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# Complex Topological $K$-theory


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# Contents

# List of notations

The following notations will be used all through this report :

| | |
|---|---|
| $\mathcal{C}_*$ | Category of pointed objects of a category $\mathcal{C}$ |
| $h\mathcal{C}$ | The homotopy category of a category $\mathcal{C}$ (when a suitable relation of homotopy makes sense) |
| $Set$ | Category of sets |
| $Top$ | Category of topological spaces |
| $\mathcal{CW}$ | Category of CW-complexes |
| $\mathcal{CW}_{\mathcal{F}}$ | Category of finite CW-complexes |
| $Mon$ | Category of monoids |
| $Grp$ | Category of groups |
| $TopGrp$ | Category of topological groups |
| $Ab$ | Category of abelian groups |
| $SemiRng$ | Category of semi-rings |
| $Rng$ | Category of rings |
| $\mathcal{V}_{\mathbb{K}^n}$ | Category of $n$-dimensional vector spaces over the field $\mathbb{K}$ |
| $Bund_{\mathbb{C}^n}$ | Category of $n$-dimensional complex vector bundles |
| $Bund_{\mathbb{C}^n}(B)$ | Category of $n$-dimensional complex vector bundles over $B$ |
| $Sp$ | Category of spectra |
| $Vect_{\mathbb{C}}(B)$ | Semi-group of the equivalence classes of complex vector bundles over $B$ |
| $Vect_{\mathbb{C}^k}(B)$ | Semi-group of the equivalence classes of complex vector $n$-bundles over $B$ |
| $[f]$ | Homotopy class of the map $f$ |
| $< (\overrightarrow{v}_1, ..., \overrightarrow{v}_n) >$ | The subspace spanned by a list $(\overrightarrow{v}_1, ..., \overrightarrow{v}_n)$ of vectors |
| $M^*$ | The conjugated transpose of a matrix $M$ |
| l.e.s. | Long exact sequence |
| s.e.s. | Short exact sequence |
| c.f. | Clutching function |

# Introduction

The aim of this project was to study a concrete example of a generalized cohomology theory. Our choice was to understand complex topological $K$-theory, which, for a space $X$, is related to complex vector bundles over this space.

Chapter 1 offers a brief description of the main algebraic and topological structures and constructions we shall be working with. In particular, the Grassmanian manifolds are defined, since our choice was to use the Grassmanian approach to classify the isomorphism classes of vector bundles.

In Chapter 2 we recall the definition of a reduced generalized cohomology theory, as well as some important results related to its representation by an $\Omega$-spectrum. We then explain what a commutative ring spectrum is, and provide a definition of a non-symetric smash product of spectra. We show that given a ring $\Omega$-spectrum, the associated cohomology theory is multiplicative, and conversely.

Complex vector bundles are the first main ingredient one would need to understand complex topological $K$-theory, so Chapter 3 is entirely devoted to their study. One of the key facts is that the direct sum and the tensor product, defined on vector spaces, pass to complex vector bundles over a space $X$.

The earliest step towards our goal is made in the fourth Chapter, where we define $K(X)$ and $\tilde{K}(X)$, the first non-reduced and reduced groups of topological $K$-theory. Each of them is given its geometrical and formal description, and it is also shown that they are both equipped with a ring structure.

An essential result that determines the 2-fold periodic structure of the complex $K$-theory is the Bott Periodicity Theorem, which states the isomorphism $K(X) \otimes K(\mathbb{S}^2) \cong K(X \times \mathbb{S}^2)$. In Chapter 5 we explain the original proof of this theorem, given by Raoul Bott, describing first all the specific tools used.

All the pieces are brought together in the last Chapter, where we show that complex topological $K$-theory is multiplicative and satisfies the axioms of a generalized reduced cohomology theory, and define the associated spectrum $KU$.

# Chapter 1

# Some basic notions

## 1.1 Algebraic structures

**Definition 1.1.1.** A **monoid** is a couple $(M, *)$, where $M$ is a set, equipped with a binary operation

$$* : M \times M \longrightarrow M$$

such that

1. $*$ *is associative:* $(a * b) * c = a(b * c)$, $\forall a, b, c \in M$;
2. $*$ *has an identity element:* $\exists e \in M$ such that $a * e = e * a = a$, $\forall a \in M$.

A monoid $M$ is said to be **commutative** if its operation $*$ is commutative.

**Definition 1.1.2.** Let $(M, *)$ and $(M', \diamond)$ be two monoids. A **morphism of monoids** $f : M \longrightarrow M'$ is a function such that

$$f(a * b) = f(a) \diamond f(b),\ \forall a, b \in M$$

and

$$f(e_1) = e_2.$$

Note that monoids form a category, denoted *Mon*.

**Definition 1.1.3.** A **semi-ring** is a triple $(R, *, +)$, where $R$ is a set, equipped with two binary operations such that

1. $(R, *)$ is a monoid with identity element denoted 1;
2. $(R, +)$ is a commutative monoid with identity element denoted 0;
3. Multiplication is distributive over addition;
4. 0 annihilates $R$, with respect to multiplication: $0 * a = a * 0 = 0$, $\forall a \in R$.

*Remark* 1.1.4. The difference between a semi-ring and a ring is that $+$ yields a group structure on the second, as opposed to only a monoidal structure on the first. In other words, a semi-ring satisfies all the axioms of a ring except the existence of additive inverses.

**Definition 1.1.5.** Let $(R, *, +)$ and $(R', *, +)$ be two semi-rings. A **morphism of semi-rings** $f : R \longrightarrow R'$ is a function such that for all $a, b \in R$

$$f(a+b) = f(a) + f(b) \text{ and } f(0) = 0',$$

$$f(a*b) = f(a) * f(b) \text{ and } f(e) = e',$$

where $e$, $e'$ denote the identities for $*$, and $0$, $0'$ denote the identities for $+$.

Semi-rings form a category, which we shall denote *SemiRng*, whose composition law and identity morphism are inherited from *Mon*.

**Definition 1.1.6.** A **graded ring** $(R_*, \mu)$ is a graded abelian group

$$R_* = \{R_n\}_{n\in\mathbb{N}}$$

equipped with an associative multiplication

$$\mu : R_* \otimes R_* \longrightarrow R_*$$

$$x \otimes y \mapsto \mu(x \otimes y).$$

Note that in degree $m$ we have

$$\mu_m : (R_* \otimes R_*)_m \longrightarrow R_m,$$

where

$$(R_* \otimes R_*)_m = \bigoplus_{i+j=m} R_i \otimes R_j.$$

**Definition 1.1.7.** Let $R_*$ be a graded ring. A **graded R-module** $(M_*, \nu)$ is given by

$$M_* = \{M_n\}_{n\in\mathbb{N}},\ M_n \in Ab \text{ for all } n \in \mathbb{N}$$

equipped with an associative action

$$\nu : M_* \otimes R_* \longrightarrow M_*$$

$$x \otimes y \mapsto \nu(x \otimes y).$$

**Definition 1.1.8.** Let $(R_*, \mu)$ be a graded ring. A **graded algebra** $(A_*, \oplus, \mu, \nu)$ is given by

$$A_* = \{A_n\}_{n\in\mathbb{N}},\ A_n \in Ab \text{ for all } n \in \mathbb{N},$$

where $(A_*, \oplus, \mu)$ is a graded ring and $(A_*, \oplus, \nu)$ is a graded R-module, such that the action $\nu : M_* \otimes R_* \longrightarrow M_*$ and the multiplication $\mu : R_* \otimes R_* \longrightarrow R_*$ are compatible, i.e., the following diagram commutes

$$\begin{tikzcd}
A_* \otimes A_* \otimes R_* \arrow[r, "\mathrm{Id}\otimes\nu"] \arrow[d, "\mu\otimes\mathrm{Id}"'] & A_* \otimes A_* \arrow[d, "\mu"] \\
A_* \otimes R_* \arrow[r, "\nu"] & A_*.
\end{tikzcd}$$

## 1.2 The Grothendieck construction

The **Grothendieck construction** gives a recipe for passing from a commutative monoid with a zero element to an abelian group; we also talk about a **group completion** in this case.

**Definition 1.2.1.** Let $(M, +)$ be a commutative monoid with identity element 0. We can associate to $M$ an abelian group, denoted $G(M)$, together with a homomorphism of the underlying monoids $G : M \longrightarrow G(M)$, satisfying the following **universal property**.

*For any abelian group $A$ and any homomorphism of the underlying monoids $f : M \longrightarrow A$, there is a unique group homomorphism $\bar{f} : G(M) \longrightarrow A$ such that $\bar{f} \circ G = f$, i.e., such that the diagram below commutes:*

$$\begin{tikzcd} M \arrow[rr, "G"] \arrow[dr, "f"'] & & G(M) \arrow[dl, dashed, "\exists! \bar{f}"] \\ & A. & \end{tikzcd}$$

$G(M)$ is called **the Grothendieck group of** $M$. Its uniqueness (up to a group isomorphism) is a consequence of the universal property. So we only have to show that such a group $G(M)$ exists.

Let us see how to build $G(M)$. There are at least three possible ways ([Ka], Ch.1), all equivalent up to isomorphism. Here we explain the construction that is the most relevant for future applications.

Consider pairs $(a, b)$ and $(c, d) \in M \times M$ and define them to be *equivalent* if and only if there exists $u \in M$ such that

$$a + d + u = b + c + u.$$

One can check that this is an equivalence relation on $M \times M$. We then write $(a, b) \sim (c, d)$ and represent by $\{a, b\}$ the equivalence class of $(a, b)$ under $\sim$.

We set

$$G(M) := M \times M \diagup \sim .$$

The **group structure** on $G(M)$ is given by the **addition**

$$+_G : G(M) \times G(M) \longrightarrow G(M)$$

defined by

$$(\{a, b\}, \{c, d\}) \mapsto \{a + c, b + d\},$$

with the corresponding **zero element** $\{0, 0\}$.
The **inverse** of $\{a, b\}$ is $\{b, a\}$.

Finally, the homomorphism of the underlying monoids

$$G : M \longrightarrow G(M),$$

is defined by

$$a \mapsto \{a, 0\}, \ \forall a \in M.$$

Given $A \in Ab$ and $f : M \longrightarrow A$ between monoids, we define

$$\bar{f} : G(M) \longrightarrow A$$

by

$$\{a, b\} \mapsto f(a) - f(b).$$

One can check that $\bar{f}$ is a well-defined abelian group homomorphism, and that it is the only one such that $\bar{f} \circ \theta = f$.

**Extension on rings**

So far, we have seen how the Grothendieck construction provides us with an abelian group from a monoid. By Remark 1.1.4, given a semi-ring $(R, *, +)$, and applying Grothendieck to $(R, +)$, we actually obtain a ring $(G(R), *_G, +_G)$, since the multiplication

$$* : R \times R \longrightarrow R$$

$$(a, b) \mapsto a * b$$

induces a multiplication on the quotient

$$*_G : G(R) \times G(R) \longrightarrow G(R)$$

defined by

$$(\{a, b\}, \{c, d\}) = \{a * c + b * d, b * c + a * d\},$$

which is linear with respect to the addition on $(G(R), +_G)$. Indeed, calculation yields

$$\begin{aligned} (\{a, b\} + \{x, y\}) * \{c, d\} &= \{a + x, b + y\} * \{c, d\} \\ &= \{ac + xc + bd + yd, bc + yc + ad + xd\} \end{aligned}$$

and

$$\begin{aligned} \{a, b\} * \{c, d\} + \{x, y\} * \{c, d\} &= \{ac + bd, bc + ad\} + \{xc + yd, yc + xd\} \\ &= \{ac + bd + xc + yd, bc + ad + yc + xd\}, \end{aligned}$$

which are the same, because $(M, +)$ is commutative.

In this way, one makes a **ring completion**. An example that comes immediately to the mind is the construction of the ring of integers $\mathbb{Z}$ from the semi-ring $\mathbb{N}$ of natural numbers.

Finally, we state a result showing that $G$ extends to morphisms.

**Lemma 1.2.2.** *The pair of maps*

$$G : SemiRng \longrightarrow Rng$$

*defined by*

$$R \mapsto G(R),$$

$$(h : R_1 \longrightarrow R_2) \mapsto G(h) : G(R_1) \longrightarrow G(R_2)$$

*where*

$$G(h)\{a, b\} = \{h(a), h(b)\}$$

*is a functor.*

The verification is straightforward.

## 1.3 Complex Stiefel and Grassmanian manifolds

*Remark* 1.3.1. The topology of all the manifolds introduced in this section is supposed to be inherited from the usual topology of $\mathbb{C}$.

**Definition 1.3.2.** We define the **complex Stiefel manifold** to be

$$W_n(\mathbb{C}^k) = \{A \in M_{k\times n}(\mathbb{C}) | A^* \cdot A = I_n,\ n \leq k\}.$$

In other words, $W_n(\mathbb{C}^k)$ is the set of all $n$-frames ($n$-tuples of ordered orthonormal vectors) in $\mathbb{C}^k$ for $n \leq k$. Topologically, it is a compact space, as a closed subspace of the product of $n$ copies of the sphere $\mathbb{S}^{k-1}$.

**Definition 1.3.3.** We define the **complex Grassmanian manifold** to be

$$G_n(\mathbb{C}^k) = \{n\text{-dimensional vector subspaces of } \mathbb{C}^k,\ n \leq k\},$$

i.e., the collection of all $n$-dimensional planes in $\mathbb{C}^k$ passing through the origin.

*Example:* The manifold $G_1(\mathbb{C}^k)$ is the collection of all lines in $\mathbb{C}^k$ through the origin, which is nothing but $\mathbb{C}\mathrm{P}^k$, the complex projective plane of dimension $k$.

To have a better understanding of these manifolds, observe that there is a natural projection

$$\begin{array}{rcl} W_n(\mathbb{C}^k) & \overset{\rho}{\twoheadrightarrow} & G_n(\mathbb{C}^k) \\ (\overrightarrow{v}_1, ..., \overrightarrow{v}_n) & \mapsto & <(\overrightarrow{v}_1, ..., \overrightarrow{v}_n)>, (\diamondsuit) \end{array}$$

which allows us to view $G_n(\mathbb{C}^k)$ as a compact space, provided with the quotient topology. A CW-structure can also be defined, so that each $G_n(\mathbb{C}^k)$

is a complex with a finite number of cells, and it can be shown that $G_n(\mathbb{C}^k)$ is a Hausdorff manifold of dimension $k(k-n)$.

The inclusions of spaces $\mathbb{C}^k \subset \mathbb{C}^{k+1} \subset ...$ induce inclusions of manifolds $W_n(\mathbb{C}^k) \subset W_n(\mathbb{C}^{k+1}) \subset ...$, as well as $G_n(\mathbb{C}^k) \subset G_n(\mathbb{C}^{k+1}) \subset ....$ We set

$$W_n(\mathbb{C}^\infty) := \varinjlim_k W_n(\mathbb{C}^k) \text{ and } G_n(\mathbb{C}^\infty) := \varinjlim_k G_n(\mathbb{C}^k),$$

giving to these two spaces the direct limit topology.

## 1.4 The Unitary group $U(k)$

Consider $\mathbb{C}$ as an $\mathbb{R}$-vector space with its standard scalar product. Let

$$U(k) = \{M_k(\mathbb{C}) | M^* \cdot M = M \cdot M^* = I_k\}$$

be the set of unitary $k \times k$ matrices with coefficients in $\mathbb{C}$.

The set $U(k)$, together with the matrix multiplication $\cdot$ , defines a group $(U(k), \cdot)$ called **the Unitary group**. It is the group of all $k$-linear transformations in $\mathbb{C}$ that preserve the standard complex norm, or, in other words, the group of isometries $Isom(\mathbb{C}^k)$. Topologically, $U(k)$ can be seen as a compact convex space, with topology induced by $\mathbb{C}$, and homeomorphic to a Euclidean space of dimension $2k^2$.

Let $\mathbb{C}^n$ be a fixed $n$-subspace of $\mathbb{C}^k$, for some $n \leq k$. One has a direct sum decomposition $\mathbb{C}^k = \mathbb{C}^n \oplus \mathbb{C}^{k-n}$, since $\mathbb{C}^{k-n}$ is the orthogonal complement of $\mathbb{C}^n$. Any subgroup $H < U(k)$ stabilizing $\mathbb{C}^n$ splits into a direct product

$$H = U(n) \times U'(k-n),$$

where $U(n)$ is the subgroup of $H$ that stabilizes $\mathbb{C}^n$ point-wise, and similarly $U'(k-n)$ for $\mathbb{C}^{k-n}$. The elements of $U(n)$ and $U'(k-n)$, regarded as subgroups of $U(k)$, can be represented respectively by matrices of the form

$$\begin{pmatrix} \sigma & 0 \\ 0 & I_{k-n} \end{pmatrix} \in U(k)$$

and

$$\begin{pmatrix} I_n & 0 \\ 0 & \sigma' \end{pmatrix} \in U(k)$$

with $\sigma \in U(n)$ and $\sigma' \in U'(k-n)$.

### Connection with $W_n(\mathbb{C}^k)$ and $G_n(\mathbb{C}^k)$

With the help of $U(k)$, the Stiefel and the Grassmanian manifolds can be viewed in a more algebraic fashion, which is very convenient. In particular, we have the homeomorphism

$$U(k) \diagup U'(k-n) \xrightarrow{\approx} W_n(\mathbb{C}^k),$$

$$A \mapsto (A\overrightarrow{v}_1, ..., A\overrightarrow{v}_n),$$

where $\overrightarrow{v}_1, ..., \overrightarrow{v}_n$ are $n$ orthonormal vectors in $\mathbb{C}^k$, $n \leq k$. Similarly, one has for $G_n(\mathbb{C}^k)$

$$U(k)\diagup(U(n) \times U'(k-n)) \xrightarrow{\approx} G_n(\mathbb{C}^k),$$

$$A \mapsto < (A\overrightarrow{v}_1, ..., A\overrightarrow{v}_n) > .$$

Therefore, ($\diamondsuit$) can be completed to give

$$U(k) \overset{q}{\twoheadrightarrow} W_n(\mathbb{C}^k) \overset{\rho}{\twoheadrightarrow} G_n(\mathbb{C}^k),$$

$$A \longmapsto (A\overrightarrow{v}_1, ..., A\overrightarrow{v}_n) \longmapsto < (A\overrightarrow{v}_1, ..., A\overrightarrow{v}_n) > .$$

Now, there are direct sequences

$$\begin{tikzcd}
 & \vdots \arrow[d, hook] & \vdots \arrow[d, hook] & \vdots \arrow[d, hook] & \\
\cdots \arrow[r, hook] & G_n(\mathbb{C}^k) \arrow[r, hook, "k_n^k"] \arrow[d, hook, "j_n^k"'] & G_{n+1}(\mathbb{C}^{k+1}) \arrow[r, hook, "k_{n+1}^{k+1}"] \arrow[d, hook, "j_{n+1}^{k+1}"] & G_{n+2}(\mathbb{C}^{k+2}) \arrow[r, hook] \arrow[d, hook, "j_{n+2}^{k+2}"] & \cdots \\
\cdots \arrow[r, hook] & G_n(\mathbb{C}^{k+1}) \arrow[r, hook, "k_n^{k-1}"] \arrow[d, hook] & G_{n+1}(\mathbb{C}^{k+2}) \arrow[r, hook, "k_{n+1}^k"] \arrow[d, hook] & G_{n+2}(\mathbb{C}^{k+3}) \arrow[r, hook] \arrow[d, hook] & \cdots \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

$$\cdots \hookrightarrow G_n(\mathbb{C}^\infty) \xhookrightarrow{i_n} G_{n+1}(\mathbb{C}^\infty) \xhookrightarrow{i_{n+1}} G_{n+2}(\mathbb{C}^\infty) \hookrightarrow \cdots,$$

where, with previous identifications, the $j_n^k$ are given for each $k$,$n \in \mathbb{N}$ by

$$A \in U(k) \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in U(k+1).$$

Similarly, the $k_n^k$ are defined for each $k$,$n \in \mathbb{N}$ by

$$A \in U(k) \mapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \in U(k+1),$$

inducing maps $i_n : G_n(\mathbb{C}^\infty) \hookrightarrow G_{n+1}(\mathbb{C}^\infty)$ in the direct limit.

Taking a direct limit over $n$ in the previous sequence leads to the definition of the following space, which will play a major role in Chapters 4 and 6.

**Definition 1.4.1.** Define the space $BU$ to be

$$BU := \varinjlim_n G_n(\mathbb{C}^\infty) = \varinjlim_n \varinjlim_k G_n(\mathbb{C}^k).$$

In the next chapter we shall discover complex vector bundles. One of the main reasons to be interested in the structure of Grassmanian manifolds is that they will appear as classifying spaces for $k$-dimensional complex vector bundles.

# Chapter 2

# Ring spectra and multiplicative cohomology theories

Under certain conditions on the spectra, the associated generalized cohomology theory takes values in graded algebras. In particular, the corresponding spectrum must be a ring spectrum.

*Remarks on notation.*

- We recall that a **spectrum** is a collection $\{(E_n, *)_{n\in\mathbb{Z}}\}$ of pointed $CW$-complexes such that maps $e_n : SE_n \xrightarrow{\approx} E_{n+1}$ are homeomorphisms for all $n \in \mathbb{Z}$.

- In this chapter we adopt different notation for a spectrum $\{(E_n, *), e_n\}$, depending basically on information required in the context. Thus base points $*$ or maps $e_n$ will sometimes be left off for simple $E$.

- We say that $E$ is an $\Omega$-spectrum if the maps $e' : E_n \longrightarrow \Omega E_{n+1}$ are weak homotopy equivalences for all $n \in \mathbb{Z}$.

- Hereafter $E^*(X, x_0)$ stands for the reduced cohomology theory associated to a spectrum $E$. Recall that $E^n(X, x_0) \in Ab$ for each $n \in \mathbb{Z}$, and note that all tensor products will be taken over $\mathbb{Z}$.

- In the case of *reduced* theories, topological spaces and spectra are always pointed. For ease of notation, however, the base point will sometimes be dropped.

## 2.1 A short reminder on reduced cohomology theories

We briefly recall the definition and the axioms for a reduced cohomology theory.

**Definition 2.1.1.** A **reduced cohomology theory** $k^*$ on $h\,Top_*$ is a collection of contravariant functors

$$k^n : h\,Top_* \longrightarrow Ab,\ n \in \mathbb{Z}$$

and natural transformations

$$\sigma^n : k^n \longrightarrow k^{n+1} \circ S,\ n \in \mathbb{Z},$$

satisfying the following axiom:

*Exactness:* For every pointed pair $(X, A, x_0) \in h\,Top_*^2$ with inclusions $i : (A, x_0) \hookrightarrow (X, x_0)$ and $j : (X, x_0) \hookrightarrow (X \cup_i CA, *)$ the induced sequence

$$k^n(X \cup_i CA, *) \xrightarrow{j^*} k^n(X, x_0) \xrightarrow{i^*} k^n(A, x_0)$$

is exact.

There is an additional axiom one can impose on reduced cohomology theories, and we shall always do so in this project.

*Wedge axiom (W):* For every collection $\{(X_\alpha, x_\alpha)\}_{\alpha\in A}$ of pointed spaces the inclusions $i_\alpha : X_\alpha \hookrightarrow \vee_{\beta\in A} X_\beta$ induce an isomorphism

$$\{i_\alpha^*\} : k^n(\vee_{\alpha\in A} X_\alpha) \xrightarrow{\cong} \prod_{\alpha\in A} k^n(X_\alpha),\ n \in \mathbb{Z}\,.$$

Let us also recall the following important results related to the representation of reduced cohomology theories by $\Omega$-spectra.

**Theorem 2.1.2.** [[Sw], Theorem 8.42] *Let $E$ be an $\Omega$-spectrum. The collection of contravariant functors*

$$k^n : h\,\mathcal{CW}_* \longrightarrow Ab$$

*given by*

$$k^n(X, x_0) = [X, x_0; E_n, *] \text{ for all } n \in \mathbb{Z}$$

*and of natural isomorphisms*

$$\sigma^n : k^n \longrightarrow k^{n+1} \circ S,\ n \in \mathbb{Z}$$

*given by*

$$[X, x_0; E_n, *] \xrightarrow{\cong} [X, x_0; \Omega E_{n+1}, *]$$

*defines a reduced cohomology theory for all $(X, x_0) \in \mathcal{CW}_*$.*

Remark that the converse exists and follows from the Brown Representability Theorem (see [Sw], Theorem 9.12).

**Theorem 2.1.3.** [[Sw], Theorem 9.27] *Let $k^*$ be a reduced cohomology theory . Then there exists an $\Omega$-spectrum $E$, such that*

$$k^n(X, x_0) = [X, x_0; E_n, *]$$

*for all $(X, x_0) \in \mathcal{CW}_*$.*

In this latter case, we say that the cohomology theory $k^*$ is **representable**, and it is represented by the spectrum $E$.

### 2.1.1 Definition of the smash product on spectra (non-symmetric version)

*Remark* 2.1.4. In this section we adopt an algebraic point of view on the spectra

Given two collections $E_* = \{E_n, *\}_{n\in\mathbb{N}}$ and $F_* = \{F_n, *\}_{n\in\mathbb{N}}$ of pointed $CW$-complexes define an associative product

$$\mu : E_* \odot F_* \longrightarrow (E \odot F)_*$$

such that

$$(E_* \odot F_*)_n := \vee_{k+l=n} E_k \wedge F_l \ \text{ for all } n \in \mathbb{N},$$

where $\vee$ and $\wedge$ denote respectively the wedge and the smash product of pointed topological spaces.

We also request the existence of the map

$$\tau : E_* \odot F_* \longrightarrow F_* \odot E_*,$$

such that $\tau_n(E_* \odot F_*)_n = (F_* \odot E_*)_n$ for all $n \in \mathbb{N}$.

In particular, the product $\odot$ is defined for $\mathbb{S}^* = \{\mathbb{S}_n, s_0\}_{n\in\mathbb{N}}$, the collection of spheres, giving a map $\mu : \mathbb{S}^* \odot \mathbb{S}^* \longrightarrow \mathbb{S}^*$. In *Top*$_*$, since $\mathbb{S}^n := \mathbb{S}^1 \wedge ... \wedge \mathbb{S}^1$, we have $\mathbb{S}^k \wedge \mathbb{S}^l \cong \mathbb{S}^{k+l}$, so the diagram

$$\begin{tikzcd}
\mathbb{S}^k \wedge \mathbb{S}^l \wedge \mathbb{S}^m \arrow[r, "\mathrm{Id} \wedge \mu_{l,m}"] \arrow[d, "\mu_{k,l} \wedge \mathrm{Id}"'] & \mathbb{S}^k \wedge \mathbb{S}_{l+m} \arrow[d, "\mu_{k,l+m}"] \\
\mathbb{S}_{k+l} \wedge \mathbb{S}_m \arrow[r, "\mu_{k+l,m}"] & \mathbb{S}_{m+n+k},
\end{tikzcd}$$

commutes strictly, by a simple rearrangement of copies of $\mathbb{S}^1$ without switching them, and makes $\mathbb{S}^*$ into a graded ring.

**Definition 2.1.5.** A collection $E_* = \{E_n, *\}_{n\in\mathbb{N}}$ of pointed $CW$-complexes is called an $\mathbb{S}^*$**-bimodule** if there exist associative actions

$$\lambda : \mathbb{S}_* \odot E^* \longrightarrow E_* \text{ and } \rho : E_* \odot \mathbb{S}^* \longrightarrow E_*$$

given for any $n = k + l$, $n \in \mathbb{N}$, by

$$\lambda_n : \mathbb{S}^k \wedge E_l \longrightarrow E_{k+l} \text{ and } \rho_n : E_k \odot \mathbb{S}^l \longrightarrow E_{k+l}.$$

The definition of an $\mathbb{S}^*$-bimodule can be summarized in the following four commutative diagrams:

$$\begin{tikzcd} E_* \odot E_* \odot E_* \arrow[r, "\mathrm{Id}\odot\mu"] \arrow[d, "\mu\odot\mathrm{Id}"'] & E_* \odot E_* \arrow[d, "\mu"] \\ E_* \odot E_* \arrow[r, "\mu"] & E_*, \end{tikzcd}$$

$$\begin{tikzcd} E_* \odot E_* \arrow[dd, "\tau"'] \arrow[dr, "\mu"] & \\ & E_*, \\ E_* \odot E_*, \arrow[ur, "\mu"'] & \end{tikzcd}$$

$$\begin{tikzcd} E_* \odot \mathbb{S}^* \arrow[rr, "\mathrm{Id}_{E_*}\odot\eta"] \arrow[dr, "\rho"'] & & E_* \odot E_* \arrow[dl, "\mu"] \\ & E_* & \end{tikzcd} \text{ and } \begin{tikzcd} \mathbb{S}^* \odot \mathbb{E}_* \arrow[rr, "\eta\odot\mathrm{Id}_{E_*}"] \arrow[dr, "\lambda"'] & & E_* \odot E_* \arrow[dl, "\mu"] \\ & E_*. & \end{tikzcd}$$

**Definition 2.1.6.** An $\mathbb{S}^*$-bimodule $E_* = \{E_n, *\}_{n\in\mathbb{N}}$ is a **spectrum** if $\lambda = \rho \circ \tau$. It is an $\Omega$-spectrum if $\lambda'_n : E_l \longrightarrow \Omega^l E_{k+l}$ is a weak homotopy equivalence for all $n \in \mathbb{N}$.

**Ring spectra**

**Definition 2.1.7.** Given spectra $E_* = \{E_n, *\}_{n\in\mathbb{N}}$, $F_* = \{F_n, *\}_{n\in\mathbb{N}}$ and $G_* = \{G_n, *\}_{n\in\mathbb{N}}$ , a **pairing** from $E_*$ and $F_*$ to $G_*$ is a collection of maps

$$\mu_{k,l} : E_k \wedge F_l \longrightarrow G_{k+l}$$

such that the diagrams

$$\begin{tikzcd} E_k \wedge F_l \wedge \mathbb{S}^1 \arrow[r, "\mu_{k,l}\wedge\mathrm{Id}"] \arrow[d, "\mathrm{Id}\wedge f_l"] & G_{k+l} \wedge \mathbb{S}^1 \arrow[d, "g_{k+l}"] \\ E_k \wedge F_{l+1} \arrow[r, "\mu_{k,l+1}"] & G_{k+l+m} \end{tikzcd}$$

and

$$\begin{tikzcd}
E_k \wedge F_l \wedge \mathbb{S}^1 \arrow[rr, "\mu_{k,l}\wedge \mathrm{Id}"] \arrow[d] & & G_{k+l} \wedge \mathbb{S}^1 \arrow[dd, "g_{k+l}"] \\
E_k \wedge \mathbb{S}^1 \wedge F_l \arrow[d, "e_k \wedge \mathrm{Id}"] & (-1)^n & \\
E_{k+1} \wedge F_l \arrow[rr, "\mu_{k+1,l}"] & & G_{k+l+1}
\end{tikzcd}$$

commute up to homotopy (for discussion on signs see the end of this Chapter).

**Definition 2.1.8.** A spectrum $E_* = \{E_n, *\}_{n\in\mathbb{N}}$ is called a commutative **ring spectrum** if there is a pairing $\mu : E_* \wedge E_* \longrightarrow E_*$ and a unit $\eta : \mathbb{S}^* \longrightarrow E_*$ such that the diagrams

$$\begin{tikzcd}
E_k \wedge E_l \wedge E_m \arrow[r, "\mathrm{Id}\wedge \mu_{l,m}"] \arrow[d, "\mu_{k,l}\wedge \mathrm{Id}"] & E_k \wedge E_{l+m} \arrow[d, "\mu_{k,l+m}"] \\
E_{k+l} \wedge E_m \arrow[r, "\mu_{k+l,m}"] & E_{k+l+m},
\end{tikzcd}$$

$$\begin{tikzcd}
E_k \wedge E_l \arrow[dr, "\mu_{k,l}"] \arrow[dd, "\tau"'] & \\
(-1)^{mn} & E_{k+l}, \\
E_l \wedge E_k \arrow[ur, "\mu_{l,k}"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
E_k \wedge \mathbb{S}^l \arrow[rr, "\mathrm{Id}\wedge \eta_l"] \arrow[dr, "\Sigma^k(e_l)"'] & & E_l \wedge E_k \arrow[dl, "\mu_{k,l}"] \\
& E_{k+l} &
\end{tikzcd}$$

commute up to homotopy (for discussion on signs see the end of this Chapter). Here $\mathbb{S}^*$ denotes the sphere spectrum.

To define a non-symmetric smash product on spectra, we refine the definition of the product $\odot$.

**Definition 2.1.9.** Let $(E_*, \odot)$ and $(F_*, \odot)$ be two $\mathbb{S}^*$-modules. We define the **smash product** between $E_*$ and $F_*$ to be the coequalizer of $(\mathrm{Id}\odot\lambda, \rho\odot \mathrm{Id})$:

$$E_* \odot \mathbb{S}^* \odot F_* \underset{\rho\odot\mathrm{Id}}{\overset{\mathrm{Id}\odot\lambda}{\rightrightarrows}} E_* \odot F_* \xrightarrow{k} E_* \odot_{\mathbb{S}^*} F_*$$

Formally, $E_* \odot_{\mathbb{S}^*} F_* := E_* \odot F_* / \sim$, where $\sim$ is the smallest relation in $E_* \odot F_*$ which contains pairs $(\mathrm{Id} \odot \lambda(e \odot s \odot f), \rho \odot \mathrm{Id}(e \odot s \odot f))$ for all $(e \odot s \odot f) \in E_* \odot \mathbb{S}^* \odot F_*$.
We denote this smash product by $E_* \wedge F_* := E_* \odot_{\mathbb{S}^*} F_*$.

**Definition 2.1.10.** Let $(E_*, \odot)$ and $(F_*, \odot)$ be two left $\mathbb{S}^*$-modules. A map $f : E_* \longrightarrow F_*$ is a **morphism of left $\mathbb{S}^*$-modules** if it respects their structure maps, i.e., if the diagram

$$\begin{tikzcd}
\mathbb{S}^* \odot E_* \arrow[r, "\lambda_E"] \arrow[d, "\mathrm{Id}_{\mathbb{S}^*} \odot f"'] & E_* \arrow[d, "f"] \\
\mathbb{S}^* \odot F_* \arrow[r, "\lambda_F"] & F_*
\end{tikzcd}$$

commutes, meaning that

$$\begin{tikzcd}
\mathbb{S}^k \wedge E_l \arrow[r, "\lambda_{n_E}"] \arrow[d, "\mathbb{S}^k \wedge f"'] & E_{k+l} \arrow[d, "f_{k+l}"] \\
\mathbb{S}^k \wedge F_l \arrow[r, "\lambda_{n_F}"] & F_{k+l}
\end{tikzcd}$$

commutes for all $k + l = n$, $n \in \mathbb{N}$.

A **morphism of right $\mathbb{S}^*$-modules** is defined similarly, using $\rho$.

**Proposition 2.1.11.** *Let $f : E_* \longrightarrow F_*$ be a morphism of left $\mathbb{S}^*$-modules and $f' : E'_* \longrightarrow F'_*$ be a morphism of right $\mathbb{S}^*$-modules. Then $f$ and $f'$ give a well-defined morphism $f' \wedge f : E'_* \wedge E_* \longrightarrow F'_* \wedge F_*$.*

### 2.1.2 External product in cohomology

The smash product on ring spectra leads to the definition of an **external product** in spectral cohomology. Recall that by definition $E^m(X, x_0) \cong [\mathbb{E}(X, x_0); \Sigma^m E, *]$, where $\mathbb{E}(X, x_0)_n := S^n(X, x_0)$, the $n$-th suspension of $(X, x_0) \in \mathcal{CW}_*$. Given $[f] \in [\mathbb{E}(X), \Sigma^m E]$ and $[g] \in [\mathbb{E}(Y), \Sigma^n E]$, let $p_{m,n}(f, g)$ denote the composite

$$\begin{tikzcd}
\mathbb{E}(X \wedge Y) \arrow[r] \arrow[rrrr, bend right, "p_{m,n}(f,g)"'] & \mathbb{E}(X) \wedge \mathbb{E}(Y) \arrow[r, "f \wedge g"] & \Sigma^m(E) \wedge \Sigma^n(E) \arrow[r, "\simeq"] & \Sigma^{m+n}(E \wedge E) \arrow[r, "\Sigma^{m+n}\mu"] & \Sigma^{m+n}E.
\end{tikzcd}$$

The external product is given by

$$p_{m,n} : E^m(X, x_0) \otimes E^n(Y, y_0) \longrightarrow E^{m+n}((X, x_0) \wedge (Y, y_0))$$

$$[f] \otimes [g] \mapsto [p_{m,n}(f,g)],$$

with associativity and commutativity coming from those of $\mu$.

Define a **graded ring multiplication** $P_{m,n}$ on $E^*(X,x_0)$ by

$$\begin{tikzcd} E^m(X,x_0) \otimes E^n(X,x_0) \arrow[r, "p_{m,n}"] \arrow[rr, bend right, "P_{m,n}"'] & E^{m+n}((X,x_0) \wedge (X,x_0)) \arrow[r, "\Delta^*"] & E^{m+n}(X,x_0), \end{tikzcd}$$

where $\Delta : X \longrightarrow X \wedge X$ is the diagonal map. Since this definition holds in particular for $(X,x_0) = (\mathbb{S}^0, s_0)$, $E^*(\mathbb{S}^0,s_0)$ is a graded ring.

Taking $(Y,y_0) = (\mathbb{S}^0, s_0)$ above, and choosing an $h \in \mathbb{E}^n(\mathbb{S}^0) \cong [\mathbb{E}(\mathbb{S}^0,s_0); \Sigma^n E, *]$, define a **graded action** $\eta_{m,n}$ of $E^*(\mathbb{S}^0,s_0)$ on $E^*(X,x_0)$ by

$$\begin{tikzcd} E^m(X,x_0) \otimes E^n(\mathbb{S}^0,s_0) \arrow[r, "p_{m,n}"] \arrow[rr, bend right, "\eta_{m,n}"'] & E^{m+n}((X,x_0) \wedge (\mathbb{S}^0,s_0)) \arrow[r, "\cong"] & E^{m+n}(X,x_0). \end{tikzcd}$$

We come up to the first main result of this section, concerning the structure of cohomology.

**Theorem 2.1.12.** *Let $(X,x_0) \in \mathcal{CW}_*$ and $E$ a commutative ring spectrum. Then $(E^*(X,x_0), \oplus, P, \eta)$ is a graded commutative algebra.*

*Sketch of the proof:* The multiplication $P_{m,n}$ is associative and commutative for every $(m,n)$, since $p_{m,n}$ is. In view of previous definitions the diagram

$$\begin{tikzcd} E^m(X,x_0) \otimes E^n(X,x_0) \otimes E^l(\mathbb{S}^0,s_0) \arrow[r, "\mathrm{Id} \otimes \eta_{n,l}"] \arrow[d, "P_{m,n} \otimes \mathrm{Id}"] & E^m(X,x_0) \otimes E^{n+l}(X,x_0) \arrow[d, "P_{m,n+l}"] \\ E^{m+n}(X,x_0) \otimes E^l(\mathbb{S}^0,s_0) \arrow[r, "\nu_{m+n,l}"] & E^{m+n+l}(X,x_0). \end{tikzcd}$$

commutes. $\square$

## 2.2 Multiplicative theories

**Definition 2.2.1.** Let $k^* : hTop_* \longrightarrow Ab$ be a reduced cohomology theory. It is called **multiplicative** if there exist natural transformations

$$\rho_{m,n} : k^m(X,x_0) \otimes k^n(Y,y_0) \longrightarrow k^{m+n}((X,x_0) \wedge (Y,y_0))$$

such that the diagrams

$$\begin{tikzcd} k^m(X,x_0) \otimes k^n(Y,y_0) \otimes k^l(Z,z_0) \arrow[r, "\rho_{m,n} \wedge \mathrm{Id}"] \arrow[d, "\mathrm{Id} \wedge \rho_{n,l}"'] & k^{m+n}((X,x_0) \wedge (Y,y_0)) \otimes k^l(Z,z_0) \arrow[d, "\rho_{m+n,l}"] \\ k^m(X,x_0) \otimes k^{n+l}((Y,y_0) \wedge (Z,z_0)) \arrow[r, "\rho_{m,n+l}"] & k^{m+n+l}((X,x_0) \wedge (Y,y_0) \wedge (Z,z_0)), \end{tikzcd}$$

$$\begin{tikzcd}
k^m(X,x_0)\otimes k^n(Y,y_0) \arrow[r, "\rho_{m,n}"] \arrow[d, "\cong"] & k^{m+n}((X,x_0)\wedge(Y,y_0)) \arrow[d, "\cong"] \\
k^n(Y,y_0)\otimes k^m(X,x_0) \arrow[r, "\rho_{n,m}"] & k^{m+n}((X,x_0)\wedge(Y,y_0))
\end{tikzcd}$$

and

$$\begin{tikzcd}
k^m(X,x_0)\otimes k^0(\mathbb{S}^0,s_0) \arrow[r, "\rho_{m,0}"] \arrow[rr, bend right, "\mathrm{Id}"'] & k^m((X,x_0)\wedge(\mathbb{S}^0,s_0)) \arrow[r, "\cong"] & k^m(X,x_0)
\end{tikzcd}$$

commute for all $(X,x_0)$, $(Y,y_0)$, $(Z,z_0)\in Top_*$.

Observe that the diagrams above make the multiplicative theory $k^*(X,x_0)$ into a $k^*(\mathbb{S}^0,s_0)$-graded commutative algebra for any $(X,x_0)\in\mathcal{CW}_*$, via the multiplication

$$k^m(X,x_0)\otimes k^n(X,x_0)\xrightarrow{\rho_{m,n}} k^{m+n}((X,x_0)\wedge(X,x_0))\xrightarrow{\Delta^*} k^{m+n}(X,x_0).$$

The last observation implies that the theorem below gives in fact the converse result of Theorem 2.1.12.

**Theorem 2.2.2.** [[H], Theorem I.4] *If an $\Omega$-spectrum $E$ represents a **multiplicative** theory $k^*$, then there exists a family of associative multiplications $\mu_{m,n}: E_m\wedge E_n\longrightarrow E_{m+n}$ and a unit $\eta:\mathbb{S}^*\longrightarrow E_*$, i.e., $E_*$ is a **ring** spectrum.*

*Idea of the proof:* In Definition 2.2.1 take $X=E_m$ and $Y=E_n$ to have

$$\begin{array}{ccc}
k^m(E_m)\otimes k^n(E_n) & \xrightarrow{\rho_{m,n}} & k^{m+n}(E_m\wedge E_n) \\
\| & & \| \\
[E_m,E_m]\otimes[E_n,E_n] & & [E_m\wedge E_n, E_{m,n}]
\end{array}$$

$$[\mathrm{Id}]\otimes[\mathrm{Id}]\mapsto \mu_{m,n}. \qquad \square$$

In conclusion, a multiplicative cohomology theory defines a ring $\Omega$-spectrum, and conversely, given a ring $\Omega$-spectrum, the associated cohomology theory is a multiplicative theory.

*Remark* 2.2.3. The definition of the smash product for ring spectra we gave in this chapter could be a rather inconvenient one, depending on one's needs, since commutativity holds only up to a sign. Considerable progress in defining "the right" smash product were made in late 90's, which had an important impact on stable homotopy theory. One can find more material on this subject in [EKMM97] and [HSS].

## 2.3 Note on signs

Here is a possible explanation of what the sign in the diagrams of Definitions 2.1.7 and 2.1.8 could mean.

Recall first that for a spectrum $E = \{(E_n, *), e_n\}$, the maps

$$e_n : (SE_n, *) \xrightarrow{\approx} (E_{n+1}, *)$$

are base-point preserving homeomorphisms for all $n \in \mathbb{N}$. The equivalence $SE_n \simeq E_n \wedge \mathbb{S}^1$ gives for all $n \in \mathbb{N}$ homotopy equivalences

$$E_0 \wedge \mathbb{S}^n \xrightarrow{\simeq} E_n.$$

Therefore, we have

$$E_m \wedge F_n \wedge \mathbb{S}^1 \simeq E_m \wedge F_0 \wedge \underbrace{\mathbb{S}^1 \wedge ... \wedge \mathbb{S}^1}_{n} \wedge \mathbb{S}^1,$$

$$E_m \wedge \mathbb{S}^1 \wedge F_n \simeq E_m \wedge \mathbb{S}^1 \wedge F_0 \wedge \underbrace{\mathbb{S}^1 \wedge ... \wedge \mathbb{S}^1}_{n}.$$

Notice that the latter circle $\mathbb{S}^1$ of the first expression has switched over $n+1$ terms, of which $n$ were $\mathbb{S}^1$, too.

Now, let us view $\mathbb{S}^2$ as a manifold of dimension 2, and for any point $(x, y) \in \mathbb{S}^2$ let us consider maps

$$\mathrm{Id} : \mathbb{S}^1 \wedge \mathbb{S}^1 \simeq \mathbb{S}^2 \longrightarrow \mathbb{S}^1 \wedge \mathbb{S}^1 \simeq \mathbb{S}^2$$

$$(x, y) \mapsto (x, y)$$

and

$$\tau : \mathbb{S}^1 \wedge \mathbb{S}^1 \simeq \mathbb{S}^2 \longrightarrow \mathbb{S}^1 \wedge \mathbb{S}^1 \simeq \mathbb{S}^2$$

$$(x, y) \mapsto (y, x).$$

We claim that $\tau$ and Id are not homotopic. This can be proved by viewing that they induce different linear maps on the tangent space $T_{(x,y)}$ in $(x, y) \in \mathbb{S}^2$. Indeed,

$$T_{(x,y)}(\mathrm{Id}) : T_{(x,y)}(\mathbb{S}^2) \longrightarrow T_{(x,y)}(\mathbb{S}^2)$$

and

$$T_{(x,y)}(\tau) : T_{(x,y)}(\mathbb{S}^2) \longrightarrow T_{(y,x)}(\mathbb{S}^2)$$

are given respectively by matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

so that

$$\det(T_{(x,y)}(\mathrm{Id})) = 1, \ \det(T_{(x,y)}(\tau)) = -1.$$

Hence

$$sgn(\det(T_{(x,y)}(\mathrm{Id})) \neq sgn(\det(T_{(x,y)}(\tau)),$$

which implies that Id and $\tau$ are not homotopic (see [Fu] 5.2.2.7) (although $\tau \circ \tau = \mathrm{Id}$).
In other words, the linear map $T_{(x,y)}(\tau)$ induced in the tangent space by

$$\tau : \mathbb{S}^1 \wedge \mathbb{S}^1 \simeq \mathbb{S}^2 \longrightarrow \mathbb{S}^1 \wedge \mathbb{S}^1 \simeq \mathbb{S}^2$$

$$(x, y) \mapsto (y, x)$$

switching the two copies of $\mathbb{S}^1$, is identity with the opposite sign.
This argument shows that the $(-1)^n$ in the diagram could possibly correspond to switching the circle $\mathbb{S}^1$ over its $n$-fold smash product in

$$E_m \wedge F_0 \wedge \underbrace{\mathbb{S}^1 \wedge ... \wedge \mathbb{S}^1}_{n} \wedge \mathbb{S}^1 \longrightarrow E_m \wedge \mathbb{S}^1 \wedge F_0 \wedge \underbrace{\mathbb{S}^1 \wedge ... \wedge \mathbb{S}^1}_{n}.$$

On the other hand, since the argument uses the manifold structure on $\mathbb{S}^2$, interchanging $\mathbb{S}^1$ and $F_0$ would not change anything.

The explanation provided remains somewhat hypothetical, but it has at least the merit to hold in the context of Definition 2.1.8. Indeed, to check the sign consider the sequence of homotopy equivalences

$$\begin{tikzcd}
E_m \wedge E_n \arrow[d, "\simeq"'] \\
E_0 \wedge \underbrace{\mathbb{S}^1 \wedge ... \wedge \mathbb{S}^1}_{m} \wedge E_0 \wedge \underbrace{\mathbb{S}^1 \wedge ... \wedge \mathbb{S}^1}_{n} \arrow[d, "\simeq"', "(-1)^?"] \\
E_0 \wedge \underbrace{\mathbb{S}^1 \wedge ... \wedge \mathbb{S}^1}_{n} \wedge E_0 \wedge \underbrace{\mathbb{S}^1 \wedge ... \wedge \mathbb{S}^1}_{m} \arrow[d, "\simeq"'] \\
E_n \wedge E_m
\end{tikzcd},$$

and count explicitly the total number of order-preserving substitutions needed to switch over the $m$-fold and the $n$-fold smash products of circles. One finds $nm$, as suggested by the power of $(-1)^{nm}$ in the second diagram of Definition 2.1.8.

# Chapter 3

# Complex vector bundles

## 3.1 The category $Bund_{\mathbb{C}}$

**Definition 3.1.1.** A **complex vector bundle** is a triple $\xi = (E, p, B)$ where $E$ and $B$ are topological spaces, and such that the following conditions are satisfied:

**(i)** the map $p : E \longrightarrow B$ is continuous and surjective;

**(ii)** for all $b \in B$, the space $p^{-1}(b)$ has the structure of a complex vector space $V$;

**(iii)** *Local triviality condition:* for all $b \in B$, there exists an open neighborhood $U_b$ of $b$ and a homeomorphism

$$\varphi_{U_b} : U_b \times V \longrightarrow p^{-1}(U_b)$$

satisfying

$$p \circ \varphi_{U_b}(b, v) = b, \text{ for all } (b, v) \in U_b \times V.$$

Moreover, we want $\varphi$ to be consistent with the vector space structure in the fibers, i.e., we want

$$\varphi_{U_b}|_{\{b\}\times V} : \{b\} \times V \longrightarrow p^{-1}(b)$$

to be an isomorphism of vector spaces for all $b \in B$.

**Terminology:**

- For any vector bundle $\xi = (E, p, B)$, the spaces $E$ and $B$ are called respectively the **total space** and the **base space**, and $p$ is the **projection** of the bundle;

- For all $b$, the space $p^{-1}(b) \in E$ is the **fiber** of the vector bundle over $b \in B$; we shall denote it by $E_b$ also.

*Remark on dimensions:* Let $\xi = (E, p, B)$ be a complex vector bundle. If for each $b \in B$, the dimension of the fiber $E_b$ is the same and constantly equal to an $n \geq 0$, we say that $\xi$ is an **complex vector $n$-bundle**, and we replace the complex vector space $V$ by $\mathbb{C}^n$ in previous definitions.

*Remark* 3.1.2. One can define real or quaternionic $n$-vector bundles in a similar way (replacing $\mathbb{C}^n$ by $\mathbb{R}^n$, resp. $\mathbb{H}^n$). However, we shall mainly concentrate on *complex* vector bundles in this project, which is why the term "complex"will sometimes be dropped, and we shall specify the field when needed.

Here are some examples of $n$-vector bundles we shall be working with.

**Example 3.1.3.** The **trivial $n$-bundle** over $B$,

$$\epsilon = (B \times \mathbb{C}^n, p, B),$$

where

$$p : B \times \mathbb{C}^n \twoheadrightarrow B$$
$$(b, v) \mapsto b$$

is the projection on the first factor.

**Example 3.1.4.** *The **universal bundle.*** Let $G_n(\mathbb{C}^k)$ be a complex Grassmanian manifold. We define

$$E_n(\mathbb{C}^k) = \{(V, v) \in G_n(\mathbb{C}^k) \times \mathbb{C}^k \,|v \in V\},$$

and a projection

$$\pi : E_n(\mathbb{C}^k) \twoheadrightarrow G_n(\mathbb{C}^k)$$

by

$$(V, v) \mapsto V.$$

The triple $\gamma_{n,k} = (E_n(\mathbb{C}^k), \pi, G_n(\mathbb{C}^k))$ is the canonical complex $n$-bundle. Its characterization as "universal"will be clarified later, although we can already point out the importance of the $\gamma_{n,k}$ bundles, due to their use in the Classification Theorem 4.1.2.

Here is another example of a bundle with a similar construction, we shall use it in some proofs.

**Example 3.1.5.** Define

$$E'_n(\mathbb{C}^k) = \{(V, u) \in G_n(\mathbb{C}^k) \times \mathbb{C}^k \,|u \in V^\perp\},$$

and the corresponding projection on the first factor

$$\pi' : E'_n(\mathbb{C}^k) \twoheadrightarrow G_n(\mathbb{C}^k).$$

The triple $\eta_{n,k} = (E'_n(\mathbb{C}^k), \pi', G_n(\mathbb{C}^k))$ is a complex $(k - n)$-bundle.

Once the "direct sum operation" on vector bundles is defined, we shall see that these two bundles are related; namely, $\gamma_{n,k} \oplus \eta_{n,k}$ is the trivial $k$-bundle over $G_n(\mathbb{C}^k)$.

*Remark* 3.1.6. The two former examples make sense if $k = \infty$, too.

A vector bundle morphism is a fiber-preserving map that is linear on each fiber. More precisely, one has the following definition.

**Definition 3.1.7.** Let $\xi = (E, p, B)$ and $\xi' = (E', p', B')$ be two vector bundles. A **morphism** of vector bundles $(f, d) : \xi \longrightarrow \xi'$ is defined by two maps $f : E \longrightarrow E'$ and $g : B \longrightarrow B'$ such that the diagram

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ B \arrow[r, "g"'] & B' \end{tikzcd}$$

commutes, i.e., $p' \circ f = g \circ p$, and such that the restriction

$$f : p^{-1}(b) \longrightarrow p^{-1}(f(b))$$

is linear for each $b \in B$.

*Remark* 3.1.8. In the previous definition we can set $B = B'$. In this case, the bundles $\xi$ and $\xi'$ are defined over the same base, and a morphism $(f, id_B)$ : $\xi \longrightarrow \xi'$ requires the triangle

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & B & \end{tikzcd}$$

to commute.

**Definition 3.1.9.** Two bundles $\xi$ and $\xi'$ over the same space $B$ are said to be **isomorphic** if there exists a bundle morphism $(f, id_B) : \xi \longrightarrow \xi'$ such that $f : E \longrightarrow E'$ is a homeomorphism and the restriction $f : p^{-1}(b) \longrightarrow (p')^{-1}(b)$ is a linear isomorphism on each fiber, for all $b \in B$.

At this point, we can talk about the **category of complex vector bundles**, which we shall denote by $Bund_{\mathbb{C}}$. Its objects and morphisms have been defined in 3.1.1 and 3.1.7. The composition law and the identity for morphisms come from $Top$ and $\mathcal{V}_{\mathbb{C}}$.

Note that for each $B \in Top$, $Bund_{\mathbb{C}}$ admits as subcategory $Bund_{\mathbb{C}}(B)$, the **category of complex vector bundles over** $B$.

Finally, remark that "dimension is preserved", i.e., for every $n \geq 0$, the $n$**-dimensional complex vector bundles** form a category too, which will be written $Bund_{\mathbb{C}^n}$.

## 3.2 Constructions on vector bundles

**Definition 3.2.1.** Let $\xi = (E, p, B)$ be a complex vector bundle and $f : Y \longrightarrow B$ a continuous map. The bundle **induced by $f$ from $\xi$**, denoted by $f^*(\xi)$, is defined as follows. Set $Y \times_B E = \{(y, e) \in Y \times B | f(y) = p(e)\}$ and consider the projections $p_Y : Y \times_B E \longrightarrow Y$ and $p_E : Y \times_B E \longrightarrow E$, then we want the diagram below to commute:

$$\begin{tikzcd} Y \times_B E \arrow[r, "p_E"] \arrow[d, "p_Y"'] & E \arrow[d, "p"] \\ Y \arrow[r, "f"'] & B. \end{tikzcd}$$

Note that $Y \times_B E := f^*(E)$, the total space of $f^*(\xi)$, is precisely the pullback of $p_Y$ and $p_E$.

The next two results will become important as we come to Chapter 3, especially for results on homotopy invariance of some functors. We do not give the proofs here.

**Proposition 3.2.2.** [Ha2], Proposition 1.7] *The restrictions of a vector bundle $p : E \longrightarrow B \times I$ over $B \times \{0\}$ and $B \times \{1\}$ are isomorphic if $B$ is compact Hausdorff.*

**Theorem 3.2.3.** [Ha2], Theorem 1.6] *Given a vector bundle $p : E \longrightarrow B$ and homotopic maps $f_0, f_1 : A \longrightarrow B$, the induced bundles $f_0^*(E)$ and $f_1^*(E)$ are isomorphic if $A$ is compact Hausdorff.*

**Definition 3.2.4.** Let $\xi = (E_1, p_1, B)$ and $\xi' = (E_2, p_2, B) \in Bund_{\mathbb{C}^n(B)}$. Define a new bundle

$$\xi \oplus \xi' := (E_1 \oplus E_2, p_1 \oplus p_2, B)$$

where

$$E_1 \oplus E_2 := \{(e_1, e_2) \in E_1 \times E_2 | p_1(e_1) = p_2(e_2)\}$$

and the projection

$$p_1 \oplus p_2 : E_1 \oplus E_2 \longrightarrow B$$

is given by

$$(e_1, e_2) \mapsto p_1(e_1) = p_2(e_2).$$

The operation $\xi \oplus \xi'$ is called the **Whitney sum of $\xi$ and $\xi'$**, and defines a product in the category $Bund_{\mathbb{C}^n}(B)$.

As promised in the previous section, we are now able to define an isomorphism

$$f : \gamma_{n,k} \oplus \eta_{n,k} \xrightarrow{\cong} \epsilon^k$$

by

$$((V,x),(V,y)) \mapsto (V, x+y),$$

where $V \in G_n(\mathbb{C}^k)$, $(V,x) \in E_n(\mathbb{C}^k)$, $(V,y) \in E'_n(\mathbb{C}^k)$. Since every $z \in \mathbb{C}^k$, there is a unique decomposition $z = x + y$ where $x \in V$ and $y \perp x$. Because this decomposition is continuous in V (direct sum for vector spaces), the map $f$ is an isomorphism over $G_n(\mathbb{C}^k)$. This result will be used in Lemma 4.3.2.

As a complement to Definition 3.2.4, observe that for $\xi$ and $\xi'$ as above, we have

$$E_1 \oplus E_2 \cong E_1 \times_B E_2,$$

by the universal property of the pullback outlined in the diagram

$$\begin{tikzcd}
E_1 \oplus E_2 \arrow[dr, dotted, "\cong", "\exists!"'] \arrow[drr, bend left, "p_{E_2}"] \arrow[ddr, bend right, "p_{E_1}"'] & & \\
& E_1 \times_B E_2 \arrow[r, "p_{E_2}"] \arrow[d, "p_{E_1}"'] & E_2 \arrow[d, "p_2"] \\
& E_1 \arrow[r, "p_1"'] & B,
\end{tikzcd}$$

and in view of Definition 3.2.4.

## 3.3 Continuous functors and operations in $Bund_{\mathbb{C}}(B)$

The Whitney sum we have just defined for vector bundles over a space $B$ is derived from the direct sum of vector spaces, and is, as we said, the categorical product on $Bund_{\mathbb{C}}(B)$. This phenomenon generalizes to other operations: in fact, every continuous operation on vector spaces defines a corresponding operation on vector bundles in a natural way. The aim of this section is to explain how to do it.

In the following definition, let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and recall that objects of $\mathcal{V}_{\mathbb{K}}$ are finite-dimensional $\mathbb{K}$-vector spaces.

**Definition 3.3.1.** A functor $F : \mathcal{V}_{\mathbb{K}} \longrightarrow \mathcal{V}_{\mathbb{K}}$ is called **continuous** if for every pair $(M,N) \in Ob\mathcal{V}_{\mathbb{K}}$, the map

$$F_{M,N} : \mathcal{V}_{\mathbb{K}}(M,N) \longrightarrow \mathcal{V}_{\mathbb{K}}(F(M),F(N))$$

is continuous with respect to the usual topology of $\mathbb{K}$.

Now we concentrate on $\mathbb{K} = \mathbb{C}$, and our goal is to associate any such functor $F$ with a functor

$$F' = F'(B) : Bund_{\mathbb{C}}(B) \longrightarrow Bund_{\mathbb{C}}(B),$$

such that if $B = \{x_0\}$ is a space restricted to a point, we have

$$F'(x_0) = F. \quad (\circledast)$$

Let $\xi = (E, \pi, B) \in Bund_{\mathbb{C}}(B)$. We define

$$F' : ObBund_{\mathbb{C}}(B) \longrightarrow ObBund_{\mathbb{C}}(B)$$
$$\xi \mapsto F'(\xi) = (E', p', B),$$

where

$$E' = F'(E) := \bigsqcup_{b \in B} F(E_b) \in Set,$$

and

$$(p' : E' \twoheadrightarrow B) := x \in F(E_b) \mapsto b.$$

Next, we need to provide $E'$ with a topology, so that $F'(\xi)$ becomes a vector bundle. For this purpose, a technical lemma is required.

**Lemma 3.3.2.** [[Ka], Lemma 4.4] *Let $U$ and $V$ be open subsets of $B$ and let $\varphi_U : E_U \xrightarrow{\approx} U \times M$ and $\varphi_V : E_V \xrightarrow{\approx} V \times N$ be local trivializations of $E$ over $U$ and $V$ respectively, where $M$ and $N$ are finite-dimensional vector spaces. Let $\varphi' : E'_U \xrightarrow{\approx} U \times F(M)$ and $\varphi' : E'_V \xrightarrow{\approx} V \times F(N)$ be the bijections induced by $F$ on each fiber. If we give $E'_U$ and $E'_V$ the topologies induced by these bijections, these two topologies agree on $E'_U \cap E'_V = E'_{U \cap V}$ and $E'_{U \cap V}$ is open in both $E'_U$ and $E'_V$.*

We are now able to define the topology on $E'$. Let $\{U_i\}$ be an open covering of $B$, and let $\varphi_i : E_{U_i} \xrightarrow{\approx} U_i \times M_i$ be a trivialization of $E$ over $U_i$, for all $i \in I$. By functoriality of $F$ the isomorphisms $\varphi_i$ induce bijections $E'_{U_i} \xrightarrow{\approx} U_i \times F(M_i)$ for all $i \in I$, and in this way $E'_{U_i}$ may be provided with a topology. In fact, we need to provide $E'$ with the largest topology making the inclusions $E'_{U_i} \hookrightarrow E'$ continuous. This is possible because of the previous lemma, because for each pair $(i, j)$ the topologies on $E'_{U_i}$ and $E'_{U_j}$ agree on $E'_{U_i \cap U_j}$, making it an open subset of $E'_{U_i}$ and $E'_{U_j}$. One can show that this topology depends neither on the choice of covering, nor on the choice of trivializations.

It remains to define the functor $F'$ on bundle morphisms. Given two bundles $\xi = (E, p, B)$, $\eta = (G, q, B)$, and a morphism $\varphi : \xi \longrightarrow \eta$, define $F'(\varphi)$ to be

$$F' : MorBund_{\mathbb{C}}(B) \longrightarrow MorBund_{\mathbb{C}}(B)$$
$$(\varphi : \xi \longrightarrow \eta) \mapsto F'(\varphi) : F'(E) \longrightarrow F'(G),$$

with $F'(\varphi) = \varphi'$ linear, defined on each fiber by

$$\varphi'_b := F(\varphi_b) : F(E_b) \longrightarrow F(G_b)$$

for all $b \in B$.

Finally, Theorem I. 1.12 in [Ka] and the discussion in Section I.4.6 in [Ka] provide arguments for the continuity of $F'(\varphi)$.

**Application**

As we said before, we are mainly concerned with proving the naturality of the direct sum and tensor product operations induced from vector spaces on vector bundles. In particular, this would imply the functoriality of these two operations in $Bund_{\mathbb{C}}(B)$, which was already clear for the Whitney sum in Definition 3.2.4, since it was defined to be the product in $Bund_{\mathbb{C}}(B)$.
The two continuous functors we shall consider are

$$S, T : \mathcal{V}_{\mathbb{C}^n}(B) \times \mathcal{V}_{\mathbb{C}^n}(B) \longrightarrow \mathcal{V}_{\mathbb{C}^n}(B)$$

$$S(M, N) = M \oplus N$$

$$T(M, N) = M \otimes N.$$

According to preceding considerations, $S'$ is defined by:

$$S' : ObBund_{\mathbb{C}}(B) \times ObBund_{\mathbb{C}}(B) \longrightarrow ObBund_{\mathbb{C}}(B)$$

$$(\xi, \eta) \mapsto S'(\xi, \eta) = (H, \pi, B),$$

with

$$H = S'(E, G)(B) := \bigsqcup_{b \in B} F(E_b, G_b) = \bigsqcup_{b \in B} E_b \oplus G_b$$

and $\pi : H \twoheadrightarrow B$ is the obvious projection on $B$.
Note that if $B = \{b\}$, $S'(E, G)(\{b\}) = F(E_b, G_b) = E_b \oplus G_b$. In other words, the fiber of $\xi \oplus \eta$ over a point $b$ is the direct sum of fibers of $\xi$ and $\eta$ over $b$, so that the condition ($\circledast$) is satisfied.

On the other hand,

$$S' : MorBund_{\mathbb{C}}(B) \times MorBund_{\mathbb{C}}(B) \longrightarrow MorBund_{\mathbb{C}}(B)$$

$$(\varphi, \psi) \mapsto S'(\varphi, \psi) : S'(E, G) \longrightarrow S'(P, Q)$$

is defined on each fiber by

$$S'(\varphi_b, \psi_b) := \varphi_b + \psi_b = S(\varphi_b, \psi_b).$$

The definition of $T'$ requires a simple substitution of $\oplus$ for $\otimes$ in the above description, and is not written for the sake of brevity. However, we note that the construction is "well-defined": if $f : \xi \oplus \eta \longrightarrow \zeta$ is a bundle map which is bilinear on each fiber $\forall b \in B$, then $f$ defines a vector bundle morphism $g : \xi \otimes \eta \longrightarrow \zeta$, obtained by the usual factorization of bilinear maps through the tensor product (of vector spaces) on each fiber.

Furthermore, all the usual properties of commutativity and distributivity of $\oplus$ and $\otimes$ in $\mathcal{V}_{\mathbb{C}}$ extend to $Bund_{\mathbb{C}}(B)$. A reformulation of Theorems V.6.2 - V.6.4 in [Hu] gives the following property.

**Proposition 3.3.3.** *Let $F, G : \mathcal{V}_{\mathbb{K}} \longrightarrow \mathcal{V}_{\mathbb{K}}$ be two continuous functors, and let $\tau : F \Longrightarrow G$ be a natural transformation, i.e.,*

$$\tau : Ob\mathcal{V}_{\mathbb{K}} \longrightarrow Mor\mathcal{V}_{\mathbb{K}}$$

$$M \mapsto \tau(M) : F(M) \longrightarrow G(M).$$

*$\tau$ induces a natural transformation $\tau' : F' \Longrightarrow G'$:*

$$\tau' : ObBund_{\mathbb{K}}(B) \longrightarrow MorBund_{\mathbb{K}}(B)$$

$$\xi \mapsto \tau'(\xi) : F'(\xi) \longrightarrow G'(\xi).$$

*In particular, if $\tau(M) : F(M) \longrightarrow G(M)$ is an isomorphism for every $M \in \mathcal{V}_{\mathbb{K}}$, so is $\tau'(\xi)$, for every $\xi \in Bund_{\mathbb{K}}(B)$.*

As an example, let us see that the tensor product is associative in $Bund_{\mathbb{C}}(B)$. Consider first

$$F, G : \mathcal{V}_{\mathbb{C}} \times \mathcal{V}_{\mathbb{C}} \times \mathcal{V}_{\mathbb{C}} \longrightarrow \mathcal{V}_{\mathbb{C}}$$

$$F(M, N, K) = (M \otimes N) \otimes K$$

$$G(M, N, K) = M \otimes (N \otimes K),$$

which are both continuous. A natural transformation between $F$ and $G$ is given by

$$\tau(M, N, K) : (M \otimes N) \otimes K \longrightarrow M \otimes (N \otimes K)$$

$$(m \otimes n) \otimes k \mapsto m \otimes (n \otimes k).$$

It is easy to verify that $\tau$ is well-defined, linear and has an inverse. Hence it is a continuous isomorphism of vector spaces.

The induced functors

$$F', G' : Bund_{\mathbb{K}}(B) \times Bund_{\mathbb{K}}(B) \times Bund_{\mathbb{K}}(B) \longrightarrow Bund_{\mathbb{K}}(B)$$

are given by

$$F'(\xi, \eta, \zeta) = (\xi \otimes \eta) \otimes \zeta,$$

$$G'(\xi, \eta, \zeta) = \xi \otimes (\eta \otimes \zeta),$$

and the induced natural transformation by

$$\tau'(\xi, \eta, \zeta) : (\xi \otimes \eta) \otimes \zeta \longrightarrow \xi \otimes (\eta \otimes \zeta).$$

Explicitly, if $\xi = (E, p, B)$, $\eta = (G, q, B)$ and $\zeta = (H, r, B)$, one gets

$$(\xi \otimes \eta) \otimes \zeta := (A, \pi, B),\ \xi \otimes (\eta \otimes \zeta) := (C, \pi', B)$$

and

$$(A, \pi, B) \xrightarrow{\tau'} (C, \pi', B)$$

where

$$A = \bigsqcup_{b \in B} (E_b \otimes G_b) \otimes H_b \text{ and } C = \bigsqcup_{b \in B} E_b \otimes (G_b \otimes H_b).$$

But for any fixed $b \in B$, the map

$$(E_b \otimes G_b) \otimes H_b \longrightarrow E_b \otimes (G_b \otimes H_b)$$

is a continuous isomorphism of vector spaces! Hence, $\tau'$ is a natural isomorphism and we obtain the associativity of the tensor product for vector bundles.

## 3.4 The semi-ring $Vect_{\mathbb{C}}(B)$

Two bundles $\xi = (E, p, B)$ and $\xi' = (E', p', B)$ over $B$ are said to be **equivalent** if they are isomorphic (see Definition 3.1.9). One can check that this relation is an *equivalence relation* in the set of all complex vector bundles over $B$. We shall write $\xi \simeq \xi'$ if $\xi$ and $\xi'$ are equivalent, and denote $[\xi]$ the equivalence class of $\xi$. We denote $Vect_{\mathbb{C}}(B)$ the set of equivalence classes of all complex vector bundles over $B$. When considering $\xi$, an $n$-vector bundle, we shall identify all fibers of $\xi$ with $\mathbb{C}^n$, which will be specified by $\xi \in Vect_{\mathbb{C}^n}(B)$.

The next Lemma says that the operations $\oplus$ and $\otimes$, defined on $Bund_{\mathbb{C}}(B)$, pass to $Vect_{\mathbb{C}}(B)$, i.e., they preserve the equivalence relation $\simeq$.

**Lemma 3.4.1.** *The pair of maps*

$$Vect_{\mathbb{C}^n} : Top \longrightarrow Set$$

*defined by*

$$X \mapsto Vect_{\mathbb{C}^n}(X)$$

*and*

$$(f : Y \longrightarrow X) \mapsto Vect_{\mathbb{C}^n}(f) : Vect_{\mathbb{C}^n}(X) \longrightarrow Vect_{\mathbb{C}}(Y)$$
$$[\xi] \mapsto [f^*(\xi)]$$

*is a contravariant functor, for every $n \in \mathbb{N}$.*

*Proof.* $Vect_{\mathbb{C}^n}$ ***is well-defined:*** Let $f : Y \longrightarrow B$, and let $\xi = (E, p, B)$ and $\xi' = (E', p', B)$, such that $\xi \simeq \xi'$. By definition there exists a homeomorphism $\phi : E \longrightarrow E'$ which makes the diagram of Definition 3.1.9 commute. We construct the induced bundles $f^*(\xi) = (Y \times_B E, p_y, Y)$ and $f^*(\xi') = (Y \times_B E', p_y, Y)$ and notice that the triangle

$$\begin{tikzcd} Y \times_B E \arrow[rr, "\phi^*"] \arrow[dr, "p_Y"'] & & Y \times_B E' \arrow[dl, "p_Y"] \\ & Y, & \end{tikzcd}$$

commutes, where $\phi^* = (id_Y, \phi) : Y \times_B E \longrightarrow Y \times_B E'$ is a homeomorphism defined by $(y, e') \mapsto (y, \phi(e))$. Indeed

$$\begin{aligned} p_Y \circ \phi^*(y, e) &= p_Y(y, \phi(e)) \\ &= y \\ &= p_Y(y, e). \end{aligned}$$

It follows that the restriction on each fiber $\phi^* : p_Y^{-1}(y) \longrightarrow (p)_Y^{-1}(y)$ is just the identity. This shows that $f^*(\xi) \simeq f^*(\xi')$ over $Y$.

***Composition:*** Let $f : Y \longrightarrow B$ and $g : Z \longrightarrow Y$. We need to see if $Vect_{\mathbb{C}^n}(f \circ g) = Vect_{\mathbb{C}^n}(g) \circ Vect_{\mathbb{C}^n}(f)$. Basically, this follows from the properties of pullbacks. Indeed, we have to check that for any $\xi \in Vect_{\mathbb{C}}(B)$, $[(f \circ g)^*(\xi)] = [g^* \circ f^*(\xi)]$. Once the two corresponding pullback diagrams are drawn, it remains to find a homeomorphism $\psi$ such that the triangle

$$\begin{tikzcd} Z \times_Y (Y \times_B E) \arrow[rr, dotted, "\psi"] \arrow[dr, "p_Z"'] & & Z \times_B E \arrow[dl, "p_Z"] \\ & Z & \end{tikzcd}$$

commutes. This can be done by showing that the diagram

$$\begin{tikzcd} Z \times_B E \arrow[r, "\alpha"] \arrow[d, "p_Z"'] & Y \times_B E \arrow[d, "p_Y"] \\ Z \arrow[r, "g"'] & Y, \end{tikzcd}$$

with

$$\alpha : Z \times_B E \longrightarrow Y \times_B E : (z, e) \mapsto (g(z), e)$$

is a pullback diagram. For any $X \in \mathit{Top}$, and continuous functions

$$f : X \longrightarrow Z : x \mapsto f(x),$$

$$(h_1, h_2) : X \longrightarrow Y \times_B E : x \mapsto (h_1(x), h_2(x))$$

in the diagram

$$\begin{tikzcd}
X \arrow[dr, dotted, "\psi"] \arrow[drr, bend left, "(h_1,h_2)"] \arrow[ddr, bend right, "f"'] & & \\
& Z \times_B E \arrow[r, "\alpha"] \arrow[d, "p_Z"'] & Y \times_B E \arrow[d, "p_Y"] \\
& Z \arrow[r, "g"'] & Y,
\end{tikzcd}$$

such that $g \circ f(x) = p_Y \circ (h_1, h_2)(x) = h_1(x)$ for all $x \in X$, one defines

$$\psi : X \longrightarrow Z \times_B E$$

$$x \mapsto (f(x), h_2(x))$$

and then checks that $f(x) = p_Z \circ \psi(x)$ and $(h_1, h_2)(x) = \alpha \circ \psi(x)$ for all $x \in X$, and that $\psi$ is unique up to isomorphism. Hence, taking $X := Z \times_Y (Y \times_B E)$, $f := pr_Z$, $(h_1, h_2) := (g \circ pr_Z, pr_E)$ one obtains the desired homeomorphism

$$\psi : Z \times_Y (Y \times_B E) \longrightarrow Z \times_B E$$

$$(z, (g(z), e)) \mapsto (z, e).$$

The last thing to verify is that the restriction of $\psi$ on each fiber for all $b \in B$, is an isomorphism. Replacing $B$ by a single point $b$ in previous diagrams, all fiber products become products and the same argument is true.

***Identity:*** The same remarks apply.

□

*Remark* 3.4.2. By Theorem 3.2.3, the contravariant functor $\mathit{Vect}_{\mathbb{C}}(B)$ is an invariant of homotopy type.

In conclusion, $(\mathit{Vect}_{\mathbb{C}}(B), \otimes, \oplus)$ admits a semi-ring structure, with the trivial 0-bundle $\epsilon^0 : B \times \{*\} \longrightarrow B$ as identity for $\oplus$, and the trivial 1-bundle $\epsilon^1 : B \times \mathbb{C} \longrightarrow B$ as identity for $\otimes$.

# Chapter 4

# The first group of topological $K$-theory, $K(X)$

## 4.1 The Classification Theorem

*Remark* 4.1.1. For psychological reasons, from now on we shall be working with vector bundles whose base space is called $X$ (and not $B$). Furthermore, $X$ will be assumed to be compact Hausdorff.

Recall Example 3.1.4, where the universal bundle $\gamma_{n,k} = (E_n(\mathbb{C}^k), \pi, G_n(\mathbb{C}^k))$ was defined. We are now ready to state the Classification Theorem mentioned before, which involves $\gamma_{n,k}$. As we shall mainly be interested in the case $k = \infty$, let us set the following notation:

$$G_n(\mathbb{C}^\infty) := G_n \ , \ E_n(\mathbb{C}^\infty) := E_n \text{ and } \gamma_{n,\infty} := \gamma_n.$$

**Theorem 4.1.2.** [[Hi], IV, Theorem 4.1] *Let $X$ be a finite-dimensional CW-complex. Then the function*

$$[X, G_n] \xrightarrow{\cong} Vect_{\mathbb{C}^n}(X)$$

$$[f] \mapsto f^*(\gamma_n)$$

*is a bijection.*

In other words, vector bundles over a fixed space $X$ are classified by homotopy classes of maps from $X$ into $G_n$. Therefore, $G_n$ is called the **classifying space** for $n$-dimensional vector bundles and the bundle $\gamma_n$ is called the **universal bundle**.

Another way to understand the statement of 4.1.2 is to realize that given an arbitrary bundle $\xi \in Vect_{\mathbb{C}^n}(X)$, one can find a map $f : X \longrightarrow G_n$ such that the class $[\xi]$ would be the same as the class of the pullback by $f$ of the universal bundle $\gamma_{k,n}$, for a sufficiently large $k$.

*Sketch of the proof:* We summarize the proof of this result, given by Hatcher in [Ha2], Theorem 1.16, making an adaptation of it to the complex case.

Let $\xi = \{E, p, X\}$ be an $n$-dimensional vector bundle. The key fact to notice is that a bundle isomorphism $\xi \cong f^*(\gamma_n)$ is equivalent to a map $g : E \longrightarrow \mathbb{C}^\infty$, called *a Gauss map*, that is a linear injection on each fiber. Indeed, suppose to have a map $f : X \longrightarrow G_n$ and a bundle isomorphism $\xi \cong f^*(\gamma_n)$. This yields a commutative diagram

$$\begin{tikzcd}
E \arrow[r, "\cong"'] \arrow[d, "p"'] \arrow[rrr, dashed, bend left, "g"] & f^*(E_n) \arrow[r, "pr_{E_n(\mathbb{C}^\infty)}"] \arrow[d, "pr_X"] & E_n(\mathbb{C}^\infty) \arrow[r, "pr"'] \arrow[d, "\pi"] & \mathbb{C}^\infty \\
X \arrow[r, "="] & X \arrow[r, "f"] & G_n(\mathbb{C}^\infty),
\end{tikzcd}$$

where

$$pr : E_n(\mathbb{C}^\infty) \longrightarrow \mathbb{C}^\infty$$

$$(V, v) \in G_n(\mathbb{C}^\infty) \times \mathbb{C}^\infty \mapsto v \in \mathbb{C}^\infty$$

is the projection. The composition across the top row is a map

$$g : E \longrightarrow \mathbb{C}^\infty$$

that is a linear injection on each fiber, because for all $x \in X$, it is given by the composition

$$g_x : p^{-1}(x) \xrightarrow{\cong} \{x\} \times_{G_n(\mathbb{C}^\infty)} E_n(\mathbb{C}^\infty) \xrightarrow{pr_{E_n(\mathbb{C}^\infty)}} E_n(\mathbb{C}^\infty) \xrightarrow{pr} \mathbb{C}^\infty$$

$$a \mapsto (p(a), (V, v)) \mapsto (V, v) \mapsto v$$

and $\dim p^{-1}(x) = n < \dim \mathbb{C}^\infty = \infty$.

*Remark* 4.1.3. Before showing the other direction, we observe that if $f : \xi \longrightarrow \xi'$ is a bundle morphism between $\xi = \{E, p, X\}$ and $\xi' = \{E', p', X'\}$, then there is an isomorphism $\xi \cong f^*(\xi')$.

To see this, note that by the universal property of pullbacks, the map $f_E$ induces an isomorphism $h_E : E \longrightarrow f^*(E')$. Hence the upper triangle in the diagram

$$\begin{tikzcd}
E \arrow[rr, "f_E"] \arrow[dr, dashed, "\exists! h_E"] \arrow[dd, "p"'] & & E' \arrow[dd, "p'"] \\
& f^*(E') \arrow[ur, "pr_{E'}"] \arrow[dd, "pr_X"] & \\
X \arrow[rr, "f_X"] \arrow[dr, "\mathrm{Id}_X"'] & & X' \\
& X. \arrow[ur, "f_X"'] &
\end{tikzcd}$$

commutes (other faces commute by hypothesis), which gives us a bundle isomorphism $h = (h_E, \mathrm{Id}_X) : \xi \longrightarrow f^*(\xi')$, as desired.

Conversely, suppose we are given a bundle $\xi = \{E, p, X\}$ and a Gauss map $g : E \longrightarrow \mathbb{C}^\infty$, which is a linear injection on each fiber. We want to show that $\xi \cong f^*(\gamma_n)$. To do this, we need to define a bundle morphism $f = (f_E, f_X)$ between $\xi$ and $\gamma_n$ such that the front face in

$$\begin{tikzcd}
E \arrow[r, "\overset{?}{\cong}"] \arrow[d, "p"'] \arrow[rrr, bend left, "g"] \arrow[rr, dashed, bend right, "f_E"] & f^*(E_n) \arrow[r, "pr_{E_n(\mathbb{C}^\infty)}"] \arrow[d] & E_n(\mathbb{C}^\infty) \arrow[r, "pr"'] \arrow[d, "\pi"] & \mathbb{C}^\infty \\
X \arrow[r, "="] \arrow[rr, dashed, bend right, "f_X"'] & X \arrow[r, "f"] & G_n(\mathbb{C}^\infty), &
\end{tikzcd}$$

commutes. We set

$$f = (f_E, f_X) : \xi \longrightarrow \gamma_n$$

where

$$\begin{aligned} f_X : X &\longrightarrow G_n(\mathbb{C}^\infty) \\ x &\mapsto \pi \circ pr^{-1} \circ g(p^{-1}(x)), \text{ a plane of dimension } n \text{ in } G_n(\mathbb{C}^\infty), \end{aligned}$$

and

$$\begin{aligned} f_E : E &\longrightarrow E_n(\mathbb{C}^\infty) \\ e &\mapsto (g(e), f_X(p(e))). \end{aligned}$$

A straightforward calculation shows that this choice makes the front face commute, and that $f_E$ is an isomorphism on each fiber, because $g$ is injective fiber-wise, and $pr^{-1} \circ \pi$ is surjective. This makes $f : \xi \longrightarrow \gamma_n$ into a bundle morphism, and by the above remark, $\xi \cong f^*(\gamma_n)$.

Therefore, to show the bijection presumed in the Classification Theorem, one needs to establish a one-to-one correspondence between $n$-dimensional bundles over $X$ and linear injections on fibers $g : E \longrightarrow \mathbb{C}^\infty$.
Given a bundle $\xi = \{E, p, X\}$, the surjectivity is proved choosing an open covering $\{U_i\}$ of $X$ together with a family of local trivializations and a partition of unity $\{\varphi_i\}$. Compositions of local trivializations with projections on $\mathbb{C}$, suitably weighted with the $\{\varphi_i\}$'s will give the coordinates of the desired map $g : E \longrightarrow \mathbb{C}^\infty$.
As for injectivity, isomorphisms of bundles $\xi \cong f_0^*(\gamma_n)$ and $\xi \cong f_1^*(\gamma_n)$ for two maps $f_0, f_1 : X \longrightarrow G_n$ will give two linear injections on fibers $g_0, g_1 : E \longrightarrow \mathbb{C}^\infty$. The rest of the proof consists in showing that $g_0$ and $g_1$ are homotopic through maps $g_t$ that are linear injections on fibers, in which case $f_0$ and $f_1$ will be homotopic via maps of the form $f_t = \pi \circ pr_t^{-1} \circ g_t(p^{-1}(x))$.
□

## 4.2 The functor $K(X)$

The purpose of introducing the Grothendieck construction on monoids (and extending it to semi-rings) in the first Chapter was to apply it afterwards to the semi-ring $(Vect_{\mathbb{C}}(X), \oplus, \otimes)$. This wiil lead us to the construction of $K(X)$, the first group of topological K-theory of a space $X$, naturally endowed with a ring structure.

**Definition 4.2.1.** Let $X \in Top$. We define the contravariant functor

$$K : Top \longrightarrow Rng$$

by the composition of functors

$$Top \xrightarrow{Vect_{\mathbb{C}}} SemiRng \xrightarrow{G} Rng$$

$$X \mapsto G \circ Vect_{\mathbb{C}}(X) := K(X)$$

$$(f : Y \longrightarrow X) \mapsto G \circ Vect_{\mathbb{C}}(f) : K(X) \longrightarrow K(Y).$$

*Remarks* 4.2.2.

- $K$ is indeed contravariant, since $Vect_{\mathbb{C}}(X)$ is.
- $K$ is an invariant of homotopy type by Remark 3.4.2.

## 4.3 The reduced group $\tilde{K}(X)$

### Defining $\tilde{K}(X)$ by a short exact sequence

For any $x_0 \in X$ consider the inclusion $i : \{x_0\} \hookrightarrow X$ and the constant map $c : X \longrightarrow \{x_0\}$. Defining $\tilde{K}(X)$ to be the kernel of $i^*$, we have a split s.e.s.

$$\begin{tikzcd} 0 \arrow[r] & \tilde{K}(X) \arrow[r, hook] & K(X) \arrow[r, "i^*"'] & K(\{x_0\}) \arrow[l, bend right, "c^*"'] \arrow[r] & 0, \end{tikzcd}$$

since $(i^* \circ c^*) = (c \circ i)^* = id_{K(\{x_0\})}$. Therefore there is an isomorphism

$$K(X) \cong \tilde{K}(X) \oplus K(\{x_0\}).$$

Every vector bundle over $x_0$ is trivial, and can be unambiguously characterized by the dimension $n$ of the fiber $p^{-1}(x_0) \cong \mathbb{C}^n$. This defines a one-to-one correspondence

$$Vect_{\mathbb{C}^n}(x_0) \xrightarrow{\cong} \mathbb{N}$$

$$\xi \mapsto n.$$

After applying the Grothendieck construction, one obtains $K(x_0) \xrightarrow{\cong} \mathbb{Z}$, so that

$$K(X) \cong \tilde{K}(X) \oplus \mathbb{Z}\,.$$

Thus the map $i^* : K(X) \longrightarrow K(x_0) \cong \mathbb{Z}$ can be regarded as the one that associates to a fiber bundle over $(X, x_0)$ the dimension of its fiber over the base point $x_0$. Note that the previous splitting depends on the choice of the base point.

### Defining $\tilde{K}(X)$ in terms of vector bundles

The aim of this paragraph is to have another geometric look on $\tilde{K}(X)$, showing that $Vect^s_{\mathbb{C}}(X) \cong \tilde{K}(X)$. Here $Vect^s_{\mathbb{C}}(X)$ denotes the set of equivalence classes of vector bundles under the *stable equivalence* relation, which is an other equivalence relation on $Bund_{\mathbb{C}}(X)$ we are about to define.

**Definition 4.3.1.** Two bundles $\xi = (E, p, X)$ and $\xi' = (E', p', X)$ over $X$ are said to be **stably equivalent**, denoted $\xi \simeq_s \xi'$, if there exist trivial bundles $\epsilon^n : X \times \mathbb{C}^n \longrightarrow X$ and $\epsilon^m : X \times \mathbb{C}^m \longrightarrow X$ such that

$$\xi \oplus \epsilon^n \simeq \xi' \oplus \epsilon^m.$$

The stable class of $\xi$ is denoted by $[\xi]_s$, and the Whitney sum is well-defined on stable classes by

$$[\xi]_s \oplus [\eta]_s = [\xi \oplus \eta]_s.$$

Indeed, if $\xi' \in [\xi]_s$ and $\eta' \in [\eta]_s$, then $\xi \oplus \epsilon^n \simeq \xi' \oplus \epsilon^m$ and $\eta \oplus \epsilon^k \simeq \eta' \oplus \epsilon^l$, so $\xi \oplus \eta \oplus \epsilon^n \oplus \epsilon^k \simeq \xi' \oplus \eta' \oplus \epsilon^m \oplus \epsilon^l$ and thus $[\xi' \oplus \eta']_s = [\xi \oplus \eta]_s$. A zero element for $\oplus$ is the stable class of the trivial 0-bundle $\epsilon^0$.
We shall denote by $Vect^s_{\mathbb{C}}(X)$ the set of all stable equivalence classes of complex vector bundles over $X$.

**Proposition 4.3.2.** *The set* $(Vect^s_{\mathbb{C}}(X), \oplus)$ *is an abelian group for every* $X \in Top$.

*Proof.* Suppose $X$ is a connected space of dimension $r$. The Whitney sum $\oplus$ endows the set $Vect^s_{\mathbb{C}}(X)$ with a monoidal structure. It remains to show the existence of inverses, i.e., given a complex $n$-bundle $\xi$ over $X$, we need to find a complex $(k-n)$-bundle $\eta$ such that $\xi \oplus \eta \simeq_s \epsilon^k$, where $\epsilon^k$ is a trivial $k$-bundle over $X$. Apply Theorem 4.1.2 for $k$ large enough to obtain a map $f : X \longrightarrow G_n$, such that $[\xi] = [f^*(\gamma_{k,n})]$. At the end of Section 2.2 we showed that $\gamma_{n,k} \oplus \eta_{n,k} \cong \epsilon^k$, with $\epsilon^k$ trivial over $G_{n,k}$. Hence $f^*(\gamma_{n,k} \oplus \eta_{n,k}) = \xi \oplus f^*(\eta_{n,k})$ is equivalent to the trivial $k$-bundle $\epsilon^k$ over $X$. Thus $\eta := f^*(\eta_{n,k})$.

If $X$ is not connected, proceed with a similar argument for each of its connected components. Note that in this case fibers of a vector bundle $\xi$ over $X$ would agree in dimension within each component of $X$, yet, they could have different dimensions from one component to another. Accordingly, suitable choices of $k$ will be required. $\square$

Now recall the Grothendieck group construction we discussed in Section 1.2. Applying it to the monoid $Vect_{\mathbb{C}}(X)$ gives the following commutative diagram

$$\begin{tikzcd}
Vect_{\mathbb{C}}(X) \arrow[rr, "G"] \arrow[dr, "\varphi"'] & & K(X) \arrow[dl, dashed, "\exists ! \bar{\varphi}"] \\
& Vect^s_{\mathbb{C}}(X), &
\end{tikzcd}$$

where

$$\varphi : Vect_{\mathbb{C}}(X) \longrightarrow Vect^s_{\mathbb{C}}(X)$$

$$[\xi] \mapsto [\xi]_s$$

is a homomorphism of monoids. The universal property provides a unique homomorphism $\bar{\varphi} : K(X) \longrightarrow Vect^s_{\mathbb{C}}(X)$ such that the diagram above commutes, i.e., $\bar{\varphi} \circ G = \varphi$.

We define $\bar{\varphi}$ as follows. Let $\{\xi, \eta\} \in K(X)$ and let $\eta'$ a vector bundle over $X$ such that $\eta \oplus \eta' = \epsilon$, a trivial vector bundle of appropriate dimension. One then has $\{\xi, \eta\} = \{\xi \oplus \eta', \epsilon\}$. Set

$$\bar{\varphi} : K(X) \longrightarrow Vect^s_{\mathbb{C}}(X)$$

$$\{\xi, \eta\} \mapsto [\xi \oplus \eta']_s.$$

Finally, we have the desired geometric description of $\tilde{K}(X)$.

**Proposition 4.3.3.** [[Hi], IV, Theorem 4.5] *For every space $X$, there is a group isomorphism*

$$Vect^s_{\mathbb{C}}(X) \cong \tilde{K}(X).$$

*Sketch of the proof:* Take $[\xi]_s \in Vect^s_{\mathbb{C}^n}(X)$ and consider $\{\xi, \epsilon^n\}$ in $K(X)$, where $n = \dim \xi$. One then shows that $\{\xi, \epsilon^n\} \in \operatorname{Ker} i^* = \tilde{K}(X)$. By definition of $\bar{\varphi}$, $\bar{\varphi}(\{\xi, \epsilon^n\}) = [\xi]_s$, hence $\bar{\varphi}|_{\tilde{K}(X)}$ is surjective.
Let $\{\xi, \eta\} \in \tilde{K}(X)$ such that $\bar{\varphi}\{\xi, \eta\} = 0$. Injectivity of $\bar{\varphi}$ follows from an easy calculation using its definition with $\eta'$. □

Recall Definition 1.4.1 that characterized the space $BU$.

**Proposition 4.3.4.** *For any well-pointed finite-dimensionnal compact space $(X, x_0)$, the map*

$$\tilde{K}(X, x_0) \xrightarrow{\cong} [X, x_0; BU \times \mathbb{Z}, *]$$

*is a one-to-one correspondence.*

*Proof.* First, note that without any assumption on the connectedness of $(X, x_0)$, classes of maps $[X, x_0; \mathbb{Z}]$ allow us to distinguish between its connected components. Hence if $(X, x_0)$ is assumed to be connected, the bijective correspondence is of the form

$$\tilde{K}(X, x_0) \xrightarrow{\cong} [X, x_0; BU, *].$$

We drop the base points for the rest of the proof. An element $a \in [X, BU]$ is represented by a map $f_n : X \longrightarrow G_n$ for some $n$, because $X$ is finite-dimensional. By the Classification Theorem 4.1.2, one can assign to $f_n$ the vector bundle $f_n^*(\gamma_n)$ over $X$, of dimension $n$. Let the inclusions $i_n : G_n \hookrightarrow \varinjlim_j G_j$ and consider the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & G_n \arrow[r, "i_{n,n+1}"] & G_{n+1} \arrow[r] & \cdots \arrow[r] & G_{n'} \arrow[r, "i_{n',n'+1}"] & G_{n'+1} \arrow[r] & \cdots \arrow[r] & G_{n''} \arrow[r, "i_{n'',n''+1}"] & G_{n''+1} \arrow[r] & \cdots \arrow[d] \\
 & X \arrow[u, "f_n"] \arrow[urrr, "f_{n'}"] \arrow[urrrrrr, dashed, "f_{n''}"] \arrow[rrrrrrrr, "a"'] & & & & & & & & \varinjlim_j G_j .
\end{tikzcd}$$

Here maps $f_n : X \longrightarrow G_n$ and $f_{n'} : X \longrightarrow G_{n'}$ correspond respectively to bundles $f_n^*(\gamma_n)$ and $f_{n'}^*(\gamma_{n'})$ of dimensions $n$ and $n'$. If both of them represent $a$, then $[a] = [i_n \circ f_n] = [i_{n'} \circ f_{n'}]$, i.e., $i_n \circ f_n \simeq i_{n'} \circ f_{n'}$ and there exist a homotopy

$$H : X \times I \longrightarrow \varinjlim_j G_j$$

such that

$$H(x, 0) = i_n \circ f_n, \; H(x, 1) = i_{n'} \circ f_{n'}.$$

In this case, since $X$ is still finite, for some $n'' \geq n, n'$ and some $G_{n''}$, one can define a homotopy

$$H'' : X \times I \longrightarrow G_{n''}$$

such that

$$H''(x, 0) = i_{n'',n''+1} \circ f_n : X \longrightarrow G_{n''} \text{ for all } x \in X$$

and

$$H''(x,1) = i_{n'',n''+1} \circ f_{n'} : X \longrightarrow G_{n''} \text{ for all } x \in X,$$

i.e., $f_n$ and $f_{n'}$ are homotopic when viewed as maps of $X$ into $G_{n''}$. Therefore, there exist trivial bundles $\epsilon^{n''-n}$ and $\epsilon^{n'-n}$ over $X$, such that

$$f_n^*(\gamma_n) \oplus \epsilon^{n''-n} \cong f_{n'}^*(\gamma_{n'}) \oplus \epsilon^{n'-n},$$

which means that $f_n^*(\gamma_n) \simeq_s f_{n'}^*(\gamma_{n'})$, i.e., the two representatives of $a$ are stably equivalent, as requested. □

## 4.4 Formal description of $K(X)$

We already know that the ring $K(X)$ consists of classes of pairs of vector bundles $\{\xi, \eta\}$ over $X$. Elements of $K(X)$ can also be thought of as formal differences $E - E'$ of vector bundles $E$ and $E'$ over $X$, as we shall immediately see.

Let $\{\xi, \eta\} \in K(X)$. Since by definition the inverse of $\{\xi, \eta\}$ is $\{\eta, \xi\}$, one writes

$$\{\xi, \eta\} = \{\xi, \epsilon^0\} \oplus \{\epsilon^0, \eta\} = \{\xi, \epsilon^0\} - \{\eta, \epsilon^0\} = G(\xi) - G(\eta) =: E - E'.$$

The equivalence relation

$$\{\xi, \eta\} = \{\xi', \eta'\} \Longleftrightarrow \text{ there exists } u \text{ such that } \xi \oplus \eta' \oplus u = \xi' \oplus \eta \oplus u,$$

thus transforms into

$$E_1 - E_1' = E_2 - E_2' \Longleftrightarrow E_1 \oplus E_2' \simeq_s E_2 \oplus E_1'. \qquad (E)$$

In terms of this new description of the elements of $K(X)$, the addition of two elements of $K(X)$ is defined by

$$(E_1 - E_1') \oplus (E_2 - E_2') := (E_1 \oplus E_2) - (E_1' \oplus E_2'), \qquad (A)$$

where $\oplus$ is the Whitney sum. The multiplication of two elements of $K(X)$ is given by

$$(E_1 - E_1') \boxtimes (E_2 - E_2') := E_1 \otimes E_2 - E_1 \otimes E_2' - E_1' \otimes E_2 + E_1' \otimes E_2', \qquad (M)$$

where $\otimes$ is the tensor product on vector bundles.

To be more precise, every element $E - E'$ of $K(X)$ can be written as a formal difference $H - \epsilon^n$ with a trivial bundle. Choose a bundle $G$ such that $E' \oplus G = \epsilon^n$. Then $E - E' = E + G - (G + E') = E \oplus G - \epsilon^n := H - \epsilon^n$.

The latter observation, combined with Propostion 4.3.3, completes our collection of relations between $K(X)$ and $\tilde{K}(X)$. There is a surjective homomorphism

$$f : K(X) \longrightarrow \tilde{K}(X)$$
$$E - \epsilon^n \mapsto [E]_s.$$

It is well defined since if $E - \epsilon^n = E' - \epsilon^m$ in $K(X)$, we have $E \oplus \epsilon^m \cong E' \oplus \epsilon^n$, hence $[E]_s = [E']_s$. The kernel is

$$\mathrm{Ker}\, f := \{E - \epsilon^n \in K(X) | [E]_s = [\epsilon^0]_s\},$$

or, equivalently,

$$\mathrm{Ker}\, f = \{\epsilon^k - \epsilon^n \in K(X)\}.$$

Indeed,

$$\begin{aligned}
E \simeq_s \epsilon^0 \quad & \Longleftrightarrow & & \exists\, a, b \text{ s.t. } E \oplus \epsilon^a \cong \epsilon^0 \oplus \epsilon^b \\
& \overset{a:=n+l;\, b:=k+n+m}{\Longleftrightarrow} & & \exists\, k, l, m, n, \text{ s.t. } E \oplus \epsilon^{n+l} \cong \epsilon^{k+n+m} \\
& \Longleftrightarrow & & \exists\, k, l, m, n, \text{ s.t. } E \oplus \epsilon^n \oplus \epsilon^l \cong \epsilon^k \oplus \epsilon^n \oplus \epsilon^m \\
& \Longleftrightarrow & & E \oplus \epsilon^n \simeq_s \epsilon^k \oplus \epsilon^n \\
& \Longleftrightarrow & & E - \epsilon^n = \epsilon^k - \epsilon^n,
\end{aligned}$$

the last equivalence arising from $(E)$.

Observe that $\mathrm{Ker}\, f = \{\epsilon^k - \epsilon^n\}$ is a subgroup of $K(X)$ isomorphic to $\mathbb{Z}$, thus to $K(x_0)$, and the s.e.s. given at the beginning of Section 4.3. can be "read from right to left":

$$\begin{tikzcd}
0 \arrow[r] & K(x_0) \arrow[r, hook] & K(X) \arrow[r, "f"'] & \tilde{K}(X) \arrow[r] \arrow[l, bend right, "g"'] & 0
\end{tikzcd}$$

with $g : \tilde{K}(X) \longrightarrow K(X) : [E]_s \mapsto E - \epsilon^n$ for an $n$-bundle $E$.

### Functoriality of $K(X)$ revisited

Now as the formal structure of $K(X)$ is settled, let us come back to Definition 4.2.1. We have

$$K : \mathit{Top} \longrightarrow \mathit{Rng}$$
$$X \mapsto K(X),$$
$$(f : X \longrightarrow Y) \mapsto f^* : K(X) \longrightarrow K(Y),$$

where

$$f^*(E - E') := f^*(E) - f^*(E')$$

and $f^*$ is the pullback of $f$.

Properties of pullbacks imply isomorphisms of vector bundles

$$f^*(E_1 \oplus E_2) \cong f^*(E_1) \oplus f^*(E_2)$$

and

$$f^*(E_1 \otimes E_2) \cong f^*(E_1) \otimes f^*(E_2),$$

making $f^*$ into a ring homomorphism. We show the last isomorphism, since we shall need it right at the beginning of the next chapter, when defining the external product in $K$-theory.

Consider vector bundles $p_1 : E_1 \longrightarrow X$, $p_2 : E_2 \longrightarrow X$ and $\pi : E_1 \otimes E_2 \longrightarrow X$ over $X$, and let $f : Y \longrightarrow X$ be a continuous map. We need to show that the diagram

$$\begin{tikzcd}
f^*(E_1) \otimes f^*(E_2) \arrow[r, "\alpha"] \arrow[d, "\beta"'] & E_1 \otimes E_2 \arrow[d, "\pi"] \\
Y \arrow[r, "f"'] & X
\end{tikzcd}$$

where the maps $\alpha$, $\beta$ are given by

$$\alpha : f^*(E_1) \otimes f^*(E_2) \longrightarrow E_1 \otimes E_2$$

$$(e_1, y) \otimes (e_2, y) \mapsto e_1 \otimes e_2$$

and

$$\beta : f^*(E_1) \otimes f^*(E_2) \longrightarrow Y$$

$$(e_1, y) \otimes (e_2, y) \mapsto y$$

is a pullback diagram.

Let $Z \in \mathit{Top}$ and let two continuous maps

$$\varphi : Z \longrightarrow Y : z \mapsto \varphi(z)$$

$$(\psi_1, \psi_2) : Z \longrightarrow E_1 \otimes E_2 : z \mapsto \psi_1(z) \otimes \psi_2(z)$$

such that

$$f \circ \varphi(z) = \pi \circ (\psi_1, \psi_2)(z), \text{ for all } z \in Z.$$

In the diagram

$$\begin{tikzcd}
Z \arrow[dr, dotted, "\Phi"] \arrow[drr, bend left, "(\psi_1,\psi_2)"] \arrow[ddr, bend right, "\varphi"'] & & \\
& f^*(E_1) \otimes f^*(E_2) \arrow[r, "\alpha"] \arrow[d, "\beta"'] & E_1 \otimes E_2 \arrow[d, "\pi"] \\
& Y \arrow[r, "f"'] & X
\end{tikzcd}$$

define $\Phi$ to be

$$\Phi : Z \longrightarrow f^*(E_1) \otimes f^*(E_2)$$

$$z \mapsto (\psi_1(z), \varphi(z)) \otimes (\psi_2(z), \varphi(z)).$$

We then have by definition

$$\star \begin{cases} \beta \circ \Phi(z) & = \beta[(\psi_1(z), \varphi(z)) \otimes (\psi_2(z), \varphi(z))] = \varphi(z) \\ \alpha \circ \Phi(z) & = \alpha[(\psi_1(z), \varphi(z)) \otimes (\psi_2(z), \varphi(z))] = \psi_1(z) \otimes \psi_2(z). \end{cases}$$

It remains to show that such a $\Phi$ is unique (up to isomorphism). Suppose there exists

$$\Phi' : Z \longrightarrow f^*(E_1) \otimes f^*(E_2)$$

such that

$$z \mapsto (a_1(z), b(z)) \otimes (a_2(z), b(z)).$$

If we want $\star$ to be satisfied, we necessarily have $\Phi'(z) = (\psi_1(z), \varphi(z)) \otimes (\psi_2(z), \varphi(z)$. Hence, the first diagram is a pullback, and, in particular, if one takes $Z := f^*(E_1 \otimes E_2)$ and defines maps

$$\varphi := pr_{E_1 \otimes E_2} : f^*(E_1 \otimes E_2) \longrightarrow E_1 \otimes E_2 : (e_1 \otimes e_2, y) \mapsto (e_1 \otimes e_2)$$

and

$$\psi := pr_Y : f^*(E_1 \otimes E_2) \longrightarrow Y : (e_1 \otimes e_2, y) \mapsto y,$$

the desired isomorphism

$$\Phi : f^*(E_1 \otimes E_2) \xrightarrow{\cong} f^*(E_1) \otimes f^*(E_2)$$

$$(e_1 \otimes e_2, y) \mapsto (e_1, y) \otimes (e_2, y)$$

follows by the universal property of the pullback.

# Chapter 5

# On Bott periodicity

*Remark* 5.0.1. As in Chapter 4, the base space $X$ is still assumed to be compact Hausdorff for all vector bundles.

## 5.1 The external product for $K(X)$

**Definition 5.1.1.** Let $\{A, p_a, X\}$ and $\{B, p_b, Y\}$ be two complex vector bundles over a fixed space $X$. Consider their pullbacks by the projections $p_X : X \times Y \longrightarrow X$ and $p_Y : X \times Y \longrightarrow Y$, as shown in the commutative diagram

$$\begin{tikzcd}
A \arrow[dr, "p_a"'] & & p_X^*(A) \arrow[ll, "pr_A"'] \arrow[dr, "p_{X\times Y}"] & & p_Y^*(B) \arrow[rr, "pr_B"] \arrow[dl] & & B \arrow[dl, "p_b"] \\
& X & & X \times Y \arrow[ll, "p_X"'] \arrow[rr, "p_Y"] & & Y. &
\end{tikzcd}$$

Using the formal product $\boxtimes$ in $K(X \times Y)$, defined in the previous Chapter, one obtains the bundle $\{p_X^*(A) \boxtimes p_Y^*(B), \pi, X \times Y\}$.

We define the **external product** by

$$\mu : K(X) \otimes K(Y) \longrightarrow K(X \times Y)$$

$$[A] \otimes [B] \mapsto \mu([A] \otimes [B]) := [p_X^*(A) \boxtimes p_Y^*(B)],$$

*Remark* 5.1.2. Rigorously speaking, elements of $K(X)$ are isomorphism classes of vector bundles over $X$, which is why we put $A$ in square brackets. However, from now on and for the sake of simplicity, we shall be writing $A$ for an element of $K(X)$ corresponding to a vector bundle $\{A, p_a, X\}$.

The multiplication on the ring $K(X) \otimes K(Y)$ is defined by

$$m : K(X) \otimes K(Y) \times K(X) \otimes K(Y) \longrightarrow K(X) \otimes K(Y)$$

$$(A \otimes B, C \otimes D) \mapsto m(A \otimes B, C \otimes D) := (A \boxtimes C) \otimes (B \boxtimes D).$$

Observe that $\mu$ is a ring homomorphism, since

$$\begin{array}{rcl}\mu(m(A\otimes B, C\otimes D)) & = & \mu((A\boxtimes C)\otimes(B\boxtimes D)) \\ & = & p_X^*(A\boxtimes C)\boxtimes p_Y^*(B\boxtimes D) \\ & \overset{(1)}{\cong} & p_X^*(A)\boxtimes p_X^*(C)\boxtimes p_Y^*(B)\boxtimes p_Y^*(D) \\ & \overset{(2)}{\cong} & p_X^*(A)\boxtimes p_Y^*(B)\boxtimes p_X^*(C)\boxtimes p_Y^*(D) \\ & = & \mu(A\otimes B)\otimes\mu(C\otimes D) \\ & = & m(\mu(A\otimes B), \mu(C\otimes D)),\end{array}$$

where the isomorphism (1) follows from the property of pullbacks we showed at the end of the previous chapter, applied to $\boxtimes$. To show (2) we need to find a bundle isomorphism $f$ between bundles $\{p_X^*(C)\boxtimes p_Y^*(B), \pi, X\times Y\}$ and $\{p_Y^*(B)\boxtimes p_X^*(C), \pi', Y\times X\}$, i.e., we want the following diagram to commute

$$\begin{tikzcd} p_X^*(C)\boxtimes p_Y^*(B) \arrow[r, "f"] \arrow[d, "\pi"] & p_Y^*(B)\boxtimes p_X^*(C) \arrow[d, "\pi'"] \\ X\times Y \arrow[r, "\cong"] & Y\times X. \end{tikzcd}$$

The bottom isomorphism is clear, and the fact that the map

$$f : p_X^*(C)\boxtimes p_Y^*(B) \xrightarrow{\cong} p_Y^*(B)\boxtimes p_X^*(C)$$

$$(x,b)\boxtimes(y,c) \mapsto (y,c)\boxtimes(x,b)$$

is an isomorphism follows from the commutativity of the formal multiplication $\boxtimes$. Indeed, the operation $\boxtimes$, defined to be a direct sum of tensor products of elements of $K(X\times Y)$, is carried out on fibers, where $\oplus$ and $\otimes$ commute, as we have seen in Chapter 3.

**Theorem 5.1.3.** [The Fundamental Product Theorem] *The homomorphism of rings*

$$\mu : K(X)\otimes K(\mathbb{S}^2) \xrightarrow{\cong} K(X\times\mathbb{S}^2)$$

*is an* ***isomorphism****.*

This result bears the name of The Fundamental Product Theorem (FPT) and is at origin of Bott Periodicity Theorem, crucial to define the structure of the spectrum $KU$, representing the topological K-theory as a generalized reduced cohomology theory. The proof of FPT is quite long and elaborate; it was originally proposed by Bott (see [Bo]). Atiyah (see [At]) offers a completed version of Bott's proof, with more details given.

The rest of this chapter is devoted to the description of all concepts and tools that will be useful to understand the main steps of Bott's proof, which will lead us to see why the map $\mu$ is an isomorphism.

## 5.2 The Fundamental Product Theorem

The strategy is the following. After defining the concept of *simple clutching functions*, we shall give an equivalent, more algebraic formulation of the FPT, which we shall be working with, rather than the original statement 5.1.3. The main tool needed will be the so called *generalized clutching functions.* We shall explain their construction and basic properties to concentrate afterwards on a particular (simpler) class of these: the *Laurent polynomial clutching functions.* The ultimate simplification will be a *linearization procedure* on polynomial functions of degree $n$, allowing a restriction to linear polynomials of degree at most 1 that have "nice properties". All this will finally bring us to the proof of the bijectivity of $\mu$, the surjectivity part being a more or less easy calculation, while the injectivity part requires some more work.

### 5.2.1 Simple clutching functions

We describe a way to construct complex vector bundles $E \longrightarrow \mathbb{S}^k$ of dimension $n$ over a sphere. Write $\mathbb{S}^k = D^k_+ \cup D^k_-$ as the union of the upper and lower hemispheres, with $D^k_+ \cap D^k_- = \mathbb{S}^{k-1}$. Given a map

$$f : \mathbb{S}^{k-1} \longrightarrow \mathrm{Aut}(\mathbb{C}^n) \cong GL_n(\mathbb{C})$$

defined by

$$\begin{aligned} z \mapsto f(z) : \mathbb{C}^n &\longrightarrow \mathbb{C}^n \\ v &\mapsto f(z)v, \end{aligned}$$

consider

$$E_f := D^k_+ \times \mathbb{C}^n \sqcup D^k_- \times \mathbb{C}^n \,/\, \sim,$$

where

$$(z, v) \in \partial D^k_- \times \mathbb{C}^n \sim (z, f(z)v) \in \partial D^k_+ \times \mathbb{C}^n\,.$$

There is a natural projection $E_f \twoheadrightarrow \mathbb{S}^k$ sending an element of an hemisphere in the disjoint union to the corresponding hemisphere of $\mathbb{S}^k$, and it defines a $n$-dimensional vector bundle. To see this one can extend the two hemispheres to open disks to define an open covering of $\mathbb{S}^k$, gluing open bands $\mathbb{S}^{k-1} \times [0, \varepsilon)$ (resp. $\mathbb{S}^{k-1} \times (-\varepsilon, 0]$) to $D^k_+$ (resp. to $D^k_-$), so that identification by $f$ occurs over the intersection $\mathbb{S}^{k-1} \times (-\varepsilon, \varepsilon)$, with $f$ defined in each slice $\mathbb{S}^{k-1} \times \{t\}$.

**Definition 5.2.1.** Such a map $f : \mathbb{S}^{k-1} \longrightarrow GL_n(\mathbb{C})$ is called a **clutching function** for the bundle $E_f$.

Tthe construction of clutching functions in the real case is absolutely analogous, replacing $GL_n(\mathbb{C})$ with $GL_n(\mathbb{R})$.

A basic property of the construction of vector bundles using clutching functions is the following.

**Lemma 5.2.2.** *If $f, g : \mathbb{S}^{k-1} \longrightarrow GL_n(\mathbb{C})$ are two clutching functions such that $f \simeq g$, then $E_f \cong E_g$, i.e., homotopic clutching functions induce isomorphic vector bundles.*

*Idea of the proof:* Given a homotopy

$$F : \mathbb{S}^{k-1} \times I \longrightarrow GL_n(\mathbb{C})$$

from $f$ to $g$

$$\begin{aligned} F(-,0) &= f : \mathbb{S}^{k-1} \longrightarrow GL_n(\mathbb{C}) \\ F(-,1) &= g : \mathbb{S}^{k-1} \longrightarrow GL_n(\mathbb{C}), \end{aligned}$$

we can construct a vector bundle over $\mathbb{S}^{k-1} \times I$ with the total space

$$E_F := D^k_+ \times \mathbb{C}^n \times I \sqcup D^k_- \times \mathbb{C}^n \times I / \sim,$$

where

$$(z, v, t) \in \partial D^k_- \times \mathbb{C}^n \times I \sim (z, F(z,t)v, t) \in \partial D^k_+ \times \mathbb{C}^n \times I.$$

Moreover, since $E_F|_{\mathbb{S}^{k-1} \times \{0\}} = E_f$ and $E_F|_{\mathbb{S}^{k-1} \times \{1\}} = E_g$, $E_f$ and $E_g$ are isomorphic by Proposition 3.2.2. □

This gives us the following proposition.

**Proposition 5.2.3.** [[Ha2], Proposition 1.11] *The map*

$$\Phi : \pi_{k-1} GL_n(\mathbb{C}) \xrightarrow{\cong} Vect_{\mathbb{C}^n}(\mathbb{S}^k)$$

$$[f] \mapsto E_f$$

*is a bijection.*

We do not sketch the proof here, referring the interested reader to Hatcher.

*Remarks* 5.2.4.

- We shall abbreviate "clutching functions" to "c.f.".
- For the time being and until the end of this chapter, $n$ denotes the trivial $n$-dimensional bundle over $\mathbb{S}^2$, i.e.,

$$n := \epsilon^n : \mathbb{S}^2 \times \mathbb{C}^n \longrightarrow \mathbb{S}^2$$

for $n \geq 1$.
Note that $n \cong \underbrace{1 \oplus 1 \oplus ... \oplus 1}_{n}$, since $\mathbb{C}^n \cong \underbrace{\mathbb{C} \oplus \mathbb{C} \oplus ... \oplus \mathbb{C}}_{n}$.

Later on, we have to be sure of $GL_n(\mathbb{C})$ being path-connected as a topological group for all $n \geq 1$.

**Lemma 5.2.5.** *For all $n \geq 1$, $GL_n(\mathbb{C})$ is path-connected.*

*Proof.* The case $n = 1$ is clear, since $GL_1(\mathbb{C}) \approx \mathbb{C}^*$, which is path-connected. Let $n > 1$, and take $A \in GL_n(\mathbb{C})$. We can write $A = CBC^{-1}$ where

$$B = \begin{pmatrix} x_1 & * & ... & & & * \\ 0 & x_2 & * & ... & & * \\ 0 & & ... & & x_{n-1} & * \\ 0 & & ... & 0 & & x_n \end{pmatrix}$$

is an upper triangular matrix, such that $x_i \neq 0$, for all $i = 1, ..., n$. Let

$$B(t) := \begin{pmatrix} x_1 & *(1-t) & ... & & & *(1-t) \\ 0 & x_2 & *(1-t) & ... & & *(1-t) \\ 0 & & ... & & x_{n-1} & *(1-t) \\ 0 & & ... & 0 & & x_n \end{pmatrix}$$

for all $t \in [0, 1]$. For each $t$, $\det(B(t)) = \det(B) \neq 0$, therefore $B(t) \in GL_n(\mathbb{C})$. Let $A(t) := CB(t)C^{-1}$ for $t \in [0, 1]$. We have $A(0) = A$ and

$$A(1) = \begin{pmatrix} x_1 & 0 & ... & & & 0 \\ 0 & x_2 & 0 & ... & & 0 \\ 0 & & ... & & 0 & 0 \\ 0 & & ... & 0 & & x_n \end{pmatrix}$$

is a diagonal matrix. Since $x_i \in \mathbb{C}$, for each $x_i$ there exists a path

$$x_i(t) : [1, 2] \longrightarrow \mathbb{C}$$

$$t \mapsto (2-t)x_i + (t-1)$$

such that

$$x_1(1) = x_i \text{ and } x_1(2) = 1,$$

and which does not pass trough 0. Indeed, for all $i$

$$(2-t)x_i + (t-1) = 0 \Longleftrightarrow t = \frac{1-2x_i}{1-x_i}$$

and the latter equation never holds if $t \in [1, 2]$ if we suppose $x_i \neq 0$. Define

$$\begin{pmatrix} x_1(t) & 0 & ... & & & 0 \\ 0 & x_2(t) & 0 & ... & & 0 \\ 0 & & ... & & x_{n-1}(t) & 0 \\ 0 & & ... & 0 & & x_n(t) \end{pmatrix} := B(t), \text{ for } t \in [1, 2].$$

This is consistent with the previous definition, since the two matrices are the same at $t = 1$, and by a pasting lemma the function $f : [0, 2] \longrightarrow GL_n(\mathbb{C})$; $t \mapsto B(t)$ is continuous. The fact that $x_i(t)$ never passes through zero is crucial to be sure that $B(t)$ remains invertible. In this way, for any $A \in GL_n(\mathbb{C})$ we have a path

$$\alpha : [1,2] \longrightarrow GL_n(\mathbb{C})$$

$$t \mapsto A(t) = CB(t)C^{-1}$$

such that

$$\alpha(1) = A \text{ and } \alpha(2) = I_n.$$

Thus, any two matrices of $GL_n(\mathbb{C})$ can be connected to the identity matrix by a path and, therefore, to each other, which implies that $GL_n(\mathbb{C})$ is path-connected for all $n \geq 1$. $\square$

**Lemma 5.2.6.** *Let $E_f \longrightarrow \mathbb{S}^k$ and $E_g \longrightarrow \mathbb{S}^k$ be $n$-dimensional vector bundles over $\mathbb{S}^k$ with c.f. $f, g : \mathbb{S}^{k-1} \longrightarrow GL_n(\mathbb{C})$. Then we have the isomorphism*

$$E_{fg} \oplus n \cong E_f \oplus E_g,$$

*where $fg : \mathbb{S}^{k-1} \longrightarrow GL_n(\mathbb{C})$ is the c.f. obtained by point-wise matrix multiplication (equivalently, by composition of automorphisms).*

*Proof.* The c.f. in question are

for $E_f$:

$$f : \mathbb{S}^{k-1} \longrightarrow GL_n(\mathbb{C})$$

$$z \mapsto f(z) : \mathbb{C}^n \longrightarrow \mathbb{C}^n$$

$$v \mapsto f(z)v;$$

for $E_g$:

$$g : \mathbb{S}^{k-1} \longrightarrow GL_n(\mathbb{C})$$

$$z \mapsto g(z) : \mathbb{C}^n \longrightarrow \mathbb{C}^n$$

$$v \mapsto g(z)v;$$

for $E_{fg}$:

$$fg : \mathbb{S}^{k-1} \longrightarrow GL_n(\mathbb{C})$$

$$z \mapsto f(z)g(z) : \mathbb{C}^n \longrightarrow \mathbb{C}^n$$

$$v \mapsto f(z)g(z)v;$$

and for $E_f \oplus E_g$:

$$f \oplus g : \mathbb{S}^{k-1} \longrightarrow GL_{2n}(\mathbb{C})$$

$$z \mapsto (f \oplus g)(z) : \mathbb{C}^{2n} \longrightarrow \mathbb{C}^{2n}$$

$$v \mapsto (f \oplus g)(z)v,$$

where $f(z)$, $g(z)$ are given by their matrices in $GL_n(\mathbb{C})$, so that the matrix for $(f \oplus g)(z)$ looks like

$$\begin{pmatrix} f(z) & 0 \\ 0 & g(z) \end{pmatrix} \in GL_{2n}(\mathbb{C}).$$

Hence, the c.f. for $E_{fg} \oplus n$ must be of the form

$$fg \oplus \mathrm{Id}_n : \mathbb{S}^{k-1} \longrightarrow GL_{2n}(\mathbb{C})$$

$$z \mapsto f(z)g(z) \oplus \mathrm{Id}_n : \mathbb{C}^{2n} \longrightarrow \mathbb{C}^{2n}$$

$$v \mapsto [f(z)g(z) \oplus I_n]v.$$

Now, the space $GL_{2n}(\mathbb{C})$ is path-connected (Lemma 5.2.5), and we know there exists a path

$$\alpha : [1,2] \longrightarrow GL_{2n}(\mathbb{C})$$

such that

$$\begin{aligned} \alpha(1) &= \text{ matrix which interchanges the two factors of } \mathbb{C}^n \times \mathbb{C}^n \\ \alpha(2) &= I_{2n}. \end{aligned}$$

We can define a homotopy

$$G : \mathbb{S}^{k-1} \times [1,2] \longrightarrow GL_{2n}(\mathbb{C})$$

$$(z,t) \mapsto G(z,t)$$

given by the matrix product $(f \oplus \mathrm{Id})(z)\alpha(3-t)(\mathrm{Id} \oplus g)(z)\alpha(3-t)$. It follows that $G_0(z) = (f \oplus g)(z)$ is (in terms of matrix multiplications)

$$I_{2n} \mapsto \begin{pmatrix} I_n & 0 \\ 0 & g(z) \end{pmatrix} \mapsto \begin{pmatrix} f(z) & 0 \\ 0 & g(z) \end{pmatrix}$$

and $G_1(z) = (fg \oplus \mathrm{Id})(z)$ is

$$I_{2n} \mapsto \begin{pmatrix} I_n & 0 \\ 0 & g(z) \end{pmatrix} \mapsto \begin{pmatrix} g(z) & 0 \\ 0 & I_n \end{pmatrix} \mapsto \begin{pmatrix} f(z)g(z) & 0 \\ 0 & I_n \end{pmatrix}.$$

The homotopy $G$ is precisely a c.f. for the bundle $E_{fg} \oplus n$ over $\mathbb{S}^{k-1} \times I$. □

*Remark* 5.2.7. Hereafter $H$ stands for the canonical line bundle over $G_1(\mathbb{C}^2) \cong \mathbb{C}\mathrm{P}^1 \cong \mathbb{S}^2$, i.e.,

$$H := \gamma_{1,2} : E_{1,2} \longrightarrow G_1(\mathbb{C}^2).$$

**Some properties of the bundle $H$**

A c.f. for $H$ is given by

$$f_H : \mathbb{S}^1 \longrightarrow GL_1(\mathbb{C})$$
$$z \mapsto f(z) : \mathbb{C} \longrightarrow \mathbb{C}$$
$$v \mapsto zv.$$

Example 1.10 in [Ha2] explains the construction of $f_H$ in details.

Here is a key property satisfied by $H$.

**Lemma 5.2.8.** [[Ha2], Example 1.13] *Let $H$ and $1$ be the canonical and the trivial bundles respectively. Then there exists an isomorphism of complex vector bundles*

$$(H \otimes H) \oplus 1 \cong H \oplus H.$$

*Proof.* Apply the formula of Lemma 5.2.6 with $n = 1$ and $k = 2$. The clutching function for 1 is

$$\mathrm{Id} : \mathbb{S}^1 \longrightarrow GL_1(\mathbb{C})$$
$$z \mapsto f(z) : \mathbb{C} \longrightarrow \mathbb{C}$$
$$v \mapsto v.$$

The matrix corresponding to the left-hand side of the isomorphism is given by

$$\begin{pmatrix} z^2 & 0 \\ 0 & 1 \end{pmatrix}$$

and the one corresponding to the right-hand side is

$$\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}.$$

The definition of the homotopy $G : \mathbb{S}^1 \times I \longrightarrow GL_2(\mathbb{C})$ follows from the general case of Lemma 5.2.6. $\square$

### 5.2.2 An equivalent formulation of FPT

Consider the polynomial ring

$$\mathbb{Z}[H] := \{\sum_{i=0}^{k} a_i H^i | a_i \in \mathbb{Z} \ , \ k \geq 0\}$$

and its ideal generated by $(H-1)^2$. The quotient $\mathbb{Z}[H]/ < (H-1)^2 >$ has as an additive basis $\{1, H\}$.

Note that in $K(X)$ Lemma 5.2.8 gives the formula

$$H^2 + 1 = 2H \Longleftrightarrow (H-1)^2 = 0.$$

Thus we have a natural ring homomorphism

$$\mathbb{Z}[H]/ < (H-1)^2 > \longrightarrow K(\mathbb{S}^2),$$

which makes sense since $(H-1)^2 = 0$ in $K(X)$ precisely.

Define the homomorphism $\tilde{\mu}$ to be the composite

$$\tilde{\mu} : K(X) \otimes \mathbb{Z}[H]/ < (H-1)^2 > \longrightarrow K(X) \otimes K(\mathbb{S}^2) \xrightarrow{\mu} K(X \times \mathbb{S}^2).$$

Now we are ready to give an equivalent formulation of FPT, we shall concentrate on.

**Theorem 5.2.9.** [FPT2] *The homomorphism of rings*

$$\tilde{\mu} : K(X, x_0) \otimes \mathbb{Z}[H]/ < (H-1)^2 > \xrightarrow{\cong} K(X \times \mathbb{S}^2, *)$$

*is an* ***isomorphism****.*

Taking $X = *$, we immediately have the structure of the first group of K-theory for the sphere $\mathbb{S}^2$.

**Corollary 5.2.10.** *The map*

$$\mathbb{Z}[H]/ < (H-1)^2 > \xrightarrow{\cong} K(\mathbb{S}^2, s_0)$$

*is an isomorphism of rings.*

*Remark* 5.2.11. As a consequence of this fact we have the following result. By the Corollary, the s.e.s.

$$0 \longrightarrow \tilde{K}(\mathbb{S}^2, *) \hookrightarrow K(\mathbb{S}^2, s_0) \xrightarrow{i^*} K(\mathbb{S}^0) \longrightarrow 0,$$

where $\tilde{K}(\mathbb{S}^2, *) = \operatorname{Ker} f$, is equivalent to the s.e.s.

$$0 \longrightarrow (H-1)^2 \hookrightarrow \mathbb{Z}[H]/ < (H-1)^2 > \xrightarrow{i^*} \mathbb{Z} \longrightarrow 0.$$

Hence $\tilde{K}(\mathbb{S}^2)$ is generated by $(H-1)$ as an abelian group, which we shall use in Chapter 5.

### 5.2.3 Generalized clutching functions

We have seen in Section 5.2.1 that simple clutching functions allowed us to construct vector bundles over $\mathbb{S}^2$. To prove the FPT2, we need to generalize this construction and to learn how to build a complex vector $n$-bundle over $X \times \mathbb{S}^2$ for any space $X$. The idea is the same and consists in gluing together two vector bundles, this time over $X \times D^2$, by means of a *generalized clutching function.*

Given a vector bundle $p : E \longrightarrow X$, consider the product vector bundle $p \times \mathrm{Id} : E \times \mathbb{S}^1 \longrightarrow X \times \mathbb{S}^1$. Let $f : E \times \mathbb{S}^1 \longrightarrow E \times \mathbb{S}^1$ be a bundle automorphism, i.e., a linear map such that the diagram

$$\begin{array}{ccc} E \times \mathbb{S}^1 & \xrightarrow{f} & E \times \mathbb{S}^1 \\ {\scriptstyle p\times\mathrm{Id}}\downarrow & & \downarrow{\scriptstyle p\times\mathrm{Id}} \\ X \times \mathbb{S}^1 & \xrightarrow{\mathrm{Id}} & X \times \mathbb{S}^1 \end{array}$$

commutes, i.e., such that for all $e \in E$, $z \in \mathbb{S}^1$ we have $p \times \mathrm{Id}(f(e,z)) = (p(e), z)$. Equivalently, if $e \in p^{-1}(x)$ then $f(e,z) \in (p \times \mathrm{Id})^{-1}(e,z)$. Using the zero section $s : X \longrightarrow E$; $x \mapsto (x, 0)$, which maps each element $x \in X$ to the zero element of the vector space $p^{-1}(x)$, one obtains for all $x \in X$ and $z \in \mathbb{S}^1$ an isomorphism

$$f((x,0),z) : p^{-1}(x) \longrightarrow p^{-1}(x)$$

on each fiber. From $E$ and $f$ we can construct a vector bundle over $X \times \mathbb{S}^2$ whose total space is given by

$$[E, f] := E \times D^2 \sqcup E \times D^2 / E \times \mathbb{S}^1 \sim_f E \times \mathbb{S}^1,$$

i.e., the two subspaces $E \times \mathbb{S}^1$ are identified via the isomorphism $f$.

**Definition 5.2.12.** Such a map $f \in \mathrm{Aut}(E \times \mathbb{S}^1)$ is called a **generalized clutching function** (g.c.f.) for the bundle $[E, f]$.

**Lemma 5.2.13.** *If $G : E \times \mathbb{S}^1 \times I \longrightarrow E \times \mathbb{S}^1$ is a homotopy of g.c.f., then $[E, G_0] \cong [E, G_1]$, where $G_0(e,z) = G(e,z,0)$ and $G_1(e,z) = G(e,z,1)$ for all $e \in E$, $z \in \mathbb{S}^1$.*

*Proof.* Given $p : E \longrightarrow X$, consider $p \times \mathrm{Id}_{\mathbb{S}^1} \times \mathrm{Id}_I : E \times \mathbb{S}^1 \times I \longrightarrow X \times \mathbb{S}^1 \times I$. We can then construct a bundle over $X \times I \times \mathbb{S}^2$ with total space

$$E' = (E \times I \times D^2)_1 \sqcup (E_1 \times I \times D^2)_2 / (E \times \{t\} \times \mathbb{S}^1)_1 \sim (E \times \{t\} \times \mathbb{S}^1)_2 \;\; t \in I,$$

where the identification is given by

$$(e,t,z) \in E \times I \times \mathbb{S}^1 \sim (G(e,t,z), t, G(e,t,z)) \in E \times I \times \mathbb{S}^1,$$

which makes sense, since for all $(e,t,z) \in E \times I \times \mathbb{S}^1$, the image $G(e,t,z)$ has two components, one in $E$ and the other in $\mathbb{S}^1$.
In this case,

$$E'|_{X\times\{0\}\times\mathbb{S}^2} = (E\times\{0\}\times D^2)_1 \sqcup (E_1\times\{0\}\times D^2)_2/(E\times\{0\}\times\mathbb{S}^1)_1 \sim (E\times\{0\}\times\mathbb{S}^1)_2$$

$$(e,0,z) \sim (\underbrace{G(e,0,z)}_{G_0(e,z)}, 0, \underbrace{G(e,0,z)}_{G_0(e,z)}),$$

i.e., $E'|_{X\times\{0\}\times\mathbb{S}^2} = [E, G_0]$ and, similarly, $E'|_{X\times\{1\}\times\mathbb{S}^2} = [E, G_1]$. Note that $G_0, G_1 \in \mathrm{Aut}(E\times\mathbb{S}^1)$, their inverses being $G_0^-$, resp. $G_1^-$, where

$$G^- : E\times\mathbb{S}^1 \times I \longrightarrow E\times\mathbb{S}^1$$

is the homotopy defined by

$$G^-(x,z,s) := G(x,z,1-t) \text{ for all } (x,z,s)\in E\times\mathbb{S}^1 \times I.$$

Hence $[E,G_0]\cong[E,G_1]$ by Proposition 3.2.2. □

**Proposition 5.2.14.** *For any vector bundle $E' \longrightarrow X\times\mathbb{S}^2$, one can find a space $E$ and a map $f\in\mathrm{Aut}(E\times\mathbb{S}^1)$, such that $E'\cong[E,f]$.*

*Proof.* Decompose $\mathbb{S}^2$ into two disks $D^+$ and $D^-$, such that $\mathbb{S}^2 = D^+\cup D^-$ and $\mathbb{S}^1 = D^+\cap D^-$, and consider the restrictions

$$E_+ := E'|_{X\times D^+},$$

$$E_- := E'|_{X\times D^-}$$

and

$$E := E'|_{X\times\{1\}}.$$

The composition

$$s\circ\pi : X\times D_\pm \longrightarrow X \longrightarrow X\times D_\pm$$

$$(x,z)\mapsto x\mapsto(x,1)$$

is homotopic to the identity map $\mathrm{Id}_{X\times D_\pm}$. In previous (and future) statements the index $_\pm$ is a shortcut meaning that they hold for both positive and negative components. Hence by Theorem 3.2.3, $\mathrm{Id}^*(E_\pm)\cong(s\circ\pi)^*(E)$ (considering the homotopy on $X\times\{1\}$) and $(s\circ\pi)^*(E)\cong E\times_X D_\pm$, as shown in the pullback diagram below

$$\begin{tikzcd} E\times D_\pm \arrow[r, "p_E"] \arrow[d, "\pi_{E\times D_\pm}"'] & E \arrow[d, "\pi_E"] \\ X\times D_\pm \arrow[r, "s\circ\pi"'] & X\times\{1\}. \end{tikzcd}$$

Call the last isomorphism $h_\pm : E_\pm \longrightarrow E\times_X D_\pm$, then the clutching function for $E'$ is given by

$$f := h_-\circ h_+^{-1}|_{E_+\times\mathbb{S}^1} : E_+\times\mathbb{S}^1 \longrightarrow E_+ \longrightarrow E_-\times\mathbb{S}^1\,.$$

□

*Remark* 5.2.15. One can assume clutching functions to be normalized, i.e., to be the identity over $X\times\{1\}$ .

**Some calculations**

*Notation:* We set $H^n = \underbrace{H \otimes ... \otimes H}_{n}$ and $\hat{H}^n$ the pullback of $H^n$ over the projection $X \times \mathbb{S}^2 \longrightarrow \mathbb{S}^2$.

To establish the following four bundle isomorphisms, we need to define bundle morphisms, and then to show that these bundle morphisms induce isomorphisms on fibers. Providing exact explicit definitions of the morphisms has appeared to be rather technical and puzzling, which is why we shall only give here an idea of how one could interpret isomorphisms between the corresponding bundle constructions in each case.

- $[E, \mathrm{Id}] \cong \mu(E \otimes 1)$

  Let $\xi = \{E, p, X\}$ be a complex vector $n$-bundle. Note that if $1_X = \{X \times \mathbb{C}, p', X\}$ denotes the trivial bundle over $X$ of dimension 1, the tensor product $\xi \otimes 1_X$ is still $\xi$, since 1 is the identity for $\otimes$ on $Vect_{\mathbb{C}}(X)$. Now, consider the following pullback diagram:

$$\begin{tikzcd}
E \arrow[dr, "p"'] & & p_X^*(E) \arrow[ll, "pr_E"'] \arrow[dr, "\pi_1"] & & p_{\mathbb{S}^2}^*(\mathbb{S}^2 \times \mathbb{C}) \arrow[rr, "pr_{\mathbb{S}^2 \times \mathbb{C}}"] \arrow[dl, "\pi_2"'] & & \mathbb{S}^2 \times \mathbb{C} \arrow[dl, "1"] \\
& X & & X \times \mathbb{S}^2 \arrow[ll, "p_X"'] \arrow[rr, "p_{\mathbb{S}^2}"] & & \mathbb{S}^2 .
\end{tikzcd}$$

  We have $\mu(E \otimes 1) := p_X^*(E) \boxtimes p_{\mathbb{S}^2}^*(\mathbb{S}^2 \times \mathbb{C}) \cong p_X^*(E) \cong E \times \mathbb{S}^2$ for the total space of the bundle $\{p_X^*(E) \boxtimes p_{\mathbb{S}^2}^*(\mathbb{S}^2 \times \mathbb{C}), \pi, X \times \mathbb{S}^2\}$. For all $(x, s) \in X \times \mathbb{S}^2$

$$\pi_1^{-1}(x, s) \cong \mathbb{C}^n \text{ and } \pi_2^{-1}(x, s) \cong \mathbb{C},$$

  therefore, each fiber satisfies $\pi^{-1}(x, s) \cong \mathbb{C}^n \otimes \mathbb{C} \cong \mathbb{C}^n$ for all $(x, s) \in X \times \mathbb{S}^2$.

  On the other hand, $[E, \mathrm{Id}]$ is an $n$-vector bundle (since $\dim(E) = n$) over $X \times \mathbb{S}^2$ with total space

$$[E, \mathrm{Id}] := F = E \times D^2 \sqcup E \times D^2 / E \times \mathbb{S}^1 \sim_{\mathrm{Id}} E \times \mathbb{S}^1 .$$

  In this clutching construction we simply glue the two hemispheres $D_1$ and $D_2$, "thickened with $E$", along their boundary $\mathbb{S}^1$ without any twisting. This gives the total space $F$ of the bundle $\{F, \pi, X \times \mathbb{S}^2\}$, and, by construction, we have $F \cong E \times \mathbb{S}^2$.

  Moreover, $\pi^{-1}(x, s) \cong \mathbb{C}^n$ for all $x \in X$, $s \in \mathbb{S}^2$, which shows the first isomorphism.

- $[1, z^n] \cong H^n$

We proceed by induction on $n$. The case $n = 1$ follows from the definition of the c.f. $f_H := [1, z]$ for $H$. Let us show the case $n = 2$. Using Lemma 5.2.8 and the fact that for any two bundles $E_1$, $E_2$ over $X$

$$[E_1, f_1] \oplus [E_2, f_2] \cong [E_1 \oplus E_2, f_1 \oplus f_2]$$

we have

$$\begin{aligned} ([1, z] \otimes [1, z]) \oplus [1, \mathrm{Id}] &\cong [1, z] \oplus [1, z] \\ &\cong [1 \oplus 1, z \oplus z] \\ &\cong [2, z \oplus z]. \end{aligned}$$

Hence we have to check that

$$[2, z \oplus z] \overset{?}{\cong} [1, z^2] \oplus 1.$$

This follows essentially from the proof of Lemma 5.2.8. Indeed, the c.f. for $[1, z^2]$ is by definition

$$z^2 : \mathbb{S}^1 \longrightarrow GL_1(\mathbb{C})$$

$$z \mapsto z^2 : \mathbb{C} \longrightarrow \mathbb{C}$$

$$v \mapsto z^2 v,$$

so that the matrix corresponding to the c.f. $[1, z^2] \oplus 1$ is

$$\begin{pmatrix} z \cdot z & 0 \\ 0 & 1 \end{pmatrix} \in GL_2(\mathbb{C}),$$

which is precisely the matrix of the left-hand side of the isomorphism of Lemma 5.2.8. Since the Lemma claims that there exists an appropriate homotopy between this c.f. and the c.f. of $[2, z \oplus z]$ given by the matrix

$$\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \in GL_2(\mathbb{C}),$$

the expression is true for $n = 2$. In fact, what we have essentially checked is $[1, z] \otimes [1, z] \cong [1, z^2]$. Does this work for all $n$?
By induction, suppose that $[1, z^{n-1}] \cong H^{n-1} \cong \underbrace{[1, z] \otimes \ldots \otimes [1, z]}_{n-1}$. Tensor both sides of

$$(H \otimes H) \oplus 1 \cong H \oplus H$$

with $H^{n-2}$ to have

$$H^{n-2} \otimes (H \otimes H \oplus 1) \cong H^{n-2} \otimes (H \oplus H)$$

$$H^n \oplus H^{n-2} \cong H^{n-1} \oplus H^{n-1}.$$

Therefore, the case $n$ would be true if

$$[1, z^n] \oplus [1, z^{n-2}] \cong [1, z^{n-1}] \oplus [1, z^{n-1}],$$

i.e., if we had a homotopy

$$G : \mathbb{S}^1 \times I \longrightarrow GL_2(\mathbb{C})$$

such that $G_0(z) = z^{n-1} \oplus z^{n-1}$ and $G_1(z) = z^n \oplus z^{n-2}$.
As in Lemma 5.2.6, consider the path

$$\alpha : [1, 2] \longrightarrow GL_2(\mathbb{C})$$

such that

$$\begin{aligned} \alpha(1) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \text{ matrix which interchanges the two factors of } \mathbb{C} \times \mathbb{C} \\ \alpha(2) &= I_2. \end{aligned}$$

Define $G$ to be the homotopy

$$G : \mathbb{S}^1 \times [1, 2] \longrightarrow GL_2(\mathbb{C})$$

$$(z, t) \mapsto G(z, t) = G_t(z)$$

given by the matrix product $h(z)(f \oplus \mathrm{Id})(z)\alpha(3-t)(\mathrm{Id} \oplus f)(z)$, where

$$f(z) : \mathbb{C} \longrightarrow \mathbb{C}$$

$$v \mapsto zv$$

and

$$h(z) : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$$
$$v \mapsto (z^{n-2} \oplus z^{n-2})v.$$

In terms of matrix multiplications we have for $G_0(z)$

$$I_2 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} \mapsto \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \mapsto \begin{pmatrix} z^{n-1} & 0 \\ 0 & z^{n-1} \end{pmatrix} \in GL_2(\mathbb{C}).$$

Similarly, $G_1(z)$ is given by

$$I_2 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} \mapsto \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} z^2 & 0 \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} z^n & 0 \\ 0 & z^{n-2} \end{pmatrix} \in GL_2(\mathbb{C}),$$

which completes the induction.

- $[E, z^n] \cong \mu(E \otimes H^n)$

  Let $p : E \longrightarrow X$ be a complex vector $n$-bundle and $H : \mathbb{S}^2 \times \mathbb{C} \longrightarrow \mathbb{S}^2$ the trivial line bundle. We have $p_H^{-1}(s) \cong \mathbb{C}$ for all $s \in \mathbb{S}^2$. Recall that by definition the bundle $H^n \xrightarrow{p_{H^n}} \mathbb{S}^2$ has its total space defined to be

$$H^n := \underbrace{H \otimes ... \otimes H}_{n} := \bigsqcup_{s \in \mathbb{S}^2} \underbrace{p_H^{-1}(s) \otimes ... \otimes p_H^{-1}(s)}_{n},$$

  so that $p_{H^n}^{-1}(s) \cong \mathbb{C}$ for all $s \in \mathbb{S}^2$,too.
  Now, consider the diagram

$$\begin{tikzcd}
E \arrow[dr, "p"'] & p_X^*(E) \arrow[l, "pr_E"'] \arrow[dr, "\pi_1"] & & p_{\mathbb{S}^2}^*(H^n) \arrow[r, "pr_{H^n}"] \arrow[dl, "\pi_2"'] & H^n \arrow[dl, "p_{H^n}"] \\
& X & X \times \mathbb{S}^2 \arrow[l, "p_X"'] \arrow[r, "p_{\mathbb{S}^2}"] & \mathbb{S}^2 . &
\end{tikzcd}$$

  By definition, $\mu(E \otimes H^n) := p_X^*(E) \boxtimes p_{\mathbb{S}^2}^*(H^n)$ is the total space of $\eta = \{p^*(E) \boxtimes p_{H^n}^*(H^n), \Pi, X \times \mathbb{S}^2\}$. Since for all $(x, s) \in X \times \mathbb{S}^2$ we have

$$\pi_1^{-1}(x, s) \cong \mathbb{C}^n \text{ and } \pi_2^{-1}(x, s) \cong \mathbb{C},$$

  in each fiber of $\eta$ we get $\Pi^{-1}(x, s) \cong \mathbb{C}^n \otimes \mathbb{C} \cong \mathbb{C}^n$.

  On the other hand, $[E, z^n]$ is an $n$-vector bundle (since $\dim(E) = n$) over $X \times \mathbb{S}^2$ with total space given by

$$[E, z^n] := K = E \times D^2 \sqcup E \times D^2 / E \times \mathbb{S}^1 \sim_{z^n} E \times \mathbb{S}^1,$$

  and with g.c.f. given by

$$f((x, 0), z) : p^{-1}(x) \longrightarrow p^{-1}(x)$$

$$v \mapsto z^n v,$$

  with $x \in X$, $z \in \mathbb{S}^1$. Each fiber of the bundle $\{K, \pi, X \times \mathbb{S}^2\}$ satisfies $(p \times \mathrm{Id})^{-1}(x, s) \cong \mathbb{C}^n$ for all $x \in X$, $s \in \mathbb{S}^2$.
  It remains to guess how one could put $\mu(E \otimes H^n)$ and $K$ in a bijective correspondence. Possibly, the bundle $\mu(E \otimes H^n)$ is constructed in a way to illustrate the "chronology" of twisting the two hemispheres, before gluing them along the equator. Each of the $n$ copies of $\mathbb{S}^2$ in $\mu(E \otimes H^n)$ would then "memorize" the next shift of, let us say, the upper hemisphere by $z$.

- $[E, z^n f] \cong [E, f] \otimes \hat{H}^n$

  To fix the names of different projections we first give the pullback diagram

$$\begin{tikzcd}
\hat{H}^n := p_{\mathbb{S}^2}^*(H^n) \arrow[r, "pr_{H^n}"] \arrow[d, "\hat{\pi}"'] & H^n \arrow[d, "p_{H^n}"] \\
X \times \mathbb{S}^2 \arrow[r, "p_{\mathbb{S}^2}"'] & \mathbb{S}^2 .
\end{tikzcd}$$

  By the previous point we know that for all $s \in \mathbb{S}^2$, $p_{H^n}^{-1}(s) \cong \mathbb{C}$, hence $\hat{\pi}^{-1}(x, s) \cong \mathbb{C}$, too, because taking the pullback of $H^n$ only changes the base space, the dimension of the new bundle $\hat{H}^n$ remaining the same.
  Since the total space of the bundle $\{[E, f] \otimes \hat{H}^n, \Pi, X \times \mathbb{S}^2\}$ is by definition

$$[E, f] \otimes \hat{H}^n := \bigsqcup_{(x,s) \in X \times \mathbb{S}^2} (p \times \mathrm{Id})^{-1}(x, s) \otimes \hat{\pi}(x, s),$$

  we have that $\Pi^{-1}(x, s) \cong \mathbb{C}^n \otimes \mathbb{C} \cong \mathbb{C}^n$ for all $(x, s) \in X \times \mathbb{S}^2$.

  On the other hand, $[E, z^n f]$ is an $n$-vector bundle over $X \times \mathbb{S}^2$, with c.f defined by

$$z^n f((x, 0), z) : p^{-1}(x) \longrightarrow p^{-1}(x)$$
$$v \mapsto z^n f(v).$$

  On each fiber we have $(p \times \mathrm{Id})^{-1}(x, s) \cong \mathbb{C}^n$ and this is isomorphic to $\Pi^{-1}(x, s)$, for all $x \in X$, $s \in \mathbb{S}^2$.

  As for finding a bundle isomorphism between $[E, z^n f]$ and $[E, f] \otimes \hat{H}^n$, the $n$-fold twisting of the upper hemisphere by $z$ is preceded with $f$ this time, and the same idea as before could may be make some sense.

### 5.2.4 Laurent polynomial clutching functions

We turn now to the description of a simple class of clutching functions that basically have the same form as Laurent polynomials, and whose coefficients are endomorphisms of $E$. The point is that an arbitrary c.f. can be reduced to a polynomial one.

**Definition 5.2.16.** Let $E \longrightarrow X$ be a vector bundle. A **Laurent polynomial clutching function** (L.p.c.f.) is a bundle automorphism

$$\ell : E \times \mathbb{S}^1 \longrightarrow E \times \mathbb{S}^1$$

such that the diagram

$$\begin{tikzcd}
E \times \mathbb{S}^1 \arrow[r, "\ell"] \arrow[d, "p\times \mathrm{Id}"'] & E \times \mathbb{S}^1 \arrow[d, "p\times \mathrm{Id}"] \\
X \times \mathbb{S}^1 \arrow[r, "\mathrm{Id}"] & X \times \mathbb{S}^1
\end{tikzcd}$$

commutes, and such that for all $x \in X$ there exists a linear transformation

$$a_i(x) : p^{-1}(x) \longrightarrow p^{-1}(x)$$

satisfying

$$\begin{aligned}\ell((x,0),z) : p^{-1}(x) &\longrightarrow p^{-1}(x)\\ v &\mapsto \sum_{|i|\leq n} z^i a_i(x) v,\end{aligned}$$

where $z \in \mathbb{S}^1$.

Note that the maps $a_i(x)$ need not to be invertible, as opposed to their linear combination $\sum_{|i|\leq n} a_i(x)z^i$, required to be an automorphism by the definition of a g.c.f.

*Terminology (Bott):*

- According to Bott, an expression of the last form is called a **Laurent series** of endomorphisms over $E$, due to its form. It can be defined for all $z \in \mathbb{C}$, but here only matter its values taken on $\mathbb{S}^1$, and we call such a series **proper** if $\ell(x,z)$ is non-singular for $z \in \mathbb{S}^1$.
- If $i \geq 0$ (i.e., no negative powers of $z$ occur in $\ell$), then $\ell$ is called a **Laurent polynomial**.
- If $\ell$ is a proper Laurent series over $E$, the vector bundle $[E,\ell]$ over $X \times \mathbb{S}^2$ is said to be obtained from $E$ by a **Laurent construction**.

*Remark* 5.2.17. Further development involves many classical tools from complex analysis, such as Laurent series, criteria for uniform and absolute convergence of complex series, etc. Though making great use of them, we shall agree to take all these results for granted, referring the reader to his own knowledge and to corresponding proofs, and leaving the pleasure of details to analysts.

**Proposition 5.2.18.** [[Ha2], Proposition 2.4] *Let $E \longrightarrow X$ be a vector bundle. Then given a c.f. $f \in \mathrm{Aut}(E \times \mathbb{S}^1)$, one can find a L.p.c.f. $\ell$ such that $[E,f] \cong [E,\ell]$.*
*Moreover, if two L.p.c.f. $\ell_0, \ell_1 \in \mathrm{Aut}(E\times\mathbb{S}^1)$ are homotopic through clutching functions, then the homotopy itself is a L.p.c.f. homotopy of the form*

$$\ell(x,z,t) : p^{-1}(x) \times I \longrightarrow p^{-1}(x)$$

$$\ell(x,z,t) = \sum_{|i|\leq n} a_i(x,t) z^i.$$

Before giving the idea of the proof, we make some additional remarks. For a compact space $X$, we wish to approximate a continuous function $f : X \times \mathbb{S}^1 \longrightarrow \mathbb{C}$ by Laurent polynomial functions of the form

$$\sum_{|n|\leq N} a_n(x) z^n = \sum_{|n|\leq N} a_n(x) e^{-in\theta},$$

where $a_n : X \longrightarrow \mathbb{C}$ is a continuous map. Motivated by Fourier series, one sets

$$a_n(x) = \frac{1}{2\pi} \int_{\mathbb{S}^1} f(x,\theta) e^{-in\theta} d\theta.$$

For $r \in \mathbb{R}^+$, let

$$u(x,r,\theta) = \sum_{n\in\mathbb{Z}} a_n(x) r^{|n|} e^{in\theta}$$

and note that for fixed $r < 1$, this series converges absolutely and uniformly as $(x,\theta)$ ranges over $X \times \mathbb{S}^1$.

**Lemma 5.2.19.** [[Ha2], Lemma 2.5] *As $r \to 1$, $u(x,r,\theta)$ converges uniformly to $f(x,\theta)$ in $x$ and $\theta$.*

It follows that taking sums of finitely many terms in the series $u(x,r,\theta)$ with $r \to 1$ will give the desired approximations to $f$ by Laurent polynomial functions.

We also need the following definition.

**Definition 5.2.20.** An Hermitian **inner product** on a complex vector bundle $p : E \longrightarrow X$ is a map

$$< . , . >: E \oplus E \longrightarrow \mathbb{C},$$

restricting in each fiber to an inner product, i.e., a positive definite symmetric bilinear form.

**Lemma 5.2.21.** [[Ha2], Proposition 1.2] *A Hermitian inner product exists for a vector bundle $p : E \longrightarrow X$ if $X$ is compact Hausdorff.*

*Idea of the proof:* Let $\{U_\alpha\}_{\alpha\in I}$, be an open cover of $X$, and let $\varphi_\beta : X \longrightarrow [0,1]$ with $supp\, \varphi_\beta \subset U_\beta$, $\beta \in I$, such that $\sum_\beta \varphi_\beta = 1$, be a partition of unity subordinate to $\{U_\alpha\}$. Since $X$ is compact, it follows that only finitely many of the $\varphi_\beta$ are non zero in a neighborhood of each $x \in X$. A Hermitian inner product for $p : E \longrightarrow X$ can be constructed by using local trivializations $\varphi_\alpha : U_\alpha \times \mathbb{C}^n \xrightarrow{\approx} p^{-1}(U_\alpha)$ to transfer the standard inner product in $\mathbb{C}^n$ to an inner product $< . , . >_\alpha: p^{-1}(U_\alpha) \oplus p^{-1}(U_\alpha) \longrightarrow \mathbb{C}^n$. To extend it to $E$, one sets for all $v, w \in E$

$$< v, w >:= \sum_\beta \varphi_\beta p(v) < v, w >_\beta . \qquad \square$$

**Sketch of the proof of Theorem 5.2.18**

***Step 1*** Choose a Hermitian inner product on $E$. It defines a norm on $E$, coming from the standard complex norm in $\mathbb{C}^n$ defined on each fiber. The set $\mathrm{End}(E\times\mathbb{S}^1)$ of endomorphisms of bundles over $X\times\mathbb{S}^1$ can be given the structure of a normed vector space with a norm defined by

$$||\alpha|| := \sup_{|((x,v),z)|=1} |\alpha((x,v),z)| = \sup_{|v|=1} |\alpha(v)|.$$

Note that here $|.|$ is the complex norm. Since $z\in\mathbb{S}^1$, $|z|=1$, and $v$ is an element of the vector space $p^{-1}(x)$, for all $x\in X$. The subspace $\mathrm{Aut}(E\times\mathbb{S}^1)$ is open in the topology defined by this norm, since it is the preimage of $(0,\infty)$ under the continuous map

$$\mathrm{End}(E\times\mathbb{S}^1)\longrightarrow[0,\infty)$$

$$\alpha\mapsto \inf_{(x,z)\in X\times\mathbb{S}^1} |\det(\alpha(x,z))|.$$

Thus to prove the first part of the proposition it suffices to show that Laurent polynomials are dense in $\mathrm{End}(E\times\mathbb{S}^1)$, i.e., that for any $\alpha\in\mathrm{End}(E\times\mathbb{S}^1)$ there exists an $\varepsilon>0$ and a Laurent polynomial $\ell$ such that

$$||\alpha-\ell||<\varepsilon. \qquad \diamond$$

In particular, this holds for $\alpha=f\in\mathrm{Aut}(E\times\mathbb{S}^1)$. In this case, given such an $f$, a sufficiently close Laurent approximation $\ell\in\mathrm{Aut}(E\times\mathbb{S}^1)$ to $f$ will be homotopic to $f$ via

$$G: E\times\mathbb{S}^1\times I\longrightarrow E\times\mathbb{S}^1$$

$$G(x,z,t)=t\ell+(1-t)f.$$

Indeed, $G(x,z,0)=f$, $G(x,z,1)=\ell$, and it remains to show that $G$ is continuous for all $t\in I$. Let $t_0,t_1\in I$ such that $|t_0-t_1|<\delta$, then

$$\begin{aligned} ||t_1\ell+(1-t_1)f-t_0\ell-(1-t_0)f &= ||(t_1-t_0)\ell-(t_1-t_0)f|| \\ &= \underbrace{|t_1-t_0|}_{<\delta}\cdot\underbrace{||f-\ell||}_{\varepsilon}<\epsilon, \end{aligned}$$

hence $G$ is (uniformly) continuous in $t$. Note finally that for any $t\in I$ the map $G(-,-,t)$ is a c.f. because it is a linear combination of c.f. $f$ and $\ell$, and since $\mathrm{Aut}(E\times\mathbb{S}^1)$ is a vector space (stability for the addition).

***Step 2*** To show $\diamond$, choose an open covering $\{U_i\}$ of $X$ together with the local trivializations $\varphi_i : U_i \times \mathbb{C}^{n_i} \longrightarrow p^{-1}(U_i)$. Let $\{\eta_i\}$ be a partition of unity subordinate to $\{U_i\}$. Via $\varphi_i$, for $x \in X_i$, the maps $f(x,z) : p^{-1}(x) \longrightarrow p^{-1}(x)$ can be viewed as matrices. The entries of these matrices define functions $X_i \times \mathbb{S}^1 \longrightarrow \mathbb{C}$, for all $i$. Thanks to the "additional considerations" we made above and Lemma 5.2.19, there exist Laurent polynomial matrices $\ell_i(x,z)$ whose entries uniformly approximate those of $f(x,z)$ for $x \in X_i$. Hence $\ell_i$ approximates $f$ in the $||\cdot||$ norm. Finally, the convex linear combination

$$\ell = \sum_i \eta_i \ell_i$$

gives a Laurent polynomial approximating the c.f. $f$ over $X \times \mathbb{S}^2$.

***Step 3*** The idea for the second part of the Proposition is the following. A homotopy

$$G : E \times \mathbb{S}^1 \times I \longrightarrow E \times \mathbb{S}^1$$

such that

$$G_0 = \ell_0, \qquad G_1 = \ell_1,$$

can be viewed as an automorphism of $E \times \mathbb{S}^1 \times I$. (details...) Hence, by *Step 1*, $G$ can be approximated by a Laurent polynomial $G_L$ (i.e., $\exists \varepsilon$ such that $||G - G_L|| < \varepsilon$ ) to give a Laurent polynomial homotopy

$$G_L : E \times \mathbb{S}^1 \times I \longrightarrow E \times \mathbb{S}^1$$

such that

$$G_{L_0} = \ell_{L_0}, \qquad G_{L_1} = \ell_{L_1}.$$

Finally, one has to find linear homotopies from $\ell_0$ to $\ell_{L_0}$ and from $\ell_1$ to $\ell_{L_1}$ (which are Laurent polynomials) to obtain a Laurent polynomial homotopy $\ell(x,z,t) : p^{-1}(x) \times I \longrightarrow p^{-1}(x)$ from $\ell_0$ to $\ell_1$:

$$\ell_0 \overset{\text{to be found}}{\rightsquigarrow} \ell_{L_0} \overset{G_L}{\rightsquigarrow} \ell_{L_1} \overset{\text{to be found}}{\rightsquigarrow} \ell_1. \qquad \square$$

### 5.2.5 Linearization procedure

Since every Laurent series is of the form $z^{-m}p$, where $p$ is a polynomial, the essence of the Laurent polynomial construction can be understood from polynomials. To simplify things even more, one uses an operation similar to the one that transforms an $n^{th}$ order differential equation to a family of first order ones, calling it a *linearization procedure*.

**Definition 5.2.22.** Let $E \longrightarrow X$ be a complex vector bundle, and consider

$$p(z) = \sum_{i=0}^{n} a_i z^i,$$

a polynomial of degree $\leq n$ over $E$, i.e., the $a_i$ are in $\mathrm{End}(E)$. Define the bundle

$$L^{n+1}(E) := \underbrace{E \oplus ... \oplus E}_{n+1}$$

and a linear polynomial

$$L^{n+1}(p) : L^{n+1}(E) \longrightarrow L^{n+1}(E)$$

given by the matrix

$$\begin{pmatrix} a_0 & a_1 & ... & & a_{n-1} & a_n \\ -z & 1 & 0 & ... & 0 & 0 \\ 0 & -z & 1 & 0 & ... & 0 \\ 0 & & ... & & & 0 \\ 0 & & ... & 0 & -z & 1 \end{pmatrix}.$$

In other words, $L^{n+1}(p)$ is an endomorphism of $L^{n+1}(E)$, and we can interpret the $(i,j)$ entry of the matrix as a linear map from the $j^{th}$ summand of $L^{n+1}$ to the $i^{th}$ summand. Entries with 1 correspond to the identity $\mathrm{Id} : E \longrightarrow E$, and $z$ stands for $z \cdot \mathrm{Id} : E \longrightarrow E$ with $z \in \mathbb{S}^1$.

By earlier observations, a L.p.c.f. can be written as $\ell = z^{-m}p$ where $p$ is a polynomial c.f., and we have $[E, \ell] \cong [E, p] \otimes \hat{H}^{-m}$ in light of previous calculations. The next proposition gives a way of reducing a polynomial c.f. of an arbitrary degree $n$ to linear c.f. i.e., of degree at most 1.

**Proposition 5.2.23.** [[Ha2], Proposition 2.6] *Let $p$ be a proper polynomial c.f. of degree at most $n$, then*

$$[E, p] \oplus [L^n(E), \mathrm{Id}] \cong [L^{n+1}(E), L^{n+1}(p)],$$

*where $L^n(p)$ is a proper linear polynomial on $L^n(E)$.*

*Sketch of the proof:* The matrix

$$A = \begin{pmatrix} a_0 & a_1 & ... & & a_{n-1} & a_n \\ -z & 1 & 0 & ... & 0 & 0 \\ 0 & -z & 1 & 0 & ... & 0 \\ 0 & & ... & & & 0 \\ 0 & & ... & 0 & -z & 1 \end{pmatrix}$$

corresponds to the right-hand side, which is clear from Definition 5.2.22, and the matrix

$$B = \begin{pmatrix} p & 0 \\ 0 & \mathrm{Id}_n \end{pmatrix}$$

to the left-hand side, which follows from earlier identities on c.f. To pass from $A$ to $B$ use a sequence of "elementary operations" adding $z$ times the first column to the second, then $z$ times the second to the third, etc. The idea is to obtain at the end zeros above the diagonal and the polynomial $p$ in the lower right corner. Secondly, for each $i \leq n$, subtract the appropriate multiple of the $i^{th}$ row from the last row, to make all the entries in the last row equal to 0, except for the final $q$.

Note that these are not quite usual elementary operations, since we are working with linear maps and not numbers. However, by restricting to a fiber of $E$ and choosing a basis in this fiber, each entry in $A$ becomes a matrix of numbers, allowing us to think in terms of usual elementary operations.

Observe that the matrix $B$ is a c.f., i.e. an *automorphism*, not only endomorphism, for $[E, p] \oplus [L^n(E), \mathrm{Id}]$, since in each fiber the extended version of $B$ has a non-zero determinant. Since elementary operations preserve the determinant, $A$ is also an automorphism of $L^{n+1}(E)$ for each $z \in \mathbb{S}^1$ and therefore determines a c.f., which is precisely $L^n(p)$.

Lastly, the c.f. associated to $A$ and $B$ are homotopic, since the "extended" elementary operations can be realized as families of continuous one-parameter operations. For example, the first operation "adding $z$ times the first column to the second" can be viewed as "adding $tz$ times the first column to the second" with $t \in [0, 1]$. We conclude using Lemma 5.2.2. □

Thus given an arbitrary vector bundle having a polynomial of degree $n$ as c.f., we can always construct a bundle whose c.f. is a family of linear c.f., and that is isomorphic to the first one, up to addition of $[L^n(E), \mathrm{Id}]$.

Let us mention one more property concerning linear bundles: we can simplify things even more by splitting a linear bundle in a sum of two bundles with "simple" c.f. Id and $z$.

**Proposition 5.2.24.** [[Ha2], Proposition 2.7] *Given a vector bundle $[E, a(x)z+b(x)]$ over $X \times \mathbb{S}^2$, there is a splitting $E \cong E_+ \oplus E_-$ such that*

$$[E, a(x)z + b(x)] \cong [E_+, \mathrm{Id}] \oplus [E_-, z].$$

*Sketch of the proof:* Hatcher gives a very complete proof of this, which is rather long, so we shall only highlight the main ideas. The first step is to reduce to the case $a(x) = \mathrm{Id}(x)$ for all $x \in E$. Next, one should note that $z + b(x)$ is invertible (in $x$) for all $z \in \mathbb{S}^1$, since $b$ is an automorphism by assumption. Therefore $b(x)$ has no eigenvalues on the unit circle $\mathbb{S}^1$. Indeed, if $z \in \mathbb{S}^1$ is an eigenvalue of $b(x)$, and $b(x)(v) = zv$, then $(-z + b(x))(v) = 0$, i.e., $z + b(x)$ is not invertible.

Next, a lemma ([Ha2] Lemma 2.8) guarantees that in this case there exist unique subbundles $E_+$ and $E_-$ of $E$ such that $E \cong E_+ \oplus E_-$ (i.e., we have the right splitting of $E$) and such that $\mathbb{S}^1$ separates the eigenvalues of $b$. Namely, those of $b|_{E_+}(x) := b_+(x)$ lie outside $\mathbb{S}^1$ and those of $b|_{E_-}(x) := b_-(x)$ inside $\mathbb{S}^1$ for each $x \in X$.

The lemma gives a splitting $[E, z + b(x)] \cong [E_+, z + b_+(x)] \oplus [E_-, z + b_-(x)]$. Finally, the importance of the distribution of eigenvalues of $b_+$ and $b_-$ becomes clear, since it enables us to define homotopies of c.f.

$$G_+ : E \times \mathbb{S}^1 \times I \longrightarrow E \times \mathbb{S}^1$$

$$G_+(x, z, t) = tz + b_+(x)$$

and

$$G_- : E \times \mathbb{S}^1 \times I \longrightarrow E \times \mathbb{S}^1$$

$$G_-(x, z, t) = z + tb_-(x),$$

showing respectively that $[E_+, z+b_+(x)] \cong [E_+, b_+(x)]$ and $[E_-, z+b_-(x)] \cong [E_-, z]$. Note that continuity of $G_+$ and $G_-$ follow from this of $b_+$ and $b_-$. The last thing one would need to convince himself of is that $[E_+, b_+(x)] \cong [E_+, \mathrm{Id}_{E_+ \times \mathbb{S}^1}]$.

*Remark* 5.2.25. Note that previous splitting preserves direct sums in the sense that

$$[E_1 \oplus E_2, (a_1 z + b_1) \oplus (a_2 z + b_2)]$$

has $(E_1 \oplus E_2)_\pm := (E_1)_\pm \oplus (E_2)_\pm$ as splitting.

## 5.3 A short sketch of the proof of FPT2

Recall that in FPT2 (Theorem 5.2.9) we were asked to show that

$$\tilde{\mu} : K(X) \otimes \mathbb{Z}[H]/ < (H-1)^2 > \xrightarrow{\cong} K(X \times \mathbb{S}^2)$$

is an isomorphism.

*Proof.* **Surjectivity**

Surjectivity of $\tilde{\mu}$ means that for any complex vector bundle $E' \longrightarrow X \times \mathbb{S}^2$ we can find bundles $\xi \otimes \eta$ in $K(X) \otimes \mathbb{Z}[H]/ < (H-1)^2 >$ such that $\tilde{\mu}(\xi \otimes \eta) = E'$. By Proposition 5.2.14, there exist a space $E$ and a c.f. $f$ such that $E' \cong [E, f]$. Therefore, working with $[E, f]$ in $K(X \times \mathbb{S}^2)$, we get

$$
\begin{aligned}
[E,f] &\overset{(1)}{=} [E, z^m p] \\
&\overset{(2)}{=} [E,p] \otimes \hat{H}^{-m} \\
&\overset{(3)}{=} [L^{n+1}(E), L^{n+1}(p)] \otimes \hat{H}^{-m} - [L^n(E), \mathrm{Id}] \otimes \hat{H}^{-m} \\
&\overset{(4)}{=} [L^{n+1}(E)_+, \mathrm{Id}_{L^{n+1}(E)_+ \times \mathbb{S}^1}] \otimes \hat{H}^{-m} + [L^{n+1}(E)_-, z] \otimes \hat{H}^{-m} \\
&\qquad\qquad\qquad\qquad - [L^n(E), \mathrm{Id}] \otimes \hat{H}^{-m} \\
&\overset{(5)}{=} \tilde{\mu}(L^{n+1}(E)_+ \otimes H^{-m}) + \tilde{\mu}(L^{n+1}(E)_+ \otimes H^{-m-1}) - \tilde{\mu}(L^n(E) \otimes H^{-m})
\end{aligned}
$$

Here (1) follows from Proposition 5.2.18, (2) and (5) have been calculated in Section 5.2.3, (3) follows from Proposition 5.2.23 and (4) is a result of Proposition 5.2.24. The last expression is in the image of $\tilde{\mu}$, hence $\tilde{\mu}$ is surjective.

**Injectivity**

To show that $\tilde{\mu}$ is injective, one constructs a map

$$\tilde{\nu} : K(X \times \mathbb{S}^2) \xrightarrow{\cong} K(X) \otimes \mathbb{Z}[H]/ < (H-1)^2 >$$

such that $\tilde{\nu} \circ \tilde{\mu} = \mathrm{Id}$. The idea is to define $\nu([E,f])$ as some linear combination of terms $E \otimes H^k$ and $L^{n+1}(E)_\pm \otimes H^k$, independent of all choices. From the matrix representations of $L^{n+1}(p)$ and $L^{n+1}(zp)$, one derives the formulas

**(1)** $[L^{n+2}(E), L^{n+1}(p)] \cong [L^{n+1}(E), L^n(p)] \oplus [E, \mathrm{Id}]$

**(2)** $[L^{n+2}(E), L^{n+1}(zp)] \cong [L^{n+1}(E), L^n(p)] \oplus [E, z]$,

with $deg(p) \leq n$. Taking into account the definitions of corresponding c.f. (seeing them as of the form $z+b(x)$), as well as the distribution of eigenvalues for $b_\pm$, it appears that the $\pm$ splittings for correction terms are

**(3)** For $[E, \mathrm{Id}] :\ E_- = 0,\ E_+ = E$

**(4)** For $[E, f] :\ E_+ = 0,\ E_- = E$.

Formulas (1) and (3) give $L^{n+2}(E)_- = L^{n+1}(E)_-$, hence the $-$ summand in independent of $n$.

Define for $deg(p) \leq n$

$$\tilde{\nu}([E, z^{-m}p]) = L^{n+1}(E)_- \otimes (H-1) + E \otimes H^{-m}.$$

We claim that it is well-defined. We have just noticed that the $-$ summand in independent of $n$, so $\nu([E, z^{-m}p])$ does not depend on $n$. One should also see that it is independent of $m$.

The last thing to verify is that $\tilde{\nu}([E, z^{-m}p])$ does not depend on the c.f. for the bundle $[E, z^{-m}p]$.
We know that every bundle over $X \times \mathbb{S}^2$ is isomorphic to a certain $[E, f]$ with $f$ a (normalized) c.f., which is unique up to homotopy. If two bundles $[E_0, f_0]$, $[E_1, f_1]$ over $X \times \mathbb{S}^2$ are such that $f_0 \simeq f_1$, then by Proposition 5.2.19 we can approximate $f_0$,$f_1$ by L.p.c.f. $\ell_0$, $\ell_1$, and these polynomial approximations are homotopic by a Laurent polynomial homotopy.
Let $z^{-m}p$ and $z^{-m'}p'$ be two different polynomial c.f for the bundle $[E, z^{-m}p]$ and consider expressions

$$\tilde{\nu}([E, z^{-m}p]) = L^{N+1}(E)_- \otimes (H-1) + E \otimes H^{-m},$$

$$\tilde{\nu}([E, z^{-m'}p']) = L^{N+1}(E)_- \otimes (H-1) + E \otimes H^{-m'},$$

for $N \geq \max\{deg(p), deg(p')\}$.
We need to show that in these two cases the bundle $L^{N+1}(E)_-$ seen respectively over $X \times \{0\}$ and $X \times \{1\}$ has homotopic c.f. Applying Proposition 5.2.23 we have

$$[E, z^{-m}p] \oplus [L^N(E), \mathrm{Id}] \cong [L^{N+1}(E), L^N z^{-m}p]$$

and

$$[E, z^{-m'}p'] \oplus [L^N(E), \mathrm{Id}] \cong [L^{N+1}(E), L^N z^{-m'}p'],$$

therefore finding a Laurent polynomial homotopy between $z^{-m}p$ and $z^{-m'}p'$ shows that $L^N z^{-m}p$ and $L^N z^{-m'}p$ are homotopic, hence the bundles $L^{N+1}(E)_-$ over $X \times \{0\}$ and $X \times \{1\}$ are isomorphic (the $\pm$ splitting is preserved thanks to Proposition 5.2.24).
Let us check that $\tilde{\nu}$ preserves sums. We have

$$\begin{aligned}\tilde{\nu}[E_1 \oplus E_2, z^{-m_1}p_1 \oplus z^{-m_2}p_2] &= L^{n+1}(E_1 \oplus E_2)_- \otimes (H-1) + (E_1 \oplus E_2) \otimes H \\ &= L^{n+1}(E_1)_- \otimes (H-1) + E_1 \otimes H \oplus \\ &\qquad L^{n+1}(E_2)_- \otimes (H-1) + E_2 \otimes H,\end{aligned}$$

by the distributivity of $\otimes$ over $\oplus$, and since the $\pm$ splitting preserves the direct sum by Remark 5.2.25 for the spaces, which implies $L^{n+1}(z^{-m_1}p_1 \oplus z^{-m_2}p_2) = L^{n+1}(z^{-m_1}p_1) \oplus L^{n+1}(z^{-m_1}p_1)$. It follows that

$$\tilde{\nu} : K(X \times \mathbb{S}^2) \xrightarrow{\cong} K(X) \otimes \mathbb{Z}[H]/ < (H-1)^2 >$$

extends to a homomorphism of groups.

Finally, it remains to verify that $\tilde{\nu} \circ \tilde{\mu} = \mathrm{Id}$. We already know that the group $\mathbb{Z}[H]/(H-1)^2$ is generated by $\{1, H\}$, and from $(H-1)^2 = 0$ we deduce $H^{-1} = H + 2$. Since by Corollary 5.2.10, $\mathbb{Z}[H]/ < (H-1)^2 > \cong K(\mathbb{S}^2)$, we can say that $K(\mathbb{S}^2)$ is generated by $\{1, H^{-1}\}$. Thus is suffices to show the identity on elements of the form $E \otimes H^{-1}$, $m \geq 0$. We have

$$\begin{aligned}\tilde{\nu} \circ \tilde{\mu}(E \otimes H^{-1}) &= \tilde{\nu}([E, z^{-m}]) \\ &= (E)_- \otimes (H-1) + E \otimes H^{-m} \\ &= E \otimes H^{-m}.\end{aligned}$$

Compare this to the formula for $\tilde{\nu}$ and remark that we dropped the $n+1$ coefficient of $L^{n+1}(E)_-$ (independence of $n$). The polynomial $p$ in $z^{-m}p$ is Id here, so that (3) applies to give $E_- = 0$. This completes the proof of Theorem 5.2.9. □

## 5.4 Deducing Bott periodicity

### 5.4.1 A l.e.s. for $\tilde{K}(X)$

*Remark* 5.4.1. Due to constructions involved, and essentially in view of the definition of *reduced* cohomology theories for *pointed* pairs, we shall necessarily be working with pointed topological spaces in this section.

*Recall:*

- The **unreduced suspension** of a topological space $X$ is defined by

$$SX := X \times I / X \times \{0\} \cup X \times \{1\}.$$

- The **reduced suspension** of a pointed topological space $(X, x_0)$ is defined by

$$\Sigma(X, x_0) := (\mathbb{S}^1 \wedge X, *) := (X \times \mathbb{S}^1, *)/(X \vee \mathbb{S}^1, *).$$

**Proposition 5.4.2.** [[Ha2], Proposition 2.9] *If $(X, x_0)$ is compact Hausdorff, and $(A, a_0) \subset (X, x_0)$ is a closed subspace, then the inclusion $i : (A, a_0) \hookrightarrow (X, x_0)$ and the quotient map $q : (X, x_0) \twoheadrightarrow (X/A, *)$ induce a s.e.s.*

$$0 \longrightarrow \tilde{K}(X/A, *) \xrightarrow{q^*} \tilde{K}(X, x_0) \xrightarrow{i^*} \tilde{K}(A, a_0) \longrightarrow 0.$$

Recall also the following result.

**Lemma 5.4.3.** [[Sw], Lemma 2.38] *If $i : (A, a_0) \hookrightarrow (X, x_0)$ denotes the inclusion, then there is a homeomorphism*

$$(X \cup_i CA/CA, *) \xrightarrow{\approx} (X/A, *),$$

*where $(CA, *)$ denotes the reduced cone over $(A, a_0)$.*

In particular, one deduces from this lemma the following sequence of homeomorphisms

$$((X \cup_i CA) \cup_j CX/CX, *) \approx (X \cup_i CA/X, *) \approx (CA/A, *) \approx (SA, *),$$

where $(SA, *)$ stands for the reduced suspension of the space $(A, a_0)$. We also have the sequence of iterated mapping cones (leaving momentarily the base points)

$$A \xrightarrow{i} X \xrightarrow{j}$$

$$\begin{tikzcd}
(X \cup_i CA) \arrow[r, "k"] \arrow[d, "q"] & (X \cup_i CA) \cup_j CX \arrow[r, "l"] \arrow[d, "q"] & ((X \cup_i CA) \cup_j CX) \cup_k C(X \cup_i CA) \arrow[d, "q"] \\
X/A \arrow[r, "f"] & SA \arrow[r, "Si"] & SX,
\end{tikzcd}$$

where the vertical maps are the quotient maps, obtained by collapsing the most recently attached cone to a point (any cone is contractible).
This allows us to extend the s.e.s. of Proposition 5.4.2 to obtain a l.e.s. of reduced K-theory groups for the pointed pair $(X, A, x_0)$

$$... \longrightarrow \tilde{K}((SX, *) \xrightarrow{(Si)^*} \tilde{K}((SA, *) \xrightarrow{f^*} \tilde{K}(X/A, *) \xrightarrow{j^*} \tilde{K}(X, x_0) \xrightarrow{i^*} \tilde{K}(A, a_0).$$

### 5.4.2 The reduced external product for $\tilde{K}(X)$

**Algebraic arguments**

Let us see how the s.e.s. obtained in Proposition 5.4.2 can be used to construct a reduced version of the external product $\mu$.
Suppose $(X, x_0)$ and $(Y, y_0)$ to be well-pointed, then the s.e.s.

$$0 \longrightarrow (X \vee Y, *) \overset{i}{\hookrightarrow} (X, x_0) \times (Y, y_0) \overset{j}{\twoheadrightarrow} (X \wedge Y, *) \longrightarrow 0$$

induces a split s.e.s. for the pointed pair $(X \times Y, X \vee Y, *)$

$$\begin{tikzcd}
0 \arrow[r] & \tilde{K}(X \wedge Y, *) \arrow[r, "j^*"] & \tilde{K}((X, x_0) \times (Y, y_0)) \arrow[r, "i^*"] & \tilde{K}(X \vee Y, *) \arrow[r] \arrow[l, bend right, "s"'] \arrow[d, phantom, "\cong"] & 0. \\
 & & & \tilde{K}(X, x_0) \oplus \tilde{K}(Y, y_0) &
\end{tikzcd} \qquad \diamond$$

The isomorphism follows from the s.e.s. of Proposition 5.4.2, since for any space $(Z, z_0) = (X, x_0) \vee (Y, y_0)$, we have $(Z/X, *) \approx (Y, y_0)$. Moreover, given two vector bundles $\{A, p_a, X\}$ and $\{B, p_b, Y\}$, the splitting $s$ is given by

$$s : \tilde{K}(X, x_0) \oplus \tilde{K}(Y, y_0) \longrightarrow \tilde{K}((X, x_0) \times (Y, y_0))$$

$$(A,B) \mapsto p_X^*(A) + p_Y^*(B).$$

Hence we have

$$\tilde{K}((X,x_0)\times(Y,y_0)) \cong \tilde{K}(X\wedge Y,*) \oplus \tilde{K}(X,x_0) \oplus \tilde{K}(Y,y_0).$$

Recall Definition 5.1.1 where the (unreduced) external product

$$\mu : K(X)\otimes K(Y) \longrightarrow K(X\times Y)$$

was defined. Since $K(X) \cong \tilde{K}(X)\oplus\mathbb{Z}$ as seen in Section 4.3, one writes

$$\begin{tikzcd}
K(X)\otimes K(Y) \arrow[r, "\mu"] \arrow[d, "\cong"'] & K(X\times Y) \arrow[d, "\cong"'] \\
(\tilde{K}(X)\oplus\mathbb{Z})\otimes(\tilde{K}(Y)\oplus\mathbb{Z}) \arrow[r] \arrow[d, "\cong"'] & \tilde{K}(X\times Y)\oplus\mathbb{Z} \arrow[d, "="'] \\
\tilde{K}(X)\otimes\tilde{K}(Y)\oplus\tilde{K}(X)\oplus\tilde{K}(Y)\oplus\mathbb{Z} \arrow[r] \arrow[d, "="'] & \tilde{K}(X\times Y)\oplus\mathbb{Z} \arrow[d, "\cong"'] \\
\tilde{K}(X)\otimes\tilde{K}(Y)\oplus\tilde{K}(X)\oplus\tilde{K}(Y)\oplus\mathbb{Z} \arrow[r] & \tilde{K}(X\wedge Y)\oplus\tilde{K}(X)\oplus\tilde{K}(Y)\oplus\mathbb{Z}
\end{tikzcd}$$

**Definition 5.4.4.** Given two vector bundles $\{A, p_a, X\}$ and $\{B, p_b, Y\}$, we define the **reduced external product** to be the ring homomorphism

$$\beta : \tilde{K}(X,x_0)\otimes\tilde{K}(Y,y_0) \longrightarrow \tilde{K}(X\wedge Y,*)$$

$$[A]_s\otimes[B]_s \mapsto \mu([A]_s\otimes[B]_s) := [p_X^*(A)\boxtimes p_Y^*(B)]_s.$$

Note that $\beta$ is defined in a very similar way to the (non-reduced) external product $\mu$, see Definition 5.1.1, the difference being that it involves the *stable* isomorphism classes of vector bundles over $X$. Further on, the same convention as in Remark 5.1.2 will be adopted, as we shall simply write $A$ and not $[A]_s$ for an element of $\tilde{K}(X,x_0)$.

The geometrical point of view explained in the next paragraph sheds some more light on the difference between these two products.

### Geometrical point of view

Let us have a closer look on $\beta$ in terms of vector bundles. Consider homomorphisms $i^* : K(X) \longrightarrow K(x_0)$ and $j^* : K(Y) \longrightarrow K(y_0)$ induced by inclusions $i : x_0 \hookrightarrow X$, $j : y_0 \hookrightarrow Y$ of the base points, and let $A$ and $B$ be elements of $\tilde{K}(X) = \mathrm{Ker}(i^*)$, respectively, $\tilde{K}(Y) = \mathrm{Ker}(j^*)$. This means that $A = \{A, p_a, X\}$ is a vector bundle over $(X,x_0)$, such that, when regarded over the base point $x_0$, it is trivial. Similarly for $B = \{B, p_b, Y\}$ defined over $(Y,y_0)$ and regarded over $y_0$.

Taking the (unreduced) external product of $A$ and $B$ gives an element $\mu(A \otimes B) = p_X^*(A) \boxtimes p_Y^*(B)$ in $K(X \times Y)$. By the pullback construction $p_X^*(A)$ is a bundle, trivial over $x_0$ and $Y$, as well as $p_X^*(B)$ is a bundle trivial over $y_0$ and $X$. Hence $p_X^*(A) \boxtimes p_Y^*(B)$ is trivial when regarded over $X \vee Y = X \times \{y_0\} \cup \{x_0\} \times Y$, which means that $p_X^*(A) \boxtimes p_Y^*(B)$ restricts to 0 in $K(X \vee Y)$. In particular, $A$ and $B$ being elements of the reduced $\tilde{K}$-groups, we have that $p_X^*(A) \boxtimes p_Y^*(B)$ lies in $\tilde{K}(X \times Y)$ and restricts to 0 in $\tilde{K}(X \vee Y)$. Applying the s.e.s. $\diamond$, where the map $j^*$ becomes a bijection, one concludes that $p_X^*(A) \boxtimes p_Y^*(B)$ corresponds to a unique element of $\tilde{K}(X \wedge Y)$. Thus taking the reduced external product of $A$ and $B$

$$\beta : \tilde{K}(X, x_0) \otimes \tilde{K}(Y, y_0) \longrightarrow \tilde{K}(X \wedge Y, *)$$

$$A \otimes B \mapsto \beta(A \otimes B) := p_X^*(A) \boxtimes p_Y^*(B)$$

yields a bundle $\{p_X^*(A) \boxtimes p_Y^*(B), \pi, X \times Y\}$ over $X \times Y$, with total space constructed in a similar way to the unreduced case (using the formal tensor product $\boxtimes$), and that becomes trivial when regarded over $X \vee Y$.

### 5.4.3 Bott periodicity

**Lemma 5.4.5.** [[Ha2], Lemma 2.10] *If $A$ is a contractible space, the quotient map $q : X \twoheadrightarrow X/A$ induces a bijection*

$$q^* : Vect_{\mathbb{C}^k}(X/A) \xrightarrow{\cong} Vect_{\mathbb{C}^k}(X)$$

*for all $k \in \mathbb{N}$.*

Since

$$\Sigma^n(X, x_0) \approx (\mathbb{S}^n \wedge X, *) \text{ for all } n \in \mathbb{N},$$

and

$$\Sigma^n(X, x_0) \approx S^n X/D^n \sim *,$$

the quotient map

$$q : (S^n X, *) \twoheadrightarrow (S^n X/D^n \sim *) \cong \Sigma^n(X, x_0)$$

induces an isomorphism on $\tilde{K}(X)$ by the previous Lemma.

**Corollary 5.4.6.** *For any well-pointed compact Hausdorff space $(X, x_0)$ there is an isomorphism*

$$\tilde{K}(S^n X, *) \xrightarrow{\cong} \tilde{K}(\Sigma^n X, *).$$

This result allows us to use either notation for the suspension of $X$, and we opt for $SX$.

**Theorem 5.4.7.** *Let $(X, x_0)$ be a well-pointed compact Hausdorff space and let $H$ denote the trivial canonical bundle over $(\mathbb{S}^2, s_0)$. The homomorphism*

$$\varphi : \tilde{K}(X, x_0) \xrightarrow{\cong} \tilde{K}(S^2X, *)$$

$$A \mapsto \beta((H-1) \otimes A)$$

*is an isomorphism of rings.*

*Proof.* The map $\varphi$ is given by the composition

$$\begin{array}{ccccc} \tilde{K}(X, x_0) & \xrightarrow{\cong} & \tilde{K}(\mathbb{S}^2, s_0) \otimes \tilde{K}(X, x_0) & \xrightarrow{\beta} & \tilde{K}(\mathbb{S}^2 \wedge X, *) \cong \tilde{K}(S^2X, *) \\ & & & & \\ A & \longmapsto & (H-1) \otimes A & \longmapsto & \beta((H-1) \otimes A). \end{array}$$

The first isomorphism follows from Remark 5.2.11 on the structure of $\tilde{K}(\mathbb{S}^2, s_0)$, and the second map is the reduced external product. Hence the fact that $\varphi$ is an isomorphism follows from the FPT2, proved in the previous section. □

**Corollary 5.4.8.** *For all $n \in \mathbb{N}$ we have*

$$\tilde{K}(\mathbb{S}^{2n+1}, s_0) = 0 \text{ and } \tilde{K}(\mathbb{S}^{2n}, s_0) = \mathbb{Z}\,.$$

*Proof.* In even dimensions we have

$$\tilde{K}(\mathbb{S}^0, s_0) \cong \tilde{K}(S^2\,\mathbb{S}^0, *) \cong \tilde{K}(\mathbb{S}^2 \wedge \mathbb{S}^0, *) \cong \tilde{K}(\mathbb{S}^2, s_0) \cong \mathbb{Z}$$

by Remark 5.2.11. In general,

$$\tilde{K}(\mathbb{S}^{2n}, s_0) \cong \tilde{K}(S^2\,\mathbb{S}^{2n}, *) \cong \tilde{K}(\mathbb{S}^{2n+2}, s_0)$$

for all $n \in \mathbb{N}$.
We compute the values of $\tilde{K}(\mathbb{S}^{2n+1}, s_0)$ in the next Chapter (see Corollary 6.1.6). □

# Chapter 6

# Extending $K(X)$ to a cohomology theory

The aim of this chapter is to show that complex $K$-theory is a multiplicative cohomology theory. We shall first extend the definition of $\tilde{K}$-groups to all dimensions, explaining why the Exactness and Wedge Axioms for a cohomology theory are then satisfied. In the second place, we shall give the definition of the $\Omega$-spectrum $KU$, associated to the complex $K$-theory.

## 6.1 Definition of $\tilde{K}$-groups for all integral dimensions

**Definition 6.1.1.** For every $(X, x_0) \in \mathcal{CW}_*$ and for all $n \geq 0$, the **complex $\tilde{K}$-groups in negative dimensions** are defined by

$$\tilde{K}^{-n}(X, x_0) := \tilde{K}(S^n X, *).$$

In particular,

$$\tilde{K}^0(X, x_0) := \tilde{K}(X, x_0).$$

Note that this definition is compatible with the l.e.s. for the pointed pair $(X, A, x_0)$ given in Section 5.4.1. Indeed, we have

$$\begin{tikzcd}
\cdots \arrow[r] & \tilde{K}(S^2X) \arrow[r, "(S^2i)^*"] \arrow[d, "\cong"'] & \tilde{K}(S^2A) \arrow[r, "(Sf)^*"] \arrow[d, "\cong"'] & \tilde{K}(S(X/A)) \arrow[r, "(Sj)^*"] \arrow[d, "\cong"'] & \tilde{K}(SX) \arrow[r, "(Si)^*"] \arrow[d, "\cong"'] & \tilde{K}(SA) \arrow[r, "f^*"] \arrow[d, "\cong"'] & {} \\
\cdots \arrow[r] & \tilde{K}^{-2}(X) \arrow[r, "\delta^{-2}"] & \tilde{K}^{-2}(A) \arrow[r, "i^*"] & \tilde{K}^{-1}(X/A) \arrow[r, "j^*"] & \tilde{K}^{-1}(X) \arrow[r, "i^*"] & \tilde{K}^{-1}(A) \arrow[r, "\delta^{-1}"] & {} \\
\cdots \arrow[r] & \tilde{K}^0(X) \arrow[r, "i^*"] \arrow[u, "\varphi", "\cong"'] & \tilde{K}^0(A) \arrow[u, "\varphi", "\cong"'] & & & &
\end{tikzcd}$$

$$
\begin{array}{ccccccc}
\xrightarrow{f^*} & \tilde{K}(X/A) & \xrightarrow{j^*} & \tilde{K}(X) & \xrightarrow{i^*} & \tilde{K}(A) \\
 & \downarrow{=} & & \downarrow{=} & & \downarrow{=} \\
\xrightarrow{\delta^{-1}} & \tilde{K}^0(X/A) & \xrightarrow{j^*} & \tilde{K}^0(X) & \xrightarrow{i^*} & \tilde{K}^0(A),
\end{array}
$$

where $\varphi$ refers to the isomorphism of Theorem 5.4.7.

*Remark* 6.1.2. In this definition the indices of $\tilde{K}$-groups are chosen to be negative, so that the connecting homomorphisms $\delta^*$ decrease dimensions, as one would expect in cohomology.

To make the sequence of $\tilde{K}$-groups into a complex reduced cohomology theory, we still need to know what happens in positive dimensions. The following theorem will bring the answer.

**Theorem 6.1.3.** [[May], Ch. 24 §2.] *The map*

$$\psi : BU \times \mathbb{Z} \xrightarrow{\simeq} \Omega^2(BU \times \mathbb{Z})$$

*is a homotopy equivalence of $H$-spaces.*

Understanding the proof of Theorem 6.1.3 requires a lot of additional tools, knowledge and time. We restrict ourselves to giving a number of important remarks one should take into consideration.

*Remarks:*

**(a)** One can view $\psi$ as the following composition

$$BU \times \mathbb{Z} \xrightarrow{\simeq_1} (\Omega SU) \times \mathbb{Z} \xrightarrow{\simeq_2} \Omega U \times \mathbb{Z} \xrightarrow{\simeq_3} \Omega^2(BU) \times \mathbb{Z} \xrightarrow{\simeq_4} \Omega^2(BU \times \mathbb{Z})$$

and then investigate why all maps are indeed homotopy equivalences.

($\simeq_1$) Bott worked with an explicit definition of $BU$ in terms of the Grassmanian manifolds and used Morse theory to prove that $\simeq_1$ is a homotopy equivalence (see [Bo1]).

($\simeq_2$) The universal cover of $U$ being $SU$, the infinite special linear group, for every $n$ there is a fibration of $H$-spaces

$$\mathbb{S}^1 \longrightarrow U(n) \longrightarrow SU(n)$$

which gives the homotopy equivalence $\Omega U \simeq (\Omega SU) \times \mathbb{Z}$.

($\simeq_3$) For all $n \geq 1$ there is an embedding

$$i_n : U(n) \hookrightarrow U(n+1) : \sigma \mapsto \begin{pmatrix} \sigma & 0 \\ 0 & 1 \end{pmatrix},$$

so that we can define the space $U := \varinjlim_n U(n)$ with the weak topology, making $U$ into a topological group, which implies the homotopy equivalence $\Omega BU \simeq U$.

($\simeq_4$) Comes from the $H$-space structure on $\Omega^2(BU)\times\mathbb{Z}$ and $\Omega^2(BU\times\mathbb{Z})$.

**(b)** From the Theorem it follows that

$$BU \simeq \Omega^2 BU \simeq \Omega(\Omega BU) \simeq \Omega U,$$

and that

$$U \simeq \Omega BU \simeq \Omega(\Omega^2 BU) \simeq \Omega^2(\Omega BU) \simeq \Omega^2 U.$$

Now, for all $n \geq 0$ we have

$$\begin{aligned}
\tilde{K}^{-n}(X,x_0) &\overset{(1)}{\cong} [S^n X, *; BU\times\mathbb{Z}, *]\\
&\overset{(2)}{\cong} [S^n X, *; \Omega U\times\mathbb{Z}, *]\\
&\overset{(3)}{\cong} [X, x_0; \Omega^{n+1} U\times\mathbb{Z}, *]\\
&\overset{(4)}{\cong} [X, x_0; \Omega^2(\Omega^{n+1} U)\times\mathbb{Z}, *]\\
&\cong [X, x_0; \Omega^{n+1+2} U\times\mathbb{Z}, *]\\
&=: \tilde{K}^{-n-2}(X,x_0).
\end{aligned}$$

Here, (1) follows from Proposition 4.3.4, (2) from $\overset{(1)}{\cong}$ and $\overset{(2)}{\cong}$, (3) holds because the suspension functor and the loop functor are adjoint, and (4) follows from the Theorem 6.1.3.

Hence, for any $(X,x_0)\in\mathcal{CW}_*$ and any $n\geq 0$, the groups $\tilde{K}^{-n}(X,x_0)$ and $\tilde{K}^{-n-2}(X,x_0)$ are isomorphic, which finally allows us to extend the definition of $\tilde{K}$-groups to positive degrees inductively.

**Definition 6.1.4.** For every $(X,x_0)\in\mathcal{CW}_*$ and for all $n\in\mathbb{Z}$, the **complex $\tilde{K}$-groups** are defined inductively by

$$\tilde{K}^n(X,x_0) \cong \tilde{K}^{n-2}(X,x_0).$$

At the beginning of this chapter we saw why the s.e.s. of Proposition 5.4.2 held for negative groups $\tilde{K}^{-n}(X)$, $n\in\mathbb{N}$. The definition we have just given guarantees that the Exactness Axiom is satisfied by $\tilde{K}^n(x)$ for all $n\in\mathbb{Z}$, making $\tilde{K}^*(-)$ into a reduced cohomology theory. From Proposition 4.3.4 and Definition 6.1.4 it follows that $\tilde{K}^*$ is an invariant of homotopy type.

*Remarks* 6.1.5.

- If $X\in\mathcal{CW}$ is a finite-dimensionnal $CW$-complex without base-point, one defines the **non-reduced complex $K$-theory** by setting

$$K^n(X) := \tilde{K}^n(X_+),$$

where $X_+$ is the disjoint union of $X$ and a point.

- The **coefficient groups** for the non-reduced theory are then given by

$$K^n(\{x_0\}) = \begin{cases} \mathbb{Z}, & \text{for } n\in\mathbb{Z} \text{ even}\\ 0, & \text{for } n\in\mathbb{Z} \text{ odd}.\end{cases}$$

We are now able to come back to the calculation of the values of $\tilde{K}$-groups for odd-dimensional spheres.

**Corollary 6.1.6.** *For all $n \in \mathbb{N}$ we have*

$$\tilde{K}(\mathbb{S}^{2n+1}, s_0) = 0.$$

*Proof.* Using Bott periodicity and Theorem 6.1.3, we have

$$\tilde{K}(\mathbb{S}^{2n+1}, s_0) \cong \tilde{K}(\mathbb{S}^1, s_0) := [\mathbb{S}^1, s_0; BU, *] \cong [\mathbb{S}^1, s_0; \Omega U, *] \cong [\mathbb{S}^2, s_0; U, *] = \pi_2(U, *).$$

According to [Hu] (VII, 12.4), $\pi_2(U(n), *) = 0$ for all $1 \leq n \leq \infty$, which gives the result. □

It remains to see why $\tilde{K}^*(-)$ satisfies the Wedge Axiom.

Let $\{X_\alpha, x_\alpha\}_{\alpha \in A}$ be a collection in *Top*$_*$. Leaving the base points, we set $X = \vee_\alpha X_\alpha$ and denote $j_\alpha : X_\alpha \hookrightarrow X$ the inclusion maps. Let $\xi_V = \{E, p, X\}$ be a vector bundle. We need to define an isomorphism of groups

$$K(X) \xrightarrow{\cong} \prod_{\alpha \in A} K(X_\alpha).$$

For every $\alpha \in A$ consider the pullback

$$\begin{tikzcd} j_\alpha^*(E) \arrow[r, "pr_{E_\alpha}"] \arrow[d, "\pi_\alpha"'] & E \arrow[d, "p"] \\ X_\alpha \arrow[r, "j_\alpha"'] & X \end{tikzcd}$$

and define

$$f : K(X) \longrightarrow \prod_{\alpha \in A} K(X_\alpha)$$
$$[\xi_V] \mapsto ([j_\alpha^*(E)])_{\alpha \in A}.$$

Conversely, a collection of vector bundles $\{E_\alpha, p_\alpha, X_\alpha\}_{\alpha \in A}$ where $E_\alpha$ and $X_\alpha$ are pointed spaces for all $\alpha \in A$, yields the bundle $\eta_V = \{\vee_{\alpha \in A} E_\alpha, \vee p_\alpha, X\}$. Define

$$g : \prod_{\alpha \in A} K(X_\alpha) \longrightarrow K(X)$$

by

$$([E_\alpha])_{\alpha \in A} \mapsto [\eta_V].$$

We have

$$K(X) \xrightarrow{f} \prod_{\alpha \in A} K(X_\alpha) \xrightarrow{g} K(X)$$
$$[\xi_V] \mapsto ([j_\alpha^*(E)])_{\alpha \in A} \mapsto [\kappa_V],$$

where $\kappa_V = \{\vee j_\alpha^*(E), k, X\}$. The bundles $\xi_V$ and $\kappa_V$ have the same base $X$, and by the universal property of the coproduct, there is a unique map $\varphi$ such that

$$\begin{tikzcd} j_\beta^*(E) \arrow[dr, "i_\beta"] \arrow[ddr, bend right, "p_{E_\beta}"'] & & j_\gamma^*(E) \arrow[dl, "i_\gamma"'] \arrow[ddl, bend left, "p_{E_\gamma}"] \\ & \vee j_\alpha^*(E) \arrow[d, dashed, "\exists!\varphi"'] & \\ & E & \end{tikzcd}$$

commutes, hence the composition $g \circ f$ is identity.
On the other hand, the pullback diagram

$$\begin{tikzcd} j_\alpha^*(\vee_{\alpha \in A} E_\alpha) \arrow[r, "pr_\vee"] \arrow[d, "l_\alpha"'] & \vee_{\alpha \in A} E_\alpha \arrow[d, "\vee p_\alpha"] \\ X_\alpha \arrow[r, "j_\alpha"'] & X \end{tikzcd}$$

gives a collection $\{l_\alpha\}_{\alpha \in A}$ of projections such that

$$\prod_{\alpha \in A} K(X_\alpha) \xrightarrow{g} K(X) \xrightarrow{f} \prod_{\alpha \in A} K(X_\alpha)$$

$$([E_\alpha])_{\alpha \in A} \mapsto [\eta_V] \mapsto ([j_\alpha^*(\vee_{\alpha \in A} E_\alpha)])_{\alpha \in A}.$$

For all $\alpha \in A$, we need to have a bundle isomorphism $E_\alpha \cong j_\alpha^*(\vee_{\alpha \in A} E_\alpha)$. By Remark 4.1.3, we know that to have this isomorphism it suffices to have bundle morphisms $j_\alpha : E_\alpha \longrightarrow \vee_{\alpha \in A} E_\alpha$ for all $\alpha \in A$, which are given precisely by the inclusions. Hence the composition $f \circ g$ is identity, too.

## 6.2 The spectrum $KU$

**Definition 6.2.1.** The **$\Omega$-spectrum** $\{(KU, *), e_i\}$**, associated to the complex $K$-theory**, has spaces defined by

$$\begin{cases} KU_{2i} := \Omega U \\ KU_{2i+1} := U, \end{cases}$$

for all $i \in \mathbb{Z}$, and (adjoints to) the structure maps

$$\begin{cases} e_{2i} : KU_{2i} \longrightarrow \Omega KU_{2i+1} \\ e_{2i+1} : KU_{2i+1} \longrightarrow \Omega KU_{2i+2} \end{cases}$$

defined respectively by

$$\begin{cases} \mathrm{Id} : \Omega U \longrightarrow \Omega U \\ e_{2i+1} : U \xrightarrow{\simeq} \Omega^2 U, \text{ called the } \textit{Bott maps}. \end{cases}$$

The spectrum $KU$ is an $\Omega$-spectrum if the maps $e_m : KU_m \longrightarrow \Omega KU_{m+1}$ are weak homotopy equivalences for every $m \in \mathbb{Z}$. This is true since for $m = 2i$ we have

$$\Omega KU_{2i} := \Omega(\Omega U) \simeq U := KU_{2i-1},$$

and for $m = 2i$

$$\Omega KU_{2i+1} := \Omega U := KU_{2i-2},$$

the homotopy equivalence following from Theorem 6.1.3, which implies that it is also a weak homotopy equivalence by the Whitehead theorem for $CW$-complexes.

## 6.3 The multiplicative structure on $KU$

We give a concise explanation of where the multiplicative structure on the spectrum $KU$ comes from, following the guideline in [May], Chapter 23 §2 and Chapter 24 §1,2.

Recall Definition 1.4.1 to see that $BU(n) := G_n(\mathbb{C}^\infty)$, hence $BU(m+n) := G_{m+n}(\mathbb{C}^\infty)$, and choosing an isomorphism $\mathbb{C}^\infty \oplus \mathbb{C}^\infty \cong \mathbb{C}^\infty$, we obtain a homeomorphism $G_{m+n}(\mathbb{C}^\infty \oplus \mathbb{C}^\infty) \approx G_{m+n}(\mathbb{C}^\infty)$ (one needs to check that the homotopy class of this homeomorphism is independent of the choice of isomorphism). These identifications give rise to maps for all $m, n \geq 0$

$$p_{m,n} : BU(m) \times BU(n) \longrightarrow BU(m+n)$$

$$(x, y) \mapsto x \oplus y,$$

where $x$ and $y$ are respectively an $m$-plane and an $n$-plane in $\mathbb{C}^\infty$. By passage to colimits over $m$ and $n$, one obtains an "addition"

$$\oplus : BU \times BU \longrightarrow BU.$$

Indeed, since $BU := G_\infty(\mathbb{C}^\infty)$, we can think of it as a space with a plane of a certain dimension in every copy of $\mathbb{C}^\infty$. Adding two such elements will still give an element in $BU$, and the zero element would be a zero-dimensional plane in $\mathbb{C}^\infty$.

To define an addition on $BU$, we basically used the direct sum of vector spaces, inducing the Whitney sum on vector bundles. Similarly, using the tensor product of vector spaces, one defines maps

$$q_{m,n} : BU(m) \times BU(n) \longrightarrow BU(mn)$$

$$(x, y) \mapsto x \otimes y.$$

Passing to colimits in this case needs some additional elaborate arguments (e.g., we need to take into consideration the bilinearity of $\otimes$), and one finally gets a "product" on $BU$:

$$\wedge : BU \wedge BU \longrightarrow BU.$$

One checks that the maps $\oplus$ and $\wedge$ are well-defined, associative and commutative up to homotopy, which will give an additive and a multiplicative $H$-space structure on $BU \times \mathbb{Z}$.

The multiplicative structure on $KU$ follows then from Definition 6.2.1 and consists in giving an explicit characterization of the maps

$$KU_{2i} \wedge KU_{2j} \longrightarrow KU_{2(i+j)},$$

$$KU_{2i+1} \wedge KU_{2j} \longrightarrow KU_{2(i+j)+1},$$

and

$$KU_{2i+1} \wedge KU_{2j+1} \longrightarrow KU_{2(i+j+1)}$$

in terms of finite-dimensional sub-planes of $\mathbb{C}^\infty$, for all $i$, $j \in \mathbb{N}$. According to [H], these maps could respectively come from

$$BU(m) \times BU(n) \longrightarrow BU(m+n)$$

$$(V, W) \mapsto V \otimes W,$$

$$\begin{aligned} U(m)\times BU(n) &\hookrightarrow U(m+n)\times BU(m+n) \longrightarrow BU(m+n) \\ (A,V) &\mapsto (\begin{pmatrix} A & 0 \\ 0 & I_n \end{pmatrix}, V\oplus\mathbb{C}^m) \mapsto \begin{pmatrix} A & 0 \\ 0 & I_n \end{pmatrix}(V\oplus\mathbb{C}^m), \end{aligned}$$

and

$$\begin{aligned} U(m)\times U(n) &\hookrightarrow U(m+n)\times U(m+n) \longrightarrow U(m+n) \\ (A,B) &\mapsto (\begin{pmatrix} A & 0 \\ 0 & I_n \end{pmatrix}, \begin{pmatrix} I_m & 0 \\ 0 & B \end{pmatrix}) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}). \end{aligned}$$

# Conclusion

The fundamental objects of study in complex $K$-theory being complex vector bundles over topological spaces, another important aspect of this theory is the existence of an additive and multiplicative structure on its groups, which allows to have both geometrical and algebraical insight into problems, useful in many contexts. The main theorem, giving its power to this theory, is the Bott Periodicity Theorem, at the origin of the 2-fold periodic structure of the complex $K$-theory. Among practical consequences, let us mention the proof of nonexistence of division algebras over $\mathbb{R}$ in dimensions other than 1, 2, 4, and 8, and the non-parallelizability of spheres other than $\mathbb{S}^1$, $\mathbb{S}^3$, and $\mathbb{S}^7$. More details on these applications could be found in the Semester Project "Characteristic Classes: theory and applications" of Philip Egger, who has been working in parallel on different aspects of this subject.

On our way to discovering the topological $K$-theory, we have come across a large variety of structures and tools, commonly used in algebraic topology, which has made working on this project all the more entertaining.

Due to a lack of time, a number of questions were not investigated, yet, they have drawn our attention and curiosity. Among these, let us mention the "right" definition of a "nice" symmetric smash product for spectra, and its consequences. Learning more about it could be an interesting follow-up to this project, hopefully.

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