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A Derivation of The Wess-Zumino-Witten Action from Chiral Anomaly Using Homotopv Ouerators

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We describe a systematic procedure for deriving the Wess-Zumino-Witten action in two dimensions, using chiral anomaly as an input. The actual calculation is carried out with a homotopy operator technique.

I. INTRODUCTION

Skyrme's original idea[1] that baryons are the solitons in the chiral non-linear u-model has recently attracted much interest[2-4], since the latter has been speculated to represent the low-energy behavior of QCD. Further developments have been made by Wess and Zumino[5] and Witten[3], who have modified the Skyrme model to incorporate the chiral anomalies. The so-called Wess-Zumino-Witten (WZW) term contains, when gauged, all the information about flavor anomalies in QCD.

Witten[3] observed that, without the WZW term, the Skyrme model will only allow processes conserving $(-1)^n$, where n is the number of the pseudoscalars. This is not one of the symmetries of QCD. He also analyzed the topological aspects of the chiral non-linear u-model, making analogy with those in the dynamics of a charged particle moving in the field of a magnetic monopole. It turns out that the WZW term is only properly expressible in terms of a certain 5-D integral. Topological consideration requires the co efficient of the WZW term to be quantized, much in the same way as monopole charge is quantized.

However, to gauge the modified Skyrme model, the conventional minimal coupling method would not work because of the topological nature of the WZW term. Witten has tried to gauge the WZW term using Noetherian methods. The resulting action becomes the sum of the original 5-D integral and a 4-D integral of a certain effective Lagrangian involving both gauge fields and Goldstone bosons. This effective Lagrangian is extremely messy, containing some twenty-more terms. Indeed, Witten's original result contained a few minor errors, which has launched many correcting papers.[6]

A different direction of constructing the modified Skyrme model, starting from the Bardeen anomaly, has been taken by Wess and Zumino[5], Rossi et al.[7], Alvarez[8], D'Yakonov and Eides[9] and Nepomechie[10], who have all obtained the ungauged WZW term. And later, Pak and Rossi[11] and Ingermanson[12] have derived the gauged WZW term. Owing to the different methods adopted, the results of Refs. (8), (10) and (12) look different from those of Refs. (7), (9) and (11).

The purpose of this work is to describe for $(QCD)_2$ a systematic procedure for deriving the WZW term using Bardeen's anomaly as an input. (We discuss the 2-D theory just for simplicity). Differential geometric approach advocated by Zumino[13], Stora[14] and others[15] are essential in this derivation.

The outline of this paper is as follows. Sec. II summarises our conventions. In Sec. III, we briefly discuss how the determinant of the Dirac operator is related to the chiral anomaly. In Sec. IV, we present the explicit derivation of the WZW term. The Appendix contains some formulae used in the text.

II. CONVENTIONS

We will work in the 2-D Euclidean space, and choose the Dirac matrices to be

$$\gamma_0 = \sigma_x, \quad \gamma_1 = \sigma_y, \quad \gamma_5 = \sigma_z .$$

The symbol "tr" will denote trace over Dirac and/or internal indices, while "Tr" will denote trace over space-time as well as Dirac and internal indices.

The vector and axial-vector gauge fields (in matrix form) are given by

$$v_\mu = -i\, v_\mu^b X^b , \qquad a_\mu = -i\, a_\mu^b X^b , \tag{2.1}$$

where X^b are the hermitian generators of SU(N), satisfying

$$\mathrm{tr}(X^a X^b) = \frac{1}{2}\delta^{ab} .$$

The field strength tensors are then given by

$$v_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu + [v_\mu, v_\nu] + [a_\mu, a_\nu],$$

$$a_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu + [v_\mu, a_\nu] + [a_\mu, v_\nu] . \tag{2.2}$$

In our calculations we will utilize differential forms[13-15], which provide an exceedingly compact notation saving us the tedious task of writing out indices explicitly. So we define the 1-forms v, a and 2-forms $\hat{v}, \hat{a}$ by

$$v \equiv v_\mu\, dx_\mu , \qquad a \equiv a_\mu\, dx_\mu ,$$

$$\hat{v} \equiv \frac{1}{2} v_{\mu\nu}\, dx_\mu dx_\nu , \qquad \hat{a} \equiv \frac{1}{2} a_{\mu\nu}\, dx_\mu dx_\nu . \tag{2.3}$$

We omit the wedge product symbols and simply regard dx_μ as an anti-commuting Grassmann object.

The fundamental field U to appear in the effective action[3,5] is an element of the coset space $SU(N)_L x SU(N)_R/SU(N)_V$, which transforms under $SU(N)_L \times SU(N)_R$ by

$$(A, B) : U \longrightarrow AUB^{-1}.$$

U can be described in terms of the Goldstone fields π^a by

$$U = e^{2i\xi} \text{ with } \xi = \pi^a X^a \tag{2.4}$$

The general theory of non-linear realization of the chiral symmetry has been described in the classic papers of Callan et al.[16] Recently, some authors[8,10] have demonstrated how to determine in such a theory the determinant of certain Dirac operators through the chiral anomaly. In this approach it is useful to consider the following l- and 2-forms depending on a real parameter t:

$$g_t \equiv v_t + a_t \gamma_5 = e^{-it\xi\gamma_5} (d+v+a \ \gamma_5)e^{it\xi\gamma_5},$$

$$g_t \equiv \hat{v}_t + \hat{a}_t \gamma_5 = e^{-it\xi\gamma_5}(\hat{v}+\hat{a} \ \gamma_5)e^{it\xi\gamma_5} . \tag{2.5}$$

Then it follows from (2.1)–(2.5) that

$$\hat{v}_t = dv_t + v_t^2 + a_t^2,$$

$$\hat{a}_t = da_t + \{v_t , a_t\} , \tag{2.6}$$

and

$$v_0 = v, \quad a_0 = a, \quad v_0 = v, \quad \hat{a}_0 = \hat{a}. \tag{2.7}$$

Finally, the covariant derivative at t of a p-form ω_p is defined by

$$D_t \omega_p = d\omega_p + v_t\omega_p - (-)^p \omega_p v_t \tag{2.8}$$

III. DETERMINANTS FROM ANOMALY

To analyze the low-energy dynamics in QCD, we perform a local chiral transformation on the quark fields in a vacuum state. Thus we assume that everywhere in the space we have an approximate vacuum state, but the parameters labelling the state are slowly varying functions of the space-time. Due to dynamical breakdown of the chiral symmetry, the parameters appearing in the chiral transformation of the quark fields describe the Goldstone particles. The difference in the path integral over the quark fields for this state and for the vacuum can be interpreted as the low-energy effective action of the Goldstone bosons[8-10,17,18] So

$$\exp -W[v+a\gamma_5,\xi,t] \equiv \frac{\int D\psi D\bar{\psi} \exp\{\int dx \bar{\psi} V_t^{-1/2} i\gamma_\mu(\partial_\mu + v_\mu + a_\mu\gamma_5) V_t^{-1/2}\psi\}}{\int D\psi D\bar{\psi} \exp\{\int dx \bar{\psi} i\gamma_\mu(\partial_\mu + v_\mu + a_\mu\gamma_5)\psi\}},$$

with

$$V_t \equiv e^{-2it\xi\gamma_5} \qquad \text{and} \qquad W[v+a\gamma_5,\xi,0] = 0.$$

To calculate W, we perform an infinitesimal chiral transformation on ψ and $\bar{\psi}$:

$$\psi \to \psi' = V_{\delta t}^{1/2}\psi \quad , \quad \bar{\psi} \to \bar{\psi}' = \bar{\psi} V_{\delta t}^{1/2},$$

and easily obtain the relation

$$W[v+a\gamma_5,\xi,t] = W[v+a\gamma_5,\xi,t-\delta t] + 2i\ \delta t\ \mathrm{Tr}\ \xi\gamma_5,$$

using the fact that under the infinitesimal chiral rotations, the fermion measure transforms as[19]

$$D\psi' D\bar{\psi}' = D\psi\ D\bar{\psi}\ e^{-2i\ \delta t\ \mathrm{Tr}\xi\gamma_5}.$$

The trace in the above equation, when suitably evaluated, yields the chiral anomaly. So the effective action W satisfies the differential equation

$$\frac{d}{dt} W[v+a\gamma_5,\xi,t] = 2i\ \mathrm{Tr}\xi\ \gamma_5.$$

Let $B_t^b[g_t] \equiv 2i\ \mathrm{Tr}\ X^b\gamma_5$ denote the chiral anomaly in the presence of the background gauge-field $g_t \equiv v_t + a_t\gamma_5$. Then

$$W[v+a\gamma_5,\xi,1] = \int_0^1 dt\ \pi^b B_t^b \equiv \int_0^1 dt \int_{R^2} \mathrm{tr}[\xi B_t], \tag{3.1}$$

where the trace operator in front of ξB_t denotes trace over SU(N) indices, and B_t is defined by

$$B_t \equiv \mathrm{tr}[2i\ \gamma_5]_{g_t},$$

with tr now denoting trace over Dirac indices. B_t is to be calculated with some kind of regularization scheme. Here we will choose a scheme such that the vector currents are conserved. Then B_t is found to be[22,23]

$$B_t = \frac{1}{2\pi}(\hat{v}_t - 2a_t^2), \tag{3.2}$$

which is the Bardeen anomaly in the 2-D Euclidean space.

IV. CALCULATION OF THE WZW TERM

Now we proceed to derive an explicit expression of the WZW action using (3.1) and (3.2). All fields appearing in ξB_t are supposed to approach constants at infinity. This allows the space-time R^2 to compactify to S^2. Let D^3 be a 3-D disk whose boundary is S^2. Then

$$W = \int_0^1 dt \int_{S^2} tr[\xi B_t] = \int_0^1 dt \int_{D^3} d(tr[\xi B_t])$$

$$= \int_0^1 dt \int_{D^3} D_t(tr[\xi B_t]) , \tag{4.1}$$

where we have used Stokes' theorem and the fact that the trace of anything is a scalar. Using the formulae in (A. 1) and (A. 2), we easiiy obtain

$$D_t(tr[\xi B_t]) = \frac{1}{2\pi} \frac{d}{dt} \Omega_t , \tag{4.2}$$

where

$$\Omega_t = -i \; tr(a_t \hat{v}_t - \frac{2}{3} a_t^3) \tag{4.3}$$

Substituting (4.2) into (4.1), we find

$$W = \frac{1}{2\pi} \int_{D^3} (\Omega_1 - \Omega_0) . \tag{4.4}$$

From the work of Witten[3], we know the gauged WZW action can be written as a 2-D integral of a certain Lagrangian involving the gauge fields and the Goldstone bosons, plus a 3-D integral (the WZW term) which cannot be reduced to a 2-D integral of a non-singular density. So we will separate out from the RHS of (4.4) a part which can be written as a 2-D integral. To carry out such a separation, it is convenient to apply the homotopy operator formalism as advertised by Zumino[13] and Stora[14]; see also Refs. (12) and (15).

For each value oft, the homotopy operator K_t is defined by

$$dK_t + K_t d = 1 , \; K_t^2 = 0, \; d^2 = 0 . \tag{4.5}$$

The action of K_t on an arbitrary polynomial $P(v_t, a_t, \hat{v}_t, \hat{a}_t)$ can be computed with the formula[13,14]

$$K_t P(v_t, a_t, \hat{v}_t \hat{a}_t) = \int_0^1 \ell_t^s \, P(v_t^s, a_t^s, \hat{v}_t^s, \hat{a}_t^s) ,$$

where ℓ_t^s is defined. to be the antiderivative operator which when acting on the s-dependent "interpolating fields"

$$v_t^s \equiv s\, v_t, \qquad a_t^s \equiv s\, a_t\,,$$

$$\hat{v}_t^s \equiv d\, v_t^s + (v_t^s)^2 + (a_t^s)^2 = s\, \hat{v}_t - s(1-s)(v_t^2 + a_t^2)\,,$$

$$\hat{a}_t^s \equiv d\, a_t^s + \{ v_t^s, a_t^s \} = s\, \hat{a}_t - s(1-s) \{ v_t, a_t \}\,, \tag{4.6}$$

gives rise to

$$\ell_t^s v_t^s = 0\,, \qquad \ell_t^s a_t^s = 0$$

$$\ell_t^s \hat{v}_t^s = \frac{d}{ds} v_t^s\, ds = v_t\, ds\,, \qquad \ell_t^s \hat{a}_t^s = a_t\, ds\,. \tag{4.7}$$

Using (4.5) and Stokes' theorem, we can achieve an initial separation of W in the following way:

$$W = \frac{1}{2\pi} \int_{D^3} [(K_1 d + dK_1)\,\Omega - (K_0 d + dK_0)\Omega_0]$$

$$= \frac{1}{2\pi} \int_{S^2} (K_1 \Omega_1 - K_0 \Omega_0) + \frac{1}{2\pi} \int_{D^3} (K_1\, d\Omega_1 - K_0\, d\Omega_0)\,. \tag{4.8}$$

Our next task is to further extract from the last 3-D integral in (4.8) another piece which can be converted to a regular 2-D integral.

Define

$$\tau \equiv [K_1\, d\Omega_1 - K_0\, d\Omega_0]_{g_0 = 0}$$

and

$$Q_t \equiv K_t\, d\Omega_t - [K_t\, d\Omega_t]_{g_0 = 0}\,. \tag{4.9}$$

Then

$$\Delta Q \equiv Q_1 - Q_0 = K_1\, d\Omega_1 - K_0\, d\Omega_0 - \tau\,,$$

and the WZW action can be written as

$$W = \frac{1}{2\pi} \int_{S^2} (K_1 \Omega_1 - K_0 \Omega_0) + \frac{1}{2\pi} \int_{D^3} (\Delta Q + \tau) \tag{4.10}$$

It is the integral of ΔQ that will be transformed into a 2-D integral. From (A. 4) we have

$$K_t d\Omega_t = d[\frac{i}{2} \mathrm{tr}(\gamma_5\, \beta_t\, g_0)] - \frac{i}{2} \mathrm{tr}[\gamma_5 (g_0 \hat{g}_0 - \frac{1}{3} g_0^3 - \frac{1}{3} \alpha_t^3)]\,,$$

where

$$\alpha_t \equiv e^{-it\xi\gamma_5}\, d(e^{it\xi\gamma_5})\,, \qquad \beta_t \equiv d(e^{it\xi\gamma_5})\, e^{-it\xi\gamma_5}\,,$$

and gt, $\hat{g}_t$ are given in (2.5). Therefore we get

$$[K_t d\Omega_t]_{g_0=0} = \frac{i}{6}\,\mathrm{tr}[\gamma_5\,\alpha_t^3]\,, \tag{4.11}$$

and hence

$$\tau = \frac{1}{6}\,\mathrm{tr}[\gamma_5\,\alpha_1^3]\ . \tag{4.12}$$

so

$$\Delta Q = d[\,\frac{i}{2}\,\mathrm{tr}(\gamma_5\,\beta_1\,g_0)\,]\ ,$$

and

$$\int_{D^3}\Delta Q = \int_{S^2}\frac{i}{2}\mathrm{tr}(\gamma_5\,\beta_1\,g_0)\,. \tag{4.13}$$

Inserting (4.12), (4.13) and (A.3) into (4. 10), we get

$$W = \frac{1}{2\pi}\int_{S^2}\frac{i}{2}\,\mathrm{tr}[a_1 v_1 - a_0 v_0 + \gamma_5\beta_1 g_0] + \frac{1}{2\pi}\int_{D^3}\frac{i}{6}\,\mathrm{tr}[\gamma_5\alpha_1^3]\,. \tag{4.14}$$

Now it is customary to keep all the gauge-field-independent terms of W in the 3-D integral. The a, v_1 term in the 2-D integral in (4.14) still contains a gauge-field-independent piece, which we will transfer back to the 3-D integral. Thus we rewrite (4.14) as

$$W = \frac{1}{2\pi}\int_{S^2}\frac{i}{2}\mathrm{tr}\,\{a_1 v_1 - [a_1 v_1]_{g_0=0} - a_0 v_0 + \gamma_5\beta_1\,g_0\}$$

$$+\frac{1}{2\pi}\int_{D^3}\frac{i}{6}\mathrm{tr}\{\gamma_5\alpha_1^3 + 3d[a_1 v_1]_{g_0=0}\}\,. \tag{4.15}$$

Carrying out the trace operation over Dirac indices and using the definitions (2.4) and (2.5), we can transform (4.15) into a more familiar form:

$$W = \frac{i}{4\pi}\int_{S^2}\mathrm{tr}[(v-a)dU^{-1}U + (v+a)U\,dU^{-1} + U^{-1}(v+a)U(v-a)$$

$$-(v+a)(v-a)] + \frac{1}{12\pi}\int_{D^3}\mathrm{tr}(dU\,U^{-1})^3 \tag{4.16}$$

(4.16) is the desired expression for the WZW action. It agrees with that obtained in Refs. (20)–(22). The procedure that we have described here can obviously be generalized without new difficulties to the case of any even dimensions, although the algebra would become much more tedious.

APPENDIX

Here we list or derive some formulae that have been used in the text.

(A. 1) From (2.5)–(2.8) we have

$$D_t v_t = dv_t + 2v_t^2 = \hat{v}_t + v_t^2 - a_t^2 ,$$

$$D_t a_t = a_t ,$$

$$D_t \hat{v}_t = [\hat{a}_t , a_t] ,$$

$$D_t \hat{a}_t = [\hat{v}_t , a_t] .$$

(A.2) Differentiate (2.5) w.r.t. t,

$$\frac{d}{dt} v_t + \frac{d}{dt} a_t \gamma_5 = i[a_t, \xi] + i[v_t, \xi]\gamma_5 + id\xi\gamma_5 ,$$

So

$$\frac{d}{dt} v_t = i[a_t, \xi] ,$$

$$\frac{d}{dt} a_t = i D_t \xi .$$

Similarly,

$$\frac{d}{dt} \hat{v}_t = i[\hat{a}_t , \xi] ,$$

$$\frac{d}{dt} \hat{a}_t = i[\hat{v}_t, \xi] .$$

(A.3) We apply the homotopy operator K_t on Ω_t of (4.3), and use (4.6), (4.7) to obtain

$$K_t \Omega_t = -i \ \mathrm{tr} \int_0^1 \ell_t^s [a_t^s \hat{v}_t^s - \frac{2}{3} (a_t^s)^3]$$

$$= i \ \mathrm{tr} \int_0^1 [a_t^s v_t] ds = \frac{1}{2} \mathrm{tr}[a_t v_t] .$$

An entirely similar calculation leads to

$$K_t d\Omega_t = -\frac{i}{2} \mathrm{tr}[\gamma_5 (g_t \hat{g}_t - \frac{1}{3} g_t^3)] .$$

(A.4) Now we will separate out from the RHS of the above equation a piece which is a total differential. We first express g_t and $\hat{g}_t$ as

$$g_t = \alpha_t + e^{-it\xi\gamma_5} g_o e^{it\xi\gamma_5} ,$$

$$\alpha_t \equiv e^{-it\xi\gamma_5} d(e^{it\xi\gamma_5}) ,$$

$$\hat{g}_t = e^{-it\xi\gamma_5} \hat{g}_o e^{it\xi\gamma_5} .$$

With some obvious abbreviations, $K_t d\Omega_t$ can be written as

$$K_t d\Omega_t = -\frac{1}{2}\,\mathrm{tr}\,\{\gamma_5[(\alpha_t + e^- g_o e^+)(e^- \hat{g}_o e^+) - \frac{1}{3}(\alpha_t + e^- g_o e^+)^3\,]$$

$$= -\frac{1}{2}\,\mathrm{tr}\,\{\gamma_5[\alpha_t e^- \hat{g}_o e^+ - \alpha_t e^- g_o^2 e^+ - \alpha_t^2 e^- g_o e^+$$

$$+ g_o \hat{g}_o - \frac{1}{3} g_o^3 - \frac{1}{3}\alpha_t^3\,]\,\} \;.$$

Using (2.5) and (2.6), we have

$$\mathrm{tr}[\gamma_5(\alpha_t e^- \hat{g}_o e^+ - \alpha_t e^- g_o^2 e^+ - \alpha_t^2 e^- g_o e^+)]$$

$$= \mathrm{tr}[\gamma_5(\alpha_t e^- dg_o e^+ - \alpha_t^2 e^- g_o e^+)]$$

$$= \mathrm{tr}[\gamma_5(de^+ e^- dg_o + de^+ de^- g_o)] \quad (\text{since } \alpha_t = e^- de^+)$$

$$= -d[\mathrm{tr}(\gamma_5\, de^+ e^- g_o)]$$

$$= -d[\mathrm{tr}(\gamma_5\, \beta_t g_o)]\;. \qquad (\text{define } \beta_t \equiv de^+ e^-)$$

So finally,

$$K_t d\Omega_t = d\,[\frac{i}{2}\,\mathrm{tr}(\gamma_5\,\beta_t g_o)] - \frac{1}{2}\,\mathrm{tr}[\gamma_5(g_o \hat{g}_o - \frac{1}{3} g_o^3 - \frac{1}{3}\alpha_t^3)]\,.$$

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