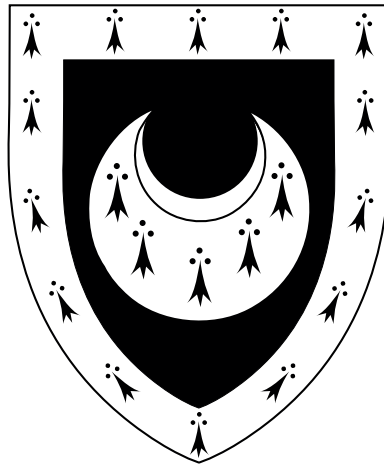


CATEGORIES OF SPACES BUILT FROM LOCAL MODELS

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This dissertation is submitted for the degree of Doctor of Philosophy

This dissertation
is dedicated
to my parents.

僅以此論文
獻給我父母

Abstract

Many of the classes of objects studied in geometry are defined by first choosing a class of nice spaces and then allowing oneself to glue these local models together to construct more general spaces. The most well-known examples are manifolds and schemes. The main purpose of this thesis is to give a unified account of this procedure of constructing a category of spaces built from local models and to study the general properties of such categories of spaces. The theory developed here will be illustrated with reference to examples, including the aforementioned manifolds and schemes.

For concreteness, consider the passage from commutative rings to schemes. There are three main steps: first, one identifies a distinguished class of ring homomorphisms corresponding to open immersions of schemes; second, one defines the notion of an open covering in terms of these distinguished homomorphisms; and finally, one embeds the opposite of the category of commutative rings in an ambient category in which one can glue (the formal duals of) commutative rings along (the formal duals of) distinguished homomorphisms. Traditionally, the ambient category is taken to be the category of locally ringed spaces, but following Grothendieck, one could equally well work in the category of sheaves for the large Zariski site—this is the so-called ‘functor of points approach’. A third option, related to the exact completion of a category, is described in this thesis.

The main result can be summarised thus: categories of spaces built from local models are extensive categories with a class of distinguished morphisms, subject to various stability axioms, such that certain equivalence relations (defined relative to the class of distinguished morphisms) have pullback-stable quotients; moreover, this construction is functorial and has a universal property.

If names be not correct, language is not in accordance with the truth of things. If language be not in accordance with the truth of things, affairs cannot be carried on to success. When affairs cannot be carried on to success, proprieties and music will not flourish. When proprieties and music do not flourish, punishments will not be properly awarded. When punishments are not properly awarded, the people do not know how to move hand or foot. Therefore a superior man considers it necessary that the names he uses may be spoken appropriately, and also that what he speaks may be carried out appropriately. What the superior man requires is just that in his words there may be nothing incorrect.

Analects, Book XIII, translated by James Legge

名不正，則言不順；
言不順，則事不成；
事不成，則禮樂不興；
禮樂不興，則刑罰不中；
刑罰不中，則民無所措手足。
故君子名之必可言也，言之必可行也。
君子於其言，無所苟而已矣。

《論語·子路》

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Declaration of originality

This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration except as declared in the preface and specified in the text. It is not substantially the same as any that I have submitted, or, is being concurrently submitted for a degree or diploma or other qualification at the University of Cambridge or any other university or similar institution except as declared in the preface and specified in the text. I further state that no substantial part of my dissertation has already been submitted, or, is being concurrently submitted for any such degree, diploma, or other qualification at the University of Cambridge or any other university or similar institution except as declared in the preface and specified in the text.

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Context

Many of the classes of objects studied in geometry are defined by first choosing a class of nice spaces and then allowing oneself to glue these local models together to construct more general spaces. The most well-known examples are manifolds and schemes. In fact, manifolds comprise a whole family of examples: after all, there are smooth manifolds, topological manifolds, complex analytic manifolds, manifolds with boundaries, manifolds with corners, etc. All of these notions of manifold are defined in roughly the same way, namely, as topological spaces equipped with a covering family of open embeddings of local models such that certain regularity conditions are satisfied. On the other hand, the traditional definition of scheme is much more involved: because the local models are not given directly as geometric objects, one has to first find a suitable geometric incarnation of the local models, which is highly non-trivial.

Another manifold-like notion is the notion of sheaf. Indeed, a sheaf on a topological space is a topological space obtained by gluing together open subspaces of the base space. This heuristic can be made precise and remains valid for Grothendieck's notion of sheaf on a site: a site is a category equipped with a notion of covering, and a sheaf on a site can be regarded as a formal colimit of a diagram in the base. Thus, it should come as no surprise that manifold-like objects can be represented by sheaves on the category of local models, in the sense that there is a fully faithful functor from the category of manifold-like objects to the category of sheaves.

There is an alternative definition of scheme based on the aforementioned sheaf representation: this is called 'the functor-of-points approach to algebraic geo-

metry’ and can be found in e.g. [Demazure and Gabriel, 1970]. The great advantage of the definition in terms of functors over the traditional one in terms of locally ringed spaces is that one no longer needs to explicitly model affine schemes—the local models—as geometric objects; instead, one can just glue together affine schemes *formally*. In exchange, one has to give up a certain sense of concreteness, but it is precisely the high level of abstraction that makes the functor-of-points approach so amenable to generalisation: one can mimic the functorial definition of ‘scheme’ in any reasonable geometric situation to get a manifold-like notion.

Of course, one should say what one means by ‘reasonable geometric situation’ here. For the purposes of defining manifold-like notions, not much is needed: it would suffice to have a class of well-behaved morphisms—akin to local homeomorphisms of topological spaces or étale morphisms of schemes—and a compatible notion of covering. Joyal and Moerdijk [JM] have previously investigated this idea, albeit without discussing the problem of defining manifold-like notions. In a sense, this thesis is a fulfilment of a suggestion of Shulman [2012]:

It should be possible to axiomatize further “open map structure”, along the lines of [JM] and [DAG 5], enabling the identification of a general class of “schemes” in [the exact completion] as the congruences where gluing happens only along “open subspaces.”

Summary

The primary goal of this thesis is to give a uniform account of manifold-like notions. Although the concept is straightforward enough, as ever, the devil is in the details. In essence, the difficulty is the tradeoff between having a bad category of nice objects and having a nice category of bad objects.^[1] It is not very hard to develop an elegant theory of manifold-like notions *if* one assumes that all the categories and functors involved behave nicely with respect to limits of finite diagrams. Unfortunately, the category of manifolds is the classical example of a bad category of nice objects—not all pullbacks exist—so such a theory would not account for manifolds. Similarly, one cannot easily account for the functor

[1] Of course, this is only an empirical observation, sometimes attributed to Grothendieck.

that sends a scheme to its underlying topological space because this functor does not preserve pullbacks.

For better or worse, the theory that *is* developed in this work *can* accommodate the two examples mentioned above. While the theory may not be as elegant as one may have hoped for at first, it is at least general enough to include well-known examples. What follows is a summary of this theory; some details have been changed or omitted in order to simplify the exposition.

First, let us fix what it means to have a category of local models. An **admissible ecumene** consists of the following data:

- An extensive category,^[2] \mathcal{C} .
- A class of morphisms in \mathcal{C} , \mathbf{E} , with the following properties:
 - Every isomorphism in \mathcal{C} is a member of \mathbf{E} .
 - \mathbf{E} is closed under composition.
 - \mathbf{E} is closed under coproduct.
 - \mathbf{E} is a quadrable class of morphisms in \mathcal{C} , i.e. \mathcal{C} has pullbacks of members of \mathbf{E} along arbitrary morphisms and, for every pullback square in \mathcal{C} of the form below,
$$\begin{array}{ccc} X' & \longrightarrow & X \\ f' \downarrow & & \downarrow f \\ Y' & \longrightarrow & Y \end{array}$$

if $f : X \rightarrow Y$ is a member of \mathbf{E} , then $f' : X' \rightarrow Y'$ is also a member of \mathbf{E} .

 - Every member of \mathbf{E} is an effective epimorphism in \mathcal{C} .
 - Every finite diagram in \mathcal{C} has an \mathbf{E} -weak limit.^[3]
- A class of morphisms in \mathcal{C} , \mathbf{D} , with the following properties:
 - Every isomorphism in \mathcal{C} is a member of \mathbf{D} .
 - \mathbf{D} is closed under composition.
 - \mathbf{D} is a quadrable class of morphisms in \mathcal{C} .

[2] See [definition 1.5.5](#).

[3] See [definition 1.4.9](#).

- For every object X in \mathcal{C} and every small set I , the codiagonal morphism $\nabla : \coprod_{i \in I} X \rightarrow X$ is a member of \mathcal{D} .
- \mathcal{D} is closed under coproduct.
- Given morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ in \mathcal{C} , if both $g : Y \rightarrow Z$ and $g \circ f : X \rightarrow Z$ are members of \mathcal{D} , then $f : X \rightarrow Y$ is also a member of \mathcal{D} .
- Given a member $f : X \rightrightarrows Y$ of \mathbf{E} and a morphism $g : Y \rightarrow Z$ in \mathcal{C} , if both $f : X \rightrightarrows Y$ and $g \circ f : X \rightarrow Z$ are members of \mathcal{D} , then $g : Y \rightarrow Z$ is also a member of \mathcal{D} .
- If $f : X \rightarrow Y$ is a member of \mathcal{D} , then there is a small set Φ of objects in $\mathcal{C}_{/X}$ with the following properties:
 - * The induced morphism $\coprod_{(U,x) \in \Phi} U \rightarrow X$ in \mathcal{C} is a member of \mathbf{E} .
 - * For every $(U,x) \in \Phi$, both $x : U \rightarrow X$ and $f \circ x : U \rightarrow Y$ are monomorphisms in \mathcal{C} that are members of \mathcal{D} .
- Every member of \mathbf{E} is also a member of \mathcal{D} .

For example, the following data define admissible ecumenae:

- (a) \mathcal{C} is the category of Hausdorff spaces, \mathcal{D} is the class of local homeomorphisms, and \mathbf{E} is the class of the class of surjective local homeomorphisms.
- (b) \mathcal{C} is the category of disjoint unions of small families of affine schemes, \mathcal{D} is the class of local isomorphisms, and \mathbf{E} is the class of surjective local isomorphisms.
- (c) \mathcal{C} is the category of disjoint unions of small families of open subspaces of euclidean spaces, \mathcal{D} is the class of local diffeomorphisms, and \mathbf{E} is the class of surjective local diffeomorphisms.

Next, we make precise what it means to have a category of spaces built from local models. We say that an admissible ecumene as above is **effective** if the following additional condition is satisfied:

- For every object X in \mathcal{C} and every tractable equivalence relation^[4] (R, d_0, d_1) on X in \mathcal{C} , there is a morphism $f : X \rightrightarrows Y$ in \mathcal{C} such that $f : X \rightrightarrows Y$ is a member of \mathbf{E} and (R, d_0, d_1) is a kernel pair of $f : X \rightarrow Y$.

[4] See [definition 2.2.14](#).

The main result of this thesis says that every admissible ecumene can be embedded in an effective admissible ecumene that is universal with respect to admissible functors.^[5] The **category of charted objects** is (the underlying category of) this universal effective admissible ecumene. For example:

- (a) For (C, D, E) as in example (a), the category of charted objects is (equivalent to) the category of locally Hausdorff spaces.
- (b) For (C, D, E) as in example (b), the category of charted objects is (equivalent to) the category of schemes.
- (c) For (C, D, E) as in example (c), the category of charted objects is (equivalent to) the category of manifolds, possibly neither second-countable nor Hausdorff.

As one might expect, every charted object can be obtained as a quotient (in the category of charted objects) of an object in C by a tractable equivalence relation (not necessarily in C). This fact is easily deduced from the explicit construction of the category of charted objects as a full subcategory of the exact completion of C relative to E , which is an exact category that C embeds into and is universal with respect to functors that preserve limits of finite diagrams and send members of E to effective epimorphisms. Since C is an extensive category and E is closed under coproduct, the exact completion is a pretopos.^[6] This is very convenient for technical purposes: recalling the tradeoff discussed in the first paragraph, what we are doing is embedding a bad category of nice objects in a nice category of bad objects, which means that many proofs boil down to showing that certain constructions on nice objects—which are guaranteed to exist in the nice category—yield nice objects.

To make a tighter connection with the previously mentioned work of Joyal and Moerdijk [JM], we propose the following definition. A **gros pretopos** consists of the following data:

- A pretopos, S .

[5] See [definition 2.5.5](#) and [theorem 2.5.7](#).

[6] See [proposition 1.5.13](#).

- A class of morphisms in \mathcal{S} , \mathcal{D} , with the following properties:
 - Every isomorphism in \mathcal{C} is a member of \mathcal{D} .
 - \mathcal{D} is closed under composition.
 - \mathcal{D} is a quadrable class of morphisms in \mathcal{S} .
 - For every object X in \mathcal{S} and every small set I , the codiagonal morphism $\nabla : \coprod_{i \in I} X \rightarrow X$ is a member of \mathcal{D} .
 - \mathcal{D} is closed under coproduct.
 - Given morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ in \mathcal{S} , if both $g : Y \rightarrow Z$ and $g \circ f : X \rightarrow Z$ are members of \mathcal{D} , then $f : X \rightarrow Y$ is also a member of \mathcal{D} .
 - Given an effective epimorphism $f : X \rightrightarrows Y$ in \mathcal{S} , a morphism $g : Y \rightarrow Z$ in \mathcal{S} and a kernel pair (R, d_0, d_1) of $f : X \rightarrow Y$ in \mathcal{S} , if $d_0, d_1 : R \rightarrow X$ and $g \circ f : X \rightarrow Z$ are all members of \mathcal{D} , then both $f : X \rightrightarrows Y$ and $g : Y \rightarrow Z$ are also members of \mathcal{D} .

The axioms on \mathcal{D} are almost the same as Joyal’s axioms for a class of étale morphisms—the difference being that we omit the descent axiom and strengthen the quotient axiom. Here, ‘gros’ is used in opposition to ‘petit’: given an object X in \mathcal{S} , the **petit pretopos** $\mathcal{D}/_X$ is the full subcategory of the slice category $\mathcal{S}/_X$ spanned by the objects (F, p) such that $p : F \rightarrow X$ is a member of \mathcal{D} .

Of course, a pretopos equipped with a class of étale morphisms as defined in [JM, §1] is a gros pretopos, but we require a bit more generality. For instance, the pretopos associated with an admissible ecumene admits the structure of a gros pretopos such that the intersection of the class of distinguished morphisms in the pretopos with the original category is the original class of distinguished morphisms—in fact, we will see two different constructions: one that yields a class of étale morphisms and one that does not.^[7] It turns out that the latter is what we need to construct the category of charted objects.

Given a full subcategory $\mathcal{C} \subseteq \mathcal{S}$, a $(\mathcal{C}, \mathcal{D})$ -**atlas** of an object Y in \mathcal{S} is an effective epimorphism $f : X \rightrightarrows Y$ in \mathcal{S} where X is an object in \mathcal{C} and $f : X \rightrightarrows Y$ is a member of \mathcal{D} , and a $(\mathcal{C}, \mathcal{D})$ -**extent** in \mathcal{S} is an object that admits a $(\mathcal{C}, \mathcal{D})$ -atlas.

[7] See [proposition 2.3.2](#) and [paragraph 2.5.3](#).

Under certain hypotheses on C and D , the category of (C, D) -extents in S admits the structure of an effective admissible ecumene. For instance, if S is the exact completion of an admissible ecumene, C is the image of the original category, and D is the induced class of local homeomorphisms, then this is part of the statement of the main result.

Although we define the category of charted objects for an admissible ecumene to be the category of extents in the exact completion with respect to a certain gros pretopos structure, we should distinguish between ‘charted object’ and ‘extent’ because there exist gros pretoposes that do not arise in this way. Indeed, whereas the petit pretopos over a charted object is guaranteed to be localic,^[8] the petit pretopos over an extent can fail to be localic. The extra generality afforded by defining extents in the setting of a general gros pretopos makes it possible to fit algebraic spaces—generalised schemes—into our framework, but exploring that possibility will be left for future work.

Outline

Abstract topology

In the first chapter, we study various aspects of what it means to be a category of spaces.

- In § 1.1, we discuss the relative point of view of Grothendieck, i.e. the idea that a morphism is a family of objects (the domain) parametrised by the base (the codomain), and we define some related terminology that will be used throughout this work.
- In § 1.2, we consider what it means for a morphism to have a property locally on the domain, locally on the base, or locally, with respect to a coverage. This is partially a generalisation of earlier work by Joyal and Moerdijk [JM, §§ 1 and 5].
- In § 1.3, we study categories with a class of morphisms that have good properties with regards to pullbacks and images, such as regular categories.

[8] See [definition 2.1.4](#) and [lemma 2.3.12\(a\)](#).

- In § 1.4, we consider the problem of adding exact quotients—i.e. coequalisers of equivalence relations that behave well under pullback—to a category with a class of covering morphisms. Specifically, we will see a sheaf-theoretic construction of the exact completion of a category equipped with a unary topology in the sense of [Shulman, 2012].
- In § 1.5, we study extensive categories and we examine properties of their exact completions. In particular, we will see that the exact completion of an extensive category equipped with a superextensive coverage is a pretopos.

Charted objects

In the second chapter, we use the concepts introduced in the first chapter to construct categories of charted objects, i.e. spaces built from local models.

- In § 2.1, we consider full subcategories of pretoposes for which the associated Yoneda representation is fully faithful and we identify the essential image of such Yoneda representations.
- In § 2.2, we define various notions of categories equipped with structure making it possible to interpret basic notions of topology such as open embeddings and local homeomorphisms.
- In § 2.3, we examine the properties of the category of extents—i.e. objects in a gros pretopos equipped with an étale atlas.
- In § 2.4, we investigate sufficient conditions for a functor between gros pretoposes to preserve atlases and extents. Specifically, we will see that a functor between gros pretoposes will preserve local homeomorphisms between extents if it preserves coproducts, some coequalisers, and pullbacks of local homeomorphisms between distinguished objects.
- In § 2.5, we characterise the category of extents by a universal property in a special case, namely, when étale morphisms coincide with local homeomorphisms and the coverage is generated by local homeomorphisms.

Specificities

In the final chapter, we see specific examples of the notions introduced in the preceding chapters.

- In §3.1, we examine three classes of continuous maps of topological spaces that arise by relativising the notion of compactness.
- In §3.2, we construct a combinatorial example of a gros pretopos based on discrete fibrations of simplicial sets, which are closely related to discrete fibrations of categories.
- In §3.3, we construct an admissible ecumene for which the charted objects are the smooth manifolds of fixed dimension and cardinality.
- In §3.4, we construct admissible ecumenae from categories of topological spaces and investigate when a topological space is representable by a charted object.
- In §3.5, we see two *prima facie* different ways of defining schemes as extents in a gros pretopos and show that they are the same.

Guide for readers

Prerequisites

I assume the reader is familiar with category theory—at least Chapters I–V and X of [CWM]. The appendix contains some material on topics not covered in op. cit.

Conventions

Following [CWM], ‘category’ always means a category with a set of objects and a set of morphisms, whereas ‘metacategory’ refers to a category with a class of objects and a class of morphisms. Nonetheless, from time to time, it is convenient to use terminology previously only defined for categories for metacategories as well. This can be justified by adopting a suitable universe axiom, but we will not do so.

Internal structure

The following is one possible reading order: §§ [I.1](#), [A.1](#), [A.2](#), [I.2](#), [3.1](#), [I.3](#), [A.3](#), [I.4](#), [I.5](#), [2.1](#), [2.2](#), [2.3](#), [3.2](#), [3.3](#), [2.4](#), [2.5](#), [3.4](#), [3.5](#). That said, because concrete examples are deferred to [chapter III](#), readers may find it helpful to peek ahead from time to time. An index for finding definitions appears at the end of the document, after the bibliography.

The material within each chapter and each section is laid out linearly; readers should avoid skipping to the middle of a section as there may be local conventions in force. Each section is divided into “paragraphs”, which are identified by a label printed in the margin.

Abstract topology

1.1 The relative point of view

SYNOPSIS. We discuss the relative point of view of Grothendieck and define related terminology.

I.1.1 ✖ Throughout this section, C is an arbitrary category.

I.1.2 ¶ The central tenet of the **relative point of view** of Grothendieck is to regard morphisms $f : X \rightarrow Y$ in C as objects (X, f) in the slice category $C_{/Y}$. In turn, objects in $C_{/Y}$ are to be regarded as “families” of objects in C indexed (or parametrised) by Y . This can be made precise in special cases: for instance, there is a canonical equivalence between $\mathbf{Set}_{/I}$ and \mathbf{Set}^I .^[1] Of course, from the relative point of view, pullback of morphisms is reindexing (or reparametrisation), so we should focus our attention on those properties and constructions which are compatible with pullback.

I.1.2(a) **DEFINITION.** A morphism $f : X \rightarrow Y$ in C is **quadrable** if it has the following property:

- For every morphism $y : Y' \rightarrow Y$ in C , there is a pullback square in C of the form below:

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow f \\ Y' & \xrightarrow{y} & Y \end{array}$$

We write $\text{qdbl } C$ for the set of all quadrable morphisms in C .

[1] Depending on the definition of \mathbf{Set}^I , this may depend on the axiom of replacement.

I.I.2(b) DEFINITION. A subset $\mathcal{F} \subseteq \text{mor } C$ is **closed under pullback** in C if it has the following property:

- For every pullback square in C of the form below,

$$\begin{array}{ccc} X' & \longrightarrow & X \\ f' \downarrow & & \downarrow f \\ Y' & \longrightarrow & Y \end{array}$$

if $f : X \rightarrow Y$ is a member of \mathcal{F} , then $f' : X' \rightarrow Y'$ is also a member of \mathcal{F} .

I.I.2(c) DEFINITION. A **quadrable class of morphisms** in C is a subset $\mathcal{F} \subseteq \text{mor } C$ with the following properties:

- Every member of \mathcal{F} is a quadrable morphism in C .
- \mathcal{F} is closed under pullback in C .

I.I.3 LEMMA. *Let \mathcal{F} be a quadrable class of morphisms in C . Given a morphism $f : X \rightarrow Y$ in C and a monomorphism $g : Y \rightarrow Z$ in C , if the composite $g \circ f : X \rightarrow Z$ is a member of \mathcal{F} , then the morphism $f : X \rightarrow Y$ is also a member of \mathcal{F} .*

Proof. The following is a pullback square in C :

$$\begin{array}{ccc} X & \xlongequal{\quad} & X \\ f \downarrow & & \downarrow g \circ f \\ Y & \xrightarrow{\quad g \quad} & Z \end{array}$$

Thus the left vertical arrow is a member of \mathcal{F} if the right vertical arrow is a member of \mathcal{F} . ■

I.I.4(a) DEFINITION. A **class of fibrations** in C is a subset $\mathcal{F} \subseteq \text{mor } C$ that satisfies the following axioms:

- \mathcal{F} is a quadrable class of morphisms in C .
- For every object X in C , $\text{id} : X \rightarrow X$ is a member of \mathcal{F} .
- \mathcal{F} is closed under composition.

I.I.4(b) DEFINITION. Let \mathcal{F} be a class of fibrations in \mathcal{C} and let S be an object in \mathcal{C} . An object (X, f) in \mathcal{C}/S is **\mathcal{F} -fibrant** if the morphism $f : X \rightarrow S$ is a member of \mathcal{F} .

We write $\mathcal{F}(S)$ for the full subcategory of \mathcal{C}/S spanned by the \mathcal{F} -fibrant objects.

I.I.4(c) EXAMPLE. $\text{iso } \mathcal{C}$, the class of isomorphisms in \mathcal{C} , is the smallest class of fibrations in \mathcal{C} .

I.I.4(d) EXAMPLE. The class of quadrable morphisms in \mathcal{C} is the largest class of fibrations in \mathcal{C} . (Note that the pullback pasting lemma implies that the class of quadrable morphisms in \mathcal{C} is closed under composition.)

I.I.4(e) EXAMPLE. The class of quadrable split epimorphisms in \mathcal{C} is a class of fibrations in \mathcal{C} .

I.I.5 ¶ Let \mathcal{F} be a class of fibrations in \mathcal{C} . Consider a commutative diagram in \mathcal{C} of the form below,

$$\begin{array}{ccccc} X & \xrightarrow{f} & Z & \xleftarrow{g} & Y \\ a \downarrow & & \downarrow c & & \downarrow b \\ A & \xrightarrow{h} & C & \xleftarrow{k} & B \end{array}$$

where the vertical arrows are members of \mathcal{F} . Suppose \mathcal{C} has pullbacks. It is not true that the induced morphism $a \times_c b : X \times_Z Y \rightarrow A \times_C B$ is a member of \mathcal{F} in general. Rather:

Pullbacks and fibrations LEMMA. *If $\langle a, f \rangle : X \rightarrow A \times_C Z$ is a member of \mathcal{F} , then $a \times_c b : X \times_Z Y \rightarrow A \times_C B$ is also a member of \mathcal{F} .*

Proof. By considering the following commutative diagram in \mathcal{S} ,

$$\begin{array}{ccccc} X \times_Z Y & \xrightarrow{a \times_c \text{id}_Y} & A \times_C Y & \longrightarrow & Y \\ \downarrow & & \text{id}_A \times_c \text{id}_C \downarrow & & \downarrow g \\ X & \xrightarrow{\langle a, f \rangle} & A \times_C Z & \longrightarrow & Z \\ & & \downarrow & & \downarrow c \\ & & A & \xrightarrow{h} & C \end{array}$$

we see that $a \times_c \text{id}_Y : X \times_Z Y \rightarrow A \times_C Y$ is a member of \mathcal{F} . On the other hand, by the pullback pasting lemma, we also have the following pullback square in \mathcal{S} ,

$$\begin{array}{ccc} A \times_C Y & \longrightarrow & Y \\ \text{id}_A \times_{\text{id}_C} b \downarrow & & \downarrow b \\ A \times_C B & \longrightarrow & B \end{array}$$

so $\text{id}_A \times_{\text{id}_C} b : A \times_C Y \rightarrow A \times_C B$ is also a member of \mathcal{F} . Hence, $a \times_c b : X \times_Z Y \rightarrow A \times_C B$ is indeed a member of \mathcal{F} . ■

1.1.6 ¶ Let \mathcal{F} be a quadrable class of morphisms in \mathcal{C} . The following can be regarded as a generalisation of the Hausdorff separation axiom, particularly when \mathcal{F} is regarded as a class of closed embeddings.

DEFINITION. An object X in \mathcal{C} is **\mathcal{F} -separated** if, for every object T in \mathcal{C} and every parallel pair $x_0, x_1 : T \rightarrow X$ in \mathcal{C} , there is an equaliser diagram in \mathcal{C} of the form below,

$$T' \xrightarrow{t} T \begin{array}{c} \xrightarrow{x_0} \\ \xrightarrow{x_1} \end{array} X$$

where $t : T' \rightarrow T$ is a member of \mathcal{F} .

EXAMPLE. An object in \mathcal{C} is subterminal if and only if it is (iso \mathcal{C})-separated.

Diagonal
criterion for
separatedness

LEMMA. Let X be an object in \mathcal{C} . Assuming $X \times X$ exists in \mathcal{C} , the following are equivalent:

- (i) X is an \mathcal{F} -separated object in \mathcal{C} .
- (ii) The diagonal $\Delta_X : X \rightarrow X \times X$ is a member of \mathcal{F} .

Proof. (i) \Rightarrow (ii). The following is an equaliser diagram in \mathcal{C} ,

$$X \xrightarrow{\Delta_X} X \times X \begin{array}{c} \longrightarrow \\ \longrightarrow \end{array} X$$

where the parallel pair of arrows are the two projections. Thus the diagonal $\Delta_X : X \rightarrow X \times X$ is a member of \mathcal{F} .

(ii) \Rightarrow (i). Given any parallel pair $x_0, x_1 : T \rightarrow X$ in \mathcal{C} , we have the following pullback square in \mathcal{C} ,

$$\begin{array}{ccc} T' & \longrightarrow & X \\ t \downarrow & & \downarrow \Delta_X \\ T & \xrightarrow{\langle x_0, x_1 \rangle} & X \times X \end{array}$$

and given any such pullback square,

$$T' \xrightarrow{t} T \begin{array}{c} \xrightarrow{x_0} \\ \xrightarrow{x_1} \end{array} X$$

is an equaliser diagram in \mathcal{C} . Note that $t : T' \rightarrow T$ is a member of \mathcal{F} , as required. ■

I.I.7 ¶ Let \mathcal{F} be a quadrable class of morphisms in \mathcal{C} . Given an object Y in \mathcal{C} , let \mathcal{F}_Y be the class of morphisms in $\mathcal{C}_{/Y}$ whose underlying morphism in \mathcal{C} is a member of \mathcal{F} . It is not hard to see that \mathcal{F}_Y is a quadrable class of morphisms in $\mathcal{C}_{/Y}$. This allows us to extend the definition of ‘separated’ from objects in \mathcal{C} to morphisms in \mathcal{C} .

DEFINITION. A morphism $f : X \rightarrow Y$ in \mathcal{C} is **\mathcal{F} -separated** if the object (X, f) in $\mathcal{C}_{/Y}$ is \mathcal{F}_Y -separated.

REMARK. Assuming 1 is a terminal object in \mathcal{C} , an object X in \mathcal{C} is \mathcal{F} -separated if and only if the unique morphism $X \rightarrow 1$ in \mathcal{C} is \mathcal{F} -separated.

Properties
of separated
morphisms

PROPOSITION.

- (i) *Assuming every isomorphism in \mathcal{C} is a member of \mathcal{F} , every monomorphism in \mathcal{C} is \mathcal{F} -separated.*
- (ii) *Given morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ in \mathcal{C} , if the composite $g \circ f : X \rightarrow Z$ is \mathcal{F} -separated, then the morphism $f : X \rightarrow Y$ is also \mathcal{F} -separated.*
- (iii) *The class of \mathcal{F} -separated morphisms in \mathcal{C} is closed under pullback in \mathcal{C} .*
- (iv) *Assuming \mathcal{F} is closed under composition, the class of \mathcal{F} -separated morphisms in \mathcal{C} is closed under composition.*

Proof. (i)–(ii). Straightforward.

(iii). Consider a pullback square in \mathcal{C} :

$$\begin{array}{ccc} X' & \xrightarrow{x} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{h} & Y \end{array}$$

Suppose $f : X \rightarrow Y$ is an \mathcal{F} -separated morphism in \mathcal{C} . We must verify that $f' : X' \rightarrow Y'$ is also an \mathcal{F} -separated morphism in \mathcal{C} .

Let $x'_0, x'_1 : T \rightarrow X'$ be morphisms in \mathcal{C} such that $f' \circ x'_0 = f' \circ x'_1$. Then $f \circ x \circ x'_0 = f \circ x \circ x'_1$, and since $f : X \rightarrow Y$ is \mathcal{F} -separated, there is an equaliser diagram in \mathcal{C} of the form below,

$$T' \xrightarrow{t} T \begin{array}{c} \xrightarrow{x \circ x'_0} \\ \xrightarrow{x \circ x'_1} \end{array} X$$

where $t : T' \rightarrow T$ is a member of \mathcal{F} . Note that the universal property of pullbacks implies that $x'_0 \circ t = x'_1 \circ t$. Thus,

$$T' \xrightarrow{t} T \begin{array}{c} \xrightarrow{x'_0} \\ \xrightarrow{x'_1} \end{array} X'$$

is also an equaliser diagram in \mathcal{C} , and this completes the proof.

(iv). Let $x_0, x_1 : T \rightarrow X$ be morphisms in \mathcal{C} such that $g \circ f \circ x_0 = g \circ f \circ x_1$. Since $g : Y \rightarrow Z$ is \mathcal{F} -separated, there is an equaliser diagram in \mathcal{C} of the form below,

$$T' \xrightarrow{t} T \begin{array}{c} \xrightarrow{f \circ x_0} \\ \xrightarrow{f \circ x_1} \end{array} Y$$

where $t : T' \rightarrow T$ is a member of \mathcal{F} . Since $f : X \rightarrow Y$ is \mathcal{F} -separated, there is an equaliser diagram in \mathcal{C} of the form below,

$$T'' \xrightarrow{t'} T' \begin{array}{c} \xrightarrow{x_0 \circ t} \\ \xrightarrow{x_1 \circ t} \end{array} X$$

where $t' : T'' \rightarrow T'$ is a member of \mathcal{F} . It is straightforward to verify that the following is an equaliser diagram in \mathcal{C} ,

$$T'' \xrightarrow{t \circ t'} T \begin{array}{c} \xrightarrow{x_0} \\ \xrightarrow{x_1} \end{array} X$$

and since \mathcal{F} is closed under composition, this completes the proof that $f : X \rightarrow Y$ is \mathcal{F} -separated. ■

I.1.8 ¶ Let \mathcal{F} be a quadrable class of morphisms.

Properties of
separated objects

PROPOSITION.

- (i) Assuming every isomorphism in \mathcal{C} is a member of \mathcal{F} , every subterminal object in \mathcal{C} is \mathcal{F} -separated.
- (ii) Given a morphism $f : X \rightarrow Y$ in \mathcal{C} , if X is a \mathcal{F} -separated object in \mathcal{C} , then $f : X \rightarrow Y$ is a \mathcal{F} -separated morphism in \mathcal{C} .
- (iii) Given objects X and Y in \mathcal{C} , if $X \times Y$ exists in \mathcal{C} and Y is \mathcal{F} -separated, then the projection $X \times Y \rightarrow X$ is \mathcal{F} -separated.
- (iv) Assuming \mathcal{F} is closed under composition, if $f : X \rightarrow Y$ is an \mathcal{F} -separated morphism in \mathcal{C} and Y is an \mathcal{F} -separated object in \mathcal{C} , then X is also an \mathcal{F} -separated object in \mathcal{C} .

Proof. Omitted. (The arguments are similar to those in the proof of [proposition I.1.7](#).) ◇

I.1.9 ¶ Let \mathcal{F} be a class of fibrations in \mathcal{C} .

LEMMA. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be morphisms in \mathcal{C} . If $g : Y \rightarrow Z$ is \mathcal{F} -separated and $g \circ f : X \rightarrow Z$ is a member of \mathcal{F} , then $f : X \rightarrow Y$ is also a member of \mathcal{F} .

Proof. Consider a pullback square in \mathcal{C} of the form below:

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{q} & Y \\ p \downarrow & & \downarrow g \\ X & \xrightarrow{g \circ f} & Z \end{array}$$

By [proposition I.1.7](#), the projection $p : X \times_Z Y \rightarrow X$ is \mathcal{F} -separated. The following is an equaliser diagram in \mathcal{C}/X ,

$$(X, \text{id}_X) \xrightarrow{\langle \text{id}_X, f \rangle} (X \times_Z Y, p) \xrightarrow[\langle p, f \circ p \rangle]{\text{id}} (X \times_Z Y, p)$$

thus $\langle \text{id}_X, f \rangle : X \rightarrow X \times_Z Y$ is a member of \mathcal{F} . Since $g \circ f : X \rightarrow Z$ is a member of \mathcal{F} , the projection $q : X \times_Z Y \rightarrow Y$ is also a member of \mathcal{F} . Thus, $f = q \circ \langle \text{id}_X, f \rangle : X \rightarrow Y$ is indeed a member of \mathcal{F} . ■

I.I.10 DEFINITION. A **class of separated fibrations** in \mathcal{C} is a subset $\mathcal{F} \subseteq \text{mor } \mathcal{C}$ that satisfies the following axioms:

- \mathcal{F} is a class of fibrations in \mathcal{C} .
- Given morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ in \mathcal{C} , assuming $g : Y \rightarrow Z$ is a member of \mathcal{F} , the composite $g \circ f : X \rightarrow Z$ is a member of \mathcal{F} if and only if the morphism $f : X \rightarrow Y$ is a member of \mathcal{F} .

Recognition principles for classes of separated fibrations

LEMMA. *Let \mathcal{F} be a class of fibrations in \mathcal{C} . The following are equivalent:*

- (i) \mathcal{F} is a class of separated fibrations in \mathcal{C} .
- (ii) *Given morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ in \mathcal{C} , if $f : X \rightarrow Y$ is a monomorphism in \mathcal{C} and both $g \circ f : X \rightarrow Z$ and $g : Y \rightarrow Z$ are members of \mathcal{F} , then $f : X \rightarrow Y$ is also a member of \mathcal{F} .*
- (iii) *Given morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow X$ in \mathcal{C} , if $g \circ f = \text{id}_X$ and $g : Y \rightarrow X$ is a member of \mathcal{F} , then $f : X \rightarrow Y$ is also a member of \mathcal{F} .*
- (iv) *If $g : Y \rightarrow Z$ is a member of \mathcal{F} , then the relative diagonal $\Delta_g : Y \rightarrow Y \times_Z Y$ is a member of \mathcal{F} .*
- (v) *Every member of \mathcal{F} is an \mathcal{F} -separated morphism in \mathcal{C} .*

Proof. (i) \Rightarrow (ii), (ii) \Rightarrow (iii), (iii) \Rightarrow (iv). Immediate.

(iv) \Leftrightarrow (v). Apply the diagonal criterion for separatedness (lemma I.I.6).

(iv) \Rightarrow (i). Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be morphisms in \mathcal{C} . Suppose both $g \circ f : X \rightarrow Z$ and $g : Y \rightarrow Z$ are members of \mathcal{F} . We must show that $f : X \rightarrow Y$ is also a member of \mathcal{F} . Since \mathcal{F} is a quadrable class of morphisms in \mathcal{C} , we have the following commutative diagram in \mathcal{C} ,

$$\begin{array}{ccccc}
 X & \xrightarrow{f} & Y & & \\
 \langle \text{id}_X, f \rangle \downarrow & & \downarrow \Delta_g & \searrow \text{id} & \\
 X \times_Z Y & \longrightarrow & Y \times_Z Y & \longrightarrow & Y \\
 \downarrow & & \downarrow & & \downarrow g \\
 X & \xrightarrow{f} & Y & \xrightarrow{g} & Z
 \end{array}$$

where every square and every rectangle is a pullback diagram in \mathcal{C} . By hypothesis, $\Delta_g : Y \rightarrow Y \times_Z Y$ is a member of \mathcal{F} , so $\langle \text{id}_X, f \rangle : X \rightarrow X \times_Z Y$ is also a member of \mathcal{F} . Similarly, $g \circ f : Y \rightarrow Z$ is a member of \mathcal{F} , so the projection $X \times_Z Y \rightarrow Y$ is also a member of \mathcal{F} . But \mathcal{F} is closed under composition, so $f : X \rightarrow Y$ is indeed a member of \mathcal{F} . ■

REMARK. Let \mathcal{F} be a class of fibrations and let $\mathcal{F}_{\text{mono}}$ be the class of monomorphisms in \mathcal{C} that are members of \mathcal{F} . Then $\mathcal{F}_{\text{mono}}$ is a class of separated fibrations in \mathcal{C} .

I.I.II ¶ Let \mathcal{F} be a class of fibrations in \mathcal{C} .

DEFINITION. A morphism in \mathcal{C} is **\mathcal{F} -perfect** if it is \mathcal{F} -separated and also a member of \mathcal{F} .

Properties
of perfect
morphisms

PROPOSITION.

- (i) *Every monomorphism in \mathcal{C} that is a member of \mathcal{F} is \mathcal{F} -perfect.*
- (ii) *The class of \mathcal{F} -perfect morphisms in \mathcal{C} is a class of separated fibrations.*

Proof. Straightforward. (Use [proposition 1.1.7](#) and [lemma 1.1.10.](#)) ◆

I.I.I2 ¶ Let S be an object in \mathcal{C} and let \mathcal{F} be a class of separated fibrations.

Limits of finite
diagrams of
separated
fibrations

PROPOSITION. *The inclusion $\mathcal{F}(S) \hookrightarrow \mathcal{C}_{/S}$ creates limits of all finite diagrams.*

Proof. It is clear that (S, id_S) is a terminal object in both $\mathcal{F}(S)$ and $\mathcal{C}_{/S}$. Similarly, since \mathcal{F} is a class of fibrations, $\mathcal{F}(S)$ has binary products and the inclusion $\mathcal{F}(S) \hookrightarrow \mathcal{C}_{/S}$ preserves binary products. Thus, it suffices to show that $\mathcal{F}(S)$ has equalisers and the inclusion $\mathcal{F}(S) \hookrightarrow \mathcal{C}_{/S}$ preserves equalisers, but this is a corollary of [lemma 1.1.10.](#) ■

I.I.I3 ¶ We briefly recall the notion of orthogonality.

DEFINITION. An object S in \mathcal{C} is **right orthogonal** to a morphism $f : X \rightarrow Y$ in \mathcal{C} if the induced map

$$\mathcal{C}(f, S) : \mathcal{C}(Y, S) \rightarrow \mathcal{C}(X, S)$$

is a bijection.

More generally, S is **right orthogonal** to a subset $\mathcal{L} \subseteq \text{mor } \mathcal{C}$ if S is right orthogonal to every member of \mathcal{L} .

Limits of objects
right ortho-
gonal to a class
of morphisms

PROPOSITION. *Let \mathcal{L} be a subset of $\text{mor } \mathcal{C}$. The full subcategory of \mathcal{C} spanned by the objects that are right orthogonal to \mathcal{L} is closed under limits of all diagrams.*

Proof. Straightforward. ◆

REMARK. In particular, every terminal object in \mathcal{C} is right orthogonal to every morphism in \mathcal{C} . (However, \mathcal{C} may not have any terminal objects at all.)

1.1.14 ¶ In consideration of the relative point of view, it behoves us to extend the notion of orthogonality from objects to morphisms.

DEFINITION. A morphism $p : Z \rightarrow S$ in \mathcal{C} is **right orthogonal** to a morphism $f : X \rightarrow Y$ in \mathcal{C} if the following is a pullback square in **SET**:

$$\begin{array}{ccc} \mathcal{C}(Y, Z) & \xrightarrow{\mathcal{C}(Y, p)} & \mathcal{C}(Y, S) \\ \mathcal{C}(f, Z) \downarrow & & \downarrow \mathcal{C}(f, S) \\ \mathcal{C}(X, Z) & \xrightarrow{\mathcal{C}(X, p)} & \mathcal{C}(X, S) \end{array}$$

More generally, $p : Z \rightarrow S$ is **right orthogonal** to a subset $\mathcal{L} \subseteq \text{mor } \mathcal{C}$ if $p : Z \rightarrow S$ is right orthogonal to every member of \mathcal{L} .

We write \mathcal{L}^\perp for the set of all morphisms in \mathcal{C} that are right orthogonal to \mathcal{L} .

REMARK. Assuming 1 is a terminal object in \mathcal{C} , an object X in \mathcal{C} is right orthogonal to \mathcal{L} if and only if the unique morphism $X \rightarrow 1$ is a member of \mathcal{L}^\perp .

Alternative
criteria for right
orthogonality

LEMMA. Let $p : Z \rightarrow S$ and $f : X \rightarrow Y$ be morphisms in C . The following are equivalent:

- (i) The morphism $p : Z \rightarrow S$ in C is right orthogonal to the morphism $f : X \rightarrow Y$ in C .
- (ii) The object (Z, S, p) in $(C \downarrow C)$ is right orthogonal to the morphism $(f, \text{id}_Y) : (X, Y, f) \rightarrow (Y, Y, \text{id}_Y)$ in $(C \downarrow C)$.
- (iii) For every morphism $q : Y \rightarrow S$ in C , the object (Z, p) in $C_{/S}$ is right orthogonal to the morphism $f : (X, q \circ f) \rightarrow (Y, q)$ in $C_{/S}$.

Proof. Straightforward. ◆

Properties of
morphisms
right ortho-
gonal to a class
of morphisms

PROPOSITION. Let \mathcal{L} be a subset of $\text{mor } C$.

- (i) Every isomorphism in C is a member of \mathcal{L}^\perp .
- (ii) The full subcategory of $(C \downarrow C)$ corresponding to \mathcal{L}^\perp is closed under limits of all diagrams.
- (iii) \mathcal{L}^\perp is closed under pullback in C .
- (iv) Given morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ in C , assuming $g : Y \rightarrow Z$ is a member of \mathcal{L}^\perp , $f : X \rightarrow Y$ is a member of \mathcal{L}^\perp if and only if $g \circ f : X \rightarrow Z$ is a member of \mathcal{L}^\perp .
- (v) Given a morphism $f : X \rightarrow Y$ in C , assuming Y is right orthogonal to \mathcal{L} , X is right orthogonal to \mathcal{L} if and only if $f : X \rightarrow Y$ is a member of \mathcal{L}^\perp .

Proof. (i) and (ii). Apply [proposition I.I.13](#) to [lemma I.I.14](#).

(iii) and (iv). Use the pullback pasting lemma.

(v). This is a consequence of the fact that the pullback of a bijection is again a bijection. ■

I.I.15 REMARK. In particular, for any subset $\mathcal{L} \subseteq \text{mor } C$, if $\mathcal{L}^\perp \subseteq \text{qdbl } C$, then \mathcal{L}^\perp is a class of separated fibrations in C .

I.I.16 ¶ It is sometimes useful to weaken the notion of right orthogonality by replacing ‘bijection’ with ‘injection’.

Right orthogonality of diagonals LEMMA. *Let $f : X \rightarrow Y$ be a morphism in \mathcal{C} and let S be an object in \mathcal{C} . Assuming the product $S \times S$ exists in \mathcal{C} , the following are equivalent:*

(i) *The induced map*

$$C(f, S) : C(Y, S) \rightarrow C(X, S)$$

is injective.

(ii) *The diagonal $\Delta_S : S \rightarrow S \times S$ is right orthogonal to $f : X \rightarrow Y$.*

Proof. Consider the diagram below:

$$\begin{array}{ccc} C(Y, S) & \xrightarrow{\Delta_{C(Y, S)}} & C(Y, S) \times C(Y, S) \\ C(f, S) \downarrow & & \downarrow C(f, S) \times C(f, S) \\ C(X, S) & \xrightarrow{\Delta_{C(X, S)}} & C(X, S) \times C(X, S) \end{array}$$

Clearly, this is a pullback square if and only if $\Delta_S : S \rightarrow S \times S$ is right orthogonal to $f : X \rightarrow Y$. On the other hand, this is a pullback square if and only if $C(f, S) : C(Y, S) \rightarrow C(X, S)$ is injective, so we are done. ■

1.2 Local properties of morphisms

SYNOPSIS. We consider variations on what it means for a morphism to have a property locally with respect to a coverage.

PREREQUISITES. §§ [I.1](#), [A.1](#), [A.2](#).

I.2.1 ¶ In general, given a topological space X and a property P of topological spaces, we say that X has property P locally if, for every open subspace $U \subseteq X$, there is a cover Φ of U such that every element of Φ has property P .

On the other hand, in the relative setting, there are at least two possible ways to interpret ‘locally’. For instance, given a continuous map $f : X \rightarrow Y$ and a property P of continuous maps, we may say that $f : X \rightarrow Y$ has property P locally on the domain if, for every open subspace $U \subseteq X$, there is a cover Φ of U such that, for every $U' \in \Phi$, the restriction $f : U' \rightarrow Y$ has property P . Or, we may say that $f : X \rightarrow Y$ has property P locally on the base if, for every open subspace $V \subseteq Y$, there is a cover Ψ of V such that, for every $V' \in \Psi$, the restriction $f : f^{-1}V' \rightarrow V'$ has property P . We could go even further by combining the two interpretations.

In this section, we study generalisations of these ideas in the abstract setting of a category with a coverage. It should be noted that the notion of having a property locally on the base is straightforwardly generalised to objects in a fibred category, as is the notion of having a property locally on the domain, but combining the two is difficult. As such, we will only discuss the case of morphisms. This is partially a generalisation of earlier work by Joyal and Moerdijk [[JM](#), §§ 1 and 5].

I.2.2 ※ Throughout this section:

- C is a category.
- J is a coverage on C .
- B is a set of morphisms in C containing all identity morphisms and closed under composition.

- \mathcal{G} is a set of morphisms in \mathcal{C} .

I.2.3(a) DEFINITION. The class \mathcal{G} is **\mathcal{B} -sifted** if it has the following property:

- If $f : X \rightarrow Y$ is a member of \mathcal{B} and $g : Y \rightarrow Z$ is a member of \mathcal{G} , then the composite $g \circ f : X \rightarrow Z$ is also a member of \mathcal{G} .

I.2.3(b) DEFINITION. The class \mathcal{G} is **\mathcal{B} -cosifted** if it has the following property:

- If $f : X \rightarrow Y$ is a member of \mathcal{G} and $g : Y \rightarrow Z$ is a member of \mathcal{B} , then the composite $g \circ f : X \rightarrow Z$ is also a member of \mathcal{G} .

I.2.3(c) DEFINITION. The class \mathcal{G} is **\mathcal{B} -bisifted** if it is both \mathcal{B} -sifted and \mathcal{B} -cosifted.

EXAMPLE. \mathcal{B} itself is \mathcal{B} -bisifted.

I.2.4 DEFINITION. A morphism $f : X \rightarrow Y$ in \mathcal{C} is **of \mathcal{G} -type $(\mathcal{B}, \mathcal{J})$ -semilocally on the domain** if it has the following property:

- There is a \mathcal{J} -covering \mathcal{B} -sink Φ on X such that, for every $(U, x) \in \Phi$, $f \circ x : U \rightarrow Y$ is a member of \mathcal{G} .

A morphism in \mathcal{C} is **of \mathcal{G} -type \mathcal{J} -semilocally on the domain** if it is of \mathcal{G} -type $(\text{mor } \mathcal{C}, \mathcal{J})$ -semilocally on the domain.

Properties of morphisms of a given type semilocally on the domain **PROPOSITION.** *Let $\hat{\mathcal{G}}$ be the class of morphisms in \mathcal{C} of \mathcal{G} -type $(\mathcal{B}, \mathcal{J})$ -semilocally on the domain.*

(i) *We have $\mathcal{G} \subseteq \hat{\mathcal{G}}$.*

(ii) *If \mathcal{G} is \mathcal{B} -cosifted, then $\hat{\mathcal{G}}$ is also \mathcal{B} -cosifted.*

(iii) *Assuming \mathcal{B} is a quadrable class of morphisms in \mathcal{C} , if \mathcal{G} is \mathcal{B} -sifted, then $\hat{\mathcal{G}}$ is also \mathcal{B} -sifted.*

(iv) *Assuming both \mathcal{B} and \mathcal{G} are quadrable classes of morphisms in \mathcal{C} , for every pullback square in \mathcal{C} of the form below,*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ f' \downarrow & & \downarrow f \\ Y' & \longrightarrow & Y \end{array}$$

if $f : X \rightarrow Y$ is a member of $\hat{\mathcal{G}}$, then $f' : X' \rightarrow Y'$ is also a member of $\hat{\mathcal{G}}$.

(v) Every morphism in C of $\hat{\mathcal{G}}$ -type $(\mathcal{B}, \mathcal{J})$ -semilocally on the domain is a member of $\hat{\mathcal{G}}$.

Proof. Straightforward. (For (iii)–(v), use [proposition A.2.14.](#)) ◆

1.2.5 DEFINITION. A morphism $f : X \rightarrow Y$ in C is **of \mathcal{G} -type $(\mathcal{B}, \mathcal{J})$ -locally on the domain** if it has the following property:

- For every object (U, x) in $C_{/X}$, if $x : U \rightarrow X$ is a member of \mathcal{B} , then $f \circ x : U \rightarrow Y$ is of \mathcal{G} -type $(\mathcal{B}, \mathcal{J})$ -semilocally on the domain.

A morphism in C is **of \mathcal{G} -type \mathcal{J} -locally on the domain** if it is of \mathcal{G} -type $(\text{mor } C, \mathcal{J})$ -locally on the domain.

Properties of
morphisms of a
given type locally
on the domain

PROPOSITION. Let $\hat{\mathcal{G}}$ be the class of morphisms in C of \mathcal{G} -type $(\mathcal{B}, \mathcal{J})$ -locally on the domain.

- (i) Every member of $\hat{\mathcal{G}}$ is a morphism in C of \mathcal{G} -type $(\mathcal{B}, \mathcal{J})$ -semilocally on the domain.
- (ii) $\hat{\mathcal{G}}$ is \mathcal{B} -sifted.
- (iii) If \mathcal{G} is \mathcal{B} -sifted, then $\mathcal{G} \subseteq \hat{\mathcal{G}}$.
- (iv) If \mathcal{G} is \mathcal{B} -cosifted, then $\hat{\mathcal{G}}$ is also \mathcal{B} -cosifted.
- (v) Assuming \mathcal{G} is \mathcal{B} -sifted and both \mathcal{B} and \mathcal{G} are quadrable classes of morphisms in C , for every pullback square in C of the form below,

$$\begin{array}{ccc} X' & \longrightarrow & X \\ f' \downarrow & & \downarrow f \\ Y' & \longrightarrow & Y \end{array}$$

if $f : X \rightarrow Y$ is a member of $\hat{\mathcal{G}}$, then $f' : X' \rightarrow Y'$ is also a member of $\hat{\mathcal{G}}$.

- (vi) Assuming both \mathcal{B} and \mathcal{G} are quadrable classes of morphisms in C , if \mathcal{G} is closed under composition, then the class of morphisms in C of \mathcal{G} -type $(\mathcal{B}, \mathcal{J})$ -locally on the domain is also closed under composition.
- (vii) Every morphism in C of $\hat{\mathcal{G}}$ -type $(\mathcal{B}, \mathcal{J})$ -locally on the domain is a member of $\hat{\mathcal{G}}$.

Proof. (i)–(iv). Straightforward.

(v). Use the pullback pasting lemma to reduce to [proposition 1.2.4](#).

(vi). Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be morphisms in \mathcal{C} of \mathcal{G} -type $(\mathcal{B}, \mathcal{J})$ -locally on the domain. Then there is a \mathcal{J} -covering \mathcal{B} -sink Φ on X such that, for every $(U, x) \in \Phi$, $f \circ x : U \rightarrow Y$ is a member of \mathcal{G} , and there is a \mathcal{J} -covering \mathcal{B} -sink Ψ on Y with the same property *mutatis mutandis*. For each $(U, x) \in \Phi$ and each $(V, y) \in \Psi$, we may choose a pullback square in \mathcal{C} of the form below,

$$\begin{array}{ccc} T & \xrightarrow{u} & U \\ v \downarrow & & \downarrow f \circ x \\ V & \xrightarrow{y} & Y \end{array}$$

where $u : T \rightarrow U$ is a member of \mathcal{B} and $v : T \rightarrow V$ is a member of \mathcal{G} ; thus, by hypothesis, $x \circ u : T \rightarrow X$ is a member of \mathcal{B} and $g \circ f \circ x \circ u = g \circ y \circ u : T \rightarrow Z$ is a member of \mathcal{G} . Hence, by [proposition A.2.14](#), $g \circ f : X \rightarrow Z$ is of \mathcal{G} -type $(\mathcal{B}, \mathcal{J})$ -locally on the domain.

(vii). Let \mathcal{G}_2 be the class of morphisms in \mathcal{C} that are of \mathcal{G} -type $(\mathcal{B}, \mathcal{J})$ -semilocally on the domain. By (i), $\hat{\mathcal{G}} \subseteq \mathcal{G}_2$, and we know that every morphism in \mathcal{C} of \mathcal{G}_2 -type $(\mathcal{B}, \mathcal{J})$ -semilocally on the domain is a member of \mathcal{G}_2 , so every morphism in \mathcal{C} of $\hat{\mathcal{G}}$ -type $(\mathcal{B}, \mathcal{J})$ -semilocally on the domain is also a member of \mathcal{G}_2 . Thus, every morphism in \mathcal{C} of $\hat{\mathcal{G}}$ -type $(\mathcal{B}, \mathcal{J})$ -locally on the domain is a member of $\hat{\mathcal{G}}$. ■

1.2.6 LEMMA. *Let $f : X \rightarrow Y$ be a morphism in \mathcal{C} . If \mathcal{B} is a quadrable class of morphisms in \mathcal{C} and \mathcal{G} is a \mathcal{B} -sifted class of morphisms in \mathcal{C} , then the following are equivalent:*

When semi-
locally on
the domain
implies locally
on the domain

(i) $f : X \rightarrow Y$ is of \mathcal{G} -type $(\mathcal{B}, \mathcal{J})$ -locally on the domain.

(ii) $f : X \rightarrow Y$ is of \mathcal{G} -type $(\mathcal{B}, \mathcal{J})$ -semilocally on the domain.

Proof. (i) \Rightarrow (ii). Immediate.

(ii) \Rightarrow (i). Let $f : X \rightarrow Y$ be a member of \mathcal{B} and let $g : Y \rightarrow Z$ be a morphism in \mathcal{C} of \mathcal{G} -type $(\mathcal{B}, \mathcal{J})$ -semilocally on the domain. We must show that $g \circ f : X \rightarrow Z$ is of \mathcal{G} -type $(\mathcal{B}, \mathcal{J})$ -semilocally on the domain.

There is a \mathcal{J} -covering \mathcal{B} -sink Ψ on Y such that, for every $(V, y) \in \Psi$, $g \circ y : V \rightarrow Z$ is a member of \mathcal{G} . For each $(V, y) \in \Psi$, choose a pullback square in \mathcal{C} of the form below:

$$\begin{array}{ccc} f^*V & \longrightarrow & V \\ f^*y \downarrow & & \downarrow y \\ X & \xrightarrow{f} & Y \end{array}$$

Since \mathcal{G} is \mathcal{B} -sifted, $g \circ f \circ f^*y : f^*V \rightarrow Z$ is a member of \mathcal{G} . Moreover, by [proposition A.2.14](#), $\{(f^*V, f^*y) \mid (V, y) \in \Psi\}$ is a \mathcal{J} -covering \mathcal{B} -sink on X . The claim follows. \blacksquare

I.2.7 DEFINITION. A morphism $f : X \rightarrow Y$ in $\mathbf{FAM}(\mathcal{C})$ is of \mathcal{G} -type if it has the following property:

- For every $(j, i) \in \text{idx } f$, the morphism $f(j, i) : X(i) \rightarrow Y(j)$ is a member of \mathcal{G} .

Properties of
matrices of
morphisms of
a given type

PROPOSITION.

- If every isomorphism in \mathcal{C} is a member of \mathcal{G} , then every coproduct injection in $\mathbf{FAM}(\mathcal{C})$ is of \mathcal{G} -type.
- If \mathcal{G} is closed under composition, then the class of morphisms in $\mathbf{FAM}(\mathcal{C})$ of \mathcal{G} -type is also closed under composition.
- If \mathcal{G} is a quadrable class of morphisms in \mathcal{C} , then the class of morphisms in $\mathbf{FAM}(\mathcal{C})$ of \mathcal{G} -type is a quadrable class of morphisms in $\mathbf{FAM}(\mathcal{C})$.

Proof. Straightforward. \blacklozenge

I.2.8 DEFINITION. A morphism $h : A \rightarrow B$ in $\mathbf{PSH}(\mathcal{C})$ is **familially of \mathcal{G} -type** if there is a pair (Φ, Ψ) with the following properties:

- Φ is a familial representation of A .
- Ψ is a familial representation of B .
- For each $(X, a) \in \Phi$, there exist a unique $(Y, b) \in \Psi$ and a unique morphism $f : X \rightarrow Y$ in \mathcal{C} such that $h(a) = b \cdot f$ and $f : X \rightarrow Y$ is a member of \mathcal{G} .

Recognition principle for morphisms familially of a given type

LEMMA. *Let $h : A \rightarrow B$ be a morphism in $\mathbf{PSH}(C)$. The following are equivalent:*

- (i) *The morphism $h : A \rightarrow B$ is familially of \mathcal{G} -type.*
- (ii) *There is a morphism $f : X \rightarrow Y$ in $\mathbf{FAM}(C)$ of \mathcal{G} -type such that $(\coprod X, \coprod Y, \coprod f) \cong (A, B, h)$ (in $(\mathbf{PSH}(C) \downarrow \mathbf{PSH}(C))$).*

Proof. Straightforward. ◆

1.2.9 DEFINITION. A morphism $h : A \rightarrow B$ in $\mathbf{PSH}(C)$ is **of \mathcal{G} -type J -semi-locally on the base** if there is a J -local generating set Ψ of elements of B that satisfies the following condition:

- For each $(Y, b) \in \Psi$, there is a representation $(X, (f, a))$ of $\text{Pb}(b \cdot -, h)$ such that the morphism $f : X \rightarrow Y$ in C is a member of \mathcal{G} .

EXAMPLE. If $f : X \rightarrow Y$ is a member of \mathcal{G} , then $h_f : h_X \rightarrow h_Y$ is of \mathcal{G} -type semilocally on the base.

Recognition principle for morphisms of presheaves of a given type semilocally on the base

LEMMA. *Let $h : A \rightarrow B$ be a morphism in $\mathbf{PSH}(C)$. The following are equivalent:*

- (i) *The morphism $h : A \rightarrow B$ is of \mathcal{G} -type semilocally on the base.*
- (ii) *There exist a family $(f_i \mid i \in I)$ where each f_i is a morphism $X_i \rightarrow Y_i$ in C that is a member of \mathcal{G} and a pullback square in $\mathbf{PSH}(C)$ of the form below,*

$$\begin{array}{ccc} \coprod_{i \in I} h_{X_i} & \longrightarrow & A \\ \coprod_{i \in I} h_f \downarrow & & \downarrow h \\ \coprod_{i \in I} h_{Y_i} & \longrightarrow & B \end{array}$$

where $\coprod_{i \in I} h_{Y_i} \twoheadrightarrow B$ is J -locally surjective.

Proof. Straightforward. ◆

Properties of morphisms of presheaves of a given type semilocally on the base

PROPOSITION.

- (i) *The class of morphisms in $\mathbf{PSH}(C)$ of \mathcal{G} -type J -semilocally on the base is closed under (possibly infinitary) coproduct in $\mathbf{PSH}(C)$.*
- (ii) *Given a pullback square in $\mathbf{PSH}(C)$ of the form below,*

$$\begin{array}{ccc} \tilde{A} & \longrightarrow & A \\ \tilde{h} \downarrow & & \downarrow h \\ \tilde{B} & \longrightarrow & B \end{array}$$

where $\tilde{B} \twoheadrightarrow B$ is J -locally surjective, if $\tilde{h} : \tilde{A} \rightarrow \tilde{B}$ is of \mathcal{G} -type J -semilocally on the base, then $h : A \rightarrow B$ is also of \mathcal{G} -type J -semilocally on the base.

Proof. Straightforward. ◆

1.2.10 DEFINITION. A morphism $h : A \rightarrow B$ in $\mathbf{PSH}(C)$ is **of \mathcal{G} -type J -locally on the base** if it has the following property:

- For every element (Y, b) of B , the projection $\text{Pb}(b \cdot -, h) \rightarrow h_Y$ is of \mathcal{G} -type J -semilocally on the base.

EXAMPLE. Assuming \mathcal{G} is a quadrable class of morphisms in C , if $f : X \rightarrow Y$ is a member of \mathcal{G} , then $h_f : h_X \rightarrow h_Y$ is of \mathcal{G} -type J -locally on the base.

Properties of morphisms of presheaves of a given type locally on the base

PROPOSITION.

- (i) *Every morphism in $\mathbf{PSH}(C)$ that is of \mathcal{G} -type J -locally on the base is also of \mathcal{G} -type J -semilocally on the base.*
- (ii) *The class of morphisms in $\mathbf{PSH}(C)$ of \mathcal{G} -type J -locally on the base is a quadrable class of morphisms in $\mathbf{PSH}(C)$.*
- (iii) *The class of morphisms in $\mathbf{PSH}(C)$ of \mathcal{G} -type J -locally on the base is closed under (possibly infinitary) coproduct in $\mathbf{PSH}(C)$.*
- (iv) *If every identity morphism in C is a member of \mathcal{G} , then every isomorphism in $\mathbf{PSH}(C)$ is of \mathcal{G} -type J -locally on the base.*

(v) Given a pullback square in $\mathbf{PSH}(C)$ of the form below,

$$\begin{array}{ccc} \tilde{A} & \longrightarrow & A \\ \tilde{h} \downarrow & & \downarrow h \\ \tilde{B} & \longrightarrow & B \end{array}$$

where $\tilde{B} \rightarrow B$ is J -locally surjective, if $\tilde{h} : \tilde{A} \rightarrow \tilde{B}$ is of \mathcal{G} -type J -locally on the base, then $h : A \rightarrow B$ is also of \mathcal{G} -type J -locally on the base.

Proof. (i). Apply lemma 1.2.13, proposition A.2.14, and the pullback pasting lemma.

(ii)–(iv). Straightforward.

(v). Let (Y, b) be an element of B and \mathcal{V} be the sieve on Y consisting of the objects (V, y) in $C_{/Y}$ such that $b \cdot y$ is in the image of $\tilde{B} \rightarrow B$. By construction, \mathcal{V} is a J -covering sieve on Y . Since $\tilde{h} : \tilde{A} \rightarrow \tilde{B}$ is of \mathcal{G} -type J -locally on the base, we may then apply the pullback pasting lemma and proposition 1.2.9 to deduce that $\text{Pb}(b \cdot -, h) \rightarrow \mathcal{h}_Y$ is of \mathcal{G} -type J -semilocally on the base. \blacksquare

1.2.11 LEMMA. Let $h : A \rightarrow B$ be a morphism in $\mathbf{PSH}(C)$. If \mathcal{G} is a quadrable class of morphisms in C , then the following are equivalent:

When semilocally on the base implies locally on the base

- (i) The morphism $h : A \rightarrow B$ is of \mathcal{G} -type J -locally on the base.
- (ii) The morphism $h : A \rightarrow B$ is of \mathcal{G} -type J -semilocally on the base.
- (iii) The subpresheaf $B' \subseteq B$ is J -dense, where B' consists of the elements (T, b) of B such that there exist a morphism $t : U \rightarrow T$ in C that is a member of \mathcal{G} and a pullback square in $\mathbf{PSH}(C)$ of the form below:

$$\begin{array}{ccc} \mathcal{h}_U & \longrightarrow & A \\ t \circ - \downarrow & & \downarrow h \\ \mathcal{h}_T & \xrightarrow{b \cdot -} & B \end{array}$$

Proof. Apply proposition A.2.14 and the pullback pasting lemma. \blacksquare

1.2.12 DEFINITION. A morphism $f : X \rightarrow Y$ in C is of \mathcal{G} -type **J-locally on the base** (resp. of \mathcal{G} -type **J-semilocally on the base**) if $h_f : h_X \rightarrow h_Y$ is a morphism in $\mathbf{PSH}(C)$ that is of \mathcal{G} -type J-locally on the base (resp. of \mathcal{G} -type J-semilocally on the base).

Properties of
morphisms of a
given type locally
on the base

PROPOSITION. Let $\hat{\mathcal{G}}$ be the class of morphisms in C of \mathcal{G} -type J-locally on the base.

- (i) If every isomorphism in C is a member of \mathcal{G} , then every isomorphism in C is also a member of $\hat{\mathcal{G}}$.
- (ii) If \mathcal{G} is a quadrable class of morphisms in C , then $\mathcal{G} \subseteq \hat{\mathcal{G}}$.
- (iii) For every pullback square in C of the form below,

$$\begin{array}{ccc} X' & \longrightarrow & X \\ f' \downarrow & & \downarrow f \\ Y' & \longrightarrow & Y \end{array}$$

if $f : X \rightarrow Y$ is a member of $\hat{\mathcal{G}}$, then $f' : X' \rightarrow Y'$ is also a member of $\hat{\mathcal{G}}$.

- (iv) Every morphism in $\mathbf{PSH}(C)$ that is of $\hat{\mathcal{G}}$ -type J-locally on the base is also of \mathcal{G} -type J-locally on the base.

Proof. Apply proposition 1.2.10. ■

1.2.13 DEFINITION. A morphism $h : A \rightarrow B$ in $\mathbf{PSH}(C)$ is **J-semilocally of \mathcal{G} -type** if there is a J-local generating set Ψ of elements of B that satisfies the following condition:

- For each $(Y, b) \in \Psi$, there is a J-local generating set $\Phi_{(Y,b)}$ of elements of $\text{Pb}(b \cdot -, h)$ such that, for every $(X, (f, a)) \in \Phi_{(Y,b)}$, the morphism $f : X \rightarrow Y$ in C is a member of \mathcal{G} .

EXAMPLE. If $f : X \rightarrow Y$ is a member of \mathcal{G} , then $h_f : h_X \rightarrow h_Y$ is J-semilocally of \mathcal{G} -type.

Recognition principle for morphisms of presheaves semilocally of a given type

LEMMA. *Let $h : A \rightarrow B$ be a morphism in $\mathbf{PSH}(C)$. The following are equivalent:*

- (i) *The morphism $h : A \rightarrow B$ is \mathcal{G} -semilocally of \mathcal{G} -type.*
- (ii) *There is a \mathcal{J} -weak pullback square in $\mathbf{PSH}(C)$ of the form below,*

$$\begin{array}{ccc} \coprod X & \longrightarrow & A \\ \Downarrow f & & \downarrow h \\ \coprod Y & \longrightarrow & B \end{array}$$

where $f : X \rightarrow Y$ is a morphism in $\mathbf{FAM}(C)$ of \mathcal{G} -type and $\coprod Y \rightarrow B$ is \mathcal{J} -locally surjective.

Proof. Straightforward. ◆

Properties of morphisms of presheaves semilocally of a given type

PROPOSITION.

- (i) *Every morphism in $\mathbf{PSH}(C)$ that is a family of \mathcal{G} -type is also \mathcal{J} -semilocally of \mathcal{G} -type.*
- (ii) *The class of morphisms in $\mathbf{PSH}(C)$ \mathcal{J} -semilocally of \mathcal{G} -type is closed under (possibly infinitary) coproduct in $\mathbf{PSH}(C)$.*
- (iii) *Given a \mathcal{J} -weak pullback square in $\mathbf{PSH}(C)$ of the form below,*

$$\begin{array}{ccc} \tilde{A} & \longrightarrow & A \\ \tilde{h} \downarrow & & \downarrow h \\ \tilde{B} & \longrightarrow & B \end{array}$$

where $\tilde{B} \rightarrow B$ is \mathcal{J} -locally surjective, if $\tilde{h} : \tilde{A} \rightarrow \tilde{B}$ is \mathcal{J} -semilocally of \mathcal{G} -type, then $h : A \rightarrow B$ is also \mathcal{J} -semilocally of \mathcal{G} -type.

Proof. Straightforward. (Use [lemma 1.2.13](#) and [proposition A.2.14](#); for (iii), also use the weak pullback pasting lemma ([lemma A.2.19](#)).) ◆

1.2.14 **DEFINITION.** A morphism $h : A \rightarrow B$ in $\mathbf{PSH}(C)$ is **\mathcal{J} -locally of \mathcal{G} -type** if it has the following property:

- For every element (Y, b) of B , the projection $\text{Pb}(b \cdot -, h) \rightarrow \hat{h}_Y$ is \mathcal{J} -semilocally of \mathcal{G} -type.

EXAMPLE. Assuming \mathcal{G} is a quadrable class of morphisms in \mathcal{C} , if $f : X \rightarrow Y$ is a member of \mathcal{G} , then $h_f : h_X \rightarrow h_Y$ is J-locally of \mathcal{G} -type.

Properties of
morphisms
of presheaves
locally of a
given type

PROPOSITION.

- (i) Every morphism in $\mathbf{PSH}(\mathcal{C})$ that is J-locally of \mathcal{G} -type is also J-semilocally of \mathcal{G} -type.
- (ii) The class of morphisms in $\mathbf{PSH}(\mathcal{C})$ J-locally of \mathcal{G} -type is a quadrable class of morphisms in $\mathbf{PSH}(\mathcal{C})$.
- (iii) The class of morphisms in $\mathbf{PSH}(\mathcal{C})$ J-locally of \mathcal{G} -type is closed under (possibly infinitary) coproduct in $\mathbf{PSH}(\mathcal{C})$.
- (iv) Assuming every identity morphism in \mathcal{C} is a member of \mathcal{G} , for every presheaf A on \mathcal{C} and every set I , the codiagonal $\coprod_{i \in I} A \rightarrow A$ is J-locally of \mathcal{G} -type.
- (v) If every identity morphism in \mathcal{C} is a member of \mathcal{G} , then every isomorphism in $\mathbf{PSH}(\mathcal{C})$ is J-locally of \mathcal{G} -type.
- (vi) Given a J-weak pullback square in $\mathbf{PSH}(\mathcal{C})$ of the form below,

$$\begin{array}{ccc} \tilde{A} & \longrightarrow & A \\ \tilde{h} \downarrow & & \downarrow h \\ \tilde{B} & \longrightarrow & B \end{array}$$

where $\tilde{B} \twoheadrightarrow B$ is J-locally surjective, if $\tilde{h} : \tilde{A} \rightarrow \tilde{B}$ is J-locally of \mathcal{G} -type, then $h : A \rightarrow B$ is also J-locally of \mathcal{G} -type.

- (vii) If every identity morphism in \mathcal{C} is a member of \mathcal{G} , then every J-locally bijective morphism in $\mathbf{PSH}(\mathcal{C})$ is J-locally of \mathcal{G} -type.

Proof. (i). Apply lemma 1.2.13, proposition A.2.14, and the weak pullback pasting lemma (lemma A.2.19).

(ii)–(v). Straightforward.

(vi). Let (Y, b) be an element of B and \mathcal{V} be the sieve on Y consisting of the objects (V, y) in $C_{/Y}$ such that $b \cdot y$ is in the image of $\tilde{B} \rightarrow B$. By construction, \mathcal{V} is a J-covering sieve on Y . Since $\tilde{h} : \tilde{A} \rightarrow \tilde{B}$ is J-locally of \mathcal{G} -type, we may then apply the weak pullback pasting lemma

and [proposition 1.2.13](#) to deduce that $\text{Pb}(b \cdot -, h) \rightarrow h_Y$ is J-semilocally of \mathcal{G} -type.

(vii). Let $h : A \rightarrow B$ be a J-locally bijective morphism of presheaves on C . Then the following is a J-weak pullback square in $\mathbf{PSH}(C)$:

$$\begin{array}{ccc} A & \xrightarrow{\text{id}} & A \\ \text{id} \downarrow & & \downarrow h \\ A & \xrightarrow{h} & B \end{array}$$

Thus, by (vi), the claim reduces to (v). ■

1.2.15 LEMMA. *Let $h : A \rightarrow B$ be a morphism in $\mathbf{PSH}(C)$. If \mathcal{G} is a quadrable class of morphisms in C , then the following are equivalent:*

When semi-locally implies locally

- (i) *The morphism $h : A \rightarrow B$ is J-locally of \mathcal{G} -type.*
- (ii) *The morphism $h : A \rightarrow B$ is J-semilocally of \mathcal{G} -type.*

Proof. Apply [proposition A.2.14](#) and the weak pullback pasting lemma ([lemma A.2.19](#)). ■

1.2.16 ¶ The following definition is a variation on the collection axiom introduced in [[JM](#), § 1].

DEFINITION. The **J-local collection axiom** for \mathcal{G} is the following:

- Given a morphism $f : X \rightarrow Y$ in C and a J-covering sieve \mathcal{U} on X , if $f : X \rightarrow Y$ is a member of \mathcal{G} , then there is a J-covering sink Ψ on Y such that, for each $(T, y) \in \Psi$, there is a J-local generating set $\Phi_{(T,y)}$ of elements of $\text{Pb}(y \circ -, f \circ -)$ such that, for each $(U, (t, x)) \in \Phi_{(T,y)}$, $t : U \rightarrow T$ is a member of \mathcal{G} and (U, x) is in \mathcal{U} .

REMARK. Assuming \mathcal{G} is closed under composition, if every J-covering sink contains a J-covering \mathcal{G} -sink, then \mathcal{G} satisfies the J-local collection axiom.

Alternative
criteria for the
local collec-
tion axiom

LEMMA. *The following are equivalent:*

- (i) \mathcal{G} satisfies the J-local collection axiom.
- (ii) For every morphism $f : X \rightarrow Y$ in \mathcal{C} , if $f : X \rightarrow Y$ is a member of \mathcal{G} , then for every J-covering sieve \mathcal{U} on X , there is a J-weak pullback square in $\mathbf{PSH}(\mathcal{C})$ of the form below,

$$\begin{array}{ccc} \coprod X' & \xrightarrow{p} & h_X \\ \coprod f' \downarrow & & \downarrow f \circ - \\ \coprod Y' & \xrightarrow{q} & h_Y \end{array}$$

where $q : \coprod Y' \rightarrow h_Y$ is J-locally surjective, $f' : X' \rightarrow Y'$ is a morphism in $\mathbf{FAM}(\mathcal{C})$ of \mathcal{G} -type, and for every element (T, x') of $\coprod X'$, $(T, p(x'))$ is in \mathcal{U} .

Proof. Straightforward. ◆

1.2.17
Composition
of morphisms
of presheaves
locally of a
given type

PROPOSITION. *If \mathcal{G} is a quadrable class of morphisms in \mathcal{C} that satisfies the J-collection axiom and is closed under composition, then the class of morphisms in $\mathbf{PSH}(\mathcal{C})$ J-locally of \mathcal{G} -type is also closed under composition.*

Proof. In view of [proposition 1.2.14](#), it suffices to prove the following statement:

- Given morphisms $h : A \rightarrow B$ and $k : B \rightarrow C$ in $\mathbf{PSH}(\mathcal{C})$, if $h : A \rightarrow B$ is J-locally of \mathcal{G} -type and $k : B \rightarrow C$ is J-semilocally of \mathcal{G} -type, then the composite $k \circ h : A \rightarrow C$ is also J-semilocally of \mathcal{G} -type.

So suppose $h : A \rightarrow B$ is J-locally of \mathcal{G} -type and $k : B \rightarrow C$ is J-semilocally of \mathcal{G} -type. By [lemma 1.2.13](#), there is a J-weak pullback square in $\mathbf{PSH}(\mathcal{C})$ of the form below,

$$\begin{array}{ccc} \coprod Y & \longrightarrow & B \\ \coprod g \downarrow & & \downarrow k \\ \coprod Z & \longrightarrow & C \end{array}$$

where $\coprod Z \rightarrow C$ is J-locally surjective and $g : Y \rightarrow Z$ is a morphism in $\mathbf{FAM}(\mathcal{C})$ of \mathcal{G} -type. The projection $(\coprod Y) \times_B A \rightarrow \coprod Y$ is also J-semilocally

of \mathcal{G} -type, so there is a J-weak pullback square in $\mathbf{Psh}(C)$ of the form below,

$$\begin{array}{ccc} \coprod X' & \longrightarrow & (\coprod Y) \times_B A \\ \coprod f' \downarrow & & \downarrow \\ \coprod Y' & \longrightarrow & \coprod Y \end{array}$$

satisfying the conditions *mutatis mutandis*. Then, using [lemma 1.2.16](#) and the hypothesis that \mathcal{G} satisfies the J-local collection axiom, we may find a J-weak pullback square in $\mathbf{Psh}(C)$ of the form below,

$$\begin{array}{ccc} \coprod Y'' & \longrightarrow & \coprod Y \\ \coprod g'' \downarrow & & \downarrow \coprod g \\ \coprod Z'' & \longrightarrow & \coprod Z \end{array}$$

where $\coprod Z'' \rightarrow \coprod Z$ is J-locally surjective, $Y'' \rightarrow Y$ factors as a morphism $Y'' \rightarrow Y'$ in $\mathbf{FAM}(C)$ followed by the J-covering morphism $Y' \rightarrow Y$ in $\mathbf{FAM}(C)$, and $g'' : Y'' \rightarrow Z''$ is a morphism in $\mathbf{FAM}(C)$ of \mathcal{G} -type. Since \mathcal{G} is a quadrable class of morphisms in C , by [proposition 1.2.7](#), there is a pullback square in $\mathbf{Psh}(C)$ of the form below,

$$\begin{array}{ccc} \coprod X'' & \longrightarrow & \coprod X' \\ \coprod f'' \downarrow & & \downarrow \coprod f' \\ \coprod Y'' & \longrightarrow & \coprod Y' \end{array}$$

where $f'' : X'' \rightarrow Y''$ is a morphism in $\mathbf{FAM}(C)$ of \mathcal{G} -type. Hence, we obtain a commutative diagram in $\mathbf{Psh}(C)$ of the form below,

$$\begin{array}{ccc} \coprod X'' & \longrightarrow & A \\ \coprod g'' \downarrow & & \downarrow h \\ \coprod Y'' & \longrightarrow & B \\ \coprod f'' \downarrow & & \downarrow k \\ \coprod Z'' & \longrightarrow & C \end{array}$$

where, by the weak pullback pasting lemma ([lemma A.2.19](#)), both squares and the outer rectangle are all J-weak pullback diagrams in $\mathbf{Psh}(C)$. Since the horizontal arrows are J-locally surjective, it follows from the hypothesis that \mathcal{G} is closed under composition that $k \circ h : A \rightarrow C$ is J-semilocally of \mathcal{G} -type, as claimed. ■

1.2.18 DEFINITION. A morphism $f : X \rightarrow Y$ in \mathcal{C} is **J-locally of \mathcal{G} -type** (resp. **J-semilocally of \mathcal{G} -type**) if $h_f : h_X \rightarrow h_Y$ is a morphism in $\mathbf{PSH}(\mathcal{C})$ that is J-locally of \mathcal{G} -type (resp. J-semilocally of \mathcal{G} -type).

LEMMA. Let $r : F \rightarrow A$ be a J-locally surjective morphism in $\mathbf{PSH}(\mathcal{C})$ and let $h : A \rightarrow B$ be a morphism in $\mathbf{PSH}(\mathcal{C})$ J-locally of \mathcal{G} -type. Assuming \mathcal{G} satisfies the J-local collection axiom, there is a J-weak pullback square in $\mathbf{PSH}(\mathcal{C})$ of the form below,

$$\begin{array}{ccc} \tilde{A} & \xrightarrow{p} & A \\ \tilde{h} \downarrow & & \downarrow h \\ \tilde{B} & \xrightarrow{q} & B \end{array}$$

where $q : \tilde{B} \twoheadrightarrow B$ is J-locally surjective, $p : \tilde{A} \rightarrow A$ factors through $r : F \rightarrow A$, and $\tilde{h} : \tilde{A} \rightarrow \tilde{B}$ is familially of \mathcal{G} -type.

Proof. By lemma 1.2.13 and proposition 1.2.14, there is a J-weak pullback square in $\mathbf{PSH}(\mathcal{C})$ of the form below,

$$\begin{array}{ccc} \coprod X & \longrightarrow & A \\ \coprod f \downarrow & & \downarrow h \\ \coprod Y & \longrightarrow & B \end{array}$$

where $\coprod Y \twoheadrightarrow B$ is J-locally surjective and $f : X \rightarrow Y$ is a morphism in $\mathbf{FAM}(\mathcal{C})$ of \mathcal{G} -type. By proposition A.2.14, the projection $\coprod X \times_A F \rightarrow \coprod X$ is J-locally surjective, so we may apply lemma 1.2.16 to obtain a J-weak pullback square in $\mathbf{PSH}(\mathcal{C})$ of the form below,

$$\begin{array}{ccc} \coprod X' & \longrightarrow & \coprod X \\ \coprod f' \downarrow & & \downarrow \coprod f \\ \coprod Y' & \longrightarrow & \coprod Y \end{array}$$

where $\coprod Y' \twoheadrightarrow \coprod Y$ is J-locally surjective, $f' : X' \rightarrow Y'$ is a morphism in $\mathbf{FAM}(\mathcal{C})$ of \mathcal{G} -type, and $\coprod X' \twoheadrightarrow \coprod X$ factors through the projection $\coprod X \times_A F \rightarrow \coprod X$. We can then take $\tilde{A} = \coprod X'$, $\tilde{B} = \coprod Y'$, and $\tilde{h} = \coprod f'$ and use the weak pullback pasting lemma (lemma A.2.19) to complete the proof. ■

- Properties of morphisms locally of a given type
- PROPOSITION.** *Let $\hat{\mathcal{G}}$ be the class of morphisms in \mathcal{C} \mathbf{J} -locally of \mathcal{G} -type.*
- (i) *If every isomorphism in \mathcal{C} is a member of \mathcal{G} , then every isomorphism in \mathcal{C} is also a member of $\hat{\mathcal{G}}$.*
 - (ii) *If \mathcal{G} is a quadrable class of morphisms in \mathcal{C} , then $\mathcal{G} \subseteq \hat{\mathcal{G}}$.*
 - (iii) *$\hat{\mathcal{G}}$ is closed under pullback in \mathcal{C} .*
 - (iv) *If \mathcal{G} satisfies the \mathbf{J} -local collection axiom, then $\hat{\mathcal{G}}$ also satisfies the \mathbf{J} -local collection axiom.*
 - (v) *If \mathcal{G} is a quadrable class of morphisms in \mathcal{C} that satisfies the \mathbf{J} -local collection axiom and is closed under composition, then $\hat{\mathcal{G}}$ is also closed under composition.*
 - (vi) *Every morphism in $\mathbf{PSH}(\mathcal{C})$ that is \mathbf{J} -locally of $\hat{\mathcal{G}}$ -type is also \mathbf{J} -locally of \mathcal{G} -type.*

Proof. Apply propositions 1.2.14 and 1.2.17, lemma 1.2.18, and the weak pullback pasting lemma (lemma A.2.19). ■

1.2.19 ¶ The following is a generalisation of Proposition 1.9 in [JM].

- 1.2.19(a) **LEMMA.** *Let $f : X \rightarrow Y$ be a monomorphism in \mathcal{C} . Assuming \mathcal{B} is a quadrable class of morphisms in \mathcal{C} , the following are equivalent:*
- Recognition principle for monomorphisms of a given type semilocally on the domain
- (i) *$f : X \rightarrow Y$ is of \mathcal{B} -type $(\mathcal{B}, \mathbf{J})$ -semilocally on the domain.*
 - (ii) *$f : X \rightarrow Y$ is of \mathcal{B} -type \mathbf{J} -semilocally on the domain.*
- Proof.* (i) \Rightarrow (ii). Immediate.

(ii) \Rightarrow (i). Let Φ be a \mathbf{J} -covering sink on X such that, for every $(U, x) \in \Phi$, $f \circ x : U \rightarrow Y$ is a member of \mathcal{B} . Then, by lemma 1.1.3, $x : U \rightarrow X$ itself is a member of \mathcal{B} . Hence, $f : X \rightarrow Y$ is indeed of \mathcal{B} -type $(\mathcal{B}, \mathbf{J})$ -semilocally on the domain. ■

- 1.2.19(b) **LEMMA.** *Let $f : X \rightarrow Y$ be a quadrable monomorphism in \mathcal{C} and let $\hat{\mathcal{B}}$ be the class of morphisms in \mathcal{C} of \mathcal{B} -type $(\mathcal{B}, \mathbf{J})$ -semilocally on the domain. The following are equivalent:*
- Recognition principle for quadrable monomorphisms semilocally of a given type
- (i) *$f : X \rightarrow Y$ is \mathbf{J} -semilocally of \mathcal{B} -type.*

(ii) $f : X \rightarrow Y$ is of $\hat{\mathcal{B}}$ -type \mathcal{J} -semilocally on the base.

Proof. (i) \Rightarrow (ii). Apply lemma 1.2.19(a).

(ii) \Rightarrow (i). Immediate. ■

1.2.20 ¶ The following is a generalisation of Proposition 1.10 in [JM].

Recognition principle for separated morphisms of a given type semilocally on the domain

LEMMA. *Let $f : X \rightarrow Y$ be a \mathcal{B} -separated morphism in \mathcal{C} . Assuming \mathcal{B} is a class of fibrations in \mathcal{C} , the following are equivalent:*

(i) $f : X \rightarrow Y$ is of \mathcal{B} -type $(\mathcal{B}, \mathcal{J})$ -semilocally on the domain.

(ii) $f : X \rightarrow Y$ is of \mathcal{B} -type \mathcal{J} -semilocally on the domain.

Proof. (i) \Rightarrow (ii). Immediate.

(ii) \Rightarrow (i). Let Φ be a \mathcal{J} -covering sink on X such that, for every $(U, x) \in \Phi$, $f \circ x : U \rightarrow Y$ is a member of \mathcal{B} . Then, by lemma 1.1.9, $x : U \rightarrow X$ itself is a member of \mathcal{B} . Hence, $f : X \rightarrow Y$ is indeed of \mathcal{B} -type $(\mathcal{B}, \mathcal{J})$ -semilocally on the domain. ■

REMARK. Since monomorphisms are always \mathcal{B} -separated, lemma 1.2.20 can be regarded as a generalisation of lemma 1.2.19(a), at least in the case where \mathcal{B} is a class of fibrations.

Perfect morphisms and locality

PROPOSITION. *Suppose \mathcal{G} is the class of morphisms in \mathcal{C} \mathcal{J} -semilocally of \mathcal{B} -type. Assume the following hypotheses:*

- \mathcal{B} is a class of fibrations in \mathcal{C} .
- Every morphism in \mathcal{C} of \mathcal{B} -type $(\mathcal{B}, \mathcal{J})$ -semilocally on the domain is a member of \mathcal{B} .
- Every quadrable morphism in \mathcal{C} of \mathcal{B} -type \mathcal{J} -semilocally on the base is a member of \mathcal{B} .

Let $f : X \rightarrow Y$ be a quadrable morphism in \mathcal{C} such that the relative diagonal $\Delta_f : X \rightarrow X \times_Y X$ is also a quadrable morphism in \mathcal{C} . The following are equivalent:

(i) $f : X \rightarrow Y$ is a \mathcal{B} -perfect morphism in \mathcal{C} .

(ii) $f : X \rightarrow Y$ is a \mathcal{G} -perfect morphism in \mathcal{C} .

(iii) $f : X \rightarrow Y$ is a member of \mathcal{G} and is \mathcal{B} -separated.

Proof. (i) \Rightarrow (ii). By [proposition 1.2.18](#), we have $\mathcal{B} \subseteq \mathcal{G}$; thus, every \mathcal{B} -perfect morphism in \mathcal{C} is also \mathcal{G} -perfect.

(ii) \Rightarrow (iii). The relative diagonal $\Delta_f : X \rightarrow X \times_Y X$ is a quadrable monomorphism, so we may apply [lemma 1.2.19\(b\)](#) to deduce that it is a member of \mathcal{B} . Thus, by [lemma 1.1.6](#), $f : X \rightarrow Y$ is indeed \mathcal{B} -separated.

(iii) \Rightarrow (i). Suppose $f : X \rightarrow Y$ is a member of \mathcal{G} . Then there is a J -covering sink Ψ on Y such that, for every $(V, y) \in \Psi$, we have a pullback square in \mathcal{C} of the form below,

$$\begin{array}{ccc} U & \xrightarrow{x} & X \\ v \downarrow & & \downarrow f \\ V & \xrightarrow{y} & Y \end{array}$$

where $v : U \rightarrow V$ is of \mathcal{B} -type J -semilocally on the domain. In addition, suppose $f : X \rightarrow Y$ is \mathcal{B} -separated. By [proposition 1.1.7](#), $v : U \rightarrow V$ is also \mathcal{B} -separated. Thus, by [lemma 1.2.20](#), $v : U \rightarrow V$ is of \mathcal{B} -type (\mathcal{B}, J) -semilocally on the domain, so $v : U \rightarrow V$ is a member of \mathcal{B} . Hence, $f : X \rightarrow Y$ is of \mathcal{B} -type J -semilocally on the base, so $f : X \rightarrow Y$ is a member of \mathcal{B} . ■

1.3 Regulated categories

SYNOPSIS. We study categories with a class of morphisms that have good properties with regards to pullbacks and images, such as regular categories.

PREREQUISITES. § 1.1.

1.3.1 ※ Throughout this section, \mathcal{C} is a category and \mathcal{F} is a class of separated fibrations in \mathcal{C} .

1.3.2(a) DEFINITION. An \mathcal{F} -**embedding** in \mathcal{C} is a monomorphism in \mathcal{C} that is a member of \mathcal{F} .

1.3.2(b) DEFINITION. An \mathcal{F} -**subobject** of an object Y in \mathcal{C} is an object (X, f) in \mathcal{C}/Y where $f : X \rightarrow Y$ is an \mathcal{F} -embedding in \mathcal{C} .

Properties of fibrant embeddings PROPOSITION. *The class of \mathcal{F} -embeddings in \mathcal{C} is a class of separated fibrations in \mathcal{C} .*

Proof. Straightforward. ◆

1.3.3 DEFINITION. An \mathcal{F} -**calypsis**^[1] in \mathcal{C} is a morphism $f : X \rightarrow Y$ in \mathcal{C} with the following property:

- If $f = m \circ e$ for some \mathcal{F} -embedding $m : Y' \rightarrow Y$ in \mathcal{C} , then $m : Y' \rightarrow Y$ is an isomorphism in \mathcal{C} .

REMARK. If every object in \mathcal{C} is \mathcal{F} -separated, then every \mathcal{F} -calypsis in \mathcal{C} is an epimorphism.

Properties of calypses PROPOSITION.

- (i) *Every extremal epimorphism in \mathcal{C} is an \mathcal{F} -calypsis in \mathcal{C} .*
- (ii) *Every \mathcal{F} -embedding in \mathcal{C} is right orthogonal to every \mathcal{F} -calypsis in \mathcal{C} .*
- (iii) *The class of \mathcal{F} -calypses in \mathcal{C} is closed under composition.*

Proof. Straightforward. ◆

[1] — from Greek «κάλυψις», covering.

I.3.4(a) DEFINITION. An \mathcal{F} -calypsis $f : X \rightarrow Y$ in \mathcal{C} is **quadrable** if it has the following properties:

- $f : X \rightarrow Y$ is a quadrable morphism in \mathcal{C} .
- For every pullback square in \mathcal{C} of the form below,

$$\begin{array}{ccc} X' & \longrightarrow & X \\ f' \downarrow & & \downarrow f \\ Y' & \longrightarrow & Y \end{array}$$

the morphism $f' : X' \rightarrow Y'$ is an \mathcal{F} -calypsis in \mathcal{C} .

I.3.4(b) DEFINITION. An **exact \mathcal{F} -image** of a quadrable morphism $f : X \rightarrow Y$ in \mathcal{C} is an \mathcal{F} -embedding $\text{im}(f) : \text{Im}(f) \rightarrow Y$ in \mathcal{C} with the following property:

- There is a (necessarily unique) *quadrable \mathcal{F} -calypsis* $\eta_f : X \rightarrow \text{Im}(f)$ in \mathcal{C} such that $\text{im}(f) \circ \eta_f = f$.

REMARK. Exact images are unique up to unique isomorphism, if they exist.

EXAMPLE. If $f : X \rightarrow Y$ is an \mathcal{F} -embedding in \mathcal{C} , then $f : X \rightarrow Y$ is its own exact \mathcal{F} -image.

I.3.5(a) DEFINITION. A quadrable morphism $f : X \rightarrow Y$ in \mathcal{C} is **\mathcal{F} -eucalyptic**^[2] if it has the following property:

- For every \mathcal{F} -embedding $m : X' \rightarrow X$ in \mathcal{C} , $f \circ m : X' \rightarrow Y$ admits an exact \mathcal{F} -image.

I.3.5(b) DEFINITION. A quadrable morphism $f : X \rightarrow Y$ in \mathcal{C} is **quadrably \mathcal{F} -eucalyptic** if it has the following property:

- For every pullback square in \mathcal{C} of the form below,

$$\begin{array}{ccc} X' & \longrightarrow & X \\ f' \downarrow & & \downarrow f \\ Y' & \longrightarrow & Y \end{array}$$

the morphism $f' : X' \rightarrow Y'$ is \mathcal{F} -eucalyptic.

[2] — from Greek «εὖ», well, and «καλύπτω», I cover.

Properties of PROPOSITION.

eucalyptic
morphisms

- (i) *Every \mathcal{F} -embedding in \mathcal{C} is quadrably \mathcal{F} -eucalyptic.*
- (ii) *The class of \mathcal{F} -eucalyptic morphisms in \mathcal{C} is closed under composition.*
- (iii) *The class of quadrably \mathcal{F} -eucalyptic morphisms in \mathcal{C} is a class of fibrations in \mathcal{C} .*

Proof. Straightforward. (For (ii), use [proposition 1.3.3](#).) ◆

1.3.6 DEFINITION. A quadrable morphism in \mathcal{C} is **\mathcal{F} -agathic**^[3] if it is both \mathcal{F} -separated and quadrably \mathcal{F} -eucalyptic.

Properties of PROPOSITION.

of agathic
morphisms

- (i) *Every \mathcal{F} -embedding in \mathcal{C} is \mathcal{F} -agathic.*
- (ii) *Assuming every quadrable \mathcal{F} -calypsis is an extremal epimorphism in \mathcal{C} , every \mathcal{F} -agathic monomorphism in \mathcal{C} is an \mathcal{F} -embedding.*
- (iii) *The class of \mathcal{F} -agathic morphisms in \mathcal{C} is a class of separated fibrations.*

Proof. (i). By [proposition 1.1.7](#), monomorphisms in \mathcal{C} are \mathcal{F} -separated; and by [proposition 1.3.5](#), \mathcal{F} -embeddings in \mathcal{C} are quadrably \mathcal{F} -eucalyptic.

(ii). Let $f : X \rightarrow Y$ be an \mathcal{F} -agathic monomorphism in \mathcal{C} . Then it factors as a quadrable \mathcal{F} -calypsis $e : X \rightarrow \text{Im}(f)$ followed by an \mathcal{F} -embedding $\text{im}(f) : \text{Im}(f) \rightarrow Y$. But $e : X \rightarrow \text{Im}(f)$ is both a monomorphism and an extremal epimorphism, so it must be an isomorphism. Hence $f : X \rightarrow Y$ is also an \mathcal{F} -embedding.

(iii). We know that the class of \mathcal{F} -agathic morphisms in \mathcal{C} is a class of fibrations in \mathcal{C} , so we may apply [lemma 1.1.10](#) to (i) to deduce the claim. ■

1.3.7 ¶ Let Y be an object in \mathcal{C} .

[3] — from Greek «ἀγαθικός», good.

DEFINITION. An **exact \mathcal{F} -union** of a set Φ of \mathcal{F} -subobjects of Y is an \mathcal{F} -subobject (\bar{X}, \bar{f}) of Y with the following property:

- For every object (T, y) in \mathcal{C}/Y , $y^{-1}(\bar{X}, \bar{f})$ is a coproduct of

$$\{y^{-1}(X, f) \mid (X, f) \in \Phi\}$$

in the category of \mathcal{F} -subobjects of T , where y^{-1} denotes pullback along $y : T \rightarrow Y$ in \mathcal{C} .

REMARK. Exact unions are unique up to unique isomorphism, if they exist.

1.3.8 ¶ There are numerous variations on the definition of ‘regular category’; we shall use the following.

DEFINITION. A **regular category** is a cartesian monoidal category \mathcal{S} with pullbacks of monomorphisms and exact \mathcal{M} -images of every morphism, where \mathcal{M} is the class of monomorphisms in \mathcal{S} .

Recognition principle for calypses in regular categories

LEMMA. Let $f : X \rightarrow Y$ be a morphism in \mathcal{S} . Assuming \mathcal{S} is a regular category, the following are equivalent:

- (i) $f : X \rightarrow Y$ is an effective epimorphism in \mathcal{S} .
- (ii) $f : X \rightarrow Y$ is a regular epimorphism in \mathcal{S} .
- (iii) $f : X \rightarrow Y$ is a strong epimorphism in \mathcal{S} .
- (iv) $f : X \rightarrow Y$ is an extremal epimorphism in \mathcal{S} .
- (v) $f : X \rightarrow Y$ is an \mathcal{M} -calypsis in \mathcal{S} , where \mathcal{M} is the class of monomorphisms in \mathcal{S} .

Proof. (i) \Rightarrow (ii), (ii) \Rightarrow (iii), (iii) \Rightarrow (iv), (iv) \Rightarrow (v). Straightforward.

(v) \Rightarrow (i). See Proposition 1.3.4 in [Johnstone, 2002, Part A]. □

REMARK. Thus, every kernel pair in a regular category is also a kernel pair of some effective epimorphism.

I.3.9 LEMMA. Consider a pullback square in \mathcal{C} :

Descent of mono-
morphisms

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{p} & X \\ \tilde{f} \downarrow & & \downarrow f \\ \tilde{Y} & \xrightarrow{q} & Y \end{array}$$

If $q : \tilde{Y} \rightarrow Y$ is a quadrable morphism in \mathcal{C} such that every pullback of $q : \tilde{Y} \rightarrow Y$ is an epimorphism in \mathcal{C} , then the following are equivalent:

- (i) $\tilde{f} : \tilde{X} \rightarrow \tilde{Y}$ is a monomorphism in \mathcal{C} .
- (ii) $f : X \rightarrow Y$ is a monomorphism in \mathcal{C} .

Proof. (i) \Rightarrow (ii). Let $x_0, x_1 : T \rightarrow X$ be a parallel pair of morphisms in \mathcal{C} . Suppose $f \circ x_0 = f \circ x_1$. Since $\tilde{f} : \tilde{X} \rightarrow \tilde{Y}$ is a monomorphism, the pullback pasting lemma implies that there exist morphisms $\tilde{x} : \tilde{T} \rightarrow \tilde{X}$ and $t : \tilde{T} \rightarrow T$ in \mathcal{C} such that both of the following are pullback squares in \mathcal{C} :

$$\begin{array}{ccc} \tilde{T} & \xrightarrow{t} & T \\ \tilde{x} \downarrow & & \downarrow x_0 \\ \tilde{X} & \xrightarrow{p} & X \end{array} \qquad \begin{array}{ccc} \tilde{T} & \xrightarrow{t} & T \\ \tilde{x} \downarrow & & \downarrow x_1 \\ \tilde{X} & \xrightarrow{p} & X \end{array}$$

But $t : \tilde{T} \rightarrow T$ is an epimorphism in \mathcal{C} , so we have $x_0 = x_1$.

(ii) \Rightarrow (i). Straightforward. ■

I.3.10 DEFINITION. A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is **regular** if \mathcal{C} is a regular category and $F : \mathcal{C} \rightarrow \mathcal{D}$ preserves limits of finite diagrams and extremal epimorphisms.

Criteria for a
regular functor to
be conservative

LEMMA. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a regular functor. The following are equivalent:

- (i) $F : \mathcal{C} \rightarrow \mathcal{D}$ reflects extremal epimorphisms.
- (ii) $F : \mathcal{C} \rightarrow \mathcal{D}$ reflects extremal epimorphisms and monomorphisms.
- (iii) $F : \mathcal{C} \rightarrow \mathcal{D}$ is conservative.

Proof. (i) \Rightarrow (ii). Since \mathcal{C} has kernel pairs and $F : \mathcal{C} \rightarrow \mathcal{D}$ preserves kernel pairs, if $F : \mathcal{C} \rightarrow \mathcal{D}$ reflects extremal epimorphisms, then $F : \mathcal{C} \rightarrow \mathcal{D}$ also reflects monomorphisms.

(ii) \Rightarrow (iii). A morphism (in any category) is an isomorphism if and only if it is both a monomorphism and an extremal epimorphism.

(iii) \Rightarrow (i). Let $f : X \rightarrow Y$ be a morphism in \mathcal{C} . Since $F : \mathcal{C} \rightarrow \mathcal{D}$ is a regular functor, $Ff : FX \rightarrow FY$ is an extremal epimorphism in \mathcal{D} if and only if $F \operatorname{im}(f) : F \operatorname{Im}(f) \rightarrow FY$ is an isomorphism in \mathcal{D} ; and since $F : \mathcal{C} \rightarrow \mathcal{D}$ is conservative, $F \operatorname{im}(f) : F \operatorname{Im}(f) \rightarrow FY$ is an isomorphism in \mathcal{D} if and only if $f : X \rightarrow Y$ is an extremal epimorphism in \mathcal{C} . \blacksquare

I.3.II ¶ Let κ be a regular cardinal.

DEFINITION. A κ -**ary coherent category** is a regular category \mathcal{S} with exact \mathcal{M} -unions of every κ -small set of \mathcal{M} -subobjects of every object, where \mathcal{M} is the class of monomorphisms in \mathcal{S} .

Initial objects in coherent categories **LEMMA.** Let \mathcal{S} be a κ -ary coherent category.

- (i) \mathcal{S} has an initial object 0 .
- (ii) For every object Y in \mathcal{S} , the unique morphism $\perp_Y : 0 \rightarrow Y$ in \mathcal{S} is a monomorphism.
- (iii) For every object X in \mathcal{S} , every morphism $X \rightarrow 0$ in \mathcal{S} is an isomorphism.

Proof. See Lemma I.4.I in [Johnstone, 2002, Part A]. \square

I.3.I2 **DEFINITION.** A **regulated category** is a pair $(\mathcal{C}, \mathcal{D})$ where \mathcal{C} is a category and \mathcal{D} is a (not necessarily full) subcategory^[4] of \mathcal{C} with the following properties:

- \mathcal{D} is a class of separated fibrations in \mathcal{C} .
- Every morphism in \mathcal{D} is an \mathcal{D} -agathic morphism in \mathcal{C} .

[4] However, abusing notation, we will also regard \mathcal{D} as a subset of $\operatorname{mor} \mathcal{C}$.

Given such, a **regulated morphism** in C is a morphism in D .

We will often abuse notation by referring to C itself as a regulated category, omitting D .

I.3.12(a) EXAMPLE. Every category is a regulated category in which the regulated morphisms are the isomorphisms.

I.3.12(b) EXAMPLE. Every regular category is a regulated category in which every morphism is regulated.

I.3.12(c) EXAMPLE. If every \mathcal{F} -agathic monomorphism in C is an \mathcal{F} -embedding in C , then (C, D) is a regulated category, where D is the subcategory of C consisting of the \mathcal{F} -agathic morphisms in C .

Recognition principle for regulated categories

LEMMA. *Let D be a subcategory of C . Assuming D is a class of separated fibrations in C , the following are equivalent:*

- (i) (C, D) is a regulated category.
- (ii) Every morphism in D factors as a quadrable D -calypsis in C followed by a D -embedding in C .

Proof. Straightforward. ◆

REMARK. In particular, if (C, D) is a regulated category, then (D, D) is also a regulated category.

I.3.I3 ✖ For the remainder of this section, (C, D) is a regulated category.

I.3.I4 PROPOSITION.

The category of regulated objects in a regulated category

- (i) For every object X in C , the slice category $\mathcal{D}_{/X}$ is a regular category in which the extremal epimorphisms are the morphisms that are quadrable D -calypses in C .
- (ii) For every morphism $f : X \rightarrow Y$ in C , the pullback functor $f^* : \mathcal{D}_{/Y} \rightarrow \mathcal{D}_{/X}$ is a regular functor.

Proof. (i). By hypothesis, every morphism in \mathcal{D}/X is \mathcal{D} -agathic as a morphism in \mathcal{C} , so it factors as a quadrable \mathcal{D} -calypsis followed by a \mathcal{D} -embedding in \mathcal{C} . Furthermore, by [proposition 1.1.12](#), the inclusion $\mathcal{D}/X \hookrightarrow \mathcal{C}/X$ creates limits. But every monomorphism in \mathcal{D}/X is a \mathcal{D} -embedding in \mathcal{C} , so it follows that every morphism in \mathcal{D}/X has an exact \mathcal{M}_X -image, where \mathcal{M}_X is the class of monomorphisms in \mathcal{D}/X . Thus, the extremal epimorphisms in \mathcal{D}/X are indeed the morphisms that are quadrable \mathcal{D} -calypsés in \mathcal{C} .

(ii). It is clear that $f^* : \mathcal{D}/Y \rightarrow \mathcal{D}/X$ preserves limits of finite diagrams, and the argument above implies that extremal epimorphisms are also preserved. ■

1.3.15 PROPOSITION. *Let $f : X \rightarrow Y$ be a quadrable morphism in \mathcal{C} . The following are equivalent:*

Recognition principle for quadrable calypsés

- (i) $f : X \rightarrow Y$ is a quadrable \mathcal{D} -calypsis in \mathcal{C} .
- (ii) $f : X \rightarrow Y$ admits an exact \mathcal{D} -image and the pullback functor $f^* : \mathcal{D}/Y \rightarrow \mathcal{D}/X$ is conservative.

Proof. (i) \Rightarrow (ii). Consider a commutative diagram in \mathcal{C} of the form below,

$$\begin{array}{ccc}
 X'' & \xrightarrow{f''} & Y'' \\
 x' \downarrow & & \downarrow y' \\
 X' & \xrightarrow{f'} & Y' \\
 x \downarrow & & \downarrow y \\
 X & \xrightarrow{f} & Y
 \end{array}$$

where the vertical arrows are morphisms in \mathcal{D} and both squares are pullback squares in \mathcal{C} . Suppose $x' : (X'', x \circ x') \rightarrow (X', x)$ is an extremal epimorphism in \mathcal{D}/X . Then, by [proposition 1.3.14](#), $x : X'' \rightarrow X'$ is a \mathcal{D} -calypsis in \mathcal{C} . Since $f : X \rightarrow Y$ is a quadrable \mathcal{D} -calypsis in \mathcal{C} , $f' : X' \rightarrow Y'$ is a \mathcal{D} -calypsis in \mathcal{C} , hence $y' : Y'' \rightarrow Y'$ is also an \mathcal{D} -calypsis in \mathcal{C} . But $y' : Y'' \rightarrow Y'$ is \mathcal{D} -eucalyptic, so it follows that $y' : (Y'', y \circ y') \rightarrow (Y', y)$ is an extremal epimorphism in \mathcal{D}/Y . Thus,

we see that $f^* : \mathcal{D}_{/Y} \rightarrow \mathcal{D}_{/X}$ reflects extremal epimorphisms. We may then apply [lemma 1.3.10](#).

(ii) \Rightarrow (i). Observe that $f^* : \mathcal{D}_{/Y} \rightarrow \mathcal{D}_{/X}$ sends the object $(\text{Im}(f), \text{im}(f))$ in $\mathcal{D}_{/Y}$ to a terminal object in $\mathcal{D}_{/X}$. Since $f^* : \mathcal{D}_{/Y} \rightarrow \mathcal{D}_{/X}$ is conservative, it follows that $\text{im}(f) : \text{Im}(f) \rightarrow Y$ is an isomorphism in \mathcal{C} , so $f : X \rightarrow Y$ is indeed a quadrable \mathcal{D} -calypsis in \mathcal{C} . \blacksquare

1.3.16 ¶ Let $(\mathcal{C}_0, \mathcal{D}_0)$ and $(\mathcal{C}_1, \mathcal{D}_1)$ be regulated categories.

DEFINITION. A **regulated functor** $(\mathcal{C}_0, \mathcal{D}_0) \rightarrow (\mathcal{C}_1, \mathcal{D}_1)$ is a functor $F : \mathcal{C}_0 \rightarrow \mathcal{C}_1$ with the following properties:

- F preserves regulated morphisms, i.e. F sends morphisms in \mathcal{D}_0 to morphisms in \mathcal{D}_1 .
- F preserves pullbacks along regulated morphisms, i.e. given a pullback square in \mathcal{C}_0 , say

$$\begin{array}{ccc} T' & \longrightarrow & T \\ \downarrow & & \downarrow x \\ X' & \longrightarrow & X \end{array}$$

if $x : T \rightarrow X$ is in \mathcal{D}_0 , then F preserves this pullback square.

- F preserves exact images of regulated morphisms, i.e. given a morphism $f : X \rightarrow Y$ in \mathcal{D}_0 , $F \text{ im}(f) : F \text{ Im}(f) \rightarrow FY$ is an exact \mathcal{D}_1 -image of $Ff : FX \rightarrow FY$.

Recognition principle for regulated functors

LEMMA. Let $F : \mathcal{C}_0 \rightarrow \mathcal{C}_1$ be a functor. Assuming F preserves regulated morphisms and pullbacks along regulated morphisms, the following are equivalent:

- $F : (\mathcal{C}_0, \mathcal{D}_0) \rightarrow (\mathcal{C}_1, \mathcal{D}_1)$ is a regulated functor.
- For every object X in \mathcal{C}_0 , the evident functor

$$F_X : (\mathcal{D}_0)_{/X} \rightarrow (\mathcal{D}_1)_{/FX}$$

given on objects by $(T, x) \mapsto (FT, Fx)$ is a regular functor.

Proof. This is a consequence of [proposition 1.3.14](#). \blacksquare

1.4 Exact quotients

SYNOPSIS. We consider the problem of freely adding exact quotients to a category with a class of covering morphisms.

PREREQUISITES. §§ I.1, I.3, A.1, A.2, A.3.

I.4.1 ¶ Let C be a category.

DEFINITION. A **strict epimorphism** (resp. **universally strict epimorphism**) in C is a morphism $f : X \rightarrow Y$ in C such that the principal sieve $\downarrow\langle f \rangle$ is strict-epimorphic (resp. universally strict-epimorphic).

Recognition principle for strict epimorphisms

LEMMA. *Let $f : X \rightarrow Y$ be a morphism in C . The following are equivalent:*

- (i) *The morphism $f : X \rightarrow Y$ is a strict epimorphism in C .*
- (ii) *For every morphism $h : X \rightarrow Z$ in C , if $h \circ x_0 = h \circ x_1$ for all elements $(T, (x_0, x_1))$ of $\text{Kr}(f \circ -)$, then there is a unique morphism $g : Y \rightarrow Z$ in C such that $g \circ f = h$.*

Proof. This is a special case of lemma A.2.6. ■

Properties of strict epimorphisms

PROPOSITION.

- (i) *Every strict epimorphism in C is an epimorphism in C .*
- (ii) *Every regular epimorphism in C is a strict epimorphism in C .*
- (iii) *A strict epimorphism in C is an effective epimorphism in C if (and only if) it admits a kernel pair in C .*

Proof. Straightforward. (Use lemma I.4.1.) ◆

I.4.2 ¶ Let X be an object in a category C .

I.4.2(a) DEFINITION. An **equivalence relation** on X in C is a tuple (R, d_0, d_1) where:

- $\langle h_{d_1}, h_{d_0} \rangle : h_R \rightarrow h_X \times h_X$ is a monomorphism in $\mathbf{PSH}(C)$.
- The image of $\langle h_{d_1}, h_{d_0} \rangle : h_R \rightarrow h_X \times h_X$ is an equivalence relation on h_X .

REMARK. In other words, an equivalence relation on X is a representation of some equivalence relation on h_X .

I.4.2(b) DEFINITION. An **exact quotient** of an equivalence relation (R, d_0, d_1) on X in \mathcal{C} is a quadrable morphism $q : X \rightarrow \bar{X}$ in \mathcal{C} with the following properties:

- The following is a pullback square in \mathcal{S} :

$$\begin{array}{ccc} R & \xrightarrow{d_0} & X \\ d_1 \downarrow & & \downarrow q \\ X & \xrightarrow{q} & \bar{X} \end{array}$$

- Every pullback of $q : X \rightarrow \bar{X}$ is an effective epimorphism in \mathcal{C} .

I.4.3 ¶ Let \mathcal{C} be a category. The following is a special case of the notion of coverage (paragraph A.2.8) and generalises the notion of weakly unary topology in the sense of [Shulman, 2012, §3].

I.4.3(a) DEFINITION. A **unary coverage** on \mathcal{C} is a subset $E \subseteq \text{mor } \mathcal{C}$ with the following properties:

- For every object X in \mathcal{C} , $\text{id} : X \rightarrow X$ is a member of E .
- For every morphism $f : X \rightarrow Y$ in \mathcal{C} and every morphism $y : T \rightarrow Y$ in \mathcal{C} , if $y : T \rightarrow Y$ is a member of E , then there is a commutative diagram in \mathcal{C} of the form below,

$$\begin{array}{ccc} U & \longrightarrow & T \\ x \downarrow & & \downarrow y \\ X & \xrightarrow{f} & Y \end{array}$$

where $x : U \rightarrow X$ is also a member of E .

I.4.3(b) DEFINITION. A unary coverage E on \mathcal{C} is **upward-closed** if it has the following property:

- Given morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ in \mathcal{C} , if $g \circ f : X \rightarrow Z$ is a member of E , then $g : Y \rightarrow Z$ is also a member of E .

REMARK. If E is an upward-closed unary coverage, then:

- (i) Every split epimorphism in C is a member of E .
- (ii) The full subcategory of $(C \downarrow C)$ spanned by the members of E is replete.

I.4.3(c) DEFINITION. A unary coverage E on C is **saturated** if it has the following property:

- For every commutative square in C of the form below,

$$\begin{array}{ccc} U & \longrightarrow & T \\ x \downarrow & & \downarrow y \\ X & \xrightarrow{f} & Y \end{array}$$

if both $x : U \rightarrow X$ and $f : X \rightarrow Y$ are members of E , then $y : T \rightarrow Y$ is also a member of E .

EXAMPLE. If C is a category with pullbacks and E is the class of universally strict epimorphisms in C , then E is a saturated unary coverage on C (by lemma A.2.3(c) and corollary A.2.3(d)).

I.4.4 ✂ For the remainder of this section, C is a category and E is a unary coverage on C . Abusing notation, we will conflate E with the associated coverage on C .

I.4.5 REMARK. Recalling paragraph A.2.13, $f : X \rightarrow Y$ is a E -covering morphism in C if and only if there is a morphism $x : T \rightarrow X$ in C such that $f \circ x : T \rightarrow Y$ is a composite of some finite (composable) sequence of members of E .

Recognition principles for saturated unary coverages

LEMMA. *Let E be a unary coverage on C . The following are equivalent:*

- (i) E is saturated.
- (ii) E is upward-closed and closed under composition.
- (iii) Every E -covering morphism in C is a member of E .

Proof. Straightforward. ◆

Properties of covering morphisms PROPOSITION. *The class of E-covering morphisms in \mathcal{C} is the smallest saturated unary coverage on \mathcal{C} that contains \mathbf{E} .*

Proof. Straightforward. ◆

I.4.6 ¶ Let A be a presheaf on \mathcal{C} .

I.4.6(a) DEFINITION. An **E-local generator** of A is an element (X, a) of A such that $\{(X, a)\}$ is an E-local generating set of elements of A .

The presheaf A is **E-locally I-generable** if it has an E-local generator.

I.4.6(b) DEFINITION. An **E-local presentation** of A is tuple (X, P, a, d_0, d_1) with the following properties:

- (X, a) is an E-local generator of A .
- $(P, (d_1, d_0))$ is an E-local generator of the kernel relation $\text{Kr}(a \cdot (-))$, where $a \cdot (-) : h_X \rightarrow A$ is the unique morphism that sends id_X to a .

The presheaf A is **E-locally I-presentable** if it admits an E-local presentation.

EXAMPLE. For every object X in \mathcal{C} , $(X, X, \text{id}_X, \text{id}_X, \text{id}_X)$ is an E-local presentation of h_X .

I.4.7 ¶ Let B be a presheaf on \mathcal{C} and let (Y, b) be an E-local generator of B .

Elements of locally I-generable presheaves LEMMA. *Let (X, a) be an element of B . There exist an E-covering morphism $p : \tilde{X} \rightarrow X$ in \mathcal{C} and a morphism $f : \tilde{X} \rightarrow Y$ in \mathcal{C} such that $b \cdot f = a \cdot p$.*

Proof. Straightforward. (Recall [remark I.4.5](#) and [paragraph A.2.13](#).) ◆

Locally I-generable subpresheaves of locally I-generable presheaves COROLLARY. *Let A be an E-locally I-generable subpresheaf of B . There is a morphism $f : \tilde{X} \rightarrow Y$ in \mathcal{C} such that $(\tilde{X}, b \cdot f)$ is an E-local generator of A .*

Proof. Let (X, a) be an E-local generator of A . By [lemma I.4.7](#), we have an E-covering morphism $p : \tilde{X} \rightarrow X$ in \mathcal{C} and a morphism $f : \tilde{X} \rightarrow Y$ in \mathcal{C} such that $b \cdot f = a \cdot p$. On the other hand, $(\tilde{X}, a \cdot p)$ is also an E-local generator of A : for every E-closed subpresheaf $A' \subseteq A$, if $a \cdot p \in A'(\tilde{X})$, then $a \in A'(X)$, so $A' = A$. ■

I.4.8 LEMMA. *Let A be an E -locally I -generable presheaf on C . For every E -local generating set Φ of elements of A , there is $(X', a') \in \Phi$ such that (X', a') is an E -local generator of A .*

Local generating
sets of locally
 I -generable
presheaves

Proof. Let (X, a) be an E -local generator of A . Recalling [remark I.4.5](#) and [paragraph A.2.13](#), since Φ is an E -local generating set of elements of A , there exist an element $(X', a') \in \Phi$, an E -covering morphism $p : \tilde{X} \rightarrow X$ in C , and a morphism $f : \tilde{X} \rightarrow X'$ in C such that $a \cdot p = a' \cdot f$. Thus, every E -closed subpresheaf of A containing (X', a') must also contain (X, a) , so (X', a') itself is an E -local generator of A . ■

I.4.9 ¶ Let $X : \mathcal{J} \rightarrow C$ be a diagram.

DEFINITION. An **E -weakly limiting cone** on X is a cone $\lambda : \Delta \tilde{X} \Rightarrow X$ in C with the following property:

- For every object T in C and every cone $\xi : \Delta T \Rightarrow X$ in C , there exist morphisms $\tilde{x} : U \rightarrow \tilde{X}$ and $t : U \rightarrow T$ in C such that $t : U \rightarrow T$ is an E -covering morphism in C and, for every object j in \mathcal{J} , $\lambda_j \circ \tilde{x} = \xi_j \circ t$.

We will often abuse notation by referring to the object \tilde{X} as an **E -weak limit** of X , omitting λ .

EXAMPLE. Every limiting cone on X is also an E -weakly limiting cone.

Recognition principle for weak limits

LEMMA. *Let \tilde{X} be an object in C and let $\lambda : \Delta \tilde{X} \Rightarrow X$ be a cone in C . The following are equivalent:*

- (i) $\lambda : \Delta \tilde{X} \Rightarrow X$ is an E -weakly limiting cone.
- (ii) (\tilde{X}, λ) is an E -local generator of the presheaf $[\mathcal{J}, C](\Delta -, X)$.

Proof. Straightforward. ◆

I.4.10 ¶ The following is due to Shulman [\[2012\]](#).

DEFINITION. The **Shulman condition** on (C, E) is the following:

- Every finite diagram in C has a E -weak limit.

REMARK. A (strongly) unary topology in the sense of [Shulman, 2012, §3] is precisely a saturated unary coverage that satisfies the Shulman condition.

PROPOSITION. *Let $X : \mathcal{J} \rightarrow \mathcal{C}$ be a diagram in \mathcal{C} . Assuming the Shulman condition on $(\mathcal{C}, \mathbb{E})$, the presheaf $\varprojlim_{\mathcal{J}} h_X$ is \mathbb{E} -locally 1-presentable.*

Proof. By lemma 1.4.9, there is an \mathbb{E} -local generator (\tilde{X}, λ) of $\varprojlim_{\mathcal{J}} h_X$. Let $R = \text{Kr}(\lambda \cdot (-)) \subseteq h_{\tilde{X}} \times h_{\tilde{X}}$. It is not hard to see that R is isomorphic to the presheaf of cones over the diagram in \mathcal{C} obtained by attaching two copies of the cone λ over the given diagram X . Thus, by hypothesis, R is \mathbb{E} -locally 1-generable. Hence, $\varprojlim_{\mathcal{J}} h_X$ is \mathbb{E} -locally 1-presentable. ■

1.4.11 LEMMA. *Let $h : A \rightarrow B$ be a morphism in $\mathbf{PSH}(\mathcal{C})$. Given \mathbb{E} -local generators (X, a) and (Y, b) of A and B , respectively, there is an element $(T, (x, y))$ of $h_X \times h_Y$ such that $x : T \rightarrow X$ is an \mathbb{E} -covering morphism in \mathcal{C} and $b \cdot y = h(a) \cdot x$. In particular, there is a commutative square in $\mathbf{PSH}(\mathcal{C})$ of the form below,*

$$\begin{array}{ccc} h_T & \xrightarrow{y \circ -} & h_Y \\ a \cdot (x \circ -) \downarrow & & \downarrow b \cdot (-) \\ A & \xrightarrow{h} & B \end{array}$$

where the vertical arrows are \mathbb{E} -locally surjective.

Proof. Straightforward. (Recall paragraph A.2.13.) ◆

1.4.12 PROPOSITION. *Let $h_0 : A_0 \rightarrow B$ and $h_1 : A_1 \rightarrow B$ be morphisms in $\mathbf{PSH}(\mathcal{C})$. Assuming the Shulman condition on $(\mathcal{C}, \mathbb{E})$, if both A_0 and A_1 are \mathbb{E} -locally 1-generable and B is \mathbb{E} -locally 1-presentable, then $\text{Pb}(h_0, h_1)$ is also \mathbb{E} -locally 1-generable.*

Proof. Choose any \mathbb{E} -local presentation of B , say (Y, Q, b, d_0, d_1) . By lemma 1.4.11, there is a commutative diagram in $\mathbf{PSH}(\mathcal{C})$ of the form below,

$$\begin{array}{ccccc} h_{X_0} & \xrightarrow{f_0 \circ -} & h_Y & \xleftarrow{f_1 \circ -} & h_{X_1} \\ a_0 \cdot (-) \downarrow & & \downarrow b \cdot (-) & & \downarrow a_1 \cdot (-) \\ A_0 & \xrightarrow{h_0} & B & \xleftarrow{h_1} & A_1 \end{array}$$

where the vertical arrows are E-locally surjective. We have the following commutative diagrams in $\mathbf{PSH}(\mathcal{C})$,

$$\begin{array}{ccccc}
 R & \longrightarrow & \text{Pb}(h_0, h_1) & \longrightarrow & B \\
 \downarrow & & \downarrow & & \downarrow \Delta_B \\
 \hat{h}_{X_0} \times \hat{h}_{X_1} & \xrightarrow{(a_0 \cdot -) \times (a_1 \cdot -)} & A_0 \times A_1 & \xrightarrow{h_0 \times h_1} & B \times B \\
 \\
 R & \longrightarrow & \text{Kr}(b \cdot -) & \longrightarrow & B \\
 \downarrow & & \downarrow & & \downarrow \Delta_B \\
 \hat{h}_{X_0} \times \hat{h}_{X_1} & \xrightarrow{(f_0 \circ -) \times (f_1 \circ -)} & \hat{h}_Y \times \hat{h}_Y & \xrightarrow{(b \cdot -) \times (b \cdot -)} & B \times B
 \end{array}$$

where every square is a pullback square in $\mathbf{PSH}(\mathcal{C})$; in particular,

$$R = \text{Pb}(h_0 \circ (a_0 \cdot -), h_1 \circ (a_1 \cdot -)) = \text{Pb}(b \cdot (f_0 \circ -), b \cdot (f_1 \circ -))$$

as subpresheaves of $\hat{h}_{X_0} \times \hat{h}_{X_1}$. Note that $R \rightarrow \text{Pb}(h_0, h_1)$ is E-locally surjective, by [proposition A.2.14](#). By definition, we have an E-locally surjective morphism $\hat{h}_Q \rightarrow \text{Kr}(b \cdot -)$ in $\mathbf{PSH}(\mathcal{C})$, so there is a pullback square in $\mathbf{PSH}(\mathcal{C})$ of the form below,

$$\begin{array}{ccc}
 \tilde{R} & \longrightarrow & \hat{h}_Q \\
 \downarrow & & \downarrow \\
 R & \longrightarrow & \text{Kr}(b \cdot -)
 \end{array}$$

where $\tilde{R} \rightarrow R$ is E-locally surjective. On the other hand, the pullback pasting lemma implies that \tilde{R} is (isomorphic to) the limit of the following diagram in $\mathbf{PSH}(\mathcal{C})$,

$$\begin{array}{ccc}
 & & \hat{h}_{X_1} \\
 & & \downarrow f_1 \circ - \\
 \hat{h}_Q & \xrightarrow{d_0 \circ -} & \hat{h}_Y \\
 \downarrow d_1 \circ - & & \\
 \hat{h}_{X_0} & \xrightarrow{f_0 \circ -} & \hat{h}_Y
 \end{array}$$

so by [lemma 1.4.9](#), the Shulman condition on $(\mathcal{C}, \mathbf{E})$ implies that \tilde{R} is E-locally 1-generable. But the composite $\tilde{R} \rightarrow R \rightarrow \text{Pb}(h_0, h_1)$ is E-locally surjective, so $\text{Pb}(h_0, h_1)$ is also E-locally 1-generable. \blacksquare

Generators of locally \mathcal{I} -presentable presheaves

COROLLARY. *Let A be a presheaf on C and let (X, a) be an element of A . Assuming the Shulman condition on (C, \mathcal{E}) , the following are equivalent:*

- (i) A is an \mathcal{E} -locally \mathcal{I} -presentable presheaf on C and (X, a) is an \mathcal{E} -local generator of A .
- (ii) There is an element (P, d_0, d_1) of $\mathcal{h}_X \times \mathcal{h}_X$ such that (X, P, a, d_0, d_1) is an \mathcal{E} -local presentation of A .

Proof. This is a special case of [proposition 1.4.12](#). ■

1.4.13 **LEMMA.** *Let B be an \mathcal{E} -locally \mathcal{I} -presentable presheaf on C and let A be a subpresheaf of B . Assuming the Shulman condition on (C, \mathcal{E}) , the following are equivalent:*

Recognition principle for locally \mathcal{I} -presentable subpresheaves

- (i) A is an \mathcal{E} -locally \mathcal{I} -presentable presheaf on C .
- (ii) A is an \mathcal{E} -locally \mathcal{I} -generable presheaf on C .

Proof. (i) \Rightarrow (ii). Immediate.

(ii) \Rightarrow (i). Choose any \mathcal{E} -local generator of A , say (X, a) . We must show that $\text{Kr}(a \cdot -)$ is \mathcal{E} -locally \mathcal{I} -generable. But we have the following pull-back square in $\mathbf{PSH}(C)$,

$$\begin{array}{ccc} \text{Kr}(a \cdot -) & \longrightarrow & \mathcal{h}_X \\ \downarrow & & \downarrow a \cdot - \\ \mathcal{h}_X & \xrightarrow{a \cdot -} & B \end{array}$$

so we may apply [proposition 1.4.12](#). ■

1.4.14 **LEMMA.** *Let $h : A \rightarrow B$ be an \mathcal{E} -locally surjective morphism in $\mathbf{PSH}(C)$. Assuming the Shulman condition on (C, \mathcal{E}) , if A is \mathcal{E} -locally \mathcal{I} -presentable, then the following are equivalent:*

Quotients of locally \mathcal{I} -presentable presheaves

- (i) B is an \mathcal{E} -locally \mathcal{I} -presentable presheaf on C .
- (ii) $\text{Kr}(h)$ is an \mathcal{E} -locally \mathcal{I} -generable presheaf on C .

Proof. (i) \Rightarrow (ii). This is a special case of [proposition 1.4.12](#).

(ii) \Rightarrow (i). Choose any E-local generator of A , say (X, a) . We have the following commutative diagram in $\mathbf{PSH}(C)$,

$$\begin{array}{ccccc}
 \mathrm{Kr}(h(a) \cdot -) & \longrightarrow & R_1 & \longrightarrow & h_X \\
 \downarrow & & \downarrow & & \downarrow^{a \cdot -} \\
 R_0 & \longrightarrow & \mathrm{Kr}(h) & \longrightarrow & A \\
 \downarrow & & \downarrow & & \downarrow^h \\
 h_X & \xrightarrow{a \cdot -} & A & \xrightarrow{h} & B
 \end{array}$$

where every square is a pullback square in $\mathbf{PSH}(C)$. Since h_X and $\mathrm{Kr}(h)$ are both E-locally \mathcal{I} -generable and A is E-locally \mathcal{I} -presentable, R_0 must also be E-locally \mathcal{I} -generable. But we also have a pullback square in $\mathbf{PSH}(C)$ of the form below,

$$\begin{array}{ccc}
 \mathrm{Kr}(h(a) \cdot -) & \longrightarrow & h_X \\
 \downarrow & & \downarrow^{a \cdot -} \\
 R_0 & \longrightarrow & A
 \end{array}$$

so $\mathrm{Kr}(h(a) \cdot -)$ has an E-local generator, say $(Q, (x_0, x_1))$. It follows that $(X, Q, h(a), x_1, x_0)$ is an E-local presentation of B . \blacksquare

1.4.15 ¶ Let A and B be presheaves on C .

1.4.15(a) **PROPOSITION.** *Assuming the Shulman condition on (C, E) , if both A and B are E-locally \mathcal{I} -generable, then $A \times B$ is also E-locally \mathcal{I} -generable.*

Products of locally \mathcal{I} -generable presheaves

Proof. Let (X, a) and (Y, b) be E-local generators of A and B , respectively. Then, by [proposition A.2.14](#), $(a \cdot -) \times (b \cdot -) : h_X \times h_Y \rightarrow A \times B$ is E-locally surjective. But [lemma 1.4.9](#) implies that $h_X \times h_Y$ is E-locally \mathcal{I} -generable, so it follows that $A \times B$ is also E-locally \mathcal{I} -generable. \blacksquare

1.4.15(b) **PROPOSITION.** *Assuming the Shulman condition on (C, E) , if both A and B are E-locally \mathcal{I} -presentable, then $A \times B$ is also E-locally \mathcal{I} -presentable.*

Products of locally \mathcal{I} -presentable presheaves

Proof. Let (X, a) and (Y, b) be E-local generators of A and B , respectively. By [corollary 1.4.12](#), both $\mathrm{Kr}(a \cdot -)$ and $\mathrm{Kr}(b \cdot -)$ are E-locally \mathcal{I} -generable. Let $h = (a \cdot -) \times (b \cdot -) : h_X \times h_Y \rightarrow A \times B$. Clearly, $\mathrm{Kr}(h) \cong$

$\text{Kr}(a \cdot -) \times \text{Kr}(b \cdot -)$, so by [proposition 1.4.15\(a\)](#), $\text{Kr}(h)$ is E-locally \mathcal{I} -generable. But [proposition A.2.14](#) implies that $h : h_X \times h_Y \rightarrow A \times B$ is E-locally surjective, and by [proposition 1.4.10](#), $h_X \times h_Y$ is E-locally \mathcal{I} -presentable, so we may apply [lemma 1.4.14](#) to deduce that $A \times B$ is also E-locally \mathcal{I} -presentable. ■

1.4.16 THEOREM. *The following are equivalent:*

Limits of
diagrams of
locally \mathcal{I} -
presentable
presheaves

- (i) (C, E) satisfies the Shulman condition.
- (ii) The full submetacategory of $\mathbf{PSH}(C)$ spanned by the E-locally \mathcal{I} -presentable presheaves on C is closed under limit of finite diagrams.

Proof. (i) \Rightarrow (ii). The terminal presheaf on C is E-locally \mathcal{I} -presentable ([proposition 1.4.10](#)), and the the product of two E-locally \mathcal{I} -presentable presheaves on C is E-locally \mathcal{I} -presentable ([proposition 1.4.15\(b\)](#)), so it suffices to verify that the equaliser of a parallel pair of morphisms between E-locally \mathcal{I} -presentable presheaves on C is E-locally \mathcal{I} -presentable. But this is a consequence of [proposition 1.4.12](#) and [lemma 1.4.13](#), so we are done.

(ii) \Rightarrow (i). Apply [lemma 1.4.9](#). ■

Morphisms
of locally \mathcal{I} -
presentable
presheaves

COROLLARY. *Let $h : A \rightarrow B$ be a morphism in $\mathbf{PSH}(C)$ and let (Y, b) be an E-local generator of B . Assuming the Shulman condition on (C, E) , if both A and B are E-locally \mathcal{I} -presentable, then there exist an E-local generator (X, a) of A and a morphism $f : X \rightarrow Y$ in C such that the diagram in $\mathbf{PSH}(C)$ shown below commutes,*

$$\begin{array}{ccc} h_X & \xrightarrow{f \circ -} & h_Y \\ a \cdot - \downarrow & & \downarrow b \cdot - \\ A & \xrightarrow{h} & B \end{array}$$

and the induced morphism $h_X \rightarrow \text{Pb}(h, b \cdot -)$ is E-locally surjective.

Proof. Apply [theorem 1.4.16](#) and [proposition A.2.14](#). ■

I.4.I7(a) DEFINITION. A **fork** in \mathcal{C} is a diagram in \mathcal{C} of the form below,

$$P \begin{array}{c} \xrightarrow{d_1} \\ \xrightarrow{d_0} \end{array} X \xrightarrow{f} Y \quad (*)$$

where $f \circ d_1 = f \circ d_0$.

I.4.I7(b) DEFINITION. The fork $(*)$ is **mid-E-exact** if the following is an E-weak pullback square in \mathcal{C} :

$$\begin{array}{ccc} P & \xrightarrow{d_0} & X \\ d_1 \downarrow & & \downarrow f \\ X & \xrightarrow{f} & Y \end{array} \quad (\dagger)$$

I.4.I7(c) DEFINITION. The fork $(*)$ is **left-exact** if (\dagger) is a pullback square in \mathcal{C} .

I.4.I7(d) DEFINITION. The fork $(*)$ is **right-E-exact** if (X, P, f, d_0, d_1) is an E-local presentation of h_Y .

I.4.I7(e) DEFINITION. The fork $(*)$ is **E-exact** if it is both left-exact and right-E-exact.

REMARK. Clearly, every left-exact fork is also mid-E-exact.

Recognition principle for right-exact forks

LEMMA. *The following are equivalent:*

- (i) *The fork $(*)$ is right-E-exact.*
- (ii) *The fork $(*)$ is mid-E-exact and $f : X \rightarrow Y$ is an E-covering morphism in \mathcal{C} .*

Proof. Straightforward. (Recall lemma I.4.9.)



I.4.I8 ¶ The sheaf condition with respect to E can be considered to be a kind of limit preservation condition. More precisely:

LEMMA. Let A be a presheaf on C . Assuming (C, E) satisfies the Shulman condition, the following are equivalent:

- (i) A is an E -sheaf on C .
- (ii) $A : C^{\text{op}} \rightarrow \mathbf{SET}$ sends right- E -exact forks in C to equaliser diagrams in \mathbf{SET} .
- (iii) For every right- E -exact fork in C of the form below,

$$R \begin{array}{c} \xrightarrow{d_0} \\ \xrightarrow{d_1} \end{array} X \xrightarrow{f} Y$$

if $f : X \rightarrow Y$ is a member of E , then the following is an equaliser diagram in \mathbf{SET} :

$$A(Y) \xrightarrow{- \cdot f} A(X) \begin{array}{c} \xrightarrow{- \cdot d_0} \\ \xrightarrow{- \cdot d_1} \end{array} A(R)$$

Proof. (i) \Rightarrow (ii). Consider a right- E -exact fork in C :

$$R \begin{array}{c} \xrightarrow{d_0} \\ \xrightarrow{d_1} \end{array} X \xrightarrow{f} Y$$

Let $a \in A(X)$ and suppose $a \cdot d_0 = a \cdot d_1$. We wish to find $a' \in A(Y)$ such that $a' \cdot f = a$. Since A satisfies the sheaf condition with respect to the principal sieve $\downarrow \langle f \rangle$, such an element a' is necessarily unique because $- \cdot f : A(Y) \rightarrow A(X)$ is injective. On the other hand, by lemma A.2.6, such a' exists if a has the following property:

- For every element $(T, (x_0, x_1))$ of $\mathcal{h}_X \times \mathcal{h}_X$, if $f \circ x_0 = f \circ x_1$, then $a \cdot x_0 = a \cdot x_1$.

However, given an element $(T, (x_0, x_1))$ of $\mathcal{h}_X \times \mathcal{h}_X$, if $f \circ x_0 = f \circ x_1$, there exist an E -covering morphism $t : U \twoheadrightarrow T$ in C and a morphism $r : U \rightarrow R$ in C such that the following diagrams in C commute,

$$\begin{array}{ccc} U & \xrightarrow{r} & R \\ t \downarrow & & \downarrow d_1 \\ T & \xrightarrow{x_0} & X \end{array} \qquad \begin{array}{ccc} U & \xrightarrow{r} & R \\ t \downarrow & & \downarrow d_0 \\ T & \xrightarrow{x_1} & X \end{array}$$

so $(a \cdot x_0) \cdot t = (a \cdot x_1) \cdot t$. Since $- \cdot t : A(T) \rightarrow A(U)$ is also injective, the claim follows.

(ii) \Rightarrow (iii). Immediate.

(iii) \Rightarrow (i). Let $f : X \twoheadrightarrow Y$ be a member of \mathbf{E} . We must show that A satisfies the sheaf condition with respect to the principal sieve $\downarrow\langle f \rangle$. Since $(\mathcal{C}, \mathbf{E})$ satisfies the Shulman condition, there is a right \mathbf{E} -exact fork in \mathcal{C} of the form below:

$$R \begin{array}{c} \xrightarrow{d_0} \\ \xrightarrow{d_1} \end{array} X \xrightarrow{f} Y$$

In particular, $- \cdot f : A(Y) \rightarrow A(X)$ is injective. Consider a commutative square of the form below,

$$\begin{array}{ccc} \downarrow\langle f \rangle & \xrightarrow{s} & \mathbf{El}(A) \\ \downarrow & & \downarrow \\ C_{/Y} & \longrightarrow & C \end{array}$$

where $C_{/Y} \rightarrow C$ and $\mathbf{El}(A) \rightarrow C$ are the respective projections. Let $(X, a) = s(X, f)$. Then $a \cdot d_0 = a \cdot d_1$, so there is a unique $a' \in A(Y)$ such that $a' \cdot f = a$. This defines a functor $C_{/Y} \rightarrow \mathbf{El}(A)$ making the evident triangles commute, and by the Yoneda lemma, it is the unique such functor. Thus A indeed satisfies the sheaf condition with respect to $\downarrow\langle f \rangle$. ■

1.4.19 ¶ The following technical results will be needed later.

1.4.19(a) LEMMA. Consider a commutative diagram in \mathcal{C} of the form below:

Covering morphisms and weak pullback squares

$$\begin{array}{ccccc} P & \xrightarrow{x_1} & X_1 & & \\ & \searrow q & & & \downarrow f_1 \\ & & Q & \xrightarrow{y_1} & Y_1 \\ x_0 \downarrow & & y_0 \downarrow & & \downarrow g_1 \\ X_0 & \xrightarrow{f_0} \twoheadrightarrow & Y_0 & \xrightarrow{g_0} & Z \end{array}$$

If the outer square is an \mathbf{E} -weak pullback square in \mathcal{C} and both $f_0 : X_0 \twoheadrightarrow Y_0$ and $f_1 : X_1 \twoheadrightarrow Y_1$ are \mathbf{E} -covering morphisms in \mathcal{C} , then the inner square is also an \mathbf{E} -weak pullback square in \mathcal{C} .

Proof. Consider a commutative square in \mathcal{C} of the form below:

$$\begin{array}{ccc} T & \xrightarrow{y'_1} & Y_1 \\ y'_0 \downarrow & & \downarrow g_1 \\ Y_0 & \xrightarrow{g_0} & Z \end{array}$$

By [remark 1.4.5](#) and [proposition 1.4.5](#), there is a commutative diagram in \mathcal{C} of the form below,

$$\begin{array}{ccccc} S & \overset{x'_1}{\dashrightarrow} & X_1 & & \\ & \searrow t & & & \downarrow f_1 \\ & & T & \xrightarrow{y'_1} & Y_1 \\ x'_0 \downarrow & & y'_0 \downarrow & & \downarrow g_1 \\ X_0 & \xrightarrow{f_0} & Y_0 & \xrightarrow{g_0} & Z \end{array}$$

where $t : S \twoheadrightarrow T$ is an E-covering morphism in \mathcal{C} . Thus, there exist an E-covering morphism $s : U \twoheadrightarrow S$ in \mathcal{C} and a morphism $p : U \rightarrow P$ in \mathcal{C} such that $x_0 \circ p = x'_0 \circ s$ and $x_1 \circ p = x'_1 \circ s$. We then have $y_0 \circ q \circ p = y'_0 \circ t \circ s$ and $y_1 \circ q \circ p = y'_1 \circ t \circ s$, and $t \circ s : U \twoheadrightarrow T$ is an E-covering morphism in \mathcal{C} , as required. ■

1.4.19(b) LEMMA. Consider a diagram in \mathcal{C} of the form below,
Exact forks
and pullbacks

$$\begin{array}{ccccc} X_1 & \xrightarrow{d_0} & X_0 & \xrightarrow{p} & \bar{X} \\ f_1 \downarrow & & d_1 \downarrow & & \downarrow \bar{f} \\ Y_1 & \xrightarrow{d_0} & Y_0 & \xrightarrow{q} & Y \end{array}$$

where:

- The top row is an E-exact fork in \mathcal{C} .
- The bottom row is a left-exact fork in \mathcal{C} .
- The two parallel squares on the left are pullback squares in \mathcal{C} .
- The square on the right commutes.

Then $(f_0, p) \cdot (-) : h_{X_0} \rightarrow \text{Pb}(q \circ -, \bar{f} \circ -)$ is an E-locally bijective morphism in $\mathbf{Psh}(\mathcal{C})$. In particular, if \mathbf{E} is a subcanonical unary coverage on \mathcal{C} , then the right square is a pullback square in \mathcal{C} .

Proof. We have the following commutative diagram in \mathcal{C} ,

$$\begin{array}{ccccc} X_1 & \xrightarrow{f_1} & Y_1 & \xrightarrow{d_0} & Y_0 \\ d_1 \downarrow & & d_1 \downarrow & & \downarrow q \\ X_0 & \xrightarrow{f_0} & Y_0 & \xrightarrow{q} & \bar{Y} \end{array}$$

where both squares are pullback squares in \mathcal{C} . Thus, by the pullback pasting lemma, in the commutative diagram in \mathcal{C} shown below,

$$\begin{array}{ccccc} X_1 & \xrightarrow{d_0} & X_0 & \xrightarrow{f_0} & Y_0 \\ d_1 \downarrow & & \downarrow p & & \downarrow q \\ X_0 & \xrightarrow{p} & \bar{X} & \xrightarrow{\bar{f}} & \bar{Y} \end{array}$$

the outer rectangle is a pullback diagram in \mathcal{C} . Hence, by lemma 1.4.19(a), the right square is an E-weak pullback square in \mathcal{C} , so $(f_0, p) \cdot (-) : \mathcal{H}_{X_0} \rightarrow \text{Pb}(q \circ -, \bar{f} \circ -)$ is E-locally surjective, by lemma 1.4.9.

We will now show that $(f_0, p) \cdot (-) : \mathcal{H}_{X_0} \rightarrow \text{Pb}(q \circ -, \bar{f} \circ -)$ is a monomorphism in $\mathbf{PSH}(\mathcal{C})$. Let T be an object in \mathcal{C} and let $x_{0,0}, x_{0,1} : T \rightarrow X_0$ be a parallel pair of morphisms in \mathcal{C} such that:

$$p \circ x_{0,0} = p \circ x_{0,1} \qquad f_0 \circ x_{0,0} = f_0 \circ x_{0,1}$$

We then have a unique morphism $x_1 : T \rightarrow X_1$ such that:

$$d_1 \circ x_1 = x_{0,0} \qquad d_0 \circ x_1 = x_{0,1}$$

On the other hand,

$$d_1 \circ f_1 \circ x_1 = f_0 \circ x_{0,0} \qquad d_0 \circ f_1 \circ x_1 = f_0 \circ x_{0,1}$$

and (by the pullback pasting lemma) we have the following pullback square in \mathcal{C} ,

$$\begin{array}{ccc} X_0 & \xrightarrow{\Delta_p} & X_1 \\ f_0 \downarrow & & \downarrow f_1 \\ Y_0 & \xrightarrow{\Delta_q} & Y_1 \end{array}$$

where the horizontal arrows are the respective relative diagonals, so $x_{0,0} = x_{0,1}$, as claimed.

Thus, $(f_0, p) \cdot (-) : h_{X_0} \rightarrow \text{Pb}(q \circ -, \bar{f} \circ -)$ is indeed \mathbf{E} -locally bijective. To complete the proof, observe that if \mathbf{E} is a subcanonical unary coverage on \mathcal{C} , then both h_{X_0} and $\text{Pb}(q \circ -, \bar{f} \circ -)$ are \mathbf{E} -sheaves on \mathcal{C} , so, in that case, by [proposition A.3.7](#), $(f_0, p) \cdot (-) : h_{X_0} \rightarrow \text{Pb}(q \circ -, \bar{f} \circ -)$ is an isomorphism. \blacksquare

I.4.19(c) LEMMA. *Consider a commutative diagram in \mathcal{C} of the form below,*

Covering morphisms and pullback squares

$$\begin{array}{ccccc} \tilde{W} & \xrightarrow{w} & W & \xrightarrow{q} & Y \\ \tilde{p} \downarrow & & p \downarrow & & \downarrow g \\ \tilde{X} & \xrightarrow{x} & X & \xrightarrow{f} & Z \end{array}$$

where:

- Both $w : \tilde{W} \rightarrow W$ and $x : \tilde{X} \rightarrow X$ are \mathbf{E} -covering morphisms in \mathcal{C} .
- Both the left square and outer rectangle are pullback diagrams in \mathcal{C} .

Then $(p, q) \cdot (-) : h_W \rightarrow \text{Pb}(f \circ -, g \circ -)$ is a \mathbf{E} -locally bijective morphism in $\mathbf{PSH}(\mathcal{C})$. In particular, if \mathbf{E} is a subcanonical unary coverage on \mathcal{C} , then the right square is a pullback square in \mathcal{C} .

Proof. By [lemma I.4.19\(a\)](#), the right square is an \mathbf{E} -weak pullback square in \mathcal{C} , so $(p, q) \cdot (-) : h_W \rightarrow \text{Pb}(f \circ -, g \circ -)$ is \mathbf{E} -locally surjective, by [lemma I.4.9](#).

We will now show that $(p, q) \cdot (-) : h_W \rightarrow \text{Pb}(f \circ -, g \circ -)$ is \mathbf{E} -locally injective. Let T be an object in \mathcal{C} and let $w_0, w_1 : T \rightarrow W$ be a parallel pair of morphisms in \mathcal{C} such that:

$$p \circ w_0 = p \circ w_1 \qquad q \circ w_0 = q \circ w_1$$

Since $x : \tilde{X} \rightarrow X$ is an \mathbf{E} -covering morphism in \mathcal{C} , there is a commutative square in \mathcal{C} of the form below,

$$\begin{array}{ccc} \tilde{T} & \xrightarrow{t} & T \\ \tilde{x} \downarrow & & \downarrow p \circ w_0 \\ \tilde{X} & \xrightarrow{x} & X \end{array}$$

where $t : \tilde{T} \rightarrow T$ is also an E-covering morphism in C . Thus, there exist unique morphisms $\tilde{w}_0, \tilde{w}_1 : \tilde{T} \rightarrow \tilde{W}$ such that:

$$\begin{aligned} \tilde{p} \circ \tilde{w}_0 &= \tilde{x} & w \circ \tilde{w}_0 &= w_0 \circ t \\ \tilde{p} \circ \tilde{w}_1 &= \tilde{x} & w \circ \tilde{w}_1 &= w_1 \circ t \end{aligned}$$

But $\langle \tilde{p} \circ -, q \circ w \circ - \rangle : \hat{h}_{\tilde{W}} \rightarrow \hat{h}_{\tilde{X}} \times \hat{h}_Y$ is a monomorphism, so $\tilde{w}_0 = \tilde{w}_1$, and therefore $w_0 \circ t = w_1 \circ t$.

Thus, $(p, q) \cdot (-) : \hat{h}_W \rightarrow \text{Pb}(f \circ -, g \circ -)$ is indeed E-locally bijective. To complete the proof, observe that if E is a subcanonical unary coverage on C , then both \hat{h}_W and $\text{Pb}(f \circ -, g \circ -)$ are E-sheaves on C , so by [proposition A.3.7](#), $(p, q) \cdot (-) : \hat{h}_W \rightarrow \text{Pb}(f \circ -, g \circ -)$ is an isomorphism in that case. ■

I.4.20 DEFINITION. An **exact category** is a regular category S with the following additional data:

- For each object X in S and each equivalence relation (R, d_0, d_1) on X , an exact quotient $q : X \rightarrow \bar{X}$ of (R, d_0, d_1) in S .

REMARK. In other words, an exact category is a regular category in which every equivalence relation is a kernel pair. (Recall [remark I.3.8](#).)

I.4.21 DEFINITION. An **E-local complex** in C is a tuple (X, P, d_0, d_1) where:

- X and P are objects in C .
- d_0 and d_1 are morphisms $P \rightarrow X$ in C .
- The E-closed support of $\langle \hat{h}_{d_1}, \hat{h}_{d_0} \rangle : \hat{h}_P \rightarrow \hat{h}_X \times \hat{h}_X$ defines an equivalence relation on \hat{h}_X .

EXAMPLE. Let X be an object in C . Then $(X, X, \text{id}_X, \text{id}_X)$ is an E-local complex in C , by [lemma A.3.4](#).

Recognition principle for local complexes

LEMMA. Let $d_0, d_1 : P \rightarrow X$ be a parallel pair of morphisms in \mathcal{C} . Assuming \mathcal{C} has E-weak pullback squares, the following are equivalent:

- (i) (X, P, d_0, d_1) is an E-local complex.
- (ii) All of the following conditions are satisfied:
 - There exist an E-covering morphism $x : \tilde{X} \twoheadrightarrow X$ in \mathcal{C} and a morphism $p : \tilde{X} \rightarrow P$ in \mathcal{C} such that $d_0 \circ p = x$ and $d_1 \circ p = x$.
 - There exist E-covering morphisms $p_0 : \tilde{P} \twoheadrightarrow P$ and $p_1 : \tilde{P} \rightarrow P$ in \mathcal{C} such that $d_0 \circ p_0 = d_1 \circ p_1$ and $d_1 \circ p_0 = d_0 \circ p_1$.
 - There is an E-weak pullback square in \mathcal{C} of the form below,

$$\begin{array}{ccc} Q & \xrightarrow{d_0} & P \\ d_2 \downarrow & & \downarrow d_0 \\ P & \xrightarrow{d_1} & X \end{array}$$

and there exist an E-covering morphism $q : \tilde{Q} \twoheadrightarrow Q$ in \mathcal{C} and a morphism $p : \tilde{Q} \rightarrow P$ in \mathcal{C} such that $d_0 \circ p = d_0 \circ d_0 \circ q$ and $d_1 \circ p = d_1 \circ d_2 \circ q$.

Proof. Straightforward. ◆

1.4.22 ¶ Let (X, P, d_0, d_1) be an E-local complex.

DEFINITION. The **E-sheaf** presented by (X, P, d_0, d_1) is the E-sheaf completion^[1] of the quotient presheaf \hat{h}_X/R where R is the E-closed support of $\langle d_1, d_0 \rangle : \hat{h}_P \rightarrow \hat{h}_X \times \hat{h}_X$.

The sheaf presented by a local complex

LEMMA. Let $Q(X, P, d_0, d_1)$ be the E-sheaf presented by (X, P, d_0, d_1) and let a be the image of the universal element (X, id_X) in $Q(X, P, d_0, d_1)$. Then (X, P, a, d_0, d_1) is an E-local presentation of $Q(X, P, d_0, d_1)$.

Proof. By lemmas A.3.3 and A.3.6, $R = \text{Kr}(a \cdot -)$, so (X, P, a, d_0, d_1) is indeed an E-local presentation of $Q(X, P, d_0, d_1)$. ■

[1] Recall proposition A.3.8(d).

1.4.23 DEFINITION. The **exact completion** of (C, E) is the category $\mathbf{Ex}(C, E)$ defined as follows:

- The objects are the E -local complexes in C .
- The morphisms $(X, P, d_0, d_1) \rightarrow (Y, Q, e_0, e_1)$ are the morphisms $Q(X, P, d_0, d_1) \rightarrow Q(Y, Q, e_0, e_1)$ in $\mathbf{SH}(C, E)$.
- Composition and identities are inherited from $\mathbf{SH}(C, E)$.

The **insertion functor** $\iota : C \rightarrow \mathbf{Ex}(C, E)$ is the evident functor that sends each object X in C to the E -local complex $(X, X, \text{id}_X, \text{id}_X)$.

REMARK. In view of lemmas 1.4.14 and 1.4.22, the evident functor $Q : \mathbf{Ex}(C, E) \rightarrow \mathbf{SH}(C, E)$ is fully faithful and essentially surjective onto the full subcategory of E -locally $\mathbb{1}$ -presentable E -sheaves on C .

The exact completion is an exact category

PROPOSITION. *If (C, E) satisfies the Shulman condition, then:*

- (i) $\mathbf{Ex}(C, E)$ is an exact category.
- (ii) The insertion functor $\iota : C \rightarrow \mathbf{Ex}(C, E)$ preserves limits of finite diagrams and sends E -covering morphisms in C to effective epimorphisms in $\mathbf{Ex}(C, E)$.

Proof. (i). By theorem 1.4.16, $\mathbf{Ex}(C, E)$ has limits of finite diagrams. Moreover, lemma 1.4.14 and theorem A.3.9 imply that every equivalence relation in $\mathbf{Ex}(C, E)$ is a kernel pair and that the class of regular epimorphisms in $\mathbf{Ex}(C, E)$ is quadrable. Thus, $\mathbf{Ex}(C, E)$ is indeed an exact category.

(ii). The preservation of limits of finite diagrams is a consequence of theorem A.3.9. For the remainder of the claim, apply lemmas A.2.18 and A.3.10 to the fact that the Yoneda embedding $C \rightarrow \mathbf{PSH}(C)$ sends E -covering morphisms in C to E -locally surjective morphisms in $\mathbf{PSH}(C)$. ■

1.4.24 ¶ Let D be a category, let J be a unary coverage on D , and assume both (C, E) and (D, J) satisfy the Shulman condition.

DEFINITION. An **admissible functor** $F : (C, E) \rightarrow (D, J)$ is a functor $F : C \rightarrow D$ with the following properties:

- $F : (C, E) \rightarrow (D, J)$ is a pre-admissible functor.
- For every E-locally \mathcal{I} -presentable E-sheaf A on C , there exist a J-locally \mathcal{I} -presentable J-sheaf $F_{\downarrow}A$ on D and a morphism $\sigma_A : A \rightarrow F^*F_{\downarrow}A$ in $\mathbf{SH}(C, E)$ such that, for every J-locally \mathcal{I} -presentable J-sheaf B on D , the following is a bijection:

$$\begin{aligned} \mathrm{Hom}_{\mathbf{SH}(D, J)}(F_{\downarrow}A, B) &\rightarrow \mathrm{Hom}_{\mathbf{SH}(C, E)}(A, F^*B) \\ h &\mapsto F^*h \circ \sigma_A \end{aligned}$$

The functor between exact completions induced by an admissible functor

LEMMA. Let $F : (C, E) \rightarrow (D, J)$ be an admissible functor. Assuming E is a subcanonical unary coverage on C :

- There exist a functor $\bar{F} : \mathbf{Ex}(C, E) \rightarrow \mathbf{Ex}(D, J)$ and an isomorphism $\eta : \iota F \Rightarrow \bar{F}\iota$ of functors $C \rightarrow \mathbf{Ex}(D, J)$ such that \bar{F} sends right-exact forks in $\mathbf{Ex}(C, E)$ to coequaliser diagrams in $\mathbf{Ex}(D, J)$.
- Moreover, any such (\bar{F}, η) is a pointwise left Kan extension of $\iota F : C \rightarrow \mathbf{Ex}(D, J)$ along $\iota : C \rightarrow \mathbf{Ex}(C, E)$.

Proof. By [proposition A.3.13](#), the restriction functor $F^* : \mathbf{SH}(D, J) \rightarrow \mathbf{SH}(C, E)$ has a left adjoint, say $F_{\downarrow} : \mathbf{SH}(C, E) \rightarrow \mathbf{SH}(D, J)$. Moreover, for every object A in $\mathbf{Ex}(C, E)$, there exist an object $\bar{F}A$ in $\mathbf{Ex}(D, J)$ and an isomorphism $\iota^*h_{\bar{F}A} \cong F_{\downarrow}h_A$ in $\mathbf{SH}(D, J)$. This defines a functor $\bar{F} : \mathbf{Ex}(C, E) \rightarrow \mathbf{Ex}(D, J)$.

By definition, we have the a natural bijection of the form below:

$$\mathrm{Hom}_{\mathbf{Ex}(D, J)}(\bar{F}A, B) \cong \mathrm{Hom}_{\mathbf{SH}(C, E)}(\iota^*h_A, F^*\iota^*h_B)$$

In particular, taking $A = \iota X$ and applying the Yoneda lemma, the functor $\iota : C \rightarrow \mathbf{Ex}(C, E)$ induces a natural map

$$\mathrm{Hom}_{\mathbf{Ex}(D, J)}(\bar{F}\iota X, B) \rightarrow \mathrm{Hom}_{\mathbf{Ex}(D, J)}(\iota F X, B)$$

and hence a natural transformation $\eta : \iota F \Rightarrow \bar{F}\iota$. The pair (\bar{F}, η) is then a pointwise left Kan extension of $\iota F : C \rightarrow \mathbf{Ex}(D, J)$ along $\iota : C \rightarrow$

$\mathbf{Ex}(C, E)$. Furthermore, because $\iota : C \rightarrow \mathbf{Ex}(C, E)$ is fully faithful, $\eta : \iota F \Rightarrow \bar{F}\iota$ is an isomorphism.

Since the Yoneda representation $\mathbf{Ex}(C, E) \rightarrow \mathbf{SH}(C, J)$ preserves right-exact forks, $\bar{F} : \mathbf{Ex}(C, E) \rightarrow \mathbf{Ex}(D, J)$ sends right-exact forks in $\mathbf{Ex}(C, E)$ to coequaliser diagrams in $\mathbf{Ex}(D, J)$. It is clear that any pair (\bar{F}, η) as in (i) is determined (up to isomorphism) by $\bar{F}\iota : C \rightarrow \mathbf{Ex}(D, J)$, so any such (\bar{F}, η) must be a pointwise left Kan extension as constructed above. ■

REMARK. We will later see a converse to the above result, i.e. that the restriction of an appropriate functor between the exact completions is admissible.

1.4.25 ¶ Assume (C, E) satisfies the Shulman condition. Let S be an exact category, let K be the class of effective epimorphisms in S , and let $F : C \rightarrow S$ be a functor with the following properties.

- $F : C \rightarrow S$ sends E -local complexes in C to K -local complexes in S .
- $F : C \rightarrow S$ sends right- E -exact forks in C to right- K -exact forks in S .

EXAMPLE. If C has limits of finite diagrams and $F : C \rightarrow S$ is a functor that preserves limits of finite diagrams and sends members of E to effective epimorphisms in S , then $F : C \rightarrow S$ has the above properties.

LEMMA. *Under the above hypotheses, $F : (C, E) \rightarrow (S, K)$ is an admissible functor.*

Proof. Recalling lemma 1.4.18, it is not hard to see that $F : (C, E) \rightarrow (S, K)$ is a pre-admissible functor. Thus, by proposition A.3.13, $F^* : \mathbf{SH}(S, K) \rightarrow \mathbf{SH}(C, E)$ has a left adjoint, say $F_! : \mathbf{SH}(C, E) \rightarrow \mathbf{SH}(S, K)$. The Yoneda embedding $S \rightarrow \mathbf{SH}(S, K)$ preserves right-exact forks, and $F_! : \mathbf{SH}(C, E) \rightarrow \mathbf{SH}(S, K)$ preserves coequalisers, so for every E -local complex (X, P, d_0, d_1) in C , $F_! : \mathbf{SH}(C, E) \rightarrow \mathbf{SH}(S, K)$ sends the E -sheaf presented by (X, P, d_0, d_1) to a representable K -sheaf on S , as required. ■

1.5 Strict coproducts

SYNOPSIS. We study extensive categories, i.e. categories in which coproducts behave like disjoint unions, and we examine the properties of exact completions of extensive categories.

PREREQUISITES. §§ [I.1](#), [I.3](#), [I.4](#), [A.1](#), [A.2](#), [A.3](#).

1.5.1 ¶ Let \mathcal{C} be a category and let κ be a regular cardinal.

DEFINITION. The κ -**ary canonical coverage** on \mathcal{C} is the coverage J where $J(X)$ is the set of κ -small sinks Φ on X with the following property:

- For every object (T, x) in $\mathcal{C}/_X$, there is a κ -small strict-epimorphic sink Θ on T such that $\downarrow(\Theta) \subseteq x^* \downarrow(\Phi)$.

REMARK. In general, the κ -ary canonical coverage may fail to be upward-closed, but it is always composition-closed (by [lemma A.2.3\(c\)](#) and [corollary A.2.3\(d\)](#)).

1.5.2 ¶ Let \mathcal{C} be a category, let Y be an object in \mathcal{C} and let Φ be a set of objects in $\mathcal{C}/_Y$.

1.5.2(a) DEFINITION. The object Y is the **disjoint union** of Φ if the following conditions are satisfied:

- The cocone $(f \mid (X, f) \in \Phi)$ is a coproduct cocone in \mathcal{C} .
- For every $(X, f) \in \Phi$, the morphism $f : X \rightarrow Y$ is a monomorphism in \mathcal{C} .
- For every commutative square in \mathcal{C} of the form below,

$$\begin{array}{ccc} T & \xrightarrow{x_1} & X_0 \\ x_0 \downarrow & & \downarrow f_0 \\ X_1 & \xrightarrow{f_1} & Y \end{array}$$

if (X_0, f_0) and (X_1, f_1) are in Φ , then either $(X_0, f_0) = (X_1, f_1)$ or T is an initial object in \mathcal{C} (or both).

1.5.2(b) DEFINITION. The cocone $(f \mid (X, f) \in \Phi)$ is a **strict coproduct cocone** in \mathcal{C} if it has the following properties:

- For every $(X, f) \in \Phi$, $f : X \rightarrow Y$ is a quadrable morphism in \mathcal{C} .
- For every object (T, y) in $\mathcal{C}/_Y$, given pullback squares in \mathcal{C} of the form below

$$\begin{array}{ccc} \mathcal{S}_{(X,f)} & \longrightarrow & X \\ t_{(X,f)} \downarrow & & \downarrow f \\ T & \xrightarrow{y} & Y \end{array}$$

for every $(X, f) \in \Phi$, the object T is a disjoint union of the following set:

$$\{(\mathcal{S}_{(X,f)}, t_{(X,f)}) \mid (X, f) \in \Phi\}$$

REMARK. In particular, when $\Phi = \emptyset$, this reduces to the notion of **strict initial object**. More explicitly, Y is a strict initial object if and only if, for every morphism $y : T \rightarrow Y$ in \mathcal{C} , T is an initial object in \mathcal{C} .

Pullbacks along strict coproduct cocones PROPOSITION. Assuming $(f \mid (X, f) \in \Phi)$ is a strict coproduct cocone, the functor

$$\mathcal{C}/_Y \rightarrow \prod_{(X,f) \in \Phi} \mathcal{C}/_X$$

defined by pullback is fully faithful.

Proof. Straightforward. ◆

1.5.3 ※ For the remainder of this section, κ is a regular cardinal.

1.5.4 ¶ Let \mathcal{C} be a κ -ary coherent category.

Recognition principle for disjoint unions in coherent categories LEMMA. Let Y be an object in \mathcal{C} and let Φ be a κ -small set of subobjects of Y . The following are equivalent:

- (i) Y is a disjoint union of Φ .
- (ii) (Y, id_Y) is a coproduct of Φ in the category of subobjects of Y and, for every commutative square in \mathcal{C} of the form below,

$$\begin{array}{ccc} T & \longrightarrow & X_1 \\ \downarrow & & \downarrow f_1 \\ X_0 & \xrightarrow{f_0} & Y \end{array}$$

if (X_0, f_0) and (X_1, f_1) are both in Φ , then either $(X_0, f_0) = (X_1, f_1)$ or T is an initial object in \mathcal{C} (or both).

Proof. (i) \Rightarrow (ii). Immediate.

(ii) \Rightarrow (i). For the case where Φ has two elements, see the proof of Proposition 1.4.3 in [Johnstone, 2002, Part A]. The general case is similar. \square

1.5.5 DEFINITION. A κ -ary extensive category is a category \mathcal{C} with the following data:

- For every family $(X_i \mid i \in I)$ of objects in \mathcal{C} where I is a κ -small set,^[1] an object $\coprod_{i \in I} X_i$ in \mathcal{C} and a strict coproduct cocone $(f_i \mid i \in I)$ in \mathcal{C} where $\text{dom } f_i = X_i$ and $\text{codom } f_i = \coprod_{i \in I} X_i$.

REMARK. In particular, a κ -ary extensive category has a (chosen) strict initial object, say 0 .

1.5.6 DEFINITION. A κ -ary pretopos is an exact category that is also a κ -ary extensive category.

1.5.6(a) LEMMA. *Every κ -ary pretopos is a κ -ary coherent category.*

Proof. Straightforward. (Define the union of a set of subobjects to be the exact image of their coproduct.) \blacklozenge

1.5.6(b) LEMMA. *Let \mathcal{C} be a κ -ary coherent category and let \mathcal{B} be a set of objects in \mathcal{C} . Assume the following hypotheses:*

Criterion for
existence of
coproducts
in coherent
categories

- \mathcal{C} is an exact category.
- For every object X in \mathcal{C} , there is an effective epimorphism $p : \tilde{X} \rightarrow X$ in \mathcal{C} where \tilde{X} is in \mathcal{B} .
- Strict coproducts of κ -small families of objects in \mathcal{B} exist in \mathcal{C} .

Then \mathcal{C} is a κ -ary pretopos.

[1] Since there is a proper class of κ -small sets, strictly speaking, one should restrict to e.g. hereditarily κ -small sets here.

Proof. Let $(X_i \mid i \in I)$ be a family of objects in \mathcal{C} . For each $i \in I$, choose an effective epimorphism $p_i : \tilde{X}_i \twoheadrightarrow X_i$ in \mathcal{C} with \tilde{X}_i in \mathcal{B} , and then choose a pullback square in \mathcal{C} of the form below:

$$\begin{array}{ccc} R_i & \xrightarrow{d_0} & \tilde{X}_i \\ d_1 \downarrow & & \downarrow p_i \\ \tilde{X}_i & \xrightarrow{p_i} & X_i \end{array}$$

Let $\tilde{X} = \coprod_{i \in I} \tilde{X}_i$. It is not hard to verify that the canonical morphism

$$\coprod_{(j,k) \in I \times I} \tilde{X}_j \times \tilde{X}_k \rightarrow \tilde{X} \times \tilde{X}$$

in \mathcal{C} is an isomorphism. Thus, for each $i \in I$, the composite $R_i \rightarrow \tilde{X}_i \times \tilde{X}_i \rightarrow \tilde{X} \times \tilde{X}$ is a monomorphism in \mathcal{C} . Let $\langle d_1, d_0 \rangle : R \rightarrow \tilde{X} \times \tilde{X}$ be the union. Then (R, d_0, d_1) is an equivalence relation on \tilde{X} , so there is an exact fork in \mathcal{C} of the form below:

$$R \begin{array}{c} \xrightarrow{d_1} \\ \xrightarrow{d_0} \end{array} \tilde{X} \xrightarrow{p} \twoheadrightarrow X$$

Moreover, for each $i \in I$, there is a unique morphism $m_i : X_i \rightarrow X$ in \mathcal{C} such that the following diagram in \mathcal{C} commutes,

$$\begin{array}{ccc} \tilde{X}_i & \xrightarrow{p_i} & X_i \\ \downarrow & & \downarrow m_i \\ \tilde{X} & \xrightarrow{p} & X \end{array} \quad (*)$$

where $\tilde{X}_i \rightarrow \tilde{X}$ is the coproduct injection. Consider a commutative diagram in \mathcal{C} of the form below,

$$\begin{array}{ccccc} R_{j,k} & \twoheadrightarrow & \bullet & \twoheadrightarrow & \tilde{X}_k \\ \downarrow & & \downarrow & & \downarrow \\ \bullet & \twoheadrightarrow & R & \xrightarrow{d_0} & \tilde{X} \\ \downarrow & & d_1 \downarrow & & \downarrow p \\ \tilde{X}_j & \twoheadrightarrow & \tilde{X} & \xrightarrow{p} & X \end{array}$$

where each square is a pullback square in \mathcal{C} . Then, either $j = k$ or $R_{j,k}$ is an initial object in \mathcal{C} (or both). In view of [lemma 1.3.11](#), it follows that,

for every pullback square in \mathcal{C} of form below,

$$\begin{array}{ccc} T & \longrightarrow & X_k \\ \downarrow & & \downarrow m_k \\ X_j & \xrightarrow{m_j} & X \end{array}$$

either $j = k$ or T is an initial object in \mathcal{C} (or both). Hence, $(*)$ is a pullback square in \mathcal{C} , and by lemmas 1.3.8 and 1.3.9, $m_i : X_i \rightarrow X$ is a monomorphism. We then apply lemma 1.5.4 to deduce that X is a disjoint union of $\{(X_i, m_i) \mid i \in I\}$, as required. ■

1.5.7 ¶ Let \mathcal{C} be a category and let $\mathbf{Fam}_\kappa(\mathcal{C})$ be full subcategory of $\mathbf{FAM}(\mathcal{C})$ spanned by those families X such that $\text{idx } X$ is a hereditarily κ -small set.

PROPOSITION. $\mathbf{Fam}_\kappa(\mathcal{C})$ is a κ -ary extensive category.

Proof. Straightforward. ◆

The free coproduct completion

THEOREM. Let \mathcal{D} be a category with κ -ary coproducts, let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor, and let $\gamma : \mathcal{C} \rightarrow \mathbf{Fam}_\kappa(\mathcal{C})$ be defined as in proposition A.1.8.

- (i) There exist a functor $\bar{F} : \mathbf{Fam}_\kappa(\mathcal{C}) \rightarrow \mathcal{D}$ that preserves κ -ary coproducts and an isomorphism $\eta : F \Rightarrow \bar{F}\gamma$ of functors $\mathcal{C} \rightarrow \mathcal{D}$.
- (ii) Moreover, any such (\bar{F}, η) is a left Kan extension of $F : \mathcal{C} \rightarrow \mathcal{D}$ along $\gamma : \mathcal{C} \rightarrow \mathbf{Fam}_\kappa(\mathcal{C})$.

Proof. (i). Straightforward.

(ii). Let $G : \mathbf{Fam}_\kappa(\mathcal{C}) \rightarrow \mathcal{D}$ be a functor and let $\varphi : F \Rightarrow G\gamma$ be a natural transformation. For every object X in $\mathbf{Fam}_\kappa(\mathcal{C})$, there is a unique morphism $\theta_X : \bar{F}(X) \rightarrow G(X)$ making the diagram in \mathcal{C} shown below commute for every $i \in \text{idx } X$,

$$\begin{array}{ccc} F(X(i)) & \xrightarrow{v_i} & \bar{F}(X) \\ \varphi_{X(i)} \downarrow & & \downarrow \theta_X \\ G(\gamma(X)) & \xrightarrow{G(u_i)} & G(X) \end{array}$$

where $u_i : \gamma(X(i)) \rightarrow X$ and $v_i : F(X(i)) \rightarrow \bar{F}(X)$ are the respective coproduct insertions. It is clear that we obtain a natural transformation $\theta : \bar{F} \Rightarrow G$, and it is unique one such that $\theta\gamma \bullet \eta = \varphi$. ■

1.5.8 ¶ Let C be a category with a strict initial object 0 and let E be a unary coverage on C .

Strict initial objects and locally \mathbb{I} -generable presheaves

LEMMA. *Let A be a E -locally \mathbb{I} -generable presheaf on C . Then $A(0)$ has a unique element.*

Proof. Let (X, a) be an E -local generator of A and let \perp_X be the unique morphism $0 \rightarrow A$ in C . Of course, $a \cdot \perp_X \in A(0)$; it remains to be shown that it is the unique element of $A(0)$. Suppose $a' \in A(0)$. Since every morphism in C with codomain 0 is an isomorphism, by [lemma 1.4.7](#), there is a morphism $f' : 0 \rightarrow X$ in C such that $a' = a \cdot f'$. But $f' = \perp_X$, so we indeed have $a' = a \cdot \perp_X$. ■

The initial locally \mathbb{I} -generable presheaf

PROPOSITION. *The representable presheaf h_0 is an initial object in the metacategory of E -locally \mathbb{I} -generable presheaves on C .*

Proof. It is clear that h_0 is E -locally \mathbb{I} -generable, and the Yoneda lemma reduces the claim to [lemma 1.5.8](#). ■

1.5.9 ✖ For the remainder of this section, C is a κ -ary extensive category.

1.5.10 **DEFINITION.** A **complemented monomorphism** in C is a monomorphism $f : X \rightarrow Y$ in C for which there is an object (X', f') in $C_{/Y}$ such that Y is a disjoint union of $\{(X, f), (X', f')\}$.

Properties of complemented monomorphisms

PROPOSITION.

- (i) *Every isomorphism in C is a complemented monomorphism in C .*
- (ii) *The class of complemented monomorphisms in C is closed under composition.*
- (iii) *The class of complemented monomorphisms in C is a quadrable class of morphisms in C .*
- (iv) *The class of complemented monomorphisms in C is closed under κ -ary coproduct in C .*

Proof. Straightforward. ◆

1.5.11 DEFINITION. A unary coverage E on C is κ -**summable** if it has the following property:

- For every family $(f_i \mid i \in I)$ of E -covering morphisms in C , if I is a κ -small set, then the coproduct

$$\coprod_{i \in I} f_i : \coprod_{i \in I} \text{dom } f_i \rightarrow \coprod_{i \in I} \text{codom } f_i$$

is also an E -covering morphism in C .

REMARK. The above is a condition on the whole class of E -covering morphisms in C , not just the members of E .

1.5.12 ¶ Let E be a κ -summable unary coverage on C .

1.5.12(a) LEMMA. *Let B be a E -locally I -generable presheaf on C . The set of E -locally I -generable E -closed subpresheaves of B (partially ordered by inclusion) has κ -ary joins.*

Joins of locally I -generable closed subpresheaves

Proof. Let (Y, b) be an E -local generator of B and let Φ be a κ -small set of elements of B . We must show that the set of E -closed subpresheaves of B generated by the members of Φ admit a join in the set of E -locally I -generable E -closed subpresheaves of B . By [corollary 1.4.7](#), we may assume that, for each $(X, a) \in \Phi$, there is some morphism $f : X \rightarrow Y$ in C such that $b \cdot f = a$. Let $\bar{X} = \coprod_{(X,a) \in \Phi} X$, let $\bar{f} : \bar{X} \rightarrow Y$ be the induced morphism in C , let $\bar{a} = b \cdot \bar{f}$, and let \bar{A} be the E -closed subpresheaf of B E -locally generated by (\bar{X}, \bar{a}) . Clearly, for every $(X, a) \in \Phi$, we have $a \in \bar{A}(X)$. It remains to be shown that \bar{A} is the smallest E -locally I -generable E -closed subpresheaf with this property.

Let A' be an E -closed subpresheaf of B E -locally generated by (X', a') and suppose, for every $(X, a) \in \Phi$, we have $a \in A'(X)$. As before, we may assume that $a' = b \cdot f'$ for some morphism $f' : X' \rightarrow Y$ in C . By [lemma 1.4.7](#), for each $(X, a) \in \Phi$, we have an E -covering morphism $p'_{(X,a)}$ in C and a morphism $f'_{(X,a)}$ in C such that $a' \cdot f'_{(X,a)} = a \cdot p'_{(X,a)}$. Let $\tilde{X} = \coprod_{(X,a) \in \Phi} \text{dom } f'_{(X,a)}$ and let $f' : \tilde{X} \rightarrow Y$ be the induced morphism in C . Then $a' \cdot f' = \bar{a} \cdot \coprod_{(X,a) \in \Phi} f'_{(X,a)}$, so $\bar{a} \in A'(X)$. Thus, $\bar{A} \subseteq A'$, so \bar{A} is the desired join. ■

1.5.12 (b) **LEMMA.** *Assuming (C, E) satisfies the Shulman condition, κ -ary joins of Pullbacks of E-locally 1-generable E-closed subpresheaves of E-locally 1-presentable presheaves on C are preserved by pullback.*

Proof. Let $h : A \rightarrow B$ be a morphism of E-locally 1-presentable presheaves on C . By [corollary 1.4.16](#), we may choose E-local generators (X, a) and (Y, b) of A and B (respectively) and a morphism $f : X \rightarrow Y$ in C such that $b \cdot f = h(a)$ and the induced morphism $h_X \rightarrow A \times_B h_Y$ is E-locally surjective. Note that, for every E-locally 1-generable subpresheaf $B' \subseteq B$, [lemma 1.4.13](#) and [theorem 1.4.16](#) imply that $h^{-1}B'$ is a E-locally 1-generable subpresheaf of A . On the other hand, by [corollary 1.4.7](#), there is a morphism $y : Y' \rightarrow Y$ in C such that B' is E-locally generated by $(Y', b \cdot y)$. We may then choose an E-weak pullback square in C of the form below,

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ x_i \downarrow & & \downarrow y \\ X & \xrightarrow{f} & Y \end{array}$$

and by the weak pullback pasting lemma ([lemma A.2.19](#)), $(X', a \cdot x)$ is an E-local generator of $A' = h^{-1}B'$.

Now, let I be a κ -small set, and for each $i \in I$, let $x_i : X'_i \rightarrow X$, $y_i : Y'_i \rightarrow Y$, and $f'_i : X'_i \rightarrow Y'_i$ be morphisms in C such that:

- $f \circ x_i = y_i \circ f'_i$.
- $(X'_i, a \cdot x_i)$ is an E-local generator of $A'_i = h^{-1}B'_i$, where B'_i is the E-closed subpresheaf of B E-locally generated by $(Y'_i, b \cdot y_i)$.

We will show that the join of $\{A'_i \mid i \in I\}$ is the pullback of the join of $\{B'_i \mid i \in I\}$. Let $X' = \coprod_{i \in I} X'_i$, let $Y' = \coprod_{i \in I} Y'_i$, let $x : X' \rightarrow X$, $y : Y' \rightarrow Y$, $f' : X' \rightarrow Y'$ be the induced morphisms in C , let A' be the E-closed subpresheaf of A E-locally generated by $(X', a \cdot x)$, and let B' be the E-closed subpresheaf of B E-locally generated by $(Y', b \cdot y)$. By [lemma 1.5.12\(a\)](#), A' is the join of $\{A'_i \mid i \in I\}$ and B' is the join of $\{B'_i \mid i \in I\}$, so $A' \subseteq h^{-1}B'$. It remains to be shown that $h^{-1}B' \subseteq A'$.

As in the first paragraph, choose an E-weak pullback square in \mathcal{C} of the form below.

$$\begin{array}{ccc} \tilde{X}' & \xrightarrow{\tilde{f}'} & Y' \\ \tilde{x}' \downarrow & & \downarrow y \\ X & \xrightarrow{f} & Y \end{array}$$

Note that $(\tilde{X}', a \cdot \tilde{x})$ is an E-local generator of $h^{-1}B'$. To complete the proof, we must verify that $a \cdot \tilde{x} \in A''(\tilde{X}')$. Since \mathcal{C} is a κ -ary extensive category, we may assume that $\tilde{X}' = \coprod_{i \in I} \tilde{X}'_i$ and $\tilde{f}' = \coprod_{i \in I} \tilde{f}'_i$ where each \tilde{f}'_i is a morphism $\tilde{X}'_i \rightarrow Y'_i$ in \mathcal{C} . Let $\tilde{x}_i : \tilde{X}'_i \rightarrow X$ be the composite of the coproduct injection $\tilde{X}'_i \rightarrow \tilde{X}$ and $\tilde{x} : \tilde{X} \rightarrow X$. Then,

$$h(a \cdot \tilde{x}_i) = b \cdot (f \circ \tilde{x}_i) = b \cdot (y_i \circ \tilde{f}'_i)$$

so we have $a \cdot \tilde{x}_i \in A'_i(\tilde{X}'_i) = h^{-1}B'_i(\tilde{X}'_i)$.

Now, (by the pullback pasting lemma) there exist E-covering morphisms $\tilde{x}'_i : \tilde{X}''_i \rightarrow \tilde{X}'_i$ and $x'_i : \tilde{X}''_i \rightarrow X'_i$ in \mathcal{C} such that $f'_i \circ x'_i = \tilde{f}'_i \circ \tilde{x}'_i$. Let $\tilde{X}'' = \coprod_{i \in I} \tilde{X}''_i$, let $\tilde{x}' = \coprod_{i \in I} \tilde{x}'_i$, and let $x' = \coprod_{i \in I} x'_i$. Since E is κ -summable, $x' : \tilde{X}'' \rightarrow X'$ is an E-covering morphism. Moreover, by construction, $x \circ x' = \tilde{x} \circ \tilde{x}'$, so $a \cdot (\tilde{x} \circ \tilde{x}') \in A'(\tilde{X}'')$. But A' is E-closed, so $a \cdot \tilde{x} \in A'(\tilde{X}')$, as required. \blacksquare

1.5.13 ¶ The following is a partial generalisation of Proposition 2.1 in [Gran and Vitale, 1998]. (Note that an extensive category satisfying the Shulman condition with respect to the trivial coverage is weakly lextensive, by Proposition 1.2 in op. cit.)

The exact completion is a pretopos

PROPOSITION. *If E is a κ -summable unary coverage on C and $(\mathcal{C}, \mathbf{E})$ satisfies the Shulman condition, then:*

- (i) $\mathbf{Ex}(\mathcal{C}, \mathbf{E})$ is a κ -ary pretopos.
- (ii) The insertion functor $\iota : \mathcal{C} \rightarrow \mathbf{Ex}(\mathcal{C}, \mathbf{E})$ preserves limits of finite diagrams and coproducts of κ -small families of objects, and sends E-covering morphisms in \mathcal{C} to effective epimorphisms in $\mathbf{Ex}(\mathcal{C}, \mathbf{E})$.

Proof. By proposition 1.4.23, $\mathbf{Ex}(\mathcal{C}, \mathbf{E})$ is a regular category, and by lemmas 1.5.12(a) and 1.5.12(b), every κ -small set of subobjects of every object has an exact union. Thus, $\mathbf{Ex}(\mathcal{C}, \mathbf{E})$ is a κ -ary coherent category.

We already know that $\iota : \mathcal{C} \rightarrow \mathbf{Ex}(\mathcal{C}, \mathbf{E})$ preserves limits of finite diagrams and sends \mathbf{E} -covering morphisms in \mathcal{C} to effective epimorphisms in $\mathbf{Ex}(\mathcal{C}, \mathbf{E})$. On the other hand, by lemma 1.5.4, it also preserves coproducts of κ -small families of objects. We then apply lemma 1.5.6(b) to deduce that $\mathbf{Ex}(\mathcal{C}, \mathbf{E})$ has strict coproducts of all κ -small families of objects. ■

1.5.14 ¶ For each object Y in \mathcal{C} , let $\mathbf{K}(Y)$ be the set of κ -small subsets $\Phi \subseteq \text{ob } \mathcal{C}_{/Y}$ such that $(f \mid (X, f) \in \Phi)$ is a (strict) coproduct cocone in \mathcal{C} .

DEFINITION. The κ -ary extensive coverage on \mathcal{C} is the coverage \mathbf{K} defined above.

PROPOSITION. \mathbf{K} as defined above is a composition-closed coverage on \mathcal{C} .

Proof. Straightforward. ◆

1.5.15 **LEMMA.** Let A be a presheaf on \mathcal{C} . The following are equivalent:

Recognition principle for sheaves on extensive sites

- (i) A is a \mathbf{K} -sheaf on \mathcal{C} .
- (ii) $A : \mathcal{C}^{\text{op}} \rightarrow \mathbf{SET}$ sends κ -ary coproducts in \mathcal{C} to κ -ary products in \mathbf{SET} .

Proof. (i) \Rightarrow (ii). Let Y be an object in \mathcal{C} , let Φ be a κ -small set of objects in $\mathcal{C}_{/Y}$, and suppose Y is a disjoint union of Φ . For each $(X, f) \in \Phi$, let $a_{(X, f)}$ be an element of $A(X)$. Since A is \mathbf{K} -separated, there is at most one element a of $A(Y)$ such that $a \cdot f = a_{(X, f)}$ for all $(X, f) \in \Phi$. Let \mathcal{U} be the sieve on Y generated by Φ . Since $A : \mathcal{C}^{\text{op}} \rightarrow \mathbf{SET}$ preserves terminal objects, for every commutative square in \mathcal{C} of the form below,

$$\begin{array}{ccc} T & \xrightarrow{x_1} & X_0 \\ x_0 \downarrow & & \downarrow f_0 \\ X_1 & \xrightarrow{f_1} & Y \end{array}$$

if (X_0, f_0) and (X_1, f_1) are in Φ , then $a_{(X_0, f_0)} \cdot x_0 = a_{(X_1, f_1)} \cdot x_1$. But A satisfies the sheaf condition with respect to \mathcal{U} , so there is indeed an element a of $A(Y)$ such that $a \cdot f = a_{(X, f)}$ for all $(X, f) \in \Phi$. Hence, the canonical map $A(Y) \rightarrow \prod_{(X, f) \in \Phi} A(X)$ is a bijection.

(ii) \Rightarrow (i). Straightforward. ■

The Yoneda embedding into sheaves on an extensive site

THEOREM.

- (i) \mathbf{K} is a subcanonical coverage on \mathcal{C} , i.e. every representable presheaf on \mathcal{C} is a \mathbf{K} -sheaf on \mathcal{C} .
- (ii) The Yoneda embedding $h_{\bullet} : \mathcal{C} \rightarrow \mathbf{SH}(\mathcal{C}, \mathbf{K})$ preserves κ -ary (strict) coproducts.

Proof. Apply lemma 1.5.15. ■

1.5.16 DEFINITION. A coverage \mathbf{J} on \mathcal{C} is **κ -ary superextensive** if it has the following properties:

- For every object X in \mathcal{C} , every element of $\mathbf{J}(X)$ is a κ -small sink on X .
- For every object X in \mathcal{C} , $\mathbf{K}(X) \subseteq \mathbf{J}(X)$, where \mathbf{K} is the κ -ary extensive coverage on \mathcal{C} .

EXAMPLE. The κ -ary extensive coverage on \mathcal{C} is κ -ary superextensive, and by lemma 1.5.15 and theorem 1.5.15, the κ -ary canonical coverage on \mathcal{C} is also κ -ary superextensive.

Recognition principle for covering sinks in superextensive coverages

LEMMA. Let X be an object in \mathcal{C} , let Φ be a κ -small set of objects in $\mathcal{C}/_X$, let $\bar{U} = \coprod_{(U,x) \in \Phi} U$, and let $\bar{x} : \bar{U} \rightarrow X$ be the induced morphism in \mathcal{C} . Assuming \mathbf{J} is a κ -ary superextensive coverage on \mathcal{C} , the following are equivalent:

- (i) Φ is a \mathbf{J} -covering sink on X .
- (ii) $\bar{x} : \bar{U} \rightarrow X$ is a \mathbf{J} -covering morphism in \mathcal{C} .

Proof. Straightforward. ◆

The unary coverage associated with a superextensive coverage

COROLLARY. Let \mathbf{J} be a coverage on \mathcal{C} and \mathbf{E} be the class of \mathbf{J} -covering morphisms in \mathcal{C} . If \mathbf{J} is a κ -ary superextensive coverage on \mathcal{C} , then \mathbf{E} is a κ -summable saturated unary coverage on \mathcal{C} .

Proof. First, we must verify that \mathbf{E} is a unary coverage. Let $f : X \rightarrow Y$ be a morphism in \mathcal{C} and let $q : \tilde{Y} \twoheadrightarrow Y$ be a \mathbf{J} -covering morphism in \mathcal{C} . Recalling paragraph A.2.13, we see that there is a κ -small \mathbf{J} -covering

sink Φ on X such that $\downarrow(\Phi) \subseteq f^* \downarrow \langle q \rangle$. Thus, by [lemma 1.5.16](#), there is a commutative square in \mathcal{C} of the form below,

$$\begin{array}{ccc} \tilde{X} & \dashrightarrow & \tilde{Y} \\ p \downarrow & & \downarrow q \\ X & \xrightarrow{f} & Y \end{array}$$

where $p : \tilde{X} \twoheadrightarrow X$ is a J-covering morphism in \mathcal{C} . It is straightforward to check that E is κ -summable. Finally, by [proposition A.2.14](#), E is indeed upward-closed and composition-closed. ■

1.5.17 ¶ Let E be a κ -summable coverage on \mathcal{C} and, for each object X in \mathcal{C} , let $J(X)$ be the set of κ -small sinks Φ on X such that the induced morphism $\coprod_{(U,x) \in \Phi} U \rightarrow X$ in \mathcal{C} is a E -covering morphism in \mathcal{C} .

Properties of
the superex-
tensive coverage
generated by
a summable
unary coverage

PROPOSITION.

- (i) J is a κ -ary superextensive composition-closed coverage on \mathcal{C} .
- (ii) A morphism in \mathcal{C} is E -covering if and only if it is J -covering.
- (iii) If E is a subcanonical unary coverage on \mathcal{C} , then J is a subcanonical coverage on \mathcal{C} .
- (iv) Assuming K is a κ -ary superextensive coverage on \mathcal{C} , if every E -covering morphism in \mathcal{C} is also K -covering, then every J -covering sink in \mathcal{C} is also K -covering.
- (v) Assuming $A : \mathcal{C}^{\text{op}} \rightarrow \mathbf{SET}$ sends κ -ary coproducts in \mathcal{C} to κ -ary products in \mathbf{SET} , A is an E -sheaf on \mathcal{C} if and only if A is a J -sheaf on \mathcal{C} .

Proof. (i). It is clear that J is a κ -ary superextensive coverage on \mathcal{C} , and J is composition-closed because E is κ -summable.

(ii). By construction, every E -covering morphism in \mathcal{C} is also J -covering. For the converse, suppose $f : X \rightarrow Y$ is a J -covering morphism in \mathcal{C} . Recalling [paragraph A.2.13](#), since J is composition-closed, there is a κ -small sink Φ on X such that the induced morphism $\coprod_{(U,x) \in \Phi} U \rightarrow X \rightarrow Y$ in \mathcal{C} is a E -covering morphism in \mathcal{C} . But that implies $f : X \rightarrow Y$ itself is a E -covering morphism in \mathcal{C} , so we are done.

(iii) and (iv). Apply lemma 1.5.16.

(v). Combine lemma 1.5.15 and (iii). ■

REMARK. Let \mathbf{K} be the κ -ary extensive coverage on \mathcal{C} and let $J'(X)$ be defined as follows:

$$J'(X) = \mathbf{K}(X) \cup \{(U, x) \in \text{ob } \mathcal{C}_{/X} \mid x \in E\}$$

Then J' is the smallest κ -ary superextensive coverage on \mathcal{C} containing E . In general, J' is strictly smaller than J , but the above proposition shows that they have the same covering sinks.

1.5.18 ¶ Let J be a subcanonical κ -ary superextensive coverage on \mathcal{C} and let \mathbf{K} be the κ -ary extensive coverage on \mathcal{C} .

LEMMA. Let $h : A \rightarrow B$ be a complemented monomorphism in $\mathbf{SH}(\mathcal{C}, J)$. If B is a representable J -sheaf, then A is also a representable J -sheaf.

Proof. By hypothesis, there is a (complemented) monomorphism $h' : A' \rightarrow B$ in $\mathbf{SH}(\mathcal{C}, J)$ such that B is the disjoint union of $\{(A, h), (A', h')\}$ in $\mathbf{SH}(\mathcal{C}, J)$. Thus, we have a unique morphism $p : B \rightarrow B \amalg B$ such that the following diagram in $\mathbf{SH}(\mathcal{C}, J)$ commutes,

$$\begin{array}{ccccc} A & \xrightarrow{h} & B & \xleftarrow{h'} & A' \\ h \downarrow & & \downarrow p & & \downarrow h' \\ B & \longrightarrow & B \amalg B & \longleftarrow & B \end{array}$$

where the bottom row is a coproduct diagram. By theorem 1.5.15, the Yoneda embedding $\mathcal{C} \rightarrow \mathbf{SH}(\mathcal{C}, J)$ preserves κ -ary coproducts, so $B \amalg B$ is a representable J -sheaf on \mathcal{C} . But coproduct injections are quadrable in \mathcal{C} and the Yoneda embedding $\mathcal{C} \rightarrow \mathbf{SH}(\mathcal{C}, J)$ preserves pullbacks, so it follows that both A and A' are indeed representable J -sheaves on \mathcal{C} . ■

PROPOSITION. *The inclusion $\mathbf{SH}(C, J) \hookrightarrow \mathbf{SH}(C, K)$ preserves κ -ary coproducts.*

Proof. Let $(A_i \mid i \in I)$ be a family of J -sheaves C where I is a κ -small set and let A be their coproduct in $\mathbf{SH}(C, J)$. It suffices to show that the coproduct cocone is jointly K -locally surjective. Moreover, since coproduct cocones are preserved by pullback, we may assume that A is a representable sheaf on C . But then lemma 1.5.18 says that each A_i is also representable, so the coproduct cocone in question is indeed jointly K -locally surjective. ■

REMARK. The above result is optimal in the following sense: if λ is a regular cardinal $> \kappa$ such that C is a non-trivial λ -ary extensive category and L is the λ -ary extensive coverage on C , then the inclusion $\mathbf{SH}(C, L) \hookrightarrow \mathbf{SH}(C, K)$ preserves κ -ary coproducts but not λ -ary coproducts.

1.5.19 ¶ In general, an exact category may not have coequalisers of all parallel pairs. However:

Coequalisers in pretoposes PROPOSITION. *Let S be a κ -ary pretopos and let K be the κ -ary canonical coverage on S . Assuming $\kappa > \aleph_0$:*

- (i) *S has coequalisers of all parallel pairs.*
- (ii) *If \mathcal{T} is a κ -ary pretopos and $F : S \rightarrow \mathcal{T}$ is a functor that preserves limits of finite diagrams, κ -ary coproducts, and exact quotients, then $F : S \rightarrow \mathcal{T}$ preserves colimits of κ -small diagrams.*
- (iii) *In particular, the Yoneda embedding $S \rightarrow \mathbf{SH}(S, K)$ preserves colimits of κ -small diagrams.*

Proof. (i) and (ii). See the proof of Lemma 1.4.19 in [Johnstone, 2002, Part A].

(iii). Since K is a κ -ary superextensive coverage on S , the Yoneda embedding $S \rightarrow \mathbf{SH}(S, K)$ preserves κ -ary coproducts, by theorem 1.5.15. The Yoneda embedding also preserves exact quotients, by lemma A.3.10. □

1.5.20 ¶ Let λ be a regular cardinal $\geq \kappa$. We define a category $\mathbf{Fam}_\lambda^\kappa(\mathcal{C})$ as follows:

- The objects are as in $\mathbf{Fam}_\lambda(\mathcal{C})$.
- The morphisms $X \rightarrow Y$ are the morphisms

$$\coprod_{i \in \text{idx } X} X(i) \rightarrow \coprod_{j \in \text{idx } Y} Y(j)$$

in $\mathbf{SH}(\mathcal{C}, \mathbf{K})$ where \mathbf{K} is the κ -ary extensive coverage on \mathcal{C} .

- Composition and identities are inherited from $\mathbf{SH}(\mathcal{C}, \mathbf{K})$.

We also define a functor $\gamma : \mathcal{C} \rightarrow \mathbf{Fam}_\lambda^\kappa(\mathcal{C})$ as follows:

- For each object X in \mathcal{C} , we have $\text{idx } \gamma(X) = \{*\}$ and $\gamma(X)(*) = X$.
- For each morphism $f : X \rightarrow Y$ in \mathcal{C} , we have $\gamma(f) = \hat{h}_f$.

PROPOSITION. $\mathbf{Fam}_\lambda^\kappa(\mathcal{C})$ is a λ -ary extensive category. Furthermore, $\gamma : \mathcal{C} \rightarrow \mathbf{Fam}_\lambda^\kappa(\mathcal{C})$ is fully faithful and preserves κ -ary (strict) coproducts.

Proof. It is clear that $\mathbf{Fam}_\lambda^\kappa(\mathcal{C})$ has λ -ary coproducts. On the other hand, coproducts in $\mathbf{PSH}(\mathcal{C})$ are always strict, so [proposition A.2.18](#) and [theorem A.3.9](#) imply that coproducts in $\mathbf{SH}(\mathcal{C}, \mathbf{K})$ are also strict. Moreover, [proposition 1.5.14](#) and [lemma A.3.11](#) imply that any morphism $X \rightarrow Y$ in $\mathbf{Fam}_\lambda^\kappa(\mathcal{C})$ must factor through the inclusion $Y' \rightarrow Y$ for some Y' where $|\text{idx } Y'| \leq \max\{|\text{idx } X|, \kappa\}$, so coproduct injections (of κ -ary coproducts) in $\mathbf{Fam}_\lambda^\kappa(\mathcal{C})$ are indeed quadrable. The remainder of the claim is a consequence of [theorem 1.5.15](#). ■

The relative
coproduct
completion

THEOREM. Let \mathcal{D} be a λ -ary extensive category and let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor that preserves κ -ary coproducts.

- There exist a functor $\bar{F} : \mathbf{Fam}_\lambda^\kappa(\mathcal{C}) \rightarrow \mathcal{D}$ that preserves λ -ary coproducts and an isomorphism $\eta : F \Rightarrow \bar{F}\gamma$ of functors $\mathcal{C} \rightarrow \mathcal{D}$.
- Moreover, any such (\bar{F}, η) is a left Kan extension of $F : \mathcal{C} \rightarrow \mathcal{D}$ along $\gamma : \mathcal{C} \rightarrow \mathbf{Fam}_\lambda^\kappa(\mathcal{C})$.

Proof. Let \mathbf{L} be the λ -ary extensive topology on \mathcal{D} . Note that the functor $F^* : \mathbf{PSH}(\mathcal{D}) \rightarrow \mathbf{PSH}(\mathcal{C})$ sends \mathbf{L} -sheaves on \mathcal{D} to \mathbf{K} -sheaves on \mathcal{C} : this is

a consequence of [lemma 1.5.15](#). Let $F_! : \mathbf{SH}(C, K) \rightarrow \mathbf{SH}(D, L)$ be (the functor part of) a left Kan extension of $h_F : C \rightarrow \mathbf{SH}(D, L)$ along the Yoneda embedding $h_\bullet : C \rightarrow \mathbf{SH}(C, K)$. The unit of this Kan extension is automatically an isomorphism, because the Yoneda embedding is fully faithful. Moreover, by the earlier observation, $F_! : \mathbf{SH}(C, K) \rightarrow \mathbf{SH}(D, L)$ is a left adjoint of the functor $F^* : \mathbf{SH}(D, L) \rightarrow \mathbf{SH}(C, K)$, so it preserves all coproducts. Since every object in $\mathbf{Fam}_\lambda^K(C)$ is a λ -ary coproduct of objects in the image of $\gamma : C \rightarrow \mathbf{Fam}_\lambda^K(C)$, this yields the desired left Kan extension of $F : C \rightarrow D$. ■

1.5.21 PROPOSITION. *With notation as in [paragraph 1.5.20](#):*

Limits of finite diagrams in the relative coproduct completion

- (i) *If C has finitary products, then $\mathbf{Fam}_\lambda^K(C)$ also has finitary products.*
- (ii) *If C has equalisers, then $\mathbf{Fam}_\lambda^K(C)$ also has equalisers.*
- (iii) *If C has pullbacks, then $\mathbf{Fam}_\lambda^K(C)$ also has pullbacks.*

Proof. (i). This is a consequence of the fact that binary products distribute over (possibly infinitary) coproducts in $\mathbf{SH}(C, K)$.

(ii). Consider an equaliser diagram in $\mathbf{SH}(C, K)$:

$$A' \hookrightarrow A \begin{array}{c} \xrightarrow{h_0} \\ \xrightarrow{h_1} \end{array} B$$

Suppose both A and B are coproducts (in $\mathbf{SH}(C, K)$) of λ -small families of representable K -sheaves on C . We must show that A' has the same property. It suffices to show that A' is representable when A is representable: the general case follows by taking coproducts. But if A is representable, then [proposition 1.5.20](#) and [lemma A.3.11](#) imply that there is a representable K -subsheaf $B' \subseteq B$ such that both $h_0, h_1 : A \rightarrow B$ factor through the inclusion $B' \hookrightarrow B$, so A' is indeed representable.

(iii). A similar argument works. ■

Chartered objects

2.1 Sites

SYNOPSIS. We consider full subcategories of pretoposes for which the associated Yoneda representation is fully faithful and we identify the essential image of such Yoneda representations.

PREREQUISITES. §§ 1.1, 1.3, 1.4, 1.5, A.1, A.2, A.3.

2.1.1 ✖ Throughout this section, \mathcal{S} is a regular category.

2.1.2 ¶ Given a full subcategory $\mathcal{C} \subseteq \mathcal{S}$, we have the Yoneda representation $h_{\bullet} : \mathcal{S} \rightarrow \mathbf{PSH}(\mathcal{C})$, which is not fully faithful in general. We will see that the following condition is sufficient.

DEFINITION. A **unary site** for \mathcal{S} is a full subcategory $\mathcal{C} \subseteq \mathcal{S}$ with the following property:

- For every object A in \mathcal{S} , there is an effective epimorphism $X \twoheadrightarrow A$ in \mathcal{S} where X is an object in \mathcal{C} .

EXAMPLE. Of course, \mathcal{S} is a unary site for \mathcal{S} .

2.1.3 ¶ Let κ be a regular cardinal and assume \mathcal{S} is a κ -ary pretopos.

DEFINITION. A **κ -ary site** for \mathcal{S} is a full subcategory $\mathcal{C}_0 \subseteq \mathcal{S}$ with the following property:

- For every object A in \mathcal{S} , there is an effective epimorphism $X \twoheadrightarrow A$ in \mathcal{S} where X is a coproduct (in \mathcal{S}) of a κ -small family of objects in \mathcal{C}_0 .

EXAMPLE. Every unary site for \mathcal{S} is also a κ -ary site for \mathcal{S} *a fortiori*.

Recognition principle for sites for pretoposes

LEMMA. Let C_0 be a full subcategory of S and let C be the full subcategory of S spanned by the objects that are coproducts (in S) of κ -small families of objects in C_0 . The following are equivalent:

- (i) C_0 is a κ -ary site for S .
- (ii) C is a unary site for S .

Proof. Straightforward. ◆

2.1.4 ¶ Let κ be a regular cardinal.

The following is a generalisation of the notion of localic topos.

DEFINITION. A κ -ary pretopos S is **localic** if the full subcategory of subterminal objects in S is a κ -ary site for S .

REMARK. The full subcategory of subterminal objects in a κ -ary pretopos has finitary meets and κ -ary joins, and moreover meets distribute over κ -ary joins.

2.1.5 PROPOSITION.

Properties of effective epimorphisms in regular categories

- (i) Every isomorphism in S is an effective epimorphism in S .
- (ii) The class of effective epimorphisms in S is a quadrable class of morphisms in S .
- (iii) The class of effective epimorphisms in S is closed under composition.
- (iv) Given morphisms $h : A \rightarrow B$ and $k : B \rightarrow C$ in S , if $k \circ h : A \rightarrow C$ is an effective epimorphism in S , then $k : B \rightarrow C$ is also an effective epimorphism in S .
- (v) The class of effective epimorphisms in S is a saturated subcanonical unary coverage on S .

Proof. (i). Immediate.

(ii). By lemma 1.3.8, the effective epimorphisms in S are the same as the extremal epimorphisms in S , which constitute a quadrable class of morphisms in S because S has exact images.

(iii). The composite of a pair of extremal epimorphisms is an extremal epimorphism (in a category with pullbacks of monomorphisms).

(iv). Straightforward.

(v). By (i)–(iv), the class of effective epimorphisms in \mathcal{S} is a saturated unary coverage on \mathcal{S} . On the other hand, by [proposition 1.4.1](#), the effective epimorphisms in \mathcal{S} are also the same as the strict epimorphisms in \mathcal{S} . Thus, we may apply [lemma A.2.11](#) to deduce that the class of effective epimorphisms is a subcanonical (unary) coverage. \blacksquare

2.1.6 \times For the remainder of this section, \mathcal{C} is a unary site for \mathcal{S} and \mathbf{E} is the class of morphisms in \mathcal{C} that are effective epimorphisms in \mathcal{S} .

2.1.7 PROPOSITION. *Let \mathbf{K} be the class of effective epimorphisms in \mathcal{S} .*

Properties of the
induced unary
coverage on
a unary site

- (i) *Every member of \mathbf{E} is a regular epimorphism in \mathcal{C} .*
- (ii) *\mathbf{E} is a saturated unary coverage on \mathcal{C} .*
- (iii) *$(\mathcal{C}, \mathbf{E})$ satisfies the Shulman condition.*
- (iv) *If $F : \mathcal{S}^{\text{op}} \rightarrow \mathbf{SET}$ is a \mathbf{K} -sheaf, then the restriction $F : \mathcal{C}^{\text{op}} \rightarrow \mathbf{SET}$ is an \mathbf{E} -sheaf.*
- (v) *In particular, \mathbf{E} is a subcanonical unary coverage on \mathcal{C} .*
- (vi) *The restriction functor $\mathbf{PSH}(\mathcal{S}) \rightarrow \mathbf{PSH}(\mathcal{C})$ sends \mathbf{K} -locally surjective morphisms to \mathbf{E} -locally surjective morphisms.*
- (vii) *The restriction functor $\mathbf{SH}(\mathcal{S}, \mathbf{K}) \rightarrow \mathbf{SH}(\mathcal{C}, \mathbf{E})$ is (half of) an equivalence of categories.*

Proof. (i). Let $f : X \rightrightarrows Y$ be an effective epimorphism in \mathcal{S} . Since \mathcal{C} is a unary site for \mathcal{S} , there is an effective epimorphism $\langle d_1, d_0 \rangle : R \rightrightarrows X \times_Y X$ in \mathcal{S} with R in \mathcal{C} . It is straightforward to verify that $f : X \rightrightarrows Y$ is a coequaliser in \mathcal{C} of the parallel pair $d_0, d_1 : R \rightarrow X$, so $f : X \rightrightarrows Y$ is indeed a regular epimorphism in \mathcal{C} .

(ii). First, we must show that \mathbf{E} is a unary coverage on \mathcal{C} .

Let $f : X \rightrightarrows Y$ be an effective epimorphism in \mathcal{S} , let T be an object in \mathcal{C} , and let $y : T \rightarrow Y$ be a morphism in \mathcal{S} . By [proposition 2.1.5](#), the

projection $T \times_Y X \rightarrow T$ is also an effective epimorphism in \mathcal{S} . Since \mathcal{C} is a unary site for \mathcal{S} , there is an effective epimorphism $\langle t, x \rangle : S \twoheadrightarrow T \times_Y X$ in \mathcal{S} with S in \mathcal{C} . In particular, we have a commutative square in \mathcal{C} of the form below,

$$\begin{array}{ccc} S & \xrightarrow{x} & X \\ t \downarrow & & \downarrow f \\ T & \xrightarrow{y} & Y \end{array}$$

where $t : S \twoheadrightarrow T$ is an effective epimorphism in \mathcal{S} , as required.

Thus, \mathbf{E} is indeed a unary coverage on \mathcal{S} . Since the class of effective epimorphisms in \mathcal{S} is a saturated coverage on \mathcal{S} , \mathbf{E} is also saturated.

(iii). Let $X : \mathcal{J} \rightarrow \mathcal{C}$ be a finite diagram. Then $\varprojlim_{\mathcal{J}} X$ exists in \mathcal{S} , so we there is an effective epimorphism $p : U \twoheadrightarrow \varprojlim_{\mathcal{J}} X$ in \mathcal{S} where U is an object in \mathcal{C} . We will show that the evident cone $\Delta U \Rightarrow X$ is an \mathbf{E} -weak limit of X in \mathcal{C} .

Let $\varphi : \Delta T \Rightarrow X$ be a cone in \mathcal{C} . By definition, it corresponds to a morphism $x : T \rightarrow \varprojlim_{\mathcal{J}} X$ in \mathcal{S} . As in (ii), we may find a commutative square in \mathcal{S} of the form below,

$$\begin{array}{ccc} S & \xrightarrow{u} & U \\ t \downarrow & & \downarrow p \\ T & \xrightarrow{x} & \varprojlim_{\mathcal{J}} X \end{array}$$

where S is an object in \mathcal{C} and $t : S \twoheadrightarrow T$ is an effective epimorphism in \mathcal{S} . This proves the claim.

(iv). In view of [proposition 1.4.1](#), the hypothesis implies that every object A in \mathcal{S} admits a strict epimorphism $X \twoheadrightarrow A$ in \mathcal{S} with X in \mathcal{C} . The claim follows, by [lemma A.2.6](#).

(v). Immediate, by (iv).

(vi). Straightforward.

(vii). Let $R : \mathbf{PSH}(\mathcal{C}) \rightarrow \mathbf{PSH}(\mathcal{S})$ be the evident functor defined on objects as follows:

$$R(P) = \mathrm{Hom}_{\mathbf{PSH}(\mathcal{C})}(h_{\bullet}, P)$$

Recalling [proposition A.1.4](#), it is not hard to verify that R is (the functor part of) a right adjoint of the restriction functor $\mathbf{PSH}(S) \rightarrow \mathbf{PSH}(C)$. Since C is a full subcategory of S , the Yoneda lemma implies that, for every presheaf P on C , the counit $R(P) \rightarrow P$ is an isomorphism. Moreover, by adjointness, [proposition A.3.7](#) and (vi) imply that $R(P)$ is a K-sheaf on S if P is a E-sheaf on C . In addition, for every object A in S , there is a right-K-exact fork in S of the form below,

$$R \rightrightarrows X \longrightarrow A$$

where R and X are in C , so by [lemma 1.4.18](#), the unit $F \rightarrow R(F)$ is an isomorphism. Thus, we indeed have an equivalence of categories. \blacksquare

2.1.8 LEMMA. *Let $h : A \rightarrow B$ be a morphism in S . The following are equivalent:*

Effective
epimorphisms in regular
categories as
locally surjective
morphisms

(i) $h_h : h_A \rightarrow h_B$ is an E-locally surjective morphism in $\mathbf{PSH}(C)$.

(ii) $h : A \rightarrow B$ is an effective epimorphism in S .

Proof. (i) \Rightarrow (ii). Since C is a unary site for S , there is an effective epimorphism $b : Y \twoheadrightarrow B$ in S with Y in C . On the other hand, $h_h : h_A \rightarrow h_B$ is E-locally surjective, so there is a commutative square in S of the form below,

$$\begin{array}{ccc} T & \xrightarrow{a} & A \\ y \downarrow & & \downarrow h \\ Y & \xrightarrow{b} & B \end{array}$$

where T is an object in C and $y : T \twoheadrightarrow Y$ is an effective epimorphism in C . By [proposition 2.1.5](#), $b \circ y : T \rightarrow B$ is an effective epimorphism in S , so $h : A \rightarrow B$ is indeed an effective epimorphism in S .

(ii) \Rightarrow (i). Let $h : A \rightarrow B$ be an effective epimorphism in S and let (Y, b) be an element of h_B . We wish to show that (Y, b) is an element of the E-closed support of $h_h : h_A \rightarrow h_B$.

Since \mathcal{C} is a unary site for \mathcal{S} , there is a commutative square in \mathcal{S} of the form below,

$$\begin{array}{ccc} X & \xrightarrow{a} & A \\ f \downarrow & & \downarrow h \\ Y & \xrightarrow{b} & B \end{array}$$

where X is an object in \mathcal{C} and $\langle a, f \rangle : X \twoheadrightarrow A \times_B Y$ is an effective epimorphism in \mathcal{S} . In particular, $f : X \twoheadrightarrow Y$ is an effective epimorphism in \mathcal{S} . Thus, (Y, b) is indeed an element of the E-closed support of $h_h : h_A \rightarrow h_B$. ■

The Yoneda representation with respect to a unary site

THEOREM. *The Yoneda representation $h_\bullet : \mathcal{S} \rightarrow \mathbf{SH}(\mathcal{C}, \mathbf{E})$ is fully faithful and preserves effective epimorphisms.*

Proof. By lemmas 2.1.8 and A.3.10, $h_\bullet : \mathcal{S} \rightarrow \mathbf{SH}(\mathcal{C}, \mathbf{E})$ preserves effective epimorphisms. Thus, for every object A in \mathcal{S} , there is a coequaliser diagram in \mathcal{S} of the form below,

$$R \rightrightarrows X \longrightarrow A$$

where R and X are objects in \mathcal{C} , such that $h_\bullet : \mathcal{S} \rightarrow \mathbf{SH}(\mathcal{C}, \mathbf{E})$ preserves this coequaliser diagram. In view of the Yoneda lemma, it follows that the Yoneda representation $h_\bullet : \mathcal{S} \rightarrow \mathbf{SH}(\mathcal{C}, \mathbf{E})$ is fully faithful. ■

2.1.9 ¶ The following gives a complete characterisation of the essential image of the Yoneda representation $\mathcal{S} \rightarrow \mathbf{SH}(\mathcal{C}, \mathbf{E})$ in the case where \mathcal{S} is an exact category.

Recognition principle for 1-presentable sheaves

LEMMA. *Let F be a E-sheaf on \mathcal{C} . Assuming \mathcal{S} is an exact category, the following are equivalent:*

- (i) *There is an object A in \mathcal{S} such that $h_A \cong F$ in $\mathbf{SH}(\mathcal{C}, \mathbf{E})$.*
- (ii) *F is a E-locally 1-presentable (as a presheaf on \mathcal{C}).*

Proof. (i) \Rightarrow (ii). It suffices to verify that h_A itself is E-locally 1-presentable. It is clear that any E-sheaf on \mathcal{C} of the form h_A is E-locally 1-generable. But \mathcal{S} has kernel pairs, so it follows that h_A is E-locally 1-presentable as well.

(ii) \Rightarrow (i). In view of [theorem 2.1.8](#) and [lemma A.3.10](#), the hypothesis implies there exist an object X in \mathcal{C} and an effective epimorphism $p : \hat{h}_X \rightarrow F$ in $\mathbf{SH}(\mathcal{C}, \mathbf{E})$ such that $\mathrm{Kr}(p)$ is \mathbf{E} -locally \mathbf{I} -generable \mathbf{E} -subsheaf of $\hat{h}_X \times \hat{h}_X$. Thus, by [lemma 1.4.13](#), $\mathrm{Kr}(p)$ is \mathbf{E} -locally \mathbf{I} -presentable, so we have an equivalence relation (R, d_0, d_1) on X in \mathcal{S} such that the \mathbf{E} -closed support of $\langle d_1 \circ -, d_0 \circ - \rangle : \hat{h}_R \rightarrow \hat{h}_X \times \hat{h}_X$ is $\mathrm{Kr}(p)$. But \mathcal{S} has exact quotients of equivalence relations and the Yoneda representation $\mathcal{S} \rightarrow \mathbf{SH}(\mathcal{C}, \mathbf{E})$ preserves them, so it follows that F is representable by an object in \mathcal{S} . \blacksquare

2.1.10 ¶ As promised, we have the following converse to [lemma 1.4.24](#).

Admissible functors by restriction

LEMMA. *Let \mathcal{T} be an exact category, let \mathcal{D} be a unary site for \mathcal{T} , let \mathbf{K} be the class of effective epimorphisms in \mathcal{S} , and let \mathbf{J} be the class of morphisms in \mathcal{D} that are effective epimorphisms in \mathcal{T} . Consider a commutative square of the form below:*

$$\begin{array}{ccc} \mathcal{C} & \hookrightarrow & \mathcal{S} \\ F \downarrow & & \downarrow \bar{F} \\ \mathcal{D} & \hookrightarrow & \mathcal{T} \end{array}$$

If \mathcal{S} is an exact category and $\bar{F} : \mathcal{S} \rightarrow \mathcal{T}$ sends right- \mathbf{K} -exact forks in \mathcal{S} to coequaliser diagrams in \mathcal{T} , then $F : (\mathcal{C}, \mathbf{E}) \rightarrow (\mathcal{D}, \mathbf{J})$ is an admissible functor.

Proof. By [lemma A.3.10](#), the inclusion $\mathcal{C} \hookrightarrow \mathcal{S}$ sends right- \mathbf{E} -exact forks in \mathcal{C} to right- \mathbf{K} -exact forks in \mathcal{S} , so by [lemma 1.4.18](#), $F : (\mathcal{C}, \mathbf{E}) \rightarrow (\mathcal{D}, \mathbf{J})$ is a pre-admissible functor. Moreover, by the Yoneda lemma, for every object A in \mathcal{S} and every object B in \mathcal{T} , we have the following natural bijection:

$$\mathcal{T}(\bar{F}A, B) \cong \mathrm{Hom}_{\mathbf{SH}(\mathcal{S}, \mathbf{K})}(\hat{h}_A, \bar{F}^* \hat{h}_B)$$

On the other hand, the restriction functor $\mathbf{SH}(\mathcal{S}, \mathbf{K}) \rightarrow \mathbf{SH}(\mathcal{C}, \mathbf{E})$ is fully faithful (and essentially surjective on objects) by [proposition 2.1.7](#), so $F : (\mathcal{C}, \mathbf{E}) \rightarrow (\mathcal{D}, \mathbf{J})$ is indeed an admissible functor. \blacksquare

2.1.11 ✖ For the remainder of this section:

- κ is a regular cardinal.
- \mathcal{S} is a κ -ary pretopos.
- \mathbf{K} is the κ -ary canonical coverage on \mathcal{S} .
- \mathcal{C}_0 is a κ -ary site for \mathcal{S} .
- For every object X in \mathcal{C}_0 , $J_0(X)$ is the set of κ -small sinks Φ on X (as an object in \mathcal{C}_0) such that the induced morphism $\coprod_{(T,x) \in \Phi} T \rightarrow X$ is an effective epimorphism in \mathcal{S} .

2.1.12 ¶ Let A be an object in \mathcal{S} , let Φ be a set of objects in $\mathcal{S}/_A$, and let \mathcal{U} be the sieve of $\mathcal{S}/_A$ generated by Φ .

2.1.12(a) LEMMA. *Suppose Φ is a κ -small sink on A . The following are equivalent:*

Recognition principle for universally strict-epimorphic sinks in pretoposes

- (i) Φ is a universally strict-epimorphic sink on A .
- (ii) Φ is a strict-epimorphic sink on A .
- (iii) The induced morphism $\coprod_{(X,a) \in \Phi} X \rightarrow A$ is an effective epimorphism in \mathcal{S} .

Proof. (i) \Rightarrow (ii). Immediate.

(ii) \Rightarrow (iii). Let $U = \coprod_{(X,a) \in \Phi} X$ and let $p : U \rightarrow A$ be the induced morphism in \mathcal{S} . By [proposition 1.3.3](#), it suffices to show that $p : U \rightarrow A$ is an extremal epimorphism in \mathcal{S} . Let $m : A' \rightarrow A$ be a monomorphism in \mathcal{S} and suppose $\mathcal{U} \subseteq \downarrow \langle m \rangle$. Then we have a commutative diagram of the form below,

$$\begin{array}{ccc}
 \mathcal{U} & \hookrightarrow & \downarrow \langle m \rangle \\
 \downarrow & \nearrow & \downarrow \\
 \mathcal{S}/_A & \xrightarrow{\text{id}} & \mathcal{S}/_A
 \end{array}$$

so $m : A' \rightarrow A$ is an isomorphism in \mathcal{S} . Thus, $p : U \rightarrow A$ is indeed an extremal epimorphism in \mathcal{S} .

(iii) \Rightarrow (i). With notation as above, if $p : U \rightarrow A$ is an effective epimorphism in \mathcal{S} , then by [propositions 1.4.1](#) and [2.1.5](#), $\{(U, p)\}$ is a universally

strict-epimorphic sink on A . On the other hand, by lemma 1.5.15, κ -ary coproduct cocones in κ -ary extensive categories are universally strict-epimorphic sinks, so lemma A.2.3(c) implies that Φ is also a universally strict-epimorphic sink. ■

2.1.12(b) LEMMA. *Suppose that, for every $(X, a) \in \Phi$, X is an object in \mathcal{C} . Let \mathcal{U}_0 be the full subcategory of \mathcal{U} spanned by the objects (X, a) such that X is an object in \mathcal{C}_0 . If F is a separated presheaf on \mathcal{S} with respect to the κ -ary canonical coverage, then, for every commutative square of the form below,*

$$\begin{array}{ccc} \mathcal{U}_0 & \xrightarrow{s} & \mathbf{El}(F) \\ \downarrow & & \downarrow \\ \mathcal{U} & \longrightarrow & \mathcal{S} \end{array}$$

where $\mathcal{U} \rightarrow \mathcal{S}$ and $\mathbf{El}(F) \rightarrow \mathcal{S}$ are the respective projections, there is a unique functor $\mathcal{U} \rightarrow \mathbf{El}(F)$ making both evident triangles commute.

Proof. It suffices to verify the following:

- For every commutative square in \mathcal{S} of the form below,

$$\begin{array}{ccc} U & \xrightarrow{x_1} & X_1 \\ x_0 \downarrow & \searrow a & \downarrow a_1 \\ X_0 & \xrightarrow{a_0} & A \end{array}$$

where (U, a) is in \mathcal{U} and both (X_0, a_0) and (X_1, a_1) are in \mathcal{U}_0 , we have $s(X_0, a_0) \cdot x_0 = s(X_1, a_1) \cdot x_1$.

But this is a consequence of the fact every object in \mathcal{S} admits a κ -small universally strict-epimorphic sink where the domains are objects in \mathcal{C}_0 and the hypothesis that F is separated. ■

2.1.13 PROPOSITION.

Properties of
the induced
coverage on a
site for a pretopos

- (i) J_0 is a composition-closed coverage on \mathcal{C}_0 .
- (ii) The restriction functor $\mathbf{PSH}(\mathcal{S}) \rightarrow \mathbf{PSH}(\mathcal{C}_0)$ sends \mathbf{K} -sheaves on \mathcal{S} to J_0 -sheaves on \mathcal{C}_0 .
- (iii) In particular, J_0 is a subcanonical coverage on \mathcal{C}_0 .

(iv) *The restriction functor $\mathbf{PSH}(\mathcal{S}) \rightarrow \mathbf{PSH}(\mathcal{C}_0)$ sends \mathbf{K} -locally surjective morphisms to \mathbf{J}_0 -locally surjective morphisms.*

(v) *The restriction functor $\mathbf{SH}(\mathcal{S}, \mathbf{K}) \rightarrow \mathbf{SH}(\mathcal{C}_0, \mathbf{J}_0)$ is (half of) an equivalence of categories.*

Proof. (i). By lemma 2.1.3 and proposition 2.1.7, \mathbf{J}_0 is a coverage on \mathcal{C}_0 , and the fact that the class of effective epimorphisms in \mathcal{S} is closed under κ -ary coproduct implies that \mathbf{J}_0 is composition-closed.

(ii). Let $F : \mathcal{S}^{\text{op}} \rightarrow \mathbf{SET}$ be a sheaf with respect to the κ -ary canonical coverage on \mathcal{S} and let $F' : (\mathcal{C}_0)^{\text{op}} \rightarrow \mathbf{SET}$ be the restriction. By lemma A.2.6, with notation as in loc. cit., to show that F' is a \mathbf{J}_0 -sheaf, it is enough verify the following:

- For every object X in \mathcal{C}_0 and every $\Phi \in \mathbf{J}_0(X)$, we have $\Gamma(\Phi, F) = \Gamma(\Phi, F')$ (as subsets of $F(X)$).

But this is a straightforward consequence of lemma 2.1.12(b).

(iii). Immediate, by (ii).

(iv). Straightforward.

(v). Let $R : \mathbf{PSH}(\mathcal{C}_0) \rightarrow \mathbf{PSH}(\mathcal{S})$ be the evident functor defined on objects as follows:

$$R(P) = \text{Hom}_{\mathbf{PSH}(\mathcal{C}_0)}(\mathfrak{h}_\bullet, P)$$

Recalling proposition A.1.4, it is not hard to verify that R is (the functor part of) a right adjoint of the restriction functor $\mathbf{PSH}(\mathcal{S}) \rightarrow \mathbf{PSH}(\mathcal{C}_0)$. Since \mathcal{C}_0 is a full subcategory of \mathcal{S} , the Yoneda lemma implies that, for every presheaf P on \mathcal{C}_0 , the counit $R(P) \rightarrow P$ is an isomorphism. Moreover, by adjointness, (iv) implies that $R(P)$ is a \mathbf{K} -separated presheaf on \mathcal{S} if P is a \mathbf{J}_0 -separated presheaf on \mathcal{S} ; so, by lemma 2.1.12(b), $R(P)$ is a \mathbf{K} -sheaf on \mathcal{S} if P is a \mathbf{J}_0 -sheaf on \mathcal{C}_0 . In addition, by the same lemma, for every \mathbf{K} -sheaf F on \mathcal{S} , the unit $F \rightarrow R(F)$ is an isomorphism. Thus, we indeed have an equivalence of categories. ■

2.1.14 LEMMA. *Let $h : A \rightarrow B$ be a morphism in \mathcal{S} . The following are equivalent:*

Effective
epimorphisms
in pretoposes as
locally surjective
morphisms

(i) $h_h : h_A \rightarrow h_B$ is a J_0 -locally surjective morphism in $\mathbf{PSH}(C_0)$.

(ii) $h : A \rightarrow B$ is an effective epimorphism in \mathcal{S} .

Proof. Omitted. (Compare lemma 2.1.8.) ◇

The Yoneda
representation
with respect to a
site for a pretopos

THEOREM. *The Yoneda representation $h_\bullet : \mathcal{S} \rightarrow \mathbf{SH}(C_0, J_0)$ is fully faithful and preserves κ -ary coproducts and effective epimorphisms.*

Proof. Consider the Yoneda embedding $h_\bullet : \mathcal{S} \rightarrow \mathbf{SH}(\mathcal{S}, \mathcal{K})$. By theorem 1.5.15, $h_\bullet : \mathcal{S} \rightarrow \mathbf{SH}(\mathcal{S}, \mathcal{K})$ preserves κ -ary coproducts. On the other hand, by theorem 2.1.8, $h_\bullet : \mathcal{S} \rightarrow \mathbf{SH}(\mathcal{S}, \mathcal{K})$ preserves effective epimorphisms. Since the Yoneda embedding is known to be fully faithful, the claim follows, by proposition 2.1.13. ■

2.1.15 ¶ Thus, we may identify \mathcal{S} with a certain full subcategory of $\mathbf{SH}(C_0, J_0)$. We will give a characterisation of the J_0 -sheaves on C_0 that are representable by an object in \mathcal{S} , but to do so, we require a preliminary result.

DEFINITION. A J_0 -sheaf on C_0 is κ -**generable** if it admits a κ -small J_0 -local generating set of elements.

Recognition principle for generable subsheaves

LEMMA. *Let B be an object in \mathcal{S} and let F be a J_0 -subsheaf of h_B . The following are equivalent:*

- (i) *There is a monomorphism $m : A \rightarrow B$ in \mathcal{S} such that F is the J_0 -closed support of $h_m : h_A \rightarrow h_B$.*
- (ii) *F is κ -generable.*

Proof. (i) \Rightarrow (ii). It suffices to show that h_A is κ -generable. Since C_0 is a κ -ary site for \mathcal{S} , there is a κ -small set Φ of objects in $\mathcal{S}/_A$ such that the induced morphism $\coprod_{(X,a) \in \Phi} X \rightarrow A$ is an effective epimorphism in \mathcal{S} and, for every $(X, a) \in \Phi$, X is an object in C_0 . Thus, by lemma 2.1.14, h_A is κ -generable.

(ii) \Rightarrow (i). In view of theorem 2.1.14 and lemma A.3.10, the hypothesis implies there exist an object X in \mathcal{C} and an effective epimorphism $p :$

$\hat{h}_X \twoheadrightarrow F$ in $\mathbf{SH}(C_0, J_0)$. Let $b : X \rightarrow B$ be the morphism in S corresponding to the composite $\hat{h}_X \twoheadrightarrow F \hookrightarrow \hat{h}_B$ and consider $\text{im}(b) : \text{Im}(b) \rightarrow B$ in S . The Yoneda representation $S \rightarrow \mathbf{SH}(C_0, J_0)$ preserves effective epimorphisms and monomorphisms, so the J_0 -closed support of $\text{im}(b) : \text{Im}(b) \rightarrow B$ is F , as desired. ■

REMARK. In particular, \hat{h}_B itself is a κ -generable J_0 -sheaf on C_0 .

2.1.16 DEFINITION. A J_0 -sheaf F on C_0 is **κ -presentable** if there is a set Φ with the following properties:

- Φ is a κ -small J_0 -local generating set of elements of F .
- For every $((X_0, a_0), (X_1, a_1)) \in \Phi \times \Phi$, the sheaf $\text{Pb}(a_0 \cdot -, a_1 \cdot -)$ is κ -generable.

REMARK. Clearly, every κ -presentable J_0 -sheaf on C_0 is also κ -generable.

Recognition principle for presentable sheaves

LEMMA. Let F be a J_0 -sheaf on C_0 . The following are equivalent:

- (i) There is an object A in S such that $\hat{h}_A \cong F$ in $\mathbf{SH}(C_0, J_0)$.
- (ii) F is a κ -presentable J_0 sheaf on C_0 .

Proof. (i) \Rightarrow (ii). It suffices to verify that \hat{h}_A itself is κ -presentable. By lemma 2.1.15, \hat{h}_A is κ -generable. Moreover, given any two elements of \hat{h}_A , say (X_0, a_0) and (X_1, a_1) , the sheaf $\text{Pb}(a_0 \cdot -, a_1 \cdot -)$ is representable by an object in S , so it is also κ -generable. Thus, \hat{h}_A is indeed κ -presentable.

(ii) \Rightarrow (i). In view of theorem 2.1.14 and lemma A.3.10, the hypothesis implies there exist an object X in C and an effective epimorphism $p : \hat{h}_X \twoheadrightarrow F$ in $\mathbf{SH}(C_0, J_0)$. It is straightforward to check that $\text{Kr}(p)$ is a κ -generable J_0 -subsheaf of $\hat{h}_X \times \hat{h}_X$, so we have an equivalence relation (R, d_0, d_1) on X in S such that the J_0 -closed support of $\langle d_1 \circ -, d_0 \circ - \rangle : \hat{h}_R \rightarrow \hat{h}_X \times \hat{h}_X$ is $\text{Kr}(p)$. But S has exact quotients of equivalence relations and the Yoneda representation $S \rightarrow \mathbf{SH}(C_0, J_0)$ preserves them, so it follows that F is representable by an object in S . ■

2.2 Ecumenae

SYNOPSIS. We define various notions of categories equipped with structure making it possible to interpret basic notions of topology such as open embeddings and local homeomorphisms.

PREREQUISITES. §§ 1.1, 1.2, 1.3, 1.4, 1.5, A.2.

2.2.1 DEFINITION. An **ecumene**^[1] is a tuple (C, \mathcal{G}, J) where:

- C is a category.
- \mathcal{G} is a class of fibrations in C .
- J is a coverage on C .
- Every morphism in C of \mathcal{G} -type (\mathcal{G}, J) -semilocally on the domain is a member of \mathcal{G} .

We will often abuse notation by referring to the category C itself as an ecumene, omitting mention of \mathcal{G} and J .

EXAMPLE. Let C be a category with pullbacks, let \mathcal{G} be the class of quadrable morphisms in C , and let J be any coverage on C . Then (C, \mathcal{G}, J) is an ecumene.

2.2.2 ¶ Let C be a category and let \mathcal{B} be a set of morphisms in C .

DEFINITION. A coverage J on C is **\mathcal{B} -adapted** if it has the following property:

- For every object X in C and every J -covering sink Φ on X , there is \mathcal{B} -sink Φ' such that $\Phi' \in J(X)$ and $\downarrow(\Phi') \subseteq \downarrow(\Phi)$.

EXAMPLE. If $\mathcal{B} = \text{mor } C$, then every coverage on C is \mathcal{B} -adapted.

REMARK. Let J be a coverage on C and let $J_{\mathcal{B}}(X)$ be the set of \mathcal{B} -sinks on X that are members of $J(X)$. If J is \mathcal{B} -adapted, then $J_{\mathcal{B}}$ is a coverage, and moreover the J -covering sinks coincide with the $J_{\mathcal{B}}$ -covering sinks.

[1] — from Greek «οικουμένη», the inhabited world.

Properties of adapted coverages

PROPOSITION. *If J is a \mathcal{B} -adapted coverage on C , then \mathcal{B} satisfies the J -local collection axiom.*

Proof. Immediate. ■

2.2.3 DEFINITION. The **descent axiom** for an ecumene (C, \mathcal{G}, J) is the following:

- Every morphism in C of \mathcal{G} -type J -semilocally on the base is a member of \mathcal{G} .

The ecumene generated by a class of fibrations

PROPOSITION. *Let C be a category, let \mathcal{B} be a class of fibrations in C , let J be a coverage on C , and let \mathcal{G} be the class of morphisms in C that are of \mathcal{B} -type (\mathcal{B}, J) -semilocally on the domain. Assuming every member of \mathcal{G} is a quadrable morphism in C :*

- (i) (C, \mathcal{G}, J) is an ecumene.

In addition, assuming J is a \mathcal{B} -adapted coverage on C :

- (ii) (C, \mathcal{G}, J) satisfies the descent axiom.
- (iii) J is a \mathcal{G} -adapted coverage on C .

Proof. (i). By [proposition 1.2.5](#) and [lemma 1.2.6](#), \mathcal{G} is a class of fibrations in C and $\mathcal{B} \subseteq \mathcal{G}$. It remains to be shown that every morphism in C of \mathcal{G} -type (\mathcal{G}, J) -semilocally on the domain is a member of \mathcal{G} .

Let $f : X \rightarrow Y$ be a morphism in C and let Φ be a J -covering sink on X such that, for every $(U, x) \in \Phi$, both $x : U \rightarrow X$ and $f \circ x : U \rightarrow Y$ are members of \mathcal{G} . Since $x : U \rightarrow X$ is a member of \mathcal{G} , there is a J -covering sink $\Phi'_{(U,x)}$ on U such that, for every $(T, u) \in \Phi'_{(U,x)}$, both $u : T \rightarrow U$ and $x \circ u : T \rightarrow X$ are members of \mathcal{B} . Thus, by [proposition A.2.14](#),

$$\Phi' = \bigcup_{(U,x) \in \Phi} \left\{ (T, x \circ u) \mid (T, u) \in \Phi'_{(U,x)} \right\}$$

is a J -covering sink on X such that, for every $(T, x') \in \Phi$, both $x' : T \rightarrow X$ and $f \circ x' : T \rightarrow Y$ are members of \mathcal{B} . Hence, $f : X \rightarrow Y$ is of \mathcal{B} -type (\mathcal{B}, J) -semilocally on the domain. The claim follows.

(ii). Let $f : X \rightarrow Y$ be a morphism in \mathcal{C} and let Ψ be a J -covering sink on Y . Suppose, for each $(V, y) \in \Psi$, there is a pullback square in \mathcal{C} of the form below,

$$\begin{array}{ccc} f^*V & \xrightarrow{f^*y} & X \\ y^*f \downarrow & & \downarrow f \\ V & \xrightarrow{y} & Y \end{array}$$

where $y^*f : f^*V \rightarrow V$ is a member of \mathcal{G} . Since \mathcal{G} is a quadrable class of morphisms and J is a \mathcal{B} -adapted coverage on \mathcal{C} , we may assume that each $y : V \rightarrow Y$ is a member of \mathcal{B} . Then $y \circ y^*f : f^*V \rightarrow Y$ is a member of \mathcal{G} and $f^*y : f^*V \rightarrow X$ is a member of \mathcal{B} . Moreover, $\{(f^*V, f^*y) \mid (V, y) \in \Psi\}$ is a J -covering sink on X , so $f : X \rightarrow Y$ is of \mathcal{G} -type (\mathcal{B}, J) -locally on the domain. Hence, by (i), $f : X \rightarrow Y$ is a member of \mathcal{G} .

(iii). Immediate. ■

REMARK. In particular, if $(\mathcal{C}, \mathcal{G}, J)$ is an ecumene and J is a \mathcal{G} -adapted coverage on \mathcal{C} , then the descent axiom is satisfied.

2.2.4 DEFINITION. A **regulated ecumene** is an ecumene $(\mathcal{C}, \mathcal{G}, J)$ with the following additional data:

- For each morphism $f : X \rightarrow Y$ in \mathcal{C} that is a member of \mathcal{G} , a monomorphism $\text{im}(f) : \text{Im}(f) \rightarrow Y$ in \mathcal{C} such that $f = \text{im}(f) \circ \eta_f$ for some J -covering morphism $\eta_f : X \rightarrow \text{Im}(f)$ in \mathcal{C} and, for every commutative diagram in \mathcal{C} of the form below,

$$\begin{array}{ccc} X' & \longrightarrow & X \\ e' \downarrow & & \downarrow \eta_f \\ I' & \longrightarrow & \text{Im}(f) \\ i' \downarrow & & \downarrow \text{im}(f) \\ Y' & \longrightarrow & Y \end{array}$$

if both squares are pullback squares in \mathcal{C} , then $e' : X' \rightarrow I'$ is an effective epimorphism in \mathcal{C} and $i' : I' \rightarrow Y'$ is a member of \mathcal{G} .

REMARK. In the above, $e' : X' \twoheadrightarrow I'$ is automatically a member of \mathcal{G} , by lemma 1.1.3.

PROPOSITION. *Let (C, \mathcal{G}, J) be a regulated ecumene.*

- (i) *Every effective epimorphism in C that is a member of \mathcal{G} is a J -covering morphism in C .*
- (ii) *(C, \mathcal{D}) is a regulated category, where \mathcal{D} is the subcategory of \mathcal{G} -perfect morphisms in C .*

Proof. (i). Let $f : X \rightarrow Y$ be an effective epimorphism in C that is a member of \mathcal{G} . Then $\text{im}(f) : \text{Im}(f) \rightarrow Y$ must be an isomorphism in C ; but $f : X \rightarrow Y$ factors as a J -covering morphism in C followed by $\text{im}(f) : \text{Im}(f) \rightarrow Y$, so $f : X \rightarrow Y$ itself is also a J -covering morphism in C .

(ii). By proposition 1.1.11, \mathcal{D} is a class of separated fibrations in C . Every effective epimorphism in C is a \mathcal{G} -calypsis *a fortiori*, so every member of \mathcal{G} is quadrably \mathcal{G} -eucalyptic. Thus, every member of \mathcal{D} is \mathcal{D} -agathic, as required. ■

2.2.5 ¶ Let κ be a regular cardinal.

DEFINITION. A κ -ary **extensive ecumene** is an ecumene (C, \mathcal{G}, J) where:

- C is a κ -ary extensive category.
- J is a κ -ary superextensive coverage on C .
- Every complemented monomorphism in C is a member of \mathcal{G} .

Recognition principle for extensive ecumena that satisfy the descent axiom

LEMMA. *Let (C, \mathcal{G}, J) be an ecumene that satisfies the descent axiom. The following are equivalent:*

- (i) *(C, \mathcal{G}, J) is a κ -ary extensive ecumene.*
- (ii) *C is a κ -ary extensive category and J is a κ -ary superextensive coverage on C .*

Proof. (i) \Rightarrow (ii). Immediate.

(ii) \Rightarrow (i). Let 0 be the initial object in \mathcal{C} . Then, for every object X in \mathcal{C} , the unique morphism $0 \rightarrow X$ is vacuously of \mathcal{G} -type (\mathcal{G}, J) -locally on the domain, so it is a member of \mathcal{G} . Similarly, given morphisms $f_0 : X_0 \rightarrow Y_0$ and $f_1 : X_1 \rightarrow Y_1$ in \mathcal{C} that are members of \mathcal{G} , $f_0 \amalg f_1 : X_0 \amalg X_1 \rightarrow Y_0 \amalg Y_1$ is of \mathcal{G} -type J -locally on the base, so it is also a member of \mathcal{G} . Since every isomorphism in \mathcal{C} is a member of \mathcal{G} , it follows that every complemented monomorphism in \mathcal{C} is a member of \mathcal{G} . ■

Properties
of extensive
ecumenae

PROPOSITION. *Let $(\mathcal{C}, \mathcal{G}, J)$ be a κ -ary extensive ecumene.*

- (i) *Given a family $(f_i \mid i \in I)$ where I is a κ -small set and each f_i is a morphism $X_i \rightarrow Y$ in \mathcal{C} , if each $f_i : X_i \rightarrow Y$ is a member of \mathcal{G} , then the induced morphism $f : \coprod_{i \in I} X_i \rightarrow Y$ is also a member of \mathcal{G} .*
- (ii) *\mathcal{G} is closed under κ -ary coproduct in \mathcal{C} .*
- (iii) *Assuming J is \mathcal{G} -adapted, a morphism $g : Y \rightarrow Z$ in \mathcal{C} is J -covering if and only if there is a morphism $f : X \rightarrow Y$ in \mathcal{C} such that $g \circ f : X \rightarrow Z$ is J -covering morphism in \mathcal{C} that is a member of \mathcal{G} .*

Proof. (i). By construction, $f : \coprod_{i \in I} X_i \rightarrow Y$ is of \mathcal{G} -type (\mathcal{G}, J) -semi-locally on the domain, so it is a member of \mathcal{G} .

(ii). Since coproduct injections are in \mathcal{G} and \mathcal{G} is closed under composition, the claim is a special case of (i).

(iii). Apply lemma 1.5.16 and (i). ■

2.2.6 ✖ For the remainder of this section, $(\mathcal{C}, \mathcal{G}, J)$ is an ecumene.

2.2.7 ¶ It is convenient to introduce some terminology for properties of morphisms related to \mathcal{G} .

2.2.7(a) DEFINITION. A morphism in \mathcal{C} is **genial** if it is a member of \mathcal{G} .

2.2.7(b) DEFINITION. A morphism in \mathcal{C} is **étale** if it is \mathcal{G} -perfect.

2.2.7(c) DEFINITION. A morphism in \mathcal{C} is **eunoic**^[2] if it is J -semilocally of \mathcal{G} -type.

[2] — from Greek «εὐνοϊκός», favourable.

REMARK. By lemma 1.2.15, eunoic morphisms are automatically J-locally of \mathcal{G} -type.

2.2.8 DEFINITION. An equivalence relation (R, d_0, d_1) on an object X in \mathcal{C} is **étale** if it has the following property:

- The projections $d_0, d_1 : R \rightarrow X$ are étale morphisms.

2.2.8(a) LEMMA. Let (R, d_0, d_1) be a kernel pair of a morphism $f : X \rightarrow Y$ in \mathcal{C} .
 Recognition principle for kernel pairs of étale morphisms
 The following are equivalent:

- (i) $f : X \rightarrow Y$ is a genial morphism in \mathcal{C} and (R, d_0, d_1) is an étale equivalence relation on X in \mathcal{C} .
- (ii) $f : X \rightarrow Y$ is an étale morphism in \mathcal{C} .

Proof. (i) \Rightarrow (ii). By lemma 1.1.10, the relative diagonal $\Delta_f : X \rightarrow R$ is étale if either $d_0 : R \rightarrow X$ or $d_1 : R \rightarrow X$ is étale, in which case $f : X \rightarrow Y$ itself is étale.

(ii) \Rightarrow (i). The class of étale morphisms in \mathcal{C} is a quadrable class of morphisms in \mathcal{C} , by proposition 1.1.11, so if $f : X \rightarrow Y$ is étale then both $d_0, d_1 : R \rightarrow X$ are also étale. ■

2.2.8(b) LEMMA. Let (R, d_0, d_1) be a kernel pair of a J-covering morphism $f : X \rightrightarrows Y$ in \mathcal{C} . Assuming $(\mathcal{C}, \mathcal{G}, J)$ satisfies the descent axiom, the following are equivalent:
 Recognition principle for kernel pairs of covering étale morphisms

- (i) (R, d_0, d_1) is an étale equivalence relation on X in \mathcal{C} .
- (ii) $f : X \rightrightarrows Y$ is an étale morphism in \mathcal{C} .

Proof. By definition, the following is a pullback square in \mathcal{C} :

$$\begin{array}{ccc} R & \xrightarrow{d_0} & X \\ d_1 \downarrow & & \downarrow f \\ X & \xrightarrow{f} & Y \end{array}$$

Since $f : X \rightrightarrows Y$ is a J-covering morphism in \mathcal{C} , the descent axiom implies that $f : X \rightrightarrows Y$ is a member of \mathcal{G} if and only if either $d_0 : R \rightarrow X$ or $d_1 : R \rightarrow X$ (or both) is a member of \mathcal{G} . Thus, the claims reduce to lemma 2.2.8(a). ■

2.2.9 ¶ The following generalises Proposition 1.6 in [JM].

DEFINITION. The ecumene (C, \mathcal{G}, J) is **étale** if every member of \mathcal{G} is an étale morphism in C .

The induced étale ecumene **PROPOSITION.** *Let \mathcal{D} be the class of étale morphisms in C .*

- (i) (C, \mathcal{D}, J) is an étale ecumene.
- (ii) If (C, \mathcal{G}, J) satisfies the descent axiom, then (C, \mathcal{D}, J) also satisfies the descent axiom.
- (iii) (C, \mathcal{G}, J) is κ -ary extensive if and only if (C, \mathcal{D}, J) is κ -ary extensive.

Proof. (i). By proposition 1.1.11, \mathcal{D} is a class of separated fibrations in C . It remains to be shown that every morphism of \mathcal{D} -type (\mathcal{D}, J) -semilocally on the domain is a member of \mathcal{D} .

Let $f : X \rightarrow Y$ be a morphism in C and let Φ be a J -covering \mathcal{D} -sink on X such that, for every $(U, x) \in \Phi$, $f \circ x : U \rightarrow Y$ is a member of \mathcal{D} . Then $f : X \rightarrow Y$ is of \mathcal{G} -type (\mathcal{G}, J) -semilocally on the domain, so by lemma 1.2.6, $f : X \rightarrow Y$ is a member of \mathcal{G} . Since $x : U \rightarrow X$ is a member of \mathcal{G} , the induced morphism $U \times_Y U \rightarrow X \times_Y X$ is also a member of \mathcal{G} . On the other hand, $f \circ x : U \rightarrow Y$ is a member of \mathcal{D} , so the relative diagonal $\Delta_{f \circ x} : U \rightarrow U \times_Y U$ is a member of \mathcal{G} . Thus, the relative diagonal $\Delta_f : X \rightarrow X \times_Y X$ is of \mathcal{G} -type (\mathcal{G}, J) -semilocally on the domain, therefore it is a member of \mathcal{G} . Hence, $f : X \rightarrow Y$ is indeed a member of \mathcal{D} .

(ii). Let $f : X \rightarrow Y$ be a morphism in C and let Ψ be a J -covering sink on Y such that, for every $(V, y) \in \Psi$, we have a pullback square in C of the form below,

$$\begin{array}{ccc} U & \xrightarrow{x} & X \\ v \downarrow & & \downarrow f \\ V & \xrightarrow{y} & Y \end{array}$$

where $v : U \rightarrow V$ is a member of \mathcal{D} . The hypothesis implies $f : X \rightarrow Y$ is a member of \mathcal{G} , and it remains to be shown that $f : X \rightarrow Y$ is \mathcal{G} -separated.

By the pullback pasting lemma, every face of the following diagram is a pullback square in \mathcal{C} :

$$\begin{array}{ccccc}
 U \times_Y U & \longrightarrow & U & & \\
 \downarrow x \times_y x & \searrow & \downarrow & \searrow & \\
 & & U & \longrightarrow & V \\
 & & \downarrow & \downarrow & \downarrow \\
 X \times_Y X & \longrightarrow & X & & \\
 & \searrow & \downarrow & \searrow & \\
 & & X & \longrightarrow & Y
 \end{array}$$

Hence, we have a pullback square in \mathcal{C} of the form below:

$$\begin{array}{ccc}
 U & \xrightarrow{x} & X \\
 \Delta_v \downarrow & & \downarrow \Delta_f \\
 U \times_Y U & \longrightarrow & X \times_Y X
 \end{array}$$

Moreover, by [proposition A.2.14](#),

$$\{ (U \times_Y U, x \times_y x) \mid (V, y) \in \Psi \}$$

is a \mathcal{J} -covering sink on $X \times_Y X$, so the relative diagonal $\Delta_f : X \rightarrow X \times_Y X$ is of \mathcal{G} -type \mathcal{J} -semilocally on the base. Thus, $\Delta_f : X \rightarrow X \times_Y X$ is a member of \mathcal{G} , as required.

(iii). Immediate, because genial monomorphisms are the same as étale monomorphisms. ■

2.2.10 ¶ Étale morphisms and eunoic morphisms are related as follows.

Recognition principle for étale morphisms

LEMMA. *Let $f : X \rightarrow Y$ be a quadrable morphism in \mathcal{C} such that the relative diagonal $\Delta_f : X \rightarrow X \times_Y X$ is also a quadrable morphism in \mathcal{C} . Assuming $(\mathcal{C}, \mathcal{G}, \mathcal{J})$ satisfies the descent axiom, the following are equivalent:*

- (i) *The morphism $f : X \rightarrow Y$ is étale.*
- (ii) *Both $f : X \rightarrow Y$ and the relative diagonal $\Delta_f : X \rightarrow X \times_Y X$ are eunoic.*

Proof. This is a special case of [proposition 1.2.20](#). ■

2.2.II ¶ It is convenient to introduce the following terminology.

2.2.II (a) **DEFINITION.** An **open embedding** in \mathcal{C} is a monomorphism in \mathcal{C} that is a member of \mathcal{G} .

2.2.II (b) **DEFINITION.** The **descent axiom for open embeddings** in $(\mathcal{C}, \mathcal{G}, \mathcal{J})$ is the following:

- Every *monomorphism* in \mathcal{C} of \mathcal{G} -type \mathcal{J} -semilocally on the base is an open embedding in \mathcal{C} .

Recognition principles for open embeddings **LEMMA.** *Let $f : X \rightarrow Y$ be a quadrable morphism in \mathcal{C} . Assuming $(\mathcal{C}, \mathcal{G}, \mathcal{J})$ satisfies the descent axiom for open embeddings, the following are equivalent:*

- (i) $f : X \rightarrow Y$ is an open embedding in \mathcal{C} .
- (ii) $f : X \rightarrow Y$ is an étale monomorphism in \mathcal{C} .
- (iii) $f : X \rightarrow Y$ is a eunoic monomorphism in \mathcal{C} .

Proof. (i) \Rightarrow (ii). The relative diagonal $\Delta_f : X \rightarrow X \times_Y X$ is an isomorphism in \mathcal{C} .

(ii) \Rightarrow (iii). Étale morphisms are eunoic *a fortiori*.

(iii) \Rightarrow (i). It suffices to verify the following:

- Every *monomorphism* in \mathcal{C} of \mathcal{G} -type \mathcal{J} -semilocally on the domain is an open embedding in \mathcal{C} .

However, by [lemma 1.2.19\(a\)](#), every such monomorphism is automatically of \mathcal{G} -type $(\mathcal{G}, \mathcal{J})$ -semilocally on the domain, hence is indeed a member of \mathcal{G} . ■

2.2.I2 ¶ Let $\mathcal{G}_{\text{mono}}$ be the class of open embeddings in \mathcal{C} .

We will see that the following is a specialisation of the notion of étale morphism.

DEFINITION. A **local homeomorphism** in \mathcal{C} is a morphism in \mathcal{C} of $\mathcal{G}_{\text{mono}}$ -type $(\mathcal{G}_{\text{mono}}, \mathcal{J})$ -semilocally on the domain.

REMARK. By lemma 1.2.6, local homeomorphisms are automatically of $\mathcal{G}_{\text{mono}}$ -type $(\mathcal{G}_{\text{mono}}, \mathbf{J})$ -locally on the domain.

PROPOSITION.

- (i) *Every open embedding in \mathcal{C} is a local homeomorphism in \mathcal{C} .*
- (ii) *The class of local homeomorphisms in \mathcal{C} is a quadrable class of morphisms in \mathcal{C} .*
- (iii) *The class of local homeomorphisms in \mathcal{C} is closed under composition.*
- (iv) *Given morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ in \mathcal{C} , if both $g : Y \rightarrow Z$ and $g \circ f : X \rightarrow Z$ are local homeomorphisms, then $f : X \rightarrow Y$ is also a local homeomorphism.*
- (v) *Every local homeomorphism in \mathcal{C} is an étale morphism in \mathcal{C} .*

Proof. (i). Immediate.

(ii). Clearly, every local homeomorphism in \mathcal{C} is of \mathcal{G} -type $(\mathcal{G}, \mathbf{J})$ -semi-locally on the domain, hence is a member of \mathcal{G} . In particular, local homeomorphisms in \mathcal{C} are quadrable. The claim follows, by proposition 1.2.4.

(iii). In view of lemma 1.2.6, we may apply proposition 1.2.5.

(iv). Let Ψ be a \mathbf{J} -covering $\mathcal{G}_{\text{mono}}$ -sink on Y such that, for every $(V, y) \in \Psi$, $g \circ y : V \rightarrow Z$ is an open embedding in \mathcal{C} . By proposition A.2.I4 and (ii), there is a \mathbf{J} -covering $\mathcal{G}_{\text{mono}}$ -sink Φ on X such that, for every $(U, x) \in \Phi$, $g \circ f \circ x : U \rightarrow Z$ is an open embedding in \mathcal{C} and factors through $g \circ y : V \rightarrow Z$ for some $(V, y) \in \Psi$. Recalling that $\mathcal{G}_{\text{mono}}$ is a class of fibrations in \mathcal{C} , by lemma 1.1.3, $f \circ x : U \rightarrow Y$ is also an open embedding in \mathcal{C} . Hence, $f : X \rightarrow Y$ is a local homeomorphism in \mathcal{C} .

(v). If $f : X \rightarrow Y$ is a local homeomorphism in \mathcal{C} , then the relative diagonal $\Delta_f : X \rightarrow X \times_Y X$ is also a local homeomorphism in \mathcal{C} , by (i), (ii), and (iv). Thus, by lemma 1.1.6, $f : X \rightarrow Y$ is indeed an étale morphism in \mathcal{C} . ■

2.2.13 PROPOSITION. *Let \mathcal{D} be the class of local homeomorphisms in \mathcal{C} .*

The ecumene of
local homeo-
morphisms

- (i) $(\mathcal{C}, \mathcal{D}, \mathcal{J})$ is an étale ecumene.
- (ii) $(\mathcal{C}, \mathcal{G}, \mathcal{J})$ satisfies the descent axiom for open embeddings if and only if $(\mathcal{C}, \mathcal{D}, \mathcal{J})$ satisfies the descent axiom for open embeddings.
- (iii) $(\mathcal{C}, \mathcal{G}, \mathcal{J})$ is κ -ary extensive if and only if $(\mathcal{C}, \mathcal{D}, \mathcal{J})$ is κ -ary extensive.

Proof. (i). By proposition 2.2.12, \mathcal{D} is a class of separated fibrations in \mathcal{C} . It remains to be shown that every morphism of \mathcal{D} -type $(\mathcal{D}, \mathcal{J})$ -semilocally on the domain is a member of \mathcal{D} .

Let $f : X \rightarrow Y$ be a morphism in \mathcal{C} and let Φ be a \mathcal{J} -covering \mathcal{D} -sink on X such that, for every $(U, x) \in \Phi$, $f \circ x : U \rightarrow Y$ is a local homeomorphism in \mathcal{C} . So, for each $(U, x) \in \Phi$, there is a \mathcal{J} -covering $\mathcal{G}_{\text{mono}}$ sink $\Theta_{(U, x)}$ on U such that, for every $(T, u) \in \Theta_{(U, x)}$, $x \circ u : T \rightarrow X$ is an open embedding in \mathcal{C} . Consider the sink Φ' defined as follows:

$$\Phi' = \bigcup_{(U, x) \in \Phi} \{(T, x \circ u) \mid (T, u) \in \Theta_{(U, x)}\}$$

By proposition A.2.14, Φ' is a \mathcal{J} -covering $\mathcal{G}_{\text{mono}}$ -sink on X . Moreover, each $f \circ x \circ u : T \rightarrow Y$ is a local homeomorphism in \mathcal{C} , so $f : X \rightarrow Y$ is of \mathcal{D} -type $(\mathcal{G}_{\text{mono}}, \mathcal{J})$ -semilocally on the domain. Thus, by proposition 1.2.4, $f : X \rightarrow Y$ itself is indeed a local homeomorphism in \mathcal{C} .

(ii) and (iii). Immediate. ■

2.2.14 ¶ We will now characterise local homeomorphisms as genial morphisms whose kernel pair have a certain property.

DEFINITION. An equivalence relation (R, d_0, d_1) on an object X in \mathcal{C} is **tractable** if it has the following properties:

- The projections $d_0, d_1 : R \rightarrow X$ are members of \mathcal{G} .
- There is a \mathcal{J} -covering $\mathcal{G}_{\text{mono}}$ -sink Φ on X such that, for every $(U, x) \in \Phi$ and every object (T, r) in $\mathcal{C}/_R$, if both $d_0 \circ r, d_1 \circ r : T \rightarrow X$ factor through $x : U \rightarrow X$, then $d_0 \circ r = d_1 \circ r$.

REMARK. By [proposition 2.2.12](#), if (R, d_0, d_1) is a tractable equivalence relation on X , then the relative diagonal $\Delta : X \rightarrow R$ is an open embedding in \mathcal{C} . However, the converse is not true in general, even when we assume that the projections $d_0, d_1 : R \rightarrow X$ are local homeomorphisms in \mathcal{C} .

2.2.14(a) LEMMA. *Let (R, d_0, d_1) be an equivalence relation on an object X in \mathcal{C} . Assuming the projections $d_0, d_1 : R \rightarrow X$ are genial morphisms in \mathcal{C} , if there is a local homeomorphism $f : X \rightarrow Y$ in \mathcal{C} such that $f \circ d_0 = f \circ d_1$, then (R, d_0, d_1) is tractable.*

A sufficient
criterion for
tractability

Proof. Let Φ be a \mathcal{J} -covering $\mathcal{G}_{\text{mono}}$ -sink on X such that, for every $(U, x) \in \Phi$, $f \circ x : U \rightarrow Y$ is an open embedding in \mathcal{C} . For every object (T, r) in $\mathcal{C}_{/R}$, we have $f \circ d_0 \circ r = f \circ d_1 \circ r$, so if both $d_0 \circ r, d_1 \circ r : T \rightarrow X$ factor through $x : U \rightarrow X$, then $d_0 \circ r = d_1 \circ r$. Hence (R, d_0, d_1) is indeed a tractable equivalence relation on X in \mathcal{C} . ■

2.2.14(b) LEMMA. *Let (R, d_0, d_1) be a kernel pair of a morphism $f : X \rightarrow Y$ in \mathcal{C} . The following are equivalent:*

Recognition prin-
ciple for kernel
pairs of local
homeomorphisms

(i) *$f : X \rightarrow Y$ is a genial morphism in \mathcal{C} and (R, d_0, d_1) is a tractable equivalence relation on X in \mathcal{C} .*

(ii) *$f : X \rightarrow Y$ is a local homeomorphism in \mathcal{C} .*

Proof. (i) \Rightarrow (ii). Let Φ be a \mathcal{J} -covering $\mathcal{G}_{\text{mono}}$ -sink on X such that, for every $(U, x) \in \Phi$ and every object (T, r) in $\mathcal{C}_{/R}$, if both $d_0 \circ r, d_1 \circ r : T \rightarrow X$ factor through $x : U \rightarrow X$, then $d_0 \circ r = d_1 \circ r$. Then $f \circ x : U \rightarrow Y$ is a monomorphism in \mathcal{C} : indeed, given $u_0, u_1 : T \rightarrow U$ in \mathcal{C} , if $f \circ x \circ u_0 = f \circ x \circ u_1$, then we may apply the hypothesis to deduce that $u_0 = u_1$. Since $f : X \rightarrow Y$ is a member of \mathcal{G} and \mathcal{G} is closed under composition, it follows that $f \circ x : U \rightarrow Y$ is an open embedding in \mathcal{C} . Hence $f : X \rightarrow Y$ is indeed a local homeomorphism in \mathcal{C} .

(ii) \Rightarrow (i). This is a special case of [lemma 2.2.14\(a\)](#). ■

2.2.14(c) **LEMMA.** *Let (R, d_0, d_1) be a kernel pair of a J-covering morphism $f : X \twoheadrightarrow Y$ in \mathcal{C} . Assuming $(\mathcal{C}, \mathcal{G}, J)$ satisfies the descent axiom for open embeddings, the following are equivalent:*

Recognition principle for kernel pairs of covering local homeomorphisms

- (i) (R, d_0, d_1) is a tractable equivalence relation on X in \mathcal{C} .
- (ii) $f : X \twoheadrightarrow Y$ is a local homeomorphism in \mathcal{C} .

Proof. (i) \Rightarrow (ii). By definition, the following is a pullback square in \mathcal{C} :

$$\begin{array}{ccc} R & \xrightarrow{d_0} & X \\ d_1 \downarrow & & \downarrow f \\ X & \xrightarrow{f} & Y \end{array}$$

Suppose $x : U \rightarrow X$ is an open embedding in \mathcal{C} such that $f \circ x : U \rightarrow Y$ is a monomorphism in \mathcal{C} . Then the projection $X \times_Y U \rightarrow X$ is also an open embedding in \mathcal{C} , and since $f : X \twoheadrightarrow Y$ is a J-covering morphism in \mathcal{C} , $f \circ x : U \rightarrow Y$ is an open embedding in \mathcal{C} . Thus, following the argument of [lemma 2.2.14\(b\)](#), we see that $f : X \twoheadrightarrow Y$ itself is a local homeomorphism in \mathcal{C} .

(ii) \Rightarrow (i). This is a special case of [lemma 2.2.14\(a\)](#). ■

2.3 Extents

SYNOPSIS. We examine the properties of categories of objects that are obtained as étale quotients of distinguished objects in pretoposes equipped with a notion of étale morphism.

PREREQUISITES. §§ 1.1, 1.2, 1.4, 1.5, 2.1, 2.2.

2.3.1 ¶ Let κ be a regular cardinal and let \mathcal{S} be a κ -ary pretopos.

The following definition is due to Joyal [JM, § 1].

DEFINITION. A **class of étale morphisms** in \mathcal{S} is a subset $\mathcal{D} \subseteq \text{mor } \mathcal{S}$ with the following properties:

A1. Every isomorphism in \mathcal{S} is a member of \mathcal{D} and \mathcal{D} is closed under composition.

A2. \mathcal{D} is a quadrable class of morphisms in \mathcal{S} .

A3. In every pullback square in \mathcal{S} of the form below,

$$\begin{array}{ccc} \tilde{A} & \longrightarrow & A \\ \tilde{h} \downarrow & & \downarrow h \\ \tilde{B} & \longrightarrow & B \end{array}$$

where $\tilde{B} \twoheadrightarrow B$ is an effective epimorphism in \mathcal{S} , if $\tilde{h} : \tilde{A} \rightarrow \tilde{B}$ is a member of \mathcal{D} , then $h : A \rightarrow B$ is also a member of \mathcal{D} .

A4. For every κ -small set I , the unique morphism $\coprod_{i \in I} 1 \rightarrow 1$ is a member of \mathcal{D} , where 1 is the terminal object of \mathcal{S} .

A5. For every family $(h_i \mid i \in I)$ where I is a κ -small set, if each $h_i : A_i \rightarrow B_i$ is a member of \mathcal{D} , then $\prod_{i \in I} h_i : \prod_{i \in I} A_i \rightarrow \prod_{i \in I} B_i$ is also a member of \mathcal{D} .

A7. If $h : A \rightarrow B$ is a member of \mathcal{D} , then the relative diagonal $\Delta_h : A \rightarrow A \times_B A$ is also a member of \mathcal{D} .

A8. Given an effective epimorphism $h : A \twoheadrightarrow B$ in \mathcal{S} and a morphism $k : B \rightarrow C$ in \mathcal{S} , if both $h : A \twoheadrightarrow B$ and $k \circ h : A \rightarrow C$ are members of \mathcal{D} , then $k : B \rightarrow C$ is also a member of \mathcal{D} .

EXAMPLE. The class of all morphisms in \mathcal{S} is a class of étale morphisms in \mathcal{S} .

Recognition
principle for
classes of étale
morphisms

LEMMA. Let \mathcal{D} be a set of morphisms in \mathcal{S} . The following are equivalent:

- (i) \mathcal{D} satisfies axioms A1, A2, A4, A5, A7, and A8.
- (ii) $(\mathcal{S}, \mathcal{D}, \mathbf{K})$, where \mathbf{K} is the κ -ary canonical coverage on \mathcal{S} , is an étale κ -ary extensive ecumene.

Moreover, \mathcal{D} satisfies axiom A3 if and only if $(\mathcal{S}, \mathcal{D}, \mathbf{K})$ satisfies the descent axiom.

Proof. (i) \Rightarrow (ii). By lemma 1.1.10, axioms A1, A2, and A7 imply that \mathcal{D} is a class of separated fibrations in \mathcal{S} . Hence, by axioms A1 and A5, for every κ -small set I , every morphism $1 \rightarrow \coprod_{i \in I} 1$ is a member of \mathcal{D} , and therefore (by axiom A2 and extensivity) every complemented monomorphism is a member of \mathcal{D} . Axioms A1, A2, A4, and A5 imply that, for every object B in \mathcal{S} and every family $(h_i \mid i \in I)$ where I is a κ -small set and each $h_i : A_i \rightarrow B$ is a member of \mathcal{D} , the induced morphism $\coprod_{i \in I} A_i \rightarrow B$ is a member of \mathcal{D} . Thus, in view of proposition 1.4.1, lemma 1.5.16, and axiom A8, every morphism in \mathcal{S} of \mathcal{D} -type $(\mathcal{D}, \mathbf{K})$ -semilocally on the domain is a member of \mathcal{D} . Hence, $(\mathcal{S}, \mathcal{D}, \mathbf{K})$ is an étale κ -ary extensive ecumene, and it is clear that axiom A3 implies that the descent axiom is satisfied.

(ii) \Rightarrow (i). Axioms A1, A2, and A7 are immediate. Axiom A8 is also straightforward to verify, given that the \mathbf{K} -covering morphisms in \mathcal{S} are precisely the effective epimorphisms in \mathcal{S} . The same argument shows that the descent axiom implies axiom A3. Finally, axioms A4 and A5 are special cases of proposition 2.2.5. ■

2.3.2 ¶ Let $(\mathcal{C}, \mathcal{D}, \mathbf{J})$ be an étale κ -ary extensive ecumene that satisfies the descent axiom, let \mathcal{E} be the class of \mathbf{J} -covering morphisms in \mathcal{C} , let $\mathcal{S} = \mathbf{Ex}(\mathcal{C}, \mathcal{E})$, and let $\iota : \mathcal{C} \rightarrow \mathcal{S}$ be the insertion functor. Assume $(\mathcal{C}, \mathcal{E})$ satisfies the Shulman condition, so that \mathcal{S} is a κ -ary pretopos. (Recall proposition 1.5.13 and corollary 1.5.16.)

The following is a variation on Theorem 5.2 and Corollary 5.3 in [JM].

The exact
completion of an
étale extensive
ecumene

PROPOSITION. *Let \mathcal{G} be the class of morphisms in \mathcal{S} corresponding to morphisms in $\mathbf{PSH}(C)$ J -semilocally of \mathcal{D} -type and let $\hat{\mathcal{D}}$ be the class of \mathcal{G} -perfect morphisms in \mathcal{S} .*

- (i) *A morphism in \mathcal{S} is a member of \mathcal{G} if and only if it corresponds to a morphism in $\mathbf{PSH}(C)$ E -semilocally of \mathcal{D} -type.*
- (ii) *$(\mathcal{S}, \mathcal{G}, \mathbf{K})$ is a κ -ary extensive ecumene that satisfies the descent axiom, where \mathbf{K} is the κ -ary canonical topology on \mathcal{S} .*
- (iii) *The insertion functor $\iota : C \rightarrow \mathcal{S}$ sends members of \mathcal{D} to members of \mathcal{G} .*
- (iv) *$\hat{\mathcal{D}}$ is a class of étale morphisms in \mathcal{S} .*
- (v) *For every object Y in C and every morphism $h : A \rightarrow \iota(Y)$ in \mathcal{S} , if $h : A \rightarrow \iota(Y)$ is a member of $\hat{\mathcal{D}}$, then there exist a morphism $f : X \rightarrow Y$ in C and an effective epimorphism $p : \iota(X) \twoheadrightarrow A$ in \mathcal{S} such that $f : X \rightarrow Y$ is a member of \mathcal{D} .*
- (vi) *For every quadrable morphism $f : X \rightarrow Y$ in C , assuming the relative diagonal $\Delta_f : X \rightarrow X \times_Y X$ is a quadrable monomorphism in C , $f : X \rightarrow Y$ is a member of \mathcal{D} if and only if $\iota(f) : \iota(X) \rightarrow \iota(Y)$ is a member of $\hat{\mathcal{D}}$.*
- (vii) *If J is a \mathcal{D} -adapted coverage on C , then \mathcal{G} and $\hat{\mathcal{D}}$ both satisfy the \mathbf{K} -local collection axiom.*

Proof. (i). It is clear that every morphism in $\mathbf{PSH}(C)$ E -semilocally of \mathcal{D} -type is also J -semilocally of \mathcal{D} -type; it remains to be shown that every member of \mathcal{G} corresponds to a morphism in $\mathbf{PSH}(C)$ E -semilocally of \mathcal{D} -type.

Let $h : A \rightarrow B$ be a morphism in $\mathbf{PSH}(C)$ where both A and B are E -locally $\mathbf{1}$ -presentable E -sheaves on C . Suppose Ψ is a J -local generating set of elements of B such that, for every $(Y, b) \in \Psi$, there is a J -local generating set $\Phi_{(Y,b)}$ of elements of $\mathbf{Pb}(b \cdot -, h)$ such that, for every $(X, (f, a)) \in \Phi$, $f : X \rightarrow Y$ is a member of \mathcal{D} . By [theorem 1.4.16](#), $\mathbf{Pb}(b \cdot -, h)$ is E -locally $\mathbf{1}$ -generable, so it is J -locally $\mathbf{1}$ -generable *a fortiori*. Thus, by [lemma A.2.16](#), replacing $\Phi_{(Y,b)}$ with a subset if necessary,

we may assume that $\Phi_{(Y,b)}$ is a κ -small set. Similarly, we may assume that Ψ is a κ -small set. But [proposition 1.5.13](#), [lemmas 1.5.15](#) and [1.5.16](#), and [corollary 1.5.16](#) imply that A and B are also J -sheaves on C , so by [proposition 2.2.5](#), we may further assume that Ψ and $\Phi_{(Y,b)}$ are singletons. Hence, $h : A \rightarrow B$ is indeed E -semilocally of D -type.

(ii) and (iii). First, note that [lemma A.3.10](#) implies that the effective epimorphisms in S are precisely the morphisms in S corresponding to E -locally surjective morphisms in $\mathbf{PSH}(C)$, which are J -locally surjective *a fortiori*. We also know that K is a κ -ary superextensive coverage on S ([lemma 1.5.15](#) and [theorem 1.5.15](#)). Thus, by [proposition 1.2.14](#) and [lemma 1.5.16](#), \mathcal{G} is a class of fibrations in S , every morphism in S K -locally of \mathcal{G} -type is a member of \mathcal{G} , and $\iota : C \rightarrow S$ sends members of D to members of \mathcal{G} . Recalling [lemma 2.2.5](#), we conclude that (S, \mathcal{G}, K) is indeed a κ -ary extensive ecumene that satisfies the descent axiom.

(iv). Recalling [lemma 2.3.1](#), this is a special case of [proposition 2.2.9](#).

(v). This is a consequence of [lemma 1.4.8](#).

(vi). Apply [proposition 1.2.20](#).

(vii). By construction, for every object A in S , there exist an object X in C and an effective epimorphism $\iota(X) \twoheadrightarrow A$ in S . Since both \mathcal{G} and \hat{D} are quadrable classes of morphisms in S , it suffices to verify the following:

- If $h : A \rightarrow B$ is an effective epimorphism in S and $k : B \rightarrow C$ is a member of \mathcal{G} where $B = \iota(Y)$ and $C = \iota(Z)$ for some objects Y and Z in C , then there is a morphism $s : A' \rightarrow A$ in S such that $h \circ s : A' \rightarrow B$ is an effective epimorphism in S that is a member of \hat{D} .

It is straightforward to further reduce to the case where $A = \iota(X)$, which is an immediate consequence of [proposition 2.2.2](#). ■

2.3.3 ¶ The following is a notion intermediate between pretoposes with a class of étale morphisms (in the sense of Joyal) and étale extensive ecumenae.

DEFINITION. A κ -ary **gros pretopos** is a pair (S, \mathcal{D}) where:

- S is a κ -ary pretopos.
- \mathcal{D} is a set of morphisms in S that satisfies axioms A1, A2, A4, A5, and A7.
- Given an effective epimorphism $h : A \twoheadrightarrow B$ in S , a morphism $k : B \rightarrow C$ in S , and a kernel pair (R, d_0, d_1) of $h : A \twoheadrightarrow B$ in S , if $d_0, d_1 : R \rightarrow A$ and $k \circ h : A \rightarrow C$ are all members of \mathcal{D} , then both $h : A \twoheadrightarrow B$ and $k : B \rightarrow C$ are members of \mathcal{D} .

Axiom A8 and
gros pretoposes

LEMMA. Let S be a κ -ary pretopos and let \mathcal{D} be a set of morphisms in S .

- (i) If (S, \mathcal{D}) is a κ -ary gros pretopos, then \mathcal{D} satisfies axiom A8.
- (ii) If (S, \mathcal{D}) is a κ -ary gros pretopos, then $(S, \mathcal{D}, \mathbf{K})$ is an étale κ -ary extensive regulated ecumene, where \mathbf{K} is the κ -ary canonical coverage on S .
- (iii) If \mathcal{D} is a class of étale morphisms in S , then (S, \mathcal{D}) is a κ -ary gros pretopos.

Proof. (i). Let $h : A \twoheadrightarrow B$ be an effective epimorphism in S that is a member of \mathcal{D} and let (R, d_0, d_1) be a kernel pair of $h : A \twoheadrightarrow B$ in S . By axiom A2, the projections $d_0, d_1 : R \rightarrow A$ are members of \mathcal{D} . Let $k : B \rightarrow C$ be a morphism in S such that $k \circ h : A \rightarrow C$ is a member of \mathcal{D} . We may then apply the hypothesis to deduce that $k : B \rightarrow C$ is a member of \mathcal{D} , as required.

(ii). By lemma 2.3.1, $(S, \mathcal{D}, \mathbf{K})$ is an étale κ -ary extensive ecumene. It remains to be shown that $(S, \mathcal{D}, \mathbf{K})$ is regulated. Since S is a κ -ary pretopos and \mathbf{K} is the κ -ary canonical coverage on S , it is enough to verify that the image of a member of \mathcal{D} is also a member of \mathcal{D} .

Let $h : A \twoheadrightarrow B$ be a member of \mathcal{D} and let (R, d_0, d_1) be a kernel pair of $h : A \twoheadrightarrow B$ in S . As before, the projections $d_0, d_1 : R \rightarrow A$ are members of \mathcal{D} . On the other hand, $h = \text{im}(h) \circ e$ where $e : A \twoheadrightarrow \text{Im}(h)$ is the coequaliser of $d_0, d_1 : R \rightarrow A$. Thus, by the hypothesis, $\text{im}(h) : \text{Im}(h) \twoheadrightarrow B$ is indeed a member of \mathcal{D} .

(iii). Let $h : A \rightarrow B$ be an effective epimorphism in \mathcal{S} and let (R, d_0, d_1) be a kernel pair of $h : A \rightarrow B$ in \mathcal{S} . Suppose $d_0, d_1 : R \rightarrow A$ are members of \mathcal{D} . Then, by axiom A3, $h : A \rightarrow B$ is also a member of \mathcal{D} . Thus, given a morphism $k : B \rightarrow C$ in \mathcal{S} such that $k \circ h : A \rightarrow C$ is a member of \mathcal{D} , axiom A8 implies that $k : B \rightarrow C$ is also a member of \mathcal{D} . ■

Gros pretopos
with local
homeomorphisms

PROPOSITION. *Let \mathcal{S} be a κ -ary pretopos, let \mathcal{K} be the κ -ary canonical coverage on \mathcal{S} , let \mathcal{G} be a set of morphisms in \mathcal{S} such that $(\mathcal{S}, \mathcal{G}, \mathcal{K})$ is a κ -ary extensive ecumene that satisfies the descent axiom, and let \mathcal{D} be the class of local homeomorphisms in \mathcal{S} .*

(i) $(\mathcal{S}, \mathcal{D})$ is a κ -ary gros pretopos.

(ii) $(\mathcal{S}, \mathcal{D}, \mathcal{K})$ satisfies the descent axiom for open embeddings.

Proof. (i). By [proposition 2.2.13](#), $(\mathcal{S}, \mathcal{D}, \mathcal{K})$ is an étale κ -ary extensive ecumene, so by [lemmas 2.2.14\(c\)](#) and [2.3.1](#), $(\mathcal{S}, \mathcal{D})$ is a κ -ary gros pretopos.

(ii). In view of [lemma 2.2.11](#), it is clear that $(\mathcal{S}, \mathcal{D}, \mathcal{K})$ satisfies the descent axiom for open embeddings if and only if $(\mathcal{S}, \mathcal{G}, \mathcal{K})$ satisfies the descent axiom for open embeddings. ■

2.3.4 ✖ For the remainder of this section, $(\mathcal{S}, \mathcal{D})$ is a κ -ary gros pretopos.

2.3.5 **DEFINITION.** The **petit κ -ary pretopos** over an object A in \mathcal{S} is the full subcategory $\mathcal{D}_{/A} \subseteq \mathcal{S}_{/A}$ spanned by the objects (F, p) where $p : F \rightarrow A$ is a member of \mathcal{D} .

Properties of the
petit pretopos
over an object

PROPOSITION.

(i) *For every object A in \mathcal{S} , the inclusion $\mathcal{D}_{/A} \hookrightarrow \mathcal{S}_{/A}$ creates limits of finite diagrams, κ -ary coproducts, and exact quotients.*

(ii) *In particular, $\mathcal{D}_{/A}$ is a κ -ary pretopos.*

(iii) *For every morphism $h : A \rightarrow B$ in \mathcal{S} , the pullback functor $h^* : \mathcal{D}_{/B} \rightarrow \mathcal{D}_{/A}$ preserves limits of finite diagrams, κ -ary coproducts, and exact forks.*

Proof. Straightforward. (Use [proposition 2.2.5](#).) ◆

2.3.6 DEFINITION. A **unary basis** for (S, \mathcal{D}) is a full subcategory $C \subseteq S$ with the following properties:

- C is a unary site for S .
- For every morphism $h : A \rightarrow B$ in S , if B is in C and $h : A \rightarrow B$ is a member of \mathcal{D} , then there is an effective epimorphism $p : X \twoheadrightarrow A$ in S such that X is an object in C and $p : X \twoheadrightarrow A$ is a member of \mathcal{D} .

EXAMPLE. Of course, S is a unary basis for (S, \mathcal{D}) .

2.3.7 DEFINITION. A **κ -ary basis** for (S, \mathcal{D}) is a full subcategory $C \subseteq S$ with the following properties:

- C is a κ -ary site for S .
- For every morphism $h : A \rightarrow B$ in S , if B is in C and $h : A \rightarrow B$ is a member of \mathcal{D} , then there is an effective epimorphism $p : X \twoheadrightarrow A$ in S such that X is a coproduct of a κ -small family of objects in C and $p : X \twoheadrightarrow A$ is a member of \mathcal{D} .

EXAMPLE. Every unary basis for (S, \mathcal{D}) is also a κ -ary basis for (S, \mathcal{D}) *a fortiori*.

Recognition principle for bases of gros pretoposes

LEMMA. Let C_0 be a full subcategory of S and let C be the full subcategory of S spanned by the objects that are coproducts (in S) of κ -small families of objects in C_0 . The following are equivalent:

- (i) C_0 is a κ -ary basis for (S, \mathcal{D}) .
- (ii) C is a unary basis for (S, \mathcal{D}) .

Proof. Straightforward. (Use lemma 2.1.3 and proposition 2.2.5.) ◆

2.3.8 ✖ For the remainder of this section, C is a unary basis for (S, \mathcal{D}) .

2.3.9 PROPOSITION. Let $\mathcal{B} = \mathcal{D} \cap \text{mor } \mathcal{C}$.

Properties of
unary bases of
gros pretoposes

(i) Given morphisms $b : Y \rightarrow B$ and $h : A \rightarrow B$ in \mathcal{S} , if $h : A \rightarrow B$ is a member of \mathcal{D} and Y is an object in \mathcal{C} , then there is a commutative square in \mathcal{S} of the form below,

$$\begin{array}{ccc} X & \xrightarrow{a} & A \\ f \downarrow & & \downarrow h \\ Y & \xrightarrow{b} & B \end{array}$$

where X is an object in \mathcal{C} and $\langle a, f \rangle : X \rightarrow A \times_B Y$ is an effective epimorphism in \mathcal{S} that is a member of \mathcal{D} .

(ii) In particular, every member of \mathcal{D} is \mathbf{K} -locally of \mathcal{B} -type.

Proof. (i). By axiom A2, the projection $A \times_B Y \rightarrow Y$ is a member of \mathcal{D} , so there indeed exist an object X in \mathcal{C} and an effective epimorphism $\langle a, f \rangle : X \twoheadrightarrow A \times_B Y$ in \mathcal{S} that is a member of \mathcal{D} .

(ii). Given a morphism $h : A \rightarrow B$ in \mathcal{S} that is a member of \mathcal{D} , there is an effective epimorphism $b : Y \twoheadrightarrow B$ in \mathcal{S} where Y is an object in \mathcal{C} . The claim then reduces to (i). ■

2.3.10 DEFINITION. A $(\mathcal{C}, \mathcal{D})$ -**atlas** of an object A in \mathcal{S} is an object (X, a) in $\mathcal{D}/_A$ such that X is an object in \mathcal{C} and $a : X \rightarrow A$ is an effective epimorphism in \mathcal{S} .

2.3.10(a) LEMMA. Let $h : A \rightarrow B$ be a morphism in \mathcal{S} and let (Y, b) be a $(\mathcal{C}, \mathcal{D})$ -atlas of B . If $h : A \rightarrow B$ is a member of \mathcal{D} , then there is a commutative diagram in \mathcal{S} of the form below,

Atlases for étale
morphisms

$$\begin{array}{ccc} X & \xrightarrow{a} & A \\ f \downarrow & & \downarrow h \\ Y & \xrightarrow{b} & B \end{array}$$

where (X, a) is a $(\mathcal{C}, \mathcal{D})$ -atlas of A and $f : X \rightarrow Y$ is a member of \mathcal{D} .

Proof. This is a corollary of [proposition 2.3.9](#). ■

2.3.10(b) **LEMMA.** *Let $h : A \rightarrow B$ be a morphism in \mathcal{S} , let (X, a) be a $(\mathcal{C}, \mathcal{D})$ -atlas of A , and let (Y, b) be a $(\mathcal{C}, \mathcal{D})$ -atlas of B .*

Atlases for morphisms

(i) *There exist a $(\mathcal{C}, \mathcal{D})$ -atlas (U, x) of X and a morphism $y : U \rightarrow Y$ in \mathcal{C} such that the following diagram in \mathcal{S} commutes:*

$$\begin{array}{ccccc} U & \xrightarrow{x} & X & \xrightarrow{a} & A \\ y \downarrow & & & & \downarrow h \\ Y & \xrightarrow{b} & & & B \end{array}$$

(ii) *Moreover, we may choose $x : U \rightarrow X$ and $y : U \rightarrow Y$ so that the induced morphism $\langle a \circ x, y \rangle : U \rightarrow A \times_B Y$ is an effective epimorphism in \mathcal{S} that is a member of \mathcal{D} .*

Proof. (i). Consider a pullback square in \mathcal{S} of the form below:

$$\begin{array}{ccc} X \times_B Y & \xrightarrow{g} & Y \\ p \downarrow & & \downarrow b \\ X & \xrightarrow{h \circ a} & B \end{array}$$

Note that $p : X \times_B Y \rightarrow X$ is an effective epimorphism in \mathcal{S} that is a member of \mathcal{D} . Since X is an object in \mathcal{C} , we can find a $(\mathcal{C}, \mathcal{D})$ -atlas $(U, \langle x, y \rangle)$ of $X \times_B Y$ such that $x = p \circ \langle x, y \rangle : U \rightarrow X$ is a member of \mathcal{D} . Thus, (U, x) is the required $(\mathcal{C}, \mathcal{D})$ -atlas of X .

(ii). The induced morphism $a \times_B \text{id}_Y : X \times_B Y \rightarrow A \times_B Y$ is an effective epimorphism in \mathcal{S} that is a member of \mathcal{D} , so the same is true of $\langle a \circ x, y \rangle : U \rightarrow A \times_B Y$. ■

2.3.II **DEFINITION.** A $(\mathcal{C}, \mathcal{D})$ -**extent**^[1] in \mathcal{S} is an object A in \mathcal{S} that admits a $(\mathcal{C}, \mathcal{D})$ -atlas.

We write $\mathbf{Xt}(\mathcal{C}, \mathcal{D})$ for the full subcategory of \mathcal{S} spanned by the $(\mathcal{C}, \mathcal{D})$ -extents in \mathcal{S} .

[1] — a back-formation from ‘extensive category’.

REMARK. We should justify the omission of S from the notation $\mathbf{Xt}(C, D)$. By [theorem 2.1.8](#), the Yoneda representation $S \rightarrow \mathbf{SH}(C, E)$ is fully faithful, with essential image given by [lemma 2.1.9](#), so S is determined up to equivalence by C and E alone. Moreover, in some cases, D is determined by $B = D \cap \text{mor } C$.

In the case where C is the category of κ -ary coproducts of objects in a κ -ary basis C_0 for (S, D) , [theorem 2.1.14](#) and [lemma 2.1.16](#) allow us to substitute $\mathbf{SH}(C_0, J_0)$ for $\mathbf{SH}(C, E)$ in the above.

Properties of
the category
of extents

PROPOSITION.

- (i) *The class of morphisms in $\mathbf{Xt}(C, D)$ that are members of D is a quadrable class of morphisms in $\mathbf{Xt}(C, D)$.*
- (ii) *Given an effective epimorphism $h : A \twoheadrightarrow B$ in S , if $h : A \rightarrow B$ is a member of D and A is a (C, D) -extent, then B is also a (C, D) -extent.*
- (iii) *If (S, D, K) satisfies the descent axiom for open embeddings, then $\mathbf{Xt}(C, D)$ is closed in S under exact quotient of tractable equivalence relations.*
- (iv) *If D satisfies axiom A3, then $\mathbf{Xt}(C, D)$ is closed in S under exact quotient of étale equivalence relations.*
- (v) *$\mathbf{Xt}(C, D)$ is closed in S under κ -ary coproduct if and only if the coproduct of every κ -small family of objects in C is a (C, D) -charted extent.*
- (vi) *If $\mathbf{Xt}(C, D)$ is closed in S under κ -ary coproduct, then $\mathbf{Xt}(C, D)$ is a κ -ary extensive category.*
- (vii) *$\mathbf{Xt}(C, D)$ is closed in S under finitary product if and only if the product of every finite family of objects in C is a (C, D) -charted extent.*
- (viii) *$\mathbf{Xt}(C, D)$ is closed in S under pullback if and only if the pullback of every cospan in C is a (C, D) -charted extent.*

Proof. (i). This is an immediate consequence of [lemma 2.3.10\(a\)](#) and the fact that D is a quadrable class of morphisms in S .

(ii). Straightforward. (Use axiom A2 and the fact that the class of effective epimorphisms in \mathcal{S} is closed under composition.)

(iii). By lemma 2.2.14(b), exact quotients of tractable equivalence relations in \mathcal{S} are local homeomorphisms in \mathcal{S} , and by proposition 2.2.12, local homeomorphisms in \mathcal{S} are members of \mathcal{D} , so the claim reduces to (ii).

(iv). By axiom A3, exact quotients of étale equivalence relations in \mathcal{S} are étale effective epimorphisms in \mathcal{S} , so this claim also reduces to (ii).

(v). Straightforward. (Use axioms A1 and A5.)

(vi). By lemma 2.3.1, every complemented monomorphism in \mathcal{S} is a member of \mathcal{D} , so by (i), the claim reduces to the fact that \mathcal{S} is a κ -ary extensive category.

(vii). The ‘only if’ direction is clear, so suppose the product of every finite family of objects in \mathcal{C} is a $(\mathcal{C}, \mathcal{D})$ -charted extent. Since the class of effective epimorphisms in \mathcal{S} that are members of \mathcal{D} is closed under finitary product in \mathcal{S} and composition, it follows that the product of any finite family of objects in $\mathbf{Xt}(\mathcal{C}, \mathcal{D})$ is also a $(\mathcal{C}, \mathcal{D})$ -charted extent.

(viii). Apply lemma 1.1.5 and lemma 2.3.10(b). ■

2.3.12 DEFINITION. An object A in \mathcal{S} is **\mathcal{D} -localic** if $\mathcal{D}_{/A}$ is a localic κ -ary pretopos.

2.3.12(a) LEMMA. *Let A be an object in \mathcal{S} . The following are equivalent:*

Recognition principle for localic objects

- (i) A is a \mathcal{D} -localic object in \mathcal{S} .
- (ii) For every object (F, p) in $\mathcal{D}_{/A}$, $p : F \rightarrow A$ is a local homeomorphism in \mathcal{S} .

Proof. Straightforward. (Recall lemma 1.5.16.) ◆

2.3.12(b) LEMMA. Let $h : A \rightarrow B$ be a morphism in \mathcal{S} . If $h : A \rightarrow B$ is a member of \mathcal{D} and B is \mathcal{D} -localic, then A is also \mathcal{D} -localic.

Proof. By lemma 2.3.12(a), it is enough to show that, for every object (F, p) in $\mathcal{D}/_A$, $p : F \rightarrow A$ is a local homeomorphism in \mathcal{S} . Since B is \mathcal{D} -localic, both $h : A \rightarrow B$ and $h \circ p : F \rightarrow B$ are local homeomorphisms in \mathcal{S} . Thus, by proposition 2.2.12, $p : F \rightarrow A$ is indeed a local homeomorphism in \mathcal{S} . ■

2.3.12(c) LEMMA. Let $h : A \twoheadrightarrow B$ be an effective epimorphism in \mathcal{S} . If $h : A \twoheadrightarrow B$ is a local homeomorphism and A is \mathcal{D} -localic, then B is also \mathcal{D} -localic.

Proof. By lemma 2.3.12(a), it is enough to show that, for every object (F, q) in $\mathcal{D}/_B$, $q : F \rightarrow B$ is a local homeomorphism in \mathcal{S} . Consider a pullback square in \mathcal{S} of the form below:

$$\begin{array}{ccc} A \times_B F & \twoheadrightarrow & F \\ p \downarrow & & \downarrow q \\ A & \xrightarrow{h} & B \end{array}$$

Since A is \mathcal{D} -localic, the projection $p : A \times_B F \rightarrow A$ is a local homeomorphism in \mathcal{S} , so by proposition 2.2.12, $h \circ p : A \times_B F \rightarrow B$ is also a local homeomorphism in \mathcal{S} . On the other hand, the projection $A \times_B F \twoheadrightarrow A$ is both an effective epimorphism and a local homeomorphism in \mathcal{S} , so $q : F \rightarrow B$ is indeed a local homeomorphism in \mathcal{S} . ■

2.3.13 DEFINITION. A morphism $h : A \rightarrow B$ in \mathcal{S} is **laminar** if there is a κ -small set Φ of objects in $\mathcal{S}/_A$ with the following properties:

- A is the disjoint union of Φ .
- For every $(U, a) \in \Phi$, $h \circ a : U \rightarrow B$ is an open embedding in \mathcal{S} .

Properties
of laminar
morphisms

PROPOSITION.

- (i) Every isomorphism in \mathcal{S} is a laminar morphism in \mathcal{S} .
- (ii) For every object A in \mathcal{S} , the unique morphism $0 \rightarrow A$ is a laminar morphism in \mathcal{S} .

- (iii) Every laminar morphism in \mathcal{S} is a local homeomorphism in \mathcal{S} .
- (iv) For every local homeomorphism $h : A \rightarrow B$ in \mathcal{S} , there is a laminar effective epimorphism $p : \tilde{A} \rightarrow A$ in \mathcal{S} such that $h \circ p : \tilde{A} \rightarrow B$ is a laminar morphism in \mathcal{S} .
- (v) The class of laminar morphisms in \mathcal{S} is a quadrable class of morphisms in \mathcal{S} .
- (vi) The class of laminar morphisms in \mathcal{S} is closed under composition.
- (vii) The class of laminar morphisms in \mathcal{S} is closed under κ -ary coproduct in \mathcal{S} .

Proof. Straightforward. ◆

2.3.14 ¶ It is clear that every object in \mathcal{C} is a $(\mathcal{C}, \mathcal{D})$ -extent. Of course, in general, not every $(\mathcal{C}, \mathcal{D})$ -extent is an object in \mathcal{C} , but we do have the following results.

2.3.14(a) PROPOSITION. Assume the following hypotheses:

Extents and exact
quotients of
étale equival-
ence relations

- \mathcal{D} is a class of étale morphisms in \mathcal{S} .
- \mathcal{C} is closed in \mathcal{S} under limit of finite diagrams.

The following are equivalent:

- (i) For every étale equivalence relation (R, d_0, d_1) on an object X in \mathcal{C} , if R is an object in \mathcal{C} , then there is an exact fork in \mathcal{S} of the form below,

$$R \begin{array}{c} \xrightarrow{d_0} \\ \xrightarrow{d_1} \end{array} X \xrightarrow{f} Y$$

where Y is a object in \mathcal{C} .

- (ii) The inclusion $\mathcal{C} \hookrightarrow \mathbf{Xt}(\mathcal{C}, \mathcal{D})$ is (fully faithful and) essentially surjective on objects.

Proof. We may assume without loss of generality that \mathcal{C} is a replete (and full) subcategory of \mathcal{S} .

- (i) \Rightarrow (ii). Let A be an object in \mathcal{S} , let (X, a) be a $(\mathcal{C}, \mathcal{D})$ -atlas of A , and let (R, d_0, d_1) be a kernel pair for $a : X \rightrightarrows A$ in \mathcal{S} . We wish to show that A is an object in \mathcal{C} .

By [proposition 2.3.11](#), R is a (C, D) -charted extent, so there is a (C, D) -atlas (\tilde{R}, r) of R . Let $(Q, \tilde{d}_0, \tilde{d}_1)$ be a kernel pair of $r : \tilde{R} \rightarrow R$. It is not hard to see that Q is (the object part of) a limit of the following diagram in S ,

$$\begin{array}{ccc}
 \tilde{R} & \xrightarrow{d_1 \circ r} & X \\
 & \searrow^{d_0 \circ r} & \nearrow \\
 & & X \\
 & \nearrow_{d_1 \circ r} & \\
 \tilde{R} & \xrightarrow{d_0 \circ r} & X
 \end{array}$$

and since both \tilde{R} and X are objects in C , Q is also an object in C . But $(Q, \tilde{d}_0, \tilde{d}_1)$ is an étale equivalence relation on \tilde{R} and $r : \tilde{R} \rightarrow R$ is an effective epimorphism in S , so R is also an object in C . The same argument (*mutatis mutandis*) shows that A is an object in C .

(ii) \Rightarrow (i). See [proposition 2.3.11](#). (Note that D satisfies axiom A3 by hypothesis.) ■

2.3.14(b)
Extents and exact
quotients of tract-
able equival-
ence relations

PROPOSITION. *Assume the following hypotheses:*

- D is the class of local homeomorphisms in S .
- C is closed in S under κ -ary coproduct.
- (S, D, K) satisfies the descent condition for open embeddings.

The following are equivalent:

(i) C has the following properties:

- Given morphisms $f : X \rightarrow Y$ and $y : Y' \rightarrow Y$ in C , if $f : X \rightarrow Y$ is a member of D , then there is a pullback square in S of the form below,

$$\begin{array}{ccc}
 X' & \dashrightarrow & X \\
 \downarrow & & \downarrow f \\
 Y' & \xrightarrow{y} & Y
 \end{array}$$

where X' is an object in C .

- For every tractable equivalence relation (R, d_0, d_1) on an object X in C , if R is an object in C , then there is an exact fork in S of the

form below,

$$R \begin{array}{c} \xrightarrow{d_0} \\ \xrightarrow{d_1} \end{array} X \xrightarrow{f} Y$$

where Y is a object in C .

(ii) The inclusion $C \hookrightarrow \mathbf{Xt}(C, D)$ is (fully faithful and) essentially surjective on objects.

Proof. We may assume without loss of generality that C is a replete (and full) subcategory of S .

(i) \Rightarrow (ii). First, observe that the hypotheses imply the following:

- For every open embedding $h : A \rightarrow B$ in S , if B is an object in C , then A is also an object in C .

Indeed, by [lemma 2.3.10\(a\)](#), there is a (C, D) -atlas (X, a) of A , and since $h : A \rightarrow B$ is a monomorphism in S , the kernel pair of $h \circ a : X \rightarrow B$ is isomorphic to the kernel pair of $a : X \rightarrow A$; but by [lemma 2.2.14\(b\)](#), the kernel pair of $h \circ a : X \rightarrow B$ is a tractable equivalence relation, so the hypotheses imply that A is indeed an object in C .

Consequently, we obtain the following results:

- If $h : A \rightarrow B$ is a laminar morphism in S and B is an object in C , then A is also an object in C .
- For every (C, D) -extent A in S , there is (C, D) -atlas (X, a) of A where $a : X \twoheadrightarrow A$ is a laminar effective epimorphism in S .

(For the second claim, use [proposition 2.3.13](#).)

Now, consider a laminar effective epimorphism $a : X \twoheadrightarrow A$ in S where X is an object in C . Let (R, d_0, d_1) be a kernel pair of $a : X \twoheadrightarrow A$ in S . Then the projections $d_0, d_1 : R \rightarrow X$ are laminar morphisms in S , so R is an object in C . But (R, d_0, d_1) is a tractable equivalence relation on X , so it follows that A is also an object in C .

Hence, every (C, D) -extent is indeed an object in C .

(ii) \Rightarrow (i). See [proposition 2.3.11](#). ■

2.4 Functoriality

SYNOPSIS. We investigate sufficient conditions for a functor between gros pretoposes to preserve atlases and extents.

PREREQUISITES. §§ [1.1](#), [1.2](#), [1.3](#), [1.4](#), [1.5](#), [2.1](#), [2.2](#), [2.3](#), [A.2](#), [A.3](#).

2.4.1 ✖ Throughout this section:

- κ is a regular cardinal.
- S_0 is a κ -ary pretopos and C_0 is a unary site for S_0 .
- S_1 is a κ -ary pretopos and C_1 is a unary site for S_1 .

2.4.2 ¶ To begin, we need to understand when and how functors $C_0 \rightarrow C_1$ extend to functors $S_0 \rightarrow S_1$.

Let J_0 (resp. J_1) be the restriction of the κ -ary canonical coverage on S_0 (resp. S_1) to C_0 (resp. C_1) and let E_0 (resp. E_1) be the class of morphisms in C_0 (resp. C_1) that are effective epimorphisms in S_0 (resp. S_1). Note that both (C_0, E_0) and (C_1, E_1) satisfy the Shulman condition, by [proposition 2.1.7](#).

2.4.2(a) **LEMMA.** *Assuming C_0 (resp. C_1) is closed in S_0 (resp. S_1) under κ -ary coproduct, if $F : C_0 \rightarrow C_1$ is a functor that preserves κ -ary coproducts and $F : (C_0, E_0) \rightarrow (C_1, E_1)$ is a pre-admissible functor, then $F : (C_0, J_0) \rightarrow (C_1, J_1)$ is a pre-admissible functor.*

Proof. Let A be a J_1 -sheaf on C_1 . Clearly, A is also an E_1 -sheaf on C_1 , so F^*A is an E_0 -sheaf on C_0 . Since C_1 is a κ -ary extensive category and J_1 is a κ -ary superextensive coverage on C_1 , by [lemmas 1.5.15](#) and [1.5.16](#), $A : C_1^{\text{op}} \rightarrow \mathbf{SET}$ sends κ -ary coproducts in C_1 to κ -ary products in \mathbf{SET} . Hence, $F^*A : C_0^{\text{op}} \rightarrow \mathbf{SET}$ sends κ -ary coproducts in C_0 to κ -ary products in \mathbf{SET} . But C_0 is a κ -ary extensive category and J_0 is a κ -ary superextensive coverage on C_0 , so it follows that F^*A is a J_0 -sheaf on C . ■

2.4.2(b) **LEMMA.** *Assuming $F : (C_0, E_0) \rightarrow (C_1, E_1)$ is an admissible functor and $F : (C_0, J_0) \rightarrow (C_1, J_1)$ is a pre-admissible functor:*

- (i) *There exist a functor $\bar{F} : S_0 \rightarrow S_1$ and an isomorphism $\eta : F \Rightarrow \bar{F}$ of functors $C_0 \rightarrow S_1$ such that $\bar{F} : S_0 \rightarrow S_1$ preserves κ -ary coproducts and sends right-exact forks in S_0 to coequaliser diagrams in S_1 .*
- (ii) *Moreover, any such (\bar{F}, η) is a pointwise left Kan extension of $F : C_0 \rightarrow S_1$ along the inclusion $C_0 \hookrightarrow S_0$.*

Proof. By lemma 2.1.9, the inclusions $C_0 \hookrightarrow S_0$ and $C_1 \hookrightarrow S_1$ induce functors $\mathbf{Ex}(C_0, E_0) \rightarrow S_0$ and $\mathbf{Ex}(C_1, E_1) \rightarrow S_1$ that are fully faithful and essentially surjective on objects, so by lemma 1.4.24, we have a pointwise left Kan extension (\bar{F}, η) of the required type. It remains to be shown that $\bar{F} : S_0 \rightarrow S_1$ preserves κ -ary coproducts.

Since $F : (C_0, E_0) \rightarrow (C_1, E_1)$ is an admissible functor, for every object A in S_0 and every object B in S_1 , we have the following natural bijection:

$$S_1(\bar{F}A, B) \cong \mathrm{Hom}_{\mathbf{SH}(C_0, E_0)}(h_A, F^*h_B)$$

On the other hand, since $F : (C_0, J_0) \rightarrow (C_1, J_1)$ is a pre-admissible functor, F^*h_B is also a J_0 -sheaf on C_0 . Thus, by theorems 1.5.15 and 2.1.13, $S_1(\bar{F}-, B) : S_0^{\mathrm{op}} \rightarrow \mathbf{SET}$ sends κ -ary coproducts in S_0 to κ -ary products in \mathbf{SET} . It follows that $\bar{F} : S_0 \rightarrow S_1$ preserves κ -ary coproducts. ■

2.4.2(c) **LEMMA.** *Let $F : C_0 \rightarrow C_1$ be a functor, let \mathcal{D}_0 be a class of fibrations in C_0 , and assume the following hypotheses:*

- *$F : C_0 \rightarrow C_1$ preserves pullbacks of members of \mathcal{D}_0 .*
- *$F : C_0 \rightarrow C_1$ sends members of $E_0 \cap \mathcal{D}_0$ to members of E_1 .*
- *For every morphism $f : X \rightarrow Y$ in C_0 , if $f : X \rightarrow Y$ is a member of E_0 , then there is a morphism $p : \tilde{X} \rightarrow X$ in C_0 such that $f \circ p : \tilde{X} \rightarrow Y$ is a member of \mathcal{D}_0 and also a member of E_0 .*

Then $F : (C_0, E_0) \rightarrow (C_1, E_1)$ is a pre-admissible functor.

Proof. Let $E'_0 = E_0 \cap \mathcal{D}_0$. By remark 1.4.5, E'_0 -covering morphisms are the same as E_0 -covering morphisms, so by proposition A.3.7, E'_0 -sheaves

are the same as E_0 -sheaves. On the other hand, every member of E'_0 has a kernel pair in C_0 , so [lemma 1.4.18](#) implies that $F : (C_0, E'_0) \rightarrow (C_1, E_1)$ is a pre-admissible functor. The claim follows. ■

2.4.2(d) LEMMA. *Let $F : C_0 \rightarrow C_1$ be a functor. Assume the following hypotheses:*

- $F : (C_0, E_0) \rightarrow (C_1, E_1)$ is a pre-admissible functor.
- C_0 has E_0 -weak pullback squares.
- $F : C_0 \rightarrow C_1$ sends E_0 -weak pullback squares in C_0 to E_1 -weak pullback squares in C_1 .

Then $F : (C_0, E_0) \rightarrow (C_1, E_1)$ is an admissible functor.

Proof. By [corollary A.3.13](#), $F : C_0 \rightarrow C_1$ sends E_0 -covering morphisms in C_0 to E_1 -covering morphisms in C_1 , so the hypotheses imply that $F : C_0 \rightarrow C_1$ sends right- E_0 -exact forks in C_0 to right- E_1 -exact forks in C_1 . Moreover, by [lemma 1.4.21](#), $F : C_0 \rightarrow C_1$ sends E_0 -local complexes in C_0 to E_1 -local complexes in C_1 , so we may apply [lemma 1.4.25](#) to complete the proof. ■

2.4.2(e) LEMMA. *Let $F : C_0 \rightarrow C_1$ be a functor. Assuming $\kappa > \aleph_0$, the following are equivalent:*

- (i) $F : (C_0, E_0) \rightarrow (C, E_1)$ is an admissible functor.
- (ii) $F : (C_0, E_0) \rightarrow (C, E_1)$ is a pre-admissible functor.

Proof. (i) \Rightarrow (ii). Immediate.

(ii) \Rightarrow (i). Let K_1 be the κ -ary canonical coverage on S_1 and let J_1 be the restriction of K_1 to C_1 . (Note that C_1 is a κ -ary site for S_1 , so J_1 is a coverage on C_1 , by [proposition 2.1.13](#).) Since $F : (C_0, E_0) \rightarrow (C, E_1)$ is a pre-admissible functor, $F : (C_0, E_0) \rightarrow (C, J_1)$ is also a pre-admissible functor *a fortiori*. Thus, by [proposition A.3.13](#), the restriction functor $F^* : \mathbf{SH}(C_1, J_1) \rightarrow \mathbf{SH}(C_0, E_0)$ has a left adjoint, say $F_! : \mathbf{SH}(C_0, E_0) \rightarrow \mathbf{SH}(C_1, J_1)$. On the other hand, by [proposition 1.5.19](#), the Yoneda representation $S_1 \rightarrow \mathbf{SH}(C_1, J_1)$ preserves coequalisers, so $F_!$ sends E_0 -locally $\mathbb{1}$ -presentable E_0 -sheaves on C_0 to J_1 -locally κ -presentable J_1 -sheaves on

\mathcal{C}_1 . Moreover, by lemma 2.1.9, J_1 -locally κ -presentable J_1 -sheaves on \mathcal{C}_1 are the same as E_1 -locally $\mathbb{1}$ -presentable E_1 -sheaves on \mathcal{C}_1 , so it follows that $F : (\mathcal{C}_0, \mathcal{E}_0) \rightarrow (\mathcal{C}, \mathcal{E}_1)$ is an admissible functor. ■

2.4.3 ✖ For the remainder of this section:

- $(\mathcal{S}_0, \mathcal{D}_0)$ is a κ -ary gros pretopos and \mathcal{C}_0 is a unary basis for $(\mathcal{S}_0, \mathcal{D}_0)$.
- $(\mathcal{S}_1, \mathcal{D}_1)$ is a κ -ary gros pretopos and \mathcal{C}_1 is a unary basis for $(\mathcal{S}_1, \mathcal{D}_1)$.
- Given a pullback square in \mathcal{S}_0 of the form below,

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow f \\ Y' & \longrightarrow & Y \end{array}$$

if $f : X \rightarrow Y$ is a member of \mathcal{D}_0 and X, Y , and Y' are all objects in \mathcal{C}_0 , then X' is also an object in \mathcal{C}_0 .

- $F : \mathcal{S}_0 \rightarrow \mathcal{S}_1$ is a functor that preserves κ -ary coproducts and sends right-exact forks in \mathcal{S}_0 to coequaliser diagrams in \mathcal{S}_1 .
- $F : \mathcal{S}_0 \rightarrow \mathcal{S}_1$ sends objects in \mathcal{C}_0 to objects in \mathcal{C}_1 .
- Given a morphism $f : X \rightarrow Y$ in \mathcal{C}_0 , if $f : X \rightarrow Y$ is a member of \mathcal{D}_0 , then $Ff : FX \rightarrow FY$ is a member of \mathcal{D}_1 .
- Given a pullback square in \mathcal{C}_0 of the form below,

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow f \\ Y' & \longrightarrow & Y \end{array}$$

if $f : X \rightarrow Y$ is a member of \mathcal{D}_0 , then $F : \mathcal{S}_0 \rightarrow \mathcal{S}_1$ preserves this pullback square.

2.4.4 REMARK. Under the above assumptions, $F : \mathcal{S}_0 \rightarrow \mathcal{S}_1$ preserves effective epimorphisms. However, $F : \mathcal{S}_0 \rightarrow \mathcal{S}_1$ may fail to preserve kernel pairs.

2.4.5 ¶ If we assume that $F : \mathcal{S}_0 \rightarrow \mathcal{S}_1$ preserves limits of finite diagrams (or at least limits of finite connected diagrams), things are much simpler; for instance, it immediately follows that $F : \mathcal{S}_0 \rightarrow \mathcal{S}_1$ preserves exact forks. However, in some examples, even $F : \mathcal{C}_0 \rightarrow \mathcal{C}_1$ fails to preserve pullbacks, so this is not a reasonable assumption to make. We instead establish the desired preservation properties of $F : \mathcal{S}_0 \rightarrow \mathcal{S}_1$ in several small steps, starting from the basic assumptions above.

We begin with the following results, which will be required later.

2.4.5(a) **LEMMA.** *Consider a weak pullback square in \mathcal{S}_0 of the form below:*

Preservation of
special weak
pullback squares

$$\begin{array}{ccc} P & \longrightarrow & A_1 \\ \downarrow & & \downarrow h_1 \\ A_0 & \xrightarrow{h_0} & B \end{array}$$

Assume the following hypotheses:

- B is an object in \mathcal{C}_0 .
- There is a $(\mathcal{C}_0, \mathcal{D}_0)$ -atlas (X_1, a_1) of A_1 in \mathcal{S}_0 such that $h_1 \circ a_1 : X_1 \rightarrow B$ is a member of \mathcal{D}_0 .

Then $F : \mathcal{S}_0 \rightarrow \mathcal{S}_1$ preserves the above weak pullback square.

Proof. Since \mathcal{C}_0 is a unary basis for $(\mathcal{S}_0, \mathcal{D}_0)$, there exist an object X_0 in \mathcal{C}_0 and an effective epimorphism $a_0 : X_0 \twoheadrightarrow A_0$ in \mathcal{S}_0 . Consider the following commutative diagram in \mathcal{S}_0 ,

$$\begin{array}{ccccc} U & \longrightarrow & P \times_{A_1} X_1 & \longrightarrow & X_1 \\ \downarrow & & \downarrow & & \downarrow a_1 \\ X_0 \times_{A_0} P & \longrightarrow & P & \longrightarrow & A_1 \\ \downarrow & & \downarrow & & \downarrow h_1 \\ X_0 & \xrightarrow{a_0} & A_0 & \xrightarrow{h_0} & B \end{array}$$

where the top left, top right, and bottom left squares are pullback squares in \mathcal{S}_0 . By the weak pullback pasting lemma (lemma A.2.19), the outer square is a weak pullback square in \mathcal{S}_0 . Since $h_1 \circ a_1 : X_1 \rightarrow B$ is a member of \mathcal{D}_0 and X_0, X_1 , and B are all objects in \mathcal{C}_0 , $F : \mathcal{S}_0 \rightarrow$

S_1 sends the outer square to a weak pullback square in S_1 . Hence, by lemma 1.4.19(a), $F : S_0 \rightarrow S_1$ also sends the inner square to a weak pullback square in S_1 . ■

2.4.5(b) LEMMA.

Preservation
of special open
embeddings

(i) If $h : A \rightarrow B$ is an open embedding in S_0 and B is an object in C_0 , then $Fh : FA \rightarrow FB$ is an open embedding in S_1 .

(ii) Given a pullback square in S_0 of the form below,

$$\begin{array}{ccc} A' & \longrightarrow & A \\ h' \downarrow & & \downarrow h \\ B' & \longrightarrow & B \end{array}$$

if $h : A \rightarrow B$ is an open embedding in S_0 and both B' and B are objects in C_0 , then $F : S_0 \rightarrow S_1$ preserves this pullback square.

Proof. (i). By lemma 2.4.5(a), the following is a weak pullback square in S_1 :

$$\begin{array}{ccc} FA & \xrightarrow{\text{id}} & FA \\ \text{id} \downarrow & & \downarrow Fh \\ FA & \xrightarrow{Fh} & FB \end{array}$$

In other words, the relative diagonal $\Delta_{Fh} : FA \rightarrow FA \times_{FB} FA$ is an effective epimorphism in S_1 . But $\Delta_{Fh} : FA \rightarrow FA \times_{FB} FA$ is also a (split) monomorphism in S_1 , so this implies it is an isomorphism. Hence, $Fh : FA \rightarrow FB$ is a monomorphism in S_1 .

It remains to be shown that $Fh : FA \rightarrow FB$ is an étale morphism in S_1 , and by lemma 1.2.19(a), it is enough to verify that $Fh : FA \rightarrow FB$ is of D_1 -type semilocally on the domain. Since C_0 is a unary basis for (S_0, D_0) , there is a (C_0, D_0) -atlas of A in S_0 , say (X, a) . Then $Fa : FX \rightarrow FA$ is an effective epimorphism in S_1 and $Fh \circ Fa : FX \rightarrow FB$ is an étale morphism in S_1 , so we are done.

(ii). We know the following is a weak pullback square in S_1 :

$$\begin{array}{ccc} FA' & \longrightarrow & FA \\ Fh' \downarrow & & \downarrow Fh \\ FB' & \longrightarrow & FB \end{array}$$

Since $Fh' : FA' \rightarrow FB'$ is a monomorphism in \mathcal{S}_1 , the induced morphism $FA' \rightarrow FB' \times_{FB} FA$ is also a monomorphism in \mathcal{S}_1 . But any monomorphism that is an effective epimorphism is an isomorphism, so the above is indeed a pullback square in \mathcal{S}_1 . \blacksquare

2.4.5(c) **LEMMA.** *Let X be an object in \mathcal{C}_0 and let (R, d_0, d_1) be an étale equivalence relation on X in \mathcal{S}_0 .*

Preservation of special étale equivalence relations

(i) *If $\langle Fd_1, Fd_0 \rangle : FR \rightarrow FX \times_{FX} FX$ is a monomorphism in \mathcal{S}_1 , then (FR, Fd_0, Fd_1) is an equivalence relation on FX in \mathcal{S}_1 .*

Furthermore, if there is an étale morphism $f : X \rightarrow Y$ in \mathcal{C}_0 such that $f \circ d_0 = f \circ d_1$, then:

- (ii) (FR, Fd_0, Fd_1) is an étale equivalence relation on FX in \mathcal{S}_1 .
- (iii) *If (R, d_0, d_1) is a kernel pair of an (étale) effective epimorphism $a : X \twoheadrightarrow A$ in \mathcal{S}_0 , then $Fa : FX \twoheadrightarrow FA$ is an étale effective epimorphism in \mathcal{S}_1 and (FR, Fd_0, Fd_1) is a kernel pair of $Fa : FX \twoheadrightarrow FA$ in \mathcal{S}_1 .*

Proof. (i). (FR, Fd_0, Fd_1) is clearly reflexive and symmetric as a relation on FX , so it suffices to verify transitivity. Consider a pullback square in \mathcal{S}_0 of the form below:

$$\begin{array}{ccc} R^2 & \xrightarrow{d_0} & R \\ d_2 \downarrow & & \downarrow d_1 \\ R & \xrightarrow{d_0} & X \end{array}$$

By lemma 2.4.5(a), its image is a weak pullback square in \mathcal{S}_1 . On the other hand, since (R, d_0, d_1) is an equivalence relation on X in \mathcal{S}_1 , there is a (necessarily unique) morphism $d_1 : R^2 \rightarrow R$ in \mathcal{S}_1 such that $d_0 \circ d_1 = d_0 \circ d_0$ and $d_1 \circ d_1 = d_1 \circ d_2$. It follows that (FR, Fd_0, Fd_1) is transitive, so we indeed have an equivalence relation on FX in \mathcal{S}_1 .

(ii). Since $\langle d_1, d_0 \rangle : R \rightarrow X \times_Y X$ is an open embedding in \mathcal{S}_0 (by lemma 1.1.10) and $X \times_Y X$ is an object in \mathcal{C}_0 , lemma 2.4.5(b) implies that $\langle Fd_1, Fd_0 \rangle : FR \rightarrow FX \times_{FY} FX$ is an open embedding in \mathcal{S}_1 . In particular, the projections $Fd_0, Fd_1 : FR \rightarrow FX$ are étale morphisms in

\mathcal{S}_1 and $\langle Fd_1, Fd_0 \rangle : FR \rightarrow FX \times FX$ is a monomorphism in \mathcal{S}_1 . Thus, by (i), (FR, Fd_1, Fd_0) is an étale equivalence relation on FX in \mathcal{S}_1 .

(iii). The following is a coequaliser diagram in \mathcal{S}_1 ,

$$FR \begin{array}{c} \xrightarrow{Fd_0} \\ \xrightarrow{Fd_1} \end{array} FX \xrightarrow{Fa} FA$$

but (FR, Fd_0, Fd_1) is an étale equivalence relation on FX in \mathcal{S}_1 , so it is indeed the kernel pair of $Fa : FX \rightrightarrows FA$. Furthermore, there is a unique morphism $y : A \rightarrow Y$ in \mathcal{S}_0 such that $y \circ a = f$, and $Ff : FX \rightarrow FY$ is an étale morphism in \mathcal{S}_1 , so both $Fa : FX \rightrightarrows FA$ and $Fy : FA \rightarrow FY$ are étale morphisms in \mathcal{S}_1 . ■

2.4.6 LEMMA. *If $h : A \rightarrow B$ is a member of \mathcal{D}_0 and B is an object in \mathcal{C}_0 , then*

Preservation
of special étale
morphisms

$Fh : FA \rightarrow FB$ is a member of \mathcal{D}_1 .

Proof. Since \mathcal{C}_0 is a unary basis for $(\mathcal{S}_0, \mathcal{D}_0)$, there is a $(\mathcal{C}_0, \mathcal{D}_0)$ -atlas of A , say (X, a) . Let (R, d_0, d_1) be a kernel pair of $a : X \rightrightarrows A$ in \mathcal{S}_0 . By lemma 2.4.5(c), (FR, Fd_0, Fd_1) is an étale equivalence relation and also a kernel pair of $Fa : FX \rightrightarrows FA$ in \mathcal{S}_1 . We know that $Fh \circ Fa : FX \rightarrow FB$ is an étale morphism in \mathcal{S}_1 , so it follows that both $Fa : FX \rightrightarrows FA$ and $Fh : FA \rightarrow FB$ are also étale morphisms in \mathcal{S}_1 . ■

Properties of the
induced functor
between special
petit pretoposes

PROPOSITION. *For every object Z in \mathcal{C}_0 , the functor $(\mathcal{D}_0)_{/Z} \rightarrow (\mathcal{D}_1)_{/FZ}$ induced by $F : \mathcal{S}_0 \rightarrow \mathcal{S}_1$ preserves:*

- (i) κ -ary coproducts,
- (ii) effective epimorphisms,
- (iii) finitary products,
- (iv) pullbacks, and
- (v) exact forks.

Proof. (i) and (ii). Since $F : \mathcal{S}_0 \rightarrow \mathcal{S}_1$ preserves κ -ary coproducts and effective epimorphisms, the same is true of the functor $(\mathcal{D}_0)_{/Z} \rightarrow (\mathcal{D}_1)_{/FZ}$, by proposition 2.3.5.

(iii). It is clear that the induced functor $(\mathcal{D}_0)_{/Z} \rightarrow (\mathcal{D}_1)_{/FZ}$ preserves terminal objects, and it is enough to verify that the functor preserves binary products.

Let $h_0 : A_0 \rightarrow Z$ and $h_1 : A_1 \rightarrow Z$ be étale morphisms in \mathcal{S}_0 , let (X_0, a_0) be a $(\mathcal{C}_0, \mathcal{D}_0)$ -atlas of A_0 , let (X_1, a_1) be a $(\mathcal{C}_0, \mathcal{D}_0)$ -atlas of A_1 , let $(R_0, d_{0,0}, d_{0,1})$ be a kernel pair of $a_0 : X_0 \rightrightarrows A_0$, and let $(R_1, d_{1,0}, d_{1,1})$ be a kernel pair of $a_1 : X_1 \rightrightarrows A_1$. It is straightforward to verify that $(R_0 \times_Z R_1, d_{0,0} \times_Z d_{1,0}, d_{0,1} \times_Z d_{1,1})$ is a kernel pair of the effective epimorphism $a_0 \times_Z a_1 : X_0 \times_Z X_1 \rightrightarrows A_0 \times_Z A_1$. On the other hand, by lemma 2.4.5(b), we see that $F : \mathcal{S}_0 \rightarrow \mathcal{S}_1$ preserves the pullback squares in the following commutative diagram in \mathcal{S}_0 ,

$$\begin{array}{ccccc}
 R_0 \times_Z R_1 & \longrightarrow & (X_0 \times_Z X_0) \times_Z R_1 & \longrightarrow & R_1 \\
 \downarrow & & \downarrow & & \downarrow \\
 R_0 \times_Z (X_1 \times_Z X_1) & \longrightarrow & (X_0 \times_Z X_0) \times_Z (X_1 \times_Z X_1) & \longrightarrow & X_1 \times_Z X_1 \\
 \downarrow & & \downarrow & & \downarrow \\
 R_0 & \longrightarrow & X_0 \times_Z X_0 & \longrightarrow & Z
 \end{array}$$

so by lemma 2.4.5(c), we have the following coequaliser diagram in \mathcal{S}_1 ,

$$FR_0 \times_{FZ} FR_1 \begin{array}{c} \xrightarrow{Fd_{0,0} \times_{FZ} Fd_{1,0}} \\ \xrightarrow{Fd_{0,1} \times_{FZ} Fd_{1,1}} \end{array} FX_0 \times_{FZ} FX_1 \xrightarrow{F(a_0 \times_Z a_1)} F(A_0 \times_Z A_1)$$

where (by abuse of notation) we have elided the canonical isomorphism $FX_0 \times_{FZ} FX_1 \cong F(X_0 \times_Z X_1)$. In particular, $F : \mathcal{S}_0 \rightarrow \mathcal{S}_1$ preserves $A_0 \times_Z A_1$.

(iv). First, we will show that $(\mathcal{D}_0)_{/Z} \rightarrow (\mathcal{D}_1)_{/FZ}$ sends pullback squares in $(\mathcal{D}_0)_{/Z}$ to weak pullback squares in $(\mathcal{D}_1)_{/FZ}$.

Let $h_0 : A_0 \rightarrow B$ be a morphism in \mathcal{S}_0 , let $h_1 : A_1 \rightarrow B$ and $z : B \rightarrow Z$ be étale morphisms in \mathcal{S}_0 , let (Y, b) be a $(\mathcal{C}_0, \mathcal{D}_0)$ -atlas of B in \mathcal{S}_0 , let $(X_1, \langle a_1, f_1 \rangle)$ be a $(\mathcal{C}_0, \mathcal{D}_0)$ -atlas of $A_1 \times_B Y$ in \mathcal{S}_0 , and let $\langle a_0, f_0 \rangle : X_0 \rightrightarrows A_0 \times_B Y$ be an effective epimorphism in \mathcal{S}_0 where X_0 is

an object in \mathcal{C}_0 . We have the following pullback square in \mathcal{S}_0 ,

$$\begin{array}{ccc} X_0 \times_B X_1 & \xrightarrow{f_0 \times_B f_1} & Y \times_B Y \\ \text{id}_{X_0} \times_Z \text{id}_{X_1} \downarrow & & \downarrow \text{id}_Y \times_Z \text{id}_Y \\ X_0 \times_Z X_1 & \xrightarrow{f_0 \times_Z f_1} & Y \times_Z Y \end{array}$$

and since $X_0 \times_Z X_1$ and $Y \times_Z Y$ are both objects in \mathcal{C}_0 and $\text{id}_Y \times_Z \text{id}_Y : Y \times_B Y \rightarrow Y \times_Z Y$ is an open embedding, $F : \mathcal{S}_0 \rightarrow \mathcal{S}_1$ preserves this pullback square, by [lemma 2.4.5\(b\)](#). On the other hand, by [lemma 2.4.5\(c\)](#), $F : \mathcal{S}_0 \rightarrow \mathcal{S}_1$ preserves kernel pairs of $b : Y \rightarrow B$ and $z \circ b : Y \rightarrow Z$. Thus, in the following commutative diagram in \mathcal{S}_1 ,

$$\begin{array}{ccccccc} F(X_0 \times_B X_1) & \longrightarrow & F(Y \times_B Y) & \longrightarrow & FB & & \\ \downarrow & & \downarrow & & \downarrow \Delta_{Fz} & \searrow Fz & \\ F(X_0 \times_Z X_1) & \longrightarrow & F(Y \times_Z Y) & \longrightarrow & FB \times_{FZ} FB & \longrightarrow & FZ \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \Delta_{Fz} \\ FX_0 \times FX_1 & \xrightarrow{Ff_0 \times Ff_1} & FY \times FY & \xrightarrow{Fb \times Fb} & FB \times FB & \xrightarrow{Fz \times Fz} & FZ \times FZ \end{array}$$

each square is a pullback square in \mathcal{S}_1 . Hence, in the commutative diagram in \mathcal{S}_1 shown below,

$$\begin{array}{ccccc} F(X_0 \times_B X_1) & \longrightarrow & FX_1 & & \\ \downarrow & \searrow & \downarrow Fa_1 & & \\ & & F(A_0 \times_B A_1) & \longrightarrow & FA_1 \\ & & \downarrow & & \downarrow Fh_1 \\ FX_0 & \xrightarrow{Fa_0} & FA_0 & \xrightarrow{Fh} & FB \end{array}$$

the outer square is a pullback square in \mathcal{S}_1 , so the inner square is indeed a weak pullback square in \mathcal{S}_1 , by [lemma 1.4.19\(a\)](#).

It follows from the above that the functor $(\mathcal{D}_0)_{/Z} \rightarrow (\mathcal{D}_1)_{/FZ}$ preserves monomorphisms and pullbacks of monomorphisms. Thus, we have the following commutative square in \mathcal{S}_1 ,

$$\begin{array}{ccc} F(A_0 \times_B A_1) & \longrightarrow & FA_0 \times_{FB} FA_1 \\ \downarrow & & \downarrow \\ F(A_0 \times_Z A_1) & \xrightarrow{\cong} & FA_0 \times_{FZ} FA_1 \end{array}$$

where the vertical arrows are monomorphisms and the horizontal arrows are the canonical comparison morphisms. It follows that the comparison $F(A_0 \times_B A_1) \rightarrow FA_0 \times_{FB} FA_1$ is both an effective epimorphism and a monomorphism, so it is indeed an isomorphism.

(v). Combine (ii) and (iv). ■

REMARK. The arguments used above can also be used to establish the universal property of the exact completion of a category with limits of finite diagrams and a subcanonical unary topology.

2.4.7 PROPOSITION. *Given a pullback square in S_0 of the form below,*

Preservation of
base changes
of special étale
morphisms

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ p \downarrow & & \downarrow q \\ X & \xrightarrow{f} & Y \end{array}$$

if $f : X \rightarrow Y$ is a morphism in C_0 and $q : B \rightarrow Y$ is a member of \mathcal{D}_0 , then $F : S_0 \rightarrow S_1$ preserves this pullback square.

Proof. There is a exact fork in S_0 of the form below,

$$R \begin{array}{c} \xrightarrow{d_0} \\ \xrightarrow{d_1} \end{array} \tilde{B} \xrightarrow{b} B$$

where $d_0, d_1 : R \rightarrow \tilde{B}$ and $b : \tilde{B} \rightarrow B$ are all étale morphisms in S_0 and \tilde{B} is an object in C_0 . Consider the following commutative diagram in S_0 ,

$$\begin{array}{ccc} X \times_Y R & \longrightarrow & R \\ \downarrow \downarrow & & d_0 \downarrow \downarrow d_1 \\ X \times_Y \tilde{B} & \longrightarrow & \tilde{B} \\ \downarrow & & \downarrow b \\ A & \xrightarrow{h} & B \\ p \downarrow & & \downarrow q \\ X & \xrightarrow{f} & Y \end{array}$$

where the squares are pullback squares in S_0 . Note that $\langle d_1, d_0 \rangle : R \rightarrow \tilde{B} \times_Y \tilde{B}$ is an open embedding (by lemma 1.1.10) and that $\tilde{B} \times_Y \tilde{B}$ is

an object in \mathcal{C}_0 . Since exact forks are preserved by pullback in \mathcal{S}_0 , by [lemma 2.4.5\(b\)](#) and [proposition 2.4.6](#), both squares in the following diagram in \mathcal{S}_1 are pullback squares in \mathcal{S}_1 ,

$$\begin{array}{ccc}
 F(X \times_Y R) & \longrightarrow & FR \\
 \downarrow & & \downarrow F\langle d_1, d_0 \rangle \\
 F(X \times_Y (\tilde{B} \times_Y \tilde{B})) & \longrightarrow & F(\tilde{B} \times_Y \tilde{B}) \\
 \downarrow & & \downarrow \\
 FX & \xrightarrow{Ff} & FY
 \end{array}$$

and we have a commutative diagram in \mathcal{S}_1 of the form below,

$$\begin{array}{ccc}
 F(X \times_Y R) & \longrightarrow & FR \\
 \downarrow \downarrow & & \downarrow \downarrow \\
 F(X \times_Y \tilde{B}) & \longrightarrow & F\tilde{B} \\
 \downarrow & & \downarrow Fb \\
 FA & \xrightarrow{Fh} & FB \\
 Fp \downarrow & & \downarrow Fq \\
 FX & \xrightarrow{Ff} & FY
 \end{array}$$

where the columns are exact forks in \mathcal{S}_1 and every rectangle with base $Ff : FX \rightarrow FY$ is a pullback diagram in \mathcal{S}_1 , except possibly the bottom square. But [lemma 1.4.19\(b\)](#) is applicable, so we have the following pullback square in \mathcal{S}_1 :

$$\begin{array}{ccc}
 F(X \times_Y \tilde{B}) & \longrightarrow & F\tilde{B} \\
 \downarrow & & \downarrow Fb \\
 FA & \xrightarrow{Fh} & FB
 \end{array}$$

We can then apply [lemma 1.4.19\(c\)](#) to complete the proof. ■

2.4.8 ¶ Under some additional assumptions, we can show that $F : \mathcal{S}_0 \rightarrow \mathcal{S}_1$ sends $(\mathcal{C}_0, \mathcal{D}_0)$ -extents in \mathcal{S}_0 to $(\mathcal{C}_1, \mathcal{D}_1)$ -extents in \mathcal{S}_1 .

2.4.8(a) **PROPOSITION.** *Assuming $F : S_0 \rightarrow S_1$ sends étale equivalence relations on objects in C_0 to equivalence relations in S_1 and \mathcal{D}_1 is a class of étale morphisms in S_1 :*

Preservation
of special
étale effective
epimorphisms

- (i) *If $h : A \rightrightarrows B$ is an étale effective epimorphism in S_0 and A is an object in C_0 , then $Fh : FA \rightrightarrows FB$ is a étale effective epimorphism in S_1 , and $F : S_0 \rightarrow S_1$ preserves kernel pairs of $h : A \rightrightarrows B$.*
- (ii) *In particular, $F : S_0 \rightarrow S_1$ sends (C_0, \mathcal{D}_0) -extents in S_0 to (C_1, \mathcal{D}_1) -extents in S_1 .*

Proof. (i). Let (R, d_0, d_1) be a kernel pair of $h : A \rightarrow B$ in S_0 . Then (R, d_0, d_1) is an étale equivalence relation on A , so lemma 2.4.6 and the hypothesis implies that (FR, Fd_0, Fd_1) is an étale equivalence relation on FA . On the other hand, the following is a coequaliser diagram in S_1 ,

$$FR \begin{array}{c} \xrightarrow{Fd_0} \\ \xrightarrow{Fd_1} \end{array} FA \xrightarrow{Fh} FB$$

so (FR, Fd_0, Fd_1) is indeed a kernel pair of $Fh : FA \rightrightarrows FB$ in S_1 . Hence, by lemma 2.2.8(b), $Fh : FA \rightrightarrows FB$ is indeed an étale morphism in S_1 .

(ii). This is an immediate consequence of (i). ■

2.4.8(b) **PROPOSITION.** *Assuming \mathcal{D}_0 is the class of local homeomorphisms in S_0 and (S_1, \mathcal{D}_1) satisfies the descent axiom for open embeddings:*

Preservation of
special effective-
epimorphic local
homeomorphisms

- (i) *If $h : A \rightrightarrows B$ is a laminar effective epimorphism in S_0 and there is a local homeomorphism $A \rightarrow X$ in S_0 where X is an object in C_0 , then $Fh : FA \rightrightarrows FB$ is also a laminar effective epimorphism in S_1 , and $F : S_0 \rightarrow S_1$ preserves kernel pairs of $h : A \rightrightarrows B$.*
- (ii) *If $h : A \rightrightarrows B$ is both an effective epimorphism and a local homeomorphism in S_0 and A is an object in C_0 , then $Fh : FA \rightrightarrows FB$ is an effective epimorphism and a local homeomorphism in S_1 .*
- (iii) *In particular, $F : S_0 \rightarrow S_1$ sends (C_0, \mathcal{D}_0) -extents in S_0 to (C_1, \mathcal{D}_1) -extents in S_1 .*

Proof. (i). First, note that the existence of an object X in \mathcal{C}_0 and a local homeomorphism $A \rightarrow X$ in \mathcal{S}_0 ensures that $(\mathcal{D}_0)_{/A} \rightarrow (\mathcal{D}_1)_{/FA}$ preserves limits of finite diagrams, by [proposition 2.4.6](#).

Let (R, d_0, d_1) be a kernel pair of $h : A \rightarrow B$ in \mathcal{S}_0 . We will now show that (FR, Fd_0, Fd_1) is a tractable equivalence relation in \mathcal{S}_1 . By hypothesis, A is the disjoint union of a κ -small set Φ of subobjects of A such that, for every $(U, a) \in \Phi$, $h \circ a : U \rightarrow B$ is an open embedding in \mathcal{S}_0 . Thus,

$$R \cong \coprod_{\substack{(U_0, a_0) \in \Phi \\ (U_1, a_1) \in \Phi}} U_0 \times_B U_1$$

as objects in $(\mathcal{S}_0)_{/A \times A}$, and since the projections $U_0 \times_B U_1 \rightarrow U_0$ and $U_0 \times_B U_1 \rightarrow U_1$ are open embeddings in \mathcal{S}_0 , the projections $d_0, d_1 : R \rightarrow A$ are laminar morphisms in \mathcal{S}_0 . Thus,

$$FR \cong \coprod_{\substack{(U_0, a_0) \in \Phi \\ (U_1, a_1) \in \Phi}} F(U_0 \times_B U_1)$$

as objects in $(\mathcal{S}_1)_{/FA \times FA}$ and, by [proposition 2.4.6](#), F sends the projections $U_0 \times_B U_1 \rightarrow U_0$ and $U_0 \times_B U_1 \rightarrow U_1$ to open embeddings in \mathcal{S}_1 , so the induced morphism $F(U_0 \times_B U_1) \rightarrow FU_0 \times FU_1$ is a monomorphism in \mathcal{S}_1 . Since the class of monomorphisms in \mathcal{S}_1 is closed under κ -ary co-product, it follows that $\langle d_1, d_0 \rangle : FR \rightarrow FA \times FA$ is a monomorphism in \mathcal{S}_1 . Moreover, $Fd_0, Fd_1 : FR \rightarrow FA$ are laminar morphisms in \mathcal{S}_1 , so by ([proposition 2.3.13](#) and) [lemma 2.4.5\(c\)](#), (FR, Fd_0, Fd_1) is an étale equivalence relation on FA in \mathcal{S}_1 . In addition, for every $(U, a) \in \Phi$, we have the following commutative diagram in \mathcal{S}_0 ,

$$\begin{array}{ccccc} U & \xrightarrow{\langle \langle a, a \rangle, \text{id}_U \rangle} & R \times_A U & \longrightarrow & U \\ \langle \text{id}_U, \langle a, a \rangle \rangle \downarrow & & \downarrow & & \downarrow a \\ U \times_A R & \longrightarrow & R & \xrightarrow{d_0} & A \\ \downarrow & & \downarrow d_1 & & \\ U & \xrightarrow{a} & A & & \end{array}$$

where every square is a pullback square in \mathcal{S}_0 . Using the pullback pasting lemma, it is not hard to see that these pullback squares are preserved by

$F : \mathcal{S}_0 \rightarrow \mathcal{S}_1$, and it follows that (FR, Fd_0, Fd_1) is a tractable equivalence relation on FA in \mathcal{S}_1 . The above also implies that, for every $(U, a) \in \Phi$, $Fh \circ Fa : FU \rightarrow FB$ is a monomorphism in \mathcal{S}_1 .

On the other hand, the following is a coequaliser diagram in \mathcal{S}_1 ,

$$FR \begin{array}{c} \xrightarrow{Fd_0} \\ \xrightarrow{Fd_1} \end{array} FA \xrightarrow{Fh} FB$$

so (FR, Fd_0, Fd_1) is indeed a kernel pair of $Fh : FA \rightarrow FB$ in \mathcal{S}_1 . Hence, by lemma 2.2.14(c), $Fh : FA \rightarrow FB$ is a local homeomorphism in \mathcal{S}_1 . In particular, for every $(U, a) \in \Phi$, $Fh \circ Fa : FU \rightarrow FB$ is an open embedding in \mathcal{S}_1 , so $Fh : FA \rightarrow FB$ is indeed a laminar effective epimorphism in \mathcal{S}_1 .

(ii). Let $h : A \rightarrow B$ be a local homeomorphism in \mathcal{S}_0 where A is an object in \mathcal{C}_0 . By proposition 2.3.13, there is a laminar effective epimorphism $p : \tilde{A} \rightarrow A$ in \mathcal{S}_0 such that $h \circ p : \tilde{A} \rightarrow B$ is a laminar morphism in \mathcal{S}_0 . Since the induced functor $(\mathcal{D}_0)_{/A} \rightarrow (\mathcal{D}_1)_{/FA}$ preserves limits of finite diagrams, κ -ary coproducts, and effective epimorphisms, $Fp : F\tilde{A} \rightarrow FA$ is a laminar effective epimorphism in \mathcal{S}_1 . Moreover, by (i), if $h : A \rightarrow B$ is an effective epimorphism in \mathcal{S}_1 , then $Fh \circ Fp : F\tilde{A} \rightarrow FB$ is also a laminar effective epimorphism in \mathcal{S} . Thus, by proposition 2.2.12, $Fh : FA \rightarrow FB$ is indeed (an effective epimorphism and) a local homeomorphism in \mathcal{S}_1 .

(iii). This is an immediate consequence of (ii). ■

2.4.9 ✂ For the remainder of this section, we make the following additional assumptions:

- Given an effective epimorphism $h : A \rightarrow B$ in \mathcal{S}_0 , if $h : A \rightarrow B$ is a member of \mathcal{D}_0 and A is an object in \mathcal{C}_0 , then $Fh : FA \rightarrow FB$ is a member of \mathcal{D}_1 .
- For every $(\mathcal{C}_0, \mathcal{D}_0)$ -extent A in \mathcal{S}_0 , there exist an effective epimorphism $p : \tilde{A} \rightarrow A$ in \mathcal{S}_0 and a morphism $x : \tilde{A} \rightarrow X$ in \mathcal{S}_0 such that X is an object in \mathcal{C}_0 , both $p : \tilde{A} \rightarrow A$ and $x : \tilde{A} \rightarrow X$ are members of \mathcal{D}_0 , and $F : \mathcal{S}_0 \rightarrow \mathcal{S}_1$ preserves kernel pairs of $p : \tilde{A} \rightarrow A$.

2.4.10 ¶ Though the above assumptions seem weak, we will see that it has some important consequences, which eventually lead to a more elegant formulation. For instance, the first assumption can be replaced with the following:

LEMMA. *If $h : A \rightarrow B$ is a member of \mathcal{D}_0 and B is a (C_0, \mathcal{D}_0) -extent in \mathcal{S}_0 , then $Fh : FA \rightarrow FB$ is a member of \mathcal{D}_1 .*

Proof. By lemma 2.3.10(a), there is a commutative square in \mathcal{S}_0 of the form below,

$$\begin{array}{ccc} X & \xrightarrow{a} \twoheadrightarrow & A \\ f \downarrow & & \downarrow h \\ Y & \xrightarrow{b} \twoheadrightarrow & B \end{array}$$

where $f : X \rightarrow Y$ is an étale morphism in C_0 and both $a : X \twoheadrightarrow A$ and $b : Y \twoheadrightarrow B$ are étale effective epimorphisms in \mathcal{S}_0 . Then, by assumption, $Ff : FX \rightarrow FY$ is an étale morphism in \mathcal{S}_1 and both $Fa : FX \twoheadrightarrow FA$ and $Fb : FY \twoheadrightarrow FB$ are étale effective epimorphisms in \mathcal{S}_1 , so $Fh : FA \rightarrow FB$ is indeed an étale morphism in \mathcal{S}_1 . ■

2.4.11 LEMMA. *Consider a pullback square in \mathcal{S}_0 of the form below:*

Preservation
of pullbacks
of special
étale effective
epimorphisms

$$\begin{array}{ccc} \tilde{A} & \xrightarrow{p} \twoheadrightarrow & A \\ \tilde{h} \downarrow & & \downarrow h \\ \tilde{B} & \xrightarrow{q} \twoheadrightarrow & B \end{array}$$

Assuming the following hypotheses:

- $p : \tilde{A} \twoheadrightarrow A$ is an étale effective epimorphism in \mathcal{S}_0 .
- $q : \tilde{B} \twoheadrightarrow B$ is an étale morphism in \mathcal{S}_0 .
- $F : \mathcal{S}_0 \rightarrow \mathcal{S}_1$ preserves kernel pairs of $p : \tilde{A} \twoheadrightarrow A$ and $q : \tilde{B} \twoheadrightarrow B$.
- $F : \mathcal{S}_0 \rightarrow \mathcal{S}_1$ preserves pullbacks of étale morphisms along $\tilde{h} : \tilde{A} \rightarrow \tilde{B}$.

Then $F : \mathcal{S}_0 \rightarrow \mathcal{S}_1$ also preserves the above pullback square.

Proof. Apply lemma 1.4.19(b). ■

Properties of the induced functor between petit pretoposes over extents

PROPOSITION. *For every (C_0, D_0) -extent E in S_0 , the functor $(D_0)_{/E} \rightarrow (D_1)_{/FE}$ induced by $F : S_0 \rightarrow S_1$ preserves:*

- (i) *limits of finite diagrams,*
- (ii) *κ -ary coproducts, and*
- (iii) *exact forks.*

Proof. (i). It suffices to prove the following:

- Given a pullback square in S_0 of the form below,

$$\begin{array}{ccc} P & \longrightarrow & A_1 \\ \downarrow & & \downarrow h_1 \\ A_0 & \xrightarrow{h_0} & B \end{array}$$

if B is a (C_0, D_0) -extent in S_0 and both $h_0 : A_0 \rightarrow B$ and $h_1 : A_1 \rightarrow B$ are members of D_0 , then $F : S_0 \rightarrow S_1$ preserves this pullback square.

Let $q : \tilde{B} \rightarrow B$ be an étale effective epimorphism in S_0 such that there exist an object Y in C_0 and a étale morphism $\tilde{B} \rightarrow Y$ in S_0 and $F : S_0 \rightarrow S_1$ preserves kernel pairs of $q : \tilde{B} \rightarrow B$. By the pullback pasting lemma, we have the following pullback square in S_0 ,

$$\begin{array}{ccc} \tilde{B} \times_B P & \longrightarrow & \tilde{B} \times_B A_1 \\ \downarrow & & \downarrow \\ \tilde{B} \times_B A_0 & \longrightarrow & \tilde{B} \end{array}$$

and by [proposition 2.4.6](#), $F : S_0 \rightarrow S_1$ preserves this pullback square.

On the other hand, by [lemma 2.3.10\(a\)](#) and [lemma 2.4.10](#), $F : S_0 \rightarrow S_1$ sends the projections $\tilde{B} \times_B P \twoheadrightarrow P$, $\tilde{B} \times_B A_0 \twoheadrightarrow A_0$, and $\tilde{B} \times_B A_1 \twoheadrightarrow A_1$ to étale effective epimorphisms in S_1 . Let (R, d_0, d_1) be a kernel pair of $q : \tilde{B} \twoheadrightarrow B$ in S_0 . Then $F : S_0 \rightarrow S_1$ preserves each of the following parallel pairs of pullback squares,

$$\begin{array}{ccc} R \times_B P & \begin{array}{c} \xrightarrow{d_0 \times_B \text{id}_P} \\ \xrightarrow{d_1 \times_B \text{id}_P} \end{array} & \tilde{B} \times_B P \\ \downarrow & & \downarrow \\ R & \begin{array}{c} \xrightarrow{d_0} \\ \xrightarrow{d_1} \end{array} & \tilde{B} \end{array}$$

$$\begin{array}{ccc}
 R \times_B A_0 & \xrightarrow[d_1 \times_B \text{id}_{A_0}]{d_0 \times_B \text{id}_{A_0}} & \tilde{B} \times_B A_0 \\
 \downarrow & & \downarrow \\
 R & \xrightarrow[d_1]{d_0} & \tilde{B}
 \end{array}
 \qquad
 \begin{array}{ccc}
 R \times_B A_1 & \xrightarrow[d_1 \times_B \text{id}_{A_1}]{d_0 \times_B \text{id}_{A_1}} & \tilde{B} \times_B A_1 \\
 \downarrow & & \downarrow \\
 R & \xrightarrow[d_1]{d_0} & \tilde{B}
 \end{array}$$

and it follows that $F : \mathcal{S}_0 \rightarrow \mathcal{S}_1$ preserves the following exact forks:

$$R \times_B P \xrightarrow[d_1 \times_B \text{id}_P]{d_0 \times_B \text{id}_P} \tilde{B} \times_B P \longrightarrow P$$

$$R \times_B A_0 \xrightarrow[d_1 \times_B \text{id}_{A_0}]{d_0 \times_B \text{id}_{A_0}} \tilde{B} \times_B A_0 \longrightarrow A_0$$

$$R \times_B A_1 \xrightarrow[d_1 \times_B \text{id}_{A_1}]{d_0 \times_B \text{id}_{A_1}} \tilde{B} \times_B A_1 \longrightarrow A_1$$

Moreover, $F : \mathcal{S}_0 \rightarrow \mathcal{S}_1$ preserves pullbacks of étale morphisms along the projections $\tilde{B} \times_B P \rightarrow \tilde{B}$, $\tilde{B} \times_B A_0 \rightarrow \tilde{B}$, and $\tilde{B} \times_B A_1 \rightarrow \tilde{B}$, so by [lemma 2.4.II](#), $F : \mathcal{S}_0 \rightarrow \mathcal{S}_1$ also preserves these pullback squares:

$$\begin{array}{ccc}
 \tilde{B} \times_B P \longrightarrow P & \tilde{B} \times_B A_0 \longrightarrow A_0 & \tilde{B} \times_B A_1 \longrightarrow A_1 \\
 \downarrow & \downarrow & \downarrow \\
 \tilde{B} \xrightarrow{q} B & \tilde{B} \xrightarrow{q} B & \tilde{B} \xrightarrow{q} B
 \end{array}$$

Thus, in the following commutative diagrams in \mathcal{S}_1 ,

$$\begin{array}{ccccc}
 F(\tilde{B} \times_B P) & \longrightarrow & F(\tilde{B} \times_B A_1) & \longrightarrow & FA_1 \\
 \downarrow & & \downarrow & & \downarrow \\
 F(\tilde{B} \times_B A_0) & \longrightarrow & F\tilde{B} & \xrightarrow{Fq} & FB
 \end{array}$$

$$\begin{array}{ccc}
 F(\tilde{B} \times_B P) & \longrightarrow & FP \\
 \downarrow & & \downarrow \\
 F(\tilde{B} \times_B A_0) & \longrightarrow & FA_0 \\
 \downarrow & & \downarrow \\
 F\tilde{B} & \xrightarrow{Fq} & FB
 \end{array}$$

all the squares are pullback squares in S_1 , and by the pullback pasting lemma, in the following commutative diagram in S_1 ,

$$\begin{array}{ccccc} F(\tilde{B} \times_B P) & \longrightarrow & FP & \longrightarrow & FA_1 \\ \downarrow & & \downarrow & & \downarrow^{Fh_1} \\ F(\tilde{B} \times_B A_0) & \longrightarrow & FA_0 & \xrightarrow{Fh_0} & FB \end{array}$$

the left square and outer rectangle are pullback diagrams in S_1 . Hence, by [lemma 1.4.19\(c\)](#), the right square is indeed a pullback square in S_1 .

(ii) and (iii). Since $F : S_0 \rightarrow S_1$ preserves κ -ary coproducts and exact forks, the same is true of the functor $(D_0)_{/E} \rightarrow (D_1)_{/FE}$, by [proposition 2.3.5](#) and (i). ■

2.4.12 PROPOSITION. *Given a pullback square in S_0 of the form below,*

Preservation of
base changes
of étale morphisms
of extents

$$\begin{array}{ccc} C & \xrightarrow{k} & D \\ p \downarrow & & \downarrow q \\ A & \xrightarrow{h} & B \end{array}$$

if $h : A \rightarrow B$ is a morphism in $\mathbf{Xt}(C_0, D_0)$ and $q : D \rightarrow B$ is a member of D_0 , then $F : S_0 \rightarrow S_1$ preserves this pullback square.

Proof. Let (Y, b) be a (C_0, D_0) -atlas of B and let $(X, \langle a, f \rangle)$ be a (C_0, D_0) -atlas of $A \times_B Y$. By the pullback pasting lemma, we have the following commutative diagram in S_0 ,

$$\begin{array}{ccccc} D \times_B X & \longrightarrow & C \times_B Y & \xrightarrow{k \times_B \text{id}_Y} & D \times_B Y \\ \downarrow & & p \times_B \text{id}_Y \downarrow & & \downarrow \\ X & \xrightarrow{\langle a, f \rangle} & A \times_B Y & \longrightarrow & Y \end{array}$$

where the squares are pullback squares in S_0 , the vertical arrows are étale morphisms in S_0 , the composite $D \times_B X \rightarrow C \times_B Y \rightarrow D \times_B Y$ is $\text{id}_D \times_B f : D \times_B X \rightarrow D \times_B Y$, and $D \times_B X \rightarrow C \times_B Y$ is an étale effective epimorphism in S_0 . Thus, by [proposition 2.4.7](#), the following is

a pullback square in \mathcal{S}_1 :

$$\begin{array}{ccc} F(D \times_B X) & \xrightarrow{F(\text{id}_D \times_B f)} & F(D \times_B Y) \\ \downarrow & & \downarrow \\ FX & \xrightarrow{Ff} & FY \end{array}$$

On the other hand, by [lemma 2.3.10\(a\)](#) and [proposition 2.4.11](#), we also have the following pullback square in \mathcal{S}_1 ,

$$\begin{array}{ccc} F(D \times_B X) & \longrightarrow & F(C \times_B Y) \\ \downarrow & & \downarrow F(p \times_B \text{id}_Y) \\ FX & \xrightarrow{F\langle a, f \rangle} & F(A \times_B Y) \end{array}$$

where the horizontal arrows are étale effective epimorphisms in \mathcal{S}_1 , so we may apply [lemma 1.4.19\(c\)](#) to deduce that the following is a pullback square in \mathcal{S}_1 :

$$\begin{array}{ccc} F(C \times_B Y) & \xrightarrow{F(k \times_B \text{id}_Y)} & F(D \times_B Y) \\ F(p \times_B \text{id}_Y) \downarrow & & \downarrow \\ F(A \times_B Y) & \longrightarrow & FY \end{array}$$

On the other hand, we have the following pullback square in \mathcal{S}_1 ,

$$\begin{array}{ccc} F(D \times_B Y) & \longrightarrow & FD \\ \downarrow & & \downarrow Fq \\ FY & \xrightarrow{Fb} & FB \end{array}$$

so the outer rectangle of the following commutative diagram in \mathcal{S}_1 is a pullback diagram in \mathcal{S}_1 :

$$\begin{array}{ccccc} F(C \times_B Y) & \longrightarrow & FC & \xrightarrow{Fk} & FD \\ F(p \times_B \text{id}_Y) \downarrow & & Fp \downarrow & & \downarrow Fq \\ F(A \times_B Y) & \longrightarrow & FA & \xrightarrow{Fh} & FB \end{array}$$

Moreover, the left square is a pullback square in \mathcal{S}_1 wherein the horizontal arrows are étale effective epimorphisms in \mathcal{S}_1 , so the right square is indeed a pullback square in \mathcal{S}_1 . ■

2.4.13 ¶ To summarise:

Properties of
the induced
functor between
categories
of extents

THEOREM. *Under the standing assumptions (2.4.3 and 2.4.9):*

- (i) $F : S_0 \rightarrow S_1$ sends (C_0, D_0) -extents in S_0 to (C_1, D_1) -extents in S_1 .
- (ii) The induced functor $F : \mathbf{Xt}(C_0, D_0) \rightarrow \mathbf{Xt}(C_1, D_1)$ preserves étale morphisms and pullbacks of étale morphisms.
- (iii) Given an exact fork in S_0 of the form below,

$$R \begin{array}{c} \xrightarrow{d_0} \\ \xrightarrow{d_1} \end{array} A \xrightarrow{h} B$$

if A is a (C_0, D_0) -extent in S_0 and $h : A \rightarrow B$ is an étale effective epimorphism in S_0 , then $F : S_0 \rightarrow S_1$ preserves this exact fork.

In particular, $F : (\mathbf{Xt}(C_0, D_0), D_0) \rightarrow (\mathbf{Xt}(C_1, D_1), D_1)$ is a regulated functor.

Proof. (i). This is an immediate consequence of [lemma 2.4.10](#) and the assumption that $F : S_0 \rightarrow S_1$ preserves effective epimorphisms.

(ii). We know $\mathbf{Xt}(C_0, D_0) \rightarrow \mathbf{Xt}(C_1, D_1)$ preserves étale morphisms. For the preservation of pullbacks of étale morphisms, see [proposition 2.4.12](#).

(iii). Apply [lemma 2.3.10\(a\)](#) and (ii). ■

2.5 Universality

SYNOPSIS. We characterise the category of extents by a universal property in a special case.

PREREQUISITES. §§ [1.1](#), [1.4](#), [1.5](#), [2.1](#), [2.2](#), [2.3](#), [2.4](#), [A.1](#), [A.2](#), [A.3](#).

2.5.1 ✖ Throughout this section, κ is a regular cardinal.

2.5.2 ¶ For convenience, we introduce the following terminology.

DEFINITION. A κ -ary **admissible ecumene** is a tuple (C, D, J) where:

- (C, D, J) is a κ -ary extensive ecumene.
- D is the class of local homeomorphisms in C .
- J is a subcanonical D -adapted coverage on C .
- (C, E) satisfies the Shulman condition, where E is the class of J -covering morphisms in C .

REMARK. By [remark 2.2.3](#) and [proposition 2.2.12](#), any κ -ary admissible ecumene is an étale ecumene that satisfies the descent axiom.

2.5.3 ¶ Let (C, D, J) be a κ -ary admissible ecumene, let E be the class of J -covering morphisms in C , and let $S = \mathbf{Ex}(C, E)$. By [corollary 1.5.16](#), E is a κ -summable saturated unary coverage on C , so by [proposition 1.5.13](#), S is a κ -ary pretopos.

Let \mathcal{G} be the class of morphisms in S corresponding to morphisms in $\mathbf{PSH}(C)$ that are J -semilocally of D -type and let K be the κ -ary canonical coverage on S . By [proposition 2.3.2](#), (S, \mathcal{G}, K) is a κ -ary extensive ecumene that satisfies the descent axiom. Let \bar{D} be the class of local homeomorphisms in S . Then (S, \bar{D}) is a gros κ -ary pretopos that satisfies the descent axiom for open embeddings, by [proposition 2.3.3](#).

DEFINITION. The **gros κ -ary pretopos** associated with a κ -ary admissible ecumene (C, D, J) is (S, \bar{D}) as defined above.

2.5.4 ¶ With notation as in [paragraph 2.5.3](#), since J is a subcanonical coverage on C , E is a subcanonical unary coverage on C , and therefore the insertion $C \rightarrow S$ is fully faithful. By abuse of notation, we will pretend that the insertion is the inclusion of a full subcategory.

By construction, C is a unary site for S . Moreover, by [proposition 2.3.2](#), C is a unary basis for (S, \bar{D}) . Recalling [lemma A.3.10](#), a morphism in C is an effective epimorphism in S if and only if it is a J -covering morphism in C . Since J is a D -adapted coverage on C , for every effective epimorphism $h : A \twoheadrightarrow B$ in S , if B is an object in C , then there is a morphism $a : X \rightarrow A$ in S such that $h \circ p : X \rightarrow B$ is both a J -covering morphism in C and a member of D . A similar argument shows that $\bar{D} \cap \text{mor } C = D$; furthermore, C is a unary basis for (S, \bar{D}) . Thus, in a further abuse of notation, we may simply write D instead of \bar{D} .

DEFINITION. A (C, D, J) -**charted object** is a (C, D) -extent in S .

Properties of
the category of
charted objects

PROPOSITION.

- (i) $(\mathbf{Xt}(C, D), D, K)$ is a κ -ary admissible ecumene.
- (ii) $\mathbf{Xt}(C, D)$ has exact quotients of tractable equivalence relations, and every effective epimorphism in $\mathbf{Xt}(C, D)$ that is a member of D is K -covering.
- (iii) If C has exact quotients of tractable equivalence relations, and every quotient of every tractable equivalence relation in C is J -covering, then the inclusion $C \hookrightarrow \mathbf{Xt}(C, D)$ is (fully faithful and) essentially surjective on objects.

Proof. (i). First, note that $C \subseteq \mathbf{Xt}(C, D) \subseteq S$, so $\mathbf{Xt}(C, D)$ is also a unary site for S . Thus, by [proposition 2.1.7](#), the Shulman condition is satisfied.

Next, by [proposition 2.3.11](#), $(\mathbf{Xt}(C, D), D, K)$ is a κ -ary extensive ecumene, and clearly, the local homeomorphisms therein are the morphisms in $\mathbf{Xt}(C, D)$ that are local homeomorphisms in S .

It remains to be shown that K is a D -adapted coverage on $\mathbf{Xt}(C, D)$. Since K is a κ -ary superextensive coverage on $\mathbf{Xt}(C, D)$, it is enough to

verify the following:

- For every K -covering morphism $h : A \twoheadrightarrow B$ in $\mathbf{Xt}(C, D)$, there is a morphism $p : \tilde{A} \rightarrow A$ in $\mathbf{Xt}(C, D)$ such that $h \circ p : \tilde{A} \rightarrow B$ is a K -covering local homeomorphism in $\mathbf{Xt}(C, D)$.

Let (Y, b) be a (C, D) -atlas of B . Then the projection $A \times_B Y \twoheadrightarrow Y$ is an effective epimorphism in \mathcal{S} , so there is a morphism $\langle a, f \rangle : X \rightarrow A \times_B Y$ in \mathcal{S} such that $f : X \rightarrow Y$ is a J -covering local homeomorphism in C . Thus, $b \circ f : X \rightarrow B$ is a K -covering local homeomorphism in $\mathbf{Xt}(C, D)$; but $h \circ a = b \circ f$, so we are done.

(ii). Since (\mathcal{S}, D, K) satisfies the descent axiom for open embeddings, by [lemma 2.2.14\(c\)](#), $\mathbf{Xt}(C, D)$ is closed in \mathcal{S} under exact quotient of tractable equivalence relations. On the other hand, D is a quadrable class of morphisms in $\mathbf{Xt}(C, D)$ and the inclusion $\mathbf{Xt}(C, D) \hookrightarrow \mathcal{S}$ preserves (these) pullbacks, so any effective epimorphism in $\mathbf{Xt}(C, D)$ that is a member of D is an exact quotient in \mathcal{S} of its kernel pair, i.e. is an effective epimorphism in \mathcal{S} .

(iii). See [proposition 2.3.14\(b\)](#). ■

2.5.5 ¶ Let (C_0, D_0, J_0) and (C_1, D_1, J_1) be κ -ary admissible ecumenae.

DEFINITION. A κ -ary **admissible functor** $F : (C_0, D_0, J_0) \rightarrow (C_1, D_1, J_1)$ is a functor $F : C_0 \rightarrow C_1$ with the following properties:

- $F : C_0 \rightarrow C_1$ preserves κ -ary coproducts.
- $F : (C_0, E_0) \rightarrow (C_1, E_1)$ is an admissible functor, where E_0 (resp. E_1) is the class of J_0 -covering (resp. J_1 -covering) morphisms in C_0 (resp. C_1).
- For every pullback square in C_0 of the form below,

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow f \\ Y' & \longrightarrow & Y \end{array}$$

if $f : X \rightarrow Y$ is a member of D_0 , then $Ff : FX \rightarrow FY$ is a member of D_1 and $F : C_0 \rightarrow C_1$ preserves this pullback square.

Properties of
admissible func-
tors between
admissible
ecumenae

PROPOSITION. *Let (S_0, D_0) and (S_1, D_1) be the gros κ -ary pretoposes associated with (C_0, D_0, J_0) and (C_1, D_1, J_1) , respectively.*

- (i) $F : (C, J_0) \rightarrow (C, J_1)$ is a pre-admissible functor.
- (ii) *There exist a functor $\bar{F} : S_0 \rightarrow S_1$ and an isomorphism $\eta : F \Rightarrow \bar{F}$ of functors $C_0 \rightarrow S_1$ such that $\bar{F} : S_0 \rightarrow S_1$ preserves κ -ary coproducts and sends right-exact forks in S_0 to coequaliser diagrams in S_1 .*
- (iii) *Moreover, any such (\bar{F}, η) is a pointwise left Kan extension of $F : C_0 \rightarrow S_1$ along the inclusion $C_0 \hookrightarrow S_0$.*
- (iv) $\bar{F} : S_0 \rightarrow S_1$ sends (C_0, D_0) -extents in S_0 to (C_1, D_1) -extents in S_1 .
- (v) $\bar{F} : (\mathbf{Xt}(C_0, D_0), D_0, K_0) \rightarrow (\mathbf{Xt}(C_1, D_1), D_1, K_1)$ is a κ -ary admissible functor, where K_0 (resp. K_1) is the restriction of the κ -ary canonical coverage on S_0 (resp. S_1).

Proof. (i). This is a special case of [lemma 2.4.2\(a\)](#).

(ii) and (iii). See [lemma 2.4.2\(b\)](#).

(iv). In view of propositions [2.3.13](#) and [2.4.8\(b\)](#), we may apply [theorem 2.4.13](#).

(v). We have seen that $\bar{F} : (\mathbf{Xt}(C_0, D_0), D_0, K_0) \rightarrow (\mathbf{Xt}(C_1, D_1), D_1, K_1)$ preserves κ -ary coproducts, local homeomorphisms, and pullbacks of local homeomorphisms, and sends right- E_0 -exact forks to coequaliser diagrams. Moreover, by [lemma 2.1.10](#),

$$\bar{F} : (\mathbf{Xt}(C_0, D_0), E_0) \rightarrow (\mathbf{Xt}(C_1, D_1), E_1)$$

is a admissible functor, where E_0 (resp. E_1) is the class of K_0 -covering (resp. K_1 -covering) morphisms in S_0 (resp. S_1). Thus,

$$\bar{F} : (\mathbf{Xt}(C_0, D_0), D_0, K_0) \rightarrow (\mathbf{Xt}(C_1, D_1), D_1, K_1)$$

is indeed a κ -ary admissible functor. ■

2.5.6 DEFINITION. An **effective κ -ary admissible ecumene** is a κ -ary admissible ecumene $(\mathcal{C}, \mathcal{D}, \mathbf{J})$ with the following additional data:

- For each tractable equivalence relation (R, d_0, d_1) on each object X in \mathcal{C} , an exact quotient $q : X \rightarrow \bar{X}$ in \mathcal{C} such that $q : X \rightarrow \bar{X}$ is a \mathbf{J} -covering morphism.

Properties of
effective admiss-
ible ecumenae

PROPOSITION. *Let $(\mathcal{C}, \mathcal{D}, \mathbf{J})$ be an effective κ -ary admissible ecumene and let $(\mathcal{S}, \mathcal{D})$ be the associated gros κ -ary pretopos.*

- (i) $(\mathcal{C}, \mathcal{D}, \mathbf{J})$ is a regulated ecumene.
- (ii) Effective epimorphisms in \mathcal{C} that are members of \mathcal{D} are \mathbf{J} -covering morphisms in \mathcal{C} .
- (iii) Every \mathbf{J} -covering morphism in \mathcal{C} that is a member of \mathcal{D} is an effective epimorphism in both \mathcal{C} and \mathcal{S} .

Proof. (i). Let $f : X \rightarrow Y$ be a local homeomorphism in \mathcal{C} and let (R, d_0, d_1) be a kernel pair of $f : X \rightarrow Y$ in \mathcal{C} . By lemma 2.2.14(b), (R, d_0, d_1) is a tractable equivalence relation on X , so it has a \mathbf{J} -covering exact quotient $q : X \rightarrow \bar{X}$ in \mathcal{C} . Let $m : \bar{X} \rightarrow Y$ be the unique morphism in \mathcal{C} such that $m \circ q = f$. Then, by proposition 1.4.23, the following is an exact fork in \mathcal{S} :

$$R \begin{array}{c} \xrightarrow{d_0} \\ \xrightarrow{d_1} \end{array} X \xrightarrow{q} \bar{X}$$

Since \mathcal{S} is a regular category, it follows that $m : \bar{X} \rightarrow Y$ is a monomorphism in \mathcal{S} ; hence, $m : \bar{X} \rightarrow Y$ is an open embedding in \mathcal{C} . Thus, we have the required factorisation of $f : X \rightarrow Y$.

(ii). With notation as above, suppose $f : X \rightarrow Y$ is also an effective epimorphism in \mathcal{C} . Then $m : \bar{X} \rightarrow Y$ is an isomorphism in \mathcal{C} , so $f : X \rightarrow Y$ is also a \mathbf{J} -covering morphism in \mathcal{C} .

(iii). Every \mathbf{J} -covering morphism in \mathcal{C} is an effective epimorphism in \mathcal{S} , and local homeomorphisms in \mathcal{C} have kernel pairs that are preserved by the inclusion $\mathcal{C} \hookrightarrow \mathcal{S}$, so the claim follows. ■

REMARK. In view of propositions 1.5.17 and 2.5.6, the coverage associated with an effective κ -ary admissible ecumene is equivalent to the κ -ary superextensive coverage generated by the class of effective epimorphisms that are local homeomorphisms.

2.5.7 THEOREM. *Let $F : (C_0, D_0, J_0) \rightarrow (C_1, D_1, J_1)$ be a κ -ary admissible functor. Assuming (C_1, D_1, J_1) is an effective κ -ary admissible ecumene:*

The universal property of the category of charted objects

- (i) *The inclusion $(C_0, D_0, J_0) \hookrightarrow (\mathbf{Xt}(C_0, D_0), D_0, K_0)$ is a κ -ary admissible functor.*
- (ii) *There exist a κ -ary admissible functor $\bar{F} : (\mathbf{Xt}(C_0, D_0), D_0, K_0) \rightarrow (C_1, D_1, J_1)$ and an isomorphism $\eta : F \Rightarrow \bar{F}$ of functors $C_0 \rightarrow C_1$.*
- (iii) *Moreover, any such (\bar{F}, η) is a pointwise left Kan extension of $F : C_0 \rightarrow C_1$ along the inclusion $C_0 \hookrightarrow \mathbf{Xt}(C_0, D_0)$.*

Proof. (i). Apply lemma 2.1.10.

(ii) and (iii). By proposition 2.5.4, the inclusion $C_1 \hookrightarrow \mathbf{Xt}(C_1, D_1)$ is fully faithful and essentially surjective on objects, so the claims reduce to proposition 2.5.5. ■

REMARK. The above theorem does most of the hard work of showing that the 2-submetacategory of effective κ -ary admissible ecumenaes is bireflective in the 2-metacategory of κ -ary admissible ecumenaes.

Specificities

3.1 Compactness

SYNOPSIS. We examine three classes of continuous maps of topological spaces that arise by relativising the notion of compactness.

PREREQUISITES. §§ 1.1, 1.2, A.2.

3.1.1 DEFINITION. A **universal topological quotient** is a continuous map $f : X \twoheadrightarrow Y$ with the following property: For every pullback square in **TOP** of the form below,

$$\begin{array}{ccc} X' & \longrightarrow & X \\ f' \downarrow & & \downarrow f \\ Y' & \longrightarrow & Y \end{array}$$

the map $f' : X' \rightarrow Y'$ is a topological quotient.

REMARK. Since the effective epimorphisms in **TOP** are precisely the topological quotients, by [proposition 1.4.1](#), the universally strict epimorphisms in **TOP** are precisely the universal topological quotients.

EXAMPLE. Every surjective open map of topological spaces is a universal topological quotient. Indeed, every surjective open map is a topological quotient, and the class of surjective open maps is a quadrable class of morphisms in **TOP**.

3.1.2 ※ Throughout this section:

- C is a full subcategory of **TOP**.
- C is closed under finitary disjoint union.

- C is closed under pullback.
- For each object X in C :
 - $J_f(X)$ is the set of all finite and jointly surjective sinks on X .
 - $J_{fq}(X)$ is the set of all finite sinks Φ on X such that the induced map $\coprod_{(U,x) \in \Phi} U \rightarrow X$ is a universal topological quotient.

3.1.3 ¶ The following terminology is non-standard.

DEFINITION. A continuous map $f : X \rightarrow Y$ is **semiproper** if it has the following property:

- For every pullback square in **TOP** of the form below,

$$\begin{array}{ccc} X' & \longrightarrow & X \\ f' \downarrow & & \downarrow f \\ Y' & \longrightarrow & Y \end{array}$$

if Y' is compact, then X' is also compact.

REMARK. Thus, from the relative point of view, a semiproper map of topological spaces is a continuous family of compact spaces.

EXAMPLE. Every continuous map from a compact space to a Hausdorff space is semiproper. Indeed, given a pullback square in **TOP** as in the definition, if X is compact and Y is Hausdorff, then the comparison map $X' \rightarrow Y' \times X$ is a closed embedding, so X' is compact when Y' is.

Properties of
semiproper maps

PROPOSITION.

- (i) Every closed embedding of topological spaces is semiproper.
- (ii) For every topological space X , the codiagonal $\nabla_X : X \amalg X \rightarrow X$ is semiproper.
- (iii) The class of semiproper maps of topological spaces is a quadrable class of morphisms in **TOP**.
- (iv) The class of semiproper maps of topological spaces is closed under composition.
- (v) The class of semiproper maps of topological spaces is closed under (possibly infinitary) coproduct in **TOP**.

(vi) Given a surjective continuous map $f : X \twoheadrightarrow Y$ and a continuous map $g : Y \rightarrow Z$, if $g \circ f : X \rightarrow Z$ is semiproper, then $g : Y \rightarrow Z$ is also semiproper.

Proof. Straightforward. ◆

COROLLARY. Let \mathcal{F}_{sp} be the class of semiproper maps in \mathcal{C} . Then every morphism in \mathcal{C} that is of \mathcal{F}_{sp} -type J_f -semilocally on the domain is semiproper.

Proof. Apply [proposition 3.1.3](#). ■

3.1.4 ¶ We will see that the following is a specialisation of the notion of semiproper map.

DEFINITION. A continuous map $f : X \rightarrow Y$ is **proper** if it has the following property:

- For every pullback square in **TOP** of the form below,

$$\begin{array}{ccc} X' & \longrightarrow & X \\ f' \downarrow & & \downarrow f \\ Y' & \longrightarrow & Y \end{array}$$

the map $f' : X' \rightarrow Y'$ is closed, i.e. the image of every closed subspace of X' is a closed subspace of Y' .

EXAMPLE. If X is a compact topological space, then the unique map $X \rightarrow 1$ is proper: this is the precisely the statement of the tube lemma.

Properties of proper maps

PROPOSITION.

- (i) An injective continuous map is proper if and only if it is a closed embedding.
- (ii) For every topological space X , the codiagonal $\nabla_X : X \amalg X \rightarrow X$ is proper.
- (iii) The class of proper maps of topological spaces is a quadrable class of morphisms in **TOP**.

- (iv) *The class of proper maps of topological spaces is closed under composition.*
- (v) *The class of proper maps of topological spaces is closed under (possibly infinitary) coproduct in **TOP**.*
- (vi) *Given a surjective continuous map $f : X \twoheadrightarrow Y$ and a continuous map $g : Y \rightarrow Z$, if $g \circ f : X \rightarrow Z$ is proper, then $g : Y \rightarrow Z$ is also proper.*
- (vii) *Given a pullback square in **TOP** of the form below,*

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & X \\ \tilde{f} \downarrow & & \downarrow f \\ \tilde{Y} & \longrightarrow & Y \end{array}$$

where $\tilde{Y} \twoheadrightarrow Y$ is a universal topological quotient, if $\tilde{f} : \tilde{X} \rightarrow \tilde{Y}$ is proper, then $f : X \rightarrow Y$ is also proper.

Proof. Straightforward. ◆

COROLLARY. *Let \mathcal{F}_p be the class of proper maps in \mathcal{C} . Then every morphism in \mathcal{C} that is J_{fq} -semilocally of \mathcal{F}_p -type is proper.*

Proof. Apply [proposition 3.1.3](#). ■

REMARK. In the language of § 2.2, what we have shown is that $(\mathcal{C}, \mathcal{F}_p, J_{fq})$ is a finitary (i.e. \aleph_0 -ary) extensive regulated ecumene that satisfies the descent axiom and in which every eunoic morphism is genial.

3.1.5 ¶ Properness is closely related to compactness. For instance, suppose X is a topological space such that the unique map $X \rightarrow 1$ is proper. Let $S = \left\{ 1 - \frac{1}{n+1} \mid n \in \mathbb{N} \right\} \cup \{1\} \subseteq \mathbb{R}$ and let $(x_n \mid n \in \mathbb{N})$ be a sequence of points of X . Consider $T = \left\{ \left(1 - \frac{1}{n+1}, x_n \right) \mid n \in \mathbb{N} \right\} \subseteq S \times X$. The closure of T is $\bar{T} = T \cup \{1\} \times A$, where A is the set of accumulation points of $(x_n \mid n \in \mathbb{N})$. Since \bar{T} is a closed subspace of $S \times X$, its image is a closed subspace of S . In particular, 1 is in the image of \bar{T} , i.e. A contains a point. Thus, every sequence in X contains a convergent subsequence,

i.e. X is sequentially compact. A similar argument using nets instead of sequences can be used to show that X is compact.

Much more generally, we have the following result.

Recognition principle for proper maps

THEOREM. *Let $f : X \rightarrow Y$ be a continuous map. The following are equivalent:*

- (i) *The map $f : X \rightarrow Y$ is proper.*
- (ii) *For every topological space T , the map $\text{id}_T \times f : T \times X \rightarrow T \times Y$ is closed.*
- (iii) *The map $f : X \rightarrow Y$ is closed and, for every $y \in Y$, $f^{-1}\{y\}$ is compact.*
- (iv) *The map $f : X \rightarrow Y$ is closed and, for every subspace $Y' \subseteq Y$, if Y' is compact, then $f^{-1}Y'$ is also compact.*
- (v) *The map $f : X \rightarrow Y$ is closed and semiproper.*

Proof. (i) \Rightarrow (ii). Immediate.

(ii) \Rightarrow (iii), (iii) \Rightarrow (i). See [tag 005R](#) in [Stacks].

(i) \Rightarrow (v). Consider a pullback square in **TOP** of the form below:

$$\begin{array}{ccc} X' & \longrightarrow & X \\ f' \downarrow & & \downarrow f \\ Y' & \longrightarrow & Y \end{array}$$

Suppose $f : X \rightarrow Y$ is proper and Y' is compact. We must show that X' is compact. Then, by [proposition 3.1.4](#), $f' : X' \rightarrow Y'$ is also proper. Since the unique map $Y' \rightarrow 1$ is proper (by the tube lemma), it follows that $X' \rightarrow 1$ is also proper. But we know (i) \Rightarrow (iii), so X' is indeed compact.

(v) \Rightarrow (iv), (iv) \Rightarrow (iii). Immediate. □

EXAMPLE. If X is a compact topological space and Y is a Hausdorff space, then every continuous map $X \rightarrow Y$ is proper: in view of [proposition 3.1.4](#) and [theorem 3.1.5](#), this is a special case of [lemma 1.1.9](#).

3.1.6 LEMMA. *Let $f : X \rightarrow Y$ be a continuous map. Assuming Y is a compactly generated Hausdorff space, the following are equivalent:*

When semiproper implies proper

- (i) *The map $f : X \rightarrow Y$ is proper.*
- (ii) *The map $f : X \rightarrow Y$ is semiproper.*
- (iii) *For every subspace $Y' \subseteq Y$, if Y' is compact, then $f^{-1}Y'$ is also compact.*

Proof. (i) \Rightarrow (ii). See [theorem 3.1.5](#).

(ii) \Rightarrow (iii). Immediate.

(iii) \Rightarrow (i). In view of the theorem, it is enough to check that $f : X \rightarrow Y$ is a closed map.

Let X' be a closed subspace of X and let Y' be its image in Y . We wish to show that Y' is a closed subspace of Y . Since Y is compactly generated, it is enough to show that $Y' \cap V$ is a closed subspace of V for all compact subspaces $V \subseteq Y$.

Let V be a compact subspace of Y . Then $f^{-1}V$ is a compact subspace of X . Since $X' \cap f^{-1}V$ is a closed subspace of $f^{-1}V$, it is compact. The image of $X' \cap f^{-1}V$ in V is $Y' \cap V$, and since V is Hausdorff, it follows that $Y' \cap V$ is indeed a closed subspace of V . ■

3.1.7 DEFINITION. A continuous map $f : X \rightarrow Y$ is **perfect** if it has the following properties:

- $f : X \rightarrow Y$ is proper.
- $f : X \rightarrow Y$ is separated, i.e. the relative diagonal $\Delta_f : X \rightarrow X \times_Y X$ is a closed embedding.

EXAMPLE. For a topological space X , the unique map $X \rightarrow 1$ is perfect if and only if X is a compact Hausdorff space.

Properties of perfect maps **PROPOSITION.**

- (i) *An injective continuous map is perfect if and only if it is a closed embedding.*

- (ii) For every topological space X , the codiagonal $\nabla_X : X \amalg X \rightarrow X$ is perfect.
- (iii) The class of perfect maps of topological spaces is a quadrable class of morphisms in **TOP**.
- (iv) The class of perfect maps of topological spaces is closed under composition.
- (v) The class of perfect maps of topological spaces is closed under (possibly infinitary) coproduct in **TOP**.
- (vi) Given continuous maps $f : X \rightarrow Y$ and $g : Y \rightarrow Z$, if both $g : Y \rightarrow Z$ and $g \circ f : X \rightarrow Z$ are perfect, then $f : X \rightarrow Y$ is also perfect.
- (vii) Given a surjective continuous map $f : X \twoheadrightarrow Y$ and a continuous map $g : Y \rightarrow Z$, if $f : X \twoheadrightarrow Y$ is proper and $g \circ f : X \rightarrow Z$ is perfect, then $g : Y \rightarrow Z$ is also perfect.
- (viii) Given a pullback square in **TOP** of the form below,

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & X \\ \tilde{f} \downarrow & & \downarrow f \\ \tilde{Y} & \longrightarrow & Y \end{array}$$

where $\tilde{Y} \twoheadrightarrow Y$ is a universal topological quotient, if $\tilde{f} : \tilde{X} \rightarrow \tilde{Y}$ is perfect, then $f : X \rightarrow Y$ is also perfect.

Proof. (i)–(vi). Straightforward. (Recall propositions 1.1.11 and 3.1.4.)

(vii). Under the hypotheses, $g : Y \rightarrow Z$ is proper. It remains to be shown that $g : Y \rightarrow Z$ is separated.

Consider the following commutative square in **TOP**:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \Delta_{g \circ f} \downarrow & & \downarrow \Delta_g \\ X \times_Z X & \xrightarrow{f \times_Z f} & Y \times_Z Y \end{array}$$

Since $f : X \twoheadrightarrow Y$ is proper, so too is $f \times_Z f : X \times_Z X \rightarrow Y \times_Z Y$. On the other hand, since $g \circ f : X \rightarrow Z$ is separated, the relative diagonal

$\Delta_{g \circ f} : X \rightarrow X \times_Z X$ is a closed embedding. Hence, $\Delta_g : Y \rightarrow Y \times_Z Y$ is indeed a closed embedding.

(viii). Under the hypotheses, the following is a pullback square in **TOP**,

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & X \\ \Delta_{\tilde{f}} \downarrow & & \downarrow \Delta_f \\ \tilde{X} \times_{\tilde{Y}} \tilde{X} & \longrightarrow & X \times_Y X \end{array}$$

and the claim follows. ■

COROLLARY. *Let \mathcal{D}_p be the class of perfect maps in \mathcal{C} . Then every morphism in \mathcal{C} that is (\mathcal{D}_p, J_{fq}) -semilocally of \mathcal{D}_p -type is perfect.*

REMARK. In the language of § 2.2, what we have shown is that $(\mathcal{C}, \mathcal{D}_p, J_{fq})$ is an étale finitary (i.e. \aleph_0 -ary) extensive regulated ecumene that satisfies the descent axiom.

3.2 Discrete fibrations

SYNOPSIS. We construct a combinatorial example of a gros pretopos based on discrete fibrations of simplicial sets.

PREREQUISITES. §§ 1.1, 2.2, 2.3, A.1.

3.2.1 ¶ Roughly speaking, a discrete fibration of simplicial sets is a morphism with a unique path lifting property, similar to covering maps in algebraic topology. However, because the edges of a simplicial set are oriented, it is perhaps better to think of discrete fibrations of simplicial sets (in the sense below) as generalisations of discrete fibrations of categories. We will see a more precise statement later.

DEFINITION. A **discrete fibration of simplicial sets** is a morphism $f : X \rightarrow Y$ in **sSET** with the following property:

- For every natural number n , the following is a pullback square in **SET**:

$$\begin{array}{ccc} X_{n+1} & \xrightarrow{f_{n+1}} & Y_{n+1} \\ d_0 \downarrow & & \downarrow d_0 \\ X_n & \xrightarrow{f_n} & Y_n \end{array}$$

Recognition principle for discrete fibrations of simplicial sets

LEMMA. *Let $f : X \rightarrow Y$ be a morphism of simplicial sets. The following are equivalent:*

- $f : X \rightarrow Y$ is a discrete fibration of simplicial sets.
- $f : X \rightarrow Y$ is right orthogonal to $\delta^0 : \Delta^n \rightarrow \Delta^{n+1}$ for every natural number n .

Proof. Straightforward. ◆

Properties of discrete fibrations of simplicial sets

PROPOSITION.

- Every isomorphism of simplicial sets is a discrete fibration of simplicial sets.
- The class of discrete fibrations of simplicial sets is a quadrable class of morphisms in **sSET**.

- (iii) *The class of discrete fibrations of simplicial sets is closed under composition.*
- (iv) *Given morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ in \mathbf{sSET} , if both $g : Y \rightarrow Z$ and $g \circ f : X \rightarrow Z$ are discrete fibrations of simplicial sets, then $f : X \rightarrow Y$ is also a discrete fibration of simplicial sets.*
- (v) *For every simplicial set X and every set I , the codiagonal $\nabla : \coprod_{i \in I} X \rightarrow X$ is a discrete fibration of simplicial sets.*
- (vi) *The class of discrete fibrations of simplicial sets is closed under (possibly infinitary) coproduct in \mathbf{sSET} .*
- (vii) *Given a pullback square in \mathbf{sSET} of the form below,*

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{p} \twoheadrightarrow & X \\ \tilde{f} \downarrow & & \downarrow f \\ \tilde{Y} & \xrightarrow{q} \twoheadrightarrow & Y \end{array}$$

where $q : \tilde{Y} \twoheadrightarrow Y$ is degreewise surjective, if $\tilde{f} : \tilde{X} \rightarrow \tilde{Y}$ is a discrete fibration of simplicial sets, then $f : X \rightarrow Y$ is also a discrete fibration of simplicial sets.

- (viii) *Given a degreewise surjective morphism $f : X \twoheadrightarrow Y$ in \mathbf{sSET} and a morphism $g : Y \rightarrow Z$ in \mathbf{sSET} , if both $f : X \twoheadrightarrow Y$ and $g \circ f : X \rightarrow Z$ are discrete fibrations of simplicial sets, then $g : Y \rightarrow Z$ is also a discrete fibration of simplicial sets.*

Proof. (i)–(iv). In view of [lemma 3.2.1](#), this is a special case of [proposition 1.1.14](#).

(v) and (vi). Straightforward.

(vii). In the given situation, we have the following commutative diagram in \mathbf{sSET} ,

$$\begin{array}{ccccc} \tilde{X}_{n+1} & \xrightarrow{d_0} & \tilde{X}_n & \xrightarrow{p_n} \twoheadrightarrow & X_n \\ \tilde{f}_{n+1} \downarrow & & \tilde{f}_n \downarrow & & \downarrow f_n \\ \tilde{Y}_{n+1} & \xrightarrow{d_0} & \tilde{Y}_n & \xrightarrow{q_n} \twoheadrightarrow & Y_n \end{array}$$

in which both squares are pullback squares in \mathbf{sSET} . Thus, by the pullback pasting lemma, the outer rectangle in the commutative diagram in \mathbf{sSET}

shown below is a pullback diagram in \mathbf{sSET} :

$$\begin{array}{ccccc} \tilde{X}_{n+1} & \xrightarrow{p_{n+1}} & X_{n+1} & \xrightarrow{d_0} & X_n \\ \tilde{f}_{n+1} \downarrow & & f_{n+1} \downarrow & & \downarrow f_n \\ \tilde{Y}_{n+1} & \xrightarrow{q_{n+1}} & Y_{n+1} & \xrightarrow{d_0} & Y_n \end{array}$$

But the left square is also a pullback square in \mathbf{sSET} , so we may apply [lemma I.4.19\(c\)](#) to deduce that the right square is indeed a pullback square in \mathbf{sSET} .

(viii). In the given situation, we have the following commutative diagram in \mathbf{sSET} ,

$$\begin{array}{ccccc} X_{n+1} & \xrightarrow{f_{n+1}} & Y_{n+1} & \xrightarrow{g_{n+1}} & Z_{n+1} \\ d_0 \downarrow & & d_0 \downarrow & & \downarrow d_0 \\ X_n & \xrightarrow{f_n} & Y_n & \xrightarrow{g_n} & Z_n \end{array}$$

where the outer rectangle and the left square are both pullback diagrams in \mathbf{sSET} . Since $f_n : X_n \rightarrow Y_n$ is surjective, it follows that the right square is a pullback square in \mathbf{sSET} . ■

3.2.2 ¶ Let X be a simplicial set. Suppose we have the following data:

- For each $x \in X_0$, a set $A(x)$.
- For each $e \in X_1$, a map $e^* : A(d_0(e)) \rightarrow A(d_1(e))$.
- For every $x \in X_0$, $s_0(x)^* = \text{id}_{A(x)}$.
- For every positive integer n and every $\sigma \in X_{n+1}$:

$$\begin{aligned} d_0(\cdots (d_{n-2}(d_{n+1}(\sigma))) \cdots)^* \circ d_0(\cdots (d_{n-2}(d_{n-1}(\sigma))) \cdots)^* \\ = d_0(\cdots (d_{n-2}(d_n(\sigma))) \cdots)^* \end{aligned}$$

(So, for instance, for every $\sigma \in X_2$, $d_1(\sigma)^* = d_2(\sigma)^* \circ d_0(\sigma)^*$.)

We may then construct a simplicial set A as follows:

- The elements of A_0 are pairs (x, a) where $x \in X_0$ and $a \in A(x)$.
- For each positive integer n , the elements of A_n are pairs (σ, a) where $\sigma \in X_n$ and $a \in A(d_0(\cdots (d_{n-1}(\sigma))))$.

- The face operators $d_0, d_1 : A_1 \rightarrow A_0$ and the degeneracy operator $s_0 : A_0 \rightarrow A_1$ are defined as follows:

$$d_0(e, a) = (d_0(e), a)$$

$$d_1(e, a) = (d_1(e), e^*(a))$$

$$s_0(x, a) = (s_0(x), a)$$

- For every positive integer n , the face operators $d_0, \dots, d_{n+1} : A_{n+1} \rightarrow A_n$ and the degeneracy operators $s_0, \dots, s_n : A_n \rightarrow A_{n+1}$ are defined as follows:

$$d_i(\sigma, a) = (d_i(\sigma), a) \quad \text{for } 0 \leq i \leq n$$

$$d_{n+1}(\sigma, a) = (d_{n+1}(\sigma), d_0(\dots (d_{n-1}(\sigma)))^*(a))$$

$$s_i(\sigma, a) = (s_i(\sigma), a) \quad \text{for } 0 \leq i \leq n$$

(It is straightforward to verify the simplicial identities.) There is an evident projection $p : A \rightarrow X$, and by construction, it is a discrete fibration of simplicial sets. In fact, every discrete fibration with codomain X arises in this fashion up to isomorphism.

On the other hand, consider the category \mathcal{A} defined inductively as follows:

- For each $x \in X_0$, x is an object in \mathcal{A} .
- For each $e \in X_1$, $e : d_1(x) \rightarrow d_0(x)$ is a morphism in \mathcal{A} .
- For each $x \in X_0$, $s_0(x) = \text{id}_x$ as morphisms in \mathcal{A} .
- For each $\sigma \in X_2$, we have $d_0(\sigma) \circ d_2(\sigma) = d_1(\sigma)$ as morphisms in \mathcal{A} .

It is clear from the description above that A defines a presheaf on \mathcal{A} . Thus:

THEOREM. *There is a functor $\mathbf{PSH}(\mathcal{A}) \rightarrow \mathbf{sSET}_{/X}$ that is fully faithful and essentially surjective onto the full subcategory spanned by the discrete fibrations of simplicial sets with codomain X .*

Proof. Straightforward, given the discussion above. ◆

3.2.3 ✖ For the remainder of this section:

- κ is an uncountable regular cardinal.
- \mathcal{S} is the category of simplicial sets X such that each X_n is hereditarily κ -small.
- \mathcal{D} is the class of discrete fibrations of simplicial sets in \mathcal{A} .

3.2.4 ¶ We will now use the above results to construct a combinatorial example of a gros pretopos.

PROPOSITION.

- (i) \mathcal{S} is a κ -ary pretopos.
- (ii) \mathcal{D} is a class of étale morphisms in \mathcal{S} .

In particular, $(\mathcal{S}, \mathcal{D})$ is a gros κ -ary pretopos.^[1]

Proof. (i). Straightforward. (The inclusion $\mathcal{S} \hookrightarrow \mathbf{sSET}$ creates limits of finite diagrams and colimits of κ -small diagrams.)

(ii). See [proposition 3.2.1](#). ■

3.2.5 REMARK. $(\mathcal{S}, \mathcal{D})$ is an example of a gros κ -ary pretopos in which the class of étale morphisms strictly contains the class of local homeomorphisms. Indeed, given [lemma 2.3.12 \(a\)](#) and [theorem 3.2.2](#), this is essentially just the observation that $\mathbf{PSH}(\mathcal{A})$ is not always a localic topos.

3.2.6 ¶ Unlike the gros pretoposes we will see later, there is no obvious candidate for a unary basis for $(\mathcal{S}, \mathcal{D})$. Nonetheless, for the sake of illustration, we may consider the following.

DEFINITION. The **strict Segal condition** on a simplicial set X is the following:

- For every positive integer n , the following is a pullback square in \mathbf{SET} :

$$\begin{array}{ccc} X_{n+1} & \xrightarrow{d_{n+1}} & X_n \\ d_0 \circ \dots \circ d_{n-1} \downarrow & & \downarrow d_0 \circ \dots \circ d_{n-1} \\ X_1 & \xrightarrow{d_1} & X_0 \end{array}$$

[1] Recall [lemma 2.3.3](#).

EXAMPLE. Let \mathcal{A} be a category. The **nerve** of \mathcal{A} is the simplicial set $N(\mathcal{A})$ defined as follows:

- $N(\mathcal{A})_0$ is the set of objects in \mathcal{A} .
- $N(\mathcal{A})_1$ is the set of morphisms in \mathcal{A} .
- For $n \geq 2$, $N(\mathcal{A})_n$ is the set of n -tuples (f_n, \dots, f_1) of morphisms in \mathcal{A} such that $f_n \circ \dots \circ f_1$ is defined in \mathcal{A} .
- The face operators $d_0, d_1 : N(\mathcal{A})_1 \rightarrow N(\mathcal{A})_0$ and the degeneracy operator $s_0 : N(\mathcal{A})_0 \rightarrow N(\mathcal{A})_1$ are defined as follows:

$$d_0(f) = \text{codom } f$$

$$d_1(f) = \text{dom } f$$

$$s_0(x) = \text{id}_x$$

- The face operators $d_0, d_1, d_2 : N(\mathcal{A})_2 \rightarrow N(\mathcal{A})_1$ and the degeneracy operators $s_0, s_1 : N(\mathcal{A})_1 \rightarrow N(\mathcal{A})_2$ are defined as follows:

$$d_0(f_1, f_0) = f_1$$

$$d_1(f_1, f_0) = f_1 \circ f_0$$

$$d_2(f_1, f_0) = f_0$$

$$s_0(f) = (f, \text{id}_{\text{dom } f})$$

$$s_1(f) = (\text{id}_{\text{codom } f}, f)$$

- For $n > 2$, the face operators $d_0, \dots, d_n : N(\mathcal{A})_n \rightarrow N(\mathcal{A})_{n-1}$ and degeneracy operators $s_0, \dots, s_{n-1} : N(\mathcal{A})_{n-1} \rightarrow N(\mathcal{A})_n$ are defined analogously.

Then $N(\mathcal{A})$ satisfies the strict Segal condition. In fact, a simplicial set satisfies the strict Segal condition if and only if it is isomorphic to $N(\mathcal{A})$ for some category \mathcal{A} .

LEMMA. Let $p : X \rightarrow Y$ be a discrete fibration of simplicial sets.

- (i) If Y satisfies the strict Segal condition, then X also satisfies the strict Segal condition.
- (ii) If $p : X \rightarrow Y$ is degreewise surjective and X satisfies the strict Segal condition, then Y also satisfies the strict Segal condition.

Proof. The pullback pasting lemma implies that the following are pullback squares in **SET**:

$$\begin{array}{ccc}
 X_n & \xrightarrow{f_n} & Y_n \\
 d_0 \circ \dots \circ d_{n-1} \downarrow & & \downarrow d_0 \circ \dots \circ d_{n-1} \\
 X_0 & \xrightarrow{f_0} & Y_0
 \end{array}
 \qquad
 \begin{array}{ccc}
 X_{n+1} & \xrightarrow{f_{n+1}} & Y_{n+1} \\
 d_0 \circ \dots \circ d_{n-1} \downarrow & & \downarrow d_0 \circ \dots \circ d_{n-1} \\
 X_1 & \xrightarrow{f_1} & Y_1
 \end{array}$$

Then (i) is another application of the pullback pasting lemma, and (ii) is consequence of lemma 1.4.19(c). ■

3.2.7 ¶ Let \mathcal{C} be the full subcategory of \mathcal{S} spanned by the simplicial sets X with the following properties:

- X satisfies the strict Segal condition.
- The map $\langle d_1, d_0 \rangle : X_1 \rightarrow X_0 \times X_0$ is injective.

PROPOSITION.

- (i) \mathcal{C} is equivalent to the metacategory of κ -small preordered sets.
- (ii) \mathcal{C} is a unary basis for $(\mathcal{S}, \mathcal{D})$.
- (iii) $\mathbf{Xt}(\mathcal{C}, \mathcal{D})$ is equivalent to the metacategory of κ -small categories in which every morphism is a monomorphism.
- (iv) A $(\mathcal{C}, \mathcal{D})$ -extent in \mathcal{S} is \mathcal{D} -localic if and only if it is an object in \mathcal{C} .

Proof. (i). A simplicial set X is isomorphic to an object in \mathcal{C} if and only if X is isomorphic to the nerve of some preordered set (considered as a category). Since $\mathbf{N} : \mathbf{CAT} \rightarrow \mathbf{sSET}$ is fully faithful, the claim follows.

(ii). Since $\kappa > \aleph_0$, every object in \mathcal{S} is a simplicial set with $< \kappa$ elements. Thus, the Yoneda lemma implies that \mathcal{C} is a unary site for \mathcal{S} . In addition, given a discrete fibration $p : X \rightarrow Y$ in \mathcal{S} , if $\langle d_1, d_0 \rangle : Y_1 \rightarrow Y_0 \times Y_0$ is injective, then $\langle d_1, d_0 \rangle : X_1 \rightarrow X_0 \times X_0$ is also injective; so, by lemma 3.2.6, if Y is an object in \mathcal{C} , then X is also an object in \mathcal{C} . Hence, \mathcal{C} is indeed a unary basis for $(\mathcal{S}, \mathcal{D})$.

(iii). It suffices to identify necessary and sufficient conditions for a category \mathcal{A} to admit a surjective discrete fibration $\tilde{\mathcal{A}} \rightarrow \mathcal{A}$ where $\tilde{\mathcal{A}}$ is a preorder category.

Clearly, if every morphism in \mathcal{A} is a monomorphism, then every slice category $\mathcal{A}_{/x}$ is a preorder category, so we may take $\tilde{\mathcal{A}} = \coprod_{x \in \text{ob } \mathcal{A}} \mathcal{A}_{/x}$.

Conversely, suppose we have a surjective discrete fibration $p : \tilde{\mathcal{A}} \rightarrow \mathcal{A}$ where $\tilde{\mathcal{A}}$ is a preorder category. Let $g : y \rightarrow z$ be a morphism in \mathcal{A} and let $f_0, f_1 : x \rightarrow y$ be a parallel pair of morphisms in \mathcal{A} such that $g \circ f_0 = g \circ f_1$. By hypothesis, there is an object \tilde{z} in $\tilde{\mathcal{A}}$ such that $p(\tilde{z}) = z$, and there exist a unique object \tilde{y} and a unique morphism $\tilde{g} : \tilde{y} \rightarrow \tilde{z}$ in $\tilde{\mathcal{A}}$ such that $p(\tilde{y}) = y$ and $p(\tilde{g}) = g$. Similarly, there exist a unique object \tilde{x} and a unique morphism $\tilde{f} : \tilde{x} \rightarrow \tilde{y}$ in $\tilde{\mathcal{A}}$ such that $p(\tilde{x}) = x$ and $p(\tilde{f}) = f_0 = f_1$. This shows $g : y \rightarrow z$ is indeed a monomorphism in \mathcal{A} .

(iv). The claim reduces to the fact that $\mathbf{Psh}(\mathcal{A})$ is localic if and only if \mathcal{A} is a preorder category. ■

3.3 Manifolds

SYNOPSIS. We construct an admissible ecumene for which the charted objects are the smooth manifolds of fixed dimension and cardinality.

PREREQUISITES. §§ 1.1, 1.4, 2.2, 2.3.

3.3.1 ¶ Manifolds are probably the best-known notion of space built from local models. Unfortunately, from a category-theoretic point of view, manifolds are somewhat awkward: for instance, the category of manifolds does not have all pullbacks. On the other hand, this can still be accommodated within the theory of charted objects with respect to an admissible ecumene—after all, an ecumene is not required to have pullbacks. In fact, even finitary products are unnecessary—we will see how to define manifolds of a fixed dimension as charted objects.

3.3.2 ✕ Throughout this section:

- n is a natural number.
- C_0 is the category of *connected* open subspaces of \mathbb{R}^n and smooth maps.
- κ is a regular cardinal such that C_0 is essentially κ -small.
- $C = \mathbf{Fam}_\kappa(C_0)$.
- D_0 is the class of local diffeomorphisms in C_0 .
- D is the class of morphisms in C that are familially of D_0 -type.
- E is the class of morphisms $f : X \rightarrow Y$ in C such that the corresponding continuous map is a surjective local homeomorphism.^[1]
- For each object X in C_0 , $J_0(X)$ is the set of open covers of X .
- For each object X in C , $J(X)$ is the set of κ -small sinks Φ on X such that the induced morphism $\coprod_{(U,x) \in \Phi} U \rightarrow X$ in C is a member of E .

[1] Here, we are using the functor $C \rightarrow \mathbf{TOP}$ induced by the forgetful functor $C_0 \rightarrow \mathbf{TOP}$.

3.3.3 PROPOSITION. (C, \mathcal{D}, J) is a κ -ary admissible ecumene, i.e.:

- (i) C is a κ -ary extensive category.
- (ii) \mathcal{D} is a class of separated fibrations in C .
- (iii) J is a subcanonical \mathcal{D} -adapted κ -ary superextensive coverage on C .
- (iv) Every morphism in C of \mathcal{D} -type (\mathcal{D}, J) -semilocally on the domain is a member of \mathcal{D} .
- (v) Every complemented monomorphism in C is a member of \mathcal{D} .
- (vi) (C, E) satisfies the Shulman condition.^[2]

Proof. (i). This is a special case of [proposition 1.5.7](#).

(ii)–(v). Straightforward.

(vi). By extensivity, it suffices to verify the following:

- Given objects X_0, \dots, X_{k-1} in C_0 , $\mathbf{El}\left(h_{X_0} \times \dots \times h_{X_{k-1}}\right)$ is an essentially κ -small category.

But this is an immediate consequence of the assumption that C_0 itself is an essentially κ -small category. ■

3.3.4 ¶ Let (S, \mathcal{D}) be the gros κ -ary pretopos associated with (C, \mathcal{D}, J) . By abuse of notation, we will consider C_0 to be a full subcategory of C and C to be a full subcategory of S . It should come as no surprise that (C, \mathcal{D}) -extents are n -dimensional manifolds, but we should be more precise about what that means.

3.3.4(a) DEFINITION. A **smooth n -dimensional atlas** of a topological space X is a set Φ with the following properties:

- Every element of Φ is a pair (U, x) where U is a connected open subspace of X and $x : U \rightarrow \mathbb{R}^n$ is an open embedding of topological spaces.
- $X = \bigcup_{(U,x) \in \Phi} U$.

[2] Note that a morphism in C is J -covering if and only if it is E -covering.

- For every $((U_0, x_0), (U_1, x_1)) \in \Phi \times \Phi$, $x_1 \circ (x_0)^{-1} : U'_0 \rightarrow U'_1$ is a diffeomorphism, where U'_0 is the image of $x_0 : U_0 \cap U_1 \rightarrow \mathbb{R}^n$ and U'_1 is the image of $x_1 : U_0 \cap U_1 \rightarrow \mathbb{R}^n$.

3.3.4(b) DEFINITION. A **n -dimensional manifold** is a pair (X, Φ) where X is a topological space and Φ is a smooth n -dimensional atlas of X .

REMARK. In particular, we do *not* require manifolds to be Hausdorff spaces, nor do we require manifolds to be second-countable.

3.3.5 ¶ Though it is more usual to first define ‘smooth map’, we will instead define the presheaf represented by a manifold directly and take for granted that the Yoneda representation is fully faithful.

DEFINITION. The **presheaf represented by an n -dimensional manifold** (X, Φ) is the presheaf $h_{(X, \Phi)}$ on \mathcal{C}_0 defined as follows:

- For each connected open subspace $T \subseteq \mathbb{R}^n$, $h_{(X, \Phi)}(T)$ is the set of all continuous maps $f : T \rightarrow X$ with the following property:
 - For every $(U, x) \in \Phi$, the map $x \circ f : T \cap f^{-1}U \rightarrow \mathbb{R}^n$ is smooth.
- The action of \mathcal{C}_0 is composition (of continuous maps).

LEMMA. Let (X, Φ) be an n -dimensional manifold. Then $h_{(X, \Phi)}$ is a J_0 -sheaf on \mathcal{C}_0 .

Proof. Straightforward. (This is essentially the fact that smoothness of maps is a local property.) ◆

3.3.6 THEOREM. The essential image of the Yoneda representation $\mathbf{Xt}(\mathcal{C}, \mathcal{D}) \rightarrow \mathbf{SH}(\mathcal{C}_0, J_0)$ is the full and replete subcategory spanned by the J_0 -sheaves represented by n -dimensional manifolds (X, Φ) where Φ is κ -small.

Proof. For ease of notation, we will identify \mathcal{S} with its essential image in $\mathbf{SH}(\mathcal{C}_0, J_0)$.

First, let (X, Φ) be an n -dimensional manifold where Φ is κ -small. We will show that $h_{(X, \Phi)}$ is in $\mathbf{Xt}(\mathcal{C}, \mathcal{D})$. It is not hard to see that Φ is J_0 -local generating set of elements of $h_{(X, \Phi)}$. In other words, we have a J_0 -locally surjective morphism $p : A \rightarrow h_{(X, \Phi)}$ where $A = \coprod_{(U, x) \in \Phi} h_{xU}$

in $\mathbf{SH}(C_0, J_0)$, where xU is the image of $x : U \rightarrow \mathbb{R}^n$. Since S is closed under κ -ary coproduct, A is an object in S . Let (R, d_0, d_1) be a kernel pair of $p : A \twoheadrightarrow h_{(X, \Phi)}$ in $\mathbf{SH}(C_0, J_0)$. Then R is a κ -ary disjoint union of representable J_0 -sheaves on C_0 , so (R, d_0, d_1) is an equivalence relation on A in S . Moreover, by lemma A.3.10, $(h_{(X, \Phi)})$ is an object in S and $p : A \twoheadrightarrow h_{(X, \Phi)}$ is an exact quotient of (R, d_0, d_1) in S . It is straightforward to verify that (R, d_0, d_1) is a tractable equivalence relation in S . Hence, by lemma 2.2.14(c), $p : A \twoheadrightarrow h_{(X, \Phi)}$ is a local homeomorphism in S , and therefore $h_{(X, \Phi)}$ is indeed a (C, D) -extent in S .

Now, let B be a (C, D) -extent in S . By definition, there is a (C, D) -atlas of B in S , say (A, p) , and by proposition 2.3.13, we may assume that $p : A \twoheadrightarrow B$ is a laminar morphism. (Note that, for every laminar morphism $h : C' \rightarrow C$ in S , if C is isomorphic to an object in C , then C' is also isomorphic to an object in C .) Suppose $A = \coprod_{i \in I} T_i$ for some family $(T_i \mid i \in I)$ of objects in C_0 where I is a κ -small set, such that each composite $T_i \rightarrow A \twoheadrightarrow B$ is an open embedding in S . Let $R = \coprod_{i_0 \in I} \coprod_{i_1 \in I} T_0 \times_B T_1$. Then (R, d_0, d_1) is a kernel pair of $p : A \twoheadrightarrow B$, where $d_0, d_1 : R \rightarrow A$ are the two evident projections. Let X be the quotient of the corresponding equivalence relation in \mathbf{TOP} . Since the projections $d_0, d_1 : R \rightarrow A$ and the relative diagonal $\Delta_p : A \rightarrow R$ correspond to open maps of topological spaces, the quotient map is a local homeomorphism of topological spaces. In particular, we obtain open embeddings $T_i \rightarrow X$, and it is straightforward to verify that their inverses comprise a smooth n -dimensional atlas Φ of X . Moreover, by the argument of the previous paragraph, we obtain $h_{(X, \Phi)} \cong B$, as desired. ■

3.3.7 ¶ Finally, we should remark that the material covered in this section does not depend very strongly on the meaning of ‘smooth’. Indeed, everything still works if we replace ‘smooth’ with ‘ m -times continuously differentiable’—then we get the category of n -dimensional manifolds and m -times continuously differentiable maps. We could even replace \mathbb{R} with \mathbb{C} and ‘smooth’ with ‘analytic’ to obtain the category of n -dimensional complex analytic manifolds and analytic maps. The dimension restriction can also

be lifted; then, by [proposition 2.3.11](#), $\mathbf{Xt}(C, \mathcal{D})$ will have finitary products (as is well known).

3.4 Topological spaces

SYNOPSIS. We construct admissible ecumenae from categories of topological spaces and investigate when a topological space is representable by a charted object.

PREREQUISITES. §§ I.1, 2.2, 2.3, 2.5, A.1, A.2, A.3.

3.4.1 ※ Throughout this section:

- κ is a regular cardinal.
- \mathcal{C} is a (small) full subcategory of the metacategory of topological spaces (and continuous maps) that is closed under finitary products, equalisers, and κ -ary disjoint unions.
- \mathcal{D} is the class of local homeomorphisms between objects in \mathcal{C} .
- For every object X in \mathcal{C} and every open subspace $U \subseteq X$, there is κ -small set Φ of open subspaces of U with the following properties:
 - For every $V \in \Phi$, V is homeomorphic to an object in \mathcal{C} .
 - $U = \bigcup_{V \in \Phi} V$.
- For each object X in \mathcal{C} , $J(X)$ is the set of κ -small sinks Φ on X in \mathcal{C} such that Φ is a jointly surjective family of open embeddings.
- \mathcal{E} is the class of surjective local homeomorphisms between objects in \mathcal{C} .

3.4.2 PROPOSITION. $(\mathcal{C}, \mathcal{D}, J)$ is a κ -ary admissible ecumene, i.e.:

- (i) \mathcal{C} is a κ -ary extensive category.
- (ii) \mathcal{D} is a class of separated fibrations in \mathcal{C} .
- (iii) J is a subcanonical \mathcal{D} -adapted κ -ary superextensive coverage on \mathcal{C} .
- (iv) Every morphism in \mathcal{C} of \mathcal{D} -type (\mathcal{D}, J) -semilocally on the domain is a member of \mathcal{D} .
- (v) Every complemented monomorphism in \mathcal{C} is a member of \mathcal{D} .

(vi) (C, E) satisfies the Shulman condition.^[1]

Proof. Straightforward. ◆

REMARK. In particular, every local homeomorphism in C in the sense of [definition 2.2.12](#) is a member of D , so there is no danger of confusion in using the phrase ‘local homeomorphism in C ’.

3.4.3 ✖ For the remainder of this section:

- $S = \mathbf{Ex}(C, E)$.
- \mathcal{G} is the class of morphisms in S corresponding to morphisms in $\mathbf{PSH}(C)$ that are J -locally of D -type.
- K is the κ -ary canonical coverage on S .
- \hat{D} is the class of \mathcal{G} -perfect morphisms in S .

Furthermore, by abuse of notation, we will identify C with the image of the insertion $C \rightarrow S$.

3.4.4 PROPOSITION.

- (i) (S, \hat{D}) is a κ -ary gros pretopos.
- (ii) Moreover, (S, \hat{D}, K) satisfies the descent axiom.
- (iii) A morphism in C is a member of D if and only if it is a member of \hat{D} .

Proof. This is a special case of [proposition 2.3.2](#). ■

3.4.5 ¶ Consider the Yoneda representation $h_\bullet : \mathbf{TOP} \rightarrow \mathbf{PSH}(C)$. Since C has pullbacks and the inclusion $C \hookrightarrow \mathbf{TOP}$ preserves them, [lemma A.2.6](#) implies that, for every topological space X , h_X is a J -sheaf on C . Thus, by [proposition A.1.4](#), for every J -sheaf A on C , there is a topological space $|A|$ and a morphism $\eta_A : A \rightarrow h_{|A|}$ in $\mathbf{SH}(C, J)$ such that the following map is a bijection for every topological space Y :

$$\mathbf{TOP}(|A|, Y) \rightarrow \mathbf{Hom}_{\mathbf{SH}(C, J)}(A, h_Y)$$

[1] Note that a morphism in C is J -covering if and only if it is E -covering.

$$f \mapsto h_f \circ \eta_A$$

Indeed, we may take $|A| = \lim_{\rightarrow (X,a):\mathbf{El}(A)} X$. This yields an adjunction:

$$\mathbf{TOP} \begin{array}{c} \xleftarrow{|-|} \\ \xrightarrow[h_\bullet]{\perp} \end{array} \mathbf{SH}(C, J)$$

It is clear (by construction) that the counit $\varepsilon_X : |h_X| \rightarrow X$ is a homeomorphism for every object X in C . We would like to know if this happens for topological spaces that are not necessarily in C .

3.4.5(a) **LEMMA.** *Let X be a topological space. The following are equivalent:*

- (i) *The counit $\varepsilon_X : |h_X| \rightarrow X$ is a homeomorphism.*
- (ii) *For every topological space Y , the following is a bijection:*

$$h_\bullet : \mathbf{TOP}(X, Y) \rightarrow \mathbf{Hom}_{\mathbf{SH}(C, J)}(h_X, h_Y)$$

Proof. Straightforward. ◆

3.4.5(b) **LEMMA.** *The functor $|-\| : \mathbf{SH}(C, J) \rightarrow \mathbf{TOP}$ preserves monomorphisms.*

Proof. Let $\Gamma : \mathbf{SH}(C, J) \rightarrow \mathbf{SET}$ be the evident functor defined on objects by $A \mapsto A(1)$. It is clear that 1 is a J -local object in C , so by lemma A.3.12, $\Gamma : \mathbf{SH}(C, J) \rightarrow \mathbf{SET}$ preserves colimits. On the other hand, proposition A.1.4 implies that $\Gamma : \mathbf{SH}(C, J) \rightarrow \mathbf{SET}$ is isomorphic to the composite of $|-\| : \mathbf{SH}(C, J) \rightarrow \mathbf{TOP}$ and the forgetful functor $\mathbf{TOP} \rightarrow \mathbf{SET}$. Since the forgetful functor $\mathbf{TOP} \rightarrow \mathbf{SET}$ is faithful, it follows that $|-\| : \mathbf{SH}(C, J) \rightarrow \mathbf{TOP}$ preserves monomorphisms. ■

3.4.5(c) **LEMMA.** *Let $f : X \rightrightarrows Y$ be a surjective local homeomorphism of topological spaces and let (R, d_0, d_1) be the kernel pair of $f : X \rightrightarrows Y$ in \mathbf{TOP} .*

- (i) *The following is an exact fork in $\mathbf{SH}(C, J)$:*

$$h_R \begin{array}{c} \xrightarrow{d_0 \circ -} \\ \xrightarrow[d_1 \circ -]{} \end{array} h_X \xrightarrow{f \circ -} h_Y$$

(ii) If both $\varepsilon_R : |\mathfrak{h}_R| \rightarrow R$ and $\varepsilon_X : |\mathfrak{h}_X| \rightarrow X$ are homeomorphisms, then $\varepsilon_Y : |\mathfrak{h}_Y| \rightarrow Y$ is also a homeomorphism.

Proof. (i). It is not hard to verify that $\mathfrak{h}_f : \mathfrak{h}_X \rightarrow \mathfrak{h}_Y$ is a \mathbf{J} -locally surjective morphism in $\mathbf{PSH}(\mathcal{C})$. Thus, by lemma A.3.10, we have the desired exact fork.

(ii). $|-| : \mathbf{SH}(\mathcal{C}, \mathbf{J}) \rightarrow \mathbf{TOP}$ preserves coequalisers, and $f : X \rightarrow Y$ is an effective epimorphism in \mathbf{TOP} , so it follows that $\varepsilon_Y : |\mathfrak{h}_Y| \rightarrow Y$ is a homeomorphism if both $\varepsilon_R : |\mathfrak{h}_R| \rightarrow R$ and $\varepsilon_X : |\mathfrak{h}_X| \rightarrow |\mathfrak{h}_Y|$ are homeomorphisms. ■

3.4.5(d) LEMMA. Let $(X_i \mid i \in I)$ be a family of topological spaces where I is a κ -small set and let $X = \coprod_{i \in I} X_i$.

(i) \mathfrak{h}_X is a coproduct of $(\mathfrak{h}_{X_i} \mid i \in I)$ in $\mathbf{SH}(\mathcal{C}, \mathbf{J})$ (with the evident coproduct injections).

(ii) If each $\varepsilon_{X_i} : |\mathfrak{h}_{X_i}| \rightarrow X_i$ is a homeomorphism, then $\varepsilon_X : |\mathfrak{h}_X| \rightarrow X$ is also a homeomorphism.

Proof. (i). Using lemma 1.5.4, it is not hard to see that the Yoneda representation $\mathfrak{h}_\bullet : \mathbf{TOP} \rightarrow \mathbf{SH}(\mathcal{C}, \mathbf{J})$ preserves κ -ary coproducts.

(ii). On the other hand, $|-| : \mathbf{SH}(\mathcal{C}, \mathbf{J})$ also preserves (κ -ary) coproducts. The claim follows. ■

3.4.5(e) LEMMA. Let X be an object in \mathcal{C} and let U be an open subspace of X .

(i) $\mathfrak{h}_U \rightarrow \mathfrak{h}_X$ is a monomorphism in $\mathbf{PSH}(\mathcal{C})$ that is \mathbf{J} -semilocally of \mathbf{D} -type.

(ii) $\varepsilon_U : |\mathfrak{h}_U| \rightarrow U$ is a homeomorphism in \mathcal{C} .

Proof. (i). By hypothesis, there is a κ -small set Φ of open subspaces of V such that:

- For each $V \in \Phi$, V is homeomorphic to an object in \mathcal{C} .
- $U = \bigcup_{V \in \Phi} V$.

It is clear that $h_U \rightarrow h_X$ is a monomorphism in $\mathbf{PSH}(\mathcal{C})$, and it follows that $h_U \rightarrow h_X$ is \mathcal{J} -semilocally of \mathcal{D} -type. Let $\bar{V} = \coprod_{V \in \Phi} V$ and let $p : \bar{V} \rightarrow U$ be the evident projection. Clearly, $p : \bar{V} \rightarrow U$ is a surjective local homeomorphism. Let (R, d_0, d_1) be the kernel pair of $p : \bar{V} \rightarrow U$. Then $R \cong \coprod_{V_0 \in \Phi} \coprod_{V_1 \in \Phi} V_0 \cap V_1$, and each $V_0 \cap V_1$ is homeomorphic to an object in \mathcal{C} , so by lemma 3.4.5(d), both $\varepsilon_R : |h_R| \rightarrow R$ and $\varepsilon_{\bar{V}} : |h_{\bar{V}}| \rightarrow \bar{V}$ are homeomorphisms. Hence, by lemma 3.4.5(c), $\varepsilon_U : |h_U| \rightarrow U$ is also a homeomorphism. ■

3.4.6 ¶ In view of the discussion above, we make the following definition.

DEFINITION. A topological space X is **of \mathcal{C} -type** if there is a κ -small set Φ of open subspaces of X with the following properties:

- For every $U \in \Phi$, U is homeomorphic to an object in \mathcal{C} .
- $X = \bigcup_{U \in \Phi} U$.

We write \mathcal{M} for the metacategory of topological spaces of \mathcal{C} -type (and continuous maps).

PROPOSITION.

- (i) \mathcal{M} is closed in \mathbf{TOP} under κ -ary disjoint union.
- (ii) Given an object X in \mathcal{M} , if U is an open subspace of X , then U is also an object in \mathcal{M} .
- (iii) For every object Y in \mathcal{M} , there exist an object X in \mathcal{C} and a surjective local homeomorphism $f : X \rightarrow Y$ such that $X \times_Y X$ is homeomorphic to a κ -ary disjoint union of open subspaces of X and $h_f : h_X \rightarrow h_Y$ is a morphism in $\mathbf{PSH}(\mathcal{C})$ that is \mathcal{J} -semilocally of \mathcal{D} -type.
- (iv) For every object Y in \mathcal{M} , the counit $\varepsilon_Y : |h_Y| \rightarrow Y$ is a homeomorphism.
- (v) The Yoneda representation $\mathcal{M} \rightarrow \mathbf{SH}(\mathcal{C}, \mathcal{J})$ is fully faithful, preserves κ -ary coproducts, and sends surjective local homeomorphisms in \mathcal{M} to effective epimorphisms in $\mathbf{SH}(\mathcal{C}, \mathcal{J})$.

Proof. (i) and (ii). Straightforward.

(iii). Let Y be an object in \mathcal{M} . By definition, there is a κ -small set Ψ of open subspaces of Y with the following properties:

- For every $V \in \Psi$, V is homeomorphic to an object in \mathcal{C} .
- $Y = \bigcup_{V \in \Psi} V$.

Since \mathcal{C} is closed under κ -ary disjoint union, $\coprod_{V \in \Psi} V$ is also homeomorphic to an object in \mathcal{C} , say X . There is an evident surjective local homeomorphism $f : X \twoheadrightarrow Y$, and it is clear that $X \times_Y X$ is homeomorphic to a κ -ary disjoint union of open subspaces of X . Moreover, by [proposition 1.2.13](#) and [lemma 3.4.5\(e\)](#), $\hat{h}_f : \hat{h}_X \rightarrow \hat{h}_Y$ is \mathbf{J} -semilocally of \mathcal{D} -type, as claimed.

(iv). Apply [lemmas 3.4.5\(c\)](#) and [3.4.5\(d\)](#) to (ii) and (iii).

(v). By [lemma 3.4.5\(a\)](#) and (iv), the Yoneda representation $\mathcal{M} \rightarrow \mathbf{SH}(\mathcal{C}, \mathbf{J})$ is fully faithful. We already know that the Yoneda representation $\mathbf{TOP} \rightarrow \mathbf{SH}(\mathcal{C}, \mathbf{J})$ preserves κ -ary coproducts and sends surjective local homeomorphisms in \mathbf{TOP} to effective epimorphisms in $\mathbf{SH}(\mathcal{C}, \mathbf{J})$, so we are done. ■

3.4.7 ¶ By [theorem 2.1.14](#), the Yoneda representation $\mathcal{S} \rightarrow \mathbf{SH}(\mathcal{C}, \mathbf{J})$ is fully faithful and preserves limits of finite diagrams, κ -ary coproducts, and exact quotients. Moreover, by [lemma 2.1.16](#), a \mathbf{J} -sheaf on \mathcal{C} is in the essential image of the Yoneda representation if and only if it is \mathbf{J} -locally κ -presentable.

LEMMA. *If $x : U \rightarrow X$ is an open embedding in \mathcal{S} and X is an object in \mathcal{C} , then $|\hat{h}_x| : |\hat{h}_U| \rightarrow |\hat{h}_X|$ is an open embedding of topological spaces.*

Proof. Since $\hat{h}_x : \hat{h}_U \rightarrow \hat{h}_X$ is a monomorphism in $\mathbf{PSH}(\mathcal{C})$ that is \mathbf{J} -semilocally of \mathcal{D} -type, there is a κ -small set Φ of objects in \mathcal{S}/U with the following properties:

- For every $(V, u) \in \Phi$, V is an object in \mathcal{C} and $x \circ u : V \rightarrow X$ is an open embedding of topological spaces.
- The induced morphism $p : \coprod_{(V, u) \in \Phi} V \rightarrow U$ in \mathcal{S} is an effective epimorphism.

Thus, $|h_p| : |h_{\prod_{(V,u) \in \Phi} V}| \twoheadrightarrow |U|$ is an effective epimorphism in **TOP** and the composite $|h_x| \circ |h_p| : |h_{\prod_{(V,u) \in \Phi} V}| \rightarrow |X|$ is a local homeomorphism of topological spaces. On the other hand, by lemma 3.4.5(b), $|h_x| : |h_U| \rightarrow |h_X|$ is an injective continuous map. Thus, $|h_x| : |h_U| \rightarrow |h_X|$ is indeed an open embedding of topological spaces. ■

THEOREM. *Let \bar{D} be the class of local homeomorphisms in S .*

- (i) *If X is an object in \mathcal{M} , then there is a (C, \bar{D}) -extent A in S such that $h_X \cong h_A$ in $\mathbf{SH}(C, J)$.*
- (ii) *If A is a (C, \bar{D}) -extent in S , then $|h_A|$ is a topological space of C -type.*
- (iii) *The functor $|h_\bullet| : \mathbf{Xt}(C, \bar{D}) \rightarrow \mathcal{M}$ is fully faithful and essentially surjective on objects.*

Proof. (i). First, consider a subspace U of an object X in C . Recalling the proof of lemma 3.4.5(e), we see that there is an open subobject A of X in S such that $h_U \cong h_A$. Thus, by proposition 3.4.6, for every object X in \mathcal{M} , there is an object A in S such that $h_X \cong h_A$. Moreover, by tracing the proof of that proposition, it is straightforward to verify that A is a (C, \bar{D}) -extent in S .

(ii). In view of lemma 3.4.7, a similar argument shows that $|h_A|$ is a topological space of C -type if A is a (C, \bar{D}) -extent in S .

(iii). Hence, by lemma 3.4.5(a) and proposition 3.4.6, $|h_\bullet| : \mathbf{Xt}(C, \bar{D}) \rightarrow \mathcal{M}$ is fully faithful and essentially surjective on objects. ■

3.4.8 ¶ By proposition 2.2.12, we have $\bar{D} \subseteq \hat{D}$, so $\mathbf{Xt}(C, \bar{D}) \subseteq \mathbf{Xt}(C, \hat{D})$. We will now see an explicit example where these inclusions are strict.

EXAMPLE. Let C be the category of topological spaces X such that the set of points of X is hereditarily κ -small. It is straightforward to check that the hypotheses of proposition 2.3.14(b) are satisfied, so each (C, \bar{D}) -extent in S is isomorphic to an object in C .

On the other hand, consider the unit circle in the complex plane:

$$S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$$

Let $R = \mathbb{Z} \times S^1$ and let $d_0, d_1 : R \rightarrow S^1$ be defined as follows:

$$d_0(n, z) = \exp(in)z$$

$$d_1(n, z) = z$$

Then $d_0, d_1 : R \rightarrow S^1$ are local homeomorphisms of topological spaces. Moreover, (R, d_0, d_1) is an equivalence relation on S^1 , but there does *not* exist a local homeomorphism $f : S^1 \rightarrow Y$ such that $f \circ d_0 = f \circ d_1$: indeed, (R, d_0, d_1) is *not* a tractable equivalence relation. (Recall [lemma 2.2.14\(a\)](#).) Nonetheless, assuming R and S^1 are objects in \mathcal{C} , an exact quotient of (R, d_0, d_1) exists in \mathcal{S} , and [lemma 2.2.8\(b\)](#) says that it is a $(\mathcal{C}, \hat{\mathcal{D}})$ -extent; but by the preceding discussion, it is *not* a $(\mathcal{C}, \bar{\mathcal{D}})$ -extent.

In this context, it is also worth noting that a $(\mathcal{C}, \bar{\mathcal{D}})$ -extent is the same thing as a $\hat{\mathcal{D}}$ -localic $(\mathcal{C}, \hat{\mathcal{D}})$ -extent. In other words, a $(\mathcal{C}, \hat{\mathcal{D}})$ -extent is a $(\mathcal{C}, \bar{\mathcal{D}})$ -extent precisely when it has enough open subobjects.

3.4.9 EXAMPLE. Let \mathcal{C} be the category of Hausdorff spaces X such that the set of points of X is hereditarily κ -small. Then, by [theorem 3.4.7](#), the essential image of $|-| : \mathbf{Xt}(\mathcal{C}, \bar{\mathcal{D}}) \rightarrow \mathbf{TOP}$ is spanned by the *locally* Hausdorff spaces X such that the set of points of X is κ -small. In particular, since there are locally Hausdorff spaces that are not Hausdorff spaces, assuming $\kappa > \aleph_0$, we have a $(\mathcal{C}, \bar{\mathcal{D}})$ -extent that is not isomorphic to any object in \mathcal{C} .

3.5 Schemes

SYNOPSIS. We see two *prima facie* different ways of defining schemes as extents in a gros pretopos and show that they are the same.

PREREQUISITES. §§ 1.1, 1.2, 2.2, 2.3, 2.5, A.1, A.2, A.3.

3.5.1 ✖ Throughout this section, rings and algebras are associative, unital, and commutative.

3.5.2 ¶ Modern algebraic geometry begins with the observation that the opposite of the category of rings is like a category of spaces. Indeed, we will see how to equip it with the structure of an étale finitary (i.e. \aleph_0 -ary) extensive ecumene in two different ways—one that starts from the notion that the formal dual of a principal localisation of a ring is analogous to a basic open subspace, and another that starts from the notion that the formal dual of a flat ring homomorphism of finite presentation is analogous to an open map. The associated gros pretoposes are also different, but we will see that schemes can be defined as objects obtained by gluing together the formal duals of rings along local homeomorphisms in either pretopos.

3.5.3 ¶ To begin, we briefly recall some commutative algebra.

PROPOSITION. $\mathbf{CRING}^{\text{op}}$ is a finitary extensive category.

Proof. Omitted. (Use the Chinese remainder theorem.)

◇

3.5.4 DEFINITION. A ring homomorphism $f : A \rightarrow B$ is **flat** if B is flat as an A -module, i.e. $B \otimes_A (-)$ preserves injective homomorphisms of modules.

Properties of
flat ring homo-
morphisms

PROPOSITION.

- (i) Every ring isomorphism is flat.
- (ii) Every principal localisation is flat.
- (iii) For every ring A , the unique homomorphism $A \rightarrow \{0\}$ is flat.
- (iv) For every ring A , the diagonal $\Delta_A : A \rightarrow A \times A$ is flat.

(v) *The class of flat ring homomorphisms is a coquadrable class of morphisms in **CRING**.*

(vi) *The class of flat ring homomorphisms is closed under composition.*

(vii) *The class of flat ring homomorphisms is closed under finitary product in **CRING**.*

Proof. Straightforward. ◆

3.5.5 DEFINITION. A ring homomorphism $f : A \rightarrow B$ is **faithfully flat** if it has the following properties:

- $f : A \rightarrow B$ is flat.
- For every pushout square in **CRING** of the form below,

$$\begin{array}{ccc} A & \longrightarrow & A' \\ f \downarrow & & \downarrow f' \\ B & \longrightarrow & B' \end{array}$$

$f' : A' \rightarrow B'$ is an injective homomorphism of rings.

Recognition
principles for
faithfully flat ring
homomorphisms

LEMMA. *Let $f : A \rightarrow B$ be a ring homomorphism. The following are equivalent:*

- (i) $f : A \rightarrow B$ is faithfully flat.
- (ii) $f : A \rightarrow B$ is flat and, for every prime ideal \mathfrak{p} of A , there is a prime ideal \mathfrak{q} of B such that $f^{-1}\mathfrak{q} = \mathfrak{p}$.
- (iii) $f : A \rightarrow B$ is flat and $B \otimes_A (-)$ reflects isomorphisms of modules.

Proof. (i) \Rightarrow (ii). Let F be the fraction field of A/\mathfrak{p} and consider the following pushout square in **CRING**:

$$\begin{array}{ccc} A & \longrightarrow & F \\ f \downarrow & & \downarrow f' \\ B & \longrightarrow & F \otimes_A B \end{array}$$

By hypothesis, $f' : F \rightarrow F \otimes_A B$ is injective, so $F \otimes_A B$ is non-zero. In particular, $F \otimes_A B$ has a prime ideal. Let \mathfrak{q} be its preimage in B . Then $f^{-1}\mathfrak{q} = \mathfrak{p}$, as desired.

(ii) \Rightarrow (iii). This is well known: see e.g. [tag 00HQ](#) in [Stacks].

(iii) \Rightarrow (i). Consider a pushout square in **CRING** of the form below:

$$\begin{array}{ccc} A & \longrightarrow & A' \\ f \downarrow & & \downarrow f' \\ B & \longrightarrow & B' \end{array}$$

We wish to show that $f' : A' \rightarrow B'$ is injective. Since $B \otimes_A (-)$ preserves kernels and reflects isomorphisms, it suffices to verify that $\text{id}_B \otimes_A f' : B \otimes_A A' \rightarrow B \otimes_A B'$ is injective. But the following is also a pushout square in **CRING**,

$$\begin{array}{ccc} B & \longrightarrow & B \otimes_A A' \\ \downarrow & & \downarrow \text{id}_B \otimes_A f' \\ B \otimes_A B & \longrightarrow & B \otimes_A B' \end{array}$$

and $B \rightarrow B \otimes_A B$ is a split monomorphism in **CRING**, so $\text{id}_B \otimes_A f' : B \otimes_A A' \rightarrow B \otimes_A B'$ is indeed injective. ■

REMARK. In the lemma above, one can also prove (i) \Rightarrow (iii) directly, thereby avoiding the use of the prime ideal theorem.

Properties of
faithfully flat ring
homomorphisms

PROPOSITION.

- (i) *Every flat split monomorphism in **CRING** is faithfully flat.*
- (ii) *The class of faithfully flat ring homomorphisms is a coquadrable class of morphisms in **CRING**.*
- (iii) *The class of faithfully flat ring homomorphisms is closed under composition.*
- (iv) *The class of faithfully flat ring homomorphisms is closed under finitary product in **CRING**.*

Proof. Straightforward. (Recall [proposition 3.5.4](#).) ◆

3.5.6 DEFINITION. A ring homomorphism $f : A \rightarrow B$ is **of finite presentation** if B is finitely presentable as an A -algebra.

Properties of ring
homomorphisms of finite
presentation

PROPOSITION.

- (i) *Every ring isomorphism is of finite presentation.*
- (ii) *Every principal localisation is of finite presentation.*
- (iii) *For every ring A , the unique homomorphism $A \rightarrow \{0\}$ is of finite presentation.*
- (iv) *For every ring A , the diagonal $\Delta_A : A \rightarrow A \times A$ is of finite presentation.*
- (v) *The class of ring homomorphisms of finite presentation is a coquadrable class of morphisms in **CRING**.*
- (vi) *The class of ring homomorphisms of finite presentation is closed under composition.*
- (vii) *The class of ring homomorphisms of finite presentation is closed under finitary product in **CRING**.*
- (viii) *Given ring homomorphisms $f : A \rightarrow B$ and $g : B \rightarrow C$, if both $f : A \rightarrow B$ and $g \circ f : A \rightarrow C$ are of finite presentation, then $g : B \rightarrow C$ is also of finite presentation.*

Proof. Straightforward. ◆

3.5.7 DEFINITION. A ring homomorphism $f : A \rightarrow B$ is **étale** if it has the following properties:

- $f : A \rightarrow B$ is a flat ring homomorphism of finite presentation.
- The relative codiagonal $\nabla_f : B \otimes_A B \rightarrow B$ is flat.

Recognition
principle for
étale ring homo-
morphisms

LEMMA. *Let $f : A \rightarrow B$ be a ring homomorphism. The following are equivalent:*

- (i) $f : A \rightarrow B$ is an étale ring homomorphism.
- (ii) $f : A \rightarrow B$ is a flat and unramified ring homomorphism of finite presentation.

Proof. (i) \Rightarrow (ii). By tag 092M in [Stacks], étale ring homomorphisms (in our sense) are formally unramified, and by tag 00UU in op. cit., a formally unramified ring homomorphism of finite presentation is unramified.

(ii) \Rightarrow (i). By [tag 08WD](#) in [Stacks], a flat unramified ring homomorphism of finite presentation is étale in their sense, and [tag 00U7](#) in op. cit. implies that an étale ring homomorphism in their sense is étale in our sense. ■

Properties of
étale ring homo-
morphisms

PROPOSITION.

- (i) *Every flat epimorphism of finite presentation in **CRING** is étale.*
- (ii) *In particular, every principal localisation is étale.*
- (iii) *For every ring A , the unique homomorphism $A \rightarrow \{0\}$ is étale.*
- (iv) *For every ring A , the diagonal $\Delta_A : A \rightarrow A \times A$ is étale.*
- (v) *The class of étale ring homomorphisms is a coquadrable class of morphisms in **CRING**.*
- (vi) *The class of étale ring homomorphisms is closed under composition.*
- (vii) *The class of étale ring homomorphisms is closed under finitary product in **CRING**.*
- (viii) *Given ring homomorphisms $f : A \rightarrow B$ and $g : B \rightarrow C$, if both $f : A \rightarrow B$ and $g \circ f : A \rightarrow C$ are étale, then $g : B \rightarrow C$ is also étale.*

Proof. Apply [lemma 1.1.10](#) and propositions [3.5.4](#) and [3.5.6](#). ■

3.5.8 ¶ We will also need some results from descent theory.

DEFINITION. A ring homomorphism is **fppf**^[1] if it is both faithfully flat and of finite presentation.

3.5.8(a) **PROPOSITION.** *Every faithfully flat ring homomorphism is an effective monomorphism in **CRING**.*

Proof. Let $f : A \rightarrow B$ be a faithfully flat ring homomorphism and let $d^0, d^1 : B \rightarrow B \otimes_A B$ be defined as follows:

$$d^0(b) = 1 \otimes b \qquad d^1(b) = b \otimes 1$$

[1] — from French « fidèlement plat de présentation finie ».

We must show that the following is an equaliser diagram in **CRING**:

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{d^0} \\ \xrightarrow{d^1} \end{array} B \otimes_A B$$

By [lemma 3.5.5](#), it is enough to verify that the following is an equaliser diagram in **CRING**,

$$B \xrightarrow{d^1} B \otimes_A B \begin{array}{c} \xrightarrow{d^1} \\ \xrightarrow{d^2} \end{array} B \otimes_A B \otimes_A B$$

where $d^1, d^2 : B \otimes_A B \rightarrow B \otimes_A B \otimes_A B$ are defined as follows:

$$d^1(b_0 \otimes b_1) = b_0 \otimes 1 \otimes b_1 \quad d^2(b_0 \otimes b_1) = b_0 \otimes b_1 \otimes 1$$

In fact, this is a split equaliser diagram, so we are done. ■

3.5.8(b) **PROPOSITION.** *Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be ring homomorphisms.*

- (i) *Assuming $g : B \rightarrow C$ is faithfully flat, $g \circ f : A \rightarrow C$ is flat if and only if $f : A \rightarrow B$ is flat.*
- (ii) *Assuming $g : B \rightarrow C$ is fppf, $g \circ f : A \rightarrow C$ is of finite presentation if and only if $f : A \rightarrow B$ is of finite presentation.*
- (iii) *Assuming $g : B \rightarrow C$ is faithfully flat and étale, $g \circ f : A \rightarrow C$ is étale if and only if $f : A \rightarrow B$ is étale.*

Proof. (i). Straightforward. (Use [proposition 3.5.4](#) and [lemma 3.5.5](#).)

(ii). For the ‘if’ direction, see [proposition 3.5.6](#); for the ‘only if’ direction, see [tag 02KK](#) in [Stacks].

(iii). For the ‘if’ direction, see [proposition 3.5.7](#); for the ‘only if’ direction, see [tag 02K6](#) in [Stacks]. □

3.5.8(c) **PROPOSITION.** *Consider a pushout square in **CRING**:*

$$\begin{array}{ccc} A & \longrightarrow & A' \\ f \downarrow & & \downarrow f' \\ B & \longrightarrow & B' \end{array}$$

Assuming $A \rightarrow A'$ is faithfully flat:

- (i) *$f' : A' \rightarrow B'$ is flat if and only if $f : A \rightarrow B$ is flat.*

(ii) $f' : A' \rightarrow B'$ is of finite presentation if and only if $f : A \rightarrow B$ is of finite presentation.

(iii) $f' : A' \rightarrow B'$ is étale if and only if $f : A \rightarrow B$ is étale.

Proof. (i). By [proposition 3.5.4](#), if $f : A \rightarrow B$ is flat, then $f' : A' \rightarrow B'$ is also flat. Conversely, by [proposition 3.5.5](#), $B \rightarrow B'$ is faithfully flat, and if $f' : A' \rightarrow B'$ is flat, then the composite $A \rightarrow B \rightarrow B'$ is also flat, so by [proposition 3.5.8\(b\)](#), $f : A \rightarrow B$ is indeed flat.

(ii). For the ‘if’ direction, see [proposition 3.5.6](#); for the ‘only if’ direction, see [tag 00QQ](#) in [Stacks].

(iii). By [proposition 3.5.7](#), if $f : A \rightarrow B$ is étale, then $f' : A' \rightarrow B'$ is étale. For the converse, suppose $f' : A' \rightarrow B'$ is étale. By (i) and (ii), $f : A \rightarrow B$ is flat and of finite presentation. Moreover, we have the following commutative diagram in **CRING**,

$$\begin{array}{ccc}
 B & \longrightarrow & B' \\
 \downarrow \downarrow & & \downarrow \downarrow \\
 B \otimes_A B & \longrightarrow & B' \otimes_{A'} B' \\
 \nabla_f \downarrow & & \downarrow \nabla_{f'} \\
 B & \longrightarrow & B'
 \end{array}$$

where every square is a pushout square in **CRING** and every horizontal arrow is a faithfully flat ring homomorphism, so the relative codiagonal $\nabla_f : B \otimes_A B \rightarrow B$ is flat. Thus, $f : A \rightarrow B$ is indeed étale. \square

3.5.9 LEMMA. *Let A be a ring, let I be a finite subset of A , and let $\tilde{A} = \prod_{a \in I} A[a^{-1}]$. The following are equivalent:*

- (i) I generates the unit ideal of A .
- (ii) The induced homomorphism $A \rightarrow \tilde{A}$ is faithfully flat.
- (iii) There is a ring homomorphism $\tilde{A} \rightarrow B$ such that the composite $A \rightarrow \tilde{A} \rightarrow B$ is faithfully flat.

Proof. (i) \Rightarrow (ii). By [proposition 3.5.4](#), $A \rightarrow \tilde{A}$ is flat. We will now show that $A \rightarrow \tilde{A}$ is injective. Since I generates the unit ideal of A , there

exist elements r_0, \dots, r_{n-1} of A and elements a_0, \dots, a_{n-1} of I such that $r_0 a_0 + \dots + r_{n-1} a_{n-1} = 1$. Suppose s is an element of A such that $s = 0$ in each $A[a_i^{-1}]$, i.e. for $0 \leq i < n$, there is a natural number k_i such that $a_i^{k_i} s = 0$ in A ; hence, for $k = k_0 + \dots + k_{n-1}$:

$$s = (r_0 a_0 + \dots + r_{n-1} a_{n-1})^k s = 0$$

Thus $A \rightarrow \tilde{A}$ is indeed injective. The same argument shows that $B \rightarrow \prod_{a \in A} B[f(a)^{-1}]$ is injective for every ring homomorphism $f : A \rightarrow B$, so $A \rightarrow \tilde{A}$ is indeed faithfully flat.

(ii) \Rightarrow (iii). Immediate.

(iii) \Rightarrow (i). Let \mathfrak{a} be the ideal of A generated by I and let \mathfrak{b} be the ideal of B generated by the image of I . We have the following pushout square in **CRING**:

$$\begin{array}{ccc} A & \longrightarrow & A/\mathfrak{a} \\ \downarrow & & \downarrow \\ B & \longrightarrow & B/\mathfrak{b} \end{array}$$

By hypothesis, $A/\mathfrak{a} \rightarrow B/\mathfrak{b}$ is injective. On the other hand, \mathfrak{b} is the unit ideal of B , so $A/\mathfrak{a} \cong \{0\}$. Hence, \mathfrak{a} is the unit ideal of A . ■

3.5.10 ¶ The following is a deep result in algebraic geometry.

Flat morphisms have open image **PROPOSITION.** *Let $f : A \rightarrow B$ be a flat ring homomorphism of finite presentation and let I be the subset of A defined as follows:*

- $a \in I$ if and only if the induced homomorphism $A[a^{-1}] \rightarrow B[f(a)^{-1}]$ is fppf.

Then $\{f(a) \mid a \in I\}$ generates the unit ideal of B .

Proof. In the language of algebraic geometry, this is the statement that a flat morphism (of schemes) of finite presentation has an open image. See [tag 01UA](#) in [Stacks]. □

3.5.II DEFINITION. An **open quasilocalisation** is an epimorphism $f : A \rightarrow B$ in **CRING** for which there is a (finite) subset $I \subseteq A$ with the following properties:

- For every $a \in I$, the induced homomorphism $A[a^{-1}] \rightarrow B[f(a)^{-1}]$ is an isomorphism.
- $\{f(a) \mid a \in I\}$ generates the unit ideal of B .

Recognition principles for open quasilocalisations

LEMMA. Let $f : A \rightarrow B$ be a ring homomorphism. The following are equivalent:

- (i) $f : A \rightarrow B$ is an open quasilocalisation.
- (ii) $f : A \rightarrow B$ is an epimorphism in **CRING**, flat, and of finite presentation.
- (iii) $f : A \rightarrow B$ is an epimorphism in **CRING** and étale.

Proof. (i) \Rightarrow (ii). By propositions 3.5.4 and 3.5.6 and lemma 3.5.9, there is a finite subset $I \subseteq A$ with the following properties:

- The induced homomorphism $B \rightarrow \prod_{a \in I} B[a^{-1}]$ is fppf.
- The composite $A \rightarrow B \rightarrow \prod_{a \in I} B[a^{-1}]$ is flat and of finite presentation.

Thus, by proposition 3.5.8(b), $f : A \rightarrow B$ is flat and of finite presentation. But open quasilocalisations are epimorphisms in **CRING** by definition, so we are done.

(ii) \Rightarrow (iii). Apply proposition 3.5.7.

(iii) \Rightarrow (i). By proposition 3.5.10, there is a finite subset $I \subseteq A$ with the following properties:

- For every $a \in I$, the induced homomorphism $A[a^{-1}] \rightarrow B[f(a)^{-1}]$ is faithfully flat.
- $\{f(a) \mid a \in I\}$ generates the unit ideal of B .

But [proposition 3.5.8\(a\)](#) implies that every faithfully flat epimorphism in **CRING** is an isomorphism, so $f : A \rightarrow B$ is indeed an open quasilocalisation. ■

Properties of
open quasi-
localisations

PROPOSITION.

- (i) *Every ring isomorphism is an open quasilocalisation.*
- (ii) *Every principal localisation is an open quasilocalisation.*
- (iii) *The class of open quasilocalisations is a coquadrable class of morphisms in **CRING**.*
- (iv) *The class of open quasilocalisations is closed under composition.*
- (v) *The class of open quasilocalisations is closed under finitary product in **CRING**.*
- (vi) *Given ring homomorphisms $f : A \rightarrow B$ and $g : B \rightarrow C$, if both $f : A \rightarrow B$ and $g \circ f : A \rightarrow C$ are open quasilocalisations, then $g : B \rightarrow C$ is also an open quasilocalisation.*

Proof. By [lemma 3.5.11](#), these reduce to [propositions 3.5.3](#) and [3.5.7](#). ■

3.5.12 ✖ For the remainder of this section:

- K is a ring.
- \mathcal{A} is a (small) full subcategory of the metacategory of K -algebras that is closed under finitary coproducts, coequalisers, and finitary products.
- For every object A in \mathcal{A} and every element $a \in A$, the principal localisation $A[a^{-1}]$ is also an object in \mathcal{A} .

3.5.13 ¶ To avoid confusion, we write $\text{Spec } A$ for the object in \mathcal{A}^{op} corresponding to an object A in \mathcal{A} , and we write $\text{Spec } f : \text{Spec } B \rightarrow \text{Spec } A$ for the morphism in \mathcal{A}^{op} corresponding to a morphism $f : A \rightarrow B$ in \mathcal{A} .

The four classical Grothendieck topologies are defined as follows:

- 3.5.13(a) DEFINITION. The **Zariski coverage** on \mathcal{A}^{op} consists of all sinks of the form below up to isomorphism,

$$\{\text{Spec } f_a : \text{Spec } A[a^{-1}] \rightarrow \text{Spec } A \mid a \in I\}$$

where A is an object in \mathcal{A} , I is a finite subset of A that generates the unit ideal, and $f_a : A \rightarrow A[a^{-1}]$ is the principal localisation.

- 3.5.13(b) DEFINITION. The **étale coverage** on \mathcal{A}^{op} is the smallest composition-closed coverage on \mathcal{A}^{op} containing the Zariski coverage as well as the singleton $\{\text{Spec } f : \text{Spec } B \rightarrow \text{Spec } A\}$ for every faithfully flat étale ring homomorphism $f : A \rightarrow B$ in \mathcal{A} .

- 3.5.13(c) DEFINITION. The **fppf coverage** on \mathcal{A}^{op} is the smallest composition-closed coverage on \mathcal{A}^{op} containing the Zariski coverage as well as the singleton $\{\text{Spec } f : \text{Spec } B \rightarrow \text{Spec } A\}$ for every fppf ring homomorphism $f : A \rightarrow B$ in \mathcal{A} .

- 3.5.13(d) DEFINITION. The **fpqc^[2] coverage** on \mathcal{A}^{op} is the smallest composition-closed coverage on \mathcal{A}^{op} containing the Zariski coverage as well as the singleton $\{\text{Spec } f : \text{Spec } B \rightarrow \text{Spec } A\}$ for every faithfully flat ring homomorphism $f : A \rightarrow B$ in \mathcal{A} .

Properties of the classical Grothendieck topologies

PROPOSITION.

(i) *The Zariski (resp. étale, fppf, fpqc) coverage on \mathcal{A}^{op} is finitary superextensive.*

(ii) *The Zariski (resp. étale, fppf, fpqc) coverage on \mathcal{A}^{op} is subcanonical.*

Proof. (i). It suffices to verify that the Zariski coverage on \mathcal{A}^{op} is finitary superextensive. Let A and B be rings and consider the projections $A \times B \rightarrow A$ and $A \times B \rightarrow B$. It is straightforward to verify that $A \times B \rightarrow A$ is isomorphic to the principal localisation $A \times B \rightarrow (A \times B)[(1, 0)^{-1}]$. Similarly, $A \times B \rightarrow B$ is isomorphic to the principal localisation $A \times B \rightarrow (A \times B)[(0, 1)^{-1}]$. Hence, every finite coproduct cocone in \mathcal{A}^{op} is in the Zariski coverage.

[2] — from French « fidèlement plat et quasi-compacte »

(ii). It suffices to verify that the fpqc coverage on \mathcal{A}^{op} is subcanonical. Since the fpqc coverage is finitary superextensive, by lemmas 1.4.18, 1.5.15 and 1.5.16, it is enough to show that every faithfully flat ring homomorphism in \mathcal{A} is an effective monomorphism in \mathcal{A} . Since the forgetful functor $\mathcal{A} \rightarrow \mathbf{CRING}$ creates cokernel pairs, this reduces to proposition 3.5.8(a). ■

3.5.14 DEFINITION. A morphism $h : X \rightarrow Y$ in $\mathbf{PSH}(\mathcal{A}^{\text{op}})$ is **étale** if both $h : X \rightarrow Y$ and the relative diagonal $\Delta_h : X \rightarrow X \times_Y X$ are fppf-semilocally of \mathcal{G} -type, where \mathcal{G} is the opposite of the class of flat ring homomorphisms of finite presentation in \mathcal{A} .

REMARK. Since \mathcal{G} is a quadrable class of morphisms in \mathcal{A}^{op} (by proposition 3.5.6), by lemma 1.2.15, every morphism in $\mathbf{PSH}(\mathcal{A}^{\text{op}})$ fppf-semilocally of \mathcal{G} -type is also fppf-locally of \mathcal{G} -type.

Properties of
étale morphisms
of presheaves

PROPOSITION.

- (i) Every fppf-locally bijective morphism in $\mathbf{PSH}(\mathcal{A}^{\text{op}})$ is étale.
- (ii) For every presheaf X on \mathcal{A}^{op} and every set I , the codiagonal morphism $\coprod_{i \in I} X \rightarrow X$ is étale.
- (iii) The class of étale morphisms in $\mathbf{PSH}(\mathcal{A}^{\text{op}})$ is a quadrable class of morphisms in $\mathbf{PSH}(\mathcal{A}^{\text{op}})$.
- (iv) The class of étale morphisms in $\mathbf{PSH}(\mathcal{A}^{\text{op}})$ is closed under composition.
- (v) The class of étale morphisms in $\mathbf{PSH}(\mathcal{A}^{\text{op}})$ is closed under (possibly infinitary) coproduct in $\mathbf{PSH}(\mathcal{A}^{\text{op}})$.
- (vi) Given a pullback square in $\mathbf{PSH}(\mathcal{A}^{\text{op}})$ of the form below,

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & X \\ \tilde{h} \downarrow & & \downarrow h \\ \tilde{Y} & \longrightarrow & Y \end{array}$$

where $\tilde{Y} \twoheadrightarrow Y$ is fppf-locally surjective, if $\tilde{h} : \tilde{X} \rightarrow \tilde{Y}$ is étale, then $h : X \rightarrow Y$ is also étale.

- (vii) Given morphisms $h : X \rightarrow Y$ and $k : Y \rightarrow Z$ in $\mathbf{PSH}(\mathcal{A}^{\text{op}})$, if both $k : Y \rightarrow Z$ and $k \circ h : X \rightarrow Z$ are étale, then $h : X \rightarrow Y$ is also étale.
- (viii) Given a fppf-locally surjective morphism $h : X \twoheadrightarrow Y$ in $\mathbf{PSH}(\mathcal{A}^{\text{op}})$ and a morphism $k : Y \rightarrow Z$ in $\mathbf{PSH}(\mathcal{A}^{\text{op}})$, if both $h : X \twoheadrightarrow Y$ and $k \circ h : X \rightarrow Z$ are étale, then $k : Y \rightarrow Z$ is also étale.
- (ix) For every morphism $f : A \rightarrow B$ in \mathcal{A} , $h^f : h^A \rightarrow h^B$ is étale if and only if $f : A \rightarrow B$ is étale.

Proof. (i)–(viii). Combine [proposition 1.1.11](#) and [remark 1.2.16](#) with [propositions 1.2.14](#) and [1.2.17](#).

(ix). Apply [proposition 1.2.20](#) to [propositions 3.5.8\(b\)](#) and [3.5.8\(c\)](#). ■

3.5.15 DEFINITION. An **open immersion** of presheaves on \mathcal{A}^{op} is a monomorphism $h : X \rightarrow Y$ in $\mathbf{PSH}(\mathcal{A}^{\text{op}})$ that is Zariski-semilocally of \mathcal{B} -type, where \mathcal{B} is the opposite of the class of morphisms in \mathcal{A} that are principal localisations up to isomorphism.

REMARK. It is clear that \mathcal{B} is a quadrable class of morphisms in \mathcal{A}^{op} . Thus, by [lemma 1.2.15](#), any morphism that is Zariski-semilocally of \mathcal{B} -type is also Zariski-locally of \mathcal{B} -type.

Recognition principles for open fppf-closed subpresheaves

LEMMA. Let A be an object in \mathcal{A} and let U be an fppf-closed subpresheaf of h^A . The following are equivalent:

- (i) The inclusion $U \hookrightarrow h^A$ is an open immersion.
- (ii) The inclusion $U \hookrightarrow h^A$ is an étale monomorphism.
- (iii) There is a (possibly infinite) set Φ of elements of U with the following properties:
 - Φ is a Zariski-local generating set of elements of U .
 - For every $(\text{Spec } B, \text{Spec } f) \in \Phi$, (B, f) is isomorphic (in \mathcal{A}^{op}) to a principal localisation of A .

Proof. (i) \Rightarrow (ii). Since every principal localisation is an étale ring homomorphism ([proposition 3.5.7](#)) and Zariski-locally surjective morphisms are fppf-locally surjective, open immersions are étale.

(ii) \Rightarrow (iii). Let I be the set of all $a \in A$ such that $(\text{Spec } A[a^{-1}], \text{Spec } f_a)$ is an element of U , where $f_a : A \rightarrow A[a^{-1}]$ is the principal localisation, and let $\Phi = \{(\text{Spec } A[a^{-1}], \text{Spec } f_a) \mid a \in I\}$. Clearly, it is enough to prove that Φ is a Zariski-local generating set of elements of U .

By [lemma 1.5.16](#) and [propositions 3.5.4, 3.5.6, and 3.5.13](#), there exist an fppf ring homomorphism $j : A \rightarrow \tilde{A}$ in \mathcal{A} and a set $\tilde{\Phi}$ of objects in \tilde{A}/\mathcal{A} with the following properties:

- $\tilde{\Phi}$ is an fppf-local generating set of elements of the preimage $\tilde{U} \subseteq \mathcal{H}^{\tilde{A}}$ of $U \subseteq \mathcal{H}^A$.
- For every $(\tilde{B}, \tilde{f}) \in \tilde{\Phi}$, $\tilde{f} : \tilde{A} \rightarrow \tilde{B}$ is flat and of finite presentation.

Note that $\tilde{f} \circ j : A \rightarrow \tilde{B}$ is flat and of finite presentation, so by [proposition 3.5.10](#), there is a finite subset $I' \subseteq A$ with the following properties:

- For every $a \in I'$, the induced homomorphism $A[a^{-1}] \rightarrow \tilde{B}[\tilde{f}(j(a))^{-1}]$ is fppf.
- $\{\tilde{f}(j(a)) \mid a \in I'\}$ generates the unit ideal of \tilde{B} .

Since $(\text{Spec } \tilde{B}, \text{Spec } (\tilde{f} \circ j))$ is an element of U and U is an fppf-closed subpresheaf of \mathcal{H}^A , it follows that $I' \subseteq I$. On the other hand, by [proposition A.2.14](#), the morphism $\tilde{U} \rightarrow U$ is fppf-locally surjective, so it follows that Φ is an fppf-local generating set of elements of U . Hence, by [lemma 3.5.9](#), Φ is also a Zariski-local generating set of elements of U .

(iii) \Rightarrow (i). Immediate. ■

Properties of
open immersions
of presheaves

PROPOSITION.

- (i) *Every isomorphism in $\mathbf{PSH}(\mathcal{A}^{\text{op}})$ is an open immersion.*
- (ii) *Every open immersion in $\mathbf{PSH}(\mathcal{A}^{\text{op}})$ is étale.*
- (iii) *The class of open immersions in $\mathbf{PSH}(\mathcal{A}^{\text{op}})$ is a quadrable class of morphisms in $\mathbf{PSH}(\mathcal{A}^{\text{op}})$.*

- (iv) *The class of open immersions in $\mathbf{Psh}(\mathcal{A}^{\text{op}})$ is closed under composition.*
- (v) *The class of open immersions in $\mathbf{Psh}(\mathcal{A}^{\text{op}})$ is closed under (possibly infinitary) coproduct in $\mathbf{Psh}(\mathcal{A}^{\text{op}})$.*
- (vi) *Given a pullback square in $\mathbf{Psh}(\mathcal{A}^{\text{op}})$ of the form below,*

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & X \\ \tilde{h} \downarrow & & \downarrow h \\ \tilde{Y} & \longrightarrow & Y \end{array}$$

where $\tilde{Y} \rightarrow Y$ is Zariski-locally surjective, if $\tilde{h} : \tilde{X} \rightarrow \tilde{Y}$ is an open immersion and $h : X \rightarrow Y$ is a monomorphism, then $h : X \rightarrow Y$ is also an open immersion.

- (vii) *Given morphisms $h : X \rightarrow Y$ and $k : Y \rightarrow Z$ in $\mathbf{Psh}(\mathcal{A}^{\text{op}})$, if both $k : Y \rightarrow Z$ and $k \circ h : X \rightarrow Z$ are open immersions, then $h : X \rightarrow Y$ is also an open immersion.*
- (viii) *For every morphism $f : A \rightarrow B$ in \mathcal{A} , $h^f : h^A \rightarrow h^B$ is an open immersion if and only if $f : A \rightarrow B$ is an open quasilocalisation.*

Proof. (i)–(vi). Apply [proposition 1.2.14](#).

(vii). This is a special case of [lemma 1.1.3](#).

(viii). Suppose $f : A \rightarrow B$ is an open quasilocalisation. Then, by definition, $\text{Spec } f : \text{Spec } B \rightarrow \text{Spec } A$ is monomorphism in \mathcal{A}^{op} that is Zariski-locally of \mathcal{B} -type, where \mathcal{B} is the opposite of the class of principal localisations, so $h^f : h^B \rightarrow h^A$ is indeed an open immersion.

For the converse, suppose $h^f : h^B \rightarrow h^A$ is an open immersion. Since the fppf coverage on \mathcal{A}^{op} is subcanonical ([proposition 3.5.13](#)), the pre-sheaf image is an fppf-closed subsheaf of h^A . Thus, we may apply [lemma 3.5.15](#) to deduce that $f : A \rightarrow B$ is an open quasilocalisation. ■

3.5.16 DEFINITION. A **local isomorphism** of presheaves on \mathcal{A}^{op} is a morphism $h : X \rightarrow Y$ in $\mathbf{PSH}(\mathcal{A}^{\text{op}})$ for which there is a set Φ of subpresheaves of X with the following properties:

- $\bigcup_{U \in \Phi} U$ is a Zariski-dense subpresheaf of X .
- For every $U \in \Phi$, both the inclusion $U \hookrightarrow X$ and the composite $U \hookrightarrow X \rightarrow Y$ are open immersions in $\mathbf{PSH}(\mathcal{A}^{\text{op}})$.

Properties of
local isomorphisms of presheaves

PROPOSITION.

- (i) *Every open immersion in $\mathbf{PSH}(\mathcal{A}^{\text{op}})$ is a local isomorphism.*
- (ii) *Every local isomorphism in $\mathbf{PSH}(\mathcal{A}^{\text{op}})$ is étale.*
- (iii) *A monomorphism in $\mathbf{PSH}(\mathcal{A}^{\text{op}})$ is an open immersion if and only if it is a local isomorphism.*
- (iv) *For every presheaf X on \mathcal{A}^{op} and every set I , the codiagonal morphism $\coprod_{i \in I} X \rightarrow X$ is a local isomorphism.*
- (v) *The class of local isomorphisms in $\mathbf{PSH}(\mathcal{A}^{\text{op}})$ is a quadrable class of morphisms in $\mathbf{PSH}(\mathcal{A}^{\text{op}})$.*
- (vi) *The class of local isomorphisms in $\mathbf{PSH}(\mathcal{A}^{\text{op}})$ is closed under composition.*
- (vii) *The class of local isomorphisms in $\mathbf{PSH}(\mathcal{A}^{\text{op}})$ is closed under (possibly infinitary) coproduct in $\mathbf{PSH}(\mathcal{A}^{\text{op}})$.*
- (viii) *Given morphisms $h : X \rightarrow Y$ and $k : Y \rightarrow Z$ in $\mathbf{PSH}(\mathcal{A}^{\text{op}})$, if both $k : Y \rightarrow Z$ and $k \circ h : X \rightarrow Z$ are local isomorphisms, then $h : X \rightarrow Y$ is also a local isomorphism.*

Proof. Straightforward given [proposition 3.5.15](#). (Compare the proof of [proposition 2.2.12](#).) ◆

3.5.17 ¶ Every Zariski-locally surjective morphism in $\mathbf{PSH}(\mathcal{A}^{\text{op}})$ is also fpqc-locally surjective. The converse is not true in general, but we do have the following result.

Recognition principle for locally surjective local isomorphisms of presheaves

LEMMA. *Let $h : X \rightarrow Y$ be a local isomorphism in $\mathbf{PSH}(\mathcal{A}^{\text{op}})$. The following are equivalent:*

- (i) $h : X \rightarrow Y$ is Zariski-locally surjective.
- (ii) $h : X \rightarrow Y$ is fpqc-locally surjective.

Proof. (i) \Rightarrow (ii). Immediate.

(ii) \Rightarrow (i). In view of propositions 3.5.16 and A.2.14, it is enough to verify the case where $Y = \hat{h}^A$ for some object A in \mathcal{A} . Let Φ be a set of subpresheaves of X with the following properties:

- $\bigcup_{U \in \Phi} U$ is a Zariski-dense subpresheaf of X .
- For every $U \in \Phi$, both the inclusion $U \hookrightarrow X$ and the composite $U \hookrightarrow X \rightarrow Y$ are open immersions in $\mathbf{PSH}(\mathcal{A}^{\text{op}})$.

Moreover, recalling proposition 3.5.15, we may assume that each $U \in \Phi$ can be represented by some principal localisation of A . This yields an fpqc-covering sink on $\text{Spec } A$. Since every fpqc-covering sink in \mathcal{A}^{op} contains a finite fpqc-covering sink, $X \rightarrow Y$ is indeed Zariski-locally surjective, by lemma 3.5.9. ■

3.5.18 ✂ For the remainder of this section:

- κ is a regular cardinal.
- $\mathcal{C} = \mathbf{Fam}_{\kappa}^{\aleph_0}(\mathcal{A}^{\text{op}})$.
- For each object X in \mathcal{C} , $J'(X)$ (resp. $J_f(X)$) is the set of κ -small sinks Φ on X in \mathcal{C} such that the induced morphism $\coprod_{(U,x) \in \Phi} U \rightarrow X$ corresponds to a Zariski-locally surjective local isomorphism (resp. fppf-locally surjective morphism) in $\mathbf{PSH}(\mathcal{A}^{\text{op}})$.
- E' , (resp. E_f) is the class of morphisms in \mathcal{C} that correspond to Zariski-locally (resp. fppf-locally) surjective morphisms in $\mathbf{PSH}(\mathcal{A}^{\text{op}})$.
- $\mathcal{S}' = \mathbf{Ex}(\mathcal{C}, E')$ and $\mathcal{S}_f = \mathbf{Ex}(\mathcal{C}, E_f)$.
- \mathcal{D} (resp. \mathcal{D}') is the class of morphisms that correspond to étale morphisms (resp. local isomorphisms) in $\mathbf{PSH}(\mathcal{A}^{\text{op}})$.

3.5.19 ¶ We will consider both \mathcal{S}' and \mathcal{S}_f as settings for algebraic geometry. First, we need to establish some basic properties.

3.5.19(a) PROPOSITION.

- (i) \mathcal{C} is a κ -ary extensive category with limits of finite diagrams.
- (ii) In particular, both $(\mathcal{C}, \mathcal{E}')$ and $(\mathcal{C}, \mathcal{E}_f)$ satisfy the Shulman condition.
- (iii) Both \mathcal{S}' and \mathcal{S}_f are κ -ary pretoposes.

Proof. (i). This is a special case of [proposition 1.5.21](#).

(ii). Immediate.

(iii). Apply [proposition 1.5.13](#). ■

3.5.19(b) PROPOSITION. $(\mathcal{C}, \mathcal{D}', \mathcal{J}')$ is a κ -ary admissible ecumene, i.e.:

- (i) \mathcal{D}' is a class of separated fibrations in \mathcal{C} .
- (ii) \mathcal{J}' is a subcanonical \mathcal{D}' -adapted κ -ary superextensive coverage on \mathcal{C} .
- (iii) Every morphism in \mathcal{C} of \mathcal{D}' -type $(\mathcal{D}', \mathcal{J}')$ -semilocally on the domain is a member of \mathcal{D}' .
- (iv) Every complemented monomorphism in \mathcal{C} is a member of \mathcal{D}' .

Moreover:

- (v) $(\mathcal{S}', \mathcal{D}')$ is the associated gros κ -ary pretopos.

Proof. (i) and (iv). See [proposition 3.5.16](#).

(ii). The Zariski coverage on \mathcal{A}^{op} is subcanonical ([proposition 3.5.13](#)), so \mathcal{J}' is a subcanonical coverage on \mathcal{C} , which is κ -ary superextensive by [proposition 1.5.17](#) and \mathcal{D}' -adapted by construction.

(iii). It suffices to verify the following:

- Given an Zariski-locally surjective morphism $h : X \rightarrow Y$ in $\mathbf{PSH}(\mathcal{A}^{\text{op}})$ and a morphism $k : Y \rightarrow Z$ in $\mathbf{PSH}(\mathcal{A}^{\text{op}})$, if both $h : X \rightarrow Y$ and $k \circ h : X \rightarrow Z$ are local isomorphisms, then $k : Y \rightarrow Z$ is also a local isomorphism.

This is straightforward. (Compare the proof of [proposition 2.2.13](#).)

(v). It is not hard to see that morphisms in S' that correspond to morphisms in $\mathbf{PSH}(C)$ that are J' -locally of D' -type are the same as morphisms in S' that correspond to morphisms in $\mathbf{PSH}(\mathcal{A}^{\text{op}})$ that are Zariski-locally of D' -type. Moreover, by [lemma 2.1.16](#), S is equivalent to the category of J' -locally κ -presentable J' -sheaves on \mathcal{A}^{op} . Thus, (S, D') is indeed the κ -ary gros pretopos associated with (C, D', J') . \blacklozenge

3.5.19(c) PROPOSITION.

- (i) D is a class of étale morphisms in S_f .
- (ii) D' is the class of local homeomorphisms in S_f .
- (iii) C is a unary basis for (S_f, D) .
- (iv) C is a unary basis for (S_f, D') .

Proof. (i). See [proposition 3.5.14](#).

(ii). By [lemma 3.5.15](#), étale monomorphisms in S_f correspond to open immersions in $\mathbf{PSH}(\mathcal{A}^{\text{op}})$, because the presheaf image of a monomorphism of fppf-sheaves is always fppf-closed. Since fppf-locally surjective morphisms are also fpqc-locally surjective, and Zariski-locally surjective morphisms are also fppf-locally surjective, we may apply [proposition 3.5.16](#) and [lemma 3.5.17](#) to deduce that local homeomorphisms in S_f correspond to local isomorphisms in $\mathbf{PSH}(\mathcal{A}^{\text{op}})$.

(iii). Use [proposition 2.3.2](#).

(iv). In view of [lemma 2.1.16](#), it suffices to verify the following:

- For every object Y in C and every fppf-subsheaf $V \subseteq h_Y$, if V is fppf-locally κ -presentable and the inclusion $U \hookrightarrow h_Y$ is an open immersion, then there exist an object X in C and a fppf-locally surjective morphism $p : h_X \twoheadrightarrow U$ such that the composite $h_X \twoheadrightarrow U \hookrightarrow h_Y$ is a local isomorphism.

This is a straightforward consequence of [lemma 3.5.15](#). \blacksquare

3.5.20 ¶ With the usual abuses of notation, we may speak of (C, D') -extents in both S' and S_f . Although these are different notions *prima facie*, fortunately, they coincide. To prove this, we will require a preliminary result.

LEMMA. *Let F be a presheaf on \mathcal{A}^{op} . The following are equivalent:*

- (i) *There exist an object Y in C and a laminar morphism $X \rightarrow Y$ in S' such that $h_X \cong F$.*
- (ii) *There exist an object Y in C and a laminar morphism $X \rightarrow Y$ in S_f such that $h_X \cong F$.*

Proof. Let K be the finitary extensive coverage on \mathcal{A}^{op} . We have the Yoneda representations $S' \rightarrow \mathbf{SH}(\mathcal{A}^{\text{op}}, K)$ and $S_f \rightarrow \mathbf{SH}(\mathcal{A}^{\text{op}}, K)$, and by [proposition 1.5.18](#) and [theorem 2.1.14](#), both preserve κ -ary coproducts. Thus, it suffices to prove the claim with ‘laminar morphism’ replaced with ‘open embedding’.

(i) \Rightarrow (ii). Let $f : X \rightarrow Y$ be an open embedding in S' where Y is an object in C . Then there is a (C, D') -atlas of X in S' , say (\tilde{X}, p) . Let (R, d_0, d_1) be a kernel pair of $f \circ p : \tilde{X} \rightarrow Y$ in C . Since $f : X \rightarrow Y$ is a monomorphism in S' , (R, d_0, d_1) is also a kernel pair of $p : \tilde{X} \rightarrow X$ in S' . Moreover, by [proposition 3.5.16](#), $f \circ p : \tilde{X} \rightarrow Y$ is a member of D' , so both projections $d_0, d_1 : R \rightarrow \tilde{X}$ are also members of D' . Let $q : \tilde{X} \rightarrow Y'$ be an exact quotient of (R, d_0, d_1) in S_f and let $m : Y' \rightarrow Y$ be the unique morphism in S_f such that $m \circ q = f \circ p$. By [proposition 3.5.19\(c\)](#), $q : \tilde{X} \rightarrow Y'$ is an étale morphism in S_f , so $m : Y' \rightarrow Y$ is also an étale morphism in S_f . On the other hand, since (R, d_0, d_1) is also a kernel pair of $m \circ q$, $m : Y' \rightarrow Y$ is a monomorphism in S_f , so it is an open embedding in S_f . Hence, $q : \tilde{X} \rightarrow Y'$ is a local homeomorphism in S_f . But then [lemma 3.5.17](#) says that $h_q : h_{\tilde{X}} \rightarrow h_{Y'}$ is Zariski-locally surjective, so we indeed have $h_X \cong h_{Y'}$.

(ii) \Rightarrow (i). Let $f : X \rightarrow Y$ be an open embedding in S_f where Y is an object in C . Then there is a (C, D') -atlas of X in S_f , say (\tilde{X}, p) . Let (R, d_0, d_1) be a kernel pair of $f \circ p : \tilde{X} \rightarrow Y$ in C , let $q : \tilde{X} \rightarrow Y'$ be an exact quotient of (R, d_0, d_1) in S' , and let $m : Y' \rightarrow Y$ be the unique

morphism in S' such that $m \circ q = f \circ p$. By the same argument as before, $m : Y' \rightarrow Y$ is an open embedding in S' . Since $p : \tilde{X} \twoheadrightarrow X$ is a member of \mathcal{D}' , $h_p : h_{\tilde{X}} \rightarrow h_X$ is Zariski-locally surjective, so we have $h_X \cong h_{Y'}$, as desired. \blacksquare

PROPOSITION. *Let F be a presheaf on \mathcal{A}^{op} . The following are equivalent:*

- (i) *There is a (C, \mathcal{D}') -extent X in S' such that $h_X \cong F$.*
- (ii) *There is a (C, \mathcal{D}') -extent X in S_f such that $h_X \cong F$.*

Proof. For ease of notation, we will identify S' and S_f with their respective essential images in $\mathbf{PSH}(\mathcal{A}^{\text{op}})$.

(i) \Rightarrow (ii). Let X be a (C, \mathcal{D}') -extent in S' . By definition, there is a (C, \mathcal{D}') -atlas of X in S' , say (\tilde{X}, p) , and by [proposition 2.3.13](#), there is a laminar effective epimorphism $\tilde{p} : U \rightarrow \tilde{X}$ in S' such that $p \circ \tilde{p} : U \rightarrow X$ is also a laminar effective epimorphism in S' . Let (R, d_0, d_1) be a kernel pair of $p \circ \tilde{p} : U \rightarrow X$ in S' . Then the projections $d_0, d_1 : R \rightarrow U$ are laminar morphisms in S' , so by [lemma 3.5.20](#), both R and U are also objects in S_f . Consider exact quotients of (R, d_0, d_1) in S_f . By [lemma 2.2.14\(c\)](#), (R, d_0, d_1) is a tractable equivalence relation in S' , and it follows that (R, d_0, d_1) is also a tractable equivalence relation in S_f . Thus, by [lemma 3.5.17](#), any exact quotient of (R, d_0, d_1) in S_f is also an exact quotient of (R, d_0, d_1) in S' , and therefore X is in S_f , as desired.

(ii) \Rightarrow (i). The same argument (*mutatis mutandis*) works. \blacksquare

REMARK. In other words, the essential images of $\mathbf{Xt}(C, \mathcal{D}') \subseteq S'$ and $\mathbf{Xt}(C, \mathcal{D}') \subseteq S_f$ in $\mathbf{PSH}(\mathcal{A}^{\text{op}})$ coincide. Thus, up to equivalence, there is no ambiguity in the notation $\mathbf{Xt}(C, \mathcal{D}')$.

3.5.21 ¶ It is more or less clear how to connect our definition of ‘ (C, \mathcal{D}') -extent’ with the functor-of-points definition of ‘scheme’ found in e.g. Demazure and Gabriel [1970], which is known to be equivalent to the definition of ‘scheme’ in terms of locally ringed spaces. We will take this for granted and instead focus on making a more precise statement about the kind of schemes that can be obtained as (C, \mathcal{D}') -extents.

3.5.21(a) DEFINITION. An \mathcal{A} -atlas of a K -scheme X is a set Φ with the following properties:

- Every element of Φ is an open subscheme of X .
- For every $U \in \Phi$, U is isomorphic to the affine K -scheme corresponding to some object in \mathcal{A} .
- $X = \bigcup_{U \in \Phi} U$.

3.5.21(b) DEFINITION. A K -scheme X is **of (\mathcal{A}, κ) -type** if there is a set Φ with the following properties:

- Φ is a κ -small \mathcal{A} -atlas.
- For every $(U_0, U_1) \in \Phi \times \Phi$, $U_0 \cap U_1$ admits a κ -small \mathcal{A} -atlas.

THEOREM. *Let \mathcal{M} be the essential image in $\mathbf{Psh}(\mathcal{A}^{\text{op}})$ of the metacategory of K -schemes of (\mathcal{A}, κ) -type. Then \mathcal{M} is also the essential image of $\mathbf{Xt}(\mathcal{C}, \mathcal{D}')$.*

Proof. Omitted. (Compare the proof of [theorem 2.4.13](#).)

◇

3.5.22 EXAMPLE. Let \mathcal{A} be the category of finitely presented K -algebras.^[3] It is straightforward to verify the following:

- (i) A K -scheme admits an \mathcal{A} -atlas if and only if it is locally of finite presentation.
- (ii) A K -scheme admits a finite \mathcal{A} -atlas if and only if it is locally of finite presentation and quasicompact.
- (iii) A K -scheme is of (\mathcal{A}, \aleph_0) -type if and only if it is of finite presentation, i.e. locally of finite presentation, quasicompact, and quasiseparated.

In particular, if $\kappa = \aleph_0$, then $\mathbf{Xt}(\mathcal{C}, \mathcal{D}')$ is equivalent to the metacategory of K -schemes of finite presentation, by [theorem 3.5.21](#).

[3] More precisely, let \mathcal{A} be the category of finitely presentable K -algebras whose underlying set is hereditarily λ -small for some cardinal λ such that the underlying set of K is λ -small.

3.5.23 EXAMPLE. Let K be a field and let \mathcal{A} be the category defined as follows:

- The objects are pairs (m, I) where m is a natural number and I is a finite subset of the polynomial ring $K[x_1, \dots, x_m]$.
- The morphisms $(m, I) \rightarrow (n, J)$ are the K -algebra homomorphisms $K[x_1, \dots, x_m]/(I) \rightarrow K[x_1, \dots, x_n]/(J)$, where (I) and (J) are the ideals generated by I and J , respectively.
- Composition and identities are inherited.

Although \mathcal{A} is not literally a subcategory of the metacategory of K -algebras, there is an evident fully faithful functor which will suffice for our purposes.

Let \mathcal{C}_1 be the category of T_1 -spaces^[4] X such that the set of points of X is hereditarily λ -small, where λ is an uncountable regular cardinal $\geq \kappa$ such that the underlying set of K is hereditarily λ -small. Given an object (m, I) in \mathcal{A} , let $F(m, I)$ be the subspace of K^m defined as follows,

$$F(m, I) = \{(x_1, \dots, x_m) \in K^m \mid \forall \varphi \in I. \varphi(x_1, \dots, x_m) = 0\}$$

where K^m is equipped with the classical Zariski topology.^[5] This defines a functor $F : \mathcal{A}^{\text{op}} \rightarrow \mathcal{C}_1$.

Let \mathcal{D}_1 be the class of local homeomorphisms of topological spaces in \mathcal{C}_1 , and let J_1 be the usual coverage on \mathcal{C}_1 . It can be shown that $F : \mathcal{A}^{\text{op}} \rightarrow \mathcal{C}_1$ preserves finitary coproducts, so by [theorem 1.5.20](#), we have an induced functor $F : \mathcal{C} \rightarrow \mathcal{C}_1$ that preserves κ -ary coproducts. It can also be shown that $F : \mathcal{C} \rightarrow \mathcal{C}_1$ sends members of \mathcal{D}' to local homeomorphisms of topological spaces and sends pullbacks of members of \mathcal{D}' to pullbacks of local homeomorphisms of topological spaces. Furthermore, $F : \mathcal{C} \rightarrow \mathcal{C}_1$ sends J' -covering morphisms in \mathcal{C} to surjective continuous maps. Hence, by [lemmas 2.4.2\(c\)](#) and [2.4.2\(e\)](#), we have a κ -ary admissible functor $F : (\mathcal{C}, \mathcal{D}', J') \rightarrow (\mathcal{C}_1, \mathcal{D}_1, J_1)$. Since $(\mathcal{C}_1, \mathcal{D}_1, J_1)$ is effective, by [theorem 2.5.7](#), we have an induced functor $F : \mathbf{Xt}(\mathcal{C}, \mathcal{D}') \rightarrow \mathcal{C}_1$ extending $F : \mathcal{C} \rightarrow \mathcal{C}_1$.

[4] — i.e. topological spaces in which every point is closed.

[5] — i.e. the topology in which every $F(m, I)$ is a closed subset of K^m .

We may think of the above as a functorial way of assigning a T_1 -space of K -rational points to every K -scheme locally of finite type that respects the geometry of schemes. In the case where K is an algebraically closed field, this can be identified with the subspace of closed points of the usual underlying topological space of a scheme.

Generalities

A.1 Presheaves

SYNOPSIS. We set up notation and terminology for working with presheaves.

A.I.1 ✖ Throughout this section, \mathcal{C} is an arbitrary category.

A.I.2 **DEFINITION.** A **presheaf** on \mathcal{C} is a contravariant functor from \mathcal{C} to **SET**. More concretely, a presheaf A on \mathcal{C} consists of the following data:

- For each object X in \mathcal{C} , a set $A(X)$.
- For each morphism $f : X \rightarrow Y$ in \mathcal{C} , a map $A(Y) \rightarrow A(X)$ sending each $a \in A(Y)$ to $a \cdot f \in A(X)$.

Moreover, these data are required to satisfy the following condition:

- For every object X in \mathcal{C} and every $a \in A(X)$, we have $a \cdot \text{id}_X = a$.
- For every composable pair $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ in \mathcal{C} and every $a \in A(Z)$, we have $(a \cdot g) \cdot f = a \cdot (g \circ f)$.

A.I.2(a) **EXAMPLE.** For each set K , we have the **constant presheaf** ΔK defined as follows:

- For each object X in \mathcal{C} , $\Delta K(X) = K$.
- For each morphism $f : X \rightarrow Y$ in \mathcal{C} and each $k \in \Delta K(Y)$, $k \cdot f = k$.

For simplicity, we write \emptyset instead of $\Delta\emptyset$ and 1 instead of $\Delta 1$.

A.I.2(b) **EXAMPLE.** For each object S in C , we have a presheaf h_S defined as follows:

- For each object X in C , $h_S(X) = C(X, S)$.
- For each morphism $f : X \rightarrow Y$ in C and each $g \in h_S(Y)$, $g \cdot f = g \circ f$.

A.I.3 DEFINITION. A **morphism of presheaves** is a natural transformation of contravariant functors. More concretely, given presheaves A and B on C , a morphism $h : A \rightarrow B$ consists of the following data:

- For each object X in C , a map $h : A(X) \rightarrow B(X)$.

Moreover, these data are required to satisfy the following condition:

- For all morphisms $f : X \rightarrow Y$ in C and all $a \in A(Y)$, we have $h(a \cdot f) = h(a) \cdot f$.

We write **PSH**(C) for the metacategory of presheaves on C .

EXAMPLE. Let X be an object in C and let $a \in A(X)$. There is a unique morphism $\varepsilon_a : h_X \rightarrow A$ such that $\varepsilon_a(\text{id}_X) = a$, namely the one defined by $\varepsilon_a(x) = a \cdot x$ for each morphism $x : T \rightarrow X$ in C . In fact, every morphism $h_X \rightarrow A$ is of this form for some $a \in A(X)$: this is the Yoneda lemma.

A.I.4 ¶ Let A be a presheaf on C .

A.I.4(a) **DEFINITION.** An **element** of A is a pair (X, a) where X is an object in C and $a \in A(X)$.

A.I.4(b) **DEFINITION.** The **category of elements** of A is the category **El**(A) defined as follows:

- The objects are the elements of A .
- The morphisms $(X, a') \rightarrow (Y, a)$ are morphisms $f : X \rightarrow Y$ in C such that $a \cdot f = a'$.^[1]
- Composition and identities are inherited from C .

[1] Strictly speaking, this is an abuse of notation, as hom-sets are supposed to be disjoint.

The **canonical projection** $P : \mathbf{El}(A) \rightarrow \mathcal{C}$ is the functor sending each object (X, a) in $\mathbf{El}(A)$ to X in \mathcal{C} and each morphism $f : (X, a') \rightarrow (Y, a)$ in $\mathbf{El}(A)$ to $f : X \rightarrow Y$ in \mathcal{C} .

The **tautological cocone** $\lambda : h_P \Rightarrow \Delta A$ is defined at the object (X, a) in $\mathbf{El}(A)$ to be the unique morphism $h_X \rightarrow A$ sending id_X to a .

EXAMPLE. Let S be an object in \mathcal{C} . The **slice category** $\mathcal{C}_{/S}$ is $\mathbf{El}(h_S)$.

PROPOSITION. Let A be a presheaf on \mathcal{C} , let $P : \mathbf{El}(A) \rightarrow \mathcal{C}$ be the canonical projection, and let $\lambda : h_P \Rightarrow \Delta A$ be the tautological cocone. Then λ is a colimiting cocone in $\mathbf{PSH}(\mathcal{C})$.

Proof. See e.g. Theorem I in [CWM, Ch. III, §7] or Proposition I in [ML–M, Ch. I, §5]. \square

A.I.5 DEFINITION. Let A be a presheaf on \mathcal{C} . A **subpresheaf** of A is a presheaf A' on \mathcal{C} that satisfies the following conditions:

- For each object X in \mathcal{C} , $A'(X) \subseteq A(X)$.
- For each morphism $f : X \rightarrow Y$ in \mathcal{C} , the following diagram commutes:

$$\begin{array}{ccc} A'(Y) & \hookrightarrow & A(Y) \\ (-)\cdot f \downarrow & & \downarrow (-)\cdot f \\ A'(X) & \hookrightarrow & A(X) \end{array}$$

i.e. for each $a \in A(Y)$, if $a \in A'(Y)$, then $a \cdot f \in A'(X)$.

We write $A' \subseteq A$ for ‘ A' is a subpresheaf of A ’.

REMARK. The set of subpresheaves of any given presheaf, partially ordered by componentwise inclusion, is a complete lattice with meet (resp. join) given by componentwise intersection (resp. union).

A.I.5(a) EXAMPLE. Let $h_0 : A_0 \rightarrow B$ and $h_1 : A_1 \rightarrow B$ be morphisms in $\mathbf{PSH}(\mathcal{C})$. The **pullback** of h_0 and h_1 is the subpresheaf $\text{Pb}(h_0, h_1) \subseteq A_0 \times A_1$ defined as follows:

$$\text{Pb}(h_0, h_1)(T) = \{(a_0, a_1) \in A_0(T) \times A_1(T) \mid h_0(a_0) = h_1(a_1)\}$$

A.I.5(b) **EXAMPLE.** Let $h_0, h_1 : A \rightarrow B$ be a parallel pair of morphisms in $\mathbf{PSH}(C)$. The **equaliser** of h_0 and h_1 is the subpresheaf $\text{Eq}(h_0, h_1) \subseteq A$ defined as follows:

$$\text{Eq}(h_0, h_1)(T) = \{a \in A(T) \mid h_0(a) = h_1(a)\}$$

A.I.5(c) **EXAMPLE.** Let $h : A \rightarrow B$ be a morphism in $\mathbf{PSH}(C)$. The **kernel relation** of h is the presheaf $\text{Kr}(h)$ defined as follows:

$$\text{Kr}(h) = \text{Pb}(h, h)$$

A.I.6 DEFINITION. Let A be a presheaf on C . A **representation** of A is an object (X, a) in $\mathbf{El}(A)$ that satisfies the following conditions:

- For every object (X', a') in $\mathbf{El}(A)$, there is a morphism $x : X' \rightarrow X$ in C such that $a \cdot x = a'$.
- Given a parallel pair $x_0, x_1 : X' \rightarrow X$ of morphisms in C , if $a \cdot x_0 = a \cdot x_1$, then $x_0 = x_1$.

A **representable presheaf** is a presheaf that admits a representation.

EXAMPLE. For each object S in C , the presheaf h_S is tautologically representable.

Recognition
principles for
representations
of presheaves

LEMMA. *Let A be a presheaf on C , let X be an object in C , and let a be an element of $A(X)$. The following are equivalent:*

- (i) (X, a) is a representation for A .
- (ii) (X, a) is a terminal object in $\mathbf{El}(A)$.
- (iii) We have an isomorphism $h_X \rightarrow A$ in $\mathbf{PSH}(C)$ given by $x \mapsto a \cdot x$.

Proof. Straightforward. ◆

A.I.7 DEFINITION. Let A be a presheaf on C . A **familial representation** of A is a subset $\Phi \subseteq \text{ob } \mathbf{El}(A)$ that satisfies the following conditions:

- For every object (X', a') in $\mathbf{El}(A)$, there is a unique element (X, a) of Φ such that there is a morphism $(X', a') \rightarrow (X, a)$ in $\mathbf{El}(A)$.

- For each $(X, a) \in \Phi$, given a parallel pair $x_0, x_1 : X' \rightarrow X$ of morphisms in \mathcal{C} , if $a \cdot x_0 = a \cdot x_1$, then $x_0 = x_1$.

A **familiably representable presheaf** is a presheaf that admits a familial representation.

LEMMA. *Let A be a presheaf on \mathcal{C} . The following are equivalent:*

- (i) *A is a familiably representable presheaf on \mathcal{C} .*
- (ii) *A is a disjoint union of a family of representable presheaves on \mathcal{C} .*

Proof. Straightforward. ◆

A.I.8 ¶ Let \mathcal{C} be a category.

A.I.8(a) DEFINITION. A **family** X of objects in \mathcal{C} is a map $X : \text{idx } X \rightarrow \text{ob } \mathcal{C}$.

A.I.8(b) DEFINITION. Let X and Y be families of objects in \mathcal{C} . A **matrix of morphisms** $X \rightarrow Y$ in \mathcal{C} is a family f (of morphisms in \mathcal{C}) that satisfies the following axioms:

- $\text{idx } f \subseteq (\text{idx } Y) \times (\text{idx } X)$.
- For each $i \in \text{idx } X$, there is a unique $j \in \text{idx } Y$ such that $(j, i) \in \text{idx } f$.
- For each $(j, i) \in \text{idx } f$, $f(j, i)$ is a morphism $X(i) \rightarrow Y(j)$ in \mathcal{C} .

A.I.8(c) DEFINITION. The **metacategory of families** of objects in \mathcal{C} is the meta-category $\mathbf{FAM}(\mathcal{C})$ defined as follows:

- The objects are the families of objects in \mathcal{C} .
- The morphisms $X \rightarrow Y$ are matrices of morphisms $X \rightarrow Y$ in \mathcal{C} .^[2]
- Composition is defined like matrix multiplication, and identities are given by the evident matrices.

REMARK. The evident projection $\text{idx} : \mathbf{FAM}(\mathcal{C}) \rightarrow \mathbf{SET}$ is a split Grothendieck fibration.

[2] Strictly speaking, this is an abuse of notation: hom-sets are supposed to be disjoint.

Properties of
the metacat-
egory of families

PROPOSITION.

(i) *There is a unique functor $\gamma : \mathcal{C} \rightarrow \mathbf{FAM}(\mathcal{C})$ with the following properties:*

- *For each object X in \mathcal{C} , we have $\text{idx } \gamma(X) = \{*\}$ and $\gamma(X)(*) = X$.*
- *For each morphism $f : X \rightarrow Y$ in \mathcal{C} , we have $\gamma(f)(*, *) = f$.*

(ii) *Moreover, $\gamma : \mathcal{C} \rightarrow \mathbf{FAM}(\mathcal{C})$ is fully faithful.*

(iii) *There is a unique functor $\Pi : \mathbf{FAM}(\mathcal{C}) \rightarrow \mathbf{PSH}(\mathcal{C})$ with the following properties:*

- *For each object X in $\mathbf{FAM}(\mathcal{C})$, we have $\Pi X = \coprod_{i \in \text{idx } X} h_{X(i)}$.*
- *For each morphism $f : X \rightarrow Y$ in $\mathbf{FAM}(\mathcal{C})$, if $(j, i) \in \text{idx } f$, then the diagram in $\mathbf{PSH}(\mathcal{C})$ shown below commutes,*

$$\begin{array}{ccc} h_{X(i)} & \longrightarrow & \Pi X \\ h_{f(j,i)} \downarrow & & \downarrow \Pi f \\ h_{Y(j)} & \longrightarrow & \Pi Y \end{array}$$

where the horizontal arrows are the evident coproduct injections.

(iv) *Moreover, $\Pi : \mathbf{FAM}(\mathcal{C}) \rightarrow \mathbf{PSH}(\mathcal{C})$ is fully faithful and essentially surjective onto the full submetacategory spanned by the familiarly representable presheaves on \mathcal{C} .*

(v) *Furthermore, $\Pi : \mathbf{FAM}(\mathcal{C}) \rightarrow \mathbf{PSH}(\mathcal{C})$ is isomorphic to the Yoneda representation induced by $\gamma : \mathcal{C} \rightarrow \mathbf{FAM}(\mathcal{C})$.*

Proof. Straightforward. (For (iv), use [lemma A.1.7.](#)) ◆

A.I.9 DEFINITION. A **discrete fibration** or **discrete cartesian fibration** is a functor $P : \mathcal{E} \rightarrow \mathcal{C}$ with the following property:

- For each object E in \mathcal{E} and each morphism $f : X \rightarrow P(E)$ in \mathcal{C} , there exist a unique object $f^* E$ and a unique morphism $\tilde{f} : f^* E \rightarrow E$ such that $P(f^* E) = X$ and $P(\tilde{f}) = f$.

Dually, a **discrete opfibration** or **discrete cocartesian fibration** is a functor $P : \mathcal{E} \rightarrow \mathcal{C}$ such that $P^{\text{op}} : \mathcal{E}^{\text{op}} \rightarrow \mathcal{C}^{\text{op}}$ is a discrete fibration.

Recognition principle for discrete fibrations

LEMMA. Let $P : \mathcal{E} \rightarrow \mathcal{C}$ be a functor. The following are equivalent:

- (i) $P : \mathcal{E} \rightarrow \mathcal{C}$ is a discrete fibration.
- (ii) The diagram below is a pullback square in **SET**:

$$\begin{array}{ccc} \text{mor } \mathcal{E} & \xrightarrow{\text{codom}} & \text{ob } \mathcal{E} \\ \text{mor } P \downarrow & & \downarrow \text{ob } P \\ \text{mor } \mathcal{C} & \xrightarrow{\text{codom}} & \text{ob } \mathcal{C} \end{array}$$

Proof. Immediate. ◆

A.I.10 ¶ Every morphism $h : A \rightarrow B$ in **PSH**(\mathcal{C}) induces an evident functor $\mathbf{El}(h) : \mathbf{El}(A) \rightarrow \mathbf{El}(B)$ making the diagram below commute,

Functoriality of the category of elements

$$\begin{array}{ccc} \mathbf{El}(A) & \xrightarrow{\mathbf{El}(h)} & \mathbf{El}(B) \\ P_A \downarrow & & \downarrow P_B \\ \mathcal{C} & \xlongequal{\quad} & \mathcal{C} \end{array}$$

where the vertical arrows are the respective canonical projections. Thus, we have a functor $\mathbf{El} : \mathbf{PSH}(\mathcal{C}) \rightarrow \mathbf{CAT}_{/\mathcal{C}}$. Moreover:

The equivalence of presheaves and discrete fibrations

PROPOSITION.

- (i) $\mathbf{El} : \mathbf{PSH}(\mathcal{C}) \rightarrow \mathbf{CAT}_{/\mathcal{C}}$ is fully faithful, and the essential image is the full subcategory spanned by the discrete fibrations with codomain \mathcal{C} .
- (ii) $\mathbf{El} : \mathbf{PSH}(\mathcal{C}) \rightarrow \mathbf{CAT}_{/\mathcal{C}}$ admits a left adjoint, namely the evident functor $|-| : \mathbf{CAT}_{/\mathcal{C}} \rightarrow \mathbf{PSH}(\mathcal{C})$ sending each object (\mathcal{E}, P) in $\mathbf{CAT}_{/\mathcal{C}}$ to the presheaf defined as follows:
 - For each object X in \mathcal{C} , $[(\mathcal{E}, P)](X)$ is the set of connected components of the comma category $(X \downarrow P)$.
 - For each morphism $f : X \rightarrow Y$ in \mathcal{C} and each object (E, g) in $(Y \downarrow P)$, $[(E, g)] \cdot f = [(E, g \circ f)]$, where $[-]$ denotes the connected component.
- (iii) $\mathbf{El} : \mathbf{PSH}(\mathcal{C}) \rightarrow \mathbf{CAT}_{/\mathcal{C}}$ admits a right adjoint, namely the evident functor $\Gamma : \mathbf{CAT}_{/\mathcal{C}} \rightarrow \mathbf{PSH}(\mathcal{C})$ sending each object (\mathcal{E}, P) in $\mathbf{CAT}_{/\mathcal{C}}$ to the presheaf defined as follows:
 - For each object X in \mathcal{C} , $\Gamma(\mathcal{E}, P)(X)$ is the set of functors $\mathcal{C}_{/X} \rightarrow \mathcal{E}$ making the evident triangle commute.

- For each morphism $f : X \rightarrow Y$ in \mathcal{C} , the action of f is precomposition (in \mathbf{CAT}) by the functor $\mathcal{C}_{/X} \rightarrow \mathcal{C}_{/Y}$ given by postcomposition (in \mathcal{C}).

Proof. Straightforward. ◆

A.I.II(a) DEFINITION. A **sieve** of \mathcal{C} is a full subcategory \mathcal{C}' such that the inclusion $\mathcal{C}' \hookrightarrow \mathcal{C}$ is a discrete fibration.

A.I.II(b) DEFINITION. A **sieve** on an object S in \mathcal{C} is a sieve of the slice category $\mathcal{C}_{/S}$.

REMARK. Explicitly, a sieve of a category \mathcal{C} is a full subcategory \mathcal{C}' such that, for every morphism $f : X \rightarrow Y$ in \mathcal{C} , if Y is in \mathcal{C}' , then X is also in \mathcal{C}' . Thus, a sieve of \mathcal{C} is essentially the same thing as a subpresheaf of the terminal presheaf 1 .

Similarly, a sieve on an object S is essentially the same thing as a subpresheaf of the representable presheaf h_S .

A.I.I2(a) DEFINITION. The sieve $\downarrow(\Phi) \subseteq \mathcal{C}$ generated by a subset $\Phi \subseteq \text{ob } \mathcal{C}$ is defined as follows:
The sieve generated by a set of objects

- For every object T in \mathcal{C} , T is in $\downarrow(\Phi)$ if and only if there is some morphism $x : T \rightarrow X$ in \mathcal{C} with $X \in \Phi$.

A.I.I2(b) DEFINITION. A **generating set of elements** of a presheaf A on \mathcal{C} is a subset $\Phi \subseteq \text{ob } \mathbf{El}(A)$ such that $\downarrow(\Phi) = \mathbf{El}(A)$.

A.I.I2(c) DEFINITION. Let κ be a regular cardinal. A presheaf A on \mathcal{C} is **κ -generable** if it admits a κ -small generating set of elements.

EXAMPLE. The **principal sieve** generated by a morphism $f : X \rightarrow Y$ in \mathcal{C} is the sieve $\downarrow\langle f \rangle$ on Y where (T, y) is in $\downarrow\langle f \rangle$ if and only if $y : T \rightarrow Y$ factors through $f : X \rightarrow Y$. By construction, $\{(X, f)\}$ is a generating set of elements of the presheaf on \mathcal{C} corresponding to $\downarrow\langle f \rangle$.

A.I.I3 DEFINITION. Let $F : C \rightarrow D$ be a functor and let \mathcal{V} be a sieve of D . The **pullback sieve** $F^*\mathcal{V}$ is the sieve of C defined as follows:

- For every object X in C , X is in $F^*\mathcal{V}$ if and only if FX is in \mathcal{V} .

EXAMPLE. Let $f : X \rightarrow Y$ be a morphism in C . There is an evident functor $\Sigma_f : C_{/X} \rightarrow C_{/Y}$ defined on objects by $(T, x) \mapsto (T, f \circ x)$. For simplicity, we write $f^*\mathcal{V}$ instead of $(\Sigma_f)^*\mathcal{V}$ in this case.

A.I.I4 ¶ Let Φ be a set of objects in C . Define a category Δ_Φ as follows:

- The objects are finite lists of elements of Φ of length ≥ 1 .
- The morphisms $(X_0, \dots, X_m) \rightarrow (Y_0, \dots, Y_n)$ are the monotone maps $\alpha : \{0, \dots, m\} \rightarrow \{0, \dots, n\}$ such that $Y_{\alpha(i)} = X_i$ for $i \in \{0, \dots, m\}$.^[3]
- Composition and identities are inherited from **SET**.

Let \mathcal{U} be the sieve of C generated by Φ . Assuming that the relevant products exist in \mathcal{U} , let $P : (\Delta_\Phi)^{\text{op}} \rightarrow \mathcal{U}$ be the functor that sends each object (X_0, \dots, X_m) in Δ_Φ to the product $X_0 \times \dots \times X_m$ in \mathcal{U} and each morphism $\alpha : (X_0, \dots, X_m) \rightarrow (Y_0, \dots, Y_n)$ in Δ_Φ to the corresponding projection $\alpha^* : Y_0 \times \dots \times Y_n \rightarrow X_0 \times \dots \times X_m$ in \mathcal{U} .

LEMMA. *The functor $P : (\Delta_\Phi)^{\text{op}} \rightarrow \mathcal{U}$ is homotopy cofinal.*

Proof. Let U be an object in \mathcal{U} . We must verify that the comma category $(U \downarrow P)$ is weakly contractible. It is clear that $(U \downarrow P)$ is inhabited. Moreover, $(U \downarrow P)$ is isomorphic to the opposite of the category of simplices of a 0-coskeletal simplicial set; but any inhabited 0-coskeletal simplicial set is a contractible Kan complex, so we are done. ■

A.I.I5 ¶ Let A be a presheaf on C .

A.I.15(a) DEFINITION. An **equivalence relation** on A is a subpresheaf $R \subseteq A \times A$ such that, for every object X in C , $R(X)$ is an equivalence relation on $A(X)$.

[3] Strictly speaking, this is an abuse of notation, as hom-sets are supposed to be disjoint.

A.I.15(b) **DEFINITION.** The **quotient** of A by an equivalence relation R is the presheaf A/R defined as follows,

$$(A/R)(X) = A(X)/R(X)$$

$$A' \cdot f = \{a \cdot f \mid a \in A'\}$$

where X is an arbitrary object in \mathcal{C} , $f : X_0 \rightarrow X_1$ is an arbitrary morphism in \mathcal{C} , and A' is an $R(X_1)$ -equivalence class in $A(X_1)$.

REMARK. A/R is indeed a well-defined presheaf on \mathcal{C} , because R is a subpresheaf of $A \times A$.

A.2 Coverages

SYNOPSIS. We define some terminology related to coverages, a variation on the notion of Grothendieck topology, and we record some basic results.

PREREQUISITES. § I.1, A.1.

A.2.1 ✖ Throughout this section, \mathcal{C} is an arbitrary category.

A.2.2 ¶ Let A be a presheaf on \mathcal{C} , let X be an object in \mathcal{C} , and let \mathcal{U} be a sieve on X .

DEFINITION. The **separation condition** (resp. **sheaf condition**) on A with respect to \mathcal{U} is the following:

- For every commutative square in **CAT** of the form below,

$$\begin{array}{ccc} \mathcal{U} & \longrightarrow & \mathbf{El}(A) \\ \downarrow & & \downarrow \\ \mathcal{C}_{/X} & \longrightarrow & \mathcal{C} \end{array}$$

where $\mathbf{El}(A) \rightarrow \mathcal{C}$ and $\mathcal{C}_{/X} \rightarrow \mathcal{C}$ are the projections, there is at most one (resp. exactly one) functor $\mathcal{C}_{/X} \rightarrow \mathbf{El}(A)$ making both evident triangles commute.

EXAMPLE. A always satisfies the sheaf condition with respect to the maximal sieve on X .

The sheaf condition as right orthogonality

LEMMA. Let S be the subpresheaf of \mathfrak{h}_X corresponding to $\mathcal{U} \subseteq \mathcal{C}_{/X}$. The following are equivalent:

- A satisfies the separation condition (resp. sheaf condition) with respect to \mathcal{U} .
- The map

$$\mathrm{Hom}_{\mathbf{PSH}(\mathcal{C})}(\mathfrak{h}_X, A) \rightarrow \mathrm{Hom}_{\mathbf{PSH}(\mathcal{C})}(S, A)$$

induced by the inclusion $S \hookrightarrow \mathfrak{h}_X$ is injective (resp. bijective).

Proof. Apply proposition A.1.10. ■

A.2.3 ¶ Let A be a presheaf on C , let X be an object in C , let \mathcal{U} and \mathcal{U}' be sieves on X , and let S and S' be the corresponding subpresheaves of h_X .

A.2.3(a)
Local character
of separation
conditions

LEMMA. Assume the following hypotheses:

- A satisfies the separation condition with respect to \mathcal{U} .
- For every object (U, x) in \mathcal{U} , A satisfies the separation condition with respect to $x^*\mathcal{U}'$.

Then A satisfies the separation condition with respect to \mathcal{U}' as well.

Proof. Let $\bar{s}_0, \bar{s}_1 : h_X \rightarrow A$ be morphisms in $\mathbf{PSH}(C)$ such that the diagram below commutes:

$$\begin{array}{ccc} S' & \hookrightarrow & h_X \\ \downarrow & & \downarrow \bar{s}_1 \\ h_X & \xrightarrow{\bar{s}_0} & A \end{array}$$

We must show that $\bar{s}_0 = \bar{s}_1$.

Let (U, x) be an object in \mathcal{U} . By definition, we have the following commutative diagram:

$$\begin{array}{ccccc} x^*S' & \hookrightarrow & h_U & & \\ \downarrow & \searrow & \searrow h_x & & \\ h_U & & S' & \hookrightarrow & h_X \\ & \searrow h_x & \downarrow & & \downarrow \bar{s}_1 \\ & & h_X & \xrightarrow{\bar{s}_0} & A \end{array}$$

Since A satisfies the separation condition with respect to $x^*\mathcal{U}'$, we deduce that $\bar{s}_0 \circ h_x = \bar{s}_1 \circ h_x$. Hence, the diagram below commutes:

$$\begin{array}{ccc} S & \hookrightarrow & h_X \\ \downarrow & & \downarrow \bar{s}_1 \\ h_X & \xrightarrow{\bar{s}_0} & A \end{array}$$

But A also satisfies the separation condition with respect to \mathcal{U} , so $\bar{s}_0 = \bar{s}_1$, as required. ■

A.2.3(b) COROLLARY. Assume the following hypotheses:

Upward-
closedness
of separation
conditions

- $\mathcal{U} \subseteq \mathcal{U}'$.
- A satisfies the separation condition with respect to \mathcal{U} .

Then A also satisfies the separation condition with respect to \mathcal{U}' .

Proof. If (U, x) is an object in \mathcal{U} , then $x^*\mathcal{U}'$ is the maximal sieve on U . Thus, the claim is a special case of lemma A.2.3(a). ■

A.2.3(c) LEMMA. Assume the following hypotheses:

Local char-
acter of sheaf
conditions

- A satisfies the sheaf condition with respect to \mathcal{U} .
- For every object (U, x) in \mathcal{U} , A satisfies the sheaf condition with respect to $x^*\mathcal{U}'$.
- For every object (U, x) in \mathcal{U}' , A satisfies the separation condition with respect to $x^*\mathcal{U}$.

Then A satisfies the sheaf condition with respect to \mathcal{U}' as well.

Proof. Let $s' : S' \rightarrow A$ be a morphism in $\mathbf{PSH}(C)$. By lemma A.2.3(a), A satisfies the separation condition with respect to \mathcal{U}' , so any extension of s' along the inclusion $S' \hookrightarrow h_X$ is unique if it exists; but it remains to be shown that such an extension exists. First, we will construct a morphism $s : S \rightarrow A$ such that the restriction to $S \cap S'$ agrees with the restriction of $s' : S' \rightarrow A$.

Let (U, x) be an object in \mathcal{U} . Since A satisfies the sheaf condition with respect to $x^*\mathcal{U}'$, there is a unique morphism $s_{(U,x)} : h_U \rightarrow A$ making the following diagram commute:

$$\begin{array}{ccccc} x^*S' & \longrightarrow & S' & \xrightarrow{s'} & A \\ \downarrow & & \downarrow & \nearrow^{s_{(U,x)}} & \\ h_U & \xrightarrow{h_x} & h_X & & \end{array}$$

Let $u : V \rightarrow U$ be a morphism in C . Then the diagram below commutes,

$$\begin{array}{ccccccc} u^*x^*S' & \longrightarrow & x^*S' & \longrightarrow & S' & \xrightarrow{s'} & A \\ \downarrow & & \downarrow & & \nearrow^{s_{(U,x)}} & & \\ h_V & \xrightarrow{h_u} & h_U & & & & \end{array}$$

and since A satisfies the separation condition with respect to $u^*x^*\mathcal{U}'$, it follows that $s_{(U,x)} \circ h_u = s_{(V,x \circ u)}$. Thus, we have a well-defined morphism $s : S \rightarrow A$ given by $s(x) = s_{(U,x)}(\text{id}_U)$ for each object (U, x) of \mathcal{U} .

Now, since A satisfies the sheaf condition with respect to \mathcal{U} , there is a (unique) morphism $\bar{s} : h_X \rightarrow A$ that extends $s : S \rightarrow A$. It remains to be shown that $\bar{s} : h_X \rightarrow A$ extends $s' : S' \rightarrow A$. Let (U, x) be an object in \mathcal{U}' . Clearly, the restriction of \bar{s} to $S \cap S'$ agrees with the restriction of s' , so the following diagram commutes,

$$\begin{array}{ccc} x^*S & \longrightarrow & S' \xrightarrow{s'} A \\ \downarrow & & \nearrow \bar{s} \\ h_U & \xrightarrow{h_x} & h_X \end{array}$$

and since A satisfies the separation condition with respect to $x^*\mathcal{U}$, it follows that $\bar{s}(x) = s'(x)$. Thus, \bar{s} is indeed an extension of s' . ■

A.2.3(d)
Upward-
closedness of
sheaf conditions

COROLLARY. Assume the following hypotheses:

- $\mathcal{U} \subseteq \mathcal{U}'$.
- A satisfies the sheaf condition with respect to \mathcal{U} .
- For every object (U, x) in \mathcal{U}' , A satisfies the separation condition with respect to $x^*\mathcal{U}$.

Then A also satisfies the sheaf condition with respect to \mathcal{U}' .

Proof. If (U, x) is an object in \mathcal{U} , then $x^*\mathcal{U}'$ is the maximal sieve on U . Thus, the claim is a special case of lemma A.2.3(c). ■

A.2.4 ¶ Let \mathcal{B} be a set of morphisms in \mathcal{C} and let X be an object in \mathcal{C} .

DEFINITION. A \mathcal{B} -sink on X is a subset Φ of $\text{ob } \mathcal{C}/_X$ with the following property:

- For every $(U, x) \in \Phi$, $x : U \rightarrow X$ is a member of \mathcal{B} .

A sink on X is a $(\text{mor } \mathcal{C})$ -sink on X .

Refinement of sinks LEMMA. Let Φ and Φ' be sinks on an object X in \mathcal{C} . The following are equivalent:

(i) $\downarrow(\Phi') \subseteq \downarrow(\Phi)$.

(ii) For every $(U, x) \in \Phi$, there is a commutative square in \mathcal{C} of the form below,

$$\begin{array}{ccc} V & \xrightarrow{u'} & U' \\ u \downarrow & & \downarrow x' \\ U & \xrightarrow{x} & X \end{array}$$

where $(U', x') \in \Phi'$.

Proof. Straightforward. ◆

A.2.5 ¶ Let X be an object in \mathcal{C} .

A.2.5(a) DEFINITION. A sieve \mathcal{U} on X is **strict-epimorphic** if it has the following property:

- For every object Y in \mathcal{C} , the representable presheaf h_Y satisfies the sheaf condition with respect to \mathcal{U} .

A.2.5(b) DEFINITION. A sieve \mathcal{U} on X is **universally strict-epimorphic** if it has the following property:

- For every object (T, x) in $\mathcal{C}/_X$, the pullback sieve $x^*\mathcal{U}$ is a strict-epimorphic sieve on T .

A.2.5(c) DEFINITION. A sink Φ on X is **strict-epimorphic** (resp. **universally strict-epimorphic**) if the sieve $\downarrow(\Phi)$ is strict-epimorphic (resp. universally strict-epimorphic).

A.2.6 ¶ Let A be a presheaf on \mathcal{C} , let X be an object in \mathcal{C} , let Φ be a sink on X , and let $\Gamma(\Phi, A)$ be the subset of $\prod_{(T,x) \in \Phi} A(T)$ consisting of the elements $(a_{(T,x)} \mid (T, x) \in \Phi)$ with the following property:

- For every commutative square in \mathcal{C} of the form below,

$$\begin{array}{ccc} U & \xrightarrow{t_1} & T_1 \\ t_0 \downarrow & & \downarrow x_1 \\ T_0 & \xrightarrow{x_0} & X \end{array}$$

if both (T_0, x_0) and (T_1, x_1) are in Φ , then $a_{(T_0, x_0)} \cdot t_0 = a_{(T_1, x_1)} \cdot t_1$.

Clearly, we have the following map:

$$\begin{aligned} \Phi^* : A(X) &\rightarrow \Gamma(\Phi, A) \\ a &\mapsto (a \cdot x \mid (T, x) \in \Phi) \end{aligned}$$

The sheaf
condition with
respect to a sink

LEMMA. *The following are equivalent:*

- (i) *A satisfies the separation condition (resp. sheaf condition) with respect to the sieve $\downarrow(\Phi)$ on X generated by Φ .*
- (ii) *The map $\Phi^* : A(X) \rightarrow \Gamma(\Phi, A)$ is injective (resp. bijective).*

Proof. Straightforward. (Compare lemma A.2.2.) ◆

A.2.7(a) DEFINITION. A **tree** on an object X in \mathcal{C} is a set Φ with the following properties:

- The elements of Φ are finite sequences (f_1, \dots, f_m) of morphisms in \mathcal{C} such that $m \geq 0$ and $\text{dom } f_i = \text{codom } f_{i+1}$ for $0 < i < m$.
- The empty sequence is in Φ .
- If $(f_1, \dots, f_m, f_{m+1}) \in \Phi$, then $(f_1, \dots, f_m) \in \Phi$.
- If $(f_1, \dots, f_m) \in \Phi$ (and $m > 0$), then $\text{codom } f_1 = X$.

A.2.7(b) DEFINITION. A **leaf** of a tree Φ on an object X in \mathcal{C} is $(f_1, \dots, f_m) \in \Phi$ with the following property:

- For every $(g_1, \dots, g_n) \in \Phi$, if $m \leq n$ and $(f_1, \dots, f_m) = (g_1, \dots, g_m)$, then $m = n$.

A.2.7(c) DEFINITION. A tree Φ on an object X in \mathcal{C} is **proper** if it satisfies the following condition:

- Every element of Φ is a prefix of some leaf of Φ .

A.2.7(d) DEFINITION. The **composite** of a tree Φ on an object X in \mathcal{C} is the sink Φ° on X defined as follows:

$$\Phi^\circ = \{ (\text{dom } f_m, f_1 \circ \dots \circ f_m) \mid (f_1, \dots, f_m) \text{ is a leaf of } \Phi \}$$

REMARK. In particular, if Φ contains only the empty sequence, then $\Phi^\circ = \{(X, \text{id}_X)\}$.

A.2.8 ¶ Following [Johnstone, 2002, §C2.1], it is convenient to introduce the following variation on the notion of Grothendieck topology.

A.2.8(a) DEFINITION. A **coverage** on \mathcal{C} consists of the following data:

- For each object X in \mathcal{C} , a set $J(X)$ of sinks on X .

These data are required to satisfy the following conditions:

- For every object X in \mathcal{C} , $\{(X, \text{id}_X)\}$ is a member of $J(X)$.
- For every morphism $f : X \rightarrow Y$ in \mathcal{C} , if $\Psi \in J(Y)$, then there is $\Phi \in J(X)$ such that $\downarrow(\Phi) \subseteq f^* \downarrow(\Psi)$.

A.2.8(b) DEFINITION. A coverage J on \mathcal{C} is **upward-closed** if it has the following property:

- For every object X in \mathcal{C} , given subsets Φ and Φ' of $\text{ob } \mathcal{C}/_X$ such that $\downarrow(\Phi') \subseteq \downarrow(\Phi)$, if $\Phi' \in J(X)$, then $\Phi \in J(X)$ as well.

A.2.8(c) DEFINITION. A coverage J on \mathcal{C} is **composition-closed** if it has the following property:

- For every object X in \mathcal{C} , given $\Phi \in J(X)$ and $\Psi_{(U,x)} \in J(U)$ for each $(U, x) \in \Phi$, we have:

$$\{(V, x \circ u) \mid (U, x) \in \Phi, (V, u) \in \Psi_{(U,x)}\} \in J(X)$$

A.2.8(d) DEFINITION. A coverage is **saturated** if it is both upward-closed and composition-closed.

A.2.8(e) EXAMPLE. The **trivial coverage** on \mathcal{C} is the coverage J where $J(X) = \{(X, \text{id}_X)\}$. This coverage is composition-closed, but it is not upward-closed in general.

A.2.8(f) EXAMPLE. The **chaotic coverage** on \mathcal{C} is the coverage J where a sink is in $J(X)$ if and only if it contains some (U, x) where $x : U \rightarrow X$ is a split epimorphism in \mathcal{C} . This coverage is saturated.

A.2.9 ※ For the remainder of this section, J is an arbitrary coverage on \mathcal{C} .

A.2.10 DEFINITION. A **J-tree** on an object X in \mathcal{C} is a tree Φ on X with the following properties:

- Φ is a proper tree on X .
- The set

$$\{(U, x) \in \text{ob } \mathcal{C}_{/X} \mid (x) \in \Phi\}$$

is either empty or a member of $J(X)$.

- For every $(f_1, \dots, f_m) \in \Phi$ (where $m > 0$), the set

$$\{(V, u) \in \text{ob } \mathcal{C}_{/\text{dom } f_m} \mid (f_1, \dots, f_m, u) \in \Phi\}$$

is either empty or a member of $J(\text{dom } f_m)$.

Recognition
principle for
composition-
closed coverages

LEMMA. *The following are equivalent:*

- (i) J is a composition-closed coverage on \mathcal{C} .
- (ii) For every object X in \mathcal{C} , the composite of every J -tree on X is a member of $J(X)$.

Proof. Straightforward. ◆

The composition-
closure of
a coverage

PROPOSITION. *For each object X in \mathcal{C} , let $\bar{J}(X)$ be the set of all sinks on X of the form Φ° for some J -tree Φ on X . Then \bar{J} is the smallest composition-closed coverage on \mathcal{C} that contains J .*

Proof. Straightforward. ◆

A.2.11(a) DEFINITION. The **canonical coverage** on \mathcal{C} is the coverage J on \mathcal{C} where $J(X)$ is the set of universally strict-epimorphic sinks on X .

A.2.11(b) DEFINITION. A **subcanonical coverage** on \mathcal{C} is a coverage J on \mathcal{C} such that every element of $J(X)$ is a universally strict-epimorphic sink on X .

REMARK. It is clear that the canonical coverage is indeed a coverage, and (by definition) it is the largest subcanonical coverage. Moreover, by lemma A.2.3(c) (and corollary A.2.3(d)), the canonical coverage is a saturated coverage.

Recognition principle for subcanonical upward-closed coverages

LEMMA. *Let J be an upward-closed coverage on C . The following are equivalent:*

- (i) *J is a subcanonical coverage on C .*
- (ii) *For every object X in C , every element of $J(X)$ is a strict-epimorphic sink on X .*

Proof. Straightforward. ◆

A.2.I2 ¶ Let B be a presheaf on C .

DEFINITION. A subpresheaf $A \subseteq B$ on C is **J-closed** if it has the following property:

- For every element (X, b) of B and every $\Phi \in J(X)$, if $b \cdot x \in A(U)$ for every $(U, x) \in \Phi$, then $b \in A(X)$.

EXAMPLE. Of course, B itself is a J-closed subpresheaf of B .

REMARK. The class of J-closed subpresheaves of B is closed under arbitrary intersections.

A.2.I3 ¶ Let $h : A \rightarrow B$ be a morphism of presheaves on C and let B' be the subpresheaf of B defined as follows:

- For every element (X, b) of B , $b \in B'(X)$ if and only if there is a J-tree Φ on X such that, for every $(T, x) \in \Phi^\circ$, there is $a \in A(T)$ such that $h(a) = b \cdot x$.

DEFINITION. The **J-closed support** of $h : A \rightarrow B$ is the subpresheaf $B' \subseteq B$ defined above.

Properties of the closed support of a morphism of presheaves

PROPOSITION.

- (i) B' as defined above is indeed a subpresheaf of B .
- (ii) B' is the smallest J -closed subpresheaf of B containing the image of $h : A \rightarrow B$.

Proof. Straightforward. (For (i), use [proposition A.2.10.](#)) ◆

A.2.14 DEFINITION. A morphism $h : A \rightarrow B$ in $\mathbf{PSH}(C)$ is **J -locally surjective** if the J -closed support of $h : A \rightarrow B$ is B itself.

Properties of local surjective morphisms

PROPOSITION.

- (i) Every epimorphism in $\mathbf{PSH}(C)$ is J -locally surjective.
- (ii) The class of J -locally surjective morphisms of presheaves on C is a class of fibrations in $\mathbf{PSH}(C)$.
- (iii) The class of J -locally surjective morphisms of presheaves on C is closed under (possibly infinitary) coproduct in $\mathbf{PSH}(C)$.
- (iv) Given morphisms $h : A \rightarrow B$ and $k : B \rightarrow C$ in $\mathbf{PSH}(C)$, if the composite $k \circ h : A \rightarrow C$ is J -locally surjective, then $k : B \rightarrow C$ is also J -locally surjective.

Proof. Straightforward. ◆

A.2.15(a) DEFINITION. A subpresheaf of a presheaf on C is **J -dense** if the inclusion is J -locally surjective.

A.2.15(b) DEFINITION. A sieve on an object X in C is **J -covering** if the corresponding subpresheaf of h_X is J -dense.

We write $\text{CSv}_J(X)$ for the set of J -covering sieves on X , partially ordered by inclusion.

A.2.15(c) DEFINITION. A sink Φ on an object X in C is **J -covering** if the sieve $\downarrow(\Phi)$ is J -covering.

A.2.15(d) DEFINITION. A morphism $x : U \rightarrow X$ in C is **J -covering** if the principal sieve $\downarrow\langle x \rangle$ is J -covering.

EXAMPLE. Assuming J is the trivial coverage on \mathcal{C} , a sink on X is J -covering if and only if it contains some (U, x) where $x : U \rightarrow X$ is a split epimorphism in \mathcal{C} .

Properties of covering sinks

PROPOSITION. For each object X in \mathcal{C} , let $\hat{J}(X)$ be the set of J -covering sinks on X . Then \hat{J} is the smallest saturated coverage on \mathcal{C} containing J .

Proof. Apply proposition A.2.I4. ■

A.2.I6 ¶ Let A be a presheaf on \mathcal{C} .

DEFINITION. A **J -local generating set of elements** of A is a set Φ of elements of A with the following property:

- For every J -closed subpresheaf $A' \subseteq A$, if Φ is contained in the set of elements of A' , then $A' = A$.

EXAMPLE. Clearly, the set of elements of A itself is a J -local generating set of elements.

Local generating sets of locally generable presheaves

LEMMA. Let κ be a regular cardinal and assume the following hypotheses:

- For every object X in \mathcal{C} , every J -covering sink on X contains a κ -small J -covering sink.
- A admits a κ -small J -local generating set of elements.

Then, for every J -local generating set Φ of elements of A , there is a κ -small subset $\Phi' \subseteq \Phi$ such that Φ' is also a J -local generating set of elements of A .

Proof. Let Θ be a κ -small J -local generating set of elements. For each $(X, a) \in \Theta$, there is a κ -small subset $\Phi'_{(X,a)} \subseteq \Phi$ such that (X, a) is contained in every J -closed subpresheaf of A containing $\Phi'_{(X,a)}$. Thus, taking $\Phi' = \bigcup_{(X,a) \in \Theta} \Phi'_{(X,a)}$, we obtain the desired κ -small J -local generating set of elements of A . ■

A.2.I7 DEFINITION. A morphism $h : A \rightarrow B$ in $\mathbf{Psh}(\mathcal{C})$ is **J -locally injective** if the relative diagonal $\Delta_h : A \rightarrow \mathbf{Kr}(h)$ is J -locally surjective.

Properties of
locally injective
morphisms

PROPOSITION.

- (i) Every monomorphism in $\mathbf{PSH}(C)$ is J -locally injective.
- (ii) The class of J -locally injective morphisms of presheaves on C is a class of separated fibrations in $\mathbf{PSH}(C)$.
- (iii) Given morphisms $h : A \rightarrow B$ and $k : B \rightarrow C$ in $\mathbf{PSH}(C)$, if the composite $k \circ h : A \rightarrow C$ is J -locally injective, then $h : A \rightarrow B$ is also J -locally injective.

Proof. (i) and (ii). Apply [proposition 1.1.7](#) to [proposition A.2.14](#).

(iii). By hypothesis, the relative diagonal $\Delta_{k \circ h} : A \rightarrow \text{Kr}(k \circ h)$ is J -locally surjective; but $\text{Kr}(h) \subseteq \text{Kr}(k \circ h)$, so by [lemma 1.1.3](#), the relative diagonal $\Delta_h : A \rightarrow \text{Kr}(h)$ is also J -locally surjective. ■

A.2.18 DEFINITION. A morphism in $\mathbf{PSH}(C)$ is **J -locally bijective** if it is both J -locally injective and J -locally surjective.

Locally injective,
locally surjective,
and locally
bijective
morphisms

LEMMA. Let $h : A \rightarrow B$ and $k : B \rightarrow C$ be morphisms in $\mathbf{PSH}(C)$.

- (i) If $h : A \rightarrow B$ is J -locally surjective and $k \circ h : A \rightarrow C$ is J -locally injective, then $k : B \rightarrow C$ is also J -locally injective.
- (ii) If $k : B \rightarrow C$ is J -locally injective and $k \circ h : A \rightarrow C$ is J -locally surjective, then $h : A \rightarrow B$ is also J -locally surjective.

Proof. (i). We have the following commutative square in $\mathbf{PSH}(C)$,

$$\begin{array}{ccc} A & \xrightarrow{\Delta_{k \circ h}} & \text{Kr}(k \circ h) \\ h \downarrow & & \downarrow \\ B & \xrightarrow{\Delta_k} & \text{Kr}(k) \end{array}$$

and by [proposition A.2.14](#), $\text{Kr}(k \circ h) \rightarrow \text{Kr}(k)$ is J -locally surjective. But $\Delta_{k \circ h} : A \rightarrow \text{Kr}(k \circ h)$ is also J -locally surjective, so it follows that $\Delta_k : B \rightarrow \text{Kr}(k)$ is J -locally surjective, as required.

(ii). We have the following pullback square in $\mathbf{PSH}(C)$,

$$\begin{array}{ccc} \text{Pb}(k \circ h, k) & \xrightarrow{q} & B \\ p \downarrow & & \downarrow k \\ A & \xrightarrow{k \circ h} & C \end{array}$$

and since $k \circ h : A \rightarrow C$ is J -locally surjective, $q : \mathbf{Pb}(k \circ h, k) \rightarrow B$ is also J -locally surjective. On the other hand, by the pullback pasting lemma, we also have a pullback square in $\mathbf{PSH}(C)$ of the form below,

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ \langle \text{id}_A, h \rangle \downarrow & & \downarrow \Delta_k \\ \mathbf{Pb}(k \circ h, k) & \longrightarrow & \mathbf{Kr}(k) \end{array}$$

and since $k : B \rightarrow C$ is J -locally injective, $\langle \text{id}_A, h \rangle : A \rightarrow \mathbf{Pb}(k \circ h, k)$ is J -locally surjective. But $h \circ p \circ \langle \text{id}_A, h \rangle = h = q \circ \langle \text{id}_A, h \rangle$, and the latter is the composite of two J -locally surjective morphisms in $\mathbf{PSH}(C)$, so $h : A \rightarrow B$ is also J -locally surjective. ■

Properties of
locally bijective
morphisms of
presheaves

PROPOSITION.

- (i) *Every isomorphism in $\mathbf{PSH}(C)$ is J -locally bijective.*
- (ii) *The class of J -locally bijective morphisms in $\mathbf{PSH}(C)$ has the 2-out-of-6 property.*

Proof. Apply propositions A.2.I4 and A.2.I7 and lemma A.2.I8. ■

A.2.I9 DEFINITION. A **J -weak pullback diagram** in $\mathbf{PSH}(C)$ is a commutative diagram in $\mathbf{PSH}(C)$ of the form below,

$$\begin{array}{ccc} P & \longrightarrow & A_1 \\ \downarrow & & \downarrow h_1 \\ A_0 & \xrightarrow{h_0} & B \end{array}$$

where the induced morphism $P \rightarrow \mathbf{Pb}(h_0, h_1)$ is J -locally surjective.

Weak pullback
pasting lemma

LEMMA. *Consider a commutative diagram in $\mathbf{PSH}(C)$ of the form below:*

$$\begin{array}{ccccc} A'' & \longrightarrow & A' & \longrightarrow & A \\ \downarrow & & \downarrow & & \downarrow \\ B'' & \longrightarrow & B' & \longrightarrow & B \end{array}$$

- (i) *If both squares are J -weak pullback diagrams in $\mathbf{PSH}(C)$, then the outer rectangle is also a J -weak pullback diagram in $\mathbf{PSH}(C)$.*

(ii) *If the right square is a pullback diagram in $\mathbf{PSH}(C)$ and the outer rectangle is a J-weak pullback diagram in $\mathbf{PSH}(C)$, then the left square is also a J-weak pullback diagram in $\mathbf{PSH}(C)$.*

Proof. Straightforward. (Use [proposition A.2.14](#) and the ordinary pullback pasting lemma.) ◆

A.3 Sheaves

SYNOPSIS. We recall some of the basic theory of sheaves on a site.

PREREQUISITES. § 1.1, A.1, A.2.

A.3.1 ✖ Throughout this section, \mathcal{C} is a category and \mathbf{J} is a coverage on \mathcal{C} .

A.3.2 DEFINITION. A presheaf A on \mathcal{C} is **J-separated** if it has the following property:

- For every object X in \mathcal{C} and every $\Phi \in \mathbf{J}(X)$, A satisfies the separation condition with respect to the sieve $\downarrow(\Phi)$ on X .

Recognition principle for separated presheaves

LEMMA. *Let A be a presheaf on \mathcal{C} . The following are equivalent:*

- (i) A is a J-separated presheaf on \mathcal{C} .
- (ii) The image of the diagonal $\Delta_A : A \rightarrow A \times A$ is a J-closed subpresheaf of $A \times A$.

Proof. In view of lemma A.2.2, this is a special case of lemma 1.1.16. ■

A.3.3 ¶ Let A be a presheaf on \mathcal{C} , let R be an equivalence relation on A , and let A/R be the quotient presheaf.

Quotients by closed equivalence relations

LEMMA. *The following are equivalent:*

- (i) R is a J-closed subpresheaf of $A \times A$.
- (ii) A/R is a J-separated presheaf on \mathcal{C} .

Proof. (i) \Rightarrow (ii). Let X be an object in \mathcal{C} , let $\bar{a}_0, \bar{a}_1 : \hat{h}_X \rightarrow A/R$ be morphisms in $\mathbf{PSH}(\mathcal{C})$, and let $q : A \rightarrow A/R$ be the quotient morphism. Since $q : A \rightarrow A/R$ is an epimorphism, we may choose $a_0, a_1 : \hat{h}_X \rightarrow A$ such that $\bar{a}_0 = q \circ a_0$ and $\bar{a}_1 = q \circ a_1$. (Here, we are using the Yoneda lemma.) Suppose \mathcal{U} is a member of $\mathbf{J}(X)$ such that we have a commutative square in $\mathbf{PSH}(\mathcal{C})$ of the form below,

$$\begin{array}{ccc} S_{\mathcal{U}} & \hookrightarrow & \hat{h}_X \\ \downarrow & & \downarrow \bar{a}_1 \\ \hat{h}_X & \xrightarrow{\bar{a}_0} & A/R \end{array}$$

where $S_{\mathcal{U}}$ is the subpresheaf of h_X corresponding to \mathcal{U} . We then have the following commutative diagram in $\mathbf{PSH}(C)$,

$$\begin{array}{ccccc} S_{\mathcal{U}} & \dashrightarrow & \text{Kr}(q) & \longrightarrow & A/R \\ \downarrow & & \downarrow & & \downarrow \Delta_{A/R} \\ h_X & \xrightarrow{\langle a_0, a_1 \rangle} & A \times A & \xrightarrow{q \times q} & (A/R) \times (A/R) \end{array}$$

where the dashed arrow exists because the right half is a pullback square. By construction, $\text{Kr}(q)$ is R , and R is J-closed in $A \times A$, so we must have $a_0 = a_1$. Hence, $\bar{a}_0 = \bar{a}_1$, as required.

(ii) \Rightarrow (i). This is a consequence of lemma A.3.2 and the fact that the preimage of a J-closed subpresheaf is also a J-closed subpresheaf. ■

REMARK. In particular, every J-closed equivalence relation on A is of the form $\text{Kr}(h)$ for some epimorphism $h : A \rightarrow B$ in $\mathbf{PSH}(C)$ where B is a J-separated presheaf on C .

A.3.4 LEMMA. *Let A be a presheaf on C , let R be a subpresheaf of $A \times A$, and let \bar{R} be the smallest J-closed subpresheaf of $A \times A$ such that $R \subseteq \bar{R}$.*

The closure of a binary relation on a presheaf

- (i) *If R is a reflexive relation on A , then \bar{R} is also a reflexive relation on A .*
- (ii) *If R is a symmetric relation on A , then \bar{R} is also a symmetric relation on A .*
- (iii) *If R is a transitive relation on A , then \bar{R} is also a transitive relation on A .*

Proof. (i) and (ii). Straightforward.

(iii). Let $R \times_A R$ and $\bar{R} \times_A \bar{R}$ be the subpresheaves of $A \times A \times A$ defined as follows:

$$\begin{aligned} R \times_A R &= (R \times A) \cap (A \times R) \\ \bar{R} \times_A \bar{R} &= (\bar{R} \times A) \cap (A \times \bar{R}) \end{aligned}$$

Clearly, $R \times_A R \subseteq \bar{R} \times_A \bar{R}$; moreover, by [proposition A.2.14](#), $R \times_A R$ is a J-dense subpresheaf of $\bar{R} \times_A \bar{R}$. Let $d_1 : \bar{R} \times_A \bar{R} \rightarrow A \times A$ be defined as follows:

$$d_1(a_0, a_1, a_2) = (a_0, a_2)$$

We then have a commutative diagram in $\mathbf{PSH}(C)$ of the form below:

$$\begin{array}{ccc} R \times_A R & \longrightarrow & \bar{R} \\ \downarrow & & \downarrow \\ \bar{R} \times_A \bar{R} & \xrightarrow{d_1} & A \times A \end{array}$$

Since $R \times_A R$ is J-dense in $\bar{R} \times_A \bar{R}$ and \bar{R} is J-closed in $A \times A$, $d_1 : \bar{R} \times_A \bar{R} \rightarrow A \times A$ factors through the inclusion $\bar{R} \hookrightarrow A \times A$. Thus, \bar{R} is indeed a transitive relation on A . ■

A.3.5 DEFINITION. A **J-sheaf** on C is a presheaf A on C with the following property:

- For every object X in C and every $\Phi \in \mathbf{J}(X)$, A satisfies the sheaf condition with respect to the sieve $\downarrow(\Phi)$ on X .

We write $\mathbf{SH}(C, \mathbf{J})$ for the full subcategory of $\mathbf{PSH}(C)$ spanned by the J-sheaves.

LEMMA. *Let A be a presheaf on C . The following are equivalent:*

- (i) A is a J-sheaf on C .
- (ii) A satisfies the sheaf condition with respect to every J-covering sieve on X .

Proof. (i) \Rightarrow (ii). Apply [lemma A.2.3\(c\)](#) and [corollary A.2.3\(b\)](#).

(ii) \Rightarrow (i). Immediate. ■

A.3.6 LEMMA. *Let $h : A \rightarrow B$ be a morphism in $\mathbf{PSH}(C)$. Assuming B is J-separated, the following are equivalent:*

Locally injective
morphisms of
presheaves
with separ-
ated codomain

- (i) $h : A \rightarrow B$ is a monomorphism in $\mathbf{PSH}(C)$.
- (ii) $h : A \rightarrow B$ is a J-locally injective morphism in $\mathbf{PSH}(C)$.

Proof. (i) \Rightarrow (ii). See [proposition A.2.17](#).

(ii) \Rightarrow (i). By [lemma A.3.2](#), the image of $\Delta_B : B \rightarrow B \times B$ is a J-closed subpresheaf of $B \times B$. It is clear that the preimage of a J-closed subpresheaf is also a J-closed subpresheaf, hence $\text{Kr}(h)$ is a J-closed subpresheaf of $A \times A$. But the relative diagonal $\Delta_h : A \rightarrow \text{Kr}(h)$ is J-locally surjective, so it is an isomorphism. Thus, $h : A \rightarrow B$ is a monomorphism in $\mathbf{PSH}(C)$. ■

A.3.7 DEFINITION. A **J-sheaf completion** of a presheaf A on C is a pair (\hat{A}, i) where \hat{A} is a J-sheaf on C and $i : A \rightarrow \hat{A}$ is a J-locally bijective morphism in $\mathbf{PSH}(C)$.

Sheaves are right PROPOSITION. *If F is a J-sheaf on C and $h : A \rightarrow B$ is a J-locally bijective orthogonal to morphism in $\mathbf{PSH}(C)$, then*

locally bijective morphisms

$$\text{Hom}_{\mathbf{PSH}(C)}(h, F) : \text{Hom}_{\mathbf{PSH}(C)}(B, F) \rightarrow \text{Hom}_{\mathbf{PSH}(C)}(A, F)$$

is a bijection.

Proof. Let $s : A \rightarrow F$ be a morphism in $\mathbf{PSH}(C)$. We must show that there is a unique morphism $\bar{s} : B \rightarrow F$ in $\mathbf{PSH}(C)$ such that $\bar{s} \circ h = s$. Since F is a J-separated presheaf on C , by [lemma A.3.2](#), $\text{Kr}(s)$ is a J-closed subpresheaf of $A \times A$. On the other hand, the relative diagonal $\Delta_h : A \rightarrow \text{Kr}(h)$ is J-locally surjective, so we have $\text{Kr}(h) \subseteq \text{Kr}(s)$. Thus, recalling [lemma A.2.18](#), we may assume without loss of generality that $h : A \rightarrow B$ is a J-locally surjective *monomorphism* in $\mathbf{PSH}(C)$.

Let (X, b) be an element of B and let \mathcal{U} be the sieve on X where (U, x) is in \mathcal{U} if and only if $b \cdot x$ is in the image of $h : A \rightarrow B$. Then, we have the following pullback square in $\mathbf{PSH}(C)$,

$$\begin{array}{ccc} S & \longrightarrow & A \\ \downarrow & & \downarrow h \\ \hat{h}_X & \xrightarrow{b \cdot -} & B \end{array}$$

where $S \subseteq \hat{h}_X$ is the subpresheaf corresponding to \mathcal{U} , so by [proposition A.2.14](#), \mathcal{U} is a J-covering sieve on X . Hence, by [lemma A.3.5](#), there

is a unique $c \in F(X)$ such that, for every (U, x) in \mathcal{U} , we have $c \cdot x = s(a)$ where a is the unique $a \in A(U)$ such that $h(a) = b \cdot x$. This defines a morphism $\bar{s} : B \rightarrow F$ extending $s : A \rightarrow F$ along $h : A \rightarrow B$, and it is straightforward to check that it is the unique such extension. ■

Sheaf completions are reflections

COROLLARY. *Let A be a presheaf on C . If (\hat{A}, i) is a J -sheaf completion of A , then (\hat{A}, i) is an initial object in the comma category $(A \downarrow \mathbf{SH}(C, J))$.*

Proof. This is an immediate consequence of [proposition A.3.7](#). ■

REMARK. In particular, J -sheaf completions are unique up to unique isomorphism.

A.3.8 ¶ Let A be a presheaf on C . For general reasons, there is a J -sheaf completion of A , but it is convenient to have a more explicit construction.

DEFINITION. The **presheaf of J -local sections** of A , or **Grothendieck plus construction** for A with respect to J , is the presheaf A^+ on C defined as follows,

$$A^+(X) = \check{H}_J^0(X, A) = \lim_{\rightarrow \mathcal{U} : \text{CS}_{V_J}(X)^{\text{op}}} \text{Hom}_{\mathbf{PSH}(C)}(S_{\mathcal{U}}, A)$$

where $S_{\mathcal{U}}$ is the subpresheaf of h_X corresponding to the sieve \mathcal{U} . The **unit** $\iota_A : A \rightarrow A^+$ is given at each object X in C by the component of the colimiting cocone corresponding to the maximal sieve on X .

REMARK. Note that the above indeed defines a presheaf on C . Moreover, [proposition A.2.14](#) implies that $\text{CS}_{V_J}(X)^{\text{op}}$ is a directed poset, so the colimit appearing in the definition is very well behaved. In addition, A^+ is clearly functorial in A .

A.3.8(a) LEMMA. *The evident endofunctor on $\mathbf{PSH}(C)$ defined by $A \mapsto A^+$ preserves limits of finite diagrams.*

Proof. Straightforward. (Use the fact that $\lim_{\rightarrow \mathcal{I}} : [\mathcal{I}, \mathbf{SET}] \rightarrow \mathbf{SET}$ preserves limits of finite diagrams when \mathcal{I} is a directed poset.) ◆

A.3.8(b) **LEMMA.** *The unit $\iota_A : A \rightarrow A^+$ is a J-locally bijective morphism in $\mathbf{PSH}(C)$.*

Proof. Let X be an object in C and let $a_0, a_1 : h_X \rightarrow A$ be morphisms in $\mathbf{PSH}(C)$. Using the Yoneda lemma and the explicit construction of filtered colimits in \mathbf{SET} , we see that $\iota_A \circ a_0 = \iota_A \circ a_1$ if and only if there is some J-covering sieve \mathcal{U} on X such that the diagram below commutes,

$$\begin{array}{ccc} S_{\mathcal{U}} & \hookrightarrow & h_X \\ \downarrow & & \downarrow a_1 \\ h_X & \xrightarrow{a_0} & A \end{array}$$

where $S_{\mathcal{U}}$ is the subsheaf of h_X corresponding to \mathcal{U} . Thus, the relative diagonal $\Delta_{\iota_A} : A \rightarrow \text{Kr}(\iota_A)$ is J-locally surjective, i.e. $\iota_A : A \rightarrow A^+$ is a J-locally injective.

On the other hand, for any morphism $b : h_X \rightarrow A^+$ in $\mathbf{PSH}(C)$, there is a J-covering sieve \mathcal{U} on X and a morphism $a : S_{\mathcal{U}} \rightarrow A$ such that the diagram below commutes:

$$\begin{array}{ccc} S_{\mathcal{U}} & \xrightarrow{a} & A \\ \downarrow & & \downarrow \iota_A \\ h_X & \xrightarrow{b} & A^+ \end{array}$$

Thus, $\iota_A : A \rightarrow A^+$ is also J-locally surjective. ■

A.3.8(c) **LEMMA.**

- (i) A^+ is a J-separated presheaf on C .
- (ii) If A is a J-separated presheaf on C , then A^+ is a J-sheaf on C .

Proof. See e.g. [ML–M, Ch. III, §5] or Proposition 2.2.6 in [Johnstone, 2002, Part C]. □

A.3.8(d) **PROPOSITION.** *Let A^{++} be the twice-iterated Grothendieck plus construction and let $\eta_A : A \rightarrow A^{++}$ be $\iota_{A^+} \circ \iota_A$. Then (A^{++}, η_A) is a J-sheaf completion of A .*

Proof. Apply proposition A.2.18 to lemmas A.3.8(b) and A.3.8(c). ■

A.3.9 ¶ To summarise:

THEOREM.

- (i) Every presheaf on C has a J -sheaf completion.
- (ii) In particular, the inclusion $\mathbf{SH}(C, J) \hookrightarrow \mathbf{PSH}(C)$ has a left adjoint.
- (iii) Moreover, any such left adjoint preserves finite limits.

Proof. (i). See [proposition A.3.8\(d\)](#).

(ii). Apply [corollary A.3.7](#).

(iii). Apply [lemma A.3.8\(a\)](#) (twice). ■

A.3.10 **LEMMA.** Let $h : A \rightarrow B$ be a morphism in $\mathbf{SH}(C, J)$. The following are equivalent:

Recognition principle for locally surjective morphisms of sheaves

- (i) $h : A \rightarrow B$ is J -locally surjective.
- (ii) $h : A \rightarrow B$ is an effective epimorphism in $\mathbf{SH}(C, J)$.
- (iii) $h : A \rightarrow B$ is an epimorphism in $\mathbf{SH}(C, J)$.

Proof. (i) \Rightarrow (ii). Let B' be the presheaf image of $h : A \rightarrow B$ and let $h' : A \rightarrow B'$ be the induced morphism. Then B' is a J -dense subpresheaf of B , so the induced morphism $(B')^{++} \rightarrow B^{++}$ is an isomorphism, by [propositions A.3.7](#) and [A.3.8\(d\)](#). On the other hand, $h' : A \rightarrow B'$ is an effective epimorphism in $\mathbf{PSH}(C)$, so $(h')^{++} : A^{++} \rightarrow (B')^{++}$ is an effective epimorphism in $\mathbf{SH}(C, J)$, by [theorem A.3.9](#). Thus, $h : A \rightarrow B$ is also an effective epimorphism in $\mathbf{SH}(C, J)$.

(ii) \Rightarrow (iii). Immediate.

(iii) \Rightarrow (i). See [Corollary 5](#) in [\[ML–M, Ch. III, §7\]](#). ■

A.3.11 **LEMMA.** Let $(A_i \mid i \in I)$ be a family of J -sheaves on C , let $A = \coprod_{i \in I} A_i$ in $\mathbf{SH}(C, J)$, and let $h_i : A_i \rightarrow A$ be the respective coproduct injection. For every element (X, a) of A , there is a J -covering sink Φ on X with the following property:

Elements of coproducts of sheaves

- For every $(U, x) \in \Phi$, there exist $i \in I$ and $a_i \in A_i(U)$ such that $h_i(a_i) = a \cdot x$.

Moreover, if J is composition-closed, then we may also assume that $\Phi \in J(X)$.

Proof. By [theorem A.3.9](#), the unit of the reflector of $\mathbf{SH}(C, J) \subseteq \mathbf{PSH}(C)$ is componentwise locally bijective. Unwinding definitions, the claim follows. (Recall [paragraph A.2.13](#).) ■

A.3.12 ¶ Let X be an object in C and let $@_X : \mathbf{SH}(C, J) \rightarrow \mathbf{SET}$ be the evaluation functor at X , i.e. the evident functor given on objects by $A \mapsto A(X)$. Assume J is a subcanonical coverage on C . By the Yoneda lemma, $@_X : \mathbf{SH}(C, J) \rightarrow \mathbf{SET}$ is a representable functor, so it preserves limits. When does it preserve colimits?

DEFINITION. An object X in C is **J-local** if every J -covering sink on X contains a split epimorphism.

LEMMA. *The following are equivalent:*

- (i) X is a J -local object in C .
- (ii) $@_X : \mathbf{SH}(C, J) \rightarrow \mathbf{SET}$ has a right adjoint, namely the evident functor $\nabla_X : \mathbf{SET} \rightarrow \mathbf{SH}(C, J)$ defined on objects as follows,

$$\nabla_X T = T^{C(X, -)}$$

with counit $@_X \nabla_X T \rightarrow T$ given by evaluation at id_X .

- (iii) $@_X : \mathbf{SH}(C, J) \rightarrow \mathbf{SET}$ preserves colimits.

Proof. (i) \Rightarrow (ii). It is straightforward to verify that we have the following adjunction,

$$\mathbf{SET} \begin{array}{c} \xleftarrow{@_X} \\ \perp \\ \xrightarrow{\nabla_X} \end{array} \mathbf{PSH}(C)$$

where the functors and the counit are defined as above. To complete the proof, it is enough to verify that $\nabla_X T$ is a J -sheaf on C for every set T . It is not hard to see that $@_X : \mathbf{PSH}(C) \rightarrow \mathbf{SET}$ sends J -locally surjective morphisms in $\mathbf{PSH}(C)$ to surjections. But $@_X : \mathbf{PSH}(C) \rightarrow \mathbf{SET}$ preserves monomorphisms, so by [lemma A.2.2](#) and adjointness, $\nabla_X T$ is indeed a J -sheaf on C .

(ii) \Rightarrow (iii). Immediate.

(iii) \Rightarrow (i). Let Φ be a covering sink on X . By lemma A.3.10, the induced morphism $\coprod_{(U,x) \in \Phi} h_U \rightarrow h_X$ in $\mathbf{SH}(C, J)$ is an (effective) epimorphism; but $@_X : \mathbf{SH}(C, J) \rightarrow \mathbf{SET}$ preserves coproducts and epimorphisms, so the induced map

$$\coprod_{(U,x) \in \Phi} C(U, X) \rightarrow C(X, X)$$

is surjective. Hence, there is $(U, x) \in \Phi$ such that $x : U \rightarrow X$ is a split epimorphism in C . ■

A.3.13 ¶ Let \mathcal{D} be a category and let K be a coverage on \mathcal{D} .

DEFINITION. A **pre-admissible functor** $F : (C, J) \rightarrow (D, K)$ is a functor $F : C \rightarrow D$ with the following property:

- For every K -sheaf B on D , $F^* B$ is an J -sheaf on C .

Pushforward of
sheaves along
pre-admissible
functors

PROPOSITION. *If $F : (C, J) \rightarrow (D, K)$ is a pre-admissible functor, then the restriction functor $F^* : \mathbf{SH}(D, K) \rightarrow \mathbf{SH}(C, J)$ has a left adjoint.*

Proof. Let A be a J -sheaf on C . By theorem A.3.9, $\mathbf{SH}(D, K)$ has colimits of all (small) diagrams, so by proposition A.1.4 (and the Yoneda lemma), there exist a K -sheaf $F_! A$ on D and a morphism $\eta_A : A \rightarrow F^* F_! A$ in $\mathbf{SH}(C, J)$ such that, for every K -sheaf B on D , the following is a bijection:

$$\begin{aligned} \mathrm{Hom}_{\mathbf{SH}(D, K)}(F_! A, B) &\rightarrow \mathrm{Hom}_{\mathbf{SH}(C, J)}(A, F^* B) \\ h &\mapsto F^* h \circ \eta_A \end{aligned}$$

Indeed, we may take $F_! A = \varinjlim_{(X,a) \in \mathrm{El}(A)} k^* h_{FX}$, where $k^* h_{FX}$ is the K -sheaf completion of the representable presheaf h_{FX} on D . ■

Pre-admissible
functors preserve
covering
morphisms

COROLLARY. *If $F : (C, J) \rightarrow (D, K)$ is a pre-admissible functor, then F sends J -covering morphisms in C to K -covering morphisms in D .*

Proof. Combine lemmas A.2.18 and A.3.10 with proposition A.3.13. ■

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