ALGANT MASTER THESIS

COMPATIBILITY OF HOMOTOPY COLIMITS AND HOMOTOPY PULLBACKS OF SIMPLICIAL PRESHEAVES

Candidate:
Edoardo LANARI

Advisor:
Dr. Matan PRASMA

Coadvisors:
Prof. Ieke MOERDIJK
Prof. Bart DE SMIT
Prof. Marco A. GARUTI

UNIVERSITÀ DEGLI STUDI DI PADOVA

UNIVERSITEIT LEIDEN

23 June 2015

Sì, L'Amore gli aveva fatto completamente dimenticare che esisteva la Morte

Dino Buzzati, *Un Amore*

Contents

Introduction

Roads are all the same:
Some go through the woods
Some penetrate the woods
But all of them lead nowhere.

Zippo, *Ask Yourself a Question*

The problem of compatibility between colimits and pullbacks is a long-standing one. It basically consists of asking under which condition, given a functor category $\mathscr{C}^{\mathbf{I}}$ and a pullback diagram of the form

$$\begin{array}{ccc} A & \xrightarrow{x} & X \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[y]{} & Y \end{array}$$

the induced square

$$\begin{array}{ccc} \operatorname{colim}_{\mathbf{I}} A & \xrightarrow{\operatorname{colim}_{\mathbf{I}} x} & \operatorname{colim}_{\mathbf{I}} X \\ {\scriptstyle \operatorname{colim}_{\mathbf{I}} u}\downarrow & & \downarrow{\scriptstyle \operatorname{colim}_{\mathbf{I}} f} \\ \operatorname{colim}_{\mathbf{I}} & \xrightarrow[\operatorname{colim}_{\mathbf{I}} y]{} & \operatorname{colim}_{\mathbf{I}} Y \end{array}$$

is again a pullback square, this time in $\mathscr{C}$ (assuming all the displayed colimits exist).
One can ask for conditions on $\mathscr{C}$ and $\mathbf{I}$, or on the map $f : X \longrightarrow Y$.
In the ordinary case, several results are well known, and a good reference is [RA].
For instance, one has the following (cf. [RA], Theorem 2.18 and Definition 2.1):

Theorem 0.0.1. *If $\mathscr{C} := \boldsymbol{Set}$ and $\boldsymbol{I}$ is filtered, then the functor*

$$\operatorname{colim} : \boldsymbol{Set}^{\boldsymbol{I}} \longrightarrow \boldsymbol{Set}$$

commutes with finite limits.

Since the forgetful functor

$$\mathbf{Ab} \longrightarrow \mathbf{Set}$$

creates filtered colimits and finite limits, it follows that the same result holds with **Ab** in place of **Set**.

Hence, it is interesting also from an algebraic point of view, and of course it is not only limited to abelian groups. Again, [RA] gives more advanced examples involving algebraic theories.
If we move to homotopy theory, we want to replace **Set** with **sSet**, i.e. we move from sets to spaces, and we want to consider homotopy colimits and homotopy pullbacks instead of their ordinary counterparts.
We will then mix both approaches, by firstly studying conditions on maps and, in the final part, by imposing conditions on the index category **I** so as to simplify the characterization of the class of maps for which the homotopy pullback along them is preserved by the homotopy colimit functor, in a suitable sense.
This work also tries, by means of a more modern approach, to overlap with the classical work in [BF]. In that paper the authors work in the category of simplicial spaces, which is of course a category of simplicial presheaves and hence fall within the horizon of this work.
They formulate a sufficient condition for a map to be what we will call a *realization-fibration*, which involves the so-called π_*-Kan condition. This thesis is also a proposal of techniques apt to avoid this condition, which is quite difficult to check in practice.

Chapter 1

Abstract Homotopy Theory

Libero è colui che ha in sé il Tutto ed il Nulla,
avendoli resi la stessa cosa.

Free are those who are inhabited by
Everything and its Opposite,
having made them the same thing.

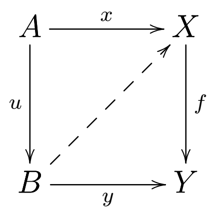
From a modern point of view, homotopy theory coincide with the study of $(\infty, 1)$-categories.
These can be modeled in various ways, the simplest of which takes the form of a category with a specified subcategory, which should be thought as a collection of arrows one would like to formally invert.
In this chapter we will review the basic definition and properties of abstract homotopy theory, using the language of model categories.
We will adopt a non-standard approach, using weak factorization systems, and morphisms between model categories (i.e. Quillen adjunctions) will be defined.
The last section of this chapter is devoted to enriched homotopy theory, which is the study of model categories whose underlying category is enriched over a monoidal (model) category in a homotopy-meaningful way, with a special emphasis on simplicial model categories.

1.1 Model Categories

Definition 1.1.1. A *relative category* is a pair $(\mathscr{C}, \mathscr{W})$ where $\mathscr{C}$ is a category and $\mathscr{W}$ is a wide subcategory, meaning it contains all the identity arrows.

This is the minimum amount of data required to do homotopy theory, but to simplify things, one often asks for more structure, which makes computations easier and provides useful tools to perform new constructions.
This enhancement of a relative category is the well-known concept of Quillen model category, which we will define in this chapter.
Throughout this section, $\mathscr{E}$ is going to be a category.

Definition 1.1.2. A map $u : A \longrightarrow B$ in $\mathscr{E}$ is said to have the *left lifting property* with respect to a map $f : X \longrightarrow Y$ in $\mathscr{E}$, and f is said to have the *right lifting property* with respect to u, if every commutative square of the form



admits a filler, as indicated by the dotted arrow.
This relation on the class of arrows of $\mathscr{E}$ will be denoted by

$$u \pitchfork f$$

Given two classes of maps $\mathscr{A}, \mathscr{B}$ in $\mathscr{E}$, we will write

$$A \pitchfork \mathscr{B}$$

to denote the fact that for any $a \in \mathscr{A},\ b \in \mathscr{B}$ it holds

$$a \pitchfork b$$

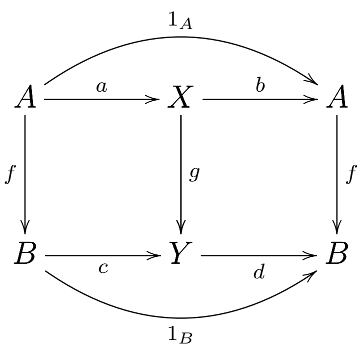
Furthermore, we will write ${}^{\pitchfork}\mathscr{A}$ to denote the class of arrows with the left lifting property with respect to any arrow in $\mathscr{A}$, and similarly for $\mathscr{A}^{\pitchfork}$.
Note that

$$\mathscr{A}^{\pitchfork} = \bigcap_{a \in \mathscr{A}} \{a\}^{\pitchfork}$$

and

$$\mathscr{A} \subset {}^{\pitchfork}\mathscr{B} \iff \mathscr{A} \pitchfork \mathscr{B} \iff \mathscr{B} \subset \mathscr{A}^{\pitchfork}$$

Definition 1.1.3. Given two arrows $f : A \longrightarrow B, g : X \longrightarrow Y$ in $\mathscr{E}$, we say that f is a retract of g if it is such as an object in the category of arrows of $\mathscr{E}$.
In other words, there must be a commutative diagram in $\mathscr{E}$ of the form:



The proof of the following properties is straightforward and left to the reader

Proposition 1.1.4. *The following facts hold:*

- *The assignments*

$$\mathscr{A} \mapsto ({}^{\pitchfork}\mathscr{A})^{\pitchfork}$$
$$\mathscr{A} \mapsto {}^{\pitchfork}(\mathscr{A}^{\pitchfork})$$

give rise to closure operators on the class of arrows of $\mathscr{E}$.
This means that they are monotone, idempotent and $\mathscr{A} \subset^{\pitchfork} (\mathscr{A})^{\pitchfork},\ \mathscr{A} \subset {}^{\pitchfork}(\mathscr{A}^{\pitchfork})$

- *Each class* ${}^{\pitchfork}\mathscr{A}$, $\mathscr{A}^{\pitchfork}$ *contains the isomorphisms, and it is closed under composition.*

- *Each class* ${}^{\pitchfork}\mathscr{A}$, $\mathscr{A}^{\pitchfork}$ *is closed under retracts, the former is closed under pushouts along any arrow and, dually, the latter is closed under pullbacks along any arrow. Moreover, the intersections* ${}^{\pitchfork}\mathscr{A} \cap \mathscr{A}$, $\mathscr{A} \cap \mathscr{A}^{\pitchfork}$ *consist of invertible arrows.*

The next definition is fundamental to the concept of model category.

Definition 1.1.5. A *weak factorization system* on $\mathscr{E}$ is a pair $(\mathscr{A}, \mathscr{B})$ of classes of maps in $\mathscr{E}$ such that:

- any arrow f in $\mathscr{E}$ can be factorized as $f = b \circ a$ with $a \in \mathscr{A}$, $b \in \mathscr{B}$;
- $\mathscr{A} = {}^{\pitchfork}\mathscr{B}$, $\mathscr{A}^{\pitchfork} = \mathscr{B}$.

The following result is proven in a straightforward way:

Proposition 1.1.6. *Given a weak factorization system* $(\mathscr{A}, \mathscr{B})$ *on* $\mathscr{E}$ *and an object* $C \in \mathscr{E}$, *we can define in a natural way a weak factorization system on the slice category*

$$\mathscr{E}/C$$

by setting

$$(\mathscr{A}_C, \mathscr{B}_C) := (\mathscr{A}, \mathscr{B})$$

and similarly for $C/\mathscr{E}$.

In what follows, $[1]$ is the poset $(\{0,1\}, <)$, with the usual ordering, so that the functor category

$$\mathscr{E}^{[1]}$$

is precisely the arrow category of $\mathscr{E}$.
We will need a more refined notion of factorization system, namely the following:

Definition 1.1.7. A triple $(\mathscr{A}, \mathscr{B}, \Theta)$ is called *functorial weak factorization system*, if $(\mathscr{A}, \mathscr{B})$ is a weak factorization system in $\mathscr{E}$ and

$$\Theta : \mathscr{E}^{[1]} \longrightarrow \mathscr{E}^{[1]} \times_{\mathscr{E}} \mathscr{E}^{[1]}$$

is a functor, which sends an arrow in $\mathscr{E}$ to a pair of composable arrows in $\mathscr{E}$ witnessing a factorization into an arrow in $\mathscr{A}$ followed by one in $\mathscr{B}$.

Definition 1.1.8. A class of arrows $\mathscr{W} \subset \mathscr{E}$ satisfies the *2-out-of-3* property if, given two composable arrows $f : A \longrightarrow B$, $g : B \longrightarrow C$, if two arrows among f, g, $g \circ f$ are in $\mathscr{W}$, then the same holds for the remaining one.

We are finally ready to give the following fundamental

Definition 1.1.9. A (Quillen) model structure on a bicomplete category $\mathscr{E}$ is a triple

$$(\mathscr{C}, \mathscr{W}, \mathscr{F})$$

of classes of maps in $\mathscr{E}$, such that:

- $\mathscr{W}$ has the 2-out-of-3 property;

- each pair $(\mathscr{C}\cap\mathscr{W},\mathscr{F})$, $(\mathscr{C},\mathscr{W}\cap\mathscr{F})$ is a functorial weak factorization system on $\mathscr{E}$.

The pair $(\mathscr{E},(\mathscr{C},\mathscr{W},\mathscr{F}))$ is called a (Quillen) model category.

Given a model category $(\mathscr{E},(\mathscr{C},\mathscr{W},\mathscr{F})$, the maps in $\mathscr{W}$ are called *weak equivalences*, those in $\mathscr{C}$ (resp. $\mathscr{C}\cap\mathscr{W}$) are called *cofibrations* (resp. *trivial cofibrations*) and those in $\mathscr{F}$ (resp. $\mathscr{F}\cap\mathscr{W}$) are called *fibrations* (resp. *trivial fibrations*).
An object $X\in\mathscr{E}$ is called *fibrant* if the unique map to the terminal object

$$X\longrightarrow *$$

is a fibration.
Dually, X is said to be *cofibrant* if the unique map from the initial object

$$\emptyset\longrightarrow X$$

is a cofibration.
Clearly, any two classes among $\mathscr{C}$, $\mathscr{W}$ and $\mathscr{F}$ determine the third, and $(\mathscr{E}^{op},(\mathscr{F}^{op},\ \mathscr{W}^{op},\ \mathscr{C}^{op}))$ is again a model category.
By factoring the map $\emptyset\longrightarrow X$ we get the so-called *cofibrant replacement* functor.
This consists of a functor

$$\mathscr{Q}(\cdot):\mathscr{C}\longrightarrow\mathscr{C}$$

such that there is a natural weak equivalence

$$q_X:\mathscr{Q}X\longrightarrow X$$

and $\mathscr{Q}X$ is cofibrant for every X in $\mathscr{C}$.
Dualizing the above construction we get the *fibrant replacement* functor

$$\mathscr{R}(\cdot):\mathscr{C}\longrightarrow\mathscr{C}$$

that comes provided with a natural weak equivalence

$$X\longrightarrow\mathscr{R}X$$

where $\mathscr{R}X$ is fibrant for any X in $\mathscr{C}$.
We give without proof the following fundamental result:

Proposition 1.1.10 ([Jo1], Proposition E.13)**.** *The class $\mathscr{W}$ of weak equivalences in a model structure is closed under retracts.*

We should now define morphisms between model categories, which ought to preserve (some part of) the given structure.

Definition 1.1.11. An adjunction between model categories (left adjoint on the left)

$$\mathscr{E}\underset{R}{\overset{L}{\rightleftarrows}}\mathscr{D}$$

is called *Quillen adjunction* if L preserves cofibrations and trivial cofibrations. If, in addition, an arrow

$$LX\longrightarrow Y$$

is a weak equivalence if and only if its adjoint map

$$X\longrightarrow RY$$

is such, then the adjunction is called *Quillen equivalence.*

Such adjunctions are to be thought as morphisms $\mathscr{E} \longrightarrow \mathscr{D}$.
The following result is useful to detect Quillen pairs, and its proof is an elementary exercise with adjuctions.

Proposition 1.1.12. *Given an adjoint pair*

$$\mathscr{E} \underset{R}{\overset{L}{\rightleftarrows}} \mathscr{D}$$

between bare categories, we have that

$$L(f) \pitchfork g \iff f \pitchfork R(g)$$

Since trivial cofibrations are characterized by having the left lifting property with respect to fibrations, the following corollary is immediate.

Corollary 1.1.13. *Given an adjoint pair*

$$\mathscr{E} \underset{R}{\overset{L}{\rightleftarrows}} \mathscr{D}$$

between model categories, L preserves trivial cofibration if and only if R preserves fibrations.

Hence a Quillen pair can also be defined as an adjoint pair in which the left adjoint preserves cofibrations and the right adjoint preserves fibrations.
We are now going to introduce the concept of properness, a stability property which will turn out to be useful when dealing with certan types of homotopy (co)limits.

Definition 1.1.14. A model category is said to be *left proper* if the pushout of a weak equivalence along any cofibration is again a weak equivalence.
This means that if we have a pushout square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ C & \xrightarrow[f']{} & D \end{array}$$

where f is a cofibration and g is a weak equivalence, then g' is a weak equivalence as well.
Dually, a model category is said to be *right proper* if the pullback of a weak equivalence along any fibration is again a weak equivalence.

Let us now recall a well-known fact.

Proposition 1.1.15. *Given a map $f : X \longrightarrow Y$ in a bicomplete category $\mathscr{E}$, one obtains two adjunctions, namely:*

$$Y \coprod_X (\cdot) : X/\mathscr{E} \rightleftarrows Y/\mathscr{E} : f^* \tag{1.1}$$

$$f_* : \mathscr{E}/X \rightleftarrows \mathscr{E}/Y : X \times_Y (\cdot) \tag{1.2}$$

An immediate corollary of Proposition 1.1.6 is the following:

Corollary 1.1.16. *Given a model category $\mathscr{E}$, for any object $C \in \mathscr{E}$ we get an induced model structure on both $\mathscr{E}/C$ and $C/\mathscr{E}$, where fibrations, weak equivalences and cofibrations are detected by the forgetful functors $\mathscr{E}/C \longrightarrow \mathscr{E}$ and $C/\mathscr{E} \longrightarrow \mathscr{E}$.*

Using these model structures and Proposition 1.1.15, we can state the following result, which proves that being left (resp. right) proper is a property which depends only on the weak equivalences of a model category.

Proposition 1.1.17 ([Re2], Proposition 2.5)**.** *Let $\mathscr{E}$ be a model category, and $f : X \longrightarrow Y$ be a map in $\mathscr{E}$.*
The following are equivalent:

- *The adjoint pair*
$$Y \coprod_X (\cdot) : X/\mathscr{E} \rightleftarrows Y/\mathscr{E} : f^*$$
is a Quillen equivalence;

- *The pushout of f along any cofibration in $\mathscr{E}$ is a weak equivalence.*

Of course, using the other adjunction 1.2 we get the analogous fact concerning right properness.

1.2 The Homotopy Category

Model categories are very useful to present a homotopy theory, and to perform calculations and constructions in an easier way.
For this reason, we will now introduce the homotopy category associated to a model category, which is a Quillen-equivalence invariant, and give some examples.
However, one should consider the fact that the homotopy category is, in a suitable sense, a truncation of the $(\infty, 1)$-category which is the real homotopy theory a model category presents (see [DK] for a detailed account of this principle).
In order to define the homotopy category of a model category, we will use a general construction which permits to formally invert a given class of arrows in a universal way. From a set-theoretical point of view, size issues may arise, but we will see that, as far as model categories are concerned, this will not be the case.

Definition 1.2.1. Given a category $\mathscr{C}$ and a class of arrows $\mathscr{W} \subset \mathscr{C}$, the *localization of $\mathscr{C}$ with respect to $\mathscr{W}$* is a pair $(\mathscr{C}[\mathscr{W}^{-1}], \gamma)$, where $\mathscr{C}[\mathscr{W}^{-1}]$ is a category and

$$\gamma : \mathscr{C} \longrightarrow \mathscr{C}[\mathscr{W}^{-1}]$$

is a functor which sends any arrow in $\mathscr{W}$ to an isomorphism, which is initial among such pairs.

More precisely, this means that given any pair $(\mathscr{M}, \delta)$, where

$$\delta : \mathscr{C} \longrightarrow \mathscr{M}$$

sends all the arrows in $\mathscr{W}$ to isomorphisms in $\mathscr{M}$, there is a unique functor

$$\mathscr{C}[\mathscr{W}^{-1}] \longrightarrow \mathscr{M}$$

such that the obvious triangle commutes.
Such localizations need not exist, basically because we lose control on the size of the hom-"sets" between pair of objects. Nevertheless, it is possible to prove the following fact:

Theorem 1.2.2 ([Ho], Theorem 1.2.10)**.** *The localization $(\mathscr{C}[\mathscr{W}^{-1}], \gamma)$ of a model category with respect to its class of weak equivalences always exists.*
Moreover, given two objects $\gamma X, \gamma Y$ in $\mathscr{C}[\mathscr{W}^{-1}]$, one has that

$$\mathscr{C}[\mathscr{W}^{-1}](\gamma X, \gamma Y) \cong \mathscr{C}(\mathscr{Q}X, \mathscr{R}Y)/\simeq$$

i.e. a quotient of the hom-set in the underlying category of the model category by a suitable homotopy relation between maps, where $\mathscr{Q}(\cdot)$, $\mathscr{R}(\cdot) : \mathscr{C} \longrightarrow \mathscr{C}$ are the cofibrant and fibrant replacement functors (respectively).

When no confusion arises, the class of weak equivalences will not be indicated, and the homotopy category will be denoted by $\mathrm{Ho}(\mathscr{C})$.

1.3 Derived Functors

The importance of Quillen adjuctions is that they can be "derived", in the sense that they descend to the homotopy category level and give rise to an adjunction as well.

Definition 1.3.1. Let $(\mathscr{C}, \mathscr{W})$ be a relative category, admitting a localization $\mathscr{C}[\mathscr{W}^{-1}]$. Given a functor $\varphi : \mathscr{C} \longrightarrow \mathscr{D}$, a *left derived functor* of φ is, whenever it exists, a choice of a right Kan extension of φ along $\gamma : \mathscr{C} \longrightarrow \mathscr{C}[\mathscr{W}^{-1}]$.

Unravelling the definition, this means we have a functor

$$\mathrm{L}\,\varphi : \mathscr{C}[\mathscr{W}^{-1}] \longrightarrow \mathscr{D}$$

together with a natural transformation

$$\varepsilon : \mathrm{L}\,\varphi \circ \gamma \Rightarrow \varphi$$

such that for any other functor

$$G : \mathscr{C}[\mathscr{W}^{-1}] \longrightarrow \mathscr{D}$$

and any other natural transformation

$$\zeta : G \circ \gamma \Rightarrow \varphi$$

there is a unique morphism of functors

$$\vartheta : G \longrightarrow \mathrm{L}\,\varphi$$

satisfying $\zeta = \varepsilon(\vartheta * \gamma)$, where $*$ denotes the horizontal composition of the involved 2-cells. The universality of right Kan extensions allows us, as it is customary, to speak of *the* left derived functor of a given functor.

Definition 1.3.2. Given a functor

$$\varphi : (\mathscr{C}, \mathscr{W}) \longrightarrow (\mathscr{D}, \mathscr{V})$$

between relative categories (not necessarily a structure-preserving one) which admits localizations with respect to the chosen classes of arrows, we define the *total left derived functor* of φ to be the left derived functor of the composition

$$\mathscr{C} \xrightarrow{\quad \varphi \quad} \mathscr{D} \xrightarrow{\quad \delta \quad} \mathscr{D}[\mathscr{V}^{-1}]$$

along $\gamma : \mathscr{C} \longrightarrow \mathscr{C}[\mathscr{W}^{-1}]$ whenever it exists.

Of course everything can be dualized in order to obtain the notion of (total) right derived functor.
We will consider the case of model categories, which are canonically relative categories by considering the weak equivalences as the distinguished class of arrows.
The following theorem is the most important of this section:

Theorem 1.3.3 ([Hir], Theorem 8.5.18)**.** *Let*

$$\mathscr{M} \underset{U}{\overset{F}{\rightleftarrows}} \mathscr{N}$$

be a Quillen adjunction between model categories.
Then the following facts hold:

- *the total left derived functor* $\mathrm{L}\,F : \mathrm{Ho}(\mathscr{M}) \longrightarrow \mathrm{Ho}(\mathscr{N})$ *exists;*
- *the total right derived functor* $\mathrm{R}\,U : \mathrm{Ho}(\mathscr{N}) \longrightarrow \mathrm{Ho}(\mathscr{M})$ *exists;*
- *we have an adjunction:*

$$\mathrm{Ho}(\mathscr{M}) \underset{\mathrm{R}\,U}{\overset{\mathrm{L}\,F}{\rightleftarrows}} \mathrm{Ho}(\mathscr{N})$$

- *if the pair is a Quillen equivalence then the derived adjunction is an equivalence of categories.*

The functor $\mathrm{L}\,F : \mathrm{Ho}(\mathscr{M}) \longrightarrow \mathrm{Ho}(\mathscr{N})$ is obtained by applying the universal property of localizations to the functor $\delta \circ F \circ \mathcal{Q} : \mathscr{M} \longrightarrow \mathrm{Ho}(\mathscr{N})$, where $\delta : \mathscr{N} \longrightarrow \mathrm{Ho}(\mathscr{N})$ is the localizing functor.
Similarly, the same universal property applied to $\gamma \circ G \circ \mathscr{R} : \mathscr{N} \longrightarrow \mathrm{Ho}(\mathscr{M})$ gives us the functor $\mathrm{R}\,G : \mathrm{Ho}(\mathscr{N}) \longrightarrow \mathrm{Ho}(\mathscr{M})$.
Loosely speaking, Quillen equivalent model categories present the same homotopy theory.

1.4 Enriched Homotopy Theory

It happens very often in the usual mathematical practice to encounter categories whose hom-sets are naturally equipped with extra algebraic structures or are objects sitting inside a richer category than that of sets.
This category which hosts the hom-objects should have some sort of inner-operation to construct composition of arrows, and this is made possible by the following definitions.

Definition 1.4.1. A *bicategory* is a 5-tuple

$$(|\mathscr{A}|, \mathscr{A}(-,-), c, 1, (\alpha, \lambda, \varrho))$$

Where:

- $|\mathscr{A}|$ is a class;
- $\mathscr{A}(-,-) : |\mathscr{A}| \times |\mathscr{A}| \longrightarrow Cat$ is a functor (witnessing the fact that for any pair of objects there is a small category of morphisms between them).
  The objects of $\mathscr{A}(A,B)$ are called *morphisms* from A to B, and the arrows are called *2-cells* (we will adopt the usual notation $\alpha : f \Rightarrow g$ to denote the fact that α is a 2-cell between the 1-cells f and g);
- $c = (c_{ABC} : \mathscr{A}(B,C) \times \mathscr{A}(A,B) \longrightarrow \mathscr{A}(A,C))_{A,B,C \in |\mathscr{A}|}$ is a family of functors (called the *composition law*);
- $1 = (1_A : A \longrightarrow A)_{A \in |\mathscr{A}|}$, i.e. an arrow $A \longrightarrow A$ for any A in $|\mathscr{A}|$, called the *identity arrow on* A;
- $\alpha = (\alpha_{ABCD} : c_{ACD} \circ (c_{ABC} \times Id) \Rightarrow c_{ABD} \circ (Id \times c_{BCD}))_{A,B,C,D \in |\mathscr{A}|}$ is a family of natural isomorphisms of functors (the *associator* for the composition);
- $$\lambda = (\lambda_{AB} : Id \Rightarrow c_{AAB} \circ (1_A \times Id))_{A,B \in |\mathscr{A}|}$$
  $$\varrho = (\varrho_{AB} : Id \Rightarrow c_{ABB} \circ (Id \times 1_B))_{A,B \in |\mathscr{A}|}$$
  are the natural isomorphisms for the identities.

These data are required to satisfy two coherence axioms: one for the associativity on four objects and one for the identity (see [Bor1], Definition 7.7.1 for the details).

The composition on objects will be denoted by the usual $\circ$, while on 2-cells we have two sort of compositions:

given two 2-cells $\alpha : f \Rightarrow g,\ \beta : g \Rightarrow h$ in $\mathscr{A}(A,B)$

we obtain a *vertical* composition

$$\beta \odot \alpha : f \Rightarrow h$$

On the other hand, given

$$\vartheta : f \longrightarrow f' \text{ in } \mathscr{A}(A,B)$$
$$\eta : g \longrightarrow g' \text{ in } \mathscr{A}(B,C)$$

we obtain a *horizontal* composition

$$\eta * \vartheta : g \circ f \Rightarrow g' \circ g'$$

A bicategory is an algebraic model for a weak 2-category.

Definition 1.4.2. A *monoidal category* is a bicategory with one object.

Unravelling the definition we see that we can think of a monoidal category as a 5-tuple

$$(\mathscr{V}, \otimes, I, \lambda, \varrho)$$

where $\mathscr{V}$ is an ordinary category (which corresponds to $\mathscr{A}(*, *)$, where $\mathscr{A}$ is the bicategory which is part of the definition of monoidal category) and

$$\otimes : \mathscr{V} \times \mathscr{V} \longrightarrow \mathscr{V}$$

is a functor which satisfies some coherence properties concerning associativity and unitality up to specified isomorphisms.
Namely, I is an object of $\mathscr{V}$ which behaves as a unit for $\otimes$, and the other data are the involved coherence isomorphisms.
If the tensor functor is commutative (in the obvious sense) then the category is called *symmetric.*
A symmetric monoidal category $\mathscr{V}$ is called *closed* if for any X in $\mathscr{V}$ the functor

$$- \otimes X : \mathscr{V} \longrightarrow \mathscr{V}$$

admits a right adjoint.

Example 1.4.3. The following are two well-known examples, encountered very often in the mathematical practice:

- Any category Mod_R of modules over a commutative ring R, endowed with the usual tensor product, is a closed symmetric monoidal category in which the unit is given by the base ring R.

- Any category with finite products is symmetric monoidal with respect to the cartesian product.
  The unit is given by the terminal object and it is closed if and only if it is cartesian closed.

A closed symmetric monoidal category is a good place to enrich categories, in the following sense:

Definition 1.4.4. Given a closed symmetric monoidal category $\mathscr{V}$, a $\mathscr{V}$-category consists of a 4-tuple

$$(|\mathscr{C}|, \mathscr{C}(-,-), \circ_{\mathscr{V}}, j)$$

where:

- $|\mathscr{C}|$ is a class, called the *class of objects* of $\mathscr{C}$;

- $\mathscr{C}(-,-) : |\mathscr{C}| \times |\mathscr{C}| \longrightarrow Ob(\mathscr{V})$ is a function, and we call $\mathscr{C}(A, B)$ the hom-object from A to B;

- $$\circ_{\mathscr{V}} = (\circ_{\mathscr{V}\,ABC} : \mathscr{C}(B, C) \otimes \mathscr{C}(A, B) \longrightarrow \mathscr{C}(A, C))_{A,B,C \in |\mathscr{C}|}$$

  is a collection of morphisms in $\mathscr{V}$, called the *composition* law;

- $$j = (j_A : I \longrightarrow \mathscr{C}(A,A))_{A\in|\mathscr{C}|}$$
  is another family of morphisms in $\mathscr{V}$, called the *identity* arrow on A.

Such that a bunch of coherence diagrams for associativity and unitality commute, see [MK] for a general reference.

Example 1.4.5. Examples of enrichments abound:

- Any ordinary category is trivially enriched over $(Set, \times)$;
- Additive categories are, by definition, enriched over the category of abelian groups;
- Mark Hovey showed in [Ho] that the homotopy category of any model category is canonically enriched over Ho(**sSet**), i.e. the homotopy category of simplicial sets (with the Kan-Quillen model structure, which will be discussed in the next chapter).

When we want to consider model categories enriched over a certain closed symmetric monoidal category, something more should be asked, in order to make this enrichment homotopy-meaningful.
Firstly, we need the following general definition:

Definition 1.4.6 ([Ho], Definition 4.1.12)**.** Given categories $\mathscr{C}, \mathscr{D}, \mathscr{E}$, an *adjunction of two variables* from $\mathscr{C} \times \mathscr{D}$ to $\mathscr{E}$ is a quintuple

$$(\otimes, \mathrm{Hom}_r, \mathrm{Hom}_l, \varphi_r, \varphi_l)$$

where:

- $$\begin{aligned} \otimes &: \mathscr{C} \times \mathscr{D} \longrightarrow \mathscr{E} \\ \mathrm{Hom}_r &: \mathscr{D}^{op} \times \mathscr{E} \longrightarrow \mathscr{C} \\ \mathrm{Hom}_l &: \mathscr{C}^{op} \times \mathscr{E} \longrightarrow \mathscr{D} \end{aligned}$$
  are functors;
- $$\mathscr{C}(C, \mathrm{Hom}_r(D,E)) \xleftarrow{\varphi_r} \mathscr{E}(C \otimes D, E) \xrightarrow{\varphi_l} \mathscr{D}(D, \mathrm{Hom}_l(C,E))$$
  are natural isomorphisms (in each of the involved variables).

If each of the involved categories $\mathscr{C}, \mathscr{D}$ and $\mathscr{E}$ are endowed with model structure, we can give the homotopy-correct version of these objects:

Definition 1.4.7 ([Ho], Definition 4.2.1)**.** An adjunction of two variables

$$(\otimes, \mathrm{Hom}_r, \mathrm{Hom}_l, \varphi_r, \varphi_l)$$

from $\mathscr{C} \times \mathscr{D}$ to $\mathscr{E}$ where the three categories are endowed with a model structure, is called a *Quillen adjunction of two variables* if given a cofibration $f : U \longrightarrow V$ in $\mathscr{C}$ and a cofibration $g : W \longrightarrow X$ in $\mathscr{D}$, the induced map

$$f \odot g : U \otimes X \coprod_{U \otimes W} V \otimes W \longrightarrow V \otimes X$$

is a cofibration in $\mathscr{E}$, which is trivial if either f or g is such.

The last property often goes under the name of *pushout product axiom.*

Remark 1.4.8. It is straightforward to prove that, if C is a cofibrant object of $\mathscr{C}$, then we have a Quillen pair

$$\mathscr{D} \underset{\mathrm{Hom}_l(C,-)}{\overset{C\otimes -}{\rightleftarrows}} \mathscr{E}$$

Similarly, if D is cofibrant in $\mathscr{D}$, then we have a Quillen pair

$$\mathscr{C} \underset{\mathrm{Hom}_r(D,-)}{\overset{-\otimes D}{\rightleftarrows}} \mathscr{E}$$

Finally, if E is a fibrant object of $\mathscr{E}$, we have the Quillen pair

$$\mathscr{D} \underset{\mathrm{Hom}_l(-,E)}{\overset{\mathrm{Hom}_r(-,E)}{\rightleftarrows}} \mathscr{C}^{op}$$

The following lemma is used very often and its proof is basically adjunctions-yoga.

Lemma 1.4.9. *Given an adjunction of two variables between model categories*

$$(\otimes, \mathrm{Hom}_r, \mathrm{Hom}_l, \varphi_r, \varphi_l) : \mathscr{C} \times \mathscr{D} \longrightarrow \mathscr{E}$$

the following facts are equivalent:

- *the adjunction is a Quillen one;*
- *given a cofibration $g : W \longrightarrow X$ in $\mathscr{D}$ and a fibration $p : Y \longrightarrow Z$ in $\mathscr{E}$, the induced map*

$$\mathrm{Hom}_{r,\square}(g,p) : \mathrm{Hom}_r(X,Y) \longrightarrow \mathrm{Hom}_r(X,Z) \times_{\mathrm{Hom}_r(W,Z)} \mathrm{Hom}_r(W,Y)$$

  is a fibration in $\mathscr{C}$, which is trivial if either g or p is such.
- *given a cofibration $f : U \longrightarrow V$ in $\mathscr{C}$ and a fibration $p : Y \longrightarrow Z$ in $\mathscr{E}$, the induced map*

$$\mathrm{Hom}_{l,\square}(f,p) : \mathrm{Hom}_l(V,Y) \longrightarrow \mathrm{Hom}_l(V,Z) \times_{\mathrm{Hom}_l(U,Z)} \mathrm{Hom}_l(U,Y)$$

  is a fibration in $\mathscr{D}$, which is trivial if either f or p is such.

The next definition is the homotopical enhancement of the notion of monoidal category.

Definition 1.4.10 ([Ho], Definition 4.2.6)**.** A *monoidal model category* is a closed monoidal category $\mathscr{C}$ (meaning that the functor $\otimes : \mathscr{C} \times \mathscr{C} \longrightarrow \mathscr{C}$ is part of an adjunction of two variables) such that:

- the adjunction of two variables is a Quillen one;
- the natural map $\mathcal{Q}S \otimes X \longrightarrow S \otimes X$ is a weak equivalence for any cofibrant X in $\mathscr{C}$, where $\mathcal{Q}(\cdot) : \mathscr{C} \longrightarrow \mathscr{C}$ is the functorial cofibrant replacement functor and S is the unit of the tensor functor.

In order to capture the action of a monoidal category (which should be thought of as a ring) acting on another category, we need to define modules.

Definition 1.4.11. Let $\mathscr{C}$ be a monoidal category. A *right $\mathscr{C}$-module structure* on a category $\mathscr{D}$ is a triple

$$(\odot, a, r)$$

where:

- $\odot : \mathscr{D} \times \mathscr{C} \longrightarrow \mathscr{D}$ is a functor;
- $a = (a_{K,L} : (- \odot K) \odot L \longrightarrow - \odot (K \otimes_{\mathscr{C}} L))_{K,L \in \mathscr{C}}$ is a family of natural isomorphisms;
- $r : - \odot S_{\mathscr{C}} \longrightarrow Id_{\mathscr{D}}$ is a natural isomorphism.

These data should satisfy associativity and unitality as in Def.4.1.6 [Ho].

Again, we need to ask a little bit more to get the right model-categorical version of modules.

Definition 1.4.12 ([Ho], Definition 4.2.18)**.** Given a monoidal model category $\mathscr{D}$, a model category $\mathscr{C}$ is said to be a *$\mathscr{D}$-module* if it is a $\mathscr{D}$ module in the above sense, and the tensor functor $\odot : \mathscr{C} \times \mathscr{D} \longrightarrow \mathscr{C}$ is part of a Quillen adjunction of two variables.
Moreover, the arrow

$$X \odot \mathcal{Q}S \longrightarrow X \odot S$$

must be a weak equivalence in $\mathscr{C}$ for any cofibrant X in $\mathscr{C}$, where $\mathcal{Q}(\cdot) : \mathscr{D} \longrightarrow \mathscr{D}$ is the functorial cofibrant replacement functor and S is the unit of the tensor functor.

In the next chapter we will describe the homotopy theory of spaces $\mathbf{sSet}_{Quillen}$, and we will see it is a monoidal model category.
Hence, the following definition makes sense, and it involves a wide class of important model categories.

Definition 1.4.13. A **sSet**-module is called a *simplicial model category.*

It is straightforward to prove that a simplicial model category is the same as a tensored and cotensored model category enriched in simplicial sets, i.e. provided with two functors

$$\otimes : \mathscr{C} \times \mathbf{sSet} \longrightarrow \mathscr{C}$$
$$\text{Hom} : \mathscr{C} \times \mathbf{sSet}^{op} \longrightarrow \mathscr{C}$$

fitting into a suitable adjunction (the adjunction of two variables in the definition), and such that the tensor functor satisfies the pushout-product axiom (see also Definition 9.1.6 of [Hir]).
We will denote the simplicial set of maps between two objects X, Y in a simplicial model category $\mathscr{M}$ by $\text{Map}_{\mathscr{M}}(X, Y)$ (dropping the $\mathscr{M}$ when no confusion can arise).

We can now give the following definition, which will be widely used throughout the rest of this work:

Definition 1.4.14 ([Hir], Definition 9.5.2)**.** Given two maps $f, g : X \longrightarrow Y$ in a simplicial model category $\mathscr{M}$, we say that they are *simplicially homotopic* if they both lie in the same path component of $\text{Map}(X, Y)$ (recall that the 0-simplices of the simplicial mapping space are precisely the morphism in $\mathscr{M}$).

Note that, by definition, being simplicially homotopic is always an equivalence relation. The next definition will yield a more intuitive characterization of simplicial homotopy.

Definition 1.4.15. A *generalized interval* is a simplicial set J presentable as the union of finitely many copies of $\Delta[1]$ with vertices identified so that its geometric realization $|J|$ is homeomorphic in **Top** to the unit interval I.

It is straightforward to prove that the following proposition holds:

Proposition 1.4.16 ([Hir], Proposition 9.5.6)**.** *Let $\mathscr{M}$ be a simplicial model category. Two maps $f, g : X \longrightarrow Y$ are simplicially homotopic if and only if there exists a generalized interval J and a map of simplicial sets $H : J \longrightarrow \mathrm{Map}(X, Y)$ such that the inclusions of the bottom and top vertices yield, respectively:*

$$H \circ i_0 = f, \;\; H \circ i_1 = g$$

A nice fact about this new notion of homotopy is the following (see Def. 1.2.4 of [Ho] for the definition of left and right homotopy between maps).
It states that for "good" objects, the notion of homotopy between maps is modeled by an enriched version thereof.

Proposition 1.4.17 ([Hir], Proposition 9.5.24)**.** *Let $\mathscr{M}$ be a simplicial model category and let X, Y be objects of $\mathscr{M}$.*

- *If $g, h : X \longrightarrow Y$ are simplicially homotopic maps in $\mathscr{M}$, then they are both left homotopic and right homotopic.*
- *If X is cofibrant and Y is fibrant, then the simplicial, left and right homotopy relations on $\mathscr{M}(X, Y)$ coincide and they are equivalence relations.*

Chapter 2

The Homotopy Theory of Spaces

La Lacerazione dell'Io fu ricucita da uno Sguardo: "Ora, So".
Interi abissi d'istante colmi, e in un attimo tutto appassì.

The Laceration of the Ego was sewn by a Glance: "Now, I Know".
Whole abysses suddenly filled up, and everything instantly withered.

In this chapter we are going to deal with the homotopy theory of spaces, modeled by simplicial sets, which can be thought of as a combinatorial version of topological spaces, from a homotopical point of view (in a sense which will be made more precise in what follows).
To do so, we will present this homotopy theory by means of a model structure on the category of simplicial sets, which can be proven to be Quillen equivalent to the Quillen model structure on topological spaces, defined in Theorem 2.4.19 of [Ho].
In the last section we will define a powerful homotopy invariant for our spaces, namely a combinatorial version of homotopy groups, which can be used to detect weak equivalences.

2.1 Model Structure on the Category of Spaces

Define Δ as the skeleton of the category of finite totally ordered sets.
Concretely, its set of objects is:

$$|\Delta| := \{[n] := (\{0, \cdots, n\}, <) : n \in \mathbb{N}\}$$

The arrows are given by order-preserving maps, i.e. functors between the categories canonically associated to these posets.
We will denote by

$$d_n^i : [n] \longrightarrow [n+1]$$
$$(\text{resp. } s_n^i : [n+1] \longrightarrow [n])$$

the map that skips the i-th term (resp. collapses the i-th and its successor to the i-th term).
We will drop the n when no confusion arises.

Definition 2.1.1. The category of *simplicial sets* is the category of presheaves over Δ, i.e.

$$\mathbf{sSet} := \mathbf{Set}^{\Delta^{op}}$$

We will denote representable presheaves in the following way:

$$\Delta[n] := \mathbf{sSet}(-, [n])$$

for each $n \in \mathbb{N}$.
We recall a well-known construction, usually named after Grothendieck:

Definition 2.1.2. Given any presheaf $K : \mathscr{C}^{op} \longrightarrow \mathbf{Set}$, we can define a pair $(\int K, F_K)$- called the *Grothendieck construction* on K- where:

- $\int K$ is the category having as objects the arrows $\mathscr{C}(-, c) \longrightarrow K$ (for each $c \in \mathscr{C}$), i.e. elements $x \in K(c)$, and as morphisms the maps $c \longrightarrow d$ in $\mathscr{C}$ making the obvious triangle commute;
- $F_K : \int K \longrightarrow \mathscr{C}$ is the forgetful functor sending $f : \mathscr{C}(-, c) \longrightarrow K$ to c.

This construction is the key ingredient in the precise formulation of the result which states that any presheaf is a colimit of representables.

Theorem 2.1.3 ([Bor1], Theorem 2.15.6)**.** *Let $K : \mathscr{C}^{op} \longrightarrow \mathbf{Set}$ be a presheaf, then one has*

$$\operatorname*{colim}_{\mathscr{C}(-,c) \longrightarrow K \in \int K} \mathscr{C}(-, c) \cong K$$

Turning to simplicial sets, i.e. presheaves on Δ, we will use the Grothendieck construction on representables to define two important families of simplicial sets, employed in the definition of the Kan-Quillen model structure on **sSet**.
Let us set:

$$\partial\Delta[n] = \operatorname*{colim}_{\Delta[m] \longrightarrow \Delta[n] \in \partial \int \Delta[n]} \Delta[m]$$

where $\partial \int \Delta[n]$ is the full subcategory of $\int \Delta[n]$ obtained by removing the object $1_{[n]}$.
Similarly, for any $0 \leq k \leq n$, we define:

$$\Lambda^k[n] = \operatorname*{colim}_{\Delta[m] \longrightarrow \Delta[n] \in \widehat{\int \Delta[n]}} \Delta[m]$$

where $\widehat{\int \Delta[n]}$ is the full subcategory of $\int \Delta[n]$ obtained by removing the objects $1_{[n]}$ and $d^k : [n-1] \longrightarrow [n]$ (the latter only if $n \geq 1$).
Given $A, C \in \mathbf{sSet}$, let $[A, C]$ denote the set of equivalence classes of maps $A \longrightarrow C$ with respect to the simplicial homotopy relation.
We are now ready to state the following result, which describes the homotopy theory of spaces:

Theorem 2.1.4 ([Jo2], Theorem 3.4.1)**.** *There is a model structure on **sSet**, which describes the homotopy theory of spaces, in which:*

- *the cofibrations are the monomorphisms;*

- *the fibrations are the maps with the right lifting property with respect to the set of inclusions*

$$\{\Lambda^k[n] \subset \Delta[n]\}_{0\leq k\leq n}$$

- *the weak equivalences are the maps $f : A \longrightarrow B$ inducing an isomorphism of sets*

$$[f, K] : [B, K] \longrightarrow [A, K]$$

  for any fibrant simplicial set K.
  Furthermore, such model structure is symmetric monoidal with respect to the cartesian product, it is simplicial (hence it is what goes under the name of ***sSet***-algebra*, or* monoidal **sSet**-model category*, see Definition 4.2.20 of [Ho]) and proper.*

Fibrant simpicial sets are usually called *Kan complexes*.
The following theorem is a general result which has proven useful in several situations. We will use it to compare the model structure on **sSet** and that on **Top** (the latter is defined in Theorem 2.4.19 of [Ho]).

Theorem 2.1.5. *Let $\mathscr{C}$ be a cocomplete category and $\mathscr{B}$ be a small category. Given a functor $F : \mathscr{B} \longrightarrow \mathscr{C}$, there is a unique (up to isomorphism) colimit preserving functor $|-|_F : \boldsymbol{Sets}^{\mathscr{B}^{op}} \longrightarrow \mathscr{C}$ such that the following diagram commutes up to an invertible 2-cell:*

$$\begin{array}{ccc} \mathscr{B} & \xrightarrow{\quad Y \quad} & \boldsymbol{Sets}^{\mathscr{B}^{op}} \\ {\scriptstyle F}\big\downarrow & \overset{\simeq}{\Longrightarrow} & \swarrow {\scriptstyle \exists! |-|_F} \\ \mathscr{C} & & \end{array}$$

where Y stands for the Yoneda embedding.
Such an $|-|_F$ can be realized as the left Kan extension of F along the Yoneda embedding Y, i.e.:

$$|G|_F = \mathrm{Lan}_Y\, F(G) = \int^{m\in\mathscr{B}} \boldsymbol{Set}^{\mathscr{B}^{op}}(Y(m), G)\cdot Fm \simeq \int^{m\in\mathscr{B}} Gm\cdot Fm$$

where $\cdot$ is the copower, i.e. the tensoring of $\mathscr{C}$ over Sets.
Finally, $|-|_F$ fits into an adjoint pair:

$$\boldsymbol{Sets}^{\mathscr{B}^{op}} \underset{N_F}{\overset{|-|_F}{\rightleftarrows}} \mathscr{C}$$

where $N_F(c) := \mathscr{C}(F(-), c)$, and this correspondence induces an equivalence of categories:

$$\mathrm{Fun}(\mathscr{B}, \mathscr{C}) \simeq \mathrm{Adj}(\boldsymbol{Sets}^{\mathscr{B}^{op}}, \mathscr{C})$$

where the latter denotes the category of adjunctions from $\boldsymbol{Sets}^{\mathscr{B}^{op}}$ to $\mathscr{C}$.

Applying this result to $F : \Delta \longrightarrow \mathbf{Top}$, where $\mathbf{Top}$ is a convenient category of topological spaces (e.g. compactly generated Hausdorff spaces) and

$$F([n]) := \{x \in \mathbb{R}^{n+1} : \sum_{i=0}^{n} x_i = 1\} = \Delta^n$$

with the obvious action on arrows, we get an adjoint pair

$$\mathbf{sSet} \underset{Sing}{\overset{|-|}{\rightleftarrows}} \mathbf{Top}$$

where $\mathrm{Sing}(X)_n := \mathbf{Top}(\Delta^n, X)$.
The functor $|-| : \mathbf{sSet} \longrightarrow \mathbf{Top}$ is called *geometric realization* and its right adjoint is the *singular complex* functor.
The following theorem asserts precisely that the homotopy theory of spaces can be modeled either by topological spaces (endowed with the model structure of Theorem 2.4.19 of [Ho]) or by simplicial sets as well.

Theorem 2.1.6. *([Ho], Theorem 3.6.7) The adjoint pair*

$$\boldsymbol{sSet} \underset{Sing}{\overset{|-|}{\rightleftarrows}} \mathbf{Top}$$

is a Quillen equivalence.
Furthermore, the geometric realization preserves finite limits and fibrations.

2.2 Homotopy groups of spaces

It is possible to define homotopy groups also for pointed Kan complexes (where a point is a map $x : \Delta[0] = * \longrightarrow K$), in such a way that, given a fibrant $K \in \mathbf{sSet}$, one gets:

$${\pi_n}^{\mathbf{sSet}}(K, x) \cong {\pi_n}^{\mathbf{Top}}(|K|, |x|)$$

as it is proven in Proposition 3.6.3 [Ho].
Moreover, a map $f : A \longrightarrow B$ is a weak equivalence if and only if it induces isomorphisms

$${\pi_n}^{\mathbf{sSet}}(K_A, x) \cong {\pi_n}^{\mathbf{sSet}}(K_B, \hat{f}(x))$$

for any vertex $x \in K_A$, where $K_A,\ K_B$ are fibrant replacements of A and B respectively.
To define such groups in a functorial way, we proceed in two steps.

Definition 2.2.1. Given a Kan complex K, we define $\pi_0(K)$ as the quotient of K_0 with respect to the relation

$$x \simeq y \iff \exists s \in K_1 : d^1(s) = x,\ d^0(s) = y$$

(which can be proven to be an equivalence relation).

We now use this path-components functor to define homotopy group functors π_n for any $n \geq 0$, in complete analogy with what one could do in the category of topological spaces.

Definition 2.2.2. Given a Kan complex K and a point $x : * \longrightarrow K$, let W_x be the Kan complex fitting into the following pullback square:

$$\begin{array}{ccc} W_x & \longrightarrow & \mathrm{Map}(\Delta[n], K) \\ \downarrow & & \downarrow \scriptstyle{\mathrm{Map}(i,K)} \\ * & \xrightarrow[x]{} & \mathrm{Map}(\partial\Delta[n], K) \end{array}$$

We set

$$\pi_n(K, x) := \pi_0(W_x)$$

Unravelling the definition, we see that the result is what one might expect.
Indeed, let $i : \partial\Delta[n] \longrightarrow \Delta[n]$ denote the canonical inclusion and let $!_X : X \longrightarrow *$ be the unique map from any space X to the terminal object.
We have that

$$\pi_n(K, x) = \{[f] : f \in \mathbf{sSet}(\Delta[n], K),\ f \circ i = x \circ !_{\Delta[n]}\}$$

where $[f]$ denotes the equivalence class with respect to the equivalence relation $\sim$ on $\mathbf{sSet}(\Delta[n], K)$ defined by

$$f \sim g \iff \exists H : \Delta[n] \times \Delta[1] \longrightarrow K$$

such that

$$H \circ i_0 = f$$
$$H \circ i_1 = g$$
$$H \circ i \times \mathrm{Id}_{\Delta[1]} = x \circ !_{\partial\Delta[n] \times \Delta[1]}$$

where $i_0 : \Delta[n] \times \{0\} \longrightarrow \Delta[n] \times \Delta[1]$, $i_1 : \Delta[n] \times \{1\} \longrightarrow \Delta[n] \times \Delta[1]$ are the canonical inclusions.

Chapter 3

Homotopy (Co)Limit Functors on Functor Categories

Gli occhi di Lei erano un frammento
di Eternità,
strappato dal Baratro del Nulla.

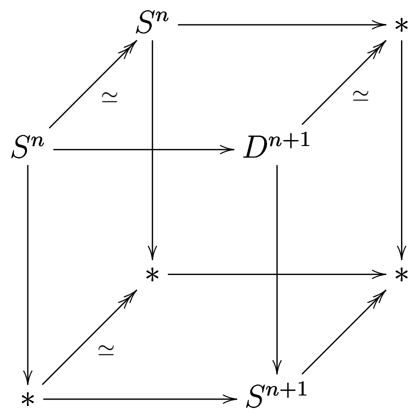
Her eyes were a fragment of Eternity,
saved from the Nothingness chasm.

In this chapter we want to briefly introduce the notion of homotopy colimit, comparing it with the ordinary colimit functor.
The former can be thought as a *homotopy-correction* of the latter, and although several models for the homotopy colimit functor are known (and they are weakly equivalent to one another), we will choose one precise model, to simplify things.
The advantage will be that our functor exists at the level of the model category, allowing for easier computations.
It is well known that, given a cocomplete category $\mathscr{C}$ and a small category $\mathbf{I}$, (a choice of) the colimit functor $\mathscr{C}^{\mathbf{I}} \longrightarrow \mathscr{C}$ fits into an adjunction

$$\operatorname{colim} : \mathscr{C}^{\mathbf{I}} \rightleftarrows \mathscr{C} : \Delta$$

where $\Delta : \mathscr{C} \longrightarrow \mathscr{C}^{\mathbf{I}}$ is the constant (or diagonal) functor.
However, the colimit functor is not always homotopy meaningful, in the sense that it does not necessarily send pointwise weak equivalences between diagrams to weak equivalences.

Example 3.0.3. A well known example of a pathological behaviour (from the homotopical point of view) of the colimit functor is the following, which lives in **Top**:



The front and back faces are pushouts, and we clearly have a pointwise weak equivalence of diagrams, which does not induce a weak equivalence on their colimits, since S^{n+1} is not contractible.

There are several ways to fix this homotopical issue, and we will not deal with the broadest generality.
One possibility is to put a suitable model structure on $\mathscr{M}^{\mathbf{I}}$, where $\mathscr{M}$ is a given model category, and $\mathbf{I}$ is a shape category we are interested in.
Unfortunately, this is not always possible, but there is a wide class of model categories for which this approach works: the so-called *cofibrantly generated* model categories.
A definition thereof can be found in [Ho] (Definition 2.1.17), and they are basically model categories for which (trival) cofibrations are generated by two sets of arrows (respectively) under a bunch of operations.
This class includes the most known examples, like chain complexes of modules over a ring or spaces, and it is possible to prove the following theorem:

Theorem 3.0.4 ([Hir], Theorem 11.6.1)**.** *Let* $\boldsymbol{I}$ *be a small category, and* $\mathscr{M}$ *a cofibrantly generated model category.*
There is a cofibrantly generated model structure on $\mathscr{M}^{\boldsymbol{I}}$*, called the* projective model structure*, such that the weak equivalences and the fibrations are the pointwise ones.*
Moreover, if $\mathscr{M}$ *is a simplicial model category, then* $\mathscr{M}^{\boldsymbol{I}}$ *can be endowed with a simplicial structure compatible with this model structure just defined.*

The simplicial structure is defined as follows:

$$\odot : \mathscr{M}^{\mathbf{I}} \times \mathbf{sSet} \longrightarrow \mathscr{M}^{\mathbf{I}}$$

is constructed pointwise, namely:

$$(F \odot K)(i) := F(i) \otimes_{\mathscr{M}} K \ \forall i \in \mathbf{I}$$

The same goes for the exponentiation:

$$(\cdot)^* : \mathscr{M}^{\mathbf{I}} \times \mathbf{sSet}^{op} \longrightarrow \mathscr{M}^{\mathbf{I}}$$

where we set

$$F^K(i) := F(i)^K$$

Mapping spaces are therefore forced to be defined as

$$\mathrm{Map}(F,G)_n := \mathscr{M}^{\mathbf{I}}(F \odot \Delta[n], G)$$

It then follows that the functor

$$- \odot \Delta[1] : \mathscr{M}^{\mathbf{I}} \longrightarrow \mathscr{M}^{\mathbf{I}}$$

is a functorial cylinder object ([Ho], Definition 1.2.4) on cofibrant objects in $\mathscr{M}^{\mathbf{I}}$.
It is straightforward to notice that the following is a Quillen pair

$$\mathrm{colim} : \mathscr{M}^{\mathbf{I}} \rightleftarrows \mathscr{M} : \Delta$$

which induces an adjunction

$$\mathrm{hocolim} : \mathrm{Ho}(\mathscr{C}^{\mathbf{I}}) \rightleftarrows \mathrm{Ho}(\mathscr{C}) : \mathrm{Ho}(\Delta)$$

Definition 3.0.5. The left adjoint functor hocolim : $\mathrm{Ho}(\mathscr{C}^{\mathbf{I}}) \longrightarrow \mathrm{Ho}(\mathscr{C})$ is called *homotopy colimit functor*, and by definition it is computed as follows:

$$\mathrm{hocolim}(F) = \gamma(\mathrm{colim}(\mathcal{Q}F))$$

where $\mathcal{Q} : \mathscr{C}^{\mathbf{I}} \longrightarrow \mathscr{C}^{\mathbf{I}}$ is the functorial cofibrant replacement and $\gamma : \mathscr{C} \longrightarrow \mathrm{Ho}(\mathscr{C})$ is the localization functor.

Notice that our functor can be lifted to a functor:

$$\mathrm{hocolim} : \mathscr{C}^{\mathbf{I}} \longrightarrow \mathscr{C}$$

by setting

$$\mathrm{hocolim}(F) = \mathrm{colim}(\mathcal{Q}F)$$

as well, and this is the form in which it will be mostly used throughout this thesis.
Since, in general, $\mathrm{Ho}(\mathscr{C}^{\mathbf{I}})$ and $\mathrm{Ho}(\mathscr{C})^I$ are not equivalent, we see that homotopy colimits possibly differ from colimits in the homotopy category.
If we restrict ourselves to the index category $\mathbb{P}$, given by

$$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & & \\ \bullet & & \end{array}$$

we get the *homotopy pushout* functor.
If $\mathscr{M}$ is a left proper model category, another simpler model for homotopy pushouts can be given.

Proposition 3.0.6 ([Hir], Proposition 19.5.3). *Let $\mathscr{M}$ be a left proper model category, and suppose given an object $F \in \mathscr{M}^{\mathbb{P}}$ of the form*

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ {\scriptstyle g}\downarrow & & \\ B & & \end{array}$$

By factoring one map (e.g. $f : A \longrightarrow C$) into a cofibration followed by a weak equivalence

$$A \xrightarrow{c(f)} \tilde{C} \xrightarrow{w(f)} C$$

we get a new diagram $\tilde{F}$ by substituting f with $c(f)$.
Setting

$$\operatorname{hocolim} F := \operatorname{colim} \tilde{F}$$

we get a left derived functor of the pushout functor, i.e. (a model of) the homotopy pushout functor.

Notice that, by construction, the following is a pushout square

$$\begin{array}{ccc} A & \xrightarrow{c(f)} & \tilde{C} \\ {\scriptstyle g}\downarrow & & \downarrow \\ B & \longrightarrow & \operatorname{hocolim} F \end{array}$$

Obviously, a dual argument applies to homotopy pullbacks.

Chapter 4

Compatibility of Homotopy Colimits and Homotopy Pullbacks of Simplicial Presheaves

Today a young man on acid realized that all matter is merely energy condensed to a slow vibration, that we are all one consciousness experiencing itself subjectively, there is no such thing as death, life is only a dream, and we are the imagination of ourselves.

Bill Hicks

In this chapter, which constitutes the central part of the thesis, we are going to study under which conditions the homotopy colimit functor on diagrams of spaces preserves homotopy cartesian squares.
More precisely, we are going to study the class of maps $p : X \longrightarrow Y$ for which homotopy cartesian squares of simplicial presheaves of the form:

$$\begin{array}{ccc} E' & \longrightarrow & E \\ {\scriptstyle p'}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ B' & \xrightarrow[i]{} & B \end{array}$$

(for any $i : B' \longrightarrow B$) are sent to homotopy cartesian squares of spaces by the homotopy colimit functor.
We will prove that this notion can be checked locally (local-to-global principles), in a sense which will be made more precise in the rest of the chapter, and that the situation becomes easier when the index category is particularly nice, i.e. it satisfies the so-called *siftedness* conditions (more precisely, a homotopical version thereof).
Homotopy descent properties of spaces will be defined and proved, and they will play a crucial role in this work.

4.1 Realization-Fibrations

Definition 4.1.1. Let $\mathbf{I}$ be a small category. A functor $P : \mathbf{I}^{op} \longrightarrow \mathbf{sSet}$ will be called -for sake of conciseness- an $\mathbf{I}$-presheaf, and the category $\mathbf{sSet}^{\mathbf{I}^{op}}$ will be denoted by $\mathrm{Psh}(\mathbf{I})$.

Given an $\mathbf{I}$-presheaf V, we will sometimes denote its homotopy colimit by $|V|_{\mathbf{I}}$. Thanks to Definition 3.0.5 it can be computed by choosing a cofibrant replacement

$$\mathcal{Q}V \longrightarrow V$$

in $\mathrm{Psh}(\mathbf{I})$, and then by setting

$$|V|_{\mathbf{I}} := \mathrm{colim}_{\mathbf{I}} \mathcal{Q}V$$

Remark 4.1.2. When we need to use a specific cofibrant replacement it will be explicitly mentioned, otherwise we adopt the one which is part of the data of the model category in question.
It can be proved that the result does not depend on the choice of the cofibrant replacement, up to weak equivalence.
The same remark holds for factorizations of arrows.

We will use a precise model for the homotopy pullback functor throughout this chapter.

Given a cospan in $\mathrm{Psh}(\mathbf{I})$ of the form:

$$\begin{array}{ccc} & & B \\ & & \downarrow{\scriptstyle f} \\ A & \xrightarrow[g]{} & C \end{array}$$

we will use the fact that $\mathrm{Psh}(\mathbf{I})$, endowed with the projective model structure, is a right proper model category, to compute the homotopy pullback of such diagram.
Indeed, thanks to the dual of Proposition 3.0.6, it is enough to replace one of the two maps by fibration and take the ordinary pullback.
More precisely, we will factor $f = p(f) \circ c(f)$, where $c(f) : B \longrightarrow \hat{B}$ is a trivial cofibration and $p(f) : \hat{B} \longrightarrow C$ is a fibration, and define:

$$A \overset{h}{\times}_C B := A \times_C \hat{B}$$

Basic properties of (left/right) proper model categories that we will use are treated in [Hir], chapter 13, as well as homotopy pullbacks in such context.

Definition 4.1.3. A commutative square

$$\begin{array}{ccc} A & \longrightarrow & C \\ \downarrow & & \downarrow{\scriptstyle f} \\ B & \longrightarrow & D \end{array}$$

is called *homotopy cartesian* if the natural arrow

$$A \longrightarrow B \overset{h}{\times}_D C$$

is a weak equivalence.

Remark 4.1.4. Just by checking the definition of homotopy (co)limit functor, i.e. a being a right (resp. left) derived functor, one sees that these functors depend only on the class of weak equivalences of $\mathscr{C}$, since this holds for the construction of the homotopy category.
Therefore, given a commutative square, the property of being cartesian depends only on the weak equivalences which are part of the model structure, and different model structures with the same underlying class of weak equivalences have isomorphic homotopy categories and isomorphic homotopy (co)limit functors.

We are interested in studying the following class of maps:

Definition 4.1.5. A map $p : E \longrightarrow B$ in $\mathrm{Psh}(\mathbf{I})$ is said to be a *realization fibration* (the class of which is denoted by $\mathbf{RF}$) if, for any homotopy cartesian square:

$$\begin{array}{ccc} E' & \longrightarrow & E \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ B' & \longrightarrow & B \end{array}$$

the square obtained by applying the homotopy colimit functor $|\cdot|_{\mathbf{I}}$ is again a homotopy cartesian square, this time in $\mathbf{sSet}$:

$$\begin{array}{ccc} |E'|_{\mathbf{I}} & \longrightarrow & |E|_{\mathbf{I}} \\ {\scriptstyle |p'|_{\mathbf{I}}}\downarrow & & \downarrow{\scriptstyle |p|_{\mathbf{I}}} \\ |B'|_{\mathbf{I}} & \longrightarrow & |B|_{\mathbf{I}} \end{array}$$

Hence, these are precisely the maps for which the homotopy colimit is compatible with the homotopy pullback along any arrow.
Let us now list some basic facts about homotopy cartesian squares, which will be frequently used throughout the rest of this work.
We will refer to the first one as "associativity of homotopy pullbacks", or "Pasting Lemma".

Lemma 4.1.6 ([Hir], Proposition 13.3.15)**.** *Given the following commutative diagram in a right proper model category, in which the right-hand side square is homotopy cartesian, we have that the outer square is homotopy cartesian if and only if the left-hand side square is such.*

$$\begin{array}{ccccc} A & \longrightarrow & C & \longrightarrow & E \\ \downarrow & & \downarrow & & \downarrow \\ B & \longrightarrow & D & \longrightarrow & F \end{array}$$

Lemma 4.1.7. *A commutative square in a right proper model category*

$$\begin{array}{ccc} A & \longrightarrow & C \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ B & \longrightarrow & D \end{array}$$

where f is a weak equivalence is homotopy cartesian if and only if g is a weak equivalence.

Proof. It follows from Proposition 13.3.10 in [Hir] that different factorizations of an arrow in a weak equivalence followed by a fibration produce weakly equivalent homotopy pullbacks (using the construction of Proposition 3.0.6).
Hence we can factor f as $1_D \circ f$, so that the given square is homotopy cartesian if and only if

$$A \longrightarrow B \times_D D \simeq B$$

is a weak equivalence, and this arrow is indeed g. □

Proposition 4.1.8. *The following properties are enjoyed by realization fibrations:*

1. *Stability under weak equivalences, i.e. given a commutative square of the form*

$$\begin{array}{ccc} E' & \xrightarrow{\simeq} & E \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ B' & \xrightarrow[\simeq]{} & B \end{array}$$

   where the horizontal arrows are weak equivalences, then p is a realization-fibration if and only if p' is such.

2. *Stability under homotopy base change, i.e. if we have a homotopy cartesian square*

$$\begin{array}{ccc} E' & \longrightarrow & E \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ B' & \longrightarrow & B \end{array}$$

   where p is in ***RF****, then p' is such.*

3. *Every weak equivalence is in* ***RF****.*

Proof. To begin with, consider a commutative square in which the horizontal arrows are weak equivalences

$$\begin{array}{ccc} F & \xrightarrow{\varphi} & E \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ C & \xrightarrow[\gamma]{} & B \end{array}$$

We want to show that p' is in **RF** if and only if p is such.
Firstly, assume p is a realization-fibration. Given a homotopy cartesian square

$$\begin{CD} F @>{a}>> E' \\ @V{q}VV @VV{p'}V \\ C @>>{b}> B' \end{CD}$$

we can paste it together with the one given by the assumptions, thus getting

$$\begin{CD} F @>>> E' @>{\simeq}>> E \\ @V{q}VV @V{p'}VV @VV{p}V \\ C @>>> B' @>>{\simeq}> B \end{CD}$$

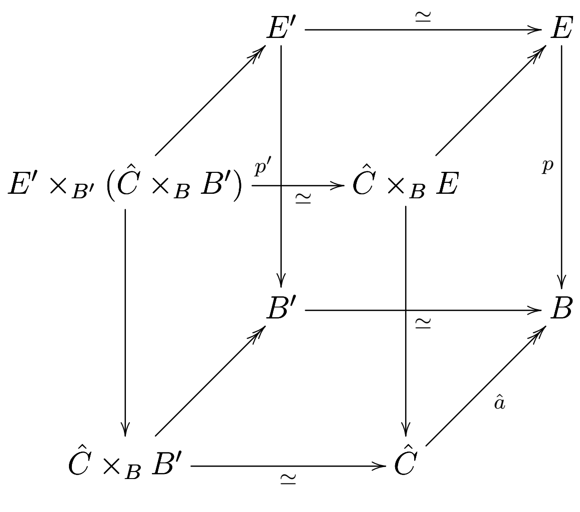
Thanks to Lemma 4.1.6, after having applied the realization functor we get the following diagram, in which the right-hand side square and the outer one are homotopy cartesian, since p is a realization-fibration:

$$\begin{CD} |F|_{\mathbf{I}} @>>> |E'|_{\mathbf{I}} @>{\simeq}>> |E|_{\mathbf{I}} \\ @V{|q|_{\mathbf{I}}}VV @V{|p'|_{\mathbf{I}}}VV @VV{|p|_{\mathbf{I}}}V \\ |C|_{\mathbf{I}} @>>> |B'|_{\mathbf{I}} @>>{\simeq}> |B|_{\mathbf{I}} \end{CD}$$

It follows from Lemma 4.1.6 that the left-hand side square is homotopy cartesian.
Conversely, assume $p' : E' \longrightarrow B'$ is a realization-fibration. Factor $b : C \longrightarrow B$ into a weak equivalence followed by a fibration:

$$C \xrightarrow{\simeq} \hat{C} \xrightarrow{\hat{a}} B$$

Construct the following commutative cube:



where the displayed weak equivalences follow from the right properness of $\mathbf{sSet}^{\mathbf{I}^{op}}$.
Now, the left-hand side square is homotopy cartesian, being a pullback square of a span

in which one arrow is a fibration. Moreover, it is weakly equivalent to the right-hand side square.
Since p' is a realization fibration, it follows that the realization of the left-hand side is homotopy cartesian, and being weakly equivalent to the realization of the right-hand side square, the latter must be homotopy cartesian as well.
Since the square

$$\begin{array}{ccc} F & \xrightarrow{a} & E' \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p'} \\ C & \xrightarrow[b]{} & B' \end{array}$$

is homotopy cartesian by assumption, we get that the natural arrow $F \longrightarrow \hat{C} \times_B E$ is a weak equivalence, so that we have the following diagram:

$$\begin{array}{ccccc} F & \xrightarrow{\simeq} & \hat{C} \times_B E & \twoheadrightarrow & E \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle p} \\ C & \xrightarrow[\simeq]{} & \hat{C} & \twoheadrightarrow & B \end{array}$$

where both squares are homotopy cartesian.
After having realized it we get two pasted squares, both homotopy cartesian: the left-hand side one thanks to Lemma 4.1.7, the right-hand side one thanks to what we have proved so far.
An application of Lemma 4.1.6 now concludes the proof.
Let us now prove the second statement. Consider a commutative diagram of the following shape, in which both squares are homotopy cartesian and p is a realization fibration:

$$\begin{array}{ccccc} F & \longrightarrow & E' & \longrightarrow & E \\ \downarrow & & {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ C & \longrightarrow & B' & \longrightarrow & B \end{array}$$

Then, by hypothesis and by the Pasting Lemma, we get that the right-hand square and the outer rectangle of the following diagram are homotopy cartesian, hence the result follows from 4.1.6.

$$\begin{array}{ccccc} |F|_{\mathbf{I}} & \longrightarrow & |E'|_{\mathbf{I}} & \longrightarrow & |E|_{\mathbf{I}} \\ \downarrow & & {\scriptstyle |p'|_{\mathbf{I}}}\downarrow & & \downarrow{\scriptstyle |p|_{\mathbf{I}}} \\ |C|_{\mathbf{I}} & \longrightarrow & |B'|_{\mathbf{I}} & \longrightarrow & |B|_{\mathbf{I}} \end{array}$$

To prove the last point it is enough to observe that a square in which an arrow is a weak equivalence is homotopy cartesian if and only if the parallel arrow is a weak equivalence too (4.1.7), and that $|\cdot|_{\mathbf{I}}$ preserves them.

□

4.2 First Local-to-Global Principle

We will now deal with concepts linked to descent properties for spaces, which will be useful to prove some closure properties involved in passing from local to global situations. In this section, $\mathbf{J}$ will always be a small category.

Definition 4.2.1. Consider two functors $E, B : \mathbf{J} \longrightarrow \mathscr{M}$, where $\mathscr{M}$ is a model category. A natural transformation $p : E \Rightarrow B$ is said to be $\mathbf{J}$-*equifibered.* if for every morphism $\alpha : J_1 \longrightarrow J_2$ in $\mathbf{J}$, the square

$$\begin{array}{ccc} E(J_1) & \xrightarrow{E(\alpha)} & E(J_2) \\ {\scriptstyle p_{J_1}}\downarrow & & \downarrow{\scriptstyle p_{J_2}} \\ B(J_1) & \xrightarrow[B(\alpha)]{} & B(J_2) \end{array}$$

is homotopy cartesian.

We will now recall a couple of results concerning descent properties for spaces. Firstly, let us state the ordinary case, which holds for any Grothendieck Topos.

Proposition 4.2.2 ([Re1], Proposition 3.7)**.** *Let* $Y : \boldsymbol{I} \longrightarrow \mathscr{E}$ *be a functor, where* $\mathscr{E}$ *is a Grothendieck topos, and suppose given an arrow* $f : A \longrightarrow \operatorname{colim}_{\boldsymbol{I}} Y$.
We obtain a diagram $X : \boldsymbol{I} \longrightarrow \mathscr{E}$ *by pulling back the natural arrows* $Y_i \longrightarrow \operatorname{colim}_{\boldsymbol{I}}$ *along* f*, i.e. we have pullback squares for any* $i \in \boldsymbol{I}$ *of the form:*

$$\begin{array}{ccc} X_i & \longrightarrow & A \\ \downarrow & & \downarrow{\scriptstyle f} \\ Y_i & \longrightarrow & \operatorname{colim}_{\boldsymbol{I}} Y \end{array}$$

In this situation the natural arrow

$$\operatorname{colim}_{\boldsymbol{I}} X \longrightarrow A$$

is an isomorphism. Hence, in particular, the following square is a pullback:

$$\begin{array}{ccc} X_i & \longrightarrow & \operatorname{colim}_{\boldsymbol{I}} X \\ \downarrow & & \downarrow{\scriptstyle f} \\ Y_i & \longrightarrow & \operatorname{colim}_{\boldsymbol{I}} Y \end{array}$$

To prove the homotopy descent properties for spaces, we need to use a couple of results from [Re1]. Let us a begin with a definition.

Definition 4.2.3. A simplicial presheaf $Y : \mathbf{I} \longrightarrow \mathbf{sSet}$ is called a *homotopy colimit diagram* if the natural map

$$\operatorname{hocolim}_{\mathbf{I}} Y \longrightarrow \operatorname{colim}_{\mathbf{I}} Y$$

is a weak equivalence.

Observe that the former definition does not depend on the cofibrant replacement chosen to compute $\mathrm{hocolim}_{\mathbf{I}} Y$.

Theorem 4.2.4 ([Re1], Theorem 1.4)**.** *Let $f : X \longrightarrow Y$ be a map in $\mathrm{Psh}(\boldsymbol{I})$, with Y homotopy colimit diagram.*

- *If for any $i \in \boldsymbol{I}$ the following square is homotopy cartesian*

$$\begin{array}{ccc} X_i & \longrightarrow & colim_{\boldsymbol{I}} X \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle colim_{\boldsymbol{I}}(f)} \\ Y_i & \longrightarrow & colim_{\boldsymbol{I}} Y \end{array}$$

  then X is a homotopy colimit diagram too.

- *If X is a homotopy colimit diagram and for any arrow $\alpha : i \longrightarrow j$ in $\boldsymbol{I}$, the square*

$$\begin{array}{ccc} X_j & \xrightarrow{X(\alpha)} & X_i \\ {\scriptstyle f_j}\downarrow & & \downarrow{\scriptstyle f_i} \\ Y_j & \xrightarrow[Y(\alpha)]{} & Y_i \end{array}$$

  is homotopy cartesian, then each diagram of the form

$$\begin{array}{ccc} X_i & \longrightarrow & colim_{\boldsymbol{I}} X \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle colim_{\boldsymbol{I}}(f)} \\ Y_i & \longrightarrow & colim_{\boldsymbol{I}} Y \end{array}$$

  is homotopy cartesian as well.

We now have all the tools to prove the next proposition, which describes what is meant by homotopy descent property.

Proposition 4.2.5. *The following homotopy-descent properties hold:*

1. *Let $V : \boldsymbol{J} \longrightarrow \boldsymbol{sSet}$ be a functor, and suppose we have a fibration $p' : E' \longrightarrow \mathrm{hocolim}_{\boldsymbol{J}} V$. Then, if we set*

$$U(\cdot) := \mathcal{Q}V(\cdot) \times_{\mathrm{hocolim}_{\boldsymbol{J}} V} E' : \boldsymbol{J} \longrightarrow \boldsymbol{sSet}$$

   there is a weak equivalence

$$\mathrm{hocolim}_{\boldsymbol{J}} U \longrightarrow E'$$

   Moreover, this implies that given any two maps $p : E \longrightarrow B$ and $j : \mathrm{hocolim}_{\boldsymbol{J}} V \longrightarrow B$, where p is a fibration, we have a weak equivalence:

$$\mathrm{hocolim}_{\boldsymbol{I}}(holim(\mathcal{Q}V(\cdot) \longrightarrow B \leftarrow E)) \longrightarrow holim(\mathrm{hocolim}_{\boldsymbol{I}} V(\cdot) \to B \leftarrow E) \qquad (4.1)$$

   In particular, if j is a weak equivalence, we get that the induced map

$$\mathrm{hocolim}_{\boldsymbol{J}} U \longrightarrow E$$

   is a weak equivalence as well.

2. *Let $f : U \longrightarrow V$ be an equifibered map in $\boldsymbol{sSet^J}$. Then for each object $i \in \boldsymbol{J}$, the square:*

$$\begin{array}{ccc} \mathcal{Q}U(i) & \to & \operatorname{hocolim}_{\boldsymbol{J}} U \\ \downarrow & & \downarrow \\ \mathcal{Q}V(i) & \to & \operatorname{hocolim}_{\boldsymbol{J}} V \end{array}$$

is homotopy cartesian in ***sSet***

Proof. Observe that, thanks to Proposition 4.2.2, we have $\operatorname{colim} U \cong E'$.

Since $p' : E' \cong \operatorname{colim}_{\mathbf{J}} U \longrightarrow \operatorname{colim} \mathcal{Q}V$ is a fibration, it follows that the square

$$\begin{array}{ccc} U_i & \to & \operatorname{colim}_{\mathbf{J}} U \cong E' \\ \downarrow & & \downarrow{\scriptstyle p} \\ \mathcal{Q}V_i & \longrightarrow & \operatorname{hocolim}_{\mathbf{J}} V \end{array}$$

is homotopy cartesian.
Moreover, $\mathcal{Q}V$ is a homotopy colimit diagram, being cofibrant (see, for instance, Theorem 11.6.8 of [Hir]). It follows that U is a homotopy colimit diagram, thanks to Theorem 4.2.4, hence the arrow

$$\operatorname{hocolim}_{\mathbf{J}} U \cong \operatorname{colim}_{\mathbf{J}} \mathcal{Q}U \longrightarrow \operatorname{colim}_{\mathbf{J}} U \cong E'$$

is a weak equivalence.
Now, consider a fibration $p : E \longrightarrow B$, and a map $j : \operatorname{hocolim}_{\mathbf{J}} V \longrightarrow B$.
The following diagram is obtained by pulling back p along j, and then pulling back the resulting arrow along the colimit inclusion $\mathcal{Q}V(j) \longrightarrow \operatorname{colim}_{\mathbf{J}} \mathcal{Q}V \cong \operatorname{hocolim}_{\mathbf{J}} V$:

$$\begin{array}{ccccc} U(j) & \longrightarrow & E' & \longrightarrow & E \\ \downarrow & & \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ \mathcal{Q}V(j) & \to & \boldsymbol{\operatorname{hocolim}_{\mathbf{J}} V} & \underset{j}{\longrightarrow} & B \end{array} \tag{4.2}$$

Thanks to what we have just proven, we know that there is a weak equivalence

$$\operatorname{hocolim}_{\mathbf{J}} U \longrightarrow E'$$

which can be written in a fancy way as

$$\operatorname{hocolim}_{\mathbf{I}}(\operatorname{holim}(\mathcal{Q}V(\cdot) \longrightarrow B \leftarrow E)) \longrightarrow \operatorname{holim}(\operatorname{hocolim}_{\mathbf{I}} V(\cdot) \to B \leftarrow E)$$

Let us prove the second point. Observe that the equifibrancy of f implies the equifibrancy of Qf, since for any map $j \longrightarrow i$ in $\mathbf{I}$,we have a commutative diagram of the form:

$$\begin{array}{ccccccc} & & U(j) & \longrightarrow & U(i) & & \\ & \overset{\simeq}{\nearrow} & \downarrow & \overset{\simeq}{\nearrow} & \downarrow & & \\ \mathcal{Q}U(j) & \longrightarrow & \mathcal{Q}U(i) & & & & \\ \downarrow & & V(j) & \longrightarrow & V(i) & & \\ & \overset{\simeq}{\nearrow} & \downarrow & \overset{\simeq}{\nearrow} & & & \\ \mathcal{Q}V(j) & \longrightarrow & \mathcal{Q}V(i) & & & & \end{array}$$

and given two weakly equivalent squares, Proposition 13.3.13 of [Hir] tells us that the front face is homotopy cartesian iff the back face is such.
Moreover, both $\mathcal{Q}U$ and $\mathcal{Q}V$ are homotopy colimit diagram, hence the following diagram is homotopy cartesian (thanks to Theorem 4.2.4):

$$\begin{array}{ccc} \mathcal{Q}U_i & \longrightarrow & \operatorname{colim}_{\mathbf{I}}\mathcal{Q}U \simeq \operatorname{hocolim}_{\mathbf{I}} U \\ \downarrow & & \downarrow \\ \mathcal{Q}V_i & \longrightarrow & \operatorname{colim}_{\mathbf{I}}\mathcal{Q}V \simeq \operatorname{hocolim}_{\mathbf{I}} V \end{array}$$

□

The following corollary gives a sufficient condition for a map to be a realization-fibration, namely that of being equifibered.

Corollary 4.2.6. *Consider a homotopy cartesian square in* $\mathbf{sSet}^{\mathbf{J}}$*, as depicted below. If f is equifibered, then the diagram obtained by applying $|\cdot|_{\mathbf{J}}$ is again homotopy cartesian.*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ Y' & \longrightarrow & Y \end{array}$$

Proof. Since $\mathbf{sSet}^{\mathbf{J}}_{proj}$ is again proper, and fibration and weak equivalences are such pointwise, it follows that a homotopy cartesian square in $\mathbf{sSet}^{\mathbf{J}}$ is pointwise such.
Applying the functor $\mathcal{Q}(\cdot)$ to the previous square we get another one for which the right-hand side map is again equifibered, thanks to the stability property of equifibered maps under weak equivalences.
In the following diagram, the two adiacent squares are homotopy cartesian (for the right-hand side square we apply Proposition 4.2.5, and for the left-hand one we just use the definition of equifibrancy):

$$\begin{array}{ccccc} \mathcal{Q}X'(j) & \longrightarrow & \mathcal{Q}X(j) & \longrightarrow & \operatorname{colim}_{\mathbf{J}}\mathcal{Q}X \\ \downarrow & & \downarrow{\scriptstyle \mathcal{Q}f_j} & & \downarrow{\scriptstyle \operatorname{hocolim}(f)} \\ \mathcal{Q}Y'(j) & \longrightarrow & \mathcal{Q}Y(j) & \longrightarrow & \operatorname{colim}_{\mathbf{J}}\mathcal{Q}Y \end{array}$$

We thus get that the outer rectangle is homotopy cartesian as well.
Now, factor the map $\operatorname{colim}_{\mathbf{J}}\mathcal{Q}X \longrightarrow \operatorname{colim}_{\mathbf{J}}\mathcal{Q}Y$ into a weak equivalence followed by a fibration:

$$\operatorname{colim}_{\mathbf{J}}\mathcal{Q}X \xrightarrow{\simeq} C \longrightarrow \operatorname{colim}_{\mathbf{J}}\mathcal{Q}Y$$

Consider the following two pullback squares pasted together (which, by construction, are also homotopy cartesian):

$$\begin{array}{ccccc} U(i) & \longrightarrow & E & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{Q}Y'(j) & \longrightarrow & \operatorname{colim}_{\mathbf{J}}\mathcal{Q}Y' & \longrightarrow & \operatorname{colim}_{\mathbf{J}}\mathcal{Q}Y \end{array}$$

Hence, the outer square is the homotopy pullback of:

$$\begin{array}{ccc} & & C \\ & & \downarrow \\ \mathcal{Q}Y'(i) & \longrightarrow & \operatorname{colim}_{\mathbf{J}} \mathcal{Q}Y \end{array}$$

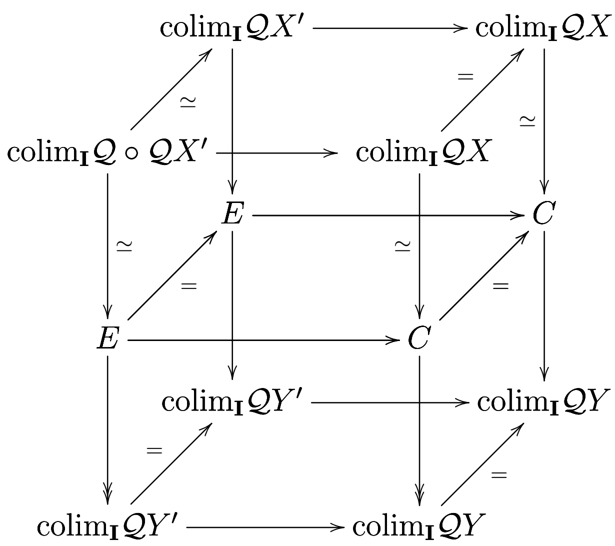
Given the fact that the following square is homotopy cartesian, as already observed, we get a natural weak equivalence $\mathcal{Q}X' \longrightarrow U$

$$\begin{array}{ccc} \mathcal{Q}X'(j) & \longrightarrow & \operatorname{colim}_{\mathbf{J}} \mathcal{Q}X \\ \downarrow & & \downarrow \\ \mathcal{Q}Y'(j) & \longrightarrow & \operatorname{colim}_{\mathbf{J}} \mathcal{Q}Y \end{array}$$

This yields weak equivalences

$$\operatorname{colim}_{\mathbf{J}} \mathcal{Q} \circ \mathcal{Q}X' \simeq \operatorname{colim}_{\mathbf{J}} \mathcal{Q}U \simeq E$$

where the last weak equivalence follows from 4.2.5.
These weak equivalences fit into the following commutative diagram:



The two front faces are homotopy cartesian by construction, hence if we paste them we get a homotpy cartesian square, and the above diagram says it is weakly equivalent to

$$\begin{array}{ccc} \operatorname{colim}_{\mathbf{J}} \mathcal{Q}X' & \longrightarrow & \operatorname{colim}_{\mathbf{J}} \mathcal{Q}X \\ \downarrow & & \downarrow \\ \operatorname{colim}_{\mathbf{J}} \mathcal{Q}Y' & \longrightarrow & \operatorname{colim}_{\mathbf{J}} \mathcal{Q}Y \end{array}$$

which has to be homotopy cartesian as well.
This means precisely that the realization of

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ Y' & \longrightarrow & Y \end{array}$$

is homotopy cartesian, which is what we wanted to prove. □

The next result is a first step in studying properties of homotopy-glueing of realization-fibrations, giving a sufficient condition for a map between diagrams of simplicial presheaves, which is pointwise a realization fibration, to be globally such.
We will thus refer to it as the *"first local-to-global principle"*.

Theorem 4.2.7. *Let $V, W : \boldsymbol{J} \longrightarrow \mathrm{Psh}(\boldsymbol{I})$ be functors and let $h : W \Rightarrow V$ be a natural transformation between them. Suppose that h is equifibered , and that for each $J \in \boldsymbol{J}$, the map $h_J : W(J) \longrightarrow V(J)$ is a realization-fibration of $\boldsymbol{I}$-presheaves.*
In this case,

$$\mathrm{hocolim}_{\boldsymbol{J}}\, h : \mathrm{hocolim}_{\boldsymbol{J}}\, W \longrightarrow \mathrm{hocolim}_{\boldsymbol{J}}\, V$$

is again in $\boldsymbol{RF}$.

Proof. We know that $\mathcal{Q}h : \mathcal{Q}W \longrightarrow \mathcal{Q}V$ is again equifibered, and $\mathcal{Q}h_j$ is a realization fibration thanks to Proposition 4.1.8. So we are left with proving that

$$\mathrm{colim}_{\mathbf{J}}\, \mathcal{Q}W \longrightarrow \mathrm{colim}_{\mathbf{J}}\, \mathcal{Q}V$$

is a realization fibration.
Define $B := \mathrm{colim}_{\mathbf{J}}\, \mathcal{Q}V = \mathrm{hocolim}_{\mathbf{J}}\, V$, and factorize $\mathrm{hocolim}_{\mathbf{J}}\, h : \mathrm{hocolim}_{\mathbf{J}}\, W \longrightarrow \mathrm{hocolim}_{\mathbf{J}}\, V$ into a weak equivalence followed by a fibration

$$\mathrm{colim}_{\mathbf{J}}\, \mathcal{Q}W \xrightarrow{\ \simeq\ j\ } E \xrightarrow{\ p\ } \mathrm{colim}_{\mathbf{J}}\, \mathcal{Q}V$$

Let $U(\cdot) := \mathcal{Q}V(\cdot) \times_B E : \mathbf{J} \longrightarrow \mathrm{Psh}(\mathbf{I})$.
As h is equifibered, thanks to the second point in Proposition 4.2.5, we see that the natural map $\mathcal{Q}W \longrightarrow U$ is a weak equivalence, since the outer square in the following diagram

$$\begin{array}{ccccc} \mathcal{Q}W(i) & & \longrightarrow & & \mathrm{colim}_{\mathbf{J}}\, \mathcal{Q}W \\ & \searrow^{\simeq} & & & \downarrow{\scriptstyle \simeq} \\ {\scriptstyle \mathcal{Q}(h)_i}\downarrow & & U(i) & \longrightarrow & E \\ & \swarrow_{f_i} & & & \downarrow{\scriptstyle p} \\ \mathcal{Q}V(i) & & \longrightarrow & & \mathrm{colim}_{\mathbf{J}}\, \mathcal{Q}V \end{array}$$

is homotopy cartesian. Thanks to the first point in Proposition 4.1.8, it follows that $f_i : U(i) \longrightarrow \mathcal{Q}V(i)$ is a realization-fibration of $\mathbf{I}$-presheaves for any $i \in \mathbf{J}$.
We want to prove that $p : E \longrightarrow B$ is a realization-fibration.

Let $\hat{i} : B' \longrightarrow B$ be a map in $\mathrm{Psh}(\mathbf{I})$. Factor it into a weak equivalence followed by a fibration

$$B' \xrightarrow{\ \simeq\ } \tilde{B} \xrightarrow{\ i\ } B$$

and construct the following reiterated pullbacks:

$$\begin{array}{ccccc}
F & \xrightarrow{\simeq} & E' & \longrightarrow & E \\
\downarrow & & \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\
B' & \xrightarrow[\simeq]{} & \tilde{B} & \xrightarrow[i]{} & B
\end{array}$$

We want to prove that the outer square remains homotopy cartesian after having applied the homotopy colimit functor to it.
To do so, it suffices to prove the same statement for the right-hand side square.
By the functoriality of the factorization systems in model category, we can factor the maps

$$U(j) \longrightarrow E, \;\; \mathcal{Q}V(j) \longrightarrow B$$

in such a way that for any $i \in \mathbf{I}$, they fit into the following commutative diagram:

$$\begin{array}{ccccc}
U(i) & \xrightarrow{\simeq} & H(i) & \longrightarrow & E \\
{\scriptstyle f_i}\downarrow & & {\scriptstyle \varrho_i}\downarrow & & \downarrow{\scriptstyle p} \\
\mathcal{Q}V(i) & \xrightarrow[\simeq]{} & T(i) & \longrightarrow & B
\end{array} \tag{4.3}$$

Notice that the induced

$$\varrho : H \longrightarrow T$$

is a realization fibration, being weakly equivalent to f.
Now, define

$$U', V' : \mathbf{I} \longrightarrow \mathbf{sSet}$$

by means of:

$$U'(\cdot) := H(\cdot) \times_B \tilde{B}, \;\; V'(\cdot) := T(\cdot) \times_B \tilde{B}$$

Thanks to diagram 4.3, we see that we obtain two pasted pullbacks

$$\begin{array}{ccccc}
U'(i) & \longrightarrow & V'(i) & \longrightarrow & \tilde{B} \\
\downarrow & & \downarrow & & \downarrow \\
H(i) & \longrightarrow & T(i) & \xrightarrow[i]{} & B
\end{array}$$

So we have a pullback square in $\mathrm{Psh}(\mathbf{J})^{\mathbf{I}}$:

$$\begin{array}{ccc}
U' & \longrightarrow & H \\
\downarrow & & \downarrow{\scriptstyle \varrho} \\
V' & \longrightarrow & T
\end{array}$$

Next, a direct application of the first part of Prop. 4.2.5 gives us a weak equivalence:

$$\mathrm{hocolim}_{\mathbf{J}}\, U = \mathrm{colim}_{\mathbf{J}} \mathcal{Q}U \longrightarrow E$$

which factors, by construction, as:

$$\mathrm{hocolim}_{\mathbf{J}}\, U \xrightarrow{\simeq} \mathrm{hocolim}_{\mathbf{J}}\, H \longrightarrow E$$

The two-out-of-three property implies the map

$$\mathrm{hocolim}_{\mathbf{J}}\, H \xrightarrow{\simeq} E$$

is a weak equivalence.
Again, by construction, we have a commutative diagram:

$$\begin{array}{ccc} \mathrm{hocolim}_{\mathbf{J}}\, \mathcal{Q}V & \xrightarrow{\simeq} & \mathrm{hocolim}_{\mathbf{J}}\, T \\ {\scriptstyle\simeq}\downarrow & & \downarrow \\ \mathrm{hocolim}_{\mathbf{J}}\, V & \xrightarrow[\simeq]{} & B \end{array}$$

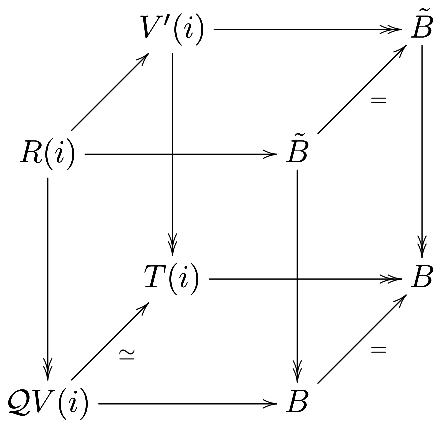
hence a weak equivalence

$$\mathrm{hocolim}_{\mathbf{J}}\, T \xrightarrow{\simeq} B$$

Let us define

$$R(\cdot) := \mathcal{Q}V(\cdot) \times_B \tilde{B} : \mathbf{J} \longrightarrow \mathrm{Psh}(\mathbf{I})$$

so that we get the following cube, where the front face and the back one are homotopy cartesian square (more precisely, pullbacks along fibrations) by construction:



It follows that we get a weak equivalence

$$R \xrightarrow{\simeq} V'$$

Which we plug into the following composable arrows:

$$\mathrm{hocolim}_{\mathbf{J}}\, R \xrightarrow{\simeq} \mathrm{hocolim}_{\mathbf{J}}\, V' \longrightarrow \tilde{B}$$

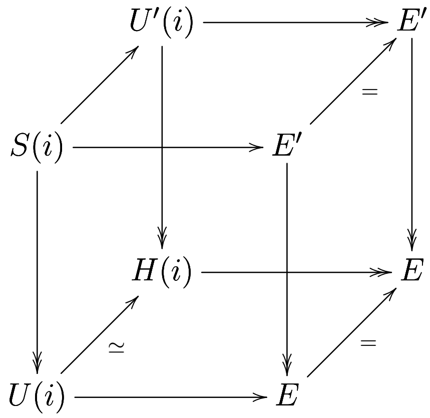
and since the composition is a weak equivalence (thanks to Proposition 4.2.5 applied to the front face of the cube), we have another weak equivalence, namely:

$$\mathrm{hocolim}_{\mathbf{J}}\, V' \xrightarrow{\ \simeq\ } \tilde{B}$$

Analogously, we set:

$$S(\cdot) := U(\cdot) \times_E E' : \mathbf{J} \longrightarrow \mathrm{Psh}(\mathbf{I})$$

which gives us another cube



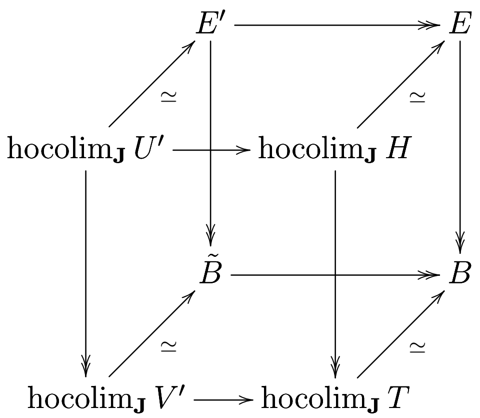
Notice that $E \cong \mathrm{colim}_{\mathbf{J}} U$ thanks to Proposition 4.2.2, so that Proposition 4.2.5 yields (similarly to the previous case) that the following composition

$$\mathrm{hocolim}_{\mathbf{J}}\, S \xrightarrow{\ \simeq\ } \mathrm{hocolim}_{\mathbf{J}}\, U' \longrightarrow E'$$

is a weak equivalence.
Hence, such is

$$\mathrm{hocolim}_{\mathbf{J}}\, U' \xrightarrow{\ \simeq\ } E'$$

Every arrow obtained so far being natural, we see that we have constructed a commutative cube:



Since ϱ is a realization fibration we get that

$$\begin{array}{ccc} \mathrm{hocolim}_{\mathbf{J}\times I^{op}}\, U' & \longrightarrow & \mathrm{hocolim}_{\mathbf{J}\times I^{op}}\, H \\ \downarrow & & \downarrow \\ \mathrm{hocolim}_{\mathbf{J}\times I^{op}}\, V' & \longrightarrow & \mathrm{hocolim}_{\mathbf{J}\times I^{op}}\, T \end{array}$$

is homotopy cartesian.
Thanks to our choice of the homotopy colimit functor we have that this square is weakly equivalent to the following:

$$\begin{array}{ccc} \mathrm{hocolim}_{I^{op}}(\mathrm{hocolim}_{\mathbf{J}} U') & \longrightarrow & \mathrm{hocolim}_{I^{op}}(\mathrm{hocolim}_{\mathbf{J}} H) \\ \downarrow & & \downarrow \\ \mathrm{hocolim}_{I^{op}}(\mathrm{hocolim}_{\mathbf{J}} V') & \longrightarrow & \mathrm{hocolim}_{I^{op}}(\mathrm{hocolim}_{\mathbf{J}} T) \end{array}$$

which is, in turn, weakly equivalent - thanks to what we have proved- to

$$\begin{array}{ccc} |E'|_{\mathbf{I}} & \longrightarrow & |E|_{\mathbf{I}} \\ \downarrow & & \downarrow \\ |B'|_{\mathbf{I}} & \longrightarrow & |B|_{\mathbf{I}} \end{array}$$

so that, finally, this last square is homotopy cartesian, which is what we had to prove.

□

4.3 Local Realization-Fibrations

Let us now see what we can grasp by a local analysis of our situation.
Here local is used in analogy with the case of spaces, where arrows from representable functors are of the form $\Delta^n \longrightarrow P$ for some space $P \in \mathbf{sSet}$.
By considering the discrete embedding

$$\mathbf{Set} \longrightarrow \mathbf{sSet}$$

we get a Yoneda functor

$$H : \mathbf{I} \longrightarrow \mathrm{Psh}(\mathbf{I})$$

for any small category $\mathbf{I}$.

Definition 4.3.1. Given maps $p : E \longrightarrow B$, $b : H_i \otimes \Delta[n] \longrightarrow B$ in $\mathrm{Psh}(\mathbf{I})$, we denote by $\mathrm{Fib}(p, b)$ the homotopy pullback of p along b.

This means that we are factoring p as a weak equivalence followed by a fibration:

$$E \xrightarrow{\ \simeq\ } \hat{E} \xrightarrow{\ \hat{p}\ } B$$

and we set

$$\mathrm{Fib}(p, b) := \hat{E} \times_B H_i \otimes \Delta[n]$$

Now, consider the reiterated homotopy pullbacks (the second square is computed by simply taking the pullback, since the vertical arrow of the cospan is a fibration by stability under base change of fibrations):

$$\begin{array}{ccccc} & & & & E \\ & & & & \big\downarrow{\scriptstyle\simeq} \\ \mathrm{Fib}(p, b\circ H_f\otimes\varrho) & \longrightarrow & \mathrm{Fib}(p,b) & \longrightarrow & \hat{E} \\ \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle\hat{p}} \\ H_j\otimes\Delta[m] & \xrightarrow[H_f\otimes\varrho]{} & H_i\otimes\Delta[n] & \xrightarrow[b]{} & B \end{array}$$

Definition 4.3.2. A map $p : E \longrightarrow B$ between **I**-presheaves is called a *local realization-fibration* (the class of which is denoted shortly by **LRF**) if for any $f : j \longrightarrow i$ in **I** and every $b : H_i \longrightarrow B$ in Psh(**I**), the map:

$$|\mathrm{Fib}(p, b \circ H_f)|_{\mathbf{I}} \longrightarrow |\mathrm{Fib}(p, b)|_{\mathbf{I}}$$

is a weak equivalence of spaces.

If we had asked for any map

$$b : H_i \otimes \Delta[n] \longrightarrow B$$

and, consequently, any

$$H_f \otimes \varrho : H_j \otimes \Delta[m] \longrightarrow H_i \otimes \Delta[n]$$

to be our "test maps", we would have got the same class of maps, as it is proved in the next proposition.

Proposition 4.3.3. *Let $p : E \longrightarrow B$ be a local realization-fibration. Then, given any $b : H_i \otimes \Delta[n] \longrightarrow B$ and any $H_f \otimes \varrho : H_j \otimes \Delta[m] \longrightarrow H_i \otimes \Delta[n]$, one has that the natural map*

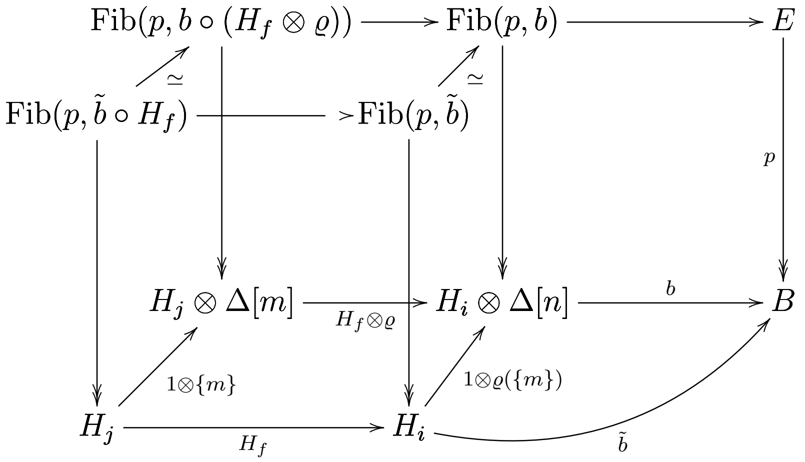
$$|\mathrm{Fib}(p, b \circ (H_f \otimes \varrho))|_{\boldsymbol{I}} \longrightarrow |\mathrm{Fib}(p, b)|_{\boldsymbol{I}}$$

is a weak equivalence.

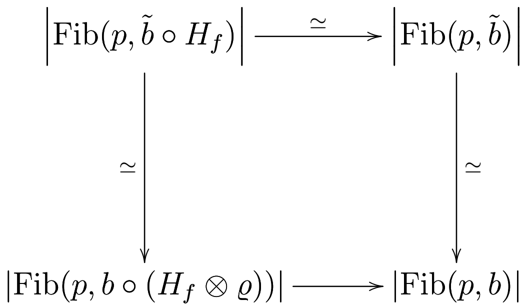
Proof. We can assume $p : E \longrightarrow B$ is a fibration, and the homotopy pullback is given by an ordinary pullback.
Suppose p is a realization-fibration, given maps

$$b : H_i \otimes \Delta[n] \longrightarrow B,\ \ H_f \otimes \varrho : H_j \otimes \Delta[m] \longrightarrow H_i \otimes \Delta[n]$$

we can consider the following diagram



where $\tilde{b} := b \circ (1 \otimes \varrho(\{m\}))$, and the front and back faces are pullbacks along fibrations, hence homotopy cartesian squares.
The displayed maps $\mathrm{Fib}(p, \tilde{b}) \longrightarrow \mathrm{Fib}(p, b)$ and $\mathrm{Fib}(p, \tilde{b} \circ H_f) \longrightarrow \mathrm{Fib}(p, b \circ (H_f \otimes \varrho))$ are weak equivalences since they are pullback of weak equivalences (thanks to Remark 1.4.8) along fibrations.
We thus get the following commutative square, where the top arrow is a weak equivalence since p is assumed to be a local realization-fibration :



Hence the bottom arrow must be a weak equivalence as well.

□

In what follows, we want to develop some tools in order to prove that the class of local realization-fibrations coincide with that of realization-fibrations.
An important result is a sort of homotopy invariance for this concept, in a sense that will be made more precise in Lemma 4.3.8.
It is easy to see that we have a natural bijection

$$\mathbf{sSet}^{I^{op}}(H_i, B) \simeq B(i)_0$$

Indeed, the isomorphism of categories

$$\mathbf{sSet}^{I^{op}} \simeq (\mathrm{Set}^{\mathbf{I}^{op}})^{\Delta^{op}}$$

together with the discreteness of H_i, tells us that (with a slight abuse of language in indicating two different objects with the same name):

$$\mathbf{sSet}^{I^{op}}(H_i, B) \simeq \mathrm{Set}^{\mathbf{I}^{op}}(H_i, B(\cdot)_0)$$

and the latter is $B(i)_0$, by Yoneda Lemma.

The next definition is a generalization of Definition 2.2.1 to any simplicial set, not necessarily a Kan-complex.

Definition 4.3.4. For any simplicial set K we define a functor

$$\tilde{\pi}_0 : \mathbf{sSet} \longrightarrow Set$$
$$\tilde{\pi}_0(K) := K_0/\simeq$$

where $\simeq$ is the equivalence relation generated by $x \simeq y$ if and only if there exists a 1-simplex $f \in K_1$ with $d_0(f) = y,\ d_1(f) = x$.

Notice that on Kan complexes it agrees with the π_0 functor, i.e. the 0-th omotopy group defined in 2.2.1.
The next result will give us the homotopy invariance of this functor as a corollary.

Lemma 4.3.5. *We have an isomorphism of functors:*

$$\tilde{\pi}_0 \cong \pi_0 \circ |\cdot|$$

where $|\cdot| : \boldsymbol{sSet} \longrightarrow Top$ *is the geometric realization functor, and the right-hand side* π_0 *is that of topological spaces.*

Proof. Firstly, notice that given a simplicial set K, each representative of a path component of the topological space $|K|$ can be chosen to be the image of a vertex in K_0.
Hence we have two relations on the same set, namely K_0, and we want to show that the two quotients are isomorphic. Let us show that the two relations coincide.
Obviously if $[x] = [y]$ in $\tilde{\pi}_0 K$, then x and y lie in the same path component of $|K|$.
Conversely, let $\alpha : I \longrightarrow |K|$ be a path joining x to y. By cellular approximation, it is homotopic (relative its boundary) to a path joining the same endpoints but contained in the 1-skeleton of $|K|$.
Now it is enough to notice that this 1-skeleton is given by glueing one interval for each non-degenerate 1 simplex of K, from which we immediately get that $[x] = [y]$ in $\tilde{\pi}_0 K$. □

Corollary 4.3.6. *Given a weak equivalence* $K \longrightarrow L$*, the induced map*

$$\tilde{\pi}_0 K \longrightarrow \tilde{\pi}_0 L$$

is an isomorphism of sets.

Proof. It is enough to observe that the geometric realization functor preserves all weak equivalences, since it is a left Quillen functor and every simplicial set is cofibrant. □

We have thus proven that our functor $\tilde{\pi}_0$ coincides (up to isomorphism) with the usual π_0 applied to a fibrant replacement in **sSet** (endowed with the Kan-Quillen model structure).
For this reason we will denote it simply by π_0, given the fact that no confusion can arise from this choice.

Definition 4.3.7. We define

$$\Theta_i(B) := \{b \in B(i)_0 : \forall\, f : j \longrightarrow i \text{ in } \mathbf{I}^{op}\ \ |\mathrm{Fib}(p, b \circ H_f)|_{\mathbf{I}} \longrightarrow |\mathrm{Fib}(p,b)|_{\mathbf{I}} \text{ is a weak equivalence of spaces}\}$$

We can prove the following:

Lemma 4.3.8. *Given $b, b' \in B(i)_0$ such that $[b] = [b']$ in $\pi_0 B(i)$, then*

$$b \in \Theta_i(B) \iff b' \in \Theta_i(B)$$

Proof. If c is a 0-simplex of a simplicial set K, and we are given a map $L \longrightarrow K$, let us denote byL_c the fiber of such map over c.
More precisely, we construct the following pullback:

$$\begin{array}{ccc} L_c & \longrightarrow & L \\ \downarrow & & \downarrow \\ \Delta[0] & \xrightarrow[c]{} & K \end{array}$$

Since each $H_i(j)$ is a discrete simplicial set, we have the natural isomorphism

$$H_i(j) \simeq \coprod_{\mathbf{I}(j,i)} \Delta[0]$$

Now, Theorem 4 (Chapter 1) of [MM] tells us that pulling back along an arrow preserves colimits, hence from the following pullback square:

$$\begin{array}{ccc} \mathrm{Fib}(p,b)(j) & \longrightarrow & \hat{E}(j) \\ \downarrow & & \downarrow \\ H_i(j) & \xrightarrow[b_j]{} & B(j) \end{array}$$

we deduce the isomorphism

$$\mathrm{Fib}(p,b)(j) \simeq \coprod_{f \in \mathbf{I}(j,i)} \hat{E}(j)_{B(f)(b)}$$

since

$$b_j : H_i(j) \simeq \coprod_{f \in \mathbf{I}(j,i)} \Delta[0] \longrightarrow B$$

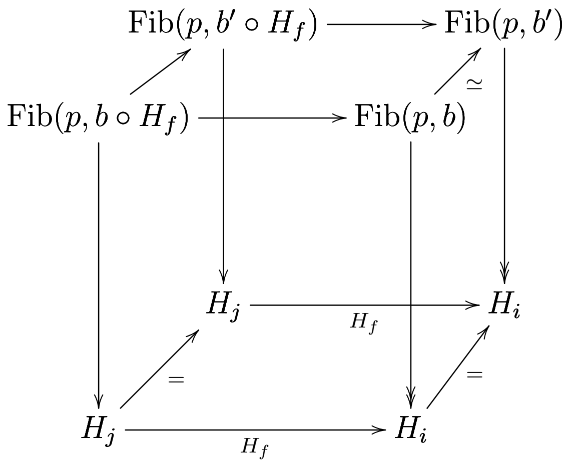
is given on the f-th addendum by picking $B(f)(b) \in B(j)_0$.
Without loss of generality we can assume not only $[b] = [b']$ in $\pi_0 B(i)$, but that there exists a 1-simplex $h \in B(i)_1$ connecting these two vertices, since this is the relation which generates $\pi_0(K)$ for a simplicial set K which is not a Kan complex.
It follows that for any $f : j \longrightarrow i$ in $\mathbf{I}$, $B(f)(h)$ is a 1-simplex of $B(j)$ connecting $B(f)(b)$ to $B(f)(b')$.
Thanks to Lemma 8.25 of [Jo1], we have a homotopy equivalence of Kan complexes

$$\hat{E}(j)_{B(f)(b)} \simeq \hat{E}(j)_{B(f)(b')}$$

which, in turn, implies

$$\mathrm{Fib}(p,b) \simeq \coprod_{f\in\mathbf{I}(j,i)} \hat{E}(j)_{B(f)(b)} \simeq \coprod_{f\in\mathbf{I}(j,i)} \hat{E}(j)_{B(f)(b')} \simeq \mathrm{Fib}(p,b')$$

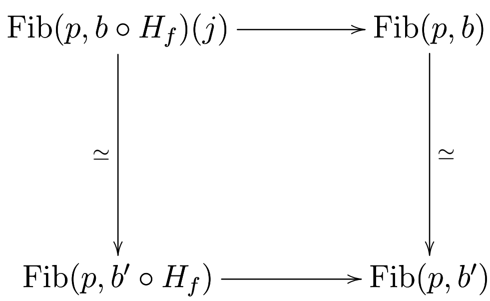
over H_i, since the coproduct of a family of weak equivalences in **sSet** is again a weak equivalence.
A proof of this latter result relies on the fact that trivial cofibrations are stable under coproducts in any model category, being the left class of a weak factorization system. Hence, thanks to Ken Brown's lemma ([Ho], Lemma 1.1.12) applied to $\mathscr{C}^{\mathbf{I}} \cong \prod_{i\in\mathbf{I}}\mathscr{C}$, where $\mathscr{C}$ is any model category and $\mathbf{I}$ is a discrete category, the coproducts of weak equivalences between cofibrant objects is a weak equivalence, in any model category.
Now, we can construct the following cube:



By Theorem 13.3.9 ([Hir]), we get that the natural map

$$\mathrm{Fib}(p, b\circ H_f) \longrightarrow \mathrm{Fib}(p, b'\circ H_f)$$

is a weak equivalence.
Now we can conclude, since the following commutative square:



implies that

$$|\mathrm{Fib}(p, b\circ H_f)|_{\mathbf{I}} \longrightarrow |\mathrm{Fib}(p,b)|_{\mathbf{I}}$$

is a weak equivalence of spaces if and only if

$$|\mathrm{Fib}(p, b'\circ H_f)|_{\mathbf{I}} \longrightarrow |\mathrm{Fib}(p,b')|_{\mathbf{I}}$$

is such, thanks to the fact that $|\cdot|_{\mathbf{I}}$ preserves weak equivalences. □

Remark 4.3.9. In analogy to the global case, let us point out some remarks:

1. Being a local realization-fibration is a property shared by weakly equivalent maps.
2. The property of being a local realization-fibration is stable under homotopy pullbacks.
3. Every realization-fibration is a **LRF**.

Proof. To begin with, let us address the second point, which is easier than the first one. Consider the following diagram, where each square is a (homotopy) pullback. We are implicitly assuming that p is a fibration and that the homotopy pullback is thus an ordinary one, which obviously can be done without loss of generality, thanks to the Definition 4.3.2:

$$\begin{array}{ccccccc} \mathrm{Fib}(p', b\circ H_f) & \longrightarrow & \mathrm{Fib}(p', b) & \longrightarrow & E' & \longrightarrow & E \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_{I'} & \longrightarrow & H_I & \xrightarrow[b]{} & B' & \xrightarrow[i]{} & B \end{array}$$

It then follows by Lemma 4.1.6 that we have a square as the next one, where the vertical arrows are weak equivalences by associativity of homotopy pullbacks, and the bottom horizontal arrow is a weak equivalence since p is in **LRF** by hypothesis.

$$\begin{array}{ccc} |\mathrm{Fib}(p', b\circ H_f)|_I & \longrightarrow & |\mathrm{Fib}(p', b)|_I \\ \simeq\big\downarrow & & \simeq\big\downarrow \\ |\mathrm{Fib}(p, i\circ b\circ H_f)|_I & \xrightarrow{\simeq} & |\mathrm{Fib}(p, i\circ b)|_I \end{array}$$

We thus get that p' is in **LRF**.
Turning to the first point, given a commutative diagram in $\mathrm{Psh}(\mathbf{I})$ like the following one, where the horizontal arrows are weak equivalences, we want to show that

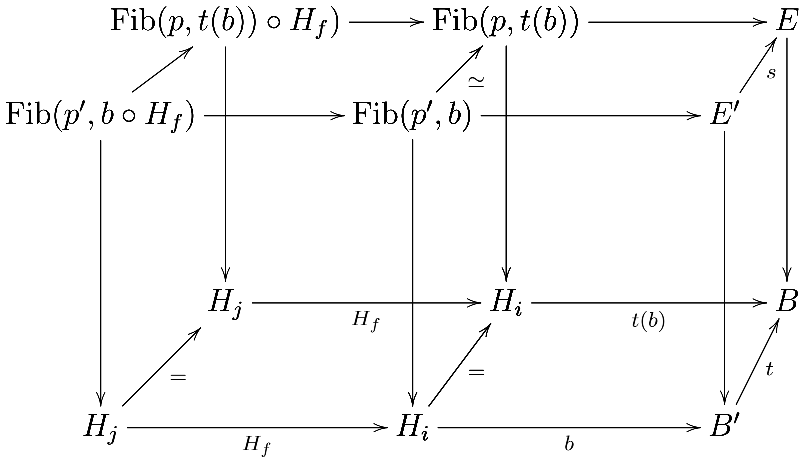
$$p \text{ is a local realization-fibration} \iff p' \text{ is such}$$

$$\begin{array}{ccc} E' & \xrightarrow{s} & E \\ p'\big\downarrow & & \big\downarrow p \\ B' & \xrightarrow[t]{} & B \end{array}$$

Using the funtoriality of the factorization systems in model categories we get a diagram of the form:

$$\begin{array}{ccc} E' & \xrightarrow{\simeq} & E \\ \simeq\big\downarrow & & \big\downarrow\simeq \\ \hat{E}' & \xrightarrow{\simeq} & \hat{E} \\ \hat{p}'\big\downarrow & & \big\downarrow\hat{p} \\ B' & \xrightarrow[\simeq]{} & B \end{array}$$

Hence, without loss of generality, p, p' can be assumed to be fibrations, and any homotopy pullback can be modeled by an ordinary one.
Throughout the rest of the proof, assume given $b : H_i \longrightarrow B$ and an arrow $f : j \longrightarrow i$ in $\mathbf{I}$.
If p is a local realization-fibration, we can construct the following commutative diagram:



The front and back faces are ordinary pullbacks, hence the diagonal arrows in the upper squares are weak equivalences, and this implication is proved.
Indeed, one has that

$$|\mathrm{Fib}(p, t(b) \circ H_f)|_{\mathbf{I}} \longrightarrow |\mathrm{Fib}(p, t(b))|_{\mathbf{I}}$$

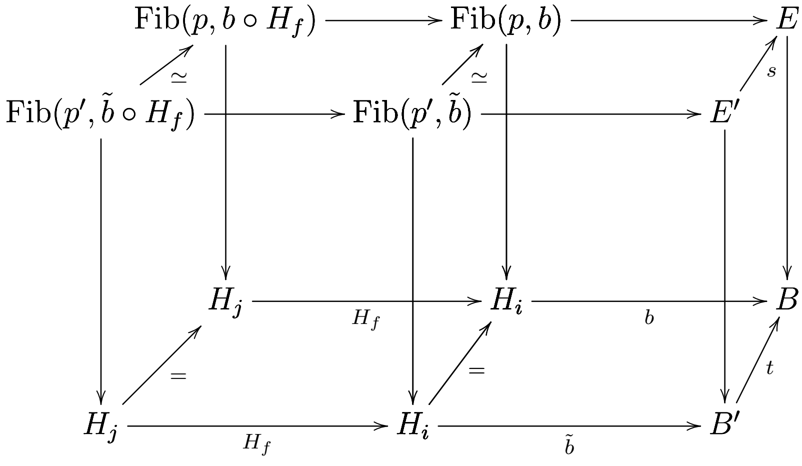
is a weak equivalence, hence

$$|\mathrm{Fib}(p', b \circ H_f)|_{\mathbf{I}} \longrightarrow |\mathrm{Fib}(p', b)|_{\mathbf{I}}$$

must be such.
Conversely, if p' is a local realization-fibration, we can proceed as follows.
The map $t_i : B'(i) \longrightarrow B(i)$ is a weak equivalence, hence it induces an isomorphism after having applied the functor $\pi_0(\cdot)$.
Since we can check the property which defines what it means to be a local realization-fibration just on a representative per each path component (as follows from Lemma 4.3.8), we can assume $b \in t_i(B'(i)_0)$, because there will always be a point in that image for each path component.
So let $\tilde{b} \in B'(i)_0$ be a vertex such that

$$b = t_i(\tilde{b})$$

We can construct the following commutative diagram, which in the same way as the previous implication gives us the desired result:



Finally, if p is a realization fibration, then the same holds for its homotopy pullback along $b : H_i \longrightarrow B$, i.e. if we factor p as

$$\hat{p} \circ c(p) : E \longrightarrow \hat{E} \longrightarrow B$$

where $c(p)$ is a weak equivalence, then by pulling back $\hat{p}$ along b, we get (thanks to Proposition 4.1.8) that

$$\mathrm{Fib}(p, b) \longrightarrow H_i$$

is again a realization fibration.
Hence, by construction and by Definition 4.1.5 the following square is homotopy cartesian:

$$\begin{array}{ccc} |\mathrm{Fib}(p, b \circ H_f)|_I & \longrightarrow & |\mathrm{Fib}(p, b)|_I \\ \downarrow & & \downarrow \\ |H_{i'}|_I & \longrightarrow & |H_i|_I \end{array}$$

The bottom map is a weak equivalence, both spaces being contractible (see Prop.19.6.10 and 14.3.14 in [Hir]), hence such is the upper map, so that p is indeed a local realization-fibration. □

4.4 Second Local-to-Global Principle

We are now going to examine a subclass of arrows for which being a realization-fibration is a local property, namely those having as codomain an object of the form $H_i \otimes \Delta[n]$, for some $i \in \mathbf{I}$ and $n \geq 0$.
To do so, we first need two preliminary facts.
Given an object $B' \in \mathrm{Psh}(\mathbf{I})$, set $\mathscr{C} := (\mathbf{I} \times \Delta \downarrow B')$, where the objects are given by pairs $((i, [n]), \alpha)$, with $\alpha : H_i \otimes \Delta[n] \longrightarrow B'$, and where an arrow

$$\tau : ((i, [n]), \alpha) \longrightarrow ((j, [m]), \beta)$$

is simply an arrow $\tau : (i, [n]) \longrightarrow (j, [m])$ in $\mathbf{I} \times \Delta$ making the following diagram commute:

$$\begin{array}{ccc} H_i \otimes \Delta[n] & \longrightarrow & H_j \otimes \Delta[m] \\ & {}_{\alpha}\searrow \quad \swarrow_{\beta} & \\ & B' & \end{array}$$

We can then apply the following theorem, which is the first step towards the result we want to prove:

Theorem 4.4.1 (Theorem 2.9,[D1]). *Following the previous notation, set:*

$$\Phi : \mathscr{C} \longrightarrow \mathrm{Psh}(\boldsymbol{I})$$
$$\Phi((i, [n]), \alpha) := H_i \otimes \Delta[n]$$

Then the natural arrow

$$\mathrm{hocolim}_{\mathscr{C}}\, \Phi \longrightarrow colim_{\mathscr{C}} \Phi \cong B'$$

is a weak equivalence.

Clearly, in our case the diagram Φ actually factors through the forgetful funtor

$$\mathrm{Psh}(\mathbf{I})/B \longrightarrow \mathrm{Psh}(\mathbf{I})$$

by simply composing (pointwise) with the given $B' \longrightarrow B$.
The second fact we need is the following:

Lemma 4.4.2. *Let p be a fibration of $\boldsymbol{I}$-presheaves and $p \in \boldsymbol{LRF}$.*
Let $\mathcal{C}(p)$ be the class of maps $f : B' \longrightarrow B$ such that the following square in ***sSet*** *is homotopy cartesian:*

$$\begin{array}{ccc} |B' \times_B E|_{\boldsymbol{I}} & \longrightarrow & |E|_{\boldsymbol{I}} \\ \downarrow & & \downarrow{\scriptstyle |p|} \\ |B'|_{\boldsymbol{I}} & \longrightarrow & |B|_{\boldsymbol{I}} \end{array}$$

Then we have:

1. *If $B'' \xrightarrow{g} B' \xrightarrow{f} B$ are two composable arrows, in which g is a weak equivalence and $f \in \mathcal{C}(p)$, then also $f \circ g \in \mathcal{C}(p)$.*

2. *Let $V : \boldsymbol{J} \longrightarrow \mathrm{Psh}(\boldsymbol{I})/B$ be a functor such that each map $V(J) \longrightarrow B$ belongs to $\mathcal{C}(p)$. Then $\mathrm{hocolim}_{\boldsymbol{J}} V \longrightarrow B$ is again in $\mathcal{C}(p)$.*

3. *Every map of the form $g : H_{i'} \otimes \Delta[m] \longrightarrow H_i \otimes \Delta[n]$ is in $\mathcal{C}(p)$.*

Proof. The proof of the first one is straightforward.
Construct the following two (homotopy) pullback squares pasted together:

$$\begin{array}{ccccc} E'' & \longrightarrow & E' & \longrightarrow & E \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle p} \\ B'' & \xrightarrow[g]{} & B' & \xrightarrow[f]{} & B \end{array}$$

Since $E' \longrightarrow B'$ is a fibration, and $\mathbf{sSet}^{\mathbf{I}^{op}}$ is right proper, we have that $E'' \longrightarrow E'$ is again a weak equivalence. Furthermore, by the assumption on f, we have that the following square is homotopy cartesian:

$$\begin{array}{ccc} |E'|_{\mathbf{I}} & \longrightarrow & |E|_{\mathbf{I}} \\ \downarrow & & \downarrow{\scriptstyle |p|} \\ |B'|_{\mathbf{I}} & \xrightarrow[|f|]{} & |B|_{\mathbf{I}} \end{array}$$

By the homotopy invariance of the homotopy colimit functor and by 4.1.7 we see that also the next square is homotopy cartesian:

$$\begin{array}{ccc} |E''|_{\mathbf{I}} & \longrightarrow & |E'|_{\mathbf{I}} \\ \Big\downarrow & & \Big\downarrow{\scriptstyle |p|} \\ |B''|_{\mathbf{I}} & \longrightarrow & |B|_{\mathbf{I}} \end{array}$$

Hence, by pasting them together, we get that

$$\begin{array}{ccc} |E''|_{\mathbf{I}} & \longrightarrow & |E|_{\mathbf{I}} \\ \Big\downarrow & & \Big\downarrow{\scriptstyle |p|} \\ |B''|_{\mathbf{I}} & \underset{|f \circ g|}{\longrightarrow} & |B|_{\mathbf{I}} \end{array}$$

is homotopy cartesian.
Turning to the second fact, Professor Rezk suggested to me the following proof.
We can consider a cofibrant replacement of V in $\text{Psh}(\mathbf{I})^{\mathbf{J}}_{proj}$, namely $\mathcal{Q}V : \mathbf{J} \longrightarrow \text{Psh}(\mathbf{I})$, so that

$$\text{hocolim}_{\mathbf{J}}\, V = \text{colim}_{\mathbf{J}}\, \mathcal{Q}V$$

where we are implicitly using the projective model structure on $\text{Psh}(\mathbf{I})^{\mathbf{J}}$, since the projective one on $\text{Psh}(\mathbf{I})$ is again cofibrantly generated (see Theorem 11.6.1 in [Hir]).
If we define

$$W(\cdot) := \mathcal{Q}V(\cdot) \times_B E : \mathbf{J} \longrightarrow \text{Psh}(\mathbf{I})$$

then we know (thanks to Prop. 4.2.2) that the following is a pullback square (hence a homotopy cartesian square too):

$$\begin{array}{ccc} \text{colim}_{\mathbf{J}} W & \longrightarrow & E \\ \Big\downarrow & & \Big\downarrow{\scriptstyle p} \\ \text{colim}_{\mathbf{J}} \mathcal{Q}V & \longrightarrow & B \end{array}$$

We want to prove the natural arrow

$$|\text{hocolim}_{\mathbf{J}}\, W|_{\mathbf{I}} \longrightarrow \text{holim}(|\text{hocolim}_{\mathbf{J}}\, V|_{\mathbf{I}} \to |B|_{\mathbf{I}} \leftarrow |E|_{\mathbf{I}})$$

is a weak equivalence, where by holim we just denote the pullback (since it is indeed also a homotopy pullback).
To be precise, our result requires that in the arrow above the domain should be given by $|\text{colim}_{\mathbf{J}} W|_{\mathbf{I}}$. The two cases are equivalent, since W is a homotopy colimit diagram, thanks to Theorem 4.2.4.
We are done if we prove this map to be weakly equivalent to

$$\text{hocolim}_{\mathbf{J}}\, |W(\cdot)|_{\mathbf{I}} \longrightarrow \text{hocolim}_{\mathbf{J}}(\text{holim}(|\mathcal{Q}V(j)|_{\mathbf{I}} \to |B|_{\mathbf{I}} \leftarrow |E|_{\mathbf{I}}))$$

since the latter is obtained by applying the functor $\mathrm{hocolim}_{\mathbf{J}}(\cdot)$ to the levelwise weak equivalence

$$|W(\cdot)| \longrightarrow \mathrm{holim}(|\mathcal{Q}V(\cdot)|_{\mathbf{I}} \to |B|_{\mathbf{I}} \leftarrow |E|_{\mathbf{I}})$$

To take care of the domain it is enough to observe that $\mathrm{hocolim}_{\mathbf{J}}(\cdot)$ commutes with $|\cdot|_{\mathbf{I}}$. Turning to the codomain, we have (thanks to Proposition 4.2.5) the weak equivalence

$$\mathrm{hocolim}_{\mathbf{J}}(\mathrm{holim}(|\mathcal{Q}V(\cdot)|_{\mathbf{I}} \longrightarrow |B|_{\mathbf{I}} \leftarrow |E|_{\mathbf{I}})) \longrightarrow \mathrm{holim}(\mathrm{hocolim}_{\mathbf{J}}\, |\mathcal{Q}V(\cdot)|_{\mathbf{I}} \to |B|_{\mathbf{I}} \leftarrow |E|_{\mathbf{I}})$$

and so we are done.
The third point follows from the observation that the map

$$g : H_{i'} \otimes \Delta[m] \longrightarrow H_i \otimes \Delta[n]$$

must be of the form $H_f \otimes \varrho$ for a unique $f : i' \longrightarrow i,\ \varrho : [m] \longrightarrow [n]$ in $\mathbf{I}$ and Δ, respectively, as a consequence of the Yoneda lemma.
Hence we can consider $b := Id_{H_i \otimes \Delta[n]}$ in the definition of local realization-fibration for p, and we can construct the following reiterated pullbacks:

$$\begin{array}{ccccc} E' & \longrightarrow & E & \xrightarrow{=} & E \\ \downarrow & & \downarrow & & \downarrow \\ H_{i'} \otimes \Delta[m] & \xrightarrow[H_f \otimes \varrho]{} & H_i \otimes \Delta[n] & \xrightarrow[=]{} & H_i \otimes \Delta[n] \end{array}$$

and thanks to Proposition 4.3.3 this gives us a weak equivalence

$$|E'|_{\mathbf{I}} \xrightarrow{\simeq} |E|_{\mathbf{I}}$$

We can conclude by noticing that, after having realized the initial square, we obtain

$$\begin{array}{ccc} |E'|_{\mathbf{I}} & \longrightarrow & |E|_{\mathbf{I}} \\ \downarrow & & \downarrow{\scriptstyle |p|} \\ |H_{i'} \otimes \Delta[m]|_{\mathbf{I}} & \xrightarrow[|H_f \otimes \varrho|]{} & |H_i \otimes \Delta[n]|_{\mathbf{I}} \end{array}$$

where the bottom map is a weak equivalence (as noticed in Remark 4.3.9) and the top one is such by what we have just proven. □

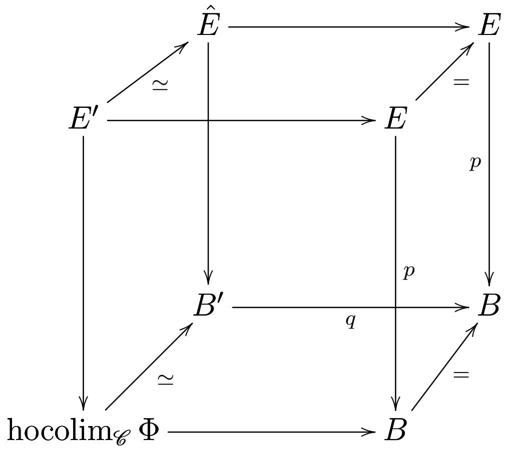
Finally, we can state and prove the abovementioned result:

Lemma 4.4.3. *Let $p : E \longrightarrow B = H_i \otimes \Delta[n]$ be a map in* $\mathrm{Psh}(\boldsymbol{I})$*, then p is a realization-fibration if and only if it is a local realization-fibration.*

Proof. Let us deal with the non trivial implication, where, in view of Proposition 4.3.9 and 4.1.8, we can assume p is a fibration of **I**-presheaves (so that a model for the homotopy pullback is just the actual pullback) and $p \in \mathbf{LRF}$.
We are now able to prove that any $q : B' \longrightarrow B$ is in $\mathcal{C}(p)$.
Using the same notation used in Theorem 4.4.1, we see that the composition

$$\mathrm{hocolim}_{\mathscr{C}}\,\Phi \xrightarrow{\simeq} B' \xrightarrow{\quad q \quad} B$$

is in $\mathcal{C}(p)$ by the last two points of Lemma 4.4.2.
Construct the following diagram, where the front and back faces are given by (homotopy) pullbacks:



The arrow $E' \longrightarrow \hat{E}$ is given by the universal property of pulbacks, and it is a weak equivalence by Theorem 13.3.9 in [Hir].
We have proven the arrow $\mathrm{hocolim}_{\mathscr{C}}\,\Phi \longrightarrow B$ is in $\mathcal{C}(p)$, hence if we apply the homotopy colimit functor to this cube we get that the front face, i.e.:

$$\begin{array}{ccc} |E'|_{\mathbf{I}} & \longrightarrow & |E|_{\mathbf{I}} \\ \downarrow & & \downarrow \\ |\mathrm{hocolim}_{\mathscr{C}}\,\Phi|_{\mathbf{I}} & \longrightarrow & |B|_{\mathbf{I}} \end{array}$$

is homotopy cartesian, and it is weakly equivalent to the back face, i.e.:

$$\begin{array}{ccc} \left|\hat{E}\right|_{\mathbf{I}} & \longrightarrow & |E|_{\mathbf{I}} \\ \downarrow & & \downarrow \\ |B'|_{\mathbf{I}} & \xrightarrow[q]{} & |B|_{\mathbf{I}} \end{array}$$

which must then be homotopy cartesian, giving $q \in \mathcal{C}(p)$.

□

We can now state and prove what we will refer to as the "second local-to-global principle".

Theorem 4.4.4. *Let* $p : E \longrightarrow B$ *be a map in* $\mathrm{Psh}(\boldsymbol{I})$*. Then* p *is a realization-fibration if and only if it is a local realization-fibration.*

Proof. One implication has already been shown, so let us prove that being a realization-fibration is a local property.
So suppose p is in **LRF**, and without loss of generality is a fibration.
By applying Theorem 4.4.1, we can find a small category $\mathbf{J}$ and a functor $V : \mathbf{J} \longrightarrow \mathrm{Psh}(\mathbf{I})/B$ in such a way that $\mathrm{hocolim}_{\mathbf{J}}\, V \longrightarrow B = \mathrm{colim}_{\mathbf{J}} V$ is a weak equivalence, and for each $j \in \mathbf{J}$, $V(j) = H_i \otimes \Delta[n]$ for some $i \in \mathbf{I}$ and $n \geq 0$.
Set $U(\cdot) : \mathbf{J} \longrightarrow \mathrm{Psh}(\mathbf{I})/E$ as the pullback $V(\cdot) \times_B E$ and call $f : U \longrightarrow V$ the natural projection map.
Proposition 4.2.5 implies that we have a commutative square of the form

$$\begin{array}{ccc} \mathrm{hocolim}_{\mathbf{J}}\, U & \xrightarrow{\simeq} & E \\ {\scriptstyle \mathrm{hocolim}_{\mathbf{J}}(f)} \downarrow & & \downarrow {\scriptstyle p} \\ \mathrm{hocolim}_{\mathbf{J}}\, V & \xrightarrow[\simeq]{} & B \end{array}$$

Hence, if we prove $\mathrm{hocolim}_{\mathbf{J}}(f)$ is a realization fibration, then we can conclude by using Proposition 4.1.8.
With the intent of applying Proposition 4.2.7, we observe that the map $f : U \longrightarrow V$ is equifibered.
Indeed, we see that, in the following diagram, the right-hand square and the outer one are homotopy cartesian by construction, for any $h : j_1 \longrightarrow j_2$ in $\mathbf{J}$:

$$\begin{array}{ccccc} U(j_1) & \xrightarrow{U(h)} & U(j_2) & \longrightarrow & E \\ {\scriptstyle f_{j_1}} \downarrow & & \downarrow {\scriptstyle f_{j_2}} & & \downarrow {\scriptstyle p} \\ V(j_1) & \xrightarrow[V(h)]{} & V(j_2) & \longrightarrow & B \end{array}$$

We thus get that, thanks to Lemma 4.1.6, that the left-hand square is homotopy cartesian as well.
Hence $f : U \Rightarrow V$ is equifibered, and since f_j is in **LRF** being the pullback of p which is in that class, we have that f_j is a realization-fibration thanks to Lemma 4.4.3.
We can now apply Proposition 4.2.7 to see that $\mathrm{hocolim}_{\mathbf{J}}\, f$ is a realization-fibration, hence such is p. □

4.5 Approximating maps by Realization-Fibrations

In this section we want to find the nearest realization-fibration to a given map.
To do so, given a map of $\mathbf{I}$-presheaves $p : E \longrightarrow B$, we look for the maximal subobject of B over which the restriction of p is a realization fibration.
Furthermore, we will see that for any map $B' \longrightarrow B$ which factors through such subobject, the realization functor $|\cdot|_{\mathbf{I}}$ preserves suitable homotopy cartesian squares.

Definition 4.5.1. Given a **I**-presheaf B, we define $\pi_0 B : \mathbf{I}^{op} \longrightarrow \mathbf{sSet}$ as

$$\pi_0 B(i) := \pi_0(B(i)) \quad \forall i \in \mathbf{I}$$

where $\pi_0(B(i))$ is seen as a discrete simplicial set.
We then construct, given $p : E \longrightarrow B$, the **I**-presheaf $\mathrm{lrf}(p)$ as the subobject of $\pi_0 B$ such that:

$$\mathrm{lrf}(p)(i) := \{[b] \in \pi_0 B(i) | \ b : H_i \longrightarrow B \text{ and } \mathrm{Fib}(p, b) \longrightarrow H_i \text{ is in RF}\}$$

This presheaf of spaces somehow measures the deviation of a map from being a realization-fibration, as made clear from the following:

Proposition 4.5.2. *A map $p : E \longrightarrow B$ of **I**-presheaves is in **RF** if and only if* $\mathrm{lrf}(p) = \pi_0 B$.

Proof. Without loss of generality p might as well be assumed to be a fibration.
If p is a realization-fibration, then any (homotopy) pullback of it is such, so that for any $b : H_I \longrightarrow B$ we have that $\mathrm{Fib}(p, b) \longrightarrow H_I$ is again a realization-fibration.
Conversely, assuming that $\mathrm{lrf}(p) = \pi_0 B$, we will show that p is a local realization-fibration, and then conclude thanks to 4.4.4.
Given any $b : H_i \longrightarrow B$ and any $H_f : H_{i'} \longrightarrow H_i$ for some $f : i \longrightarrow i'$ in **I**, consider the following diagram:

$$\begin{array}{ccccc}
\mathrm{Fib}(p, b \circ H_f) & \longrightarrow & \mathrm{Fib}(p, b) & \longrightarrow & E \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle p} \\
H_{i'} & \xrightarrow[H_f]{} & H_i & \xrightarrow[b]{} & B
\end{array}$$

The central vertical arrow is a realization fibration by hypothesis, hence the following diagram is homotopy cartesian:

$$\begin{array}{ccc}
|\mathrm{Fib}(p, b \circ H_f)|_I & \longrightarrow & |\mathrm{Fib}(p, b)|_I \\
\downarrow & & \downarrow \\
|H_{I'}|_I & \xrightarrow{|H(f)|_{\mathbf{I}}} & |H_I|_I
\end{array}$$

Now, since the bottom horizontal arrow is a weak equivalence, the top one must be such, from which the thesis follows. □

Definition 4.5.3. Let $B_{\mathbf{RF}(p)}$ be the **I**-presheaf defined by:

$$B_{\mathbf{RF}(p)}(\cdot) := B(\cdot) \times_{\pi_0 B(\cdot)} \mathrm{lrf}(p)$$

The map $\mathbf{RF}(p) : E_{\mathbf{RF}(p)} \longrightarrow B_{\mathbf{RF}(p)}$ which is defined by the following pasted pullback diagrams is called the ***RF** approximation to p*

$$\begin{array}{ccccc}
E_{\mathbf{RF}(p)} & \xrightarrow{\mathbf{RF}(p)} & B_{\mathbf{RF}(p)} & \longrightarrow & \mathrm{lrf}(p) \\
\downarrow & & \downarrow & & \downarrow \\
E & \xrightarrow[p]{} & B & \longrightarrow & \pi_0 B
\end{array}$$

The map $B \longrightarrow \pi_0 B$ is given, for any $I \in \mathbf{I}$, by $B(i)_n \longrightarrow B(i)_0 \longrightarrow \pi_0 B(i)$, where the first arrow is any face operator (it is easy to check that such map is well defined).
Notice also that $B_{\mathbf{RF}(p)}$ is a subobject of B, consisting of the union of some of its component.

Theorem 4.5.4. *Consider a homotopy cartesian square of* $\boldsymbol{I}$*-presheaves*

$$\begin{array}{ccc} E' & \longrightarrow & E \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ B' & \underset{i}{\longrightarrow} & B \end{array}$$

Then p' *is a realization-fibration if and only if* i *factors through* $B_{\boldsymbol{RF}(p)} \hookrightarrow B$.
In particular, $\boldsymbol{RF}(p) : E_{\boldsymbol{RF}(p)} \longrightarrow B_{\boldsymbol{RF}(p)}$ *is a realization-fibration.*

Proof. Without loss of generality p can be assumed to be a fibration, so that the second statement immediately follows from the first one by considering the pullback square

$$\begin{array}{ccc} E_{\mathbf{RF}(p)} & \longrightarrow & E \\ \downarrow & & \downarrow{\scriptstyle p} \\ B_{\mathbf{RF}(p)} & \longrightarrow & B \end{array}$$

Now, suppose p' is a realization-fibration, hence it is equivalently a local one.
To see that i factors through $B_{\mathbf{RF}(p)}$ we proceed as follows: given an element $b \in \pi_0 B(j)$ which comes from $i_j : B'(j) \longrightarrow B(j)$, we represent it as an arrow $b : H_j \longrightarrow B$, which factorizes trough B'.
We then have the following pullbacks glued together:

$$\begin{array}{ccccc} \mathrm{Fib}(p,b) & \longrightarrow & E' & \longrightarrow & E \\ {\scriptstyle p''}\downarrow & & {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ H_j & \longrightarrow & B' & \underset{i}{\longrightarrow} & B \end{array}$$

Hence that element $b \in \pi_0 B(j)$ is actually in $\mathrm{lrf}(p)(j)$, since p'' is a realization-fibration, being the (homotopy) pullback of p, which is such by assumption.
This concludes the first half of the proof.
Turning to the reversed implication, suppose i factors through $B_{\mathbf{RF}(p)}$.
Then $b \circ i : H_j \longrightarrow B$ does the same, hence by definition the pullback of p along it, namely p'', must be a realization fibration.
This, in turn, yields that p' is also in $\mathbf{RF}$, because given two (homotopy) pullbacks as follows:

$$\begin{array}{ccccc} \mathrm{Fib}(p', b \circ H_f) & \longrightarrow & \mathrm{Fib}(p', b) & \longrightarrow & E' \\ \downarrow & & {\scriptstyle p''}\downarrow & & \downarrow{\scriptstyle p'} \\ H_{i'} & \underset{H_f}{\longrightarrow} & H_j & \longrightarrow & B' \end{array}$$

the square which results by applying $|\cdot|_{\mathbf{I}}$ to the left-hand side of the diagram is again homotopy cartesian, p'' being a realization-fibration, hence $|\mathrm{Fib}(p', b\circ H_f)|_{\mathbf{I}} \longrightarrow |\mathrm{Fib}(p', b)|_{\mathbf{I}}$ is a weak equivalence (since we have already said that $|H_{i'}|_{\mathbf{I}} \longrightarrow |H_j|_{\mathbf{I}}$ is a weak equivalence, and we are thus applying Lemma 4.1.7).
It follows that p' is a local realization fibration. □

Notice that, given a homotopy cartesian square in $\mathrm{Psh}(\mathbf{I})$

$$\begin{array}{ccc} E' & \longrightarrow & E \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ B' & \xrightarrow[i]{} & B \end{array}$$

if $i : B' \longrightarrow B$ factors through $B_{\mathbf{RF}(p)}$, then the homotopy cartesian diagram

$$\begin{array}{ccc} E' & \longrightarrow & E_{\mathbf{RF}(p)} \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle \mathbf{RF}(p)} \\ B' & \longrightarrow & B_{\mathbf{RF}(p)} \end{array}$$

remains such after having applied the homotopy colimit functor, since

$$\mathbf{RF}(p) : E_{\mathbf{RF}(p)} \longrightarrow B_{\mathbf{RF}(p)}$$

is a realization-fibration.

4.6 Homotopy-Sifted Categories

Definition 4.6.1. Given a small category $\mathscr{D}$, we say that $\mathscr{D}^{op}$ is *homotopy-sifted* (though for the rest of the section we will simply call them *sifted*) if the canonical maps (the first of which is induced by the product projections) :

$$|X \times Y|_{\mathscr{D}} \longrightarrow |X|_{\mathscr{D}} \times |Y|_{\mathscr{D}}$$
$$|*|_{\mathscr{D}} \longrightarrow \Delta[0]$$

are weak equivalences for any X, Y in $\mathrm{Psh}(\mathscr{D})$, where $*$ denotes its terminal object.

An important example is given by Δ. Indeed, we can check that Δ^{op} is sifted by recalling that the diagonal of a simplicial space models its homotopy colimit (see Cor. 18.7.7 in [Hir]), hence it is enought to observe that:

$$\mathrm{diag}(X \times Y) \cong \mathrm{diag}(X) \times \mathrm{diag}(Y),\ \forall X, Y \in \mathrm{Psh}(\Delta)$$

and that

$$\mathrm{diag}(*) = \Delta[0]$$

We will give a sufficient condition for maps between simplicial presheaves on a sifted category to be realization-fibrations.
It resembles the definition of a fiber bundle in the classical context of spaces, namely the property of looking locally as a projection.
Let us first define the trivial version thereof:

Definition 4.6.2. Given a sifted category $\mathscr{D}^{op}$, a map $p : E \longrightarrow B$ in $\mathrm{Psh}(\mathscr{D})$ is called a *trivial fiber bundle* if there is a zig-zag of weak equivalences in $\mathrm{Psh}(\mathscr{D})/B$ from p to a projection map $p' : B \times C \longrightarrow B$ for some $\mathscr{D}$-presheaf C.

In the next proposition we list some nice properties enjoyed by the class of trivial fiber bundles, similar to that of realization-fibrations.

Proposition 4.6.3. *Let $\mathscr{D}^{op}$ be a sifted category.*

1. *Given a commutative square in $\mathrm{Psh}(\mathscr{D})$ of the form*

$$\begin{array}{ccc} E' & \xrightarrow{\simeq} & E \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ B' & \xrightarrow[\simeq]{} & B \end{array}$$

   where the horizontal arrows are weak equivalences, p is a trivial fiber bundle if and only if p' is such.

2. *If we have a homotopy cartesian square in $\mathrm{Psh}(\mathscr{D})$ of the form*

$$\begin{array}{ccc} E' & \longrightarrow & E \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ B' & \xrightarrow[i]{} & B \end{array}$$

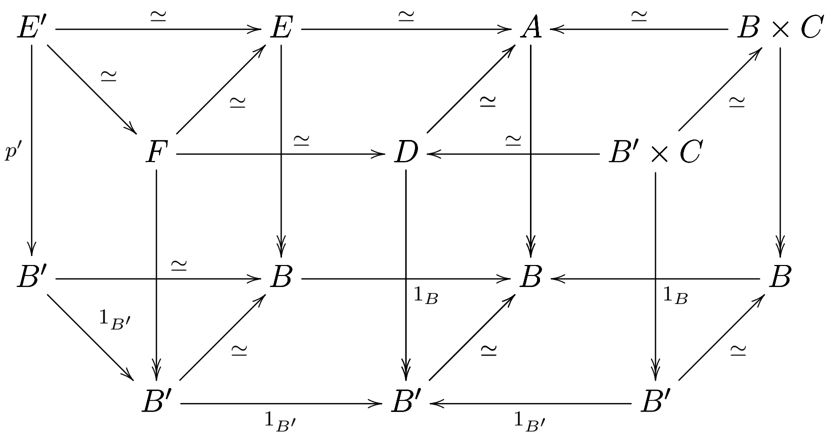
   where p is a trivial fiber bundle, then p' is such.

3. *Every weak equivalence is a trivial fiber bundle.*

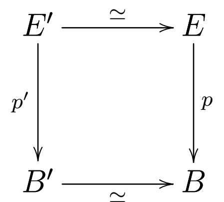
Proof. Let us address the first point. The claim is obvious if $B' \longrightarrow B$ is the identity on B, by definition of trivial fiber bundle map.
This allows us to restrict ourselves to the case where the zig-zag involves only fibrations of the form $D \longrightarrow B$, just by factoring the maps into weak equivalences followed by fibrations.
Now, assume p is a trivial fiber bundle map, hence we have a zig-zag of weak equivalences in $\mathrm{Psh}(\mathscr{D})$ between p and a projection $B \times C \longrightarrow B$. Without loss of generality C is fibrant, so that the projection is a fibration.
By simply pulling back all the arrows along $i : B' \longrightarrow B$ we get the desired result: an example will clarify this process.
Assume the zig-zag is given by

$$E \xrightarrow{\simeq} A \xleftarrow{\simeq} B \times C$$

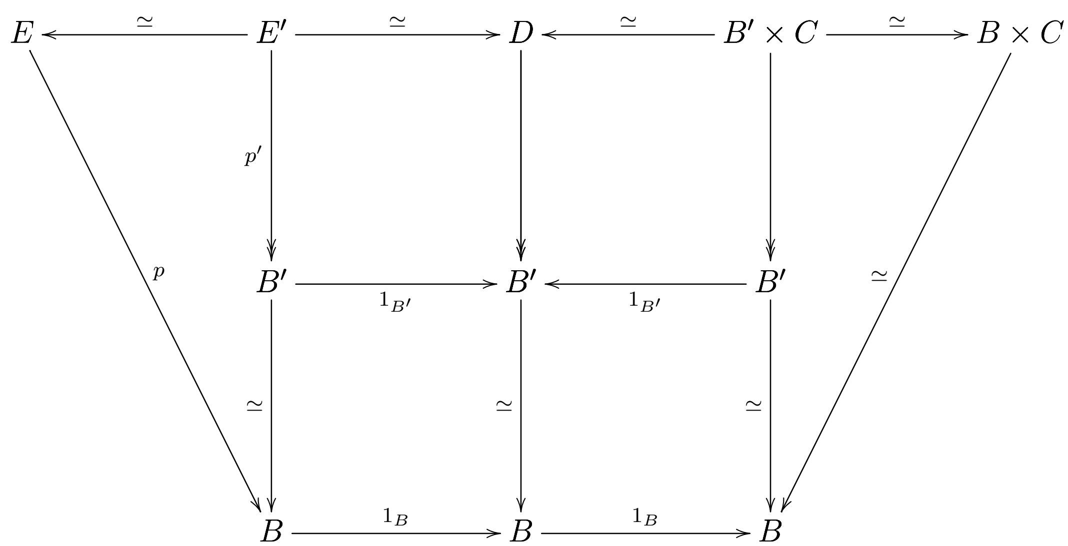
We can consider the following diagram, where the new objects are defined by pulling back the objects over B along i:



The induced map $E' \longrightarrow F$ is a weak equivalence since



is homotopy cartesian, and the maps $B' \times C \longrightarrow D$, $F \longrightarrow D$ are such thanks to Prop. 13.3.14 of [Hir].
The diagram shows the existence of the desired zig-zag for p'.
To prove the other implication, compose the structure maps of the objects over B' of the given zig-zag with the weak equivalence $i : B' \longrightarrow B$.
For instance, using the same type of zig-zag as above, consider:



Turning to the second point, thanks to what we have just proved we can assume p is a fibration, and the square is a pullback.
Now, using exactly the same trick as in the first implication of the first point, we get the desired result.
The third point is proved by observing that if $E \longrightarrow B$ is a weak equivalence, then it is weakly equivalent (over B) to the projection $B \times * \longrightarrow B$. $\square$

The same properties are enjoyed by the following class of maps too (and the proof is straightforward and similar to the others already encountered).

Definition 4.6.4. A map between $\mathscr{D}$-presheaves $p : E \longrightarrow B$ is called a *fiber bundle* if for every homotopy cartesian square of the form:

$$\begin{array}{ccc} E' & \longrightarrow & E \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ H_d & \xrightarrow[b]{} & B \end{array}$$

the map q is a trivial fiber bundle map.

It follows immediately from the homotopy base change stability that any trivial fiber bundle map is locally such.
Given a fibration $p : E \longrightarrow B$, let us consider the subobject $\mathrm{lproj}(p) \subset \pi_0 B$ which is the subobject of $\pi_0 B$ defined by

$$\mathrm{lproj}(p)(i) := \{[b] \in \pi_0 B(i) |\ b : H_i \longrightarrow B \text{ and } \mathrm{Fib}(p, b) \longrightarrow H_i \text{ is a trivial fiber bundle map}\}$$

The following result is analogous to Theorem 4.5.4, and the same holds for the proof.

Proposition 4.6.5. *Consider $p : E \longrightarrow B$, $f : B' \longrightarrow B$ maps of $\mathscr{D}$-presheaves, where p is a fibration. The pullback of p along f is a fiber bundle map if and only if $f(\pi_0 B') \subset \mathrm{lproj}(p)$. In particular, p itself is a fiber bundle if and only if $\pi_0 B = \mathrm{lproj}(p)$*

Proof. Assume $f(\pi_0 B') \subset \mathrm{lproj}(p)$, and suppose given a pullback square

$$\begin{array}{ccc} E' & \longrightarrow & E \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ B' & \xrightarrow[f]{} & B \end{array}$$

Then we have to show that for each $b : H_d \longrightarrow B'$ and for any homotopy cartesian square of the form

$$\begin{array}{ccc} F & \longrightarrow & E' \\ {\scriptstyle p''}\downarrow & & \downarrow{\scriptstyle p'} \\ H_d & \xrightarrow[b]{} & B' \end{array}$$

the map $p'' : F \longrightarrow H_d$ is a trivial fiber bundle map.
By the previous remarks, it is enough to show that it holds assuming the above square is an actual pullback.
By hypothesis, $f \circ b$ represents an element in $\mathrm{lproj}(p)(d) \subset \pi_0 B(d)$, hence the pullback of p along $f \circ b$, i.e. p'' must be a trivial fiber bundle map.
Conversely, assume that $p' : E' \longrightarrow B'$ is a fiber bundle map. Then the pullback of p' along any map $H_d \longrightarrow B'$ must be a trivial fiber bundle map by definition, hence any element of $\pi_0 B'(d)$ is sent to $\mathrm{lproj}(p)$ by f, for any $d \in \mathscr{D}$.
The second fact follows by setting $f := 1_B$ in the first one. □

The following lemma establishes a link between projections and realization fibrations, and we make use of the siftedness hypothesis.

Lemma 4.6.6. *If $p : B \times C \longrightarrow B$ is a projection of $\mathscr{D}$-presheaves and $\mathscr{D}^{op}$ is sifted, then p is a realization fibration.*

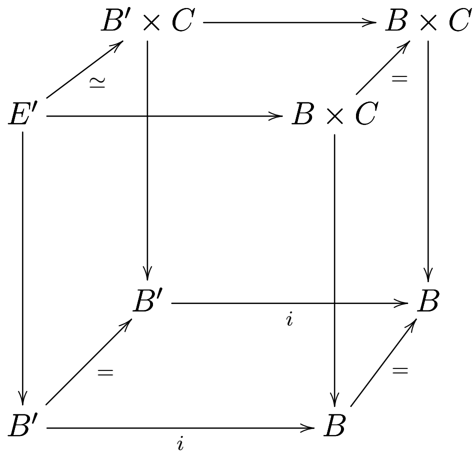
Proof. Thanks to the stability under weak equivalences of realization-fibrations, we can assume C is fibrant.
Hence p is a fibration, and given a homotopy cartesian square:

$$\begin{array}{ccc} E' & \longrightarrow & B \times C \\ \downarrow & & \downarrow{\scriptstyle \pi_B} \\ B' & \underset{i}{\longrightarrow} & B \end{array}$$

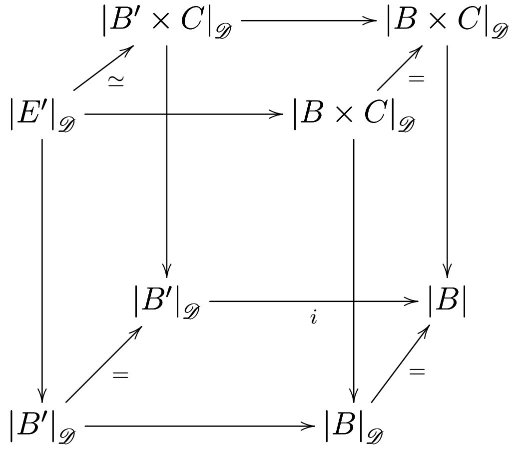
we thus have, by assumption, a natural weak equivalence over B':

$$E' \xrightarrow{\simeq} B' \times C$$

the latter being the pullback of p along i.
Everything fits into the following commutative diagram:



Applying the functor $|\cdot|_{\mathscr{D}}$ to such cube, we obtain:



Now, by definition of siftedness, we see that the back face is weakly equivalent to

$$\begin{array}{ccc} |B'|_{\mathscr{D}} \times |C|_{\mathscr{D}} & \longrightarrow & |B|_{\mathscr{D}} \times |C|_{\mathscr{D}} \\ \downarrow & & \downarrow \\ |B'|_{\mathscr{D}} & \longrightarrow & |B|_{\mathscr{D}} \end{array}$$

which is homotopy cartesian, hence such must be the former, by Prop.13.3.13 in [Hir]. The same proposition implies the front face is homotopy cartesian, hence we are done. □

We will extend this result to fiber bundles in the following:

Theorem 4.6.7. *Let $p : E \longrightarrow B$ be a map of $\mathscr{D}$-presheaves, with D^{op} sifted. Then* $\mathrm{lproj}(p) \subset \mathrm{lrf}(p)$*, so that, in particular, all fiber bundles in* $\mathrm{Psh}(\mathscr{D})$ *are realization-fibrations.*

Proof. The homotopy pullback of p along a map $H_i \longrightarrow B$ representing an element in $\mathrm{lproj}(p)$ is, by definition, weakly equivalent to a map of the form $H_i \times C \longrightarrow H_i$, which is a realization fibration by Lemma 4.6.6.
The second point follows by observing that, thanks to Proposition 4.6.5, p is a fiber bundle if and only if $\mathrm{lproj}(p) = \pi_0 B$, and the first point we have just proved implies that in this case $\mathrm{lrf}(p) = \pi_0 B$.
Therefore, p is a realization-fibration by Proposition 4.5.2. □

Chapter 5

Application to Simplicial Spaces

You can't lose what you don't put in the middle.
But you can't win much either.

Michael McDermott, *Rounders*

In this chapter we will apply the previous results, especially the last part on homotopy-sifted categories, to the case of simplicial spaces.
We will thus focus on the case $\mathscr{D} := \Delta$, so that the category $\mathrm{Psh}(\mathscr{D})$ is indeed that of simplicial spaces.
In the end, we will state the classical results of Bousfield and Friedlander in this framework, and a generalized (in a suitable sense) version of these will be proved.
We want to investigate what $\mathrm{lproj}(p) \subset \pi_0 B$ looks like.
Firstly, notice that if $B \in \mathbf{sSet}^{\Delta^{op}}$, then $\pi_0 B$ is a simplicial set, being the composition

$$\pi_0 \circ B : \Delta^{op} \longrightarrow \mathbf{sSet} \longrightarrow \mathbf{Sets}$$

Lemma 5.0.8. *Let $p : E \longrightarrow B$ be a fibration, and let $V(p) \subset \pi_0 B$ denote the union of all path components of $\pi_0 B$ which are not in the image of $\pi_0(p) : \pi_0 E \longrightarrow \pi_0 B$. Then $V(p) \subset$ lproj(p).*

Proof. If $b : H_{[m]} \longrightarrow B$ represents an element in $V(p)$, then pulling back p along b gives, by definition, the empty simplicial space.
The unique map $\emptyset \longrightarrow H_{[m]}$ is obviously a projection map, being equal to

$$\emptyset \times H_{[m]} \longrightarrow H_{[m]}$$

hence $V(p) \subset \mathrm{lproj}(p)$. □

Lemma 5.0.9. *Let $p : E \longrightarrow B$ be a map of simplicial spaces. Then every 0-simplex of $\pi_0 B$ is contained in lproj(p).*

Proof. It is enough to observe that $H_{[0]}$ is the terminal object in the category $\mathbf{sSet}^{\Delta^{op}}$, so that every map $q : F \longrightarrow H_{[0]}$ is a projection map. □

The following proposition gives a sufficient condition on B for $p : E \longrightarrow B$ to be a realization-fibration.

Proposition 5.0.10. *Let $p : E \longrightarrow B$ be a map in $\boldsymbol{sSet}^{\Delta^{op}}$, with $\pi_0 B$ discrete simplicial set.*
Then p is a realization-fibration.

Proof. Consider a map $b : H_{[m]} \longrightarrow B$ representing an element of

$$\pi_0 B([m]) \simeq \pi_0 B([0]) = \mathrm{lproj}(p)([0])$$

The above displayed isomorphism implies that the map

$$b : H_{[m]} \longrightarrow B$$

factors through a map

$$\tilde{b} : H_{[0]} \longrightarrow B$$

As before, assume p is a fibration, so that the homotopy pullback of p along b is an ordinary pullback.
By associativity of (homotopy) pullbacks, it can be computed in two steps, namely:

$$\begin{array}{ccccc} E'' & \longrightarrow & E' & \longrightarrow & E \\ {\scriptstyle p''}\downarrow & & {\scriptstyle p'}\downarrow & & \downarrow \\ H_{[m]} & \longrightarrow & H_{[0]} & \longrightarrow & B \end{array}$$

where each square is a pullback.
Now, p' is a fiber bundle by Lemma 5.0.9, so by the stability of fiber bundles under homotopy base change, we get that p'' is again a fiber bundle, hence a realization fibration for Theorem 4.6.7.
So, given a homotopy cartesian square

$$\begin{array}{ccc} G & \longrightarrow & E \\ {\scriptstyle p'}\downarrow & & \downarrow \\ H_{[m]} & \xrightarrow[b]{} & B \end{array}$$

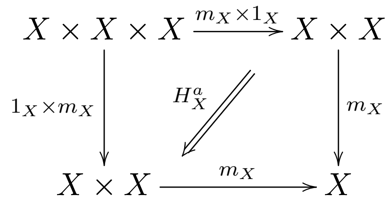
we have that $G \longrightarrow H_{[m]}$ is weakly equivalent to the previous defined p'', hence it is a realization fibration.
It follows immediately that p is a local realization-fibration, hence it is a realization fibration thanks to Theorem 4.4.4.

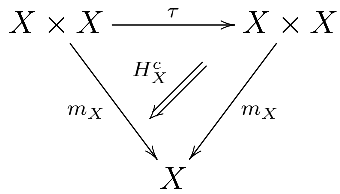
□

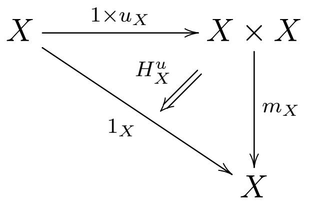
5.1 Commutative H-Group Objects

In this section we want to study a (internal) version of commutative group objects up to a suitable notion of homotopy. The tools developed here will be used in the last part of the chapter to establish a parallelism between classical results on realization-fibrations and the more modern approach presented here.

Definition 5.1.1. Given a model category $\mathscr{M}$, a commutative *H-group object* in $\mathscr{M}$ is a sestuple $(X, u, m, H^a_X, H^c_X, H^u_X)$, where X is a fibrant and cofibrant object of $\mathscr{M}$, $u : * \longrightarrow X$ is a map from the terminal object of $\mathscr{M}$ and $m : X \times X \longrightarrow X$ is a map, such that these data satisfy the axiom of a commutative monoid in $\mathrm{Ho}(\mathscr{M})$, and this has to be witnessed by the prescribed homotopies.
More precisely, we have:







Where $\tau : X \times X \longrightarrow X \times X$ is the shuffle isomorphism, and we have considered only the right unit axiom thanks to the commutativity hypothesis.
Moreover, we ask the so-called *shearing map*

$$s := (\pi_X, m_X) : X \times X \longrightarrow X \times X$$

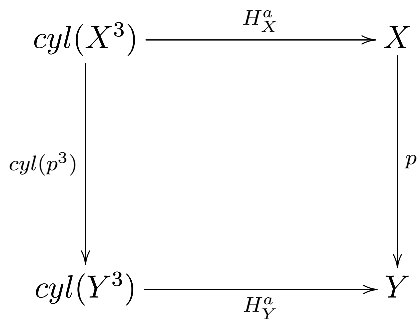
to be a weak equivalence.

When no confusion arises, we will just indicate the underlying object in place of the sestuple constituting the commutative H-group structure.

Definition 5.1.2. A morphism between commutative H-group objects

$$\tilde{f} : (X, u_X, m_X, H^a_X, H^c_X, H^u_X) \longrightarrow (Y, u_Y, m_Y, H^a_Y, H^c_Y, H^u_Y)$$

is a map $f : X \longrightarrow Y$ in $\mathscr{M}$ that commutes (strictly, for our purposes) with the structure maps m_X, m_Y, u_X, u_Y, and such that the following squares are strictly commutative (where, without loss of generality, we have assumed our homotopies to be modeled by a fixed functorial cylinder object $cyl : \mathscr{M} \longrightarrow \mathscr{M}$):



$$
\begin{array}{ccc}
cyl(X^2) & \xrightarrow{H^c_X} & X \\
{\scriptstyle cyl(p^2)}\downarrow & & \downarrow{\scriptstyle p} \\
cyl(Y^2) & \xrightarrow[H^c_Y]{} & Y
\end{array}
$$

$$
\begin{array}{ccc}
cyl(X) & \xrightarrow{H^u_X} & X \\
{\scriptstyle cyl(p)}\downarrow & & \downarrow{\scriptstyle p} \\
cyl(Y) & \xrightarrow[H^u_Y]{} & Y
\end{array}
$$

Let us now consider the case where $\mathscr{M} := \mathbf{sSet}^{\Delta^{op}}$, and the functorial cylinder object is given by its simplicial structure, i.e.:

$$(\cdot) \otimes \Delta[1] : \mathbf{sSet}^{\Delta^{op}} \longrightarrow \mathbf{sSet}^{\Delta^{op}}$$

(see Lemma 9.5.14 of [Hir] for a reference).
Given an H-group morphism $p : X \longrightarrow Y$ which is also a fibration, we set

$$\ker(p) := X \times_Y *$$

i.e., with $G := ker(p)$, we have a pullback square:

$$
\begin{array}{ccc}
G & \xrightarrow{i} & X \\
\downarrow & & \downarrow{\scriptstyle p} \\
* & \xrightarrow[u_Y]{} & Y
\end{array}
$$

We want to endow G with a natural H-group structure.
To do so, consider the following diagram:

$$
\begin{array}{ccccc}
G \times G & \xrightarrow{i \times i} & X \times X & & \\
 & \searrow & & \searrow{\scriptstyle m_X} & \\
 & & G & \xrightarrow{i} & X \\
 & & \downarrow & & \downarrow{\scriptstyle p} \\
 & & * & \xrightarrow{u_Y} & Y
\end{array}
$$

Since the outer part commutes, the universal property of pullbacks yields a map

$$m_G : G \times G \longrightarrow G$$

such that

$$i \circ m_G = m_X \circ (i \times i)$$

Similarly, we get a map

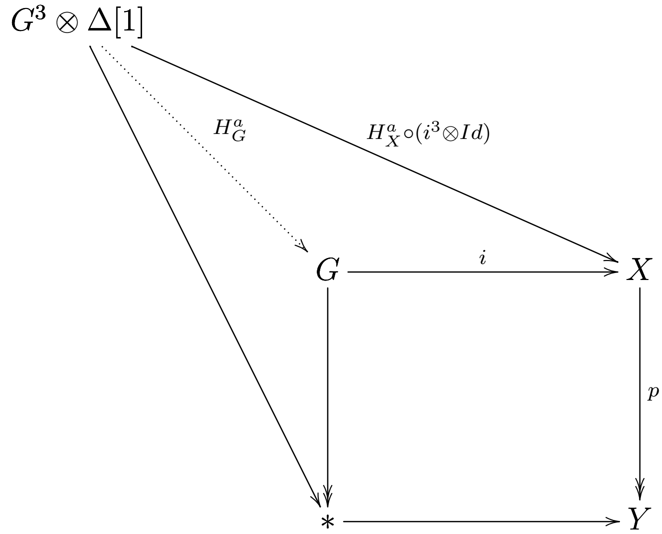
$$u_G : * \longrightarrow G$$

such that

$$i \circ u_G = u_X$$

Lemma 5.1.3. *Let* $p : X \longrightarrow Y$ *be a fibration and a commutative H-group objects morphism in simplicial spaces.*
Then $G := \ker(p)$ *is naturally a commutative H-group object as well.*

Proof. We just need to construct the missing homotopies, witnessing the fact that (G, u_G, m_G) is a commutative monoid in $Ho(\mathbf{sSet}^{\Delta^{op}})$, and we have to show that the shearing map for G is a weak equivalence.
Let us deal with the homotopy associativity, the other cases being similar.
If we consider the following solid diagram, we see that we have a unique factorization (thanks to the universal property of pullbacks) rendering the whole diagram commutative:



Furthermore, notice that:

$$(i \circ H^a{}_G)_{|G^3 \otimes \{0\}} = (H^a_X \circ (i^3 \otimes Id))_{|G^3 \otimes \{0\}} = m_X \circ (m_X \times 1_X) \circ i^3 = m_X \circ (m_X \circ (i \times i) \times i) =$$

$$= m_X \circ (i \circ m_G \times i) = m_X \circ (i \times i) \circ (m_G \times 1_G) = i \circ m_G \circ (m_G \times 1_G)$$

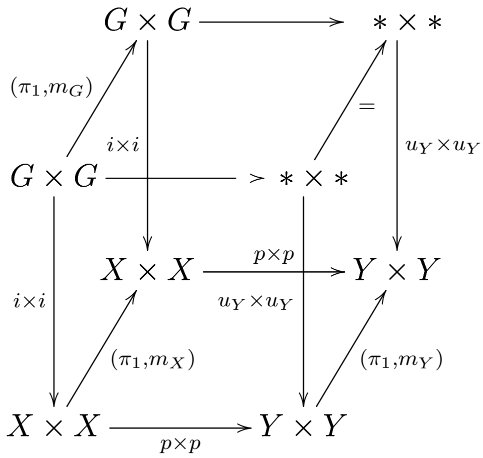
And, similarly,

$$(i \circ H^a{}_G)_{|G^3 \otimes \{1\}} = i \circ m_G \circ (1_G \times m_G)$$

Since i is a monomorphism, it follows that:

$$H^a_G : m_G \circ (m_G \times 1_G) \simeq m_G \circ (1_G \times m_G)$$

hence G is homotopy associative.
The other homotopies are obtained in an analogous manner.
Turning to the shearing map, consider the following commutative diagram:



The front face is homotopy cartesian, as well as the back one, and

$$(\pi_1, m_X) : X \times X \longrightarrow X \times X,\ (\pi_1, m_Y) : Y \times Y \longrightarrow Y \times Y$$

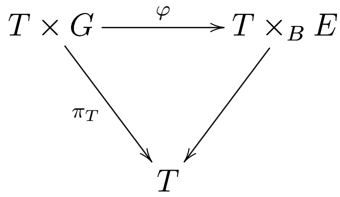
are weak equivalences by hypothesis.
It then follows that also

$$(\pi_1, m_G) : G \times G \longrightarrow G \times G$$

is a weak equivalence (Prop.13.3.14, [Hir]).

$\square$

Lemma 5.1.4. *Let $p : E \longrightarrow B$ be a fibration and a commutative H-group objects morphism in simplicial spaces, and set $G := \ker(p)$.*
Let $b : T \longrightarrow B$ be a map of simplicial spaces. If there exists a map $e : T \longrightarrow E$ such that $p \circ e = b$, then there is a commutative diagram:

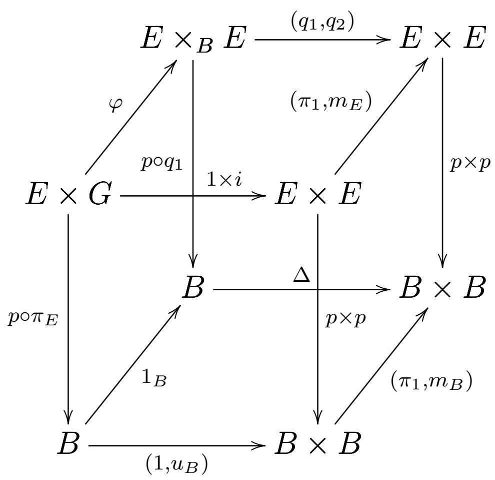


such that φ is a weak equivalence.

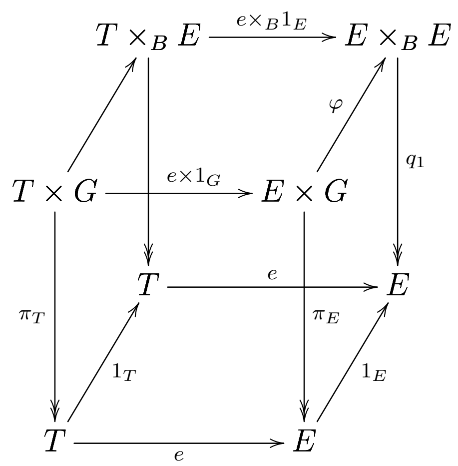
Proof. Let us first address the case $b = p : E \longrightarrow B$ and $e := 1_E$.
The following squares are easily seen to be pullbacks (hence homotopy cartesian, since they are along fibrations):

$$\begin{array}{ccc} E \times G & \xrightarrow{1 \times i} & E \times E \\ \downarrow{\scriptstyle p \circ \pi_E} & & \downarrow{\scriptstyle p \times p} \\ B & \xrightarrow[(1, u_B)]{} & B \times B \end{array} \qquad \begin{array}{ccc} E \times_B E & \xrightarrow{(q_1, q_2)} & E \times E \\ \downarrow{\scriptstyle p \circ q_1} & & \downarrow{\scriptstyle p \times p} \\ B & \xrightarrow[\Delta]{} & B \times B \end{array}$$

Where $q_1, q_1 : E \times_B E \longrightarrow E$ are the pullback projections.
Hence, the following commutative cube shows that the induced map $\varphi : E \times G \longrightarrow E \times_B E$ is a weak equivalence, since the displayed shearing map are such:



Furthermore, by construction, one has that $q_1 \circ \varphi = \pi_E : E \times G \longrightarrow E$, as required.
Now, the general case follows by pulling back along e.
More precisely, the front and back faces of the following cube are pullbacks (hence, as above, homotopy cartesian squares too):



We thus get a weak equivalence $\hat{\varphi} : T \times G \longrightarrow T \times_B E$ over T. $\square$

Thanks to the last lemma we can prove the following:

Theorem 5.1.5. *Let $p : E \longrightarrow B$ be a fibration and a morphism of commutative H-group objects in simplicial spaces.*
Let $F(p) \subset \pi_0 B$ be the image of the map $\pi_0(p) : \pi_0 E \longrightarrow \pi_0 B$, and let $V(p)$ be the union of path components of the simplicial set $\pi_0 B$ which do not touch $F(p)$.
Then:

$$\mathrm{lproj}(p) = F(p) \cup V(p)$$

Proof. Pick a representative $d : H_{[m]} \longrightarrow B$ for an element $d \in F(p)([m])$. By hypothesis there exists an arrow $d' : H_{[m]} \longrightarrow E$ such that $p(d')$ and d represent the same element in $\pi_0 B([m])$.
We have shown in Lemma 4.3.8 that this implies that the fiber over $p \circ d'$ and d, respectively, are weakly equivalent over $H_{[m]}$. The previous lemma implies that the fiber over

$p(d')$ is a trivial fiber bundle, hence such is the fiber over d, by Proposition 4.6.3.
We have thus proven that $F(p) \subset \mathrm{lproj}(p)$.
Lemma 5.0.8 implies $V(p) \subset \mathrm{lproj}(p)$, so it remains to show that $\mathrm{lproj}(p) \subset F(p) \cup V(p)$.
Suppose $b \in \mathrm{lproj}(p)$ is represented by an arrow $b : H_{[m]} \longrightarrow B$.
By definition, the pullback of b along p is weakly equivalent (over $H_{[m]}$) to a projection of the form

$$q : H_{[m]} \times C \longrightarrow H_{[m]}$$

If $C = \emptyset$ then $b \in V(p)$, just by definition, while otherwise the map q admits a section. Passing to the π_0 we then see that $b \in F(p)$(i.e. it represents an element in the image of $\pi_0(p)$), and we can conclude the proof.

□

The following corollary is now immediate (thanks to Theorem 4.6.5):

Corollary 5.1.6. *Let $p : E \longrightarrow B$ be a fibration and a morphism of commutative H-group objects in simplicial spaces, such that*

$$\pi_0(p) : \pi_0 E \longrightarrow \pi_0 B$$

is surjective onto all the components which touch its image.
Then p is a fiber bundle, hence a realization-fibration.

We will now characterize realization-fibrations of simplicial spaces by using Theorem 4.4.4 .

Theorem 5.1.7. *A map $p : E \longrightarrow B$ of simplicial spaces is a realization-fibration if and only if for all $b : H_{[n]} \longrightarrow B$ and all $f : [0] \longrightarrow [n]$ in Δ, the induced map*

$$|\mathrm{Fib}(p, b \circ H_f)|_\Delta \longrightarrow |\mathrm{Fib}(p, b)|_\Delta$$

is a weak equivalence of spaces.

Proof. One implication is trivial, let us focus on the other one.
We want to prove p to be a local realization-fibration, and then use Theorem 4.4.4 to conclude.
Given any $u : [m] \longrightarrow [n]$ in Δ, pick any map $i : [0] \longrightarrow [m]$ in Δ so as to obtain

$$H_{[0]} \xrightarrow{H_i} H_{[m]} \xrightarrow{H_u} H_{[n]} \xrightarrow{b} B, \qquad H_{u \circ i} : H_{[0]} \longrightarrow H_{[n]}$$

Computing the homotopy pullback of these maps along p we get the following diagram:

$$\mathrm{Fib}(p, b \circ H_{u\circ i}) \xrightarrow{g} \mathrm{Fib}(p, b \circ H_u) \xrightarrow{h} \mathrm{Fib}(p, b), \qquad h \circ g : \mathrm{Fib}(p, b \circ H_{u\circ i}) \longrightarrow \mathrm{Fib}(p, b)$$

Now, $|h \circ g|_\Delta$ and $|g|_\Delta$ are weak equivalences by hypothesis, so that it follows by the two out of three property of weak equivalences in a model category that $|h|_\Delta$ is a weak equivalence, and we can conclude. □

5.2 The π_*-Kan condition

There is a classical result by Bousfield and Friedlander which deals with our problem in a less modern language, and in the context of simplicial spaces only.
The original reference is [BF], while a bit more modern treatment can be found in [GJ].
Throughout this section we will assume our simplicial spaces are pointwise fibrant, although everything works up to a fibrant replacement.
Given such a simplicial space X, let us define a new simplicial set $\pi_n X$(for each $n \geq 0$) by setting for each $n \geq 0$

$$\pi_n X(m) := \coprod_{x \in X(m,0)} \pi_n(X_m, x)$$

Obviously we have a map of simplicial sets $\pi_n X \longrightarrow X(\bullet, [0])$, which we can extend to a map

$$\Pi : \pi_n X \longrightarrow X$$

by thinking the domain as a discrete simplicial space.
By construction, the fiber of the map

$$\Pi_m : \pi_n X([m]) \longrightarrow X([m], \bullet)$$

over a vertex $x \in X([m],[0])$ is precisely $\pi_n(X_m, x)$.
For any $0 \leq i \leq m$ let $H_{\Lambda^i[m]} \subset H_{[m]}$ be the functor which represents the i-th horn, i.e.

$$H_{\Lambda^i[m]} := \underset{f:[i] \longrightarrow [m] \in \widehat{\int \Delta[m]}}{\operatorname{colim}} H_{[i]}$$

where $\widehat{\int \Delta[m]}$ is the full subcategory of $\in \Delta[m]$ without the objects corresponding to $1_{[m]}$ and $d^i : [m-1] \longrightarrow [m]$.
Obviously, we have a natural inclusion

$$j : H_{\Lambda^i[m]} \longrightarrow H_{[m]}$$

for any $0 \leq i \leq m$.
We can now formulate the following:

Definition 5.2.1. A simplicial space X satisfies the π_*-Kan condition if for any $m, t \geq 1$ and for any $0 \leq i \leq m$, any square of the following form admits a diagonal lift:

$$\begin{array}{ccc} H_{\Lambda^i[m]} & \longrightarrow & \pi_t X \\ {\scriptstyle j}\downarrow & \nearrow & \downarrow{\scriptstyle \Pi} \\ H_{[m]} & \underset{v}{\longrightarrow} & X \end{array}$$

In [GJ] and [BF] some sufficient conditions are given in order for a simplicial space X to satisfy the π_*-Kan condition, namely:

- A group object X in the category of simplicial spaces satisfies the π_*-Kan condition ([[BF], Section B.3]).

- A levelwise connected simplicial space X satisfies the π_*-Kan condition ([[BF], Section B.3]).

The classical theorem is the following:

Theorem 5.2.2 ([BF], Theorem B4)**.** *Let $f : X \longrightarrow Y$ be a map in $\boldsymbol{sSet}^{\Delta^{op}}$. If X and Y satisfy the π_*-Kan condition, and if $\pi_0(f) : \pi_0 X \longrightarrow \pi_0 Y$ is a Kan fibration of simplicial sets, then f is a realization-fibration.*

It is possible to establish some sort of parallelism between this approach and the one we have presented so far.

Proposition 5.2.3. *A map $p : E \longrightarrow B$, where E and B are levelwise connected (hence they satisfy the π_*-Kan condition), is a realization fibration.*

Proof. By hypothesis, $\pi_0 B$ is a discrete simplicial set, hence p must be a realization-fibration thanks to Prop. 5.0.10. □

Proposition 5.2.4. *Let X, Y be commutative group objects in the category of simplicial spaces (thus, a fortiori, H-group objects), so that they satisfy the π_*-Kan condition.*
If $p : X \longrightarrow Y$ is a fibration and a map of H-group objects, and $\pi_0(p) : \pi_0 X \longrightarrow \pi_0 Y$ is a fibration, then p is a realization-fibration.

Proof. We will obtain that p is a realization fibration provided that $\pi_0(p)$ is surjective on the path components it touches, thanks to Corollary 5.1.6.
This follows from Lemma 5.2.5. □

Lemma 5.2.5. *Given a fibration $p : E \longrightarrow B$ in **sSet**, then it is (levelwise) surjective on the path components it reaches.*
*More precisely, if $I \subset \pi_0 B$ parametrizes the path components which are touched by p, then the left-hand side map in the following pullback square is an epimorphism in **sSet** (where $B_i \subset B$ is the path component corresponding to $i \in I$):*

$$\begin{array}{ccc} E' & \longrightarrow & E \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ \coprod_{i \in I} B_i & \longrightarrow & B \end{array}$$

Proof. Of course it suffices to show that a Kan fibration between simplicial sets which induces a surjection on π_0 is an epimorphism.
So assume $p : E \longrightarrow B$ is such: given a 0-vertex $b \in B_0$ we can find a 0-vertex $e \in E_0$ and a 1-simplex $f : p(e) \longrightarrow b$ by the assumption of surjectivity on components.
Now we can use the lifting property of p in the following commutative square:

$$\begin{array}{ccc} \Lambda^1[1] & \xrightarrow{\;e\;} & E \\ {\scriptstyle h^1_1}\downarrow & & \downarrow{\scriptstyle p} \\ \Delta[1] & \longrightarrow & B \end{array}$$

so that we get a lift e' for the vertex b.
So far we have proven the surjectivity on 0-simplices, but this is all we need.
Indeed, given an n-simplex $\beta : \Delta[n] \longrightarrow B$, in order to find a counterimage via p, just pick a lift x of a vertex (say, the nth one) of β, and consider the following commutative diagram:

$$\begin{array}{ccc} \Delta[0] & \xrightarrow{\quad x \quad} & E \\ \downarrow & & \downarrow p \\ \Delta[n] & \xrightarrow[\quad \beta \quad]{} & B \end{array}$$

The left-hand side map is a trivial cofibration, hence the diagrams admits a diagonal filler, which is a counterimage of β via p. □

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