CHAPTER 1

BASIC TOPOLOGY

Topology, sometimes referred to as "the mathematics of continuity", or "rubber sheet geometry", or "the theory of abstract topological spaces", is all of these, but, above all, it is a *language*, used by mathematicians in practically all branches of our science. In this chapter, we will learn the basic words and expressions of this language as well as its "grammar", i.e. the most general notions, methods and basic results of topology. We will also start building the "library" of examples, both "nice and natural" such as manifolds or the Cantor set, other more complicated and even pathological. Those examples often possess other structures in addition to topology and this provides the key link between topology and other branches of geometry. They will serve as illustrations and the testing ground for the notions and methods developed in later chapters.

1.1. Topological spaces

The notion of topological space is defined by means of rather simple and abstract axioms. It is very useful as an "umbrella" concept which allows to use the geometric language and the geometric way of thinking in a broad variety of vastly different situations. Because of the simplicity and elasticity of this notion, very little can be said about topological spaces in full generality. And so, as we go along, we will impose additional restrictions on topological spaces, which will enable us to obtain meaningful but still quite general assertions, useful in many different situations in the most varied parts of mathematics.

1.1.1. Basic definitions and first examples.

DEFINITION 1.1.1. A *topological space* is a pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T}$ is a family of subsets of X (called the *topology* of X) whose elements are called *open sets* such that

(1) $\varnothing, X \in \mathcal{T}$ (the empty set and X itself are open),
(2) if $\{O_\alpha\}_{\alpha \in A} \subset \mathcal{T}$ then $\bigcup_{\alpha \in A} O_\alpha \in \mathcal{T}$ for any set A (the union of any number of open sets is open),
(3) if $\{O_i\}_{i=1}^k \subset \mathcal{T}$, then $\bigcap_{i=1}^k O_i \in \mathcal{T}$ (the intersection of a finite number of open sets is open).

If $x \in X$, then an open set containing x is said to be an (*open*) *neighborhood* of x.

We will usually omit $\mathcal{T}$ in the notation and will simply speak about a "topological space X" assuming that the topology has been described.

The complements to the open sets $O \in \mathcal{T}$ are called *closed* sets .

EXAMPLE 1.1.2. Euclidean space $\mathbb{R}^n$ acquires the structure of a topological space if its open sets are defined as in the calculus or elementary real analysis course (i.e a set $A \subset \mathbb{R}^n$ is open if for every point $x \in A$ a certain ball centered in x is contained in A).

EXAMPLE 1.1.3. If all subsets of the integers $\mathbb{Z}$ are declared open, then $\mathbb{Z}$ is a topological space in the so–called discrete topology.

EXAMPLE 1.1.4. If in the set of real numbers $\mathbb{R}$ we declare open (besides the empty set and $\mathbb{R}$) all the half-lines $\{x \in \mathbb{R} | x \geq a\}$, $a \in \mathbb{R}$, then we do not obtain a topological space: the first and third axiom of topological spaces hold, but the second one does not (e.g. for the collection of all half lines with positive endpoints).

EXAMPLE 1.1.5. Example 1.1.2 can be extended to provide the broad class of topological spaces which covers most of the natural situations.

Namely, a *distance function* or a *metric* is a function of two variables on a set X (i,e, a function of the Cartesian product $X \times X$ of X with itself) which is nonnegative, symmetric, strictly positive outside the diagonal, and satisfies the triangle inequality (see Definition 3.1.1). Then one defines an (open) ball or radius $r > 0$ around a point $x \in X$ as the set of all points at a distance less that r from X, and an open subset of X as a set which together with any of its points contains some ball around that point. It follows easily from the properties of the distance function that this defines a topology which is usually called the *metric topology*. Naturally, different metrics may define the same topology. We postpone detailed discussion of these notions till Chapter 3 but will occasionally notice how natural metrics appear in various examples considered in the present chapter.

The *closure* $\bar{A}$ of a set $A \subset X$ is the smallest closed set containing A, that is, $\bar{A} := \bigcap\{C \mid A \subset C$ and C closed$\}$. A set $A \subset X$ is called *dense* (or *everywhere dense*) if $\bar{A} = X$. A set $A \subset X$ is called *nowhere dense* if $X \setminus \bar{A}$ is everywhere dense.

A point x is said to be an *accumulation point* (or sometimes *limit point*) of $A \subset X$ if every neighborhood of x contains infinitely many points of A.

A point $x \in A$ is called an *interior point* of A if A contains an open neighborhood of x. The set of interior points of A is called the *interior* of A and is denoted by $\operatorname{Int} A$. Thus a set is open if and only if all of its points are interior points or, equivalently $A = \operatorname{Int} A$.

A point x is called a *boundary point* of A if it is neither an interior point of A nor an interior point of $X \setminus A$. The set of boundary points is called the *boundary* of A and is denoted by ∂A. Obviously $\bar{A} = A \cup \partial A$. Thus a set is closed if and only if it contains its boundary.

EXERCISE 1.1.1. Prove that for any set A in a topological space we have $\partial \overline{A} \subset \partial A$ and $\partial(\operatorname{Int} A) \subset \partial A$. Give an example when all these three sets are different.

A sequence $\{x_i\}_{i\in\mathbb{N}} \subset X$ is said to *converge* to $x \in X$ if for every open set O containing x there exists an $N \in \mathbb{N}$ such that $\{x_i\}_{i>N} \subset O$. Any such point x is called a *limit* of the sequence.

EXAMPLE 1.1.6. In the case of Euclidean space $\mathbb{R}^n$ with the standard topology, the above definitions (of neighborhood, closure, interior, convergence, accumulation point) coincide with the ones familiar from the calculus or elementary real analysis course.

EXAMPLE 1.1.7. For the real line $\mathbb{R}$ with the discrete topology (all sets are open), the above definitions have the following weird consequences: any set has neither accumulation nor boundary points, its closure (as well as its interior) is the set itself, the sequence $\{1/n\}$ does not converge to 0.

Let $(X, \mathcal{T})$ be a topological space. A set $D \subset X$ is called *dense* or *everywhere dense* in X if $\bar{D} = X$. A set $A \subset X$ is called *nowhere dense* if $X \setminus \bar{A}$ is everywhere dense.

The space X is said to be *separable* if it has a finite or countable dense subset. A point $x \in X$ is called *isolated* if the one–point set $\{x\}$ is open.

EXAMPLE 1.1.8. The real line $\mathbb{R}$ in the discrete topology is *not* separable (its only dense subset is $\mathbb{R}$ itself) and each of its points is isolated (i.e. is not an accumulation point), but $\mathbb{R}$ *is* separable in the standard topology (the rationals $\mathbb{Q} \subset \mathbb{R}$ are dense).

1.1.2. Base of a topology. In practice, it may be awkward to list *all* the open sets constituting a topology; fortunately, one can often define the topology by describing a much smaller collection, which in a sense generates the entire topology.

DEFINITION 1.1.9. A *base* for the topology $\mathcal{T}$ is a subcollection $\beta \subset \mathcal{T}$ such that for any $O \in \mathcal{T}$ there is a $B \in \beta$ for which we have $x \in B \subset O$.

Most topological spaces considered in analysis and geometry (but not in algebraic geometry) have a *countable base*. Such topological spaces are often called *second countable*.

A *base of neighborhoods of a point* x is a collection $\mathcal{B}$ of open neighborhoods of x such that any neighborhood of x contains an element of $\mathcal{B}$.

If any point of a topological space has a countable base of neighborhoods, then the space (or the topology) is called *first countable*.

EXAMPLE 1.1.10. Euclidean space $\mathbb{R}^n$ with the standard topology (the usual open and closed sets) has bases consisting of all open balls, open balls of rational radius, open balls of rational center and radius. The latter is a countable base.

EXAMPLE 1.1.11. The real line (or any uncountable set) in the discrete topology (all sets are open) is an example of a first countable but not second countable topological space.

PROPOSITION 1.1.12. *Every topological space with a countable space is separable.*

PROOF. Pick a point in each element of a countable base. The resulting set is at most countable. It is dense since otherwise the complement to its closure would contain an element of the base. □

1.1.3. Comparison of topologies. A topology $\mathcal{S}$ is said to be *stronger* (or *finer*) than $\mathcal{T}$ if $\mathcal{T} \subset \mathcal{S}$, and *weaker* (or *coarser*) if $\mathcal{S} \subset \mathcal{T}$.

There are two extreme topologies on any set: the weakest *trivial topology* with only the whole space and the empty set being open, and the strongest or finest *discrete topology* where all sets are open (and hence closed).

EXAMPLE 1.1.13. On the two point set D, the topology obtained by declaring open (besides D and $\varnothing$) the set consisting of one of the points (but not the other) is strictly finer than the trivial topology and strictly weaker than the discrete topology.

PROPOSITION 1.1.14. *For any set X and any collection $\mathcal{C}$ of subsets of X there exists a unique weakest topology for which all sets from $\mathcal{C}$ are open.*

PROOF. Consider the collection $\mathcal{T}$ which consist of unions of finite intersections of sets from $\mathcal{C}$ and also includes the whole space and the empty set. By properties (2) and (3) of Definition 1.1.1 in any topology in which sets from $\mathcal{C}$ are open the sets from $\mathcal{T}$ are also open. Collection $\mathcal{T}$ satisfies property (1) of Definition 1.1.1 by definition, and it follows immediately from the properties of unions and intersections that $\mathcal{T}$ satisfies (2) and (3) of Definition 1.1.1. □

Any topology weaker than a separable topology is also separable, since any dense set in a stronger topology is also dense in a weaker one.

EXERCISE 1.1.2. How many topologies are there on the 2–element set and on the 3–element set?

EXERCISE 1.1.3. On the integers $\mathbb{Z}$, consider the *profinite* topology for which open sets are defined as unions (not necessarily finite) of arithmetic progressions (non-constant and infinite in both directions). Prove that this defines a topology which is neither discrete nor trivial.

EXERCISE 1.1.4. Define *Zariski* topology in the set of real numbers by declaring complements of finite sets to be open. Prove that this defines a topology which is coarser than the standard one. Give an example of a sequence such that all points are its limits.

EXERCISE 1.1.5. On the set $\mathbb{R} \cup \{*\}$, define a topology by declaring open all sets of the form $\{*\} \cup G$, where $G \subset \mathbb{R}$ is open in the standard topology of $\mathbb{R}$.

(a) Show that this is indeed a topology, coarser than the discrete topology on this set.

(b) Give an example of a convergent sequence which has two limits.

1.2. Continuous maps and homeomorphisms

In this section, we study, in the language of topology, the fundamental notion of continuity and define the main equivalence relation between topological spaces – homeomorphism. We can say (in the category theory language) that now, since the objects (topological spaces) have been defined, we are ready to define the corresponding morphisms (continuous maps) and isomorphisms (topological equivalence or homeomorphism).

Categorical language: preface, appendix reference?

1.2.1. Continuous maps. The topological definition of continuity is simpler and more natural than the ε, δ definition familiar from the elementary real analysis course.

DEFINITION 1.2.1. Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be topological spaces. A map $f\colon X \to Y$ is said to be *continuous* if $O \in \mathcal{S}$ implies $f^{-1}(O) \in \mathcal{T}$ (preimages of open sets are open):

f is an *open map* if it is continuous and $O \in \mathcal{T}$ implies $f(O) \in \mathcal{S}$ (images of open sets are open);

f is *continuous at the point* x if for any neigborhood A of $f(x)$ in Y the preimage $f^{-1}(A)$ contains a neighborhood of x.

A function f from a topological space to $\mathbb{R}$ is said to be *upper semicontinuous* if $f^{-1}(-\infty, c) \in \mathcal{T}$ for all $c \in \mathbb{R}$:

lower semicontinuous if $f^{-1}(c, \infty) \in \mathcal{T}$ for $c \in \mathbb{R}$.

EXERCISE 1.2.1. Prove that a map is continuous if and only if it is continuous at every point.

Let Y be a topological space. For any collection $\mathcal{F}$ of maps from a set X (without a topology) to Y there exists a unique weakest topology on

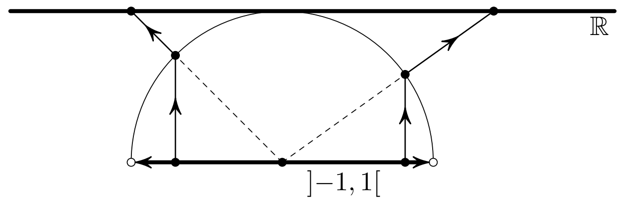

FIGURE 1.2.1. The open interval is homeomorphic to the real line

X which makes all maps from $\mathcal{F}$ continuous; this is exactly the weakest topology with respect to which preimages of all open sets in Y under the maps from $\mathcal{F}$ are open. If $\mathcal{F}$ consists of a single map f, this topology is sometimes called the *pullback topology* on X under the map f.

EXERCISE 1.2.2. Let p be the orthogonal projection of the square K on one of its sides. Describe the pullback topology on K. Will an open (in the usual sense) disk inside K be an open set in this topology?

1.2.2. Topological equivalence. Just as algebraists study groups up to isomorphism or matrices up to a linear conjugacy, topologists study (topological) spaces up to homeomorphism.

DEFINITION 1.2.2. A map $f : X \to Y$ between topological spaces is a *homeomorphism* if it is continuous and bijective with continuous inverse.

If there is a homeomorphism $X \to Y$, then X and Y are said to be *homeomorphic* or sometimes *topologically equivalent*.

A property of a topological space that is the same for any two homeomorphic spaces is said to be a *topological invariant* .

The relation of being homeomorphic is obviously an equivalence relation (in the technical sense: it is reflexive, symmetric, and transitive). Thus topological spaces split into equivalence classes, sometimes called *homeomorphy classes*. In this connection, the topologist is sometimes described as a person who cannot distinguish a coffee cup from a doughnut (since these two objects are homeomorphic). In other words, two homeomorphic topological spaces are identical or indistinguishable from the intrinsic point of view in the same sense as isomorphic groups are indistinguishable from the point of view of abstract group theory or two conjugate $n \times n$ matrices are indistinguishable as linear transformations of an n-dimensional vector space without a fixed basis.

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EXAMPLE 1.2.3. The figure shows how to construct homeomorphisms between the open interval and the open half-circle and between the open half-circle and the real line $\mathbb{R}$, thus establishing that the open interval is homeomorphic to the real line.

EXERCISE 1.2.3. Prove that the sphere $\mathbb{S}^2$ with one point removed is homeomorphic to the plane $\mathbb{R}^2$.

EXERCISE 1.2.4. Prove that any open ball is homeomorphic to $\mathbb{R}^3$.

EXERCISE 1.2.5. Describe a topology on the set $\mathbb{R}^2 \cup \{*\}$ which will make it homeomorphic to the sphere $\mathbb{S}^2$.

To show that certain spaces are homeomorphic one needs to exhibit a homeomorphism; the exercises above give basic but important examples of homeomorphic spaces; we will see many more examples already in the course of this chapter. On the other hand, in order to show that topological spaces are not homeomorphic one need to find an invariant which distinguishes them. Let us consider a very basic example which can be treated with tools from elementary real analysis.

EXAMPLE 1.2.4. In order to show that closed interval is not homeomorphic to an open interval (and hence by Example 1.2.3 to the real line) notice the following. Both closed and open interval as topological spaces have the property that the only sets which are open and closed at the same time are the space itself and the empty set. This follows from characterization of open subsets on the line as finite or countable unions of disjoint open intervals and the corresponding characterization of open subsets of a closed interval as unions of open intervals and semi-open intervals containing endpoints. Now if one takes any point away from an open interval the resulting space with induced topology (see below) will have two proper subsets which are open and closed simultaneously while in the closed (or semi-open) interval removing an endpoint leaves the space which still has no non-trivial subsets which are closed and open.

In Section 1.6 we will develop some of the ideas which appeared in this simple argument systematically.

The same argument can be used to show that the real line $\mathbb{R}$ is not homeomorphic to Euclidean space $\mathbb{R}^n$ for $n \geq 2$ (see Exercise 1.10.7). It is not sufficient however for proving that $\mathbb{R}^2$ is not homeomorphic $\mathbb{R}^3$. Nevertheless, we feel that we intuitively understand the basic structure of the space $\mathbb{R}^n$ and that topological spaces which locally look like $\mathbb{R}^n$ (they are called (n-dimensional) *topological manifolds*) are natural objects of study in topology. Various examples of topological manifolds will appear in the course of this chapter and in Section 1.8 we will introduce precise definitions and deduce some basic properties of topological manifolds.

1.3. Basic constructions

1.3.1. Induced topology. If $Y \subset X$, then Y can be made into a topological space in a natural way by taking the *induced topology*

$$\mathcal{T}_Y := \{O \cap Y \mid O \in \mathcal{T}\}.$$

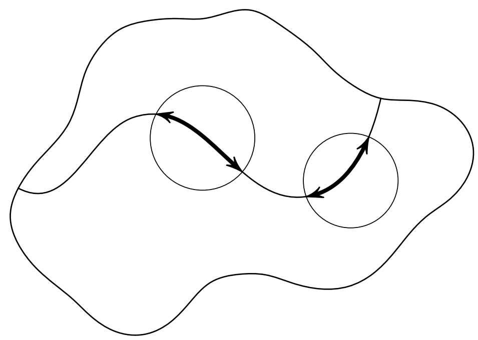

FIGURE 1.3.1. Induced topology

EXAMPLE 1.3.1. The topology induced from $\mathbb{R}^{n+1}$ on the subset

$$\{(x_1, \dots, x_n, x_{n+1}) : \sum_{i=1}^{n+1} x_i^2 = 1\}$$

produces the (standard, or unit) *n–sphere* $\mathbb{S}^n$. For $n = 1$ it is called the (*unit*) *circle* and is sometimes also denoted by $\mathbb{T}$.

EXERCISE 1.3.1. Prove that the boundary of the square is homeomorphic to the circle.

EXERCISE 1.3.2. Prove that the sphere $\mathbb{S}^2$ with any two points removed is homeomorphic to the infinite cylinder $C := \{(x, y, z) \in \mathbb{R}^3 | x^2+y^2 = 1\}$.

EXERCISE 1.3.3. Let $S := \{(x, y, z) \in \mathbb{R}^3 \mid z = 0,\ x^2 + y^2 = 1\}$. Show that $\mathbb{R}^3 \setminus S$ can be mapped continuously onto the circle.

1.3.2. Product topology. If $(X_\alpha, \mathcal{T}_\alpha)$, $\alpha \in A$ are topological spaces and A is any set, then the *product topology* on $\prod_{\alpha \in A} X$ is the topology determined by the base

$$\Big\{ \prod_\alpha O_\alpha \mid O_\alpha \in \mathcal{T}_\alpha,\ O_\alpha \neq X_\alpha \text{ for only finitely many } \alpha \Big\}.$$

EXAMPLE 1.3.2. The standard topology in $\mathbb{R}^n$ coincides with the product topology on the product of n copies of the real line $\mathbb{R}$.

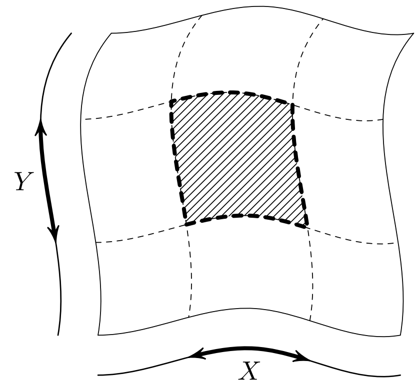

FIGURE 1.3.2. Basis element of the product topology

EXAMPLE 1.3.3. The product of n copies of the circle is called the *n–torus* and is usually denoted by $\mathbb{T}^n$. The n– torus can be naturally identified with the following subset of $\mathbb{R}^{2n}$:

$$\{(x_1, \dots x_{2n}) : x_{2i-1}^2 + x_{2i}^2 = 1, \ i = 1, \dots, n.\}$$

with the induced topology.

EXAMPLE 1.3.4. The product of countably many copies of the two–point space, each with the discrete topology, is one of the representations of the *Cantor set* (see Section 1.7 for a detailed discussion).

EXAMPLE 1.3.5. The product of countably many copies of the closed unit interval is called the *Hilbert cube*. It is the first interesting example of a Hausdorff space (Section 1.4) "too big" to lie inside (that is, to be homeomorphic to a subset of) any Euclidean space $\mathbb{R}^n$. Notice however, that not only we lack means of proving the fact right now but the elementary invariants described later in this chapter are not sufficient for this task either.

1.3.3. Quotient topology. Consider a topological space $(X, \mathcal{T})$ and suppose there is an equivalence relation $\sim$ defined on X. Let π be the natural projection of X on the set $\hat{X}$ of equivalence classes. The *identification space* or *quotient space* $X/\!\sim \,:= (\hat{X}, \mathcal{S})$ is the topological space obtained by calling a set $O \subset \hat{X}$ open if $\pi^{-1}(O)$ is open, that is, taking on $\hat{X}$ the finest topology for which π is continuous. For the moment we restrict ourselves to "good" examples, i.e. to the situations where quotient topology is natural in some sense. However the reader should be aware that even very natural equivalence relations often lead to factors with bad properties ranging from the trivial topology to nontrivial ones but lacking basic separation properties (see Section 1.4). We postpone description of such examples till Section 1.9.2.

EXAMPLE 1.3.6. Consider the closed unit interval and the equivalence relation which identifies the endpoints. Other equivalence classes are single points in the interior. The corresponding quotient space is another representation of the circle.

The product of n copies of this quotient space gives another definition of the n–torus.

EXERCISE 1.3.4. Describe the representation of the n–torus from the above example explicitly as the identification space of the unit n–cube I^n:

$$\{(x_1, \ldots, x_n) \in \mathbb{R}^n : 0 \leq x_i \leq 1,\ i = 1, \ldots n.$$

EXAMPLE 1.3.7. Consider the following equivalence relation in punctured Euclidean space $\mathbb{R}^{n+1} \setminus \{0\}$:

$$(x_1, \ldots, x_{n+1}) \sim (y_1, \ldots, y_{n+1}) \text{ iff } y_i = \lambda x_i \text{ for all } i = 1, \ldots, n+1$$

with the same real number λ. The corresponding identification space is called the *real projective n–space* and is denoted by $\mathbb{R}P(n)$.

A similar procedure in which λ has to be positive gives another definition of the n–sphere $\mathbb{S}^n$.

EXAMPLE 1.3.8. Consider the equivalence relation in $\mathbb{C}^{n+1} \setminus \{0\}$:

$$(x_1, \ldots, x_{n+1}) \sim (y_1, \ldots, y_{n+1}) \text{ iff } y_i = \lambda x_i \text{ for all } i = 1, \ldots, n+1$$

with the same complex number λ. The corresponding identification space is called the *complex projective n–space* and is detoted by $\mathbb{C}P(n)$.

EXAMPLE 1.3.9. The map $E : [0,1] \to \mathbb{S}^1, E(x) = \exp 2\pi i x$ establishes a homeomorphism between the interval with identified endpoints (Example 1.3.6) and the unit circle defined in Example 1.3.1.

EXAMPLE 1.3.10. The identification of the equator of the 2-sphere to a point yields two spheres with one common point.

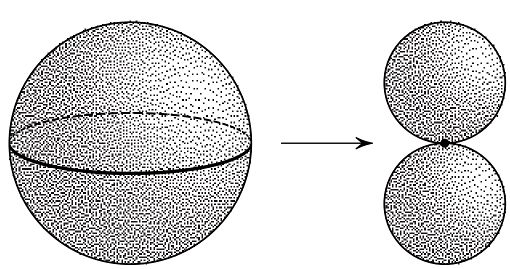

FIGURE 1.3.3. The sphere with equator identified to a point

EXAMPLE 1.3.11. Identifying the short sides of a long rectangle in the natural way yields the lateral surface of the cylinder (which of course is homeomorphic to the annulus), while the identification of the same two sides in the "wrong way" (i.e., after a half twist of the strip) produces the famous Möbius strip. We assume the reader is familiar with the failed experiments of painting the two sides of the Möbius strip in different colors or cutting it into two pieces along its midline. Another less familiar but amusing endeavor is to predict what will happen to the physical object obtained by cutting a paper Möbius strip along its midline if that object is, in its turn, cut along its own midline.

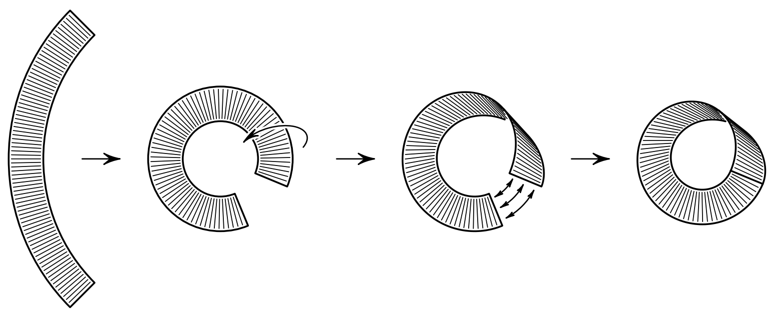

FIGURE 1.3.4. The Möbius strip

EXERCISE 1.3.5. Describe a homeomorphism between the torus $\mathbb{T}^n$ (Example 1.3.3) and the quotient space described in Example 1.3.6 and the subsequent exercise.

EXERCISE 1.3.6. Describe a homeomorphism between the sphere $\mathbb{S}^n$ (Example 1.3.1) and the second quotient space of Example 1.3.7.

EXERCISE 1.3.7. Prove that the real projective space $\mathbb{R}P(n)$ is homeomorphic to the quotient space of the sphere S^n with respect to the equivalence relation which identifies pairs of opposite points: x and $-x$.

EXERCISE 1.3.8. Consider the equivalence relation on the closed unit ball $\mathbb{D}^n$ in $\mathbb{R}^n$:

$$\{(x_1, \ldots, x_n) : \sum_{i=1}^{n} x_i^2 \leq 1\}$$

which identifies all points of $\partial\mathbb{D}^n = \mathbb{S}^{n-1}$ and does nothing to interior points. Prove that the quotient space is homeomorphic to $\mathbb{S}^n$.

EXERCISE 1.3.9. Show that $\mathbb{C}P(1)$ is homeomorphic to $\mathbb{S}^2$.

DEFINITION 1.3.12. The *cone* $\mathrm{Cone}(X)$ over a topological space X is the quotient space obtained by identifying all points of the form $(x, 1)$ in the product $(X \times [0, 1]$ (supplied with the product topology).

The *suspension* $\Sigma(X)$ of a topological space X is the quotient space of the product $X \times [-1, 1]$ obtained by identifying all points of the form $x \times 1$ and identifying all points of the form $x \times -1$. By convention, the suspension of the empty set will be the two-point set $\mathbb{S}^0$.

The *join* $X * Y$ of two topological spaces X and Y , roughly speaking, is obtained by joining all pairs of points (x, y), $x \in X$, $y \in Y$, by line segments and supplying the result with the natural topology; more precisele, $X * Y$ is the quotient space of the product $X \times [-1, 1] \times Y$ under the following identifications:

$(x, -1, y) \sim (x, -1, y')$ for any $x \in X$ and all $y, y' \in Y$,
$(x, 1, y) \sim (x', 1, y)$ for any $y \in Y$ and all $x, x' \in X$.

EXAMPLE 1.3.13. (a) $\mathrm{Cone}(*) = \mathbb{D}^1$ and $\mathrm{Cone}(\mathbb{D}^{n-1}) = \mathbb{D}^n$ for $n > 1$.
(b) The suspension $\Sigma(\mathbb{S}^n)$ of the n-sphere is the $(n + 1)$-sphere $\mathbb{S}^{n+1}$.
(c) The join of two closed intervals is the 3-simplex (see the figure).

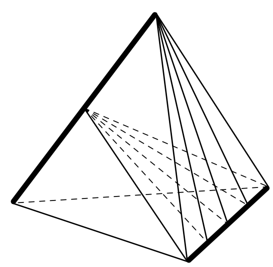

FIGURE 1.3.5. The 3-simplex as the join of two segments

EXERCISE 1.3.10. Show that the cone over the sphere $\mathbb{S}^n$ is (homeomorphic to) the disk $\mathbb{D}^{n+1}$.

EXERCISE 1.3.11. Show that the join of two spheres $\mathbb{S}^k$ and $\mathbb{S}^l$ is (homeomorphic to) the sphere $\mathbb{S}^{k+l+1}$.

EXERCISE 1.3.12. Is the join operation on topological spaces associative?

1.4. Separation properties

Separation properties provide one of the approaches to measuring how fine is a given topology.

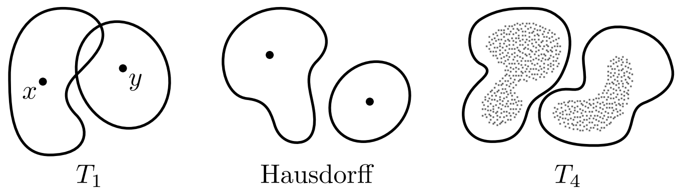

FIGURE 1.4.1. Separation properties

1.4.1. T1, Hausdorff, and normal spaces. Here we list, in decreasing order of generality, the most common separation axioms of topological spaces.

DEFINITION 1.4.1. A topological space $(X, \mathcal{T})$ is said to be a

(T1) *space* if any point is a closed set. Equivalently, for any pair of points $x_1, x_2 \in X$ there exists a neighborhood of x_1 not containing x_2;

(T2) or *Hausdorff space* if any two distinct points possess nonintersecting neighborhoods;

(T4) or *normal space* if it is Hausdorff and any two closed disjoint subsets possess nonintersecting neighborhoods. [1]

It follows immediately from the definition of induced topology that any of the above separation properties is inherited by the induced topology on any subset.

On the picture the interior does not looks closed but the exterior does

EXERCISE 1.4.1. Prove that in a (T2) space any sequence has no more than one limit. Show that without the (T2) condition this is no longer true.

EXERCISE 1.4.2. Prove that the product of two (T1) (respectively Hausdorff) spaces is a (T1) (resp. Hausdorff) space.

REMARK 1.4.2. We will see later (Section 1.9) that even very naturally defined equivalence relations in nice spaces may produce quotient spaces with widely varying separation properties.

The word "normal" may be understood in its everyday sense like "commonplace" as in "a normal person". Indeed, normal topological possess many properties which one would expect form commonplaces notions of continuity. Here is an examples of such property dealing with extension of maps:

THEOREM 1.4.3. [Tietze] *If X is a normal topological space, $Y \subset X$ is closed, and $f\colon Y \to [-1, 1]$ is continuous, then there is a continuous*

[1] Hausdorff (or (T1)) assumption is needed to ensure that there are enough closed sets; specifically that points are closed sets. Otherwise trivial topology would satisfy this property.

extension of f to X, i.e., a continuous map $F : X \to [-1, 1]$ such that $F|_Y = f$.

The proof is based on the following fundamental result, traditionally called Urysohn Lemma, which asserts existence of many continuous maps from a normal space to the real line and thus provided a basis for introducing measurements in normal topological spaces (see Theorem 3.5.1) and hence by Corollary 3.5.3 also in compact Hausdorff spaces.

THEOREM 1.4.4. [Urysohn Lemma] *If X is a normal topological space and A, B are closed subsets of X, then there exists a continuous map $u : X \to [0, 1]$ such that $u(A) = \{0\}$ and $u(B) = \{1\}$.*

PROOF. Let V be en open subset of X and U any subset of X such that $\overline{U} \subset V$. Then there exists an open set W for which $\overline{U} \subset W \subset \overline{W} \subset V$. Indeed, for W we can take any open set containing $\overline{U}$ and not intersecting an open neighborhood of $X \setminus V$ (such a W exists because X is normal).

Applying this to the sets $U := A$ and $V := X \setminus B$, we obtain an "intermediate" open set A_1 such that

$$A \subset A_1 \subset X \setminus B, \tag{1.4.1}$$

where $\overline{A_1} \subset X \setminus B$. Then we can introduce the next intermediate open sets A_1' and A_2 so as to have

$$A \subset A_1' \subset A_1 \subset A_2 \subset X \setminus B, \tag{1.4.2}$$

where each set is contained, together with its closure, in the next one.

For the sequence (1.4.1), we define a function $u_1 : X \to [0, 1]$ by setting

$$u_1(x) = \begin{cases} 0 & \text{for } x \in A\ , \\ 1/2 & \text{for } x \in A_1 \setminus A, \\ 1 & \text{for } X \setminus A_1. \end{cases}$$

For the sequence (1.4.2), we define a function $u_2 : X \to [0, 1]$ by setting

$$u_2(x) = \begin{cases} 0 & \text{for } x \in A\ , \\ 1/4' & \text{for } x \in A_1' \setminus A\ , \\ 1/2 & \text{for } x \in A_1 \setminus A_1', \\ 3/4 & \text{for } x \in A_2 \setminus A_1, \\ 1 & \text{for } x \in X \setminus A_2. \end{cases}$$

Then we construct a third sequence by inserting intermediate open sets in the sequence (1.4.2) and define a similar function u_3 for this sequence, and so on.

maybe insert a picture

Obviously, $u_2(x) \geq u_1(x)$ for all $x \in X$. Similarly, for any $n > 1$ we have $u_{n+1}(x) \geq u_n(x)$ for all $x \in X$, and therefore the limit function $u(x) := \lim_{n \to infty} u_n(x)$ exists. It only remains to prove that u is continuous.

Suppose that at the nth step we have constructed the nested sequence of sets corresponding to the function u_n

$$A \subset A_1 \subset \ldots A_r \subset X \setminus B,$$

where $\overline{A_i} \subset A_{i+1}$. Let $A_0 := \operatorname{int} A$ be the interior of A, let $A_{-1} := \varnothing$, and $A_{r+1} := X$. Consider the open sets $A_{i+1} \setminus \overline{A_{i-1}}$, $i = 0, 1, \ldots, r$. Clearly,

$$X = \bigcup_{i=0}^{r} (\bar{A}_i \setminus \overline{A_{i-1}}) \subset \bigcup_{i=0}^{r} (A_{i+1} \setminus \overline{A_{i-1}}),$$

so that the open sets $A_{i+1} \setminus \overline{A_{i-1}}$ cover the entire space X.

On each set $A_{i+1} \setminus \overline{A_{i-1}}$ the function takes two values that differ by $1/2^n$. Obviously,

$$|u(x) - u_n(x)| \leq \sum_{k=n+1}^{\infty} 1/2^k \;=\; 1/2^n.$$

For each point $x \in X$ let us choose an open neighborhood of the form $A_{i+1} \setminus \overline{A_{i-1}}$. The image of the open set $A_{i+1} \setminus \overline{A_{i-1}}$ is contained in the interval $(u(x) - \varepsilon, u(x) + \varepsilon)$, where $\varepsilon < 1/2^n$. Taking $\varepsilon \to \infty$, we see that u is continuous. □

Now let us deduce Theorem 1.4.3 from the Urysohn lemma.

To this end, we put

$$r_k \;:=\; \frac{1}{2}\left(\frac{2}{3}\right)^k, \quad k = 1, 2, \ldots.$$

Let us construct a sequence of functions $f_1, f_2, \ldots$ on X and a sequence of functions $g_1, g_2, \ldots$ on Y by induction. First, we put $f_1 := f$. Suppose that the functions $f_1, \ldots, f_k$ have been constructed. Consider the two closed disjoint sets

$$A_k := \{x \in X \mid f_k(x) \leq -r_k\} \quad \text{and} \quad B_k := \{x \in X \mid f_k(x) \geq r_k\}.$$

Applying the Urysohn lemma to these sets, we obtain a continuous map $g_k : Y \to [-r_k, r_k]$ for which $g_k(A_k) = \{-r_k\}$ and $g_k(B_k) = \{r_k\}$. On the set A_k, the functions f_k and g_k take values in the interval $]-3r_k, -r_k[$; on the set A_k, they take values in the interval $]r_k, 3r_k[$; at all other points of the set X, these functions take values in the interval $]-r_k, r_k[$.

Now let us put $f_{k+1} := f_k - g_k|_X$. The function f_{k+1} is obviously continuous on X and $|f_{k+1}(x)| \le 2r_k = 3r_{k+1}$ for all $x \in X$.

Consider the sequence of functions $g_1, g_2, \dots$ on Y. By construction, $|g_k(y)| \le r_k$ for all $y \in Y$. The series

$$\sum_{k=1}^{\infty} r_k = \frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^k$$

converges, and so the series $\Sigma_{k=1}^{\infty} g_k(x)$ converges uniformly on Y to some continuous function

$$F(x) := \sum_{k=1}^{\infty} g_k(x).$$

Further, we have

$$(g_1+\dots+g_k) = (f_1-f_2)+(f_2-f_3)+\dots+(f_k-f_{k+1}) = f_1-f_{k+1} = f-f_{k+1}.$$

But $\lim_{k\to\infty} f_{k+1}(y) = 0$ for any $y \in Y$, hence $F(x) = f(x)$ for any $x \in X$, so that F is a continuous extension of f.

It remains to show that $|F(x)| \le 1$. We have

$$|F(x)| \le \sum_{k=1}^{\infty} |g_k(x)| \le \sum_{k=1}^{\infty} r_k = \sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^k$$
$$= \sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^k = \frac{1}{3}\left(1-\frac{2}{3}\right)^{-1} = 1. \qquad \square$$

COROLLARY 1.4.5. *Let $X \subset Y$ be a closed subset of a normal space Y and let $f : X \to \mathbb{R}$ be continuous. Then f has a continuous extension $F : Y \to \mathbb{R}$.*

PROOF. The statement follows from the Tietze theorem and the Urysohn lemma by appropriately using the rescaling homeomorphism

$$g : \mathbb{R} \to (-\pi/2, \pi/2) \quad \text{given by} \quad g(x) := \arctan(x). \quad \square$$

Most natural topological spaces which appear in analysis and geometry (but not in some branches of algebra) are normal. The most important instance of non-normal topology is discussed in the next subsection.

1.4.2. Zariski topology. The topology that we will now introduce and seems pathological in several aspects (it is non-Hausdorff and does not possess a countable base), but very useful in applications, in particular in algebraic geometry. We begin with the simplest case which was already mentioned in Example 1.1.4

DEFINITION 1.4.6. The *Zariski topology* on the real line $\mathbb{R}$ is defined as the family $\mathcal{Z}$ of all complements to finite sets.

PROPOSITION 1.4.7. *The Zariski topology given above endows $\mathbb{R}$ with the structure of a topological space $(\mathbb{R}, \mathcal{Z})$, which possesses the following properties:*

(1) *it is a* (T1) *space;*
(2) *it is separable;*
(3) *it is not a Hausdorff space;*
(4) *it does not have a countable base*.

PROOF. All four assertions are fairly straightforward:

(1) the Zariski topology on the real line is (T1), because the complement to any point is open;

(2) it is separable, since it is weaker than the standard topology in $\mathbb{R}$;

(3) it is not Hausdorff, because any two nonempty open sets have nonempty intersection;

(4) it does not have a countable base, because the intersection of all the sets in any countable collection of open sets is nonemply and thus the complement to any point in that intersection does not contain any element from that collection. □

The definition of Zariski topology on $\mathbb{R}$ (Definition 1.4.6) can be straightforwardly generalized to $\mathbb{R}^n$ for any $n \geq 2$, and the assertions of the proposition above remain true. However, this definition is not the natural one, because it generalizes the "wrong form" of the notion of Zariski topology. The "correct form" of that notion originally appeared in algebraic geometry (which studies zero sets of polynomials) and simply says that closed sets in the Zariski topology on $\mathbb{R}$ are sets of zeros of polynomials $p(x) \in \mathbb{R}[x]$. This motivates the following definitions.

DEFINITION 1.4.8. The *Zariski topology* is defined

- in Euclidean space $\mathbb{R}^n$ by stipulating that the sets of zeros of all polynomials are closed;
- on the unit sphere $S^n \subset \mathbb{R}^{n+1}$ by taking for closed sets the sets of zeros of homogeneous polynomials in $n+1$ variables;
- on the real and complex projective spaces $\mathbb{R}P(n)$ and $\mathbb{C}P(n)$ (Example 1.3.7, Example 1.3.8) via zero sets of homogeneous polynomials in $n+1$ real and complex variables respectively.

EXERCISE 1.4.3. Verify that the above definitions supply each of the sets $\mathbb{R}^n$, S^n, $\mathbb{R}P(n)$, and $\mathbb{C}P(n)$ with the structure of a topological space satisfying the assertions of Proposition 1.4.7.

1.5. Compactness

The fundamental notion of compactness, familiar from the elementary real analysis course for subsets of the real line $\mathbb{R}$ or of Euclidean space $\mathbb{R}^n$, is defined below in the most general topological situation.

1.5.1. Types of compactness. A family of open sets $\{O_\alpha\} \subset \mathcal{T}$, $\alpha \in A$ is called an *open cover* of a topological space X if $X = \bigcup_{\alpha \in A} O_\alpha$, and is a finite open cover if A is finite.

DEFINITION 1.5.1. The space $(X, \mathcal{T})$ is called
• *compact* if every open cover of X has a finite subcover;
• *sequentially compact* if every sequence has a convergent subsequence;
• *σ–compact* if it is the union of a countable family of compact sets.
• *locally compact* if every point has an open neighborhood whose closure is compact in the induced topology.

It is known from elementary real analysis that for subsets of a $\mathbb{R}^n$ compactness and sequential compactness are equivalent. This fact naturally generalizes to metric spaces (see Proposition 3.6.4).

PROPOSITION 1.5.2. *Any closed subset of a compact set is compact.*

PROOF. If K is compact, $C \subset K$ is closed, and Γ is an open cover for C, then $\Gamma_0 := \Gamma \cup \{K \smallsetminus C\}$ is an open cover for K, hence Γ_0 contains a finite subcover $\Gamma' \cup \{K \smallsetminus C\}$ for K; therefore Γ' is a finite subcover (of Γ) for C. □

PROPOSITION 1.5.3. *Any compact subset of a Hausdorff space is closed.*

PROOF. Let X be Hausdorff and let $C \subset X$ be compact. Fix a point $x \in X \smallsetminus C$ and for each $y \in C$ take neighborhoods U_y of y and V_y of x such that $U_y \cap V_y = \varnothing$. Then $\bigcup_{y \in C} U_y \supset C$ is a cover of C and has a finite subcover $\{U_{x_i} \mid 0 \le i \le n\}$. Hence $N_x := \bigcap_{i=0}^n V_{y_i}$ is a neighborhood of x disjoint from C. Thus

$$X \smallsetminus C = \bigcup_{x \in X \smallsetminus C} N_x$$

is open and therefore C is closed. □

PROPOSITION 1.5.4. *Any compact Hausdorff space is normal.*

PROOF. First we show that a closed set K and a point $p \notin K$ can be separated by open sets. For $x \in K$ there are open sets O_x, U_x such that $x \in O_x$, $p \in U_x$ and $O_x \cap U_x = \varnothing$. Since K is compact, there is a finite subcover $O := \bigcup_{i=1}^n O_{x_i} \supset K$, and $U := \bigcap_{i=1}^n U_{x_i}$ is an open set containing p disjoint from O.

Now suppose K, L are closed sets. For $p \in L$, consider open disjoint sets $O_p \supset K$, $U_p \ni p$. By the compactness of L, there is a finite subcover $U := \bigcup_{j=1}^m U_{p_j} \supset L$, and so $O := \bigcap_{j=1}^m O_{p_j} \supset K$ is an open set disjoint from $U \supset L$. □

DEFINITION 1.5.5. A collection of sets is said to have the *finite intersection property* if every finite subcollection has nonempty intersection.

PROPOSITION 1.5.6. *Any collection of compact sets with the finite intersection property has a nonempty intersection.*

PROOF. It suffices to show that in a compact space every collection of closed sets with the finite intersection property has nonempty intersection. Arguing by contradiction, suppose there is a collection of closed subsets in a compact space K with empty intersection. Then their complements form an open cover of K. Since it has a finite subcover, the finite intersection property does not hold. □

EXERCISE 1.5.1. Show that if the compactness assumption in the previous proposition is omitted, then its assertion is no longer true.

EXERCISE 1.5.2. Prove that a subset of $\mathbb{R}$ or of $\mathbb{R}^n$ is compact iff it is closed and bounded.

1.5.2. Compactifications of non-compact spaces.

DEFINITION 1.5.7. A compact topological space K is called a *compactification* of a Hausdorff space $(X, \mathcal{T})$ if K contains a dense subset homeomorphic to X.

The simplest example of compactification is the following.

DEFINITION 1.5.8. The *one-point compactification* of a noncompact Hausdorff space $(X, \mathcal{T})$ is $\hat{X} := (X \cup \{\infty\}, \mathcal{S})$, where

$$\mathcal{S} := \mathcal{T} \cup \{(X \cup \{\infty\}) \smallsetminus K \mid K \subset X \text{ compact}\}.$$

EXERCISE 1.5.3. Show that the one-point compactification of a Hausdorff space X is a compact (T1) space with X as a dense subset. Find a necessary and sufficient condition on X which makes the one-point compactification Hausdorff.

EXERCISE 1.5.4. Describe the one-point compactification of $\mathbb{R}^n$.

Other compactifications are even more important.

EXAMPLE 1.5.9. Real projective space $\mathbb{R}P(n)$ is a compactification of the Euclidean space $\mathbb{R}^n$. This follows easily form the description of $\mathbb{R}P(n)$ as the identification space of a (say, northern) hemisphere with pairs of opposite equatorial points identified. The open hemisphere is homeomorphic to $\mathbb{R}^n$ and the attached "set at infinity" is homeomorphic to the projective space $\mathbb{R}P(n-1)$.

EXERCISE 1.5.5. Describe the complex projective space $\mathbb{C}P(n)$ (see Example 1.3.8) as a compactification of the space $\mathbb{C}^n$ (which is of course homeomorphic to $\mathbb{R}^{2n}$). Specifically, identify the set of added "points at infinity" as a topological space. and desribe open sets which contain points at infinity.

1.5.3. Compactness under products, maps, and bijections. The following result has numerous applications in analysis, PDE, and other mathematical disciplines.

THEOREM 1.5.10. *The product of any family of compact spaces is compact.*

PROOF. Consider an open cover $\mathcal{C}$ of the product of two compact topological spaces X and Y. Since any open neighborhood of any point contains the product of opens subsets in x and Y we can assume that every element of $\mathcal{C}$ is the product of open subsets in X and Y. Since for each $x \in X$ the subset $\{x\} \times Y$ in the induced topology is homeomorphic to Y and hence compact, one can find a finite subcollection $\mathcal{C}_x \subset \mathcal{C}$ which covers $\{x\} \times Y$.

For $(x, y) \in X \times Y$, denote by p_1 the projection on the first factor: $p_1(x, y) = x$. Let $U_x = \bigcap_{C \in \mathcal{O}_x} p_1(C)$; this is an open neighborhood of x and since the elements of $\mathcal{O}_x$ are products, $\mathcal{O}_x$ covers $U_x \times Y$. The sets $\mathcal{U}_x$, $x \in X$ form an open cover of X. By the compactness of X, there is a finite subcover, say $\{U_{x_1}, \dots, U_{x_k}\}$. Then the union of collections $\mathcal{O}_{x_1}, \dots, \mathcal{O}_{x_k}$ form a finite open cover of $X \times Y$.

For a finite number of factors, the theorem follows by induction from the associativity of the product operation and the case of two factors. The proof for an arbitrary number of factors uses some general set theory tools based on axiom of choice. □

PROPOSITION 1.5.11. *The image of a compact set under a continuous map is compact.*

PROOF. If C is compact and $f\colon C \to Y$ continuous and surjective, then any open cover Γ of Y induces an open cover $f_*\Gamma := \{f^{-1}(O) \mid O \in \Gamma\}$ of C which by compactness has a finite subcover $\{f^{-1}(O_i) \mid i = 1, \dots, n\}$. By surjectivity, $\{O_i\}_{i=1}^n$ is a cover for Y. □

A useful application of the notions of continuity, compactness, and separation is the following simple but fundamental result, sometimes referred to as *invariance of domain*:

PROPOSITION 1.5.12. *A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.*

PROOF. Suppose X is compact, Y Hausdorff, $f\colon X \to Y$ bijective and continuous, and $O \subset X$ open. Then $C := X \setminus O$ is closed, hence compact, and $f(C)$ is compact, hence closed, so $f(O) = Y \setminus f(C)$ (by bijectivity) is open. □

Using Proposition 1.5.4 we obtain

COROLLARY 1.5.13. *Under the assumption of Proposition 1.5.12 spaces X and Y are normal.*

EXERCISE 1.5.6. Show that for noncompact X the assertion of Proposition 1.5.12 no longer holds.

1.6. Connectedness and path connectedness

There are two rival formal definitions of the intuitive notion of connectedness of a topological space. The first is based on the idea that such a space "consists of one piece" (i.e., does not "fall apart into two pieces"), the second interprets connectedness as the possibility of "moving continuously from any point to any other point".

1.6.1. Definition and invariance under continuous maps.

DEFINITION 1.6.1. A topological space $(X, \mathcal{T})$ is said to be

• *connected* if X cannot be represented as the union of two nonempty disjoint open sets (or, equivalently, two nonempty disjoint closed sets);

• *path connected* if for any two points $x_0, x_1 \in X$ there exists a path joining x_0 to x_1, i.e., a continuous map $c\colon [0,1] \to X$ such that $c(i) = x_i,\ i = \{0,1\}$.

PROPOSITION 1.6.2. *The continuous image of a connected space X is connected.*

PROOF. If the image is decomposed into the union of two disjoint open sets, the preimages of theses sets which are open by continuity would give a similar decomposition for X. □

PROPOSITION 1.6.3. (1) *Interval is connected*

(2) *Any path-connected space is connected.*

PROOF. (1) Any open subset X of an interval is the union of disjoint open subintervals. The complement of X contains the endpoints of those intervals and hence cannot be open.

(2) Suppose X is path-connected and let $x = X_0 \cup X_1$, where X_0 and X_1 are open and nonempty. Let $x_0 \in X_0,\ x_1 \in X_1$ and $c\colon [0,1] \to X$ is a continuous map such that $c(i) = x_i,\ i \in \{0,1\}$. By Proposition 1.6.2 the image $c([0,1])$ is a connected subset of X in induced topology which is decomposed into the union of two nonempty open subsets $c([0,1]) \cap X_0$ and $c([0,1]) \cap X_1$, a contradiction. □

REMARK 1.6.4. Connected space may not be path-connected as is shown by the union of the graph of $\sin 1/x$ and $\{0\} \times [-1,1]$ in $\mathbb{R}^2$ (see the figure).

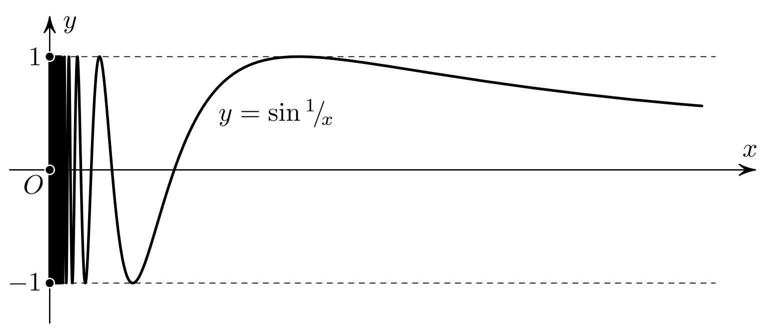

FIGURE 1.6.1. Connected but not path connected space

PROPOSITION 1.6.5. *The continuous image of a path connected space X is path connected.*

PROOF. Let $f : X \to Y$ be continuous and surjective; take any two points $y_1, y_2 \in Y$. Then by surjectivity the sets $f^{-1}(y_i),\ i = 1,2$ are nonempty and we can choose points $x_i \in f^{-1}(y_1),\ i = 1,2$. Since X is path connected, there is a path $\alpha : [0,1] \to X$ joining x_1 to x_2. But then the path $f \circ \alpha$ joins y_1 to y_2. □

1.6.2. Products and quotients.

PROPOSITION 1.6.6. *The product of two connected topological spaces is connected.*

PROOF. Suppose $X,\ Y$ are connected and assume that $X \times Y = A \cup B$, where A and B are open, and $A \cap B = \varnothing$. Then either $A = X_1 \times Y$ for some open $X_1 \subset X$ or there exists an $x \in X$ such that $\{x\} \times Y \cap A \neq \varnothing$ and $\{x\} \times Y \cap B \neq \varnothing$.

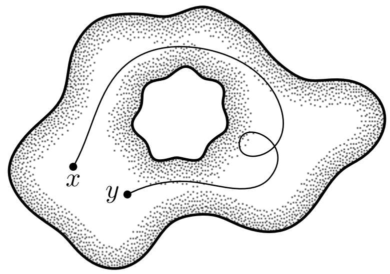

FIGURE 1.6.2. Path connectedness

The former case is impossible, else we would have $B = (X \setminus X_1) \times Y$ and so $X = X_1 \cup (X \setminus X_1)$ would not be connected.

In the latter case, $Y = p_2(\{x\} \times Y \cap A) \cup p_2(\{x\} \times Y \cap B)$ (where $p_2(x, y) = y$ is the projection on the second factor) that is, $\{x\} \times Y$ is the union of two disjoint open sets, hence not connected. Obviously p_2 restricted to $\{x\} \times Y$ is a homeomorphism onto Y, and so Y is not connected either, a contradiction. □

PROPOSITION 1.6.7. *The product of two path-connected topological spaces is connected.*

PROOF. Let $(x_0, y_0),\ (x_1, y_1) \in X \times Y$ and $c_X,\ c_Y$ are paths connecting x_0 with x_1 and y_0 with y_1 correspondingly. Then the path $c\colon [0, 1] \to X \times Y$ defined by

$$c(t) = (c_X(t), c_Y(t))$$

connects (x_0, y_0) with (x_1, y_1). □

The following property follows immediately from the definition of the quotient topology

PROPOSITION 1.6.8. *Any quotient space of a connected topological space is connected.*

1.6.3. Connected subsets and connected components. A subset of a topological space is *connected* (*path connected*) if it is a connected (path connected) space in the induced topology.

A *connected component* of a topological space X is a maximal connected subset of X.

A *path connected component* of X is a maximal path connected subset of X.

PROPOSITION 1.6.9. *The closure of a connected subset $Y \subset X$ is connected.*

PROOF. If $\bar{Y} = Y_1 \cup Y_2$, where $Y_1,\ Y_2$ are open and $Y_1 \cap Y_2 = \varnothing$, then since the set Y is dense in its closure $Y = (Y \cap Y_1) \cup (Y \cap Y_2)$ with both $Y \cap Y_1$ and $Y \cap Y_1$ open in the induced topology and nonempty. □

COROLLARY 1.6.10. *Connected components are closed.*

PROPOSITION 1.6.11. *The union of two connected subsets $Y_1, Y_2 \subset X$ such that $Y_1 \cap Y_2 \neq \varnothing$, is connected.*

PROOF. We will argue by contradiction. Assume that $Y_1 \cap Y_2$ is the disjoint union of of open sets Z_1 and Z_2. If $Z_1 \supset Y_1$, then $Y_2 = Z_2 \cup (Z_1 \cap Y_2)$ and hence Y_2 is not connected. Similarly, it is impossible that $Z_2 \supset Y_1$. Thus $Y_1 \cap Z_i \neq \varnothing,\ i = 1, 2$ and hence $Y_1 = (Y_1 \cap Z_1) \cup (Y_1 \cap Z_2)$ and hence Y_1 is not connected. □

1.6.4. Decomposition into connected components. For any topological space there is a unique *decomposition into connected components* and a unique *decomposition into path connected components*. The elements of these decompositions are equivalence classes of the following two equivalence relations respectively:

(i) x is equivalent to y if there exists a connected subset $Y \subset X$ which contains x and y.

In order to show that the equivalence classes are indeed connected components, one needs to prove that they are connected. For, if A is an equivalence class, assume that $A = A_1 \cup A_2$, where A_1 and A_2 are disjoint and open. Pick $x_1 \in A_1$ and $x_2 \in A_2$ and find a closed connected set A_3 which contains both points. But then $A \subset (A_1 \cup A_3) \cup A_2$, which is connected by Proposition 1.6.11. Hence $A = (A_1 \cup A_3) \cup A_2)$ and A is connected.

(ii) x is equivalent to y if there exists a continuous curve $c\colon [0, 1] \to X$ with $c(0) = x,\ c(1) = y$

REMARK 1.6.12. The closure of a path connected subset may be fail to be path connected. It is easy to construct such a subset by looking at Remark 1.6.4

1.6.5. Arc connectedness. Arc connectedness is a more restrictive notion than path connectedness: a topological space X is called *arc connected* if, for any two distinct points $x, y \in X$ there exist an arc joining them, i.e., there is an injective continuous map $h : [0, 1] \to X$ such that $h(0) = x$ and $h(1) = y$.

It turns out, however, that arc connectedness is not a much more stronger requirement than path connectedness – in fact the two notions coincide for Hausdorff spaces.

THEOREM 1.6.13. *A Hausdorff space is arc connected if and only if it is path connected.*

PROOF. Let X be a path-connected Hausdorff space, $x_0, x_1 \in X$ and $c\colon [0,1] \to X$ a continuous map such that $c(i) = x_i,\ i = 0,1$. Notice that the image $c([0,1])$ is a compact subset of X by Proposition 1.5.11 even though we will not use that directly. We will change the map c within this image by successively cutting of superfluous pieces and rescaling what remains.

Consider the point $c(1/2)$. If it coincides with one of the endpoints x_o or x_1 we define $c_1(t)$ as $c(2t-1)$ or $c(2t)$ correspondingly. Otherwise consider pairs $t_0 < 1/2 < t_1$ such that $c(t_0) = c(t_1)$. The set of all such pairs is closed in the product $[0,1] \times [0,1]$ and the function $|t_0 - t_1|$ reaches maximum on that set. If this maximum is equal to zero the map c is already injective. Otherwise the maximum is positive and is reached at a pair (a_1, b_1). we define the map c_1 as follows

$$c_1(t) = \begin{cases} c(t/2a_1),\ if\ \ 0 \le t \le a_1, \\ c(1/2),\ if\ a_1 \le t \le b_1, \\ c(t/2(1-b_1) + (1-b_1)/2),\ if\, b_1 \le t \le 1. \end{cases}$$

Notice that $c_1([0,1/2))$ and $c_1((1/2,1])$ are disjoint since otherwise there would exist $a' < a_1 < b_1 < b'$ such that $c(a') = c(b')$ contradicting maximality of the pair (a_1, b_1).

Now we proceed by induction. We assume that a continuous map $c_n\colon [0,1] \to\ c([0,1])$ has been constructed such that the images of intervals $(k/2^n, (k+1)/2^n),\ k = 0,\dots,2^n-1$ are disjoint. Furthermore, while we do not exclude that $c_n(k/2^n) = c_n((k+1)/2^n)$ we assume that $c_n(k/2^n) \ne c_n(l/2^n)$ if $|k-l| > 1$.

We find a_n^k, b_n^k maximizing the difference $|t_0 - t_1|$ among all pairs

$$(t_0, t_1):\ \ k/2^n \le t_0 \le t_1 \le (k+1)/2^n$$

and construct the map c_{n+1} on each interval $[k/2^n, (k+1)/2^n]$ as above with c_n in place of c and a_n^k, b_n^k in place of a_1, b_1 with the proper renormalization. As before special provision are made if c_n is injective on one of the intervals (in this case we set $c_{n+1} = c_n$) of if the image of the midpoint coincides with that of one of the endpoints (one half is cut off that the other renormalized). □

EXERCISE 1.6.1. Give an example of a path connected but not arc connected topological space.

1.7. Totally disconnected spaces and Cantor sets

On the opposite end from connected spaces are those spaces which do not have any connected nontrivial connected subsets at all.

1.7.1. Examples of totally disconnected spaces.

DEFINITION 1.7.1. A topological space $(X, \mathcal{T})$ is said to be *totally disconnected* if every point is a connected component. In other words, the only connected subsets of a totally disconnected space X are single points.

Discrete topologies (all points are open) give trivial examples of totally disconnected topological spaces. Another example is the set

$$\left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \right\}$$

with the topology induced from the real line. More complicated examples of compact totally disconnected space in which isolated points are dense can be easily constructed. For instance, one can consider the set of rational numbers $\mathbb{Q} \subset \mathbb{R}$ with the induced topology (which is not locally compact).

The most fundamental (and famous) example of a totally disconnected set is the Cantor set, which we now define.

DEFINITION 1.7.2. The (standard middle-third) *Cantor set* $C(1/3)$ is defined as follows:

$$C(1/3); = \left\{x \in \mathbb{R} : x = \sum_{i=1}^{\infty} \frac{x_i}{3^i},\ x_i \in \{0, 2\},\ i = 1, 2, \dots\right\}.$$

Geometrically, the construction of the set $C(1/3)$ may be described in the following way: we start with the closed interval $[0, 1]$, divide it into three equal subintervals and throw out the (open) middle one, divide each of the two remaining ones into equal subintervals and throw out the open middle ones and continue this process *ad infinitum*. What will be left? Of course the (countable set of) endpoints of the removed intervals will remain, but there will also be a much larger (uncountable) set of remaining "mysterious points", namely those which do not have the ternary digit 1 in their ternary expansion.

1.7.2. Lebesgue measure of Cantor sets. There are many different ways of constructing subsets of $[0, 1]$ which are homeomorphic to the Cantor set $C(1/3)$. For example, instead of throwing out the middle one third intervals at each step, one can do it on the first step and then throw out intervals of length $\frac{1}{18}$ in the middle of two remaining interval and inductively throw out the interval of length $\frac{1}{2^n 3^{n+1}}$ in the middle of each of 2^n intervals which remain after n steps. Let us denote the resulting set $\hat{C}$

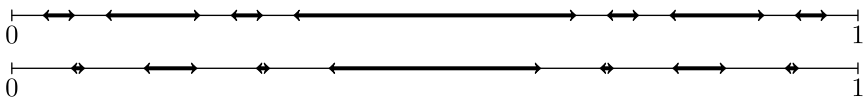

FIGURE 1.7.1. Two Cantor sets

EXERCISE 1.7.1. Prove (by computing the infinite sum of lengths of the deleted intervals) that the Cantor set $C(1/3)$ has Lebesgue measure 0 (which was to be expected), whereas the set $\hat{C}$, although nowhere dense, has *positive* Lebesgue measure.

1.7.3. Some other strange properties of Cantor sets. Cantor sets can be obtained not only as subsets of $[0,1]$, but in many other ways as well.

PROPOSITION 1.7.3. *The countable product of two point spaces with the discrete topology is homeomorphic to the Cantor set.*

PROOF. To see that, identify each factor in the product with $\{0,2\}$ and consider the map

$$(x_1, x_2, \dots) \mapsto \sum_{i=1}^{\infty} \frac{x_i}{3^i}, \quad x_i \in \{0,2\},\ i = 1, 2, \dots.$$

This map is a homeomorphism between the product and the Cantor set. □

PROPOSITION 1.7.4. *The product of two* (*and hence of any finite number*) *of Cantor sets is homeomorphic to the Cantor set.*

PROOF. This follows immediately, since the product of two countable products of two point spaces can be presented as such a product by mixing coordinates. □

EXERCISE 1.7.2. Show that the product of countably many copies of the Cantor set is homeomorphic to the Cantor set.

The Cantor set is a compact Hausdorff with countable base (as a closed subset of $[0,1]$), and it is *perfect* i.e. has no isolated points. As it turns out, it is a universal model for compact totally disconnected perfect Hausdorff topological spaces with countable base, in the sense that any such space is homeomorphic to the Cantor set $C(1/3)$. This statement will be proved later by using the machinery of metric spaces (see Theorem 3.6.7). For now we restrict ourselves to a certain particular case.

PROPOSITION 1.7.5. *Any compact perfect totally disconnected subset A of the real line $\mathbb{R}$ is homeomorphic to the Cantor set.*

PROOF. The set A is bounded, since it is compact, and nowhere dense (does not contain any interval), since it is totally disconnected. Suppose $m = \inf A$ and $M = \sup A$. We will outline a construction of a strictly monotone function $F : [0,1] \to [m,M]$ such that $F(C) = A$. The set $[m,M] \setminus A$ is the union of countably many disjoint intervals without common ends (since A is perfect). Take one of the intervals whose length is maximal (there are finitely many of them); denote it by I. Define F on the interval I as the increasing linear map whose image is the interval $[1/3, 2/3]$. Consider the longest intervals I_1 and I_2 to the right and to the left to I. Map them linearly onto $[1/9.2/9]$ and $[7/9, 8/9]$, respectively. The complement $[m,M] \setminus (I_1 \cup I \cup I_2)$ consists of four intervals which are mapped linearly onto the middle third intervals of $[0,1] \setminus ([1/9.2/9] \cup [1/3, 2/3] \cup [7/9, 8/9]$ and so on by induction. Eventually one obtains a strictly monotone bijective map $[m,M] \setminus A \to [0,1] \setminus C$ which by continuity is extended to the desired homeomorphism. □

EXERCISE 1.7.3. Prove that the product of countably many finite sets with the discrete topology is homeomorphic to the Cantor set.

1.8. Topological manifolds

At the other end of the scale from totally disconnected spaces are the most important objects of algebraic and differential topology: the spaces which locally look like a Euclidean space. This notion was first mentioned at the end of Section 1.2 and many of the examples which we have seen so far belong to that class. Now we give a rigorous definition and discuss some basic properties of manifolds.

1.8.1. Definition and some properties. The precise definition of a topological manifold is as follows.

DEFINITION 1.8.1. A *topological manifold* is a Hausdorff space X with a countable base for the topology such that every point is contained in an open set homeomorphic to a ball in $\mathbb{R}^n$ for some $n \in \mathbb{N}$. A pair (U, h) consisting of such a neighborhood and a homeomorphism $h\colon U \to B \subset \mathbb{R}^n$ is called a *chart* or a system of *local coordinates*.

picture illustrating the definition

REMARK 1.8.2. Hausdorff condition is essential to avoid certain pathologies which we will discuss laler.

Obviously, any open subset of a topological manifold is a topological manifold.

If X is connected, then n is constant. In this case it is called the *dimension* of the topological manifold. Invariance of the dimension (in other words, the fact that $\mathbb{R}^n$ or open sets in those for different n are not homeomorphic) is one of the basic and nontrivial facts of topology.

PROPOSITION 1.8.3. *A connected topological manifold is path connected.*

PROOF. Path connected component of any point in a topological manifold is open since if there is a path from x to y there is also a path from x to any point in a neighborhood of y homeomorphic to $\mathbb{R}^n$. For, one can add to any path the image of an interval connecting y to a point in such a neighborhood. If a path connected component is not the whole space its complement which is the union of path connected components of its points is also open thus contradicting connectedness. □

1.8.2. Examples and constructions.

EXAMPLE 1.8.4. The n–sphere $\mathbb{S}^n$, the n–torus $\mathbb{T}^n$ and the real projective n–space $\mathbb{R}P(n)$ are examples of n dimensional connected topological manifolds; the complex projective n–space $\mathbb{C}P(n)$ is a topological manifold of dimension $2n$.

EXAMPLE 1.8.5. Surfaces in 3-space, i.e., compact connected subsets of $\mathbb{R}^3$ locally defined by smooth functions of two variables x, y in appropriately chosen coordinate systems (x, y, z), are examples of 2-dimensional manifolds.

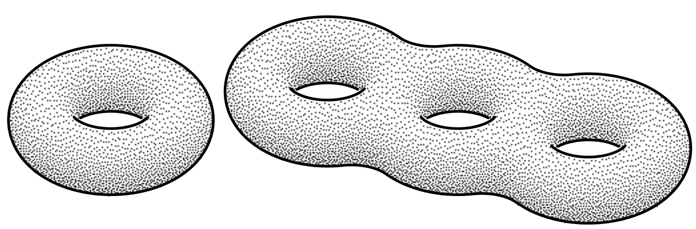

FIGURE 1.8.1. Two 2-dimensional manifolds

EXAMPLE 1.8.6. Let $F : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function and let c be a noncritical value of F, that is, there are no critical points at which the value of F is equal to c. Then $F^{-1}(c)$ (if nonempty) is a topological manifold of dimension $n - 1$. This can be proven using the Implicit Function theorem from multivariable calculus.

Among the most important examples of manifolds from the point of view of applications, are configuration spaces and phase spaces of mechanical systems (i.e., solid mobile instruments obeying the laws of classical mechanics). One can think of the configuration space of a mechanical system as a topological space whose points are different "positions" of the system, and neighborhoods are "nearby" positions (i.e., positions that can be obtained from the given one by motions of "length" smaller than a fixed number). The phase space of a mechanical system moving in time is obtained from its configuration space by supplying it with all possible velocity vectors. There will be numerous examples of phase and configuration spaces further in the course, here we limit ourselves to some simple illustrations.

EXAMPLE 1.8.7. The configuration space of the mechanical system consisting of a rod rotating in space about a fixed hinge at its extremity is the 2-sphere. If the hinge is fixed at the midpoint of the rod, then the configuration space is $\mathbb{R}P^2$.

EXERCISE 1.8.1. Prove two claims of the previous example.

EXERCISE 1.8.2. The *double pendulum* consists of two rods AB and CD moving in a vertical plane, connected by a hinge joining the extremities B and C, while the extremity A is fixed by a hinge in that plane. Find the configuration space of this mechanical system.

EXERCISE 1.8.3. Show that the configuration space of an asymmetric solid rotating about a fixed hinge in 3-space is $\mathbb{R}P^3$.

EXERCISE 1.8.4. On a round billiard table, a pointlike ball moves with uniform velocity, bouncing off the edge of the table according to the law saying that the angle of incidence is equal to the angle of reflection (see the figure). Find the phase space of this system.

FIGURE ?? Billiards on a circular table

Another source of manifolds with interesting topological properties and usually additional geometric structures is geometry. Spaces of various geometric objects are endowed with a the natural topology which is often generated by a natural metric and also possess natural groups of homeomorphisms.

The simplest non-trival case of this is already familiar.

EXAMPLE 1.8.8. The real projective space $\mathbb{R}P(n)$ has yet another description as the space of all lines in $\mathbb{R}^{n+1}$ passing through the origin. One can define the distance d between two such line as the smallest of four angles between pairs of unit vectors on the line. This distance generates the same topology as the one defined before. Since any invertible linear transformation of $\mathbb{R}^{n+1}$ takes lines into lines and preserves the origin it naturally acts by bijections on $\mathbb{R}P(n)$. Those bijections are homeomorphisms but in general they do not preserve the metric described above or any metric generating the topology.

EXERCISE 1.8.5. Prove claims of the previous example: (i) the distance d defines the same topology on the space $\mathbb{R}^{n+1}$ as the earlier constructions; (ii) the group $GL(n+1,\mathbb{R})$ of invertible linear transformations of $\mathbb{R}^{n+1}$ acts on $\mathbb{R}P(n)$ by homeomorphisms.

There are various modifications and generalizations of this basic example.

EXAMPLE 1.8.9. Consider the space of all lines in the Euclidean plane. Introduce topology into it by declaring that a base of neighborhoods of a given line L consist of the sets $N_L(a, b, \epsilon)$ where $a, b \in L$, $\epsilon > 0$ and $N_L(a, b, \epsilon)$ consist of all lines L' such that the interval of L between a and b lies in the strip of width ϵ around L'

EXERCISE 1.8.6. Prove that this defines a topology which makes the space of lines homeomorphic to the Möbius strip.

EXERCISE 1.8.7. Describe the action of the group $GL(2,\mathbb{R})$ on the Möbius strip coming from the linear action on $\mathbb{R}^2$.

This is the simplest example of the family of *Grassmann manifolds* or *Grassmannians* which play an exceptionally important role in several branches of mathematics including algebraic geometry and theory of group representation. The general Grassmann manifold $G_{k,n}$ (over $\mathbb{R}$) is defined for $i \leq k < n$ as the space of all k-dimensional affine subspaces in $\mathbb{R}^n$. In order to define a topology we again define a base of neighborhoods of a given k-space L. Fix $\epsilon > 0$ and $k+1$ points $x_1, \dots, x_{k+1} \in L$. A neighborhood of L consists of all k-dimensional spaces L' such that the convex hull of points $x_1, \dots, x_{k+1}$ lies in the ϵ-neighborhood of L'.

EXERCISE 1.8.8. Prove that the Grassmannian $G_{k,n}$ is a topological manifold. Calculate its dimension. [2]

Another extension deals with replacing $\mathbb{R}$ by $\mathbb{C}$ (and also by quaternions).

EXERCISE 1.8.9. Show that the complex projective space $\mathbb{C}P(n)$ is homeomorphic to the space of all lines on $\mathbb{C}^{n+1}$ with topology defined by a distance similarly to the case of $\mathbb{R}P(n)$

EXERCISE 1.8.10. Define complex Grassmannians, prove that they are manifolds and calculate the dimension.

1.8.3. Additional structures on manifolds. It would seem that the existence of local coordinates should make analysis in $\mathbb{R}^n$ an efficient tool in the study of topological manifolds. This, however, is not the case, because global questions cannot be treated by the differential calculus unless the coordinates in different neighborhoods are connected with each other via *differentiable* coordinate transformations. Notice that continuous functions may be quite pathological form the “normal” commonplace point of view. This requirement leads to the notion of *differentiable manifold*, which will be introduced in Chapter 4 and further studied in Chapter 10. Actually, all the manifolds in the examples above are differentiable, and it has been proved that all manifolds of dimension $n \leq 3$ have a differentiable structure, which is unique in a certain natural sense. For two-dimensional manifolds we will prove this later in Section 5.2.3; the proof for three–dimensional manifolds goes well beyond the scope of this book.

Furthermore, this is no longer true in higher dimensions: there are manifolds that possess no differentiable structure at all, and some that have more than one differentiable structure.

Another way to make topological manifolds more manageable is to endow them with a polyhedral structure, i.e., build them from simple geometric “bricks” which must fit together nicely. The bricks used for this purpose are n-simplices, shown on the figure for $n = 0, 1, 2, 3$ (for the formal definition for any n, see ??).

[2]Remember that we cannot as yet prove that dimension of a connected topological manifold is uniquely defined, i.e. that the same space cannot be a topological manifold of two different dimensions since we do not know that $\mathbb{R}^n$ for different n are not homeomorphic. The question asks to calculate dimension as it appears in the proof that the spaces are manifolds.

FIGURE ?? Simplices of dimension 0,1,2,3.

A PL-structure on an n-manifold M is obtained by representing M as the union of k-simplices, $0 \leq k \leq n$, which intersect pairwise along simplices of smaller dimensions (along "common faces"), and the set of all simplices containing each vertex (0-simplex) has a special "disk structure". This representation is called a *triangulation*. We do not give precise definitions here, because we do not study n-dimensional PL-manifolds in this course, except for $n = 1, 2$, see ?? and ??. In chapter ?? we study a more general class of topological spaces with allow a triangulation, the *simplicial complexes*.

Connections between differentiable and PL structures on manifolds are quite intimate: in dimension 2 existence of a differentiable structure will be derived from simplicial decomposition in **??**. Since each two-dimensional simplex (triangle) possesses the natural smooth structure and in a triangulation these structures in two triangles with a common edge argee along the edge, the only issue here is to "smooth out" the structure around the corners of triangles forming a triangulation.

Conversely, in any dimension any differentiable manifold can be triangulated. The proof while ingenuous uses only fairly basic tools of differential topology.

Again for large values of n not all topological n-manifolds possess a PL-structure, not all PL-manifolds possess a differentiable structure, and when they do, it is not necessarily unique. These are deep and complicated results obtained in the 1970ies, which are way beyond the scope of this book.

1.9. Orbit spaces for group actions

An important class of quotient spaces appears when the equivalence relation is given by the action of a group X by homeomorphisms of a topological space X.

1.9.1. Main definition and nice examples. The notion of a group acting on a space, which formalizes the idea of symmetry, is one of the most important in contemporary mathematics and physics.

DEFINITION 1.9.1. An *action* of a group G on a topological space X is a map $G \times X \to X$, $(g, x) \mapsto xg$ such that

(1) $(xg)h = x(g \cdot h)$ for all $g, h \in G$;

(2) $(x)e = x$ for all $x \in X$, where e is the unit element in G.

The equivalence classes of the corresponding identification are called *orbits* of the action of G on X.

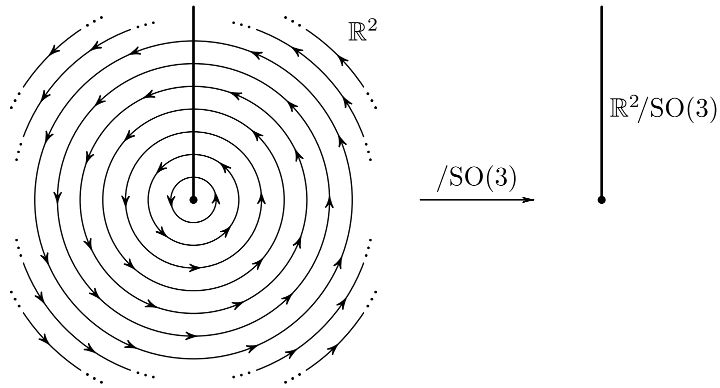

FIGURE 1.9.1. Orbits and identification space of $SO(2)$ action on $\mathbb{R}^2$

The identification space in this case is denoted by X/G and called the *quotient of* X *by* G or the *orbit space* of X under the action of G.

We use the notation xg for the point to which the element g takes the point x, which is more convenient than the notation $g(x)$ (nevertheless, the latter is also often used). To specify the chosen notation, one can say that G acts on X *from the right* (for our notation) or *from the left* (when the notation $g(x)$ or gx is used).

Usually, in the definition of an action of a group G on a space X, the group is supplied with a topological structure and the action itself is assumed continuous. Let us make this more precise.

A *topological group* G is defined as a topological Hausdorff space supplied with a continuous group operation, i.e., an operation such that the maps $(g, h) \mapsto gh$ and $g \mapsto g^{-1}$ are continuous. If G is a finite or countable group, then it is supplied with the discrete topology. When we speak of the action of a topological group G on a space X, we tacitly assume that the map $X \times G \to X$ is a continuous map of topological spaces.

EXAMPLE 1.9.2. Let X be the plane $\mathbb{R}^2$ and G be the rotation group $SO(2)$. Then the orbits are all the circles centered at the origin and the origin itself. The orbit space of $\mathbb{R}^2$ under the action of $SO(2)$ is in a natural bijective correspondence with the half-line $\mathbb{R}_+$.

The main issue in the present section is that in general the quotient space even for a nice looking group acting on a good (for example, locally compact normal with countable base) topological space may not have good separation properties. The (T1) property for the identification space is easy to ascertain: every orbit of the action must be closed. On the other hand, there does not seem to be a natural necessary and sufficient condition for the quotient space to be Hausdorff. Some useful sufficient conditions will appear in the context of metric spaces.

Still, lots of important spaces appear naturally as such identification spaces.

EXAMPLE 1.9.3. Consider the natural action of the integer lattice $\mathbb{Z}^n$ by translations in $\mathbb{R}^n$. The orbit of a point $p \in \mathbb{R}^n$ is the copy of the integer lattice $\mathbb{Z}^n$ translated by the vector p. The quotient space is homeomorphic to the torus $\mathbb{T}^n$.

An even simpler situation produces a very interesting example.

EXAMPLE 1.9.4. Consider the action of the cyclic group of two elements on the sphere S^n generated by the central symmetry: $Ix = -x$. The corresponding quotient space is naturally identified with the real projective space $\mathbb{R}P(n)$.

EXERCISE 1.9.1. Consider the cyclic group of order q generated by the rotation of the circle by the angle $2\pi/q$. Prove that the identification space is homeomorphic to the circle.

EXERCISE 1.9.2. Consider the cyclic group of order q generated by the rotation of the plane $\mathbb{R}^2$ around the origin by the angle $2\pi/q$. Prove that the identification space is homeomorphic to $\mathbb{R}^2$.

1.9.2. Not so nice examples. Here we will see that even simple actions on familiar spaces can produce unpleasant quotients.

EXAMPLE 1.9.5. Consider the following action A of $\mathbb{R}$ on $\mathbb{R}^2$: for $t \in \mathbb{R}$ let $A_t(x, y) = (x+ty, y)$. The orbit space can be identified with the union of two coordinate axis: every point on the x-axis is fixed and every orbit away from it intersects the y-axis at a single point. However the quotient topology is weaker than the topology induced from $\mathbb{R}^2$ would be. Neighborhoods of the points on the y-axis are ordinary but any neighborhood of a point on the x-axis includes a small open interval of the y-axis around the origin. Thus points on the x-axis cannot be separated by open neighborhoods and the space is (T1) (since orbits are closed) but not Hausdorff.

An even weaker but still nontrivial separation property appears in the following example.

EXAMPLE 1.9.6. Consider the action of $\mathbb{Z}$ on $\mathbb{R}$ generated by the map $x \to 2x$. The quotient space can be identified with the union of the circle and an extra point p. Induced topology on the circle is standard. However, the only open set which contains p is the whole space! See Exercise 1.10.21.

Finally let us point out that if all orbits of an action are dense, then the quotient topology is obviously trivial: there are no invariant open sets other than $\varnothing$ and the whole space. Here is a concrete example.

EXAMPLE 1.9.7. Consider the action T of $\mathbb{Q}$, the additive group of rational number on $\mathbb{R}$ by translations: put $T_r(x) = x + r$ for $r \in \mathbb{Q}$ and $x \in \mathbb{R}$. The orbits are translations of $\mathbb{Q}$, hence dense. Thus the quotient topology is trivial.

1.10. Problems

EXERCISE 1.10.1. How many non-homeomorphic topologies are there on the 2–element set and on the 3–element set?

EXERCISE 1.10.2. Let $S := \{(x, y, z) \in \mathbb{R}^3 \,|\, z = 0,\ x^2 + y^2 = 1\}$. Show that $\mathbb{R}^3 \setminus S$ can be mapped continuously onto the circle.

EXERCISE 1.10.3. Consider the product topology on the product of countably many copies of the real line. (this product space is sometimes denoted $\mathbb{R}^\infty$).

a) Does it have a countable base?

b) Is it separable?

EXERCISE 1.10.4. Consider the space $\mathcal{L}$ of all bounded maps $\mathbb{Z} \to \mathbb{Z}$ with the topology of pointwise convergece.

a) Describe the open sets for this topology.

b) Prove that $\mathcal{L}$ is the countable union of disjoint closed subsets each homeomorphic to a Cantor set.

Hint: Use the fact that the countable product of two–point spaces with the product topology is homeomorphic to a Cantor set.

EXERCISE 1.10.5. Consider the *profinite* topology on $\mathbb{Z}$ in which open sets are defined as unions (not necessarily finite) of (non-constant and infinite in both directions) arithmetic progressions. Show that it is Hausdorff but not discrete.

EXERCISE 1.10.6. Let $\mathbb{T}^\infty$ be the product of countably many copies of the circle with the product topology. Define the map $\varphi : \mathbb{Z} \to \mathbb{T}^\infty$ by

$$\varphi(n) = (\exp(2\pi in/2), \exp(2\pi in/3), \exp(2\pi in/4), \exp(2\pi in/5), \dots)$$

Show that the map φ is injective and that the pullback topology on $\varphi(\mathbb{Z})$ coincides with its profinite topology.

EXERCISE 1.10.7. Prove that $\mathbb{R}$ (the real line) and $\mathbb{R}^2$ (the plane with the standard topology) are not homeomorphic.
Hint: Use the notion of connected set.

EXERCISE 1.10.8. Prove that the interior of any convex polygon in $\mathbb{R}^2$ is homeomorphic to $\mathbb{R}^2$.

EXERCISE 1.10.9. A topological space $(X, \mathcal{T})$ is called *regular* (or $(T3)$- space) if for any closed set $F \subset X$ and any point $x \in X \setminus F$ there exist disjoint open sets U and V such that $F \subset U$ and $x \in V$. Give an example of a Hausdorff topological space which is not regular.

EXERCISE 1.10.10. Give an example of a regular topological space which is not normal.

EXERCISE 1.10.11. Prove that any open convex subset of $\mathbb{R}^2$ is homeomorphic to $\mathbb{R}^2$.

EXERCISE 1.10.12. Prove that any compact topological space is sequentially compact.

EXERCISE 1.10.13. Prove that any sequentially compact topological space with countable base is compact.

EXERCISE 1.10.14. A point x in a topological space is called *isolated* if the one-point set $\{x\}$ is open. Prove that any compact separable Hausdorff space without isolated points contains a closed subset homeomorphic to the Cantor set.

EXERCISE 1.10.15. Find all different topologies (up to homeomorphism) on a set consisting of 4 elements which make it a connected topological space.

EXERCISE 1.10.16. Prove that the intersection of a nested sequence of compact connected subsets of a topological space is connected.

EXERCISE 1.10.17. Give an example of the intersection of a nested sequence of compact path connected subsets of a Hausdorff topological space which is not path connected.

EXERCISE 1.10.18. Let $A \subset \mathbb{R}^2$ be the set of all vectors (x, y) such that $x + y$ is a rational number and $x - y$ is an irrational number. Show that $\mathbb{R}^2 \setminus A$ is path connected.

EXERCISE 1.10.19. Prove that any compact one–dimensional manifold is homeomorphic to the circle.

EXERCISE 1.10.20. Let $f : \mathbb{S}^1 \to \mathbb{R}^2$ be a continuous map for which there are two points $a, b \in \mathbb{S}^1$ such that $f(a) = f(b)$ and f is injective on $\mathbb{S}^1 \setminus \{a\}$. Prove that $\mathbb{R}^2 \setminus f(\mathbb{S}^1)$ has exactly three connected components.

EXERCISE 1.10.21. Consider the one–parameter group of homeomorphisms of the real line generated by the map $x \to 2x$. Consider three separation properties: (T2), (T1), and

(T0) For any two points there exists an open set which contains one of them but not the other (but which one is not given in advance).

Which of these properties does the quotient topology possess?

EXERCISE 1.10.22. Consider the group $SL(2,\mathbb{R})$ of all 2×2 matrices with determinant one with the topolology induced from the natural coordinate embedding into $\mathbb{R}^4$. Prove that it is a topological group.