Equivariantly Twisted Cohomology Theories

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Twisted cohomology theories

A twisted cohomology theory is a functorial algebraic invariant of topological spaces that behaves similarly to a cohomology theory, but depends on additional local information. Examples:

- Ordinary cohomology $H^*(X;\mathcal{A})$ with coefficients in a local coefficient system $\mathcal{A}$
- Twisted K-theory $K_E^*(X)$ with coefficients in a $U(1)$-gerbe E.

A $U(1)$-gerbe E is a higher categorical version of a line bundle and determines a class $[E] \in H^3(X;\mathbf{Z})$.

$$K_E^*(X) = \text{Grothendieck}[E\text{-twisted vector bundles}]$$

Another point of view: E is a principal $\mathrm{PU}(\mathcal{H})$-bundle over X for a fixed Hilbert space $\mathcal{H}$.

$$K_E^*(X) = \pi_{-*}\Gamma(E \times_{\mathrm{PU}(\mathcal{H})} \mathrm{Fred}(\mathcal{H}))$$

Twisted cohomology theories are represented by parametrized spectra

R: a ring spectrum
X: a topological space

Twisted R theory is a cohomology theory $R^*_\tau(-)$ defined on the category (Spaces)$/X$ that depends on a choice of local twisting τ on X.

The twisted theory $R^*_\tau(-)$ is represented by a parametrized spectrum R_τ over X:

$$\tau \colon X \longrightarrow BGL_1 R \quad \rightsquigarrow \quad R_\tau = R \wedge_{\Sigma^\infty_+ GL_1 R} \Sigma^\infty_X E(\tau)$$

$E(\tau)$ is the "principal $GL_1 R$-bundle" over X classified by τ.

In the case of twisted K-theory $K^*_E(-)$, the $U(1)$-gerbe $[E] = [E(\tau)] \in H^3(X;\mathbf{Z})$ is classified by a map

$$\tau \colon X \longrightarrow K(\mathbf{Z},3) \simeq BPU(\mathcal{H}) \simeq BCP^\infty_\otimes \subset BGL_1 K.$$

Goal: a framework for equivariantly twisted cohomology theories

G: compact Lie group
X: a G-space
R: G-ring spectrum

My goal is to set up a framework to define and work with G-equivariant twisted R-theory.

This parametrized cohomology theory should:

- be represented by a parametrized G-spectrum R_τ over X
- agree with R when $X = *$
- depend on an equivariant twist classified by a G-map:

$$\tau \colon X \longrightarrow B_G \mathrm{GL}_1 R$$

- recover previous definitions (for example, of twisted equivariant K-theory).

Monoidal presentations of A_∞ G-spaces

$\mathcal{I}_G$: the category with

- objects: G inner product spaces V
- morphisms: linear isometries $V \longrightarrow W$

An $\mathcal{I}_G$-space is a G-equivariant functor $X\colon \mathcal{I}_G \longrightarrow G$-spaces. There is a symmetric monoidal product $\boxtimes$ on $\mathcal{I}_G$-spaces such that the functor

$$X \longmapsto \operatorname{hocolim}_V X(V)$$

induces Quillen equivalences:

$$(\mathcal{I}_G\text{-spaces}) \simeq (G\text{-spaces})$$
$$(\boxtimes\text{-monoids}) \simeq (A_\infty\ G\text{-spaces})$$

I apologize for being evil, but for this talk I will treat these equivalences as if they were *equalities*.

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Isotropy subgroups of $\Pi \rtimes G$

Working towards a definition of $B_G \mathrm{GL}_1 R$, our general setup is:

Π: a grouplike A_∞ G-space (think $\Pi = \mathrm{GL}_1 R$)
$\Pi \rtimes G$: the product A_∞ G-space determined by $G \circlearrowleft \Pi$

$$\Pi \text{ acts on } X \text{ through } G\text{-maps} \iff \Pi \rtimes G \text{ acts on } X$$

Π acts freely on X when the isotropy subgroups of $\Pi \rtimes G$ are of the form:

$$H_\alpha = \{(\alpha(h), h) \in \Pi \rtimes G \mid h \in H\}$$

for some subgroup $H < G$ and 1-cocycle $\alpha\colon H \longrightarrow \Pi$.

The monoidal model for the A_∞ space Π allows us to make sense of the cocycle condition

$$\alpha(g) \cdot {}^g\alpha(h) = \alpha(gh).$$

The construction of $E_G\Pi \longrightarrow B_G\Pi$

$\mathcal{O}$: the orbit category of $(\Pi \rtimes G)$-spaces of the form $(\Pi \rtimes G)/H_\alpha$
Define

$$E_G\Pi = B(*, \mathcal{O}, R) = \underset{[(\Pi \rtimes G)/H_\alpha] \in \mathcal{O}}{\text{hocolim}} (\Pi \rtimes G)/H_\alpha$$

Then $E_G\Pi \longrightarrow B_G\Pi = E_G\Pi/\Pi$ is the universal “principal Π G-bundle". More generally, we can define

$$E_{\mathcal{F}}\Pi \longrightarrow B_{\mathcal{F}}\Pi$$

for any family $\mathcal{F}$ of isotropy “subgroups”:

$$\mathcal{F} \subset \{H_\alpha < \Pi \rtimes G\}$$

Definition: Equivariant twists of a G ring spectrum R

An equivariant twist for R-theory is a G-map

$$\tau\colon X \longrightarrow B_G \mathrm{GL}_1 R.$$

By pulling back† the universal bundle $E_G \mathrm{GL}_1 R$, there is an associated $\mathrm{GL}_1 R$-bundle $E(\tau) \longrightarrow X$.

Definition

The τ-twisted R-cohomology of X is given by the homotopy classes of sections of the parametrized G-spectrum R_τ classified by τ:

$$R^\star_\tau(X) = \pi_{-\star}\Gamma(R \wedge_{\Sigma^\infty_+ \mathrm{GL}_1 R} \Sigma^\infty_X E(\tau))$$

$R^\star_\tau(-)$ is an $RO(G)$-graded cohomology theory defined on G-spaces$/X$.

The G-homotopy type of $B_G\mathrm{GL}_1 R$

Let $H < G$. If $\alpha \colon H \longrightarrow \Pi$ is the 1-cocycle with associated $H_\alpha < \Pi \rtimes G$, then define

$$\Pi^{H_\alpha} = \text{“}\{\pi \in \Pi \mid \pi \cdot \alpha(h) \simeq \alpha(h) \cdot {}^h\pi\}\text{”}$$

Π^{H_α} is an A_∞ space with (non-equivariant) delooping $B(\Pi^{H_\alpha})$. Letting α run over equivalence classes of 1-cocycles, we get:

$$(B_G\Pi)^H = \coprod_{[H_\alpha] \in H^1(H;\Pi)} B(\Pi^{H_\alpha})$$

Work in progress: understand the C_2-homotopy type of

$$B_{C_2}\mathrm{GL}_1 K\mathbf{R}.$$

Twisted equivariant K-theory

Returning to full generality, the (non-equivariant) principal Π-bundle

$$EG \times_G E_{\mathcal{F}}\Pi \longrightarrow EG \times_G B_{\mathcal{F}}\Pi$$

is classified by a map $EG \times_G B_{\mathcal{F}}\Pi \longrightarrow B\Pi$, which induces

$$[X, B_{\mathcal{F}}\Pi] \longrightarrow [EG \times_G X, B\Pi].$$

By naturality for the inclusion $\mathrm{PU}(\mathcal{H}) \simeq CP^\infty \subset \mathrm{GL}_1 K$:

$$\begin{array}{ccc} [X, B_{\mathcal{F}}\mathrm{PU}(\mathcal{H})]^G & \longrightarrow & [EG \times_G X, B\mathrm{PU}(\mathcal{H})] \\ \downarrow & & \downarrow \\ [X, B_{\mathcal{F}}\mathrm{GL}_1 K]^G & \longrightarrow & [EG \times_G X, B\mathrm{GL}_1 K] \end{array}$$

We will use this diagram to compare Borel twists with equivariant twists.

Use the family of isotropy subgroups

$$\mathcal{F} = \{H_\alpha < \mathrm{PU}(\mathcal{H}) \rtimes G \mid \alpha\colon H \longrightarrow \mathrm{PU}(\mathcal{H}) \text{ is stable}\}$$

$\widetilde{H}$: the S^1-central extension determined by α

α is stable if the image of

$$\text{index}\colon \mathrm{Fred}(\mathcal{H})^\alpha \longrightarrow R_\alpha(H) = R(\widetilde{H})$$

is a subgroup containing all representations of $\widetilde{H}$ on which S^1 acts by multiplication.

The top map is an isomorphism [Lück-Uribe]:

$$\begin{array}{ccc} [X, B_{\mathcal{F}}\mathrm{PU}(\mathcal{H})]^G & \xrightarrow{\cong} & [EG \times_G X, B\mathrm{PU}(\mathcal{H})] \ni \tau \\ \downarrow & & \downarrow \\ f(\tau) \in [X, B_{\mathcal{F}}\mathrm{GL}_1 K]^G & \longrightarrow & [EG \times_G X, B\mathrm{GL}_1 K] \end{array}$$

Twisted equivariant K-theory vs. equivariantly twisted K_G-theory

Starting with

$$[\tau] \in H^3(EG \times_G X; \mathbf{Z}) \cong [EG \times_G X, BPU(\mathcal{H})],$$

we get an equivariant twist $f(\tau)\colon X \longrightarrow B_{\mathcal{F}}\mathrm{GL}_1 K$.

For K the complex K-theory spectrum with trivial G-action, our definition of $f(\tau)$-twisted K-theory agrees with that of Atiyah-Segal, Hopkins-Freed-Teleman, Lupercio-Uribe:

$$K^*_\tau(X) \cong K^*_{f(\tau)}(X)$$

Q: what about twists of fully equivariant K-theory K_G?


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