodyne (resp. mid anodyne, left anodyne, right anodyne) for any $n \geq 0$.

The theorem is proved in H.0.20. We need a series of intermediate combinatorial results.

Definition H.0.15. *Let $\mathcal{P}(X)$ be the poset of simplicial subsets of a simplicial set X. We shall say that a partial order relation $\sqsubseteq$ on $\mathcal{P}(X)$ is* stable *if it satisfies the following two conditions:*

- $A \sqsubseteq B \quad \Rightarrow \quad A \subseteq B$;
- $A \cap B \sqsubseteq A \quad \Rightarrow \quad B \sqsubseteq A \cup B$.

The following two lemmas will be used repeatedly in the paper.

Lemma H.0.16. *Let $\mathcal{A} \subseteq \mathbf{S}$ be a saturated class of maps. If X is a simplicial set and $E \subseteq X$ is a simplicial subset, consider the relation $A \sqsubseteq B$ on $\mathcal{P}(X)$ which is defined by the following two conditions:*

- *$A \subseteq B$;*
- *the inclusion $A \cup E \subseteq B \cup E$ belongs to $\mathcal{A}$.*

The relation $A \sqsubseteq B$ is stable.

Proof: If $A, B \subseteq X$ and $A \cap B \sqsubseteq A$, then the inclusion $(A \cap B) \cup E \subseteq A \cup E$ belongs to $\mathcal{A}$. But the square

$$\begin{array}{ccc} (A \cap B) \cup E & \longrightarrow & B \cup E \\ \downarrow & & \downarrow \\ A \cup E & \longrightarrow & A \cup B \cup E \end{array}$$

is a pushout. It follows that the inclusion $B \cup E \subseteq A \cup B \cup E$ belongs to $\mathcal{A}$, since a saturated class is closed under cobase change. This proves that $B \sqsubseteq A \cup B$. ∎

Recall that a poset $\mathcal{L}$ is called an *inf-lattice* if every finite subset $S \subseteq \mathcal{L}$ has an infimum,

$$\inf(S) = \bigwedge_{a \in S} a.$$

The infimum of the empty set is a largest element of $\mathcal{L}$. A finite inf-lattice is also a sup-lattice by a classical elementary result, but we shall not use this fact.

Lemma H.0.17. *Let X be a simplicial set, $\mathcal{L}$ be a finite inf-lattice and let $\epsilon : \mathcal{L} \to \mathcal{P}(X)$ be a map preserving infima. For any $a \in \mathcal{L}$, let us put*

$$\dot{\epsilon}(a) = \bigcup_{b<a} \epsilon(b).$$

Suppose that we have $\dot{\epsilon}(a) \sqsubseteq \epsilon(a)$ for every $a \in \mathcal{L}$, where $\sqsubseteq$ is a stable partial order on $\mathcal{P}(X)$. Then we have $\emptyset \sqsubseteq X$.

Proof: Notice first that if μ denotes the largest element of $\mathcal{L}$, then we have $\epsilon(\mu) = X$ since ϵ preserves the infimum of the empty set. Let $\mathcal{L}'$ be the poset of lower sections of $\mathcal{L}$ (a subset $S \subseteq \mathcal{L}$ is a lower section if $a \leq b \in S \Rightarrow a \in S$). If $S \in \mathcal{L}'$, let us put

$$\bar{\epsilon}(S) = \bigcup_{s \in S} \epsilon(s).$$

This defines a map $\bar{\epsilon} : \mathcal{L}' \to \mathcal{P}(X)$ which preserves (finite) unions. Let us show that it preserves (finite) intersections. We have $\epsilon(\mu) = X$, where μ denotes the largest element of $\mathcal{L}$. It follows that we have $\bar{\epsilon}(\mathcal{L}) = X$. This shows that $\bar{\epsilon}$ preserves the infimum of the empty set in $\mathcal{L}'$ Let us verify that $\bar{\epsilon}$ preserves binary intersections. If $A, B \in \mathcal{L}'$, it is easy to see that we have

$$A \cap B = A \wedge B = \{a \wedge b : a \in A, b \in B\}.$$

Therefore,

$$\begin{aligned}
\bar{\epsilon}(A) \cap \bar{\epsilon}(B) &= \Big(\bigcup_{a\in A} \epsilon(a)\Big) \cap \Big(\bigcup_{b\in B} \epsilon(b)\Big) \\
&= \bigcup_{a\in A}\bigcup_{b\in B} \epsilon(a) \cap \epsilon(b) \\
&= \bigcup_{a\in A}\bigcup_{b\in B} \epsilon(a \wedge b) = \bar{\epsilon}(A \cap B)
\end{aligned}$$

This shows that $\bar{\epsilon}$ preserves finite intersections. Let Σ be the set of lower sections $A \in \mathcal{L}'$ such that we have $\bar{\epsilon}(S) \sqsubseteq \bar{\epsilon}(A)$ for every lower section $S \subseteq A$. We wish to prove that Σ is closed under finite unions. Clearly, $\emptyset \in \Sigma$. If $A, B \in \Sigma$, let us show that $A \cup B \in \Sigma$. For this, we have to show that for every lower section $S \subseteq A \cup B$ we have $\bar{\epsilon}(S) \sqsubseteq \bar{\epsilon}(A \cup B)$ But we have $\bar{\epsilon}(S \cap A) \sqsubseteq \bar{\epsilon}(A)$ since $A \in \Sigma$. Hence we have $\bar{\epsilon}(S) \cap \bar{\epsilon}(A) \sqsubseteq \bar{\epsilon}(A)$, since $\bar{\epsilon}$ preserves intersection. It follows that we have $\bar{\epsilon}(S) \sqsubseteq \bar{\epsilon}(S) \cup \bar{\epsilon}(A)$, since the relation $\sqsubseteq$ is stable by hypothesis. This shows that we have $\bar{\epsilon}(S) \sqsubseteq \bar{\epsilon}(S \cup A)$, since $\bar{\epsilon}$ preserves union. Similarly, if $T \subseteq A \cup B$ is a lower section, then $\bar{\epsilon}(T) \sqsubseteq \bar{\epsilon}(T \cup B)$. In particular, if $S \subseteq A \cup B$ is a lower section and we take $T = S \cup A$, we obtain that $\bar{\epsilon}(S \cup A) \sqsubseteq \bar{\epsilon}(A \cup B)$, since $S \cup A \cup B = A \cup B$. But we saw that $\bar{\epsilon}(S) \sqsubseteq \bar{\epsilon}(S \cup A)$. It follows by transitivity that $\bar{\epsilon}(S) \sqsubseteq \bar{\epsilon}(A \cup B)$. We have proved that $A \cup B \in \Sigma$. For the rest of the proof we shall use the maps $i, j : \mathcal{L} \to \mathcal{L}'$ defined by putting

$$i(a) = \{x \in \mathcal{L} : x \leq a\} \quad \text{and} \qquad j(a) = \{x \in \mathcal{L} : x < a\}.$$

The section $j(a)$ is the largest section properly included in $i(a)$. Observe that $\bar{\epsilon}(i(a)) = \epsilon(a)$ and that $\bar{\epsilon}(j(a)) = \dot{\epsilon}(a)$. We shall prove by induction on $a \in \mathcal{L}$ that $i(a) \in \Sigma$. The induction hypothesis is that we have $i(b) \in \Sigma$ for every $b < a$. Let us then prove that $i(a) \in \Sigma$. For this, we have to show that for every lower section $S \subseteq i(a)$ we have $\bar{\epsilon}(S) \sqsubseteq \bar{\epsilon}(i(a))$. This is clear if $S = i(a)$. Otherwise we have $S \subseteq j(a)$, since $j(a)$ is the largest section properly included in $i(a)$. But we have $i(b) \in \Sigma$ for every $b < a$, by the induction hypothesis on a. Hence we have $j(a) \in \Sigma$ since

$$j(a) = \bigcup_{b<a} i(b)$$

and since Σ is closed under finite unions. This shows that $\bar{\epsilon}(S) \sqsubseteq \bar{\epsilon}(j(a))$. But we have $\bar{\epsilon}(j(a)) \sqsubseteq \bar{\epsilon}(i(a))$ by the hypothesis of the lemma since $\bar{\epsilon}(j(a)) = \dot{\epsilon}(a)$ and

$\bar{\epsilon}(i(a)) = \epsilon(a)$. Hence we have $\bar{\epsilon}(S) \sqsubseteq \bar{\epsilon}(i(a))$ by transitivity, We have proved that $i(a) \in \Sigma$. In particular, $X = i(\mu) \in \Sigma$, where μ is the largest element of $\mathcal{L}$. Thus, $\emptyset = \bar{\epsilon}(\emptyset) \sqsubseteq \bar{\epsilon}(X) = X$. ■

Recall that a *simplicial complex* is a pair (S, Σ), where S is a set and Σ is a set of non-empty finite subsets of S satisfying the following two conditions: (i) for every $s \in S$ we have $\{s\} \in \Sigma$; (ii) every non-empty subset of an element of Σ belongs to Σ. An element $\sigma \in \Sigma$ is called a *simplex* of the simplicial complex; the *dimension* of σ is its cardinality minus one. A subset $\Sigma' \subseteq \Sigma$ is called a *subcomplex* of (S, Σ) if every non-empty subset of an element of Σ' belongs to Σ'; the pair (S', Σ') is then a simplicial complex, where $S' = \{s \in S : \{s\} \in \Sigma'\}$. The set of subcomplexes of (S, Σ) is closed under union and intersection. We shall denote by $\emptyset$ the empty subcomplex. If (S, Σ) is a simplicial complex and $A \in \Sigma$, we shall denote by $\Delta[A]$ the subcomplex of non-empty subsets of A. If $\dim(A) > 0$, we shall denote by $\partial\Delta[A]$ the union of the subcomplexes $\Delta[A\backslash\{a\}]$ for $a \in A$; if $\dim(A) = 0$ we shall put $\partial\Delta[A] = \emptyset$. If $B \subseteq A$ and $\dim(A) > 0$, we shall denote by $\Lambda^B[A]$ the union of the subcomplexes $\Delta[A\backslash\{a\}]$ for $a \in A\backslash B$. If $B \subseteq C \subseteq A$, then $\Lambda^C[A] \subseteq \Lambda^B[A]$.

We shall say that simplicial set X is *regular* if the vertices of every non-degenerate simplex of X are distinct. For example, the nerve of a poset is regular. If X is regular and $x : \Delta[n] \to X$, let us denote by $|x|$ the image of the map $x_0 : [n] \to X_0$. If $\Sigma_X = \{|x| : x \in X\}$, then the pair $C(X) = (X_0, \Sigma_X)$ is a simplicial complex. If $S \subseteq X$ is a simplicial subset, then $C(S) = (S_0, \Sigma_S)$ is a subcomplex of $C(X)$. The map $S \mapsto C(S)$ induces a bijection between the simplicial subset of X and the subcomplexes of $C(X)$. If P is a poset, then a simplex $A \in \Sigma_P$ is a finite non-empty chain of P. The pair $C(P) = (P, \Sigma_P)$ is called the *complex of chains* of P.

The simplicial set $\Delta[m, n] = \Delta[m] \times \Delta[n]$ is the nerve of the poset $[m] \times [n]$ equipped with the product ordering ($(x_1, x_2) \leq (y_1, y_2)$ iff $x_1 \leq y_1$ and $x_2 \leq y_2$). We shall often identify the simplicial set $\Delta[m, n]$ with the complex $C([m] \times [n])$. We have

$$\Delta[m, n] = \bigcup_{A \in \mathrm{Max}(m,n)} \Delta[A],$$

where $\mathrm{Max}(m, n)$ is the set of maximal chains of the poset $[m] \times [n]$, since every chain is contained in a maximal chain. We shall denote by $\preceq$ the partial order on $[m] \times [n]$ defined by putting $(i, j) \preceq (r, s)$ iff $i \geq r$ and $j \leq s$. We call the relation $\preceq$ the *transverse partial order*. The minimum element for this partial order is the south-east corner $(m, 0)$ of the rectangle $[m] \times [n]$, and its maximum element is the north-west corner $(0, n)$. If $A \in \mathrm{Max}(m, n)$, we shall say that an element $(i, j) \in A$

is an *upper corner* of A if the chain

$$\begin{array}{ccc} (i,j) & \longrightarrow & (i+1,j) \\ \uparrow & & \\ (i,j-1) & & \end{array}$$

is included in A. The upper corners of A are exactly the maximal elements of the set $A \setminus \{(0,0),(m,n)\}$ for the transverse partial order. We shall denote by $uc(A)$ the set of upper corners of A. Similarly, we shall say that an element $(i,j) \in A$ is a *lower corner* of A if the chain

$$\begin{array}{ccc} & & (i,j+1) \\ & & \uparrow \\ (i-1,j) & \longrightarrow & (i,j) \end{array}$$

is included in A. The lower corners of A are the minimal elements of the set $A \setminus \{(0,0),(m,n)\}$ for the transverse partial order. We shall denote by $lc(A)$ the set of lower corners of A. We shall say that a subset $S \subseteq [m] \times [n]$ is a *transverse section* if it is a lower section for the partial ordering $\preceq$ and if it contains the elements $(0,0)$ and (m,n). Notice that the second condition is equivalent to the requirement that a transverse section contains the border chain

$$(0,0) < (1,0) < (2,0) < \cdots < (m,0) < (m,1) < \cdots < (m,n)$$

Every subset $A \subseteq [n] \times [m]$ is contained in a smallest transverse section of $[m] \times [n]$, called its *shadow* and denoted $sh(A)$. Let us denote by $\mathrm{Tr}(m,n)$ the set of transverse sections of $[m] \times [n]$. The map $A \mapsto sh(A)$ induces a bijection between $\mathrm{Max}(m,n)$ and $\mathrm{Tr}(m,n)$. If $A \in \mathrm{Max}(m,n)$, then $sh(A) = sh(uc(A))$. If $S \in \mathrm{Tr}(m,n)$, let us denote by $C(S)$ the subcomplex of $C([m] \times [n]) = \Delta[m,n]$ whose simplicies are the chains of $[m] \times [n]$ included in S. Observe that every maximal simplex of $C(S)$ is also maximal in $C([m] \times [n])$. Thus,

$$C(S) = \bigcup_{A \in \mathrm{Max}(m,n), A \subseteq S} \Delta[A].$$

Let us put

$$\dot{C}(S) = \bigcup_{T \in \mathrm{Tr}(m,n), T \subset S} C(T).$$

If $S = Sh(B)$ and $T \in \mathrm{Tr}(m,n)$, then we have $T \subset S$ iff we have $T \subseteq sh(B \backslash \{b\})$ for some $b \in uc(B)$. It follows that

$$\dot{C}(S) = \bigcup_{b \in uc(B)} C(sh(B \setminus \{b\})).$$

Lemma H.0.18. *If* $B \in \mathrm{Max}(m,n)$ *and* $S = sh(B)$*, then we have*

$$C(S) = \dot{C}(S) \cup \Delta[B] \quad \text{and} \quad \dot{C}(S) \cap \Delta[B] = \Lambda^{B \setminus uc(B)}[B].$$

Proof: If $A \in \mathrm{Max}(m,n)$, then we have $sh(A) \subset Sh(B)$ iff we have $A \subseteq Sh(B)$ and $A \neq B$. It follows that we have

$$\dot{C}(S) = \bigcup_{A \in \mathrm{Max}(m,n), A \subseteq S, A \neq B} \Delta[A].$$

Thus, $C(S) = \dot{C}(S) \cup \Delta[B]$. Let us prove the second formula. A non-empty subset $E \subseteq B$ is a simplex of $\dot{C}(S)$ iff we have $E \subseteq B \setminus \{b\}$ for some $b \in up(B)$. This proves the second formula. ∎

Lemma H.0.19. *If* $m, n > 0$*,* $B \in \mathrm{Max}(m,n)$ *and* $S = sh(B)$*, then*

$$\Delta[B] \cap (\dot{C}(S) \cup \partial\Delta[m,n]) = \Lambda^{lc(B)}[B].$$

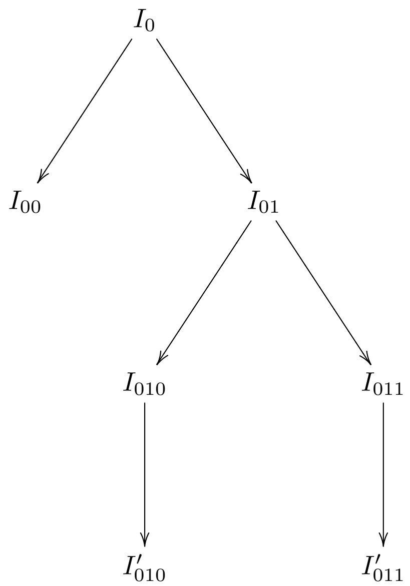
Moreover, if $0 < k \leq m$ *and* t *is the lowest element of* B *on the column* $\{k\} \times [n]$*, then*

$$\Delta[B] \cap (\dot{C}(S) \cup \Lambda^{k}[m,n]) = \Lambda^{lc(B) \cup \{t\}}[B].$$

Proof: Let us prove the first formula. For this, we have to establish the following two inclusions:

$$\begin{aligned} I_0 \quad &: \quad \Delta[B] \cap (\dot{C}(S) \cup \partial\Delta[m,n]) \subseteq \Lambda^{lc(B)}[B], \\ I_1 \quad &: \quad \Lambda^{lc(B)}[B] \subseteq \Delta[B] \cap (\dot{C}(S) \cup \partial\Delta[m,n]). \end{aligned}$$

Let us start with I_0. We shall consider various cases organised in a tree:



Obviously, we have

$$\Delta[B] \cap (\dot{C}(S) \cup \partial\Delta[m,n]) = (\Delta[B] \cap \dot{C}(S)) \cup (\Delta[B] \cap \partial\Delta[m,n]).$$

Hence the inclusion I_0 is equivalent to the conjunction of the following two inclusions:

$$\begin{array}{lcl} I_{00} & : & \Delta[B] \cap \dot{C}(S) \subseteq \Lambda^{lc(B)}[B], \\ I_{01} & : & \Delta[B] \cap \partial\Delta[m,n] \subseteq \Lambda^{lc(B)}[B]. \end{array}$$

Let us prove the inclusion I_{00}. By H.0.18 we have

$$\Delta[B] \cap \dot{C}(S) = \Lambda^{B\backslash uc(B)}[B].$$

But we have also $\Lambda^{B\backslash uc(B)}[B] \subseteq \Lambda^{lc(B)}[B]$ since we have $lc(B) \subseteq B\backslash uc(B)$. The inclusion I_{00} is proved. Let us prove the inclusion I_{01}. We have

$$\partial\Delta[m,n] = (\partial\Delta[m] \times \Delta[n]) \cup (\Delta[m] \times \partial\Delta[n]).$$

Hence the inclusion I_{01} is equivalent to the conjunction of the following two inclusions:

$$\begin{array}{lcl} I_{010} & : & \Delta[B] \cap (\partial\Delta[m] \times \Delta[n]) \subseteq \Lambda^{lc(B)}[B], \\ I_{011} & : & \Delta[B] \cap (\Delta[m] \times \partial\Delta[n]) \subseteq \Lambda^{lc(B)}[B]. \end{array}$$

Let us first prove the inclusion I_{010}. For this, we need to show that we have

$$I'_{010} : \Delta[B] \cap (\partial_i \Delta[m] \times \Delta[n]) \subseteq \Lambda^{lc(B)}[B]$$

for every $i \in [m]$. We have

$$\Delta[B] \cap (\partial_i \Delta[m] \times \Delta[n]) \subseteq \Delta[B \setminus (\{i\} \times [n])]$$

since $([m] \setminus \{i\}) \times [n] = ([m] \times [n]) \setminus (\{i\} \times [n])$. The intersection $B \cap (\{i\} \times [n])$ is non-empty since B is a maximal chain. If $b \in B \cap (\{i\} \times [n])$, then

$$\Delta[B \setminus (\{i\} \times [n])] \subseteq \Delta[B \backslash \{b\}].$$

If b is the maximum element of B on the column $\{i\} \times [n]$, then we have $\Delta[B\backslash\{b\}] \subseteq \Lambda^{lc(B)}[B]$, since $b \notin lc(B)$. The inclusion I'_{010} is proved, hence also the inclusion I_{010}. Let us now prove the inclusion I_{011}. For this, we need to show that we have

$$I'_{011} : \Delta[B] \cap (\Delta[m] \times \partial_j \Delta[n]) \subseteq \Lambda^{lc(B)}[B].$$

for every $j \in [n]$. As above, we have

$$\Delta[B] \cap (\Delta[m] \times \partial \Delta[n]) \subseteq \Delta[B \setminus ([m] \times \{j\})] \subseteq \Delta[B\backslash\{b\}] \subseteq \Lambda^{lc(B)}[B],$$

where b is the minimum element of B on the line $[m] \times \{j\}$. The inclusion I'_{011} is proved. The proof of the inclusion I_0 is complete. Let us now prove the inclusion I_1. Obviously, $\Lambda^{lc(B)}[B] \subseteq \Delta[B]$. Hence it suffices to show that we have

$$\Lambda^{lc(B)}[B] \subseteq \dot{C}(S) \cup \partial \Delta[m, n].$$

For this, we need to show that for every $b \in B \backslash lc(B)$ we have

$$\Delta[B\backslash\{b\}] \subseteq \dot{C}(S) \cup \partial \Delta[m, n].$$

This is clear if $b \in uc(B)$ since we have $\Delta[B\backslash\{b\}] \subseteq C(sh(B \setminus \{b\})) \subseteq \dot{C}(S)$ in this case. It remains to consider the cas where $b \in B\backslash(lc(B) \cup uc(B))$, in which case we shall prove that $\Delta[B\backslash\{b\}] \subseteq \partial\Delta[m, n]$. If $b = (i, j)$, then we have either $B \cap ([m] \times \{j\}) = \{b\}$ or $B \cap (\{i\} \times [n]) = \{b\}$ since $b \notin uc(B) \cup lc(B)$. In the first case we have

$$\Delta[B\backslash\{b\}] \subseteq \Delta[m] \times \partial_j \Delta[n] \subseteq \Delta[m] \times \partial\Delta[n] \subseteq \partial\Lambda^k[m, n].$$

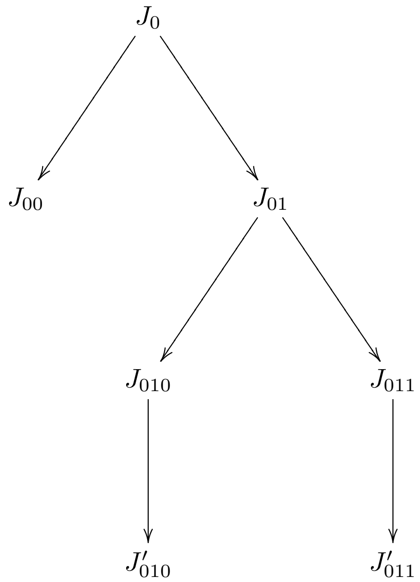
In the second case we have

$$\Delta[B\backslash\{b\}] \subseteq \partial_i \Delta[m] \times \Delta[n] \subseteq \Delta[m] \times \partial\Delta[n] \subseteq \partial\Delta[m, n].$$

The inclusion I_1 is proved. Hence the first formula of the lemma is proved. Let us prove the second formula. For this, we have to prove the following two inclusions:

$$\begin{aligned} J_0 &: \Delta[B] \cap (\dot{C}(S) \cup \partial\Lambda^k[m, n]) \subseteq \Lambda^{lc(B)\cup\{t\}}[B], \\ J_1 &: \Lambda^{lc(B)\cup\{t\}}[B] \subseteq \Delta[B] \cap (\dot{C}(S) \cup \Lambda^k \Delta[m, n]). \end{aligned}$$

Let us start with J_0. We shall consider various cases organised in a tree:



Obviously,

$$\Delta[B] \cap (\dot{C}(S) \cup \Lambda^k \Delta[m,n]) = (\Delta[B] \cap \dot{C}(S)) \cup (\Delta[B] \cap \Lambda^k \Delta[m,n]).$$

Hence the inclusion J_0 is equivalent to the conjunction of the following two inclusions:

$$\begin{array}{lcl} J_{00} & : & \Delta[B] \cap \dot{C}(S) \subseteq \Lambda^{lc(B)\cup\{t\}}[B], \\ J_{01} & : & \Delta[B] \cap \Lambda^k \Delta[m,n] \subseteq \Lambda^{lc(B)\cup\{t\}}[B]. \end{array}$$

We start with the inclusion J_{00}. By H.0.18 we have $\Delta[B] \cap \dot{C}(S) = \Lambda^{B\backslash uc(B)}[B]$. Observe that $t \notin uc(B)$ since t is the lowest element of B on the column $\{k\} \times [n]$. Thus, $lc(B) \cup \{t\} \subseteq B\backslash uc(B)$ since $lc(B) \cap uc(B) = \emptyset$. It follows that

$$\Lambda^{B\backslash uc(B)}[B] \subseteq \Lambda^{lc(B)\cup\{t\}}[B].$$

The inclusion J_{00} is proved. Let us first consider the inclusion J_{01}. By definition, we have

$$\Lambda^k \Delta[m,n] = (\Lambda^k[m] \times \Delta[n]) \cup (\Delta[m] \times \partial\Delta[n]).$$

Hence the inclusion J_{01} is equivalent to the conjunction of the following two inclusions:

$$\begin{array}{lcl} J_{010} & : & \Delta[B] \cap (\Lambda^k[m] \times \Delta[n]) \subseteq \Lambda^{lc(B)\cup\{t\}}[B], \\ J_{011} & : & \Delta[B] \cap (\Delta[m] \times \partial\Delta[n]) \subseteq \Lambda^{lc(B)\cup\{t\}}[B]. \end{array}$$

We first consider the inclusion J_{010}. For this, we need to show that if $i \in [m]$ and $i \neq k$, then we have

$$J'_{010} : \Delta[B] \cap (\partial_i \Delta[m] \times \Delta[n]) \subseteq \Lambda^{lc(B)\cup\{t\}}[B].$$

If b be the maximum element of B on the column $\{i\} \times [n]$, then we have $b \notin lc(B)$. Moreover, $b \neq t$ since $i \neq t$. Thus,

$$\Delta[B] \cap (\partial_i \Delta[m] \times \Delta[n]) \subseteq \Delta[B \setminus (\{i\} \times [n])] \subseteq \Delta[B \setminus \{b\}] \subseteq \Lambda^{lc(B)\cup\{t\}}[B].$$

The inclusion J'_{010} is proved. Let us now prove the inclusion J_{011}. For this, we need to show that for every $j \in [n]$ we have

$$J'_{011} : \Delta[B] \cap (\Delta[m] \times \partial_j \Delta[n]) \subseteq \Lambda^{lc(B)\cup\{t\}}[B].$$

If b be the the minimum element of B on the line $[m] \times \{j\}$ then we have $b \notin lc(B)$. Let us show that $b \neq t$. Otherwise, the element $b = t$ is the minimum element of B on the line $[m] \times \{j\}$ and on the column $\{i\} \times [n]$. Thus, $t = (0, 0)$ since B is a maximal chain. But this contradict the hypothesis that $k > 0$. Thus, $b \notin lc(B) \cup \cup\{t\}$ and this shows that we have $\Delta[B\backslash\{b\}] \subseteq \Lambda^{lc(B)\cup\{t\}}[B]$. Thus,

$$\Delta[B] \cap (\Delta[m] \times \partial\Delta[n]) \subseteq \Delta[B \setminus ([m] \times \{j\})] \subseteq \Delta[B\backslash\{b\}] \subseteq \Lambda^{lc(B)\cup\{t\}}[B].$$

The inclusion J'_{011} is proved, hence also the inclusion J_{011}. The proof of the inclusion J_0 is complete. Let us now prove the inclusion J_1. Obviously,

$$\Lambda^{lc(B)\cup\{t\}}[B] \subseteq \Delta[B].$$

Hence it suffices to show that we have

$$J_{11} : \Lambda^{lc(B)\cup\{t\}}[B] \subseteq \dot{C}(S) \cup \Lambda^k[m, n].$$

For this, we need to show that for every $b \in B\backslash(lc(B) \cup \{t\})$ we have

$$J'_{11} : \Delta[B\backslash\{b\}] \subseteq \dot{C}(S) \cup \Lambda^k[m, n].$$

This is clear if $b \in uc(B)$, since in this case we have $C(sh(B \setminus \{b\})) \subseteq \dot{C}(S)$ and since we have $\Delta[B\backslash\{b\}] \subseteq C(sh(B \setminus \{b\}))$ for every $b \in B$. It remains to consider the cas where $b \notin lc(B) \cup uc(B) \cup \{t\}$, in which case we shall prove that $\Delta[B\backslash\{b\}] \subseteq \Lambda^k[m, n]$. But if $b = (i, j)$, we have either $B \cap ([m] \times \{j\}) = \{b\}$ or $B \cap (\{i\} \times [n]) = \{b\}$, since $b \notin uc(B) \cup lc(B)$. In the first case we have

$$\Delta[B\backslash\{b\}] \subseteq \Delta[m] \times \partial_j \Delta[n] \subseteq \Lambda^k[m, n].$$

It remains to consider the second case. Let us show that $i \neq k$. Otherwise

$$t \in B \cap (\{k\} \times [n]) = B \cap (\{i\} \times [n]) = \{b\}$$

and this contradicts the hypothesis that $b \neq t$. Thus, $i \neq k$. It follows that

$$\Delta[B\backslash\{b\}] \subseteq \partial_i \Delta[m] \times \Delta[n] \subseteq \partial\Lambda^k[m, n].$$

The inclusion J'_{11} is proved, hence also the inclusions J_{11} and J_1. The two inclusions J_0 and J_1 are proved. Hence the second formula is proved. ∎

If P is a poset, we shall say that an inclusion $U \subseteq V$ between two subcomplexes of $C(P)$ is *anodyne* (resp. *left anodyne, mid anodyne, right anodyne*) if this is true of the corresponding inclusion of simplicial subsets of NP.

Theorem H.0.20. *If $0 \leq k \leq m$ (resp. $0 < k < m$, $0 \leq k < m$, $0 < k \leq m$), then the inclusion*

$$\Lambda^k[m,n] \subset \Delta[m,n]$$

is anodyne (resp. mid anodyne, left anodyne, right anodyne).

Proof: We can suppose that $n > 0$ since the result is trivial otherwise. If $0 < k < m$, let us prove that the inclusion $\Lambda^k[m,n] \subset \Delta[m,n]$ is mid anodyne. We shall use lemma H.0.17 with $X = \Delta[m,n]$ and with $\mathcal{L}$ the poset $\mathrm{Tr}(m,n)$ of transversal sections of $[m] \times [n]$. The map

$$\epsilon : \mathcal{L} \to \mathcal{P}(X)$$

is defined by putting $\epsilon(S) = C(S)$ for each $S \in \mathrm{Tr}(m,n)$. Obviously, $C(S \cap T) = C(S) \cap C(T)$ and $C([m] \times [n]) = X$. This means that ϵ preserves finite intersection. The map $\dot{\epsilon} : \mathcal{L} \to \mathcal{P}(X)$ is given by

$$\begin{aligned} \dot{\epsilon}(S) &= \bigcup_{T \in \mathcal{L}, T \subset S} C(T) \\ &= \dot{C}(S) \end{aligned}$$

We shall define a stable relation $U \sqsubseteq V$ between the subsets of X by using lemma H.0.16 with $E = \Lambda^k[m,n]$ and with $\mathcal{A}$ the class of mid anodyne maps. Let us see that we have $\dot{C}(S) \sqsubseteq C(S)$ for every $S \in \mathrm{Tr}(m,n)$. For this we have to show that the inclusion

$$\dot{C}(S) \cup \Lambda^k[m,n] \sqsubseteq C(S) \cup \Lambda^k[m,n]$$

is mid anodyne. But we have $S = Sh(B)$ for some maximal chain $B \in \mathrm{Max}(m,n)$. By lemma H.0.18 we have $C(S) = \Delta[B] \cup \dot{C}(S)$. Hence the square

$$\begin{array}{ccc} \Delta[B] \cap (\dot{C}(S) \cup \Lambda^k[m,n]) & \longrightarrow & \dot{C}(S) \cup \Lambda^k[m,n] \\ \downarrow & & \downarrow \\ \Delta[B] & \longrightarrow & C(S) \cup \Lambda^k[m,n] \end{array}$$

is a pushout. It thus suffices to show that the inclusion

$$\Delta[B] \cap (\dot{C}(S) \cup \Lambda^k[m,n]) \subseteq \Delta[B]$$

is mid anodyne. But we have

$$\Delta[B] \cap (\dot{C}(S) \cup \Lambda^k[m,n]) = \Lambda^{lc(B) \cup \{t\}}[B]$$

by lemma H.0.19. Hence it suffices to show that the inclusion

$$\Lambda^{lc(B)\cup\{t\}}[B] \subseteq \Delta[B]$$

is mid anodyne. For this, it suffices to show by 2.12 that $(0,0) \notin lc(B) \cup \{t\})$ and that $(m,n) \notin lc(B) \cup \{t\})$. We have $(0,0) \notin lc(B)$ and $(m,n) \notin lc(B)$ since the verticies $(0,0)$ and (m,n) are never a lower corner. Moreover, we have $(0,0) \neq t$ and $(m,n) \neq t$ since $0 < k < m$. The relation $\dot{C}(S) \sqsubseteq C(S)$ is proved. It then follows by lemma H.0.17 that we have $\emptyset \sqsubseteq X$ This proves that the inclusion $\Lambda^k[m,n] \subset \Delta[m,n]$ is mid anodyne. Let us now show that if $0 < k \leq m$, then the inclusion $\Lambda^k[m,n] \subset \Delta[m,n]$ is right anodyne. The result is true if $0 < k < m$ by what we just proved, since a mid anodyne map is right anodyne. Hence it suffices to consider the case $k = m$. We can suppose $n > 0$ since the result is trivial otherwise. As above, we shall use lemma H.0.17 but with the relation $U \sqsubseteq V$ defined by using the class $\mathcal{A}$ of right anodyne maps. As above, the problem is reduced to proving that the inclusion $\Lambda^{lc(B)\cup\{t\}}[B] \subseteq \Delta[B]$ is right anodyne for any maximal chain $B \in \mathrm{Max}(m,n)$. For this, it suffices to show by 2.12 that $(0,0) \notin lc(B) \cup \{t\}$. We have $(0,0) \notin lc(B)$ since $(0,0)$ is never a lower corner. Moreover, $(0,0) \neq t$ since $k > 0$. This proves that the inclusion $\Lambda^k[m,n] \subset \Delta[m,n]$ is right anodyne. Dually, if $0 \leq k < m$, then the inclusion $\Lambda^k[m,n] \subset \Delta[m,n]$ is left anodyne. Taken together, the results imply that the inclusion $\Lambda^k[m,n] \subset \Delta[m,n]$ is anodyne for every $0 \leq k \leq m$. Of course, this last result is classical. ■

Let us denote by P' the sub-poset of the poset $P = [m] \times [n]$ obtained by removing the element $(0,n)$, and let $C(P')$ be the complex of chains of P'. We shall identify $C(P')$ with a simplicial subset of $C(P) = \Delta[m,n]$.

Proposition H.0.21. *If $m, n > 0$, then the inclusion*

$$\Lambda^m[m,n] \subset C(P') \cup \Lambda^m[m,n]$$

is mid anodyne.

Proof: The proof is similar to the proof of lemma H.0.20. Notice that P' is a transversal section of P. We shall use lemma H.0.17 with $X = C(P')$ and with $\mathcal{L}$ the poset of transversal sections $S \subseteq P'$. The map $\epsilon : \mathcal{L} \to \mathcal{P}(X)$ is defined by putting $\epsilon(S) = C(S)$. The map ϵ preserves finite intersection. We have $\dot{\epsilon}(S) = \dot{C}(S)$. The relation $U \sqsubseteq V$ on $\mathcal{P}(X)$ is defined by using lemma H.0.16 with $E = \Lambda^m[m,n]$ and with $\mathcal{A}$ the class mid anodyne maps. Let us show that we have $\dot{C}(S) \sqsubseteq C(S)$ for every $S \in \mathcal{L}$. For this we need to show that the inclusion

$$\dot{C}(S) \cup \Lambda^m[m,n] \subseteq C(S) \cup \Lambda^m[m,n]$$

is mid anodyne for every $S \in \mathcal{L}$. But $S = Sh(B)$ for a unique maximal chain

$B \subseteq P'$ and we have $C(S) = \dot{C}(S) \cup \Delta[B]$ by H.0.18. Hence the square

$$\begin{array}{ccc} \Delta[B] \cap (\dot{C}(S) \cup \Lambda^k[m,n]) & \longrightarrow & \dot{C}(S) \cup \Lambda^k[m,n] \\ \downarrow & & \downarrow \\ \Delta[B] & \longrightarrow & C(S) \cup \Lambda^k[m,n] \end{array}$$

is a pushout. It thus suffices to show that the inclusion

$$\Delta[B] \cap (\dot{C}(S) \cup \Lambda^k[m,n]) \subseteq \Delta[B]$$

is mid anodyne. Il $lc(B)$ is the set of lower corners of B and t is the lowest element of B on the vertical line $\{m\} \times [n]$, then we have

$$\Delta[B] \cap (\dot{C}(S) \cup \Lambda^k[m,n]) = \Lambda^{lc(B)\cup\{t\}}[B]$$

by lemma H.0.19 since $m > 0$. Hence it suffices to show that the inclusion

$$\Lambda^{lc(B)\cup\{t\}}[B] \subseteq \Delta[B]$$

is mid anodyne. For this, we shall use lemma 2.12(iii) and proposition 3.18. We have $(0,0) \notin lc(B)$ since $(0,0)$ cannot be a lower corner. We have $(0,0) \neq t$ since $m > 0$. Thus, $(0,0) \notin lc(B) \cup \{t\}$. We now distinguish two cases: we have either $(m, n-1) \in B$ or $(m-1, n) \in B$ since B is a maximal chain. In the first case we have $t \neq (m,n)$ since t is the lowest element of B on the vertical line $\{m\} \times [n]$. Moreover, we have $(m,n) \notin lc(B)$ since (m,n) cannot be a lower corner. Thus, $(m,n) \notin lc(B) \cup \{t\}$. It then follows from lemma 2.12 that the inclusion $\Lambda^{lc(B)\cup\{t\}}[B] \subseteq \Delta[B]$ is mid anodyne. It remains to consider the case where $(m-1,n) \in B$. Notice that we have $t = (m,n)$ since $B \cap (\{m\} \times [n]) = \{(m,n)\}$ in this case. Let us put $B' = B \setminus \{(m,n)\}$. The element $(m-1,n)$ is maximal in B' and we have $(m-1,n) \notin lc(B)$ since $(m-1,n)$ cannot be a lower corner. The set $lc(B)$ is non-empty since we have $B \subseteq P'$. It then follows from lemma 2.12(iii) that the inclusion $\Lambda^{lc(B)}[B'] \subseteq \Delta[B']$ is mid anodyne. But we have $\Delta[B] = \Delta[B'] \star 1$ and $\Lambda^{lc(B)\cup\{t\}}[B] = \Lambda^{lc(B)}[B'] \star 1$ by 3.8 since $t = (m,n)$. This shows by 3.18 that the inclusion $\Lambda^{lc(B)\cup\{t\}}[B] \subseteq \Delta[B]$ is mid anodyne. Hence we have $\dot{C}(S) \sqsubseteq C(S)$ for every $S \in \mathcal{L}$. The hypothesis of Lemma H.0.17 are satisfied. Thus, $\emptyset \sqsubseteq X$. This means that the inclusion $\Lambda^m[m,n] \subset C(P') \cup \Lambda^m[m,n]$ is mid anodyne. ∎

Bibliography

[AR] [1] J. Adamek and J. Rosicky, *Locally Presentable and Accessible Categories*, London Math. Soc. Lecture Note Ser. 189, Cambridge University Press, Cambridge, 1994.

[B1] J. E. Bergner, *A model category structure on the category of simplicial categories*, to appear in Trans. Math. Soc., available at arXiv:math.AT/0406507.

[B2] J. E. Bergner, *Three models for the homotopy theory of homotopy theories*, preprint available at arXiv:math.AT/0504334.

[B3] J. E. Bergner, *A characterisation of fibrant Segal categories*, preprint available at arXiv:math.AT/0603400.

[B4] J. E. Bergner, *Rigidification of algebras over multisorted theories*, arXiv:math.AT/0508152

[B5] J. E. Bergner, *A survey of* $(\infty, 1)$*-categories*, preprint. July 14, 2006.

[BC] D. Bourn and J.M. Cordier, *A general formulation of homotopy limits*, Jour. Pure Appl. Alg. 29(1983), 129-141.

[BD] G. Biedermann and W. Dwyer, *Homotopy nilpotent groups*, arXiv:math.AT/0709.3925v2, Dec 2007

[Bi1] G. Biedermann, *L-stable functors*, arXiv:math.AT/0704.2576,

[Bi2] G. Biedermann, *On the homotopy theory of* n*-types*, arXiv:math.AT/0604514

[Bor] F. Borceux, *Handbook of Categorical Algebra, I, II, III*, Encyclopedia of Mathematics and its Applications. vol 50, 51, 52. Cambridge University Press (1994).

[Bous] A. K. Bousfield, *Constructions of factorisation systems in categories*, J. Pure and Applied Algebra 9 (1976/77), no.2, 207-220.

[BK] A. K. Bousfield and D. M. Kan, *Homotopy limits, completions and localisations*, Lectures Notes in Math. vol. 304, Springer Verlag, 1972.

[BV] M. Boardman and R. Vogt, *Homotopy invariant algebraic structures on topological spaces*, Lectures Notes in Math. vol. 347, Springer Verlag, 1973.

[CF] C. Casacuberta and A. Frei, *Saturated classes of morphisms*, Theory and Applications of Categories 7 (2000), 43-46.

[CSS] C. Casacuberta, D.Scevenels and J. Smith, *Implications of large-cardinal principles in homotopical localisation*, Advances in Mathematics, 197 (2005), 120-139.

[CC] C. Casacuberta and B. Chorny, *The orthogonal subcategory problem in homotopy theory*, Contemporary Mathematics 399, American Mathematical Society, Providence (2006), 41-53.

[Ci1] D.C. Cisinski, *Théories homotopiques dans les topos*, Journal of Pure and Applied Algebra 174 (2002) 43-82.

[Ci2] D.C. Cisinski,*Les prfaisceaux comme modles des types d'homotopie*, Astrisque 308, Soc. Math. France (2006), xxiv+392 pages.

[Ci3] D.C. Cisinski,*Batanin higher groupoids and homotopy types*, in Categories in Algebra, Geometry and Physics, Street Festschrift, A. Davydov et al. eds., Contemp. Math. 431, Amer. Math. Soc., pp. 171-186 (2007).

[C] J. M. Cordier, *Sur la notion de diagramme homotopiquement coherent* , Proc. 3ieme Colloque sur les Catégories, Amiens (1980).

[CP] J. M. Cordier and T. Porter, *Vogt's theorem on categories of homotopy coherent diagrams*, Math. Proc. Camb. Phil. Soc. (1986), 100, 65-90.

[Da] B. Day,

[Du] D. Dugger, *Universal homotopy theories*, Adv. in Math. 164, (2001), no. 1, 144-176.

[DHKS] W.G. Dwyer, P.S. Hirschhorn, D.M. Kan and J.H. Smith, *Homotopy limit functors on model categories and homotopical categories*, Mathematical Surveys and Monographs 113, American Mathematical Society, Providence, RI, 1999.

[DK1] W.G. Dwyer and D. Kan, *Functions complexes in homotopical algebra* , Topology 19 (1980), 427-440.

[DK2] W.G. Dwyer and D. Kan, *Simplicial localisations of categories*, J. of Pure and Appl. Algebra. vol 17 (1980), 427-440.

[DS] W.G. Dwyer and J. Spalinski, *Homotopy theories and model categories*, Handbook of Algebraic Topology, Elsevier, 1995.

[EM] S. Eilenberg and S. MacLane, *General theory of natural equivalences*, Trans. Amer. Math. Soc., 58, (1945)

[Eh] C. Ehresmann, *Espece de structures locales*, Séminaire de topologie et de géomtrie différentielle, Paris (1961)

[GZ] P. Gabriel and M. Zisman, *Calculus of fractions and homotopy theory*, Springer-Verlag, New York, 1963.

[GU] P. Gabriel and F. Ulmer, *Lokal prsentierbare Kategories*, Lecture Notes in Math., vol. 221, Springer-Verlag, Berlin and New York, 1971.

[GJ] P. G. Goerss and J.F. Jardine, *Simplicial homotopy theory*, Progress in Mathematics, vol 174, Birkhauser, 1999.

[G] J.Gray, *Closed categories, lax limits and homotopy limits*, J. Pure and Applied Algebra 19 (1980) 127-158

[Gro] A.Grothendieck, *Pursuing Stacks*, Manuscript 1983, To appear in Documents Mathématiques, edited by Georges Maltsiniotis.

[He] A. Heller, *Homotopy theories*, Memoir of the American Math. Soc. 383, 1988.

[Hi] P. S. Hirschhorn, *Model Categories and Their Localisations*, Mathematical Surveys and Monographs, 99, American Mathematical Society, Providence, RI, 2003.

[HS] A. Hirschowitz and C. Simpson, *Descente pour les n-champs*, preprint available at arXiv:math. AG/9807049.

[Ho] M. Hovey, *Model Categories*, Mathematical Surveys and Monographs 63, American Mathematical Society, Providence, RI, 1999.

[Jo] P. Johnstone, *Sketches of an Elephant: a Topos Theory Compendium*, Oxford University Press, 2002

[J1] A. Joyal, *Quasi-categories and Kan complexes*, J. of Pure and Appl. Algebra. vol 175 (2002), 207-222.

[J2] A. Joyal, *The theory of quasi-categories I.* in preparation.

[J3] A. Joyal, *The theory of quasi-categories II.* in preparation.

[J4] A. Joyal, *Quasi-categories vs Simplicial categories*, in preparation.

[J5] A. Joyal, *Higher quasi-categories*, in preparation.

[JT1] A. Joyal and M. Tierney, *Strong stacks and classyfying spaces*, Lectures Notes in Math. vol. 1588, Springer Verlag, 1991.

[JT2] A. Joyal and M. Tierney, *Quasi-categories vs Segal spaces*, Categories in Algebra, Geometry and Mathematical Physics. AMS Contemporary Mathematics 431. A. Davydov. M. Batanin, M. Johnson, S.Lack, A. Neeman editors.

[JT3] A. Joyal and M. Tierney, *Element of simplicial homotopy theory*, In preparation.

[Kan] D. Kan, *Adjoint functors*, Trans. Amer. Math. Soc. 87; 330-346(1958)

[Kelly] M. Kelly, *Basic concepts of enriched category theory*, LMS Lecture note series. 64, Cambridge university Press, 1982.

[KP] K. H. Kamps and T. Porterf, *Abstract homotopy and simple homotopy theory*, World Scientific, 1997.

[Lair] C. Lair, *Sur le genre d'esquissibilté des catégories modélables (accessibles) possédant les produits de deux*, Diagrammes **35** (1996), 25-52.

[Law1] W. Lawvere, *Functorial semantics of algebraic theories*, Proc. Nat. Acad, Sc. 50(1963), 869-872.

[Law2] F. W Lawvere, *Equality in hyperdoctrines and comprehension scheme as an adjoint functor*, Proc. Symp. Pure Math. Vol XVII, AMS (1970).

[Law3] W. Lawvere, *Ordinal sum ?*,

[Le] T. Leinster, *Higher Operads, Higher Categories*, arXiv:math.CT/0305049v1, 2003.

[Lu1] J. Lurie, *Higher Topos Theory*, arXiv:math.CT/0608040, 2007.

[Lu2] J. Lurie, *Stable∞-Categories*, arXiv:math.CT/0608228, 2007.

[Lu3] J. Lurie, *Derived Algebraic Geometry II: Non-commutative Algebra*, arXiv:math.CT/0702299, 2007.

[Lu4] J. Lurie, *Derived Algebraic Geometry III: Commutative Algebra*, arXiv:math.CT/0703204, 2007.

[Lu5] J. Lurie, *Derived Algebraic Geometry IV: Deformation Theory*, arXiv:math.CT/0709.3091, 2007.

[Mac] S. MacLane, *Categories for the Working Mathematicians*, Graduate Texts in Mathematics 5, Springer-Verlag(1971).

[Mal1] G. Maltsiniotis, *Introduction à la théorie des dérivateurs d'après Grothendieck*, unpublished manuscript available at http://www.math.jussieu.fr/ maltsin/.

[Mal2] G. Maltsiniotis, *La théorie de l'"homotopie de Grothendieck*, Asterisque 301, 2005.

[Mal3] G. Maltsiniotis, *Infini groupoides d'aprs Grothendieck*, Preprint. http://www.institut.math.jussieu.fr/ maltsin/ps/infgrart.pdf

[Mal4] G. Maltsiniotis, *Infini catgories non strictes, une nouvelle définition*, Preprint. http://www.institut.math.jussieu.fr/ maltsin/ps/infctart.pdf.

[May1] P. May, *A concise Course in Algebraic Topology* Chicago Lect. Notes in Math., The University of Chigago Press. 1999.

[May2] P. May, *The Geometry of Iterated loop spaces*, Lect. Notes in Math. 271, Springer Verlag, 1972.

[MP] M. Makkai and R.Paré *Acessible Categories: The Foundations of Categorical Model Theory* Contemporary Mathematics, vol 104, AMS (1989), 176 p.

[MW1] I. Moerdijk, I. Weiss *Dendroidal Sets*, arXiv:math,AT/0701293, 2007.

[MW2] I. Moerdijk, I. Weiss *On inner Kan complexes in the category of dendroidal sets* arXiv:math,AT/0701295, 2007.

[Ni] J.P. Nichols-Barrer, *On quasi-categories as a foundation for Higher Algebraic Stacks* PhD Thesis, MIT, June 2001.

[P] R. Pellissier. *Catégories enrichies faibles.* Thèse, Université de Nice-Sophia Antipolis. June 2002, arXiv:math/0308246, 2003.

[Q] D. Quillen, *Homotopical algebra*, Lectures Notes in Math. vol. 43, Springer Verlag, 1967.

[Ree] C.L. Reedy, *Homotopy theories of model categories*, unpublished manuscript, available at http://www.math.mit.edu/psh.

[Rezk1] C. Rezk, *A model for the homotopy theory of homotopy theory*, Trans. Amer. Math Soc., 353(2001), no. 3, 973-1007.

[Rezk2] C. Rezk, *Toposes and homotopy toposes*, unpublished manuscript. Dec 2005.

[Ro] A. Roig, *Model category structures in bifibered categories*, JPAA 95 (1994) 203-223.

[Ros] J. Rosicky, *On homotopy varieties*, Advances in Mathematics, Vol 214, N.2, 1 Oct 2007.

[Sch1] S. Schwede, *Spectra in model categories and applications to the algebraic cotangent complex*, JPAA 120 (1997) 77-104.

[Sch2] S. Schwede, *Stable homotopy of algebraic theories*, Topology 40 (2001) 1-41

[S1] G.B. Segal, *Classifying spaces and spectral sequences*, IHES Publ. Math. 34, 105-112(1968).

[S2] G. Segal, *Categories and cohomology theories*, Topology 13 (1974) 293-312.

[Si1] C. Simpson, *On the Breen-Baez-Dolan stabilisation hypothesis for Tamsamani weak n-categories* arXiv:math.CT/9810058, 1998.

[Si2] C. Simpson, *A Giraud-type characterisation of the simplicial category associated to closed model categories as ∞-pretopoi* arXiv:math.CT/990316,1999.

[Si3] C. Simpson, *A closed model structure for n-categories, internal Hom, n-stacks and generalised Seifert-Van Kampen* arXiv:math.AG/9704006v2, 2000.

[Si4] C. Simpson, *Some properties of the theory of n-categories* arXiv:math.CT/0110273, 2001.

[SGA] A. Grothendieck, J.L. Verdier *Topos theory and etale cohomology, SGA4*,

[SW] R. Street and R. Walters, *The comprehensive factorisation of a functor*, Bulletin American Math. Soc. **79** (1973) 936-941.

[Sta] J,D, Stasheff, *Homotopy associativity of H-spaces I, II,* Trans. Amer. Math, Soc. 108 (1963), 275-312.

[Ta] Z. Tamsamani *On non-strict notions of n-category and n-groupoid via multisimplicial sets*, arXiv:math.AG/9512006, 1995.

[T1] B. Töen *Homotopical and Higher Categorical Structures in Algebraic Geometry*, Habilitation thesis, May 2003. Unpublished manuscript, preprint available at arXiv:math.AG/0312262.

[TV3] B. Toen and G. Vezzosi, *Segal topoi and stacks over Segal categories*, arXiv:math.AG/0212330

[TV] B. Toen and G. Vezzosi, *Homotopical algebraic geometry I: topos theory*, preprint available at arXiv:math.AG/0207028

[V] R.M. Vogt, *Homotopy limits and colimits*, Math.Z. 134. 11-52 (1973)

[W] M. Weber, *Generic Morphisms, parametric representations and weakly cartesian monads*, Theory and Applications of Categories, Vol. 13, No. 14, 2004, pp. 191234.

Index of terminology

Index of notation