An Introduction to Simplicial Homotopy Theory

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This book is intended mainly for graduate students, and will include abstract homotopical algebra as well as a concrete treatment of the basic homotopy theory of simplicial sets. This first chapter establishes, with a new proof, the classical Quillen model structure on simplicial sets. Further chapters will treat covering spaces and the fundamental groupoid (including the Van Kampen Theorem), bundles and classifying spaces, simplicial groups and groupoids and the Dwyer-Kan Theorem, $K(\pi, n)'s$ and Postnikov towers, Quillen model structures on diagrams and sheaves, homotopy limits and colimits, and bisimplicial sets. There will be appendices on basic category theory, cartesian closed categories and compactly generated spaces, introductory topos theory with torsors and descent, CW-complexes and geometric realization, and abstract homotopy theory.

Chapter 1

The homotopy theory of simplicial sets

In this chapter we introduce simplicial sets and study their basic homotopy theory. A simplicial set is a combinatorial model of a topological space formed by gluing simplices together along their faces. This topological space, called the *geometric realization* of the simplicial set, is defined in section 1. Its properties are established in Appendix D. In section 2 we discuss the nerve of a small category.

The rest of the chapter is concerned with developing the basic ingredients of homotopy theory in the context of simplicial sets. Our principal goal is to establish the existence of the classical Quillen homotopy structure, which will then be applied, in various ways, throughout the rest of the book. Thus, we give the general definition of a Quillen structure in section 3 and state the main theorem. In section 4 we study fibrations and the extremely useful concept of anodyne extension due to Gabriel and Zisman [ ]. Section 5 is concerned with the homotopy relation between maps. Next, section 6 contains an exposition of the theory of minimal complexes and fibrations. These are then used in section 7 to establish the main theorem, which is the existence of the classical Quillen structure. The proof we give is different from those in the literature, [ ] or [ ]; from [ ], for example, in that it is purely combinatorial, making no use of geometric realization. However, none seems to be able to avoid the use of minimal fibrations. In section 8, we introduce the homotopy groups of a Kan complex, establish the long exact sequence of a fibration, and prove Whitehead's Theorem. We treat Milnor's Theorem in section 9, which shows that the category of Kan complexes and homotopy classes of maps is equivalent to the category of CW-complexes and homotopy classes of maps. Finally, in section 10, we show that the weak equivalences we used in the proof of the Quillen structure are the same as the classical ones.

1.1 Simplicial sets and their geometric realizations

The *simplicial category* $\mathbf{\Delta}$ has objects $[n] = \{0, \ldots, n\}$ for $n \geq 0$ a nonnegative integer. A map $\alpha : [n] \to [m]$ is an order preserving function.

Geometrically, an *n-simplex* is the convex closure of $n+1$ points in general position in a euclidean space of dimension at least n. The *standard, geometric n-simplex* Δ_n is the convex closure of the standard basis $e_0, \ldots, e_n$ of $\mathbf{R}^{n+1}$. Thus, the points of Δ_n consists of all combinations

$$p = \sum_{i=0}^{n} t_i e_i$$

with $t_i \geq 0$, and $\sum_{i=0}^{n} t_i = 1$. We can identify the elements of $[n]$ with the vertices $e_0, \ldots, e_n$ of Δ_n. In this way a map $\alpha : [n] \to [m]$ can be linearly extended to a map $\Delta_\alpha : \Delta_n \to \Delta_m$. That is,

$$\Delta_\alpha(p) = \sum_{i=0}^{n} t_i e_{\alpha(i)}$$

Clearly, this defines a functor $r : \mathbf{\Delta} \to Top$.

A *simplicial set* is a functor $X : \mathbf{\Delta}^{op} \to Set$. To conform with traditional notation, when $\alpha : [n] \to [m]$ we write $\alpha^* : X_m \to X_n$ instead of $X_\alpha : X[m] \to X[n]$.

Many examples arise from classical simplicial complexes. Recall that a *simplicial complex* K is a collection of non-empty, finite subsets (called *simplices*) of a given set V (of *vertices*) such that any non-empty subset of a simplex is a simplex. An *ordering* on K consists of a linear ordering $O(\sigma)$ on each simplex σ of K such that if $\sigma' \subseteq \sigma$ then $O(\sigma')$ is the ordering on σ' induced by $O(\sigma)$. The choice of an ordering for K determines a simplicial set by setting

$$K_n = \{(a_0, \ldots, a_n) | \sigma = \{a_0, \ldots, a_n\} \text{ is a simplex of } K,$$

$$\text{and } a_0 \leq a_1 \leq \ldots \leq a_n \text{ in the ordering } O(\sigma).\}$$

For $\alpha : [n] \to [m]$, $\alpha^* : K_m \to K_n$ is $\alpha^*(a_0, \ldots, a_m) = (a_{\alpha(0)}, \ldots, a_{\alpha(n)})$.

Remark: An $\alpha : [n] \to [m]$ in $\mathbf{\Delta}$ can be decomposed uniquely as $\alpha = \varepsilon\eta$, where $\varepsilon : [p] \to [m]$ is injective, and $\eta : [n] \to [p]$ is surjective. Moreover, if $\varepsilon^i : [n-1] \to [n]$ is the injection which skips the value $i \in [n]$, and $\eta^j : [n+1] \to [n]$ is the surjection covering $j \in [n]$ twice, then $\varepsilon = \varepsilon^{i_s} \ldots \varepsilon^{i_1}$ and $\eta = \eta^{j_t} \ldots \eta^{j_1}$ where $m \geq i_s > \ldots > i_1 \geq 0$, and $0 \leq j_t < \ldots < j_1 < n$ and $m = n - t + s$. The decomposition is unique, the i's in $[m]$ being the values not taken by α, and the j's being the elements of $[m]$ such that $\alpha(j) = \alpha(j+1)$. The $\varepsilon^{i's}$ and $\eta^{j's}$ satisfy the following relations:

$$\begin{array}{rcll} \varepsilon^j\varepsilon^i & = & \varepsilon^i\varepsilon^{j-1} & i<j \\ \eta^j\eta^i & = & \eta^i\eta^{j+1} & i\le j \\ \eta^j\varepsilon^i & = & \begin{cases} \varepsilon^i\eta^{j-1} & i<j \\ id & i=j \text{ or } i=j+1 \\ \varepsilon^{i-1}\eta^j & i>j+1 \end{cases} \end{array}$$

Thus, a simplicial set X can be considered to be a graded set $(X_n)_{n\geq 0}$ together with functions $d^i = \varepsilon^{i*}$ and $s^j = \eta^{j*}$ satisfying relations dual to those satisfied by the $\varepsilon^{i's}$ and $\eta^{j's}$. Namely,

$$\begin{array}{rcll} d^id^j & = & d^{j-1}d^i & i<j \\ s^is^j & = & s^{j+1}s^i & i\le j \\ d^is^j & = & \begin{cases} s^{j-1}d^i & i<j \\ id & i=j \text{ or } i=j+1 \\ s^jd^{i-1} & i>j+1 \end{cases} \end{array}$$

This point of view is frequently adopted in the literature.

The category of simplicial sets is $[\mathbf{\Delta}^{op}, Set]$, which we often denote simply by $\mathbf{S}$. Again for traditional reasons, the representable functor $\mathbf{\Delta}(, [n])$ is written $\Delta[n]$ and is called the *standard (combinatorial) n-simplex*. Conforming to this usage, we use $\Delta : \mathbf{\Delta} \to \mathbf{S}$ for the Yoneda functor, though if $\alpha : [n] \to [m]$, we write simply $\alpha : \Delta[n] \to \Delta[m]$ instead of $\Delta\alpha$.

Remark: We have

$$\Delta[n]_m = \Delta([m],[n]) = \{(a_0, \dots , a_m)|0 \leq a_i \leq a_j \leq n \text{ for } i \leq j\}$$

Thus, $\Delta[n]$ is the simplicial set associated to the simplicial complex whose n-simplices are all non-empty subsets of $\{0, \dots , n\}$ with their natural orders. the *boundary* of this simplicial complex has all *proper* subsets of $\{0, \dots , n\}$ as simplices. Its associated simplicial set is a simplicial (n-1)-sphere $\dot{\Delta}[n]$ called the *boundary* of $\Delta[n]$. Clearly, we have

$$\dot{\Delta}[n]_m = \{\alpha : [n] \to [m]|\alpha \text{ is not surjective}\}$$

$\dot{\Delta}[n]$ can also be described as the union of the (n-1)-faces of $\Delta[n]$. That is,

$$\dot{\Delta}[n] = \bigcup_{i=0}^{n} \Delta^i[n]$$

where $\Delta^i[n] = im(\varepsilon^i : \Delta[n-1] \to \Delta[n])$. Recall that the union is calculated pointwise, as is any colimit (or limit) in $\mathbf{\Delta} - Set$ [A 4.4].

Using the universal property (A.5.4) of $\mathbf{\Delta} - Set$, the functor $r : \mathbf{\Delta} \to Top$ can be extended to a functor $r^\sharp : \mathbf{S} \to Top$, called the *geometric realization.* Following Milnor [ ], we write $|X|$ instead of $r^\sharp X$ for the geometric realization of a simplicial set X. Thus, we have a commutative triangle

$$\begin{tikzcd} \mathbf{\Delta} \arrow[rr, "\Delta"] \arrow[dr, "r"'] & & \mathbf{S} \arrow[dl, "|\ \ |"] \\ & Top & \end{tikzcd}$$

where

$$|X| = \varinjlim_{\Delta[n]\to X} \Delta_n$$

As in A.5.4, $r^\sharp = |\ \ |$ has a right adjoint $r_\sharp$. For any topological space T, $r_\sharp T$ is the *singular complex* sT of T. That is,

$$(r_\sharp T)[n] = Top(\Delta_n, T) = (sT)_n$$

Using the fact that a left adjoint preserves colimits (A.4.5), we see that the geometric realization functor $|\ \ | : \mathbf{S} \to Top$ is colimit preserving. A consequence of this is that $|X|$ is a CW-complex. Furthermore, if Top is replaced by Top_c - the category of compactly generated spaces - then $|\ \ |$ is also left-exact, i.e. preserves all finite limits. See Appendix B for the basic properties of Top_c, and Appendix D for CW-complexes and the proofs of the above facts.

1.2 Simplicial sets and categories

Denote by Cat the category of small categories and functors. There is a functor $\mathbf{\Delta} \to Cat$ which sends $[n]$ into $[n]$ regarded as a category via its natural ordering. Again, by the universal property of $\mathbf{\Delta} - Set$ this functor can be extended to a functor $R : \mathbf{S} \to Cat$ so as to give a commutative triangle

$$\begin{tikzcd} \mathbf{\Delta} \arrow[rr, "\Delta"] \arrow[dr] & & \mathbf{S} \arrow[dl, "R"] \\ & Cat & \end{tikzcd}$$

where

$$RX = \varinjlim_{\Delta[n]\to X} [n]$$

As before, R has a right adjoint N. If $\boldsymbol{A}$ is a small category, $N\boldsymbol{A}$ is the *nerve* of $\boldsymbol{A}$ and

$$(N\boldsymbol{A})_n = Cat([n], \boldsymbol{A})$$

An n-simplex x of a simplicial set X is said to be *degenerate* if there is a surjection $\eta : [n] \to [m]$ with $m < n$ and an m-simplex y such that $x = \eta * y$. Otherwise, we say x is *non-degenerate.* Consider the case when X is the nerve NP of a partially ordered set P. Then an n-simplex of NP is an order-preserving mapping $x : [n] \to P$ which is non-degenerate iff it is injective. Since N preserves monomorphisms, the singular n-simplex $\Delta[n] \to NP$ associated to x is also injective. Thus, the image of a non-degenerate n-simplex of NP is a standard n-simplex.

Suppose P is finite. Call a totally ordered subset c of P a *chain* of P. Then there are a finite number $c_1 \dots c_r$ of maximal chains of P, and every chain c is contained in some c_i. If c_i contains $n_i + 1$ elements we can associate to it a unique non-degenerate simplex $x_i : [n_i] \to P$ whose image in P is c_i. Each non-degenerate simplex of NP is a face of some x_i. The x_i together yield a commutative diagram

$$\textstyle\sum_{1\le i<j\le r}[n_{ij}] \underset{\nu}{\overset{\mu}{\rightrightarrows}} \sum_{1\le i\le r}[n_i] \longrightarrow P$$

where $n_{ij} + 1$ is the number of elements in $c_i \cap c_j$, and μ, respectively ν, is defined by the inclusion of $c_i \cap c_j$ in c_i, respectively c_j. Moreover, applying N, we obtain an *exact* diagram

$$\textstyle\sum_{1\le i<j\le r}\Delta[n_{ij}] \rightrightarrows \sum_{1\le i\le r}\Delta[n_i] \longrightarrow NP$$

in $\mathbf{S}$, i.e. a diagram which is both a coequalizer and a kernel pair. To see this, simply evaluate the diagram at any $m \ge 0$, and check that the result is exact as a diagram in *Set*. We call this the *finite presentation of* NP *corresponding to the maximal chains of* P.

We examine in detail the example $P = [p] \times [q]$ as this will be of use to us later. We claim first that a maximal chains of $[p] \times [q]$ can be pictured as a path from $(0,0)$ to (p,q) in the lattice of points (m,n) in the plane with integral coordinates where at each point (i,j) on the path, the next point is either immediately to the right, or up. An example follows for $p = 3, q = 2$.

$$\begin{tikzcd}
(0,2) & (1,2) & (2,2) & (3,2) \\
(0,1) & (1,1) \arrow[r] & (2,1) \arrow[r] & (3,1) \arrow[u] \\
(0,0) \arrow[r] & (1,0) \arrow[u] & (2,0) & (3,0)
\end{tikzcd}$$

The number of elements in each of these chains is $p + q + 1$ and they are clearly the maximal ones, since any maximal chain must contain $(0,0)$ and (p,q), and whenever it contains (i,j) it must contain either $(i+1,j)$ or $(i,j+1)$.

We can identify these chains with (p,q)-*shuffles* $(\sigma;\tau)$, which are partitions of the set $(1,2,\dots,p+q)$ into two disjoint subsets $\sigma = (\sigma_1 < \dots < \sigma_p)$ and $\tau = (\tau_1 < \dots < \tau_q)$. Such a partition describes a shuffling of a pack of p cards through a pack of q cards, putting the cards of the first pack in the positions $\sigma_1 < \dots < \sigma_p$, and those of the second in the positions $\tau_1 < \dots < \tau_q$. The identification proceeds as follows. Given a maximal chain c, lets the σ's be the sums of the coordinates of the right-hand endpoints of the horizontal segments. The τ's are then the sums of the coordinates of the upper endpoints of the vertical segments. Thus, the shuffle associated to the maximal chain above is $(1,3,4;2,5)$. On the other hand, given a (p,q) shuffle $(\sigma;\tau)$, form a chain by selecting $(0,0)$, then $(0,1)$ if $1 \in \sigma$ and $(0,1)$ if $1 \in \tau$. In general, if (i,j) has been selected, select $(i+1,j)$ if $i+j+1 \in \sigma$ and $(i,j+1)$ if $i+j+1 \in \tau$. Since the correspondence is clearly $1-1$, and there are $\binom{p+q}{p}$ such shuffles, this is also the number of maximal chains. We thus obtain the presentation

$$\textstyle\sum_{1\leq i<j\leq r} \Delta[n_{ij}] \rightrightarrows \sum_{1\leq i\leq r} \Delta[p+q] \longrightarrow \Delta[p]\times\Delta[q]$$

of $N([p]\times[q]) \simeq \Delta[p]\times\Delta[q]$, where $r = \binom{p+q}{p}$. Taking the geometric realization of this presentation yields a triangulation of $\Delta_p \times \Delta_q$ into $\binom{p+q}{p}$ (p+q)-simplices.

1.3 Quillen homotopy structures

Let $\mathcal{K}$ be a category with finite limits and colimits. A *Quillen homotopy structure* on $\mathcal{K}$ (also called a *Quillen model structure* on $\mathcal{K}$) consists of three classes of mappings of $\mathcal{K}$ called *fibrations*, *cofibrations*, and *weak equivalences*. These are subject to the following axioms:

Q1.(Saturation) If $f : X \to Y$ and $g : Y \to Z$ are mappings of $\mathcal{K}$, and any two of f, g or gf are weak equivalences, then so is the third. This is sometimes called the "three for two" property.

Q2.(Retracts) Let

$$\begin{array}{ccccc} X & \xrightarrow{i} & X' & \xrightarrow{u} & X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[j]{} & Y' & \xrightarrow[v]{} & Y \end{array}$$

be a commutative diagram with $ui = id_X$ and $vj = id_Y$. Then if f' is a fibration, cofibration or weak equivalence, so is f.

Q3.(Lifting) If

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

is a commutative diagram in which i is a cofibration and f is a fibration, then if i or f is a weak equivalence, there is a dotted lifting making both triangles commute.

Q4.(Factorization) Any map $f : X \to Y$ can be factored as

$$\begin{tikzcd} X \arrow[rr, "i"] \arrow[dr, "f"'] & & E \arrow[dl, "p"] \\ & Y & \end{tikzcd}$$

where i is a cofibration and p is a fibration in two ways: one in which i is a weak equivalence, and one in which p is a weak equivalence.

The homotopy structure is said to be *proper*, if, in addition, the following axiom is satisfied.

Q5. If

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "w'"'] & X \arrow[d, "w"] \\ Y' \arrow[r, "f"'] & Y \end{tikzcd}$$

is a pullback diagram, in which f is a fibration and w is a weak equivalence, then w' is a weak equivalence. Dually, the pushout of a weak equivalence by a cofibration is a weak equivalence.

An example of a Quillen homotopy structure is obtained by taking $\mathcal{K}$ to be Top_c. A map $f : X \to Y$ is a weak equivalence if $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is a bijection, and for $n \geq 1$ and $x \in X$, $\pi_n(f) : \pi_n(X, x) \to \pi_n(Y, fx)$ is an isomorphism. Fibrations are Serre fibrations, i.e. maps $p : E \to X$ with the *covering homotopy property* (CHP) for each n-simplex Δ_n, $n \geq 0$. This means that if $h : \Delta_n \times I \to X$ is a homotopy ($I = [0, 1]$), and $f : \Delta_n \to E$ is such that $pf = h_0$, then there is a "covering homotopy" $\bar{h} : \Delta_n \times I \to E$ such that $\bar{h}_0 = f$, and $p\bar{h} = h$. Cofibrations are mappings $i : A \to B$ having the *left lifting property* (LLP) with respect to those fibrations $p : E \to X$ which are also weak equivalences. That is, if

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & E \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

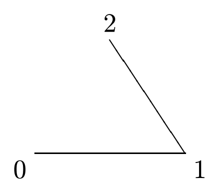
is a commutative diagram where p is a fibration and a weak equivalence, then there is a dotted lifting making both triangles commute. Details of the proof can be found in Quillen [ ] (for the case Top which is the same as Top_c), or see exercise [ ] at the end of the chapter.

Our principal example is in $\mathbf{S}$, the category of simplicial sets. Here, the weak equivalences are *geometric homotopy equivalences*, by which we mean a map $f : X \to Y$ such that $|f| : |X| \to |Y|$ is a homotopy equivalence, i.e. there is a map $f' : |Y| \to |X|$ such that $f'|f|$ is homotopic to $id_{|X|}$, and $|f|f'$ is homotopic to $id_{|Y|}$. The cofibrations are monomorphisms.

To define the fibrations, recall that the i^{th} face of $\Delta[n]$ $(n \geq 1, 0 \leq i \leq n)$ is $\Delta^i[n] = im(\varepsilon^i : \Delta[n-1] \to \Delta[n])$. The k^{th} *horn* of $\Delta[n]$ is

$$\Lambda^k[n] = \bigcup_{i \neq k} \Delta^i[n]$$

The geometric realization of $\Lambda^k[n]$ is the union of all those (n-1)-dimensional faces of Δ_n that contain the k^{th} vertex of Δ_n. For example,



is the geometric realization of $\Lambda^1[2]$.

Definition 1.3.1 *A Kan fibration is a map $p : E \to X$ of simplicial sets having the right lifting property (RLP) with respect to the inclusions of the horns $\Lambda^k[n] \to \Delta[n]$ for $n \geq 1$, and $0 \leq k \leq n$.*

That is, if

$$\begin{tikzcd}
\Lambda^k[n] \arrow[r] \arrow[d] & E \arrow[d, "p"] \\
\Delta[n] \arrow[r] \arrow[ur, dashed] & X
\end{tikzcd}$$

is a commutative diagram with $n \geq 1$, and $0 \leq k \leq n$, then there is a dotted lifting making both triangles commute. We express this by saying "any horn in E which can be filled in X, can be filled in E". For example, if $p : E \to X$ is a Serre fibration in Top_c, then $sp : sE \to sX$ is a Kan fibration in $\mathbf{S}$. This is so, because a diagram

$$\begin{tikzcd}
\Lambda^k[n] \arrow[r] \arrow[d] & sE \arrow[d, "sp"] \\
\Delta[n] \arrow[r] & sX
\end{tikzcd}$$

is equivalent to a diagram

$$\begin{tikzcd} {|\Lambda^k[n]|} \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ {|\Delta[n]|} \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

and $|\Lambda^k[n]| \to |\Delta[n]|$ is homeomorphic to $\Delta_{n-1} \to \Delta_{n-1} \times I$.

The rest of the chapter will be devoted to the proof of the following theorem.

Theorem 1.3.1 *The fibrations, cofibrations and weak equivalences defined above form a proper Quillen homotopy structure on* $\mathbf{S}$.

1.4 Anodyne extensions and fibrations

A class $\mathcal{A}$ of monomorphisms is said to be *saturated* if it satisfies the following conditions:

(i.) $\mathcal{A}$ contains all isomorphisms.

(ii.) $\mathcal{A}$ is closed under pushouts. That is, if

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & A' \arrow[d, "i'"] \\ B \arrow[r, "f"'] & B' \end{tikzcd}$$

is a pushout diagram, and $i \in \mathcal{A}$, then $i' \in \mathcal{A}$.

(iii.) $\mathcal{A}$ is closed under retracts. That is, if

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "i"] & A' \arrow[r, "u"] \arrow[d, "i'"] & A \arrow[d, "i"] \\ B \arrow[r, "k"'] & B' \arrow[r, "v"'] & B \end{tikzcd}$$

is a commutative diagram with $uj = id_A$, $vk = id_B$ and $i' \in \mathcal{A}$, then $i \in \mathcal{A}$.

(iv.) $\mathcal{A}$ is closed under coproducts. That is, if $(A_l \xrightarrow{i_l} B_l | l \in L)$ is a family of monomorphisms with $i_l \in \mathcal{A}$ for each $l \in L$, then

$$\sum_{l\in L} i_l : \sum_{l\in L} A_l \longrightarrow \sum_{l\in L} B_l$$

is in $\mathcal{A}$.

(v.) $\mathcal{A}$ is closed under countable composition. That is, if

$$(A_n \to A_{n+1} | n = 1, 2, \dots)$$

is a countable family of morphisms of $\mathcal{A}$, then

$$\mu_1 : A_1 \longrightarrow \varinjlim_{n \geq 1} A_n$$

is a morphism of $\mathcal{A}$.

The intersection of all saturated classes containing a given set of monomorphisms Γ is called the *saturated class generated by* Γ.

For example, if $m : A \to X$ is an arbitrary monomorphism of $\mathbf{S}$, then (D.4.9)

$$\begin{array}{ccc} \sum_{e(X-A)_n} \dot{\Delta}[n] & \longrightarrow & \sum_{e(X-A)_n} \Delta[n] \\ \downarrow & & \downarrow \\ Sk^{n-1}(X) \cup A & \longrightarrow & Sk^n(X) \cup A \end{array}$$

is a pushout for $n \geq -1$. Furthermore,

$$X = \varinjlim_{n \geq -1} (Sk^n(X) \cup A) \quad \text{and} \quad \mu_{-1} : Sk^{-1}(X) \cup A \longrightarrow \varinjlim_{n \geq -1} (Sk^n(X) \cup A)$$

is $m : A \to X$. Thus, the saturated class generated by the family

$$(\dot{\Delta}[n] \to \Delta[n] | n \geq 0)$$

is the class of all monomorphisms.

The saturated class $\mathcal{A}$ generated by the family

$$(\Lambda_k[n] \to \Delta[n] | 0 \leq k \leq n, n \geq 1)$$

is called the class of *anodyne extensions.*

Evidently, from the nature of conditions (i.) - (v.) we can conclude

Proposition 1.4.1 *A map $p : E \to X$ is a (Kan) fibration iff it has the right lifting property (RLP) with respect to all anodyne extensions.*

Definition 1.4.1 *A map $p : E \to X$ is said to be a trivial fibration if it has the RLP with respect to the family $(\dot{\Delta}[n] \to \Delta[n] | n \geq 0)$ or, equivalently as above, with respect to the family of all monomorphisms.*

Theorem 1.4.1 *Any map $f : X \to Y$ of $\mathbf{S}$ can be factored as*

$$\begin{array}{ccc} X & \xrightarrow{i} & E \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle p} & \\ & Y & \end{array}$$

where i is anodyne and p is a fibration.

Proof: Consider the set L of all commutative diagrams

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \Delta[n] \arrow[r] & Y \end{tikzcd}$$

with $n \geq 1$ and $0 \leq k \leq n$. Summing over L yields a commutative diagram

$$\begin{tikzcd} \sum_L \Lambda^k[n] \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\ \sum_L \Delta[n] \arrow[r] & Y \end{tikzcd}$$

with i anodyne. In the pushout

$$\begin{tikzcd} \sum_L \Lambda^k[n] \arrow[r] \arrow[d, "i"'] & X \arrow[d, "i^0"] \\ \sum_L \Delta[n] \arrow[r] & X^1 \end{tikzcd}$$

i^0 is anodyne, and we have a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "i^0"] \arrow[dr, "f"'] & & X^1 \arrow[dl, "f^1"] \\ & Y & \end{tikzcd}$$

Now repeat the process with f^1, obtaining

$$\begin{tikzcd} X \arrow[r, "i^0"] \arrow[dr, "f"'] & X^1 \arrow[r, "i^1"] \arrow[d, "f^1"] & X^2 \arrow[dl, "f^2"] \\ & Y & \end{tikzcd}$$

with i^1 anodyne, etc. Let $X^0 = X$, and $f^0 = f$. Putting $E = \varinjlim_{n \geq 0} X^n$ let $p : E \to Y$ be the map induced by f^n on each X^n. Writing i for the map $\mu_0 : X \to E$, we obtain a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "i"] \arrow[dr, "f"'] & & E \arrow[dl, "p"] \\ & Y & \end{tikzcd}$$

where i is anodyne. It remains to show that p is a fibration. So let

$$\begin{tikzcd}
\Lambda^k[n] \arrow[r, "h"] \arrow[d] & E \arrow[d, "p"] \\
\Delta[n] \arrow[r] & Y
\end{tikzcd}$$

be a commutative diagram with $n \geq 1$ and $0 \leq k \leq n$. $\Lambda^k[n]$ has only finitely many non-degenerate simplices, so h factors through X^n for some $n \geq 0$. But then we have a lifting into X^{n+1}

$$\begin{tikzcd}
\Lambda^k[n] \arrow[r, "h"] \arrow[d] & X^n \arrow[r, "i^n"] \arrow[d, "f^n"'] & X^{n+1} \arrow[dl, "f^{n+1}"] \\
\Delta[n] \arrow[r] \arrow[urr, dashed] & Y &
\end{tikzcd}$$

and hence a dotted lifting in

$$\begin{tikzcd}
\Lambda^k[n] \arrow[r, "h"] \arrow[d] & E \arrow[d, "p"] \\
\Delta[n] \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

■

Corollary 1.4.1 *$i : A \to B$ is anodyne if it has the LLP with respect to the class of all fibrations.*

Proof: Factor i in the form

$$\begin{tikzcd}
A \arrow[rr, "j"] \arrow[dr, "i"'] & & E \arrow[dl, "p"] \\
& B &
\end{tikzcd}$$

where j is anodyne and p is a fibration. Since i has the LLP with respect to p, we can find a dotted lifting in

$$\begin{tikzcd}
A \arrow[r, "j"] \arrow[d, "i"'] & E \arrow[d, "p"] \\
B \arrow[r, "id_B"'] \arrow[ur, dashed, "k"] & B
\end{tikzcd}$$

But then i is a retract of j, so i is anodyne. ■

Theorem 1.4.2 *Any map $f : X \to Y$ of $\mathbf{S}$ can be factored as*

$$\begin{tikzcd} X \arrow[rr, "i"] \arrow[dr, "f"'] & & E \arrow[dl, "p"] \\ & Y & \end{tikzcd}$$

where i is a monomorphism, and p is a trivial fibration.

Proof: Repeat the proof of Theorem 1.4.1 using the family $(\dot{\Delta}[n] \to \Delta[n] | n \geq 0)$ instead of the family $(\Lambda_k[n] \to \Delta[n] | 0 \leq k \leq n, n \geq 1)$. ∎

Definition 1.4.2 *A simplicial set X is called a Kan complex if $X \to 1$ is a fibration.*

As an example, we have the singular complex sT of any topological space T. Another important example is provided by the following theorem.

Theorem 1.4.3 (Moore) *Any group G in $\mathbf{S}$ is a Kan complex.*

Theorem 1.4.3, together with the following lemma, provides many examples of fibrations.

Lemma 1.4.1 *The property of being a fibration, or trivial fibration, is local. That is, if $p : E \to X$ and there exists a surjective map $q : Y \to X$ such that in the pullback*

$$\begin{tikzcd} E' \arrow[r] \arrow[d, "p'"'] & E \arrow[d, "p"] \\ Y \arrow[r, "q"'] & X \end{tikzcd}$$

p' is a fibration, or trivial fibration, then p is a fibration, or trivial fibration.

The straightforward proof is left as an excersise.

Definition 1.4.3 *A bundle with fiber F in $\mathbf{S}$ is a mapping $p : E \to X$ such that for each n-simplex $\Delta[n] \to X$ of X, there is an isomorphism ϕ in*

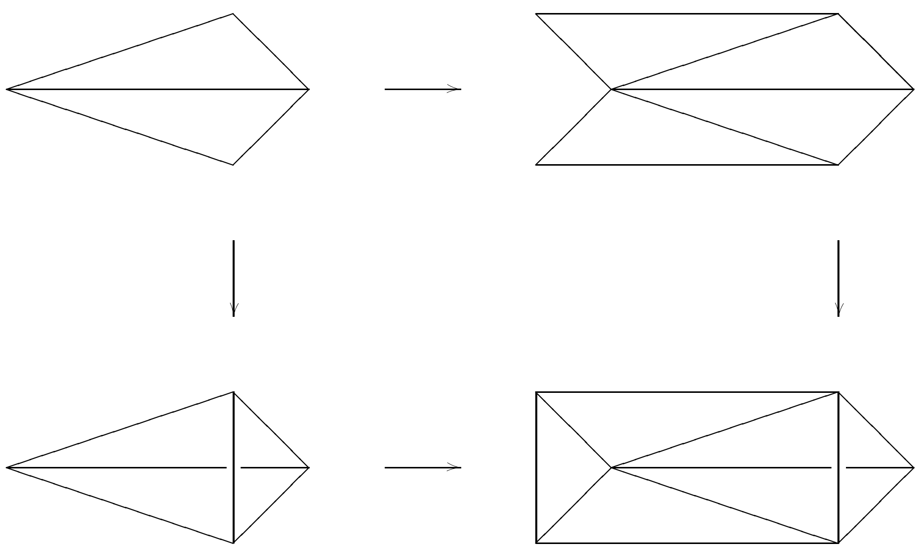
$$\begin{tikzcd} \Delta[n] \times F \arrow[rr, "\phi"] \arrow[dr, "\pi_1"'] & & \Delta[n] \times_X E \arrow[dl, "\pi_1"] \\ & \Delta[n] & \end{tikzcd}$$

Let $Y = \sum_{\Delta[n]\to X} \Delta[n]$. Then the cannonical map $Y \to X$ is surjective. Since $\pi_1 : Y \times F \to Y$ is clearly a fibration when F is a Kan complex, it follows from Lemma 1.4.1 that a bundle with Kan fiber F is a fibration. In particular, a *principal G-bundle*, which is a bundle with fiber a group G, is a fibration.

Proof of Theorem 1.4.3: We give a new proof of the theorem, which perhaps involves less extensive use of the simplicial identities than the classical one. Thus, let G be a group and let $f : \Lambda^k[n] \to G$. We want to extend f to $\Delta[n]$ and we proceed by induction on n. The case $n = 1$ is obvious, since each $\Lambda^k[1]$ is $\Delta[0]$ and a retract of $\Delta[1]$. For the inductive step, let $\Lambda^{k,k-1}[n]$ be $\Lambda^k[n]$ with the $(k-1)$-st face removed (if $k = 0$ use $\Lambda^{0,1}[n]$). Then there is a commutative diagram of inclusions

$$\begin{tikzcd}
\Lambda^{k,k-1}[n] \arrow[r] \arrow[d] & \Lambda^{k-1}[n-1] \times \Delta[1] \arrow[d] \\
\Delta[n] \arrow[r] & \Delta[n-1] \times \Delta[1]
\end{tikzcd}$$

whose geometric realizations in dimension 3 look like



Now f restricted to $\Lambda^{k,k-1}[n]$ can be extended to $\Lambda^{k-1}[n-1]\times\Delta[1]$, since the inclusion $\Lambda^{k,k-1}[n] \to \Lambda^{k-1}[n-1] \times \Delta[1]$ is an anodyne extension of dimension $n-1$. Moreover, this extension can be further extended to $\Delta[n-1] \times \Delta[1]$ by exponential adjointness and induction, since $G^{\Delta[1]}$ is a group. Restricting this last extension to $\Delta[n]$ we see that f restricted to $\Lambda^{k,k-1}[n]$ can be extended to $\Delta[n]$.

We are thus in the following situation. We have two subcomplexes $\Lambda^{k,k-1}[n]$ and $\Delta[n-1]$ of $\Delta[n]$ where the inclusion $\Delta[n-1] \to \Delta[n]$ is ε^{k-1}. Furthermore, we have a map $f : \Lambda^k[n] = \Lambda^{k,k-1}[n] \cup \Delta[n-1] \to G$ whose restriction to $\Lambda^{k,k-1}[n]$ can be extended to $\Delta[n]$. Now put $r = \eta^{k-1} : \Delta[n] \to \Delta[n-1]$. Then $r|\Delta[n-1] = id$ and it is easy to see that $r : \Delta[n] \to \Delta[n-1] \to \Lambda^k[n]$ maps $\Lambda^{k,k-1}[n]$ into itself. The following Lemma then completes the proof.

Lemma 1.4.2 *Let A and B be subcomplexes of C, and $r : C \to B$ a mapping such that $r = id$ on B and $r : C \to B \to A \cup B$ maps A into itself. Let $f : A \cup B \to G$ where G is a group. Then if $f|A$ can be extended to C, f can be extended to C*

Proof: Extend $f|A$ to $g : C \to G$, and define $h : C \to G$ by $h(x) = g(x)g(r(x))^{-1}f(r(x))$. ∎

1.5 Homotopy

Definition 1.5.1 *Let X be a simplicial set. In the coequaliser*

$$\begin{tikzcd} X_1 \arrow[r, shift left, "d^0"] \arrow[r, shift right, "d^1"'] & X_0 \arrow[r] & \pi_0(X) \end{tikzcd}$$

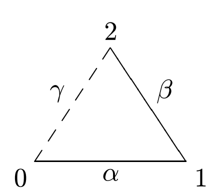
$\pi_0(X)$ is called the set of connected components of X.

We remark that this $\pi_0(X)$ is the same as the set of connected components of X considered as a set-valued functor and defined in A.5, i.e. $\pi_0(X) = \varinjlim X$. For the proof, see excercise [ ] at the end of the chapter.

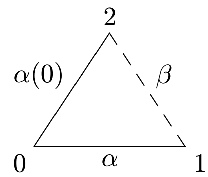
Let us write the relation on X_0 determined by d^0 and d^1 as $x \sim y$, saying "x is connected to y by a path". That is, writing I for $\Delta[1]$ and $(0) \subseteq I$, respectively $(1) \subseteq I$, for the images of $\varepsilon_1 : \Delta[0] \to \Delta[1]$ and $\varepsilon_0 : \Delta[0] \to \Delta[1]$, then $x \sim y$ iff there is a map $\alpha : I \to X$ such that $\alpha(0) = x$, and $\alpha(1) = y$. In general, $x \sim y$ is not an equivalence relation, so that if we denote the map $X_0 \to \pi_0(X)$ by $x \mapsto \bar{x}$ then $\bar{x} = \bar{y}$ iff there is a "path of length n' connecting x and y", i.e, we have a diagram of the form

$$x \to x_1 \leftarrow x_2 \to x_3 \cdots x_{n-1} \to y$$

$x \sim y$ *is* an equivalence relation, however, when X is a Kan complex. For suppose $\alpha : I \to X$ and $\beta : I \to X$ are such that $\alpha(1) = \beta(0)$. Let $s : \Lambda^1[2] \to X$ be the unique map such that $s\varepsilon^0 = \beta$ and $s\varepsilon^2 = \alpha$. A picture is given by



If X is Kan, there is a $t : \Delta[2] \to X$ extending s, and $\gamma = t\varepsilon^1$ connects $\alpha(0)$ to $\beta(1)$, showing transitivity. Using $\Lambda^0[2]$ and the constant (degenerate) path $\alpha(0)$ as in



yields the symmetry.

A useful fact is the following.

Proposition 1.5.1 *If X and Y are simplicial sets, the canonical mapping*

$$\pi_0(X \times Y) \longrightarrow \pi_0(X) \times \pi_0(Y)$$

is a bijection.

Proof: The canonical mapping, in the notation defined above, is given by $\overline{(x,y)} \mapsto (\bar{x}, \bar{y})$. It is clearly surjective. If $(\bar{x_1}, \bar{y_1}) = (\bar{x_2}, \bar{y_2})$ then $\bar{x_1} = \bar{x_2}$, and $\bar{y_1} = \bar{y_2}$. Thus, there is a path of length n connecting x_1 to x_2, and a path of length m connecting y_1 to y_2. By using constant paths, we we may assume $n = m$, so that $\overline{(x_1, y_1)} = \overline{(x_2, y_2)}$ and the map is injective. ■

Definition 1.5.2 *If $f, g : X \to Y$, we say f is homotopic to g if there is a map $h : X \times I \to Y$ such that $h_0 = f$ and $h_1 = g$.*

Clearly, we can interpret a homotopy h as a path in Y^X such that $h(0) = f$ and $h(1) = g$. As above, the relation of homotopy among maps is not an equivalence relation in general (see excercise [ ] for an example), but it is when Y^X is Kan. We will show below that Y^X is Kan when Y is, which, by adjointness, amounts to showing that each $\Lambda^k[n] \times X \to \Delta[n] \times X$ is anodyne.

We denote by $ho(\mathbf{S})$ the category of Kan complexes and homotopy classes of maps. That is, its objects are Kan complexes, and the set of morphisms between two Kan complexes X and Y is $[X, Y] = \pi_0(Y^X)$. Composition is defined as follows. Given X, Y and Z, there is a map $Y^X \times Z^Y \to Z^X$, which is the exponential transpose of the mapping

$$Y^X \times Z^Y \times X \simeq Z^Y \times Y^X \times X \stackrel{id \times ev}{\to} Z^Y \times Y \stackrel{ev}{\to} Z$$

The composition in $ho(\mathbf{S})$, $[X, Y] \times [Y, Z] \to [X, Z]$, is obtained by applying π_0 to this map, using Proposition 1.5.1.

Notice that an isomorphism of $ho(\mathbf{S})$ is a *homotopy equivalence*, i.e. $X \simeq Y$ in $ho(\mathbf{S})$ iff there are mappings $f : X \to Y$ and $f' : Y \to X$ such that $ff' \sim id_Y$ and $f'f \sim id_X$.

$ho(\mathbf{S})$ has a number of different descriptions, as we will see. For example, it is equivalent to the category of CW-complexes and homotopy classes of maps.

Returning to the problem of showing that Y^X is Kan when Y is, we will, in fact, prove the following more general result. Let $k : Y \to Z$ be a monomorphism and suppose $p : E \to X$. Denote the pullback of X^k and p^Y by

$$\begin{tikzcd} (k,p) \arrow[r] \arrow[d] & E^Y \arrow[d, "p^Y"] \\ X^Z \arrow[r, "X^k"'] & X^Y \end{tikzcd}$$

(k, p) is the "object of diagrams" of the form

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "k"'] & E \arrow[d, "p"] \\ Z \arrow[r] & X \end{tikzcd}$$

The commutative diagram

$$\begin{tikzcd} E^Z \arrow[r, "E^k"] \arrow[d, "P^Z"'] & E^Y \arrow[d, "p^Y"] \\ X^Z \arrow[r, "X^k"'] & X^Y \end{tikzcd}$$

gives rise to a map $k|p : E^Z \to (k, p)$, and we have

Theorem 1.5.1 *If $p : E \to X$ is a fibration, then $k|p : E^Z \to (k, p)$ is a fibration, which is trivial if either k is anodyne, or p is trivial.*

Proof: Let $i : A \to B$ be a monomorphism. The problem of finding a dotted lifting in

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & E^Z \arrow[d, "k|p"] \\ B \arrow[r] \arrow[ur, dashed] & (k,p) \end{tikzcd}$$

coincides with the problem of finding a dotted lifting in the adjoint transposed diagram

$$\begin{tikzcd} (A \times Z) \cup (B \times Y) \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ B \times Z \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

which exists for any i if p is trivial. By Theorem 1.5.2, below, the left-hand vertical map is anodyne if either i or k is, which will complete the proof. ■

Taking k to be the identity yields, in particular,

Corollary 1.5.1 *If $p : E \to X$ is a fibration, so is $p^Y : E^Y \to X^Y$ for any Y.*

This, of course, generalizes the original statement that X^Y is a Kan complex when X is.

Let $i : A \to B$, and $k : Y \to Z$ be monomorphisms. Then we have $i \times Z : A \times Z \to B \times Z$ and $B \times k : B \times Y \to B \times Z$, and we write $i \star k$ for the inclusion $(A \times Z) \cup (B \times Y) \to B \times Z$.

Theorem 1.5.2 *(Gabriel-Zisman) If i is anodyne, so is $i \star k$.*

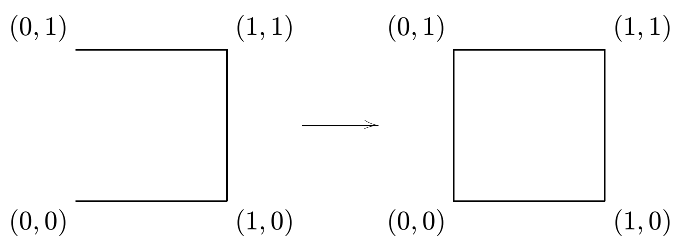
For the proof of Theorem 1.5.2 we need the following auxiliary result. Recall that we denoted by $\mathcal{A}$ the class of anodyne extensions, which is the saturated class generated by all the inclusions $\Lambda^k[n] \to \Delta[n]$ for $n \geq 1$ and $0 \leq k \leq n$. Denote by a the anodyne extension $(e) \to \Delta[1]$ and by i_n the inclusion $\dot{\Delta}[n] \to \Delta[n]$. Now let $\mathcal{B}$ be the saturated class generated by all the inclusions

$$a \star i_n : ((e) \times \Delta[n]) \cup (\Delta[1] \times \dot{\Delta}[n]) \longrightarrow \Delta[1] \times \Delta[n]$$

for $e = 0, 1$ $n \geq 1$. $\Delta[1] \times \Delta[n]$ is called a *prism* and $((e) \times \Delta[n]) \cup (\Delta[1] \times \dot{\Delta}[n])$ an *open prism*. For example, the geometric realization of the inclusion

$$a \star i_1 : ((1) \times \Delta[1]) \cup (\Delta[1] \times \dot{\Delta}[1]) \longrightarrow \Delta[1] \times \Delta[1]$$

is



Finally, we denote by $\mathcal{C}$ the saturated class generated by all inclusions

$$a \star m : ((e) \times Y) \cup (\Delta[1] \times X) \longrightarrow \Delta[1] \times Y$$

where $m : X \to Y$ is a monomorphism of $\mathbf{S}$ and $e = 0, 1$.

Theorem 1.5.3 *$\mathcal{B} \subseteq \mathcal{A}$ and $\mathcal{A} = \mathcal{C}$*

With Theorem 1.5.3, whose proof we give shortly, we can complete the proof of Theorem 1.5.2.

Proof of Theorem 1.5.2: Let $k : Y \to Z$ be an arbitrary monomorphism of $\mathbf{S}$, and denote by $\mathcal{D}$ the class of all monomorphisms $i : A \to B$ such that $i \star k : (A \times Z) \cup (B \times Y) \to B \times Z$ is anodyne. $\mathcal{D}$ is clearly saturated, so it suffices to show that $\mathcal{C} \subseteq \mathcal{D}$ since $\mathcal{A} = \mathcal{C}$.

Thus, let $m' : Y' \to Z'$ be a monomorphism of $\mathbf{S}$, and consider the inclusion $a \star m' : ((e) \times Z') \cup (\Delta[1] \times Y') \longrightarrow \Delta[1] \times Z'$ of $\mathcal{C}$. Then

$$(a \star m') \star k : (((e) \times Z') \cup (\Delta[1] \times Y')) \times Z \cup (\Delta[1] \times Z') \times Y \longrightarrow (\Delta[1] \times Z') \times Z$$

is isomorphic to

$$a \star (m' \star k) : ((e) \times Z' \times Z) \cup \Delta[1] \times (Y' \times Z \cup Z' \times Y) \longrightarrow \Delta[1] \times (Z' \times Z)$$

which is in $\mathcal{C}$, and hence anodyne. It follows that $a \star m'$ is in $\mathcal{D}$, which proves the theorem. ∎

Proof of Theorem 1.5.3:

$\mathcal{B} \subseteq \mathcal{A}$: Taking $e = 1$, we want to show the inclusion

$$a \star i_n : ((1) \times \Delta[n]) \cup (\Delta[1] \times \dot{\Delta}[n]) \longrightarrow \Delta[1] \times \Delta[n]$$

is anodyne. From section 2, we know that the top dimensional non-degenerate simplices of $\Delta[1] \times \Delta[n]$ correspond, under the nerve N, to the injective order-preserving maps $\sigma_j : [n+1] \to [1] \times [n]$ whose images are the maximal chains

$$((0,0), \ldots, (0,j), (1,j), \ldots, (1,n))$$

for $0 \leq j \leq n$. As to the faces of the σ_j, we see that $d^{j+1}\sigma_j = d^{j+1}\sigma_{j+1}$ with image

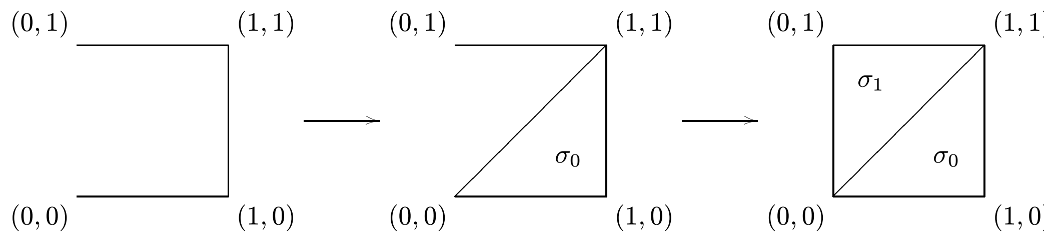
$$((0,0), \ldots, (0,j), (1,j+1), \ldots, (1,n))$$

for $0 \leq j \leq n$. Also, $d^i\sigma_j \in \Delta[1] \times \dot{\Delta}[n]$ for $i \neq j, j+1$, $d^0\sigma_0 \in (1) \times \Delta[n]$ and $d^{n+1}\sigma_n \in (0) \times \Delta[n]$.

So, we first attach σ_0 to the open prism $((1) \times \Delta[n]) \cup (\Delta[1] \times \dot{\Delta}[n])$ along $\Lambda^1[n+1]$ since all the faces $d^i\sigma_0$ except $d^1\sigma_0$ are already there. Next we attach σ_1 along $\Lambda^2[n+1]$ since now $d^1\sigma_1 = d^1\sigma_0$ is there, and only $d^2\sigma_1$ is lacking. In general, we attach σ_j along $\Lambda^{j+1}[n+1]$ since $d^j\sigma_j = d^j\sigma_{j-1}$ was attached the step before, and only $d^{j+1}\sigma_j$ is lacking. Thus, we see that the inclusion of the open prism in the prism is a composite of $n+1$ pushouts of horns. For example, the filling of the inclusion

$$a \star i_1 : ((1) \times \Delta[1]) \cup (\Delta[1] \times \dot{\Delta}[1]) \longrightarrow \Delta[1] \times \Delta[1]$$

above proceeds as follows:



When $e = 0$, we first attach σ_n, and work backwards to σ_0.

$\mathcal{C} \subseteq \mathcal{A}$: We proved above that $a \star i_n$ is anodyne. Thus, $a \star i_n$ has the left lifting property (LLP) with respect to any fibration $p : E \to Z$. By adjointness, i_n has the LLP with respect to $a|p$. But then any member of the saturated class generated by the i_n, i.e. any monomorphism $m : X \to Y$, has the LLP with respect to $a|p$. Thus, again by adjointness, $a \star m$ has the LLP with respect to p, and hence is anodyne.

$\mathcal{A} \subseteq \mathcal{C}$: For $0 \le k < n$, let $s^k : [n] \to [1] \times [n]$ be the injection $s^k(i) = (1, i)$ and $r^k : [1] \times [n] \to [n]$ the surjection given by $r^k(1, i) = i$, and $r^k(0, i) = i$ for $i \le k$, $r^k(0, i) = k$ $i \ge k$. Clearly, $r^k s^k = id_{[n]}$. It is easy to check that $Ns^k : \Delta[n] \to \Delta[1] \times \Delta[n]$ induces a map $\Lambda^k[n] \to ((0) \times \Delta[n]) \cup (\Delta[1] \times \Lambda^k[n])$ and $Nr^k : \Delta[1] \times \Delta[n] \to \Delta[n]$ a map $((0) \times \Delta[n]) \cup (\Delta[1] \times \Lambda^k[n]) \to \Lambda^k[n]$. It follows that we have a retract

$$\begin{tikzcd}
\Lambda^k[n] \arrow[r] \arrow[d] & ((0) \times \Delta[n]) \cup (\Delta[1] \times \Lambda^k[n]) \arrow[r] \arrow[d] & \Lambda^k[n] \arrow[d] \\
\Delta[n] \arrow[r] & \Delta[1] \times \Delta[n] \arrow[r] & \Delta[n]
\end{tikzcd}$$

The middle vertical map is in $\mathcal{C}$, so the horns $\Lambda^k[n] \to \Delta[n]$ for $k < n$ are in $\mathcal{C}$. For $k = n$, or in general $k > 0$, we use the inclusion $u^k : [n] \to [1] \times [n]$ given by $u^k(i) = (0, i)$ and the retraction $v^k : [1] \times [n] \to [n]$ given by $v^k(0, i) = i$, $v^k(1, i) = k$ if $i \le k$ and $v^k(1, i) = i$ for $i \ge k$. ■

An important consequence of Theorem 1.5.2 is the *covering homotopy extension property* (CHEP) for fibrations, which is the statement of the following proposition.

Proposition 1.5.2 *Let $p : E \to X$ be a fibration, and $h : Z \times I \to X$ a homotopy. Suppose that $Y \to Z$ is a monomorphism, and $h' : Y \times I \to E$ is a lifting of h to E on $Y \times I$, i.e. the diagram*

$$\begin{tikzcd}
Y \times I \arrow[r, "h'"] \arrow[d] & E \arrow[d, "p"] \\
Z \times I \arrow[r, "h"'] & X
\end{tikzcd}$$

commutes. Suppose further that $f : Z \times (e) \to E$ *is a lifting of* $h_e (e = 0, 1)$ *to* E*, i.e.*

$$\begin{tikzcd}
Z \times (e) \arrow[r, "f"] \arrow[d] & E \arrow[d, "p"] \\
Z \times I \arrow[r, "h"'] & X
\end{tikzcd}$$

commutes. Then there is a homotopy $\bar{h} : Z \times I \to E$*, which lifts* h*, i.e.* $p\bar{h} = h$*, agrees with* h' *on* $Y \times I$*, and is such that* $\bar{h}_e = f$*.*

Proof: The given data provides a commutative diagram

$$\begin{tikzcd}
(Y \times I) \cup (Z \times (e)) \arrow[r] \arrow[d] & E \arrow[d, "p"] \\
Z \times I \arrow[r, "h"'] \arrow[ur, dashed, "\bar{h}"] & X
\end{tikzcd}$$

which has a dotted lifting $\bar{h}$ by Theorem 1.5.2, since $(e) \to I$ is anodyne. ∎

We establish now several applications of the CHEP, which will be useful later. To begin, let $i : A \to B$ be a monomorphism of **S**.

Definition 1.5.3 *A is said to be a strong deformation retract of* B *if there is a retraction* $r : B \to A$ *and a homotopy* $h : B \times I \to B$ *such that* $ri = id_A$*,* $h_0 = id_B$*,* $h_1 = ir$*, and* h *is "stationary on* A*", meaning*

$$\begin{tikzcd}
A \times I \arrow[r] \arrow[dr, "\pi_1"'] & B \times I \arrow[r, "h"] & B \\
& A \arrow[ur, "i"'] &
\end{tikzcd}$$

commutes.

Proposition 1.5.3 *If* $i : A \to B$ *is anodyne, and* A *and* B *are Kan complexes, then* A *is a strong deformation retract of* B*.*

Proof: We get a retraction as a dotted filler in

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "id_A"'] & B \arrow[dl, dashed, "r"] \\ A & \end{tikzcd}$$

The required homotopy $h : B \times I \to B$ is obtained as the exponential transpose of the dotted lifting in the diagram

$$\begin{tikzcd} A \arrow[r, "\overline{i\pi_1}"] \arrow[d, "i"'] & B^I \arrow[d] \\ B \arrow[r, "(id_B, ir)"'] \arrow[ur, dashed] & B^{((0)+(1))} \end{tikzcd}$$

where the right-hand vertical mapping is a fibration by Theorem 1.5.1. ∎

Proposition 1.5.4 *If $i : A \to B$ is a monomorphism such that A is a strong deformation retract of B, then i is anodyne.*

Proof: Let $r : B \to A$ be a retraction, and $h : B \times I \to B$ a strong deformation between id_B and ir. If a diagram

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ B \arrow[r, "b"'] \arrow[ur, dashed] & X \end{tikzcd}$$

is given with p a fibration, consider

$$\begin{tikzcd} A \times I \arrow[r, "\pi_1"] \arrow[d, "i \times I"'] & A \arrow[r, "a"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ B \times I \arrow[r, "h"'] & B \arrow[r, "b"'] & X \end{tikzcd}$$

A lifting of bh at 1 is provided by $ar : B \to E$, so lift all of bh by the CHEP, and take the value of the lifted homotopy at 0 as a dotted filler in the original diagram. i is then anodyne by Corollary 1.4.1 ∎

Proposition 1.5.5 *A fibration $p : E \to X$ is trivial iff p is the dual of a strong deformation retraction. That is, iff there is an $s : X \to E$ and $h : E \times I \to E$ such that $ps = id_X$, $h_0 = id_E$, $h_1 = sp$, and*

$$\begin{tikzcd} E \times I \arrow[r, "h"] \arrow[d, "\pi_1"'] & E \arrow[d, "p"] \\ E \arrow[r, "p"'] & X \end{tikzcd}$$

commutes.

Proof: If p is trivial, construct s and h as dotted liftings in

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ X \arrow[r, "="'] \arrow[ur, dashed, "s"] & X \end{tikzcd}$$

and

$$\begin{tikzcd} E \times ((0)+(1)) \arrow[r, "(id_X, sp)"] \arrow[d] & E \arrow[d, "p"] \\ E \times I \arrow[r, "p\pi_1"'] \arrow[ur, dashed, "h"] & X \end{tikzcd}$$

On the other hand, if p is the dual of a strong deformation retraction as above, and a diagram

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ B \arrow[r, "b"'] & X \end{tikzcd}$$

is given, with i an arbitrary monomorphism, lift provisionally by sb, then adjust by the CHEP as in Proposition 1.5.4. ∎

We can use Proposition 1.5.5 to obtain

Proposition 1.5.6 *A fibration $p : E \to X$ is trivial iff p is a homotopy equivalence.*

Proof: Suppose p is a homotopy equivalence, i.e. there is a map $s : X \to E$ together with homotopies $k : X \times I \to X$ and $h : E \times I \to E$ such that $k_0 = ps$, $k_1 = id_X$, $h_0 = sp$, $h_1 = id_E$. First, let k' be a lifting in the diagram

$$\begin{tikzcd} X \times (0) \arrow[r, "s"] \arrow[d] & E \arrow[d, "p"] \\ X \times I \arrow[r, "k"'] \arrow[ur, dashed, "k'"] & X \end{tikzcd}$$

Then $s' = k'_1$ is such that $ps' = id_X$, so we may assume from the beginning that $ps = id_X$. Now we have two maps $I \to E^E$. Namely the adjoint transposes of h and sph, which agree at 0, giving a diagram

$$\begin{tikzcd} \Lambda^0[2] \arrow[r, "a"] \arrow[d] & E^E \arrow[d, "p^E"] \\ \Delta[2] \arrow[r, "b=ph\eta^1"'] \arrow[ur, dashed, "\alpha"] & X^E \end{tikzcd}$$

Since p is a fibration, we can find a dotted filler α. Then, $\alpha\varepsilon^0 = h'$ is a homotopy between id_E and sp, which is fiberwise, i.e.

$$\begin{tikzcd} E \times I \arrow[r, "h'"] \arrow[d, "\pi_1"'] & E \arrow[d, "p"] \\ E \arrow[r, "p"'] & X \end{tikzcd}$$

commutes. Thus, p is trivial by Proposition 1.5.5. The converse follows immediately from Proposition 1.5.5. ∎

Proposition 1.5.7 *Let $p : E \to X$ be a fibration, and $i : A \to X$ a monomorphism. If A is a strong deformation retract of X, then in*

$$\begin{tikzcd} p^{-1}(A) \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ A \arrow[r, "i"'] & X \end{tikzcd}$$

$p^{-1}(A)$ is a strong deformation retract of E.

Proof: Let $h : X \times I \to X$ denote the deformation of X into A. Then we have two commutative diagrams

$$\begin{tikzcd} E \times (0) \arrow[rr, "id"] \arrow[d] & & E \arrow[d, "p"] \\ E \times I \arrow[r, "p\times id"'] & X \times I \arrow[r, "h"'] & X \end{tikzcd}$$

and

$$\begin{tikzcd} p^{-1}(A) \times I \arrow[r, "\pi_1"] \arrow[d] & p^{-1}(A) \arrow[r] & E \arrow[d, "p"] \\ E \times I \arrow[r, "p\times id"'] & X \times I \arrow[r, "h"'] & X \end{tikzcd}$$

These provide a diagram

$$\begin{tikzcd}
(p^{-1}(A)\times I)\cup(E\times(0)) \arrow[rr] \arrow[d] & & E \arrow[d, "p"] \\
E\times I \arrow[r, "p\times id"'] \arrow[urr, dashed, "k"] & X\times I \arrow[r, "h"'] & X
\end{tikzcd}$$

A dotted lifting k, which exists by the CHEP, provides a deformation of E into $p^{-1}(A)$. ∎

As applications of Proposition 1.5.7 we have the following.

Corollary 1.5.2 *Let $p : E \to X \times I$ be a fibration. Denote $p^{-1}(X \times (0))$ and $p^{-1}(X \times (1))$ by $p_0 : E_0 \to X$ and $p_1 : E_1 \to X$. Then p_0 and p_1 are fiberwise homotopy equivalent. That is, there are mappings*

$$\begin{tikzcd}
E_0 \arrow[rr, "f"] \arrow[dr, "p_0"'] & & E_1 \arrow[dl, "p_1"] \\
& X &
\end{tikzcd}$$

and

$$\begin{tikzcd}
E_1 \arrow[rr, "g"] \arrow[dr, "p_1"'] & & E_0 \arrow[dl, "p_0"] \\
& X &
\end{tikzcd}$$

together with homotopies $h : E_0 \times I \to E_0$ and $k : E_1 \times I \to E_1$ such that $h_0 = id_{E_0}$, $h_1 = gf$, $k_0 = id_{E_1}$, $k_1 = fg$ and the diagrams

$$\begin{tikzcd}
E_0\times I \arrow[r, "h"] \arrow[d, "\pi_1"'] & E_0 \arrow[d, "p_0"] \\
E_0 \arrow[r, "p_0"'] & X
\end{tikzcd}$$

and

$$\begin{tikzcd}
E_1\times I \arrow[r, "k"] \arrow[d, "\pi_1"'] & E_1 \arrow[d, "p_1"] \\
E_1 \arrow[r, "p_1"'] & X
\end{tikzcd}$$

commute.

Proof: $X \times (0)$ and $X \times (1)$ are both strong deformation retracts of $X \times I$. Thus, by Proposition 1.5.7, E_0 and E_1 are strong deformation retracts of E.

The inclusions and retractions of E_0 and E_1 yield homotopy equivalences f and g. It follows easily from the proof of Proposition 1.5.7 that the homotopies h and k are fiberwise. ∎

Corollary 1.5.3 *Let $p : E \to Y$ be a fibration and $f, g : X \to Y$ two homotopic maps Then the pullbacks $f^*(E)$ and $g^*(E)$ are fiberwise homotopy equivalent.*

Proof: It suffices to consider the case of a homotopy $h : X \times I \to Y$ such that $h_0 = f$ and $h_1 = g$. For this, take the pullback

$$\begin{tikzcd} h^*(E) \arrow[r] \arrow[d, "q"'] & E \arrow[d, "p"] \\ X \times I \arrow[r, "h"'] & Y \end{tikzcd}$$

and apply Corollary 1.5.2 to q. ∎

Corollary 1.5.4 *Let X be connected and $p : E \to X$ a fibration. Then any two fibers of p are homotopy equivalent.*

Proof: It is enough to show that the fibers over the endpoints of any path $\alpha : I \to X$ are homotopy equivalent. For this, apply Corollary 1.5.3 to the inclusion of the endpoints of α. ∎

1.6 Minimal complexes

Let X be a simplicial set and $x, y : \Delta[n] \to X$ two n-simplices of X such that $x|\dot{\Delta}[n] = y|\dot{\Delta}[n] = a$. We say *$x$ is homotopic to y mod $\dot{\Delta}[n]$*, written $x \sim y \, mod \, \dot{\Delta}[n]$, if there is a homotopy $h : \Delta[n] \times I \to X$ such that $h_0 = x$, $h_1 = y$ and h is "stationary on $\dot{\Delta}[n]$", meaning

$$\begin{tikzcd} \dot{\Delta}[n] \times I \arrow[r, "\pi_1"] \arrow[d] & \dot{\Delta}[n] \arrow[d, "a"] \\ \Delta[n] \times I \arrow[r, "h"'] & X \end{tikzcd}$$

commutes. It is easy to see that $x \sim y \, mod \, \dot{\Delta}[n]$ is an equivalence relation when X is a Kan complex.

Definition 1.6.1 *Let X be a Kan complex. X is said to be minimal if $x \sim y \, mod \, \dot{\Delta}[n]$ entails $x = y$.*

Our main goal in this section is to prove the following theorem, and its corresponding version for fibrations.

Theorem 1.6.1 *Let X be a Kan complex. Then there exists a strong deformation retract X' of X which is minimal.*

In the proof of Theorem 1.6.1 we will need a lemma.

Lemma 1.6.1 *Let x and y be two degenerate n-simplices of a simplicial set X. Then $x|\dot{\Delta}[n] = y|\dot{\Delta}[n]$ implies $x = y$.*

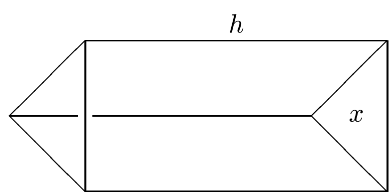
Proof: Let $x = s^i d^i x$ and $y = s^j d^j y$. If $i = j$, we are done. If, say, $i < j$, we have $x = s^i d^i x = s^i d^i y = s^i d^i s^j d^j y = s^i s^{j-1} d^i d^j y = s^j s^i d^i d^j y$. Thus, $x = s^j z$, where $z = s^i d^i d^j y$. Hence, $d^j x = d^j s^j z = z$ and $x = s^j d^j x = s^j d^j y = y$. ∎

Proof of Theorem 1..1: We construct X' skeleton by skeleton. For $Sk^0 X'$ take one representative in each equivalence class of $\pi_0(X)$. Suppose we have defined $Sk^{n-1}X'$. To define $Sk^n X'$ we take one representative in each equivalence class among those n-simplices of X whose restrictions to $\dot{\Delta}[n]$ are contained in $Sk^{n-1}X'$, choosing a degenerate one wherever possible. Lemma 1.6.1 shows that X'_n contains all degenerate simplices from X'_{n-1}.

For the deformation retraction, suppose we have $h : Sk^{n-1}X \times I \to X$. Consider the pushout

$$\begin{tikzcd}
\sum_{e(X)_n} \dot{\Delta}[n] \times I \arrow[r] \arrow[d] & \sum_{e(X)_n} \Delta[n] \times I \arrow[d] \\
Sk^{n-1}X \times I \arrow[r] & Sk^n X \times I
\end{tikzcd}$$

To extend h to $Sk^n X \times I$ we must define it on each $\Delta[n] \times I$ consistent with its given value on $\dot{\Delta}[n] \times I$. Thus, let $x \in e(X)_n$. Since h is already defined on the boundary of x, we have an open prism



in X whose $(n-1)$-simplices in the open end belong to X'. Since X is Kan, we can fill the prism obtaining a new n-simplex y at the other end whose boundary is in X'. Now take a homotopy mod $\dot{\Delta}[n]$ to get into X'. This defines the retraction r and the homotopy h. ∎

Theorem 1.6.2 *Let X and Y be minimal complexes and $f : X \to Y$ a homotopy equivalence. Then f is an isomorphism.*

The proof of Theorem 1.6.2 follows immediately from the following lemma.

Lemma 1.6.2 *Let X be a minimal complex and $f : X \to X$ a map homotopic to id_X. Then f is an isomorphism.*

Proof: We show first that $f_n : X_n \to X_n$ is injective by induction on n, letting X_{-1} be empty. Thus, let $x, x' : \Delta[n] \to X$ be such that $f(x) = f(x')$. By induction, $x|\dot{\Delta}[n] = x'|\dot{\Delta}[n] = a$. Let $h : X \times I \to X$ satisfy $h_0 = f$ and $h_1 = id_X$. Then the homotopy $h(x \times I)$ is $f(x)$ at 0 and x at 1. Similarly, $h(x' \times I)$ is $f(x')$ at 0 and x' at 1. Since $f(x) = f(x')$, we obtain a map $\Delta[n] \times \Lambda^0[2] \to X$. Let $\dot{\Delta}[n] \times \Delta[2] \to X$ be the map $h(a \times \eta^1)$. These agree on the intersection of their domains, so we obtain a diagram

$$\begin{tikzcd}
(\Delta[n] \times \Lambda^0[2]) \cup (\dot{\Delta}[n] \times \Delta[2]) \arrow[r] \arrow[d] & X \\
\Delta[n] \times \Delta[2] \arrow[ur, dashed, "k"] &
\end{tikzcd}$$

which has a dotted filler k by Theorem 1.5.2. $k(id \times \varepsilon^0)$ is a homotopy between x and x' *mod* $\dot{\Delta}[n]$. Since X is minimal, $x = x'$.

Now assume that $f_m : X_m \to X_m$ is surjective for $m < n$, and let $x : \Delta[n] \to X$ be an n-simplex of X. By induction, and the first part of the proof, each $x\varepsilon^i$ is uniquely of the form $f(y_i)$ for $y_i : \Delta[n-1] \to X$. Hence, we obtain a map $y : \dot{\Delta}[n] \to X$ such that $f(y) = x|\dot{\Delta}[n]$. The maps $h(a \times I)$ and $x \times (0)$ agree on their intersections giving a diagram

$$\begin{tikzcd}
(\Delta[n] \times (0)) \cup (\dot{\Delta}[n] \times I) \arrow[r] \arrow[d] & X \\
\Delta[n] \times I \arrow[ur, dashed, "k"] &
\end{tikzcd}$$

which has a dotted filler k. k at 0 is x, and k at 1 is some n-simplex z. The homotopy $h(z \times I)$ is $f(z)$ at 0 and z at 1. Thus, as above, we obtain a diagram

$$\begin{tikzcd}
(\Delta[n] \times \Lambda^1[2]) \cup (\dot{\Delta}[n] \times \Delta[2]) \arrow[r] \arrow[d] & X \\
\Delta[n] \times \Delta[2] \arrow[ur, dashed, "l"] &
\end{tikzcd}$$

which has a dotted filler l by Theorem 1.4.1. $k(id \times \varepsilon^1)$ is a homotopy between x and $f(z)$ *mod* $\dot{\Delta}[n]$. Since X is minimal, $x = f(z)$. ∎

We discuss now the corresponding matters for fibrations. Thus, let $p : E \to X$ be a map and $e, e' : \Delta[n] \to E$ two n-simplices of E such that $e|\dot{\Delta}[n] = e'|\dot{\Delta}[n] = a$, and $pe = pe' = b$. We say *e is fiberwise homotopic to e' mod* $\dot{\Delta}[n]$, written $e \sim_f e'$ *mod* $\dot{\Delta}[n]$, if there is a homotopy $h : \Delta[n] \times I \to E$ such that $h_0 = e$, $h_1 = e'$, $h = a$ on $\dot{\Delta}[n]$ as before, meaning

$$\begin{tikzcd}
\dot{\Delta}[n] \times I \arrow[r, "\pi_1"] \arrow[d] & \dot{\Delta}[n] \arrow[d, "a"] \\
\Delta[n] \times I \arrow[r, "h"'] & X
\end{tikzcd}$$

commutes, and h is "fiberwise", meaning

$$\begin{tikzcd}
\Delta[n] \times I \arrow[r, "h"] \arrow[d, "\pi_1"'] & E \arrow[d, "p"] \\
\Delta[n] \arrow[r, "b"'] & X
\end{tikzcd}$$

commutes. As before, it is easy to see that $e \sim_f e' \; mod \; \dot{\Delta}[n]$ is an equivalence relation when p is a fibration.

Definition 1.6.2 *A fibration $p : E \to X$ is said to be minimal if $e \sim_f e' mod \dot{\Delta}[n]$ entails $e = e'$.*

Notice that minimal fibrations are stable under pullback.

Theorem 1.6.3 *Let $p : E \to X$ be a fibration. Then there is a subcomplex E' of E such that p restricted to E' is a minimal fibration $p' : E' \to X$ which is a strong, fiberwise deformation retract of p.*

Proof: The proof is essentially the same as that for Theorem 1.5.1, with $x \sim y \; mod \; \dot{\Delta}[n]$ replaced by $e \sim_f e' \; mod \; \dot{\Delta}[n]$. ∎

Theorem 1.6.4 *Let $p : E \to X$ and $q : G \to X$ be minimal fibrations and $f : E \to G$ a map such that $qf = p$. Then if f is a fiberwise homotopy equivalence, f is an isomorphism.*

Proof: Again, the proof is essentially the same as that for Theorem 1.5.2, with $x \sim y \; mod \; \dot{\Delta}[n]$ replaced by $e \sim_f e' \; mod \; \dot{\Delta}[n]$. ∎

Theorem 1.6.5 *A minimal fibration is a bundle.*

Proof: Let $p : E \to X$ be a minimal fibration, and $x : \Delta[n] \to X$ an n-simplex of X. Pulling back p along x yields a minimal fibration over $\Delta[n]$ so it suffices to show that any minimal fibration $p : E \to \Delta[n]$ is isomorphic to one of the form $\pi_1 : \Delta[n] \times F \to \Delta[n]$.

For this, define $c : [n] \times [1] \to [n]$ as follows: $c(i, 0) = i$ and $c(i, 1) = n$ for $0 \leq i \leq n$. $Nc = h$ is a homotopy $\Delta[n] \times I \to \Delta[n]$ between the identity of $\Delta[n]$ and the constant map at the n^{th} vertex of $\Delta[n]$. From Corollary 1.5.3 it follows that $p : E \to \Delta[n]$ is fiberwise homotopy equivalent to $\pi_1 : \Delta[n] \times F \to \Delta[n]$ where F is the fiber of p over the n^{th} vertex of $\Delta[n]$. By Theorem 1.6.4, p is isomorphic to π_1 over $\Delta[n]$. ∎

1.7 The Quillen homotopy structure

Here we establish Theorem 1.3.1, or rather a modified version thereof.

Definition 1.7.1 *Let $f : X \to Y$ be a mapping in* $\mathbf{S}$*. We say f is a weak equivalence if for each Kan complex K, $[f, K] : [Y, K] \to [X, K]$ is a bijection.*

For example, a homotopy equivalence is a weak equivalence. Thus, a trivial fibration is a weak equivalence by Proposition 1.5.6. Or, if $i : A \to B$ is anodyne and K is Kan, then $K^i : K^B \to K^A$ is a trivial fibration by Theorem 1.5.1, hence a homotopy equivalence by Proposition 1.5.6. Thus, $\pi_0(K^i) = [i, K]$ is a bijection and i is a weak equivalence. Also, if $f : X \to Y$ is a weak equivalence with X and Y Kan, then f is a homotopy equivalence since f becomes an isomorphism in $ho(\mathbf{S})$.

Notice that Definition 1.7.1 is equivalent to saying that if $X \to \overline{X}$ and $Y \to \overline{Y}$ are anodyne extensions with $\overline{X}$ and $\overline{Y}$ Kan, and $\overline{f}$ is any dotted filler in

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & \overline{X} \arrow[d, dashed, "\overline{f}"] \\ Y \arrow[r] & \overline{Y} \end{tikzcd}$$

then $\overline{f}$ is a homotopy equivalence.

Now, in $\mathbf{S}$ we take as *fibrations* the Kan fibrations, as *cofibrations* the monomorphisms, and as *weak equivalences* the ones given in Definition 1.7.1. Then the main theorem of this chapter is the following.

Theorem 1.7.1 *The fibrations, cofibrations and weak equivalences defined above form a proper Quillen homotopy structure on* $\mathbf{S}$*.*

Remark: We will show in section 10 that the weak equivalences of Definition 1.7.1 coincide with the goemetric homotopy equivalences of section 3, so that Theorem 1.7.1 is, in fact, the same as Theorem 1.3.1.

The proof of Theorem 1.7.1 is based on the following two propositions, which we establish first.

Proposition 1.7.1 *A fibration $p : E \to X$ is trivial iff p is a weak equivalence.*

Proposition 1.7.2 *A cofibration $i : A \to B$ is anodyne iff i is a weak equivalence.*

Proof of Proposition 1.7.1: Let $p : E \to X$ be a fibration and a weak equivalence. By Theorem 1.6.3 there is a minimal fibration $p' : E' \to X$ which is a strong deformation retract of p, and hence also a weak equivalence. Let $X \to \overline{X}$ be an anodyne extension with $\overline{X}$ Kan (Theorem 1.4.1). By Theorem 1.6.5, p' is a bundle. Using Lemma 1.7.1 below, we can extend p' uniquely to a bundle $\overline{p} : \overline{E} \to \overline{X}$ in such a way as to have a pullback

$$\begin{tikzcd} E' \arrow[r] \arrow[d, "p'"'] & \overline{E} \arrow[d, "\overline{p}"] \\ X \arrow[r] & \overline{X} \end{tikzcd}$$

with $E' \to \overline{E}$ anodyne. Now $\overline{p}$ is a fibration since its fiber is a minimal Kan complex, so $\overline{E}$ is Kan. Furthermore, $\overline{p}$ is a weak equivalence since all the other maps in the diagram are. Thus, $\overline{p}$ is a homotopy equivalence. So, by Proposition 1.5.6, $\overline{p}$ is a trivial fibration. It follows that p' is also trivial, hence a homotopy equivalence. But this shows that p is also a homotopy equivalence, and thus a trivial fibration. ∎

Proof of Proposition 1.7.2: Let $i : A \to B$ be a cofibration and a weak equivalence. Factor i as

$$\begin{tikzcd} A \arrow[rr, "j"] \arrow[dr, "i"'] & & E \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

where j is anodyne and p is is a fibration. p is a weak equivalence since i and j are. Thus, by Proposition 1.7.1, p is a trivial fibration. But then there is a dotted lifting s in

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ B \arrow[r, "id_B"'] \arrow[ur, dashed, "s"] & B \end{tikzcd}$$

so that i is a retract of j and hence anodyne. ∎

Lemma 1.7.1 *Let $A \to B$ be an anodyne extension and $p : E \to A$ a bundle. Then there is a pullback diagram*

$$\begin{tikzcd} E \arrow[r] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ A \arrow[r] & B \end{tikzcd}$$

such that $p' : E' \to B$ is a bundle, and $E \to E'$ is anodyne. Furthermore, such an extension p' of p is unique up to isomorphism.

Proof: Let $\mathcal{E}$ be the class of all monomorphisms having the unique extension property above. We will show that $\mathcal{E}$ contains the horn inclusions and is saturated, hence contains all anodyne extensions.

For the horn inclusions, let C be a *contractible* simplicial set, i.e. one provided with an anodyne map $c : 1 \to C$. Let $p : E \to C$ be a principal G-bundle over C. Picking a point $e : 1 \to E$ such that $pe = c$, we have a dotted lifting in the diagram

$$\begin{tikzcd} 1 \arrow[r, "e"] \arrow[d, "x"'] & E \arrow[d, "p"] \\ C \arrow[r, "id_C"'] \arrow[ur, dashed] & C \end{tikzcd}$$

so that p has a section and hence is trivial. In case $p : E \to C$ is a bundle with fiber F, $Iso(C \times F, E) \to C$ is a principal $Aut(F)$-bundle, and hence has a section. But such a section is a trivialization of p. Thus, any bundle over C is trivial. In particular, any bundle $p : E \to \Lambda^k[n]$ is trivial (the kth vertex is an anodyne point), and hence can be extended uniquely as a trivial bundle over $\Lambda^k[n] \to \Delta[n]$.

$\mathcal{E}$ clearly contains all the isomorphisms. Let us see that its maps are stable under pushout. Thus, let $A \to B$ be in $\mathcal{E}$, and let $A \to A'$ be an arbitrary map. Form the pushout

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A' \arrow[d] \\ B \arrow[r] & B' \end{tikzcd}$$

and suppose $p' : E' \to A'$ is a bundle. Pull back p' to a bundle $p : E \to A$ and extend p as

$$\begin{tikzcd} E \arrow[r] \arrow[d, "p"'] & G \arrow[d, "q"] \\ A \arrow[r] & B \end{tikzcd}$$

with $E \to G$ anodyne since $A \to B$ is in $\mathcal{E}$. Now form the pushout

$$\begin{tikzcd} E \arrow[r] \arrow[d] & E' \arrow[d] \\ G \arrow[r] & G' \end{tikzcd}$$

giving a cube

$$\begin{tikzcd}
& E \arrow[rr] \arrow[dl] \arrow[dd] & & E' \arrow[dl] \arrow[dd] \\
G \arrow[rr, crossing over] \arrow[dd] & & G' & \\
& A \arrow[rr] \arrow[dl] & & A' \arrow[dl] \\
B \arrow[rr] & & B' \arrow[from=uu, crossing over] &
\end{tikzcd}$$

where $q' : G' \to B'$ is the natural induced map. In this cube, the left-hand and back faces are pullbacks.Hence, by the lemma following this proof, we can conclude that the front and right-hand faces are also. Thus, the pullback of q' over the surjection $B + A' \to B'$ is the bundle $q + p' : G + E' \to B + A'$. It follows that q' is a bundle and unique, for any other bundle which extends p' must pull back over $B \to B'$ to q by the uniqueness of q. Clearly, $E' \to G'$ is anodyne.

Now let

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d] & A \arrow[d] \\
B \arrow[r] & B' \arrow[r] & B
\end{tikzcd}$$

be a retract with $A' \to B'$ in $\mathcal{E}$. Pushing out $A' \to B'$ along $A' \to A$, and using the stability of $\mathcal{E}$ under pushouts, we see that it is enough to consider retracts of the form

$$\begin{tikzcd}
& A \arrow[dl] \arrow[dr] & \\
B' \arrow[rr, shift left, "r"] & & B \arrow[ll, shift left, "i"]
\end{tikzcd}$$

$ri = id_B$, with $A \to B'$ in $\mathcal{E}$. Thus, let $p : E \to A$ be a bundle. Extend p to $p' : E' \to B'$, then take the pullback $p'' : E'' \to B$ of p' along i, yielding a diagram of pullbacks

$$\begin{tikzcd}
E \arrow[r] \arrow[d, "p"'] & E' \arrow[d, "p'"'] & E'' \arrow[l] \arrow[d, "p''"] \\
A \arrow[r] & B' & B \arrow[l, "i"']
\end{tikzcd}$$

with $E \to E'$ anodyne. Pulling back p'' back along $A \to B$ gives the pullback of p' along $A \to B'$, i.e. p. Thus, p'' is a bundle extending p. But any bundle over B which extends p pulls back along r to a bundle over B' which extends p, so it must be p' by uniqueness. Thus p'' is unique and we have a retract

$$\begin{tikzcd}
 & E \arrow[dl] \arrow[dr] & \\
E' \arrow[rr, shift left] & & E'' \arrow[ll, shift left]
\end{tikzcd}$$

so that $E \to E''$ is anodyne.

We leave the straightforward verification of coproducts and countable composites as an excercise for the reader. ∎

Lemma 1.7.2 *Let*

$$\begin{tikzcd}
 & E \arrow[rr] \arrow[dl] \arrow[dd] & & E' \arrow[dl] \arrow[dd] \\
G \arrow[rr, crossing over] \arrow[dd] & & G' & \\
 & A \arrow[rr] \arrow[dl] & & A' \arrow[dl] \\
B \arrow[rr] & & B' \arrow[from=uu, crossing over] &
\end{tikzcd}$$

be a commutative cube in **S**, *whose left-hand and back faces are pullbacks. If $A \to B$ is a monomorphism, and the top and bottom faces are pushouts, then the right-hand and front faces are pullbacks.*

Proof: It is enough to prove the lemma in the category of sets. In that case, the right-hand and front faces are

$$\begin{tikzcd}
E' \arrow[r] \arrow[d] & E' + (G - E) \arrow[d] \\
A' \arrow[r] & A' + (B - A)
\end{tikzcd}$$

and

$$\begin{tikzcd}
E + (G - E) \arrow[r] \arrow[d] & E' + (G - E) \arrow[d] \\
A + (B - A) \arrow[r] & A' + (B - A)
\end{tikzcd}$$

In these diagrams, $G - E$ maps to $B - A$ since the left-hand face of the cube is a pullback. Thus, these two faces are the coproducts of

$$\begin{tikzcd}
E' \arrow[r, "id"] \arrow[d] & E' \arrow[d] \\
A' \arrow[r, "id"] & A'
\end{tikzcd}$$

and

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & (G-E) \arrow[d] \\ 0 \arrow[r] & (B-A) \end{tikzcd}$$

and

$$\begin{tikzcd} E \arrow[r] \arrow[d] & E' \arrow[d] \\ A \arrow[r] & A' \end{tikzcd}$$

and

$$\begin{tikzcd} (G-E) \arrow[r, "id"] \arrow[d] & (G-E) \arrow[d] \\ (B-A) \arrow[r, "id"] & (B-A) \end{tikzcd}$$

respectively. Now use the fact that the coproduct of two pullbacks in Sets is a pullback. ∎

Remark: Notice that Lemma 1.7.1 shows that weak equivalences are stable under pullback along a bundle. In fact, let $w : A \to B$ be a weak equivalence and $p' : E' \to B$ a bundle. Factor w as $w = pi$ where i is a cofibration and p is a trivial fibration (Theorem 1.4.2). i is a weak equivalence since w and p are, hence anodyne by Proposition 1.7.2. Trivial fibrations are stable under any pullback, and anodyne extensions are stable under pullback along a bundle by Lemma 1.7.1, so the result follows.

Proof of Theorem 1.7.1: Q1 and Q2 are clear. Q3 follows immediately from Propositions 1.7.1 and 1.7.2, and Q4 follows from Theorems 1.4.1 and 1.4.2.

For Q5, let $w : A \to B$ be a weak equivalence and $p : E \to B$ a fibration. By Theorem 1.6.3 there is a minimal fibration $p_0 : E_0 \to B$ which is a strong fiberwise deformation retract of p. Let $p' : E' \to A$ be the pullback of p along w and $p'_0 : E'_0 \to A$ the pullback of p_0. Then p'_0 is a strong fiberwise deformation retract of p', and the map $E'_0 \to E_0$ is a weak equivalence by the remark following Lemma 1.7.1. Thus $E' \to E$ is a weak equivalence. Dually, let

$$\begin{tikzcd} A \arrow[r, "w"] \arrow[d, "i"'] & B \arrow[d, "i'"] \\ A' \arrow[r, "w'"'] & B' \end{tikzcd}$$

be a pushout diagram with w a weak equivalence and i a cofibration. If K is a Kan complex, the diagram

$$\begin{tikzcd} K^{B'} \arrow[r, "K^{i'}"] \arrow[d, "K^{w'}"'] & K^B \arrow[d, "K^w"] \\ K^{A'} \arrow[r, "K^i"'] & K^A \end{tikzcd}$$

is a pullback with K^i a fibration by Theorem 1.5.3. K^w is a weak equivalence (factor w as a cofibration followed by a trivial fibration to see this) so $K^{w'}$ is a weak equivalence by the first part of Q5. Thus w' is a weak equivalence, which proves the theorem. ∎

1.8 Homotopy groups and Whitehead's Theorem

Let $\dot{\Delta}[n] \to \Delta[n]$, $n \geq 1$, be the inclusion of the boundary, and

$$\begin{tikzcd} \dot{\Delta}[n] \arrow[r] \arrow[d] & \Delta[n] \arrow[d] \\ 1 \arrow[r, "b"'] & \dot{\Delta}[n]/\Delta[n] \end{tikzcd}$$

a pushout. If X is a Kan complex, and $x \in X_0$, we write $\pi_n(X, x)$, $n \geq 1$, for the set of homotopy classes of maps $\dot{\Delta}[n]/\Delta[n] \to X$, which take b to x, modulo homotopies which are constantly equal to x at b. At the moment, the $\pi_n(X, x)$ are simply pointed sets, the point being the class $[x]$ of the constant map at x. In our final version we will, in fact, show combinatorially that the $\pi_n(X, x)$, $n \geq 1$, are groups, which are abelian for $n \geq 2$. We will not use this here, however, though we remark that in section 10 we will show that $\pi_n(X, x) \simeq \pi_n(|X|, |x|)$ so this will follow, albeit unsatisfactorially.

Let $p : E \to X$ be a fibration with X Kan, and write $i : F \to E$ for the inclusion $p^{-1}(x) \to E$. Let $e \in F_0$. If $\alpha : \Delta[n] \to X$ represents an element of $\pi_n(X, x)$, let γ denote a dotted filler in the diagram

$$\begin{tikzcd} \Lambda^0[n] \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ \Delta[n] \arrow[r, "\alpha"'] \arrow[ur, dashed, "\gamma"] & X \end{tikzcd}$$

where $\Lambda^0[n] \to E$ is constant at e. Then $d^0\gamma$ sends $\dot{\Delta}[n-1]$ to x, and is independent up to homotopy of the choice of γ and the choice of α. It thus defines a function

$$\partial : \pi_n(X, x) \to \pi_{n-1}(F, e)$$

called the *boundary map.*

Theorem 1.8.1 *The boundary map is a homomorphism for $n \geq 2$, and the sequence*

$$\ldots \to \pi_n(F, e) \stackrel{\pi_n i}{\to} \pi_n(E, e) \stackrel{\pi_n p}{\to} \pi_n(X, x) \stackrel{\partial}{\to} \ldots \pi_1(X, x) \stackrel{\partial}{\to} \pi_0(F) \stackrel{\pi_0 i}{\to} \pi_0(E) \stackrel{\pi_0 p}{\to} \pi_0(X)$$

is exact as a sequence of pointed sets, in the sense that for each map, the set of elements of its domain which are sent to the point is the image of the preceding map.

Again, we will prove this in detail combinatorially in the final version. We remark only that, as above, it will follow later from Quillen's theorem that $|p|$ is a Serre fibration, which is proved in Appendix D.

Let $f : X \to Y$ be a mapping of simplicial sets. We will call f, just in this chapter, a *homotopy isomorphism* if $\pi_0 f : \pi_0(X) \to \pi_0(Y)$ is a bijection, and $\pi_n f : \pi_n(X, x) \to \pi_n(Y, fx)$ is an isomorphism for $n \geq 1$ and $x \in X_0$.

Lemma 1.8.1 *Let X be a minimal Kan complex such that $X \to 1$ is a homotopy isomorphism. Then $X \to 1$ is an isomorphism.*

Proof: Clearly, $X \to 1$ induces an isomorphism $Sk^{-1}X \to Sk^{-1}1$. Suppose $Sk^{n-1}X \to Sk^{n-1}1$ is an isomorphism, with inverse represented by a basepoint $x : 1 \to X$. Let $\sigma : \Delta[n] \to X$ be an n-simplex of X. $\sigma|\dot{\Delta}[n] = x$, so σ represents an element of $\pi_n(X, x)$. $\pi_n(X, x) = [x]$, so $\sigma \sim x \; mod \dot{\Delta}[n]$. But then $\sigma = x$. ∎

Corollary 1.8.1 *Let X be a Kan complex, and $p : E \to X$ a minimal fibration. If p is a homotopy isomorphism, then p is an isomorphism.*

Proof: The homotopy exact sequence of p as above has the form

$$\ldots \to \pi_{n+1}(E, e) \stackrel{\simeq}{\to} \pi_{n+1}(X, pe) \to \pi_n(F, e) \to \pi_n(E, e) \stackrel{\simeq}{\to} \pi_n(X, pe) \ldots$$

$$\pi_1(E, e) \stackrel{\simeq}{\to} \pi_1(X, pe) \to \pi_0(F) \to \pi_0(E) \stackrel{\simeq}{\to} \pi_0(X)$$

It follows that $\pi_0(F) = [e]$ and $\pi_n(F, e) = [e]$ for $n \geq 1$. By Lemma 1.8.1, the fiber F over each component of X is a single point, so p is a bijection. ∎

Theorem 1.8.2 *(Whitehead) Let X and Y be Kan complexes and $f : X \to Y$ a homotopy isomorphism. Then f is a homotopy equivalence.*

Proof: Factor f as

$$\begin{tikzcd} X \arrow[rr, "i"] \arrow[dr, "f"'] & & E \arrow[dl, "p"] \\ & Y & \end{tikzcd}$$

where i is anodyne and p is a fibration. Y is Kan, so E is and i is a strong deformation retract by Proposition 1.5.3. By Theorem 1.6.3, let

$$\begin{tikzcd} E' \arrow[rr] \arrow[dr, "p'"'] & & E \arrow[dl, "p"] \\ & Y & \end{tikzcd}$$

be a minimal fibration which is a strong, fiberwise deformation retract of p. Since p induces isomorphisms on π_n for $n \geq 0$ so does p'. Thus p' is an isomorphism by Corollary 1.8.1. It follows that p is a homotopy equivalence, so the same is true of f. ∎

1.9 Milnor's Theorem

Our goal in this section is to prove the following theorem.

Theorem 1.9.1 *(Milnor) Let X be a Kan complex, and let $\eta X : X \to s|X|$ be the unit of the adjunction $|\ | \dashv s$. Then ηX is a homotopy equivalence.*

The proof of Theorem 1.9.1 uses some properties of the path space of X, so we establish these first. To begin, since $(0) + (1) \to I$ is a cofibration and X is a Kan complex, $(p_0, p_1) : X^I \to X \times X$ is a fibration. Let $x : 1 \to X$ be a basepoint. Define PX as the pullback

$$\begin{tikzcd} PX \arrow[r] \arrow[d, "p_1"'] & X^I \arrow[d, "(p_0,p_1)"] \\ 1 \times X \arrow[r, "x \times id_X"'] & X \times X \end{tikzcd}$$

$(0) \to I$ is anodyne, so $p_0 : X^I \to X$ is a trivial fibration, again by Theorem 1.5.1. The diagram

$$\begin{tikzcd} PX \arrow[r] \arrow[d] & X^I \arrow[d, "p_0"] \\ 1 \arrow[r, "x"'] & X \end{tikzcd}$$

is a pullback, so $PX \to 1$ is a trivial fibration and hence a homotopy equivalence. Let ΩX denote the fiber of p_1 over x, and write x again for the constant path at x. Then the homotopy exact sequence of the fibration p_1 has the form

$$\ldots \to \pi_n(PX, x) \to \pi_n(X, x) \xrightarrow{\partial} \pi_{n-1}(\Omega X, x) \to \pi_{n-1}(PX, x) \ldots$$

$$\pi_1(PX, x) \to \pi_1(X, x) \xrightarrow{\partial} \pi_0(\Omega X) \to \pi_0(PX) \to \pi_0(X)$$

where $\pi_0(PX) = [x]$ and $\pi_n(PX, x) = [x]$ for $n \geq 1$. It follows that the boundary map induces isomorphisms $\pi_1(X, x) \to \pi_0(\Omega X)$ and $\pi_n(X, x) \to \pi_{n-1}(\Omega X, x)$ for $n \geq 2$.

Proof of Theorem 1.9.1: First, let X be connected. Then any two vertices of $|X|$ can be joined by a path. But then any two points of $|X|$ can be joined by a path, since any point is in the image of the realization of a simplex, which is connected. Thus $|X|$ is connected, as is $s|X|$. Otherwise, X is the coproduct of its connected components. Since $s|\ |$ preserves coproducts, it follows that $\pi_0\eta X : \pi_0(X) \to \pi_0(s|X|)$ is a bijection. Assume by induction that for any Y, and any $y \in Y_0$ $\pi_m\eta X : \pi_m(Y, y) \to \pi_m(s|Y|, |y|)$ is an isomorphism for $m \leq n-1$. By naturality we have a diagram

$$\begin{tikzcd}
\Omega X \arrow[r, "\eta\Omega X"] \arrow[d] & s|\Omega X| \arrow[d] \\
PX \arrow[r, "\eta PX"] \arrow[d, "p_1"'] & s|PX| \arrow[d, "s|p_1|"] \\
X \arrow[r, "\eta X"'] & s|X|
\end{tikzcd}$$

By Quillen's theorem (see Appendix B), $|p_1|$ is a Serre fibration, so $s|p_1|$ is a Kan fibration. Since $PX \to 1$ is a homotopy equivalence, so is $s|PX| \to 1$. Hence we have a commutative diagram

$$\begin{tikzcd}
\pi_n(X, x) \arrow[r, "\pi_n\eta X"] \arrow[d, "\simeq"'] & \pi_n(s|X|, |x|) \arrow[d, "\simeq"] \\
\pi_{n-1}(\Omega X, x) \arrow[r, "\pi_{n-1}\eta\Omega X"'] & \pi_{n-1}(s|\Omega X|, x)
\end{tikzcd}$$

$\pi_{n-1}\eta\Omega X$ is an isomorphism by induction, so $\pi_n\eta X$ is an isomorphism. By Whitehead's Theorem, ηX is a homotopy equivalence. ∎

An entirely similar argument, using the topological path space, shows that if T is a topological space, then the counit $\varepsilon T : |sT| \to T$ is a topological weak equivalence. Thus, if T is a CW-complex εT is a homotopy equivalence by the topological Whitehead Theorem. Since $|\ |$ and s both clearly preserve the homotopy relation between maps, we see that they induce an equivalence

$$ho(Top_c) \rightleftarrows ho(\mathbf{S})$$

where $ho(Top_c)$ is the category of CW-complexes and homotopy classes of maps.

1.10 Some remarks on weak equivalences

Here we collect all the possible definitions we might have given for weak equivalence, and show they are all the same. We begin with a lemma.

Lemma 1.10.1 *If $j : C \to D$ is a mapping in Top_c which has the left lifting property with respect to the Serre fibrations, then C is a strong deformation retreact of D.*

Proof: Δ_n is a retract of $\Delta_n \times I$, so every space in Top_c is fibrant. Also, if T is a space and X a simplicial set, we see easily that $s(T^{|X|}) \simeq (sT)^X$. From this it follows that the singular complex of $(p_0, p_1) : T^I \to T \times T$ is a Kan fibration, so it itself is a Serre fibration. Now we obtain the retraction r as a lifting in

$$\begin{tikzcd} C \arrow[r, "id_C"] \arrow[d, "j"'] & C \\ D \arrow[ur, dashed, "r"'] & \end{tikzcd}$$

and the strong deformation h as the exponential transpose of a lifting in

$$\begin{tikzcd} C \arrow[r, "\overline{j\pi_1}"] \arrow[d, "j"'] & D^I \arrow[d, "{(p_0,p_1)}"] \\ D \arrow[ur, dashed, "\overline{h}"] \arrow[r, "{(id_D, jr)}"'] & D \times D \end{tikzcd}$$

■

As a consequence we obtain immediately

Proposition 1.10.1 *If $i : A \to B$ is an anodyne extension then $|i| : |A| \to |B|$ and $|A|$ is a strong deformation retract of $|B|$.*

Proposition 1.10.2 *Let X be an arbitrary simplicial set. Then $\eta X : X \to s|X|$ is a weak equivalence.*

Proof: Let $i : X \to \overline{X}$ be an anodyne extension with $\overline{X}$ Kan. Then in the diagram

$$\begin{tikzcd} X \arrow[r, "\eta X"] \arrow[d, "i"'] & s|X| \arrow[d, "s|i|"] \\ \overline{X} \arrow[r, "\eta \overline{X}"'] & s|\overline{X}| \end{tikzcd}$$

$|i|$ is a homotopy equivalence by the above, $\eta\overline{X}$ is a homotopy equivalence by Milnor's Theorem, and i is a weak equivalence, so ηX is a weak equivalence. ∎

Proposition 1.10.3 *$w : X \to Y$ is a weak equivalence in the sense of Definition 1.7.1 iff w is a geometric homotopy equivalence.*

Proof: Factor w as

$$\begin{tikzcd} X \arrow[rr, "i"] \arrow[dr, "w"'] & & E \arrow[dl, "p"] \\ & Y & \end{tikzcd}$$

where i is a cofibration and p is a trivial fibration. Since w and p are weak equivalences so is i. i is anodyne by Proposition 1.7.2, so $|i|$ is a homotopy equivalence. p is a homotopy equivalence by Proposition 1.5.6, so $|p|$ is. Thus $|w|$ is a homotopy equivalence.

On the other hand, suppose $w : X \to Y$ is a goemetric homotopy equivalence. Then in the diagram

$$\begin{tikzcd} X \arrow[r, "\eta X"] \arrow[d, "w"'] & s|X| \arrow[d, "s|w|"] \\ Y \arrow[r, "\eta Y"'] & s|Y| \end{tikzcd}$$

ηX and ηY are weak equivalences as above, and $|w|$ is a homotopy equivalence, so $s|w|$ is. Thus w is a weak equivalence. ∎

Let $w : U \to V$ be a topological weak equivalence. From the diagram

$$\begin{tikzcd} |sU| \arrow[r, "|sw|"] \arrow[d, "\varepsilon U"'] & |sV| \arrow[d, "\varepsilon V"] \\ U \arrow[r, "w"'] & V \end{tikzcd}$$

we see that $|sw|$ is a weak equivalence since εU, εV and w are. By the topological Whitehead Theorem, it follows that $|sw|$ is a homotopy equivalence. Thus sw is a geometric homotopy equivalence, and a weak equivalence by the above. Thus, both s and $|\ |$ preserve weak equivalences. Since $\varepsilon T : |sT| \to T$ and $\eta X : X \to s|X|$ are weak equivalences, we see that s and $||$ induce an equivalence

$$Top_c[W^{-1}] \rightleftarrows \mathbf{S}[W^{-1}]$$

where W stands for the class of weak equivalences in each case. This is not surprising, of course, since by Appendix E we have that $ho(Top_c)$ is equivalent to $Top_c[W^{-1}]$ and $ho(\mathbf{S})$ is equivalent to $\mathbf{S}[W^{-1}]$.

When X is a Kan complex, the homotopy equivalence $\eta X : X \to s|X|$ provides a bijection $\pi_0(X) \to \pi_0(|X|)$ and an isomorphism $\pi_n(X, x) \to \pi_n(|X|, |x|)$ for $n \geq 1$ and $x \in X_0$. As remarked above, this shows that the $\pi_n(X, x)$ are groups for $n \geq 1$ and abelian for $n \geq 2$, though this is certainly not the way to see that. In any case, for X an arbitrary simplicial set we can *define* $\pi_n(X, x)$ as $\pi_n(X, x) = \pi_n(|X|, |x|)$ as this is consistent with the case when X is Kan.

Finally, let $f : X \to Y$ be a homotopy isomorphism, i.e. $\pi_0 f : \pi_0(X) \to \pi_0(Y)$ is a bijection, and $\pi_n f : \pi_n(X, x) \to \pi_n(Y, fx)$ is an isomorphism for $n \geq 1$ and $x \in X_0$. Now, if f is a geometric homotopy equivalence, f is a homotopy isomorphism. On the other hand, if f is a homotopy isomorphism then f is a geometric homotopy equivalence by the topological Whitehead Theorem. Thus, the classes of weak equivalences, geometric homotopy equivalences and homotopy isomorphisms all coincide.