G_2 conifolds: A survey

Spiro Karigiannis

Department of Pure Mathematics
University of Waterloo

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Results include separate joint works with
Jason Lotay (University College London)
Dominic Joyce (University of Oxford).

Physics?

M-theory?

M-theory is an 11-dimensional theory that is "compactified" on a 7-dimensional manifold M^7, which (for supersymmetry) admits a *parallel spinor*. Such manifolds are known as G_2 manifolds, and have Riemannian holonomy contained in the exceptional Lie group $\mathrm{G}_2 \subseteq \mathrm{SO}(7)$.

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- G_2 manifolds are always Ricci-flat. They are 7-dimensional analogues of Calabi-Yau 3-folds, which are the 6-dimensional compactification spaces in 10-dimensional string theory.
- They possess 3-dimensional "instantons": *associative submanifolds*, the analogue of J-holomorphic curves.
- They possess 4-dimensional "branes": *coassociative submanifolds*, the analogue of special Lagrangian 3-folds.

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- SYZ: a Calabi-Yau 3-fold X should admit a *fibration* $f : X \to B$ over a real 3-manifold B, whose generic fibre is a *special Lagrangian torus*. To obtain the mirror, one "dualizes the non-singular fibres", then does something(?) to compactify and obtain "the mirror" $\widehat{X}$.

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- A notion of "mirror symmetry" should(?) also exist for G_2 manifolds. Work of Acharya and others suggests that a G_2 manifold M should admit a fibration $f : M \to B$ over a real 3-manifold B, whose generic fibres are *coassociative tori or K3's*.

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- *No one really knows how to do this yet.*

Singularities necessary for good physics?

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M theory and Singularities of Exceptional Holonomy Manifolds

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Abstract

M theory compactifications on G_2 holonomy manifolds, whilst supersymmetric, require *singularities* in order to obtain non-Abelian gauge groups, chiral fermions and other properties necessary for a realistic model of particle physics. We review recent progress in understanding the physics of such singularities. Our main aim is to describe the techniques which have been used to develop our understanding of M theory physics near these singularities. In parallel, we also describe similar sorts of singularities in $Spin(7)$ holonomy manifolds which correspond to the properties of three dimensional field theories. As an application, we review how various aspects of strongly coupled gauge theories, such as confinement, mass gap and non-perturbative phase transitions may be given a simple explanation in M theory.

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- The simplest singularities that can be considered in physics are *isolated conical singularities*.

G_2 manifolds

Manifolds with G_2 structure

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- A G_2 structure exists if and only if M is *orientable* and *spin*, which is equivalent to $w_1(M) = 0$ and $w_2(M) = 0$.
- A G_2 structure is encoded by a "non-degenerate" 3-form φ which nonlinearly determines a Riemannian metric g_φ and an orientation. We thus have a Hodge star operator $*_\varphi$ and dual 4-form $\psi = *_\varphi \varphi$.

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- A G_2 structure is encoded by a "non-degenerate" 3-form φ which nonlinearly determines a Riemannian metric g_φ and an orientation. We thus have a Hodge star operator $*_\varphi$ and dual 4-form $\psi = *_\varphi \varphi$.
- On a manifold (M, φ) with G_2 structure, each tangent space $T_p M$ can be canonically identified with the *imaginary octonions* $\mathbb{O} \cong \mathbb{R}^7$.

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Properties of G_2 manifolds:

- The holonomy $\mathrm{Hol}(g_\varphi)$ is contained in G_2. If $\mathrm{Hol}(g_\varphi) = \mathrm{G}_2$, then (M, φ) is called an irreducible G_2 manifold. A *compact* G_2 manifold is irreducible if and only if $\pi_1(M)$ is finite.

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- The metric g_φ is Ricci-flat.
- G_2 manifolds admit a parallel spinor. They play the role in M-theory that Calabi-Yau 3-folds play in string theory.
- A G_2 structure is torsion-free if and only if $d\varphi = 0$ and $d{*_\varphi}\varphi = 0$. (Fernàndez–Gray, 1982.) Both φ and $*_\varphi\varphi$ are calibrations.

Comparison with Kähler and Calabi-Yau geometry

- G_2 manifolds are very similar to Kähler manifolds.
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- Both admit calibrated *submanifolds* and *connections*.
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- However, unlike G_2 manifolds, *not all* Kähler manifolds are Ricci-flat. Those are the *Calabi-Yau* manifolds.
- By the Calabi-Yau theorem, we have a topological characterization of the Ricci-flat Kähler manifolds.
- We are still *very far* from knowing sufficient topological conditions for existence of a torsion-free G_2 structure.

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- Therefore, Kähler geometry can be thought of as 'decoupling' into complex geometry and symplectic geometry.

- However, if M admits a G_2 structure, the 3-form φ determines the metric g in a nonlinear way:

$$(u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \varphi = C\, g_\varphi(u, v)\, \mathrm{vol}_\varphi$$

- Thus, we cannot 'decouple' G_2 geometry in any way.

Examples of G_2 manifolds

Complete noncompact examples

- Bryant–Salamon (1989): these examples are total spaces of vector bundles $\Lambda^2_-(S^4)$, $\Lambda^2_-(\mathbb{CP}^2)$, $\not{S}(S^3)$; they are all asymptotically conical: far away from the base of the bundle, they "look like" *metric cones*.
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- These examples are all explicit cohomogeneity one G_2 manifolds — they have enough "symmetry" so that the nonlinear PDE reduces to a system of fully nonlinear ODEs, which can often be solved exactly.
- It can be shown (using the Bochner–Weitzenböck formula) that *compact* examples cannot have *any* symmetry. So the construction of compact examples is necessarily much more difficult.

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- Joyce–Karigiannis (2014?) — glueing a 3-dimensional family of Eguchi-Hanson spaces

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Theorem (Joyce, 1994)

Let M be a compact manifold with a closed G_2 structure φ such that the torsion is sufficiently small. (One needs good control of the L^{14} norm of the torsion and some other estimates.) Then there exists a torsion-free G_2 structure $\widetilde{\varphi}$ close to φ in the C^0 norm, with $[\widetilde{\varphi}] = [\varphi]$ in $H^3(M, \mathbb{R})$.

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These constructions provide thousands of examples, but they are likely only a very small part of the "landscape."

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Ingredients [2] and [3] require compactness of M, and thus need to be modified in any noncompact setting.

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G_2 cones

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- The *link* Σ of a G_2 cone C is necessarily a compact strictly nearly Kähler 6-manifold (also called a *Gray manifold*.)
- These are almost Hermitian manifolds (Σ, J, g, ω) with $c_1(\Sigma) = 0$, such that

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- There are only three known compact examples, all homogeneous, but there are expected to exist *many examples*.

Asymptotically conical (AC) G_2 manifolds

Definition

We say (N, φ_N) is an AC G_2 manifold of rate $\nu < 0$, asymptotic to the G_2 cone (C, φ_C), if outside of a compact set $K \subseteq N$, we have $N \setminus K \cong (R, \infty) \times \Sigma$, and

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- There are three known examples, the Bryant–Salamon manifolds, asymptotic to the three known G_2 cones.
- $\Lambda^2_-(S^4)$ and $\Lambda^2_-(\mathbb{CP}^2)$ have rate $\nu = -4$.
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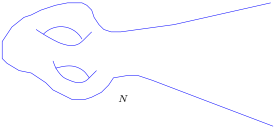

Conically singular (CS) G_2 manifolds

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Let $\overline{M}$ be a topological space with $M = \overline{M} \setminus \{x_1, \ldots, x_n\}$ a noncompact smooth 7-manifold. We say (M, φ_M) is an CS G_2 manifold of rate $(\nu_1, \ldots, \nu_n)$, where $\nu_i > 0$, asymptotic to the G_2 cones (C_i, φ_{C_i}), if outside of a compact set $K \subseteq M$, we have $M \setminus K \cong \bigsqcup_{i=1}^{n} (0, R) \times \Sigma_i$, and

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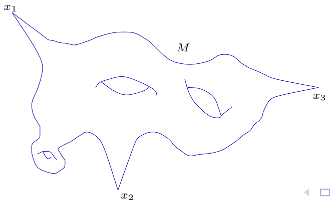

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- A way to desingularize them is to cut out a neighbourhood of the singular points, and glue in *AC* G_2 *manifolds*, such as the Bryant–Salamon examples.

Desingularization of CS G_2 manifolds

Theorem (Karigiannis, Geometry & Topology, 2009)

Let M be a CS G_2 manifold with isolated conical singularities $x_1, \ldots, x_n$, modelled on G_2 cones $C_1, \ldots, C_n$. Suppose that $N_1, \ldots, N_n$ are AC G_2 manifolds modelled on the same G_2 cones, with all rates $\nu_i \leq -3$.

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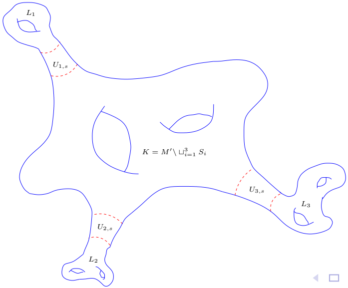

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There are natural maps $\Upsilon^k : H^k(M) \to \oplus_{i=1}^n H^k(\Sigma_i)$. Let $K_i(\lambda)$ be the space of *homogeneous* closed and coclosed 3-forms on C_i of rate λ.

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- *In the AC case with $\nu < -4$, the moduli space may be obstructed, and its virtual dimension* v-$\dim \mathcal{M}_\nu$ *is*

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- The proof uses the Lockhart–McOwen machinery of weighted Sobolev spaces and its associated Fredholm theory, plus new Hodge-theoretic results in this context, and other G_2 specific ingredients (surjectivity of Dirac operator, L^2 harmonic 1-forms are parallel, more ...)

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[4] Statements [2] and [3] will be true *in general* if certain conjectures about the spectrum of the Laplacian on forms are true for *all* compact strictly nearly Kähler 6-manifolds.

A new construction of compact G_2 manifolds

(which may possibly generalize to construct compact CS G_2 manifolds)

[Step 1] Construct an orbifold $\widehat{M}$

- Let $(N^6, g, \omega, \Omega, J)$ be a compact Calabi-Yau manifold admitting an *antiholomorpic isometric involution* τ:

$$\tau^*(g) = g, \qquad \tau^*(\omega) = -\omega, \qquad \tau^*(\Omega) = \overline{\Omega}, \qquad \tau^*(J) = -J.$$

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- Define $\sigma : M \to M$ by $\sigma(p, \theta) = (\tau(p), -\theta)$. Then σ is an involution of M such that $\sigma^*(\varphi) = \varphi$. The quotient space $\widehat{M} = M/\langle\sigma\rangle$ is a G_2 orbifold, with singularities locally of the form $\mathbb{R}^3 \times (\mathbb{C}^2/\{\pm 1\})$.

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- The singular set $L^3 = A^3 \times \{\pm 1\}$, where $A^3 = \mathrm{Fix}(\tau)$ is a compact special Lagrangian submanifold of N^6, and L is totally geodesic in M.

[Step 2] Glue in a family of Eguchi-Hanson spaces

- We want to cut out a neighbourhood of the singular locus L in $\widehat{M}$ and glue in a noncompact smooth manifold to get a smooth compact 7-manifold $\widetilde{M}$, which hopefully will admit a *closed* G_2 *structure with small enough torsion*, to to apply Joyce's existence theorem.

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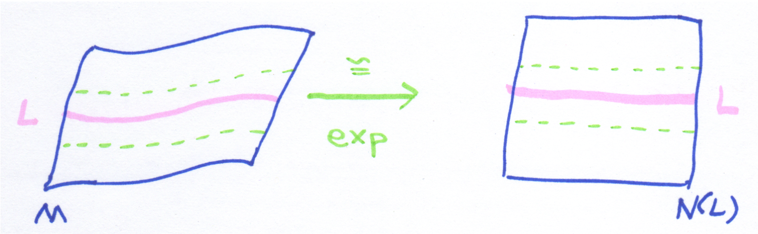

- The submanifold L is an associative submanifold. This implies that, *given a nonvanishing* 1*-form* α *on* L, the normal bundle $N(L)$ is actually a $\mathbb{C}^2$ bundle over L, and the above diffeomorphism descends to identify $\widehat{M}$ with $P = N(L)/\{\pm 1\}$ near L.

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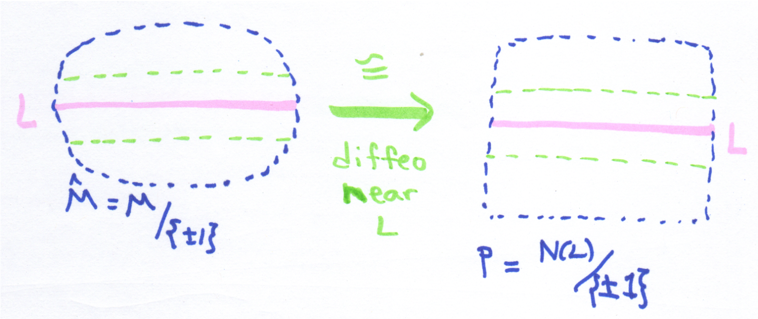

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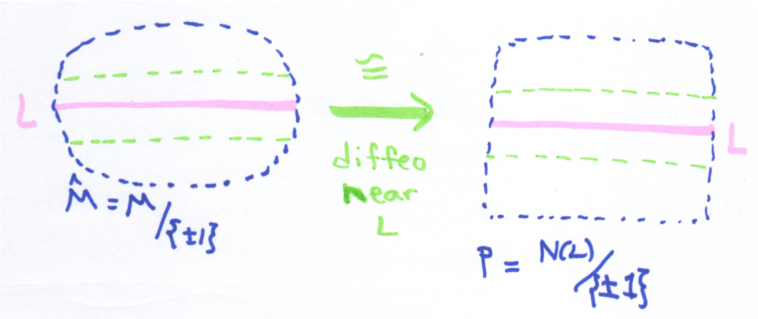

- The fibres of $P = N(L)/\{\pm 1\}$ are $\mathbb{C}^2/\{\pm 1\}$. We resolve P to $\widetilde{P}$ with a 'fibre-wise blow-up', replacing each fibre with $\widetilde{\mathbb{C}^2/\{\pm 1\}} \cong T^*S^2$.

- Each fibre T^*S^2 admits an $S^2 \times (0, \infty)$ family of *Eguchi-Hanson metrics* (holonomy $\mathrm{SU}(2)$ metrics) that are parametrized by a choice of complex structure on $\mathbb{R}^4 = \mathbb{H}$ (a unit vector in $\mathbb{R}^3$) and a scaling.

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- We can use α to construct a closed G_2 structure $\varphi_{\widetilde{P}}$ on $\widetilde{P}$ with small torsion, but for the torsion to have any chance of being small enough, it is necessary that $d\alpha = 0$ and $d^*\alpha = 0$. For now, let us assume that we have such a nowhere vanishing harmonic 1-form α.

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- We construct a compact smooth manifold $\widetilde{M}$ as follows. Far from the zero section, identify P with $\widehat{M}$ using the *exponential map*. Close to the zero section, identify P with $\widetilde{P}$ using the *resolution map*.

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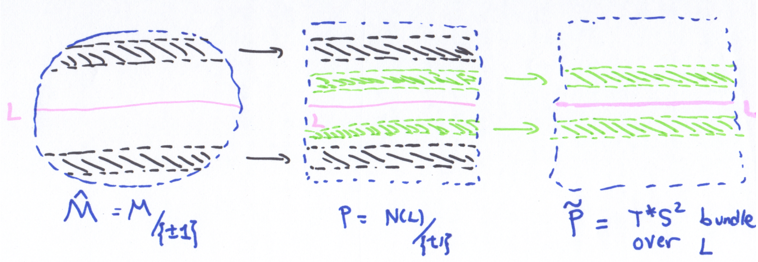

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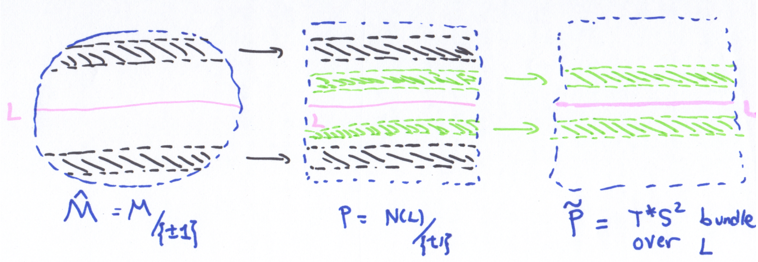

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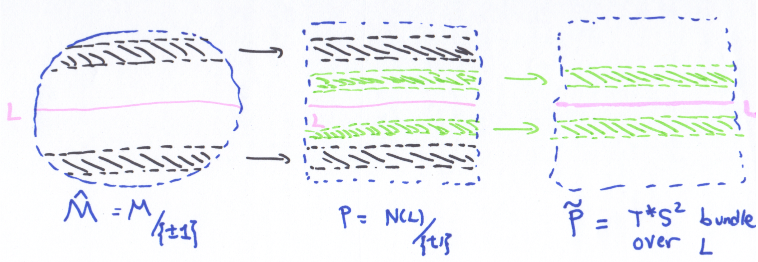

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- We want to construct a closed G_2 structure $\widetilde{\varphi}$ on $\widetilde{M}$ by interpolating between $\varphi_{\widehat{M}}$ and $\varphi_{\widetilde{P}}$ using $\overline{\varphi}$. We use the metric $\overline{g}$ of $\overline{\varphi}$ to measure the torsion of $\widetilde{\varphi}$, since we cannot compare $\widehat{M}$ and $\widetilde{P}$ directly.

- In fact, the G_2 structures $\overline{\varphi}$ on P and $\varphi_{\widetilde{P}}$ on $\widetilde{P}$ are *not closed*, so these have to be slightly modified, using smooth cut-off functions, to "closed versions" before we can construct $\widetilde{\varphi}$ on $\widetilde{M}$ by interpolation.

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- The major problem is that the space $\widetilde{P}$ that we are "glueing in" does not have a natural torsion-free G_2 structure.

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- The major problem is that the space $\widetilde{P}$ that we are "glueing in" does not have a natural torsion-free G_2 structure.
- Also, the fact that we need to introduce an "intermediary" manifold with G_2 structure $(P, \overline{\varphi})$ and use its metric $\overline{g}$ to measure the size of the torsion creates additional complications. The G_2 structure $\overline{\varphi}$ is not a priori close enough to φ_M.

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- We need to perform two *corrections* to solve these problems.

[Step 4] 1^{st} correction: bend horizontal and vertical

- We begin with the easier correction: modifying the identification between M and $N(L)$ so that the canonical G_2 structure $\overline{\varphi}$ on $P = N(L)/\{\pm 1\}$ is close enough to φ_M on $\widehat{M}$.

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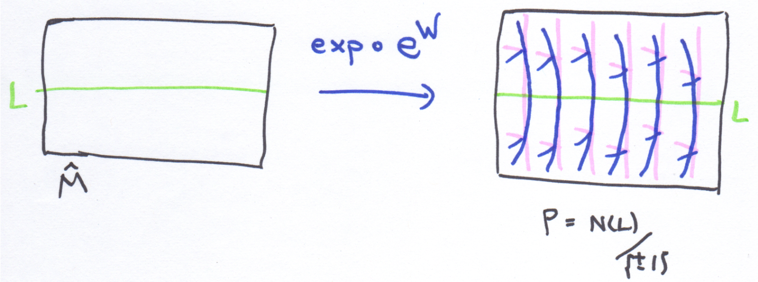

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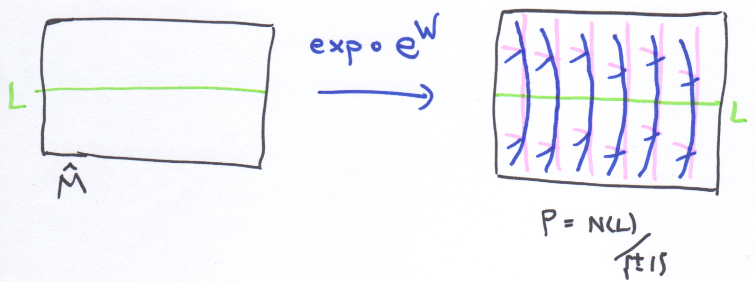

- These can in fact be chosen to make $\overline{\varphi}$ close enough to φ_M.

[Step 5] 2^{nd} correction: solving a PDE on E-H space

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- This is done using Lockhart–McOwen theory of Fredholm operators on noncompact manifolds with "well-behaved" geometry at infinity.
- The theory says that such an equation can be solved if and only if σ has appropriate asymptotic behaviour at infinity, which it does.

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However, if N is near the "large complex structure limit" of the moduli space, from mirror symmetry arguments we expect it to contain a special Lagrangian torus that is *close to being flat*, so it will admit such 1-forms.

- Generically, a harmonic 1-form α on L has isolated zeroes. Then we can resolve M to $\widetilde{M}$ except for a finite number of singular points. In fact, near the singular points, $\widetilde{M}$ is *topologically* a cone over $\mathbb{CP}^3$.

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This is work in progress, with Jason Lotay.
We have done (i). We are working on (ii).

Thank you for your attention.