AN OVERVIEW AND PROOF OF THE LEFSCHETZ FIXED-POINT THEOREM

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Abstract. The Lefschetz Fixed-Point Theorem provides a method of proving the existence of a fixed-point for self-maps on simplicial complexes. In this paper we prove the Lefschetz Fixed-Point Theorem. We also prove the Hopf Trace Formula and the Simplicial Approximation Theorem, two facts that provide the basis for our proof of the Lefschetz Fixed-Point Theorem.

Contents

1. Introduction

An n-sphere has only two non-trivial homology groups: $H_n(S^n)$ and $H_0(S^n)$, and thus maps from n-spheres to themselves only induce a homomorphism on $H_n(S^n)$, the map $f_*\colon H_n(S^n) \to H_n(S^n)$, and the homomorphism on $H_0(S^n$, which is always the identity map, $id\colon H_0(S^n) \to H_0(S^n)$. Further, since $H_n(S^n) \approx \mathbb{Z}$, that homomorphism must take the form $f_*(x) = dx$. This allow us to define d as the degree of the map from a n-sphere to itself; further, we can extend the properties of degree to prove the Brouwer Fixed-Point Theorem for disks.

Theorem 1.1 (Brouwer Fixed-Point Theorem)**.** *Every map $f\colon D^n \to D^n$ has at least one fixed point.*

We would like to generalize both of these notions: we will generalize the idea of degree with the Lefschetz number, which gives us information about the induced homomorphisms on all the homology groups of the space, and we will generalize the Brouwer Fixed-Point Theorem with the Lefschetz Fixed-Point Theorem.

2. The Lefschetz Fixed Point Theorem

Definition 2.1 (Lefschetz Number)**.** *Let X be a finite simplicial complex. Then, for a map $f\colon X \to X$, define the number $\tau(f)$ such that*

$$\tau(f) = \sum_n (-1)^n \mathrm{tr}(f_*\colon H_n(X) \to H_n(X)).$$

Then, we call $\tau(f)$ the Lefschetz number of f.

Theorem 2.2 (Lefschetz Fixed-Point Theorem)**.** *Let X be a finite simplicial complex. For any map $f\colon X \to X$ with Lefschetz number $\tau(f)$, if $\tau(f) \neq 0$, then it must be true that f has at least one fixed point.*

I would like to note a few obvious implications of the theorem before proceeding with a proof. First, if a space is homotopy equivalent to a point, such as all D^n, or if a space has equivalent homology groups to a point ignoring torsion, as are all $\mathbb{R}\mathrm{P}^{2n}$, then any map from that space to itself has a Lefschetz number of 1 and thus has a fixed point. Second, the Lefschetz number of the identity map is always the Euler-characteristic $\chi(X)$ of the space, and a flow on a space is a homotopy $f_t(x)$ where f_0 is the identity map: therefore, if a space has non-zero Euler-characteristic, all flows on that space must have a fixed-point. This gives us our hairy ball theorem: since n-spheres have an Euler-characteristic of 2 if n is even and 0 if n is odd, a flow on any even n-sphere must have a fixed point.

We require two lemmas before we begin our proof. First, we need the Simplicial Approximation Theorem, a property of maps on finite simplicial complexex.

Theorem 2.3 (Simplicial Approximation Theorem)**.** *If K is a finite simplicial complex and L is any simplicial complex, then any map $f\colon K \to L$ is homotopic to some map $g\colon K' \to L$ where K' is an iterated barycentric subdivision of K. Furthermore, $f(\sigma) \subset \mathrm{st}(g(\sigma))$. Proof in appendix 3.2*

Second, we need the Hopf Trace Formula, an algebraic property that will supply us with an alternate method to calculate the Lefschetz number of a map.

Lemma 2.4 (Hopf Trace Formula)**.** *Let C_* be the chain complex*

$$0 \to C_k \to C_{k-1} \to \ldots \to C_0 \to 0$$

where each C_k is finitely generated and abelian, and let $H_k, \ldots H_0$ be the corresponding homology groups where

$$H_n = \mathrm{Ker}(\delta_n\colon C_n \to C_{n-1}) \setminus \mathrm{Im}(\delta_{n+1}\colon C_{n+1} \to C_n),$$

and let $\phi_\#$ be a chain map on C_* that induces homologies $\phi_*\colon H_n \to H_n$ for each H_n. Then,

$$\sum_n (-1)^n \mathrm{tr}(\phi_*\colon H_n \to H_n) = \sum_n (-1)^n \mathrm{tr}(\phi_\#\colon C_n \to C_n).$$

If $H_k, \ldots, H_0$ are the homology groups on some simplicial complex X and ϕ_* the homologies induced by some $\phi\colon X \to X$, this implies that

$$\tau(\phi) = \sum_n (-1)^n \mathrm{tr}(\phi_\#\colon C_n \to C_n)$$

Proof in appendix 3.1.

Now we can begin our proof.

Proof of Theorem 2.2. To prove the Lefschetz fixed-point theorem, we will try to prove than any map f from a finite simplicial complex X to itself that has no fixed points must have a Lefschetz number of zero.

By Lemma 2.4, we can calculate the Lefschetz number of a map by taking the traces of the induced chain maps on some chain complex instead of the induced maps between homology groups. So, we can prove that a map has a Lefschetz number of zero by showing that for the cellular chain complex implied by a CW-complex K on X, the map takes no generator of any $H(K^n, K^{n-1})$ in to itself. Since those generators correspond to the cells of each K^n, if we choose K to be a simplicial complex, those generators will be simplices. Therefore, we will use the fact that f takes no point to itself to try to find some simplicial complex K such that we can use Theorem 2.3 to approximate f by a simplicial map g that takes no simplex of any K^n to itself.

First we will construct a subdivision L of X such that no simplex of L contains both x and $f(x)$. We note that since X is compact, if d is a metric on X, then, $d(x, f(x))$ being greater than 0 for all x, there exists some smallest distance $\delta > 0$ such that $\delta \leq d(x, f(x))$ for all $x \in X$. Then, we iteratively barycentrically divide X until we obtain a simplicial complex L such that for all simplices $\sigma \in L$, the diameter of $\mathrm{st}(\sigma)$ is less than $\frac{\delta}{2}$.

Now, we've constructed a subdivision L of X with simplices small enough that f takes no point in any simplex to the same simplex—however, f does not takes simplices to simplices and so does not induce "nice" maps when we use our simplicial complex to create a cellular chain complex. Fortunately, by Theorem 2.3, since f is a map between finite simplicial complexes, f can be approximated by a homotopic map $g\colon K \to L$, where K is a subdivision of L that takes simplices to simplices. And, for all $\sigma \in K$ and for any $x \in \sigma$, all points in $g(\sigma)$ are within a distance of $\frac{\delta}{2}$ of $f(x)$, while all points in σ are within a distance of $\frac{\delta}{2}$ of x, so, since x and $f(x)$ are at least a distance of δ apart, no point is in both σ and $g(\sigma)$.

Since g is simplicial from K to L, g maps each simplex of K^n to some simplex of L^n. Since K is a subdivision of L, each simplex in L^n corresponds to some subcomplex of K^n. Therefore g induces a mapping from each K^n to K^n. Thus, using K as a CW-complex on X, g induces chain maps $g_*\colon H_n(K^n, K^{n-1}) \to H_n(K^n, K^{n-1})$ on the resulting cellular chain complex. Each $H_n(K^n, K^{n-1})$ is free abelian with the simplices of K^n as its basis: therefore, since g takes no simplex of any K^n to itself, no g_* takes any basis element of K^n to itself; thus $\mathrm{tr}(g_*\colon H_n(K^n, K^{n-1}) \to H_n(K^n, K^{n-1})) = 0$ for all n. By Theorem 2.4, this implies $\tau(g) = 0$; and, since g is homotopic to f, $\tau(f) = \tau(g) = 0$.

Thus, since if f is fixed-point free, $\tau(f) = 0$, if $\tau(f) \neq 0$, f has at least one fixed point. □

3. Appendix

3.1. **Proof of Hopf Trace Formula.**

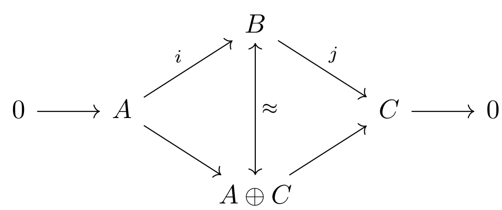
Proof. Our proof rests on a simple property of maps of short exact sequences to themselves: given maps α, β, and γ that form the following commutative diagram:

$$\begin{CD} 0 @>>> A @>>> B @>>> C @>>> 0 \\ @. @VV{\alpha}V @VV{\beta}V @VV{\gamma}V @. \\ 0 @>>> A @>>> B @>>> C @>>> 0 \end{CD}$$

we have

$$\operatorname{tr}(\alpha) + \operatorname{tr}(\gamma) = \operatorname{tr}(\beta).$$

To prove this, we start from the fact that trace ignores torsion: therefore for the rest of this proof, let A, B, and C refer to the free-abelian components of those groups. We then use the fact that all short exact sequences of free-abelian groups split, so we have the following commutative diagram:



where the maps on the bottom take $a \in A$ to $(a, 0)$ and $(0, c)$ to $c \in C$. Let A have a basis $\{a_1, a_2 \ldots, a_i\}$ and C have a basis $\{c_1, c_2, \ldots, c_j\}$: then, $A \oplus C$ has $\{(a_1, 0), \ldots, (a_i, 0)\} \cup \{(0, c_1), \ldots, (0, c_j)\}$ as a basis, so the union of the equivalents of $\{(a_1, 0), \ldots, (a_i, 0)\}$ and of $\{(0, c_1), \ldots, (0, c_j)\}$ forms a basis for B. Let the bases that map to $(a_n, 0)$ be called a'_n and those that map to $(0, c_m)$ be called c'_m. Our commutative diagram tells us that the bases a'_n are the images of the bases a_n under the map i and that j takes the bases c'_n to the bases c_n of C.

Thus, returning to our maps from the sequence to itself, since the set of a'_n and c'_n together form a basis for B, we can calculate the trace of β by adding the sum of the number of times β takes each a'_n to itself over all a'_n to the sum of the number of times β takes each c'_n to itself over all c'_n. We note that our commutative diagram tells us that for all a_n, $i\alpha(a_n) = \beta i(a_n)$. Thus, if α maps a_n to $k_1 a_1 + \cdots + k_n a_n + \cdots + k_i a_i$, then, since $i(a_n) = a'_n$, β maps a'_n to $\beta i(a_n) = i\alpha(a_n) = k_1 a'_1 + \cdots + k_n a'_n + \cdots + k_i a i'$. Thus if α maps a_n to itself k_n times, β also maps a'_n to itself k_n times, so the first sum in our calculation for $\operatorname{tr}(\beta)$ is merely $\operatorname{tr}(\alpha)$. Similarly, since $j\beta(c'_n) = \gamma j(c'_n)$ and $c_n = j(c'_n)$, our second sum is equal to $\operatorname{tr}(\gamma)$. Thus we have $\operatorname{tr}(\beta) = \operatorname{tr}(\alpha) + \operatorname{tr}(\gamma)$.

Now that we have this fact, we will take two commutative diagrams of maps from short exact sequences to themselves formed from the chain maps induced by ϕ. These diagrams will relate C_n and H_n with $Z_n = \operatorname{Ker}(\delta_n \colon C_n \to C_{n-1})$ and $B_n = \operatorname{Im}(\delta_{n+1} \colon C_{n+1} \to C_n)$. By the definitions of Z_n and B_n, we have the following short exact sequence and its ϕ induced maps, where γ_n is the induced map $\phi_{\#} \colon C_n \to C_n$:

$$\begin{CD} 0 @>>> Z_n @>{i}>> C_n @>{\delta_n}>> B_{n-1} @>>> 0 \\ @. @VV{\zeta_n}V @VV{\gamma_n}V @VV{\beta_{n-1}}V @. \\ 0 @>>> Z_n @>{i}>> C_n @>{\delta_n}>> B_{n-1} @>>> 0 \end{CD}$$

And, from the definition of H_n as $Z_n \setminus B_n$, we have the following short exact sequence, where η_n is the induced map $\phi_* \colon H_n \to H_n$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B_n & \longrightarrow & Z_n & \longrightarrow & H_n & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \beta_n} & & \downarrow{\scriptstyle \zeta_n} & & \downarrow{\scriptstyle \eta_n} & & \\ 0 & \longrightarrow & B_n & \longrightarrow & Z_n & \longrightarrow & H_n & \longrightarrow & 0 \end{array}$$

This gives us the following relations between the traces of mappings:

$$\operatorname{tr}(\gamma_n) = \operatorname{tr}(\zeta_n) + \operatorname{tr}(\beta_{n-1})$$
$$\operatorname{tr}(\zeta_n) = \operatorname{tr}(\beta_n) + \operatorname{tr}(\eta_n)$$

and thus,

$$\operatorname{tr}(\gamma_n) = \operatorname{tr}(\beta_n) + \operatorname{tr}(\eta_n) + \operatorname{tr}(\beta_{n-1})$$

so we have

$$\begin{aligned} &\textstyle\sum_n (-1)^n \operatorname{tr}(\phi_\# \colon C_n \to C_n) \\ &= \textstyle\sum_n (-1)^n \operatorname{tr}(\gamma_n) \\ &= \textstyle\sum_n (-1)^n \operatorname{tr}(\beta_n) + \operatorname{tr}(\eta_n) + \operatorname{tr}(\beta_{n-1}) \\ &= \textstyle\sum_n (-1)^n \operatorname{tr}(\eta_n) \\ &= \textstyle\sum_n (-1)^n \operatorname{tr}(\phi_* \colon H_n \to H_n) \end{aligned}$$

□

3.2. **Proof of Simplicial Approximation Theorem.**

Proof. We start with a lemma that will help us determine that a set of vertices do indeed define a simplex:

Lemma 3.1. *For vertices $v_1, \ldots, v_n$ of a simplicial complex X, either the intersection $\operatorname{st}(v_1) \cap \cdots \cap \operatorname{st}(v_n) = \emptyset$, or $v_1, \ldots, v_n$ are the vertices of a simplex of X and the intersection is the star of that simplex.*

Proof. For each vertex v_i, its star $\operatorname{st}(v_i)$ is the union of the interiors of all simplices with v_i as a vertex. Since the interiors of simplices on a simplicial complex don't overlap, $\operatorname{st}(v_1) \cap \cdots \cap \operatorname{st}(v_n)$ is the union of the interiors of all simplices that are in all $\operatorname{st}(v_1), \ldots, \operatorname{st}(v_n)$. That is, it is the union of the interiors of all simplices that have $v_1, \ldots, v_n$ in its set of vertices. If the intersection is non-empty, then there exists at least one simplex that has $v_1, \ldots, v_n$ as a subset of its set of vertices, and therefore the simplex σ with vertices $v_1, \ldots, v_n$ exists as a face of that simplex. Furthermore, the union contains the interiors of all simplices that have those vertices as a subset and therefore σ as a face, so it is in fact $\operatorname{st}(\sigma)$. □

Now that we have this lemma, we want to subdivide K until for each simplex of our subdivision, we can approximate its vertices to vertices of L with intersecting stars: this implies that the simplex maps to a simplex of L. We know that the preimages of the open stars of the vertices of L form an open cover on K. Therefore, it has a Lebesgue number ε such that for each set with diameter less than ε, there is some vertex whose open star's preimage wholly contains that set.

We can iteratively barycentrically subdivide K until we obtain K' where each simplex of K' has a diameter less than $\frac{\varepsilon}{2}$. Then, for each vertex v of K', the diameter of $\operatorname{St}(v)$ is less than ε, so $f(\operatorname{St}(v))$ is wholly contained in the open star of some vertex of L. Call that vertex $g(v)$: this gives us a map $g \colon K'^0 \to L^0$ where $f(\operatorname{St}(v)) \subset \operatorname{st}(g(v))$.

Now take any simplex σ of K'. For all vertices v_i of σ, $\sigma \in \mathrm{St}(v_i)$. Therefore $f(\sigma) \subset \mathrm{st}(g(v_i))$ for all i, so $f(\sigma) \subset \bigcap_i \mathrm{st}(g(v_i)) \neq \emptyset$. That means that $\{g(v_i)\}$ is the set of vertices of a simplex in L: g maps the vertices of each simplex in K' to the vertices of a simplex in L. Further, if we extend g so that it linearly maps each σ to the simplex defined by the image of σ's vertices, $f(\sigma) \subset \bigcap_i \mathrm{st}(g(v_i)) = \mathrm{st}(g(\sigma))$. Therefore, each $f(x)$ is in a simplex that has $g(\sigma)$ as a face, so, since simplices are convex, $f(x)$ and $g(x)$ are connected by a linear path. Therefore, since f is given as continuous and g, being linear within each simplex, must also be continuous, we can construct a homotopy $f_t(x) = (1-t)f(x) + tg(x)$ between f and g along those linear paths, so f and g are homotopic.

□

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4. Bibliography

References

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