Frobenius Algebras in Functor Categories

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Abstract

There is an adjunction between monoids in a monoidal category and strong monads over it. This dissertation examines this adjunction and its variants. In particular we extend it to an equivalence between Frobenius algebras in a monoidal category and strong Frobenius monads over it.

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Chapter 1

Introduction

The language of (symmetric) monoidal categories has been used in quantum information theory in the program of categorical quantum mechanics (see e.g. [2]). Within this framework one can define a category-theoretical notion of a Frobenius algebra. A Frobenius algebra consists of a single object equipped with both a monoid and comonoid structure such that these two structures interact in the right way. Frobenius algebras turn out to be important as they can for example be used to characterize orthogonal bases in **Fhilb** [6].

Our main results (theorem 8.2 and corollary 8.4) allow one to view Frobenius algebras from a different yet equivalent point of view, as certain kinds of monads, namely strong Frobenius monads. Hence everything that can be done with Frobenius algebras can be done with monads of this sort instead. This ties well with so-called monadic computation (see e.g. [14] and follow-up work) where monads are used to model sequential computation in functional programming.

It is part of category theorists' folklore that for any (symmetric) monoidal category **C** there is an adjunction between monoids in **C** and strong monads over **C**, and commutativity is preserved by both of these functors (see e.g. [18]). Dually there is a similar adjunction between comonoids and costrong comonads. We extend this to an equivalence between Frobenius algebras in **C** and strong Frobenius monads over **C**.

Another way to think of this result is as an analogue to the Eilenberg-Watts theorem ([7] and [17]) of homological algebra. When restricted to endofunctors, this theorem characterises cocontinuous functors as those that arise as tensoring with a fixed monoid. Our result characterises functors that arise as tensoring with a Frobenius algebra. In a sense, the main result is simultaneously more and less general than the Eilenberg-Watts theorem. It is more general since it holds in any monoidal category and not just in Abelian ones (see [8] for another categorical generalisation). This is fortunate as the usual proof of the Eilenberg-Watts theorem works for cancellative semirings and cancellative semimodules but fails when trying to extend it further to arbitrary semirings. On the other hand it is also less general in that we restrict our attention to functors that arise as tensoring with a fixed Frobenius algebra instead of an object of a more mundane kind.

It is well known that adjoint pairs of functors give rise to monads, and in fact every monad can be seen as arising this way. This leads one to ask if something similar can be said about the Frobenius case. Ross Street [16] showed that Frobenius monads correspond exactly to ambijunctions, i.e. adjoint pairs of functors where both functors are both left and right adjoints to each other.

In chapter 2 we develop the necessary background of monoidal categories, while in chapter 3 we recap the necessary backround on monads. Chapter 4 introduces the adjunction between objects and strong functors. All other adjunctions examined here build on top of this or its dual. After this we prove the results concerning monoids and strong monads in chapters 5 and 6. Chapter 7 examines the dual versions of these, while the final chapter presents the Frobenius version of this adjunction, which turns out to be an equivalence. The proof of lemma 2.17 is in the appendix, as the proof is tedious and not central to the text.

The language used is that of traditional category theory. In particular, most of the proofs are done by chasing diagrams instead of using the graphical calculus of monoidal categories (see [15]) or the language of string diagrams. The main reason for doing so is notational uniformity. This comes at a cost: the graphical calculus is quite efficient when reasoning within a monoidal category, and it would have streamlined and clarified some of the proofs. In particular the proof now relegated to the appendix would have been much simpler in this notation. Likewise the fact that most structure is preserved when an object A is considered as a functor $- \otimes A$ is almost trivial graphically – just do the relevant operations "on the side".

However, the algebraic proof of this isn't significantly worse, and for many other results in this dissertation the graphical calculus wouldn't have been useful at all. The reason for this is that besides the monoidal structure we often reason about functors and natural transformations, and the graphical calculus doesn't handle this that well. While string diagrams are efficient when reasoning about the latter, they have trouble interacting with the monoidal structure. There are graphical approaches trying to combine the two (e.g. [12] and [13]), but they also have their disadvantages when using them to carry out the reasoning at hand. Hence it seems prudent to use just the traditional diagrams so as to be able to stick to a single notation troughout and not to have to worry about soundness and completeness properties. The only exception to this is the use of the free monoidal category construction, which we use as a source of examples. When discussing this construction we use the graphical calculus freely.

Chapter 2

Monoidal categories

Definition 2.1. A *monoidal category* consists of the following data:

- An underlying category $\mathbf{C}$
- A functor $\mathbf{C} \times \mathbf{C} \to \mathbf{C}$, which write in infix notation as $(C, D) \mapsto C \otimes D$ and $(f, g) \mapsto f \otimes g$. $C \otimes D$ is called the *tensor product* of C and D, and likewise $f \otimes g$ is called the tensor product of f and g.
- An object $I \in ob(C)$, called the *tensor unit*
- Natural isomorphisms α, λ and ρ with components

$$\alpha_{A,B,C} \colon A \otimes (B \otimes C) \to (A \otimes B) \otimes C,$$
$$\lambda_C \colon I \otimes C \to C, \qquad \rho_C \colon C \otimes I \to C.$$

  The components of α are called associators, the components of λ are called left unitors and the components of ρ are called right unitors.

This data must satisfy two axioms. First of all, the following *pentagon diagram* must commute for all objects A, B, C and D:

$$\begin{tikzcd}
A \otimes (B \otimes (C \otimes D)) \arrow[r, "\mathrm{id}_A \otimes \alpha_{B,C,D}"] \arrow[d, "\alpha_{A,B,C\otimes D}"'] & A \otimes ((B \otimes C) \otimes D) \arrow[r, "\alpha_{A,B\otimes C,\otimes D}"] & (A \otimes (B \otimes C)) \otimes D \arrow[d, "\alpha_{A,B,C} \otimes \mathrm{id}_D"] \\
(A \otimes B) \otimes (C \otimes D) \arrow[rr, "\alpha_{A\otimes B,C,D}"'] & & ((A \otimes B) \otimes C) \otimes D
\end{tikzcd}$$

The following *triangle diagram* must commute for every object A and B:

$$\begin{tikzcd}
A \otimes (I \otimes B) \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "\mathrm{id}_A \otimes \lambda_B"'] & & (A \otimes I) \otimes B \arrow[dl, "\rho_A \otimes \mathrm{id}_B"] \\
& A \otimes B &
\end{tikzcd}$$

When there is no risk of confusion, we speak of a monoidal category $\mathbf{C}$ instead of a monoidal category $(\mathbf{C}, \otimes, I, \alpha, \rho, \lambda)$. If it so happens that all the natural isomorphisms required from a monoidal category are in fact identities, we say that the monoidal category in question is a *strict monoidal category.*

Besides the triangle and the pentagon diagram there are many other diagrams one can build out of α, λ, ρ and their inverses. The reason to choose these two as axioms for a monoidal category is that everything else you might want to hold follows from them. The so-called coherence theorem states roughly that every diagram built out of $\alpha, \lambda, \rho, \mathrm{id}$ and their inverses by composition and tensoring commutes in any monoidal category. This isn't the exact statement, as for example it might happen in some monoidal category that some objects in the diagram might be equal "by accident" and thus yield a non-intended counterexample. While proving the commutativity of certain diagrams we use the coherence theorem rather freely. For the exact statement and proof, see [11, section VII.2]

Sometimes the tensor product not only has a unit, but is also commutative. The following definition captures this:

Definition 2.2. A *symmetric monoidal category* is a monoidal category $\mathbf{C}$ equipped with a natural isomorphism γ with components $\gamma_{A,B} \colon A \otimes B \to B \otimes A$ that satisfies $\gamma_{B,A} \circ \gamma_{A,B} = \mathrm{id}_{A \otimes B}$ for every $A, B \in ob(\mathbf{C})$, and also for all objects A, B and C the following *hexagon diagram* commutes:

$$\begin{tikzcd}
& (A \otimes B) \otimes C \arrow[r, "\gamma_{A\otimes B,C}"] & C \otimes (A \otimes B) \arrow[dr, "\alpha_{C,A,B}"] & \\
A \otimes (B \otimes C) \arrow[ur, "\alpha_{A,B,C}"] \arrow[dr, "\mathrm{id}_A \otimes \gamma_{B,C}"'] & & & (C \otimes A) \otimes B \\
& A \otimes (C \otimes B) \arrow[r, "\alpha_{A,C,B}"'] & (A \otimes C) \otimes B \arrow[ur, "\gamma_{A,C} \otimes \mathrm{id}_B"'] &
\end{tikzcd}$$

There is also a version of the coherence theorem for symmetric monoidal categories. Formulating it exactly is slightly trickier, since for example even though $\gamma_{A,A}$ and $\mathrm{id}_{A\otimes A}$ have the same domain and codomain we don't expect them to be equal in general. If one thinks of $\gamma_{A,B}$ as exchanging A and B, then one more or less expects every two maps built with the same codomain and domain from the structural natural transformations and identities to agree whenever the two maps agree on which object was permuted where. That is, whenever the resulting permutations on objects agree, the maps should as well. We also use this version of the coherence theorem as needed.

Here are some standard examples of (symmetric) monoidal categories:

- The category **Set** of sets, with cartesian product as the tensor product and a fixed singleton $\{*\}$ as its unit.
- Various categories of "sets with extra structure", e.g. topological spaces **Top**, groups **Grp** and posets **Pos** with the the cartesian product giving the tensor product. In fact, every category with binary products and a terminal object can be given the structure of a monoidal category.
- The category **Rel**, the objects of which are sets and morphisms are relations. Again equipped with cartesian product as the tensor product.
- The category $\mathbf{Vect}_k$ of vector spaces over a field k, with the tensor product of vector spaces as the monoidal tensor functor
- The category **Hilb** of (complex) Hilbert spaces, where the tensor product of Hilbert spaces is the monoidal tensor.

Given any categories $\mathbf{C}$ and $\mathbf{D}$, we denote the category of functors $\mathbf{C} \to \mathbf{D}$ by $[\mathbf{C}, \mathbf{D}]$. There is an obvious binary operation on objects of $[\mathbf{C}, \mathbf{C}]$, namely composition $(G, F) \mapsto G \circ F$. We would like to extend it into a functor $[\mathbf{C}, \mathbf{C}] \times [\mathbf{C}, \mathbf{C}] \to \mathbf{C}$, resulting in a monoidal category. The morphism part of this is a special case of the following:

Definition 2.3. Given functors $F, G \colon \mathbf{C} \to \mathbf{D}$, $H, K \colon \mathbf{D} \to \mathbf{E}$ and natural transformations $\sigma \colon F \to G$, $\tau \colon H \to K$, the *Godement product* of σ and τ is defined by the equation

$$(\tau * \sigma)_C = \tau_{G(C)} \circ H(\sigma_C) \ (= K(\sigma_C) \circ \tau_{F(C)})$$

In this situation, $\tau * F$ means $\tau * \mathrm{id}_F$ and likewise $H * \sigma$ means $\mathrm{id}_H * \sigma$.

The following two lemmas (or variants thereof) concerning properties of the Godement product are standard, see e.g [4, chapter 1.3]. We provide proofs for the sake of self-containment.

Lemma 2.4. *In the above situation $\tau * \sigma$ is a natural transformation $H \circ F \to K \circ G$*

Proof. The naturality asserts that the outer rectangle in the diagram

$$\begin{tikzcd}
HF(C) \arrow[r, "H(\sigma_C)"] \arrow[d, "HF(f)"'] & HG(C) \arrow[r, "\tau_{G(C)}"] \arrow[d, "HG(f)"'] & KG(C) \arrow[d, "KG(f)"] \\
HF(D) \arrow[r, "H(\sigma_D)"'] & HG(D) \arrow[r, "\tau_{G(D)}"'] & KG(D)
\end{tikzcd}$$

commutes. But the square on the left commutes as σ is natural, and the one on the right commutes as τ is. $\square$

Lemma 2.5. *(i) If $F \colon \mathbf{C} \to \mathbf{D}$ and $G \colon \mathbf{D} \to \mathbf{E}$, then $\mathrm{id}_G * \mathrm{id}_F = \mathrm{id}_{G \circ F}$*

(ii) If $F, H, L \colon \mathbf{C} \to \mathbf{D}$, $G, K, M \colon \mathbf{D} \to \mathbf{E}$ are functors and $\sigma \colon F \to H$ $\gamma \colon H \to L$, $\tau \colon G \to K$ and $\delta \colon K \to M$ are natural transformations, then $(\delta * \gamma) \circ (\tau * \sigma) = (\delta \circ \tau) * (\gamma \circ \sigma)$

(iii) If $G, F \colon \mathbf{C} \to \mathbf{D}$ and $\sigma \colon G \to F$, then $\sigma * \mathrm{id}_{\mathbf{C}} = \sigma = \mathrm{id}_{\mathbf{D}} * \sigma$.

(iv) If $F, H \colon \mathbf{B} \to \mathbf{C}$, $G, K \colon \mathbf{C} \to \mathbf{D}$ and $L, M \colon \mathbf{D} \to \mathbf{E}$ are functors and $\gamma \colon F \to H$, $\sigma \colon G \to K$ and $\tau : L \to M$ are natural transformations, then $\tau * (\sigma * \gamma) = (\tau * \sigma) * \gamma$

Proof. Just calculations with the definitions:

1. For any $C \in ob(\mathbf{C})$,

$$(\mathrm{id}_G * \mathrm{id}_F)_C = (\mathrm{id}_G)_{F(C)} \circ G((\mathrm{id}_F)_C) = \mathrm{id}_{GF(C)} \circ \mathrm{id}_{GF(C)} = \mathrm{id}_{GF(C)} = (\mathrm{id}_{G\circ F})_C$$

2. For any $C \in ob(\mathbf{C})$,

$$\begin{aligned}((\delta * \gamma) \circ (\tau * \sigma))_C &= ((\delta * \gamma)_C \circ (\tau * \sigma)_C \\ &= \delta_{L(C)} \circ K(\gamma_C) \circ \tau_{H(C)} \circ G(\sigma_C) \\ &= \delta_{L(C)} \circ \tau_{L(C} \circ G(\gamma_C) \circ G(\sigma_C) \\ &= (\delta \circ \tau)_{L(C)} \circ G((\gamma \circ \sigma)_C) \\ &= ((\delta \circ \tau) * (\gamma \circ \sigma))_C\end{aligned}$$

3. For any $C \in ob(\mathbf{C})$,

$$(\sigma * \mathrm{id}_{\mathbf{C}})_C = \sigma_{\mathrm{id}_{\mathbf{C}}(C)} \circ F(\mathrm{id}_C) = \sigma_C = \mathrm{id}_G(C) \circ \mathrm{id}_{\mathbf{D}}(\sigma_C) = (\mathrm{id}_{\mathbf{D}} * \sigma)_C$$

4. For any $B \in ob(\mathbf{B})$,

$$\begin{aligned}(\tau * (\sigma * \gamma))_B &= \tau_{KH(B)} \circ L((\sigma * \gamma)_B) \\ &= \tau_{KH(B)} \circ L(\sigma_{H(B)} \circ G(\gamma_B)) \\ &= \tau_{KH(B)} \circ L(\sigma_{H(B)}) \circ LG(\gamma_B)) \\ &= (\tau * \sigma)_{H(B)} \circ LG(\gamma_B) \\ &= ((\tau * \sigma) * \gamma)_B\end{aligned}$$

□

1. and 2. show that composition of functors and the Godement product of natural transformations is a functor $[\mathbf{C}, \mathbf{C}] \times [\mathbf{C}, \mathbf{C}] \to \mathbf{C}$. Even more is true:

Proposition 2.6. $[\mathbf{C}, \mathbf{C}]$ *is a strict monoidal category if tensor product is taken to be composition on objects and the Godement product on morphisms.*

Proof. By the previous lemma, $\circ$ is associative and has a unit, namely $\mathrm{id}_{\mathbf{C}}$. Taking identity transformations of the appropriate type gives us all the data required of a strict monoidal category. As the natural tranformations in question all are identities, all the triangle and pentagon diagrams commute. □

Another example that will be useful later on is that of a free monoidal category. Our development of the notion follows closely that of [1]:

Definition 2.7. Given a category $\mathbf{C}$, the *free monoidal category generated by* $\mathbf{C}$ is denoted by $\mathbf{C}^*$. It has as its objects all tuples $(A_1, \dots A_n)$ of objects of $\mathbf{C}$, including the empty sequence which we denote by I. Given objects $\bar{A} := (A_1, \dots A_n)$ and $\bar{B} := (B_1, \dots B_m)$, the set of morphisms $\bar{A} \to \bar{B}$ is given by

$$\hom_{\mathbf{C}^*}(\bar{A}, \bar{B}) = \begin{cases} \emptyset \text{ if } m \neq n \\ \{(f_1, \dots f_n) \colon f_i \in \hom_{\mathbf{C}}(A_i, B_i)\} \text{ otherwise} \end{cases}$$

Composition of morphisms is done pointwise. The tensor product is given by concatenation, i.e. $(A_1, \dots A_n) \otimes (B_1, \dots B_m) = (A_1, \dots A_n, B_1, \dots B_m)$ on objects and $(f_1, \dots f_n) \otimes (g_1, \dots g_m) = (f_1, \dots f_n, g_1, \dots g_m)$ on morphisms. This makes $\mathbf{C}^*$ into a strict monoidal category.

One can represent a morphism $\bar{f} = \bar{A} \to \bar{B}$ pictorially by drawing the individual morphisms next to each other:

$$\begin{array}{cccc} B_1 & B_2 & \dots & B_n \\ f_1 \uparrow & f_2 \uparrow & & f_n \uparrow \\ A_1 & A_2 & \dots & A_n \end{array}$$

The word 'free' in 'free monoidal category' is there for a reason. It comes from this construction having a similar universal property as say free groups: write $i \colon \mathbf{C} \to \mathbf{C}^*$ for the obvious inclusion. If $\mathbf{D}$ is any monoidal category and $F \colon \mathbf{C} \to \mathbf{D}$ is an arbitrary functor, then there is a unique monoidal functor $\hat{F} \colon \mathbf{C}^* \to \mathbf{D}$ satisfying $\hat{F} \circ i = F$.

A nice concequence of this is that endofunctors on $\mathbf{C}$ and natural transformations between them lift to $\mathbf{C}^*$. As we haven't defined monoidal functors, we don't take this route. Instead, we define the lift explicitly.

Definition 2.8. If $F \colon \mathbf{C} \to \mathbf{C}$ is a functor, define $F^* \colon \mathbf{C}^* \to \mathbf{C}^*$ by $F^*(A_1, \dots A_n) = (F(A_1), \dots F(A_n))$ on objects and by $F^*(f_1, \dots f_n) = (F(f_1), \dots F(f_n))$ on morphisms. If $\sigma \colon F \to G$ is a natural transformation, then define $\sigma^* \colon F^* \to G^*$ by $\sigma^*_{\bar{A}} = (\sigma_{A_1}, \dots \sigma_{A_n})$.

Clearly this operation is a functor $[C, C] \to [C^*, C^*]$. Furthermore it is clear from the definitions that $(\sigma * \tau)^* = \sigma^* * \tau^*$. This observation will be useful later.

In the monoidal category of Hilbert spaces one can take adjoints of morphisms. The properties of this are captured in the abstract concept of a dagger:

Definition 2.9. A *dagger* on a category $\mathbf{C}$ is a contravariant functor $\dagger\colon \mathbf{C} \to \mathbf{C}$ that is the identity on objects, and satisfies $\dagger \circ \dagger = \mathrm{id}_{\mathbf{C}}$. We write $f^\dagger$ for its action on morphisms.

Given a dagger $\dagger\colon \mathbf{C} \to \mathbf{C}$, an isomorphism f is *unitary* provided $f^\dagger = f^{-1}$.

A monoidal category $\mathbf{C}$ equipped with a dagger is called a *dagger monoidal category*, if all the components of α, ρ and λ are unitary and $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$ holds for all morphisms f and g. A symmetric monoidal category with a dagger is dagger symmetric monoidal in case it is dagger monoidal and γ is unitary.

Besides **Hilb**, also **Rel** can be given the structure of a dagger symmetric monoidal category, when we define the dagger of a relation to be its inverse.

Given a dagger $\dagger$ on $\mathbf{C}$, we would like to make $[\mathbf{C}, \mathbf{C}]$ into a dagger monoidal category. An obvious way to try to do this would be to apply the dagger pointwise to natural transformations, i.e. if $\sigma\colon F \to G$ in $[C, C]$, define $\sigma^\dagger$ by $(\sigma^\dagger)_C = (\sigma_C)^\dagger$. However, it's not clear that this results in a natural transformation. The solution is to take the subcategory of $[\mathbf{C}, \mathbf{C}]$ on which this works:

Definition 2.10. Given a dagger $\dagger$ on $\mathbf{C}$, let $[\mathbf{C}, \mathbf{C}]_\dagger$ be the full subcategory of $[\mathbf{C}, \mathbf{C}]$ that has as its objects those functors $F\colon \mathbf{C} \to \mathbf{C}$ that commute with $\dagger$, i.e. satisfy $F(f^\dagger) = (F(f))^\dagger$ for all morphisms f.

Indeed this does work: if $\sigma\colon F \to G$ is a natural transformation and F and G commute with the dagger, then $\sigma^\dagger$, defined by $(\sigma^\dagger)_C = (\sigma_C)^\dagger$ is a natural transformation $G \to F$. To see this, consider a morphism $f\colon C \to D$ in $\mathbf{C}$. As σ is natural, the diagram

$$\begin{tikzcd}
G(C) & F(C) \arrow[l, "\sigma_C"'] \\
G(D) \arrow[u, "G(f^\dagger)"] & F(D) \arrow[l, "\sigma_D"] \arrow[u, "F(f^\dagger)"']
\end{tikzcd}$$

commutes. As $\dagger$ is involutive and F and G commute with it, applying $\dagger$ to this diagram results in the commutative diagram

$$\begin{tikzcd}
G(C) \arrow[r, "\sigma_C^\dagger"] \arrow[d, "G(f)"'] & F(C) \arrow[d, "F(f)"] \\
G(D) \arrow[r, "\sigma_D^\dagger"'] & F(D)
\end{tikzcd}$$

proving that $\sigma^\dagger$ is natural. As all the other properties required from a dagger follow from $\dagger\colon \mathbf{C} \to \mathbf{C}$ having those, the assignment $(\sigma\colon F \to G) \mapsto (\sigma^\dagger\colon G \to F)$ is a dagger on $[\mathbf{C}, \mathbf{C}]_\dagger$.

If F and G commute with the dagger, so does $G \circ F$. Thus $\circ$ makes $[\mathbf{C}, \mathbf{C}]_\dagger$ into a monoidal category. Consider two natural transformations $\sigma\colon F \to G$, $\tau\colon H \to K$ in $[\mathbf{C}, \mathbf{C}]_\dagger$. Then

$$(\tau * \sigma)^\dagger_C = ((\tau_{G(C)} \circ H(\sigma_C))^\dagger = H(\sigma^\dagger_C) \circ \tau^\dagger_{G(C)} = (\tau^\dagger * \sigma^\dagger)_C$$

so that $(\tau * \sigma)^\dagger = \tau^\dagger * \sigma^\dagger$. As the associators and unitors are identities, they are unitary pointwise. Hence we have proved that

Proposition 2.11. $[\mathbf{C}, \mathbf{C}]_\dagger$ *is dagger monoidal.*

One of the motivating reasons of coming up with the concept of a monoidal category in the first place is that we can then define monoids in them:

Definition 2.12. A *monoid* in a monoidal category $\mathbf{C}$ consists of an object $M \in ob(\mathbf{C})$ and morphisms $m\colon M \otimes M \to M$, $u\colon I \to M$ such that

$$m \circ (\mathrm{id}_M \otimes m) = m \circ (m \otimes \mathrm{id}_M) \circ \alpha_{M,M,M}$$

and

$$m \circ (\mathrm{id}_M \otimes u) \circ \rho_M^{-1} = \mathrm{id}_M = M \circ (u \otimes \mathrm{id}_M) \circ \lambda_M^{-1}.$$

The morphism m is called the multiplication of the monoid, and u is called the unit of the monoid. We speak of the monoid M instead of the monoid (M, m, u), when there is no risk of confusion.

If we're working in a symmetric monoidal category, a monoid M is called commutative in case $m \circ \gamma_{M,M} = m$.

In **Set**, monoids in the above sense correspond to monoids in the usual sense, and commutativity in the above sense is just commutativity as usual. In **Top** monoids are topological monoids.

As is usual in category theory, the monoids come with a notion of morphism between monoids:

Definition 2.13. Given monoids (M, m_M, u_M) and (N, m_N, u_N) in $\mathbf{C}$, a morphism $f\colon M \to N$ is a *monoid homorphism* from the monoid M to the monoid N, provided the equations $f \circ u_M = u_N$ and $f \circ m_M = m_N \circ (f \otimes f)$ hold.

Obviously id_M is a monoid homomorphism $M \to N$. Furthermore, if $f\colon M \to N$ and $g\colon N \to O$ are monoid homomorphisms, then $g \circ f \circ u_M = g \circ u_N = u_O$ and $g \circ f \circ m_M = g \circ m_N \circ (f \otimes f) = m_O \circ (g \otimes g) \circ (f \otimes f) = m_O \circ ((g \circ f) \otimes (g \circ f))$, showing that $g \circ f$ is also a monoid homomorphism. Hence we get a category:

Definition 2.14. Given a monoidal category $\mathbf{C}$, the category $\mathrm{Mon}(\mathbf{C})$ has monoids in $\mathbf{C}$ as its objects, monoid homomorphisms as its morphisms and it inherits composition of morphisms from that of $\mathbf{C}$. If $\mathbf{C}$ is also symmetric, then $\mathrm{cMon}(\mathbf{C})$ is the full subcategory of $\mathrm{Mon}(\mathbf{C})$ having commutative monoids as its objects.

If one reverses all the arrows in the above, we get the notions of comonoid, homomorphisms between them and the category $\mathrm{Comon}(\mathbf{C})$. To be explicit, we have the following:

Definition 2.15.

- A *comonoid* in a monoidal category $\mathbf{C}$ consists of an object $M \in ob(\mathbf{C})$ and morphisms $d\colon M \to M \otimes M$, $e\colon M \to I$ such that

$$\alpha_{M,M,M} \circ (\mathrm{id}_M \otimes d) \circ d = (d \otimes \mathrm{id}_M) \circ d$$

and

$$\rho_M \circ (\mathrm{id}_M \otimes e) \circ d = \mathrm{id}_M = \lambda_M \circ (e \otimes \mathrm{id}_M) \circ d.$$

The morphism d is called the comultiplication of the comonoid, and e is called the counit of the monoid. A comonoid M in a symmetric monoidal category is called cocommutative in case $\gamma_{M,M} \circ d = d$.

- Given comonoids (M, d_M, e_M) and (N, d_N, e_N) in $\mathbf{C}$, a morphism $f\colon M \to N$ is a *comonoid homorphism* from the comonoid M to the comonoid N, provided the equations $e_N \circ f = e_M$ and $(f \otimes f) \circ d_M = d_N \circ f$ hold.

- Given a monoidal category $\mathbf{C}$, the category $\mathrm{Comon}(\mathbf{C})$ has comonoids in $\mathbf{C}$ as its objects, comonoid homomorphisms as its morphisms and it inherits composition of morphisms from that of $\mathbf{C}$. In case $\mathbf{C}$ is symmetric monoidal the category of cocommutative comonoids is denoted by $\mathrm{cComon}(\mathbf{C})$.

Lastly, it is possible for an object to have both a monoid and a comonoid structure on it. This isn't particularly interesting in itself unless these structures interact somehow. This leads to the concept of a Frobenius algebra:

Definition 2.16. If (M, m, u) is a monoid in $\mathbf{C}$, and (M, d, e) is a comonoid in $\mathbf{C}$, then (M, m, u, d, e) is a *Frobenius algebra* in $\mathbf{C}$ if it satisfies the *Frobenius law*:

$$(m \otimes \mathrm{id}_M) \circ (\mathrm{id}_M \otimes d) = (\mathrm{id}_M \otimes m) \circ (d \otimes \mathrm{id}_M)$$

A Frobenius algebra in a symmetric monoidal category is called commutative in case the underlying monoid is commutative and the underlying comonoid is cocommutative. If $\mathbf{C}$ is dagger monoidal, the Frobenius algebra M is a †-Frobenius algebra in case $d = m^\dagger$ and $e = u^\dagger$, i.e. the underlying comonoid is obtained by daggering the underlying monoid. A Frobenius algebra is special if $m \circ d = \mathrm{id}_M$. Notions such as commutative dagger special Frobenius algebra mean exactly what one would expect them to mean.

As an example, consider the category **Rel** with the † given by taking inverses of relations. Let G be any group. View the multiplication $m\colon G \times G \to G$ and the function $u\colon \{*\} \to G : * \mapsto 1_G$ as relations. Then (G, m, u) is a monoid in **Rel** and since **Rel** is dagger monoidal, $(G, m^\dagger, u^\dagger)$ is a comonoid. It is easy to check that the Frobenius law is satisfied, so that $(G, m, u, m^\dagger, u^\dagger)$ is a dagger Frobenius algebra. It is also special, and it is commutative in case G is commutative.

The obvious way of defining morphisms between Frobenius algebras is to consider morphisms that are homomorphisms with respect to the underlying monoid and comonoid structures. Call such a morphism a *strict morphism* of Frobenius algebras. This notion is a bit too stringent, as witnessed in the following lemma.

Lemma 2.17. *If $f\colon M \to N$ is a strict morphism of Frobenius algebras, then f is an isomorphism in $\mathbf{C}$ and its inverse is also a strict morphism of Frobenius algebras*

Proof. The proof is tedious, straightforward and not central to the text. Hence the proof is in the appendix. □

This means that if one uses strict morphisms to define the category of Frobenius algebras in $\mathbf{C}$, the resulting category would be a groupoid. To avoid this, we define morphisms of Frobenius algebras as follows.

Definition 2.18. If (A, m^A, u^A, d^A, e^A) and (B, m^B, u^B, d^B, e^B) are Frobenius algebras and $f\colon A \to B$, then f is a Frobenius homomorphism from A to B if $f \circ m^A = m^A \circ (f \otimes f)$ and $(f \otimes f) \circ d^A = d^B \circ f$.

The category Frob($\mathbf{C}$) has Frobenius algebras in $\mathbf{C}$ as its objects, Frobenius homomorphisms as its morphisms and it inherits composition of morphisms from that of $\mathbf{C}$.

Chapter 3

Monads

Usually in introductory textbooks on category theory one defines monads and comonads as certain kinds of endofunctors $\mathbf{C} \to \mathbf{C}$. As $[\mathbf{C}, \mathbf{C}]$ is a monoidal category, we can define these and other notions using terminology we developed for monoidal categories.

Definition 3.1.

- A *monad* on a category $\mathbf{C}$ is a monoid in $[\mathbf{C}, \mathbf{C}]$, and $\mathrm{Monad}(\mathbf{C}) = \mathrm{Mon}([\mathbf{C}, \mathbf{C}])$
- A *comonad* on a category $\mathbf{C}$ is a comonoid in $[\mathbf{C}, \mathbf{C}]$, and $\mathrm{Comonad}(\mathbf{C}) = \mathrm{Comon}([\mathbf{C}, \mathbf{C}])$
- A *Frobenius monad* on a category $\mathbf{C}$ is a Frobenius algebra in $[\mathbf{C}, \mathbf{C}]$, and $\mathrm{FrobMonad}(\mathbf{C}) = \mathrm{Frob}([\mathbf{C}, \mathbf{C}])$
- A *dagger Frobenius monad* on a dagger category $\mathbf{C}$ is a dagger Frobenius algebra in $[\mathbf{C}, \mathbf{C}]_\dagger$.
- A (dagger) Frobenius monad is special if it is special as a Frobenius algebra in $[\mathbf{C}, \mathbf{C}]$.

This definition gives rise to the usual notions of monad and co-monad. Unwinding the definition, a monad is a functor $T \colon \mathbf{C} \to \mathbf{C}$ together with natural transformations $\mu \colon T \circ T \to T$, $\eta \colon \mathrm{id} \to T$ satisfying $\mu \circ (\mu * T) = \mu \circ (T * \mu)$ and $\mu \circ (T * \eta) = \mu \circ (\eta * T) = \mathrm{id}_T$.

One example of a monad on $\mathbf{Set}$ is given by the covariant powerset functor $\mathcal{P}$, where μ is given by union and η is defined as $\eta_A : a \mapsto \{a\}$.

We recall the definition of an adjoint pair of functors:

Definition 3.2. An adjunction between two categories $\mathbf{C}$ and $\mathbf{D}$ consists of functors two functors $F\colon C \to D$, $G\colon \mathbf{D} \to C$ and of two natural transformations $\eta\colon \mathrm{id}_C \to G \circ F$ $\epsilon\colon F \circ G \to \mathrm{id}_{\mathbf{D}}$ such that the equations

$$\mathrm{id}_F = (\epsilon * F) \circ (F * \eta)$$
$$\mathrm{id}_G = (G * \epsilon) \circ (\eta * G)$$

hold. In this situation F is called the left adjoint and G is called the right adjoint. We write this as $(\epsilon, \eta)\colon F \dashv G$ or simply as $F \dashv G$ when η and ϵ are clear from context.

It is a standard result whenever we have an adjunction $F \dashv G$ we also have a monad structure on $G \circ F$ given by $(G \circ F, G * \epsilon * F, \eta)$. Furthermore, every monad arises this way. One way of proving this is to consider the Kleisli category of a monad:

Definition 3.3. The *Kleisli category* of a monad (T, μ, η) on $\mathbf{C}$, denoted by $\mathbf{C}_T$, has as its objects those of $\mathbf{C}$, and morphisms $f\colon A \to B$ in $\mathbf{C}_T$ are morphisms $f\colon A \to T(B)$ in $\mathbf{C}$. Given two morphisms $f\colon A \to B$ and $g\colon B \to C$ in $\mathbf{C}_T$, their composite $g \circ f$ in $\mathbf{C}_T$ is defined to taking the composite

$$A \xrightarrow{f} T(B) \xrightarrow{T(g)} T^2(C) \xrightarrow{\mu} T(C)$$

in $\mathbf{C}$.

There are functors $\mathbf{C} \to \mathbf{C}_T$ and $\mathbf{C}_T \to \mathbf{C}$ defined by

$$(A \xrightarrow{f} B) \longmapsto (A \xrightarrow{f} B \xrightarrow{\eta} T(B))$$

and

$$(A \longrightarrow B = A \xrightarrow{f} T(B)) \longmapsto (T(A) \xrightarrow{T(f)} T^2(B) \xrightarrow{\mu} T(B))$$

respectively, and these form an adjunction, the functor $\mathbf{C} \to \mathbf{C}_T$ being the left adjoint. The proofs that $\mathbf{C}_T$ is a valid category, that these operations are functors and adjoint to each other, are all standard and can be found in any standard textbook on category theory, see e.g. [5, chapter4].

The reason we introduced the Kleisli category is the following lemma.

Lemma 3.4. *If T is a dagger Frobenius monad on $\mathbf{C}$, then the dagger on $\mathbf{C}$ gives rise to one on $\mathbf{C}_T$.*

Proof. Let $(T, \mu, \eta, \mu^\dagger, \eta^\dagger)$ be a dagger Frobenius monad. We claim that the operation

$$(A \xrightarrow{f} T(B)) \longmapsto (B \xrightarrow{\eta} T(B) \xrightarrow{\mu^\dagger} T^2(B) \xrightarrow{T(f^\dagger)} T(A))$$

is a dagger on $\mathbf{C}_T$. It is contravariant on morphisms and the identity on objects, so we just need to check that it is involutive and functorial. As T commutes with the dagger, applying this operation twice to $f\colon A \to T(B)$ results in the morphism

$$A \xrightarrow{\eta} T(A) \xrightarrow{\mu^\dagger} T^2(A) \xrightarrow{T^2(f)} T^3(B) \xrightarrow{T(\mu)} T^2(B) \xrightarrow{T(\eta^\dagger)} T(B)$$

To see that this equals f, consider the diagram

$$\begin{tikzcd}
A \arrow[r, "\eta"] \arrow[dr, "f"'] & T(A) \arrow[r, "\mu^\dagger"] \arrow[dr, "T(f)"] \arrow[d, phantom, "\text{(i)}"] & T^2(A) \arrow[r, "T^2(f)"] \arrow[d, phantom, "\text{(ii)}"] & T^3(B) \arrow[r, "T(\mu)"] \arrow[d, phantom, "\text{(iii)}"] & T^2(B) \arrow[ddd, "T(\eta^\dagger)"] \\
& T(B) \arrow[r, "\eta"] \arrow[drrr, "\mathrm{id}"'] \arrow[dr, phantom, "\text{(iv)}"] & T^2(B) \arrow[ur, "\mu^\dagger"] \arrow[r, "T(\mu^\dagger)"] \arrow[dr, "\mathrm{id}"] & T^3(B) \arrow[ur, "\mu"] \arrow[d, "T^2(\eta^\dagger)"] \arrow[r, phantom, "\text{(vi)}"] \arrow[dl, phantom, "\text{(v)}"] & {} \\
& & & T^2(B) \arrow[dr, "\mu"] & \\
& & & & T(B)
\end{tikzcd}$$

Region (i) is just the naturality square of η. The commutativity of region (ii) follows from the naturality of $\mu^\dagger$. and (iii) is an instance of the Frobenius law of T. (iv) is one of the monad laws and (v) one of the comonad ones. Finally, (vi) commutes by naturality of μ. Hence the diagram commutes, proving the claim.

Next we check that this operation operation is functorial. First of all, identities $A \to A$ in $\mathbf{C}_T$ are arrows $\eta\colon A \to T(A)$ in $\mathbf{C}$. This daggers to

$$A \xrightarrow{\eta} T(A) \xrightarrow{\mu^\dagger} T^2(A) \xrightarrow{T(\eta^\dagger)} T(A)$$

which equals η_A since $T(\eta^\dagger) \circ T(\mu^\dagger) = \mathrm{id}$ by the comonoid axioms. Hence identities are preserved.

To see that composition is preserved, consider morphisms $f\colon A \to T(B)$ and $g\colon B \to T(C)$. Their composite is

$$A \xrightarrow{f} T(B) \xrightarrow{T(g)} T^2(C) \xrightarrow{\mu} T(C)$$

which daggers to

$$C \xrightarrow{\eta} T(C) \xrightarrow{\mu^\dagger} T^2(C) \xrightarrow{T(\mu^\dagger)} T^3(C) \xrightarrow{T^2(g^\dagger)} T^2(B) \xrightarrow{T(f^\dagger)} T(A)$$

.

On the other hand, the dagger of f is

$$B \xrightarrow{\eta} T(B) \xrightarrow{\mu^\dagger} T^2(B) \xrightarrow{T(f^\dagger)} T(A)$$

and that of g is

$$C \xrightarrow{\eta} T(C) \xrightarrow{\mu^\dagger} T^2(C) \xrightarrow{T(g^\dagger)} T(B)$$

and these compose to

$$\begin{tikzcd}
C \arrow[r, "\eta"] & T(C) \arrow[r, "\mu^\dagger"] & T^2(C) \arrow[r, "T(g^\dagger)"] & T(B) \arrow[r, "T(\eta)"] & T^2(B) \arrow[r, "T(\mu^\dagger)"] & T^3(B) \arrow[r, "T^2(f^\dagger)"] & T^2(A) \arrow[d, "\mu"] \\
 & & & & & & T(A)
\end{tikzcd}$$

To see that these are equal, i.e. that the composite of the daggers equals the dagger of the composite, consider the diagram

$$\begin{tikzcd}
C \arrow[r, "\eta"] & T(C) \arrow[r, "\mu^\dagger"] \arrow[dl, "\mu^\dagger"'] \arrow[d, phantom, "(i)"] & T^2(C) \arrow[r, "T(g^\dagger)"] \arrow[dl, "\mu^\dagger"'] & T(B) \arrow[r, "T(\eta)"] \arrow[d, phantom, "(ii)"] & T^2(B) \arrow[dl, "\mu^\dagger"'] \arrow[d, "T(\mu^\dagger)"] \\
T^2(C) \arrow[r, "T(\mu^\dagger)"] & T^3(C) \arrow[r, "T^2(g^\dagger)"] & T^2(B) \arrow[r, "T^2(\eta)"] \arrow[dr, "\mathrm{id}"'] \arrow[dr, phantom, "(iv)"] & T^3(B) \arrow[d, "T(\mu)"] \arrow[r, phantom, "(iii)"] & T^3(B) \arrow[dl, "\mu"'] \arrow[d, "T^2(f^\dagger)"] \\
 & & & T^2(B) \arrow[r, phantom, "(v)"] \arrow[dr, "T(f^\dagger)"'] & T^2(A) \arrow[d, "\mu"] \\
 & & & & T(A)
\end{tikzcd}$$

Now region (i) commutes by the associativity of the comonad. Commutativity of (ii) follows from $\mu^\dagger$ being natural and (iii) is just the Frobenius law of T. (iv) is one of the unit laws of the monad and (v) commutes as μ is natural. Hence the whole diagram commutes, concluding the proof. $\square$

Recall that a faithful functor is a functor $U\colon \mathbf{C}' \to \mathbf{C}$ that is injective on hom-sets, i.e. for every $f, g\colon A \to B$ $U(f) = U(g)$ implies $f = g$. In this situation we say that $(\mathbf{C}', U)$ (or just $\mathbf{C}'$ whenever U is obvious from context), is a concrete category over $\mathbf{C}$. The usual context for this is when every object of $\mathbf{C}'$ is an object of $\mathbf{C}$ with some extra structure and morphisms of $\mathbf{C}'$ are morphisms of $\mathbf{C}$ preserving this structure. Examples of this are various categories of sets with extra structure (topological spaces, groups, etc.) together with the forgetful functors to the category of sets. The standard reference for concrete categories is [3]. With this in mind, we can define the following notion, used primarily to state the results obtained in this dissertation succintly:

Definition 3.5. Let $(\mathbf{C}', U_{\mathbf{C}}), (\mathbf{D}', U_{\mathbf{D}})$ be concrete categories over $\mathbf{C}$ and $\mathbf{D}$ respectively. We say that a functor $F\colon \mathbf{C} \to \mathbf{D}$ can be lifted if there is a functor $F'\colon \mathbf{C}' \to \mathbf{D}'$ such that $U_{\mathbf{D}} \circ F' = F \circ U_{\mathbf{C}}$, i.e. the diagram

$$\begin{tikzcd} \mathbf{C}' \arrow[r, "F'"] \arrow[d, "U_{\mathbf{C}}"'] & \mathbf{D}' \arrow[d, "U_{\mathbf{D}}"] \\ \mathbf{C} \arrow[r, "F"'] & \mathbf{D} \end{tikzcd}$$

commutes.

If $(\epsilon, \eta)\colon F \dashv G$ is an adjunction, where $F\colon \mathbf{C} \to \mathbf{D}$ and $G\colon \mathbf{D} \to \mathbf{C}$, we say that the adjunction can be lifted to an adjuction between $\mathbf{C}'$ and $\mathbf{D}'$ if F and G can be lifted to F' and G' respectively and there are natural transformations η', ϵ' satisfying $U_{\mathbf{C}}(\eta'_A) = \eta_{U\_C(A)}$ and $U_{\mathbf{D}}(\epsilon'_B) = \epsilon_{U_D(B)}$ forming an adjunction $(\epsilon', \eta')\colon F' \dashv G'$. In case G' and F' form an equivalence of categories, we say that the adjunction can be lifted to an equivalence.

In the usual context where concrete categories arise, the interpration for this is that for every object of $\mathbf{C}$ equipped with the extra structure making it an object of $\mathbf{C}'$, the image of it under F inherits from this a structure making it an object of $\mathbf{D}'$, and morphisms respect the induced structure. In this context being able to lift an adjunction means in addition to this that the natural transformations of the adjunction respect the induced extra structure. Examples of this will be seen in subsequent chapters.

Chapter 4

The strength adjunction

As monoidal categories are categories with some extra structure, one might expect that there are interesting classes of functors interacting with this structure in some way or another. One such class is that of strong functors.

Definition 4.1. Given a monoidal category $\mathbf{C}$, a strong functor $\mathbf{C} \to \mathbf{C}$ is a functor $F \colon \mathbf{C} \to \mathbf{C}$ equipped with a natural transformation st with components

$$st_{A,B} \colon A \otimes F(B) \to F(A \otimes B)$$

which satisfies $(F * \lambda) \circ st = \lambda$ and $(F * \alpha) \circ st \circ (id \otimes st) = st \circ \alpha$, or in diagrams,

$$\begin{tikzcd}
I \otimes F(A) \arrow[r, "st_{I,A}"] \arrow[dr, "\lambda_A"'] & F(I \otimes A) \arrow[d, "F(\lambda_A)"] \\
 & F(A)
\end{tikzcd}$$

and

$$\begin{tikzcd}
A \otimes (B \otimes F(C)) \arrow[r, "\mathrm{id}_A \otimes st_{B,C}"] \arrow[d, "\alpha_{A,B,F(C)}"'] & A \otimes F(B \otimes C) \arrow[r, "st_{A,B\otimes C}"] & F(A \otimes (B \otimes C)) \arrow[d, "F(\alpha_{A,B,C})"] \\
(A \otimes B) \otimes F(C) \arrow[rr, "st_{A\otimes B,C}"'] & & F((A \otimes B) \otimes C)
\end{tikzcd}$$

must commute for every object A, B and C.

A morphism of strong functors $F, G\colon \mathbf{C} \to \mathbf{C}$ is a natural transformation $\sigma\colon F \to G$ which satisfies $\sigma \circ st^F = st^G \circ (\mathrm{id} \otimes \sigma)$, ie. for which the diagram

$$\begin{tikzcd}
A \otimes F(B) \arrow[r, "st^F_{A,B}"] \arrow[d, "\mathrm{id}_A \otimes \sigma_B"'] & F(A \otimes B)) \arrow[d, "\sigma_{A\otimes B}"] \\
A \otimes G(B) \arrow[r, "st^G_{A,B}"'] & G(A \otimes B)
\end{tikzcd}$$

commutes. We denote the category of strong functors $\mathbf{C} \to \mathbf{C}$ and their morphisms by $\mathrm{str}(\mathbf{C})$.

Let $T\colon \mathbf{C} \to \mathbf{C}$ be a functor and $\eta\colon \mathrm{id}_{\mathbf{C}} \to T$ a natural transformation. Then we can use η to make $T^*\colon \mathbf{C}^* \to \mathbf{C}^*$ into a strong functor by setting $st_{\bar{A},\bar{B}} := \eta_{\bar{A}} \otimes \mathrm{id}_{\bar{B}}$, so pictorially $st_{\bar{A},\bar{B}}$ is the morphism

$$\begin{tikzcd}
T(A_1) & \dots & T(A_n) & T(B_1) & \dots & T(B_m) \\
A_1 \arrow[u, "\eta_{A_1}"] & \dots & A_n \arrow[u, "\eta_{A_n}"] & T(B_1) \arrow[u, "\mathrm{id}"] & \dots & T(B_m) \arrow[u, "\mathrm{id}"]
\end{tikzcd}$$

The first strength axiom is trivial as both maps evaluate to the identity on $\bar{A}$, and the other one boils down to the fact that

$$\begin{tikzcd}
T(A_1) & \dots & T(A_n) & T(B_1) & \dots & T(B_m) & T(C_1) & \dots & T(C_k) \\
A_1 \arrow[u, "\eta_{A_1}"] & \dots & A_n \arrow[u, "\eta_{A_n}"] & T(B_1) \arrow[u, "\mathrm{id}"] & \dots & T(B_m) \arrow[u, "\mathrm{id}"] & T(C_1) \arrow[u, "\mathrm{id}"] & \dots & T(C_k) \arrow[u, "\mathrm{id}"] \\
A_1 \arrow[u, "\mathrm{id}"] & \dots & A_n \arrow[u, "\mathrm{id}"] & B_1 \arrow[u, "\eta_{B_1}"] & \dots & B_m \arrow[u, "\eta_{B_m}"] & T(C_1) \arrow[u, "\mathrm{id}"] & \dots & T(C_k) \arrow[u, "\mathrm{id}"]
\end{tikzcd}$$

equals

$$\begin{tikzcd}
T(A_1) & \dots & T(A_n) & T(B_1) & \dots & T(B_m) & T(C_1) & \dots & T(C_k) \\
A_1 \arrow[u, "\eta_{A_1}"] & \dots & A_n \arrow[u, "\eta_{A_n}"] & B_1 \arrow[u, "\eta_{B_1}"] & \dots & B_m \arrow[u, "\eta_{B_m}"] & T(C_1) \arrow[u, "\mathrm{id}"] & \dots & T(C_k) \arrow[u, "\mathrm{id}"]
\end{tikzcd}$$

As another example of a strong functor, consider an object C in a monoidal category, and the functor $-\otimes C$. If we set $st_{A,B} = \alpha_{A,B,C}$, we have a map of the right type. The first of the strength equations follows by coherence and the latter is just the pentagon diagram, so this makes $-\otimes C$ into a strong functor. Given a morphism $f\colon C \to D$, the family $(\mathrm{id}_A \otimes f)_{A\in ob(\mathbf{C})}$ is a natural transformation $-\otimes C \to -\otimes D$, as $\otimes$ is a functor. To see that $-\otimes f$ is a morphism of strong functors, note that the diagram

$$\begin{tikzcd}
A \otimes (B \otimes C) \arrow[r, "\alpha_{A,B,C}"] \arrow[d, "\mathrm{id}_A \otimes (\mathrm{id}_B \otimes f)"'] & (A \otimes B) \otimes C \arrow[d, "\mathrm{id}_{A\otimes B} \otimes f"] \\
A \otimes (B \otimes D) \arrow[r, "\alpha_{A,B,D}"'] & (A \otimes B) \otimes D
\end{tikzcd}$$

commutes as $\mathrm{id}_{A\otimes B} = \mathrm{id}_A \otimes \mathrm{id}_B$ and α is natural. Hence we have a functor $\mathbf{F}\colon \mathbf{C} \to \mathrm{str}(\mathbf{C})$ defined by $\mathbf{F}(C) = -\otimes C$ and $\mathbf{F}(f) = -\otimes f$. We also have a functor $\mathbf{G}\colon \mathrm{str}(\mathbf{C}) \to \mathbf{C}$ defined by $\mathbf{G}(F) = F(I)$ and $\mathbf{G}(\sigma) = \sigma_I$.

Theorem 4.2. $\mathbf{F} \dashv \mathbf{G}$

Proof. As $\mathbf{G} \circ \mathbf{F}(A)$ is $I \otimes A$, the natural transformation $\boldsymbol{\eta} := \lambda^{-1}$ is a natural isomorphism $\mathrm{id}_{\mathbf{C}} \to \mathbf{G}\circ\mathbf{F}$. We also need a natural transformation $\boldsymbol{\epsilon}\colon \mathbf{F}\circ\mathbf{G} \to \mathrm{id}_{\mathrm{str}(\mathbf{C})}$. The components of $\boldsymbol{\epsilon}$ are morphisms of strong functors of the form $-\otimes F(I) \to F(-)$. Hence a natural candidate for $\boldsymbol{\epsilon}_F$ is $(F * \rho) \circ str^F_{-,I}$, i.e. $(\boldsymbol{\epsilon}_F)_A$ is the composite $A \otimes F(I) \to F(A \otimes I) \to F(A)$. For every functor F, the map $\boldsymbol{\epsilon}_F$ is a natural transformation as it is a composite of natural transformations, so we just need to check that it is a morphism of strong functors. As $st^{-\otimes F(I)}_{A,B} = \alpha_{A,B,F(I)}$, we wish to prove that the outer rectangle in

$$\begin{tikzcd}
A \otimes (B \otimes F(I)) \arrow[rr, "\alpha_{A,B,F(I)}"] \arrow[d, "\mathrm{id}_A \otimes st^F_{B,I}"'] & & (A \otimes B) \otimes F(I) \arrow[d, "st^F_{A\otimes B,I}"] \\
A \otimes (F(B \otimes I)) \arrow[r, "st^F_{A,B\otimes I}"] \arrow[d, "\mathrm{id}_A \otimes F(\rho_B)"'] & F(A \otimes (B \otimes I)) \arrow[r, "F(\alpha_{A,B,I})"] \arrow[dr, "F(\mathrm{id}_A \otimes \rho_B)"] & F((A \otimes B) \otimes I) \arrow[d, "F(\rho_{A\otimes B})"] \\
A \otimes F(B) \arrow[rr, "st^F_{A,B}"'] & & F(A \otimes B)
\end{tikzcd}$$

commutes. The upper rectangle commutes as F is strong. To see that the lower rectangle commutes, note that the triangle inside commutes, as it is just F applied to a diagram that commutes by coherence. The left half of the lower rectangle commutes just because st^F is a natural transformation. Hence the whole diagram commutes and $\boldsymbol{\epsilon}_F$ is a morphism in the category $\mathrm{str}(\mathbf{C})$. To see that $\boldsymbol{\epsilon}$ defines a natural transformation, let $\tau\colon F \to G$ be a morphism of strong functors. Consider the diagram

$$\begin{tikzcd}
A \otimes F(I) \arrow[r, "st^F_{A,I}"] \arrow[d, "\mathrm{id}_A \otimes \tau_I"'] & F(A \otimes I) \arrow[r, "F(\rho_A)"] \arrow[d, "\tau_{A\otimes I}"'] & F(A) \arrow[d, "\tau_A"] \\
A \otimes G(I) \arrow[r, "st^G_{A,I}"'] & G(A \otimes I) \arrow[r, "G(\rho_A)"'] & G(A)
\end{tikzcd}$$

The square on the left commutes as τ is a morphism of strong functors, and the square on the right commutes as τ is natural. As A was arbitrary, this means that the diagram

$$\begin{tikzcd}
\mathbf{F} \circ \mathbf{G}(F) \arrow[r, "\boldsymbol{\epsilon}_F"] \arrow[d, "\mathbf{F} \circ \mathbf{G}(\tau)"'] & F \arrow[d, "\tau"] \\
\mathbf{F} \circ \mathbf{G}(G) \arrow[r, "\boldsymbol{\epsilon}_G"'] & G
\end{tikzcd}$$

commutes, so that $\boldsymbol{\epsilon}$ is indeed natural.

Finally, we prove the equations showing that we have an adjunction.

$$\begin{aligned}
(\mathbf{G} * \boldsymbol{\epsilon}) \circ (\boldsymbol{\eta} * \mathbf{G})_F &= \mathbf{G}(\boldsymbol{\epsilon}_F) \circ \boldsymbol{\eta}_{\mathbf{G}(F)} \\
&= F(\rho_I) \circ st^F_{I,F(I)} \circ \lambda^{-1}_{F(I)} \\
&= F(\rho_I) \circ F(\lambda_I^{-1}) \text{ by the strength triangle} \\
&= F(\lambda_I) \circ F(\lambda_I^{-1}) \text{ by coherence} \\
&= \mathrm{id}_{F(I)} = (\mathrm{id}_{\mathbf{G}})_F
\end{aligned}$$

$$
\begin{aligned}
(\boldsymbol{\epsilon} * \mathbf{F}) \circ (\mathbf{F} * \boldsymbol{\eta})_A &= \boldsymbol{\epsilon}_{\mathbf{F}(A)} \circ \mathbf{F}(\boldsymbol{\eta}_A) \\
&= \mathbf{F}(A)(\rho) \circ st^{\mathbf{F}(A)}_{,I} \circ (- \otimes \boldsymbol{\eta}_A) \\
&= (\rho_- \otimes \mathrm{id}_A) \circ \alpha_{-,I,A} \circ (- \otimes \lambda_A^{-1}) \\
&= - \otimes \mathrm{id}_A \text{ by the triangle diagram} \\
&= \mathrm{id}_{\mathbf{F}(A)} = (\mathrm{id}_{\mathbf{F}})_A
\end{aligned}
$$

□

We call this the strength adjunction. Our results consist of lifting this and its dual in various circumstances.

Chapter 5

Monads and monoids

It might so happen that the underlying functor of a monad is a strong functor. To make this situation interesting, we have to require for the strength map to interact with the monad structure. This results in the notion of a strong monad.

Definition 5.1. Let $\mathbf{C}$ be a monoidal category. Then a *strong monad* on $\mathbf{C}$ is a tuple (T, μ, η, st) such that (T, μ, η) is a monad on $\mathbf{C}$, (T, st) is a strong functor, $st \circ (\mathrm{id} \otimes \eta) = \eta$ and $\mu \circ T(st) \circ st = st \circ (\mathrm{id} \otimes \mu)$, or in diagrams

$$\begin{tikzcd}
A \otimes B \arrow[r, "\mathrm{id}_A \otimes \eta_B"] \arrow[dr, "\eta_{A\otimes B}"'] & A \otimes T(B) \arrow[d, "st_{A,B}"] \\
& T(A \otimes B)
\end{tikzcd}$$

and

$$\begin{tikzcd}
A \otimes TT(B) \arrow[r, "st_{A,T(B)}"] \arrow[d, "\mathrm{id}_A \otimes \mu_B"'] & T(A \otimes T(B)) \arrow[r, "T(st_{A,B})"] & TT(A \otimes B) \arrow[d, "\mu_{A\otimes B}"] \\
A \otimes T(B) \arrow[rr, "st_{A,B}"'] & & T(A \otimes B)
\end{tikzcd}$$

commute. A morphism of strong monads is a natural transformation, which is a morphism of the underlying monads and the underlying strong functors. We denote the category of strong monads by $\mathrm{strMonad}(\mathbf{C})$.

The powerset monad on **Set** is strong: just define $st_{A,B} \colon A \times \mathcal{P}(B) \to \mathcal{P}(A \times B)$ by $st_{A,B}(a, X) = \{a\} \times X$. In fact, every monad on **Set** is strong.

As another example, given a monad (T, μ, η) on $\mathbf{C}$ we can lift it to a strong monad on $\mathbf{C}^*$: as the operation $(\sigma\colon F \to G) \mapsto (\sigma^*\colon F^* \to G^*)$ is functorial and preserves the Godement product, (T^*, μ^*, η^*) is a monad on $\mathbf{C}^*$. We already know that setting $st_{\bar{A},\bar{B}} = \eta_{\bar{A}} \otimes \mathrm{id}_{\bar{B}}$ makes T into a strong functor. To see that it also a strong monad, note that the axiom concerning η and strength boils down to the fact that

$$\begin{tikzcd}
T(A_1) & \ldots & T(A_n) & T(B_1) & \ldots & T(B_m) \\
A_1 \arrow[u, "\eta_{A_1}"] & \ldots & A_n \arrow[u, "\eta_{A_n}"] & T(B_1) \arrow[u, "\mathrm{id}"] & \ldots & T(B_m) \arrow[u, "\mathrm{id}"] \\
A_1 \arrow[u, "\mathrm{id}"] & \ldots & A_n \arrow[u, "\mathrm{id}"] & B_1 \arrow[u, "\eta_{B_1}"] & \ldots & B_m \arrow[u, "\eta_{B_m}"]
\end{tikzcd}$$

equals

$$\begin{tikzcd}
T(A_1) & \ldots & T(A_n) & T(B_1) & \ldots & T(B_m) \\
A_1 \arrow[u, "\eta_{A_1}"] & \ldots & A_n \arrow[u, "\eta_{A_n}"] & B_1 \arrow[u, "\eta_{B_1}"] & \ldots & B_m \arrow[u, "\eta_{B_m}"]
\end{tikzcd}$$

To see that also the other axiom holds, note that

$$\begin{tikzcd}
T(A_1) & \ldots & T(A_n) & T(B_1) & \ldots & T(B_m) \\
T^2(A_1) \arrow[u, "\mu_{A_1}"] & \ldots & T^2(A_n) \arrow[u, "\mu_{A_n}"] & T^2(B_1) \arrow[u, "\mu_{B_1}"] & \ldots & T^2(B_m) \arrow[u, "\mu_{B_m}"] \\
T(A_1) \arrow[u, "T(\eta_{A_1})"] & \ldots & T(A_n) \arrow[u, "T(\eta_{A_n})"] & T^2(B_1) \arrow[u, "\mathrm{id}"] & \ldots & {}^2(B_m) \arrow[u, "\mathrm{id}"] \\
A_1 \arrow[u, "\eta_{A_1}"] & \ldots & A_n \arrow[u, "\eta_{A_n}"] & T^2(B_1) \arrow[u, "\mathrm{id}"] & \ldots & T^2(B_m) \arrow[u, "\mathrm{id}"]
\end{tikzcd}$$

equals

$$\begin{tikzcd}
T(A_1) & \dots & T(A_n) & T(B_1) & \dots & T(B_m) \\
A_1 \arrow[u, "\eta_{A_1}"] & \dots & A_n \arrow[u, "\eta_{A_n}"] & T(B_1) \arrow[u, "\mathrm{id}"] & \dots & T(B_m) \arrow[u, "\mathrm{id}"] \\
A_1 \arrow[u, "\mathrm{id}"] & \dots & A_n \arrow[u, "\mathrm{id}"] & T^2(B_1) \arrow[u, "\mu_{B_1}"] & \dots & T^2(B_m) \arrow[u, "\mu_{B_m}"]
\end{tikzcd}$$

because $\mu \circ T(\eta) = \mathrm{id}_T$ by the monad axioms.

The rest of the chapter is dedicated to lifting the strength adjunction to one between monoids and strong monoids. We begin with considering how to lift the functor $\mathbf{F}$.

Given a monoid (M, m, u) in $\mathbf{C}$, is there a way to use m and u to get a (strong) monad structure on the functor $\mathbf{F}(M) = - \otimes M$? As $\mathbf{F}(M) \circ \mathbf{F}(M) = (- \otimes M) \otimes M$, an obvious choice for the multiplication is $\mu_A := (\mathrm{id}_A \otimes m) \circ \alpha^{-1}_{A,M,M}$ and for the unit $\eta_A := (\mathrm{id}_A \otimes u) \circ \rho^{-1}_A$. To see that this indeed gives a strong monad structure on $\mathbf{F}(M)$, we need to check that the multiplication and the unit are morphisms of strong functors, that the monad laws are satisfied and that the the monad structure interacts with the strength giving us a strong monad. For the first one, it suffices to check that $\alpha^{-1}_{-,M,M}$ and ρ^{-1} are morphisms of strong functors, as we've already seen $- \otimes f$ to be a morphism of strong functors for any morphism f. As $st^{G \circ F} = G(st^F) \circ st^G$, this boils down to checking that

$$\begin{tikzcd}
A \otimes ((B \otimes M) \otimes M) \arrow[r, "(\alpha_{A,B,M} \otimes \mathrm{id}_M) \circ \alpha_{A,B \otimes M,M}"] \arrow[d, "\mathrm{id}_A \otimes \alpha^{-1}_{B,M,M}"'] & ((A \otimes B) \otimes M) \otimes M \arrow[d, "\alpha^{-1}_{A \otimes B,M,M}"] \\
A \otimes (B \otimes (M \otimes M)) \arrow[r, "\alpha_{A,B,M \otimes M}"'] & (A \otimes B) \otimes (M \otimes M)
\end{tikzcd}$$

and

$$\begin{tikzcd}
A \otimes B \arrow[r, "\mathrm{id}_{A,B}"] \arrow[d, "\mathrm{id}_A \otimes \rho^{-1}_B"'] & A \otimes B \arrow[d, "\rho^{-1}_{A \otimes B}"] \\
A \otimes (B \otimes I) \arrow[r, "\alpha_{A,B,I}"'] & (A \otimes B) \otimes I
\end{tikzcd}$$

commute, which follows from coherence.

Then we check the monad laws. Consider the diagram

$$\begin{tikzcd}
 & (A \otimes M) \otimes I \arrow[r, "\mathrm{id} \otimes u"] \arrow[d, "\alpha^{-1}"'] & (A \otimes M) \otimes M \arrow[d, "\alpha^{-1}"] \\
A \otimes M \arrow[ur, "\rho^{-1}"] \arrow[r, "\mathrm{id} \otimes \rho^{-1}"] \arrow[drr, "\mathrm{id}"'] & A \otimes (M \otimes I) \arrow[r, "\mathrm{id} \otimes (\mathrm{id} \otimes u)"] & A \otimes (M \otimes M) \arrow[d, "\mathrm{id} \otimes m"] \\
 & & A \otimes M
\end{tikzcd}$$

The triangle on the left commutes by coherence, the rectangle commutes as α^{-1} is natural and the lower triangle commutes as M is a monoid. Hence the whole diagram commutes establishing that $\mu_A \circ (\eta * \mathbf{F}(M))_A = (\mathrm{id}_{\mathbf{F}(M)})_A$. As A was arbitrary, this means that $\mu \circ (\eta * \mathbf{F}(M)) = \mathrm{id}_{\mathbf{F}(M)}$.

The diagram

$$\begin{tikzcd}
 & (A \otimes I) \otimes M \arrow[r, "(\mathrm{id} \otimes u) \otimes \mathrm{id}"] \arrow[d, "\alpha^{-1}"'] & (A \otimes M) \otimes M \arrow[d, "\alpha^{-1}"] \\
A \otimes M \arrow[ur, "\rho^{-1} \otimes \mathrm{id}"] \arrow[r, "\mathrm{id} \otimes \lambda^{-1}"] \arrow[drr, "\mathrm{id}"'] & A \otimes (I \otimes M) \arrow[r, "\mathrm{id} \otimes (u \otimes \mathrm{id})"] & A \otimes (M \otimes M) \arrow[d, "\mathrm{id} \otimes m"] \\
 & & A \otimes M
\end{tikzcd}$$

commutes for similar reasons, proving that the other unit law $\mu \circ (\mathbf{F}(M) * \eta) = \mathrm{id}_{\mathbf{F}(M)}$ holds.

To see that μ is associative, consider the diagram

$$\begin{tikzcd}
((A \otimes M) \otimes M) \otimes M \arrow[r, "\alpha^{-1} \otimes \mathrm{id}"] \arrow[dd, "\alpha^{-1}"'] & (A \otimes (M \otimes M)) \otimes M \arrow[r, "(\mathrm{id} \otimes m) \otimes \mathrm{id}"] \arrow[d, "\alpha^{-1}"] & (A \otimes M) \otimes M \arrow[d, "\alpha^{-1}"] \\
& A \otimes ((M \otimes M) \otimes M) \arrow[r, "\mathrm{id} \otimes (m \otimes \mathrm{id})"] \arrow[d, "\mathrm{id} \otimes \alpha^{-1}"] & A \otimes (M \otimes M) \arrow[dd, "\mathrm{id} \otimes m"] \\
(A \otimes M) \otimes (M \otimes M) \arrow[r, "\alpha^{-1}"] \arrow[d, "\mathrm{id} \otimes m"'] & A \otimes (M \otimes (M \otimes M)) \arrow[d, "\mathrm{id} \otimes (\mathrm{id} \otimes m)"] & \\
(A \otimes M) \otimes M \arrow[r, "\alpha^{-1}"'] & A \otimes (M \otimes M) \arrow[r, "\mathrm{id} \otimes m"'] & A \otimes M
\end{tikzcd}$$

The rectangles in the bottom left and top right corners commute as α^{-1} is natural. The rectangle in the upper left corner commutes by coherence, and the remaining square by the associativity of m. Hence the whole square commutes, showing that

$$\mu \circ (\mathbf{F}(M) * \mu) = \mu \circ (\mu * \mathbf{F}(M))$$

i.e. that μ is associative. Thus $\mathbf{F}(M)$ is a monad.

To see that the monad is strong, we first check that $st \circ (\mathrm{id} \otimes \eta) = \eta$. Consider the diagram

$$\begin{tikzcd}
A \otimes B \arrow[r, "\mathrm{id} \otimes \rho^{-1}"] \arrow[dr, "\rho^{-1}"'] & A \otimes (B \otimes I) \arrow[r, "\mathrm{id} \otimes (\mathrm{id} \otimes u)"] \arrow[d, "\alpha"] & A \otimes (B \otimes M) \arrow[d, "\alpha"] \\
& (A \otimes B) \otimes I \arrow[r, "(\mathrm{id} \otimes \mathrm{id}) \otimes u"'] & (A \otimes B) \otimes M
\end{tikzcd}$$

The triangle on the left commutes by coherence and the rectangle on the right by naturality of α. Hence the whole diagram commutes showing that $st \circ (\mathrm{id} \otimes \eta) = \eta$.

Then we prove that $\mu \circ T(st) \circ st = st \circ (\mathrm{id} \otimes \mu)$ by considering the diagram

$$\begin{tikzcd}
A \otimes ((B \otimes M) \otimes M) \arrow[r, "\alpha"] \arrow[d, "\mathrm{id} \otimes \alpha^{-1}"'] & (A \otimes (B \otimes M)) \otimes M \arrow[r, "\alpha \otimes \mathrm{id}"] & ((A \otimes B) \otimes M) \otimes M \arrow[d, "\alpha^{-1}"] \\
A \otimes (B \otimes (M \otimes M)) \arrow[rr, "\alpha"] \arrow[d, "\mathrm{id} \otimes (\mathrm{id} \otimes m)"'] & & (A \otimes B) \otimes (M \otimes M) \arrow[d, "\mathrm{id} \otimes m"] \\
A \otimes (B \otimes M) \arrow[rr, "\alpha"'] & & (A \otimes B) \otimes M
\end{tikzcd}$$

The upper rectangle commutes by coherence and the lower as α is natural. Hence the outer rectangle commutes, implying $\mu \circ T(st) \circ st = st \circ (\mathrm{id} \otimes \mu)$.

Furthermore, if $f : M \to N$ is a monoid homomorphism, then $- \otimes f$ is morphism of strong functors for which the diagrams

$$\begin{tikzcd}
A \arrow[r, "\rho^{-1}"] & A \otimes I \arrow[r, "\mathrm{id} \otimes u_M"] \arrow[dr, "\mathrm{id} \otimes u_N"'] & A \otimes M \arrow[d, "\mathrm{id} \otimes f"] \\
& & A \otimes N
\end{tikzcd}$$

and

$$\begin{tikzcd}
(A \otimes M) \otimes M \arrow[r, "\alpha^{-1}"] \arrow[d, "(\mathrm{id} \otimes f) \otimes f"'] & A \otimes (M \otimes M) \arrow[d, "\mathrm{id} \otimes m_M"] \arrow[ddl, "\mathrm{id} \oplus (f \oplus f)"'] \\
(A \otimes N) \otimes N \arrow[d, "\alpha^{-1}"'] & A \otimes M \arrow[d, "\mathrm{id} \otimes f"] \\
A \otimes (N \otimes N) \arrow[r, "\mathrm{id} \otimes m_N"'] & A \otimes N
\end{tikzcd}$$

commute. This means that $- \otimes f \circ \eta^M = \eta^N$ and $\mu^N \circ (- \otimes f) \otimes f = (- \otimes f) \circ \mu^M$ so that $- \otimes f$ is a morphism of strong monads. Hence we've shown that the map $\mathbf{F}_{\mathrm{Mon}}$ defined by $\mathbf{F}_{\mathrm{Mon}}(M, m, u) = (- \otimes M, \mu^M, \eta^M, st^{\mathbf{F}(M)})$ and $\mathbf{F}_{\mathrm{Mon}}(f) = - \otimes f$ is a functor $\mathrm{Mon}(\mathbf{C}) \to \mathrm{strMonad}(\mathbf{C})$.

Going in the other direction we lift $\mathbf{G}$ by showing that for any strong monad (T, μ, η, st) the maps $m := \mu_I \circ T(\rho_{T(I)}) \circ st_{T(I),I}$ and $u := \eta_I$ define a monoid structure on $T(I)$, and that for any morphism $\sigma : S \to T$ of strong monads, σ_I is a monoid homomorphism. Essentially the monoid axioms follow from the monad ones.

We begin with the unit laws. Consider the diagram

$$\begin{tikzcd}
I \otimes T(I) \arrow[r, "\eta \otimes \mathrm{id}"] \arrow[dr, "st"] \arrow[ddr, "\lambda"'] & T(I) \otimes T(I) \arrow[dr, "st"] & \\
& T(I \otimes I) \arrow[r, "T(\eta \otimes \mathrm{id})"] \arrow[d, "T(\rho)"] & T(T(I) \otimes I) \arrow[d, "T(\rho)"] \\
& T(I) \arrow[r, "T(\eta)"] \arrow[dr, "\mathrm{id}"'] & T^2(I) \arrow[d, "\mu"] \\
& & T(I)
\end{tikzcd}$$

The parrallelogram on the top commutes as st is natural. The triangle on the left commutes as $\rho_I = \lambda_I$ by coherence and T is a strong functor. The rectangle in the middle commutes as ρ is natural. Finally, the triangle on the bottom commutes as T is a monad. Hence the whole diagram commutes giving us

$$m \circ (u \otimes \mathrm{id}_{T(I)}) \circ \lambda^{-1}_{T(I)} = \mathrm{id}_{T(I)}$$

For the other unit law, consider the diagram

$$\begin{tikzcd}
& T(I) \otimes T(I) \arrow[d, "st"] \\
T(I) \otimes I \arrow[ur, "\mathrm{id} \otimes \eta"] \arrow[r, "\eta"] \arrow[d, "\rho"'] & T(T(I) \otimes T(I) \arrow[d, "T(\rho)"] \\
T(I) \arrow[r, "\eta"] \arrow[dr, "\mathrm{id}"'] & T^2(I) \arrow[d, "\mu"] \\
& T(I)
\end{tikzcd}$$

The triangle on the top commutes as T is a strong monad. The rectangle in the middle commutes as η is natural, and the triangle on the bottom commutes because

T is a monad. Hence

$$m \circ (\mathrm{id}_{T(I)} \otimes u) \circ \rho^{-1}_{T(I)} = \mathrm{id}_{T(I)}.$$

To claim that m is associative is to claim that the outer rectangle in the diagram

$$\begin{tikzcd}
T(I) \otimes (T(I) \otimes T(I)) \arrow[r, "\mathrm{id} \otimes st"] \arrow[ddddddd, "\alpha"'] \arrow[dddddr, phantom, "\text{(i)}"] & T(I) \otimes T(T(I) \otimes I) \arrow[r, "\mathrm{id} \otimes T(\rho)"] \arrow[d, "st"] \arrow[dr, phantom, "\text{(ii)}"] & T(I) \otimes T^2(I) \arrow[r, "\mathrm{id} \otimes \mu_I"] \arrow[d, "st"] \arrow[ddr, phantom, "\text{(iii)}"] & T(I) \otimes T(I) \arrow[dd, "st"] \\
 & T(T(I) \otimes (T(I) \otimes I)) \arrow[r, "T(\mathrm{id} \otimes \rho)"] \arrow[dddd, "T(\alpha)"] \arrow[dr, phantom, "\text{(iv)}"] & T(T(I) \otimes T(I)) \arrow[d, "T(st)"] & \\
 & & T^2(T(I) \otimes I) \arrow[r, "\mu_{T(I) \otimes I}"] \arrow[d, "T^2(\rho)"] \arrow[dr, phantom, "\text{(vi)}"] & T(T(I) \otimes I) \arrow[d, "T(\rho)"] \\
 & & T^3(I) \arrow[r, "\mu_{T(I)}"] \arrow[d, "T(\mu_I)"] \arrow[dr, phantom, "\text{(vii)}"] & T^2(I) \arrow[d, "T\mu_I"] \\
 & & T^2(I) \arrow[r, "\mu_I"] & T(I) \\
 & T((T(I) \otimes T(I)) \otimes I) \arrow[uuuur, "T(\rho)"] \arrow[d, "T(st \otimes \mathrm{id})"] \arrow[uur, phantom, "\text{(v)}"] & & T^2(I) \arrow[u, "\mu_I"'] \\
 & T(T(T(I) \otimes I) \otimes I) \arrow[r, "T(T(\rho) \otimes \mathrm{id})"] \arrow[dr, phantom, "\text{(ix)}"] & T(T^2(I) \otimes I) \arrow[r, "T(\mu \otimes \mathrm{id})"] \arrow[uuuu, bend left, "T(\rho)"] \arrow[uur, phantom, "\text{(viii)}"] & T(T(I) \otimes I) \arrow[u, "T(\rho)"'] \\
(T(I) \otimes T(I)) \otimes T(I) \arrow[r, "st \otimes \mathrm{id}"] \arrow[uur, "st"] & T(T(I) \otimes I) \otimes T(I) \arrow[r, "T(\rho) \otimes \mathrm{id}"] & T^2(I) \otimes T(I) \arrow[r, "\mu_I \otimes \mathrm{id}"] & T(I) \otimes T(I) \arrow[u, "st"']
\end{tikzcd}$$

commutes. Region (i) commutes as T is a strong functor. (ii) and (ix) commute because st is natural. (iii) is just the fact that T is a strong monad. (iv) is an instance of coherence. (v) commutes because ρ is natural and (vi) because μ is. (vii) is associativity of the monad T and (viii) commutes because ρ is natural. Hence the whole diagram commutes and thus m is associative.

Given a morphism $\sigma : S \to T$ of strong monads, we show that σ_I is a monoid homomorphism. By definition $\sigma \circ \eta^S = \eta^T$, so σ_I preserves the unit. Consider the diagram

$$\begin{tikzcd}
S(I)\otimes S(I) \arrow[r, "st^S"] \arrow[d, "\mathrm{id}\otimes\sigma"'] \arrow[dr, phantom, "\text{(i)}"] & S(S(I)\otimes I) \arrow[r, "S(\rho)"] \arrow[d, "\sigma"'] \arrow[dr, phantom, "\text{(ii)}"] & S^2(I) \arrow[r, "\mu^S"] \arrow[d, "\sigma"] & S(I) \arrow[dd, "\sigma"] \\
S(I)\otimes T(I) \arrow[r, "st^T"] \arrow[d, "\sigma\otimes\mathrm{id}"'] \arrow[dr, phantom, "\text{(iii)}"] & T(S(I)\otimes I) \arrow[r, "T(\rho)"] \arrow[d, "T(\sigma\otimes\mathrm{id})"'] \arrow[dr, phantom, "\text{(iv)}"] & TS(I) \arrow[d, "T(\sigma)"] \arrow[r, phantom, "\text{(v)}"] & {} \\
T(I)\otimes T(I) \arrow[r, "st^T"'] & T(T(I)\otimes I) \arrow[r, "T(\rho)"'] & T^2(I) \arrow[r, "\mu^T"'] & T(I)
\end{tikzcd}$$

(i) commutes as σ is a morphism of strong functors and (ii) because it is natural. (iii) commutes because st^T is natural and (iv) because ρ is. Finally, (v) comutes because σ is a morphism of monads. Hence the outer rectangle commutes which means that

$$\sigma_I \circ m^S = m^T \circ (\sigma_I \otimes \sigma_I)$$

so that σ_I is indeed a monoid homomorphism.

Hence the map $\mathbf{G}_{\text{Mon}}$ defined by $\mathbf{G}_{\text{Mon}}(T, \mu, \eta, st) = (T(I), m, u)$ and $\mathbf{G}_{\text{Mon}}(\sigma) = \sigma_I$ is a functor strMonad$(\mathbf{C}) \to$ Mon$(\mathbf{C})$. Define $\boldsymbol{\eta}$ and $\boldsymbol{\epsilon}$ as we defined them in the proof of theorem 4.2. To see that they are natural transformations $\boldsymbol{\eta}\colon \mathrm{id}_{\text{Mon}(\mathbf{C})} \to \mathbf{G}_{\text{Mon}} \circ \mathbf{F}_{\text{Mon}}$ and $\boldsymbol{\epsilon}\colon \mathbf{F}_{\text{Mon}} \circ \mathbf{G}_{\text{Mon}} \to \mathrm{id}$, we only have to check that $\boldsymbol{\eta}_M$ is always a monoid homomorphism and that $\boldsymbol{\epsilon}_T$ is always a morphism of strong monads.

For the former, we wish to show that λ_M^{-1} is a homomorphism from a monoid (M, m, u) to the monoid $\mathbf{G}_{\text{Mon}} \circ \mathbf{F}_{\text{Mon}}(M)$. The latter has $I \otimes M$ as its underlying object, the map $(\mathrm{id}_I \otimes u) \otimes \rho_I^{-1}$ as its unit and the composite

$$(I\otimes M)\otimes(I\otimes M) \xrightarrow{\alpha} ((I\otimes M)\otimes I)\otimes M \xrightarrow{\rho\otimes\mathrm{id}} (I\otimes M)\otimes M \xrightarrow{\alpha^{-1}} I\otimes(M\otimes M) \xrightarrow{\mathrm{id}\otimes m} I\otimes M$$

as its multiplication. λ_M^{-1} preserves the unit as $\lambda_I^{-1} = \rho_I^{-1}$ due to coherence and because

$$\begin{tikzcd}
I \arrow[r, "u"] \arrow[d, "\lambda_I^{-1}"'] & M \arrow[d, "\lambda_M^{-1}"] \\
I\otimes I \arrow[r, "\mathrm{id}\otimes u"'] & I\otimes M
\end{tikzcd}$$

commutes by naturality of λ^{-1}. Furthermore, the top half of

$$\begin{tikzcd}
M \otimes M \arrow[r, "\lambda^{-1} \otimes \lambda^{-1}"] \arrow[dddd, "m"'] \arrow[rrdd, "\lambda^{-1}"] & (I \otimes M) \otimes (I \otimes M) \arrow[r, "\alpha"] & ((I \otimes M) \otimes I) \otimes M \arrow[d, "\rho \otimes \mathrm{id}"] \\
 & & (I \otimes M) \otimes M \arrow[d, "\alpha^{-1}"] \\
 & & I \otimes (M \otimes M) \arrow[dd, "\mathrm{id} \otimes m"] \\
 & & \\
M \arrow[rr, "\lambda^{-1}"'] & & I \otimes M
\end{tikzcd}$$

commutes by coherence and the bottom half by naturality of λ^{-1} so that λ_M^{-1} is indeed a monoid homomorphism. Note that if one replaces λ^{-1} with λ throughout in the previous two diagrams, they commute nevertheless. This means that λ is also a monoid homorphism so that λ_M^{-1} is in fact an isomorphism of monoids.

Then we show that $\boldsymbol{\epsilon}_T$ is a morphism of strong monads. As we already know it to be a morphism of the underlying strong functors, it suffices to show that it is a morphism of monads. Given a strong monad (T, μ, η, st), the monad $\mathbf{F}_{\mathrm{Mon}} \circ \mathbf{G}_{\mathrm{Mon}}(T)$ has $- \otimes T(I)$ as its underlying functor, the maps

$$(A \otimes T(I)) \otimes T(I) \xrightarrow{\alpha^{-1}} A \otimes (T(I) \otimes T(I)) \xrightarrow{\mathrm{id} \otimes st} A \otimes T(T(I) \otimes I) \xrightarrow{\mathrm{id} \otimes T(\rho)} A \otimes T^2(I) \xrightarrow{\mathrm{id} \otimes \mu} A \otimes T(I)$$

as its multiplication and the maps $(\mathrm{id}_A \otimes \eta_I) \circ \rho_A^{-1}$ as its unit.

The small triangle on the right in

$$\begin{tikzcd}
A \arrow[r, "\rho^{-1}"] \arrow[dr, "\mathrm{id}"'] & A \otimes I \arrow[r, "\mathrm{id} \otimes \eta"] \arrow[d, "\rho"] \arrow[dr, "\eta"] & A \otimes T(I) \arrow[d, "st"] \\
 & A \arrow[dr, "\eta"'] & T(A \otimes I) \arrow[d, "T(\rho)"] \\
 & & T(A)
\end{tikzcd}$$

commutes as T is a strong monad, the triangle on the left commutes by definition and the parallelogram on the bottom commutes as η is natural. Hence the outermost triangle commutes showing that the unit is preserved by $\boldsymbol{\epsilon}$. To see that this holds also for the multplication, consider the diagram

$$
\begin{tikzcd}
(A\otimes T(I))\otimes T(I) \arrow[r, "\alpha^{-1}"] \arrow[d, "st"'] \arrow[dr, phantom, "\text{(i)}"] & A\otimes(T(I)\otimes T(I)) \arrow[r, "\mathrm{id}\otimes st"] & A\otimes T(T(I)\otimes I) \arrow[r, "\mathrm{id}\otimes T(\rho)"] \arrow[dl, "st"'] \arrow[d, phantom, "\text{(ii)}"] & A\otimes T^2(I) \arrow[d, "\mathrm{id}\otimes\mu"] \arrow[dl, "st"'] \\
T((A\otimes T(I))\otimes I) \arrow[r, "T(\alpha^{-1})"] \arrow[rr, bend right, "T(\rho)"'] & T(A\otimes(T(I)\otimes I)) \arrow[r, "T(\mathrm{id}\otimes\rho)"] \arrow[d, phantom, "\text{(v)}"] & T(A\otimes T(I)) \arrow[d, "T(st)"'] \arrow[dr, phantom, "\text{(iii)}"] & A\otimes T(I) \arrow[d, "st"] \\
 & & T^2(A\otimes I) \arrow[r, "\mu"] \arrow[d, "T^2(\rho)"'] \arrow[dr, phantom, "\text{(iv)}"] & T(A\otimes I) \arrow[d, "T(\rho)"] \\
 & & T^2(A) \arrow[r, "\mu"'] & T(A)
\end{tikzcd}
$$

Region (i) commutes as T is a strong functor and (ii) because st is natural. Region (iii) commutes as T is a strong monad, (iv) because μ is natural and (v) due to coherence. Hence the whole diagram commutes which means that

$$(\boldsymbol{\epsilon}_T)_A \circ \mathrm{id}_A \otimes (\mu_I \circ \rho_{T(I)} \circ st_{T(I),I}) \circ \alpha^{-1}_{A,T(I),T(I)} = \mu_A \circ (\boldsymbol{\epsilon}_T * \boldsymbol{\epsilon}_T)_A$$

so that $\boldsymbol{\epsilon}_T$ preserves the multiplication as well.

This shows that $\boldsymbol{\epsilon}_T$ is a morphism of strong monads. We've shown that $\boldsymbol{\epsilon}$ and $\boldsymbol{\eta}$ are natural transformations $\boldsymbol{\eta}\colon \mathrm{id}_{\mathrm{Mon}(\mathbf{C})} \to \mathbf{G}_{\mathrm{Mon}} \circ \mathbf{F}_{\mathrm{Mon}}$ and $\boldsymbol{\epsilon}\colon \mathbf{F}_{\mathrm{Mon}} \circ \mathbf{G}_{\mathrm{Mon}} \to \mathrm{id}$, and as they're defined by the same equations as in the proof of theorem 4.2 the exact same calculations show that we have an adjunction. Hence we've proved the following theorem:

Theorem 5.2. *The strength adjunction lifts to an adjunction between monoids in* $\mathbf{C}$ *and strong monads on* $\mathbf{C}$.

Chapter 6

Commutativity

In the case of symmetric monoidal categories, there's also a notion of commutativity for strong monads. Given a strong monad (T, μ, η, st), one can define a natural transformation st' with components of the form $T(A) \otimes B \to T(A \otimes B)$ by setting $st'_{A,B}$ to be the composite

$$T(A) \otimes B \xrightarrow{\gamma} B \otimes T(A) \xrightarrow{st} T(B \otimes A) \xrightarrow{T(\gamma)} T(A \otimes B)$$

With this at hand, one can define two possibly different natural transformations with components of the form $TA \otimes TB \to T(A \otimes B)$:

$$dst_{A,B} := \mu_{A \otimes B} \circ T(st'_{A,B}) \circ st_{T(A),B}$$
$$dst'_{A,B} := \mu_{A \otimes B} \circ T(st_{A,B}) \circ st'_{A,T(B)}$$

A strong monad is called commutative if $dst = dst'$. Commutative monads correspond to so-called monoidal monads, see. e.g. [9] and [10]. We prove next that if T is commutative, then so is the monoid structure on $T(I)$, and likewise if M is commutative, then $- \otimes M$ is as well.

Theorem 6.1. *If the monoid M is commutative, then so is the strong monad structure on $- \otimes M$*

Proof. Consider the diagram

$$\begin{tikzcd}
(A \otimes M_A) \otimes (B \otimes M_B) \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & ((A \otimes M) \otimes B) \otimes M_B \arrow[r, "\gamma \otimes \mathrm{id}"] & (B \otimes (A \otimes M_A)) \otimes M_B \arrow[d, "\alpha \otimes \mathrm{id}"] \\
(B \otimes M_B) \otimes (A \otimes M_A) \arrow[d, "\alpha"'] & & ((B \otimes A) \otimes M_A) \otimes M_B \arrow[d, "(\gamma \otimes \mathrm{id}) \otimes \mathrm{id}"] \\
((B \otimes M_B) \otimes A) \otimes M_A \arrow[d, "\gamma \otimes \mathrm{id}"'] & & ((A \otimes B) \otimes M_A) \otimes M_B \arrow[d, "\alpha^{-1}"] \\
(A \otimes (B \otimes M_B)) \otimes M_A \arrow[d, "\alpha \otimes \mathrm{id}"'] & & (A \otimes B) \otimes (M_A \otimes M_B) \arrow[d, "\mathrm{id} \otimes m"] \arrow[dl, "\mathrm{id} \otimes \gamma"'] \\
((A \otimes B) \otimes M_B) \otimes M_A \arrow[r, "\alpha^{-1}"'] & (A \otimes B) \otimes (M_B \otimes M_A) \arrow[r, "\mathrm{id} \otimes m"'] & (A \otimes B) \otimes M
\end{tikzcd}$$

where $M_A = M_B = M$, the subscripts being there just to keep track of where each copy of M is sent by γ. The composite going through the top right corner is $dst_{A,B}$, and the other composite is $dst'_{A,B}$, so it suffices to prove that the diagram commutes. But the triangle commutes as M is a commutative monoid, and the rest commutes by coherence. $\square$

Before we prove that commutativity of a strong monad T implies the commutativity of the induced monoid structure on $T(I)$ we have the following lemma:

Lemma 6.2. *For any strong monad T on a symmetric monoidal category*

(i) $T(\rho_A) \circ st'_{A,I} = \rho_{T(A)}$

(ii) $dst_{B,A} \circ \gamma_{T(A),T(B)} = T(\gamma_{A,B}) \circ dst'_{A,B}$

(iii) $T(\rho_I) \circ dst_{I,I} = \mu_I \circ T(\rho_{T(I)}) \circ st_{T(I),I}$

Proof. For (i) consider the diagram

$$\begin{tikzcd}
T(A) \otimes I \arrow[r, "\gamma"] \arrow[drrr, "\rho"'] & I \otimes T(A) \arrow[r, "st"] \arrow[drr, "\lambda"'] & T(I \otimes A) \arrow[r, "T(\gamma)"] \arrow[dr, "T(\lambda)"] & T(A \otimes I) \arrow[d, "T(\rho)"] \\
& & & T(A)
\end{tikzcd}$$

The leftmost and rightmost triangles commute by coherence and the one in the middle as T is a strong functor. Hence the whole diagram commutes proving (i).

For (ii), consider the diagram

$$\begin{tikzcd}
T(A)\otimes T(B) \arrow[r, "\gamma"] & T(B)\otimes T(A) \arrow[r, "st"] & T(T(B)\otimes A) \arrow[r, "T(\gamma)"] & T(A\otimes T(B)) \arrow[r, "T(st)"] & T^2(A\otimes B) \arrow[dl, "\mu"'] \arrow[d, "T^2(\gamma)"] \\
 & & & T(A\otimes B) \arrow[dr, "T(\gamma)"'] & T^2(B\otimes A) \arrow[d, "\mu"] \\
 & & & & T(B\otimes A)
\end{tikzcd}$$

The triangle on the right commutes by naturality of μ. Using the definitions of st', dst and dst one sees that the path along the top and right is $dst_{B,A} \circ \gamma_{T(A),T(B)}$ and the path along the left the left of the triangle is $T(\gamma_{A,B}) \circ dst'_{A,B}$, proving (ii).

Finally, to prove (iii), consider the diagram

$$\begin{tikzcd}
T(I)\otimes T(I) \arrow[r, "st"] & T(T(I)\otimes I) \arrow[r, "T(st')"] \arrow[dr, "T(\rho)"'] & T^2(I\otimes I) \arrow[r, "\mu"] \arrow[d, "T^2(\rho)"] & T(I\otimes I) \arrow[d, "T(\rho)"] \\
 & & T^2(I) \arrow[r, "\mu"'] & T(I)
\end{tikzcd}$$

The triangle commutes by part (i) and the square as μ is natural. Hence the diagram commutes proving (iii). □

We're now in a position to prove the following:

Theorem 6.3. *If T is commutative, then so is the induced monoid structure on $T(I)$*

Proof. By part (iii) of the previous lemma $T(\rho_I) \circ dst_{I,I}$ equals the multiplication on $T(I)$ so it suffices to prove that $T(\rho_I) \circ dst_{I,I} \circ \gamma_{T(I),T(I)} = T(\rho_I) \circ dst_{I,I}$. To do this, consider the diagram

$$\begin{tikzcd}
T(I) \otimes T(I) \arrow[r, "\gamma"] \arrow[d, "dst'"'] & T(I) \otimes T(I) \arrow[d, "dst"] \\
T(I \otimes I) \arrow[r, "T(\gamma)"'] \arrow[dr, "T(\lambda)"'] & T(I \otimes I) \arrow[d, "T(\rho)"] \\
& T(I)
\end{tikzcd}$$

The rectangle commutes by part (ii) of the previous lemma, and the triangle on the bottom by coherence. Hence the diagram commutes. As $\lambda_I = \rho_I$ by coherence and $dst = dst'$ by commutativity of the monad T, this implies the desired result. $\square$

Corollary 6.4. *The strength adjunction lifts to an adjunction between commutative monoids and commutative strong monads.*

Chapter 7

Costrength

One can dualize essentially everything in the previous two chapters by replacing the strength maps with costrength maps and reversing all the natural isomorphisms that are part of the monoidal data. For example, we have the notion of a costrong functor:

Definition 7.1. Given a monoidal category $\mathbf{C}$, a costrong functor $\mathbf{C} \to \mathbf{C}$ is a functor $F\colon \mathbf{C} \to \mathbf{C}$ equipped with a natural transformation cst with components

$$cst_{A,B}\colon F(A \otimes B) \to A \otimes F(B)$$

for which the diagrams

$$\begin{tikzcd}
I \otimes F(A) \arrow[dr, "\lambda_A"'] & F(I \otimes A) \arrow[l, "cst_{I,A}"'] \arrow[d, "F(\lambda_A)"] \\
& F(A)
\end{tikzcd}$$

and

$$\begin{tikzcd}
A \otimes (B \otimes F(C)) \arrow[d, "\alpha_{A,B,F(C)}"'] & A \otimes F(B \otimes C) \arrow[l, "\mathrm{id}_A \otimes cst_{B,C}"'] & F(A \otimes (B \otimes C)) \arrow[l, "cst_{A,B\otimes C}"'] \arrow[d, "F(\alpha_{A,B,C})"] \\
(A \otimes B) \otimes F(C) & & F((A \otimes B) \otimes C) \arrow[ll, "cst_{A\otimes B,C}"]
\end{tikzcd}$$

commute for every object A, B and C.

A morphism of costrong functors $F, G\colon \mathbf{C} \to \mathbf{C}$ is a natural transformation $\sigma\colon F \to G$ such that

$$\begin{tikzcd}
A \otimes F(B) \arrow[d, "\mathrm{id}_A \otimes \sigma_B"'] & F(A \otimes B)) \arrow[l, "cst^F_{A,B}"'] \arrow[d, "\sigma_{A\otimes B}"] \\
A \otimes G(B) & G(A \otimes B) \arrow[l, "cst^G_{A,B}"]
\end{tikzcd}$$

commutes. We denote the category of costrong functors $\mathbf{C} \to \mathbf{C}$ and their morphisms by $\mathrm{costr}(\mathbf{C})$.

Just as in the last chapter, for any object C the functor $- \otimes C$ is costrong. Hence we have the functors $\mathbf{F}_c\colon \mathbf{C} \to \mathrm{costr}(\mathbf{C})$ and $\mathbf{G}_c\colon \mathrm{costr}(\mathbf{C}) \to \mathbf{C}$ defined on objects by $\mathbf{F}_c(C) = - \otimes \mathbf{C}$ and $\mathbf{G}_c\colon (F) = F(I)$. Dualizing the proof of theorem 4.2 we get the theorem below. Note that now the adjunction goes the other way around. The reason for this is that while $\mathbf{G} \circ \mathbf{F}$ is naturally isomorphic with $\mathrm{id}_{\mathbf{C}}$, and likewise for $\mathbf{G}_c \circ \mathbf{F}_c$ and id_C, the natural transformation $\mathbf{F} \circ \mathbf{G} \to \mathrm{id}_{\mathrm{str}(\mathbf{C})}$ was defined using the strength maps. With costrong functors, we get a natural transformation in the opposite direction, with components $cst^F \circ F(\rho_A^{-1})\colon F(A) \to F(A \otimes I) \to A \otimes F(I)$. The following theorem results:

Theorem 7.2. $\mathbf{G}_c \dashv \mathbf{F}_c$

We call this the costrength adjunction.

Likewise one can dualize the notion of a strong monad. Here one has in fact several choices: if one replaces the multiplication and unit in the diagrams of definition 5.1 with comultiplication and counit, one gets the notion of a strong comonad. If one just replaces the strengths by costrenghts, one gets the notion of a costrong monad. If one does both, one gets the notion of a costrong comonad. To illustrate this, we give explicitly the definition of a costrong comonad:

Definition 7.3. A *costrong comonad* on $\mathbf{C}$ is a tuple $(T, \delta, \epsilon, cst)$ such that (T, δ, ϵ) is a comonad on $\mathbf{C}$, (T, cst) is a costrong functor, and the diagrams

$$\begin{tikzcd}
A \otimes B & A \otimes T(B) \arrow[l, "\mathrm{id}_A \otimes \epsilon_B"'] \\
 & T(A \otimes B) \arrow[ul, "\epsilon_{A\otimes B}"] \arrow[u, "cst_{A,B}"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
A \otimes TT(B) & T(A \otimes T(B)) \arrow[l, "cst_{A,T(B)}"'] & TT(A \otimes B) \arrow[l, "T(cst_{A,B})"'] \\
A \otimes T(B) \arrow[u, "\mathrm{id}_A \otimes \delta_B"] & & T(A \otimes B) \arrow[ll, "cst_{A,B}"] \arrow[u, "\delta_{A\otimes B}"']
\end{tikzcd}$$

commute.

A morphism of costrong comonads is a natural transformation which is a morphism of the underlying comonads and the underlying costrong functors. We denote the category of costrong monads by costrComonad($\mathbf{C}$).

Similarly as before, a comonad (T, δ, ϵ) on $\mathbf{C}$ lifts to a costrong comonad on $\mathbf{C}^*$.

Reversing all the arrows, we can define functors $\mathbf{F}_{\text{coMon}} \colon \text{Comon}(\mathbf{C}) \to \text{costrComonad}(\mathbf{C})$ and $\mathbf{G}_{\text{coMon}} \colon \text{costrComonad}(\mathbf{C}) \to \text{Comon}(\mathbf{C})$ by

$$\begin{aligned}
\mathbf{F}_{\text{cMon}}(M, d, e) &= (- \otimes M, \alpha_{-,M,M} \circ - \otimes d, \rho_{-} \circ - \otimes e, \alpha^{-1}_{-,M,M}) \\
\mathbf{F}_{\text{cMon}}(f) &= - \otimes f
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{G}_{\text{cMon}}(T, \delta, \epsilon, costr) &= (T(I), costr_{T(I),I} \circ T(\rho^{-1}_{T(I)} \circ \delta_I, \epsilon_I) \\
\mathbf{G}_{\text{cMon}}(\sigma) &= \sigma_I
\end{aligned}$$

To see that these functors are well defined, one just reverses all the arrows in the diagrams that showed that $\mathbf{F}_{\text{Mon}}$ and $\mathbf{G}_{\text{Mon}}$ were well defined. Similarly one just dualizes the proof of theorem 5.2 to show that

Theorem 7.4. *The costrength adjunction can be lifted to an adjunction between comonoids and costrong comonads*

Likewise, in a symmetric monoidal setting one has the maps $cst'_{A,B} \colon T(A \otimes B) \colon T(A) \otimes B$ defined by $\gamma_{B,T(A)} \circ cst_{B,A} \circ T(\gamma_{A,B})$ with which one can define two natural transformations $dcst, dcst'$ as follows

$$\begin{aligned}
dcst_{A,B} &:= cst_{T(A),B} \circ T(cst'_{A,B}) \circ \delta_{A\otimes B} \\
dcst'_{A,B} &:= cst'_{A,T(B)} \circ T(cst_{A,B}) \circ \delta_{A\otimes B}
\end{aligned}$$

A costrong comonad is called cocommutative in case $dcst = dcst'$. Reversing all the arrows in the relevant proofs one notices that cocommutative comonoids yield cocommutative costrong comonads and vice versa. Hence we have the following corollary.

Corollary 7.5. *The costrength adjunction can be lifted to an adjunction between cocommutative costrong comonads and cocommutative comonoids.*

Chapter 8

Strength with Frobenius

One would expect that an analogue of the previous theorems should also hold for Frobenius monoids in $\mathbf{C}$ and Frobenius monads equipped with an appropriate amount of strength and (co)strength. The least amount of strength one might assume would consist of having a Frobenius monad with strength and costrength morphisms making the underlying monad strong and the underlying comonad costrong. Indeed, this suffices to guarantee that the operation $T \mapsto T(I)$ takes the underlying monad to a monoid and the underlying comonad to a comonoid. However, one should also show that the Frobenius law is preserved. When proving this, stronger assumptions come naturally into the picture. Hence we have the following definition:

Definition 8.1. A *strong Frobenius monad* on $\mathbf{C}$ is a tuple $(T, \mu, \eta, \delta, \epsilon, st)$ such that st is a natural isomorphism, (T, μ, η, st) is a strong monad, $(T, \delta, \epsilon, st^{-1})$ is a costrong comonad and $(T, \mu, \eta, \delta, \epsilon)$ is a Frobenius monad.

A morphism of strong Frobenius monads is a natural transformation, which is a morphism of the underlying Frobenius monads and the underlying strong and costrong functors. We denote the category of strong Frobenius monads by strFrobMonad($\mathbf{C}$).

Analogously to previous chapters, we wish to define functors

$$\mathbf{F}_{\text{Frob}} \colon \text{Frob}(\mathbf{C}) \to \text{strFrobMonad}(\mathbf{C})$$
$$\mathbf{G}_{\text{Frob}} \colon \text{strFrobMonad}(\mathbf{C}) \to \text{Frob}(\mathbf{C})$$

such that $\mathbf{F}_{\text{Frob}}(M) = - \otimes M$ and $\mathbf{G}_{\text{Frob}}(F) = F(I)$ on the underlying objects. As everything else has already been checked, it suffices to prove that $\mathbf{F}_{\text{Frob}}$ and $\mathbf{G}_{\text{Frob}}$ preserve the Frobenius law. To see that $\mathbf{G}$ does so, consider the diagram 8.1 If the outer rectangle commutes, then $\mathbf{G}_{\text{Frob}}$ preserves the Frobenius law. To see that it indeed commutes, we note that region (i) commutes because T is a Frobenius monad,

$$
\begin{tikzcd}
T(I)\otimes T(T(I)\otimes I) \arrow[r, "id\otimes st^{-1}"] \arrow[dr, "st"] \arrow[ddr, phantom, "\text{(x)}"] & T(I)\otimes(T(I)\otimes T(I)) \arrow[rr, "\alpha"] \arrow[drr, phantom, "\text{(xii)}"] & & (T(I)\otimes T(I))\otimes T(I) \arrow[r, "st\otimes\mathrm{id}"] \arrow[dr, phantom, "\text{(ix)'}"] & T(T(I)\otimes I)\otimes T(I)) \arrow[dd, "T(\rho)\otimes\mathrm{id}"] \\
 & T(T(I)\otimes(T(I)\otimes I)) \arrow[r, "T(\alpha)"] \arrow[dr, phantom, "\text{(xi)}"] & T((T(I)\otimes T(I))\otimes I) \arrow[ur, "st^{-1}"] \arrow[dr, phantom, "\text{(vii)'}"] & T(T(I)\otimes I)\otimes T(I) \arrow[u, "st^{-1}\otimes\mathrm{id}"] \arrow[dr, "T(\rho)\otimes\mathrm{id}"] \arrow[d, phantom, "\text{(viii)'}"] & \\
T(I)\otimes T^2(I) \arrow[uu, "\mathrm{id}\otimes T(\rho^{-1})"] \arrow[r, "st"] & T(T(I)\otimes T(I)) \arrow[u, "T(\mathrm{id}\otimes\rho^{-1})"] \arrow[ur, "T(\rho^{-1})"] \arrow[dr, phantom, "\text{(vi)'}"] & T(T(T(I)\otimes I)\otimes I) \arrow[u, "T(st^{-1}\otimes\mathrm{id})"] \arrow[ur, "st^{-1}"] \arrow[r, "T(T(\rho)\otimes\mathrm{id})"] \arrow[dr, phantom, "\text{(iii)}"] & T(T^2(I)\otimes I) \arrow[r, "st^{-1}"] \arrow[d, "T(\mu\otimes\mathrm{id})"] \arrow[ddr, phantom, "\text{(iv)}"] & T^2(I)\otimes T(I) \arrow[dd, "\mu\otimes\mathrm{id}"] \\
 & T^2(T(I)\otimes I) \arrow[u, "T(st^{-1})"] \arrow[ur, "T(\rho^{-1})"] \arrow[r, "T^2(\rho)"] \arrow[d, phantom, "\text{(ii)}"] & T^3(I) \arrow[dr, "T(\mu_I)"] \arrow[dd, phantom, "\text{(i)}"] & T(T(I)\otimes I) \arrow[dr, "st^{-1}"] & \\
T(I)\otimes T(I) \arrow[uu, "\mathrm{id}\otimes\delta"] \arrow[dr, "st"] \arrow[dd, "\delta\otimes\mathrm{id}"] \arrow[uur, phantom, "\text{(v)'}"] \arrow[ddr, phantom, "\text{(iv)'}"] & T^2(I) \arrow[ur, "\delta_{T(I)}"] \arrow[dr, "T(\delta_I)"] \arrow[dr, phantom, "\text{(iii)'}"] & & T^2(I) \arrow[u, "T(\rho^{-1})"] \arrow[d, phantom, "\text{(ii)'}"] \arrow[ddr, phantom, "\text{(v)}"] & T(I)\otimes T(I) \\
 & T(T(I)\otimes I) \arrow[uu, bend left, "\delta"] \arrow[u, "T(\rho)"] \arrow[d, "T(\delta\otimes\mathrm{id})"] & T^3(I) \arrow[ur, "\mu_{T(I)}"] \arrow[r, "T^2(\rho^{-1})"] \arrow[dr, phantom, "\text{(vi)}"] & T^2(T(I)\otimes I) \arrow[uu, bend right, "\mu"'] & \\
T^2(I)\otimes T(I) \arrow[r, "st"] \arrow[dr, "T(\rho^{-1})\otimes\mathrm{id}"] \arrow[dd, "T(\rho^{-1})\otimes\mathrm{id}"] \arrow[ddr, phantom, "\text{(ix)}"] & T(T^2(I)\otimes I) \arrow[r, "T(T(\rho^{-1})\otimes\mathrm{id})"] \arrow[d, phantom, "\text{(viii)}"] \arrow[dr, phantom, "\text{(vii)}"] & T(T(T(I)\otimes I)\otimes I) \arrow[ur, "T(\rho)"] \arrow[dr, phantom, "\text{(xi)'}"] & T(T(I)\otimes T(I)) \arrow[u, "T(st)"] \arrow[r, "st^{-1}"] \arrow[dr, phantom, "\text{(x)'}"] & T(I)\otimes T^2(I) \arrow[uu, "\mathrm{id}\otimes\mu"] \\
 & T(T(I)\otimes I)\otimes T(I) \arrow[ur, "st"] & T((T(I)\otimes T(I))\otimes I) \arrow[u, "T(st\otimes\mathrm{id})"] \arrow[ur, "T(\rho)"] \arrow[r, "T(\alpha^{-1})"] \arrow[dr, phantom, "\text{(xii)'}"] & T(T(I)\otimes(T(I)\otimes I)) \arrow[u, "T(\mathrm{id}\otimes\rho)"] \arrow[dr, "st^{-1}"] & \\
T(T(I)\otimes I)\otimes T(I) \arrow[r, "st^{-1}\otimes\mathrm{id}"'] & (T(I)\otimes T(I))\otimes T(I) \arrow[u, "st\otimes\mathrm{id}"] \arrow[ur, "st"] \arrow[rr, "\alpha^{-1}"'] & & T(I)\otimes(T(I)\otimes T(I)) \arrow[r, "\mathrm{id}\otimes st"'] & T(I)\otimes T(T(I)\otimes I) \arrow[uu, "\mathrm{id}\otimes T(\rho)"]
\end{tikzcd}
$$

Figure 8.1: The diagram proving that $\mathbf{G}_{\mathrm{Frob}}$ preserves the Frobenius law

(ii) because δ is natural, (iii) because ρ^{-1} is natural, (iv) because st^{-1} is natural, (v) is a consequence of T being a strong monad, (vi) commutes as ρ is natural, (vii) and (viii) because st is natural, (ix) commutes trivially and (x) because st is natural. (ii)'-(x)' commute for dual reasons.

Consider now the functor $\mathbf{F}_{\text{Frob}}$. To see claim that it preserves the Frobenius law is to say that for every A the outer rectangle in the diagram 8.2 commutes. To see that it does, note that region (i) commutes because M is a Frobenius algebra. Regions (ii) and (iii) commute because α^{-1} is natural and (iv) and (v) because α is. Finally, (vi) and (vii) commute by the coherence theorem, showing that $\mathbf{F}_{\text{Frob}}$ does indeed preserve the Frobenius law.

What has been shown before implies that $\boldsymbol{\eta} := \lambda^{-1}$ is a natural transformation $\mathrm{id}_{\text{Frob}(\mathbf{C})} \to \mathbf{G}_{\text{Frob}} \circ \mathbf{F}_{\text{Frob}}$, each component of which is in fact a strict morphism of Frobenius algebras. Now consider the family of maps $\boldsymbol{\epsilon}_T := (T * \rho) \circ str^T_{-,I}$. We already know that it is a morphism of the underlying monads and strong functors. We check that it is also a morphism of the underlying costrong functors and of the underlying comonads. As it is already a morphism of the underlying strong functors and as for both of the functors $- \otimes T(I)$ and for T the strength and the costrength are inverses of each other, it's also a morphism of costrong functors. To show that $\boldsymbol{\epsilon}$ is a morphism of the underlying comonads, consider the diagram

$$\begin{tikzcd}
A \otimes T(I) \arrow[d, "st_{A,I}"'] \arrow[dr, "\mathrm{id} \otimes \epsilon_I"] & \\
T(A \otimes I) \arrow[r, "\epsilon_{A\otimes I}"'] \arrow[d, "T(\rho_A)"'] & A \otimes I \arrow[d, "\rho_A"] \\
T(A) \arrow[r, "\epsilon_A"'] & A
\end{tikzcd}$$

The lower rectangle commutes because ϵ is natural and the triangle commutes because T is a costrong monad and st is an isomorphism. Hence the whole diagram commutes, showing that $\boldsymbol{\epsilon}$ preserves the counit. To see that it preserves the comultiplication, consider the diagram 8.3:

To assert that the outer rectangle in diagram 8.3 commutes is to claim that $\delta \circ \boldsymbol{\epsilon}_T = (\boldsymbol{\epsilon} * \boldsymbol{\epsilon})_T \circ \delta^*$, where δ^* is the comultiplication of $- \otimes T(I)$. In the diagram, (i) commutes because T is a strong comonad, (ii) commutes by definition, (iii) commutes as st is

$$
\begin{tikzcd}
(A\otimes M)\otimes M \arrow[r, "(\mathrm{id}\otimes d)\otimes \mathrm{id}"] \arrow[dr, "\alpha^{-1}"'] \arrow[dd, "\mathrm{id}\otimes d"'] & (A\otimes (M\otimes M))\otimes M \arrow[rr, "\alpha\otimes \mathrm{id}"] \arrow[dr, "\alpha^{-1}"] & & ((A\otimes M)\otimes M)\otimes M \arrow[r, "\alpha^{-1}"] & (A\otimes M)\otimes (M\otimes M) \arrow[ddddd, "\mathrm{id}\otimes m"] \\
& A\otimes (M\otimes M) \arrow[r, "\mathrm{id}\otimes (d\otimes \mathrm{id})"'] \arrow[ddd, "\mathrm{id}\otimes (\mathrm{id}\otimes d)"] & A\otimes ((M\otimes M)\otimes M) \arrow[r, "\mathrm{id}\otimes \alpha^{-1}"'] \arrow[ddd, phantom, "\text{(i)}"] & A\otimes (M\otimes (M\otimes M)) \arrow[ur, "\alpha"] \arrow[ddd, "\mathrm{id}\otimes (\mathrm{id}\otimes m)"'] & \\
(A\otimes M)\otimes (M\otimes M) \arrow[d, "\alpha"'] \arrow[ddr, "\alpha^{-1}"] & & & & \\
((A\otimes M)\otimes M)\otimes M \arrow[dd, "\alpha^{-1}\otimes \mathrm{id}"'] & & & & \\
& A\otimes (M\otimes (M\otimes M)) \arrow[r, "\mathrm{id}\otimes \alpha"] & A\otimes ((M\otimes M)\otimes M) \arrow[r, "\mathrm{id}\otimes (m\otimes \mathrm{id})"] \arrow[dll, "\alpha"] & A\otimes (M\otimes M) \arrow[dr, "\alpha"] & \\
(A\otimes (M\otimes M))\otimes M \arrow[rrrr, "(\mathrm{id}\otimes m)\otimes \mathrm{id}"'] & & & & (A\otimes M)\otimes M
\arrow[from=1-2, to=2-2, phantom, "\text{(ii)}"]
\arrow[from=2-1, to=3-2, phantom, "\text{(iii)}"]
\arrow[from=1-3, to=2-4, phantom, "\text{(vii)}"]
\arrow[from=4-1, to=5-2, phantom, "\text{(vi)}"]
\arrow[from=5-3, to=6-4, phantom, "\text{(iv)}"]
\arrow[from=2-5, to=5-5, phantom, "\text{(v)}"]
\end{tikzcd}
$$

Diagram 8.2: The diagram proving that $\mathbf{F}_{\mathrm{Frob}}$ preserves the Frobenius law

$$
\begin{tikzcd}
A \otimes T(I) \arrow[r, "\mathrm{id} \otimes \delta"] \arrow[dddd, "st"'] & A \otimes T^2(I) \arrow[r, "\mathrm{id} \otimes T(\rho^{-1})"] \arrow[d, "\mathrm{id}"] & A \otimes T(T(I) \otimes I) \arrow[r, "\mathrm{id} \otimes st^{-1}"] \arrow[d, phantom, "\text{(ii)}"] & A \otimes (T(I) \otimes T(I)) \arrow[dl, "\mathrm{id} \otimes st"] \arrow[d, "\alpha"] \\
& A \otimes T^2(I) \arrow[ddrr, bend right, "\mathrm{st}"'] \arrow[d, phantom, "\text{(i)}"] \arrow[dr, phantom, "\text{(iii)}"] & A \otimes T(T(I) \otimes I) \arrow[l, "\mathrm{id} \otimes T(\rho)"'] \arrow[d, "st"] \arrow[dr, phantom, "\text{(iv)}"] & (A \otimes T(I)) \otimes T(I) \arrow[d, "st"] \\
& {} & T(A \otimes (T(I) \otimes I)) \arrow[r, "T(\alpha)"] \arrow[dr, "T(\mathrm{id} \otimes \rho)"'] & T((A \otimes T(I)) \otimes I) \arrow[d, "T(\rho)"] \\
& & & T(A \otimes T(I)) \arrow[d, "T(st)"] \\
T(A \otimes I) \arrow[rrr, "\delta"'] \arrow[d, "T(\rho)"] \arrow[drrr, phantom, "\text{(vi)}"] & & & T^2(A \otimes I) \arrow[d, "T^2(\rho)"] \\
T(A) \arrow[rrr, "\delta"'] & & & T^2(A)
\end{tikzcd}
$$

Figure 8.3: The diagram proving that $\boldsymbol{\epsilon}$ preserves the comultiplication

natural, (iv) because T is a strong functor, (v) by coherence and finally (vi) by naturality of δ. Hence $\boldsymbol{\epsilon}$ preserves the comultiplication.

As every component of both $\boldsymbol{\epsilon}$ and $\boldsymbol{\eta}$ are strict morphisms of Frobenius algebras, they are also isomorphisms by lemma 2.17. Hence both $\boldsymbol{\epsilon}$ and $\boldsymbol{\eta}$ are natural isomorphisms. We've in fact proved the bulk of our main result.

Theorem 8.2. *The strength adjunction lifts to an equivalence between Frobenius algebras and strong Frobenius monads.*

Definition 8.3. In the symmetric monoidal setting, we call a strong Frobenius monad commutative in case the underlying strong monad is commutative and the underlying costrong comonad is cocommutative. If the underlying category is dagger monoidal, a dagger strong Frobenius monad is a dagger Frobenius monad which is also strong and for which st is unitary. Notions such as dagger strong commutative special Frobenius monad mean what one would expect.

In a dagger monoidal category, for any object M the functor $- \otimes M$ commutes with the dagger. Furthermore $st = \alpha_{-,-,M}$ is unitary. If M is a dagger Frobenius algebra, the comonad structure is the dagger of the monad structure, so that $- \otimes M$ is a dagger strong Frobenius algebra. Going in the other direction, if T is a dagger strong Frobenius monad, the st being unitary and the underlying category being dagger monoidal implies that the comonoid structure is the dagger of the monoid structure, so that $T(I)$ is dagger Frobenius.

Furthermore, if (M, m, u, d, e) is special, then for any A the diagram

$$\begin{tikzcd}
A \otimes M \arrow[r, "\mathrm{id} \otimes d"] \arrow[d, "\mathrm{id}"'] & A \otimes (M \otimes M) \arrow[r, "\alpha"] \arrow[d, "\mathrm{id}"'] & (A \otimes M) \otimes M \arrow[dl, "\alpha^{-1}"] \\
A \otimes M & A \otimes (M \otimes M) \arrow[l, "\mathrm{id} \otimes m"] &
\end{tikzcd}$$

commutes showing that $- \otimes M$ is special. On the other hand, if $(T, \mu, \eta, \delta, \epsilon, st)$ is a strong special Frobenius monad, then the commutativity of the diagram

$$\begin{tikzcd}
T(I) \arrow[r, "\delta"] \arrow[d, "\mathrm{id}"'] & T^2(I) \arrow[r, "T(\rho^{-1})"] \arrow[d, "\mathrm{id}"'] & T(T(I) \otimes I) \arrow[r, "st^{-1}"] \arrow[d, "\mathrm{id}"'] & T(I) \otimes T(I) \arrow[dl, "st"] \\
T(I) & T^2(I) \arrow[l, "\mu"] & T(T(I) \otimes I) \arrow[l, "T(\rho)"] &
\end{tikzcd}$$

implies that $T(I)$ is a special Frobenius algebra.

These remarks, along with earlier results give us the rest of our main result:

Corollary 8.4. *The strength adjunction lifts to an equivalence between special Frobenius algebras and strong special Frobenius monads. If* $\mathbf{C}$ *is symmetric, the strength adjunction lifts to an equivalence between commutative (special) Frobenius algebras and strong commutative (special) Frobenius monads. If* $\mathbf{C}$ *is dagger monoidal, it lifts to an equivalence between dagger (special) Frobenius algebras and dagger strong (special) Frobenius monads. If* $\mathbf{C}$ *is dagger symmetric monoidal, it lifts to an equivalence between commutative dagger (special) Frobenius algebras and commutative dagger (special) Frobenius monads.*

In definition 8.5 the assumption that st is a natural isomorphism might seem unreasonably strong. To show that one cannot hope to prove something like the previous theorem, say with $\mathbf{F}_{\text{Frob}}$ and $\mathbf{G}_{\text{Frob}}$ forming an adjoint pair instead of an equivalence, we proceed as follows: we discuss the strongest obvious weakening of definition 8.5, and produce an example with that definition at play where the morphism ϵ fails to preserve comultiplication. This means that the components of ϵ aren't morphisms in the relevant category. As the choice for the counit is unique once the functors are chosen, we know that a similar adjunction fails with the weaker definition. Dually, one could with more or less the same example show that the variant of ϵ using cst instead of st fails to preserve the multiplication, so the adjunction fails either way.

The obvious weakening of definition 8.5 is the following:

Definition 8.5. A *rather strong Frobenius monad* on $\mathbf{C}$ is a tuple $(T, \mu, \eta, \delta, \epsilon, st, cst)$ such that (T, μ, η, st) is a strong monad, $(T, \delta, \epsilon, cst)$ is a costrong comonad and $(T, \mu, \eta, \delta, \epsilon)$ is a Frobenius monad.

A morphism of rather strong Frobenius monads is a natural transformation, which is a morphism of the underlying Frobenius monads and the underlying strong and costrong functors. We denote the category of rather strong Frobenius monads by rstrFrobMonad($\mathbf{C}$).

To produce a counterexample, consider a Frobenius monad $(T, \mu.\eta, \delta, \epsilon)$ on a category $\mathbf{C}$. As we've already seen, the monad part lifts to a strong monad on $\mathbf{C}^*$ and the comonad part to a costrong comonad. Furthermore, as $-^*$ is functorial and preserves the Godement product, the Frobenius law is also preserved. Now, unwinding the construction, the comultiplication on $- \otimes T^*(I)$ has as its component on an object $\bar{A}$ the identity on $\bar{A}$, and on the other hand $(\epsilon_T)_{\bar{A}} = \eta(\bar{A})$. Hence the question whether

ϵ preserves multiplication boils down to wether $T(\eta_A) \circ \eta_A = \delta_A \circ \eta_A$. If for any A this doesn't hold, ϵ isn't a morphism of the underlying comonads. If C is a monoidal category and (M, m, u, d, e) is a Frobenius algebra for which $u \otimes u \circ \rho_I^{-1} \neq d \circ u$, then it is easy to check that $- \otimes M$ is such a Frobenius monad. Hence to provide a counterexample it suffices to find such an Frobenius algebra. For this purpose, let G be any non-trivial group, and consider the Frobenius algebra $(G, m, u, m^\dagger, u^\dagger)$ in **Rel** induced by the group structure. Then $u \otimes u \circ \rho_I^{-1}$ is the relation $\{(*, (1_G, 1_G))\}$ but $m^\dagger \circ u = \{(*, (g, g^{-1})) \mid g \in G\}$ which is different as G is nontrivial, giving us a counterexample.

Appendix A

Strict morphisms are invertible

We prove lemma 2.17:

Proof. Let $f\colon M \to N$ be a strict morphism between Frobenius algebras (M, m_M, u_M, d_M, e_M) and (N, m_N, u_N, d_N, e_N). Define $g\colon N \to M$ by

$$g := \lambda_M \circ (e_N \otimes \mathrm{id}) \circ (m_N \circ \mathrm{id}) \circ \alpha \circ \mathrm{id} \otimes (f \otimes \mathrm{id}) \circ (\mathrm{id} \otimes d_M) \circ (\mathrm{id} \otimes u_M) \circ \rho_N^{-1}.$$

We wish to show that

(i) $f \circ g = \mathrm{id}_N$ and $g \circ f = \mathrm{id}_M$

(ii) $e_M \circ g = e_N$ and $g \circ u_N = u_M$

(iii) $g \circ m_N = m_M \circ (g \otimes g)$ and $(g \otimes g) \circ d_N = d_M \circ e$

Of each of these three statements we just prove the first half, the other one being similar. First we show that $f \circ g = \mathrm{id}_N$ by proving that the diagram A.1 commutes Region (i) commutes by coherence. (ii) commutes because $\otimes$ is a functor $\mathbf{C} \times \mathbf{C} \to \mathbf{C}$. (iii) commutes because f is a monoid homomorphism and (iv) because f is a comonoid homomorphism. (v) commutes by functoriality of $\otimes$. (vi) and (vii) commute because α^{-1} is natural. (viii) commutes because N is a comonad and (ix) is just the Frobenius law of N. (x) commutes because α is natural and (xi) commutes because N is a monoid. (xii) and (xiii) follow from the fact that $\otimes$ is a functor, and finally (xiv) commutes as λ is natural.

Then we show that $e_M \circ g = e_N$ by proving that the diagram A.2 commutes. (i) commutes because N is a monoid and (ii) as f is a monoid homomorphism. (iii) commutes because ρ^{-1} is natural. (iv) commutes because M is a comonoid. The commutativity of (v) follows from functoriality of $\otimes$. (vi) commutes because α is natural and (vii) commutes by coherence. (viii) again is naturality of ρ^{-1} and (ix)

$$
\begin{tikzcd}
N \arrow[r, "\rho^{-1}"] \arrow[ddddd, "\mathrm{id}"'] & N \otimes I \arrow[r, "\mathrm{id} \otimes u_M"] \arrow[dd, "\lambda^{-1} \otimes \mathrm{id}"'] \arrow[dr, "\mathrm{id} \otimes u_N"'] \arrow[dr, phantom, "\text{(iii)}" near start, shift left=4] & N \otimes M \arrow[r, "\mathrm{id} \otimes d_M"] \arrow[d, "\mathrm{id} \otimes f"'] \arrow[dr, phantom, "\text{(iv)}"] & N \otimes (M \otimes M) \arrow[r, "\mathrm{id} \otimes (f \otimes \mathrm{id})"] \arrow[d, "\mathrm{id} \otimes (f \otimes f)"'] \arrow[dr, phantom, "\text{(v)}"] & N \otimes (N \otimes M) \arrow[dl, "\mathrm{id} \otimes (\mathrm{id} \otimes f)"] \arrow[d, "\alpha"] \\
{} \arrow[r, phantom, "\text{(i)}"] & {} \arrow[r, phantom, "\text{(ii)}"] & N \otimes N \arrow[r, "\mathrm{id} \otimes d_N"] \arrow[d, "d_N \otimes \mathrm{id}"] \arrow[ddl, "\lambda^{-1} \otimes \mathrm{id}"] \arrow[dr, phantom, "\text{(ix)}"] \arrow[d, phantom, "\text{(viii)}", shift right=8] & N \otimes (N \otimes N) \arrow[d, "\alpha"'] \arrow[r, phantom, "\text{(x)}"] & (N \otimes N) \otimes M \arrow[dl, "\mathrm{id} \otimes f"'] \arrow[d, "m_N \otimes \mathrm{id}"] \\
& (I \otimes N) \otimes I \arrow[d, "\mathrm{id} \otimes u_N"'] \arrow[dd, bend right=60, "\alpha^{-1}"'] & (N \otimes N) \otimes N \arrow[dl, "(e_N \otimes \mathrm{id}) \otimes \mathrm{id}"] \arrow[d, "\alpha^{-1}"] & (N \otimes N) \otimes N \arrow[d, "m_N \otimes \mathrm{id}"'] \arrow[dr, phantom, "\text{(xiii)}"] & N \otimes M \arrow[d, "e_N \otimes \mathrm{id}"] \\
& (I \otimes N) \otimes N \arrow[dr, "\alpha^{-1}"] \arrow[d, phantom, "\text{(vi)}"] \arrow[r, phantom, "\text{(vii)}"] & N \otimes (N \otimes N) \arrow[r, "\mathrm{id} \otimes m_N"] \arrow[d, "e_N \otimes \mathrm{id}"] \arrow[dr, phantom, "\text{(xii)}"] & N \otimes N \arrow[d, "e_N \otimes \mathrm{id}"'] & I \otimes M \arrow[dl, "\mathrm{id} \otimes f"'] \arrow[d, "\lambda"] \\
& I \otimes (N \otimes I) \arrow[r, "\mathrm{id} \otimes (\mathrm{id} \otimes u_N)"] \arrow[rr, bend right=20, "\mathrm{id} \otimes \rho"'] & I \otimes (N \otimes N) \arrow[r, "\mathrm{id} \otimes m_N"] \arrow[d, phantom, "\text{(xi)}"] & I \otimes N \arrow[dr, "\lambda"'] \arrow[r, phantom, "\text{(xiv)}"] & M \arrow[d, "f"] \\
N \arrow[rrrr, "\mathrm{id}"'] & & {} & & N
\end{tikzcd}
$$

Diagram A.1: Proof of $f \circ g = \mathrm{id}_N$

$$
\begin{tikzcd}
N \arrow[r, "\rho^{-1}"] \arrow[ddr, "\mathrm{id}"'] \arrow[dr, phantom, "\text{(i)}"] & N \otimes I \arrow[r, "\mathrm{id} \otimes u_M"] \arrow[d, "\mathrm{id} \otimes u_N"'] \arrow[dr, phantom, "\text{(ii)}"] & N \otimes M \arrow[r, "\mathrm{id} \otimes d_M"] \arrow[dl, "\mathrm{id} \otimes f"'] \arrow[d, "\mathrm{id} \otimes \rho^{-1}"] \arrow[dr, phantom, "\text{(iv)}"] & N \otimes (M \otimes M) \arrow[r, "\mathrm{id} \otimes (f \otimes \mathrm{id})"] \arrow[dl, "\mathrm{id} \otimes (\mathrm{id} \otimes e_M)"'] \arrow[dr, phantom, "\text{(v)}"] & N \otimes (N \otimes M) \arrow[dl, "\mathrm{id} \otimes (\mathrm{id} \otimes e_M)"'] \arrow[d, "\alpha"] \\
& N \otimes N \arrow[d, "m_N"] \arrow[rr, bend right, "\mathrm{id} \otimes \rho^{-1}"'] \arrow[drr, "\rho^{-1}"'] \arrow[r, phantom, "\text{(iii)}"] & N \otimes (M \otimes I) \arrow[r, "\mathrm{id} \otimes (f \otimes \mathrm{id})"] \arrow[dr, phantom, "\text{(vii)}"] & N \otimes (N \otimes I) \arrow[d, "\alpha"] \arrow[r, phantom, "\text{(vi)}"] & (N \otimes N) \otimes M \arrow[dl, "\mathrm{id} \otimes e_M"'] \arrow[d, "m_N \otimes \mathrm{id}"] \\
& N \arrow[ddrr, "\rho^{-1}"] \arrow[dddrrr, bend right, "e_N"'] \arrow[r, phantom, "\text{(viii)}"] \arrow[ddrr, phantom, "\text{(x)}"'] & & (N \otimes N) \otimes I \arrow[d, "m_N \otimes \mathrm{id}"] \arrow[r, phantom, "\text{(ix)}"] & N \otimes M \arrow[d, "e_N \otimes \mathrm{id}"] \\
& & & N \otimes I \arrow[d, "e_N \otimes \mathrm{id}"] & I \otimes M \arrow[dl, "\mathrm{id} \otimes e_M"'] \arrow[d, "\lambda"] \\
& & & I \otimes I \arrow[dr, "\lambda"] \arrow[r, phantom, "\text{(xi)}"] & M \arrow[d, "e_M"] \\
& & & & I
\end{tikzcd}
$$

Diagram A.2: The diagram proving that $e_M \circ g = e_N$

Region	Reason why commutes
(i)	Coherence
(ii)	λ^{-1} natural
(iii)	Coherence
(iv)	α^{-1} natural
(v)	α^{-1} natural
(vi)	α natural
(vii)	M is a monoid
(viii)	α natural
(ix)	Functoriality of $\otimes$
(x)	M is Frobenius
(xi)	Coherence
(xii)	α^{-1} natural
(xiii)	α^{-1} natural
(xiv)	α natural
(xv)	α natural
(xvi)	Functoriality of $\otimes$
(xvii)	Functoriality of $\otimes$
(xviii)	λ natural
(xix)	Coherence

Table A.1: Table explaining why diagram A.3 commutes.

the functoriality of $\otimes$. (x) commutes as N is a comonoid and finally (xi) commutes by naturality of λ.

Finally, we show that $g \circ m_N = m_M \circ (g \otimes g)$. Note that $(g \otimes g) = (g \otimes \mathrm{id}_M) \circ (\mathrm{id}_N \otimes g)$. Consider now diagram A.3. To see that it commutes, it suffices to check that each numbered region of it commutes. This is done in table A.1.

Next we introduce some notation to avoid the cluttering of diagrams. Define A^n and ${}^nA \otimes B$ inductively for positive integers by

$$A^1 := A \qquad\qquad {}^1A \otimes B := A \otimes B$$
$$A^{n+1} := A^n \otimes A \qquad\qquad {}^{n+1}A \otimes B : = A \otimes ({}^nA \otimes B)$$

For example, $A^{\otimes 3} = (A \otimes A) \otimes A$, and ${}^{\otimes 3}A \otimes B^{\otimes 2} = A \otimes (A \otimes (A \otimes (B \otimes B)))$. This notation extends to morphisms in the obvious way. With this in mind, we turn our attention to diagram A.4

Table A.2 explains why each numbered region of diagram A.4 commutes. Finally, consider the diagram A.5. To prove that $g \circ m_N = m_M \circ (g \otimes g)$ it suffices to prove that this diagram commutes. But region (i) commutes as ρ^{-1} is natural and region (ii) by functoriality of $\otimes$. (iii) and (vii) commute by coherence and (iv), (v) and (viii)

Region	Reason why commutes
(i)	Frobenius law of N
(ii)	α natural
(iii)	Coherence
(iv)	α natural
(v)	Functoriality of $\otimes$
(vi)	α^{-1} natural
(vii)	N is a comonoid
(viii)	f is a morphism of comonoids
(ix)	α^{-1} natural
(x)	N is a comonoid
(xi)	Functoriality of $\otimes$
(xii)	Functoriality of $\otimes$
(xiii)	Functoriality of $\otimes$
(xiv)	λ natural
(xv)	Coherence
(xvi)	λ natural
(xvii)	Coherence

Table A.2: Table explaining why diagram A.4 commutes.

by naturality of α. (vi) follows from commutativity of diagram A.4 by tensoring on the left with N. (ix) commutes because M is a monoid and (x) follows from the commutativity of diagram A.3.

$\square$

$$
\begin{tikzcd}
N \otimes M \arrow[r, "\rho^{-1} \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes d_M"'] \arrow[dr, "\mathrm{id} \otimes \lambda^{-1}"] & (N \otimes I) \otimes M \arrow[r, "(\mathrm{id} \otimes u_M) \otimes \mathrm{id}"] \arrow[d, "\alpha^{-1}"] \arrow[r, phantom, "\text{(iv)}", shift right=6ex] & (N \otimes M) \otimes M \arrow[r, "(\mathrm{id} \otimes d_M) \otimes \mathrm{id}"] \arrow[d, "\alpha^{-1}"] & (N \otimes (M \otimes M)) \otimes M \arrow[r, "(\mathrm{id} \otimes (f \otimes \mathrm{id})) \otimes \mathrm{id}"] \arrow[dd, "\alpha \otimes \mathrm{id}"] \arrow[ddl, "\alpha^{-1}"'] & (N \otimes (N \otimes M)) \otimes M \arrow[d, "\alpha \otimes \mathrm{id}"] \arrow[dl, phantom, "\text{(vi)}"] \\
N \otimes (M \otimes M) \arrow[d, "\mathrm{id} \otimes \lambda^{-1}"] \arrow[dd, bend right=40, "(*)"'] \arrow[dddd, bend right=60, "\mathrm{id}"'] \arrow[r, phantom, "\text{(ii)}"] \arrow[d, phantom, "\text{(i)}", shift right=5ex] & N \otimes (I \otimes M) \arrow[r, "\mathrm{id} \otimes (u_M \otimes \mathrm{id})"] \arrow[dl, "\mathrm{id} \otimes (\mathrm{id} \otimes d_M)"] \arrow[d, phantom, "\text{(ix)}"] & N \otimes (M \otimes M) \arrow[dl, "\mathrm{id} \otimes (\mathrm{id} \otimes d_M)"'] \arrow[d, "\mathrm{id} \otimes (d_M \otimes \mathrm{id})"] \arrow[r, phantom, "\text{(v)}"] & & ((N \otimes N) \otimes M) \otimes M \arrow[dd, "(m_N \otimes \mathrm{id}) \otimes \mathrm{id}"] \arrow[ddl, "\alpha^{-1}"] \\
N \otimes (I \otimes (M \otimes M)) \arrow[r, "\mathrm{id} \otimes (u_M \otimes \mathrm{id})"] \arrow[d, "\mathrm{id} \otimes \alpha"] \arrow[dr, phantom, "\text{(viii)}"] & N \otimes (M \otimes (M \otimes M)) \arrow[d, "\mathrm{id} \otimes \alpha"] \arrow[dr, phantom, "\text{(x)}"] & N \otimes ((M \otimes M) \otimes M) \arrow[d, "\mathrm{id} \otimes \alpha^{-1}"] \arrow[r, phantom, "\text{(xi)}"] & ((N \otimes M) \otimes M) \otimes M \arrow[uur, "((\mathrm{id} \otimes f) \otimes \mathrm{id}) \otimes \mathrm{id}"] \arrow[ddl, "\alpha^{-1}"'] \arrow[d, phantom, "\text{(xii)}"] & \text{(xiii)} \\
N \otimes ((I \otimes M) \otimes M) \arrow[r, "\mathrm{id} \otimes ((u_M \otimes \mathrm{id}) \otimes \mathrm{id})"] \arrow[d, phantom, "\text{(vii)}"] & N \otimes ((M \otimes M) \otimes M) \arrow[d, "\mathrm{id} \otimes (m_M \otimes \mathrm{id})"] & N \otimes (M \otimes (M \otimes M)) \arrow[dl, "\mathrm{id} \otimes (\mathrm{id} \otimes m_M)"'] \arrow[d, "\alpha"] \arrow[dl, phantom, "\text{(xv)}", shift left=3ex] & (N \otimes N) \otimes (M \otimes M) \arrow[d, "m_N \otimes \mathrm{id}"] \arrow[ddl, "\mathrm{id} \otimes m_M"'] & (N \otimes M) \otimes M \arrow[d, "(e_N \otimes \mathrm{id}) \otimes \mathrm{id}"'] \\
N \otimes (M \otimes M) \arrow[r, "\mathrm{id}"'] & N \otimes (M \otimes M) \arrow[d, "\alpha"'] \arrow[dd, bend right=60, "\mathrm{id} \otimes (f \otimes \mathrm{id})"'] \arrow[d, phantom, "\text{(xiv)}", shift right=5ex] & (N \otimes M) \otimes (M \otimes M) \arrow[ur, "(\mathrm{id} \otimes f) \otimes \mathrm{id}"] \arrow[dl, "\mathrm{id} \otimes m_M"'] \arrow[d, phantom, "\text{(xvi)}"] & N \otimes (M \otimes M) \arrow[d, "e_N \otimes \mathrm{id}"] & (I \otimes M) \otimes M \arrow[d, "\lambda \otimes \mathrm{id}"] \arrow[dl, "\alpha^{-1}"'] \arrow[d, phantom, "\text{(xix)}", shift right=5ex] \\
& (N \otimes M) \otimes M \arrow[r, "(\mathrm{id} \otimes f) \otimes \mathrm{id}"'] & (N \otimes N) \otimes M \arrow[d, "m_N \otimes \mathrm{id}"] \arrow[r, phantom, "\text{(xvii)}"] & I \otimes (M \otimes M) \arrow[r, "\lambda"] \arrow[d, "\mathrm{id} \otimes m_M"] \arrow[dr, phantom, "\text{(xviii)}"] & M \otimes M \arrow[d, "m_M"] \\
(*)\colon \mathrm{id} \otimes (\lambda^{-1} \otimes \mathrm{id}) & N \otimes (N \otimes M) \arrow[ur, "\alpha"] & N \otimes M \arrow[r, "e_N \otimes \mathrm{id}"'] & I \otimes M \arrow[r, "\lambda"'] & M
\end{tikzcd}
$$

Diagram A.3: First diagram in the proof of $g \circ m_N = m_M \circ (g \otimes g)$

$$
\begin{tikzcd}
N \arrow[r, "\rho^{-1}"] & N\otimes I \arrow[r, "\mathrm{id}\otimes u_M"] & N\otimes M \arrow[r, "\mathrm{id}\otimes d_M"] & N\otimes(M\otimes M) \arrow[r, "\mathrm{id}\otimes(f\otimes \mathrm{id})"] \arrow[dl, "\mathrm{id}\otimes(f\otimes\mathrm{id})"'] \arrow[d, "\mathrm{id}\otimes(d_M\otimes\mathrm{id})"'] \arrow[dd, bend left, "\mathrm{id}\otimes\ (\mathrm{id}\otimes d_M)"] & {}^2N\otimes M \arrow[d, "\alpha"] \arrow[dddl, "{}^2\mathrm{id}\oplus d_M"'] \\
 & & {}^2N\otimes M \arrow[dl, "\alpha"'] \arrow[d, "\mathrm{id}\otimes(d_N\otimes\mathrm{id})"'] \arrow[r, phantom, "\text{(viii)}"] & N\otimes(M^2\otimes M) \arrow[dl, "\mathrm{id}\otimes((f\otimes f)\otimes\mathrm{id})"'] \arrow[d, "\mathrm{id}\otimes\alpha^{-1}"'] \arrow[d, phantom, shift left=6ex, "\text{(x)}"] & N^2\otimes M \arrow[d, "m_N\otimes\mathrm{id}"] \arrow[dddl, "\mathrm{id}\oplus d_M"'] \\
N^2\otimes M \arrow[ddddd, "\mathrm{id}"'] & N^2\otimes M \arrow[l, "\mathrm{id}"'] \arrow[d, "(\mathrm{id}\otimes d_N)\otimes\mathrm{id}"'] \arrow[r, phantom, "\text{(ii)}"] & N\otimes(N^2\otimes M) \arrow[dl, "\alpha"'] \arrow[d, "\mathrm{id}\otimes\alpha^{-1}"'] \arrow[r, phantom, "\text{(ix)}"] & N\otimes(M\otimes M^2) \arrow[dl, "\mathrm{id}\otimes(f\otimes(f\otimes\mathrm{id}))"'] \arrow[d, phantom, "\text{(xi)}"] & N\otimes M \arrow[d, "e_N\otimes\mathrm{id}"] \\
 & (N\otimes N^2)\otimes M \arrow[dd, "\alpha\otimes\mathrm{id}"'] \arrow[r, phantom, "\text{(iii)}"] & {}^3N\otimes M \arrow[d, "\alpha"] & {}^2N\otimes M^2 \arrow[l, "{}^2\mathrm{id}\otimes(f\otimes\mathrm{id})"'] \arrow[d, phantom, "\text{(xii)}"] & I\otimes M \arrow[d, "\lambda"] \arrow[ddl, "\mathrm{id}\oplus d_M"'] \\
 \arrow[r, phantom, "\text{(i)}"] & {} & N^2\otimes(N\otimes M) \arrow[dl, "\alpha"'] \arrow[d, "m_N\otimes\mathrm{id}"] & N^2\otimes M^2 \arrow[l, "\mathrm{id}\otimes(f\otimes\mathrm{id})"'] \arrow[d, phantom, "\text{(xiii)}"] & M \arrow[d, "d_M"] \\
 & N^3\otimes M \arrow[dr, "(m_N\otimes id)\otimes id"] & {}^2N\otimes M \arrow[dr, "e_N\otimes id"] \arrow[d, phantom, "\text{(iv)}"] & I\otimes(M\otimes M) \arrow[d, "\mathrm{id}\otimes(f\otimes\mathrm{id})"] \arrow[r, phantom, "\text{(xiv)}"] & M\otimes M \arrow[d, "f\otimes\mathrm{id}"] \\
 & (N\otimes N^2)\otimes M \arrow[r, "(\mathrm{id}\otimes m_N)\otimes\mathrm{id}"] \arrow[dr, "(e_N\otimes id)\otimes id"] & N^2\otimes M \arrow[dr, "(e_N\otimes id)\otimes id"] \arrow[d, phantom, "\text{(v)}"] & I\otimes(N\otimes M) \arrow[r, "\lambda"] \arrow[d, "\alpha"'] \arrow[d, phantom, shift left=6ex, "\text{(xv)}"] & N\otimes M \\
N^2\otimes M \arrow[r, "(d_N\otimes\mathrm{id})\otimes\mathrm{id}"] \arrow[drrr, bend right, "\mathrm{id}"'] & N^3\otimes M \arrow[u, "\alpha^{-1}\otimes\mathrm{id}"] \arrow[dr, "((e_N\otimes id)\otimes id)\otimes id"] \arrow[ur, phantom, "\text{(vi)}"] \arrow[d, phantom, "\text{(vii)}"] & (I\otimes N^2)\otimes M \arrow[r, "(\mathrm{id}\otimes m_N)\otimes\mathrm{id}"] \arrow[dr, "\lambda\otimes id"] & (I\otimes N)\otimes M \arrow[ur, "\lambda\otimes id"] \arrow[d, phantom, "\text{(xvi)}"] & \\
 & {} & ((I\otimes N)\otimes N)\otimes M \arrow[u, "\alpha^{-1}\otimes\mathrm{id}"'] \arrow[u, phantom, shift left=6ex, "\text{(xvii)}"] \arrow[r, "((\lambda\otimes\mathrm{id})\otimes\mathrm{id}"] & N^2\otimes M \arrow[uur, "m_N\oplus id"'] &
\end{tikzcd}
$$

Diagram A.4: Second diagram in the proof of $g \circ m_N = m_M \circ (g \otimes g)$

$$
\begin{tikzcd}
 & N \arrow[r, "\rho^{-1}"] \arrow[d, phantom, "(i)"] & N \otimes I \arrow[r, "\mathrm{id} \otimes u_M"] & N \otimes M \arrow[r, "\mathrm{id} \otimes d_M"] \arrow[d, phantom, "(ii)"] & N \otimes (M \otimes M) \arrow[r, "\mathrm{id} \otimes (f \otimes \mathrm{id})"] & N \otimes (N \otimes M) \arrow[d, "\alpha"] \arrow[dl, phantom, "(v)"] \\
N \otimes N \arrow[ur, "m_N"] \arrow[r, "\rho^{-1}"] \arrow[dr, "\mathrm{id} \oplus \rho^{-1}"'] \arrow[ddd, "\mathrm{id} \otimes g"'] & N^2 \otimes I \arrow[ur, "m_N \otimes \mathrm{id}"] \arrow[r, "\mathrm{id} \otimes u_M"] \arrow[dl, phantom, "(iii)"] & N^2 \otimes M \arrow[r, "\mathrm{id} \otimes d_M"] \arrow[d, phantom, "(iv)"] & N^2 \otimes M^2 \arrow[r, "\mathrm{id} \otimes (f \otimes \mathrm{id})"] & N^2 \otimes (N \otimes M) \arrow[ur, "m_N \otimes \mathrm{id}"] \arrow[d, "\alpha"] & (N \otimes N) \otimes M \arrow[ddd, "m_N \otimes \mathrm{id}"] \\
 & {}^2N \otimes I \arrow[u, "\alpha"'] \arrow[d, "{}^2\mathrm{id} \otimes u_M"'] & & {}^3N \otimes M \arrow[ur, "\alpha"] \arrow[d, "\mathrm{id} \otimes \alpha"] \arrow[dr, phantom, "(vii)"] & (N^2 \otimes N) \otimes M \arrow[ur, "(m_N \otimes \mathrm{id}) \otimes \mathrm{id}"] \arrow[dr, phantom, "(ix)"] & \\
 & {}^2N \otimes M \arrow[r, "{}^2\mathrm{id} \otimes d_M"] & {}^2N \otimes M^2 \arrow[ur, "{}^2\mathrm{id} \otimes (f \otimes \mathrm{id})"] \arrow[d, phantom, "(vi)"] & N \otimes (N^2 \otimes M) \arrow[r, "\alpha"] \arrow[d, "\mathrm{id} \otimes (m_N \otimes \mathrm{id})"] \arrow[dr, phantom, "(viii)"] & (N \otimes N^2) \otimes M \arrow[u, "\alpha \otimes \mathrm{id}"'] \arrow[d, "(\mathrm{id} \otimes m_N) \otimes \mathrm{id}"] & \\
N \otimes M \arrow[r, "\mathrm{id} \otimes d_m"] \arrow[d, "g \otimes \mathrm{id}"'] & N \otimes M^2 \arrow[rr, "\mathrm{id} \otimes (f \otimes \mathrm{id})"] & & N \otimes (N \otimes M) \arrow[r, "\alpha"] \arrow[d, phantom, "(x)"] & (N \otimes N) \otimes M \arrow[r, "m_N \otimes \mathrm{id}"] & N \otimes M \arrow[d, "e_N \otimes \mathrm{id}"] \\
M \otimes M \arrow[rrrr, "m_M"'] & & & & M & I \otimes M \arrow[l, "\lambda"]
\end{tikzcd}
$$

Diagram A.5: Final diagram in the proof of $g \circ m_N = m_M \circ (g \otimes g)$

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