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G-SMOOTHING THEORY

R. LASHOF AND M. ROTHENBERG

Introduction. If G is a finite group then smoothing theory **[L3]** for topological manifolds can be carried over to G-smoothing theory for G-manifolds with only small changes.

1. *Replace vector bundles by G-vector bundles.* A G-vector bundle is a vector bundle $p: E \to X$ such that E and X are G-spaces and p and the 0-section s are G-maps, and G acts on E through vector bundle maps. The tangent bundle of a smooth G-manifold is a G-vector bundle.

2. *Replace R^n bundles by G-R^n bundles.* If M is a G-manifold then the tangent microbundle is a G-microbundle. However to prove a G-Kister theorem that G-microbundles contain G-R^n bundles unique up to equivalence one needs to know that the G-microbundle is locally equivalent to a G-vector bundle. This will be true if M is locally G-smoothable in the sense of Bredon **[B1]**. It is easy to show that a G-manifold M is locally smoothable if its tangent microbundle is locally linear. A standard construction gives classifying spaces for G-vector bundles and G-locally linear R^n bundles.

3. *G-isotopy extension theorem.* Let M be a locally smoothable G-manifold and $K \subset M$ a G-invariant compact subspace. Consider the semisimplicial complex $E(K, M)$ of G-embeddings $f: \Delta^i \times U \to \Delta^i \times M$, f commuting with projection on the i-simplex Δ^i, when U is a G-neighborhood of K and we identify f and f': $\Delta^i \times U' \to \Delta^i \times M$ if they agree on some smaller neighborhood of K. Then r: Homeo$(M) \to E(K, M)$ is a fibration, where $H(M)$ is the ss-complex of G-homeomorphisms of M. This may be proved using the work of Siebenmann **[S1]**.

4. *G-immersion theorem.* Let M be a locally smoothable G-manifold. For $H \subset G$ let $M_{(H)} = (x \in M | G_x$ is conjugate to $H)$. Let $M^H =$ fixed point set under H. M^H is a locally flat submanifold. Let $M^{(H)} = GM^H$.

LEMMA. *Let $(H_j)_{j \in J}$ be the orbit types of M. For each $j \in J$, let $[M_i^j | i \in I(j)]$ be the*

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G-components of $M^{(H_j)}$. *The minimal elements of this partially ordered set under inclusion are the topologically closed G-components of the* $M_{(H_j)}$, *and hence are G-submanifolds.*

DEFINITION (BIERSTONE). A smooth G-manifold is said to have a good handle-bundle decomposition if there is a normal direction in each handle bundle on which G acts trivially.

THEOREM (BIERSTONE). *A smooth G-manifold has a good handle-bundle decomposition if and only if the minimal elements of the* (M_i^j) *are nonclosed (as manifolds).*

DEFINITION. If a locally smooth G-manifold M has nonclosed minimal elements in (M_i^j), we say that M satisfies the *Bierstone condition*. Bierstone **[B2]** proves a G-Gromov Theorem for smooth G-manifolds satisfying the Bierstone condition. In particular, he proves

THEOREM. $I(M^n, N^n) \to^d \mathbf{R}(TM, TN)$ *is a homotopy equivalence when* M^n *satisfies the Bierstone condition,* $I(M, N)$ *is the space of G-immersions and* $\mathbf{R}(TM, TN)$ *is the space of G-vector bundle maps.*

A topological version of this is true for locally smoothable G-manifolds where $\mathbf{R}(TM, TN)$ is G-$\mathbf{R}^n$ bundle maps and the spaces are treated semisimplicially. This uses the G-isotopy extension theorem above.

5. *G-smoothing up to sliced concordance.* As in the case $G = (e)$, the immersion theory has the immediate consequence that if M is a locally smoothable G-manifold satisfying the Bierstone condition, then the sliced concordance classes of G-smoothings of M are in 1-1 correspondence with the isotopy classes of reductions of its tangent bundle to a G-vector bundle. Two smoothings M_α, M_β of M are said to be *sliced concordant* if there exists a smoothing $(M \times I)_\gamma$ of $M \times I$ such that the projection Π_2 on I is a submersion (and hence $\Pi_2^{-1}(t)$ is a smooth submanifold for each $t \in I$) and $(M \times 0)_\gamma = M_\alpha$, $(M \times 1)_\gamma = M_\beta$.

Gerald Anderson **[A1]** also outlines a proof of the result along the same lines but he does not mention the G-isotopy extension theorem and may have a different argument in mind.

6. *G-engulfing theorem.* In order to pass from a smoothing theory up to sliced concordance to a general smoothing theory up to isotopy, one uses an engulfing lemma (this is where the restriction $n \neq 4$ comes into ordinary smoothing theory).

THEOREM. *Let* M^n *be a compact smooth G-manifold and* $h: M \times \mathbf{R} \to V^{n+1}$ *be a G-homeomorphism onto a smooth G-manifold such that* $h|M \times 0$ *and* $h|\partial M \times \mathbf{R}$ *are smooth. Then if for each* $H \subset G$, $\dim V^H \neq 4$, *there is a G-diffeomorphism* $f: M \times \mathbf{R} \to V$ *such that* $f|M \times 0 = h|M \times 0$ *and* $f|\partial M \times \mathbf{R} = h|\partial M \times \mathbf{R}$.

7. *G-smoothing theorem.*

THEOREM. *Let* M^n *be a locally smoothable G-manifold such that for any* $H \subset G$, $\dim M^H \neq 4$. *Then the G-isotopy classes of G-smoothings of M are in* 1-1 *correspondence with the isotopy classes of reductions of the tangent bundle to a G-vector bundle.* (*If* $\partial M \neq \emptyset$, *we need* $\dim(\partial M)^H \neq 4$.)

In terms of classifying spaces we have

$$\begin{array}{ccc} & & BO_n(G) \\ & \overset{\tilde{t}}{\nearrow} & \downarrow p \\ M & \xrightarrow{t} & B\,\mathrm{Top}_n(G), \end{array}$$

where t and $\tilde{t}$ are G-maps. Thus our problem reduces to G-obstruction theory. This comes down to studying the obstructions to lifting the map on fixed point sets for $H \subset G$.

$[B\,\mathrm{Top}_n(G)]^H$ is the disjoint union of $B(\mathrm{Top}_n^\rho)$, where $\rho\colon H \to O(n)$ is a representation and $\mathrm{Top}_n^\rho \subset \mathrm{Top}_n$ is the subgroup of homeomorphisms commuting with $\rho(h)$, $h \in H$. The sum is over one ρ from each Top_n equivalence class of representations. The fibre of $p_n^H\colon [BO_n(G)]^H \to [B\,\mathrm{Top}_n(G)]^H$ above $x \in B\,\mathrm{Top}_n^\rho$ is the disjoint union of $\mathrm{Top}_n^{\rho^1}/O_n^{\rho^1}$ when ρ^1 is Top_n equivalent to ρ and we pick one ρ^1 from each O_n equivalence class. For groups H of odd prime power order, Schultz has shown that Top_n equivalence implies O_n equivalence and hence $(p_n^H)^{-1}(x) = \mathrm{Top}_n^\rho/O_n^\rho$. In any case, our obstructions lie in the homotopy groups of $\mathrm{Top}_n^\rho/O_n^\rho$, $\rho\colon H \to O(n)$.

8. $\pi_i\mathrm{Top}_n^\rho/O_n^\rho$, *$\rho$ semifree.* We consider a stable representation $\alpha\colon H \to O_n \subset O_{n+1} \subset \cdots \subset O_{n+k}$, where the action of H on S^{n-1} is free.

THEOREM. *There are fibrations (up to homotopy equivalence):*

(1) $P^\alpha(S^{n+k}) \to \mathrm{Top}_{n+k}^\alpha/O_{n+k}^\alpha \to \mathrm{Top}_{n+k+1}^\alpha/O_{n+k+1}^\alpha$,

(2) $P^\alpha(S^{n+k} \bmod S^k) \to \mathrm{Top}_{n+k}^\alpha/\mathrm{Top}_k \to \mathrm{Top}_{n+k+1}^\alpha/\mathrm{Top}_{k+1}$,

(3) $P^\alpha(S^{n+k} \bmod S^k) \to P^\alpha(S^{n+k}) \to P(S^k)$,

where $P^\alpha(S^{n+k}) \subset P(S^{n+k})$ is the subgroup of pseudoisotopies (or concordances) which commute with α.

THEOREM (D. ANDERSON AND W. C. HSIANG).

$$P^\alpha(S^{n+k} \bmod S^k) \simeq P_b(L \times R^{k+1}),$$

where $P_b(L \times R^{k+1})$ is the group of bounded pseudoisotopies of $L \times R^{k+1}$, $L = S^{n-1}/\alpha$.

THEOREM (D. ANDERSON AND W. C. HSIANG [**A2**]). *For $n + k \geqq 6$, $\Pi = \Pi_1(L)$.*

$$\Pi_i P_b(L \times R^{k+1}) = \begin{cases} K_{-k+1+i}Z(\Pi), & 0 \leqq i < k-1, \\ \tilde{K}_0[Z(\Pi)], & i = k-1, \\ Wh_1(\Pi), & i = k, \\ \Pi_{i-k-1}P(L \times D^{k+1}), & i \geqq k+1. \end{cases}$$

REMARKS. For Π Abelian, $K_{-S}[Z(\Pi)] = 0$ for $S > 1$.

For Π Abelian + prime power order, $K_{-1}[Z(\Pi)] = 0$.

For Π cyclic of order p, $\tilde{K}_0[Z(\Pi)]$ = class group of $Q(e^{2\pi i/p})$.

For Π finite $\tilde{K}_0[Z(\Pi)]$ is finite.

For Π free Abelian $\tilde{K}_0[Z(\Pi)] = 0$.

The above results show that to compute $\Pi_i\mathrm{Top}_{n+k}^\alpha/O_{n+k}^\alpha$, at least up to extensions, one can either compute $\Pi_i\mathrm{Top}^\alpha/O^\alpha$ and work backwards or begin at $\Pi_i\mathrm{Top}_n^\alpha/O_n^\alpha$ and work up. We have had some success in both approaches, but before stating the results we first note an immediate consequence for stability of G-smoothings.

9. *Stability theorems.*

DEFINITION. Let M be a G-manifold. M is said to have a spine of codim $\geqq r$ if the cohomological dimension of $M^H/G \leqq \dim M^H - r$.

EXAMPLE 1. $M \times R^r$, where G acts trivially on R^r, has a spine of codim r.

EXAMPLE 2. If M^H and ∂M^H are noncompact for all $H \subset G$, then M has a spine of codim $\geqq 1$.

DEFINITION. Let $\mathscr{S}_G(M)$ be the isotopy classes of G-smoothings of M. Let $\bar{\mathscr{S}}_G(M)$ be the isotopy classes of stable G-smoothings, i.e., of $M \times R^s$, s arbitrarily large. Let $M_0 = M - \partial M$.

Consider the following statements:

A_r: M spine codim r then $\mathscr{S}_G(M) \to \bar{\mathscr{S}}_G(M)$ is epi,

B_r: M spine codim r then $\mathscr{S}_G(M) \to \bar{\mathscr{S}}_G(M)$ is bijective,

C_r: M spine codim r then $\mathscr{S}_G(M) \to \bar{\mathscr{S}}_G(M_0)$ is epi.

THEOREM. *Let M be a locally smoothable G-manifold, G finite and acting semifreely. Then if* $\dim M - \dim M^G \neq 2$ *and* $\dim M^G \geqq 5$:

(1) *If G is finite Abelian, then* A_2, B_3, C_2 *are true.*

(2) *If further G is of prime power order, then* A_1, B_2, C_1 *are true.*

(3) *If further G is of prime order and the class group of* $Z[e^{2\pi i/o(G)}] = 0$ *then* A_0, B_1, C_0 *are true.*

(4) *If* $G = Z_2, Z_3$ *then* A_0, B_0, C_0 *are true.*

10. *Computation of $\Pi_i \operatorname{Top}^\alpha_{n+k}$.* First note that

$$\Pi_i \operatorname{Top}^\alpha_{n+k} \simeq \Pi_i \operatorname{Top}_k \oplus \Pi_i \operatorname{Top}^\alpha_{n+k}/\operatorname{Top}_k.$$

Write $T^\alpha_{n,k} = \operatorname{Top}^\alpha_{n+k}/\operatorname{Top}_k$. Then

(a) One has an exact sequence, $n + k \geqq 6$, $k \geqq 0$:

$$\begin{aligned} 0 &\to \Pi_{k+1}\tilde{A}(L) \to \Pi_{k+1}T^\alpha_{n+1,\,k+1} \to Wh_1(\Pi) \to \Pi_k T^\alpha_{n,\,k} \to \Pi_k T^\alpha_{n+1,\,k+1} \\ &\to \tilde{K}_0 Z(\Pi) \to \Pi_{k-1} T^\alpha_{n,k} \to \Pi_{k-1} T^\alpha_{n+1,k+1} \to K_{-1}[Z(\Pi)] \to \cdots. \end{aligned}$$

Here $\tilde{A}(L)$ = block automorphisms of $L = S^{n-1}/\alpha$. Also

$$\Pi_1(L) = \begin{cases} H & \text{if } n \geqq 3, \\ Z & \text{if } n = 2, \\ e & \text{if } n = 1. \end{cases}$$

(b) For $n + i \geqq 5$, $i \geqq 1$, there is an exact sequence **[A3]** ($L_* = L \cup$ point)

$$\begin{aligned} &\to [\Sigma^{n+1}(L_*), G/\operatorname{Top}] \to \mathscr{L}^s_{n+1}(\Pi) \to \Pi_i[\mathscr{H}(L)/A(L)] \\ &\to [\Sigma^i(L_*), G/\operatorname{Top}] \to \ldots \to [\Sigma(L_*), G/\operatorname{Top}] \to \mathscr{L}^s_n(\Pi), \end{aligned}$$

where $\mathscr{H}(L)$ is the space of homotopy equivalences of L and $\mathscr{L}^s_n(\Pi)$ is the Wall group. Thus (a) and (b) determine $\Pi_i \operatorname{Top}^\alpha_{n+k}/\operatorname{Top}_k$, $0 \leqq i \leqq k$, up to extensions, $n + k \geqq 6$. (For $i = 0$, one needs a special argument.) For $i \geqq k + 1$, we have

$$\Pi_i \operatorname{Top}^\alpha_{n+k}/\operatorname{Top}_k \simeq \Pi_{i-k-1} A(L \times D^{k+1}),$$

where $A(L \times D^{k+1})$ = homeomorphism of $L \times D^{k+1}$ fixed on $L \times S^k$. Results of Hatcher and Wagoner give results on $\Pi_0 A(L \times D^{k+1})$. The higher homotopy groups are still unknown. However, for G-smoothing, G acting semifreely, this is sufficient.

11. *Computation of* Π_i $\mathrm{Top}^\alpha/O^\alpha$, $o(H)$ *prime power.*

THEOREM. *If* $n + k \geqq 8$ *and* $k \geqq 5$, *there exists an exact sequence*

$$\to H_{n+k}(\tilde{K}_0) \to C_\alpha^{n+k} \to [S_\alpha^{n+k}, \mathrm{Top}/O]_H \to H_{n+k-1}(K_0) \to \cdots.$$

Here

(a) $[S_\alpha^{n+k}, \mathrm{Top}/O]$ = equivariant homotopy classes of base-pointed maps = stable H-smoothings of S_α^{n+k} which give standard α action on $D_\alpha^{n+k} \subset S_\alpha^{n+k}$.

(b) $W \in C_\alpha^{n+k}$ if W is a smooth H-manifold, W homeomorphic to S^{n+k}, W^H homeomorphic to S^k. If $x \in W^H$, the action of H on W_x is given by α. We can take the H-connected sum along the fixed point set. Identify $W = 0$ if $W = \partial V$ where ∂V is homeomorphic to D^{n+1} and V^H is homeomorphic to D^{m+1}. C_α^{n+k} is determined up to extensions by homotopy groups of spheres and Wall groups, etc. (Rothenberg [**R1**]). C_α^{n+k} is finitely generated, rank is known if H is cyclic.

(c) A differential $d_i\colon \tilde{K}_0 \to \tilde{K}_0$ can be defined so that $d_i d_{i+1} = 0$. If $o(H)$ is odd, it is conjectured that $d_i(x) = x + (-1)^i x$. If this is the case then

$$H_{\text{even}}(\tilde{K}_0) = \text{elements of order 2},$$
$$H_{\text{odd}}(\tilde{K}_0) = \tilde{K}_0/2\tilde{K}_0.$$

THEOREM. *There is an exact sequence*

$$\to [(D^{n+k+1} \times L, S^{n+k} \times L), (\mathrm{Top}/O, *)] \to [S_\alpha^{n+k}, \mathrm{Top}/O]_H \to [S^k, \mathrm{Top}^\alpha/O^\alpha]$$
$$\to [(D^{n+k} \times L, S^{n+k-1} \times L), (\mathrm{Top}/O, *)] \to \cdots.$$

Thus $\Pi_k(\mathrm{Top}^\alpha/O^\alpha)$ *is finitely generated with rank* = *rank* C_α^{n+k}.

1. *G*-**bundles.** Let $p\colon E \to X$ be a locally trivial bundle with fibre F and group A. p is called a *G-bundle*, or more precisely a *G-A bundle* if E and X are G-spaces, p is a G-map, and G acts on E through A-bundle maps. Two G-A bundles over X are called *G-A equivalent* if they are A-equivalent via a G-equivariant map.

EXAMPLE 1. A G-vector bundle of dimension n is simply a G-L_n bundle, L_n the group of linear isomorphisms of R^n.

If $p\colon E \to X$ is a G-A bundle, G acts on the associated principal A-bundle P through bundle maps. That is, G acts on the left and A acts on the right of P and these actions commute. Conversely, if $p\colon P \to X$ is a principal G-A bundle and A acts on the left of F, then $E = P \times_A F$ is a G-A bundle with fibre F. Two G-A bundles with fibre F are G-A equivalent if and only if their associated principal G-A bundles are G-A equivalent.

In order to prove a covering homotopy property or to produce a classifying space for ordinary bundles the local triviality property is essential. For G-A bundles we will need a G-local triviality condition for the same purpose. Before defining this condition we recall the local structure of a completely regular G-space X (see Bredon [**B1**]): For any $x \in X$ there is a G_x-invariant subspace V_x containing x, called a *slice* through x, such that $\psi_x\colon G \times_{G_x} V_x \to X$, $\psi_x[g, v] = gv$ is a homeomorphism onto a G-invariant neighborhood of the orbit Gx. The G-invariant neighborhood, GV_x, is called a *tube* about Gx. For any G-space X we define a *G-chart* to be a pair (V, H) where $H \subset G$, V is an H-invariant subspace of X and the map $\psi\colon G \times_H V \to X$, $\psi[g, v] = gv$, is a homeomorphism onto an open set. (Note that V need not be a slice, i.e., H may not be G_x for any $x \in V$.) A *G-atlas* is a family $\{(V_\alpha, H_\alpha)\}$ of G-

charts such that $\{GV_\alpha\}$ covers X. If X is paracompact, then any cover by G-invariant open sets has a refinement which is a G-atlas.

We note that the preimage under a G-map of a G-chart is a G-chart. That is, if (V, H) is a G-chart in X and $f\colon Y \to X$ is a G-map, then $f^{-1}(V, H) = (f^{-1}(V), H)$. This means $f^{-1}(V)$ is H-invariant and $G \times_H f^{-1}(V) \to^{\psi} Gf^{-1}(V) = f^{-1}(GV)$ is a homeomorphism.

We now describe the appropriate generalization of product bundle: Let $H \subset G$ and $\rho\colon H \to A$ be a representation. If A acts on the left of F and H on the left of V, then H acts on $V \times F$ by $h(v, y) = (hv, \rho(h)y)$. We denote by $1^H_\rho(V)$ the G-A bundle over $G \times_H V$ with fibre F, given by $p\colon G \times_H (V \times F) \to G \times_H F$, $p[g, (v, y)] = [g, v]$. (Note that p is trivial as an A-bundle.)

DEFINITION. A G-A bundle $p\colon E \to X$ with fibre F is called *G-A locally trivial* (or simply *G-locally trivial* if A is fixed) if there is a G-atlas $\{(V_\alpha, H_\alpha)\}$ on X such that $E|GV_\alpha$ is G-A equivalent to $1^{H_\alpha}(V_\alpha)$ for some representation $\rho_\alpha\colon H_\alpha \to A$ (under the identification $\psi_\alpha\colon G \times_{H_\alpha} V_\alpha \to GV_\alpha$).

If X is completely regular this is equivalent to Bierstone's definition **[B2]**: For each $x \in X$, there is a G_x-invariant neighborhood U_x such that $p^{-1}(U_x)$ is G_x-A equivalent to $U_x \times F$ with G_x action $h(u, y) = (hu, \rho_x(h)y)$, where $u \in U_x$, $h \in G_x$, $y \in F$ and $\rho_x\colon G_x \to A$ is a representation.

If $p\colon E \to X$ is a G-A bundle and $f\colon Y \to X$ is a G-map the induced bundle $f^*(p)\colon f^*E \to Y$ is a G-A bundle. Further if p is G-A locally trivial, then $f^*(p)$ is G-A locally trivial.

A G-A bundle is G-A locally trivial if and only if the associated principal bundle is G-A locally trivial.

The following is essentially due to Bierstone and Wasserman.

THEOREM 1. *Any G-L_n bundle over a completely regular X is G-locally trivial.*

PROOF. Let $p\colon E \to X$ be the associated G-vector bundle. Let $x \in X$ and let $\bar{\varphi}\colon \bar{U}_x \times R^n \to p^{-1}(\bar{U}_x)$ be a local trivialization.

We can assume $\bar{U}_x$ is G_x-invariant, since any neighborhood contains a G_x-invariant one as an L_n-bundle. Define $\rho_x\colon G_x \to L_n$ by $\rho_x(h)y = \bar{\varphi}_x^{-1}h\bar{\varphi}_x y$, $h \in G_x$, $y \in R^n$. Now let $\varphi\colon \bar{U} \times R^n \to p^{-1}(\bar{U})$ be the map obtained by averaging over G_x; i.e.

$$\varphi_u y = \frac{1}{|G_x|} \sum h^{-1}\bar{\varphi}_{hu}(\rho_x(h)y), \qquad u \in \bar{U}_x.$$

Then φ_u is linear, $\varphi_x = \bar{\varphi}_x$ and $g\varphi(u, y) = \varphi(gu, \rho_x(g)y)$, $g \in G_x$. Since $\varphi_x = \bar{\varphi}_x$ is an isomorphism, φ_u is an isomorphism for u in some smaller G_x-invariant neighborhood $U_x \subset \bar{U}_x$. Hence $\varphi|U_x \times R^n\colon U_x \times R^n \to p^{-1}(U_x)$ is a G_x-L_n trivialization, and p is G-L_n locally trivial. Q.E.D.

Just as for ordinary bundles, one may ask if a G-A bundle reduces to a G-B bundle, $B \subset A$. If A/B has local cross-section in A, this is true if and only if the associated G-A bundle with fibre A/B has a G-cross-section. A G-vector bundle over a paracompact base can always be given a G-invariant Riemannian metric, and so reduces to a G-O_n bundle, O_n the orthogonal group. This reduction is unique up to equivalence. In particular, if follows from Theorem 1 that G-O_n bundles over paracompact spaces are G-O_n locally trivial.

DEFINITION 2. A G-A bundle $p\colon E \to X$ is called *numerable* if there is a trivializing

G-partition of unity, i.e., there exists a G-partition of unity subordinate to a G-atlas $\{(V_\alpha, H_\alpha)\}$ such that $E|GV_\alpha$ is equivalent to $1^{H_\alpha}(V_\alpha)$, some $\rho_\alpha\colon H_\alpha \to A$.

Bierstone **[B3]** proves that a G-locally trivial bundle over a paracompact base satisfies the G-covering homotopy property. We generalize this slightly to:

THEOREM 2. *A numerable G-A bundle satisfies the G-covering homotopy property.*

To show this one follows the proof for ordinary numerable bundles as given in Husemoller **[H1]**, simply substituting G-partitions of unity for ordinary partitions of unity. This comes down to showing that a G-bundle E over $X \times I$, when G acts trivially on the I-factor, is G-equivalent to $E_0 \times I$, where $E_0 = E/X \times (0)$. To see that Husemoller's argument works one needs to observe the following lemma and corollary.

LEMMA 3. *Let X be a G-space and suppose $X \times I = (W, H)$, i.e., $X \times I$ may be identified with $G \times_H W$ via $[g, w] \to gw$. Then $W = W_0 \times I$, where $W_0 = W \cap (X \times (0))$.*

PROOF. The map $W_0 \times I \subset G \times_H W \xrightarrow{\pi} G/H$ has image eH since G/H is discrete and I is connected. Hence $W_0 \times I \subset W$. To see that $W \subset W_0 \times I$ first note that since $X \times (0)$ is G-invariant, $X \times (0) = G \times_H W_0$ and $X \times I = G \times_H (W_0 \times I)$. Hence if $w \in W$, $w = g(w_0, t)$, some $g \in G$, $w_0 \in W_0$, $t \in I$. Since $(w_0, t) \in W$, we must have $g \in H$; and $g(w_0, t) = (gw_0, t)$, $gw_0 \in W_0$. Hence $w \in W_0 \times I$ and $W = W_0 \times I$.

COROLLARY 4. *With $X \times I = (W, H)$ as in the above lemma, $1^H_\rho(W) = 1^H_\rho(W_0) \times I$ as a G-A bundle.*

Of course, Theorem 2 has the:

COROLLARY 5. *If $p\colon E \to X$ is a numerable G-A bundle and $f_i\colon Y_i \to X$, $i = 1, 2$, are G-homotopic G-maps, then $f_1^*(p)$ and $f_2^*(p)$ are G-A equivalent.*

DEFINITION 3. A universal G-A bundle $p\colon P \to X$ is a G-A numerable bundle such that G-A equivalence classes of G-A numerable bundles over any G-space Y correspond to G-homotopy classes, $[Y, X]_G$, of G-maps of Y into X, the correspondence being given by induced bundles.

Now for ordinary bundles a universal A-bundle is characterized as a numerable contractible principal A-bundle **[D1]**. By a slight refinement of this argument one may prove:

THEOREM 6. *A numerable principal G-A bundle $p\colon P \to X$ is universal if and only if it satisfies: For each $H \subset G$ and representation $\rho\colon H \to A$ consider P as an H-space under the action $p \to hp\rho(h)^{-1}$, $h \in H$, $p \in P$. Then*

(1) $P^H \neq \emptyset$.

(2) *For any $p_H \in P^H$, P is H-contractible to p_H.*

PROOF. We first show the conditions are necessary:

(1) Consider the G-A bundle $\pi\colon G \times_H A \to G/H$, H acting through $\rho\colon H \to A$. Let $e_1 \in G$, $e_2 \in A$ be the unit elements. Then $[e_1, e_2] \in G \times_H A$ and $h[e_1, e_2] = [h, e_2] = [e_1, \rho(h)e_2] = [e_1, e_2]\rho(h)$. Since p is universal, there is a G-A bundle map $\varphi\colon G \times_H A \to P$. Then

$$h\varphi[e_1, e_2] = \varphi(h[e_1, e_2]) = \varphi([e_1, e_2]\rho(h)) = (\varphi[e_1, e_2])\rho(h).$$

Hence $\varphi[e_1, e_2] \in P^H$, and (1) is satisfied.

(2) Consider the G-A bundle $\pi: G \times_H (P \times A) \to G \times_H P$, H acting on P as above. Define the G-A bundle maps $\lambda_i: G \times_H (P \times A) \to P$, $i = 0, 1$, by $\lambda_0[g, p, a] = gpa$ and $\lambda_1[g, p, a] = gp_H a$, $p_H \in P^H$. By the universality of P there is a G-A homotopy $\lambda_t: G \times_H (P \times A) \to P$, $0 \leqq t \leqq 1$, between λ_0 and λ_1. In particular, $p = \lambda_0[e_1, p, e_2]$ is deformed to $\lambda_1[e_1, p_H, e_2] = p_H$. But

$$\lambda_t[e_1, hp\rho(h)^{-1}, e_2] = \lambda_t[h, p, \rho(h)^{-1}] = h\lambda_t[e_1, p, e_2]\rho(h)^{-1}.$$

Hence λ_t defines an H contraction of P to p_H.

The proof of sufficiency is just the same as in Dold **[D1]**, except that we replace the section extension property by the equivariant section property. In particular, if $p': P' \to X'$ is a G-A bundle, G-A bundle maps of P' into P correspond to G-sections of the associated bundle $P' \times_A P$ over X' with fibre P. If p' is numerable and (V, H) is a chart, $P' \times_A P|GV \simeq G \times_H (V \times P)$. But G-sections of a bundle of the form $\pi: G \times_H (V \times P) \to G \times_H V$ are in 1-1 correspondence with H-sections of $\pi: V \times P \to V$. The sufficiency follows from the H-contractibility of P. Q.E.D.

If we restrict our attention to a class of numerable G-A bundles for which the local trivializations are given by representations ρ in some designated subset S of all representations, then a numerable G-A bundle with local trivializations in S will be universal for the class of bundles if and only if (1) and (2) are satisfied with respect to representations in S. In particular, we are interested in the case where S consists of representations into some subgroup $B \subset A$. These bundles will be denoted as *G-(A, B) bundles*. We have in mind the case where $A = \mathrm{Top}_n$, the group of homeomorphisms of R^n fixing $0 \in R^n$ and $B = O_n$. The associated bundles with fibre R^n will be called locally linear G-R^n bundles. In fact, a G-R^n bundle $p: E \to X$, with X paracompact, will be locally linear if and only if for each $x \in X$ there exists a G-invariant neighborhood U of x such that $E|U$ is G-R^n equivalent to a G-vector bundle.

THEOREM 7. *Universal G-(A, B) bundles exist (see Lu* **[L1]**).

PROOF. We first note some properties of infinite joins.

(1) If $\pi_i: P_i \to X_i$, $i = 1, 2, 3, \ldots$, is any sequence of principal G-A bundles then the infinite join $P = *_i P_i$ is a G-A bundle.

In fact a point in P is of the form $\sum \lambda_i p_i$, where λ_i are the join coordinates, $\lambda_i \geqq 0$ with only a finite number nonzero, and $\sum \lambda_i = 1$. The actions of G and A on P are given by $g(\sum \lambda_i p_i) = \sum \lambda_i(gp_i)$ and $(\sum \lambda_i p_i)a = \sum \lambda_i (p_i a)$. Let $\pi: P \to X$ be the quotient map under the A-action, i.e., $X = P/A$.

Then as shown in Husemoller **[H1]**, π is a locally trivial, in fact numerable, A-bundle. We will show

(2) If each π_i is a numerable G-(A, B) bundle, so is π.

Consider a chart $(V, H) \subset X_j$. Let $\tilde{W} = \{\sum \lambda_i p_i | \lambda_j > 0$ and $p_j \in \pi_j^{-1}(V)\}$. Then $\tilde{W}$ is A-invariant and $G\tilde{W} = \{\sum \lambda_i p_i | \lambda_j > 0$ and $p_j \in \pi_j^{-1}(GV)\}$ is A-invariant. Also $\pi(G\tilde{W}) = G\pi(\tilde{W})$. Let $W = \pi(\tilde{W})$. Now $r_j: G\tilde{W} \to p_j$, $r_j(\sum \lambda_i p_i) = p_j$, is a G-A map and induces a G-map $\bar{r}_j: GW \to GV \subset X_j$, with $\bar{r}_j^{-1}(V) = W$. Hence (W, H) is a chart in X.

If $P_j|GV = 1_\rho^H(V)$, we claim $P|GW = 1_\rho^H(W)$. This is equivalent to showing that if $\varphi_j: V \times A \to \pi_j^{-1}(V)$ is an H-A equivalence with H-action $h(v, a) = (hv, \rho(h)a)$, then there is an H-A equivalence φ: $\tilde{W} \times A \to \tilde{W} = \pi^{-1}(W)$ with H-action $h(w, a) = (hw, \rho(h)a)$. Following Husemoller, define $\tilde{s}$: $\tilde{W} \to \tilde{W}$, $\tilde{s}(\sum \lambda_i p_i) = \sum \lambda_i p_i a^{-1}$, where $\varphi_j^{-1}(p_j) = (v, a)$. Then s is A-invariant and induces a cross-section s: $W \to \tilde{W}$. Define φ: $W \times A \to \tilde{W}$ by $\varphi(w, a) = s(w)a$. Then φ is an A-equivalence. To see that φ is an H-A equivalence note that

$$\begin{aligned} hs(\pi(\textstyle\sum \lambda_i p_i)) &= h\tilde{s}(\textstyle\sum \lambda_i p_i) = \textstyle\sum \lambda_i h p_i a^{-1} \\ &= \textstyle\sum \lambda_i h p_i \rho(h^{-1})\rho(h) a^{-1} = (\tilde{s}(\textstyle\sum \lambda_i h p_i \rho(h)^{-1}))\rho(h) \\ &= (sh\pi(\textstyle\sum \lambda_i p_i))\rho(h), \end{aligned}$$

i.e., $hs(w) = (s(hw))\rho(h)$ and $h\varphi(w, a) = s(hw)\rho(h)a = \varphi(hw, \rho(h)a)$. Hence $P|GW = 1_\rho^H(W)$.

Taking an atlas of charts in each X_j, the reunion of all the corresponding charts in X will be an atlas in X. Now the λ_i: $P \to [0, 1]$, being A-invariant, induce $\bar{\lambda}_i$: $X \to [0, 1]$. Let $\mu_i(x) = \max(0, \lambda_i(x) - \sum_{j<i} \bar{\lambda}_j(x))$. Then $\{\mu_i^{-1}(0, 1]\}$ is locally finite and $\nu_i = \mu_i / \sum_j \mu_j$ is a locally finite partition of unity subordinate to $\{\bar{\lambda}_i^{-1}(0, 1]\}$. Let $\{\alpha_i^k\}$ be a partition of unity on X_i subordinate to the trivializing atlas. Let $\beta_i^k = \alpha_i^k \circ \bar{r}_i: \bar{\lambda}_i^{-1}(0, 1] \to [0, 1]$, $\bar{r}_i$ as above. Then $\{\beta_i^k \nu_i\}$ is a partition of unity subordinate to the trivializing atlas on X. Hence π is a numerable G-(A, B) bundle.

(3) For any representation ρ: $H \to B$, $P^H = *_i P_i^H$, assuming $P_i^H \neq \emptyset$, all i.

Now choose a representative H from each conjugacy class of subgroups of G and a representative ρ: $H \to B$ from each A-equivalence class of representations of H in B. Let $\{\rho_\beta\}$ be this set. Let $E_\beta = G \times_{H_\beta} A$, ρ_β: $H_\beta \to B$, and let $E = \coprod_\beta E_\beta$. Let p_β: $G \times_{H_\beta} A \to G/H_\beta$ be the projection and p: $E \to \coprod_\beta G/H_\beta$ be $\coprod p_\beta$. Then E is a numerable G-(A, B) bundle. Further for any subgroup H of G and representative ρ: $H \to B$, $E^H \neq \emptyset$. In fact, $[e_1, e_2] \in G \times_{H_\beta} A$ satisfies $h[e_1, e_2] = [e_1, e_2]\rho_\beta(h)$ and $[g, e_2]$ satisfies $ghg^{-1}[g, e_2] = [g, e_2]\rho_\beta(h)$, $h \in H$. Finally $[e_1, a]$ satisfies $h[e_1, a] = [e_1, a]a^{-1}\rho_\beta(h)a$. So all subgroups and representatives have fixed points.

For each i, let $P_i = E$ and $\pi_i = p$. Then $P = *_i P_i$ satisfies property (1) of Theorem 6. Now let $p_H \in P^H$. Then p_H belongs to some finite join $*_{i \leqq k} P_i^H \subset P$. Let $P^k = \{\sum \lambda_i p_i \in P | \lambda_i = 0 \text{ for } i \leqq k\}$. Then it is easy to construct (see **[H1]**) a G-A deformation of P into P^k. For any representation ρ: $H \to B$ this deformation will be in particular an H-deformation. Now $p_H * P^k \subset P$ and since p_H is a fixed point, the contraction of this cone to the vertex p_H will be an H-deformation. Hence P satisfies (2) of Theorem 6, and P is universal. Q.E.D.

REMARK. If X is a G-space and (V, H) is a chart (i.e., $G \times_H V \to GV$ is a homeomorphism) then (gV, gHg^{-1}) is a chart with the same image (i.e., $G \times_{gHg^{-1}} V \to GgV = GV$ is a homeomorphism). Further, the trivial G-A bundle $1_\rho^H(V)$ over GV is equivalent to $1_{\rho \circ c(g^{-1})}^{gHg^{-1}}(gV)$, where $c(g)$ means conjugation by g.

The next theorem gives information on the universal base space for G-(A, B) bundles. In order to state it we need some notation: Let $H \subset G$. For any representation ρ: $H \to B$, let $A^\rho = \{a \in A | \rho(h) a \rho(h)^{-1} = a\}$. Then A^ρ is a closed subgroup of A. Let R be a collection of representations of H in B containing exactly one representation from each A-equivalence class.

THEOREM 8. *Let $\pi: P \to X$ be a universal G-(A, B) bundle. Then $X^H = \coprod_{\rho \in R} BA^\rho$ (disjoint union), where BA^ρ is a universal base space for the topological group A^ρ.*

PROOF. Let $P^\rho = \{p \in P \mid hp = p\rho(h), h \in H\}$.

(1) P^ρ is nonempty and contractible (by Theorem 6).

(2) For any $p \in P^\rho$, $\pi^{-1}(\pi(p)) \cap P^\rho = pA^\rho$, i.e., $h(pa) = (pa)\rho(h) \Leftrightarrow p\rho(h)\, a = pa\rho\,(h) \Leftrightarrow \rho(h)a = a\rho(h)$.

(3) $\pi(P^\rho) \subset X^H$.

(4) If $x \in X^H$, $x \in \pi(P^\rho)$, some $\rho \in R$.

In fact, by the remark above, there is a trivializing chart (V, H') with $x \in V$ and $P \mid GV = 1^{H'}_{\rho'}(V)$ $\rho': H' \to B$. Now $H \subset G_x \subset H'$. Since $\pi^{-1}(V) \simeq V \times A$ with H' action, $h'(v, a) = (h'v, \rho'(h')a)$; $(x, e_2) \in V \times A$ and $h(x, e_2) = (x, e_2)\rho'(h)$. Hence $x \in \pi(P^\rho)$ where $\rho = \rho' \mid H$. If $\rho \notin R$, then $c(a) \circ \rho \in R$, some $a \in A$. But if $p \in P^\rho$, $pa^{-1} \in P^{c(a)\circ\rho}$. Hence $x \in \pi(P^\rho)$, some $\rho \in R$.

(5) $\pi \mid P^\rho$ is a locally trivial principal A^ρ-bundle and $\pi(P^\rho)$ is open in X^H.

If $x \in \pi\,(P^\rho)$, then we may choose a trivializing chart (V, H') such that $x \in V$ and $P \mid GV = 1^{H'}_{\rho'}$, $\rho': H' \to B$ and $\rho' \mid H' = \rho$. Now if $U = V \cap X^H$ we claim $\pi^{-1}(U) \cap P^\rho = U \times A^\rho \subset V \times A$. In fact, $\pi^{-1}(U) = U \times A$ with $h'(u, a) = (h'u, \rho(h')a)$. If $(u, a) \in P^\rho$, $h(u, a) = (u, a)\,\rho(h) = (hu, \rho(h)a) = (u, \rho(h)a)$ by (3). Hence $\rho(h)a = a\rho(h)$ and $a \in A^\rho$. Since V is open in X, U is open in X^H. But $(u, e_2) \in P^\rho$ and $U \subset \pi(P^\rho)$. Hence P^ρ is locally trivial and $\pi(P^\rho)$ is open in X^H.

(6) $\pi(P^\rho) \cap \pi(P^{\rho'}) = \emptyset$, $\rho, \rho' \in R$, $\rho \neq \rho'$.

If $x \in \pi(P^{\rho'}) \cap \pi(P^{\rho'})$ and $p \in \pi^{-1}(x) \cap P^\rho$, then $pa \in P^{\rho'}$, some $a \in A$. Hence $\rho = c(a) \circ \rho'$. Contradiction.

By (1), (4), (5) and (6), $\pi(P^\rho) = BA^\rho$ and $X^H = \coprod_{\rho \in R} BA^\rho$.

Notation. We will denote the classifying space for G-$O(n)$ bundles by $BO_n(G)$ and for G-(Top_n, O_n) bundles by $B\,\mathrm{Top}_n(G)$. This last will cause no confusion because only locally linear R^n-bundles will be used. Note that since G-vector bundles are locally linear R^n-bundles, there is a G-map of $BO_n(G)$ into $B\,\mathrm{Top}_n(G)$ defined up to G-homotopy. More explicitly:

LEMMA 9. *If $\pi: E\,\mathrm{Top}_n(G) \to B\,\mathrm{Top}_n(G)$ is a universal G-(Top_n, O_n) bundle then the quotient map $q: E\,\mathrm{Top}_n(G) \to E\,\mathrm{Top}_n(G)/O_n$ is a universal G-O_n bundle and we can take $BO_n(G) = E\,\mathrm{Top}_n(G)/O_n$. With this choice $BO_n(G)$ is a numerable G-(Top_n, O_n) bundle over $B\,\mathrm{Top}_n(G)$ with fibre Top_n/O_n.*

PROOF. q is a principal G-O_n bundle. To see that it is G-locally trivial let (V, H) be a trivializing chart for π. Then $\pi^{-1}(GV) \simeq G \times_H (V \times \mathrm{Top}_n)$, $\rho: H \to O_n$, and $\pi^{-1}(GV)/O_n \simeq G \times_H (V \times \mathrm{Top}_n/O_n)$. Now $\mathrm{Top}_n \to \mathrm{Top}_n/O_n$ has local cross-sections since O_n is compact Lie. Since Top_n and hence Top_n/O_n is metrizable, Top_n is an H-O_n locally trivial bundle over Top_n/O_n and hence has a trivializing H partition of unity. It is easy to see that this implies that q is a numerable G-O_n bundle. Since $E\,\mathrm{Top}_n(G)$ satisfies (1) and (2) of Theorem 6 as a G-(Top_n, O_n) bundle it satisfies (1) and (2) as a G-O_n bundle and q is universal.

THEOREM 10. *Let $p_n: BO_n(G) \to B\,\mathrm{Top}_n(G)$ be the G-bundle of Lemma 9. For any $H \subset G$, let $p_n^H: [BO_n(G)]^H \to [B\,\mathrm{Top}_n(G)]^H$ be the restriction to fixed point sets. Then p_n^H is a numerable bundle such that the fibre over the component $B\,\mathrm{Top}_n^\rho$, $\rho: H \to$*

O_n, of $[B\ \mathrm{Top}_n(G)]^H$ is $\coprod_{\rho' \in S} \mathrm{Top}_n^{\rho'}/O_n^{\rho'}$, where S consists of one representation $\rho'\colon H \to O_n$ from each O_n-equivalence class which is Top_n *equivalent to* ρ.

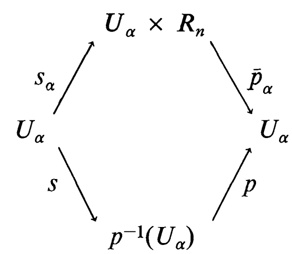
PROOF. If $z \in E\ \mathrm{Top}_n(G)$ is such that $q(z) \in [BO_n(G)]^H = [E\ \mathrm{Top}_n(G)/O_n]^H$, then $hz = z\rho'(h)$, $\rho' : H \to O_n$ a representation. But if $\pi(z) \in B\,\mathrm{Top}_n^\rho$, ρ' is Top_n equivalent to ρ (see step (6) of the proof of Theorem 8). Thus

$$p_n^{-1}(B\ \mathrm{Top}_n^\rho) \cap [BO_n(G)]^H = \coprod_{\rho' \in S} BO_n^{\rho'},$$

where $BO_n^{\rho'} = (E\,\mathrm{Top}_n(G))^{\rho'}/O_n^{\rho'}$, $(E\,\mathrm{Top}_n(G))^{\rho'} = \{z \in E\,\mathrm{Top}_n(G) \mid hz = z\rho'(h)\}$. Since for each $\rho' \in S$, $B\,\mathrm{Top}_n^\rho = B\,\mathrm{Top}_n^{\rho'} = (E\,\mathrm{Top}_n\,(G))^{\rho'}/\mathrm{Top}_n^{\rho'}$ and the quotient map is a numerable bundle (see proof of Theorem 8), $p_n^H | B\ \mathrm{Top}_n^\rho$ is a numerable bundle with fibre $\coprod_{\rho' \in S} \mathrm{Top}_n^{\rho'}/O_n^{\rho'}$.

2. *G*-microbundles. Before defining G-microbundles we recall the definition of microbundles without G-action.

DEFINITION 1. An R^n-microbundle μ over X is a diagram of maps and spaces $X \xrightarrow{s} E \xrightarrow{p} X$ such that $ps = \mathrm{id}_X$ and there exist an open covering $\{U_\alpha\}$ of X and open embeddings $\varphi_\alpha\colon U_\alpha \times R^n \to p^{-1}(U_\alpha)$ such that



commutes, where $s_\alpha(x) = (x, 0)$ and $p_\alpha(x, y) = x$. μ is called *numerable* if there exists a partition of unity subordinate to $\{U_\alpha\}$.

If E^0 is a neighborhood of sX in E, $X \xrightarrow{s} E^0 \xrightarrow{p} X$ is again a microbundle. However, E^0 may not be a numerable microbundle even if E is, as the following example from **[H2]** shows:

EXAMPLE 1. Let X be a denumerable set and let $x_0 \in X$ be a fixed element. Define a topology on X by requiring $U \subset X$ to be open if U is empty or contains x_0. Then X is connected and so any continuous function $f\colon X \to R$ is constant. Consider the trivial microbundle $\varepsilon\colon X \xrightarrow{s} X \times R \xrightarrow{p} X$, $s(x) = (x, 0)$, $p(x, r) = x$. Writing $X = \{x_i, i = 0, 1, 2, \cdots\}$, let $E^0 \subset X \times R$ be the set $E^0 = \bigcup_i x_i \times (-1/i, 1/i)$. Then $E^0 = \bigcup_i U_i \times (-1/i, 1/i)$, where $U_i \subset X$ is the open set $U_i = \{x_j, 0 \leqq j \leqq i\}$. Hence E^0 is an open neighborhood of $s(X) = X \times 0$. We claim E^0 is not a numerable microbundle. Suppose there existed a partition of unity $\{\lambda_\alpha\}$ and open embeddings $\varphi_\alpha\colon W_\alpha \times R \to p^{-1}(W_\alpha) \cap E^0$, $W_\alpha = \lambda_\alpha^{-1}(0, 1]$, such that $p\varphi_\alpha = p_\alpha$ and $\varphi_\alpha s_\alpha = s$ as above. Define $f_\alpha\colon W_\alpha \to R$ by $f_\alpha(x) = pr_2\varphi_\alpha(x, 1)$. Then f_α is a continuous positive function and so is $f = \sum \lambda_\alpha f_\alpha\colon X \to R$. Then on the one hand, f is a constant function, and on the other hand $\{(x, r) \in X \times R \mid r \leqq f(x)\}$ must be in E^0. Contradiction.

We will say that a neighborhood E^0 of sX in the numerable microbundle μ: $X \xrightarrow{s} E \xrightarrow{p} X$ is a microbundle neighborhood if $X \longrightarrow E^0 \xrightarrow{p} X$ is a numerable