Derived Algebraic Geometry XIII: Rational and p-adic Homotopy Theory

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Contents

Introduction

The principal goal of algebraic topology is to study topological spaces X by means of invariants associated to X, such as the homology and cohomology groups $\mathrm{H}_*(X; \mathbf{Z})$ and $\mathrm{H}^*(X; \mathbf{Z})$. Ideally, we would like to devise invariants $X \mapsto F(X)$ which satisfy the following requirements:

(a) The invariant F is algebraic in nature: that is, it assigns to each space X something like a group or a vector space, which is amenable to study using methods of abstract algebra.

(b) The invariant F is powerful: that is, $F(X)$ contains a great deal of useful information about X.

There is a natural tension between these requirements: the simpler an object $F(X)$ is, the less information we should expect it to contain. Nevertheless, there are some invariants which do a good job of meeting both objectives. One example is provided by Sullivan's approach to rational homotopy theory. To every topological space X, Sullivan associates a *polynomial deRham complex* $A(X)$, which is a commutative differential graded algebra over the field $\mathbf{Q}$ of rational numbers. This is an object of a reasonably algebraic nature (albeit not quite so simple as a group or a vector space), which at the same time captures the entire rational homotopy type of X: for example, if X is a simply connected space whose homotopy groups $\{\pi_n X\}_{n \geq 2}$ are finite-dimensional vector spaces over $\mathbf{Q}$, then we can functorially recover X (up to homotopy equivalence) from $A(X)$.

As a chain complex, Sullivan's polynomial deRham complex $A(X)$ is quasi-isomorphic with the complex of singular cochains $C^*(X; \mathbf{Q})$. The advantage of $A(X)$ of $C^*(X; \mathbf{Q})$ is that $A(X)$ is equipped with a multiplication which is commutative at the level of cochains, whereas the multiplication on $C^*(X; \mathbf{Q})$ (given by the Alexander-Whitney construction) is only commutative at the level of cohomology. We can therefore think of $A(X)$ as a remedy for the failure of the multiplication on $C^*(X; \mathbf{Q})$ to be commutative at the level of cochains. This is specific to the case of rational coefficients. If p is a prime number and $\mathbf{F}_p$ is the finite field with p elements, then there is no way to replace the chain complex $C^*(X; \mathbf{F}_p)$ by a commutative differential graded algebra over $\mathbf{F}_p$, which is functorially quasi-isomorphic to $C^*(X; \mathbf{F}_p)$. However, a different remedy is available over $\mathbf{F}_p$: although the multiplication on $C^*(X; \mathbf{F}_p)$ is not commutative, it is commutative up to *coherent homotopy.* More precisely, $C^*(X; \mathbf{F}_p)$ has the structure of an $\mathbb{E}_\infty$-algebra over $\mathbf{F}_p$. Moreover, Mandell has used this observation to develop a "p-adic" counterpart of rational homotopy theory. For example, he has shown that if X is a simply connected space whose homotopy groups are finitely generated modules over $\mathbf{Z}_p$, then X can be functorially recovered from $C^*(X; \mathbf{F}_p)$, together with its $\mathbb{E}_\infty$-algebra structure (see [54]).

Our goal in this paper is to give an exposition of rational and p-adic homotopy theory from the ∞-categorical point of view, emphasizing connections with the earlier papers in this series. We will begin with the case of rational homotopy theory. Sullivan's work on the subject was preceded by Quillen, who showed that the homotopy theory of rational spaces can be described in terms of the homotopy theory of differential graded Lie algebras over $\mathbf{Q}$. This result of Quillen was the impetus for later work of many authors, relating differential graded Lie algebras to the study of deformation problems in algebraic geometry. In [49], we made this relationship explicit by constructing an equivalence of ∞-categories $\mathrm{Lie}_k \simeq \mathrm{Moduli}_k$, where k is any field of characteristic zero. Here Lie_k denotes the ∞-category of differential graded Lie algebras over k, and Moduli_k the ∞-category of formal moduli problems over k. In §1, we will apply this result (in the special case $k = \mathbf{Q}$) to recover Quillen's results. Along the way, we will discuss Sullivan's approach to rational homotopy theory and its relationship with the theory of coaffine stacks developed in [47].

In §2 we will turn our attention to the case of p-adic homotopy theory. We will say that a space X is *p-finite* if it has finitely many connected components and finitely many nonzero homotopy groups, each of which is a finite p-group (Definition 2.4.1). If X is a p-finite space, then Mandell shows that X can be recovered as the mapping space

$$\mathrm{Map}_{\mathrm{CAlg}_{\mathbf{F}_p}}(C^*(X; \mathbf{F}_p), \overline{\mathbf{F}}_p) \simeq \mathrm{Map}_{\mathrm{CAlg}_{\overline{\mathbf{F}}_p}}(C^*(X; \overline{\mathbf{F}}_p), \overline{\mathbf{F}}_p)$$

where CAlg_k denotes the ∞-category of $\mathbb{E}_\infty$-algebras over k and $\overline{\mathbf{F}}_p$ denotes the algebraic closure of $\mathbf{F}_p$. Here it is important to work over $\overline{\mathbf{F}}_p$ rather than $\mathbf{F}_p$: the functor $X \mapsto C^*(X; \mathbf{F}_p)$ is not fully faithful (even when restricted to p-finite spaces). We can explain this point as follows: if k is an arbitrary field of characteristic p, then the homotopy theory of $\mathbb{E}_\infty$-algebras over k is most naturally related not to the homotopy theory of (p-finite) spaces, but to the theory of (p-finite) spaces equipped with a continuous action of the absolute Galois group $\mathrm{Gal}(\overline{k}/k)$. More generally, we will show that if k is a commutative ring in which p is nilpotent, then there is a fully faithful embedding from $\mathcal{S}\mathrm{hv}_k^{p-\mathrm{fc}}$ into CAlg_k^{op} (Corollary 2.6.12); here $\mathcal{S}\mathrm{hv}_k^{p-\mathrm{fc}}$ denotes the ∞-category of *p-constructible* étale sheaves (of spaces) on $\operatorname{Spec} R$ (see Definition 2.4.1).

In order to apply the results of §2 to the study of an arbitrary space X, we need to study the problem of *approximating* X by p-finite spaces. For this, it is convenient to introduce the notion of a *p-profinite space*. By definition, a p-profinite space is a Pro-object in the ∞-category $\mathcal{S}^{p-\mathrm{fc}}$ of p-finite spaces. The collection of p-profinite spaces can be organized into an ∞-category $\mathcal{S}^{\mathrm{Pro}(p)} = \mathrm{Pro}(\mathcal{S}^{p-\mathrm{fc}})$. In §3 we will study the ∞-category $\mathcal{S}^{\mathrm{Pro}(p)}$, and show that it behaves in many respects like the usual ∞-category of spaces. Using Corollary 2.6.12, we will construct a fully faithful embedding

$$\mathcal{S}^{\mathrm{Pro}(p)} \to \mathrm{CAlg}_k^{op}$$

$$X \mapsto C^*(X; k),$$

where k is any separably closed field (Proposition 3.1.16). Moreover, if k is algebraically closed, we can explicitly describe the essential image of this functor (Theorem 3.5.8). We then recover some results of [54] by restricting our attention to p-profintie spaces of finite type (Corollary 3.5.15).

Notation and Terminology

We will use the language of ∞-categories freely throughout this paper. We refer the reader to [43] for a general introduction to the theory, and to [44] for a development of the theory of structured ring spectra from the ∞-categorical point of view. We will also assume that the reader is familiar with the formalism of spectral algebraic geometry developed in the earlier papers in this series. For convenience, we will adopt the following reference conventions:

(T) We will indicate references to [43] using the letter T.

(A) We will indicate references to [44] using the letter A.

(V) We will indicate references to [45] using the Roman numeral V.

(VII) We will indicate references to [46] using the Roman numeral VII.

($VIII$) We will indicate references to [47] using the Roman numeral VIII.

(IX) We will indicate references to [48] using the Roman numeral IX.

(X) We will indicate references to [49] using the Roman numeral X.

(XI) We will indicate references to [50] using the Roman numeral XI.

(XII) We will indicate references to [51] using the Roman numeral XII.

For example, Theorem T.6.1.0.6 refers to Theorem 6.1.0.6 of [43].

If $\mathcal{C}$ is an ∞-category, we let $\mathcal{C}^{\simeq}$ denote the largest Kan complex contained in $\mathcal{C}$: that is, the ∞-category obtained from $\mathcal{C}$ by discarding all non-invertible morphisms. We will say that a map of simplicial sets $f : S \to T$ is *left cofinal* if, for every right fibration $X \to T$, the induced map of simplicial sets $\mathrm{Fun}_T(T, X) \to \mathrm{Fun}_T(S, X)$ is a homotopy equivalence of Kan complexes (in [43], we referred to a map with this property as *cofinal*). We will say that f is *right cofinal* if the induced map $S^{op} \to T^{op}$ is left cofinal: that is, if f induces a homotopy equivalence $\mathrm{Fun}_T(T, X) \to \mathrm{Fun}_T(S, X)$ for every *left* fibration $X \to T$. If S

and T are ∞-categories, then f is left cofinal if and only if for every object $t \in T$, the fiber product $S \times_T T_{t/}$ is weakly contractible (Theorem T.4.1.3.1).

Throughout this paper, we let CAlg denote the ∞-category of $\mathbb{E}_\infty$-rings. If R is an $\mathbb{E}_\infty$-ring, we let $\mathrm{CAlg}_R = \mathrm{CAlg}(\mathrm{Mod}_R)$ denote the ∞-category of $\mathbb{E}_\infty$-algebras over R. We let $\mathrm{Spec}^{\text{ét}} R$ denote the affine spectral Deligne-Mumford stack associated to R. This can be identified with the pair $(\mathcal{S}\mathrm{hv}_R^{\text{ét}}, \mathcal{O})$, where $\mathcal{S}\mathrm{hv}_R^{\text{ét}} \subseteq \mathrm{Fun}(\mathrm{CAlg}_R^{\text{ét}}, \mathcal{S})$ is the full subcategory spanned by those functors which are sheaves with respect to the étale topology, and $\mathcal{O}$ is the sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{S}\mathrm{hv}_R^{\text{ét}}$ determined by the forgetful functor $\mathrm{CAlg}_R^{\text{ét}} \to \mathrm{CAlg}$.

1 Rational Homotopy Theory

Definition 1.0.1. Let Y be a simply connected space. We say that Y is *rational* if, for each $n \geq 2$, the homotopy group $\pi_n Y$ is a vector space over the field $\mathbf{Q}$ of rational numbers (since Y is simply connected, the homotopy groups $\pi_n(Y, y)$ are canonically independent of the choice of base point $y \in Y$). We let $\mathcal{S}^{\mathrm{rat}}$ denote the full subcategory of $\mathcal{S}$ spanned by the simply connected rational spaces, and $\mathcal{S}^{\mathrm{rat}}_*$ the ∞-category of pointed objects of $\mathcal{S}^{\mathrm{rat}}$ (that is, the ∞-category of pointed simply connected rational spaces).

Definition 1.0.2. Let k be a field of characteristic zero, and let Lie_k denote the ∞-category of differential graded Lie algebas over k (Definition X.2.1.14). We will say that a differential graded Lie algebra $\mathfrak{g}_*$ is *connected* if the graded Lie algebra $\mathrm{H}_*(\mathfrak{g}_*)$ is concentrated in positive degrees: that is, if the homology groups $\mathrm{H}_n(\mathfrak{g}_*)$ vanish for $n \leq 0$ (here our notation indicates passage to the homology groups of the underlying chain complex of $\mathfrak{g}_*$, rather than the Lie algebra homology of $\mathfrak{g}_*$). We let $\mathrm{Lie}_k^{\geq 1}$ denote the full subcategory of Lie_k spanned by the connected differential graded Lie algebras.

Quillen's work on rational homotopy theory establishes a close connection between rational spaces and differential graded Lie algebras. We can formulate his main result as follows:

Theorem 1.0.3 (Quillen)**.** *The ∞-category $\mathcal{S}^{\mathrm{rat}}_*$ of rational pointed spaces is equivalent to the ∞-category $\mathrm{Lie}_{\mathbf{Q}}^{\geq 1}$ of connected differential graded Lie algebras over the field $\mathbf{Q}$ of rational numbers.*

In §X.2, we studied a different interpretation of the homotopy theory of differential graded Lie algebras: for any field k of characteristic zero, there is a canonical equivalence of ∞-categories $\Psi : \mathrm{Lie}_k \to \mathrm{Moduli}_k$, where Moduli_k denotes the ∞-category of formal moduli problems over k (Theorem X.2.0.2). In this section, we will explore the relationship between this statement and Quillen's work. To this end, we will associate to every field k of characteristic zero an ∞-category $\mathrm{RType}(k)$, which we refer to as the ∞-category of *k-rational homotopy types*. By definition, $\mathrm{RType}(k)$ is a full subcategory of the ∞-category $\mathrm{Fun}(\mathrm{CAlg}_k^{\mathrm{cn}}, \mathcal{S})$. Our main results can be summarized as follows:

- (a) If $k = \mathbf{Q}$ is the field of rational numbers, then the evaluation map $X \mapsto X(\mathbf{Q})$ induces an equivalence of ∞-categories $\mathrm{RType}(\mathbf{Q}) \to \mathcal{S}^{\mathrm{rat}}$ (Theorem 1.3.6). Restricting to pointed objects, we obtain an equivalence $\mathrm{RType}(\mathbf{Q})_* \to \mathcal{S}^{\mathrm{rat}}_*$.
- (b) For any field k of characteristic zero, and k-rational homotopy type X, and any base point $\eta \in X(k)$, we can associate a formal moduli problem $X^\vee \in \mathrm{Moduli}_k$, which we call the *formal completion* of X at the point k. The construction $(X, \eta) \mapsto X^\vee$ induces a fully faithful embedding $\mathrm{RType}(k)_* \to \mathrm{Moduli}_k^{\geq 2}$, where $\mathrm{Moduli}_k^{\geq 2}$ denotes the full subcategory of Moduli_k spanned by those formal moduli problems having a 2-connective tangent complex (Theorem 1.5.3).
- (c) For every field k of characteristic zero, the equivalence $\Psi : \mathrm{Lie}_k \to \mathrm{Moduli}_k$ restricts to an equivalence $\mathrm{Lie}_k^{\geq 1} \to \mathrm{Moduli}_k^{\geq 2}$.

Taking k to be the field of rational numbers, assertions (a), (b), and (c) give a diagram of equivalences

$$\mathcal{S}^{\mathrm{rat}}_* \xleftarrow{\alpha} \mathrm{RType}(\mathbf{Q})_* \xrightarrow{\beta} \mathrm{Moduli}_{\mathbf{Q}}^{\geq 2} \xleftarrow{\gamma} \mathrm{Lie}_{\mathbf{Q}}^{\geq 1},$$

thereby giving a proof of Theorem 1.0.3.

Let us now outline the contents of this section. We will begin in §1.1 by reviewing a convergence criterion for the cohomological Eilenberg-Moore spectral sequence, which plays an important role in our study of rational and p-adic homotopy theory. In §1.2, we define the ∞-category $\mathrm{RType}(k)$ of k-rational homotopy types and establish some of its formal properties. In §1.3, we review the Sullivan model for rational homotopy theory, and use it to construct the equivalence $\mathrm{RType}(\mathbf{Q}) \to \mathcal{S}^{\mathrm{rat}}$ described in (a). The functor $\beta : \mathrm{RType}(k)_* \to \mathrm{Moduli}_k^{\geq 2}$ of (b) will be constructed in §1.5. The proof that β is an equivalence will require some basic facts about the homotopy theory of differential graded Lie algebras, which we review in §1.4.

1.1 Cohomological Eilenberg-Moore Spectral Sequences

Notation 1.1.1. Let $\mathcal{C}$ be an ∞-category. For every Kan complex X, we let $\mathcal{C}^X$ denote the ∞-category $\mathrm{Fun}(X, \mathcal{C})$ of functors from X to $\mathcal{C}$. If $f : X \to Y$ is a map of Kan complexes, then composition with f induces a functor $\mathcal{C}^Y \to \mathcal{C}^X$, which we will denote by f_*.

Assume that, for each vertex $y \in Y$, the ∞-category $\mathcal{C}$ admits colimits indexed by the Kan complex $X \times_Y Y_{/y}$. Then the functor $f^* : \mathcal{C}^Y \to \mathcal{C}^X$ admits a left adjoint, given by left Kan extension along f; we will denote this functor by $f_!$. If for each $y \in Y$ the ∞-category $\mathcal{C}$ admits limits indexed by $X \times_Y Y_{y/}$, then f^* admits a right adjoint (given by right Kan extension along f), which we will denote by f_*.

Let k be a field and let Mod_k denote the ∞-category of k-module spectra. For every Kan complex X, we let Mod_k^X denote the ∞-category $\mathrm{Fun}(X, \mathrm{Mod}_k)$. The symmetric monoidal structure on Mod_k induces a symmetric monoidal structure on Mod_k^X. We will denote the unit object of Mod_k^X by k_X (that is, $k_X \in \mathrm{Mod}_k^X$ denotes the constant functor $X \to \mathrm{Mod}_k$ taking the value k).

Remark 1.1.2. In the situation of Notation 1.1.1, the functor $f_* : \mathrm{Mod}_k^X \to \mathrm{Mod}_k^Y$ is lax symmetric monoidal. In particular, $A = f_*(k_X)$ is a commutative algebra object of Mod_k^Y, and f_* determines a functor $\mathrm{Mod}_k^X \to \mathrm{Mod}_A(\mathrm{Mod}_k^Y)$.

Notation 1.1.3. Let X be a Kan complex and let $f : X \to \Delta^0$ be the projection map. For each $\mathcal{F} \in \mathrm{Mod}_k^X$, we set

$$C_*(X; \mathcal{F}) = f_! \, \mathcal{F} \qquad C^*(X; \mathcal{F}) = f_* \, \mathcal{F}.$$

In the special case $\mathcal{F} = k_X$, we will denote $C_*(X; k_X)$ and $C^*(X; k_X)$ by $C_*(X; k)$ and $C^*(X; k)$, respectively. We regard $C^*(X; k)$ as a commutative algebra object of Mod_k.

Remark 1.1.4. If we identify Mod_k with the underlying ∞-category of the model category $\mathrm{Vect}^{\mathrm{dg}}_(k)$ of chain complexes of k-vector spaces, then $C_*(X; k)$ and $C^*(X; k)$ can be represented by the objects of $\mathrm{Vect}^{\mathrm{dg}}_(k)$ given by the usual k-valued chain and cochain complexes associated to the simplicial set X. In particular, we have canonical isomorphisms of k-vector spaces

$$\pi_n C_*(X; k) \simeq \mathrm{H}_n(X; k) \qquad \pi_n C^*(X; k) \simeq \mathrm{H}^{-n}(X; k).$$

Lemma 1.1.5. *Suppose we are given a homotopy pullback diagram of Kan complexes σ :*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g"] & Y' \arrow[d, "g'"] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

Let $\mathcal{C}$ be an ∞-category, and assume that for each $y \in Y$ the ∞-category $\mathcal{C}$ admits limits indexed by the Kan complex $X \times_Y Y_{y/}$. Then the diagram of ∞-categories

$$\begin{tikzcd} \mathcal{C}^Y \arrow[r] \arrow[d] & \mathcal{C}^X \arrow[d] \\ \mathcal{C}^{Y'} \arrow[r] & \mathcal{C}^{X'} \end{tikzcd}$$

is right adjointable.

Proof. Let $\mathcal{F} : X \to \mathcal{C}$ be a functor; we wish to show that the canonical map $g'^* f_* \mathcal{F} \to f'_* g^* \mathcal{F}$ is an equivalence in $\mathcal{C}^{Y'}$. Unwinding the definition, we must show that for each vertex $y' \in Y'$, the map

$$\varprojlim(\mathcal{F} | X \times_Y Y_{g'(y)/}) \to \varprojlim(\mathcal{F} | X' \times_{Y'} Y'_{y/})$$

is an equivalence in $\mathcal{C}$. This follows from the fact that the map $X' \times_{Y'} Y'_{y/} \to X \times_Y Y_{g'(y)/}$ is a homotopy equivalence, since we have assumed that σ is a homotopy pullback diagram. $\square$

Remark 1.1.6. Let X be a Kan complex. The stable ∞-category Mod_k^X admits an accessible t-structure $((\mathrm{Mod}_k^X)_{\geq 0}, (\mathrm{Mod}_k^X)_{\leq 0})$, where $(\mathrm{Mod}_k^X)_{\geq 0}$ is the full subcategory of Mod_k^X spanned by those functors $\mathcal{F} : X \to \mathrm{Mod}_k$ such that $\mathcal{F}(x) \in (\mathrm{Mod}_k)_{\geq 0}$ for all $x \in X$, and $(\mathrm{Mod}_k)_{\leq 0}$ is defined similarly. The heard of Mod_k^X can be identified with the category of functors from the fundamental groupoid of X to the category of vector spaces over k. If X is connected and contains a vertex x, then we can identify the heart of Mod_k^X with the category of representations (in k-vector spaces) of the fundamental group $\pi_1(X, x)$.

If $f : X \to Y$ is a map of Kan complexes, then the functor $f^* : \mathrm{Mod}_k^Y \to \mathrm{Mod}_k^X$ is t-exact. It follows that the functor $f_* : \mathrm{Mod}_k^X \to \mathrm{Mod}_k^Y$ is left t-exact and the functor $f_! : \mathrm{Mod}_k^X \to \mathrm{Mod}_k^Y$ is right t-exact.

Definition 1.1.7. Let X be a connected Kan complex containing a vertex x. We will say that an object $\mathcal{F} \in \mathrm{Mod}_k^X$ is *nilpotent* if the following conditions are satisfied:

(1) For each integer n, regard $V = \pi_n \mathcal{F}$ as a representation of the fundamental group $G = \pi_1(X, x)$. Then V has a finite filtration by subrepresentations

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_m = V$$

such that each quotient V_i / V_{i-1} is isomorphic to k (endowed with the trivial G-action).

(2) The homotopy groups $\pi_n \mathcal{F}$ vanish for $n \gg 0$.

Remark 1.1.8. In the situation of Definition 1.1.7, the condition that $\mathcal{F}$ be nilpotent does not depend on the choice of vertex $x \in X$.

Proposition 1.1.9. *Let $f : X \to Y$ be a map of Kan complexes. Assume that Y is connected and let $\mathcal{F} \in \mathrm{Mod}_k^Y$ be nilpotent. Then the canonical map*

$$\theta_{\mathcal{F}} : C^*(Y; \mathcal{F}) \otimes_{C^*(Y;k)} C^*(X; k) \to C^*(X; f^* \mathcal{F})$$

is an equivalence in Mod_k.

Proof. Let $\mathcal{C} \subseteq \mathrm{Mod}_k^Y$ be the full subcategory spanned by those objects $\mathcal{F} \in \mathrm{Mod}_k^Y$ for which $\theta_{\mathcal{F}}$ is an equivalence. It is clear that $\mathcal{C}$ is a stable subcategory of Mod_k^Y which contains k_Y.

Let $\mathcal{F} \in \mathrm{Mod}_k^Y$ be nilpotent; we wish to show that $\mathcal{F} \in \mathcal{C}$. Replacing $\mathcal{F}$ by a shift if necessary, we can assume that $\mathcal{F} \in (\mathrm{Mod}_k^Y)_{\leq 0}$. Let $K_{\mathcal{F}}$ be the cofiber of the map $\theta_{\mathcal{F}}$. We will show that $K_{\mathcal{F}} \in (\mathrm{Mod}_k)_{\leq -n}$ for every integer n, so that $K_{\mathcal{F}} \simeq 0$. To prove this, we observe that the fiber sequence

$$\tau_{\geq -n} \mathcal{F} \to \mathcal{F} \to \tau_{\leq -n-1} \mathcal{F}$$

in Mod_k^Y determines a fiber sequence

$$K_{\tau_{\geq -n} \mathcal{F}} \to K_{\mathcal{F}} \to K_{\tau_{\leq -n-1} \mathcal{F}}$$

in Mod_k. For each $n \geq 0$, the truncation $\tau_{\geq -n} \mathcal{F}$ is a successive extension of (shifted) copies of k_Y, so that $\tau_{\geq -n} \mathcal{F} \in \mathcal{C}$ and therefore $K_{\tau_{\geq -n} \mathcal{F}} \simeq 0$. It follows that $K_{\mathcal{F}} \simeq K_{\tau_{\leq -n-1} \mathcal{F}}$. We may therefore replace $\mathcal{F}$ by $\tau_{\leq -n-1} \mathcal{F}$ and thereby assume that $\pi_i \mathcal{F} \simeq 0$ for $i \geq -n$.

To prove that $K_{\mathcal{F}} \in (\mathrm{Mod}_k)_{\leq -n}$, it will suffice to show that $C^*(X; f^* \mathcal{F})$ and $C^*(Y; \mathcal{F}) \otimes_{C^*(Y;k)} C^*(X; k)$ belong to $(\mathrm{Mod}_k)_{\leq -n-1}$. In the first case, this follows from the left t-exactness of the functor $C^*(X; \bullet)$. In the second case, it follows from the left t-exactness of the functor $C^*(Y; \bullet)$ together with Corollary VIII.4.1.11 (note that $C^*(Y; k)$ is a coconnective k-algebra, since we have assumed that Y is connected). $\square$

Corollary 1.1.10. *Let k be a field, and let $f : X \to Y$ be a Kan fibration between Kan complexes. Assume that:*

(*a*) *The set of components $\pi_0 Y$ is finite.*

(*b*) *For each point $y \in Y$ and each $n \geq 0$, the vector space $V = \mathrm{H}^n(X_y; k)$ admits a finite filtration*

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_m \subseteq 0$$

by $\pi_1(Y, y)$-invariant subspaces such that each quotient V_i/V_{i-1} is a one-dimensional vector space over k endowed with the trivial action of $\pi_1(Y, y)$.

Then, for any map $g : Y' \to Y$, the canonical map

$$\theta : C^*(Y'; k) \otimes_{C^*(Y;k)} C^*(X; k) \to C^*(Y' \times_Y X; k)$$

is an equivalence.

Proof. Using (a), we can immediately reduce to the case where Y is connected. Using Proposition 1.1.9, we can identify θ with the canonical map $C^*(Y'; g^* f_* k_X) \to C^*(Y'; f'_* g'^* k_X)$, where $f' : Y' \times_Y X \to Y'$ and $g' : Y' \times_Y X \to X$ denote the projection maps. It follows from Lemma 1.1.5 that θ is an equivalence. □

Remark 1.1.11. Under the hypotheses of Corollary 1.1.10, Proposition A.7.2.1.19 gives a spectral sequence $\{E_r^{p,q}\}_{r \geq 2}$ with $E_2^{p,*} \simeq \mathrm{Tor}_p^{\mathrm{H}^*(Y;k)}(\mathrm{H}^*(X; k), \mathrm{H}^*(Y'; k))$ which converges to $\mathrm{H}^*(Y' \times_Y X; k)$. This spectral sequence is called the *cohomological Eilenberg-Moore spectral sequence.*

Corollary 1.1.12 (KünnethFormula)**.** *Let k be a field, let X and Z be Kan complexes, and assume that the cohomology groups $\mathrm{H}^n(X; k)$ are finite dimensional for $n \geq 0$. Then the canonical map*

$$C^*(X; k) \otimes_k C^*(Z; k) \to C^*(X \times Z; k)$$

is an equivalence of $\mathbb{E}_\infty$-algebras over k.

Proof. Apply Corollary 1.1.10 in the case $Y = \Delta^0$, $Y' = Z$. □

Remark 1.1.13. In the situation of Corollary 1.1.12, the spectral sequence of Remark 1.1.11 degenerates, and we obtain the usual Künnethisomorphisms

$$\mathrm{H}^n(X \times Z; k) \simeq \bigoplus_{i+j=n} \mathrm{H}^i(X; k) \otimes_k \mathrm{H}^j(Z; k).$$

For $n \geq 1$ and A an abelian group, we let $K(A, n)$ denote the associated Eilenberg-MacLane space: it is characterized up to equivalence by the requirements that $K(A, n)$ be connected and have homotopy groups

$$\pi_i K(A, n) \simeq \begin{cases} A & \text{if } i = n \\ 0 & \text{otherwise.} \end{cases}$$

When A is a field k, we have a tautological cohomology class

$$\eta_n \in \mathrm{H}^n(K(A, n); k),$$

which classifies a map $k[-n] \to C^*(K(A, n), k)$ in Mod_k and therefore a map $\mathrm{Sym}^*(k[-n]) \to C^*(K(A, n); k)$ in CAlg_k.

Proposition 1.1.14. *For each $n \geq 1$, the map*

$$\phi : \mathrm{Sym}^*(\mathbf{Q}[-n]) \to C^*(K(\mathbf{Q}, n); \mathbf{Q})$$

is an equivalence of $\mathbb{E}_\infty$-rings.

Lemma 1.1.15. *Let k be a field, let A be a coconnective $\mathbb{E}_1$-algebra over k. Let $f : M \to M'$ be a morphism in RMod_A and let $N \in \mathrm{LMod}_A$. Assume that N is nonzero, that $\pi_i M \simeq \pi_i M' \simeq \pi_i N \simeq 0$ for $i > 0$, and that the induced map $M \otimes_A N \to M' \otimes_A N$ is an equivalence. Then f is an equivalence.*

Proof. Let K be the fiber of f, and assume (for a contradiction) that $K \neq 0$. Then there exists some largest integer n such that $\pi_n K \neq 0$. Since $N \neq 0$, there exists some largest integer m such that $\pi_m N \neq 0$. Corollary VIII.4.1.11 implies that the map

$$(\pi_n)K \otimes_k (\pi_m N) \to \pi_{m+n}(K \otimes_A N) \simeq 0$$

is injective, which is a contradiction. $\square$

Proof of Proposition 1.1.14. We proceed by induction on n. Consider first the case $n = 1$. The space $K(\mathbf{Q}, 1)$ is given by the filtered limit of the diagram

$$K(\mathbf{Z}, 1) \to K(\frac{1}{2}\mathbf{Z}, 1) \to K(\frac{1}{6}\mathbf{Z}, 1) \to K(\frac{1}{24}\mathbf{Z}, 1) \to \cdots$$

Each space appearing in this diagram can be identified with the circle S^1, and each map in the diagram induces an isomorphism on rational cohomology. It follows that

$$\mathrm{H}^i(K(\mathbf{Q}, 1); \mathbf{Q}) \simeq \mathrm{H}^i(S^1; \mathbf{Q}) \simeq \begin{cases} \mathbf{Q} & \text{if } i \in \{0, 1\} \\ 0 & \text{otherwise.} \end{cases}$$

For $m \geq 0$, we can identify $\mathrm{Sym}^m(\mathbf{Q}[-1])$ with $V[-m]$, where V is the mth exterior power of $\mathbf{Q}$ (as a vector space over $\mathbf{Q}$). It follows that $\mathrm{Sym}^m(\mathbf{Q}[-1]) \simeq 0$ for $m \geq 2$, so that the map $\psi : \bigoplus_{i \leq 1} \mathrm{Sym}^i(\mathbf{Q}[-1]) \to \mathrm{Sym}^*(\mathbf{Q}[-1])$ is an equivalence. It is easy to see that $\phi \circ \psi$ is an equivalence, so that ϕ is an equivalence as well.

Now suppose that $n \geq 2$. We have a homotopy pullback diagram of spaces

$$\begin{tikzcd} K(\mathbf{Q}, n-1) \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & K(\mathbf{Q}, n). \end{tikzcd}$$

The inductive hypothesis guarantees that each cohomology group $\mathrm{H}^i(K(\mathbf{Q}, n-1); \mathbf{Q})$ is finite dimensional, so that Corollary 1.1.10 gives an equivalence

$$C^*(K(\mathbf{Q}, n-1); \mathbf{Q}) \simeq \mathbf{Q} \otimes_{C^*(K(\mathbf{Q}, n); \mathbf{Q})} \mathbf{Q}.$$

Let $V = C^*(K(\mathbf{Q}, n); \mathbf{Q}) \otimes_{\mathrm{Sym}^*(\mathbf{Q}[-n])} \mathbf{Q}$. It follows from Corollary VIII.4.1.11 that $\pi_i V \simeq 0$ for $i > 0$. Using the inductive hypothesis, we deduce that

$$C^*(K(\mathbf{Q}, n-1); \mathbf{Q}) \simeq \mathrm{Sym}^*(\mathbf{Q}[1-n]) \simeq \mathbf{Q} \otimes_{\mathrm{Sym}^*(\mathbf{Q}[-n])} \mathbf{Q} \simeq \mathbf{Q} \otimes_{C^*(K(\mathbf{Q}, n); \mathbf{Q})} V.$$

Since $K(\mathbf{Q}, n)$ is connected, $C^*(K(\mathbf{Q}, n); \mathbf{Q})$ is coconnective and Lemma 1.1.14 implies ϕ induces an equivalence $V \simeq \mathbf{Q}$. Using Lemma 1.1.14 again, we deduce that ϕ is itself an equivalence. $\square$

Corollary 1.1.16. *Let V be a finite-dimensional vector space over $\mathbf{Q}$ and let $V^\vee$ denote the dual space of V. Then for $n \geq 1$, the canonical map of $\mathbb{E}_\infty$-algebras over $\mathbf{Q}$*

$$\mathrm{Sym}^*(V^\vee[-n]) \to C^*(K(V, n); \mathbf{Q})$$

is an equivalence.

1.2 k-Rational Homotopy Types

Let k be a field of characteristic zero, which we regard as fixed throughout this section. We let CAlg_k denote the ∞-category of $\mathbb{E}_\infty$-algebras over k, $\mathrm{CAlg}_k^{\mathrm{cn}}$ the full subcategory of CAlg_k spanned by the connective $\mathbb{E}_\infty$-algebras over k, and CAlg_k^0 the full subcategory of $\mathrm{CAlg}_k^{\mathrm{cn}}$ spanned by the discrete $\mathbb{E}_\infty$-algebras over k (so that CAlg_k^0 is equivalent to the nerve of the ordinary category of commutative k-algebras).

Definition 1.2.1. Let $X : \mathrm{CAlg}_k^{\mathrm{cn}} \to \mathcal{S}$ be a functor. We will say that X is a *k-rational homotopy type* if the following conditions are satisfied:

- (a) The functor X is a left Kan extension of its restriction $X_0 = X|\,\mathrm{CAlg}_k^0$ to discrete $\mathbb{E}_\infty$-algebras.
- (b) For every $R \in \mathrm{CAlg}_k^0$, the space $X_0(R)$ is simply connected.
- (c) For every integer $n \geq 2$, there exists a vector space V over k such that the functor $R \mapsto \pi_n X_0(R)$ is given by $R \mapsto V \otimes_k R$.

We let $\mathrm{RType}(k)$ denote the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}_k^{\mathrm{cn}}, \mathcal{S})$ spanned by the k-rational homotopy types. We will refer to $\mathrm{RType}(k)$ as the *∞-category of k-rational homotopy types*.

Example 1.2.2. If V is a vector space over k, we define a functor $\underline{K}(V,n) : \mathrm{CAlg}_k^{\mathrm{cn}} \to \mathcal{S}$ by the formula

$$\underline{K}(V,n)(R) = \Omega^{\infty-n}(V \otimes_k R).$$

If $n \geq 2$, then $\underline{K}(V,n)$ is a k-rational homotopy type. If V is finite-dimensional, then $\underline{K}(V,0)$ is the functor corepresented by the discrete $\mathbb{E}_\infty$-algebra $\mathrm{Sym}^* V^\vee$.

Our goal in this section is to study the ∞-category $\mathrm{RType}(k)$ and establish some of its properties.

Remark 1.2.3. Let Vect_k denote the category of vector spaces over k, let $\mathcal{A}b$ denote the category of abelian groups, and define a functor $T : \mathrm{Vect}_k \to \mathrm{Fun}(\mathrm{CAlg}_k^0, \mathcal{A}b)$ by the formula

$$T(V)(R) = V \otimes_k R.$$

If V is finite dimensional, then T carries V to the functor represented by the group scheme $\mathrm{Spec}\, V^\vee$. In particular, T carries finite-dimensional vector spaces to compact objects of $\mathrm{Fun}(\mathrm{CAlg}_k^0, \mathcal{A}b)$. Since k has characteristic zero, the functor T is fully faithful when restricted to finite dimensional vector spaces. Since T commutes with filtered colimits, we conclude that T is fully faithful in general.

Now suppose that $X : \mathrm{CAlg}_k^{\mathrm{cn}} \to \mathcal{S}$ is a k-rational homotopy type and $n \geq 2$ an integer. Then there exists a vector space V over k such that the functor $R \mapsto X(R)$ is given by $V \otimes_k R$ for $R \in \mathrm{CAlg}_k^0$. The above argument shows that the vector space V is determined by X (up to unique isomorphism). We will emphasize this dependence by writing $V = \pi_n X$.

Definition 1.2.4. Let X be a k-rational homotopy type. We will say that X is of *finite type* if the vector spaces $\pi_n X$ are finite-dimensional for each $n \geq 2$.

Remark 1.2.5. Recall that a functor $X : \mathrm{CAlg}_k^{\mathrm{cn}} \to \mathcal{S}$ is said to be a *coaffine stack* if it is corepresentable by a coconnective $\mathbb{E}_\infty$-algebra $A \in \mathrm{CAlg}_k$. We will say that a coaffine stack X is *simply connected* if $\pi_{-1} A \simeq 0$. Using Propositions VIII.4.4.8 and VIII.4.4.6, we see that an arbitrary functor $X : \mathrm{CAlg}_k^{\mathrm{cn}} \to \mathcal{S}$ is a simply connected coaffine stack if and only if it satisfies conditions (a) and (b) of Definition 1.2.1, together with the following version of (c):

- (c') For every integer $n \geq 2$, there exists a vector space W over k such that the functor $R \mapsto \pi_n X_0(R)$ is given by $R \mapsto \mathrm{Hom}_k(W, R)$. Here X_0 denotes the restriction $X|\,\mathrm{CAlg}_k^0$.

We say that a simply connected coaffine stack X is of *finite type* if the vector spaces W appearing in (c') are finite-dimensional. The following conditions on a functor $X : \mathrm{CAlg}_k^{\mathrm{cn}} \to \mathcal{S}$ are equivalent:

(i) The functor X is a k-rational homotopy type of finite type.

(ii) The functor X is a coaffine stack which is simply connected and of finite type.

In particular, if X is a k-rational homotopy type of finite type, then X is a coaffine stack.

We now establish some basic formal properties of k-rational homotopy types.

Proposition 1.2.6. *Let k be a field of characteristic zero and let $X : \mathrm{CAlg}_k^{\mathrm{cn}} \to \mathcal{S}$ be a k-rational homotopy type. Then X commutes with sifted colimits.*

Lemma 1.2.7. *Let $A_\bullet$ be a simplicial abelian group, and suppose that the unnormalized chain complex*

$$\cdots \to A_2 \to A_1 \to A_0$$

is an acyclic resolution of some abelian group A. Then for each integer $n \geq 0$, the canonical map

$$|K(A_\bullet, n)| \to K(A, n)$$

is a homotopy equivalence.

Proof. We proceed by induction on n. If $n = 0$, then $|K(A_\bullet, n)|$ can be identified with the geometric realization $|A_\bullet|$, and its homotopy groups are computed by the chain complex

$$\cdots \to A_2 \to A_1 \to A_0.$$

Now suppose that $n > 0$, and consider the map $\theta_n : |K(A_\bullet, n)| \to K(A, n)$. Using Corollary A.5.1.3.7, we see that the map $\Omega(\theta_n) : \Omega|K(A_\bullet, n)| \to \Omega K(A, n)$ can be identified with θ_{n-1}, which is a homotopy equivalence by the inductive hypothesis. Since the spaces $|K(A_\bullet, n)|$ and $K(A, n)$ are both connected, we conclude that θ_n is a homotopy equivalence. □

Proof of Proposition 1.2.6. Let Poly_k denote the full subcategory of $\mathrm{CAlg}_k^{\mathrm{cn}}$ spanned by those k-algebras of the form $k[x_1, \ldots, x_n]$ for some $n \geq 0$. Then Poly_k is a subcategory of compact projective generators for $\mathrm{CAlg}_k^{\mathrm{cn}}$: that is, the inclusion $\mathrm{Poly}_k \hookrightarrow \mathrm{CAlg}_k^{\mathrm{cn}}$ extends to an equivalence $\mathcal{P}_\Sigma(\mathrm{Poly}_k) \simeq \mathrm{CAlg}_k^{\mathrm{cn}}$ (see Proposition A.7.1.4.20). Let $X_0 = X|\mathrm{Poly}_k$ and let $X' : \mathrm{CAlg}_k^{\mathrm{cn}} \to \mathcal{S}$ be a functor which extends X_0 and commutes with sifted colimits. Then X' is a left Kan extension of X_0, so that the identity transformation from X' to itself extends to a natural transformation $\alpha : X' \to X$. We wish to show that α is an equivalence, or equivalently that X is a left Kan extension of X_0. Since X is a left Kan extension of $X|\mathrm{CAlg}_k^0$, it will suffice to show that $X|\mathrm{CAlg}_k^0$ is a left Kan extension of X_0 (Proposition T.4.3.2.8). For this, it suffices to show that for every object $R \in \mathrm{CAlg}_k^0$, the canonical map $X'(R) \to X(R)$ is an equivalence.

We first treat the case where $R = \mathrm{Sym}^* V$ for some k-vector space V. Then $R \simeq \varinjlim \mathrm{Sym}^* V_\alpha$, where the colimit is taken over the filtered partially ordered set of all finite-dimensional subspaces $V_\alpha \subseteq V$. We then have a commutative diagram

$$\begin{tikzcd} \varinjlim X'(\mathrm{Sym}^* V_\alpha) \arrow[r] \arrow[d] & X'(R) \arrow[d] \\ \varinjlim X(\mathrm{Sym}^* V_\alpha) \arrow[r] & X(R). \end{tikzcd}$$

Since each $\mathrm{Sym}^* V_\alpha$ belongs to Poly_k, the left vertical map is a homotopy equivalence. The upper horizontal map is a homotopy equivalence because the functor X' commutes with filtered colimits. It will therefore suffice to show that the lower horizontal map is a homotopy equivalence. Since the domain and codomain of this map are both simply connected, we are reduced to proving that the map

$$\varinjlim \pi_n X(\mathrm{Sym}^* V_\alpha) \to \pi_n X(R)$$

is an isomorphism for each $n \geq 2$. This is clear, since the functor $W \mapsto (\pi_n X) \otimes_k (\mathrm{Sym}^* W)$ commutes with filtered colimits.

Now suppose that $R \in \mathrm{CAlg}^0_k$ is arbitrary. Choose a representation of R as the geometric realization of a simplicial object $R_\bullet$ of $\mathrm{CAlg}^{\mathrm{cn}}_k$, where each R_n a polynomial algebra (on a possibly infinite set of generators) over k. We then have a commutative diagram

$$\begin{tikzcd} |X'(R_\bullet)| \arrow[r] \arrow[d] & X'(R) \arrow[d] \\ |X(R_\bullet)| \arrow[r] & X(R). \end{tikzcd}$$

The upper horizontal map is a homotopy equivalence because X' commutes with sifted colimits, and the left vertical map is a homotopy equivalence by the previous step in the proof. We are therefore reduced to proving that the map $\theta : |X(R_\bullet)| \to X(R)$ is a homotopy equivalence.

To show that θ is a homotopy equivalence, it suffices to show that θ induces a homotopy equivalence $\tau_{\leq m}|X(R_\bullet)| \to \tau_{\leq m} X(R)$ for each integer $m \geq 0$. Note that $\tau_{\leq m}|X(R_\bullet)|$ is equivalent to the m-truncation of $|\tau_{\leq m} X(R_\bullet)|$. It will therefore suffice to show that each of the maps

$$\theta_m : |\tau_{\leq m} X(R_\bullet)| \to \tau_{\leq m} X(R)$$

is a homotopy equivalence. We proceed by induction on m, the case $m \leq 1$ being trivial. We have a simplicial diagram of fiber sequences

$$\tau_{\leq m} X(R_\bullet) \to \tau_{\leq m-1} X(R_\bullet) \to K((\pi_m X) \otimes_k R_\bullet, m+1).$$

Since each of the spaces $K((\pi_m X) \otimes_k R_\bullet, m+1)$ is connected, Lemma A.5.3.6.17 gives a fiber sequence of geometric realizations

$$|\tau_{\leq m} X(R_\bullet)| \to |\tau_{\leq m-1} X(R_\bullet)| \to |K((\pi_m X) \otimes_k R_\bullet, m+1)|.$$

Consequently, to show that θ_m is an equivalence, it will suffice to show that the vertical maps in the diagram

$$\begin{tikzcd} |\tau_{\leq m-1} X(R_\bullet)| \arrow[r] \arrow[d] & |K((\pi_m X) \otimes -k R_\bullet, m+1)| \arrow[d] \\ \tau_{\leq m-1} X(R) \arrow[r] & K((\pi_m X) \otimes_k R_\bullet, m+1) \end{tikzcd}$$

are homotopy equivalences. For the left vertical map this follows from the inductive hypothesis, and for the right vertical map it follows from Lemma 1.2.7. $\square$

Corollary 1.2.8. *Let X be a k-rational homotopy type. For each $R \in \mathrm{CAlg}^{\mathrm{cn}}_k$, the space $X(R)$ is simply connected.*

Proof. Write R as a geometric realization $|R_\bullet|$, where each R_n is discrete. Proposition 1.2.6 implies that $X(R) \simeq |X(R_\bullet)|$. Since each of the spaces $X(R_n)$ is simply connected, we conclude that $X(R)$ is simply connected. $\square$

Corollary 1.2.9. *Let X be k-rational homotopy type, and suppose we are given a pullback diagram σ :*

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & A \arrow[d] \\ B' \arrow[r] & B \end{tikzcd}$$

in $\mathrm{CAlg}_k^{\mathrm{cn}}$. *Suppose that the right vertical map induces a surjection* $\pi_0 A \to \pi_0 B$. *Then the diagram* $X(\sigma)$

$$\begin{tikzcd} X(A') \arrow[r] \arrow[d] & X(A) \arrow[d] \\ X(B') \arrow[r] & X(B) \end{tikzcd}$$

is also a pullback square.

Proof. Choose a base point $\eta \in X(k)$. Using η, we can lift X to a functor $\mathrm{CAlg}_k^{\mathrm{cn}} \to \mathcal{S}_*$ taking values in the ∞-category of pointed spaces. Choose a simplicial object $B_\bullet$ of $\mathrm{CAlg}_k^{\mathrm{cn}}$ whose geometric realization is equivalent to B, where each B_n is discrete. Set

$$A_\bullet = A \times_B B_\bullet \qquad B'_\bullet = B' \times_B B_\bullet \qquad A'_\bullet = A' \times_B B_\bullet.$$

Then σ can be obtained as the geometric realization of a simplicial diagram $\sigma_\bullet$:

$$\begin{tikzcd} A'_\bullet \arrow[r] \arrow[d] & A_\bullet \arrow[d] \\ B'_\bullet \arrow[r] & B_\bullet. \end{tikzcd}$$

Proposition 1.2.6 implies that X commutes with sifted colimits. It follows that $X(\sigma)$ can be identified with the geometric realization $|X(\sigma_\bullet)|$. We wish to show that $|X(\sigma_\bullet)|$ is a pullback diagram. Each of the spaces $X(B_\bullet)$ is connected (and equipped with a base point determined by our chosen point $\eta \in X(k)$). Applying Lemma A.5.3.6.17, we are reduced to proving that each of the diagrams $X(\sigma_n)$ is a pullback square. We may therefore replace B by B_n and thereby reduce to the case where B is discrete. Using a similar argument, we may suppose that A and B' are also discrete (so that $A' \simeq A \times_B B'$ is likewise discrete). Let Y denote the fiber product $X(A) \times_{X(B)} X(B')$. The homotopy groups of Y fit into a long exact sequence

$$\pi_{n+1} X(A) \times \pi_{n+1} X(B') \to \pi_{n+1} X(B) \to \pi_n Y \to \pi_n X(A) \times \pi_n X(B') \to \pi_n X(B).$$

Since X is a k-rational homotopy type, we can rewrite this sequence as

$$(\pi_{n+1} X) \otimes_k (A \oplus B') \xrightarrow{\alpha} (\pi_{n+1} X) \otimes_k B \to \pi_n Y \to (\pi_n X) \otimes_k (A \oplus B') \xrightarrow{\beta} (\pi_n X) \otimes_k B.$$

Since the map $A \to B$ is surjective, α is a surjection for each $n \geq 1$. It follows that Y is simply connected and that for each $n \geq 2$ we have a canonical isormophism $\pi_n Y \simeq \ker(\beta) \simeq (\pi_n X) \otimes_k A'$. From this, we immediately deduce that the map $X(A') \to Y$ is a homotopy equivalence. $\square$

Corollary 1.2.10. *Let* X *be a* k*-rational homotopy type and let* $f : A \to B$ *be a morphism in* $\mathrm{CAlg}_k^{\mathrm{cn}}$. *If* f *is* n*-connective for some* $n \geq 0$, *then the induced map* $X(A) \to X(B)$ *is* $(n+2)$*-connective.*

Proof. We proceed by induction on n. If $n > 0$, then the diagonal map $\delta : A \to A \times_B A$ is $(n-1)$-connective. The inductive hypothesis then implies that $X(\delta) : X(A) \to X(A \times_B A)$ is $(n+1)$-connective. Using Corollary 1.2.9 we can identify $X(\delta)$ with the diagonal map $X(A) \to X(A) \times_{X(B)} X(A)$. Since $X(f) : X(A) \to X(B)$ is surjective on connected components (both spaces are simply connected, by Corollary 1.2.8), the $(n+1)$-connectivity of $X(A) \to X(A) \times_{X(B)} X(A)$ implies that $(n+2)$-connectivity of $X(f)$.

It remains to treat the case $n = 0$. Since $X(A)$ and $X(B)$ are simply connected, it suffices to show that the map $X(f)$ induces a surjection $\pi_2 X(A) \to \pi_2 X(B)$. Choose an equivalence $B \simeq |B_\bullet|$ for some simplicial object $B_\bullet$ of $\mathrm{CAlg}_k^{\mathrm{cn}}$, where each B_n is a polynomial algebra over k (possibly on an infinite set of generators). Proposition 1.2.6 implies that $X(B)$ is equivalent to the geometric realization of the simplicial space $X(B_\bullet)$, where each $X(B_n)$ is simply connected. It follows that the map $\pi_2 X(B_0) \to \pi_2 X(B)$ is surjective. Since f is connective, the map $B_0 \to B$ factors through A, so that the map $\pi_2 X(A) \to \pi_2 X(B)$ is also surjective. $\square$

Corollary 1.2.11. *Let X be a k-rational homotopy type. For every $A \in \mathrm{CAlg}_k^{\mathrm{cn}}$, the canonical map $X(A) \to \varprojlim X(\tau_{\leq n} A)$ is a homotopy equivalence.*

We now turn our attention to the compactness properties of k-rational homotopy types of finite type.

Lemma 1.2.12. *Let X be a k-rational homotopy type of finite type. Then, for each $n \geq 0$, the functor*

$$V \mapsto \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}_k^{\mathrm{cn}}, \mathcal{S})}(X, \underline{K}(V, n))$$

commutes with filtered colimits,

Proof. Since X is a coaffine stack (Remark 1.2.5), Proposition VIII.4.4.4 implies that X is given by the geometric realization of a simplicial object $X_\bullet$ of $\mathrm{Fun}(\mathrm{CAlg}_k^{conn}, \mathcal{S})$, where each X_n is the set-valued functor corepresented by some discrete k-algebra R^n. Since $\underline{K}(V, n)$ is n-truncated, the mapping space

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}_k^0, \mathcal{S})}(X, \underline{K}(V, n))$$

is equivalent to the finite limit

$$\varprojlim_{[m] \in \mathbf{\Delta}_{\leq n+1}} \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}_k^{\mathrm{cn}}, \mathcal{S})}(X_m, \underline{K}(V, n)).$$

It will therefore suffice to show that each of the functors

$$V \mapsto \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}_k^{\mathrm{cn}}, \mathcal{S})}(X_m, \underline{K}(V, n)) \simeq K(R^m \otimes_k V, n)$$

commutes with filtered colimits, which is clear. □

Definition 1.2.13. Let X be a k-rational homotopy type and let $n \geq 0$ be an integer. We will say that X is *n-truncated* if, for each $R \in \mathrm{CAlg}_k^0$, the space $X(R)$ is n-truncated. We let $\mathrm{RType}(k)_{\leq n}$ denote the full subcategory of $\mathrm{RType}(k)$ spanned by the the n-truncated k-algebraic homotopy types.

Remark 1.2.14. The inclusion functor $\mathrm{RType}(k)_{\leq n} \hookrightarrow \mathrm{RType}(k)$ admits a left adjoint $X \mapsto \tau_{\leq n} X$, where $\tau_{\leq n} X$ denotes a left Kan extension of the functor $\mathrm{CAlg}_k^0 \to \mathcal{S}$ given by $R \mapsto \tau_{\leq n} X(R)$.

Remark 1.2.15. Let X be an n-truncated k-rational homotopy type and let $V = \pi_n X$. To each $R \in \mathrm{CAlg}_k^0$ we can associate a fiber sequence

$$X(R) \to \tau_{\leq n-1} X(R) \to K(V \otimes_k R, n+1),$$

depending functorially on R. Taking left Kan extensions, we obtain a commutative diagram of k-rational homotopy types σ

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \tau_{\leq n-1} X \arrow[d] \\ * \arrow[r] & \underline{K}(V, n+1). \end{tikzcd}$$

Let $\mathcal{C} \subseteq \mathrm{CAlg}_k^{\mathrm{cn}}$ denote the full subcategory spanned by those connective $\mathbb{E}_\infty$-algebras A for which the diagram

$$\begin{tikzcd} X(A) \arrow[r] \arrow[d] & (\tau_{\leq n-1} X)(A) \arrow[d] \\ * \arrow[r] & \underline{K}(V, n+1)(A). \end{tikzcd}$$

is a pullback square. Using Lemma A.5.3.6.17, we see that $\mathcal{C}$ is closed under sifted colimits. Since $\mathcal{C}$ contains CAlg_k^0, it follows that $\mathcal{C} = \mathrm{CAlg}_k^{\mathrm{cn}}$. That is, we have a fiber sequence of functors

$$X \to \tau_{\leq n-1} X \to \underline{K}(V, n+1).$$

Lemma 1.2.16. *Let k be a field of characteristic zero. Then the ∞-category* $\mathrm{RType}(k)$ *is closed under filtered colimits in* $\mathrm{Fun}(\mathrm{CAlg}_k^{\mathrm{cn}}, \mathcal{S})$*. Moreover, for each $n \geq 2$, the functor $X \mapsto \pi_n X$ commutes with filtered colimits. In particular, of each of the subcategories* $\mathrm{RType}(k)_{\leq n}$ *is also closed under filtered colimits.*

Proof. Let $\{X_\alpha\}$ be a filtered diagram of k-rational homotopy types, and let X denote its colimit in $\mathrm{Fun}(\mathrm{CAlg}_k^{\mathrm{cn}}, \mathcal{S})$. It is clear that X satisfies conditions (a) and (b) of Definition 1.2.1. Moreover, for $R \in \mathrm{CAlg}_k^0$, we have a canonical isomorphism

$$\pi_n X(R) \simeq \varinjlim \pi_n(X_\alpha(R)) \simeq (\pi_n X_\alpha) \otimes_k R \simeq V \otimes_k R,$$

where V denotes the direct limit $\varinjlim \pi_n X_\alpha$ (see Remark 1.2.3). It follows that X is a k-rational homotopy type and that the canonical map

$$\varinjlim \pi_n X_\alpha = V \to \pi_n X$$

is an isomorphism of k-vector spaces. □

Lemma 1.2.17. *Let k be a field of characteristic zero, let $n \geq 0$, and let X be an n-truncated k-algebraic homotopy type. If X is of finite type, then X is a compact object of* $\mathrm{RType}(k)_{\leq n}$*.*

Proof. Let $\theta : \mathrm{RType}(k) \to \mathcal{S}$ denote the functor corepresented by X. We will prove that for each integer m, the restriction $\theta | \mathrm{RType}(k)_{\leq m}$ commutes with filtered colimits. We proceed by induction on m, the case $m \leq 1$ being trivial. To carry out the inductive step, suppose we are given a filtered diagram $\{Y_\alpha\}$ in $\mathrm{RType}(k)_{\leq m}$ having colimit Y, and set $V_\alpha = \pi_n Y_\alpha$. We then have a fiber sequence of functors

$$Y_\alpha \to \tau_{\leq n-1} Y_\alpha \to \underline{K}(V_\alpha, n+1)$$

depending functorially on α. Set $V = \varinjlim V_\alpha$ so that $V \simeq \pi_n Y$. We then have another fiber sequence $Y \to \tau_{\leq n-1} Y \to \underline{K}(V, n+1)$. Since θ commutes with limits, we have a map of fiber sequences

$$\begin{array}{ccccc}
\varinjlim \theta(Y_\alpha) & \longrightarrow & \varinjlim \theta(\tau_{\leq n-1} Y_\alpha) & \longrightarrow & \varinjlim \theta(\underline{K}(V_\alpha, n+1)) \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\
\theta(Y) & \longrightarrow & \theta(\tau_{\leq n-1} Y) & \longrightarrow & \theta(\underline{K}(V, n+1)).
\end{array}$$

The map β is a homotopy equivalence by the inductive hypothesis, and γ is a homotopy equivalence by Lemma 1.2.12. It follows that α is a homotopy equivalence, as desired. □

For each integer $n \geq 0$, we let $\mathrm{RType}(k)^{\mathrm{ft}}_{\leq n}$ denote the intersection $\mathrm{RType}(k)^{\mathrm{ft}} \cap \mathrm{RType}(k)_{\leq n}$: that is, the ∞-category of k-rational homotopy types whcih are n-truncated and of finite type.

Proposition 1.2.18. *Let k be a field of characteristic zero. For each integer n, the inclusion* $\mathrm{RType}(k)^{\mathrm{ft}}_{\leq n} \hookrightarrow \mathrm{RType}(k)_{\leq n}$ *induces an equivalence of ∞-categories* $\theta : \mathrm{Ind}(\mathrm{RType}(k)^{\mathrm{ft}}_{\leq n}) \simeq \mathrm{RType}(k)_{\leq n}$*.*

Remark 1.2.19. Since $\mathrm{RType}(k)^{\mathrm{ft}}_{\leq n}$ is closed under retracts in $\mathrm{RType}(k)_{\leq n}$, it follows from Proposition 1.2.18 and Lemma 1.2.17 that an object $X \in \mathrm{RType}(k)_{\leq n}$ is compact if and only if it is finite type.

Remark 1.2.20. One can show that the ∞-category $\mathrm{RType}(k)$ itself is compactly generated, but we will not need this.

Proof. Using Lemma 1.2.17 and Proposition T.5.1.3.1, we deduce that θ is fully faithful. Let $\mathcal{C} \subseteq \mathrm{RType}(k)_{\leq n}$ denote the essential image of θ. We wish to show that $\mathcal{C}$ contains every n-truncated k-algebraic homotopy type X. The proof proceeds by induction on n, the case $n \leq 1$ being trivial. Set $V = \pi_n X$ so that we have a fiber sequence of functors

$$X \to \tau_{\leq n-1} X \to \underline{K}(V, n+1).$$

Using the inductive hypothesis, we can write $\tau_{\leq n-1} X$ as a filtered colimit $\varinjlim Y_\alpha$, where each Y_α is $(n-1)$-truncated and of finite type. For each α, let X_α denote the fiber of the composite map

$$Y_\alpha \to \tau_{\leq n-1} X \to \underline{K}(V, n+1).$$

Then $X \simeq \varinjlim X_\alpha$. Since $\mathcal{C}$ is closed under filtered colimits, it will suffice to prove that each X_α belongs to $\mathcal{C}$. Using Lemma 1.2.12, we see that there exists a finite-dimensional subspace $W \subseteq V$ such that the map $Y_\alpha \to \underline{K}(V, n+1)$ factors through $\underline{K}(W, n+1)$. Write $V \simeq W \oplus W'$, so that we have an equivalence of functors

$$X_\alpha \simeq Z_\alpha \times \underline{K}(W', n)$$

where Z_α denotes the fiber of the map $Y_\alpha \to \underline{K}(W, n+1)$. Then Z_α is of finite type. Since $\mathcal{C}$ is closed under products, we are reduced to proving that $\underline{K}(W', n)$ belongs to $\mathcal{C}$. This is clear, since $\underline{K}(W', n) \simeq \varinjlim \underline{K}(W'_\beta, n)$, where W'_β ranges over all finite-dimensional subspaces of W'. ☐

Proposition 1.2.21. *Let k be a field of characteristic zero. Then the canonical map from* $\mathrm{RType}(k)$ *to the homotopy limit of the tower*

$$\cdots \to \mathrm{RType}(k)_{\leq 3} \xrightarrow{\tau_{\leq 2}} \mathrm{RType}(k)_{\leq 2} \to \mathrm{RType}(k)_{\leq 1} \simeq *$$

is an equivalence.

Proof. Let $\mathrm{RType}'(k)$ denote the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}^0_k, \mathcal{S})$ spanned by those functors which satisfy conditions (b) and (c) of Definition 1.2.1, and for each integer $n \geq 0$ define $\mathrm{RType}'(k)_{\leq n}$ similarly. Using Proposition T.4.3.2.15, we see that the restriction maps

$$\mathrm{RType}(k) \to \mathrm{RType}'(k) \qquad \mathrm{RType}(k)_{\leq n} \to \mathrm{RType}'(k)_{\leq n}$$

are equivalences. It will therefore suffice to show that $\mathrm{RType}'(k)$ is equivalent to the homotopy inverse limit of the tower of ∞-categories

$$\cdots \to \mathrm{RType}'(k)_{\leq 3} \xrightarrow{\tau_{\leq 2}} \mathrm{RType}'(k)_{\leq 2} \to \mathrm{RType}'(k)_{\leq 1} \simeq *.$$

This follows from the observation that $\mathrm{Fun}(\mathrm{CAlg}^0_k, \mathcal{S})$ is given by the homotopy inverse limit of the tower $\{\mathrm{Fun}(\mathrm{CAlg}^0_k, \tau_{\leq n} \mathcal{S})$. ☐

1.3 Rational Homotopy Theory and $\mathbb{E}_\infty$-Algebras

Let Y be a simply connected rational space. We will say that X is of *finite type* if the homotopy group $\pi_n X$ is a finite-dimensional vector space over $\mathbf{Q}$ for each $n \geq 2$. Sullivan has shown that the homotopy theory of rational spaces of finite type can be described algebraically, using commutative differential graded algebras over $\mathbf{Q}$. His main result can be formulated as follows:

Theorem 1.3.1 (Sullivan)**.** *Let $\mathbf{Q}$ be the field of rational numbers. Let $\mathcal{S}^{\mathrm{rat}}_{\mathrm{ft}}$ denote the full subcategory of $\mathcal{S}$ spanned by those spaces X which are simply connected and such that each homotopy group $\pi_i X$ is a finite dimensional vector space over $\mathbf{Q}$. Then the construction $X \mapsto C^*(X; \mathbf{Q})$ determines a fully faithful embedding $\mathcal{S}^{\mathrm{rat}}_{\mathrm{ft}} \to \mathrm{CAlg}_{\mathbf{Q}}$, whose essential image is the collection of those $\mathbf{Q}$-algebras A satisfying the following conditions:*

(a) The $\mathbf{Q}$-vector spaces $\pi_i A$ are finite dimensional for all i.

(b) We have $\pi_i A \simeq \begin{cases} 0 & \text{if } i > 0 \\ \mathbf{Q} & \text{if } i = 0 \\ 0 & \text{if } i = -1. \end{cases}$

In this section, we will review the proof of Theorem 1.3.1, and deduce from it an analogous description of the *entire* ∞-category $\mathcal{S}^{\mathrm{rat}}$ of rational spaces (Theorem 1.3.6).

Remark 1.3.2. Let k be a field. The construction $X \mapsto C^*(X;k)$ determines a functor from $\mathcal{S}$ to CAlg_k^{op}. This functor admits a right adjoint, given by the formula $A \mapsto \mathrm{Map}_{\mathrm{CAlg}_k}(A,k)$. In particular, for every space X we have a canonical unit map $X \to \mathrm{Map}_{\mathrm{CAlg}_k}(C^*(X;k),k)$, which carries a vertex $x \in X$ to the evaluation map $C^*(X;k) \to C^*(\{x\};k) \simeq k$.

Proposition 1.3.3. *Let X be a simply connected rational space of finite type. Then the unit map*

$$u_X : X \to \mathrm{Map}_{\mathrm{CAlg}_{\mathbf{Q}}}(C^*(X;\mathbf{Q}),\mathbf{Q})$$

is a homotopy equivalence.

Proof. The map $X \to \tau_{\leq n}X$ induces an isomorphism on homotopy groups in degrees $\leq n$, hence an isomorphism $\mathrm{H}^i(X;\mathbf{Q}) \to \mathrm{H}^i(\tau_{\leq n}X;\mathbf{Q})$ for $i \leq n$. It follows that $C^*(X;\mathbf{Q}) \simeq \varinjlim C^*(\tau_{\leq n}X;\mathbf{Q})$, so that u_X can be identified with the limit of the maps $\{u_{\tau_{\leq n}X}\}_{n\geq 0}$. It will therefore suffice to prove that each of the maps $u_{\tau_{\leq n}X}$ is a homotopy equivalence. We proceed by induction on n, the case $n=1$ being obvious. Let $n>1$ and set $V = \pi_n X$, so that V is a finite-dimensional vector space over $\mathbf{Q}$. We have a pullback diagram of spaces

$$\begin{tikzcd}
\tau_{\leq n}X \arrow[r] \arrow[d] & * \arrow[d] \\
\tau_{\leq n-1}X \arrow[r] & K(V,n+1)
\end{tikzcd}$$

so that Corollary 1.1.10 yields a pushout diagram

$$\begin{tikzcd}
C^*(K(V,n+1),\mathbf{Q}) \arrow[r] \arrow[d] & \mathbf{Q} \arrow[d] \\
C^*(\tau_{\leq n-1}X;\mathbf{Q}) \arrow[r] & C^*(\tau_{\leq n}X;\mathbf{Q}).
\end{tikzcd}$$

Consequently, to show that $u_{\tau_{\leq n}X}$ is a homotopy equivalence, it suffices to show that $u_{\tau_{\leq n-1}X}$ and $u_{K(V,n+1)}$ are homotopy equivalences. In the first case, this follows from the inductive hypothesis. In the second, it follows from Corollary 1.1.16. $\square$

Corollary 1.3.4. *Let X be a simply connected rational space of finite type, and let Y be an arbitrary space. Then the functor $C^*(\bullet;\mathbf{Q})$ induces a homotopy equivalence*

$$\mathrm{Map}_{\mathcal{S}}(Y,X) \to \mathrm{Map}_{\mathrm{CAlg}_{\mathbf{Q}}}(C^*(X;\mathbf{Q}),C^*(Y;\mathbf{Q})).$$

Proof of Theorem 1.3.1. Corollary 1.3.4 implies that $C^*|\,\mathcal{S}^{\mathrm{rat}}_{\mathrm{ft}}$ is fully faithful. We next show that if $X \in \mathcal{S}^{\mathrm{rat}}_{\mathrm{ft}}$, then $C^*(X;\mathbf{Q})$ satisfies conditions (a) and (b). Since X is simply connected, condition (b) is obvious. We wish to prove that $\mathrm{H}^{-i}(X;\mathbf{Q})$ is finite dimensional for every integer i. Replacing X by $\tau_{\leq n}X$ for $n \geq i$, we can assume that X is n-truncated, and we proceed by induction on n. If $n=1$, then $X \simeq *$ and the result is obvious. Otherwise, set $V = \pi_n X$ so that we have a pullback diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & * \arrow[d] \\
\tau_{\leq n-1}X \arrow[r] & K(V,n+1)
\end{tikzcd}$$

and therefore (by Corollaries 1.1.10 and 1.1.16) a pushout diagram of $\mathbf{Q}$-algebras

$$\begin{tikzcd}
\mathrm{Sym}^*(V[-n-1]) \arrow[r] \arrow[d] & \mathbf{Q} \arrow[d] \\
C^*(\tau_{\leq n-1}X;\mathbf{Q}) \arrow[r] & C^*(X;\mathbf{Q}).
\end{tikzcd}$$

The desired result now follows from the inductive hypothesis and Lemma X.4.1.16.

We now show that if $A \in \mathrm{CAlg}_{\mathbf{Q}}$ satisfies conditions (a) and (b), then A belongs to the essential of $C^*|\,\mathcal{S}^{\mathrm{rat}}_{\mathrm{ft}}$. Let $X = \mathrm{cSpec}\,A$ be the coaffine stack determined by A. Proposition VIII.4.4.18 implies that X has finite type, so that each homotopy group $\pi_i X$ is representable by a commutative unipotent algebraic group over $\mathbf{Q}$ and therefore $\pi_i X(\mathbf{Q})$ is a finite dimensional vector space over $\mathbf{Q}$ for each i. Proposition VIII.4.4.12 implies that $X(\mathbf{Q})$ is simply connected, so that $X(\mathbf{Q}) \in \mathcal{S}^0$. Let $A' = C^*(X(\mathbf{Q});\mathbf{Q})$, and let X' be the coaffine stack determined by A'. The canonical map $A \to A'$ induces a map of coaffine stacks $v : X' \to X$. Since $C^*|\,\mathcal{S}^0$ is fully faithful, the induced map $X'(\mathbf{Q}) \to X(\mathbf{Q})$ is a homotopy equivalence. For each $i \geq 1$, we have a map $\pi_i(v) : \pi_i X' \to \pi_i X$ of unipotent algebraic groups over $\mathbf{Q}$ which induces an isomorphism on $\mathbf{Q}$-points. It follows that $\pi_i(v)$ is an isomorphism of algebraic groups and therefore that v is an equivalence of coaffine stacks, so that $A \simeq A' = C^*(X(\mathbf{Q});\mathbf{Q})$ lies in the essential image of $C^*|\,\mathcal{S}^{\mathrm{rat}}_{\mathrm{ft}}$ as desired. □

In the language of coaffine stacks, Theorem 1.3.1 has the following interpretation:

Theorem 1.3.5. *Let $\mathcal{C} \subseteq \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_{\mathbf{Q}}, \mathcal{S})$ denote the full subcategory spanned by the simply connected coaffine stacks of finite type. The evaluation map $X \mapsto X(\mathbf{Q})$ determines an equivalence of ∞-categories $\mathcal{C} \to \mathcal{S}^{\mathrm{rat}}_{\mathrm{ft}}$.*

Proof. It is clear that if $X \in \mathcal{C}$, the $X(\mathbf{Q}) \in \mathcal{S}^{\mathrm{rat}}_{\mathrm{ft}}$. Let $\mathcal{D}$ denote the full subcategory of $\mathrm{CAlg}_{\mathbf{Q}}$ spanned by those $\mathbb{E}_\infty$-algebras $A \in \mathrm{CAlg}_{\mathbf{Q}}$ satisfying conditions (a) and (b) of Theorem 1.3.1. Using Propositions VIII.4.4.12 and VIII.4.4.18, we see that the construction $A \mapsto \mathrm{cSpec}\,A$ induces an equivalence of ∞-categories $\mathcal{D}^{op} \to \mathcal{C}$. It will therefore suffice to prove that the composite functor $\mathcal{D}^{op} \to \mathcal{C} \to \mathcal{C}'$ is an equivalence of ∞-categories. Unwinding the definitions, we see that this composite functor is given by

$$A \mapsto \mathrm{Map}_{\mathrm{CAlg}_{\mathbf{Q}}}(A, \mathbf{Q}).$$

We now observe that this construction is right adjoint to the equivalence of ∞-categories appearing in Theorem 1.3.1. □

We now use the theory of k-rational homotopy types developed in §1.2 to remove the finiteness restrictions from Theorem 1.3.5.

Theorem 1.3.6. *Let $\mathbf{Q}$ denote the field of rational numbers. Then the construction $X \mapsto X(\mathbf{Q})$ induces an equivalence of ∞-categories $e : \mathrm{RType}(\mathbf{Q}) \to \mathcal{S}^{\mathrm{rat}}$.*

Proof. We first show that the functor $X \mapsto X(\mathbf{Q})$ is fully faithful. Fix $\mathbf{Q}$-rational homotopy types X and Y; we wish to show that the canonical map

$$u_{X,Y} : \mathrm{Map}_{\mathrm{RType}(\mathbf{Q})}(X, Y) \to \mathrm{Map}_{\mathcal{S}}(X(\mathbf{Q}), Y(\mathbf{Q}))$$

is a homotopy equivalence. Note that $u_{X,Y}$ is given by the homotopy limit of the tower of maps $\{u_{X,\tau_{\leq n}Y}\}_{n\geq 0}$. It will therefore suffice to show that each of the maps $u_{X,\tau_{\leq n}Y}$ is a homotopy equivalence. We proceed by induction on n, the case $n \leq 1$ being trivial. To carry out the inductive step, write $\pi_n Y(R) \simeq V \otimes_{\mathbf{Q}} R$ for some $\mathbf{Q}$-vector space V. We then have a fiber sequence of functors

$$\tau_{\leq n}Y \to \tau_{\leq n-1}Y \to \underline{K}(V, n+1).$$

The inductive hypothesis implies that $u_{X,\tau_{\leq n-1}Y}$ is a homotopy equivalence. We are therefore reduced to proving that $u_{X,\underline{K}(V,n+1)}$ is a homotopy equivalence. In other words, we may assume without loss of generality that Y has the form $\underline{K}(V,m)$ for some $\mathbf{Q}$-vector space V and some $m \geq 3$.

Replacing X by $\tau_{\leq m}X$, we may assume without loss of generality that X is m-truncated. Applying Proposition 1.2.18, we can write X as a filtered colimit $\varinjlim X_\alpha$, where each X_α is an m-truncated $\mathbf{Q}$-rational homotopy type of finite type. Then $u_{X,Y}$ is equivalent to the limit of the maps $u_{X_\alpha,Y}$, so it will suffice to show that each $u_{X_\alpha,Y}$ is a homotopy equivalence. Replacing X by X_α, we are reduced to the case where X has finite type.

Write V as a direct limit $\varinjlim V_\beta$ of finite-dimensional subspaces of V. We have a commutative diagram

$$\begin{tikzcd}
\varinjlim \mathrm{Map}_{\mathrm{RType}(\mathbf{Q})}(X, \underline{K}(V_\beta, m)) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{RType}(\mathbf{Q})}(X, \underline{K}(V, m)) \arrow[d, "u_{X,\underline{K}(V,m)}"] \\
\varinjlim \mathrm{Map}_{\mathcal{S}}(X(\mathbf{Q}), K(V_\beta, m)) \arrow[r] & \mathrm{Map}_{\mathcal{S}}(X(\mathbf{Q}), K(V, m)).
\end{tikzcd}$$

Using Lemma 1.2.12, we see that the upper horizontal map is a homotopy equivalence. We claim that the lower horizontal map is also a homotopy equivalence. Using Whitehead's theorem, we are reduced to proving that the canonical map

$$\gamma : \varinjlim_\beta \mathrm{H}^i(X(\mathbf{Q}); V_\beta) \to \mathrm{H}^i(X(\mathbf{Q}); V)$$

is an isomorphism for each $i \leq m$. Using the universal coefficient formula, we can identify γ with the canonical map

$$\varinjlim_\beta \mathrm{Hom}_{\mathbf{Q}}(\mathrm{H}_i(X(\mathbf{Q}); \mathbf{Q}), V_\beta) \to \mathrm{Hom}_{\mathbf{Q}}(\mathrm{H}_i(X(\mathbf{Q}); \mathbf{Q}), V).$$

This map is an isomorphism since the group $\mathrm{H}_i(X(\mathbf{Q}); \mathbf{Q})$ is a finite-dimensional vector space (by Theorem 1.3.1). Consequently, we can identify $u_{X,\underline{K}(V,m)}$ with the filtered colimit of the maps $u_{X,\underline{K}(V_\beta,m)}$. Replacing V by V_β, we can reduce to the case where $Y = \underline{K}(V,m)$ for some finite-dimensional vector space V. In this case, both X and Y can be extended to coaffine stacks (Remark 1.2.5), so that Theorem 1.3.5 guarantees that $u_{X,Y}$ is a homotopy equivalence.

To complete the proof, it will suffice to show that if Z is a simply connected space whose homotopy groups are rational vector spaces, then Z has the form $X(\mathbf{Q})$ for some $\mathbf{Q}$-rational homotopy type X. We will construct X as the inverse limit of a tower

$$\cdots \to X_4 \to X_3 \to X_2 \to X_1 \simeq *,$$

where each X_n satisfies $X_n(\mathbf{Q}) \simeq \tau_{\leq n}Z$. The construction proceeds by recursion. Assume that X_{n-1} has been constructed. The existence of a homotopy equivalence $X_{n-1}(\mathbf{Q}) \simeq \tau_{\leq n-1}Z$ guarantees that X_{n-1} is $(n-1)$-truncated. Let $V = \pi_n Z$, so that we have a fiber sequence $\tau_{\leq n}Z \to \tau_{\leq n-1}Z \xrightarrow{\gamma} K(V, n+1)$. Using the first part of the proof, we see that γ is obtained from a map of $\mathbf{Q}$-rational homotopy types $\widetilde{\gamma} : X_{n-1} \to \underline{K}(V, n+1)$. We now define X_n to be fiber of the map $\widetilde{\gamma}$. Using the fact that X_{n-1} is $(n-1)$-truncated, we immediately deduce that X_n is a $\mathbf{Q}$-rational homotopy type. By construction, we have $X_n(\mathbf{Q}) \simeq \tau_{\leq n}Z$. □

1.4 Differential Graded Lie Algebras

Let k be a field of characteristic zero, which we regard as fixed throughout this section. In this section, we will study connectivity and finiteness properties of differential graded Lie algebras over k.

Notation 1.4.1. Let $\mathfrak{g}_*$ be a differential graded Lie algebra over k. We let $\mathrm{H}_*(\mathfrak{g}_*)$ denote the homology of the underlying chain complex of $\mathfrak{g}_*$ (so that $\mathrm{H}_*(\mathfrak{g}_*)$ has the structure of a graded Lie algebra). We let $C^*(\mathfrak{g}_*)$ denote the cohomological Chevalley-Eilenberg complex of $\mathfrak{g}_*$ (see Construction X.2.2.13), which we regard as an augmented commutative differential graded algebra over k. Our notation is potentially confusing: take note that the cohomology groups of $C^*(\mathfrak{g}_*)$ are *not* given by the k-linear duals of the vector spaces $\mathrm{H}_n(\mathfrak{g}_*)$.

We will generally not distinguish between the commutative differential graded algebra $C^*(\mathfrak{g}_*)$ and its image in the ∞-category $\mathrm{CAlg}_k^{\mathrm{aug}}$. Recall that the construction $C^* : \mathrm{Lie}_k \to (\mathrm{CAlg}_k^{\mathrm{aug}})^{op}$ admits a right adjoint $\mathfrak{D} : (\mathrm{CAlg}_k^{\mathrm{aug}})^{op} \to \mathrm{Lie}_k$, which we refer to as *Koszul duality*.

Definition 1.4.2. Let $\mathfrak{g}_*$ be a differential graded Lie algebra and let n be an integer. We will say that $\mathfrak{g}_*$ is:

- *n-connective* if the underlying chain complex of $\mathfrak{g}_*$ is n-connective: that is, if the homology groups $\mathrm{H}_m(\mathfrak{g}_*)$ vanish for $m < n$.
- *connective* if it is 0-connective; that is, if $\mathrm{H}_m(\mathfrak{g}_*) \simeq 0$ for $m < 0$.
- *connected* if it is 1-connective: that is, if $\mathrm{H}_m(\mathfrak{g}_*) \simeq 0$ for $m \leq 0$.
- *n-truncated* if the underlying chain complex of $\mathfrak{g}_*$ is n-truncated: that is, if the homology groups $\mathrm{H}_m(\mathfrak{g}_*)$ vanish for $m > n$.
- *truncated* if it is n-truncated for some integer n: that is, if $\mathrm{H}_m(\mathfrak{g}_*) \simeq 0$ for $m \gg 0$.
- *of finite type* if the homology groups $\mathrm{H}_m(\mathfrak{g}_*)$ are finite-dimensional for each integer m.

The main result of this section can be formulated as follows:

Theorem 1.4.3. *Let $\mathfrak{g}_*$ be differential graded Lie algebra which is connected, truncated, and of finite type. Then the unit map $\mathfrak{g}_* \to \mathfrak{D}C^*(\mathfrak{g}_*)$ is an equivalence.*

Remark 1.4.4. The conclusion of Theorem 1.4.3 remains valid if we assume only that $\mathfrak{g}_*$ is connected and of finite type: the condition of truncatedness is superfluous. However, we will not need this stronger result.

The proof of Theorem 1.4.3 will require a few remarks about Postnikov towers of a differential graded Lie algebra.

Construction 1.4.5. Let $\mathfrak{g}_*$ be a differential graded Lie algebra, given by a chain complex

$$\cdots \to \mathfrak{g}_2 \xrightarrow{d_2} \mathfrak{g}_1 \xrightarrow{d_1} \mathfrak{g}_0 \xrightarrow{d_0} \mathfrak{g}_{-1} \xrightarrow{d_{-1}} \mathfrak{g}_{-2} \to \cdots .$$

For each integer $n \geq 0$, we let $\tau_{\geq n}\mathfrak{g}_*$ denote the subcomplex of $\mathfrak{g}_*$, given by

$$\cdots \to \mathfrak{g}_{n+2} \xrightarrow{d_{n+2}} \mathfrak{g}_{n+1} \xrightarrow{d_{n+1}} \ker(d_n) \to 0 \to \cdots .$$

We note that $\tau_{\geq n}\mathfrak{g}_*$ is stable under the Lie bracket (using the assumption that $n \geq 0$), and therefore inherits the structure of a differential graded Lie algebra. By construction we have

$$\mathrm{H}_m(\tau_{\geq n}\mathfrak{g}_*) \simeq \begin{cases} \mathrm{H}_m(\mathfrak{g}_*) & \text{if } m \geq n \\ 0 & \text{if } m < n. \end{cases}$$

From this we deduce the following:

(*a*) Every quasi-isomorphism of differential graded Lie algebras $\mathfrak{h}_* \to \mathfrak{g}_*$ induces a quasi-isomorphism $\tau_{\geq n}\mathfrak{h}_* \to \tau_{\geq n}\mathfrak{g}_*$. Consequently, $\tau_{\geq n}$ determines a functor from the ∞-category Lie_k to itself, which we will also denote by $\tau_{\geq n}$.

(*b*) If $\mathfrak{g}_*$ is n-connective, then the inclusion $\tau_{\geq n}\mathfrak{g}_* \hookrightarrow \mathfrak{g}_*$ is a quasi-isomorphism. In particular, every n-connective differential graded Lie algebra $\mathfrak{g}_*$ is quasi-isomorphic to a differential graded Lie algebra which is concentrated in homological degrees $\geq n$.

Remark 1.4.6. Let $n \geq 0$ be an integer. We let $\mathrm{Lie}_k^{\geq n}$ denote the full subcategory of Lie_k consisting the n-connective differential graded Lie algebras over k. The truncation functor $\tau_{\geq n}$ defines a map $\mathrm{Lie}_k \to \mathrm{Lie}_k^{\geq n}$, equipped with a natural transformation $\alpha : \tau_{\geq n} \to \mathrm{id}$. For every differential graded Lie algebra $\mathfrak{g}_*$, the canonical maps

$$\alpha_{\tau_{\geq n}\mathfrak{g}_*}, \tau_{\geq n}(\alpha_{\mathfrak{g}_*}) : \tau_{\geq n}\tau_{\geq n}\mathfrak{g}_* \to \tau_{\geq n}\mathfrak{g}_*$$

are equivalences (in fact, they are given by *isomorphisms* in the ordinary category $\mathrm{Lie}_k^{\mathrm{dg}}$). It follows that $\tau_{\geq n}$ determines a colocalization functor on the ∞-category Lie_k (see Proposition T.5.2.7.4) with having essential image $\mathrm{Lie}_k^{\geq n}$. That is, we can identify $\tau_{\geq n}$ with a right adjoint to the inclusion $\mathrm{Lie}_k^{\geq n} \hookrightarrow \mathrm{Lie}_k$.

From this we conclude that $\mathrm{Lie}_k^{\geq n}$ is closed under small colimits in Lie_k. Since the functor $\tau_{\geq n}$ commutes with filtered colimits, its essential image $\mathrm{Lie}_k^{\geq n}$ is an accessible subcategory of Lie_k, and therefore a presentable ∞-category.

Remark 1.4.7. Let $\mathrm{Free} : \mathrm{Mod}_k \to \mathrm{Lie}_k$ denote a left adjoint to the forgetful functor $\theta : \mathrm{Lie}_k \to \mathrm{Mod}_k$. Note that Free can be identified with the left derived functor of the left Quillen functor $F : \mathrm{Vect}_k^{\mathrm{dg}} \to \mathrm{Lie}_k^{\mathrm{dg}}$, which assigns to each differential graded vector space V_* the free differential graded Lie algebra generated by V_*. Note that if V_* is concentrated in degrees $\geq n$ for some $n \geq 0$, then so is $F(V_*)$. It follows that for $n \geq 0$, the adjunction

$$\mathrm{Mod}_k \underset{\theta}{\overset{\mathrm{Free}}{\rightleftarrows}} \mathrm{Lie}_k$$

restricts to an adjunction

$$(\mathrm{Mod}_k)_{\geq n}, \mathrm{Lie}_k^{\geq n} \rightleftarrows .$$

The forgetful functor θ is conservative and preserves small sifted colimits (Proposition X.2.1.16). Since the ∞-category $(\mathrm{Mod}_k)_{\geq n}$ has compact projective generators (given by $V[n]$, where V is a finite-dimensional vector space over k), the ∞-category $\mathrm{Lie}_k^{\geq n}$ has a set of compact projective generators given by $\mathrm{Free}(V[n])$, where V ranges over isomorphism classes of finite-dimensional vector spaces over k (Proposition A.7.1.4.12).

Remark 1.4.8. Let $m \geq n \geq 0$ be integers, and let $\mathfrak{g}_*$ be an n-connective differential graded Lie algebra over k. The following conditions are equivalent:

- (a) When viewed as an object of the ∞-category $\mathrm{Lie}_k^{\geq n}$, $\mathfrak{g}_*$ is $(m-n)$-truncated. That is, for every n-connective differential graded Lie algebra $\mathfrak{h}_*$, the mapping space $\mathrm{Map}_{\mathrm{Lie}_k}(\mathfrak{h}_*, \mathfrak{g}_*)$ is $(m-n)$-truncated.
- (b) For every finite-dimensional vector space V, the mapping space $\mathrm{Map}_{\mathrm{Lie}\,k}(\mathrm{Free}(V[n]), \mathfrak{g}_*)$ is $(m-n)$-truncated.
- (c) The differential graded Lie algebra $\mathfrak{g}_*$ is m-truncated in the sense of Definition 1.4.2: that is, $\mathrm{H}_p(\mathfrak{g}_*) \simeq 0$ for $p > m$.

The equivalence of (a) and (b) follows from the fact that $\mathrm{Lie}_k^{\geq n}$ is generated by the objects $\mathrm{Free}(V[n])$ under small colimits, and the equivalence of (b) and (c) follows from the existence of isomorphisms

$$\mathrm{Hom}_k(V, \mathrm{H}_p(\mathfrak{g}_*)) \simeq \pi_{p-n} \mathrm{Map}_{\mathrm{Lie}_k}(\mathrm{Free}(V[n]), \mathfrak{g}_*).$$

Let $(\mathrm{Lie}_k^{\geq n})^{\leq m}$ denote the full subcategory of $\mathrm{Lie}_k^{\geq n}$ spanned by those objects which satisfy the equivalent conditions of Remark 1.4.8. Then the inclusion functor $(\mathrm{Lie}_k^{\geq n})^{\leq m} \hookrightarrow (\mathrm{Lie}_k^{\geq n})$ admits a left adjoint, which we will denote by $\tau_{\leq m}$. Using Remark T.5.5.8.26, we see that for each $\mathfrak{g}_* \in \mathrm{Lie}_k^{\geq n}$, the truncation $\tau_{\leq m}\mathfrak{g}_*$ is characterized up to equivalence by the requirement that there exist a map $\mathfrak{g}_* \to \tau_{\leq m}\mathfrak{g}_*$ which induces an isomorphism on homology in degrees $\leq m$, and where $\tau_{\leq m}\mathfrak{g}_*$ has trivial homology in degrees $\geq m$. From this description, we see that $\tau_{\leq m}\mathfrak{g}_*$ does not depend on the integer n. It will be convenient to have a more explicit construction of $\tau_{\leq m}\mathfrak{g}_*$:

Construction 1.4.9. Let $\mathfrak{g}_*$ be a differential graded Lie algebra and suppose that $\mathfrak{g}_m \simeq 0$ for $m < 0$. Let us denote the underlying chain complex of $\mathfrak{g}_*$ by

$$\cdots \to \mathfrak{g}_2 \xrightarrow{d_2} \mathfrak{g}_1 \xrightarrow{d_1} \mathfrak{g}_0 \to 0.$$

For each integer $n \geq 0$, we let $T_m \mathfrak{g}_*$ denote the quotient chain complex given by

$$\cdots \to 0 \to \mathrm{coker}(d_{n+1}) \to \mathfrak{g}_{n-1} \to \cdots \to \mathfrak{g}_0 \to 0.$$

Then $T_m \mathfrak{g}_*$ inherits the structure of a differential graded Lie algebra. We have an evident map $\mathfrak{g}_* \to T_m \mathfrak{g}_*$ whcih identifies $T_m \mathfrak{g}_*$ with the truncation $\tau_{\leq m} \mathfrak{g}_*$ defined above. This map is a quasi-isomorphism if and only if $\mathfrak{g}_*$ is m-truncated.

Remark 1.4.10. Let $n \geq 0$ be an integer. Since $\mathrm{Lie}_k^{\geq n}$ is projectively generated, Postnikov towers in $\mathrm{Lie}_k^{\geq n}$ are convergent (see Remark T.5.5.8.26). It follows that $\mathrm{Lie}_k^{\geq n}$ is equivalent to the homotopy inverse limit of the tower of ∞-categories

$$\cdots \to (\mathrm{Lie}_k^{\geq n})^{\leq n+2} \to (\mathrm{Lie}_k^{\geq n})^{\leq n+1} \to (\mathrm{Lie}_k^{\geq n})^{\leq n}.$$

Notation 1.4.11. Let V be a vector space over k. For each integer n, we let $V[n]$ denote the chain complex consisting of the single vector space V, concentrated in homological degree n. We will regard $V[n]$ as a differential graded Lie algebra over k, where the differential and Lie bracket on $V[n]$ are zero.

Proposition 1.4.12. *Let $\mathfrak{g}_*$ be a differential graded Lie algebra over k. Assume that $\mathfrak{g}_*$ is connected and n-truncated for some integer $n \geq 1$, and let $V = \mathrm{H}_n(\mathfrak{g}_*)$. Then there exists a fiber sequence*

$$\mathfrak{g}_* \to \mathfrak{h}_* \to V[n+1]$$

in the ∞-category Lie_k, where $\mathfrak{h}_*$ is connected and $(n-1)$-truncated.

Proof. Using Constructions 1.4.5 and 1.4.9, we may reduce to the case where $\mathfrak{g}_m = 0$ unless $1 \leq m \leq n$. Then we can regard V as a subspace of $\mathfrak{g}_n$; let $i : V \hookrightarrow \mathfrak{g}_n$ denote the inclusion map. Let $\mathfrak{h}_*$ denote the chain complex

$$\cdots \to 0 \to V \xrightarrow{i} \mathfrak{g}_n \to \mathfrak{g}_{n-1} \to \cdots.$$

The Lie algebra structure on $\mathfrak{g}_*$ extends uniquely to a Lie algebra structure on $\mathfrak{h}_*$ (satisfying $[v, x] = 0$ for all $v \in V$). As a differential graded Lie algebra, $\mathfrak{h}_*$ is equipped with an evident map $\mathfrak{h}_* \to V[n+1]$. This map is fibration (with respect to the model structure of Proposition X.2.1.10) with fiber $\mathfrak{g}_*$. Since the ∞-category $\mathrm{Lie}_k^{\mathrm{dg}}$ is right proper, we obtain a homotopy fiber sequence

$$\mathfrak{g}_* \to \mathfrak{h}_* \to V[n+1]$$

in $\mathrm{Lie}_k^{\mathrm{dg}}$, hence a fiber sequence in the underlying ∞-category Lie_k. Since $\mathrm{H}_m(\mathfrak{h}_*) \simeq \begin{cases} \mathrm{H}_m(\mathfrak{g}_*) & \text{if } m \neq n \\ 0 & \text{if } m = n, \end{cases}$, we conclude that $\mathfrak{h}_*$ is connected and $(n-1)$-truncated. $\square$

Remark 1.4.13. In the proof of Proposition 1.4.12, suppose that the vector space V is finite-dimensional, so that $C^*(V[n+1])$ can be identified with the symmetric algebra $\mathrm{Sym}^*(V^\vee[-n-2])$ (equipped with the trivial differential). The diagram

$$\begin{tikzcd} \mathrm{Sym}^*(V^\vee[-n-2]) \arrow[r] \arrow[d] & C^*(\mathfrak{h}_*) \arrow[d] \\ k \arrow[r] & C^*(\mathfrak{g}_*) \end{tikzcd}$$

is a pushout diagram in the ordinary category of commutative differential graded algebras over k. Since $C^*(\mathfrak{h}_*)$ is cofibrant when viewed as a differential graded module over $\mathrm{Sym}^*(V^\vee)[-n-2]$, this diagram is also a homotopy pushout square. It follows that we have a cofiber sequence

$$C^*(V[n+1]) \to C^*(\mathfrak{h}_*) \to C^*(\mathfrak{g}_*)$$

in the ∞-category $\mathrm{CAlg}_k^{\mathrm{aug}}$.

Corollary 1.4.14. *Let $\mathfrak{g}_*$ be a differential graded Lie algebra over k. Assume that there exists an integer $n > 0$ such that $\mathrm{H}_m(\mathfrak{g})_* \simeq 0$ for $m \neq n$. Then $\mathfrak{g}_*$ is equivalent (in the ∞-category Lie_k) to $V[n]$, where $V = \mathrm{H}_n(\mathfrak{g}_*)$.*

Proof. Since $n > 0$, we see that $\mathfrak{g}_*$ is connected and n-truncated. Applying Proposition 1.4.12 we obtain a fiber sequence

$$\mathfrak{g}_* \to 0 \to V[n+1],$$

from which we immediately deduce that $\mathfrak{g}_*$ is equivalent to $V[n]$. □

Proof of Theorem 1.4.3. Assume that $\mathfrak{g}_*$ is connected, n-truncated, and of finite type. We wish to prove that the unit map $u_{\mathfrak{g}_*} : \mathfrak{g}_* \to \mathfrak{D}C^*(\mathfrak{g}_*)$ is an equivalence of differential graded Lie algebras over k. The proof proceeds by induction on n, the case $n \leq 0$ being trivial. To carry out the inductive step, we apply Proposition 1.4.12 to choose a fiber sequence

$$\mathfrak{g}_* \to \mathfrak{h}_* \to V[n+1]$$

where $\mathfrak{h}_*$ is $(n-1)$-truncated. Since $\mathfrak{g}_*$ has finite type, the vector space V is finite-dimensional and $\mathfrak{h}_*$ is also finite type. Applying Remark 1.4.13, we obtain a cofiber sequence $C^*(V[n+1]) \to C^*(\mathfrak{h}_*) \to C^*(\mathfrak{g}_*)$. It follows that we have a commutative diagram of fiber sequences

$$\begin{tikzcd}
\mathfrak{g}_* \arrow[r] \arrow[d, "u_{\mathfrak{g}_*}"] & \mathfrak{h}_* \arrow[r] \arrow[d, "u_{\mathfrak{h}_*}"] & V[n+1] \arrow[d, "u_{V[n+1]}"] \\
\mathfrak{D}C^*(\mathfrak{g}_*) \arrow[r] & \mathfrak{D}C^*(\mathfrak{h}_*) \arrow[r] & \mathfrak{D}C^*(V[n+1]).
\end{tikzcd}$$

Since the map $u_{\mathfrak{h}_*}$ is an equivalence by the inductive hypothesis, we are reduced to proving that $u_{V[n+1]}$ is an equivalence.

Let $A = \mathrm{Sym}^*(V^\vee[-n-2]) \in \mathrm{CAlg}_k^{\mathrm{aug}}$. Then the cotangent complex $L_{A/k}$ is equivalent to $V^\vee[-n-2] \otimes_k A$. Applying Proposition X.2.3.9, we see that the underlying k-module of $\mathfrak{D}(A)$ is equivalent to $V[n+1]$. Applying Corollary 1.4.14 we obtain an equivalence of differential graded Lie algebras $\mathfrak{D}(A) \simeq V[n+1]$. Note that $C^*(V[n+1])$ is a commutative differential graded algebra with trivial differential, given by the product $\prod_{n \geq 0} \mathrm{Sym}^*(V^\vee[-n-2])$. Since $n \neq 2$, the canonical map $v : A \to C^*\mathfrak{D}(A) \simeq C^*(V[n+1])$ is an equivalence in $\mathrm{CAlg}_k^{\mathrm{aug}}$. Since the composition

$$V[n+1] \stackrel{u_{V[n+1]}}{\to} \mathfrak{D}C^*(V[n+1]) \stackrel{\mathfrak{D}(v)}{\to} A$$

is homotopic to the the identity map, we conclude that $u_{V[n+1]}$ is also an equivalence, as desired. □

We will also need the following characterization of compact objects in the setting of differential graded Lie algebras:

Proposition 1.4.15. *Let $m \geq n > 0$. Then the ∞-category $(\mathrm{Lie}_k^{\geq n})^{\leq m}$ is compactly generated. Moreover, an object $\mathfrak{g}_* \in (\mathrm{Lie}_k^{\geq n})^{\leq m}$ is compact if and only if it is of finite type.*

Proof. Remark 1.4.7 implies that $\mathrm{Lie}_k^{\geq n}$ is compactly generated. Since the inclusion $(\mathrm{Lie}_k^{\geq n})^{\leq m} \hookrightarrow \mathrm{Lie}_k^{\geq n}$ commutes with filtered colimits, the truncation functor $\tau_{\leq m}$ carries compact objects of of $(\mathrm{Lie}_k^{\geq n})^{\leq m}$. Since every object of $\mathrm{Lie}_k^{\geq n}$ can be written as a filtered colimit of compact objects, we deduce that $(\mathrm{Lie}_k^{\geq n})^{\leq m}$ is compactly generated.

We now show that every object of $(\mathrm{Lie}_k^{\geq n})^{\leq m}$ which has finite type is compact. This can be formulated as follows:

$(*)$ Let $\mathfrak{g}_* \in \mathrm{Lie}_k^{\geq n}$ be of finite type. Then $\tau_{\leq m}\mathfrak{g}_*$ is a compact object of $(\mathrm{Lie}_k^{\geq n})^{\leq m}$.

To prove $(*)$, we let $\mathcal{C}$ denote the full subcategory of $(\mathrm{Mod}_k)_{\geq n}$ spanned by those k-modules V whose homotopy groups are finite-dimensional as k-vector spaces, and let $\mathcal{D}$ denote the full subcategory of $\mathrm{Lie}_k^{\geq n}$ spanned by those objects which are of finite type. Note that $\mathcal{C}$ is closed under the formation of geometric realizations in $(\mathrm{Mod}_k)_{\geq n}$. Since the forgetful functor $\mathrm{Lie}_k^{\geq n} \to (\mathrm{Mod}_k)_{\geq n}$ preserves geometric realizations of simplicial objects (Proposition X.2.1.16), we conclude that $\mathcal{D}$ is closed under geometric realizations in $\mathrm{Lie}_k^{\geq n}$. Let $\mathrm{Free} : (\mathrm{Mod}_k)_{\geq n} \to \mathrm{Lie}_k^{\geq n}$ denote the left adjoint to the forgetful functor (see Remark 1.4.7). We claim that Free carries $\mathcal{C}$ into $\mathcal{D}$. To prove this, let $V \in \mathcal{C}$; we wish to show that $\mathrm{Free}(V)$ has finite type. Using Remark X.2.1.8, we are reduced to proving that the universal enveloping algebra $U(\mathrm{Free}(V))$ has finite type. The conclusion now follows from the observation that $U(\mathrm{Free}(V))$ is equivalent to the tensor algebra $\bigoplus_{p\geq 0} V^{\otimes p}$. The forgetful functor and its left adjoint Free therefore restrict to give a pair of adjoint functors $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$. Since is closed under geometric realizations in $\mathrm{Lie}_k^{\geq n}$. We have an evident forgetful functor $G : \mathcal{D} \to \mathcal{C}$, which is conservative and preserves geometric realizations. Let $\mathfrak{g}_*$ be as in $(*)$. Using Proposition A.6.2.2.11, we can write $\mathfrak{g}_*$ as the geometric realization of a simplical object $X_\bullet$ of $\mathrm{Lie}_k^{\geq n}$, where each X_q has the form $F(V)$ for some $V \in \mathcal{C}$. Then $\tau_{\leq m}\mathfrak{g}_*$ is given by the geometric realization of $\tau_{\leq m}X_\bullet$ in the ∞-category $(\mathrm{Lie}_k^{\geq n})^{\leq m}$. Since $(\mathrm{Lie}_k^{\geq n})^{\leq m}$ is equivalent to an $(m-n+1)$-category (Remark 1.4.8), we $\tau_{\leq m}\mathfrak{g}_*$ is equivalent to the finite colimit $\varinjlim_{[q]\in \mathbf{\Delta}^{op}_{\leq m-n+1}} \tau_{\leq m}X_q$ (see the proof of Proposition A.1.3.2.9). It will therefore suffice to show that each $\tau_{\leq m}X_q$ is a compact object $(\mathrm{Lie}_k^{\geq n})^{\leq m}$. We may therefore replace $\mathfrak{g}_*$ by X_q and thereby reduce to the case where $\mathfrak{g}_* = F(V)$ for some $V \in \mathcal{C}$. Replacing V by $\tau_{\leq m}V$, we can reduce to the case where V is perfect as a k-module. Then V is a compact object of $(\mathrm{Mod}_k)_{\geq n}$. Since the forgetful functor $\mathrm{Lie}_k^{\geq n} \to (\mathrm{Mod}_k)_{\geq n}$ commutes with filtered colimits, $F(V)$ is a compact object of $\mathrm{Lie}_k^{\geq n}$. It follows that $\tau_{\leq m}F(V)$ is a compact object of $(\mathrm{Lie}_k^{\geq n})^{\leq m}$, as desired. This completes the proof of $(*)$.

We now complete the proof by showing that every compact object $\mathfrak{g}_*$ of $(\mathrm{Lie}_k^{\geq n})^{\leq m}$ has finite type. We begin by constructing a transfinite sequence of differential graded Lie algebras $\mathfrak{g}(\alpha)_*$ equipped with a compatible family of maps $\rho_\alpha : \mathfrak{g}(\alpha)_* \to \mathfrak{g}_*$. We set $\mathfrak{g}(0)_* = 0$, and $\mathfrak{g}(\lambda)_* = \varinjlim_{\beta<\lambda} \mathfrak{g}(\beta)_*$ when λ is a limit ordinal. Suppose that $\mathfrak{g}(\alpha)_*$ has been defined and that the map ρ_α is not a quasi-isomorphism. Let q be the smallest integer such that the map $u : \mathrm{H}_q(\mathfrak{g}(\alpha)_*) \to \mathrm{H}_q(\mathfrak{g}_*)$ induced by ρ_α is not an isomorphism. If u fails to be surjective, then we choose a cycle $y \in \mathfrak{g}_q$ whose homology class does not lie in the image of u. We then define $\mathfrak{g}(\alpha+1)_*$ to be the differential graded Lie algebra obtained from $\mathfrak{g}(\alpha)_*$ by freely adjoining a generator Y_α in degree q satisfying $dY = 0$, and define $\rho_{\alpha+1}$ to be the unique extension of ρ_α satisfying $\rho_{\alpha+1}(Y) = y$. If u is surjective but fails to be injective, we can choose a cycle $z \in \mathfrak{g}(\alpha)_q$ and an element $y \in \mathfrak{g}_{q+1}$ satisfying $dy = \rho_\alpha(z)$. We then define $\mathfrak{g}(\alpha+1)_*$ to be the differential graded Lie algebra obtained from $\mathfrak{g}(\alpha)_*$ by freely adjoining an element Y in degree $q+1$ satisfying $dY_\alpha = z$, and $\rho_{\alpha+1}$ to be the unique extension of ρ_α satisfying $\rho_{\alpha+1}(Y) = y$.

If β is a sufficiently large ordinal, then ρ_β is a quasi-isomorphism. Note that, as a graded Lie algebra, $\mathfrak{g}(\beta)_*$ is freely generated by the elements $\{Y_\alpha\}_{\alpha<\beta}$. Let us say that a subset $S \subseteq \{\alpha : \alpha < \beta\}$ is *saturated* if, for each $\gamma \in S$, the element dY_γ belongs to the Lie subalgebra of $\mathfrak{g}(\beta)_*$ generated by the elements $\{Y_\alpha : \alpha \in S\}$. In this case, we let $\mathfrak{g}(S)_*$ denote the differential graded Lie subalgebra of $\mathfrak{g}(\beta)_*$ generated by the elements $\{Y_\alpha : \alpha \in S\}$. Note that $\mathfrak{g}(\beta)_*$ is given by the filtered colimit

$$\varinjlim_{S} \mathfrak{g}(S)_*$$

where S ranges over all finite saturated subsets of $\{\alpha : \alpha < \beta\}$ (where this limit is formed in the ordinary category $\mathrm{Lie}_k^{\mathrm{dg}}$). Since the collection of weak equivalences in $\mathrm{Lie}_k^{\mathrm{dg}}$ is closed under filtered colimits, we have an equivalence $\varinjlim_S \mathfrak{g}(S)_* \to \mathfrak{g}_*$ in the ∞-category $\mathrm{Lie}_k^{\geq n}$. Then $\mathfrak{g}_* \tau_{\leq m} \mathfrak{g}_*$ is equivalent to the filtered colimit of the objects $\tau_{\leq m}\mathfrak{g}(S)_*$. Since $\mathfrak{g}_*$ is a compact object of $(\mathrm{Lie}_k^{\geq n})^{\leq m}$, we conclude that $\mathfrak{g}_*$ is equivalent to a retract of $\tau_{\leq m}\mathfrak{g}(S)_*$ for some finite saturated subset $S \subseteq \{\alpha : \alpha < \beta\}$. It will therefore suffice to show that $\mathfrak{g}(S)_*$ is of finite type. This is clear, since each of the vector spaces $\mathfrak{g}(S)_q$ is finite dimensional over k. $\square$

1.5 Comparison with Formal Moduli Problems

Let k be a field of characteristic zero and let $\mathrm{RType}(k)$ be the ∞-category of k-rational homotopy types. We let $\mathrm{RType}(k)_*$ denote the ∞-category of pointed objects of $\mathrm{RType}(k)$. In other words, $\mathrm{RType}(k)_*$ is the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S}_*)$ spanned by those functors which satisfy (a), (b), and (c) of Definition 1.2.1. We will refer to the objects of $\mathrm{RType}(k)_*$ as *pointed k-rational homotopy types.*

Construction 1.5.1. Let $\mathrm{CAlg}_k^{\mathrm{sm}}$ denote the full subcategory of $\mathrm{CAlg}_k^{\mathrm{aug}}$ spanned by the small $\mathbb{E}_\infty$-algebras over k (see Proposition X.1.1.11). If $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}_*$ is a pointed k-rational homotopy type, we let $X^\vee$ denote the functor $\mathrm{CAlg}_k^{\mathrm{sm}} \to \mathcal{S}$ given by the formula $X^\vee(A) = \mathrm{fib}(X(A) \to X(k))$. Here the fiber is taken over the chosen base point of $X(k)$. We will refer to $X^\vee$ as the *formal completion* of X.

Proposition 1.5.2. *Let X be a pointed k-rational homotopy type. Then the formal completion $X^\vee$ is a formal moduli problem over k. Moreover, the tangent complex $T_{X^\vee}$ is 2-connective, and we have canonical isomorphisms $\pi_n T_{X^\vee} \simeq \pi_n X$ for $n \geq 2$.*

Proof. It is clear that the space $X^\vee(k)$ is contractible. Suppose we are given a pullback diagram σ :

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & B' \arrow[d, "\phi"] \\ A \arrow[r] & B \end{tikzcd}$$

in $\mathrm{CAlg}_k^{\mathrm{sm}}$, where ϕ induces a surjection $\pi_0 B' \to \pi_0 B$. We wish to show that $X^\vee(\sigma)$ is a pullback diagram of spaces: that is, that the upper horizontal morphism in the diagram τ :

$$\begin{tikzcd} X^\vee(A') \arrow[r] \arrow[d] & X^\vee(B') \times_{X^\vee(B)} X^\vee(A) \arrow[d] \\ X(A') \arrow[r] & X(B') \times_{X(B)} X(A) \end{tikzcd}$$

is a homotopy equivalence. Since τ is a pullback square, it suffices to show that the bottom horizontal map is a homotopy equivalence, which follows from Corollary 1.2.9. This completes the proof that $X^\vee$ is a k-rational homotopy type.

Let us now study the tangent complex of $X^\vee$. For each $m \geq 0$, let $k \oplus k[m]$ denote the trivial square-zero extension of k by $k[m]$. We then have $\Omega^{\infty - m} T_{X^\vee} \simeq X^\vee(k \oplus k[m]) \simeq \mathrm{fib}(X(k \oplus k[m]) \to X(k))$. Since $X(k \oplus k[m])$ and $X(k)$ are simply connected (Corollary 1.2.8), we conclude that $\Omega^{\infty-m} T_{X^\vee}$ is connected. It follows that $\pi_n T_{X^\vee} \simeq 0$ for $n \leq 0$. Let $k[\epsilon]/(\epsilon^2) \simeq k \oplus k[0]$ denote the ring of dual numbers, so that $\pi_n T_{X^\vee} \simeq \pi_n X^\vee(k[\epsilon]/(\epsilon^2))$ for $n \geq 0$. The fiber sequence

$$X^\vee(k[\epsilon]/(\epsilon^2)) \to X(k[\epsilon]/(\epsilon^2)) \to X(k)$$

gives a long exact sequence of homotopy groups

$$(\pi_{n+1} X) \otimes_k k[\epsilon]/(\epsilon^2) \to \pi_{n+1}(X) \to \pi_n X^\vee(k[\epsilon]/(\epsilon^2)) \to (\pi_n X) \otimes_k k[\epsilon]/(\epsilon^2) \to \pi_n X.$$

For $n \geq 1$, this sequence gives an isomorphism

$$\pi_n T_{X^\vee} \simeq \pi_n X \otimes_k (\epsilon k[\epsilon]/(\epsilon^2)) \simeq \pi_n X.$$

□

Let Moduli_k denote the ∞-category of formal moduli problems over k, and $\mathrm{Moduli}_k^{\geq 2}$ the full subcategory of Moduli_k spanned by those formal moduli problems Y for which T_Y is 2-connective. Proposition 1.5.2 implies that the formal completion construction $X \mapsto X^\vee$ determines a functor $\mathrm{RType}(k)_* \to \mathrm{Moduli}_k^{\geq 2}$. Our goal in this section is to prove the following result:

Theorem 1.5.3. *Let k be a field of characteristic zero. Then the formal completion functor $X \mapsto X^\vee$ induces an equivalence of ∞-categories*

$$\theta : \mathrm{RType}(k)_* \to \mathrm{Moduli}_k^{\geq 2}.$$

Proof of Theorem 1.0.3. Combining Theorem X.2.0.2, Theorem 1.5.3, and Theorem 1.3.6, we obtain a diagram of categorical equivalences

$$\mathcal{S}^{\mathrm{rat}} \leftarrow \mathrm{RType}(\mathbf{Q}) \to \mathrm{Moduli}_{\mathbf{Q}}^{\geq 2} \leftarrow \mathrm{Lie}_{\mathbf{Q}}^{\geq 2}.$$

□

Remark 1.5.4. Let Y be a simply connected rational space, so that $Y \simeq X(\mathbf{Q})$ for some $\mathbf{Q}$-rational homotopy type X (uniquely determined up to equivalence, by Theorem 1.3.6). Let $X^\vee$ denote the formal completion of X, so that $X^\vee$ is the formal moduli problem associated to $\mathfrak{g}_*$ for some differential graded Lie algebra $\mathfrak{g}_*$ over $\mathbf{Q}$. Using Theorem X.2.0.2 and Proposition 1.5.2, we see that for for each $n \geq 2$ there are canonical isomorphisms

$$\pi_n Y \simeq \pi_n X(\mathbf{Q}) \simeq \pi_n X \simeq \pi_n T_{X^\vee} \simeq \mathrm{H}_{n-1}(\mathfrak{g}_*).$$

That is, the homology groups of the differential graded Lie algebra $\mathfrak{g}_*$ can be identified with the homotopy groups of Y. With more effort, one can show that this is an isomorphism of graded Lie algebras (where the Lie bracket on $\pi_* Y$ is defined by means of the classical Whitehead product).

Proof of Theorem 1.5.3. For each $m \geq 2$, let $(\mathrm{Moduli}_k^{\geq 2})_{\leq m}$ denote the full subcategory of $\mathrm{Moduli}_k^{\geq 2}$ spanned by those formal moduli problems Y for which the tangent complex T_Y is m-truncated. It follows from Proposition 1.5.2 that θ induces a functor $\theta_m : (\mathrm{RType}(k)_{\leq m})_* \to (\mathrm{Moduli}_k^{\geq 2})_{\leq m}$. Using Theorems X.2.0.2, Proposition 1.2.21, and Remark 1.4.10, we see that θ can be identified with the limit of the tower of functors $\{\theta_m\}_{m \geq 2}$. It will therefore suffice to show that each θ_m is an equivalence of ∞-categories.

Let $\mathcal{C}$ denote the full subcategory of Moduli_k spanned by those formal moduli problems Y for which the homotopy groups $\pi_n T_Y$ are finite-dimensional vector spaces over k, which vanish unless $2 \leq n \leq m$. Proposition 1.5.2 implies that θ_m restricts to a functor $\theta' : (\mathrm{RType}(k)_{\leq m}^{\mathrm{ft}})_* \to \mathcal{C}$. Using Proposition 1.4.15 and Theorem X.2.0.2, we see the inclusion $\mathcal{C} \hookrightarrow (\mathrm{Moduli}_k^{\geq 2})_{\leq m}$ induces an equivalence of ∞-categories $\mathrm{Ind}(\mathcal{C}) \simeq (\mathrm{Moduli}_k^{\geq 2})_{\leq m}$. Proposition 1.2.18 implies that the inclusion $(\mathrm{RType}(k)_{\leq m}^{\mathrm{ft}})_* \hookrightarrow (\mathrm{RType}_{\leq m})_*$ induces an equivalence of ∞-categories $\mathrm{Ind}((\mathrm{RType}(K)_{\leq m}^{\mathrm{ft}})_*) \to (\mathrm{RType}_{\leq m})_*$. Since the functor θ_m commutes with filtered colimits, we are reduced to proving that θ' is an equivalence of ∞-categories.

Let $\mathcal{D}$ denote the full subcategory of Lie_k spanned by those differential graded Lie algebras which are connected, m-truncated, and of finite type. For $\mathfrak{g}_* \in \mathcal{D}$, then $C^*(\mathfrak{g}_*)$ is a 2-coconnective $\mathbb{E}_\infty$-algebra over k having finite type. We let $T(\mathfrak{g}_*)$ denote the associated coaffine stack $\mathrm{cSpec}\, C^*(\mathfrak{g}_*)$: that is, $T(\mathfrak{g}_*)$ denote the functor $\mathrm{CAlg}_k^{\mathrm{cn}} \to \mathcal{S}$ given by $R \mapsto \mathrm{Map}_{\mathrm{CAlg}_k}(C^*(\mathfrak{g}_*), R)$. Since $C^*(\mathfrak{g}_*)$ has finite type, we can regard $T(\mathfrak{g}_*)$ as a k-rational homotopy type, which is equipped with a canonical base point determined by the augmentation on $C^*(\mathfrak{g})$. We may therefore regard the construction $\mathfrak{g}_* \to T(\mathfrak{g}_*)$ as a functor

$$T : \mathcal{D} \to \mathrm{RType}(k)_*.$$

We next claim that the diagram σ :

$$\begin{tikzcd} \mathcal{D} \arrow[r] \arrow[d, "T"] & \mathrm{Lie}_k \arrow[d, "\Psi"] \\ \mathrm{RType}(k)_* \arrow[r, "\theta"] & \mathrm{Moduli}_k \end{tikzcd}$$

commutes up to homotopy. To prove this, we note that the two maps from $\mathcal{D}$ to Moduli_k determined by this diagram can be identified with functors

$$\mathcal{D} \times \mathrm{CAlg}_k^{\mathrm{sm}} \to \mathcal{S}$$

given by

$$(\mathfrak{g}_*, R) \mapsto \mathrm{Map}_{\mathrm{Lie}_k}(\mathfrak{D}(R), \mathfrak{g}_*) \qquad (\mathfrak{g}_*, R) \mapsto \mathrm{Map}_{\mathrm{CAlg}_k^{\mathrm{aug}}}(C^*(\mathfrak{g})_*, R).$$

To prove that these functors are equivalent, it suffices to show that the canonical maps

$$\mathrm{Map}_{\mathrm{CAlg}_k^{\mathrm{aug}}}(C^*(\mathfrak{g}_*), R) \to \mathrm{Map}_{\mathrm{CAlg}_k^{\mathrm{aug}}}(C^*(\mathfrak{g}_*), C^*\mathfrak{D}(R)) \simeq \mathrm{Map}_{\mathrm{Lie}_k}(\mathfrak{D}(R), \mathfrak{D}C^*(\mathfrak{g}_*)) \leftarrow \mathrm{Map}_{\mathrm{Lie}_k}(\mathfrak{D}(R), \mathfrak{g}_*)$$

are homotopy equivalences for $\mathfrak{g}_* \in \mathcal{D}$ and $R \in \mathrm{CAlg}_k^{\mathrm{sm}}$. This follows from the fact that the unit maps $R \mapsto C^*\mathfrak{D}(R)$ and $\mathfrak{g}_* \to \mathfrak{D}C^*(\mathfrak{g}_*)$ are equivalences (in the first case, this follows from Proposition X.1.3.5 and Theorem X.2.3.1; in the second case, it follows from Theorem 1.4.3).

Using the commutativity of the diagram σ together with Proposition 1.5.2, we see that for $\mathfrak{g}_* \in \mathcal{D}$, we have canonical isomorphisms $\mathrm{H}_n(\mathfrak{g}_*) \simeq \pi_n T(\mathfrak{g}_*)$. It follows that the functor T carries $\mathcal{D}$ into the full subcategory $(\mathrm{RType}(k)^{\mathrm{ft}}_{\leq m})_* \subseteq \mathrm{RType}(k)_*$. The diagram

$$\begin{tikzcd} & (\mathrm{RType}(k)^{\mathrm{ft}}_{\leq m})_* \arrow[dr, "\theta'"] & \\ \mathcal{D} \arrow[ur, "T"] \arrow[rr, "\Psi"] & & \mathcal{C} \end{tikzcd}$$

commutes up to homtopy, where the horizontal map is an equivalence of ∞-categories (Theorem X.2.0.2). We are therefore reduced to proving that T induces an equivalence of ∞-categories $\mathcal{D} \to (\mathrm{RType}(k)^{\mathrm{ft}}_{\leq m})_*$.

We first show that the functor T is fully faithful. According to Corollary VIII.4.4.7, the functor $A \mapsto \mathrm{cSpec}\, A$ is fully faithful when restricted to coconnective $\mathbb{E}_\infty$-algebras over k. It will therefore suffice to show that the functor $C^* : \mathrm{Lie}_k \to \mathrm{CAlg}_k^{\mathrm{aug}}$ is fully faithful when restricted to $\mathcal{D}$. Equivalently, we wish to show that for every pair of objects $\mathfrak{g}_*, \mathfrak{h}_* \in \mathcal{D}$, the canonical map

$$\mathrm{Map}_{\mathrm{Lie}_k}(\mathfrak{g}_*, \mathfrak{h}_*) \to \mathrm{Map}_{\mathrm{CAlg}_k^{\mathrm{aug}}}(C^*(\mathfrak{g}_*), C^*(\mathfrak{g}_*)) \simeq \mathrm{Map}_{\mathrm{Lie}_k}(\mathfrak{g}_*, \mathfrak{D}C^*(\mathfrak{h}_*))$$

is an equivalence. This follows immediately from Theorem 1.4.3.

It remains to prove that T induces an essentially surjective functor $\mathcal{D} \to (\mathrm{RType}(k)^{\mathrm{ft}}_{\leq m})_*$. Let X be a pointed k-rational homotopy type which is m-truncated and of finite type. According to Remark 1.2.5, X is a coaffine stack, having the form $\mathrm{cSpec}\, A$ for some coconnective $\mathbb{E}_\infty$-algebra A over k. The base point of X determines an augmentation on A. Let $\mathfrak{g}_* = \mathfrak{D}(A)$. For each $R \in \mathrm{CAlg}_k^{\mathrm{sm}}$, we have a canonical homotopy equivalence

$$\mathrm{Map}_{\mathrm{CAlg}_k^{\mathrm{aug}}}(A, R) \simeq \mathrm{Map}_{\mathrm{CAlg}_k^{\mathrm{aug}}}(A, C^*\mathfrak{D}(R)) \simeq \mathrm{Map}_{\mathrm{Lie}_k}(\mathfrak{D}(R), \mathfrak{D}(A)),$$

so that the formal completion $X^\vee$ of X is represented by the differential graded Lie algebra $\mathfrak{D}(A)$. Since X is m-truncated and of finite type, Proposition 1.5.2 implies that the homotopy groups $\pi_n T_{X^\vee}$ are finite-dimensional and vanish unless $2 \leq n \leq m$, so that $\mathfrak{g}_* \in \mathcal{D}$. We will complete the proof by showing that $X \simeq T(\mathfrak{g}_*)$. Let $Y = T(\mathfrak{g}_*) \simeq \mathrm{cSpec}\, C^*(\mathfrak{g}_*)$. Then the unit map $A \to C^*(\mathfrak{g}_*)$ induces a map of coaffine stacks $\beta : Y \to X$; we will prove that this map is an equivalence. Since $C^*(\mathfrak{g}_*)$ is of finite type, Y is a k-rational homotopy type. It will therefore suffice to show that β induces a homotopy equivalence $Y(B) \to X(B)$ for each $B \in \mathrm{CAlg}^0_k$. Since both of these spaces are simply connected, we are reduced to proving that the map $\pi_n Y(B) \to \pi_n X(B)$ is an isomorphism for each $n \geq 2$. Equivalently, we wish to show that β induces an isomorphism $\pi_n Y \to \pi_n X$ of vector spaces over k. Using Proposition 1.5.2, we see that this is equivalent to the requirement that β induces an equivalence between the formal completions $Y^\vee \to X^\vee$, which is an immediate consequence of our construction. $\square$

2 $\mathbb{E}_\infty$-Algebras in Positive Characteristic

Let R be an $\mathbb{E}_\infty$-ring, and let $\mathcal{S}\mathrm{hv}^{\text{ét}}_R \subseteq \mathrm{Fun}(\mathrm{CAlg}^{\text{ét}}_R, \mathcal{S})$ denote the ∞-category of étale sheaves on R. Then the affine spectral Deligne-Mumford stack $\mathrm{Spec}^{\text{ét}} R$ is given by $(\mathcal{S}\mathrm{hv}^{\text{ét}}_R, \mathcal{O})$, where $\mathcal{O}$ is a sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{S}\mathrm{hv}^{\text{ét}}_R$. We will identify $\mathcal{O}$ with a functor $(\mathcal{S}\mathrm{hv}^{\text{ét}}_R)^{op} \to \mathrm{CAlg}_R$. It is natural to ask the following question:

Question 2.0.5. Given an object $U \in \mathcal{S}\mathrm{hv}^{\text{ét}}_R$, to what extent can U be recovered from the $\mathbb{E}_\infty$-algebra $\mathcal{O}(U)$?

For example, suppose that R is an algebraically closed field. Then the global sections functor

$$\Gamma : \mathcal{S}\mathrm{hv}^{\text{ét}}_R \to \mathcal{S}$$

is an equivalence of ∞-categories. For $U \in \mathcal{S}\mathrm{hv}^{\text{ét}}_R$, we have a canonical equivalence $\mathcal{O}(U) \simeq C^*(\Gamma(U); R)$. Question 2.0.5 then asks to what extent a space X can be recovered from the cochain algebra $C^*(X; R)$. When R is a field of characteristic zero, this question can be attacked using the ideas of §1.

In this section, we will focus instead on the case where R has characteristic p. We begin by introducing some tools for studying the ∞-category CAlg_R. Suppose that R is a commutative ring and let A be an $\mathbb{E}_\infty$-algebra over R. Then $\pi_* A$ has the structure of a graded-commutative algebra over R. We can regard $\pi_* A$ as a concrete invariant of A, which can be studied by means of classical ($\mathbf{Z}$-graded) commutative algebra. However, if we are given a prime number p such that $p = 0$ in R, then there is a richer story. In this case, the commutative ring $\pi_* R$ is equipped with additional "power operations," (which, in the case where $A = C^*(X; R)$, reduce to the classical *Steenrod operations* on the cohomology of spaces). We will recall the construction of these power operations in §2.1 and §2.2.

In the situation of Question 2.0.5, it is generally not reasonable to expect that an object $U \in \mathcal{S}\mathrm{hv}^{\text{ét}}_R$ can be recovered from $\mathcal{O}(U) \in \mathrm{CAlg}_R$ unless U satisfies a reasonable finiteness condition. In §2.3 we will introduce the notion of a *finitely constructible* object of a coherent ∞-topos $\mathcal{X}$. Our main result (Theorem 2.3.24) asserts that an object $U \in \mathcal{S}\mathrm{hv}^{\text{ét}}_R$ is finitely constructible if and only if there exists a finite stratification of $\mathrm{Spec}^{\mathrm{Z}} R$ such that the restriction of U to each stratum is locally constant, and the stalks of U have finitely many finite homotopy groups.

Now let p be a prime number. We will say that an object $U \in \mathcal{S}\mathrm{hv}^{\text{ét}}_R$ is *p-constructible* if it is finitely constructible, and the homotopy groups of each stalk of U are finite p-groups (see Definition 2.4.1). Our main result (Corollary 2.6.12) can be formulated as follows:

($*$) Let R be a commutative ring, let p be a prime number which is nilpotent in R, and let $\mathcal{O}$ be the structure sheaf of $\mathrm{Spec}^{\text{ét}} R$. Then the functor $U \mapsto \mathcal{O}(U)$ induces a fully faithful embedding from the ∞-category of p-constructible objects of $\mathcal{S}\mathrm{hv}^{\text{ét}}_R$ to the ∞-category CAlg^{op}_R of $\mathbb{E}_\infty$-algebras over R.

Remark 2.0.6. The formulation of Corollary 2.6.12 is actually a little more general than ($*$): we will consider more generally the case where R is an $\mathbb{E}_\infty$-ring which is *p-thin* (see Definition 2.4.2).

The proof of ($*$) relies heavily on the excellent behavior of the functor $U \mapsto \mathcal{O}(U)$. More specifically, we will need to know the following:

(a) Let $f : R \to R'$ be a map of commutative rings in which p is nilpotent, let $\mathcal{O}$ be the structure sheaf of $\mathrm{Spec}^{\text{ét}} R$ and $\mathcal{O}'$ the structure sheaf of $\mathrm{Spec}^{\text{ét}} R'$. If $U \in \mathcal{S}\mathrm{hv}^{\text{ét}}_R$ is p-constructible, then the canonical map

$$R' \otimes_R \mathcal{O}(U) \to \mathcal{O}'(f^* U)$$

is an equivalence of $\mathbb{E}_\infty$-algebras over R'.

(b) If p is nilpotent in R and $U \in \mathcal{S}\mathrm{hv}_R^{\text{ét}}$ is p-constructible, then $\mathcal{O}(U)$ is a compact object of CAlg_R.

We will prove (a) in §2.4 (Theorem 2.4.9), and (b) in §2.5 (Theorem 2.5.1). In §2.6, we will combine (a) and (b) to give the proof of $(*)$.

Remark 2.0.7. The results of this section were inspired by Mandell's work on p-adic homotopy theory (see [54]). The main ingredient in our proof of $(*)$ is Mandell's "generators and relations" description of the $\mathbb{E}_\infty$-algebra $C^*(X; \mathbf{F}_p)$ in the case where X is an Eilenberg-MacLane space $K(\mathbf{Z}/p\,\mathbf{Z}, n)$ (Theorem 2.2.17), which we formulate (without proof) in §2.2. We can recover Mandell's applications of this result by specializing to the case where R is a separably closed field of characteristic p. We will make a detailed study of this case in §3.

2.1 Norm Maps

Let M be an abelian group equipped with an action of a group G. We can associate to M the subgroup $M^G = \{x \in M : (\forall g \in G)[g(x) = x]\}$ consisting of G-invariant elements, as well as the quotient group $M_G = M/K$, where K is the subgroup of M generated by all elements of the form $g(x) - x$. When the group G is finite, there is a canonical norm map

$$Nm : M_G \to M^G,$$

which is induced by the map from M to itself given by $x \mapsto \sum_{g \in G} g(x)$. In this section, we will describe an analogous construction in the ∞-categorical setting and the closely *Tate construction*.

Notation 2.1.1. Let $\mathcal{C}$ be an ∞-category and X a Kan complex. We let $\mathcal{C}^X$ denote the ∞-category $\mathrm{Fun}(X, \mathcal{C})$ of all maps from X to $\mathcal{C}$. If $f : X \to Y$ is a map of Kan complexes, then composition with f induces a map $f^* : \mathcal{C}^Y \to \mathcal{C}^X$. If we assume that $\mathcal{C}$ admits small limits and colimits, then f^* admits both a right and left adjoint, which we denote by f_* and $f_!$, respectively.

Example 2.1.2. Let G be a group and BG its classifying space (which we regard as a Kan complex). Recall that if $\mathcal{C}$ is an ∞-category, then a *G-equivariant object* of $\mathcal{C}$ is an object of $\mathcal{C}^{BG}$. Let $f : BG \to \Delta^0$ be the projection map. If $\mathcal{C}$ admits small limits and colimits, then we have functors $f_*, f_! : \mathcal{C}^{BG} \to \mathcal{C}$. We will denote these functors by $M \mapsto M^G$ and $M \mapsto M_G$, respectively.

We can now formulate our problem more precisely. Let G be a finite group, $f : BG \to \Delta^0$ the projection map, and $\mathcal{C}$ be a sufficiently nice ∞-category. We wish to associate to the pair $(G, \mathcal{C})$ a natural transformation $Nm : f_! \to f_*$. That is, we wish to construct a natural map $M_G \to M^G$ for each G-equivariant object $M \in \mathcal{C}$. It will be convenient to construct this natural transformation more generally for any map $f : X \to Y$ having reasonably simple homotopy fibers. We will proceed in several steps, each time allowing slightly more general homotopy fibers.

Construction 2.1.3. Let $\mathcal{C}$ be an ∞-category which has both an initial object and a final object. It follows that for any map of Kan complexes $f : X \to Y$ with (-1)-truncated homotopy fibers, the pullback functor $f^* : \mathcal{C}^Y \to \mathcal{C}^X$ admits left and right adjoints $f_!$ and f_*, given by left and right Kan extension along f.

Let $X \times_Y X$ denote the homotopy fiber product of X with itself over Y and let $\delta : X \to X \times_Y X$ be the diagonal map. Since f is (-1)-truncated, δ is a homotopy equivalence. It follows that the Kan extension functors $\delta_!, \delta_* : \mathcal{C}^X \to \mathcal{C}^{X \times_Y X}$ are both homotopy inverse to δ^*, so there is a canonical equivalence $\delta_* \to \delta_!$. Let $p_0, p_1 : X \times_Y X \to X$ be the projection onto the first and second factor, respectively. We have a natural transformation of functors

$$p_0^* \to \delta_* \delta^* p_0^* \simeq \delta_* \simeq \delta_! \simeq \delta_! \delta^* p_1^* \to p_1^*$$

which is adjoint to a natural transformation $\beta : \mathrm{id}_{\mathcal{C}^X} \to (p_0)_* p_1^*$. Since we have a homotopy pullback diagram

$$\begin{tikzcd} X \times_Y X \arrow[r, "p_0"] \arrow[d, "p_1"] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y, \end{tikzcd}$$

Lemma 1.1.5 implies that the canonical map $f^* f_* \to (p_0)_* p_1^*$ is an equivalence, so that β determines a natural transformation $\mathrm{id}_{\mathcal{C}^X} \to f^* f_*$, which is in turn adjoint to a map $Nm_f : f_! \to f_*$. We will refer to Nm_f as the *norm map* determined by f.

Example 2.1.4. Let $\mathcal{C}$ be an ∞-category with initial and final objects and let $f : X \to Y$ be a homotopy equivalence of Kan complexes. Then the natural transformation $Nm_f : f_! \to f_*$ of Construction 2.1.3 is the equivalence determined by the observation that $f_!$ and f_* are both homotopy inverse to f^*: in other words, it is determined by the requirement that the induced map $f^* f_! \to f^* f_*$ is homotopy inverse to the composition of counit and unit maps

$$f^* f_* \to \mathrm{id}_{\mathcal{C}^X} \to f^* f_!.$$

Example 2.1.5. Let $Y = \Delta^0$ and let $\mathcal{C}$ be an ∞-category with initial and final objects. If $f : X \to Y$ is a (-1)-truncated map of Kan complexes, then X is either empty or contractible. If X is contractible, then the norm map Nm_f is the equivalence described in Example 2.1.4. If $X = \emptyset$, then $\mathcal{C}^X \simeq \Delta^0$ and the functors $f_!$ and f_* can be identified with initial and final objects of $\mathcal{C} \simeq \mathcal{C}^Y$, respectively. In this case, the norm map Nm_f is determined up to a contractible space of choices, since it is a map from an initial object of $\mathcal{C}$ to a final object of $\mathcal{C}$.

Proposition 2.1.6. *Let $\mathcal{C}$ be an ∞-category with an initial and final object. The following conditions are equivalent:*

(1) *For every map of Kan complexes $f : X \to Y$ with (-1)-truncated homotopy fibers, the norm map $Nm_f : f_! \to f_*$ is an equivalence.*

(2) *Condition* (1) *holds whenever $Y = \Delta^0$.*

(3) *The ∞-category $\mathcal{C}$ is pointed.*

Proof. The equivalence of (1) and (2) is easy, and the equivalence of (2) and (3) follows from Example 2.1.5. ☐

When the hypotheses of Proposition 2.1.6 are satisfied, it is possible to perform a more elaborate version of Construction 2.1.3.

Construction 2.1.7. Let $\mathcal{C}$ be an ∞-category which admits finite products and coproducts. It follows that for any map of Kan complexes $f : X \to Y$ whose homotopy fibers are 0-truncated and have finitely many path components, the pullback functor $f^* : \mathcal{C}^Y \to \mathcal{C}^X$ admits left and right adjoints $f_!$ and f_*, given by left and right Kan extension along f.

Let $X \times_Y X$ denote the homotopy fiber product of X with itself over Y and let $\delta : X \to X \times_Y X$ be the diagonal map. Since f is 0-truncated, the map δ is (-1)-truncated, so that Construction 2.1.3 defines a norm map $Nm_\delta : \delta_! \to \delta_*$. Assume that $\mathcal{C}$ is pointed. Proposition 2.1.6 implies that Nm_δ is an equivalence, and therefore admits a homotopy inverse $Nm_\delta^{-1} : \delta_* \to \delta_!$.

Let $p_0, p_1 : X \times_Y X \to X$ be the projection onto the first and second factor, respectively. We have a natural transformation of functors

$$p_0^* \to \delta_* \delta^* p_0^* \overset{Nm_\delta^{-1}}{\to} \delta_* \simeq \delta_! \simeq \delta_! \delta^* p_1^* \to p_1^*$$

which is adjoint to a natural transformation $\beta : \mathrm{id}_{\mathcal{C}^X} \to (p_0)_* p_1^*$. Since we have a homotopy pullback diagram

$$\begin{tikzcd} X \times_Y X \arrow[r, "p_0"] \arrow[d, "p_1"] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y, \end{tikzcd}$$

Lemma 1.1.5 implies that the canonical map $f^* f_* \to (p_0)_* p_1^*$ is an equivalence, so that β determines a map $\mathrm{id}_{\mathcal{C}^X} \to f^* f_*$, which is adjoint to a natural transformation $Nm_f : f_! \to f_*$. We will refer to Nm_f as the *norm map* determined by f.

Remark 2.1.8. In the situation of Construction 2.1.7, assume that f is (-1)-truncated. Then our definition of Nm_f is unambiguous: in other words, the natural transformations $Nm_f : f_! \to f_*$ described in Constructions 2.1.3 and 2.1.7 agree. This follows immediately from Example 2.1.4.

Remark 2.1.9. Suppose we are given a homotopy pullback diagram

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "p'"] & Y' \arrow[d, "p"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where the homotopy fibers of f are 0-truncated and have finitely many homotopy groups. Let $\mathcal{C}$ be a pointed ∞-category which admits finite products and coproducts. Using Lemma 1.1.5, it is not difficult to show that the natural transformation $f'_! \circ p'^* \to f'_* \circ p'^*$ determined by $Nm_{f'}$ is homotopic to the composition

$$f'_! \circ p'^* \to p^* \circ f_! \xrightarrow{Nm_f} p^* \circ f_* \to f'_* \circ p'^*.$$

Example 2.1.10. Let $\mathcal{C}$ be a pointed ∞-category which admits finite products and coproducts, let S be a finite set (regarded as a discrete simplicial set), and let $f : S \to \Delta^0$ be the canonical projection map. We can identify objects of $\mathcal{C}^S$ with tuples $C = (C_s \in \mathcal{C})_{s \in S}$. The norm map $f_!(C) \to f_*(C)$ can be identified with the map

$$\coprod_{s \in S} C_s \to \prod_{t \in S} C_t,$$

which classifies a collection of maps $\phi_{s,t} : C_s \to C_t$ in $\mathcal{C}$ where $\phi_{s,t} = \mathrm{id}$ if $s = t$ and the zero map otherwise.

Proposition 2.1.11. *Let $\mathcal{C}$ be a pointed ∞-category which admits finite products and coproducts. The following conditions are equivalent:*

(1) *For every map of Kan complexes $f : X \to Y$ whose homotopy fibers are discrete and have finitely many connected components, the norm map $Nm_f : f_! \to f_*$ is an equivalence.*

(2) *Condition* (1) *holds whenever $Y = \Delta^0$.*

(3) *For every finite collection of objects $\{C_s \in \mathcal{C}\}_{s \in S}$, the map*

$$\coprod_{s \in S} C_s \to \prod_{t \in S} C_t$$

described in Example 2.1.10 is an equivalence.

Proof. The implication (1) $\Rightarrow$ (2) is obvious and the converse follows from Remark 2.1.9. The equivalence of (2) and (3) follows from Example 2.1.10. □

Definition 2.1.12. We will say that an ∞-category $\mathcal{C}$ is *semiadditive* if it satisfies the equivalent conditions of Proposition 2.1.11.

Remark 2.1.13. Let $\mathcal{C}$ be a semiadditive ∞-category. Suppose we are given a pair of objects $C, D \in \mathcal{C}$ and a finite collection of maps $\{\phi_s : C \to D\}_{s \in S}$. Then we can define a new map $\phi : C \to D$ by the composition

$$C \to \prod_{s \in S} C \stackrel{(\phi_s)_{s\in S}}{\longrightarrow} \prod_{s \in S} D \simeq \coprod_{s \in S} D \to D,$$

where the first map is the diagonal of C and the last the codiagonal of D. This construction determines a map

$$\prod_{s \in S} \mathrm{Map}_{\mathcal{C}}(C, D) \to \mathrm{Map}_{\mathcal{C}}(C, D),$$

which endows $\mathrm{Map}_{\mathcal{C}}(C, D)$ with the structure of a commutative monoid up to homotopy. We will denote the image of a collection of morphisms $(\phi_s)_{s \in S}$ by $\sum_{s \in S} \phi_s$.

It is possible to make a much stronger assertion: the addition on $\mathrm{Map}_{\mathcal{C}}(C, D)$ is not only commutative and associative up to homotopy, but up to coherent homotopy. That is, each mapping space in $\mathcal{C}$ can be regarded as a commutative algebra object of $\mathcal{S}$, and the composition of morphisms in $\mathcal{C}$ is multilinear. Since we do not need this for the time being, we omit the proof.

Remark 2.1.14. Let $\mathcal{C}$ be an ∞-category which admits finite products and coproducts. Since products and coproducts in $\mathcal{C}$ are also products and coproducts in the homotopy category $\mathrm{h}\mathcal{C}$, we see that $\mathcal{C}$ is semiadditive if and only if (the nerve of) the category $\mathrm{h}\mathcal{C}$ is semiadditive.

Example 2.1.15. Let $\mathcal{A}$ be an additive category (see Definition A.1.1.2.1). Then the ∞-category $\mathrm{N}(\mathcal{A})$ is semiadditive.

Example 2.1.16. Let $\mathcal{C}$ be a stable ∞-category. Then the homotopy category $\mathrm{h}\mathcal{C}$ is additive (Lemma A.1.1.2.8). Combining this with Example 2.1.15 and Remark 2.1.14, we deduce that $\mathcal{C}$ is semiadditive.

Definition 2.1.17. Let X be a Kan complex. We will say that X is a *finite groupoid* if the following conditions are satisfied:

(1) The set of connected components $\pi_0 X$ is finite.

(2) For every point $x \in X$, the fundamental group $\pi_1(X, x)$ is finite.

(3) The homotopy groups $\pi_n(X, x)$ vanish for $n \geq 2$.

More generally, we say that a map of Kan complexes $f : X \to Y$ is a *relative finite groupoid* if the homotopy fibers of f are finite groupoids.

Construction 2.1.18. Let $\mathcal{C}$ be a semiadditive ∞-category which admits limits and colimits indexed by finite groupoids. It follows that for any map of Kan complexes $f : X \to Y$ which is a relative finite groupoid, the pullback functor $f^* : \mathcal{C}^Y \to \mathcal{C}^X$ admits left and right adjoints $f_!$ and f_*, given by left and right Kan extension along f.

Let $X \times_Y X$ denote the homotopy fiber product of X with itself over Y and let $\delta : X \to X \times_Y X$ be the diagonal map. Since f is a relative finite groupoid, the homotopy fibers of δ are homotopy equivalent to finite discrete spaces. Construction 2.1.7 defines a norm map $Nm_\delta : \delta_! \to \delta_*$. Since $\mathcal{C}$ is semiadditive, the natural transformation Nm_δ is an equivalence and therefore admits a homotopy inverse $Nm_\delta^{-1} : \delta_* \to \delta_!$.

Let $p_0, p_1 : X \times_Y X \to X$ be the projection onto the first and second factor, respectively. We have a natural transformation of functors

$$p_0^* \to \delta_* \delta^* p_0^* \stackrel{Nm_\delta^{-1}}{\to} \delta_* \simeq \delta_! \simeq \delta_! \delta^* p_1^* \to p_1^*$$

which is adjoint to a natural transformation $\beta : \mathrm{id}_{\mathcal{C}^X} \to (p_0)_* p_1^*$. Since we have a homotopy pullback diagram

$$\begin{tikzcd} X \times_Y X \arrow[r, "p_0"] \arrow[d, "p_1"] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y, \end{tikzcd}$$

Lemma 1.1.5 implies that the canonical map $f^* f_* \to (p_0)_* p_1^*$ is an equivalence, so that β determines a map $\mathrm{id}_{\mathcal{C}^X} \to f^* f_*$, which is adjoint to a natural transformation $Nm_f : f_! \to f_*$. We will refer to Nm_f as the *norm map* determined by f.

Remark 2.1.19. In the situation of Construction 2.1.18, assume that f is 0-truncated. Then the definition of Nm_f given in Construction 2.1.18 agrees with that given in Construction 2.1.7 (and, if f is (-1)-truncated, with that given in Construction 2.1.3): this follows easily from Remark 2.1.8.

Remark 2.1.20. In the situation of Construction 2.1.18, suppose we are given a homotopy pullback diagram

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "p'"] & Y' \arrow[d, "p"] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

Using Lemma 1.1.5, we deduce that the natural transformation $f'_! \circ p'^* \to f'_* \circ p'^*$ determined by $Nm_{f'}$ is homotopic to the composition

$$f'_! \circ p'^* \to p^* \circ f_! \xrightarrow{Nm_f} p^* \circ f_* \to f'_* \circ p'^*.$$

Example 2.1.21. Let G be a finite group. Then the classifying space BG is a finite groupoid. Let $f : BG \to \Delta^0$ be the projection map. If $\mathcal{C}$ is semiadditive ∞-category which admits limits and colimits indexed by finite groupoids, then Construction 2.1.18 determines a natural transformation $Nm_f : f_! \to f_*$. In particular, for every G-equivariant object $M \in \mathcal{C}$, we obtain a canonical map $Nm : M_G \to M^G$.

Remark 2.1.22. Let $\mathcal{C}$ be a semiadditive ∞-category which admits limits and colimits indexed by finite groupoids, and let G be a finite group. Let M be a G-equivariant object of $\mathcal{C}$, and abuse notation by identifying M with its image in $\mathcal{C}$. We have canonical maps $e : M \to M_G$ and $e' : M^G \to M$. Unwinding the definitions, we see that the composition

$$M \xrightarrow{e} M_G \xrightarrow{Nm} M^G \xrightarrow{e'} M$$

is given by $\sum_{g \in G} \phi_g$, where for each $g \in G$ we let $\phi_g : M \to M$ be the map given by evaluation on the 1-simplex of BG corresponding to g; here the sum is formed with respect to the addition described in Remark 2.1.13.

If $\mathcal{C}$ is equivalent to the nerve of an ordinary category, then the map $M \to M_G$ is a categorical epimorphism and the map $M^G \to M$ is a categorical monomorphism. It follows that the map $Nm : M_G \to M^G$ is determined (up to homotopy) by the formula $e' \circ Nm \circ e \simeq \sum_g \phi_g$. Moreover, the map Nm exists by virtue of the observation that the map $\sum_g \phi_g : M \to M$ is invariant under left and right composition with the maps ϕ_g.

Definition 2.1.23. Let $\mathcal{C}$ be a stable ∞-category which admits limits and colimits indexed by finite groupoids. Let G be a finite group, and let M be a G-equivariant object of $\mathcal{C}$. We will denote the cofiber of the norm map $Nm : M_G \to M^G$ by M^{tG}. We refer to the formation $M \mapsto M^{tG}$ as the *Tate construction.*

Remark 2.1.24. Let $\mathcal{C}$ be a semiadditive ∞-category which admits limits and colimits indexed by finite groupoids. Assume that for every finite group G and every G-equivariant object M of $\mathcal{C}$, the norm map $Nm : M_G \to M^G$ is an equivalence (if $\mathcal{C}$ is stable, this is equivalent to the requirement that the Tate construction M^{tG} vanish). It follows that for every relative finite groupoid $f : X \to Y$, the norm map $Nm_f : f_! \to f_*$ is an equivalence of functors from $\mathcal{C}^X$ to $\mathcal{C}^Y$. We can then repeat Construction 2.1.18 to define a norm map $Nm_f : f_! \to f_*$ for maps $f : X \to Y$ whose homotopy fibers are finite 2-groupoids. If $\mathcal{C}$ also admits limits and colimits indexed by finite 2-groupoids, then we can repeat Construction 2.1.18 to define a norm map $Nm_f : f_! \to f_*$ whenever f is a relative finite 2-groupoid. This condition is satisfied, for example, if $\mathcal{C}$ is a $\mathbf{Q}$-linear ∞-category (here $\mathbf{Q}$ denotes the field of rational numbers), but is generally not satisfied for stable ∞-categories defined in positive or mixed characteristics. However, it is always satisfies in the setting of $K(n)$-local stable homotopy theory. We will study this construction in more detail in [30].

Example 2.1.25. Let $\mathcal{C}$ be a semiadditive ∞-category which admits limits and colimits indexed by finite groupoids. Let G be a finite group, let $i : \Delta^0 \to BG$ be the inclusion of the base point and let $f : BG \to \Delta^0$ be the projection map. Let $X \in \mathcal{C} \simeq \mathcal{C}^{\Delta^0}$ and let $N = i_! M \in \mathcal{C}^{BG}$, so that $N \simeq i_! M \simeq\simeq \coprod_{g \in G} M \simeq \prod_{g \in G} M \simeq i_* N$. Unwinding the definitions, we see that the norm map $f_!(N) \to f_*(N)$ is given by the composition

$$f_!(N) = f_! i_! M \simeq (\mathrm{id}_! \, M) \simeq (\mathrm{id}_* \, M) \simeq f_* i_*(M) \simeq f_*(N)$$

and is therefore an equivalence. If $\mathcal{C}$ is stable, we conclude that the Tate construction N^{tG} is a zero object of $\mathcal{C}$.

2.2 Power Operations on $\mathbb{E}_\infty$-Algebras

Let k be a field and let A be an $\mathbb{E}_\infty$-algebra over k. Then the homotopy groups $\pi_* A$ have the structure of a graded commutative ring. However, we can say much more in the case where k has characteristic $p > 0$. In this case, the homotopy groups $\pi_* A$ are equipped with *power operations* (which reduce to the classical *Steenrod operations* in the case where $A = C^*(X; k)$ is the $\mathbb{E}_\infty$-algebra of cochains on a space X). In this section, we will review the construction of these power operations and use them to formulate (without proof) a theorem of Mandell, giving a generators-and-relations presentation of $C^*(X; k)$ in the case where X is an Eilenberg-MacLane space $K(\mathbf{Z}/p\,\mathbf{Z}, n)$.

Construction 2.2.1. Given a group G, we let EG denote the nerve of the category whose objects are the elements of G, where there is a unique isomorphism between every pair of objects. Then EG is a contractible Kan complex with a free action of the group G.

Let $n \geq 0$ be an integer and let Σ_n be the symmetric group on n letters. For every simplicial set K, we let $K^{(n)}$ denote the quotient $(K^n \times E\Sigma_n)/\Sigma_n$. We refer to $K^{(n)}$ as the *nth extended power of K*. It is a model for the homotopy coinvariants for the action of the symmetric group Σ_n on K^n. It follows that if K is an ∞-category, we can identify $K^{(n)}$ with the nth symmetric power $\mathrm{Sym}^n(K)$ in the ∞-category $\mathcal{C}\mathrm{at}_\infty$.

Let $\mathcal{C}$ be a symmetric monoidal ∞-category. Then we can identify $\mathcal{C}$ with a commutative algebra object of $\mathcal{C}\mathrm{at}_\infty$. It follows that $\mathcal{C}$ is equipped with a canonical map $\theta : \mathcal{C}^{(n)} \simeq \mathrm{Sym}^n(\mathcal{C}) \to \mathcal{C}$. In particular, for every diagram $K \to \mathcal{C}$, composition with θ yields a map $K^{(n)} \to \mathcal{C}$.

In particular, if $C \in \mathcal{C}$ is an object classified by a map $\Delta^0 \to \mathcal{C}$, we obtain a map $f : B\Sigma_n \simeq (\Delta^0)^{(n)} \to \mathcal{C}$ which classifies a Σ_n-equivariant object of $\mathcal{C}$, which we will denote by $C^{\otimes n}$. We let $\mathrm{Sym}^n(C) \in \mathcal{C}$ denote a colimit of f (provided that such a colimit exists).

Construction 2.2.2. Let $\mathcal{C}$ be a symmetric monoidal ∞-category. Assume that $\mathcal{C}$ is stable and admits limits and colimits indexed by finite groupoids. Suppose we are given a homomorphism of finite groups $\eta : G \to \Sigma_n$, so that η induces a diagonal map $\delta : \mathcal{C} \times BG \to \mathcal{C}^{(n)}$. Composing δ with the extended power map $\mathcal{C}^{(n)} \to \mathcal{C}$ of Construction 2.2.1, we obtain a map $\mathcal{C} \to \mathrm{Fun}(BG, \mathcal{C})$. We let $\widehat{T}_\eta : \mathcal{C} \to \mathcal{C}$ be the composition of this functor with the Tate cohomology construction $\mathrm{Fun}(BG, \mathcal{C}) \to \mathcal{C}$ of Definition 2.1.23. In other words, $\widehat{T}_\eta : \mathcal{C} \to \mathcal{C}$ is the functor given by $C \mapsto (C^{\otimes n})^{tG}$, where G acts on $C^{\otimes n}$ via the representation η.

Let p be a prime number, and let $\eta : \mathbf{Z}/p\mathbf{Z} \to \Sigma_p$ correspond to the action of $\mathbf{Z}/p\mathbf{Z}$ on itself by translation. In this case, we will denote the functor $\widehat{T}_\eta : \mathcal{C} \to \mathcal{C}$ by $\widehat{T}_p$.

Proposition 2.2.3. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category. Assume that $\mathcal{C}$ is stable, admits limits and colimits indexed by finite groupoids, and that the tensor product on $\mathcal{C}$ is exact in each variable. For every prime number p, the functor $\widehat{T}_p : \mathcal{C} \to \mathcal{C}$ of Construction 2.2.2 is exact.*

Proof. It is easy to see that $\widehat{T}_p$ preserves zero objects. It will therefore suffice to show that if σ :

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd}$$

is an exact triangle in $\mathcal{C}$, the $\widehat{T}_p(\sigma)$ is an exact triangle in $\mathcal{C}$. We can identify f with a map $\Delta^1 \to \mathcal{C}$, so that f determines an extended power $f^{(p)} : (\Delta^1)^{(p)} \to \mathcal{C}$. We can identify the objects of the ∞-category $(\Delta^1)^p$ with p-tuples $(i_1, \ldots, i_p)$, where each i_j belongs to the linearly ordered set $[1] = \{0, 1\}$. For $0 \leq j \leq p$, let $\overline{K}_{\leq j}$ denote the full subcategory of $(\Delta^1)^p$ spanned by those tuples $(i_1, \ldots, i_p)$ such that $i_1 + \cdots + i_p \leq j$ and $\overline{K}_j$ the full subcategory spanned by those tuples where $i_1 + \cdots + i_p = j$. These subsets are invariant under the action of the group $G = \mathbf{Z}/p\mathbf{Z}$ by cyclic permutations. We let $K_{\leq j}$ denote the quotient $(\overline{K}_{\leq j} \times EG)/G$ and define K_j similarly. Then $f^{(p)}$ determines a map $F_{\leq p} : K_{\leq p} \to \mathcal{C}$, which restricts to diagrams $F_{\leq j} : K_{\leq j} \to \mathcal{C}$ for each $j \geq 0$. Let $F'_{\leq j} : BG \to \mathcal{C}$ be a left Kan extension of $F_{\leq j}$ along the projection map $K_{\leq j} \to BG$, and for $j > 0$ let F'_j be the cofiber of the map $F'_{\leq j-1} \to F'_{\leq j}$. Unwinding the definitions, we obtain identifications

$$\widehat{T}_p(X) \simeq (F'_{\leq 0})^{tG} \qquad \widehat{T}_p(Y) = (F'_{\leq p})^{tG} \qquad \widehat{T}_p(Z) = (F'_p)^{tG}.$$

It follows that the fiber of the map $\widehat{T}_p(Y) \to \widehat{T}_p(Z)$ can be identified with $(F'_{\leq p-1})^{tG}$. We wish to show that the evident transformation $F'_{\leq 0} \to F'_{\leq p-1}$ induces an equivalence after applying the Tate construction. For this, it suffices to show more generally that $F'_{\leq j-1} \to F'_{\leq j}$ induces an equivalence of Tate constructions for $0 < j < p$: that is, we wish to show that ${F'_j}^{tG} \simeq 0$. Let $U : K_{\leq j} \to \mathcal{C}$ be a left Kan extension of $F_{\leq j-1}$ and let F_j denote the cofiber of the canonical map $U \to F_{\leq j}$. It follows that F'_j can be identified with the left Kan extension of F_j along the map $K_{\leq j} \to BG$. We observe that F_j is a left Kan extension of the restriction $F_j | K_j$. It follows that F'_j is a left Kan extension of $F_j | K_j$ along the map $K_j \to BG$. Since p is prime and $0 < j < p$, the simplicial set K_j is a union of finitely many contractible Kan complexes, indexed by the collection of G-orbits on the set of subsets of $\{1, \ldots, p\}$ which have cardinality j. It follows from Example 2.1.25 that ${K'_j}^{tG} \simeq 0$. $\square$

Remark 2.2.4. Let $\mathcal{C}$ be a symmetric monoidal ∞-category. Assume that $\mathcal{C}$ is stable and admits limits and colimits indexed by finite 1-groupoids. For every homomorphism of finite groups $\eta : G \to \Sigma_n$ and every commutative algebra object $A \in \mathrm{CAlg}(\mathcal{C})$, we obtain a canonical map

$$\widehat{T}_\eta(A)[-1] \simeq (A^{\otimes n})^{tG}[-1] \to A^{\otimes n}_G \to \mathrm{Sym}^n(A) \to A.$$

This construction is functorial in A, and determines a natural transformation $\widehat{T}_\eta[-1] \circ \theta \to \theta$, where θ denotes the forgetful functor $\mathrm{CAlg}(\mathcal{C}) \to \mathcal{C}$.

Example 2.2.5. Let κ be an $\mathbb{E}_\infty$-ring and let p be a prime number. Let $\mathrm{Mod}_k = \mathrm{LMod}_k$ be the ∞-category of k-module spectra, which we will view as *left* modules over k. The construction $M \mapsto \widehat{T}_p(M)$ is an exact functor from LMod_k to itself, which does not commute with filtered colimits. Let $T_p : \mathrm{LMod}_k \to \mathrm{LMod}_k$ be a left Kan extension of $\widehat{T}_p | \mathrm{LMod}_k^{\mathrm{perf}}$, where $\mathrm{LMod}_k^{\mathrm{perf}}$ is the full subcategory of LMod_k spanned by the perfect k-modules. Then T_p commutes with colimits, and we have a natural transformation $T_p \to \widehat{T}_p$ which induces an equivalence $T_p(M) \to \widehat{T}_p(M)$ whenever $M \in \mathrm{LMod}_k$ is perfect. Using Proposition A.7.1.2.4, we deduce that T_p is given by the formula $M \mapsto B \otimes_k M$, where $B \in {}_k\mathrm{BMod}_k(\mathrm{Sp})$ is a k-bimodule spectrum. Unwinding the definitions, we obtain an equivalence of left k-modules $B \simeq T_p(k) \simeq \widehat{T}_p(k) \simeq k^{tG}$, where $G = \mathbf{Z}/p\,\mathbf{Z}$ acts trivially on $k \simeq k^{\otimes p} \in \mathrm{LMod}_k$.

Construction 2.2.6. Suppose that k is a discrete commutative ring and let $G = \mathbf{Z}/p\,\mathbf{Z}$. Then we have canonical isomorphisms $\pi_i k^{tG} \simeq \widehat{\mathrm{H}}^{-i}(G;k)$, where $\widehat{\mathrm{H}}^{i}(G;\bullet)$ denotes the usual Tate cohomology functor. In particular, we have a canonical isomorphism of $\pi_1 k^{tG} \simeq \widehat{\mathrm{H}}^{-1}(G;k)$ with the kernel of the norm map

$$k \simeq \mathrm{H}_0(G;k) \to \mathrm{H}^0(G;k) \simeq k$$

, which is given by multiplication by p. If we assume that k is a ring of characteristic p (that is, that $p = 0$ in k), then the element $1 \in k$ gives a canonical element of $\pi_1 k^{tG}$, which induces a map of *right* k-module spectra $k \to B[-1]$. It follows that for each $A \in \mathrm{CAlg}_k$ we obtain a canonical map of spectra

$$A \simeq k \otimes_k A \to B[-1] \otimes_k A \simeq T_p(A)[-1] \to \widehat{T}_p(A)[-1] = (A^{\otimes p})^{tG}[-1] \to A^{\otimes p}_G \to \mathrm{Sym}^p(A) \to A.$$

We will denote this map by $P^0 : A \to A$. This construction is functorial in A, so that we can regard P^0 as an endomorphism of the forgetful functor $\mathrm{CAlg}_k \to \mathrm{Sp}$.

Remark 2.2.7. Let k be a commutative ring of characteristic p and let A be an $\mathbb{E}_\infty$-algebra over k. Then the map of spectra $P^0 : A \to A$ induces a homomorphism of abelian groups $\pi_n A \to \pi_n A$ for every integer n, which we will also denote by P^0. Note that a class $\eta \in \pi_n A$ is represented by a map of k-modules $k[n] \to A$, and that $P^0(\eta)$ corresponds to the composition

$$k[n] \to T_p(k[n])[-1] \simeq \widehat{T}_p(k[n]) \simeq (k[pn]^{tG})[-1] \to k[pn]_G \to A^{\otimes p}_G \to \mathrm{Sym}^p(A) \to A.$$

If $n > 0$, then the map $k[n] \to k[pn]_G$ is nullhomotopic (since $\pi_n k[pn]_G \simeq \mathrm{H}_{n-pn}(G;k) \simeq 0$), so that $P^0(\eta) = 0$. If $n = 0$, then the map $P^0 : \pi_0 A \to \pi_0 A$ coincides with the Frobenius map $x \mapsto x^p$.

Warning 2.2.8. In the situation of Construction 2.2.6, the map $P^0 : A \to A$ is a morphism of spectra, but not of k-module spectra. For example. if A is discrete, then Remark 2.2.7 allows us to identify P^0 with the Frobenius map $x \mapsto x^p$. This map is generally not k-linear: for $\lambda \in k$, we have $(\lambda x)^p = \lambda^p x^p$, which is generally not equal to λx^p.

Remark 2.2.9. Construction 2.2.6 can be generalized: given any class $x \in \widehat{\mathrm{H}}^{n-1}(\mathbf{Z}/p\,\mathbf{Z};k)$, we obtain an associated map $P(x) : A \to A[n]$, which induces group homomorphisms $\pi_m A \to \pi_{m-n} A$. These operations depend functorially on A and generate an algebra (the *extended Steenrod algebra*) of "power operations" which act on the homotopy groups of every $\mathbb{E}_\infty$-algebra over k.

Warning 2.2.10. If k is a discrete commutative ring of characteristic p and A is a discrete commutative k-algebra, then the Frobenius map $P^0 : A \to A$ is a ring homomorphism. If A is not discrete, then the map $P^0 : A \to A$ usually cannot be refined to a map of $\mathbb{E}_\infty$-rings. In fact, P^0 need not even induce a ring homomorphism at the level of homotopy groups. We have instead a Cartan formula

$$P^0(xy) = \sum_{i+j=0} P^i(x) P^j(y),$$

where P^n denotes the power operation obtained from a suitable generator of the Tate cohomology group $\widehat{\mathrm{H}}^{(2p-2)n-1}(G;k)$ (see Remark 2.2.9).

Remark 2.2.11. Let k be a commutative ring of characteristic p and let A be an $\mathbb{E}_\infty$-algebra over k. We can identify the ∞-category Mod_A of A-modules with the stabilization of the ∞-category $\mathrm{CAlg}^{\mathrm{aug}}_A$ of augmented A-algebras. It follows that P^0 induces a natural transformation from the forgetful functor $\mathrm{Mod}_A \to \mathrm{Sp}$ to itself. Proposition A.7.1.2.4 implies that the ∞-category of colimit-preserving functors $\mathrm{Mod}_A \to \mathrm{Sp}$ can be identified with the ∞-category Mod_A. Under this equivalence, the forgetful functor $\mathrm{Mod}_A \to \mathrm{Sp}$ correponds to A, regarded as a module over itself. It follows that P^0 determines an endomorphism of A as an A-module, which is given by multiplication by an element $\eta \in \pi_0 A$. This element is characterized by the requirement that for every A-module M and every element $(a, x) \in \pi_n(A \oplus M)$, we have $P^0(a, x) = (P^0(a), \eta x)$. Taking $M = A[1]$, $n = 1$, and x to be a generator of $\pi_1 M \simeq \pi_0 A$, we conclude that $\eta = 0$ (since P^0 acts trivially on $\pi_1(A \oplus M)$, by Remark 2.2.7).

Notation 2.2.12. Let k be a commutative ring. For every Kan complex X, we let $C^*(X; k)$ denote the $\mathbb{E}_\infty$-algebra over k of cochains on X. If X is equipped with a base point, then there is a natural augmentation on the algebra $C^*(X; k)$; we will denote the fiber of this augmentation by $C^*_{\mathrm{red}}(X; k)$. We have canonical isomorphisms

$$\pi_i C^*(X; k) \simeq \mathrm{H}^{-i}(X; k) \qquad \pi_i C^*_{\mathrm{red}}(X; k) \simeq \mathrm{H}^{-i}_{\mathrm{red}}(X; k),$$

where $\mathrm{H}^{-i}(X; k)$ and $\mathrm{H}^{-i}_{\mathrm{red}}(X; k)$ denote the cohomology and reduced cohomology groups of X, respectively.

Suppose now that X is an Eilenberg-MacLane space $K(\mathbf{Z}/p\mathbf{Z}; n)$ (when $n = 0$, this means that X is homotopy equivalent to $\mathbf{Z}/p\mathbf{Z}$, as a discrete space). The identity map id_X determines a cohomology class $\eta_n \in \mathrm{H}^n_{\mathrm{red}}(X; \mathbf{Z}/p\mathbf{Z})$. Multiplication by η_n determines a k-module homomorphism $\phi_n : k \to \mathrm{H}^n_{\mathrm{red}}(X; k)$. Elementary obstruction theory shows that this map is an isomorphism for $n > 0$, and that the cohomology groups $\mathrm{H}^i(X; k)$ vanish for $0 < i < n$.

We will need the following well-known fact concerning the behavior of the power operation P^0:

Proposition 2.2.13. *Let k be a commutative ring of characteristic p. Let $n \geq 0$ and let X be an Eilenberg-MacLane space $K(\mathbf{Z}/p\mathbf{Z}, n)$. Then the diagram*

$$\begin{tikzcd}
k \arrow[r, "F"] \arrow[d, "\phi_n"] & k \arrow[d, "\phi_n"] \\
\mathrm{H}^n_{\mathrm{red}}(X; k) \arrow[r, "P^0"] & \mathrm{H}^n_{\mathrm{red}}(X; k)
\end{tikzcd}$$

is commutative, where $F : k \to k$ denotes the Frobenius map $x \mapsto x^p$ and ϕ_n is defined as in Notation 2.2.12.

Proof. We proceed by induction on n. When $n = 0$, the desired result follows from Remark 2.2.7. Assume that $n > 0$, and let Y be the Eilenberg-MacLane space $K(\mathbf{Z}/p\mathbf{Z}, n-1)$. We have a pullback diagram of pointed spaces

$$\begin{tikzcd}
Y \arrow[r] \arrow[d] & * \arrow[d] \\
* \arrow[r] & X
\end{tikzcd}$$

which induces a diagram of spectra

$$\begin{tikzcd}
C^*_{\mathrm{red}}(Y) & 0 \arrow[l] \\
0 \arrow[u] & C^*_{\mathrm{red}}(X) \arrow[l] \arrow[u]
\end{tikzcd}$$

given by a map $C^*_{\mathrm{red}}(X) \to C^*_{\mathrm{red}}(Y)[-1]$. From this we obtain a transgression map $t : \mathrm{H}^n_{\mathrm{red}}(X; k) \to \mathrm{H}^{n-1}_{\mathrm{red}}(Y; k)$. Note that $\phi_{n-1} \simeq t \circ \phi_n$. It follows that t is injective (and an isomorphism if $n > 1$). It follows from naturality that the diagram

$$\begin{tikzcd}
\mathrm{H}^n_{\mathrm{red}}(X; k) \arrow[r, "P^0"] \arrow[d, "t"] & \mathrm{H}^n_{\mathrm{red}}(X; k) \arrow[d, "t"] \\
\mathrm{H}^{n-1}_{\mathrm{red}}(Y; k) \arrow[r, "P^0"] & \mathrm{H}^{n-1}_{\mathrm{red}}(Y; k)
\end{tikzcd}$$

commutes. We are therefore reduced to verifying the commutativity of the diagram

$$\begin{tikzcd} k \arrow[r, "F"] \arrow[d, "\phi_{n-1}"] & k \arrow[d, "\phi_{n-1}"] \\ \mathrm{H}^{n-1}_{\mathrm{red}}(Y;k) \arrow[r, "P^0"] & \mathrm{H}^{n-1}_{\mathrm{red}}(Y;k), \end{tikzcd}$$

which follows from the inductive hypothesis. □

Corollary 2.2.14. *Let $k = \mathbf{F}_p$ and let $X \in \mathcal{S}$ be arbitrary. Then $P^0 : \mathrm{H}^n(X;k) \to \mathrm{H}^n(X;k)$ is the identity map.*

Proof. Every cohomology class $\eta \in \mathrm{H}^n(X;k)$ classifies a map $X \to K(\mathbf{Z}/p\,\mathbf{Z}, n)$. We may therefore replace X by $K(\mathbf{Z}/p\,\mathbf{Z}, n)$ and thereby reduce to the situation of Proposition 2.2.13. □

Construction 2.2.15. Let k be a commutative ring of characteristic $p > 0$, let

$$\theta : \mathrm{CAlg}_k = \mathrm{CAlg}(\mathrm{Mod}_k(\mathrm{Sp})) \to \mathrm{Sp}$$

be the forgetful functor, and let $\mathrm{Free} : \mathrm{Sp} \to \mathrm{CAlg}_k$ be a left adjoint to θ. Let P^0 be the endomorphism of θ defined in Construction 2.2.6. Since $\mathrm{Fun}(\mathrm{CAlg}_k, \mathrm{Sp})$ is a stable ∞-category, we can also regard the difference $\mathrm{id} - P^0$ as an endomorphism of θ. Passing to left adjoints, we obtain a natural transformation $\wp : \mathrm{Free} \to \mathrm{Free}$, which we will refer to as the *Artin-Schreier transformation*.

Example 2.2.16. Let k be a commutative ring of characteristic $p > 0$ and let S denote the sphere spectrum. Then $\mathrm{Free}(S) \simeq k\{x\}$ is the free $\mathbb{E}_\infty$-algebra over k on one generator; in particular, $k\{x\}$ is a connective $\mathbb{E}_\infty$-ring with $\pi_0 k\{x\} \simeq k[x]$. Using Remark 2.2.7, we deduce that the Artin-Schreier transformation $\wp : \mathrm{Free} \to \mathrm{Free}$ induces a map $k\{x\} \to k\{x\}$ which is determined, up to homotopy, by the requirement that $x \mapsto x - x^p \in k[x] \simeq \pi_0 k\{x\}$.

Note that the free algebra functor $\mathrm{Free} : \mathrm{Sp} \to \mathrm{CAlg}_k$ carries the zero spectrum to the object $k \simeq \mathrm{Free}(0)$. It follows that Free induces a functor

$$\mathrm{Sp} \simeq \mathrm{Sp}_{/0} \to (\mathrm{CAlg}_k)_{/k} = \mathrm{CAlg}_k^{\mathrm{aug}},$$

which we will also denote by Free.

If $A \in \mathrm{CAlg}_k^{\mathrm{aug}}$ and I is the augmentation ideal of A, then we have a canonical homotopy equivalence

$$\mathrm{Map}_{\mathrm{CAlg}_k^{\mathrm{aug}}}(\mathrm{Free}(M), A) \simeq \mathrm{Map}_{\mathrm{Sp}}(M, I)$$

for every spectrum M. In particular, if the augmentation ideal I satisfies $\pi_m I \simeq 0$ for $m > -n$, then the mapping space $\mathrm{Map}_{\mathrm{CAlg}_k^{\mathrm{aug}}}(\mathrm{Free}(S^{-n}), A)$ homotopy equivalent to the discrete space $\pi_{-n} I$. It follows that if X is an Eilenberg-MacLane space $K(\mathbf{Z}/p\,\mathbf{Z}, n)$, then the mapping space $\mathrm{Map}_{\mathrm{CAlg}_k^{\mathrm{aug}}}(\mathrm{Free}(S^{-n}), C^*(X;k))$ is homotopy equivalent to the discrete set $\mathrm{H}^n_{\mathrm{red}}(X;k)$. The morphism $\phi_n : k \to \mathrm{H}^n_{\mathrm{red}}(X;k)$ of Notation 2.2.12 determines a map $k \to \mathrm{Map}_{\mathrm{CAlg}_k^{\mathrm{aug}}}(\mathrm{Free}(S^{-n}), C^*(X;k))$, which is a homotopy equivalence if $n > 0$ (and the inclusion of a summand when $n = 0$).

The element $1 \in k$ determines a map of augmented k-algebras $\mathrm{Free}(S^{-n}) \to C^*(X;k)$. Using Proposition 2.2.13, we see that the composition

$$\mathrm{Free}(S^{-n}) \xrightarrow{\wp} \mathrm{Free}(S^{-n}) \to C^*(X;k)$$

is given by $0 \in \mathrm{H}^n_{\mathrm{red}}(X;k)$: that is, it factors through the augmentation map $\mathrm{Free}(S^{-n}) \to k$. Since the mapping space $\mathrm{Map}_{\mathrm{CAlg}_k^{\mathrm{aug}}}(\mathrm{Free}(S^{-n}), C^*(X;k))$ is homotopy equivalent to a discrete space, this choice of factorization is essentially unique. We obtain a commutative diagram

$$\begin{tikzcd} \mathrm{Free}(S^{-n}) \arrow[r, "\wp"] \arrow[d] & \mathrm{Free}(S^{-n}) \arrow[d] \\ k \arrow[r] & C^*(X;k) \end{tikzcd}$$

of $\mathbb{E}_\infty$-algebras over k. We will need the following fundamental result:

Theorem 2.2.17 (Mandell)**.** *Let k be a commutative ring of characteristic p and let $X = K(\mathbf{Z}/p\mathbf{Z}, n)$ be an Eilenberg-MacLane space. Then the diagram*

$$\begin{tikzcd} \mathrm{Free}(S^{-n}) \arrow[r, "\wp"] \arrow[d] & \mathrm{Free}(S^{-n}) \arrow[d] \\ k \arrow[r] & C^*(X;k) \end{tikzcd}$$

is a pushout square in CAlg_k*.*

For a proof, we refer the reader to Theorem 6.2 of [54].

Remark 2.2.18. Theorem 2.2.17 asserts that for an Eilenberg-MacLane space $X = K(\mathbf{Z}/p\mathbf{Z}, n)$, the cochain algebra $C^*(X;k)$ can be described as the $\mathbb{E}_\infty$-algebra $\mathrm{Free}(S^{-n}) \otimes_{\mathrm{Free}(S^{-n})} k$, which is generated by a single element η (in degree $-n$) which is fixed by the operation P^0. The proof given in [54] proceeds by explicitly computing the homotopy groups of the algebra $\mathrm{Free}(S^{-n}) \otimes_{\mathrm{Free}(S^{-n})} k$ and comparing them with the cohomology groups of X. It would be desirable to have a less computationally intensive proof, but we do not know of one.

2.3 Finitely Constructible Sheaves

Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_\mathcal{X})$ be a spectral Deligne-Mumford stack. Then we can think of $\mathcal{X}$ as the ∞-category of $\mathcal{S}$-valued sheaves on $\mathfrak{X}$ (with respect to the étale topology). In this section, we will introduce a full subcategory $\mathcal{X}^{\mathrm{fc}}$, which we call the ∞-*category of finitely constructible sheaves on* $\mathcal{X}$. Roughly speaking, a sheaf $\mathcal{F} \in \mathcal{X}$ is finitely constructible if it is truncated, its stalks have finite homotopy groups, and it is locally constant along each stratum of a suitable stratification of $\mathfrak{X}$ (actually, we demand the existence of this stratification only locally on $\mathfrak{X}$). However, it will be convenient to begin with a much more general definition.

Definition 2.3.1. Let $\mathcal{X}$ be a coherent ∞-topos. We will say that an object $X \in \mathcal{X}$ is *finitely constructible* if it is coherent and n-truncated for some integer n. We let $\mathcal{X}^{\mathrm{fc}}$ denote the full subcategory of $\mathcal{X}$ spanned by the finitely constructible objects.

Example 2.3.2. A space X is a finitely constructible object of the ∞-topos $\mathcal{S}$ if and only if it satisfies the following conditions:

(1) The space X is n-truncated for some integer n.

(2) The set $\pi_0 X$ is finite.

(3) For each vertex $x \in X$ and each integer $m \geq 1$, the group $\pi_m(X, x)$ is finite.

We will say that a space X is π-*finite* if it satisfies these conditions.

Remark 2.3.3. Let $\mathcal{X}$ be a coherent ∞-topos. If $X \in \mathcal{X}$ is finitely constructible and $U \in \mathcal{X}$ is a coherent object, then $X \times U$ is finitely constructible as an object of $\mathcal{X}_{/U}$.

Definition 2.3.4. Let $\mathcal{X}$ be a locally coherent ∞-topos. We will say that an object $X \in \mathcal{X}$ is *finitely constructible* if, for every coherent object $U \in \mathcal{X}$, the product $X \times U$ is a finitely constructible object of $\mathcal{X}_{/U}$. (When $\mathcal{X}$ is coherent, this definition is equivalent to Definition 2.3.1, by virtue of Remark 2.3.3.)

Finite constructibility is a local condition:

Proposition 2.3.5. *Let $\mathcal{X}$ be a locally coherent ∞-topos and let $X \in \mathcal{X}$ be an object. Suppose that there exists a covering $\{U_\alpha \in \mathcal{X}\}_{\alpha \in A}$ of $\mathcal{X}$ such that each $U_\alpha \in \mathcal{X}$ is coherent and each product $U_\alpha \times X$ is a finitely constructible object of $\mathcal{X}_{/U_\alpha}$. Then $X \in \mathcal{X}$ is finitely constructible.*

Proof. We may assume without loss of generality that $\mathcal{X}$ is coherent. Since $\mathcal{X}$ is quasi-compact, we may assume without loss of generality that the set A is finite. It follows that there exists an integer n such that each $U_\alpha \times X$ is an n-truncated object of $\mathcal{X}_{/U_\alpha}$. It follows from Proposition T.6.2.3.17 that X is an n-truncated object of $\mathcal{X}$. The coherence of X follows from Corollary VII.3.11. $\square$

Proposition 2.3.6. *Let $f : (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ be a map of spectral Deligne-Mumford stacks. Then the pullback functor f^* carries finitely constructible objects of $\mathcal{Y}$ to finitely constructible objects of $\mathcal{X}$.*

Proof. The assertion is local on $\mathcal{X}$ and $\mathcal{Y}$, by Proposition 2.3.5. We may therefore assume that $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ and $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ are affine. Since it is clear that f^* preserves n-truncatedness, it suffices to show that f^* preserves coherence. This follows from Example VIII.1.4.6. $\square$

Notation 2.3.7. If R is an $\mathbb{E}_\infty$-ring, we let $\mathcal{S}\mathrm{hv}_R^{\text{ét}}$ denote the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}_R^{\text{ét}}, \mathcal{S})$ spanned by those functors which are sheaves with respect to the étale topology. This is a coherent ∞-topos; we will denote its subcategory of finitely constructible objects by $\mathcal{S}\mathrm{hv}_R^{\mathrm{fc}}$. It follows from Proposition 2.3.6 that we can regard the construction $R \mapsto \mathcal{S}\mathrm{hv}_R^{\mathrm{fc}}$ as a covariant functor from CAlg to the ∞-category $\mathcal{C}\mathrm{at}_\infty$ of small ∞-categories. We will denote this functor by $\mathcal{S}\mathrm{hv}^{\mathrm{fc}}$.

We can now formulate the first main result of this section.

Theorem 2.3.8. *The functor $\mathcal{S}\mathrm{hv}^{\mathrm{fc}} : \mathrm{CAlg} \to \mathcal{C}\mathrm{at}_\infty$ preserves small filtered colimits.*

The proof of Theorem 2.3.8 will require a number of preliminaries. First, we establish some compactness properties of finitely constructible sheaves.

Proposition 2.3.9. *Let $\mathcal{X}$ be an n-coherent ∞-topos for some $n \geq 0$, and let $\Gamma : \mathcal{X} \to \mathcal{S}$ be the global sections functor (that is, Γ is the functor corepresented by the final object $\mathbf{1} \in \mathcal{X}$). Then the restriction of Γ to $\tau_{\leq n-1}\,\mathcal{X}$ commutes with filtered colimits.*

Corollary 2.3.10. *Let $\mathcal{X}$ be a locally n-coherent ∞-topos. Then:*

(1) *If $U \in \mathcal{X}$ is an n-coherent object, then $\tau_{\leq n-1}U$ is a compact object of $\tau_{\leq n-1}\,\mathcal{X}$.*

(2) *The ∞-category $\tau_{\leq n-1}\,\mathcal{X}$ is generated, under small colimits, by objects of the form $\tau_{\leq n-1}U$, where $U \in \mathcal{X}$ is n-coherent.*

(3) *The ∞-category $\tau_{\leq n-1}\,\mathcal{X}$ is compactly generated.*

Proof. Assertion (1) follows immediately from Proposition 2.3.9. Consider an arbitrary object $X \in \mathcal{X}$. Since $\mathcal{X}$ is locally coherent, we can choose a hypercovering $X_\bullet$ of X such that each X_m is a coproduct of n-coherent objects of $\mathcal{X}$. If we assume that $X \in \tau_{\leq n-1}\,\mathcal{X}$, then the map

$$\tau_{\leq n-1}|X_\bullet| \to \tau_{\leq n-1}X \simeq X$$

is an equivalence. Consequently, X is the geometric realization of a simplicial object of $\tau_{\leq n-1}\,\mathcal{X}$, each term of which is a coproduct of objects having the form $\tau_{\leq n-1}U$, where U is n-coherent. This proves (2). Assertion (3) follows immediately from (1) and (2). $\square$

Remark 2.3.11. Let $\mathcal{X}$ be a locally $(n+1)$-coherent ∞-topos. Let $\mathcal{C}$ be the smallest full subcategory of $\tau_{\leq n}\,\mathcal{X}$ which is closed under finite colimits and contains $\tau_{\leq n}U$ for every $(n+1)$-coherent object $U \in \mathcal{X}$. Then $\mathcal{C}$ is the full subcategory of $\tau_{\leq n}\,\mathcal{X}$ spanned by the compact objects. To see this, we first note that every object of $\mathcal{C}$ is compact in $\tau_{\leq n}\,\mathcal{X}$. It follows from Proposition T.5.3.5.11 that the inclusion $\mathcal{C} \hookrightarrow \tau_{\leq n}\,\mathcal{X}$ extends to a fully faithful embedding $\phi : \mathrm{Ind}(\mathcal{C}) \to \tau_{\leq n}\,\mathcal{X}$. Proposition T.5.5.1.9 implies that ϕ preserves small colimits, so that ϕ is an equivalence of ∞-categories by Corollary 2.3.10. It follows that the ∞-category of compact objects of $\tau_{\leq n}\,\mathcal{X}$ can be identified with an idempotent completion of $\mathcal{C}$. In particular, every object $X \in \tau_{\leq n}\,\mathcal{X}$ is the colimit (in $\tau_{\leq n}\,\mathcal{X}$) of a diagram $p : \mathrm{Idem} \to \mathcal{C}$, where Idem is the simplicial set of Definition T.4.4.5.2. Since $\tau_{\leq n}\,\mathcal{X}$ is equivalent to an $(n+1)$-category, X is the colimit of the restriction of p to the $(n+1)$-skeleton of Idem, which is a finite simplicial set. Since $\mathcal{C}$ is closed under finite colimits, we conclude that $X \in \mathcal{C}$ as desired.

Corollary 2.3.12. *Let $f : (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ be a morphism of spectral Deligne-Mumford stacks (or spectral schemes) and let $n \geq 0$ be an integer. Suppose that f is n-quasi-compact. Then:*

(1) *The pullback functor f^* carries n-coherent objects to n-coherent objects.*

(2) *The pullback functor f^* carries compact objects of $\tau_{\leq n-1}\,\mathcal{Y}$ to compact objects of $\tau_{\leq n-1}\,\mathcal{X}$.*

(3) *The pushforward functor $f_* : \tau_{\leq n-1}\,\mathcal{X} \to \tau_{\leq n-1}\,\mathcal{X}$ commutes with filtered colimits.*

Proof. Assertion (1) follows from the definition of n-quasi-compactness (see Proposition VIII.1.4.4). Assertion (2) follows from (1) and Remark 2.3.11, and assertion (3) from (2) and Proposition T.5.5.7.2. $\square$

Remark 2.3.13. Proposition 2.3.9 implies that if $\mathcal{F}$ is a finitely constructible object of a coherent ∞-topos $\mathcal{X}$, then $\mathcal{F}$ is a compact object of $\tau_{\leq n}\,\mathcal{X}$ for some $n \geq 0$. It follows that the ∞-category $\mathcal{X}^{\mathrm{fc}}$ is essentially small.

We now turn to the proof of Proposition 2.3.9.

Lemma 2.3.14. *Let $n \geq 0$ be an integer, and let $\mathcal{X}$ be an ∞-topos which we assume to be locally $(n-1)$-coherent if $n > 0$. Let $f : U \to X$ be a morphism in $\mathcal{X}$. If f is $(n-2)$-truncated, X is n-coherent, and U is $(n-1)$-coherent, then U is n-coherent.*

Proof. We proceed by induction on n. In the case $n = 0$, the map f is an equivalence and the n-coherence of U follows from the n-coherence of X. Assume therefore that $n > 0$, so that U is quasi-compact. We wish to show that if we are given maps $V_1 \to U$, $V_2 \to U$, where V_1 and V_2 are $(n-1)$-coherent objects of $\mathcal{X}$, then the fiber product $V_1 \times_U V_2$ is also $(n-1)$-coherent. Since U is $(n-1)$-coherent, the fiber product $V_1 \times_U V_2$ is automatically $(n-2)$-coherent. The map $V_1 \times_U V_2 \to V_1 \times_X V_2$ is a pullback of the diagonal map $U \to U \times_X U$ and therefore $(n-3)$-truncated. Since X is n-coherent, the fiber product $V_1 \times_X V_2$ is $(n-1)$-coherent, and the desired result follows from the inductive hypothesis. $\square$

Remark 2.3.15. Let $\mathcal{X}$ be a coherent ∞-topos and let $U \in \mathcal{X}$ be an n-truncated object. Then U is coherent if and only if it is $(n+1)$-coherent. This follows by applying Lemma 2.3.14 in the case where X is a final object of $\mathcal{X}$.

Proof of Proposition 2.3.9. We proceed by induction on n. In the case $n = 0$, our assumption guarantees that $\mathcal{X}$ is quasi-compact and the desired result follows immediately from the definition. Let us therefore assume that $n > 0$. Let $\mathcal{J}$ be a small filtered ∞-category, let $U, U' : \mathrm{Fun}(\mathcal{J}, \tau_{\leq n-1}\,\mathcal{X}) \to \mathcal{S}$ be given by

$$U(F) = \varinjlim_{J \in \mathcal{J}} \Gamma(F(J)) \qquad U'(F) = \Gamma(\varinjlim_{J \in \mathcal{J}} F(J)).$$

There is an evident natural transformation $\beta_F : U(F) \to U'(F)$; we wish to show that β_F is a homotopy equivalence. Assume for the moment that β_F is surjective on connected components. It then suffices to show that for every pair of points $\eta, \eta' \in U(F)$, the map β_F induces a homotopy equivalence of path spaces

$\phi : \{\eta\} \times_{U(F)} \{\eta'\} \to \{\eta\} \times_{U'(F)} \{\eta'\}$. Since $\mathcal{J}$ is filtered, we may assume without loss of generality that η and η' are the images of points $\eta_0, \eta'_0 \in \Gamma(F(J))$. Since $\mathcal{J}$ is filtered, the map $\mathcal{J}_{J/} \to \mathcal{J}$ is left cofinal; we may therefore replace $\mathcal{J}$ by $\mathcal{J}_{J/}$ and thereby assume that J is a final object of $\mathcal{J}$. In this case, η_0 and η'_0 determine natural transformations $* \to F$, where $*$ denotes the constant functor $\mathcal{J} \to \mathcal{X}$ taking the value $\mathbf{1}$. Let $F' = * \times_F *$. Unwinding the definitions, we see that ϕ can be identified with the map $\beta_{F'}$. The desired result then follows from the inductive hypothesis, since $\mathcal{X}$ is $(n-1)$-truncated and F' takes values in $\tau_{\leq n-2}\,\mathcal{X}$.

It remains to prove that β_F is surjective on connected components. Choose a point $\eta \in U'(F)$, corresponding to a map $\alpha : \mathbf{1} \to \varinjlim_{J \in \mathcal{J}} F(J)$. We wish to show that α factors (up to homotopy) through $F(J)$ for some $J \in \mathcal{J}$. Note that the map $\coprod_{J \in \mathcal{J}} F(J) \to \varinjlim_{J \in \mathcal{J}} F(J)$ is an effective epimorphism. Since $\mathcal{X}$ is locally $(n-1)$-coherent, there exists a collection of $(n-1)$-coherent objects $\{U_i \in \mathcal{X}\}_{i \in I}$ such that $\coprod_{i \in I} U_i \to \mathbf{1}$ is an effective epimorphism and each of the composite maps $U_i \to \mathbf{1} \to \varinjlim_{J \in \mathcal{J}} F(J)$ factors through $F(J_i)$, for some $J_i \in \mathcal{J}$. Since $\mathcal{X}$ is quasi-compact, we can assume that the set I is finite. Let $U = \coprod_{i \in I} U_i$. Since $\mathcal{J}$ is filtered, there exists an object $J_0 \in \mathcal{J}$ and maps $J_i \to J_0$ for $i \in I$, so that the composite map $U \to \mathbf{1} \xrightarrow{\alpha} \varinjlim_{J \in \mathcal{J}} F(J)$ factors through $F(J_0)$. Since $\mathcal{J}$ is filtered, the map $\mathcal{J}_{J_0/} \to \mathcal{J}$ is left cofinal; we may therefore replace $\mathcal{J}$ by $\mathcal{J}_{J_0/}$ and thereby reduce to the case where J_0 is an initial object of $\mathcal{J}$. Let $F_0 : \mathcal{J} \to \mathcal{X}$ be the constant functor taking the value $F(\mathcal{J}_0)$, and let $F_\bullet$ be the simplicial object of $\mathrm{Fun}(\mathcal{J}, \mathcal{X})$ given by the Čech nerve of the map $F_0 \to F$. Let $U_\bullet$ be the Čech nerve of the map $U \to \mathbf{1}$, so that we obtain a map of simplicial objects of $\mathcal{X}$ $\gamma : U_\bullet \to \varinjlim_{J \in \mathcal{J}} F_\bullet(J)$. We will prove the following assertion by induction on $m \geq 0$:

$(*)$ Let $\mathbf{\Delta}_{s,\leq m}$ denote the subcategory of $\mathbf{\Delta}$ whose objects are linearly ordered sets $[j]$ for $j \leq m$, and whose morphisms are given by injective maps $[j] \to [j']$. Let $U_\bullet^{\leq m}$ be the restriction of $U_\bullet$ to $\mathrm{N}(\mathbf{\Delta}_{s,\leq m})^{op}$, define $F_\bullet^{\leq m}$ similarly, and let $\gamma^{\leq m} : U_\bullet^{\leq m} \to \varinjlim_{J \in \mathcal{J}} F_\bullet^{\leq m}(J)$ be the map induced by γ. Then there exists an object $J_m \in \mathcal{J}$ such that $\gamma^{\leq m}$ factors through $F_\bullet^{\leq m}(J_m)$.

Assertion $(*)$ is obvious when $m = 0$, since the functor F_0 is constant. Assume that $\gamma^{\leq m-1}$ factors through $F_\bullet^{\leq m-1}(J_{m-1})$ for some $J_{m-1} \in \mathcal{J}$. Replacing $\mathcal{J}$ by $\mathcal{J}_{J_{m-1}/}$, we may assume that J_{m-1} is an initial object of $\mathcal{J}$, so that we have a canonical map $\delta_J : U_\bullet^{\leq m-1} \to F_\bullet^{\leq m-1}(J)$ for all $J \in \mathcal{J}$. Let $M(J) \in \mathcal{X}$ denote the mth matching object of $F_\bullet(J)$, for each $j \in J$, so that δ_J determines a map $\theta : U_m \to M(J)$. Using Proposition T.A.2.9.14, we see that promoting δ_J to a natural transformation $U_\bullet^{\leq m} \to F_\bullet^{\leq m}(J)$ is equivalent to choosing a point of the mapping space $\mathrm{Map}_{\mathcal{X}_{/M(J)}}(U_m, F_m(J)) \simeq \mathrm{Map}_{\mathcal{X}_{/U_m}}(U_m, F_m(J) \times_{M(J)} U_m)$. Consequently, to prove $(*)$, it suffices to show that the map

$$\varinjlim \mathrm{Map}_{\mathcal{X}_{/U_m}}(U_m, F_m(J) \times_{M(J)} U_m) \to \mathrm{Map}_{\mathcal{X}_{/U_m}}(U_m, \varinjlim F_m(J) \times_{M(J)} U_m)$$

is a homotopy equivalence. Since $\mathcal{X}$ is n-coherent and $U \in \mathcal{X}$ is $(n-1)$-coherent, the ∞-topos $\mathcal{X}_{/U_m}$ is $(n-1)$-coherent. By the inductive hypothesis, it suffices to show that the the objects $F_m(J) \times_{M(J)} U_m$ are $(n-2)$-truncated objects of $\mathcal{X}_{/U_m}$. For this, it suffices to show that the map $F_m(J) \to M(J)$ is $(n-2)$-truncated. This map is a pullback of the diagonal $F(J) \to F(J)^{\partial \Delta^m}$, and therefore $(n-m-1)$-truncated (since $F(J)$ is assumed to be $(n-1)$-truncated). This completes the proof of $(*)$.

Applying $(*)$ in the case $m = n$ and composing with the natural map $\varinjlim F_\bullet^{\leq n} \to F$, we deduce the existence of an object $J_n \in \mathcal{J}$ and a commutative diagram σ :

$$\begin{tikzcd}
\varinjlim U_\bullet^{\leq n} \arrow[r] \arrow[d] & F(J_n) \arrow[d] \\
\mathbf{1} \arrow[r, "\eta"] & \varinjlim_{J \in \mathcal{J}} F(J).
\end{tikzcd}$$

Since the map $U \to \mathbf{1}$ is an effective epimorphism, we deduce that $\tau_{\leq n-1} \varinjlim U_\bullet^{\leq n} \simeq |U_\bullet| \simeq \mathbf{1}$. Applying the truncation functor $\tau_{\leq n-1}$ to the diagram σ, we conclude that η factors through $F(J_n)$. $\square$

Lemma 2.3.16. *Let $U : \mathrm{CAlg} \to \mathcal{C}\mathrm{at}_\infty$ be the functor given by $U(R) = \mathrm{CAlg}^{\text{ét}}_R$. Then U commutes with filtered colimits.*

Proof. Suppose we are given a filtered diagram $F : \mathcal{J} \to \mathrm{CAlg}$ having colimit R. We wish to prove that the canonical map

$$\phi : \varinjlim_J \mathrm{CAlg}^{\text{ét}}_{F(J)} \to \mathrm{CAlg}^{\text{ét}}_R$$

is an equivalence of ∞-categories. Since every étale R-algebra is a compact object of CAlg_R, Lemma XII.2.3.4 implies that ϕ is fully faithful. It follows from Proposition VII.8.10 that ϕ is essentially surjective. □

For the next statement, we let ${}^{\mathrm{L}}\mathcal{T}\mathrm{op}$ denote the subcategory of $\widehat{\mathcal{C}\mathrm{at}}_\infty$ whose objects are ∞-topoi and whose morphisms are functors $f^* : \mathcal{X} \to \mathcal{Y}$ which preserve small colimits and finite limits.

Lemma 2.3.17. *Let $\mathcal{S}\mathrm{hv}^{\acute{e}t} : \mathrm{CAlg} \to {}^{\mathrm{L}}\mathcal{T}\mathrm{op}$ be the functor given by $R \mapsto \mathcal{S}\mathrm{hv}^{\acute{e}t}_R$. Then $\mathcal{S}\mathrm{hv}^{\acute{e}t}$ commutes with filtered colimits.*

Proof. Let $\mathcal{X}$ be an ∞-topos and let $F : \mathcal{J} \to \mathrm{CAlg}$ be a filtered diagram having colimit R. We wish to show that the canonical map

$$\theta : \mathrm{Map}_{{}^{\mathrm{L}}\mathcal{T}\mathrm{op}}(\mathcal{S}\mathrm{hv}^{\text{ét}}_R, \mathcal{X}) \to \varprojlim_J \mathrm{Map}_{{}^{\mathrm{L}}\mathcal{T}\mathrm{op}}(\mathcal{S}\mathrm{hv}^{\text{ét}}_{F(J)}, \mathcal{X})$$

is a homotopy equivalence. We have a commutative diagram

$$\begin{tikzcd}
\mathrm{Map}_{{}^{\mathrm{L}}\mathcal{T}\mathrm{op}}(\mathcal{S}\mathrm{hv}^{\text{ét}}_R, \mathcal{X}) \arrow[r, "\theta"] \arrow[d, "\phi"] & \varprojlim_J \mathrm{Map}_{{}^{\mathrm{L}}\mathcal{T}\mathrm{op}}(\mathcal{S}\mathrm{hv}^{\text{ét}}_{F(J)}, \mathcal{X}) \arrow[d, "\psi"] \\
\mathrm{Map}_{\widehat{\mathcal{C}\mathrm{at}}_\infty}((\mathrm{CAlg}^{\text{ét}}_R)^{op}, \mathcal{X}) \arrow[r, "\theta'"] & \varprojlim_J \mathrm{Map}_{\widehat{\mathcal{C}\mathrm{at}}_\infty}((\mathrm{CAlg}^{\text{ét}}_{F(J)})^{op}, \mathcal{X})
\end{tikzcd}$$

where θ' is a homotopy equivalence by Lemma 2.3.16 and the vertical maps are fully faithful by Proposition T.6.2.3.20. To complete the proof, it suffices to show that if we are given a functor $f : (\mathrm{CAlg}^{\text{ét}}_R)^{op} \to \mathcal{X}$ such that $\theta'(f)$ lies in the essential image of ψ, then f belongs to the essential image of ϕ. In view of Proposition T.6.2.3.20, we must verify two conditions:

(1) The functor f is left exact. That is, for every finite simplicial set K and every map $p : K \to (\mathrm{CAlg}^{\text{ét}}_R)^{op}$, we must show that the canonical map $f(\varprojlim(p)) \to \varprojlim(f \circ p)$ is an equivalence in $\mathcal{X}$. Since K is finite, Lemma 2.3.16 implies that p is homotopic to a composition

$$K \xrightarrow{p'} (\mathrm{CAlg}^{\text{ét}}_{F(J)})^{op} \xrightarrow{p''} (\mathrm{CAlg}^{\text{ét}}_R)^{op}$$

for some $J \in \mathcal{J}$. Since p'' is left exact, it suffices to show that $f \circ p''$ is left exact. This follows from our assumption that $\theta'(f)$ belongs to the essential image of ψ (using Proposition T.6.2.3.20).

(2) For every collection of morphisms $\{g_\alpha : R' \to R'_\alpha\}_{\alpha \in A}$ in $\mathrm{CAlg}^{\text{ét}}_R$ which generate a covering sieve on R' (with respect to the étale topology), the induced map $\coprod_{\alpha \in A} f(R'_\alpha) \to f(R')$ is an effective epimorphism in $\mathcal{X}$. Without loss of generality, we may assume that the set A is finite. Using Lemma 2.3.16, we may assume that there is an object $J \in \mathcal{J}$, an object $R'_0 \in \mathrm{CAlg}^{\text{ét}}_{F(J)}$, and morphisms $g_{\alpha,0} : R'_0 \to R'_{\alpha,0}$ in $\mathrm{CAlg}^{\text{ét}}_{F(J)}$ such that each g_α is the image of g_{α_0}. Let X denote the Zariski spectrum $\mathrm{Spec}^{\mathrm{Z}}(\pi_0 R'_0)$. Since the maps $\pi_0 R'_0 \to \pi_0 R'_{\alpha,0}$ are étale , the image of each $\mathrm{Spec}^{\mathrm{Z}}(\pi_0 R'_{\alpha,0})$ is an open subset $U_\alpha \subseteq X$. Let $X' = X - \bigcup_{\alpha \in A} U_\alpha$, so that X' is a closed subset of X defined by an ideal $I \subseteq \pi_0 R'_0$. Since the morphisms g_α are a covering of R', the image of I generates the unit ideal in $\pi_0 R'$. It follows that there is a map $J \to J'$ in $\mathcal{J}$ such that the image of I in $\pi_0(R'_0 \otimes_{F(J)} F(J'))$ generates the unit ideal.

Replacing J by J', we can assume that $X' = \emptyset$, so that the maps $g_{\alpha,0}$ generate a covering sieve in $\mathrm{CAlg}^{\text{ét}}_{F(J)}$. Since $\theta'(f)$ belongs to the essential image of ψ, the induced functor

$$f_J : (\mathrm{CAlg}^{\text{ét}}_{F(J)})^{op} \to (\mathrm{CAlg}^{\text{ét}}_R)^{op} \xrightarrow{f} \mathcal{X}$$

satisfies the hypotheses of Proposition T.6.2.3.20, which implies that the map

$$\coprod_{\alpha \in A} f(R'_\alpha) \simeq \coprod_{\alpha \in A} f_J(R'_{\alpha,0}) \to f_J(R'_0) \simeq f(R')$$

is an effective epimorphism as desired.

□

Lemma 2.3.18. *Let $\mathcal{J}$ be a filtered ∞-category and let $F : \mathcal{J} \to \mathrm{CAlg}$ be a diagram having a colimit R.*

(1) *The canonical map*

$$\mathcal{S}\mathrm{hv}^{\acute{e}t}_R \to \varprojlim_{J \in \mathcal{J}} \mathcal{S}\mathrm{hv}^{\acute{e}t}_{F(J)}$$

is an equivalence of ∞-categories; here the limit is taken with respect to the pushforward functors $\pi_* : \mathcal{S}\mathrm{hv}^{\acute{e}t}_{F(J)} \to \mathcal{S}\mathrm{hv}^{\acute{e}t}_{F(J')}$ associated to maps $J' \to J$ in $\mathcal{J}$.

(2) *For each $n \geq -2$, the canonical map*

$$\tau_{\leq n} \mathcal{S}\mathrm{hv}^{\acute{e}t}_R \to \varprojlim_{J \in \mathcal{J}} \tau_{\leq n} \mathcal{S}\mathrm{hv}^{\acute{e}t}_{F(J)}$$

is an equivalence of ∞-categories.

Proof. Assertion (1) is a reformulation of Lemma 2.3.17 (see Theorem T.6.3.3.1), and assertion (2) follows immediately from (1). □

Lemma 2.3.19. *Let $\mathcal{J}$ be a filtered ∞-category and let $F : \mathcal{J} \to \mathrm{CAlg}$ be a diagram having a colimit R. For every $\mathbb{E}_\infty$-ring R, let $\mathcal{S}\mathrm{hv}^{c,\leq n}_R$ denote the full subcategory of $\mathcal{S}\mathrm{hv}^{\acute{e}t}_R$ spanned by the compact objects of $\tau_{\leq n} \mathcal{S}\mathrm{hv}^{\acute{e}t}_R$. Then the canonical map*

$$\varinjlim_{J \in \mathcal{J}} \mathcal{S}\mathrm{hv}^{c,\leq n}_{F(J)} \to \mathcal{S}\mathrm{hv}^{c,\leq n}_R$$

is an equivalence of ∞-categories.

Proof. Since each $\tau_{\leq n} \mathcal{S}\mathrm{hv}^{\text{ét}}_R$ is compactly generated (Corollary 2.3.10), the desired result follows from Lemma 2.3.18 (see Lemma A.6.3.7.9 and Remark A.6.3.7.10). □

Proof of Theorem 2.3.8. Let R be an $\mathbb{E}_\infty$-ring. For $n \geq -2$, we define a full subcategory $\mathcal{S}\mathrm{hv}^{\mathrm{fc},\leq n}_R \subseteq \mathcal{S}\mathrm{hv}^{\text{ét}}_R$ as follows:

(a) If $n \geq 0$, an object $\mathcal{F} \in \mathcal{S}\mathrm{hv}^{\text{ét}}_R$ belongs to $\mathcal{S}\mathrm{hv}^{\mathrm{fc},\leq n}_R$ if and only if $\mathcal{F}$ is coherent and n-truncated.

(b) An object $\mathcal{F} \in \mathcal{S}\mathrm{hv}^{\text{ét}}_R$ belongs to $\mathcal{S}\mathrm{hv}^{\mathrm{fc},\leq -1}_R$ if and only if it is coherent and there exists a (-1)-truncated map $\mathcal{F} \to \mathcal{F}'$, where $\mathcal{F}'$ is corepresentable by an étale R-algebra.

(c) IIf $n = -2$, then an object $\mathcal{F} \in \mathcal{S}\mathrm{hv}^{\text{ét}}_R$ belongs to $\mathcal{S}\mathrm{hv}^{\mathrm{fc},\leq -2}_R$ if and only if it is corepresentable by an étale R-algebra.

To prove that $\mathcal{S}\mathrm{hv}^{\mathrm{fc}}_R \simeq \varinjlim_J \mathcal{S}\mathrm{hv}^{\mathrm{fc}}_{F(J)}$, it will suffice to show that for each $n \geq -2$, the map

$$\theta : \varinjlim_J \mathcal{S}\mathrm{hv}^{\mathrm{fc}, \leq n}_{F(J)} \to \mathcal{S}\mathrm{hv}^{\mathrm{fc}, \leq n}_R$$

is an equivalence of ∞-categories. It follows from Lemma 2.3.19 that θ is fully faithful. We prove the essential surjectivity by induction on n. The case $n = -2$ follows from Lemma 2.3.16. Assume therefore that $n \geq -1$, and let $\mathcal{F} \in \mathcal{S}\mathrm{hv}^{\mathrm{fc}, \leq n}_R$. Since $\mathcal{F}$ is quasi-compact, we can choose an effective epimorphism $u : \mathcal{F}_0 \to \mathcal{F}$, where $\mathcal{F}_0$ is representable by an étale R-algebra. Let $\mathcal{F}_\bullet$ be the Čech nerve of u. Note that the map $v : \mathcal{F}_1 \to \mathcal{F}_0 \times \mathcal{F}_0$ is $(n-1)$-truncated if $n \geq 0$. If $n = -1$, then $\mathcal{F}_1$ is representable and v is (-1)-truncated. It follows that $\mathcal{F}_n$ belongs to $\mathcal{S}\mathrm{hv}^{\mathrm{fc}, \leq n-1}_R$. Choose $m \geq \max\{n+2, 2\}$ and let $\mathcal{F}'_\bullet$ be the m-skeletal category object $\mathcal{F}_\bullet \,|\, \mathrm{N}(\mathbf{\Delta}_{\leq m})^{op}$. Since $\mathrm{N}(\mathbf{\Delta}_{\leq m})^{op}$ is a finite simplicial set, the inductive hypothesis implies that $\mathcal{F}'_\bullet$ is equivalent to a composition

$$\mathrm{N}(\mathbf{\Delta}_{\leq m})^{op} \xrightarrow{\mathcal{G}'_\bullet} \mathcal{S}\mathrm{hv}^{\mathrm{fc}, \leq n-1}_{F(J)} \xrightarrow{\phi^*} \mathcal{S}\mathrm{hv}^{\mathrm{fc}, \leq n-1}_R$$

for some functor $\mathcal{G}'_\bullet$. For each $j \leq m$, the morphism

$$\beta_j : \mathcal{G}'_j \to \mathcal{G}'_1 \times_{\mathcal{G}'_0} \cdots \times_{\mathcal{G}'_0} \mathcal{G}'_1$$

is such that $\phi^*(\beta_j)$ is an equivalence. Altering our choice of J if necessary, we may assume that each β_j is an equivalence: that is, $\mathcal{G}'_\bullet$ is an m-skeletal category object of $\mathcal{S}\mathrm{hv}^{\mathrm{fc}, \leq n-1}_{F(J)}$. Let $\mathcal{G}_\bullet : \mathrm{N}(\mathbf{\Delta})^{op} \to \mathcal{S}\mathrm{hv}^{\mathrm{fc}, \leq n-1}_{F(J)}$ be a right Kan extension of $\mathcal{G}'_\bullet$. Since ϕ^* is left exact, the image $\phi^* \mathcal{G}_\bullet$ is a right Kan extension of $\mathcal{F}'_\bullet$ so that $\phi^* \mathcal{G}_\bullet \simeq \mathcal{F}_\bullet$ by Proposition XII.2.1.4. Since ϕ^* preserves colimits, we obtain an equivalence

$$\phi^* |\, \mathcal{G}_\bullet \,| \simeq |\phi^* \mathcal{G}_\bullet \,| \simeq |\, \mathcal{F}_\bullet \,| \simeq \mathcal{F}.$$

It will therefore suffice to show that $\mathcal{G} = |\, \mathcal{G}_\bullet \,|$ belongs to $\mathcal{S}\mathrm{hv}^{\mathrm{fc}, \leq n}_{F(J)}$.

For $0 \leq i \leq 2$, the inclusion $\Lambda^2_i \hookrightarrow \Delta^2$ induces a map $\gamma_i : \mathcal{G}_2 \to \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1$. Since $\mathcal{F}_\bullet$ is a groupoid object of $\mathcal{C}$, the image $\phi^*(\gamma_i)$ is an equivalence. Altering our choice of J if necessary, we may assume that each γ_i is an equivalence: that is, $\mathcal{G}_\bullet$ is a groupoid object of $\mathcal{S}\mathrm{hv}^{\text{ét}}_{F(J)}$. Since $\mathcal{S}\mathrm{hv}^{\text{ét}}_{F(J)}$ is an ∞-topos, this groupoid object is effective; it follows that the natural map $\mathcal{G}_1 \to \mathcal{G}_0 \times_{\mathcal{G}} \mathcal{G}_0$ is an equivalence.

We next prove that $\mathcal{G}$ is coherent. Since the map $\psi : \mathcal{G}_0 \to \mathcal{G}$ is an effective epimorphism and $\mathcal{G}_0$ is coherent, it will suffice to show that ψ is relatively coherent (Proposition VII.3.9). Because ψ is an effective epimorphism, we are reduced to proving that the projection map $\mathcal{G}_0 \times_{\mathcal{G}} \mathcal{G}_0 \to \mathcal{G}_0$ is relatively coherent (Corollary VII.3.11). This follows from the coherence of $\mathcal{G}_0$ and $\mathcal{G}_1$.

Suppose next that $n = -1$, so that there exists a (-1)-truncated map $q : \mathcal{F} \to \overline{\mathcal{F}}$, where $\overline{\mathcal{F}}$ is representable by an étale R-algebra. Altering J if necessary, we can use Lemma 2.3.16 to guarantee that $\overline{\mathcal{F}} = \phi^* \overline{\mathcal{G}}$, where $\overline{\mathcal{G}}$ is representable by an étale $F(J)$-algebra. The map q determines a finite diagram

$$\mathcal{F}_1 \rightrightarrows \mathcal{F}_0 \longrightarrow \overline{\mathcal{F}}.$$

Using the inductive hypothesis and altering our choice of J, we can assume that this diagram is the image under ϕ^* of a diagram

$$\mathcal{G}_1 \rightrightarrows \mathcal{G}_0 \longrightarrow \overline{\mathcal{G}}.$$

Since $\overline{\mathcal{G}}$ is discrete, this diagram determines a map $q' : \mathcal{G} \to \overline{\mathcal{G}}$. Let $\delta : \mathcal{G} \to \mathcal{G} \times_{\overline{\mathcal{G}}} \mathcal{G}$ be the diagonal map. Since q is (-1)-truncated, $\phi^*(\delta)$ is an equivalence. Altering our choice of J, we may assume that δ is an equivalence. It follows that q' is (-1)-truncated so that $\mathcal{G} \in \mathcal{S}\mathrm{hv}^{\mathrm{fc}}_{\leq -1}((\mathrm{CAlg}^{\text{ét}}_{F(J)})^{op})$ as desired.

Now suppose that $n = 0$. We wish to show that $\mathcal{G}$ is discrete. By assumption, each $\mathcal{G}_i$ is discrete; it will therefore suffice to show that $\mathcal{G}_1$ is an equivalence relation on $\mathcal{G}_0$: that is, the map $r : \mathcal{G}_1 \to \mathcal{G}_0 \times \mathcal{G}_0$ is (-1)-truncated. Let $\delta : \mathcal{G}_1 \to \mathcal{G}_1 \times_{\mathcal{G}_0 \times \mathcal{G}_0} \mathcal{G}_1$ be the diagonal map. Since $\mathcal{F}$ is discrete, $\phi^*(r)$ is (-1)-truncated and therefore $\phi^*(\delta)$ is an equivalence. Altering our choice of J if necessary, we may assume that δ is an equivalence, which implies that r is (-1)-truncated.

Suppose now that $n > 0$. We wish to show that $\mathcal{G}$ is n-truncated. We have an effective epimorphism $\mathcal{G}_0 \to \mathcal{G}$, where $\mathcal{G}_0$ is n-truncated (in fact, $(n-1)$-truncated). It will therefore suffice to show that the map $\mathcal{G}_0 \to \mathcal{G}$ is $(n-1)$-truncated. This is equivalent to the statement that the projection map $\mathcal{G}_0 \times_{\mathcal{G}} \mathcal{G}_0 \to \mathcal{G}_0$ is $(n-1)$-truncated, which follows immediately from the fact that both $\mathcal{G}_0$ and $\mathcal{G}_1$ are n-truncated. □

Our final goal in this section is to obtain a structure theorem (Theorem 2.3.24) for finitely constructible objects of $\mathcal{S}\mathrm{hv}_R^{\text{ét}}$, where R is a commutative ring. We begin by studying some prototypical examples.

Definition 2.3.20. Let $\mathcal{X}$ be an ∞-topos, and let $q^* : \mathcal{S} \to \mathcal{X}$ be a left adjoint to the global sections functor $\Gamma : \mathcal{X} \to \mathcal{S}$. We will say that an object $X \in \mathcal{X}$ is *finite constant* if there exists an equivalence $X \simeq q^*Y$, where $Y \in \mathcal{S}$ is π-finite (see Example 2.3.2). We say that $X \in \mathcal{X}$ is *finite locally constant* if there exists a collection of objects $\{U_\alpha \in \mathcal{X}\}_{\alpha \in A}$ such that $X \times U_\alpha$ is a finite constant object of the ∞-topos $\mathcal{X}_{/U_\alpha}$ for each $\alpha \in A$, and the objects $\{U_\alpha\}_{\alpha \in A}$ cover $\mathcal{X}$ (that is, the map $\coprod_\alpha U_\alpha \to \mathbf{1}$ is an effective epimorphism, where $\mathbf{1}$ denotes the final object of $\mathcal{X}$).

Lemma 2.3.21. *Let $\mathcal{X}$ be a locally coherent ∞-topos, and let $X \in \mathcal{X}$ be a finite locally constant object. Then X is finitely constructible.*

Proof. Using Lemma 2.3.5, we can reduce to the case where X is finite constant and $\mathcal{X}$ is coherent. Let $q^* : \mathcal{S} \to \mathcal{X}$ be a geometric morphism and assume that $X = \pi^*Y$ for some π-finite space Y. Then X is n-truncated; we must show that X is coherent. We prove that X is m-coherent using induction on m. When $m = 0$, we must show that X is quasi-compact. Choose a finite set S and an effective epimorphism $S \to Y$ in $\mathcal{S}$. Then we have an effective epimorphism $f^*S \to X$ in $\mathcal{X}$, so it will suffice to show that f^*S is finitely constructible. Since f^*S is a coproduct of finitely many copies of the final object of $\mathcal{X}$, the desired result follows from the coherence of $\mathcal{X}$.

Now suppose that $m > 0$. According to Proposition VII.3.9, it will suffice to show that the map $f^*S \to X$ is relatively m-coherent. Using Corollary VII.3.11, we are reduced to showing that the projection map $f^*S \times_X f^*S \to f^*S$ is relatively $(m-1)$-coherent. In other words, we must show that for every element $s \in S$, the fiber product $f^*S \times_X f^*\{s\}$ is an $(m-1)$-coherent object of $\mathcal{X}$. This follows from the inductive hypothesis, since $f^*S \times_X f^*\{s\} \simeq f^*(S \times_Y \{s\})$. □

Proposition 2.3.22. *Let $\mathcal{X}$ be an $(n+1)$-coherent ∞-topos, let $U \in \mathcal{X}$ be a quasi-compact (-1)-truncated object, and let*

$$i^* : \mathcal{X} \to \mathcal{X}/U \qquad j^* : \mathcal{X} \to \mathcal{X}_{/U}$$

be the associated geometric morphisms. Then an object $X \in \mathcal{X}$ is n-coherent if and only if i^*X and j^*X are n-coherent objects of $\mathcal{X}/U$ and $\mathcal{X}_{/U}$, respectively. In particular, the ∞-topoi $\mathcal{X}_{/U}$ and $\mathcal{X}/U$ are n-coherent.

Proof. We first prove that U is $(n+1)$-coherent. For this, we show that U is m-coherent for $m \leq n+1$ using induction on m. In the case $m = 0$, this follows from our hypothesis that U is quasi-compact. If $m \geq 1$, we must show that for every pair of $(m-1)$-coherent objects $V, V' \in \mathcal{X}_{/U}$, the fiber product $V \times_U V'$ is $(m-1)$-coherent. Since U is (-1)-truncated, we have an equivalence $V \times_U V' \simeq V \times V'$, and the desired result follows from our assumption that $\mathcal{X}$ is $(n+1)$-coherent.

Since $\mathcal{X}$ is $(n+1)$-coherent and U is n-coherent, the product $X \times U \in \mathcal{X}$ is n-coherent whenever X is n-coherent. In other words, j^*X is n-coherent whenever X is n-coherent. We prove the remaining assertions by induction on n.

Suppose first that $n = 0$ and that X is quasi-compact; we must show that i^*X is quasi-compact. Suppose we are given a collection of maps $\{V_\alpha \to i^*X\}_{\alpha \in A}$ which induce an effective epimorphism $\coprod_{\alpha \in A} V_\alpha \to i^*X$ in $\mathcal{X}/U$. We wish to prove that there exists a finite subset $A_0 \subseteq A$ such that the map $\coprod_{\alpha \in A_0} V_\alpha \to i^*X$ is also an effective epimorphism. Let us identify $\mathcal{X}/U$ with a full subcategory of $\mathcal{X}$, so that we have a canonical map $X \to i^*X$. For each $\alpha \in A$, let $V'_\alpha = X \times_{i^*X} V_\alpha \in \mathcal{X}$. If A is empty, there is nothing to prove. Otherwise, the map $\coprod_{\alpha \in A} V'_\alpha \to X$ is an effective epimorphism (because it becomes an effective epimorphism after pullback along i or j). Since X is quasi-compact, there exists a finite subset $A_0 \subseteq A$ such that the map $\coprod_{\alpha \in A_0} V'_\alpha \to X$ is an effective epimorphism. Applying the functor i^*, we deduce that $\coprod_{\alpha \in A_0} V_\alpha \to i^* X$ is an effective epimorphism.

Now assume that $n = 0$ and that $i^* X$ and $j^* X$ are quasi-compact objects of $\mathcal{X}/U$ and $\mathcal{X}_{/U}$, respectively. We wish to show that X is quasi-compact. Suppose we are given an effective epimorphism $\coprod_{\alpha \in A} V_\alpha \to X$ in $\mathcal{X}$. Then the induced maps

$$\coprod_{\alpha \in A} i^* V_\alpha \to i^* X \qquad \coprod_{\alpha \in A} j^* V_\alpha \to j^* X$$

are effective epimorphisms in $\mathcal{X}/U$ and $\mathcal{X}_{/U}$, respectively. Using the quasi-compactness of $i^* X$ and $j^* X$, we conclude that there is a finite subset $A_0 \subseteq A$ such that the maps

$$\coprod_{\alpha \in A_0} i^* V_\alpha \to i^* X \qquad \coprod_{\alpha \in A_0} j^* V_\alpha \to j^* X$$

are effective epimorphisms, from which it follows that the map $\coprod_{\alpha \in A_0} V_\alpha \to X$ is also an effective epimorphism.

Now suppose that $n > 0$ and that $X \in \mathcal{X}$ is such that the objects $i^* X \in \mathcal{X}/U$ and $j^* X \in \mathcal{X}_{/U}$ are n-coherent. We wish to show that X is n-coherent. Choose $(n-1)$-coherent objects $Y, Y' \in \mathcal{X}$ equipped with maps $Y \to X \leftarrow Y'$; we need to show that the fiber product $Y \times_X Y'$ is $(n-1)$-coherent. The inductive hypothesis implies that $i^* Y$, $i^* Y'$, $j^* Y$ and $j^* Y'$ are $(n-1)$-coherent, so that the fiber products $i^*(Y \times_X Y') \simeq i^* Y \times_{i^* X} i^* Y'$ and $j^*(Y \times_X Y') \simeq j^* Y \times_{j^* X} j^* Y'$ are $(n-1)$-coherent. Applying the inductive hypothesis again, we conclude that $Y \times_X Y'$ is $(n-1)$-coherent.

We now complete the proof by showing that if X is n-coherent, then $i^* X$ is an n-coherent object of $\mathcal{X}/U$. Let $\mathcal{C} \subseteq (\mathcal{X}/U)_{/i^* X}$ be the full subcategory spanned by those morphisms $i^* V \to i^* X$ obtained by applying i^* to a morphism $V \to X$ in $\mathcal{X}$, where V is $(n-1)$-coherent. Since X is n-coherent, the subcategory $\mathcal{C}$ is stable under products, and the inductive hypothesis implies that $\mathcal{C}$ consists of $(n-1)$-coherent objects of $(\mathcal{X}_{/U})_{/i^* X}$. Consequently, to prove that $i^* X$ is n-coherent, it will suffice to show that for every object $X' \in (\mathcal{X}/U)_{/i^* X}$, there exists an effective epimorphism $\coprod W_\alpha \to X'$, where each W_α belongs to $\mathcal{C}$ (Corollary VII.3.10). To prove this, choose an effective epimorphism $\coprod W_\alpha \to X' \times_{i^* X} X$ in $\mathcal{X}$, where each W_α is $(n-1)$-coherent. We now complete the proof by taking $V_\alpha = i^* W_\alpha$. □

Corollary 2.3.23. *Let $\mathcal{X}$ be a coherent ∞-topos containing a quasi-compact (-1)-truncated object U, and let $i^* : \mathcal{X} \to \mathcal{X}/U$ and $j^* : \mathcal{X} \to \mathcal{X}_{/U}$ be as in Proposition 2.3.22. Then an object $X \in \mathcal{X}$ is finitely constructible if and only if both $i^* X$ and $j^* X$ are finitely constructible.*

We can now state our main result:

Theorem 2.3.24. *Let R be a commutative ring and let $\mathcal{F} \in \mathrm{Shv}_R^{\acute{e}t}$. The following conditions are equivalent:*

(1) *There exists a finite sequence of elements $x_1, x_2, \ldots, x_n \in R$ which generate the unit ideal and a collection of Galois extensions $(R/(x_1, \ldots, x_{i-1}))[x_i^{-1}] \to R_i$ (see Definition XI.4.16), such that the pullback functors $\phi_i^* : \mathrm{Shv}_R^{\acute{e}t} \to \mathrm{Shv}_{R_i}^{\acute{e}t}$ carry $\mathcal{F}$ to finite constant objects $\phi_i^* \mathcal{F} \in \mathrm{Shv}_{R_i}^{\acute{e}t}$.*

(2) *There exists a finite sequence of elements $x_1, x_2, \ldots, x_n \in R$ which generate the unit ideal such that, if $R_i = R/(x_1, \ldots, x_{i-1})[x_i^{-1}]$, then the pullback maps $\phi_i^* : \mathrm{Shv}_R^{\acute{e}t} \to \mathrm{Shv}_{R_i}^{\acute{e}t}$ carry $\mathcal{F}$ to finite locally constant objects $\phi_i^* \mathcal{F} \in \mathrm{Shv}_{R_i}^{\acute{e}t}$.*

(3) *There exists a finite sequence of elements $x_1, x_2, \ldots, x_n \in R$ which generate the unit ideal such that, if $R_i = R/(x_1, \ldots, x_{i-1})[x_i^{-1}]$, then the pullback maps $\phi_i^* : \mathrm{Shv}_R^{\acute{e}t} \to \mathrm{Shv}_{R_i}^{\acute{e}t}$ carry $\mathcal{F}$ to finitely constructible objects $\phi_i^* \mathcal{F} \in \mathrm{Shv}_{R_i}^{\acute{e}t}$.*

(4) *The sheaf $\mathcal{F}$ is finitely constructible.*

Proof. The implication (1) $\Rightarrow$ (2) is obvious, the implication (2) $\Rightarrow$ (3) follows from Lemma 2.3.21, and the implication (3) $\Rightarrow$ (4) follows from Corollary 2.3.23. We will prove that (4) $\Rightarrow$ (1).

Assume that $\mathcal{F}$ is finitely constructible. Write R as the union of its finitely generated subrings R_α. Using Theorem 2.3.8, we may suppose that $\mathcal{F}$ is the image of a finitely constructible object $\mathcal{F}_\alpha \in \mathcal{S}\mathrm{hv}^{\mathrm{fc}}_{R_\alpha}$ for some index α. Replacing R by R_α and $\mathcal{F}$ by $\mathcal{F}_\alpha$ we can reduce to the case where R is a finitely generated algebra over the ring $\mathbf{Z}$ of natural numbers; in particular, R is Noetherian.

Choose a sequence of elements $x_1, x_2, \ldots \in R$ of maximal length having the following properties:

(a) For each $i > 0$, the element x_i does not belong to the ideal generated by $\{x_j\}_{j<i}$.

(b) For each $i > 0$, there exists a Galois covering $(R/(x_1, \ldots, x_{i-1}))[x_i^{-1}] \to R_i$ such that the image of $\mathcal{F}$ in $\mathcal{S}\mathrm{hv}^{\text{ét}}_{R_i}$ is finite constant.

Since R is Noetherian, condition (a) guarantees that the sequence has some finite length n. We will complete the proof by showing that the elements $x_1, x_2, \ldots, x_n \in R$ generate the unit ideal. Replacing R by $R/(x_1, \ldots, x_n)$, we may assume that $n = 0$; we wish to show that $R = 0$. Let I be the nilradical of R. Since $\mathcal{S}\mathrm{hv}^{\text{ét}}_R \simeq \mathcal{S}\mathrm{hv}^{\text{ét}}_{R/I}$, we may replace R by R/I and thereby assume that R is reduced.

Assume that $R \neq 0$. Since R is Noetherian, we can choose a minimal prime ideal $\mathfrak{p} \subseteq R$. Let k denote the localization $R_{\mathfrak{p}}$. Since R is reduced, k is a field. Let k' be a separable closure of k. Then $\mathcal{S}\mathrm{hv}^{\text{ét}}_{k'} \simeq \mathcal{S}$ and the image of $\mathcal{F}$ in $\mathcal{S}\mathrm{hv}^{\text{ét}}_{k'}$ corresponds to an object $Y \in \mathcal{S}$. Since $\mathcal{F}$ is finitely constructible, the space Y is π-finite. Let $\mathcal{F}_0$ be the image of $\mathcal{F}$ in $\mathcal{S}\mathrm{hv}^{\text{ét}}_k$ and let $\mathcal{F}'_0 \in \mathcal{S}\mathrm{hv}^{\text{ét}}_k$ be the finite constant sheaf associated to Y. Then $\mathcal{F}_0$ and $\mathcal{F}'_0$ are both finitely constructible and they have equivalent images in $\mathcal{S}\mathrm{hv}^{\text{ét}}_{k'}$. Using Theorem 2.3.8, we conclude that $\mathcal{F}_0$ and $\mathcal{F}'_0$ become equivalent in $\mathcal{S}\mathrm{hv}^{\text{ét}}_{k''}$ for some finite separable extension k'' of k. Enlarging k'' is necessary, we may assume that it is Galois over k.

Choose an element $x \in R - \mathfrak{p}$ and a Galois extension $R[x^{-1}] \to R'$ such that $k'' \simeq R' \otimes_{R[x^{-1}]} k$. Let $\mathcal{F}_1$ be the image of $\mathcal{F}$ in $\mathcal{S}\mathrm{hv}^{\text{ét}}_{R'}$, and let $\mathcal{F}'_1$ be the finite constant object of $\mathcal{S}\mathrm{hv}^{\text{ét}}_{R'}$ corresponding to π-finite space Y. Note that k'' is isomorphic to a filtered colimit of localizations $R'[y^{-1}]$, where y ranges over elements of $R - \mathfrak{p}$. Since the images of $\mathcal{F}_1$ and $\mathcal{F}'_1$ in $\mathcal{S}\mathrm{hv}^{\text{ét}}_{k''}$ are equivalent, Theorem 2.3.8 implies that there exists an element $y \in R - \mathfrak{p}$ such that the images of $\mathcal{F}_1$ and $\mathcal{F}'_1$ in $\mathcal{S}\mathrm{hv}^{\text{ét}}_{R'[y^{-1}]}$ are equivalent. Then $R'[y^{-1}]$ is a Galois covering of $R[(xy)^{-1}]$ over which the sheaf $\mathcal{F}$ becomes finite constant, contradicting our assumption that $n = 0$. $\square$

Corollary 2.3.25. *Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a spectral algebraic space which is quasi-compact and quasi-separated, and let $\mathcal{F} \in \mathcal{X}$ be an object. The following conditions are equivalent:*

(1) *There exists a sequence of quasi-compact open substacks*

$$\emptyset = \mathfrak{U}_0 \subseteq \mathfrak{U}_1 \subseteq \cdots \subseteq \mathfrak{U}_n = \mathfrak{X}$$

such that, if $\mathfrak{Y}_i$ denotes the reduced closed substack of $\mathfrak{U}_i$ complementary to $\mathfrak{U}_{i-1}$, then there is a surjective finite étale map $\mathfrak{Z}_i \to \mathfrak{Y}_i$ such that the restriction of $\mathcal{F}$ to $\mathfrak{Z}_i$ is finite constant.

(2) *There exists a sequence of quasi-compact open substacks*

$$\emptyset = \mathfrak{U}_0 \subseteq \mathfrak{U}_1 \subseteq \cdots \subseteq \mathfrak{U}_n = \mathfrak{X}$$

such that, if $\mathfrak{Y}_i$ denotes the reduced closed substack of $\mathfrak{U}_i$ complementary to $\mathfrak{U}_{i-1}$, then the restriction of $\mathcal{F}$ to $\mathfrak{Y}_i$ is finite locally constant.

(3) *There exists a sequence of quasi-compact open substacks*

$$\emptyset = \mathfrak{U}_0 \subseteq \mathfrak{U}_1 \subseteq \cdots \subseteq \mathfrak{U}_n = \mathfrak{X}$$

such that, if $\mathfrak{Y}_i$ denotes the reduced closed substack of $\mathfrak{U}_i$ complementary to $\mathfrak{U}_{i-1}$, then the restriction of $\mathcal{F}$ to $\mathfrak{Y}_i$ is finitely constructible.

(4) *The sheaf $\mathcal{F}$ is finitely constructible.*

Proof. The implication (1) $\Rightarrow$ (2) is obvious, the implication (2) $\Rightarrow$ (3) follows from Lemma 2.3.21, and the implication (3) $\Rightarrow$ (4) follows from Corollary 2.3.23. To prove that (4) $\Rightarrow$ (1), we first use Theorem XII.1.3.8 to reduce to the case where $\mathfrak{X} = \mathrm{Spec}^{\text{ét}} R$ is affine. Replacing R by $\pi_0 R$, we may suppose that R is discrete, in which case the desired result follows immediately from Theorem 2.3.24. □

2.4 A Universal Coefficient Theorem

To every space X and every $\mathbb{E}_\infty$-ring k, we can associate an $\mathbb{E}_\infty$-ring $C^*(X;k)$ of k-valued cochains on X. The functor $X \mapsto C^*(X;k)$ is canonically determined by the following pair of properties:

(a) If $X = *$ consists of a single point, then $C^*(X;k) \simeq k$.

(b) The construction $X \mapsto C^*(X;k)$ carries colimit in $\mathcal{S}$ to limits in CAlg.

Write $\mathrm{Spec}^{\text{ét}} k = (\mathcal{S}\mathrm{hv}_k^{\text{ét}}, \mathcal{O})$, and let $q^* \, \mathcal{S} \to \mathcal{S}\mathrm{hv}_k^{\text{ét}}$ be a left adjoint to the global sections functor (so that q^* carries a space $X \in \mathcal{S}$ to the sheafification of the constant presheaf $\mathrm{CAlg}_k^{\text{ét}} \to \mathcal{S}$ taking the value X). The functor $X \mapsto \mathcal{O}(q^* X)$ satisfies conditions (a) and (b), so that we have a functorial equivalence $C^*(X;k) \simeq \mathcal{O}(q^* X)$. We may therefore think of the structure sheaf $\mathcal{O}$ of $\mathrm{Spec}^{\text{ét}} k$ as a kind of generalized version of cohomology with coefficients in k: rather than being defined on spaces, it is defined on étale sheaves of spaces over k. Our goal in this section is to initiate the study of this cohomology theory.

Our main result (Theorem 2.4.9) asserts that, under some reasonable hypotheses, it satisfies a universal coefficient formula. Before we can formulate this result, we need some definitions.

Definition 2.4.1. Let p be a prime number. We will say that a space X is *p-finite* if it satisfies the following conditions:

(1) The space X is n-truncated for some integer n.

(2) The set $\pi_0 X$ is finite.

(3) For each vertex $x \in X$ and each integer $m \geq 1$, the group $\pi_m(X, x)$ is a finite p-group.

Let R be a connective $\mathbb{E}_\infty$-ring. We say that an object $X \in \mathcal{S}\mathrm{hv}_R^{\text{ét}}$ is *p-constructible* if it is finitely constructible and, for every separably closed field k and every map $\phi : R \to k$, the stalk $\phi^* X \in \mathcal{S}\mathrm{hv}_k^{\text{ét}} \simeq \mathcal{S}$ is a p-finite space.

Definition 2.4.2. Let p be a prime number and let R be an $\mathbb{E}_\infty$-ring. We will say that R is *p-thin* if the following conditions are satisfied:

(1) The $\mathbb{E}_\infty$-ring R is connective and truncated: that is, $\pi_n R$ vanishes for $n < 0$ and $n \gg 0$.

(2) The prime number p is nilpotent in the commutative ring $\pi_0 R$.

(3) The $\mathbb{E}_\infty$-ring R lies in the essential image of the forgetful functor $\mathrm{SCR} \to \mathrm{CAlg}$; here SCR denotes the ∞-category of simplicial commutative rings.

Remark 2.4.3. In this paper, we will prove many results about p-constructible objects of $\mathcal{S}\mathrm{hv}_R^{\text{ét}}$ under the assumption that R is p-thin. We conjecture that these results are valid more generally under the assumption that R satisfies conditions (1) and (2) of Definition 2.4.2.

Remark 2.4.4. Let R be a p-thin $\mathbb{E}_\infty$-ring, and let $x \in \pi_0 R$. Then R admits the structure of a simplicial commutative ring, and x determines a map $\mathbf{Z}[X] \to R$ in the ∞-category of simplicial commutative rings. Let us regard $\mathbf{Z}$ as a $\mathbf{Z}[X]$-module, with X acting by 0. Then

$$R_0 = R \otimes_{\mathbf{Z}[X]} \mathbf{Z} \qquad R_1 = R \otimes_{\mathbf{Z}[X]} \mathbf{Z}[X^{\pm 1}]$$

are p-thin $\mathbb{E}_\infty$-rings.

Remark 2.4.5. Let R be a p-thin $\mathbb{E}_\infty$-ring. Then every étale R-algebra R' is also p-thin. The only nontrivial point is to show that R' admits the structure of a simplicial commutative ring. This follows from the classification of étale morphisms in both SCR and CAlg (see Theorem A.7.5.0.6 and Corollary V.4.3.12).

Definition 2.4.6. Let $f : R \to R'$ be a map of $\mathbb{E}_\infty$-rings. We say that f is *formally étale* if the relative cotangent complex $L_{R'/R}$ is trivial. In this case, we will also say that R' is *formally étale* over R.

Remark 2.4.7. Let R be an $\mathbb{E}_\infty$-ring and let $A, A' \in \mathrm{CAlg}_R$. Then $A \times A'$ is formally étale over R if and only if A and A' are formally étale over R.

Remark 2.4.8. Suppose we are given a pushout diagram of $\mathbb{E}_\infty$-rings

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R' \arrow[d] \\ A \arrow[r] & A'. \end{tikzcd}$$

Then $L_{A'/A} \simeq L_{R'/R} \otimes_{k'} A'$. Consequently, if R' is formally étale over R, then A' is formally étale over A.

We can now formulate our main result:

Theorem 2.4.9. *Let R be an $\mathbb{E}_\infty$-ring, p a prime number, and $X \in \mathcal{S}\mathrm{hv}_R^{\acute{e}t}$. Assume that R is p-thin and that X is p-constructible. Write* $\mathrm{Spec}^{\acute{e}t} R = (\mathcal{S}\mathrm{hv}_R^{\acute{e}t}, \mathcal{O})$*. Then:*

(a) *The $\mathbb{E}_\infty$-ring $\mathcal{O}(X)$ is formally étale over R.*

(b) *If $M \in (\mathrm{Mod}_R)_{\leq 0}$ and $\mathcal{M}$ denotes the associated quasi-coherent sheaf on* $\mathrm{Spec}^{\acute{e}t} R$*, then the canonical map $\mathcal{O}(X) \otimes_R M \to \mathcal{M}(X)$ is an equivalence.*

Corollary 2.4.10. *Let $\phi : R \to R'$ be a map of $\mathbb{E}_\infty$-rings, and let p be a prime number. Assume that R is p-thin and that R' is truncated. Write* $\mathrm{Spec}^{\acute{e}t} R = (\mathcal{S}\mathrm{hv}_R^{\acute{e}t}, \mathcal{O})$ *and* $\mathrm{Spec}^{\acute{e}t} R' = (\mathcal{S}\mathrm{hv}_{R'}^{\acute{e}t}, \mathcal{O}')$*. For every p-constructible $X \in \mathcal{S}\mathrm{hv}_R^{\acute{e}t}$, the canonical map $R' \otimes_R \mathcal{O}(X) \to \mathcal{O}'(\phi^* X)$ is an equivalence, and $\mathcal{O}'(\phi^* X)$ is formally étale over R'.*

The proof of Theorem 2.4.9 will occupy our attention throughout this section. We begin by establishing an analogue of Theorem 2.3.24 for p-constructible sheaves.

Proposition 2.4.11. *Let R be a commutative ring and let $X \in \mathcal{S}\mathrm{hv}_R^{\acute{e}t}$. Then X is p-constructible if and only if the following condition is satisfied:*

($*$) *There exists a finite sequence of elements $x_1, x_2, \ldots, x_n \in R$ which generate the unit ideal and a collection of maps $(R/(x_1, \ldots, x_{i-1}))[x_i^{-1}] \to R_i$ which are finite étale and faithfully flat, such that each pullback map $\phi_i^* : \mathcal{S}\mathrm{hv}_R^{\acute{e}t} \to \mathcal{S}\mathrm{hv}_{R_i}^{\acute{e}t}$ carries X to the constant sheaf associated to a p-finite space.*

Proof. Suppose first that X is p-constructible. Using Theorem 2.3.24, we can choose a finite sequence of elements $x_1, x_2, \ldots, x_n \in R$ which generate the unit ideal and a collection of maps $(R/(x_1, \ldots, x_{i-1}))[x_i^{-1}] \to R_i$ which are finite étale and faithfully flat, such that the pullback maps $\phi_i^* : \mathcal{S}\mathrm{hv}_R^{\acute{e}t} \to \mathcal{S}\mathrm{hv}_{R_i}^{\acute{e}t}$ carry X to the constant sheaves associated to certain π-finite spaces $Y_i \in \mathcal{S}$. We may assume without loss of generality that x_i does not belong to the radical of the ideal generated by $\{x_j\}_{j<i}$, so that each of the rings R_i is nonzero. It follows that there exist ring homomorphisms $R_i \to k_i$, where each k_i is a separably closed field. Then $Y_i \simeq \eta_i^* X$ where η_i denotes the composite map $R \to R_i \to k_i$. Since X is p-constructible, we conclude that each Y_i is p-finite.

Conversely, assume that ($*$) is satisfied. Theorem 2.3.24 implies that $\mathcal{F}$ is finitely constructible. If $\eta : R \to k$ is a ring homomorphism where k is a separably closed field, then η factors through some R_i (where R_i is as in the statement of (2)), so that $\eta^* X \in \mathcal{S}$ is p-finite. $\square$

The proof of Theorem 2.4.9 will proceed by dévissage, using Proposition 2.4.11 to reduce to the case of the constant sheaf associated to a p-finite space. We therefore begin by studying the constant case. Our first step is to prove the following:

Proposition 2.4.12. *Let k be a commutative ring, let p be a prime number which vanishes in k, and let X be a p-finite space. Then the cochain algebra $C^*(X;k)$ formally étale over k.*

Lemma 2.4.13. *Let k be a commutative ring, p a prime number which vanishes in k, and let $X = K(\mathbf{Z}/p\mathbf{Z}, n)$ be an Eilenberg-MacLane space. Then the cochain algebra $C^*(X;k)$ is formally étale over k.*

Proof. Let Free : Sp $\to$ CAlg$_k$ denote the free algebra functor, so that Theorem 2.2.17 provides a pushout diagram of $\mathbb{E}_\infty$-algebras over k

$$\begin{tikzcd} \mathrm{Free}(S^{-n}) \arrow[r, "\wp"] \arrow[d] & \mathrm{Free}(S^{-n}) \arrow[d] \\ k \arrow[r] & C^*(X;k). \end{tikzcd}$$

Let $R = R' = \mathrm{Free}(S^{-n})$, and regard $\wp$ as a map from R to R'. Let $\eta \in \pi_{-n}R$ and $\eta' \in \pi_{-n}R'$ be the generators, so that $\wp(\eta) = \eta' - P^0(\eta')$. We have canonical maps $\alpha : R \to R \oplus R[-n]$, $\beta : R' \to R' \oplus R'[-n]$, characterized up to homotopy by

$$\alpha(\eta) = (\eta, 1) \in \pi_{-n}(R \oplus R[-n]) \qquad \beta(\eta') = (\eta', 1) \in \pi_{-n}(R' \oplus R'[-n]).$$

These maps induce equivalences $L_{R/k} \simeq R[-n]$, $L_{R'/k} \simeq R'[-n]$.

Let $\phi : R \oplus R[-n] \to R' \oplus R'[-n]$ be the map induced by $\wp$. Consider the diagram

$$\begin{tikzcd} R \arrow[r, "\wp"] \arrow[d, "\alpha"] & R' \arrow[d, "\beta"] \\ R \oplus R[-n] \arrow[r, "\phi"] & R' \oplus R'[-n]. \end{tikzcd}$$

It follows from Remark 2.2.11 that this diagram commutes up to homotopy, so that $\wp$ induces the identity map

$$R'[-n] \simeq R[-n] \otimes_R R' \simeq L_{R/k} \otimes_R R' \to L_{R'/k} \simeq R'[-n].$$

It follows that $L_{R'/R} \simeq 0$, so that R' is formally étale over R. Using Remark 2.4.8, we deduce that $C^*(X;k)$ is formally étale over k. □

Lemma 2.4.14. *Suppose we are given maps of $\mathbb{E}_\infty$-algebras $k \to k' \to k''$, where k' is formally étale over k. Then k'' is formally étale over k if and only if it is formally étale over k'.*

Proof. This follows by inspecting the fiber sequence

$$L_{k'/k} \otimes_{k'} k'' \to L_{k''/k} \to L_{k''/k'}.$$

□

Lemma 2.4.15. *Suppose we are given a pushout diagram of $\mathbb{E}_\infty$-algebras*

$$\begin{tikzcd} k \arrow[r] \arrow[d] & k' \arrow[d] \\ A \arrow[r] & A'. \end{tikzcd}$$

If k' and A are formally étale over k, then A' is formally étale over k.

Proof. Combine Lemma 2.4.14 and Remark 2.4.8. $\square$

Lemma 2.4.16. *Let X be a connected p-finite space. Then there exists a finite sequence of maps*

$$X = X_0 \to X_1 \to \cdots \to X_n = *$$

and pullback diagrams

$$\begin{tikzcd} X_i \arrow[r] \arrow[d] & X_{i+1} \arrow[d] \\ * \arrow[r] & K(\mathbf{Z}/p\,\mathbf{Z}, m_i) \end{tikzcd}$$

for some integers $m_i \geq 2$.

Proof. Choose a base point $x \in X$. Since X is p-finite, it is m-truncated for some m. We proceed by induction on m and on the order p^a of the finite p-group $\pi_m(X, x)$. If $m = 0$ then X is contractible and there is nothing to prove. Assume therefore that $m > 0$. If $a = 0$ then X is $(m-1)$-truncated and the desired result follows from the inductive hypothesis. Assume that $a > 0$, so that the order of the group $\pi_m(X, x)$ is divisible by p. Let $G = \pi_1(X, x)$, and let $\pi_m(X, x)^G$ denote the group of G-fixed points of $\pi_m(X, x)$. Since G is a finite p-group, the order of $\pi_m(X, x)^G$ is congruent to the order of $\pi_m(X, x)$ modulo p. Since $a > 0$, we conclude that the order of the group $\pi_m(X, x)^G$ is divisible by p so we can choose a cyclic subgroup $Z \subseteq \pi_m(X, x)^G$ of order p. The theory of principal fibrations implies that there exists a fiber sequence of spaces

$$X \xrightarrow{f} X_1 \to K(\mathbf{Z}/p\,\mathbf{Z}, m+1)$$

where f induces isomorphisms $\pi_i(X, x) \simeq \pi_i(X_1, f(x))$ for $i \neq m$ and a surjection $\pi_m(X, x) \to \pi_m(X_1, f(x))$ having kernel Z. The desired result now follows by applying the inductive hypothesis to X_1. $\square$

Proof of Proposition 2.4.12. Using Remark 2.4.8, we can reduce to the case where $k = \mathbf{F}_p$ (in particular, k is a field). Remark 2.4.7 allows us to assume that X is connected. Choose a finite sequence of maps

$$X = X_0 \to X_1 \to \cdots \to X_n = *$$

as in Lemma 2.4.16. By Lemma 2.4.14, it suffices to show that each of the maps $C^*(X_{i+1}; k) \to C^*(X_i; k)$ is formally étale . Corollary 1.1.10 implies that we have a pushout diagram of $\mathbb{E}_\infty$-rings

$$\begin{tikzcd} C^*(K(\mathbf{Z}/p\,\mathbf{Z}; m) \arrow[r] \arrow[d] & k \arrow[d] \\ C^*(X_i; k) \arrow[r] & C^*(X_{i+1}; k). \end{tikzcd}$$

Using Lemma 2.4.15, we reduce to proving that k is formally étale over $C^*(\mathbf{Z}/p\,\mathbf{Z}; k)$. This follows from Lemmas 2.4.14 and 2.4.13. $\square$

We now turn to the verification of condition (b) in Theorem 2.4.9. Recall that if k is a connective $\mathbb{E}_k$-ring, then a k-module M is said to have Tor-*amplitude* $\leq m$ if the tensor product functor $N \mapsto M \otimes_k M$ carries $(\mathrm{Mod}_k)_{\leq 0}$ into $(\mathrm{Mod}_k)_{\leq m}$.

Proposition 2.4.17. *Let R be a connective $\mathbb{E}_\infty$-ring. Let M and N be R-modules. Assume that $M \in (\mathrm{Mod}_R)_{\leq n}$ and that N has* Tor-*amplitude $\leq m$, so that $M \otimes_R N \in (\mathrm{Mod}_R)_{\leq n+m}$. Write $\mathrm{Spec}^{ét} R = (\mathcal{S}\mathrm{hv}^{ét}_R, \mathcal{O})$, and let $\mathcal{M}$ and $\mathcal{N}$ be the quasi-coherent sheaves on $\mathrm{Spec}^{ét} R$ associated to M and N, respectively. Then:*

(1) *The homotopy groups $\pi_i \mathcal{M}(X)$ vanish for $i > n$ and every $X \in \mathcal{S}\mathrm{hv}^{ét}_R$.*

(2) *If $X \in \mathcal{S}\mathrm{hv}_R^{\text{ét}}$ is coherent, then the canonical map*

$$\mathcal{M}(X) \otimes_R N \to (\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N})(X)$$

is an equivalence of R-modules.

Proof. We first prove (1). Let $\mathcal{X}$ denote the full subcategory of $\mathcal{S}\mathrm{hv}_R^{\text{ét}}$ spanned by those objects X such that $\mathcal{M}(X)$ is n-truncated. Then $\mathcal{X}$ is closed under small colimits in $\mathcal{S}\mathrm{hv}_R^{\text{ét}}$. To prove that $\mathcal{X} = \mathcal{S}\mathrm{hv}_R^{\text{ét}}$, it suffices to show that $\mathcal{X}$ contains every representable sheaf $X \in \mathcal{S}\mathrm{hv}_R^{\text{ét}}$. This is clear, since if X is represented by an étale R-algebra R', then $\mathcal{M}(X) \simeq R' \otimes_R M$ is n-truncated by virtue of our assumption on M and the fact that R' is flat over R.

We now prove (2). Since N has Tor-amplitude $\leq m$, the module $\mathcal{M}(X) \otimes_R N$ is $(m+n)$-truncated by (1). Note that $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N}$ is the quasi-coherent sheaf associated to the tensor product $M \otimes_R N$ which is $(n+m)$-truncated, so that $(\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N})(X)$ is $(n+m)$-truncated for all $X \in \mathcal{S}\mathrm{hv}_R^{\text{ét}}$ by (1). Let K_X denote the fiber of the map $\mathcal{M}(X) \otimes_R N \to (\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N})(X)$, so that K_X is $(n+m)$-truncated for all X.

We will prove by induction on r that if X is r-coherent, then $\pi_i K_X \simeq 0$ for $i \geq n+m-r$. The assertion is obvious if $r < 0$. Assume that $r \geq 0$, so that X is quasi-compact. We may therefore choose an effective epimorphism $u : X_0 \to X$, where X_0 is representable. Let $X_\bullet$ denote the Čech nerve of u. Let $M' = \varprojlim_{[j] \in \mathbf{\Delta}^{op}_{\leq r}} \mathcal{M}(X_j)$. Since each $\mathcal{M}(X_i)$ is n-truncated, the map $\mathcal{M}(X) \to M'$ has $(n-r-1)$-truncated fibers, so that the induced map $\mathcal{M}(X) \otimes_k N \to M' \otimes_R N$ has $(n+m-r-1)$-truncated fibers. Let $P = \varprojlim_{[j] \in \mathbf{\Delta}^{op}_{\leq r}} (\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N})(X_j)$; the same reasoning shows that the map $(\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N})(X) \to P$ has $(n+m-r-1)$-truncated fibers. It follows that the map $K_X \to K'$ has $(n+m-r-1)$-truncated fibers, where $K' = \varinjlim_{[j] \in \mathbf{\Delta}^{op}_{\leq r}} K_{X_j}$. It will therefore suffice to show that K' is $(n+m-r-1)$-truncated.

Since X is r-coherent, each X_j is $(r-1)$-coherent and therefore K_{X_j} is $(n+m-r)$-truncated by the inductive hypothesis. It follows that K' is $(n+m-r)$-truncated, and that $\pi_{n+m-r} K'$ is the inverse limit of the system of abelian groups $\{\pi_{n+m-r} K_{X_j}\}_{[j] \in \mathbf{\Delta}^{op}_{\leq r}}$. In particular, the map $\pi_{n+m-r} K' \to \pi_{n+m-r} K_{X_0}$ is injective. Since X_0 is representable, $K_{X_0} \simeq 0$ so that $\pi_{n+m-r} K' \simeq 0$ and K' is $(n+m-r-1)$-truncated, as desired. □

Corollary 2.4.18 (Universal Coefficient Formula)**.** *Let X be a π-finite space and let $\phi : R \to R'$ be a map of $\mathbb{E}_\infty$-rings. Assume that k and k' are connective and truncated, and that there exists a map of $\mathbb{E}_\infty$-rings $\mathbf{Z} \to R$ (this is automatic, for example, if R is discrete). Then the induced map $C^*(X; R) \otimes_R R' \to C^*(X; R')$ is an equivalence.*

Proof. Using the commutative diagram

$$\begin{tikzcd} & C^*(X;R) \otimes_R R' \arrow[dr] & \\ C^*(X;\mathbf{Z}) \otimes_{\mathbf{Z}} R' \arrow[ur] \arrow[rr] & & C^*(X;R') \end{tikzcd}$$

we can reduce to the case where R is the ring $\mathbf{Z}$ of integers. In this case, R' has finite Tor-amplitude over R, so the desired result follows from Proposition 2.4.17. □

In what follows, it will be convenient to give a name to those sheaves which satisfy the conclusions of Theorem 2.4.9.

Definition 2.4.19. Let R be a connective $\mathbb{E}_\infty$-ring and write $\mathrm{Spec}^{\text{ét}} R = (\mathcal{S}\mathrm{hv}_R^{\text{ét}}, \mathcal{O})$. We will say that an object $X \in \mathcal{S}\mathrm{hv}_R^{\text{ét}}$ is *pseudo-affine* if it satisfies the following conditions:

(a) The $\mathbb{E}_\infty$-ring $\mathcal{O}(X)$ is formally étale over R.

(b) If $M \in (\mathrm{Mod}_R)_{\leq 0}$ and $\mathcal{M}$ is the associated quasi-coherent sheaf on $\mathrm{Spec}^{\text{ét}} R$, then the canonical map $\mathcal{O}(X) \otimes_R M \to \mathcal{M}(X)$ is an equivalence.

Example 2.4.20. Let R be a connective $\mathbb{E}_\infty$-ring. Then every affine object $X \in \mathcal{S}\mathrm{hv}_R^{\text{ét}}$ is pseudo-affine. If X is corepresentable by an étale R-algebra R', then $\mathcal{O}(X) \simeq R'$ is obviously formally étale over R, and condition (b) of Definition 2.4.19 is satisfied for *all* R-modules M.

Remark 2.4.21. Let R be a connective $\mathbb{E}_\infty$-ring and suppose that $X \in \mathcal{S}\mathrm{hv}_R^{\text{ét}}$ satisfies condition (a) of Proposition 2.4.19. Then X is pseudo-affine if and only if it satisfies the following weaker version of (b):

(b') If M is a discrete R-module and $\mathcal{M}$ denotes the associated quasi-coherent sheaf on $\mathcal{X}$, then the canonical map $\mathcal{O}(X) \otimes_R M \to \mathcal{M}(X)$ is an equivalence.

To prove this, we first note that (b') and Proposition 2.4.17 imply that $\mathcal{O}(X)$ has Tor-amplitude ≤ 0 over R (see Proposition A.7.2.5.23). Let us now suppose that M is an arbitrary object of $(\mathrm{Mod}_R)_{\leq 0}$ and let K_M denote the fiber of the map $\mathcal{O}(X) \otimes_R M \to \mathcal{M}(X)$. Since $\mathcal{O}(X)$ has Tor-amplitude ≤ 0, the tensor product $\mathcal{O} \otimes_R M$ is 0-truncated. Since $\mathcal{M}(X)$ is 0-truncated (by Proposition 2.4.17), we conclude that K_M is 0-truncated for all $M \in (\mathrm{Mod}_R)_{\leq 0}$. We now prove that K_M is $(-r)$-truncated using induction on r. We have a cofiber sequence of R-modules $\pi_0 M \to M \to \tau_{\leq -1} M$, hence another cofiber sequence $K_{\pi_0 M} \to K_M \to K_{\tau_{\leq -1} M}$. Since $K_{\pi_0 M} \simeq 0$ by virtue of assumption (b), we conclude that $K_M \simeq K_{\tau_{\leq -1} M}$. Since $\tau_{\leq -1} M \in (\mathrm{Mod}_R)_{\leq -1}$, the inductive hypothesis implies that $K_M \simeq K_{\tau_{\leq -1} M}$ is $(-r)$-truncated as desired.

Theorem 2.4.9 asserts that if R is a p-thin $\mathbb{E}_\infty$-ring, then every p-constructible object of $\mathcal{S}\mathrm{hv}_R^{\text{ét}}$ is pseudo-affine. In order to verify this, we will need to establish some stability properties for the collection of pseudo-affine sheaves.

Lemma 2.4.22. *Let k be a field of characteristic p, let Y be a p-finite space, and let $X \in \mathcal{S}\mathrm{hv}_k^{\acute{e}t}$ be the constant sheaf associated to Y. Then X is pseudo-affine.*

Proof. Let $\phi^* : \mathcal{S} \to \mathcal{S}\mathrm{hv}_k^{\text{ét}}$ be a geometric morphism (that is, a left adjoint to the functor given by evaluation at k), and write $\mathrm{Spec}^{\text{ét}} k = (\mathcal{S}\mathrm{hv}_k^{\text{ét}}, \mathcal{O})$, so that we have an equivalence $C^*(Z; k) \simeq \mathcal{O}(\phi^* Z)$ depending functorially on Z. Then $\mathcal{O}(X) \simeq C^*(Y; k)$ is formally étale over k by Proposition 2.4.12. Since k is a field, every object $M \in (\mathrm{Mod}_k)_{\leq 0}$ has Tor-amplitude ≤ 0. Condition (b) of Definition 2.4.19 now follows immediately from Proposition 2.4.17. □

Suppose we are given a map $f : R \to R'$ of connective $\mathbb{E}_\infty$-rings, and write

$$\mathrm{Spec}^{\text{ét}} R = (\mathcal{S}\mathrm{hv}_R^{\text{ét}}, \mathcal{O}) \qquad \mathrm{Spec}^{\text{ét}} R' = (\mathcal{S}\mathrm{hv}_{R'}^{\text{ét}}, \mathcal{O}').$$

Suppose that M' is a R'-module, and let M denote the same spectrum regarded as a R-module. Let $\mathcal{M}$ and $\mathcal{M}'$ be the associated quasi-coherent sheaves on $\mathrm{Spec}^{\text{ét}} R$ and $\mathrm{Spec}^{\text{ét}} R'$. Then $\mathcal{M}$ is the pushforward of $\mathcal{M}'$: in other words, for each $X \in \mathcal{S}\mathrm{hv}_R^{\text{ét}}$ we have a canonical equivalence $\mathcal{M}(X) \simeq \mathcal{M}'(f^* X)$. If X is pseudo-affine and M is truncated, then $\mathcal{M}(X) \simeq \mathcal{O}(X) \otimes_R M$ and we obtain an equivalence $\alpha : \mathcal{O}(X) \otimes_R M \to \mathcal{M}'(f^* X)$. Assume that R' is n-truncated for some n. Applying this argument to the module $M' = R'$, we obtain an equivalence $\mathcal{O}(X) \otimes_R R' \simeq \mathcal{O}'(X)$, so that $\mathcal{O}'(X)$ has finite Tor-amplitude over R' and is formally étale over R' by Remark 2.4.8. In this case, for any truncated R'-module M', we can identify α with the map

$$\mathcal{O}(X) \otimes_R M \simeq (\mathcal{O}(X) \otimes_R R') \otimes_{R'} M' \simeq \mathcal{O}'(f^* X) \otimes_{R'} M' \to \mathcal{M}'(f^*(X)).$$

This proves the following:

Lemma 2.4.23. *Let $f : R \to R'$ be a map of connective $\mathbb{E}_\infty$-rings. Assume that R' is n-truncated for some n. Then the pullback functor $f^* : \mathcal{S}\mathrm{hv}_R^{\acute{e}t} \to \mathcal{S}\mathrm{hv}_{R'}^{\acute{e}t}$ carries pseudo-affine objects of $\mathcal{S}\mathrm{hv}_R^{\acute{e}t}$ to pseudo-affine objects of $\mathcal{S}\mathrm{hv}_{R'}^{\acute{e}t}$.*

If R is a commutative ring and p is a prime number which vanishes in R, then there is a map of commutative rings $\mathbf{F}_p \to R$, where $\mathbf{F}_p$ denotes the field with p elements. Combining Lemmas 2.4.22 and 2.4.23, we obtain:

Lemma 2.4.24. *Let R be a commutative ring, p a prime number which vanishes in R, Y a p-finite space, and $X \in \mathcal{S}\mathrm{hv}_R^{\acute{e}t}$ the constant sheaf associated to Y. Then X is pseudo-affine.*

We next observe that the property of being pseudo-affine is local:

Lemma 2.4.25. *Let R be a connective $\mathbb{E}_\infty$-ring, let $\{\phi_\alpha : R \to R_\alpha\}_{\alpha \in A}$ be a finite collection of étale maps of $\mathbb{E}_\infty$-rings such that the map $R \to \prod_\alpha R_\alpha$ is faithfully flat, and let $X \in \mathcal{S}\mathrm{hv}_R^{\acute{e}t}$ be coherent. Then X is pseudo-affine if and only if each pullback $\phi_\alpha^* X \in \mathcal{S}\mathrm{hv}_{R_\alpha}^{\acute{e}t}$ is pseudo-affine.*

Proof. Write $\mathrm{Spec}^{\text{ét}} R = (\mathcal{S}\mathrm{hv}_R^{\text{ét}}, \mathcal{O})$ and $\mathrm{Spec}^{\text{ét}} R_\alpha = (\mathcal{S}\mathrm{hv}_{R_\alpha}^{\text{ét}}, \mathcal{O}_\alpha)$. Using Proposition 2.4.17, we deduce that $\mathcal{O}_\alpha(\phi_\alpha^* X) \simeq \mathcal{O}(X) \otimes_R R_\alpha$ so we have equivalences

$$L_{\mathcal{O}_\alpha(\phi_\alpha^* X)/R_\alpha} \simeq L_{\mathcal{O}_{\mathcal{X}}(X)/R} \otimes_R R_\alpha.$$

Since the map $R \to \prod_\alpha Rk_\alpha$ is faithfully flat, we deduce that $\mathcal{O}(X)$ is formally étale over R if and only if each $\mathcal{O}_\alpha(\phi_\alpha^* X)$ is formally étale over R_α.

Assume now that X is pseudo-affine and let $\alpha \in A$. We wish to prove that $\phi_\alpha^* X$ is pseudo-affine. We have already verified condition (a) of Definition 2.4.19. To prove (b), let $M' \in (\mathrm{Mod}_{R_\alpha})_{\leq 0}$, let M be its image in Mod_R, and let $\mathcal{M}$ and $\mathcal{M}'$ denote the associated quasi-coherent sheaves on $\mathrm{Spec}^{\text{ét}} R$ and $\mathrm{Spec}^{\text{ét}} R_\alpha$, respectively. We wish to show that the canonical map $\beta : \mathcal{O}_\alpha(\phi_\alpha^*(X)) \otimes_{R_\alpha} M' \to \mathcal{M}'(\phi_\alpha^*(X))$ is an equivalence. Using Proposition 2.4.17, we can identify β with the map $\mathcal{O}(X) \otimes_R M \to \mathcal{M}(X)$, which is an equivalence by virtue of our assumption that X is pseudo-affine.

Conversely, assume that each ϕ_α^* is pseudo-affine. We must show that X satisfies condition (b) of Definition 2.4.19. Let $M \in (\mathrm{Mod}_R)_{\leq 0}$ and let $\mathcal{M}$ be the associated quasi-coherent sheaf on $\mathcal{X}$; we wish to show that the map $\mathcal{O}(X) \otimes_R M \to \mathcal{M}(X)$ is an equivalence. Since $R \to \prod_\alpha R_\alpha$ is faithfully flat, it suffices to show that for each $\alpha \in A$, the induced map

$$\gamma : (\mathcal{O}(X) \otimes_R R_\alpha) \otimes_{R_\alpha} (R_\alpha \otimes_R M) \to R_\alpha \otimes_R \mathcal{M}(X).$$

Let $M' = R_\alpha \otimes_R M$ and let $\mathcal{M}'$ be the associated quasi-coherent sheaf on $\mathrm{Spec}^{\text{ét}} R_\alpha$.

Since R_α is flat over R, Proposition 2.4.17 allows us to identify γ with the map

$$\mathcal{O}_\alpha(\phi_\alpha^* X) \otimes_{R_\alpha} M' \to \mathcal{M}'(\phi_\alpha^* X),$$

which is an equivalence by virtue of our assumption that $\phi_\alpha^* X$ is pseudo-affine. $\square$

Lemma 2.4.26. *Let p be a prime number, R a p-thin $\mathbb{E}_\infty$-ring, and $X \in \mathcal{S}\mathrm{hv}_R^{\acute{e}t}$ a sheaf which is locally constant and p-constructible. Then X is pseudo-affine.*

Proof. Using Lemma 2.4.25, we can reduce to the case where X is the constant sheaf associated to a p-finite space Y. Write $\mathrm{Spec}^{\text{ét}} R = (\mathcal{S}\mathrm{hv}_R^{\text{ét}}, \mathcal{O})$. Let M be an R-module and let $\mathcal{M}$ denote the corresponding quasi-coherent sheaf on $\mathrm{Spec}^{\text{ét}} R$. We will say that M is *good* if it satisfies the following conditions:

(i) The canonical map $\mathcal{O}(X) \otimes_R M \to \mathcal{M}(X)$ is an equivalence.

(ii) The tensor product $L_{\mathcal{O}(X)/R} \otimes_R M$ vanishes.

To prove that X is pseudo-coherent, it will suffice to show that (i) is satisfied whenever M is discrete, and condition (ii) is satisfied when $M = R$ (see Remark 2.4.21). Since R is connective and truncated, it can be obtained as a successive extension of finitely many discrete R-modules. Because the collection of good R-modules is closed under extensions, it will suffice to show that every discrete R-module M is good. Choose $m \geq 0$ such that $p^m = 0$ in $\pi_0 R$. The module M then has a finite filtration

$$0 = p^m M \subseteq p^{m-1} M \subseteq \cdots \subseteq M.$$

Consequently, to show that M is good, it will suffice to show that each quotient $p^{a-1}M/p^a M$ is good. We may therefore assume without loss of generality that M is a module over the commutative ring $k = (\pi_0 R)/(p)$. Using Corollary 2.4.18, we can replace R by k and thereby reduce to the situation of Lemma 2.4.24. □

Proof of Theorem 2.4.9. Using Theorem 2.3.24, we may assume that there is a finite sequence of elements $x_1, \ldots, x_m \in \pi_0 R$ which generate the unit ideal in R such that, if we let $R_i = (\pi_0 R)/(x_1, \ldots, x_{i-1})[x_i^{-1}]$, then the map of $\mathbb{E}_\infty$-rings $\phi_i : R \to R_i$ is such that $\phi_i^* X \in \mathcal{S}\mathrm{hv}^{\text{ét}}_{R_i}$ is locally constant and p-constructible. Let us assume that m is chosen as small as possible; we proceed by induction on m. If $m = 0$, then $R \simeq 0$ and there is nothing to prove. Let us therefore assume that $m > 0$. Let $\mathbf{Z}[y]$ denote the polynomial ring in one variable over $\mathbf{Z}$. Since R admits the structure of a simplicial commutative ring, there exists a map of $\mathbb{E}_\infty$-rings $\mathbf{Z}[y] \to R$ such that $y \mapsto x_1 \in \pi_0 R$. Consider also the map of commutative rings $\mathbf{Z}[y] \to \mathbf{Z}$ given by $y \mapsto 0$, and let R' denote the $\mathbb{E}_\infty$-ring given by $R \otimes_{\mathbf{Z}[y]} \mathbf{Z}$. Then R' is also p-thin, and $\pi_0 R' \simeq (\pi_0 R)/(x_1)$. Let $\psi' : R \to R'$ and $\psi'' : R \to R[x_1^{-1}]$ be the canonical maps. Then $\psi'^* X \in \mathcal{S}\mathrm{hv}^{\text{ét}}_{R'}$ is pseudo-affine by the inductive hypothesis, and $\psi''^* X \in \mathcal{S}\mathrm{hv}^{\text{ét}}_{R[x_1^{-1}]}$ is pseudo-affine by Lemma 2.4.26. Let $\mathcal{O}'$ and $\mathcal{O}''$ denote the structures sheaves of $\mathrm{Spec}^{\text{ét}} R'$ and $\mathrm{Spec}^{\text{ét}} R[x_1^{-1}]$, respectively.

We now prove that X is pseudo-affine. Let $A = \mathcal{O}(X)$. We first show that A is formally étale over R. Consider the relative cotangent complex $L_{A/R}$. Note that R' has Tor-amplitude ≤ 1 over R (since $\mathbf{Z}$ has Tor-amplitude ≤ 1 over $\mathbf{Z}[y]$). It follows from Proposition 2.4.17 that the canonical map $A \otimes_R R' \to \mathcal{O}'(\psi'^* X)$ is an equivalence, so that the inductive hypothesis implies that $A \otimes_R R'$ is formally étale over R'. It follows that the tensor product $L_{A/R} \otimes_R R'$ vanishes. Note that as an R-module, R' is given by the cofiber of the map $x_1 : R \to R$. We therefore have a fiber sequence

$$L_{A/R} \xrightarrow{x_1} L_{A/R} \to L_{A/R} \otimes_R R',$$

so that multiplication by x_1 is an equivalence from $L_{A/R}$ to itself. It follows that the canonical map $L_{A/R} \to L_{A/R} \otimes_R R[x_1^{-1}] \simeq L_{A[x_1^{-1}]/R[x_1^{-1}]}$ is an equivalence. Since $R[x_1^{-1}]$ is flat over R, Proposition 2.4.17 gives an equivalence $A[x_1^{-1}] \simeq \mathcal{O}''(\psi''^* X)$. Since $\psi''^* X$ is pseudo-affine, we conclude that $A[x_1^{-1}]$ is formally étale over $R[x_1^{-1}]$, so that $L_{A/R} \simeq 0$.

Now suppose that $M \in (\mathrm{Mod}_R)_{\leq 0}$, and let $\mathcal{M}$ be the associated quasi-coherent sheaf on $\mathrm{Spec}^{\text{ét}} R$. We wish to prove that the canonical map $u : A \otimes_R M \to \mathcal{M}(X)$ is an equivalence. Let K denote the fiber of u; we wish to show that $K \simeq 0$. Let $M' = R' \otimes_R M$ and let $\mathcal{M}'$ be the associated quasi-coherent sheaf on $\mathrm{Spec}^{\text{ét}} R'$. Using Proposition 2.4.17, we can identify $K \otimes_R R'$ with the fiber of the map $u' : (A \otimes_R R') \otimes_{R'} M' \to \mathcal{M}'(\psi'^* X)$. Since $\psi'^* X$ is pseudo-affine, the map u' is an equivalence. Using the fiber sequence

$$K \xrightarrow{x_1} K \to K \otimes_R R',$$

we deduce that multiplication by x_1 induces an equivalence from K to itself, so that the map $K \to K \otimes_R R[x_1^{-1}]$ is an equivalence. It will therefore suffice to show that $K \otimes_R R[x_1^{-1}] \simeq 0$: that is, that the map

$$u''' : A[x_1^{-1}] \otimes_R M \to \mathcal{M}(X) \otimes_R R[x_1^{-1}]$$

is an equivalence. This follows from Proposition 2.4.17, together with our assumption that $\psi''^* X$ is mild. □

2.5 Compactness of Relative Cochain Algebras

Let R be a connective $\mathbb{E}_\infty$-ring, write $\mathrm{Spec}^{\text{ét}} R = (\mathcal{S}\mathrm{hv}^{\text{ét}}_R, \mathcal{O})$, and let $X \in \mathcal{S}\mathrm{hv}^{\text{ét}}_R$. In §2.4, we proved that if R is p-thin and X is p-constructible, then $\mathcal{O}(X)$ is well-behaved as an $\mathbb{E}_\infty$-algebra over R: it is formally étale (Theorem 2.4.9) and its definition is compatible with base change in R (Corollary 2.4.10). In this section, we will continue our study of R-algebras having the form $\mathcal{O}(X)$. Our main result can be stated as follows:

Theorem 2.5.1. *Let R be an $\mathbb{E}_\infty$-ring, let p be a prime number such that R is p-thin, and let $X \in \mathcal{S}\mathrm{hv}_R^{\acute{e}t}$ be p-constructible. If we let $\mathcal{O}$ denote the structure sheaf of $\mathrm{Spec}^{\acute{e}t} R$, then $\mathcal{O}(X)$ is a compact object of CAlg_R.*

The proof of Theorem 2.5.1 will require several preliminary results. We begin by studying the behavior of $\mathcal{O}(X)$ as a functor of X.

Proposition 2.5.2. *Let R be an $\mathbb{E}_\infty$-ring which is p-thin and let $\mathcal{O}$ denote the structure sheaf of $\mathrm{Spec}^{\acute{e}t} R$. Let τ :*

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$

be a pullback square of p-constructible objects of $\mathcal{S}\mathrm{hv}_R^{\acute{e}t}$. Then the diagram σ :

$$\begin{tikzcd} \mathcal{O}(X') & \mathcal{O}(X) \arrow[l] \\ \mathcal{O}(Y') \arrow[u] & \mathcal{O}(Y) \arrow[l] \arrow[u] \end{tikzcd}$$

is a pushout square in CAlg_R.

Proof. Using Theorem 2.3.24, we conclude that there exists a sequence of elements $x_1, \ldots, x_m \in \pi_0 R$ which generate the unit ideal such that, if we set $k_i = (\pi_0 R)/(x_1, \ldots, x_{i-1})[x_i^{-1}]$, the image of X in $\mathcal{S}\mathrm{hv}_{k_i}^{\text{ét}}$ is locally constant. Choose m as small as possible. We proceed by induction on m.

If $m = 0$, then $R \simeq 0$ and there is nothing to prove. Assume therefore that $m > 0$. Since R is p-thin, it admits the structure of a simplicial commutative ring. Choose a map of simplicial commutative rings $\mathbf{Z}[t] \to R$ which carries t to x_1. Let R' denote the tensor product $R \otimes_{\mathbf{Z}[t]} \mathbf{Z}$, where $\mathbf{Z}[t] \to \mathbf{Z}$ is given by $t \mapsto 0$. Then R' is p-thin and $\pi_0 R' \simeq R/(x_1)$. Using the inductive hypothesis and Corollary 2.4.10, we deduce that the diagram

$$\begin{tikzcd} \mathcal{O}(X') \otimes_R R' & \mathcal{O}(X) \otimes_R R' \arrow[l] \\ \mathcal{O}(Y') \otimes_R R' \arrow[u] & \mathcal{O}(Y) \otimes_R R' \arrow[l] \arrow[u] \end{tikzcd}$$

is a pushout square of $\mathbb{E}_\infty$-algebras over R'. If we let K denote the total homotopy fiber of σ (regarded as an R-module), then $K \otimes_R R' \simeq 0$. It follows that multiplication by x_1 induces an equivalence from K to itself, so that $K \simeq K \otimes_R R[x_1^{-1}]$. It will therefore suffice to show that σ is a pullback square after tensoring with $R[x_1^{-1}]$. Using Corollary 2.4.10, we can replace R by $R[x_1^{-1}]$ and thereby reduce to the case where X is locally constant.

Using the same argument, we may reduce to the case where Y and Y' are locally constant. Let $\phi^* : \mathcal{S} \to \mathcal{S}\mathrm{hv}_R^{\text{ét}}$ be a geometric morphism. The assertion that σ is a pushout square is local with respect to the étale topology. We may therefore assume without loss of generality that X, Y, and Y' are constant: that is, $X = \phi^* X_0$, $Y = \phi^* Y_0$, and $Y' = \phi^* Y_0'$ for some p-finite spaces $X_0, Y_0, Y_0' \in \mathcal{S}$.

Choose an integer r such that X_0 and Y_0 are r-truncated. Since X_0 is p-finite, Proposition 2.3.9 implies that X_0 is a compact object of $\tau_{\leq r} \mathcal{S}$, so there exists a compact object $K \in \mathcal{S}$ such that $X_0 = \tau_{\leq r} K$. The map $\alpha = X \to Y$ induces a morphism $\phi^* K \to \phi^* X_0 \to \phi^* Y_0$. Since K is finite, we may assume (after passing to an étale covering of R if necessary) that this morphism is induced by a map of spaces $\beta : K \to Y_0$. Since Y_0 is r-truncated, β we deduce that β factors (in an essentially unique way) through some map $\beta' : X_0 \to Y_0$, and that α can be identified with $\phi^*(\beta')$. Similarly, we may assume that the map $Y' \to Y$ arises from a map of p-finite spaces $Y'_0 \to Y_0$. Then the diagram τ is obtained from a pullback diagram of p-finite spaces τ_0:

$$\begin{tikzcd} X'_0 \arrow[r] \arrow[d] & X_0 \arrow[d] \\ Y'_0 \arrow[r] & Y_0. \end{tikzcd}$$

We may therefore identify σ with the diagram

$$\begin{tikzcd} C^*(X'_0; R) & C^*(X_0; R) \arrow[l] \\ C^*(Y'_0; R) \arrow[u] & C^*(Y_0; R). \arrow[l] \arrow[u] \end{tikzcd}$$

Let K denote the fiber of the map $C^*(X_0; R) \otimes_{C^*(Y_0;R)} C^*(Y'_0; R) \to C^*(X'_0; R)$, and let us say that an R-module M is *good* if $M \otimes_R K \simeq 0$. We wish to prove that R is good. Since R is truncated and the collection of good R-modules is closed under extensions, it will suffice to show that every discrete R-module is good. Since p is nilpotent in R, every discrete R-module M admits a finite filtration

$$0 = p^m M \subseteq p^{m-1} M \subseteq \cdots \subseteq pM \subseteq M.$$

It therefore suffices to show that every module over the commutative ring $(\pi_0 R)/(p)$ is good. Using Corollary 2.4.18, we can replace R by $(\pi_0 R)/(p)$, and thereby reduce to the case where R is a commutative ring in which $p = 0$. In that case, there exists a ring homomorphism $\mathbf{F}_p \to R$ (where $\mathbf{F}_p$ denotes the finite field with p elements). Using Corollary 2.4.18 again, we can reduce to the case where $R = \mathbf{F}_p$. In this case, σ is a pushout diagram by virtue of Corollary 1.1.10. □

Corollary 2.5.3. *Let R be an $\mathbb{E}_\infty$-ring which is p-thin, let $\mathcal{O}$ be the structure sheaf of $\mathrm{Spec}^{\acute{e}t} R$, and let $X \in \mathcal{S}\mathrm{hv}_R^{\acute{e}t}$ be p-constructible. If X is m-truncated, then $\mathcal{O}(X)$ is an m-truncated object of CAlg_R^{op}. In other words, for every $\mathbb{E}_\infty$-algebra A over R, the mapping space $\mathrm{Map}_{\mathrm{CAlg}_R}(\mathcal{O}(X), A)$ is m-truncated.*

Proof. Combine Proposition 2.5.2 with Proposition T.5.5.6.16. □

Lemma 2.5.4. *Let $R \to \overline{R}$ be a square-zero extension of $\mathbb{E}_\infty$-rings, and let A be a formally étale R-algebra. Then R is a compact object of CAlg_R if and only if $\overline{A} = A \otimes_R \overline{R}$ is a compact object of $\mathrm{CAlg}_{\overline{R}}$.*

Proof. The "only if" direction is obvious (and does not require the assumption that A is formally étale over R). For the converse, assume that $\overline{A}$ is a compact object of $\mathrm{CAlg}_{\overline{R}}$. Suppose we are given a filtered diagram $\{B_\alpha\}$ in CAlg_R having colimit B. Let $\overline{B}_\alpha = B_\alpha \otimes_R \overline{R}$, and let $\overline{B} = B \otimes_R \overline{R}$. Then each B_α is a square-zero extension of $\overline{B}_\alpha$, so the canonical maps

$$\mathrm{Map}_{\mathrm{CAlg}_R}(A, B_\alpha) \to \mathrm{Map}_{\mathrm{CAlg}_R}(A, \overline{B}_\alpha) \simeq \mathrm{Map}_{\mathrm{CAlg}_{\overline{R}}}(\overline{A}, \overline{B}_\alpha)$$

are homotopy equivalences. Similarly, the natural map $\mathrm{Map}_{\mathrm{CAlg}_R}(A, B) \to \mathrm{Map}_{\mathrm{CAlg}_{\overline{R}}}(\overline{A}, \overline{B})$ is a homotopy equivalence. We have a commutative diagram

$$\begin{tikzcd} \varinjlim \mathrm{Map}_{\mathrm{CAlg}_R}(A, B_\alpha) \arrow[r] \arrow[d] & \varinjlim \mathrm{Map}_{\mathrm{CAlg}_{\overline{R}}}(\overline{A}, \overline{B}_\alpha) \arrow[d] \\ \mathrm{Map}_{\mathrm{CAlg}_R}(A, B) \arrow[r] & \mathrm{Map}_{\mathrm{CAlg}_{\overline{R}}}(\overline{A}, \overline{B}) \end{tikzcd}$$

in which the horizontal maps are homotopy equivalences, and the right vertical map is a homotopy equivalence by virtue of our assumption that $\overline{A}$ is a compact object of $\mathrm{CAlg}_{\overline{R}}$. It follows that the left vertical map is a homotopy equivalence, as desired. □

Notation 2.5.5. Let R be an $\mathbb{E}_\infty$-ring and let $x \in \pi_0 R$. If M is an R-module, we let $M[x^{-1}]$ denote the tensor product $M \otimes_R R[x^{-1}]$ and $M^\wedge_{(x)}$ the completion of M with respect to the ideal (x) generated by x (see §XII.4.1). Note that if A is an $\mathbb{E}_\infty$-algebra over R, then we can also regard $A[x^{-1}]$ and $A^\wedge_{(x)}$ as $\mathbb{E}_\infty$-algebras over R.

Lemma 2.5.6. *Let R be an $\mathbb{E}_\infty$-ring, let $x \in \pi_0 R$, and let M be an R-module. Then the diagram*

$$\begin{tikzcd} M \arrow[r] \arrow[d] & M^\wedge_{(x)} \arrow[d] \\ M[x^{-1}] \arrow[r] & M^\wedge_{(x)}[x^{-1}] \end{tikzcd}$$

is a pullback square of R-modules.

Proof. We freely use the terminology of §XII.4.1. Let $N = M^\wedge_{(x)} \times_{M^\wedge_{(x)}[x^{-1}]} M[x^{-1}]$; we wish to show that the canonical map $\phi : M \to N$ is an equivalence. Let K denote the fiber of ϕ. Then we have a fiber sequence

$$K \to K' \to K'',$$

where K' is the fiber of the map $M \to M[x^{-1}]$ and K'' is the fiber of the map $M^\wedge_{(x)} \to M^\wedge_{(x)}[x^{-1}]$. Since K' and K'' are both (x)-nilpotent, we conclude that K is (x)-nilpotent. We also have a fiber sequence

$$K \to L \to L',$$

where L is the fiber of the completion map $M \to M^\wedge_{(x)}$ and L' is the fiber of the map $M[x^{-1}] \to M^\wedge_{(x)}[x^{-1}]$. The definition of completion guarantees that L is (x)-local. Since L' is the fiber of a map between (x)-local R-modules, it is (x)-local. It follows that K is (x)-local. Since K is also (x)-nilpotent, the identity map $\mathrm{id} : K \to K$ is nullhomotopic, so that $K \simeq 0$ and $M \simeq N$. $\square$

Lemma 2.5.7. *Let R be an $\mathbb{E}_\infty$-ring, let $x \in \pi_0 R$, and suppose we are given a filtered diagram of R-modules $\{M_\alpha\}$ having a colimit M. Then the diagram σ :*

$$\begin{tikzcd} \varinjlim (M_\alpha)^\wedge_{(x)} \arrow[r] \arrow[d] & M^\wedge_{(x)} \arrow[d] \\ \varinjlim (M_\alpha)^\wedge_{(x)}[x^{-1}] \arrow[r] & M^\wedge_{(x)}[x^{-1}] \end{tikzcd}$$

is a pullback square.

Proof. The diagram σ is a filtered colimit of diagrams σ_n :

$$\begin{tikzcd} \varinjlim (M_\alpha)^\wedge_{(x)} \arrow[r] \arrow[d, "x^n"] & M^\wedge_{(x)} \arrow[d, "x^n"] \\ \varinjlim (M_\alpha)^\wedge_{(x)} \arrow[r] & M^\wedge_{(x)} \end{tikzcd}$$

in which the vertical maps are induced by multiplication by x^n. Let τ_n denote the diagram

$$\begin{tikzcd} \varinjlim M_\alpha \arrow[r] \arrow[d, "x^n"] & M \arrow[d, "x^n"] \\ \varinjlim M_\alpha \arrow[r] & M. \end{tikzcd}$$

The diagrams τ_n are evidently pullback squares (since the horizontal maps are equivalences). There are transformations $\tau_n \to \sigma_n$ which induce equivalences after passing to the fibers of the vertical maps. It follows that each σ_n is a pullback square. Passing to the limit, we deduce that σ is a pullback square. □

Lemma 2.5.8. *Let $f : R \to R'$ be a finite étale map between commutative rings and let $\mathcal{O}$ and $\mathcal{O}$ denote the structure sheaves of $\operatorname{Spec}^{ét} R$ and $\operatorname{Spec}^{ét} R'$, respectively. Let $X' \in \mathcal{S}\mathrm{hv}^{ét}_{R'}$ and let $A \in \mathrm{CAlg}_R$. Suppose that X' is p-constructible for some prime number p which vanishes in R. Then the canonical map*

$$\mathrm{Map}_{\mathrm{CAlg}_R}(\mathcal{O}(f_* X'), A) \to \mathrm{Map}_{\mathrm{CAlg}_{R'}}(R' \otimes_R \mathcal{O}(f_* X'), R' \otimes_R A) \to \mathrm{Map}_{\mathrm{CAlg}_{R'}}(\mathcal{O}'(X'), R' \otimes_R A)$$

is a homotopy equivalence.

Proof. Since f is finite étale , there exists an étale covering $\{R \to R_\alpha\}$ such that each tensor product $R' \otimes_R R_\alpha$ is equivalent to $R_\alpha^{n_\alpha}$ for some integer $n_\alpha \geq 0$. Since the property of being p-constructible is local and stable under finite products, we conclude that $f_* X'$ is p-constructible. Using Corollary 2.4.10, we see that $\mathcal{O}(f_* X')$ and $\mathcal{O}'(X')$ are compatible with base change in R. It follows that the desired conclusion is local with respect to the étale topology on R. We may therefore replace R by R_α and thereby reduce to the case where $R' \simeq R^n$ for some integer n. In this case, X' corresponds to an n-tuple of p-constructible objects $X_i \in \mathcal{S}\mathrm{hv}^{\text{ét}}_R$, and $\mathcal{O}'(X') \simeq \prod_{1 \leq i \leq n} \mathcal{O}(X_i)$. It follows that

$$\mathrm{Map}_{\mathrm{CAlg}_{R'}}(\mathcal{O}'(X'), R' \otimes_R A) \simeq \mathrm{Map}_{\mathrm{CAlg}_{R'}}(\prod_{1 \leq i \leq n} \mathcal{O}(X_i), A^n) \simeq \prod_{1 \leq i \leq n} \mathrm{Map}_{\mathrm{CAlg}_R}(\mathcal{O}(X_i), A).$$

Consequently, we are reduced to showing that $\mathcal{O}(f_* X')$ is a coproduct of the objects $\mathcal{O}(X_i)$ in CAlg_R. Since $f_* X' \simeq \prod_{1 \leq i \leq n} X_i$, this follows from Proposition 2.5.2. □

Lemma 2.5.9. *Let $\mathcal{C}$ be a full subcategory of the ∞-category $\mathcal{S}$ of spaces. Assume that $\mathcal{C}$ is closed under retracts and finite limits, and that $\mathcal{C}$ contains the Eilenberg-MacLane space $K(\mathbf{Z}/p\,\mathbf{Z}, n)$ for every integer $n \geq 0$. Then $\mathcal{C}$ contains every p-finite space.*

Proof. Let X be a p-finite space. Let m be the smallest integer such that X is m-truncated; we proceed by induction on m. The case $m = 0$ is trivial and left to the reader. Assume that $m > 0$ and write X as a disjoint union $X_1 \coprod \cdots \coprod X_n$ where each X_i is connected, and choose a base point x_i in each X_i. We proceed by induction on the order of the group $\prod_{1 \leq i \leq n} \pi_m(X_i, x_i)$. Since X is not $(m-1)$-truncated, the group $\pi_m(X_j, x_j)$ does not vanish for some j, and therefore has order divisible by p. Since $\pi_1(X_j, x_j)$ is a finite p-group, the number of $\pi_1(X_j, x_j)$-fixed elements of $\pi_m(X_j, x_j)$ is congruent to the order of $\pi_m(X_j, x_j)$ modulo p, and therefore divisible by p. It follows that there exists a subgroup $N \subseteq \pi_m(X_j, x_j)$ of order p on which $\pi_1(X_j, x_j)$ acts trivially. It follows that there is a fiber sequence of spaces

$$X_j \xrightarrow{\phi} X'_j \xrightarrow{\eta_j} K(\mathbf{Z}/p\,\mathbf{Z}, m+1),$$

with

$$\pi_i(X'_j, \phi(x_j)) \simeq \begin{cases} \pi_i(X'_j, x_j) & \text{if } i \neq m \\ \pi_i(X'_j, x_j)/N & \text{if } i = m. \end{cases}$$

Let Y be the disjoint union of X'_j with $\coprod_{i \neq j} X_i$. The inductive hypothesis implies that $Y \in \mathcal{C}$. Let $\eta : Y \to K(\mathbf{Z}/p\,\mathbf{Z}, m+1)$ be a map which is given by η_j on the component X'_j, and is nullhomotopic on every other component, and let Y' be the homotopy fiber of η. Since $\mathcal{C}$ contains $K(\mathbf{Z}/p\,\mathbf{Z}, m+1)$ and is stable under finite limits, we conclude that $Y' \in \mathcal{C}$. Note that Y' homotopy equivalent to the disjoint union of X_j with $\coprod_{i \neq j}(X_i \times K(\mathbf{Z}/p\,\mathbf{Z}, m+1))$. It follows that X is a retract of Y', so that $X \in \mathcal{C}$ as desired. □

Lemma 2.5.10. *Let $\phi : R \to A$ be a map of $\mathbb{E}_\infty$-rings which exhibits A as an m-truncated object of CAlg_R^{op}. Suppose we are given a finite collection of morphisms $\{R \to R_\alpha\}$ such that the induced map $R \to \prod R_\alpha$ is faithfully flat, and each tensor product $R_\alpha \otimes_R A$ is a compact object of CAlg_{R_α}. Then A is a compact object of CAlg_R.*

Proof. Let $R^0 = \prod_\alpha R_\alpha$, so that $R \to R^0$ is faithfully flat and $R^0 \otimes_R A$ is a compact object of CAlg_{R^0}. Let $R^\bullet$ be the Čech nerve (in CAlg^{op}) of the map $R \to R^0$, and set $A^\bullet = A \otimes_R R^\bullet$. Let $F : \mathrm{CAlg}_R \to \mathcal{S}$ be the functor corepresented by A. Then F is the totalization of a complicial object $F^\bullet$ of $\mathrm{Fun}(\mathrm{CAlg}_R, \mathcal{S})$, where F^n is given by the formula $F^n(B) = \mathrm{Map}_{\mathrm{CAlg}_R}(A, R^n \otimes_R B) \simeq \mathrm{Map}_{\mathrm{CAlg}_{R^n}}(R^n \otimes_R A, R^n \otimes_R B)$. Since each $R^n \otimes_R A$ is a compact object of CAlg_{R^n}, we conclude that each F^n commutes with filtered colimits. Since A is an m-truncated object of $(\mathrm{CAlg}_R)^{op}$, each of the functors F^i takes values in $\tau_{\leq m}\,\mathcal{S}$, so that F is equivalent to the finite limit $\varprojlim_{[n] \in \mathbf{\Delta}_{\leq m+1}} F^n$. It follows that F commutes with filtered colimits. $\square$

Lemma 2.5.11. *Let R be a commutative ring, let $\mathcal{O}$ denote the structure sheaf of $\mathrm{Spec}^{ét} R$, and let $X \in \mathcal{S}\mathrm{hv}_R^{ét}$. Assume that X is locally constant and p-constructible for some prime number p which vanishes in R. Then $\mathcal{O}(X)$ is a compact object of CAlg_R.*

Proof. Using Lemma 2.5.10, Corollary 2.5.3, and Corollary 2.4.10, we can reduce to the case where X is the constant sheaf associated to a p-finite space X_0. In this case, we can identify $\mathcal{O}(X)$ with $C^*(X_0; R) \simeq C^*(X_0; \mathbf{F}_p) \otimes_{\mathbf{F}_p} R$. It will therefore suffice to show that $C^*(X_0; \mathbf{F}_p)$ is a compact object of $\mathrm{CAlg}_{\mathbf{F}_p}$. Using Corollary 1.1.10, we see that the collection of p-finite spaces X_0 for which $C^*(X_0; \mathbf{F}_p)$ is compact in $\mathrm{CAlg}_{\mathbf{F}_p}$ is closed under retracts and finite limits. Using Lemma 2.5.9, we are reduced to proving that each of the cochain algebras $C^*(K(\mathbf{Z}/p\,\mathbf{Z}, n); \mathbf{F}_p)$ is a compact object of $\mathrm{CAlg}_{\mathbf{F}_p}$, which follows immediately from Theorem 2.2.17. $\square$

Proposition 2.5.12. *Let R be a Noetherian commutative ring, let p be a prime number which vanishes in R. Choose an element $x \in R$ and let $\mathcal{O}$ denote the structure sheaf of $\mathrm{Spec}^{ét} R[x^{-1}]$. Let $X \in \mathcal{S}\mathrm{hv}_{R[x^{-1}]}^{ét}$ be p-constructible, and suppose there is a Galois covering $\phi : R[x^{-1}] \to R'_0$ such that the image of X in $\mathcal{S}\mathrm{hv}_{R'_0}^{ét}$ is constant. Then the construction*

$$A \mapsto \mathrm{Map}_{\mathrm{CAlg}_{R[x^{-1}]}}(\mathcal{O}(X), A^\wedge_{(x)}[x^{-1}])$$

determines a functor $\mathrm{CAlg}_R \to \mathcal{S}$ which commutes with filtered colimits.

Proof. Let $\mathcal{S}\mathrm{hv}_{R[x^{-1}]}^{ét} \underset{\phi_*}{\overset{\phi^*}{\rightleftarrows}} \mathcal{S}\mathrm{hv}_{R'_0}^{ét}$ be the geometric morphism determined by ϕ, and let $T : \mathcal{S}\mathrm{hv}_{R[x^{-1}]}^{ét} \to \mathcal{S}\mathrm{hv}_{R[x^{-1}]}^{ét}$ be the monad given by $\phi_* \circ \phi^*$. For every object $Y \in \mathcal{S}\mathrm{hv}_{R[x^{-1}]}^{ét}$, we can associate an augmented cosimplicial object $Y^\bullet$, given informally by $[n] \mapsto T^{n+1}(Y)$. This augmented cosimplicial object is ϕ^*-split, so that the map $\phi^* Y \to \varprojlim \phi^* Y^\bullet$ is an equivalence. If Y is n-truncated for some integer n, then we have $\phi^* Y \simeq \varprojlim \phi^* Y^\bullet$ in the ∞-category $\tau_{\leq n}\,\mathcal{S}\mathrm{hv}_{R'_0}^{ét}$ which is equivalent to an $(n+1)$-category. It follows that $\phi^* Y$ is equivalent to the finite limit $\varprojlim_{[m] \in \mathbf{\Delta}_{\leq n+1}} \phi^* Y^m$. Since ϕ^* is left exact and conservative, we conclude that $Y \simeq \varprojlim_{[m] \in \mathbf{\Delta}_{\leq n+1}} Y^m$.

Let us say that a p-constructible sheaf $Y \in \mathcal{S}\mathrm{hv}_R^{ét}$ is *good* if the construction

$$A \mapsto \mathrm{Map}_{\mathrm{CAlg}_{R[x^{-1}]}}(\mathcal{O}(Y), A^\wedge_{(x)}[x^{-1}])$$

commutes with filtered colimits. Proposition 2.5.2 implies that the collection of good objects of $\mathcal{X}$ is closed under finite limits. It follows that to prove that Y is good, it suffices to show that each $Y^m \simeq T^m(Y)$ is good.

Let G denote a Galois group for the covering $R[x^{-1}] \to R'_0$. It follows that G acts on the ∞-topos $\mathcal{S}\mathrm{hv}_{R'_0}^{ét}$. Unwinding the definitions, we see that for each $Z \in \mathcal{S}\mathrm{hv}_{R'_0}^{ét}$, we have $\phi^* \phi_* Z \simeq \prod_{g \in G} Z^g$. In particular, if Z is the constant sheaf associated to a p-finite space Z_0, then $\phi^* \phi_* Z$ is the constant sheaf associated to the p-finite space $Z_0^{|G|} \simeq \prod_{g \in G} Z_0$. Since $\phi^* X$ is constant and p-constructible, we deduce that each $\phi^*(T^n X)$ is constant and p-constructible. Consequently, to prove that X is good, it will suffice to show that $\phi_* Z$ is good whenever $Z \in \mathcal{X}'$ is the constant sheaf associated to a p-finite space $Z_0 \in \mathcal{S}$. In other words, we must show that the functor

$$A \mapsto \mathrm{Map}_{\mathrm{CAlg}_{R[x^{-1}]}}(\mathcal{O}(\phi_* Z), A^{\wedge}_{(x)}[x^{-1}])$$

commutes with filtered colimits. Let $\mathcal{O}'$ denote the structure sheaf of $\mathrm{Spec}^{\text{ét}} R'_0$. Using Lemma 2.5.8, we obtain canonical homotopy equivalences

$$\begin{aligned} \mathrm{Map}_{\mathrm{CAlg}_{R[x^{-1}]}}(\mathcal{O}(\phi_* Z), A^{\wedge}_{(x)}[x^{-1}]) &\simeq \mathrm{Map}_{\mathrm{CAlg}_{R'_0}}(\mathcal{O}'(Z), R'_0 \otimes_R A^{\wedge}_{(x)}) \\ &\simeq \mathrm{Map}_{\mathrm{CAlg}_{R'_0}}(C^*(Z_0; R'_0), R'_0 \otimes_R A^{\wedge}_{(x)}). \end{aligned}$$

By assumption, the map $R[x^{-1}] \to R'_0$ is finite étale. It follows that the map $R \to R'_0$ is quasi-finite, and therefore admits a factorization $R \to R' \to R'_0$ where R' is a finite R-module and the map of schemes $\mathrm{Spec}^{\mathrm{Z}} R'_0 \to \mathrm{Spec}^{\mathrm{Z}} R'$ is an open immersion (Theorem VII.7.11). It follows that $j : \mathrm{Spec}^{\mathrm{Z}} R'_0 \to \mathrm{Spec}^{\mathrm{Z}} R'[x^{-1}]$ is also an open immersion. But R'_0 is finitely generated as a module over $R[x^{-1}]$ and therefore over $R'[x^{-1}]$, so the image of j is closed. It follows that $R'_0 \simeq R'[x^{-1}, e^{-1}]$ for some idempotent element $e \in R'[x^{-1}]$. Then $x^m e$ is the image of some element $\overline{e} \in R'$. Replacing R' by $R'/(x^m - \overline{e})$, we can reduce to the case where $e = 1$, so that $R'_0 \simeq R'[x^{-1}]$.

The ring R' is finitely presented as an R-algebra. Write $R' = R[y_1, \ldots, y_m]/(f_1, \ldots, f_k)$. Let $\psi : \mathbf{Z}[z_1, \ldots, z_k] \to R[y_1, \ldots, y_m]$ be the map which sends z_i to the polynomial f_i and let R'' denote the tensor product $R[y_1, \ldots, y_m] \otimes_{\mathbf{Z}[z_1, \ldots, z_k]} \mathbf{Z}$, where $\mathbf{Z}[z_1, \ldots, z_k] \to \mathbf{Z}$ is the map which sends each z_i to $0 \in \mathbf{Z}$. Then R'' is an $\mathbb{E}_\infty$-algebra over R with $\pi_i R'' \simeq \mathrm{Tor}_i^{\mathbf{Z}[z_1, \ldots, z_k]}(R[y_1, \ldots, y_m], \mathbf{Z})$. In particular, we deduce that $\pi_0 R'' \simeq R'$. Each $\pi_i R''$ is a finite module over $R[y_1, \ldots, y_m]$ which is annihilated by each f_i, hence a finite module over R' and therefore also a finite module over R. It follows that R'' is an almost perfect R-module (Proposition A.7.2.5.17). Since $\mathbf{Z}$ has Tor-amplitude $\leq k$ over $\mathbf{Z}[z_1, \ldots, z_k]$, the algebra R'' has Tor-amplitude $\leq k$ over $R[y_1, \ldots, y_m]$ and is therefore of finite Tor-amplitude over R. Using Proposition A.7.2.5.23, we conclude that R'' is perfect as an R-module.

Corollary 2.4.10 gives an equivalence $C^*(Z_0, R'_0) \simeq C^*(Z_0, R'') \otimes_{R''} R'_0$. Consequently, we wish to prove that the functor

$$A \mapsto \mathrm{Map}_{\mathrm{CAlg}_{R''}}(C^*(Z_0; R''), R'_0 \otimes_R A^{\wedge}_{(x)})$$

commutes with filtered colimits. Note that we have an equivalence $R'_0 \simeq \pi_0 R''[x^{-1}]$. Since R'' is k-truncated, it follows that the map $R''[x^{-1}] \to R'_0$ is a composition of square-zero extensions (Corollary A.7.4.1.28). Since $C^*(Z_0; R'')$ is formally étale over R'', we have canonical homotopy equivalences

$$\mathrm{Map}_{\mathrm{CAlg}_{R''}}(C^*(Z_0; R''), R''[x^{-1}] \otimes_R A^{\wedge}_{(x)} \to \mathrm{Map}_{\mathrm{CAlg}_{R''}}(C^*(Z_0; R''), R'_0 \otimes_R A^{\wedge}_{(x)}).$$

Because R'' is perfect as an R-module, the canonical map $R'' \otimes_R A^{\wedge}_{(x)} \to (R'' \otimes_R A)^{\wedge}_{(x)}$ is an equivalence for each $A \in \mathrm{CAlg}_R$. It will therefore suffice to show that the construction

$$B \mapsto \mathrm{Map}_{\mathrm{CAlg}_{R''}}(C^*(Z_0; R''), B^{\wedge}_{(x)}[x^{-1}])$$

determines a functor $\theta_{Z_0} : \mathrm{CAlg}_{R''} \to \mathcal{S}$ which commutes with filtered colimits.

Using Proposition 2.5.2, we see that the functor $Z_0 \mapsto \theta_{Z_0}$ commutes with finite limits. It follows that the collection of p-finite spaces Z_0 for which θ_{Z_0} commutes with filtered colimits is closed under retracts and finite limits. Using Lemma 2.5.9, we are reduce to proving that θ_{Z_0} commutes with filtered colimits in the case where Z_0 is an Eilenberg-MacLane space $K(\mathbf{Z}/p\mathbf{Z}, n)$.

The sequence of spaces $\{K(\mathbf{Z}/p\mathbf{Z}, n)\}_{n \geq 0}$ is a spectrum object in the ∞-category of p-finite spaces. It follows from Proposition 2.5.2 that for every $\mathbb{E}_\infty$-algebra B over R'', we obtain a spectrum $U(B)$, given by $\Omega^{\infty - n} U(B) = \mathrm{Map}_{\mathrm{CAlg}_{R''}}(C^*(K(\mathbf{Z}/p\mathbf{Z}, n), R''), B)$. Theorem 2.2.17 implies that each $C^*(K(\mathbf{Z}/p\mathbf{Z}, n); R'')$ is a compact object of $\mathrm{CAlg}_{R''}$, so that U commutes with filtered colimits. Suppose we are given a filtered diagram $\{B(\alpha)\}$ of objects of $\mathrm{CAlg}_{R''}$ having colimit B. We wish to show that the induced map

$$\varinjlim U(B(\alpha)^{\wedge}_{(x)}[x^{-1}] \to U(B^{\wedge}_{(x)})[x^{-1}]$$

is an equivalence. Since U commutes with filtered colimits, this is equivalent to the assertion that U carries the map $\varinjlim B(\alpha)^{\wedge}_{(x)}[x^{-1}] \to B^{\wedge}_{(x)}[x^{-1}]$ to an equivalence of spectra. According to Lemma 2.5.6, we have a pullback diagram

$$\begin{tikzcd}
\varinjlim B(\alpha)^{\wedge}_{(x)} \arrow[r] \arrow[d] & \varinjlim B(\alpha)^{\wedge}_{(x)}[x^{-1}] \arrow[d] \\
B^{\wedge}_{(x)} \arrow[r] & B^{\wedge}_{(x)}[x^{-1}]
\end{tikzcd}$$

in $\mathrm{CAlg}_{R''}$, hence a pullback diagram of spectra

$$\begin{tikzcd}
U(\varinjlim B(\alpha)^{\wedge}_{(x)}) \arrow[r] \arrow[d, "\psi"] & U(\varinjlim B(\alpha)^{\wedge}_{(x)}[x^{-1}]) \arrow[d] \\
U(B^{\wedge}_{(x)}) \arrow[r] & U(B^{\wedge}_{(x)}[x^{-1}]).
\end{tikzcd}$$

It will therefore suffice to show that ψ is an equivalence. Using the fact that U commutes with filtered colimits, we can identify ψ with the canonical map

$$\varinjlim U(B(\alpha)^{\wedge}_{(x)}) \to U(B^{\wedge}_{(x)}).$$

Consider the map of commutative rings $\mathbf{Z}[t] \to R$ given by $t \mapsto x$. For $n \geq 0$, set $R^n = R \otimes_{\mathbf{Z}[t]} \mathbf{Z}[t]/(t^n)$. $\mathbf{Z}[t]/(t^{n+1}) \to \mathbf{Z}[t]/(t^n)$ is a square-zero extension. It follows that for $B \in \mathrm{CAlg}_{R''}$, we have a tower of square zero extensions

$$\cdots \to B \otimes_R R^3 \to B \otimes_R R^2 \to B \otimes_R R^1$$

with limit $B^{\wedge}_{(x)}$. Since each $C^*(K(\mathbf{Z}/p\,\mathbf{Z}, n), R'')$ is formally étale over R'', it follows that the induced map $U(B^{\wedge}_{(x)}) \to U(B \otimes_R R^1)$ is an equivalence of spectra. Consequently, we can identify ψ with the map

$$\varinjlim U(B(\alpha) \otimes_R R^1) \to U(B \otimes_R R^1).$$

This map is an equivalence by virtue of the fact that U commutes with filtered colimits. □

Proof of Theorem 2.5.1. Let R be p-thin. Then the map $R \to (\pi_0 R)/(p)$ is a composition of finitely many square-zero extensions. Using Lemma 2.5.4, we are reduced to proving that $\mathcal{O}(X) \otimes_R ((\pi_0 R)/(p))$ is a compact $(\pi_0 R)/(p)$-algebra. Using Corollary 2.4.10, we can replace R by $(\pi_0 R)/(p)$ and thereby reduce to the case where R is a commutative ring in which $p = 0$. Then R is the union finitely generated subrings R_i. Using Theorem 2.3.8, we can assume that X lies in the essential image of one of the pullback functors $\mathcal{S}\mathrm{hv}^{\text{ét}}_{R_i} \to \mathcal{S}\mathrm{hv}^{\text{ét}}_{R}$. Using Corollary 2.4.10, we can replace R by R_i and thereby reduce to the case where R is Noetherian.

Let S be the collection of all ideals $I \subseteq R$ such that $\mathcal{O}(X) \otimes_R R/I$ is not a compact object of $\mathrm{CAlg}_{R/I}$. We wish to show that S is empty. Assume otherwise; then, since R is Noetherian, there is a maximal element $I \in S$. Replacing R by R/I, we may assume S does not contain any nonzero ideals of R.

If $R = 0$, there is nothing to prove. Otherwise, since X is p-constructible, Theorem 2.3.24 implies that there exists a nonzero element $x \in R$ and a Galois extension $R[x^{-1}] \to R'_0$ such that the image of X in $\mathcal{S}\mathrm{hv}^{\text{ét}}_{R'_0}$ is a constant sheaf.

Choose a map of commutative rings $\mathbf{Z}[t] \to R$ carrying t to x. For every $\mathbb{E}_\infty$-algebra A over R, let A^n denote the relative tensor product $B \otimes_{\mathbf{Z}[t]} \mathbf{Z}[t]/(t^n)$. Using Proposition XII.4.2.7, we deduce that that (x)-adic completion of A can be identified with the limit of the tower

$$\cdots \to A^3 \to A^2 \to A^1.$$

Note that each map in this tower is a square-zero extension.

Let $F : \mathrm{CAlg}_R \to \mathcal{S}$ be the functor corepresented by $\mathcal{O}(X)$. Let $\{A_\alpha\}$ be a filtered diagram in CAlg_R having colimit A. It follows from Lemma 2.5.6 that the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A^{\wedge}_{(x)} \arrow[d] \\ A[x^{-1}] \arrow[r] & A^{\wedge}_{(x)}[x^{-1}] \end{tikzcd}$$

is a pullback square in CAlg_R. Since F is a corepresentable functor, the canonical map

$$F(A) \to F(A[x^{-1}]) \times_{F(A^{\wedge}_{(x)}[x^{-1}])} F(A^{\wedge}_{(x)})$$

is a homotopy equivalence. We wish to prove that the canonical map

$$\beta : \varinjlim F(A_\alpha) \to F(A[x^{-1}]) \times_{F(A^{\wedge}_{(x)}[x^{-1}])} F(A^{\wedge}_{(x)})$$

is a homotopy equivalence. Since each A_α fits into a pullback diagram

$$\begin{tikzcd} A_\alpha \arrow[r] \arrow[d] & (A_\alpha)^{\wedge}_{(x)} \arrow[d] \\ A_\alpha[x^{-1}] \arrow[r] & (A_\alpha)^{\wedge}_{(x)}[x^{-1}], \end{tikzcd}$$

we can identify β with the map

$$\varinjlim (F(A_\alpha[x^{-1}]) \times_{F((A_\alpha)^{\wedge}_{(x)}[x^{-1}])} F((A_\alpha)^{\wedge}_{(x)}) \to F(A[x^{-1}]) \times_{F(A^{\wedge}_{(x)}[x^{-1}]} F(A^{\wedge}_{(x)}).$$

It will therefore suffice to show that the maps

$$\beta_0 : \varinjlim F(A_\alpha[x^{-1}]) \to F(A[x^{-1}])$$

$$\beta_1 : \varinjlim F((A_\alpha)^{\wedge}_{(x)}[x^{-1}]) \to F(A^{\wedge}_{(x)}[x^{-1}])$$

$$\beta_2 : \varinjlim F((A_\alpha)^{\wedge}_{(x)}) \to F(A^{\wedge}_{(x)})$$

are homotopy equivalences.

By construction, the image of X in $\mathcal{S}\mathrm{hv}^{\text{ét}}_{R[x^{-1}]})$ is locally constant. Lemma 2.5.11 and Corollary 2.4.10 imply that β_0 is a homotopy equivalence. Similarly, Proposition 2.5.12 implies that β_1 is a homotopy equivalence. We complete the proof by showing that β_2 is a homotopy equivalence. Since the functor F is corepresentable, we have $F(A^{\wedge}_{(x)}) \simeq \varprojlim F(A^n)$. Since each of the maps $A^{n+1} \to A^n$ is a square-zero extension and $\mathcal{O}(X)$ is formally étale over R, the tower $\{F(A^n)\}$ is constant, so that the evident map $F(A^{\wedge}_{(x)}) \to F(A^1)$ is a homotopy equivalence. Similarly, each of the maps $F((A_\alpha)^{\wedge}_{(x)}) \to F(A^1_\alpha)$ is a homotopy equivalence. We may therefore identify β_2 with the canonical map $\varinjlim F(A^1_\alpha) \to F(A^1)$. To prove that this map is a homotopy equivalence, it suffices to show that $\mathcal{O}(X) \otimes_R R^1$ is a compact object of CAlg_{R^1}. Note that R^1 is a square-zero extension of $\pi_0 R^1 \simeq R/(x)$. According to Lemma 2.5.4, we are reduced to proving that $\mathcal{O}(X) \otimes_R R/(x)$ is a compact object of $\mathrm{CAlg}_{R/(x)}$. This follows from the observation that (x) is a nonzero ideal in R. $\square$

2.6 Affine Behavior of p-Constructible Morphisms

In §2.4, we introduced the definition of a *p-constructible* object of an ∞-topos of the form $\mathcal{S}\mathrm{hv}_R^{\text{ét}}$. We can relativize this notion as follows:

Definition 2.6.1. Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a spectral Deligne-Mumford stack and let p be a prime number. We will say that an object $U \in \mathcal{X}$ is *p-constructible* if, for every étale morphism $f : \operatorname{Spec}^{\text{ét}} R \to \mathfrak{X}$, the pullback $f^* U$ is a p-constructible object of the ∞-topos $\mathcal{S}\mathrm{hv}_R^{\text{ét}}$ (see Definition 2.4.1).

We say that a morphism of spectral Deligne-Mumford stacks $g : \mathfrak{Y} \to \mathfrak{X}$ is *p-constructible* if $\mathfrak{Y}$ is equivalent to a spectral Deligne-Mumford stack of the form $(\mathcal{X}_{/U}, \mathcal{O}_{\mathcal{X}} |U)$, where $U \in \mathcal{X}$ is p-constructible.

Remark 2.6.2. If $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a spectral Deligne-Mumford stack, then an object $U \in \mathcal{X}$ is p-constructible if and only if U is finitely constructible and, for every separably closed field k and every map $f : \operatorname{Spec}^{\text{ét}} k \to \mathfrak{X}$, the stalk $f^* U \in \mathcal{S}\mathrm{hv}_k^{\text{ét}} \simeq \mathcal{S}$ is a p-finite space.

Remark 2.6.3. Every p-constructible morphism of spectral Deligne-Mumford stacks in étale.

Remark 2.6.4. If $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \simeq \operatorname{Spec}^{\text{ét}} R$ is affine, then an object $U \in \mathcal{X}$ is p-constructible in the sense of Definition 2.6.1 if and only if it is p-constructible in the sense of Definition 2.4.1.

Remark 2.6.5. Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a spectral Deligne-Mumford stack and let $U \in \mathcal{X}$ be an object. Using Proposition 2.3.5, we see that the condition that U be p-constructible is local for the étale topology.

Definition 2.6.6. Let $\mathfrak{X}$ be a spectral Deligne-Mumford stack and p a prime number. We will say that $\mathfrak{X}$ is *p-thin* if there exists a surjective étale map $\coprod \operatorname{Spec}^{\text{ét}} R_\alpha \to \mathfrak{X}$, where each R_α is a p-thin $\mathbb{E}_\infty$-ring.

Warning 2.6.7. If R is a p-thin $\mathbb{E}_\infty$-ring, then the affine spectral Deligne-Mumford stack $\operatorname{Spec}^{\text{ét}} R$ is p-thin. It seems likely that the converse is false, because condition (3) of Definition 2.4.2 is somewhat unnatural.

Remark 2.6.8. Let $f : \mathfrak{Y} \to \mathfrak{X}$ be an étale map between spectral Deligne-Mumford stacks. If $\mathfrak{X}$ is p-thin, then $\mathfrak{Y}$ is also p-thin (see Remark 2.4.5). If f is surjective, then the converse holds.

Example 2.6.9. Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a spectral Deligne-Mumford stack. Suppose that $\mathcal{O}_{\mathcal{X}}$ is discrete (that is, we can think of $\mathcal{O}_{\mathcal{X}}$) as a sheaf of commutative rings on $\mathcal{X}$) and that the map $p^n : \mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{\mathcal{X}}$ is zero for $n \gg 0$. Then $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is p-thin.

We can now formulate the first main result of this section.

Theorem 2.6.10. *Let p be a prime number, let $q : \mathfrak{Y} = (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \to \mathfrak{X}$ be a p-constructible morphism of p-thin spectral Deligne-Mumford stacks, and let $f : \mathfrak{Z} = (\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}) \to \mathfrak{X}$ be an ∞-quasi-compact morphism between spectral Deligne-Mumford stacks. If $\mathfrak{Z}$ is p-thin, then the canonical map*

$$\mathrm{Map}_{\mathrm{Stk}_{/\mathfrak{X}}}(\mathfrak{Z}, \mathfrak{Y}) \to \mathrm{Map}_{\mathrm{CAlg}(\mathrm{QCoh}(\mathfrak{X}))}(q_* \mathcal{O}_{\mathcal{Y}}, f_* \mathcal{O}_{\mathcal{Z}})$$

is a homotopy equivalence.

Remark 2.6.11. In the situation of Theorem 2.6.10, the quasi-coherence of the pushforwards $q_* \mathcal{O}_{\mathcal{Y}}$ and $f_* \mathcal{O}_{\mathcal{Z}}$ follows from Corollary VIII.2.5.22.

Corollary 2.6.12. *Let R be a p-thin $\mathbb{E}_\infty$-ring, and let $\mathcal{O}$ denote the structure sheaf of $\operatorname{Spec}^{\text{ét}} R$. Then the construction $U \mapsto \mathcal{O}(U)$ induces a fully faithful embedding from the full subcategory $\mathcal{S}\mathrm{hv}_R^{p-\mathrm{fc}} \subseteq \mathcal{S}\mathrm{hv}_R^{\text{ét}}$ spanned by the p-constructible sheaves to the ∞-category CAlg_R^{op}.*

To prove Theorem 2.6.10, we are free to work locally on $\mathfrak{X}$ and $\mathfrak{Z}$, and thereby reduce to the case where each is the spectrum of a p-thin $\mathbb{E}_\infty$-ring. Using Theorem 2.4.9, we can replace $\mathfrak{Y}$ by $\mathfrak{Z} \times_{\mathfrak{X}} \mathfrak{Z}$, we can reduce to the case where $\mathfrak{X} = \mathfrak{Z} = \operatorname{Spec}^{\text{ét}} R$. In this case, we are reduced to proving the following:

Theorem 2.6.13. *Let R be a p-thin $\mathbb{E}_\infty$-ring, let $\mathcal{O}$ denote the structure sheaf of $\operatorname{Spec}^{ét} R$, and let $X \in \mathcal{S}\mathrm{hv}_R^{ét}$ be a p-constructible sheaf. Then the canonical map $X(R) \to \mathrm{Map}_{\mathrm{CAlg}_R}(\mathcal{O}(X), R)$ is a homotopy equivalence.*

Proof. The functor $\mathcal{O} : \mathcal{S}\mathrm{hv}_R^{ét} \to \mathrm{CAlg}_R^{op}$ preserves small colimits. It follows from Remark T.5.5.2.10 that $\mathcal{O}$ admits a right adjoint $\mathcal{S}\mathrm{ol} : \mathrm{CAlg}_R^{op} \to \mathcal{S}\mathrm{hv}_R^{ét}$. Unwinding the definitions, we see that $\mathcal{S}\mathrm{ol}$ carries an object $A \in \mathrm{CAlg}_R^{op}$ to the sheaf $\mathcal{S}\mathrm{ol}(A)$ given by $\mathcal{S}\mathrm{ol}(A)(R') = \mathrm{Map}_{\mathrm{CAlg}_R}(A, R')$. Theorem 2.6.13 is an immediate consequence of the following assertion:

$(*)$ If $X \in \mathcal{S}\mathrm{hv}_R^{ét}$ is p-constructible, then the unit map $u_X : X \to \mathcal{S}\mathrm{ol}(\mathcal{O}(X))$ is an equivalence in $\mathcal{S}\mathrm{hv}_R^{ét}$.

We first treat the case where R is a separably closed field of characteristic p. In this case, we have an equivalence of ∞-topoi $\mathcal{S}\mathrm{hv}_R^{ét} \simeq \mathcal{S}$. We can identify p-constructible sheaves $X \in \mathcal{S}\mathrm{hv}_R^{ét}$ with p-finite spaces, so that $\mathcal{O}(X) \simeq C^*(X; R)$. Note that $\mathcal{S}\mathrm{ol}(\mathcal{O}(X))$ can be identified with the mapping space $\mathrm{Map}_{\mathrm{CAlg}_R}(C^*(X; R), R)$. Using Proposition 2.5.2, we see that the construction $X \mapsto \mathcal{S}\mathrm{ol}(\mathcal{O}(X))$ commutes with finite limits. Consequently, to prove that u_X is an equivalence for every p-finite space X, it suffices to treat the case where X is an Eilenberg-MacLane space $K(\mathbf{Z}/p\mathbf{Z}, m)$ (see Lemma 2.5.9). Using Theorem 2.2.17, we can identify $\mathcal{S}\mathrm{ol}(\mathcal{O}(X))$ with the homotopy fiber of the map $K(R, m) \xrightarrow{A} K(R, m)$, where A is induced by the Artin-Schreier map $x \mapsto x - x^p$ from R to itself. Since R is a separably closed field, the Artin-Scheier map is surjective with kernel $\mathbf{F}_p \subseteq R$, so that $\mathcal{S}\mathrm{ol}(\mathcal{O}(X)) \simeq K(\mathbf{F}_p, m)$. Since composite map $K(\mathbf{Z}/p\mathbf{Z}, m) \xrightarrow{u_X} K(\mathbf{F}_p, m) \to K(R, m)$ is given by the canonical embedding of $\mathbf{Z}/p\mathbf{Z}$ into R, we conclude that u_X is a homotopy equivalence as desired.

We now treat the general case. Write $\operatorname{Spec}^{ét} \pi_0 R = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ and let $\phi^* : \mathcal{S}\mathrm{hv}_R^{ét} \to \mathcal{X}$ be the equivalence of ∞-topoi induced by the truncation map $R \mapsto \pi_0 R$. Since R is truncated, the map $R \to \pi_0 R$ is a composition of square-zero extensions. Since $\mathcal{O}(X)$ is formally étale over R (Theorem 2.4.9), Corollary 2.4.10 implies that $\phi_* \mathcal{S}\mathrm{ol}(\mathcal{O}(X)) \simeq \mathcal{S}\mathrm{ol}(\mathcal{O}_{\mathcal{X}}(\pi_* X))$. We may therefore replace R by $\pi_0 R$ and thereby reduce to the case where R is discrete.

Define full subcategories $\mathcal{S}\mathrm{hv}_R^{\mathrm{fc}, \leq m} \subseteq \mathcal{S}\mathrm{hv}_R^{ét}$ as in the proof of Theorem 2.3.8. We will prove that the unit map u_X is an equivalence for all p-constructible sheaves X in $\mathcal{S}\mathrm{hv}_R^{\mathrm{fc}, \leq m}$ using induction on m. When $m = -2$, the sheaf X is representable by an étale R-algebra R', and the desired result follows from Theorem VII.5.14. Suppose that $m \geq -1$. Since X is quasi-compact, we can choose a representable sheaf X_0 and an effective epimorphism $f : X_0 \to X$. Let $X_\bullet$ denote the Čech nerve of f. Using Proposition 2.5.2, we deduce that $\mathcal{S}\mathrm{ol}(\mathcal{O}(X_\bullet))$ is the Čech nerve of the map $\mathcal{S}\mathrm{ol}(\mathcal{O}(X_0)) \to \mathcal{S}\mathrm{ol}(\mathcal{O}(X))$. Each X_i belongs to $\mathcal{S}\mathrm{hv}_R^{\mathrm{fc}, \leq m-1}$, so that u_{X_i} is an equivalence by the inductive hypothesis. It follows that $X_\bullet$ can be identified with the Čech nerve of the composite map $X_0 \to X \xrightarrow{u_X} \mathcal{S}\mathrm{ol}(\mathcal{O}(X))$. Consequently, to show that u_X is an equivalence, it will suffice to show that u_X is an effective epimorphism. Unwinding the definitions, this amounts to the following assertion:

$(*')$ Let R' be an étale R-algebra suppose we are given a map of R-algebras $\phi : \mathcal{O}(X) \to R'$. Then, for every maximal ideal $\mathfrak{m} \subseteq R'$, there exists an étale map $\psi : R' \to A$ such that $A/\mathfrak{m}A \neq 0$, and the composition $\psi \circ \phi$ is homotopic to the map given by a point $\eta \in X(A)$.

Let $\phi : \mathcal{O}(X) \to R'$ be as in $(*')$. Using Theorem 2.3.8, we deduce that there exists a subring $R_0 \subseteq R$ which is finitely generated over $\mathbf{Z}$ such that X is the restriction of a p-constructible sheaf $Y \in \mathcal{S}\mathrm{hv}_{R_0}^{ét}$. Enlarging R_0 if necessary, we can assume that R' is given by $R \otimes_{R_0} R'_0$ for some étale R_0-algebra R'_0. Let $\mathcal{O}''$ denote the structure sheaf of $\operatorname{Spec}^{ét} R_0$. Using Theorem 2.5.1, we can assume that ϕ arises from a map $\phi_0 : \mathcal{O}''(Y) \to R'_0$. To prove $(*)$, we may replace R by R_0, X by Y, and R' by R'_0. We may therefore assume without loss of generality that the commutative ring R is finitely generated over $\mathbf{Z}$; in particular, R is Noetherian.

For every map of commutative rings $\psi : R' \to A$, let X_A denote the image of X in the ∞-topos ∞-topos $\mathcal{S}\mathrm{hv}_A^{ét}$, and let $F(A)$ denote the homotopy fiber of the map of spaces $X_A(A) \to \mathrm{Map}_{\mathrm{CAlg}_R}(\mathcal{O}(X), A)$ over the point given by $\psi \circ \phi$. Theorem 2.3.8 implies that the functor $A \mapsto X_A(A)$ commutes with filtered colimits, and Theorem 2.5.1 implies that the functor $A \mapsto \mathrm{Map}_{\mathrm{CAlg}_R}(\mathcal{O}(X), A)$ commutes with filtered colimits. It follows that F commutes with filtered colimits. We wish to prove that for every maximal ideal $\mathfrak{m} \subseteq R'$, there exists an étale R'-algebra A such that $A/\mathfrak{m}A \neq 0$ and $F(A)$ is nonempty.

Let k denote the residue field $R'/\mathfrak{m}$ and let $\overline{k}$ denote a separable closure of k. Using the first part of the proof together with Corollary 2.4.10, we deduce that $F(\overline{k})$ is nonempty. Since F commutes with filtered colimits, there exists a finite separable extension $\widetilde{k}$ of k such that $F(\widetilde{k}) \neq \emptyset$. Using the structure theory of étale morphism (Proposition VII.8.10), we see that there exists an étale R'-algebra $\widetilde{R}$ such that $\widetilde{R}/\mathfrak{m}\widetilde{R} \simeq \widetilde{k}$. In particular, $\mathfrak{m}\widetilde{R}$ is a maximal ideal of $\widetilde{R}$. Let $\widetilde{R}^{\wedge}$ denote the completion of $\widetilde{R}$ at $\mathfrak{m}\widetilde{R}$. Then $\widetilde{R}^{\vee}$ is the inverse limit of a tower of square-zero extensions

$$\cdots \rightarrow \widetilde{R}/\mathfrak{m}^3\widetilde{R} \rightarrow \widetilde{R}/\mathfrak{m}^2\widetilde{R} \rightarrow \widetilde{R}/\mathfrak{m}\widetilde{R}.$$

Since $\mathcal{O}_{\mathcal{X}}(X)$ is formally étale over R, we conclude that the reduction map $\mathrm{Map}_{\mathrm{CAlg}_R}(\mathcal{O}_{\mathcal{X}}(X), \widetilde{R}^{\vee}) \rightarrow \mathrm{Map}_{\mathrm{CAlg}_R}(\mathcal{O}_{\mathcal{X}}(X), \widetilde{k})$ is a homotopy equivalence. Since $\widetilde{R}^{\wedge}$ is a complete local Noetherian ring, it is Henselian (Proposition VII.7.16) so that the map $X_{\widetilde{R}^{\wedge}}(\widetilde{R}^{\wedge}) \rightarrow X_{\widetilde{k}}(\widetilde{k})$ is a homotopy equivalence (Proposition XI.3.22). It follows that the map $F(\widetilde{R}^{\wedge}) \rightarrow F(\widetilde{k})$ is a homotopy equivalence. In particular, $F(\widetilde{R}^{\wedge})$ is nonempty.

Since $\widetilde{R}$ is an étale R-algebra, it is a finitely generated $\mathbf{Z}$-algebra, and therefore an excellent ring (see [56]). It follows that the map $\widetilde{R} \rightarrow \widetilde{R}^{\vee}$ is geometrically regular. Using Popescu's theorem ([67]), we see that $\widetilde{R}^{\vee}$ can be obtained as filtered colimit of smooth $\widetilde{R}$-algebras. Since F commutes with filtered colimits, we conclude that there exists a smooth $\widetilde{R}$-algebra B equipped with a map $B \rightarrow \widetilde{R}^{\vee}$ such that $F(B)$ is nonempty. Since $\widetilde{R}^{\vee}/\mathfrak{m}\widetilde{R}^{\vee} \simeq \widetilde{k} \neq 0$, we conclude that $B/\mathfrak{m}B \neq 0$. We can therefore choose a map of rings $B \rightarrow A$ such that $A/\mathfrak{m}A \neq 0$ and A is étale over $\widetilde{R}$ (and therefore also over R'). Since $F(B) \neq \emptyset$, we conclude that $F(A) \neq \emptyset$, as required by $(*)$. □

Suppose we are given a p-constructible morphism $q : (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \rightarrow \mathfrak{X}$. Assume that the structure sheaf of $\mathfrak{X}$ is truncated, so that $\mathcal{O}_{\mathcal{Y}}$ is also truncated and therefore $q_* \mathcal{O}_{\mathcal{Y}}$ is a quasi-coherent sheaf on $\mathfrak{X}$ (Corollary VIII.2.5.22). It follows from Theorem 2.6.13 that, under some mild hypotheses, we can recover $\mathfrak{Y}$ as a sort of relative spectrum of the quasi-coherent sheaf of $\mathbb{E}_\infty$-rings $q_* \mathcal{O}_{\mathcal{Y}}$ on $\mathfrak{X}$. In other words, p-constructible morphisms behave, in some respects, as if they are affine. We formulate another result to this effect.

Notation 2.6.14. Let $q : \mathfrak{Y} = (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \rightarrow \mathfrak{X}$ be a map of spectral Deligne-Mumford stacks. Assume that q is ∞-quasi-compact and that the structure sheaf $\mathcal{O}_{\mathcal{Y}}$ is truncated. Then $q_* \mathcal{O}_{\mathcal{Y}}$ is a commutative algebra object of $\mathrm{QCoh}(\mathfrak{X})$ (Corollary VIII.2.5.22). We let $\mathrm{Mod}_{q_* \mathcal{O}_{\mathcal{Y}}}(\mathrm{QCoh}(\mathfrak{X})_{\leq 0})$ denote the full subcategory of $\mathrm{Mod}_{q_* \mathcal{O}_{\mathcal{Y}}}(\mathrm{QCoh}(\mathfrak{X}))$ spanned by those $q_* \mathcal{O}_{\mathcal{Y}}$-modules whose image in $\mathrm{QCoh}(\mathfrak{X})$ belongs to $\mathrm{QCoh}(\mathfrak{X})_{\leq 0}$, so that q_* induces a functor

$$\mathrm{QCoh}(\mathfrak{Y})_{\leq 0} \rightarrow \mathrm{Mod}_{q_* \mathcal{O}_{\mathcal{Y}}}(\mathrm{QCoh}(\mathfrak{X})_{\leq 0}).$$

Theorem 2.6.15. *Let p be a prime number and let $q : \mathfrak{Y} = (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \rightarrow \mathfrak{X}$ be a p-constructible morphism between p-thin spectral Deligne-Mumford stacks. Then the pushforward functor q_* induces an equivalence of ∞-categories*

$$\mathrm{QCoh}(\mathfrak{Y})_{\leq 0} \rightarrow \mathrm{Mod}_{q_* \mathcal{O}_{\mathcal{Y}}}(\mathrm{QCoh}(\mathfrak{X})_{\leq 0}).$$

The proof of Theorem 2.6.15 will require some preliminaries.

Lemma 2.6.16. *Let R be a p-thin $\mathbb{E}_\infty$-ring and let $\mathfrak{X}$ be a spectral Deligne-Mumford stack over R. Let $x \in \pi_0 R$ be an element, define $R_0, R_1 \in \mathrm{CAlg}_R$ as in Remark 2.4.4, and set*

$$\mathfrak{X}_0 = \mathrm{Spec}^{\acute{e}t} R_0 \times_{\mathrm{Spec}^{\acute{e}t} R} \mathfrak{X} \qquad \mathfrak{X}_1 = \mathrm{Spec}^{\acute{e}t} R_1 \times_{\mathrm{Spec}^{\acute{e}t} R} \mathfrak{X},$$

so that we have a closed immersion $i : \mathfrak{X}_0 \rightarrow \mathfrak{X}$ and an open immersion $j : \mathfrak{X}_1 \rightarrow \mathfrak{X}$. Let $\mathcal{F}$ be a quasi-coherent sheaf on $\mathfrak{X}$. Then:

(1) *The sheaf $\mathcal{F}$ belongs to $\mathrm{QCoh}(\mathfrak{X})_{\leq 0}$ if and only if $j^* \mathcal{F} \in \mathrm{QCoh}(\mathfrak{X}_1)_{\leq 0}$ and $i^* \mathcal{F} \in \mathrm{QCoh}(\mathfrak{X}_0)_{\leq 1}$.*

(2) *The sheaf $\mathcal{F}$ is zero if and only if $j^* \mathcal{F}$ and $i^* \mathcal{F}$ are zero.*

(3) *A map $\alpha : \mathcal{F} \to \mathcal{F}'$ in $\mathrm{QCoh}(\mathfrak{X})$ is an equivalence if and only if $i^* \mathcal{F}$ and $j^* \mathcal{F}$ are equivalences.*

Proof. The implication (2) $\Rightarrow$ (3) is obvious, and the implication (1) $\Rightarrow$ (2) follows from the fact that the t-structure on $\mathrm{QCoh}(\mathfrak{X})$ is right complete. Assertion (1) is local on $\mathfrak{X}$; we may therefore suppose that $\mathfrak{X} = \mathrm{Spec}^{\text{ét}} A$ is affine. Then $\mathfrak{X}_0 \simeq \mathrm{Spec}^{\text{ét}} A_0$ and $\mathfrak{X}_1 \simeq \mathrm{Spec}^{\text{ét}} A_1$ are also affine. We will identify x with its image in $\pi_0 A$, so that $A_1 \simeq A[x^{-1}]$ and we have a fiber sequence of A-modules

$$A \xrightarrow{x} A \to A_0.$$

Let us identify $\mathcal{F}$ with an A-module M, so that $j^* \mathcal{F}$ can be identified with $M[\frac{1}{x}] = A[\frac{1}{x}] \otimes_A M$ and $i^* \mathcal{F}$ with the cofiber $\mathrm{cofib}(M \xrightarrow{x} M)$. It is now clear that $\mathcal{F} \in \mathrm{QCoh}(\mathfrak{X})_{\leq 0}$ implies that $j^* \mathcal{F} \in \mathrm{QCoh}(\mathfrak{X}_1)_{\leq 0}$ and $i^* \mathcal{F} \in \mathrm{QCoh}(\mathfrak{X}_0)_{\leq 1}$. To prove the converse, let us assume that $M[\frac{1}{x}] \in (\mathrm{Mod}_A)_{\leq 0}$ and that $\mathrm{cofib}(M \xrightarrow{x} M) \in (\mathrm{Mod}_A)_{\leq 1}$. We wish to show that $M \in (\mathrm{Mod}_A)_{\leq 0}$. Form a fiber sequence

$$K \to M \to M[\frac{1}{x}];$$

we will show that $K \in (\mathrm{Mod}_A)_{\leq 0}$. Note that $K \simeq Q[-1] \otimes_A M$, where Q denotes the cofiber of the map $A \to A[\frac{1}{x}]$. Then $Q \simeq \varinjlim Q_m$, where Q_m is the cofiber of the map $x^m : A \to A$. It will therefore suffice to prove that $Q_m \otimes_A M \in (\mathrm{Mod}_A)_{\leq 1}$ for each m. When $m = 1$, this follows from our assumption on $\mathrm{cofib}(M \xrightarrow{x} M)$. The general case follows by induction on m, using the existence of fiber sequences

$$Q_m \to Q_{m+1} \to Q_1.$$

□

Lemma 2.6.17. *Let G be a finite p-group and let V be an $\mathbf{F}_p$-vector space equipped with an action of G. If $V \neq 0$, then V contains a nonzero element which is fixed by the action of G.*

Proof. Replacing V by the G-submodule generated by any nonzero element $v \in V$, we may suppose that V is a finite set. Every nontrivial orbit for the action of G on V has cardinality divisible by p. Since the cardinality of V is divisible by p, we conclude that the cardinality of the set of G-fixed points is divisible by p. In particular, $0 \in V$ cannot be the only fixed point. □

Proof of Theorem 2.6.15. The assertion is local on $\mathfrak{X}$. We may therefore assume without loss of generality that $\mathfrak{X} = \mathrm{Spec}^{\text{ét}} R$, where R is a p-thin $\mathbb{E}_\infty$-ring.

Consider the pushforward functor $G : \mathrm{Mod}_{\mathcal{O}_\mathcal{Y}} \to \mathrm{Mod}_{q_* \mathcal{O}_\mathcal{Y}}$ (in the setting of sheaves which are not necessarily quasi-coherent), and let

$$G_0 : \mathrm{QCoh}(\mathfrak{Y})_{\leq 0} \to \mathrm{Mod}_{q_* \mathcal{O}_\mathcal{Y}}(\mathrm{QCoh}(\mathfrak{X})_{\leq 0})$$

denote the restriction of G. We wish to show that G_0 is an equivalence of ∞-categories. The functor G admits a left adjoint F, given by

$$F(\mathcal{M}) = \mathcal{O}_\mathcal{Y} \otimes_{q^* q_* \mathcal{O}_\mathcal{Y}} q^* \mathcal{M}.$$

From this description it is clear that F carries quasi-coherent sheaves of $q_* \mathcal{O}_\mathcal{Y}$-modules to quasi-coherent sheaves on $\mathfrak{Y}$. The main step of the proof is to establish the following:

$(*)$ Let $\mathcal{M} \in \mathrm{Mod}_{q_* \mathcal{O}_\mathcal{Y}}(\mathrm{QCoh}(\mathfrak{X})_{\leq 0})$. Then $F(\mathcal{M}) \in \mathrm{QCoh}(\mathfrak{X})_{\leq 0}$. Moreover, the unit map $\mathcal{M} \to (G \circ F)(\mathcal{M})$ is an equivalence.

Write $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_\mathcal{X})$ and $\mathfrak{Y} = (\mathcal{X}_{/U}, \mathcal{O}_\mathcal{X} |U)$ for some p-constructible sheaf $U \in \mathcal{X} \simeq \mathcal{S}\mathrm{hv}_R^{\text{ét}}$. By repeated use of Lemma 2.6.16, we are reduced to proving $(*)$ in the special case where U is locally constant and $p = 0$ in $\pi_0 R$. Since the assertion is local on $\mathcal{X}$, we may reduce further to the case where U is the constant sheaf associated to a p-finite space K. In this case, we can identify $q_* \mathcal{O}_\mathcal{Y}$ with the $\mathbb{E}_\infty$-ring $C^*(K; R)$. We can now rephrase $(*)$ as follows:

$(*')$ Let M be a $C^*(K;R)$-module with $\pi_m M \simeq 0$ for $m > 0$. Then, for each $x \in K$, the module $C^*(\{x\};R) \otimes_{C^*(K;R)} M$ belongs to $(\mathrm{Mod}_R)_{\leq 0}$. Moreover, the canonical map

$$M \to \varprojlim_{x \in K} (C^*(K;\{x\}) \otimes_{C^*(K;R)} M)$$

is an equivalence.

Since R is strongly commutative, it admits the structure of a simplicial commutative ring. In particular, we have a map of simplicial commutative rings $\mathbf{Z} \to R$. Let $e_0, e_p : \mathbf{Z}[X] \to \mathbf{Z}$ be the maps carrying the variable X to 0 and p, respectively. Since $p = 0$ in $\pi_0 R$, the diagram

$$\begin{tikzcd} \mathbf{Z}[X] \arrow[r, "e_0"] \arrow[d, "e_p"] & \mathbf{Z} \arrow[d] \\ \mathbf{Z} \arrow[r] & R \end{tikzcd}$$

commutes up to homotopy (in the ∞-category of simplicial commutative rings), so there exists a map $\mathbf{F}_p \simeq (\mathbf{Z} \otimes_{\mathbf{Z}[X]} \mathbf{Z}) \to R$. Corollary 2.4.18 supplies an equivalence of $\mathbb{E}_\infty$-rings

$$C^*(K;R) \simeq R \otimes_{\mathbf{F}_p} C^*(K;\mathbf{F}_p).$$

It will therefore suffice to prove $(*')$ in the special case where $R = \mathbf{F}_p$.

By breaking K up into its connected components, we may reduce to the case where K is connected. In this case, $C^*(K;\mathbf{F}_p)$ is a coconnective $\mathbb{E}_\infty$-algebra over $\mathbf{F}_p$, in the sense of Definition VIII.4.1.1. Using Proposition VIII.4.1.9 (and Remark VIII.4.1.10), we may assume that M is given by the colimit of a sequence of $C^*(K;\mathbf{F}_p)$-modules

$$0 = M(0) \to M(1) \to M(2) \to \cdots,$$

where each $M(n)$ fits into a fiber sequence

$$M(n) \to M(n+1) \to C^*(K;\mathbf{F}_p) \otimes_{\mathbf{F}_p} V(n)$$

for some $V(n) \in (\mathrm{Mod}_{\mathbf{F}_p})_{\leq 1}$. Fix a point $x \in K$, and consider the corresponding map $C^*(K;\mathbf{F}_p) \to C^*(\{x\};\mathbf{F}_p) \simeq \mathbf{F}_p$. Then

$$\mathbf{F}_p \otimes_{C^*(K;\mathbf{F}_p)} M \simeq \varinjlim \mathbf{F}_p \otimes_{C^*(K;\mathbf{F}_p)} M(n).$$

Consequently, to prove that $\pi_m(\mathbf{F}_p \otimes_{C^*(K;\mathbf{F}_p)} M)$ vanishes for $m > 0$, it will suffice to prove that

$$\pi_m(\mathbf{F}_p \otimes_{C^*(K;\mathbf{F}_p)} M(n)) \simeq 0$$

for $m > 0$ and each $n \geq 0$. We proceed by induction on n, the case $n = 0$ being trivial. The inductive step follows from the observation that

$$\pi_m(\mathbf{F}_p \otimes_{C^*(K;\mathbf{F}_p)} C^*(K;\mathbf{F}_p) \otimes_{\mathbf{F}_p} V(n)) \simeq \pi_m V(n) \simeq 0$$

for $m > 0$.

Let N denote the local system of $\mathbf{F}_p$-module spectra on K given by $x \mapsto C^*(\{x\};\mathbf{F}_p) \otimes_{C^*(K;\mathbf{F}_p)} M$. To complete the proof of $(*')$, we wish to show that the canonical map $\theta_M : M \to C^*(K;N)$ is an equivalence. For each $n \geq 0$, let $N(n)$ denote the local system of $\mathbf{F}_p$-module spectra on K given by $x \mapsto C^*(\{x\};\mathbf{F}_p) \otimes_{C^*(K;\mathbf{F}_p)} M(n)$. Then Then $N \simeq \varinjlim N(n)$, and the above argument shows that each $N(n)$ takes values in $(\mathrm{Mod}_{\mathbf{F}_p})_{\leq 0}$. Since K is p-finite, the functor $C^*(K;\bullet)$ commutes with filtered colimits when restricted to $(\mathrm{Mod}^K_{\mathbf{F}_p})_{\leq 0}$. It will therefore suffice to show that each of the maps $\theta_{M(n)}$ is an equivalence. We proceed by induction on n, the case $n = 0$ being trivial. To carry out the inductive step, it suffices to show that each of the maps $\theta_{C^*(K;\mathbf{F}_p) \otimes_{\mathbf{F}_p} V(n)}$ is an equivalence. Since $\mathbf{F}_p$ is a field, we can write $V(n)$ as a direct sum of $\mathbf{F}_p$-modules of the form $\mathbf{F}_p[m]$, where $m \leq 0$. Using the fact that the functor $C^*(K; \bullet)$ commutes with direct sums when restricted to $(\mathrm{Mod}^K_{\mathbf{F}_p})_{\leq 0}$, we are reduced to proving that $\theta_{C^*(K;\mathbf{F}_p)[m]}$ is an equivalence for $m \leq 0$, which is clear. This completes the proof of $(*')$ (and therefore also the proof of $(*)$).

It follows from $(*)$ that the functor G_0 admits a left adjoint $F_0 : \mathrm{Mod}_{q_* \mathcal{O}_{\mathcal{Y}}}(\mathrm{QCoh}(\mathfrak{X})_{\leq 0}) \to \mathrm{QCoh}(\mathfrak{Y})_{\leq 0}$, and that the counit map $\mathrm{id} \to G_0 \circ F_0$ is an equivalence. In particular, F_0 is fully faithful. To complete the proof that G_0 is an equivalence of ∞-categories, it will suffice to show that G_0 is conservative. Suppose that $\mathcal{M} \in \mathrm{QCoh}(\mathfrak{Y})_{\leq 0}$ and that $q_* \mathcal{M} \simeq 0$; we wish to prove that $\mathcal{M} \simeq 0$. We may again use Lemma 2.6.16 to reduce to the case where U is locally constant and $p = 0$ in $\pi_0 R$. The assertion is local on $\mathcal{X}$, so we may pass to an étale cover and thereby reduce to the case where U is the constant sheaf determined by a p-finite space K. In this case, we can identify $\mathcal{M}$ with a local system of R-modules on K. We wish to prove that if $\mathcal{M} \neq 0$, then $C^*(K; \mathcal{M}) \neq 0$. Since $\mathcal{M}$ is truncated, there exists a largest integer n such that $\pi_n \mathcal{M} \neq 0$. Choose a point $x \in K$ such that $\pi_n \mathcal{M}(x) \neq 0$. Let $V = \pi_n \mathcal{M}(x)$. We will view V as an $\mathbf{F}_p$-vector space equipped with an action of the fundamental group $G = \pi_1(K, x)$. Since $\pi_m \mathcal{M} \simeq 0$ for all $m > n$, the vector space $\mathrm{H}^0(G; V)$ appears as a direct summand in $\pi_n C^*(K; \mathcal{M})$. Since G is a finite p-group, Lemma 2.6.17 implies that $C^*(K; \mathcal{M}) \neq 0$, as desired. □

3 p-Profinite Homotopy Theory

Let p be a prime number and let $\mathcal{S}^{p-\mathrm{fc}}$ denote the ∞-category of p-finite spaces (Definition 2.4.1). We let $\mathcal{S}^{\mathrm{Pro}(p)}$ denote the ∞-category $\mathrm{Pro}(\mathcal{S}^{p-\mathrm{fc}})$. We will refer to the objects of $\mathcal{S}^{\mathrm{Pro}(p)}$ as *p-profinite spaces.*

Our goal in this section is to develop the homotopy theory of p-profinite spaces. Our main result can be stated as follows:

- $(*)$ Let k be an algebraically closed field of characteristic p. Then the functor $X \mapsto C^*(X; k)$ on p-finite spaces extends to a fully faithful embedding $\mathcal{S}^{\mathrm{Pro}(p)} \to \mathrm{CAlg}_k^{op}$. Moreover, the essential image of this embedding has a simple algebraic description.

Let us now outline the organization of this section. We begin in §3.1 by reviewing the definition of $\mathrm{Pro}(\mathcal{C})$, where $\mathcal{C}$ is an ∞-category. With an eye toward some later applications, we treat the case where $\mathcal{C}$ is an accessible ∞-category which is not essentially small. We then extend Theorem 2.6.10 to the setting of proconstructible sheaves. Specializing to the case of an algebraically closed (or, more generally, a separably closed) field k, we obtain the fully faithful embedding $\mathcal{S}^{\mathrm{Pro}(p)} \to \mathrm{CAlg}_k^{op}$ described in $(*)$ (Proposition 3.1.16).

In §3.2, we begin our study of p-profinite spaces in proper. To every p-profinite space X we can associate an ordinary space $\mathrm{Mat}(X) = \mathrm{Map}_{\mathcal{S}^{\mathrm{Pro}(p)}}(*, X)$, which we call the *materialization* of X. Given a point $\eta \in \mathrm{Mat}(X)$, we define

$$\pi_0 X = \pi_0 \mathrm{Mat}(X) \qquad \pi_n(X, \eta) = \pi_n(\mathrm{Mat}(X), \eta)$$

for $n \geq 1$. We will see that $\pi_0 X$ is equipped with the structure of a totally disconnected compact Hausdorff space, and that each $\pi_n(X, \eta)$ has the structure of a p-profinite group. Moreover, these homotopy groups control the structure of X: if $f : X \to Y$ is a map of p-profinite spaces which induces an isomorphism on all homotopy groups, then f is an equivalence (Theorem 3.2.2).

Let X be a p-profinite space. We will say that X is of *finite type* if it is simply connected (that is, the materialization $\mathrm{Mat}(X)$ is simply connected) and the homotopy groups $\pi_n X$ are finitely generated modules over the ring $\mathbf{Z}_p$ of p-adic integers, for all $n \geq 2$. In §3.3, we will show that the materialization functor $\mathrm{Mat} : \mathcal{S}^{\mathrm{Pro}(p)} \to \mathcal{S}$ is fully faithful when restricted to p-profinite spaces of finite type (Theorem 3.3.3).

The homotopy theory of p-profinite spaces behaves in many respects like the usual homotopy theory of spaces. For example, to every p-profinite space X on can associate a Postnikov tower

$$\cdots \to \tau_{\leq 2} X \to \tau_{\leq 1} X \to \tau_{\leq 0} X$$

having limit X. In §3.4, we will study the relative version of this Postnikov tower. That is, we will introduce the notions of *n-connective* and *n-truncated* morphisms in $\mathcal{S}^{\mathrm{Pro}(p)}$, and show that they behave much like the usual theory of n-connective and n-truncated morphisms of spaces. In particular, every map $f : X \to Y$ of p-profinite spaces admits an essentially unique factorization

$$X \xrightarrow{f'} Z \xrightarrow{f''} Y,$$

where f' is n-truncated and f'' is $(n-1)$-connective (Theorem 3.4.2).

In §3.5 we will complete the proof of $(*)$ by characterizing the essential image of the embedding $\mathcal{S}^{\mathrm{Pro}(p)} \to \mathrm{CAlg}_k^{\mathrm{op}}$ (at least in the case where k is algebraically closed; see Theorem 3.5.8). We will also characterize the essential image of the p-profinite spaces of finite type. Combining this the results of §3.3, we recover the following result of Mandell (see Corollary 3.5.15):

$(*')$ Let k be an algebraically closed field of characteristic p, and let $\mathcal{C}$ denote the full subcategory of $\mathcal{S}$ spanned by those spaces X which are simply connected and for which each homotopy group $\pi_n X$ is a finitely generated module over the ring $\mathbf{Z}_p$ of p-adic integers. Then the functor $X \mapsto C^*(X; k)$ extends to a fully faithful embedding $\mathcal{C} \to \mathrm{CAlg}_k^{op}$. Moreover, the essential image of this embedding admits a simple algebraic description.

For any p-thin $\mathbb{E}_\infty$-ring R, Corollary 2.6.12 supplies a fully faithful embedding $\mathcal{S}\mathrm{hv}_R^{p-\mathrm{fc}} \to \mathrm{CAlg}_R^{op}$. The proofs of $(*)$ and $(*')$ rest ultimately on a simple special case of this: namely, the case where R is an algebraically closed field of characteristic p. In §3.6, we will describe a different specialization of Corollary 2.6.12: we allow R to be arbitrary, but restrict our attention to sheaves which are constant. This leads to an algebraic description of the p-adic étale homotopy type of $\mathrm{Spec}^{\text{ét}} R$ (or, more generally, any p-thin spectral Deligne-Mumford stack: see Theorem 3.6.3).

Remark 3.0.18. For another approach to the homotopy theory of profinite spaces, we refer the reader to [80].

3.1 Pro-Constructible Sheaves and p-Profinite Spaces

Let R be an $\mathbb{E}_\infty$-ring and p a prime number. We let $\mathcal{S}\mathrm{hv}_R^{p-\mathrm{fc}}$ denote the full subcategory of $\mathcal{S}\mathrm{hv}_R^{\text{ét}}$ spanned by the p-constructible sheaves. If R is p-thin, then Corollary 2.6.12 gives a fully faithful embedding from $\mathcal{S}\mathrm{hv}_R^{p-\mathrm{fc}}$ into the ∞-category CAlg_R^{op}. In this section, we will consider an extension of this embedding to Pro-objects of $\mathcal{S}\mathrm{hv}_R^{p-\mathrm{fc}}$. We begin with a general review of the theory of Pro-objects.

Definition 3.1.1. Let $\mathcal{C}$ be an accessible ∞-category which admits finite limits. We let $\mathrm{Pro}(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{S})^{op}$ spanned by those functors $F : \mathcal{C} \to \mathcal{S}$ which are accessible and preserve finite limits. We will refer to $\mathrm{Pro}(\mathcal{C})$ as the *∞-category of* Pro-*objects* of $\mathcal{C}$.

Example 3.1.2. Let $\mathcal{C}$ be a small ∞-category which admits finite limits. Then $\mathcal{C}$ is accessible if and only if it is idempotent complete (Proposition T.5.4.2.17). Moreover, any functor $F : \mathcal{C} \to \mathcal{S}$ is automatically accessible. It follows that $\mathrm{Pro}(\mathcal{C})$ is the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{S})^{op}$ spanned by those functors which preserve finite limits. We therefore have $\mathrm{Pro}(\mathcal{C}) \simeq \mathrm{Ind}(\mathcal{C}^{op})^{op}$, where $\mathrm{Ind}(\mathcal{C}^{op})$ is defined as in §T.5.3.5.

Example 3.1.3. Let $\mathcal{C}$ be the nerve of an accessible category $\mathcal{C}_0$. Then $\mathrm{Pro}(\mathcal{C})$ is itself equivalent to the nerve of a category, which we will denote by $\mathrm{Pro}(\mathcal{C}_0)$. We can identify $\mathrm{Pro}(\mathcal{C}_0)$ with the usual category of Pro-objects of $\mathcal{C}_0$: that is, it is a category whose objects are small filtered diagrams $\{X_\alpha\}$ taking values in $\mathcal{C}_0$, with morphisms given by

$$\mathrm{Hom}_{\mathrm{Pro}(\mathcal{C}_0)}(\{X_\alpha\}, \{Y_\beta\}) = \varprojlim_\beta \varinjlim_\alpha \mathrm{Hom}_{\mathcal{C}_0}(X_\alpha, Y_\beta).$$

Remark 3.1.4. If $\mathcal{C}$ is an accessible ∞-category which admits finite limits, then every corepresentable functor $\mathcal{C} \to \mathcal{S}$ is accessible and preserves finite limits. It follows that the Yoneda embedding determines a functor $j : \mathcal{C} \to \mathrm{Pro}(\mathcal{C})$, which we will also refer to as the Yoneda embedding.

Remark 3.1.5. Let $\mathcal{C}$ be an accessible ∞-category which admits finite limits. Then the collection of left-exact accessible functors from $\mathcal{C}$ to $\mathcal{S}$ is closed under filtered colimits. It follows that the ∞-category $\mathrm{Pro}(\mathcal{C})$ admits small filtered limits. Moreover, filtered limits are computed "pointwise": that is, the inclusion $\mathrm{Pro}(\mathcal{C}) \hookrightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{S})^{op}$ preserves small filtered limits.

The ∞-category $\mathrm{Pro}(\mathcal{C})$ can be characterized by the following universal property:

Proposition 3.1.6. *Let $\mathcal{C}$ be an accessible ∞-category which admits finite limits, let $\mathcal{D}$ be an ∞-category which admits small filtered limits, and let $\mathrm{Fun}'(\mathrm{Pro}(\mathcal{C}), \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}(\mathrm{Pro}(\mathcal{C}), \mathcal{D})$ spanned by those functors which preserve small filtered limits. Then composition with the Yoneda embedding restricts to an equivalence of ∞-categories*

$$\mathrm{Fun}'(\mathrm{Pro}(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D}).$$

We can state Proposition 3.1.6 more informally as follows: the ∞-category $\mathrm{Pro}(\mathcal{C})$ is obtained from $\mathcal{C}$ by freely adjoining filtered limits.

Proof. Let $\widehat{\mathcal{S}}$ denote the ∞-category of spaces which are not necessarily small, let $\mathcal{E}$ denote the smallest full subcategory of $\mathrm{Fun}(\mathcal{C}, \widehat{\mathcal{S}})$ which contains the essential image of the Yoneda embedding and is closed under small filtered colimits, and let $\mathrm{Fun}'(\mathcal{E}^{op}, \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{E}^{op}, \mathcal{D})$ spanned by those functors which preserve small filtered limits. Using Remark T.5.3.5.9, we see that composition with the Yoneda embedding induces an equivalence of ∞-categories $\mathrm{Fun}'(\mathcal{E}^{op}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$. It will therefore suffice to show that $\mathcal{E}$ is equivalent to $\mathrm{Pro}(\mathcal{C})^{op}$ (as subcategories of $\mathrm{Fun}(\mathcal{C}, \widehat{\mathcal{S}})$. Using Remarks 3.1.4 and 3.1.5, we see that $\mathcal{E}$ is contained in the essential image of $\mathrm{Pro}(\mathcal{C})^{op}$. We will complete the proof by verifying the following:

- $(*)$ Let $F : \mathcal{C} \to \mathcal{S}$ be an accessible functor which preserves finite limits. Then F can be written as a small filtered colimit $\varinjlim F_\alpha$, where each of the functors $F_\alpha : \mathcal{C} \to \mathcal{S}$ is corepresentable by an object of $\mathcal{C}$.

To prove $(*)$, choose a regular cardinal κ such that $\mathcal{C}$ is κ-accessible and the functor $F : \mathcal{C} \to \mathcal{S}$ preserves κ-filtered colimits. Let $\mathcal{C}^\kappa$ denote the full subcategory of $\mathcal{C}$ spanned by the κ-compact objects. Enlarging κ if necessary, we may assume $\mathcal{C}^\kappa$ is closed under finite limits. Let F^κ denote the restriction of F to $\mathcal{C}^\kappa$. Since $\mathcal{C}^\kappa$ is essentially small and F^κ is left exact, we can write F^κ as a filtered colimit of functors F^κ_α, where each F^κ_α is corepresentable by an object of $\mathcal{C}^\kappa$ (Corollary T.5.3.5.4). Since the Yoneda embedding $h^\kappa : (\mathcal{C}^\kappa)^{op} \to \mathrm{Fun}(\mathcal{C}^\kappa, \mathcal{S})$ is fully faithful, we can write $F^\kappa_\alpha = h^\kappa(C_\alpha)$ for some filtered diagram $\{C_\alpha\}$ in $(\mathcal{C}^\kappa)^{op}$. Let $h : \mathcal{C}^{op} \to \mathrm{Fun}(\mathcal{C}, \mathcal{S})$ denote the Yoneda embedding for $\mathcal{C}$, and let $F' = \varinjlim_\alpha h(C_\alpha)$. We will complete the proof of $(*)$ by showing that $F' \simeq F$. By construction, F and F' have the same restriction to $\mathcal{C}^\kappa$. Since $\mathcal{C}$ is κ-accessible, it will suffice to show that the functors F and F' preserve small κ-filtered colimits (Proposition T.5.3.5.10). For the functor F, this follows by assumption. To show that F' commutes with κ-filtered colimits, it will suffice to show that each $h(C_\alpha)$ commutes with κ-filtered colimits; this follows from the fact that each C_α is κ-compact. □

Remark 3.1.7. Let $f : \mathcal{C} \to \mathcal{D}$ be a functor between accessible ∞-categories which admit small limits. Then the composite functor $\mathcal{C} \xrightarrow{f} \mathcal{D} \to \mathrm{Pro}(\mathcal{D})$ induces a map $\mathrm{Pro}(f) : \mathrm{Pro}(\mathcal{C}) \to \mathrm{Pro}(\mathcal{D})$, which commutes with small filtered limits. If f is fully faithful, then $\mathrm{Pro}(f)$ is fully faithful. If f is accessible and preserves finite limits, then composition with f induces a functor $F : \mathrm{Pro}(\mathcal{D}) \to \mathrm{Pro}(\mathcal{C})$. It is not difficult to see that F (when defined) is a right adjoint to $\mathrm{Pro}(f)$.

Notation 3.1.8. Let R be an $\mathbb{E}_\infty$-ring, let p be a prime number, and let $\mathcal{S}\mathrm{hv}_R^{p-\mathrm{fc}}$ denote the full subcategory of $\mathcal{S}\mathrm{hv}_R^{\text{ét}}$ spanned by the p-constructible sheaves. We let $\mathcal{S}\mathrm{hv}_R^{\mathrm{Pro}(p)}$ denote the ∞-category $\mathrm{Pro}(\mathcal{S}\mathrm{hv}_R^{p-\mathrm{fc}})$. We will refer to $\mathcal{S}\mathrm{hv}_R^{\mathrm{Pro}(p)}$ as the *p-profinite sheaves* on $\mathrm{Spec}^{\text{ét}} R$.

Let $\mathcal{O}$ denote the structure sheaf of $\mathrm{Spec}^{\text{ét}} R$, and regard $\mathcal{O}$ as a functor from $\mathcal{S}\mathrm{hv}_R^{\text{ét}}$ to $\mathrm{CAlg}_R^{\text{ét}}$. Using Proposition 3.1.6, we see that $\mathcal{O}$ admits an essentially unique extension

$$\widehat{\mathcal{O}} : \mathcal{S}\mathrm{hv}_R^{\mathrm{Pro}(p)} \to \mathrm{CAlg}_R^{op}$$

which commutes with small filtered limits.

We can now prove an extension of Corollary 2.6.12:

Theorem 3.1.9. *Let p be a prime number, let R be an $\mathbb{E}_\infty$-ring which is p-thin, and let $\widehat{\mathcal{O}} : \mathcal{S}\mathrm{hv}_R^{\mathrm{Pro}(p)} \to \mathrm{CAlg}_R^{op}$ be the functor described in Notation 3.1.8. Then:*

(1) *The functor $\widehat{\mathcal{O}}$ commutes with small limits.*

(2) *For every pro-p sheaf $X \in \mathcal{S}\mathrm{hv}_R^{\mathrm{Pro}(p)}$, the $\mathbb{E}_\infty$-ring $\widehat{\mathcal{O}}(X)$ is formally étale over R.*

(3) *The functor $\widehat{\mathcal{O}} : \mathcal{S}\mathrm{hv}_R^{\mathrm{Pro}(p)} \to \mathrm{CAlg}_R^{op}$ is fully faithful.*

Proof. Assertion (1) follows from Propositions T.5.5.1.9 and 2.5.2, and assertion (2) follows from Theorem 2.4.9 (note that the collection of formally étale R-algebras is closed under small colimits in CAlg_R). We prove (3). Let $\mathcal{O} : \mathcal{S}\mathrm{hv}_R^{\text{ét}} \to \mathrm{CAlg}_R^{op}$ be the structure sheaf of $\mathrm{Spec}^{\text{ét}} R$. If $X \in \mathcal{S}\mathrm{hv}_R^{p-\mathrm{fc}}$, then $\mathcal{O}(X)$ is a compact object of CAlg_R by Theorem 2.5.1. In view of Proposition T.5.3.5.11, it will suffice to show that the restriction $\mathcal{O} \,|\, \mathcal{S}\mathrm{hv}_R^{p-\mathrm{fc}}$ is fully faithful. We conclude by invoking Corollary 2.6.12. $\square$

If $f : R \to R'$ is a map of $\mathbb{E}_\infty$-rings, then the pullback functor $f^* : \mathcal{S}\mathrm{hv}_R^{\text{ét}} \to \mathcal{S}\mathrm{hv}_{R'}^{\text{ét}}$ carries $\mathcal{S}\mathrm{hv}_R^{p-\mathrm{fc}}$ into $\mathcal{S}\mathrm{hv}_{R'}^{p-\mathrm{fc}}$, and therefore induces a functor $\widehat{f}^* : \mathcal{S}\mathrm{hv}_R^{\mathrm{Pro}(p)} \to \mathcal{S}\mathrm{hv}_{R'}^{\mathrm{Pro}(p)}$. Note that $\widehat{f}^*$ is an equivalence whenever f^* is an equivalence. In particular, we see that $\mathcal{S}\mathrm{hv}_R^{\mathrm{Pro}(p)}$ depends only on the underlying commutative ring $\pi_0 R$.

Corollary 3.1.10. *Let R be an $\mathbb{E}_\infty$-ring, and suppose we are given a flat hypercovering $R \to R^\bullet$ of R. If p is nilpotent in $\pi_0 R$, then the canonical map*

$$\mathcal{S}\mathrm{hv}_R^{\mathrm{Pro}(p)} \to \varprojlim \mathcal{S}\mathrm{hv}_{R^\bullet}^{\mathrm{Pro}(p)}$$

is fully faithful.

Proof. We may assume without loss of generality that R is discrete (so that $R^\bullet$ is also discrete). Then R is p-thin, and each R^n is p-thin. We have a commutative diagram of ∞-categories

$$\begin{tikzcd} \mathcal{S}\mathrm{hv}_R^{\mathrm{Pro}(p)} \arrow[r] \arrow[d] & \varprojlim \mathcal{S}\mathrm{hv}_R^{\mathrm{Pro}(p)} \arrow[d] \\ \mathrm{CAlg}_R^{op} \arrow[r] & \varprojlim \mathrm{CAlg}_{R^\bullet}^{op} . \end{tikzcd}$$

The vertical functors are fully faithful by Theorem 3.1.9, and the bottom horizontal map is an equivalence of ∞-categories by Theorem VII.5.14. It follows that the functor $\mathcal{S}\mathrm{hv}_R^{\mathrm{Pro}(p)} \to \varprojlim \mathcal{S}\mathrm{hv}_{R^\bullet}^{\mathrm{Pro}(p)}$ is fully faithful, as desired. $\square$

Warning 3.1.11. In the situation of Corollary 3.1.10, the functor $\mathcal{S}\mathrm{hv}_R^{\mathrm{Pro}(p)} \to \varprojlim \mathcal{S}\mathrm{hv}_R^{\mathrm{Pro}(p)}$ is generally not essentially surjective.

We now specialize Theorem 3.1.9 to the case where R is a separably closed field.

Definition 3.1.12. Let p be a prime number and let $\mathcal{S}^{p-\mathrm{fc}}$ denote the full subcategory of $\mathcal{S}$ spanned by the p-finite spaces (Definition 2.4.1). A *p-profinite space* is a pro-object of the ∞-category $\mathcal{S}^{p-\mathrm{fc}}$. We let $\mathcal{S}^{\mathrm{Pro}(p)}$ denote the ∞-category $\mathrm{Pro}(\mathcal{S}^{p-\mathrm{fc}})$ of p-profinite spaces.

Remark 3.1.13. The Yoneda embedding is a fully faithful functor $j : \mathcal{S}^{p-\mathrm{fc}} \to \mathcal{S}^{\mathrm{Pro}(p)}$. We will generally abuse notation by identifying a p-finite space X with its image $j(X) \in \mathcal{S}^{\mathrm{Pro}(p)}$.

Notation 3.1.14. Let k be a field of characteristic p. Corollary 1.1.10 implies that the construction $X \mapsto C^*(X;k)$ defines a functor $F_0 : (\mathcal{S}^{p-\mathrm{fc}}) \to \mathrm{CAlg}_k^{op}$ which commutes with finite limits. Using Propositions T.5.3.5.10 and T.5.5.1.9, we see that F_0 admits an essentially unique extension to a functor $F : \mathcal{S}^{\mathrm{Pro}(p)} \to \mathrm{CAlg}_k^{op}$ which commutes with small limits. We will denote this functor also by $X \mapsto C^*(X;k)$.

Remark 3.1.15. Let k be a separably closed field, so that the global sections functor induces an equivalence of ∞-categories $\mathcal{S}\mathrm{hv}_k^{\text{ét}} \simeq \mathcal{S}$. This restricts to an equivalence $\mathcal{S}\mathrm{hv}_k^{p-\mathrm{fc}} \simeq \mathcal{S}^{p-\mathrm{fc}}$, which gives rise to an equivalence of Pro-objects $\mathcal{S}\mathrm{hv}_k^{\mathrm{Pro}(p)} \simeq \mathcal{S}^{\mathrm{Pro}(p)}$. Under this equivalence, the functor $X \mapsto C^*(X;k)$ of Notation 3.1.14 coincides with the functor $\widehat{\mathcal{O}} : \mathcal{S}\mathrm{hv}_k^{\mathrm{Pro}(p)} \to \mathrm{CAlg}_k^{op}$ appearing in Theorem 3.1.9.

Combining Theorem 3.1.9 with Remark 3.1.15, we obtain the following result:

Proposition 3.1.16. *Let k be a separably closed field of characteristic $p > 0$. Then the construction $X \mapsto C^*(X;k)$ determines a fully faithful embedding from the ∞-category $\mathcal{S}^{\mathrm{Pro}(p)}$ of p-profinite spaces to the ∞-category CAlg_k^{op}.*

Remark 3.1.17. Proposition 3.1.16 does not really require the full strength of the hypothesis that k is separably closed. It suffices to assume that the Galois cohomology of k is trivial modulo p: equivalently, that that Artin-Schreier map $x \mapsto x - x^p$ is a surjection from k to itself.

We will study the essential image of the embedding $\mathcal{S}^{\mathrm{Pro}(p)} \to \mathrm{CAlg}_k^{op}$ in §3.5.

3.2 Whitehead's Theorem

Let p be a prime number. In §3.1, we introduced the ∞-category $\mathcal{S}^{\mathrm{Pro}(p)}$ of p-profinite spaces. Our goal in this section is to show that the ∞-category $\mathcal{S}^{\mathrm{Pro}(p)}$ behaves, in many respects, like the usual ∞-category $\mathcal{S}$ of spaces. In particular, we will see that there is a good theory of connective and truncated objects in $\mathcal{S}^{\mathrm{Pro}(p)}$, and that every object of $\mathcal{S}^{\mathrm{Pro}(p)}$ has a convergent Postnikov tower.

We begin by establishing a direct connection of $\mathcal{S}^{\mathrm{Pro}(p)}$ with the usual ∞-category of spaces.

Definition 3.2.1. Let p be a prime number and let $\mathcal{S}^{\mathrm{Pro}(p)} = \mathrm{Pro}(\mathcal{S}^{p-\mathrm{fc}})$ denote the ∞-category of p-profinite spaces. We let $\mathrm{Mat} : \mathcal{S}^{\mathrm{Pro}(p)} \to \mathcal{S}$ denote the functor corepresented by the final object $* \in \mathcal{S}^{\mathrm{Pro}(p)}$, so that $\mathrm{Mat}(X) \simeq \mathrm{Map}_{\mathcal{S}^{\mathrm{Pro}(p)}}(*, X)$. We will refer to Mat as the *materialization functor*.

Let X be a p-profinite space. A *point* of X is a point of the materialization $\mathrm{Mat}(X)$. We let $\pi_0 X$ denote the set $\pi_0 \mathrm{Mat}(X)$ of all path components of the materialization of X. Given a point η of X and an integer $n \geq 1$, we let $\pi_n(X, \eta)$ denote the homotopy group $\pi_n(\mathrm{Mat}(X), \eta)$.

We can formulate our main result as follows:

Theorem 3.2.2 (Whitehead's Theorem for p-Profinite Spaces)**.** *Let $f : X \to Y$ be a map of p-profinite spaces. Then f is an equivalence if and only if the following conditions are satisfied:*

(1) *The map f induces a bijection $\pi_0 X \to \pi_0 Y$.*

(2) *For every point η of X and every $n \geq 1$, the map f induces an isomorphism of profinite p-groups $\pi_n(X, \eta) \to \pi_n(Y, f(\eta))$.*

Remark 3.2.3. Theorem 3.2.2 can be reformulated as follows: for every prime number p, the materialization functor $\mathrm{Mat} : \mathcal{S}^{\mathrm{Pro}(p)} \to \mathcal{S}$ is conservative.

Before giving the proof of Theorem 3.2.2, let us describe an application. Since the ∞-category $\mathcal{S}$ is an ∞-topos, the functor $\mathcal{S}^{op} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ which carries each space X to the ∞-category $\mathcal{S}_{/X}$ of spaces over X preserves small limits (see Theorem T.6.1.3.9). It follows that for every Kan complex X, we have a canonical equivalence $\mathcal{S}_{/X} \simeq \mathrm{Fun}(X, \mathcal{S})$, depending functorially on X. We can describe this equivalence informally as follows: it assigns to a map of spaces $Y \to X$ the functor $X \to \mathcal{S}$ which carries a point $x \in X$ to the homotopy fiber $Y_x = Y \times_X \{x\}$. We now prove that an analogous statement holds in the setting of p-profinite spaces:

Proposition 3.2.4. *Let p be a prime number and let X be a p-finite space. Then:*

(1) *Every diagram $e : X \to \mathcal{S}^{\mathrm{Pro}(p)}$ admits a colimit $\varinjlim(e) \in \mathcal{S}^{\mathrm{Pro}(p)}$. Consequently, the formation of colimits determines a functor $\varinjlim : \mathrm{Fun}(X, \mathcal{S}^{\mathrm{Pro}(p)}) \to \mathcal{S}^{\mathrm{Pro}(p)}$.*

(2) *Let e_0 denote a final object of $\mathrm{Fun}(X, \mathcal{S}^{\mathrm{Pro}(p)})$ (that is, the constant functor from X to $\mathcal{S}^{\mathrm{Pro}(p)}$ taking the value $* \in \mathcal{S}^{\mathrm{Pro}(p)}$). Then there is a canonical equivalence $\varinjlim(e_0) \simeq X$ in $\mathcal{S}^{\mathrm{Pro}(p)}$.*

(3) *The functor $\varinjlim$ induces an equivalence of ∞-categories*

$$F : \mathrm{Fun}(X, \mathcal{S}^{\mathrm{Pro}(p)}) \simeq \mathrm{Fun}(X, \mathcal{S}^{\mathrm{Pro}(p)})_{/e_0} \to \mathcal{S}^{\mathrm{Pro}(p)}_{/X}.$$

Proof. Since the ∞-category $\mathcal{S}^{p-\mathrm{fc}}$ admits finite limits, the ∞-category $(\mathcal{S}^{\mathrm{Pro}(p)})^{op} \simeq \mathrm{Ind}((\mathcal{S}^{p-\mathrm{fc}})^{op})$ is presentable (Theorem T.5.5.1.1). It follows that $\mathcal{S}^{\mathrm{Pro}(p)}$ admits all small limits and colimits, which proves (1). If we regard e_0 as a diagram $X \to \mathcal{S}^{p-\mathrm{fc}}$, then it has colimit X. Assertion (2) now follows from the observation that the Yoneda embedding $j : \mathcal{S}^{p-\mathrm{fc}} \to \mathrm{Fun}(\mathcal{S}^{p-\mathrm{fc}}, \mathcal{S})^{op}$ preserves all colimits which exist in $\mathcal{S}^{p-\mathrm{fc}}$ (Proposition T.5.1.3.2).

We now prove (3). We first note that the functor F admits a right adjoint $G : \mathcal{S}^{\mathrm{Pro}(p)}_{/X} \to \mathrm{Fun}(X, \mathcal{S})$. Unwinding the definitions, we see that G carries a map of p-profinite spaces $Y \to X$ to the functor $G(Y) \in \mathrm{Fun}(X, \mathcal{S}^{\mathrm{Pro}(p)})$ given on objects by the formula $G(Y)(x) = Y \times_X \{x\}$. We first show that the unit map $u : \mathrm{id} \to G \circ F$ is an equivalence of functors from $\mathrm{Fun}(X, \mathcal{S}^{\mathrm{Pro}(p)})$ to itself. To prove this, it will suffice to show that for every functor $e : X \to \mathcal{S}^{\mathrm{Pro}(p)}$ and every point $x \in X$, the canonical map $u_{e,x} : e(x) \to \varinjlim(e) \times_X \{x\}$ is an equivalence of p-profinite spaces.

Let k be a separably closed field of characteristic p, so that the functor $Y \mapsto C^*(Y; k)$ is a fully faithful embedding $\mathcal{S}^{\mathrm{Pro}(p)} \to \mathrm{CAlg}_k^{op}$ which preserves small colimits (Proposition 3.1.16). Let $\overline{e} : X \to \mathrm{CAlg}_k$ be the functor given by $\overline{e}(y) = C^*(e(y); k)$ (using the identification $X \simeq X^{op}$). We are then reduced to proving that $u_{e,x}$ induces an equivalence

$$(\varprojlim \overline{e}) \otimes_{C^*(X;k)} C^*(\{x\}, k) \to \overline{e}(x)$$

of $\mathbb{E}_\infty$-algebras over k. Using Proposition 1.1.9, we are reduced to proving that for every point $y \in X$ and every integer $n \geq 0$, the action of the fundamental group $G = \pi_1(X, y)$ on the vector space $V = \mathrm{H}^n(e(y); k)$ is nilpotent. Let $k[G]$ denote the group algebra of G over k, and let $I \subseteq k[G]$ be its augmentation ideal. Then V has a G-invariant filtration

$$V \supseteq IV \supseteq I^2V \supseteq \cdots,$$

and the action of G is trivial on each quotient $I^mV/I^{m+1}V$. To complete the proof, it suffices to show that this filtration is finite. This follows from the fact that I is a nilpotent ideal (since G is a finite p-group and k is a field of characteristic p).

Since the natural transformation u is an equivalence, the functor F is fully faithful. To complete the proof that F is an equivalence of ∞-categories, it will suffice to show that the right adjoint G is conservative. Suppose we are given a morphism $Y \to Y'$ in $\mathcal{S}^{\mathrm{Pro}(p)}_{/X}$ which induces an equivalence $Y \times_X \{x\} \to Y' \times_X \{x\}$ for each $x \in X$. For each $x \in X$, we have homotopy equivalences

$$\mathrm{Mat}(Y) \times_X \{x\} \simeq \mathrm{Mat}(Y \times_X \{x\}) \simeq \mathrm{Mat}(Y' \times_X \{x\}) \simeq \mathrm{Mat}(Y') \times_X \{x\}.$$

It follows that the map $\mathrm{Mat}(Y) \to \mathrm{Mat}(Y')$ is a homotopy equivalence of spaces. Invoking Theorem 3.2.2, we conclude that the map $Y \to Y'$ is an equivalence. $\square$

We now turn to the proof of Theorem 3.2.2. We begin with some general remarks about filtered limits of spaces.

Proposition 3.2.5. *Let $\mathcal{J}$ be a filtered ∞-category and let $X : \mathcal{J}^{op} \to \mathcal{S}$ be a diagram of spaces indexed by $\mathcal{J}^{op}$. Assume that:*

(a) For every object $J \in \mathcal{J}$, the set $\pi_0 X(J)$ is nonempty and finite.

(b) For every object $J \in \mathcal{J}$, every point $\eta \in X(J)$, and every integer $n \geq 1$, the group $\pi_n(X(J), \eta)$ is finite.

Then the limit $\varprojlim_{J \in \mathcal{J}^{op}} X(J)$ is nonempty.

Proof. According to Proposition T.5.3.1.16, there exists a filtered partially ordered set A and a left cofinal map $\mathrm{N}(A) \to \mathcal{J}$. We may therefore replace $\mathcal{J}$ by $\mathrm{N}(A)$ and thereby assume that $\mathcal{J}$ is the nerve of a filtered partially ordered set.

If $n \geq 0$, we will say that X is *n-truncated* if the space $X(\alpha)$ is n-truncated for each $\alpha \in A$. We first prove that $\varprojlim_{\alpha \in A^{op}} X(\alpha)$ is nonempty under the additional assumption that X is n-truncated. Our proof proceeds by induction on n; the case $n = 0$ follows from Lemma XII.A.1.8.

Suppose that X is n-truncated for $n > 0$. Let $X' : \mathrm{N}(A)^{op} \to \mathcal{S}$ denote the composition of X with the truncation functor $\tau_{\leq n-1} : \mathcal{S} \to \mathcal{S}$. Our inductive hypothesis implies that the limit $\varprojlim_{\alpha \in A^{op}} X'(\alpha)$ is nonempty. We will prove that X is nonempty by showing that the map $\theta : \varprojlim_{\alpha \in A^{op}} X(\alpha) \to \varprojlim_{\alpha \in A^{op}} X'(\alpha)$ is surjective on connected components. To this end, suppose we are given a point $\eta \in \varprojlim_{\alpha \in A^{op}} X'(\alpha)$, so that η determines a natural transformation $X'_0 \to X'$, where X'_0 denotes the constant functor $\mathrm{N}(A)^{op} \to \mathcal{S}$ taking the value Δ^0. Let $X_0 = X \times_{X'} X'_0$. To prove that the homotopy fiber of θ over η is nonempty, we must show that $\varprojlim_{\alpha \in A^{op}} X_0(\alpha)$ is nonempty. Note that for each $\alpha \in A$, the space $X_0(\alpha)$ is an n-gerbe: that is, it is both n-truncated and n-connective. In particular, since $n > 0$, each of the spaces $X_0(\alpha)$ is connected.

Let $\mathcal{B}$ denote the collection of all finite subsets $B \subseteq A$ which contain a largest element. Let K denote the simplicial subset of $\mathrm{N}(A)$ given by the union of all the vertices. For each $B \in \mathcal{B}$, let $K_B \subseteq \mathrm{N}(A)$ denote the union $K \cup \mathrm{N}(B)$. Regard $\mathcal{B}$ as a partially ordered set with respect to inclusions, and define a functor $Y : \mathrm{N}(\mathcal{B})^{op} \to \mathcal{S}$ by the formula $Y(B) = \varprojlim (X_0 | K_B^{op})$ (see §T.4.2.3). Using Proposition T.4.2.3.8, we obtain a homotopy equivalence $\varprojlim_{B \in \mathcal{B}^{op}} Y(B) \simeq \varprojlim_{\alpha \in A} X_0(\alpha)$. It will therefore suffice to show that $\varprojlim_{B \in \mathcal{B}^{op}} Y(B)$ is nonempty. Let $M = \varprojlim (X_0 | K) = \prod_{\alpha \in A} X_0(\alpha)$ and let $Z : \mathrm{N}(\mathcal{B}^{op}) \to \mathcal{S}$ be the constant functor taking the value M. Note that $\mathcal{B}$ is filtered, so that $\varprojlim_{B \in \mathcal{B}^{op}} Z(B) \simeq M$. We have an evident natural transformation of functors $Y \to Z$ which induces a map $\theta' : \varprojlim_{B \in \mathcal{B}^{op}} Y(B) \to \varprojlim_{B \in \mathcal{B}^{op}} Z(B) \simeq M$. Since M is nonempty, it will suffice to show that the homotopy fibers of θ' are nonempty.

Choose a point $\zeta \in M$, corresponding to a collection of points $\{\zeta_\alpha \in X_0(\alpha)\}_{\alpha \in A}$. The point ζ determines a natural transformation of functors $Z_0 \to Z$, where $Z_0 : \mathrm{N}(\mathcal{B})^{op} \to \mathcal{S}$ is the constant functor taking the value Δ^0. Let $Y_0 = Y \times_Z Z_0$, so that the homotopy fiber of θ' over the point ζ is given by $\varprojlim_{B \in \mathcal{B}^{op}} Y_0(B)$. Fix an element $B \in \mathcal{B}$, so that B is a subset of A which contains a largest element β. We have homotopy equivalences $Y(B) \simeq X_0(\beta) \times \prod_{\alpha \notin B} X_0(\alpha)$ and $Z(B) \simeq \prod_{\alpha \in A} X_0(\alpha)$. For each $\alpha \in B$, let ζ'_α denote the image of ζ_β under the map $X_0(\beta) \to X_0(\alpha)$. Unwinding the definitions, we see that $Y_0(B)$ can be identified with the product over all $\alpha \in B - \{\alpha\}$ of the space of paths joining ζ_α with ζ'_α in $X_0(\alpha)$. Since each $X_0(\alpha)$ is a connected n-truncated space with finite homotopy groups, we conclude that $Y_0(B)$ is a nonempty $(n-1)$-truncated space with finite homotopy groups. Since $\mathcal{B}$ is filtered, it follows from the inductive hypothesis that $\varprojlim_{B \in \mathcal{B}^{op}} Y_0(B)$ is nonempty.

We now treat the case of a general functor $X : \mathrm{N}(A)^{op} \to \mathcal{S}$. For each integer n, let $\tau_{\leq n} X$ denote the composition of X with the truncation functor $\tau_{\leq n} : \mathcal{S} \to \mathcal{S}$. Then X is the limit of the tower

$$\cdots \to \tau_{\leq 2} X \to \tau_{\leq 1} X \to \tau_{\leq 0} X,$$

so that $\varprojlim_{\alpha \in A^{op}} X(\alpha)$ is given by the limit of the tower of spaces $\{\varprojlim_{\alpha \in A^{op}} \tau_{\leq n} X(\alpha)\}_{n \geq 0}$. The above arguments show that $\varprojlim_{\alpha \in A^{op}} \tau_{\leq 0} X(\alpha)$ is nonempty and that each of the transition maps $\varprojlim_{\alpha \in A^{op}} \tau_{\leq n} X(\alpha) \to \varprojlim_{\alpha \in A^{op}} \tau_{\leq n-1} X(\alpha)$ has nonempty homotopy fibers, from which it immediately follows that the tower $\{\varprojlim_{\alpha \in A^{op}} \tau_{\leq n} X(\alpha)\}_{n \geq 0}$ has nonempty limit. $\square$

Corollary 3.2.6. *Let $\mathcal{J}$ and X be as in Proposition 3.2.5, and let $n \geq 0$ be an integer such that each $X(J)$ is n-connective. Then $X = \varprojlim_{J \in \mathcal{J}} X(J)$ is n-connective.*

Proof. We proceed by induction on n. If $n = 0$, the desired result follows from Proposition 3.2.5. Assume therefore that $n > 0$. The inductive hypothesis implies that X is nonempty. It will therefore suffice to show that, for every pair of points $\eta, \eta' \in X$, the path space $\{\eta\} \times_X \{\eta'\}$ is $(n-1)$-connective. Let $X_0, X_1 : \mathcal{J}^{op} \to \mathcal{S}$ denote the constant functor taking the value Δ^0, so that η and η' determine natural transformations $X_0 \to X \leftarrow X_1$ and we have a homotopy equivalence $\{\eta\} \times_X \{\eta'\} \simeq \varprojlim_{J \in \mathcal{J}^{op}} (X_0 \times_X X_1)(J)$. Since $X_0 \times_X X_1$ takes $(n-1)$-connective values, the inductive hypothesis implies that $\{\eta\} \times_X \{\eta'\}$ is $(n-1)$-connective. □

Corollary 3.2.7. *Let p be a prime number, and let $F : \mathcal{S}^{\mathrm{Pro}(p)} \to \mathrm{N}(\mathcal{S}\mathrm{et})$ denote the functor given by $X \mapsto \pi_0 \mathrm{Map}_{\mathcal{S}^{\mathrm{Pro}(p)}}(*, X)$. Then F commutes with filtered limits.*

Proof. Let us abuse notation by identifying the ∞-category $\mathcal{S}^{p-\mathrm{fc}}$ of p-finite spaces with a full subcategory of the ∞-category $\mathcal{S}^{\mathrm{Pro}(p)} = \mathrm{Pro}(\mathcal{S}^{p-\mathrm{fc}})$ of p-profinite spaces, so that $F_0 = F|\,\mathcal{S}^{p-\mathrm{fc}}$ is the functor given by $X \mapsto \pi_0 X$. Let $F' : \mathcal{S}^{\mathrm{Pro}(p)} \to \mathrm{N}(\mathcal{S}\mathrm{et})$ be a right Kan extension of F_0, so that the identification $F'|\,\mathcal{S}^{p-\mathrm{fc}} = F|\,\mathcal{S}^{p-\mathrm{fc}}$ extends to a natural transformation $u : F \to F'$. Since F' commutes with filtered limits, it will suffice to show that u is an equivalence. To this end, consider a p-profinite space K, which we can assume is given by the limit of a diagram $X : \mathcal{J}^{op} \to \mathcal{S}^{p-\mathrm{fc}}$ for some filtered ∞-category $\mathcal{J}$. We wish to show that the canonical map

$$\theta : \pi_0(\varprojlim_{J \in \mathcal{J}^{op}} X(J)) \to \varprojlim_{J \in \mathcal{J}^{op}} \pi_0 X(J)$$

is a bijection. Choose a point $\eta \in \varprojlim_{J \in \mathcal{J}^{op}} \pi_0 X(J)$. Then η determines, for each $J \in \mathcal{J}$, a connected component $X_\eta(J)$ of $X(J)$. We can regard X_η itself as a functor $\mathcal{J}^{op} \to \mathcal{S}^{p-\mathrm{fc}}$. Note that $\theta^{-1}\{\eta\}$ can be identified with the set of path components of the limit $V_\eta = \varprojlim_{J \in \mathcal{J}^{op}} X_\eta(J)$. To prove that θ is a bijection, it will suffice to show that each of the spaces V_η is connected. This follows from Lemma 3.2.6, since each $X_\eta(J)$ is connected by construction. □

Remark 3.2.8. Let X be a p-profinite space. It follows from Corollary 3.2.7 that if X is given as a filtered limit of p-finite spaces (or even p-profinite spaces) $\varprojlim_\alpha X_\alpha$, then we have $\pi_0 X \simeq \varprojlim_\alpha \pi_0 X_\alpha$. Every point $\eta \in X$ determines a compatible system of points $\eta_\alpha \in X_\alpha$, and Corollary 3.2.7 furnishes a canonical isomorphism $\pi_n(X, \eta) \simeq \varprojlim_\alpha \pi_n(X_\alpha, \eta_\alpha)$.

Remark 3.2.9. Let $X = \varprojlim X_\alpha$ be a p-profinite space. Using Corollary 3.2.7, we deduce that $\pi_0 X$ can be identified with the Stone space associated to the profinite set given by the diagram $\{\pi_0 X_\alpha\}$. In particular, the inverse limit topology endows $\pi_0 X$ with the structure of a compact, totally disconnected Hausdorff space.

To discuss the higher homotopy groups of a p-profinite space, it will be convenient to employ the language of profinite groups. For the reader's convenience, we give a brief review of the theory here.

Definition 3.2.10. A *profinite group* is a topological group G whose underlying topological space is compact, Hausdorff, and totally disconnected. We let PFinGp denote the category whose objects are profinite groups and whose morphisms are continuous group homomorphisms.

Example 3.2.11. Every finite group G can be regarded as a profinite group (when endowed with the discrete topology). That is, we can regard the category FinGp of finite groups as a full subcategory of PFinGp.

Proposition 3.2.12. *The inclusion* $\mathrm{FinGp} \hookrightarrow \mathrm{PFinGp}$ *extends to an equivalence of categories* $\mathrm{Pro}(\mathrm{FinGp}) \simeq \mathrm{PFinGp}$.

Proof. Since the category of compact, totally disconnected Hausdorff spaces is closed under projective limits (in the larger category of all topological spaces; see Lemma XII.A.1.7), the category PFinGp is closed under projective limits in the category of all topological groups. It follows that PFinGp admits small projective limits, so that the inclusion $i : \mathrm{FinGp} \hookrightarrow \mathrm{PFinGp}$ extends to a functor $F : \mathrm{Pro}(\mathrm{FinGp}) \to \mathrm{PFinGp}$ which commutes with filtered limits (Proposition T.5.3.5.10). We first claim that F is fully faithful. According to Proposition T.5.3.5.11, it will suffice to show that every finite group G is compact when viewed as an object of PFinGp^{op}. In other words, we claim that if we are given a filtered system of profinite groups H_α having limit H, then the canonical map

$$\varinjlim \mathrm{Hom}_{\mathrm{PFinGp}}(H_\alpha, G) \to \mathrm{Hom}_{\mathrm{PFinGp}}(H, G)$$

is bijective. Proposition XII.A.1.6 implies that G is compact when viewed as an object of the opposite of the category of compact, totally disconnected Hausdorff spaces. Consequently, every continuous group homomorphism $f : H \to G$ factors as a composition

$$H \to H_\alpha \xrightarrow{f'} G,$$

for some index α. We must show that it is possible to choose α so that f' is a group homomorphism. To prove this, we consider the pair of maps

$$u, v : H_\alpha \times H_\alpha \to G$$

given by $u(x, y) = f'(xy)$, $v(x, y) = f'(x)f'(y)$. Then u and v induce the same map from $H \times H$ into G. Using Proposition XII.A.1.6 we conclude that there exists a map of indices $\beta \to \alpha$ such that u and v agree on $H_\beta \times H_\beta$. Replacing α by β, we may assume that $u = v$ so that f' is a group homomorphism, as desired.

It remains to prove that the functor F is essentially surjective. Fix a profinite group G, and let S be the partially ordered set of open normal subgroups $G_0 \subseteq G$. Then S^{op} is filtered (since the collection of open normal subgroups is closed under finite intersections). We may therefore view the inverse system $\{G/G_0\}_{G_0 \in A}$ as an object $\overline{G} \in \mathrm{Pro}(\mathrm{FinGp})$. We will complete the proof by showing that the natural map $\phi : G \to F(\overline{G})$ is an isomorphism of profinite groups.

Note that every nonempty open subset of $F(\overline{G})$ contains the inverse image of some element of G/G_0, where G_0 is an open normal subgroup of G. From this we immediately deduce that ϕ has dense image. Since G is compact and $F(\overline{G})$ is Hausdorff, it follows that ϕ is a quotient map: that is, ϕ induces an isomorphism of profinite groups $G/\ker(\phi) \to F(\overline{G})$. We will complete the proof by showing that $\ker(\phi)$ is trivial.

Choose a non-identity element $x \in G$; we wish to show that there exists an open normal subgroup of G which does not contain x. Since G is totally disconnected, there exists a closed and open subset $Y \subseteq G$ which contains the identity element but does not contain x. Let $Y^+ = \{gyg^{-1} : g \in G, y \in Y\}$. Then Y^+ image of a continuous map $G \times Y \to G$. Since G is compact, we conclude that Y^+ is compact and therefore a closed subset of G. As a union of conjugates of Y, Y^+ is also an open subset of G. Let $G_0 = \{g \in G : gY^+ = Y^+\}$. Then G_0 is a subgroup of G which does not contain x. Since Y^+ is conjugation-invariant, the subgroup G_0 is normal. Moreover, the complement of G_0 is given by the image of $(G - Y^+) \times Y^+$ under the continuous map $(g, h) \mapsto (gh^{-1})$. Since the product is a compact set, we conclude that $G - G_0$ is compact, so that G_0 is an open subgroup of G. $\square$

Definition 3.2.13. Let G be a profinite group and p a prime number. We say that G is *p-profinite* if, for every open neighborhood $U \subseteq G$ containing the identity, there exists an integer $n \geq 0$ such that $\{g^{p^n} : g \in G\} \subseteq U$.

Proposition 3.2.14. *Let p be a prime number, let $\mathrm{FinGp}_p \subseteq \mathrm{FinGp}$ denote the category of finite p-groups, let $\mathrm{FinGp}^{\mathrm{ab}} \subseteq \mathrm{FinGp}$ be the category of finite abelian groups, and $\mathrm{FinGp}^{\mathrm{ab}}_p = \mathrm{FinGp}_p \cap \mathrm{FinGp}^{\mathrm{ab}}$ the category of finite abelian p-groups. Then:*

(1) *The equivalence $\mathrm{Pro}(\mathrm{FinGp}) \simeq \mathrm{PFinGp}$ restricts to an equivalence of $\mathrm{Pro}(\mathrm{FinGp}^{\mathrm{ab}})$ with the full subcategory of PFinGp spanned by the profinite abelian groups.*

(2) *The equivalence $\mathrm{Pro}(\mathrm{FinGp}) \simeq \mathrm{PFinGp}$ restricts to an equivalence of $\mathrm{Pro}(\mathrm{FinGp}_p)$ with the full subcategory of PFinGp spanned by the p-profinite groups.*

(3) *The equivalence* $\mathrm{Pro}(\mathrm{FinGp}) \simeq \mathrm{PFinGp}$ *restricts to an equivalence of* $\mathrm{Pro}(\mathrm{FinGp}_p^{\mathrm{ab}})$ *with the full subcategory of* PFinGp *spanned by the p-profinite abelian groups.*

Proof. We will prove (2); the proofs of (1) and (3) are similar. We first show that if G is the inverse limit of a filtered system G_α of finite p-groups, then G is p-profinite. To prove this, let U be an open neighborhood of the identity in G. Then U contains the kernel of the map $\phi_\alpha : G \to G_\alpha$ for some index α. Let p^n be the order of G_α, so that $g^{p^n} = e$ for all $g \in G_\alpha$. It follows that $\overline{g}^{p^n} \in \ker(\phi_\alpha) \subseteq U$ for all $\overline{g} \in G$.

Conversely, suppose that G is a p-profinite group. The proof of Proposition 3.2.12 shows that we can write G as the filtered inverse limit $\varprojlim G/G_0$, where G_0 ranges over all open normal subgroups of G. It will therefore suffice to show that G/G_0 is a finite p-group. Assume otherwise: then G/G_0 contains an element g which is not annihilated by any power of p. Let $\overline{g}$ be an element of G representing g. Then $\overline{g}^{p^n} \notin G_0$ for all $n \geq 0$, contradicting our assumption that G is p-profinite. □

Example 3.2.15. Let X be a p-profinite space, and choose a base point $\eta \in \mathrm{Mat}(X)$. Then we can identify $\pi_n(X, \eta)$ with $\pi_0 \Omega^n(X)$, where $\Omega^n(X)$ denotes the n-fold loop space of X in the ∞-category $\mathcal{S}^{\mathrm{Pro}(p)}$. It follows that $\pi_n(X, \eta)$ is itself endowed with a totally disconnected compact Hausdorff topology. Write $X = \varprojlim X_\alpha$ and let η_α denote the image of η in X_α. Then $\pi_n(X, \eta) = \varprojlim \pi_n(X_\alpha, \eta_\alpha)$. It follows that $\pi_n(X, \eta)$ has the structure of a p-profinite group (which is abelian if $n \geq 2$).

For each integer $n \geq 0$, the truncation functor $\tau_{\leq n}$ carries the ∞-category $\mathcal{S}^{p-\mathrm{fc}}$ of p-finite spaces to itself. This extends uniquely to a functor $\mathcal{S}^{\mathrm{Pro}(p)} \to \mathcal{S}^{\mathrm{Pro}(p)}$ which commutes with filtered limits. We will denote this extension also by $\tau_{\leq n}$. This is a localization functor from $\mathcal{S}^{\mathrm{Pro}(p)}$ to itself; we will denote its essential image by $\tau_{\leq n} \mathcal{S}^{\mathrm{Pro}(p)}$. We will say that a p-profinite space X is *n-truncated* if it belongs to $\tau_{\leq n} \mathcal{S}^{\mathrm{Pro}(p)}$: that is, if the canonical map $X \to \tau_{\leq n} X$ is an equivalence.

Example 3.2.16. When $n = 0$, the truncation $\tau_{\leq 0} X$ is equivalent to $\pi_0 X$, regarded as a profinite set via the topology described in Remark 3.2.8.

Remark 3.2.17. For each $n \geq 0$, we will generally abuse notation by identifying the truncation functor $\tau_{\leq n}$ defined above with a functor from the ∞-category $\mathcal{S}^{\mathrm{Pro}(p)}$ of p-profinite spaces to itself. For every p-finite space X, the tower

$$\cdots \to \tau_{\leq 2} X \to \tau_{\leq 1} X \to \tau_{\leq 0} X$$

is eventually constant, and its limit is equivalent to X. It follows that for $X \in \mathcal{S}^{\mathrm{Pro}(p)}$, the tower

$$\cdots \to \tau_{\leq 2} X \to \tau_{\leq 1} X \to \tau_{\leq 0} X$$

also has limit X (though it is generally not eventually constant).

The Postnikov tower of a p-profinite space X is particularly useful in the case where X is simply connected; in this case, one can show that each of the maps $\tau_{\leq n+1} X \to \tau_{\leq n} X$ is a principal fibration. When working with p-profinite spaces which are not simply connected, it will be convenient to consider a refinement of the Postnikov tower.

Definition 3.2.18. Let G and H be groups, and suppose we are given a group homomorphism $\rho : G \to \mathrm{Aut}(H)$. We define a sequence of normal subgroups

$$\cdots \subseteq H_2^\rho \subseteq H_1^\rho \subseteq H_0^\rho = H$$

by induction as follows: for each integer k, we let H_{k+1}^ρ be the smallest normal subgroup of H which contains $h^{-1}\rho(g)(h)$ for each $h \in H_k^\rho$ and each $g \in G$.

If X is a space equipped with a base point η and $n \geq 1$ is an integer, then we let ρ_η denote the associated map $\pi_1(X, \eta) \to \mathrm{Aut}(\pi_n(X, \eta))$. Let $X \in \mathcal{S}$, and suppose we are given integers $n \geq 1$, $c \geq 0$. We will say that X is *(n, c)-truncated* if it is n-truncated and, for every point $\eta \in X$, the subgroup $\pi_n(X, \eta)_c^{\rho_\eta} \subseteq \pi_n(X, \eta)$ is trivial.

Example 3.2.19. Let G be a group, and let $\rho : G \to \mathrm{Aut}(G)$ classify the action of G on itself by conjugation. Then the sequence of subgroups

$$\cdots \subseteq G_2^\rho \subseteq G_1^\rho \subseteq G_0^\rho = G$$

is the lower central series of the group G.

Example 3.2.20. A space $X \in \mathcal{S}$ is $(n,0)$-truncated if and only if it is $(n-1)$-truncated. If X is simply connected and $c > 0$, then X is (n,c)-truncated if and only if it is n-truncated.

Lemma 3.2.21. *Let X be a p-finite space. If X is n-truncated, then it is (n,c) truncated for some integer $c \gg 0$.*

Proof. Since X has finitely many connected components, we may assume without loss of generality that X is connected. Choose a point $\eta \in X$, and let $\rho : \pi_1(X,\eta) \to \mathrm{Aut}(\pi_n(X,\eta))$ be the canonical map. We wish to show that $\pi_n(X,\eta)_c^\rho$ is trivial for $c \gg 0$. Since $\pi_n(X,\eta)$ is a finite group, it will suffice to show that the containments $\pi_n(X,\eta)_{c+1}^\rho \subseteq \pi_n(X,\eta)_c^\rho$ are proper whenever $\pi_n(X,\eta)_c^\rho \neq 0$. To prove this, it suffices to construct a nonzero $\pi_1(X,\eta)$-equivariant map from $\pi_n(X,\eta)$ into the the abelian group $\mathbf{Z}/p\mathbf{Z}$ (which we regard as endowed with a trivial action of $\pi_1(X,\eta)$). Since $\pi_n(X,\eta)_c^\rho$ is a nonzero finite p-group, it has a nonzero abelianization A. We now apply Lemma 2.6.17 to choose a nonzero fixed point for the action of $\pi_1(X,\eta)$ on the finite abelian group $\mathrm{Hom}(A, \mathbf{Z}/p\mathbf{Z})$. $\square$

Notation 3.2.22. Let p be a prime number, and let $n \geq 1$ and $c \geq 0$ be integers. We let $\mathcal{S}_{(n,c)}^{p-\mathrm{fc}}$ denote the full subcategory of $\mathcal{S}$ spanned by those spaces which are p-finite and (n,c)-truncated.

The inclusion $\mathcal{S}_{(n,c)}^{p-\mathrm{fc}} \subseteq \mathcal{S}^{p-\mathrm{fc}}$ admits a left adjoint, which we will denote by $\tau_{\leq n}^c : \mathcal{S}^{p-\mathrm{fc}} \to \mathcal{S}_{(n,c)}^{p-\mathrm{fc}}$. This functor extends to a functor $\mathcal{S}^{\mathrm{Pro}(p)} \to \mathrm{Pro}(\mathcal{S}_{(n,c)}^{p-\mathrm{fc}}$ which commutes with filtered limits. We will denote this extension also by $\tau_{\leq n}^c$; it is left adjoint to the fully faithful embedding $\mathrm{Pro}(\mathcal{S}_{(n,c)}^{p-\mathrm{fc}}) \to \mathcal{S}^{\mathrm{Pro}(p)}$.

Example 3.2.23. When $c = 0$, the truncation functor $\tau_{\leq n} c$ coincides with the functor $\tau_{\leq n-1}$ constructed above.

Remark 3.2.24. Let $n \geq 1$ be an integer. For any p-finite space X, Lemma 3.2.21 implies that the tower of truncations

$$\cdots \to \tau_{\leq n}^2 X \to \tau_{\leq n}^1 X \to \tau_{\leq n}^0 X = \tau_{\leq n-1} X$$

is eventually constant, and its limit is an n-truncation of X. It follows that when X is p-profinite, the tower

$$\cdots \to \tau_{\leq n}^2 X \to \tau_{\leq n}^1 X \to \tau_{\leq n}^0 X = \tau_{\leq n-1} X$$

has limit $\tau_{\leq n} X$, though it is not eventually constant in general.

Let X be an arbitrary p-profinite space. Our next goal is to show that, for every pair of integers $n \geq 1$ and $c \geq 0$, the canonical map $\tau_{\leq n}^{c+1} X \to \tau_{\leq n}^c X$ behaves like a principal fibration. To formulate this result more precisely, we need to introduce a bit of notation.

Definition 3.2.25. A *finite bundle of abelian groups* consists of the following data:

(a) A map of finite sets $\phi : A \to S$.

(b) An abelian group structure on each fiber $\phi^{-1}\{s\}$.

We will generally abuse notation and indicate a finite bundle of abelian groups simply as a map $\phi : A \to S$.

If $\phi : A \to S$ and $\phi' : A' \to S'$ are finite bundles of abelian groups, then a *morphism of finite bundles of abelian groups* from ϕ to ϕ' is a commutative diagram of finite sets

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "F"] & S \arrow[d, "f"] \\ A' \arrow[r, "\phi'"] & S' \end{tikzcd}$$

such that, for every element $s \in S$, the induced map $F_s : \phi^{-1}\{s\} \to \phi'^{-1}\{f(s)\}$ is a group homomorphism.

We let Bun denote the category whose objects are finite bundles of abelian groups, and Pro(Bun) the category of Pro-objects of Bun (see Example 3.1.3). We will refer to the objects of Pro(Bun) as *profinite bundles of abelian groups*.

If p is a prime number, we let Bun_p denote the full subcategory of Bun spanned by those objects $\phi : A \to S$ such that each fiber $\phi^{-1}\{s\}$ is a p-group. We will refer to the objects of Bun_p as *finite bundles of abelian p-groups*, and objects of $\mathrm{Pro}(\mathrm{Bun}_p)$ as *profinite bundles of abelian p-groups*.

Remark 3.2.26. We can think of a finite bundle of abelian groups as a pair $(\phi : A \to S, m : A \times_S A \to A)$, where ϕ is a morphism in the category $\mathcal{S}\mathrm{et}^{\mathrm{fin}}$ of finite sets and m exhibits A as an abelian group object of $\mathcal{S}\mathrm{et}^{\mathrm{fin}}_{/S}$. Stone duality supplies an equivalence of the category $\mathrm{Pro}(\mathcal{S}\mathrm{et}^{\mathrm{fin}})$ with the category of totally disconnected Hausdorff spaces (Proposition XII.A.1.6). Consequently, we obtain a fully faithful embedding from $\mathrm{Pro}(\mathcal{S}\mathrm{et}^{\mathrm{fin}})$ to the category whose objects pairs $(\phi : A \to S, m : A \times_S A \to A)$, where ϕ is a continuous map between totally disconnected compact Hausdorff spaces, and $m : A \times_S A \to A$ is a continuous map which endows each fiber of ϕ with the structure of a topological abelian group.

Construction 3.2.27. If $\phi : A \to S$ is a finite bundle of abelian groups, then we can identify A with an abelian group object of the topos $\mathcal{S}\mathrm{et}_{/S}$. For each $n \geq 0$, we let $K(\phi, n)$ denote the associated Eilenberg-MacLane object of the ∞-topos $\mathcal{S}_{/S}$. More concretely, we have $K(\phi, n) = \coprod_{s \in S} K(A_s, n)$, where A_s denotes the abelian group $\phi^{-1}\{s\}$. Note that if ϕ is a finite bundle of abelian p-groups, then the space $K(\phi, n)$ is p-finite. The construction $\phi \mapsto K(\phi, n)$ determines a functor $\mathrm{N}(\mathrm{Bun}_p) \to \mathcal{S}^{p-\mathrm{fc}}$, which extends to a functor $\mathrm{N}(\mathrm{Pro}(\mathrm{Bun}_p)) \to \mathcal{S}^{\mathrm{Pro}(p)}$ which commutes with filtered limits. We will denote this extended functor by $\phi \mapsto \widehat{K}(\phi, n)$.

Example 3.2.28. Let A be a profinite abelian p-group: that is, A is a topological abelian group gives as a filtered limit $\varprojlim A_\alpha$, where each A_α is a finite abelian p-group. Then we can regard the map $\phi : A \to *$ as a profinite bundle abelian p-groups. We will denote the p-profinite space $\widehat{K}(\phi, n)$ simply by $\widehat{K}(A, n)$. We note that there is a canonical homotopy equivalence of spaces $\mathrm{Mat}(\widehat{K}(A, n)) \simeq K(A, n)$.

Construction 3.2.29. Let p be a prime number, and fix integers $n, c \geq 1$. We define a full subcategory $\mathcal{C}^p_{n,c} \subseteq \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{S})$ as follows. A commutative diagram σ :

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"] & X_0 \arrow[d, "g'"] \\ Y \arrow[r] & Y' \end{tikzcd}$$

in $\mathcal{S}$ belongs to $\mathcal{C}^p_{n,c}$ if and only if the following conditions are satisfied:

(*i*) The space X is p-finite and (n, c)-truncated.

(*ii*) The map g exhibits X_0 as a 0-truncation of X.

(*iii*) The map f exhibits Y as an $(n, c-1)$-truncation of X.

(*iv*) The map g' induces a bijection $\pi_0 X_0 \to \pi_0 Y'$.

(*v*) The diagram σ is a pullback square.

It follows from these conditions that $Y' \simeq K(\phi, n+1)$, where $\phi : A \to \pi_0 X$ is an object of Bun_p such that, for every point $\eta \in X$ having homotopy class $[\eta] \in \pi_0 X$, the fiber $\phi^{-1}\{[\eta]\}$ can be identified with the subgroup $\pi_n(X, \eta)^{\rho_\eta}_{c-1} \subseteq \pi_n(X, \eta)$ (note that this group is abelian and canonically independent of the choice of point η in its homotopy class).

Using obstruction theory, one can show that evaluation at $(0,0) \in \Delta^1 \times \Delta^1$ induces an equivalence of ∞-categories $\mathcal{C}^p_{n,c} \to \mathcal{S}^{p-\mathrm{fin}}_{n,c}$. In other words, we can functorially associate to every (n,c)-truncated p-finite space X a pullback square

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & \pi_0 X \arrow[d] \\
\tau^{c-1}_{\leq n} X \arrow[r] & K(\phi, n+1),
\end{tikzcd}$$

where $\phi \in \mathrm{Bun}_p$ is defined as above. This construction admits an essentially unique extension to a functor $\mathcal{S}^{\mathrm{Pro}(p)}_{(n,c)} \to \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{S}^{\mathrm{Pro}(p)})$ which commutes with filtered limits, and carries an (n,c)-truncated p-profinite space X to a pullback square of p-profinite spaces

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & \tau_{\leq 0} X \arrow[d] \\
\tau^{c-1}_{\leq n} X \arrow[r] & \widehat{K}(\phi, n+1).
\end{tikzcd}$$

Proof of Theorem 3.2.2. The "only if" direction is obvious. To prove the converse, let us suppose that $f : X \to Y$ is a map of p-profinite spaces satisfying conditions (1) and (2); we wish to show that f is a homotopy equivalence. We note that f is a filtered limit of maps $f_{\leq n} : \tau_{\leq n} X \to \tau_{\leq n} Y$; it will therefore suffice to show that each $f_{\leq n}$ is an equivalence. For this, we proceed by induction on n.

We begin by treating the case $n = 0$. We are given that f induces a bijection $(\pi_0 f) : \pi_0 X \to \pi_0 Y$. As explained in Example 3.2.9, $\pi_0 X$ and $\pi_0 Y$ can be identified with the Stone spaces of the profinite sets $\tau_{\leq 0} X$ and $\tau_{\leq 0} Y$. Since $\pi_0 f$ is a continuous bijection between compact Hausdorff spaces, it is a homeomorphism. Using Proposition XII.A.1.6, we deduce that $f_{\leq 0}$ induces an isomorphism of profinite sets.

Now suppose that $n \geq 1$ and that $f_{\leq n-1}$ is an equivalence; we wish to show that $f_{\leq n}$ is an equivalence. Note that $f_{\leq n}$ is the limit of a tower of maps

$$f^c_{\leq n} : \tau^c_{\leq n} X \to \tau^c_{\leq n} Y.$$

It will therefore suffice to show that each $f^c_{\leq n}$ is an equivalence. We proceed by induction on c. In the case $c = 0$, we observe that $f^0_{\leq n} \simeq f_{\leq n-1}$. For the inductive step, suppose that $c > 0$. Using Construction 3.2.29, we obtain a pullback square of morphisms in $\mathcal{S}^{\mathrm{Pro}(p)}$:

$$\begin{tikzcd}
f^c_{\leq n} \arrow[r] \arrow[d] & f_{\leq 0} \arrow[d] \\
f^{c-1}_{\leq n} \arrow[r] & (\psi : K(\phi_X, n+1) \to \widehat{K}(\phi_Y, n+1))
\end{tikzcd}$$

where $\phi_X : A_X \to \pi_0 X$ and $\phi_Y : A_Y \to \pi_0 Y$ are objects of $\mathrm{Pro}(\mathrm{Bun}_p)$ and ψ is induced by f. It will therefore suffice to show that ψ is an equivalence in $\mathcal{S}^{\mathrm{Pro}(p)}$. As explained in Example 3.2.26, we can identify ϕ_X and ϕ_Y with maps of totally disconnected compact Hausdorff spaces; it will therefore suffice to show that the vertical maps in the diagram

$$\begin{tikzcd}
A_X \arrow[r] \arrow[d] & \pi_0 X \arrow[d] \\
A_Y \arrow[r] & \pi_0 Y
\end{tikzcd}$$

are homeomorphisms. The right vertical map is a homeomorphism by the argument given above. To prove that the left vertical map is a homeomorphism, it will suffice to show that it is bijective. That is, we must show that for each $[\eta] \in \pi_0 X$ having image $[\eta'] \in \pi_0 Y$, the induced map $\theta : \phi_X^{-1}\{[\eta]\} \to \phi_Y^{-1}\{[\eta']\}$ is an isomorphism of profinite groups. Unwinding the definitions, we can identify θ with the map

$$\overline{\pi_n(X,\eta)^{\rho_\eta}_{c-1}} / \overline{\pi_n(X,\eta)^{\rho_\eta}_{c}} \to \overline{\pi_n(Y,\eta)^{\rho_{\eta'}}_{c-1}} / \overline{\pi_n(Y,\eta')^{\rho_{\eta'}}_{c}}$$

(here $\overline{H}$ denote the closure of H, if H is a subgroup of $\pi_n(X,\eta)$ or $\pi_n(Y,\eta')$). Assumption (2) guarantees that f induces isomorphisms of profinite groups $\pi_1(X,\eta) \to \pi_1(Y,\eta')$ and $\pi_n(X,\eta) \to \pi_n(Y,\eta')$, from which it follows immediately that θ is an isomorphism. $\square$

3.3 p-Profinite Spaces of Finite Type

Let X be a p-profinite space. In §3.2, we defined the materialization $\mathrm{Mat}(X) = \mathrm{Map}_{\mathcal{S}^{\mathrm{Pro}(p)}}(*, X) \in \mathcal{S}$, and proved that the functor $X \mapsto \mathrm{Mat}(X)$ is conservative (Theorem 3.2.2). Our goal in this section is to show that if X satisfies some reasonable finiteness conditions, then the passage from X to $\mathrm{Mat}(X)$ does not lose any information: that is, we can recover X as the *p-profinite completion* of $\mathrm{Mat}(X)$.

We begin by precisely formulating the relevant finiteness conditions. Assume that X is a simply connected p-profinite space. Theorem 3.2.2 implies that the structure of X is "controlled" by its homotopy groups $\{\pi_n X\}_{n \geq 0}$, each of which is a p-profinite abelian group. We therefore begin by reviewing some finiteness conditions in the setting of p-profinite algebra.

Proposition 3.3.1. *Let p be a prime number and let A be a p-profinite abelian group. The following conditions are equivalent:*

(1) *The set $\mathrm{Hom}(A, \mathbf{F}_p)$ of continuous group homomorphisms from A to $\mathbf{F}_p$ is finite.*

(2) *The quotient A/pA is a finite abelian group.*

(3) *There exists a surjection of p-profinite abelian groups $\mathbf{Z}_p^n \to A$ for some integer n.*

(4) *The pro-p group A is isomorphic to a direct sum of finitely many p-profinite abelian groups of the form $\mathbf{Z}_p$ or $\mathbf{Z}/p^k\mathbf{Z}$.*

Proof. The implication (3) ⇒ (4) follows from the structure theory of finitely generated modules over the discrete valuation ring $\mathbf{Z}_p$, and the implication (4) ⇒ (1) is obvious. We next show that (1) ⇒ (2). Let pA denote the image of the multiplication-by-p map $p : A \to A$. Since A is compact, its image $pA \subseteq A$ is a closed subgroup of A, so the quotient A/pA inherits the structure of a p-profinite group. Let $\mathrm{Vect}^{\mathrm{fin}}_{\mathbf{F}_p}$ denote the category of finite-dimensional vector spaces over the finite field $\mathbf{F}_p$. If we write $A = \varprojlim A_\alpha$ where each α is a finite abelian p-group, then $A/pA \simeq \varprojlim A_\alpha/pA_\alpha$ is given as a filtered inverse limit of objects of $\mathrm{Vect}^{\mathrm{fin}}_{\mathbf{F}_p}$, so we can regard A/pA as a pro-object in the category $\mathrm{Vect}^{\mathrm{fin}}_{\mathbf{F}_p}$. Vector space duality determines an equivalence of categories

$$(\mathrm{Vect}^{\mathrm{fin}}_{\mathbf{F}_p})^{op} \simeq \mathrm{Vect}^{\mathrm{fin}}_{\mathbf{F}_p},$$

which extends to an equivalence of categories

$$\mathrm{Pro}(\mathrm{Vect}^{\mathrm{fin}}_{\mathbf{F}_p})^{op} \simeq \mathrm{Ind}(\mathrm{Vect}^{\mathrm{fin}}_{\mathbf{F}_p}) \simeq \mathrm{Vect}_{\mathbf{F}_p}.$$

The image of A/pA under this equivalence is given by $\mathrm{Hom}(A, \mathbf{F}_p)$. Condition (1) implies that $\mathrm{Hom}(A, \mathbf{F}_p) \in \mathrm{Vect}^{\mathrm{fin}}_{\mathbf{F}_p} \subseteq \mathrm{Vect}_{\mathbf{F}_p}$: that is, the Ind-object obtained by applying vector space duality to A/pA is constant. It follows that A/pA is equivalent to a constant Pro-object of $\mathrm{Vect}^{\mathrm{fin}}_{\mathbf{F}_p}$: in other words, A/pA is a finite-dimensional $\mathbf{F}_p$-vector space.

We complete the proof by showing that (2) ⇒ (3). Choose a basis $\{\overline{x}_i\}_{1 \leq i \leq n}$ for the $\mathbf{F}_p$-vector space A/pA. Each $\overline{x}_i$ can be lifted to an element $x_i \in A$, which determines a group homomorphism $\phi_i : \mathbf{Z} \to A$. Since A is a p-profinite group, the map ϕ_i factors through the p-profinite completion $\mathbf{Z}_p$ of $\mathbf{Z}$. We therefore obtain a sequence of continuous maps $\overline{\phi}_i : \mathbf{Z}_p \to A$. Since A is abelian, we can add these homomorphisms to obtain a continuous group homomorphism $\overline{\phi} : \mathbf{Z}_p^n \to A$. We claim that $\overline{\phi}$ is surjective. Since $\mathbf{Z}_p^n$ is compact, it will suffice to show that the image of $\overline{\phi}$ is dense. In other words, we must show that if $\psi : A \to A'$ is a continuous surjection for some finite p-group A', then the composite map $\psi \circ \overline{\phi} : \mathbf{Z}_p^n \to A'$ is surjective. Since the action of p is nilpotent on A', it suffices to show that the composite map $\theta : \mathbf{Z}_p^n \to A'/pA'$ is surjective (by Nakayama's lemma). This is clear, since θ is a composition of surjections

$$\mathbf{Z}_p^n \to \mathbf{F}_p^n \simeq A/pA \to A'/pA'.$$

$\square$

Definition 3.3.2. Let p be a prime number. We will say that a p-profinite abelian group A is *topologically finitely generated* if it satisfies the equivalent conditions of Proposition 3.3.1. We will say that a p-profinite space X is *simply connected* if its materialization $\mathrm{Mat}(X)$ is simply conneceted. We say that X has *finite type* if X is simply connected and each homotopy group $\pi_n X$ is topologically finitely generated.

We can now formulate the main result of this section.

Theorem 3.3.3. *Let X and Y be p-profinite spaces. If X is of finite type, then the canonical map* $\mathrm{Map}_{\mathcal{S}^{\mathrm{Pro}(p)}}(X,Y) \to \mathrm{Map}_{\mathcal{S}}(\mathrm{Mat}(X), \mathrm{Mat}(Y))$ *is a homotopy equivalence. In particular, the materialization functor* $\mathrm{Mat} : \mathcal{S}^{\mathrm{Pro}(p)} \to \mathcal{S}$ *is fully faithful when restricted to p-profinite spaces of finite type.*

To prove Theorem 3.3.3, we need to study cohomology groups in the p-profinite setting.

Definition 3.3.4. Let X be a p-profinite space. For every field k of characteristic p, we let $\mathrm{H}^*(X;k)$ denote the graded abelian group given by $\mathrm{H}^n(X;k) = \pi_{-n} C^*(X;k)$, where $C^*(X;k)$ is the $\mathbb{E}_\infty$-algebra over k introduced in Proposition 3.1.14.

Remark 3.3.5. If X is p-finite, then the groups $\mathrm{H}^n(X;k)$ introduced in Definition 3.3.4 agree with the usual cohomology groups of X with coefficients in K. In particular, we have a canonical isomorphism

$$\mathrm{H}^n(X; \mathbf{F}_p) \simeq \pi_0 \mathrm{Map}_{\mathcal{S}^{\mathrm{Pro}(p)}}(X, K(\mathbf{Z}/p\,\mathbf{Z}, n)).$$

Since both sides of this equivalence are compatible with the formation of filtered limits, we deduce that there is a canonical isomorphism $\mathrm{H}^n(X;\mathbf{F}_p) \simeq \pi_0 \mathrm{Hom}_{\mathcal{S}^{\mathrm{Pro}(p)}}(X, K(\mathbf{Z}/p\,\mathbf{Z}, n))$ for *every* p-profinite space X.

Remark 3.3.6. If X is a p-finite space, then $\mathrm{H}^0(X;\mathbf{F}_p)$ can be identified with the $\mathbf{F}_p$-vector space of all maps $\pi_0 X \to \mathbf{F}_p$. It follows that if X is a p-profinite space, then $\mathrm{H}^0(X;\mathbf{F}_p)$ can be identified with $\mathbf{F}_p$-vector space of all *continuous* maps $\pi_0 X \to \mathbf{F}_p$. This is a p-Boolean algebra (Definition XII.A.1.9) from which we can functorially recover the topological space $\pi_0 X$ (it is given by the Zariski spectrum of $\mathrm{H}^0(X;\mathbf{F}_p)$; see Proposition XII.A.1.12). In particular, $\mathrm{Mat}(X)$ is connected if and only if the unit map $\mathbf{F}_p \to \mathrm{H}^0(X;\mathbf{F}_p)$ is an isomorphism.

Remark 3.3.7. Let X be a p-profinite space. Every map from X to $K(\mathbf{Z}/p\,\mathbf{Z}, n)$ in the ∞-category $\mathcal{S}^{\mathrm{Pro}(p)}$ induces a map from $\mathrm{Mat}(X)$ to $\mathrm{Mat}(K(\mathbf{Z}/p\,\mathbf{Z}, n)) \simeq K(\mathbf{Z}/p\,\mathbf{Z}, n)$ in the ∞-category of spaces. Letting n vary, we obtain a map of graded rings

$$\mathrm{H}^*(X;\mathbf{F}_p) \to \mathrm{H}^*(\mathrm{Mat}(X);\mathbf{F}_p).$$

We will deduce Theorem 3.3.3 from the following pair of results:

Proposition 3.3.8. *Let X be a p-profinite space of finite type. Then the map of cohomology rings* $\theta : \mathrm{H}^*(X;\mathbf{F}_p) \to \mathrm{H}^*(\mathrm{Mat}(X);\mathbf{F}_p)$ *is an isomorphism. /*

Proposition 3.3.9. *Let X and Y be p-profinite spaces, and suppose that the map*

$$\mathrm{H}^*(X;\mathbf{F}_p) \to \mathrm{H}^*(\mathrm{Mat}(X);\mathbf{F}_p)$$

is an isomorphism. Then the canonical map $\theta : \mathrm{Map}_{\mathcal{S}^{\mathrm{Pro}(p)}}(X,Y) \to \mathrm{Map}_{\mathcal{S}}(\mathrm{Mat}(X), \mathrm{Mat}(Y))$ *is a homotopy equivalence.*

We begin with the proof of Proposition 3.3.8, which will require some preliminary results.

Proposition 3.3.10. *Let $n \geq 1$, and consider the tower of spaces $\{K(\mathbf{Z}/p^a\mathbf{Z}, n)\}_{a\geq 0}$ having limit $K(\mathbf{Z}_p, n)$. For every integer $m \geq 0$, the pro-system of abelian groups $\mathrm{H}_m(K(\mathbf{Z}/p^a\mathbf{Z}, n); \mathbf{F}_p)$ is equivalent to the constant pro-system with value $\mathrm{H}_m(K(\mathbf{Z}_p, n); \mathbf{F}_p)$.*

Proof. We first treat the case $n = 1$. We have a fiber sequence of simply connected spaces

$$K(\mathbf{Z}, 1) \to K(\mathbf{Z}_p, 1) \to K(\mathbf{Z}_p/\mathbf{Z}, 1).$$

Since multiplication by p is invertible on $\mathbf{Z}_p/\mathbf{Z}$, we have $\mathrm{H}_m(K(\mathbf{Z}_p/\mathbf{Z}, 1); \mathbf{F}_p) \simeq 0$ for $m > 0$. Using the Serre spectral sequence, we deduce that the map $K(\mathbf{Z}, 1) \to K(\mathbf{Z}_p, 1)$ induces an isomorphism on homology groups with coefficients in k. It will therefore suffice to show that, for each $m \geq 0$, the pro-system $\{\mathrm{H}_m(K(\mathbf{Z}/p^a\mathbf{Z}, 1); \mathbf{F}_p)\}_{a\geq 0}$ is equivalent to the constant pro-system taking the value

$$\mathrm{H}_m(K(\mathbf{Z}, 1); \mathbf{F}_p) \simeq \begin{cases} \mathbf{F}_p & \text{if } 0 \leq m \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

For each $a \geq 1$, let $\epsilon_a \in \mathrm{H}^1(K(\mathbf{Z}/p^a\mathbf{Z}, 1); \mathbf{F}_p)$ classify the unit map $\mathbf{Z}/p^a\mathbf{Z} \to \mathbf{Z}/p\mathbf{Z} \simeq \mathbf{F}_p$, and let $\eta_a \in \mathrm{H}^2(K(\mathbf{Z}/p^a\mathbf{Z}, 1); \mathbf{F}_p)$ classify the central extension

$$0 \to \mathbf{F}_p \to \mathbf{Z}/p^{a+1}\mathbf{Z} \to \mathbf{Z}/p^a\mathbf{Z} \to 0.$$

Then $\mathrm{H}^*(K(\mathbf{Z}/p^a\mathbf{Z}, 1); \mathbf{F}_p)$ has a basis given by the products η^i and $\epsilon_a\eta_a^i$ for $i \geq 0$. Note that the image of η_a in $\mathrm{H}^2(K(\mathbf{Z}/p^{a+1}\mathbf{Z}, 1); \mathbf{F}_p)$ is zero (since the central extension classified by η_a splits over $\mathbf{Z}/p^{a+1}\mathbf{Z}$), and the image of ϵ_a in $\mathrm{H}^1(K(\mathbf{Z}/p^{a+1}\mathbf{Z}, 1))$ is ϵ_{a+1}. It follows that the maps

$$\mathrm{H}^m(K(\mathbf{Z}/p\mathbf{Z}, 1); \mathbf{F}_p) \to \mathrm{H}^m(K(\mathbf{Z}/p^2\mathbf{Z}, 1); \mathbf{F}_p) \to \mathrm{H}^m(K(\mathbf{Z}/p^3\mathbf{Z}, 1); \mathbf{F}_p) \to \cdots$$

are isomorphisms for $m \leq 1$ and zero for $m > 1$. Passing to dual spaces, we conclude that the tower

$$\cdots \to \mathrm{H}_m(K(\mathbf{Z}/p^3\mathbf{Z}, 1); \mathbf{F}_p) \to \mathrm{H}_m(K(\mathbf{Z}/p^2\mathbf{Z}, 1); \mathbf{F}_p) \to \mathrm{H}_m(K(\mathbf{Z}/p\mathbf{Z}, 1); \mathbf{F}_p)$$

consists of isomorphisms for $m \leq 1$ and zero maps for $m > 1$. We now complete the proof (in the case $n = 1$) by observing that the maps $\mathrm{H}_m(K(\mathbf{Z}, 1); \mathbf{F}_p) \to \mathrm{H}_m(K(\mathbf{Z}/p\mathbf{Z}, 1); \mathbf{F}_p)$ are isomorphisms for $m \leq 1$.

We now treat the general case. Since $\mathrm{Mod}_{\mathbf{F}_p}$ is a presentable symmetric monoidal ∞-category, there is a unique colimit-preserving symmetric monoidal functor $\mathcal{S} \to \mathrm{Mod}_{\mathbf{F}_p}$. Let us denote this functor by $X \mapsto C_*(X)$. Note that the homology groups of a space X are given by the formula $\mathrm{H}_m(X; \mathbf{F}_p) = \pi_m C_*(X)$. It will therefore suffice to prove the following:

$(*)$ For every integer m, the Pro-object $\{\tau_{\leq m} C_*(K(\mathbf{Z}/p^a\mathbf{Z}, n))\}_{a\geq 0}$ of $\mathrm{Mod}_{\mathbf{F}_p}$ is equivalent to the constant $\mathbf{F}_p$-module spectrum $\tau_{\leq m} C_*(K(\mathbf{Z}_p, n))$.

Since the functor $X \mapsto \tau_{\leq m} C_*(X)$ is a successive extension of the functors $X \mapsto \mathrm{H}_a(X; \mathbf{F}_p)$ for $0 \leq a \leq m$, the proof given above shows that $(*)$ holds when $n = 1$. To prove $(*)$ in general, we proceed by induction on n. Suppose that $n > 1$. Let $X_\bullet$ be a Čech nerve of the map $* \to K(\mathbf{Z}_p, n)$, so that $X_\bullet$ is a group object of $\mathcal{S}$. Since C_* is a symmetric monoidal functor, we deduce that $C_*(X_1)$ is an associative algebra object of $\mathrm{Mod}_{\mathbf{F}_p}$; here $X_1 = * \times_{K(\mathbf{Z}_p, n)} * \simeq K(\mathbf{Z}_p, n-1)$. Since $K(\mathbf{Z}_p, n)$ is connected, the canonical map $|X_\bullet| \to K(\mathbf{Z}_p, n)$ is an equivalence. Because C_* preserves colimits, we deduce that $C_*(K(\mathbf{Z}_p, n))$ is given by $|C_*(X_\bullet)|$. Unwinding the definitions, we see that this geometric realization corresponds to the bar construction $\mathbf{F}_p \otimes_{C_*(K(\mathbf{Z}_p, n-1))} \mathbf{F}_p$. Similar reasoning yields an equivalence $C_*(K(\mathbf{Z}/p^a\mathbf{Z}, n)) \simeq \mathbf{F}_p \otimes_{C_*(K(\mathbf{Z}/p^a\mathbf{Z}, n-1))} \mathbf{F}_p$ for each $a \geq 0$. We have a commutative diagram of Pro-objects

$$\begin{tikzcd}
\mathbf{F}_p \otimes_{C_*(K(\mathbf{Z}_p, n-1))} \mathbf{F}_p \arrow[r] \arrow[d] & \{\mathbf{F}_p \otimes_{C_*(K(\mathbf{Z}/p^a\mathbf{Z}, n-1); \mathbf{F}_p} \mathbf{F}_p\} \arrow[d] \\
C_*(K(\mathbf{Z}_p, n)) \arrow[r] & \{C_*(K(\mathbf{Z}/p^a\mathbf{Z}, n))\}
\end{tikzcd}$$

where the vertical maps are equivalences. Using the inductive hypothesis, we deduce that upper horizontal map becomes an equivalence after applying the truncation functor $\tau_{\leq m}$. It follows that the lower horizontal map becomes an equivalence after applying the truncation functor $\tau_{\leq m}$ as well, which proves $(*)$. $\square$

Remark 3.3.11. Using the proof of Proposition 3.3.10, we see that the homology groups $\mathrm{H}_m(K(\mathbf{Z}_p, n); \mathbf{F}_p)$ are finite dimensional vector spaces over $\mathbf{F}_p$ for all $m \geq 0$, $n \geq 1$. In the case $n = 1$, this follows by inspection. In the general case, we have a pushout diagram of $\mathbb{E}_\infty$-algebras over $\mathbf{F}_p$

$$\begin{tikzcd} C_*(K(\mathbf{Z}_p, n-1)) \arrow[r] \arrow[d] & \mathbf{F}_p \arrow[d] \\ \mathbf{F}_p \arrow[r] & C_*(K(\mathbf{Z}_p, n)). \end{tikzcd}$$

The inductive hypothesis and Proposition A.7.2.5.31 imply that $\mathbf{F}_p$ is almost of finite presentation over $C_*(K(\mathbf{Z}_p, n-1))$, so that $C_*(K(\mathbf{Z}_p, n))$ is almost of finite presentation over $\mathbf{F}_p$ and therefore the homology groups of $K(\mathbf{Z}_p, n)$ are finite dimensional over $\mathbf{F}_p$ by Proposition A.7.2.5.31.

Similar reasoning using the pushout diagram

$$\begin{tikzcd} C_*(K(\mathbf{Z}_p, n)) \arrow[r] \arrow[d] & C_*(K(p^a \mathbf{Z}_p, n)) \arrow[d] \\ \mathbf{F}_p \arrow[r] & C_*(K(\mathbf{Z}_p / p^a \mathbf{Z}_p, n); \mathbf{F}_p), \end{tikzcd}$$

shows that the homology groups $\mathrm{H}_m(K(\mathbf{Z} / p^a \mathbf{Z}, n); \mathbf{F}_p)$ are finite-dimensional for $n \geq 1$, $m, a \geq 0$.

Proof of Proposition 3.3.8. Let us say that a p-profinite space X is *good* if the map $\theta : \mathrm{H}^*(X; \mathbf{F}_p) \rightarrow \mathrm{H}^*(\mathrm{Mat}(X); \mathbf{F}_p)$ is an isomorphism. Proposition 3.3.10 implies that $\widehat{K}(\mathbf{Z}_p, n)$ is good for each $n \geq 1$, and it is obvious that any p-finite space is good. Suppose we are given a pullback diagram of p-profinite spaces

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ X' \arrow[r] & Y'. \end{tikzcd}$$

Using Corollary 1.1.10 and Proposition 2.5.2, we deduce that if X', Y, and Y' are good, Y' is simply connected, and the homotopy fiber of the map $\mathrm{Mat}(Y) \rightarrow \mathrm{Mat}(Y')$ has finite-dimensional cohomology in each degree (with coefficients in $\mathbf{F}_p$), then X is also good. Taking $Y = \widehat{K}(\mathbf{Z}_p, n)$ for $n \geq 1$ and $Y' = *$ (and using Remark 3.3.11), we conclude that if X' is good then $X = X' \times \widehat{K}(\mathbf{Z}_p, n)$ is good. Similarly, if X' is good then $X = X' \times K(\mathbf{Z} / p^a \mathbf{Z}, n)$ is good for any integer a. Combining these observations with Proposition 3.3.1, we deduce that $\widehat{K}(A, n)$ is good whenever A is topologically finitely generated.

Let X be an arbitrary p-profinite space, and write $X = \varprojlim \{\tau_{\leq n} X\}$. The map $\mathrm{Mat}(X) \rightarrow \mathrm{Mat}(\tau_{\leq n} X)$ has $(n+1)$-connective homotopy fibers, and therefore induces an isomorphism $\mathrm{H}^m(\mathrm{Mat}(\tau_{\leq n} X); \mathbf{F}_p) \rightarrow \mathrm{H}^m(\mathrm{Mat}(X); \mathbf{F}_p)$ for $m \leq n$. We therefore have a commutative diagram

$$\begin{tikzcd} \varinjlim \mathrm{H}^*(\tau_{\leq n} X; \mathbf{F}_p) \arrow[r] \arrow[d] & \varinjlim \mathrm{H}^*(\mathrm{Mat}(\tau_{\leq n} X); \mathbf{F}_p) \arrow[d] \\ \mathrm{H}^*(X; \mathbf{F}_p) \arrow[r] & \mathrm{H}^*(\mathrm{Mat}(X); \mathbf{F}_p). \end{tikzcd}$$

where the vertical maps are isomorphisms. Consequently, to prove that X is good, it suffices to show that each truncation $\tau_{\leq n} X$ is good.

Assume now that X has finite type. We prove by induction on n that the truncation $\tau_{\leq n}X$ is good, the case $n = 1$ being trivial. If $n > 1$, then the simple-connectivity of X and Construction 3.2.29 give a pullback square of p-profinite spaces

$$\begin{tikzcd}
\tau_{\leq n}X \arrow[r] \arrow[d] & * \arrow[d] \\
\tau_{\leq n-1}X \arrow[r] & \widehat{K}(A, n+1)
\end{tikzcd}$$

where $A = \pi_n X$ is topologically finitely generated. Since $\tau_{\leq n-1}X$ is good by the inductive hypothesis and $\mathrm{H}^*(K(A,n); \mathbf{F}_p) \simeq \mathrm{H}^*(\widehat{K}(A,n); \mathbf{F}_p)$ is finite-dimensional in each degree (this follows from Remark 3.3.11 in the case where $A \simeq \mathbf{Z}_p$ or $A \simeq \mathbf{Z}/p^a\mathbf{Z}$, and the general case follows from Proposition 2.5.2), we conclude that $\tau_{\leq n}X$ is good. □

Remark 3.3.12. Let A be a profinite abelian p-group which is topologically finitely generated, and let $n \geq 1$. Then the cohomology groups $\mathrm{H}^m(\widehat{K}(A,m); \mathbf{F}_p)$ are finite dimensional over $\mathbf{F}_p$. To see this, we can use Proposition 2.5.2 and Proposition 3.3.1 to reduce to the case where A is a cyclic module over $\mathbf{Z}_p$. Using Proposition 3.3.8, we are reduced to proving that $\mathrm{H}^m(K(A,n); \mathbf{F}_p)$ is finite dimensional, which follows from Remark 3.3.11.

Remark 3.3.13. The proof of Proposition 3.3.8 does not require the full strength of our assumption that X is simply connected: the same result holds for a large class of p-profinite spaces which are *nilpotent* in a suitable sense.

We now turn to the proof of Proposition 3.3.9.

Notation 3.3.14. Let X be a space. Consider the functor $\mathcal{S}^{p-\mathrm{fc}} \to \mathcal{S}$ given by $K \mapsto \mathrm{Map}_{\mathcal{S}}(X, K)$. This functor is left exact, and can therefore be identified with an object $X_p^\vee \in \mathcal{S}^{\mathrm{Pro}(p)} \subseteq \mathrm{Fun}(\mathcal{S}^{p-\mathrm{fc}}, \mathcal{S})$. We will refer to $X_p^\vee$ as the *p-profinite completion* of X. The construction $X \mapsto X_p^\wedge$ determines a functor from $\mathcal{S} \to \mathcal{S}^{\mathrm{Pro}(p)}$, which is left adjoint to the materialization functor $\mathrm{Mat} : \mathcal{S}^{\mathrm{Pro}(p)} \to \mathcal{S}$. Note that we have a canonical isomorphism of cohomology groups $\mathrm{H}^*(X; \mathbf{F}_p) \simeq \mathrm{H}^*(X_p^\wedge; \mathbf{F}_p)$.

Lemma 3.3.15. *Let $f : X \to Y$ be a map of p-profinite spaces. The following conditions are equivalent:*

(1) *The map f is an equivalence.*

(2) *The map f induces an isomorphism of cohomology rings $\mathrm{H}^*(Y; \mathbf{F}_p) \to \mathrm{H}^*(X; \mathbf{F}_p)$.*

Proof. The implication (1) $\Rightarrow$ (2) is trivial. Conversely, if (2) is satisfied then f induces an equivalence $C^*(Y; k) \to C^*(X; k)$ whenever k is a separably closed field of characteristic p, from which it follows that f is an equivalence (Theorem 3.5.8). □

Proof of Proposition 3.3.9. We wish to show that the map

$$\mathrm{Map}_{\mathcal{S}^{\mathrm{Pro}(p)}}(X, Y) \to \mathrm{Map}_{\mathcal{S}}(\mathrm{Mat}(X), \mathrm{Mat}(Y)) \simeq \mathrm{Map}_{\mathcal{S}^{\mathrm{Pro}(p)}}(\mathrm{Mat}(X)_p^\wedge, Y)$$

is a homotopy equivalence. For this, it suffices to show that the counit map $v : \mathrm{Mat}(X)_p^\wedge(X) \to X$ is an equivalence of p-profinite spaces. Using Lemma 3.3.15, we are reduced to proving checking that the map of cohomology groups

$$\mathrm{H}^*(X; \mathbf{F}_p) \to \mathrm{H}^*(\mathrm{Mat}(X)_p^\wedge; \mathbf{F}_p) \to \mathrm{H}^*(\mathrm{Mat}(X); \mathbf{F}_p)$$

is an isomorphism, which follows from our hypothesis. □

3.4 Connectivity in p-Profinite Homotopy Theory

Let $f : X \to Y$ be a map of spaces, and let $n \geq -2$ be an integer. Recall that f is said to be *n-truncated* if, for each $y \in Y$, the homotopy fiber $X_y = X \times_Y \{y\}$ is an n-truncated space: that is, if the truncation map $X_y \to \tau_{\leq n} X_y$ is an equivalence. We say that f is *$(n+1)$-connective* if each homotopy fiber X_y of f is $(n+1)$-connective: that is, if the truncation $\tau_{\leq n} X_y$ is contractible. According to Example T.5.2.8.16, every map of spaces $f : X \to Y$ admits an essentially unique factorization

$$X \xrightarrow{f'} Z \xrightarrow{f''} Y,$$

where the map f' is $(n+1)$-connective and f'' is n-truncated. Our goal in this section is to construct an analogous factorization in the case where f is a map of p-profinite spaces.

Definition 3.4.1. Let p be a prime number and let $n \geq -2$ be an integer. We will say that a map $f : X \to Y$ of p-profinite spaces is *n-truncated* if it is given as a filtered limit of morphisms $f_\alpha : X_\alpha \to Y_\alpha$, where each f_α is an n-truncated map of p-finite spaces. If $n \geq -1$, we say that f is *n-connective* if it is given as a filtered limit of morphisms $f_\beta : X_\beta \to Y_\beta$, where each f_β is an n-connective map of p-finite spaces.

We say that a p-profinite space X is *n-connective* if the constant map $X \to *$ is n-connective, and *n-truncated* if the constant map $X \to *$ is n-truncated. We say that X is *connected* if it is 1-connective, and *simply connected* if it is 2-connective.

Recall that a *factorization system* on an ∞-category $\mathcal{C}$ is a pair (S_L, S_R) with the following properties (Definition T.5.2.8.8):

(1) Both S_L and S_R are collections of morphisms of $\mathcal{C}$ which are closed under the formation of retracts.

(2) Every morphism in S_L is left orthogonal to every morphism in S_R (see Definition T.5.2.8.1).

(3) Every morphism $f : X \to Y$ in $\mathcal{C}$ can be obtained as a composition $X \xrightarrow{f'} Z \xrightarrow{f''} Y$, where $f' \in S_L$ and $f'' \in S_R$.

We refer the reader to §T.5.2.8 for a general discussion of factorization systems in ∞-categories. The starting point for our discussion in this section is the following observation:

Theorem 3.4.2. *Let p be a prime number and let $n \geq -2$ be an integer. Let $\hat{S}_L$ denote the collection of $(n+1)$-connective morphisms in $\mathcal{S}^{\mathrm{Pro}(p)}$, and let $\hat{S}_R$ denote the collection of n-truncated morphisms in $\mathcal{S}^{\mathrm{Pro}(p)}$. Then the pair $(\hat{S}_L, \hat{S}_R)$ is a factorization system on the ∞-category $\mathcal{S}^{\mathrm{Pro}(p)}$.*

Theorem 3.4.2 is an immediate consequence of the following pair of results:

Proposition 3.4.3. *Let p be a prime number and $n \geq -2$ an integer. Let S_L be the collection of $(n+1)$-connective morphisms between p-finite spaces and let S_R be the collection of n-truncated morphisms between p-finite spaces. Then the pair (S_L, S_R) is a factorization system on the ∞-category $\mathcal{S}^{p-\mathrm{fc}}$ of p-finite spaces.*

Proposition 3.4.4. *Let $\mathcal{C}$ be an essentially small ∞-category equipped with a factorization system (S_L, S_R). Let $\mathcal{C}_L$ denote the full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{C})$ spanned by those morphisms belonging to S_L, and $\mathcal{C}_R$ the full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{C})$ spanned by those morphisms belonging to S_R. Then the inclusions $\mathcal{C}_L, \mathcal{C}_R \hookrightarrow \mathrm{Fun}(\Delta^1, \mathcal{C})$ determine fully faithful embeddings*

$$\mathrm{Pro}(\mathcal{C}_L), \mathrm{Pro}(\mathcal{C}_R) \to \mathrm{Pro}(\mathrm{Fun}(\Delta^1, \mathcal{C})) \simeq \mathrm{Fun}(\Delta^1, \mathrm{Pro}(\mathcal{C}))$$

(see Proposition T.5.3.5.15). Let $\hat{S}_L$ and $\hat{S}_R$ denote the collections of morphisms in $\mathrm{Pro}(\mathcal{C})$ which belong to the essential images of these embeddings. Then $(\hat{S}_L, \hat{S}_R)$ is a factorization system on $\mathrm{Pro}(\mathcal{C})$.

Proof of Proposition 3.4.3. According to Example T.5.2.8.16, the collections of $(n+1)$-connective and n-truncated morphisms in $\mathcal{S}$ determine a factorization system on $\mathcal{S}$. The only nontrivial point is to show that if $f : X \to Y$ is a map of p-finite spaces and we factor f as a composition $X \xrightarrow{f'} Z \xrightarrow{f''} Y$ where f' is $(n+1)$-connective and f'' is n-truncated, then the space Z is also p-finite. Since Y is p-finite, it will suffice to show that for each $y \in Y$, the homotopy fiber $Z_y = Z \times_Y \{y\}$ is p-finite. For this, we observe that Z_y is given by the truncation $\tau_{\leq n} X_y$, where X_y denotes the homotopy fiber $X \times_Y \{y\}$. □

Proof of Proposition 3.4.4. Since ∞-categories $\mathrm{Pro}(\mathcal{C}_L)$ and $\mathrm{Pro}(\mathcal{C}_R)$ are idempotent complete, the sets $\hat{S}_L$ and $\hat{S}_R$ are clearly stable under retracts. Let $\mathcal{D}$ denote the full subcategory of $\mathrm{Fun}(\Delta^2, \mathcal{C})$ spanned by those diagrams

$$\begin{tikzcd} & Z \arrow[dr, "f''"] & \\ X \arrow[ur, "f'"] \arrow[rr, "f"] & & Y \end{tikzcd}$$

where $f' \in S_L$ and $f'' \in S_R$. According to Proposition T.5.2.8.17, the inclusion $\Delta^1 \simeq \Delta^{\{0,2\}} \hookrightarrow \Delta^2$ induces an equivalence of ∞-categories $\mathcal{D} \to \mathrm{Fun}(\Delta^1, \mathcal{C})$. It follows that the induced map $\mathrm{Pro}(\mathcal{D}) \to \mathrm{Pro}(\mathrm{Fun}(\Delta^1, \mathcal{C})) \simeq \mathrm{Fun}(\Delta^1, \mathrm{Pro}(\mathcal{C}))$ is an equivalence. From this, we conclude that every morphism $f : X \to Y$ in $\mathrm{Pro}(\mathcal{C})$ factors as a composition $X \xrightarrow{f'} Z \xrightarrow{f''} Y$, where $f' \in \hat{S}_L$ and $f'' \in \hat{S}_R$.

It remains to prove that every morphism in $\hat{S}_L$ is left orthogonal to every morphism in $\hat{S}_R$. To prove this, suppose we are given a filtered diagram $\{f_\alpha : A_\alpha \to B_\alpha\}$ in $\mathcal{C}_L$ and a filtered diagram $\{g_\beta : X_\beta \to Y_\beta\}$ in $\mathcal{C}_R$, having limits given by morphisms $f : A \to B$ and $g : X \to Y$ in $\mathrm{Pro}(\mathcal{C})$. We wish to show that the diagram

$$\begin{tikzcd} \mathrm{Map}_{\mathrm{Pro}(\mathcal{C})}(B, X) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{Pro}(\mathcal{C})}(A, X) \arrow[d] \\ \mathrm{Map}_{\mathrm{Pro}(\mathcal{C})}(B, Y) \arrow[r] & \mathrm{Map}_{\mathrm{Pro}(\mathcal{C})}(B, X) \end{tikzcd}$$

is a pullback square of spaces. Since the collection of pullback diagrams in $\mathcal{S}$ is closed under filtered colimits and small limits, it suffices to prove that for every pair of indices α and β, the diagram of spaces

$$\begin{tikzcd} \mathrm{Map}_{\mathrm{Pro}(\mathcal{C})}(B_\alpha, X_\beta) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{Pro}(\mathcal{C})}(A_\alpha, X_\beta) \arrow[d] \\ \mathrm{Map}_{\mathrm{Pro}(\mathcal{C})}(B_\alpha, Y_\beta) \arrow[r] & \mathrm{Map}_{\mathrm{Pro}(\mathcal{C})}(B_\alpha, X_\beta) \end{tikzcd}$$

is a pullback square. This follows from the fact that f_α is left orthogonal to g_β in the ∞-category $\mathcal{C}$. □

Example 3.4.5. Let X be a p-profinite space and let $n \geq -2$ be an integer. Theorem 3.4.2 implies that the map $X \to *$ admits an (essentially unique) factorization $X \xrightarrow{f} Y \to *$, where f is $(n+1)$-connective and Y is n-truncated. Unwinding the proof of Proposition 3.4.4, we see that Y can be identified with the truncation $\tau_{\leq n} X$ studied in §3.2.

For applications of Theorem 3.4.2, it is useful to have some alternate descriptions of the classes of n-connective and n-truncated morphisms in $\mathcal{S}^{\mathrm{Pro}(p)}$.

Proposition 3.4.6. *Let p be a prime number, let $f : X \to Y$ be a morphism of p-profinite spaces, and let $n \geq -2$ be an integer. The following conditions are equivalent:*

(1) *The morphism f is n-truncated, in the sense of Definition 3.4.1.*

(2) *The morphism f exhibits X as an n-truncated object of the ∞-category $\mathcal{S}^{\mathrm{Pro}(p)}_{/Y}$: that is, for every p-profinite space Z, the map of spaces $\mathrm{Map}_{\mathcal{S}^{\mathrm{Pro}(p)}}(Z, X) \to \mathrm{Map}_{\mathcal{S}^{\mathrm{Pro}(p)}}(Z, Y)$ is n-truncated.*

(3) *The induced map of materializations $\mathrm{Mat}(X) \to \mathrm{Mat}(Y)$ is n-truncated.*

Lemma 3.4.7. *Let $f : X \to Y$ be an n-connective map of p-profinite spaces. Then the induced map $\mathrm{Mat}(X) \to \mathrm{Mat}(Y)$ is also n-connective.*

Proof. Write f as a filtered limit of n-connective maps $f_\alpha : X_\alpha \to Y_\alpha$ between p-finite spaces. Let $\eta \in \mathrm{Mat}(Y)$, so that η determines a compatible family of points $\eta_\alpha \in Y_\alpha$. We wish to prove that the homotopy fiber $\mathrm{Mat}(X)_\eta = \mathrm{Mat}(X) \times_{\mathrm{Mat}(Y)} \{\eta\}$ is n-connective. This homotopy fiber is given as the limit of a filtered system n-connective, p-finite spaces $X_\alpha \times_{Y_\alpha} \{\eta_\alpha\}$, and is therefore n-connective by Corollary 3.2.6. □

Proof of Proposition 3.4.6. Suppose first that condition (1) is satisfied: that is, f is given as the limit of a filtered diagram of n-truncated morphisms $f_\alpha : X_\alpha \to Y_\alpha$ between p-finite spaces. We will prove that (2) is satisfied. Let Z be an arbitrary p-profinite space. Then the map $\theta : \mathrm{Map}_{\mathcal{S}^{\mathrm{Pro}(p)}}(Z, X) \to \mathrm{Map}_{\mathcal{S}^{\mathrm{Pro}(p)}}(Z, Y)$ is a filtered limit of maps

$$\theta_\alpha : \mathrm{Map}_{\mathcal{S}^{\mathrm{Pro}(p)}}(Z, X_\alpha) \to \mathrm{Map}_{\mathcal{S}^{\mathrm{Pro}(p)}}(Z, Y_\alpha).$$

Consequently, to prove that θ is n-truncated, it will suffice to show that each θ_α is n-truncated. Write Z as a filtered limit of p-finite spaces Z_β, so that θ_α is a filtered colimit of maps

$$\theta_{\alpha,\beta} : \mathrm{Map}_{\mathcal{S}}(Z_\beta, X_\alpha) \to \mathrm{Map}_{\mathcal{S}}(Z_\beta, Y_\alpha).$$

It will therefore suffice to show that each $\theta_{\alpha,\beta}$ is n-truncated, which follows immediately from our assumption that f_α is n-truncated.

The implication (2) $\Rightarrow$ (3) is obvious. We will complete the proof by showing that (3) $\Rightarrow$ (1). Using Theorem 3.4.2, we can factor f as a composition $X \xrightarrow{f'} Z \xrightarrow{f''} Y$ where f' is $(n+1)$-connective and f'' is n-truncated. The first part of the proof shows that the map of materializations $\mathrm{Mat}(Z) \to \mathrm{Mat}(Y)$ is n-truncated, so that f' induces an n-truncated map $\mathrm{Mat}(X) \to \mathrm{Mat}(Z)$. Lemma 3.4.7 implies that $\mathrm{Mat}(X) \to \mathrm{Mat}(Z)$ is $(n+1)$-connective. It follows that the map from $\mathrm{Mat}(X)$ to $\mathrm{Mat}(Z)$ is a homotopy equivalence. Using Theorem 3.2.2, we deduce that f' is an equivalence of p-profinite spaces, so that f is n-truncated as desired. □

We now study some consequences of Theorem 3.2.2.

Corollary 3.4.8. *Let p be a prime number, let $n \geq -2$ be an integer, and let $X \in \mathcal{S}^{\mathrm{Pro}(p)}$ be a p-profinite space. The following conditions are equivalent:*

(1) *The p-profinite space X belongs to the essential image of the localization functor $\tau_{\leq n} : \mathcal{S}^{\mathrm{Pro}(p)} \to \mathcal{S}^{\mathrm{Pro}(p)}$.*

(2) *The p-profinite space X is n-truncated, in the sense of Definition 3.4.1.*

(3) *For every p-profinite space Y, the mapping space $\mathrm{Map}_{\mathcal{S}^{\mathrm{Pro}(p)}}(Y, X)$ is n-truncated.*

(4) *The space $\mathrm{Mat}(X)$ is n-truncated. That is, for every point $\eta \in X$ and every $m > n$, the homotopy group $\pi_m(X, \eta)$ is trivial.*

Proof. The equivalence (1) $\Leftrightarrow$ (2) follows from Example 3.4.5, and the equivalences (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) follow from Proposition 3.4.6. □

We now prove an analogue of Proposition 3.4.6 for n-connective morphisms in $\mathcal{S}^{\mathrm{Pro}(p)}$.

Proposition 3.4.9. *Let p be a prime number, let $f : X \to Y$ be a morphism of p-profinite spaces, and let $n \geq -1$ be an integer. The following conditions are equivalent:*

(1) *The morphism f is n-connective, in the sense of Definition 3.4.1.*

(2) *The induced map of materializations* $\mathrm{Mat}(X) \to \mathrm{Mat}(Y)$ *is* n*-connective.*

(3) *The induced map of cohomology groups* $\mathrm{H}^i(Y; \mathbf{F}_p) \to \mathrm{H}^i(X; \mathbf{F}_p)$ *is injective when* $i = n$ *and an isomorphism for* $i < n$.

Proof. Using Theorem 3.4.2, we can write f as a composition $X \xrightarrow{f'} Z \xrightarrow{f''} Y$ where f' is n-connective and f'' is $(n-1)$-truncated. Using Lemma 3.4.7 and Proposition 3.4.6, we see that the map $\mathrm{Mat}(f') : \mathrm{Mat}(X) \to \mathrm{Mat}(Z)$ is n-connective and the map $\mathrm{Mat}(f'') : \mathrm{Mat}(Z) \to \mathrm{Mat}(Y)$ is $(n-1)$-truncated. It follows that $\mathrm{Mat}(f)$ is n-connective if and only if $\mathrm{Mat}(f'')$ is an equivalence. Applying Theorem 3.2.2, we see that $\mathrm{Mat}(f)$ is n-connective if and only if f'' is an equivalence, from which we deduce that (1) $\Leftrightarrow$ (2).

Assume now that (1) is satisfied; we will prove (3). If $m < n$, then the map $g : K(\mathbf{Z}/p\mathbf{Z}, m) \to *$ is $(n-1)$-truncated. It follows that f is left orthogonal to g, so that composition with f induces a homotopy equivalence

$$\mathrm{Map}_{\mathcal{S}^{\mathrm{Pro}(p)}}(Y, K(\mathbf{Z}/p\mathbf{Z}, m)) \to \mathrm{Map}_{\mathcal{S}^{\mathrm{Pro}(p)}}(X, K(\mathbf{Z}/p\mathbf{Z}, m)).$$

Passing to connected components, we obtain an isomorphism $\mathrm{H}^m(Y; \mathbf{F}_p) \to \mathrm{H}^m(X; \mathbf{F}_p)$. To prove that the map $\mathrm{H}^n(Y; \mathbf{F}_p) \to \mathrm{H}^n(X; \mathbf{F}_p)$ is injective, it suffices to show that every lifting problem depicted in the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & * \arrow[d] \\ Y \arrow[r] & K(\mathbf{Z}/p\mathbf{Z}, n), \end{tikzcd}$$

admits a solution. This is clear, since f is n-connective and the right vertical map is $(n-1)$-truncated.

We now prove that (3) $\Rightarrow$ (1). Note that assertion (1) is vacuous when $n = -1$. When $n = 0$, it is equivalent to the assertion that the map $\pi_0 X \to \pi_0 Y$ is surjective (by the first part of the proof). Using Remark 3.3.6, we see that this is equivalent to the injectivity of the map $\mathrm{H}^0(Y; \mathbf{F}_p) \to \mathrm{H}^0(X; \mathbf{F}_p)$. Let us therefore assume that $n > 0$, and that f satisfies (3). To prove that f is n-connective, it will suffice to show that every $(n-1)$-truncated morphism $g : U \to V$ in $\mathcal{S}^{\mathrm{Pro}(p)}$ is right orthogonal to f (Theorem 3.4.2 and Proposition T.5.2.8.11). Since the collection of those morphisms g which are right orthogonal to f is closed under limits (Proposition T.5.2.8.6), we may assume without loss of generality that U and V are p-finite spaces. Then the map $\mathrm{H}^0(Y; \mathbf{F}_p) \to \mathrm{H}^0(X; \mathbf{F}_p)$ is an isomorphism, so Remark 3.3.6 implies that f induces a homeomorphism $\pi_0 X \to \pi_0 Y$. We may therefore reduce to the case where U and V are connected. To complete the proof, it will suffice to verify the following assertion for all $m < n$:

($*$) If $g : U \to V$ is an m-truncated morphism between connected p-finite spaces, then g is right orthogonal to f.

We prove ($*$) using induction on m. If $m = 0$, then U is a connected covering space of V. Take $G = \pi_1 V$, so that there exists a subgroup $H \subseteq G$ and a pullback diagram of spaces

$$\begin{tikzcd} U \arrow[r] \arrow[d] & BH \arrow[d] \\ V \arrow[r] & BG. \end{tikzcd}$$

It will therefore suffice to show that the map $BH \to BG$ is right orthogonal to f. We proceed by induction on the order of G. If G is trivial, there is nothing to prove. Otherwise, we can choose a central subgroup $Z \subseteq G$ of order p. We then have a pullback diagram

$$\begin{tikzcd} BZH \arrow[r, "\iota"] \arrow[d] & BG \arrow[d] \\ BZH/Z \arrow[r, "\iota_0"] & BG/Z. \end{tikzcd}$$

Since ι_0 is right orthogonal to f by the inductive hypothesis, we conclude that ι is right orthogonal to f. It will therefore suffice to show that the map $BH \to BZH$ is right orthogonal to f. Using the pullback square

$$\begin{tikzcd} BH \arrow[r] \arrow[d] & BZH \arrow[d] \\ * \arrow[r] & BZH/H, \end{tikzcd}$$

we are reduced to proving that the map $* \to BG'$ is right orthogonal to f, where G' is either a trivial group (if $Z \subseteq H$) or the cyclic group $\mathbf{Z}/p\mathbf{Z}$ (if $Z \not\subseteq H$). In the first case, the result is obvious. In the second case, we must show that the map

$$\mathrm{Map}_{\mathcal{S}^{\mathrm{Pro}(p)}}(Y, K(\mathbf{Z}/p\mathbf{Z}, 1)) \to \mathrm{Map}_{\mathcal{S}^{\mathrm{Pro}(p)}}(X, K(\mathbf{Z}/p\mathbf{Z}, 1))$$

is the inclusion of a summand, which follows from our assumption that the map $\mathrm{H}^i(Y; \mathbf{F}_p) \to \mathrm{H}^i(X; \mathbf{F}_p)$ is injective for $m = 1$ and bijective for $m = 0$.

We now prove $(*)$ in the case where $m > 0$. The map g factors as a composition

$$U \xrightarrow{g'} V' \xrightarrow{g''} V$$

where g'' is $(m-1)$-truncated and g' is m-connective. Using the inductive hypothesis, we see that g'' is right orthogonal to f. We may therefore replace g by g' and thereby reduce to the case where g is m-connective. In this case, we have a homotopy fiber sequence

$$K(G, m) \to U \xrightarrow{g} V,$$

for some finite p-group G equipped with an action of the fundamental group $\pi_1 U$. We proceed by induction on the order of G. If G is trivial, then g is a homotopy equivalence and there is nothing to prove. Otherwise, we can apply Lemma 2.6.17 to the subgroup of central elements of order p in G to obtain a cyclic subgroup $Z \subseteq G$ isomorphic to $\mathbf{Z}/p\mathbf{Z}$, on which the group $\pi_1 U$ acts trivially. We then have a map of fiber sequences

$$\begin{tikzcd} K(G, m) \arrow[r] \arrow[d] & U \arrow[r, "g"] \arrow[d] & V \arrow[d, "\mathrm{id}"] \\ K(G/Z, m) \arrow[r] & U' \arrow[r] & V. \end{tikzcd}$$

The map $U' \to V$ is right orthogonal to f by the inductive hypothesis. We are therefore reduced to proving that the map $U \to U'$ is right orthogonal to f. This map is a principal fibration, fitting into a pullback square

$$\begin{tikzcd} U \arrow[r] \arrow[d] & * \arrow[d] \\ U' \arrow[r] & K(\mathbf{Z}/p\mathbf{Z}, m+1). \end{tikzcd}$$

We are therefore reduced to proving that the map $* \to K(\mathbf{Z}/p\mathbf{Z}, m+1)$ is right orthogonal to f. Unwinding the definitions, this amounts to the assertion that the map $\mathrm{H}^i(Y; \mathbf{F}_p) \to \mathrm{H}^i(X; \mathbf{F}_p)$ is an isomorphism for $i \leq m$ and an injection for $i = m+1$. This follows from (3), since $m < n$. $\square$

Corollary 3.4.10. *Let X be a p-profinite space and let $n \geq -1$ be an integer. The following conditions are equivalent:*

(1) *The p-profinite space X is n-connective.*

(2) *The space* $\mathrm{Mat}(X)$ *is* n*-connective.*

(3) *The map* $\mathbf{F}_p \to \mathrm{H}^0(X; \mathbf{F}_p)$ *is an isomorphism, and the cohomology groups* $\mathrm{H}^m(X; \mathbf{F}_p)$ *are trivial for* $0 < m < n$.

Remark 3.4.11. If X is a simply connected p-profinite space, then the pro-p-groups $\pi_n(X, \eta)$ are canonically independent of the choice of base point $\eta \in X$. In this case, we will simply denote them by $\pi_n(X)$.

We conclude by applying some of the ideas above to obtain a cohomological characterization of the p-profinite spaces having finite type, which will play an important role in §3.5.

Theorem 3.4.12. *Let* k *be a field of characteristic* $p > 0$ *and let* X *be a* p*-profinite space. Then* X *is of finite type if and only if the following conditions are satisfied:*

(1) *The canonical map* $\mathbf{F}_p \to \mathrm{H}^0(X; \mathbf{F}_p)$ *is an isomorphism.*

(2) *The cohomology group* $\mathrm{H}^1(X; \mathbf{F}_p)$ *is trivial.*

(3) *For each* $n \geq 2$*, the cohomology group* $\mathrm{H}^n(X; \mathbf{F}_p)$ *is a finite-dimensional vector space over* $\mathbf{F}_p$.

Our proof of Theorem 3.4.12 rests on the following basic calculation:

Proposition 3.4.13. *Let* X *be a* p*-profinite space which is* n*-connective for some* $n \geq 1$*. There are canonical isomorphisms*

$$\mathrm{H}^m(X; \mathbf{F}_p) \simeq \begin{cases} \mathbf{F}_p & \text{if } m = 0 \\ 0 & \text{if } 0 < m < n \\ \mathrm{Hom}(\pi_n X, \mathbf{F}_p) & \text{if } m = n. \end{cases}$$

Here $\mathrm{Hom}(\pi_n X, \mathbf{F}_p)$ *denotes the collection of continuous group homomorphisms from* $\pi_n(X, \eta)$ *to* $\mathbf{F}_p$ *(where we regard* $\mathbf{F}_p$ *as endowed with the discrete topology), where* η *is an arbitrarily chosen point of* X.

Proof. Choose a point $\eta \in X$. Using Corollary 3.4.10, we can write X as the limit of a filtered diagram of p-finite spaces X_α, each of which is n-connective. Let η_α denote the image of η in X_α, and let $\pi_n X_\alpha$ denote the finite p-group $\pi_n(X_\alpha, \eta_\alpha)$. Since each X_α is n-connective, the Hurewicz and universal coefficient theorems of classical homotopy theory give isomorphisms

$$\mathrm{H}^m(X_\alpha; \mathbf{F}_p) \simeq \begin{cases} \mathbf{F}_p & \text{if } m = 0 \\ 0 & \text{if } 0 < m < n \\ \mathrm{Hom}(\pi_n X_\alpha, \mathbf{F}_p) & \text{if } m = n. \end{cases}$$

Since $\mathrm{H}^*(X; \mathbf{F}_p) \simeq \varinjlim \mathrm{H}^*(X_\alpha; \mathbf{F}_p)$, we obtain canonical isomorphisms

$$\mathrm{H}^m(X; \mathbf{F}_p) \simeq \begin{cases} \mathbf{F}_p & \text{if } m = 0 \\ 0 & \text{if } 0 < m < n \\ \varinjlim \mathrm{Hom}(\pi_n X_\alpha, \mathbf{F}_p) & \text{if } m = n. \end{cases}$$

It now suffices to observe that the profinite group $\pi_n X$ is given as the limit of the filtered system of finite groups $\pi_n X$, so that $\mathrm{Hom}(\pi_n X, \mathbf{F}_p) \simeq \varinjlim \mathrm{Hom}(\pi_n X_\alpha, \mathbf{F}_p)$. $\square$

Corollary 3.4.14. *Let* X *be a* p*-profinite space which is* n*-connective for some* $n \geq 2$*. The following conditions are equivalent:*

(1) *The* p*-profinite group* $\pi_n X$ *is topologically finitely generated.*

(2) *The cohomology group* $\mathrm{H}^n(X; \mathbf{F}_p)$ *is finite dimensional over* $\mathbf{F}_p$.

Proof. Combine Propositions 3.3.1 and 3.4.13. □

Proof of Theorem 3.4.12. Suppose first that X is of finite type. We will show that conditions (1), (2), and (3) are satisfied. Conditions (1) and (2) follow from the simple-connectivity of X (Corollary 3.4.10). We will prove (3). We have a commutative diagram

$$\begin{tikzcd}
\mathrm{H}^n(\tau_{\leq n}X;\mathbf{F}_p) \arrow[r] \arrow[d] & \mathrm{H}^n(\mathrm{Mat}(\tau_{\leq n}X);\mathbf{F}_p) \arrow[d] \\
\mathrm{H}^n(X;\mathbf{F}_p) \arrow[r] & \mathrm{H}^n(\mathrm{Mat}(X);\mathbf{F}_p).
\end{tikzcd}$$

Here the vertical maps are isomorphisms by Proposition 3.3.8, and the left vertical map is an equivalence because the map $\mathrm{Mat}(X) \to \mathrm{Mat}(\tau_{\leq n}X)$ is $(n+1)$-connective. It follows that the right vertical map is an equivalence. We may therefore replace X by $\tau_{\leq n}X$ and thereby reduce to the case where X is m-truncated for some integer m. We proceed by induction on m. If $m = 1$, then $X \simeq *$ and there is nothing to prove. Assume therefore that $m \geq 2$, so we have a pullback diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & * \arrow[d] \\
\tau_{\leq m-1}X \arrow[r] & \widehat{K}(A, m+1)
\end{tikzcd}$$

where $A = \pi_m X$. It follows from Remark 3.3.12 that the cohomology groups $\mathrm{H}^i(\widehat{K}(A, m+1);\mathbf{F}_p)$ are finite dimensional, and from the inductive hypothesis that the cohomology groups $\mathrm{H}^i(\tau_{\leq m-1}X;\mathbf{F}_p)$ are finite-dimensional for all $i \geq 0$. Proposition 2.5.2 implies that the canonical map

$$C^*(\tau_{\leq m-1}X;\mathbf{F}_p) \otimes_{C^*(\widehat{K}(A,m+1);\mathbf{F}_p)} \mathbf{F}_p \to C^*(X;F_p)$$

is an equivalence of $\mathbb{E}_\infty$-algebras over $\mathbf{F}_p$. Using Lemma X.4.1.16, we deduce that the cohomology groups $\mathrm{H}^i(X;\mathbf{F}_p)$ are finite-dimensional for all $i \geq 0$.

Now suppose that conditions (1), (2), and (3) are satisfied. We wish to show that X is of finite type. It follows from (1) and (2) that X is simply connected (Corollary 3.4.10). We must show that each homotopy group $\pi_n(X)$ is topologically finitely generated. We proceed by induction on n, the case $n \leq 1$ being vacuous. Assume that the homotopy groups $\pi_m X$ are topologically generated for $m < n$, so that $\tau_{\leq n-1}X$ is of finite type. Choose a point $\eta \in \tau_{\leq n-1}X$ and consider the associated fiber sequence

$$F \to X \to \tau_{\leq n-1}X.$$

The first part of the proof shows that the cohomology groups $\mathrm{H}^*(\tau_{\leq n-1}X;\mathbf{F}_p)$ satisfy conditions (1), (2), and (3). Proposition 2.5.2 implies that the canonical map

$$C^*(X;\mathbf{F}_p) \otimes_{C^*(\tau_{\leq n-1}X;\mathbf{F}_p)} \mathbf{F}_p \to C^*(F;\mathbf{F}_p)$$

is an equivalence of $\mathbb{E}_\infty$-algebras over $\mathbf{F}_p$. Using Lemma X.4.1.16, we deduce that the cohomology groups $\mathrm{H}^i(F;\mathbf{F}_p)$ are finite-dimensional for $i \geq 0$. In particular, the group $\mathrm{H}^n(F;\mathbf{F}_p)$ is finite-dimensional. Since F is n-connective, Corollary 3.4.14 implies that $\pi_n F \simeq \pi_n X$ is topologically finitely generated. □

Remark 3.4.15. Let $X \in \mathcal{S}$ be a simply connected space, and suppose that the cohomology groups $\mathrm{H}^n(X;\mathbf{F}_p)$ are finite-dimensional over $\mathbf{F}_p$ for each $n \geq 0$. It follows from Theorem 3.4.12 that the p-profinite completion $X_p^\wedge$ is of finite type. Combining this with Proposition 3.3.8, we conclude that the map of spaces $u : X \to \mathrm{Mat}(X_p^\wedge)$ induces an isomorphism $\mathrm{H}^*(\mathrm{Mat}(X_p^\wedge);\mathbf{F}_p) \to \mathrm{H}^*(X;\mathbf{F}_p)$. Choose a point $\eta \in X$ and let F denote the homotopy fiber of u over the image of η. Using the Serre spectral sequence, we deduce that the cohomology groups $\mathrm{H}^i(F; \mathbf{F}_p)$ vanish for $i > 0$. It follows that multiplication by p is invertible on the homotopy groups of F. If we assume that the homotopy groups of X are finitely generated abelian groups, then the long exact sequence

$$\pi_n(F, \eta) \to \pi_n X \to \pi_n X_p^\wedge \to \pi_{n-1}(F, \eta)$$

implies that each homotopy group $\pi_n X_p^\wedge$ is given by the p-adic completion of $\pi_n X$.

3.5 p-adic Homotopy Theory

Let k be a separably closed field of characteristic $p > 0$. According to Proposition 3.1.16, the construction $X \mapsto C^*(X; k)$ determines a fully faithful embedding from the ∞-category $\mathcal{S}^{\mathrm{Pro}(p)}$ of p-profinite spaces to the ∞-category CAlg_k^{op} of $\mathbb{E}_\infty$-algebras over k. Our first goal in this section is to describe the essential image of this embedding. First, we need to introduce some terminology.

Definition 3.5.1. Let k be a field of characteristic p and let V be a vector space over k. A map $\sigma : V \to V$ is *Frobenius-semilinear* if it satisfies the conditions

$$\sigma(x + y) = \sigma(x) + \sigma(y) \qquad \sigma(\lambda x) = \lambda^p \sigma(x)$$

for $x, y \in V$, $\lambda \in k$. In this case, we let V^σ denote the subset $\{x \in V : \sigma(x) = x\}$.

Example 3.5.2. Let k be a field of characteristic p and let $A \in \mathrm{CAlg}_k$ be an $\mathbb{E}_\infty$-algebra over k. For every integer n, the operation $P^0 : \pi_n A \to \pi_n A$ of Construction 2.2.6 is Frobenius-semilinear.

Lemma 3.5.3. *Let k be a field of characteristic p, let V be a vector space over k, and let $\sigma : V \to V$ be a Frobenius-semilinear map. Then V^σ is a vector space over the field $\mathbf{F}_p$. Moreover, the canonical k-linear map $k \otimes_{\mathbf{F}_p} V^\sigma \to V$ is injective.*

Proof. It suffices to show that if $x_1, \ldots, x_n \in V^\sigma$ are linearly independent over $\mathbf{F}_p$, then they are linearly independent over k. Suppose otherwise. Then there exists a nontrivial dependence relation $\sum \lambda_i x_i = 0$ where $\lambda_i \in k$. Let us assume that n has been chosen as small as possible, so that each coefficient λ_i is nonzero. Without loss of generality, we may assume that $\lambda_1 = 1$. Applying the map σ, we deduce that $\sum \lambda_i^p x_i = 0$. Subtracting, we obtain a dependence relation $\sum (\lambda_i - \lambda_i^p) x_i = 0$. Note that $\lambda_1 - \lambda_1^p = 0$. By minimality, we conclude that each difference $\lambda_i - \lambda_i^p = 0$. This implies that each coefficient λ_i belongs to $\mathbf{F}_p$, contradicting our assumption that the elements x_i are linearly independent over $\mathbf{F}_p$. $\square$

Definition 3.5.4. Let k be a field of characteristic p, V a vector space over k, and $\sigma : V \to V$ a Frobenius-semilinear map. We will say that σ is *solvable* if the map $k \otimes_{\mathbf{F}_p} V^\sigma \to V$ is an isomorphism. We will say that an $\mathbb{E}_\infty$-algebra A over k is *solvable* if, for every integer n, the Frobenius-semilinear map $P^0 : \pi_n A \to \pi_n A$ is solvable.

Remark 3.5.5. Let k be a field of characteristic p and let A be a solvable $\mathbb{E}_\infty$-algebra over k. For $n > 0$, the operation $P^0 : \pi_n A \to \pi_n A$ is trivial (Remark 2.2.7). Since $\pi_n A$ is generated (as a k-vector space) by elements which are fixed by P^0, we conclude that $\pi_n A \simeq 0$.

Remark 3.5.6. Let k be a field of characteristic p. Suppose we are given a filtered diagram of k-vector spaces $\{V_\alpha\}$ equipped with a compatible collection of Frobenius-semilinear maps $\sigma_\alpha : V_\alpha \to V_\alpha$, so that the filtered colimit $V = \varinjlim V_\alpha$ inherits a Frobenius-semilinear map $\sigma : V \to V$. If each σ_α is solvable, then σ is solvable.

Example 3.5.7. Let X be a space such that each cohomology group $\mathrm{H}^n(X; \mathbf{F}_p)$ is finite-dimensional. For any field k of characteristic p, the universal coefficient theorem gives an isomorphism $k \otimes_{\mathbf{F}_p} \mathrm{H}^n(X; \mathbf{F}_p) \to \mathrm{H}^n(X; k)$, and the image of the map $\mathrm{H}^n(X; \mathbf{F}_p) \to \mathrm{H}^n(X; k)$ consists of elements which are fixed by the operation $P^0 : \mathrm{H}^n(X; k) \to \mathrm{H}^n(X; k)$ (Corollary 2.2.14). It follows that $C^*(X; k) \in \mathrm{CAlg}_k$ is solvable.

Theorem 3.5.8. *Let k be a separably closed field of characteristic $p > 0$. Then the construction $X \mapsto C^*(X;k)$ determines a fully faithful embedding $F : (\mathcal{S}^{\mathrm{Pro}(p)})^{op} \to \mathrm{CAlg}_k$, whose essential image is the collection of solvable $\mathbb{E}_\infty$-algebras over k.*

Proof. We have already seen that F is fully faithful (Proposition 3.1.16). Let $\mathcal{C} \subseteq \mathrm{CAlg}_k$ be the essential image of F and let $\mathcal{C}' \subseteq \mathrm{CAlg}_k$ be the full subcategory spanned by the solvable $\mathbb{E}_\infty$-algebras over k. Note that $\mathcal{C}'$ is closed under filtered colimits by Remark 3.5.6 and contains $C^*(X;k)$ for every p-finite space X by Example 3.5.7. It follows that $\mathcal{C} \subseteq \mathcal{C}'$. We will complete the proof by showing that $\mathcal{C}' \subseteq \mathcal{C}$.

Let $A \in \mathrm{CAlg}_k$ be solvable. For every integer n let V_A^n be the kernel of the map $(1-P^0) : \pi_{-n}A \to \pi_{-n}A$. Then we have an isomorphism of vector spaces $\pi_{-n}A \simeq k \otimes_{\mathbf{F}_p} V_A^n$, under which the operation P^0 corresponds to the map $k \otimes_{\mathbf{F}_p} V_A^n \to k \otimes_{\mathbf{F}_p} V_A^n$ given by the Frobenius map on k. Since k is separably closed, the Artin-Schreier map $x \mapsto x - x^p$ is a surjection from k to itself, so that $(1 - P^0) : \pi_{-n}A \to \pi_{-n}A$ is surjective for every integer n. If $n \geq 0$, then Theorem 2.2.17 gives an exact sequence of abelian groups

$$\pi_{1-n}A \xrightarrow{1-P^0} \pi_{1-n}A \to \pi_0 \mathrm{Map}_{\mathrm{CAlg}_k}(C^*(K(\mathbf{Z}/p\mathbf{Z},n);k),A) \to \pi_{-n}A \xrightarrow{1-P^0} \pi_{-n}A$$

and therefore a canonical isomorphism $\pi_0 \mathrm{Map}_{\mathrm{CAlg}_k}(C^*(K(\mathbf{Z}/p\mathbf{Z},n);k),A) \simeq V_A^n$.

Since F preserves small colimits, it admits a right adjoint G (Proposition T.5.5.2.9). Let $\phi : A \to B$ be a map of solvable $\mathbb{E}_\infty$-algebras over k such that $G(\phi)$ is an equivalence. Then for every integer $n \geq 0$, composition with ϕ gives a bijection

$$\begin{aligned} \pi_0 \mathrm{Map}_{\mathrm{CAlg}_k}(C^*(K(\mathbf{Z}/p\mathbf{Z},n);k),A) &\simeq \pi_0 \mathrm{Map}_{\mathcal{S}^{\mathrm{Pro}(p)}}(G(A),K(\mathbf{Z}/p\mathbf{Z},n)) \\ &\to \pi_0 \mathrm{Map}_{\mathcal{S}^{\mathrm{Pro}(p)}}(G(B),K(\mathbf{Z}/p\mathbf{Z},n)) \\ &\simeq \pi_0 \mathrm{Map}_{\mathrm{CAlg}_k}(C^*(K(\mathbf{Z}/p\mathbf{Z},n);k),B), \end{aligned}$$

It follows that ϕ induces an isomorphism of $\mathbf{F}_p$-vector spaces $V_A^n \to V_B^n$ and therefore an isomorphism of k-vector spaces $\pi_{-n}A \to \pi_{-n}B$. For $n > 0$, we have $\pi_n A \simeq 0 \simeq \pi_n B$ by Remark 3.5.5. It follows that the map $\phi : A \to B$ is an equivalence.

Now let $A \in \mathrm{CAlg}_k$ be an arbitrary solvable $\mathbb{E}_\infty$-algebra over k, and consider the counit map $v : (F \circ G)(A) \to A$. Note that $(F \circ G)(A) \in \mathcal{C} \subseteq \mathcal{C}'$ and that $G(v)$ is an equivalence (since F is fully faithful). It follows from the above argument that v is an equivalence, so that $A \simeq F(G(A))$ belongs to the essential image of F. ☐

We now wish to specialize Theorem 3.5.8 to the setting of p-profinite spaces of finite type. In this case we have a particularly convenient description of solvability:

Proposition 3.5.9. *Let k be an algebraically closed field of characteristic p, let V be a finite-dimensional vector space over k, and let $F : V \to V$ be a Frobenius-semilinear endomorphism. The following conditions are equivalent:*

(1) *The map F is injective.*

(2) *The map F is bijective.*

(3) *The Frobenius-semilinear automorphism F is solvable.*

Lemma 3.5.10. *Let k be a separably closed field of characteristic p, let V be a finite-dimensional vector space over k, and let $F : V \to V$ be a Frobenius-semilinear endomorphism. If F is injective and V is nonzero, then V^F is nonzero.*

Proof. Choose a nonzero element $v \in V$. Since V is finite-dimensional, the elements $\{v, F(v), F^2(v), \ldots\}$ cannot all be linearly independent. Thus there exists a nonzero dependence relation

$$\sum_{0 \leq i \leq n} \lambda_i F^i(v) = 0.$$

Replacing v by $F^i(v)$ if necessary, we may assume that the coefficient λ_0 is nonzero. Dividing by $-\lambda_0$, we may assume that $\lambda_0 = 1$: that is, we have

$$v = \sum_{1 \leq i \leq n} -\lambda_i F^i(v).$$

We may assume that n is chosen as small as possible: it follows that the set $\{v, F(v), \ldots, F^{n-1}(v)\}$ is linearly independent, and therefore $\lambda_n \neq 0$. Since $v \neq 0$, we must have $n > 0$.

Note that

$$f(x) = x^{p^n} + \lambda_1^{p^{n-1}} x^{p^{n-1}} + \lambda_2^{p^{n-2}} x^{p^{n-2}} + \cdots + \lambda_n x$$

is a separable polynomial of degree $p^n > 1$, and therefore has p^n distinct roots in the field k. Consequently, there exists a nonzero element $a \in k$ such that $f(a) = 0$. Let

$$w = av + (a^p + a\lambda_1)F(v) + (a^{p^2} + a^p\lambda_1^p + a\lambda_2)F^2(v) + \cdots + (a^{p^{n-1}} + a^{p^{n-2}}\lambda_1^{p^{n-2}} + \cdots + a\lambda_{n-1})F^{n-1}(v).$$

Since the elements $\{F^i(v)\}_{0 \leq i < n}$ are linearly independent and $a \neq 0$, w is a nonzero element of V. An explicit calculation gives

$$w - F(w) = av + \sum_{0 < i < n} a\lambda_i F^i(v) + (a\lambda_n - f(a))F^n(v) = a(v + \lambda_1 F(v) + \cdots + \lambda_n F^n(v)) = 0,$$

so that w is a nonzero element of V^F. $\square$

Proof of Proposition 3.5.9. The implication (3) $\Rightarrow$ (2) follows from the fact that k is perfect, and the implication (2) $\Rightarrow$ (1) is obvious. Assume that (1) is satisfied; we will prove (3). Lemma 3.5.3 implies that the map of vector spaces $u : V^F \otimes_{\mathbf{F}_p} k \to V$ is injective, so that $V^F \otimes_{\mathbf{F}_p} k$ is finite-dimensional over k and therefore V^F is finite-dimensional over $\mathbf{F}_p$. Choose a basis $\{v_1, v_2, \ldots, v_m\}$ for V^F over $\mathbf{F}_p$.

Let W be the cokernel of u, and note that F induces a Frobenius-semilinear endormorphism F' of W. We first claim that F' is injective. To prove this, suppose that $F'(w) = 0$ for some $w \in W$; we will prove that $w = 0$. Let $\overline{w} \in V$ be a representative for w, so that $F(\overline{w})$ belongs to the image of u and therefore has the form $\sum_{1 \leq i \leq m} c_i v_i$ for some coefficients $c_i \in k$. Since k is algebraically closed, we can choose elements $b_i \in k$ satisfying $c_i = b_i^p$. Then

$$F(\overline{w} - \sum_{1 \leq i \leq m} b_i v_i) = F(\overline{w}) - \sum_{1 \leq i \leq m} b_i^p v_i = 0.$$

Since F is injective, we deduce that $\overline{w} = \sum_{1 \leq i \leq m} b_i v_i$ belongs to the image of u, so that $w = 0$.

We next claim that $W^{F'} = 0$. To prove this, suppose that $w \in W$ satisfies $F'(w) = w$. Let $\overline{w} \in V$ be a representative of w. Arguing as above, we deduce that

$$F(\overline{w}) = \overline{w} + \sum_{1 \leq i \leq m} c_i v_i$$

for some coefficients $c_i \in k$. Since k is algebraically closed, we can choose elements $a_i \in k$ such that $a_i^p = a_i - c_i$. Then Then

$$F(\overline{w} + \sum_{1 \leq i \leq m} a_i v_i) = F(\overline{w}) + \sum_{1 \leq i \leq m} a_i^p v_i = \overline{w} + \sum_{1 \leq i \leq m} (c_i + a_i^p) v_i = \overline{w} + \sum_{1 \leq i \leq m} a_i v_i,$$

so that $\overline{w} + \sum_{1 \leq i \leq m} a_i v_i \in V^F$. It follows that $\overline{w}$ belongs to the image of u and therefore $w = 0$.

Invoking Lemma 3.5.10, we conclude that $W = 0$. It follows that the map u is surjective and therefore an isomorphism. $\square$

Theorem 3.5.11. *Let k be an algebraically closed field of characteristic $p > 0$, and let $\mathcal{S}^{\mathrm{Pro}(p)}_{\mathrm{ft}}$ be the full subcategory of $\mathcal{S}^{\mathrm{Pro}(p)}$ spanned by the p-profinite spaces of finite type. Then the construction $X \mapsto C^*(X;k)$ determines a fully faithful embedding $(\mathcal{S}^{\mathrm{Pro}(p)}_{\mathrm{ft}})^{op} \to \mathrm{CAlg}_k$, whose essential image is the collection of $\mathbb{E}_\infty$-algebras A over k satisfying the following conditions:*

(1) *The unit map $k \to \pi_0 A$ is an isomorphism.*

(2) *The homotopy group $\pi_{-1} A$ vanishes.*

(3) *For every integer n, the homotopy group $\pi_n A$ is a finite-dimensional vector space over k.*

(4) *The map $P^0 : \pi_n A \to \pi_n A$ is injective for every integer n.*

Moreover, if these conditions are satisfied, then the map $P^0 : \pi_n A \to \pi_n A$ is bijective for every integer n.

Proof. Combine Proposition 3.5.9, Theorem 3.4.12, and Theorem 3.5.8. □

We would now like to apply Theorem 3.5.11 to obtain information about the usual homotopy theory of spaces, rather than the homotopy theory of p-profinite spaces. For this, we need to characterize the essential image of the fully faithful embedding described in Theorem 3.3.3.

Proposition 3.5.12. *Let p be a prime number and let $X \in \mathcal{S}$. The following conditions are equivalent:*

(1) *The space X is simply connected, and each homotopy group $\pi_n X$ is a finitely generated module over $\mathbf{Z}_p$.*

(2) *There exists a p-profinite space Y of finite type such that $X \simeq \mathrm{Mat}(Y)$.*

Remark 3.5.13. In the situation of Proposition 3.5.12, the p-profinite space Y is canonically determined by X: in fact, it is given by the p-profinite completion $X^\wedge_p$ of X.

Proof. Let $\mathcal{C} \subseteq \mathcal{S}$ be the full subcategory spanned by those spaces which satisfy condition (1), and let $\mathcal{C}' \subseteq \mathcal{S}$ be the full subcategory spanned by those objects which satisfy condition (2). We wish to prove that $\mathcal{C} = \mathcal{C}'$. The containment $\mathcal{C}' \subseteq \mathcal{C}$ is obvious. To prove the converse, we first make the following observation:

($*$) If we are given a small diagram $p : K \to \mathcal{C}'$ whose limit $\varprojlim(p)$ belongs to $\mathcal{C}$, then $\varprojlim(p)$ belongs to $\mathcal{C}'$.

To prove ($*$), we let $\mathcal{S}^{\mathrm{Pro}(p)}_{\mathrm{ft}}$ denote the full subcategory of $\mathcal{S}^{\mathrm{Pro}(p)}$ spanned by the p-profinite spaces of finite type. Theorem 3.3.3 implies that the materialization functor induces an equivalence of ∞-categories $\mathrm{Mat}_{\mathrm{ft}} : \mathcal{S}^{\mathrm{Pro}(p)}_{\mathrm{ft}} \to \mathcal{C}'$. Consequently, any diagram $p : K \to \mathcal{C}'$ is equivalent to a composition

$$K \xrightarrow{\overline{p}} \mathcal{S}^{\mathrm{Pro}(p)}_{\mathrm{ft}} \overset{\mathrm{Mat}_{\mathrm{ft}}}{long} \to \mathcal{C}'.$$

Let X denote the limit $\varprojlim(\overline{p})$ (formed in the ∞-category $\mathcal{S}^{\mathrm{Pro}(p)}$). Since the materialization functor Mat commutes with limits, we deduce that $\mathrm{Mat}(X) \simeq \varprojlim(p) \in \mathcal{C}$, so that X is of finite type and therefore $\varprojlim(p) = \mathrm{Mat}(X) \in \mathcal{C}'$.

Now suppose that X is an arbitrary object of $\mathcal{C}$; we wish to show that $X \in \mathcal{C}'$. We can realize X as the limit of its Postnikov tower

$$\cdots \to \tau_{\leq 2} X \to \tau_{\leq 1} X \simeq *.$$

Using ($*$), we are reduced to proving that each truncation $\tau_{\leq n} X$ belongs to $\mathcal{C}'$. We proceed by induction on n. When $n = 1$, the space $\tau_{\leq n} X$ is contractible and the result is obvious. Assume therefore that $n > 1$, so that we have a pullback diagram

$$\begin{tikzcd} \tau_{\leq n} X \arrow[r] \arrow[d] & * \arrow[d] \\ \tau_{\leq n-1} X \arrow[r] & K(A, n+1) \end{tikzcd}$$

where $A = \pi_n X$. Using $(*)$, we are reduced to proving that $K(A, n+1)$ and $\tau_{\leq n-1} X$ belong to $\mathcal{C}'$. In the second case, this follows from the inductive hypothesis. In the first case, we observe that $K(A, n+1)$ is the materialization of $\widehat{K}(A, n+1)$. □

We can summarize Proposition 3.5.12 and Theorem 3.3.3 as follows:

Corollary 3.5.14. *Let $\mathcal{S}_{\mathrm{ft}}^{\mathrm{Pro}(p)}$ denote the full subcategory of $\mathcal{S}^{\mathrm{Pro}(p)}$ spanned by the p-profinite spaces of finite type. Then the materialization functor* $\mathrm{Mat} : \mathcal{S}^{\mathrm{Pro}(p)} \to \mathcal{S}$ *induces a fully faithful embedding $\mathcal{S}_{\mathrm{ft}}^{\mathrm{Pro}(p)} \to \mathcal{S}$, whose essential image is spanned by those spaces X which are simply connected and such that each homotopy group $\pi_n X$ is a finitely generated module over $\mathbf{Z}_p$.*

Corollary 3.5.15 (Mandell)**.** *Let k be an algebraically closed field of characteristic $p > 0$. Let $\mathcal{C}$ denote the full subcategory of $\mathcal{S}$ spanned by those spaces X which are simply connected and for which each homotopy group $\pi_n(X)$ has the structure of a finitely generated module over $\mathbf{Z}_p$. Then the construction $X \mapsto C^*(X; k)$ induces a fully faithful embedding $\mathcal{C} \to \mathrm{CAlg}_k^{op}$, whose essential image is the collection of $\mathbb{E}_\infty$-algebras A over k satisfying conditions* (1) *through* (4) *of Theorem 3.5.11.*

Proof. Note that if the cohomology groups $\mathrm{H}^n(X; \mathbf{F}_p)$ are finite dimensional for each $n \geq 0$, the canonical map $C^*(X; \mathbf{F}_p) \otimes_{\mathbf{F}_p} k \to C^*(X; k)$ is an equivalence. Using Notation 3.3.14, we conclude that the map $C^*(X_p^\wedge; k) \to C^*(X; k)$ is also an equivalence. The desired result now follows by combining Theorem 3.5.11 with Corollary 3.5.14. □

3.6 Étale Homotopy Theory

Let X be a scheme. In [1], Artin and Mazur introduce a Pro-object of the homotopy category $\mathrm{h}\mathcal{S}$, which they call the *étale homotopy type* of X. By a slight variation on this construction, one can obtain a p-profinite space $\mathrm{Sh}^p(X)$, which we call the *p-profinite étale homotopy type of X*. In this section, we will use Theorem 2.6.13 to obtain a very simple description of $\mathrm{Sh}^p(X)$ in the case when p is nilpotent in the structure sheaf of X.

We begin by reviewing the definition of the étale homotopy type.

Notation 3.6.1. Let $\mathcal{S}^{<\infty}$ denote the full subcategory of $\mathcal{S}$ spanned by the truncated spaces, let $\mathcal{S}^{\mathrm{fc}} \subseteq \mathcal{S}^{<\infty}$ denote the full subcategory spanned by the π-finite spaces (Example 2.3.2), and for each prime number p let $\mathcal{S}^{p-\mathrm{fc}} \subseteq \mathcal{S}^{\mathrm{fc}}$ denote the full subcategory spanned by the p-finite spaces (Definition 2.4.1).

The ∞-categories $\mathcal{S}^{\mathrm{fc}}$ is essentially small and idempotent complete, and the ∞-category $\mathcal{S}^{<\infty}$ is accessible. Consequently, we can define ∞-categories of Pro-objects $\mathrm{Pro}(\mathcal{S}^{\mathrm{fc}})$, and $\mathrm{Pro}(\mathcal{S}^{<\infty})$. We will refer to $\mathrm{Pro}(\mathcal{S}^{\mathrm{fc}})$ as the *∞-category of profinite spaces* and $\mathrm{Pro}(\mathcal{S}^{<\infty})$ as the *∞-category of protruncated spaces.* Using Remark 3.1.7, we obtain fully faithful embeddings

$$\mathrm{Pro}(\mathcal{S}^{p-\mathrm{fc}}) \hookrightarrow \mathrm{Pro}(\mathcal{S}^{\pi-\mathrm{fc}}) \hookrightarrow \mathrm{Pro}(\mathcal{S}^{<\infty}) \hookrightarrow \mathrm{Pro}(\mathcal{S}).$$

Moreover, each of these functors admits a left adjoint which is given by restriction.

Let $\mathcal{X}$ be an ∞-topos, let $f_* : \mathcal{X} \to \mathcal{S}$ be a geometric morphism (which is unique up to a contractible space of choices), and f^* its left adjoint. Then $f_* f^* : \mathcal{S} \to \mathcal{S}$ is a left-exact accessible functor from the ∞-category $\mathcal{S}$ to itself, which we can identify with an object of $\mathrm{Pro}(\mathcal{S})$. We will denote this object by $\mathrm{Sh}(\mathcal{X})$ and refer to it as the *shape* of $\mathcal{X}$ (see §T.7.1.6). Restricting the functor $\mathrm{Sh}(X)$ to the subcategories

$$\mathcal{S}^{p-\mathrm{fc}} \subset \mathcal{S}^{\mathrm{fc}} \subset \mathcal{S}^{<\infty},$$

we obtain pro-objects

$$\mathrm{Sh}^p(\mathcal{X}) \in \mathrm{Pro}(\mathcal{S}^{p-\mathrm{fc}}) \qquad \mathrm{Sh}^{\mathrm{fc}}(\mathcal{X}) \in \mathrm{Pro}(\mathcal{S}^{\mathrm{fc}}) \qquad \mathrm{Sh}^{<\infty}(\mathcal{X}) \in \mathrm{Pro}(\mathcal{S}^{<\infty}).$$

We will refer to $\mathrm{Sh}^p(\mathcal{X})$ as the *p-profinite shape of* $\mathcal{X}$, to $\mathrm{Sh}^{\mathrm{fc}}(\mathcal{X})$ as the *profinite shape of* $\mathcal{X}$, and to $\mathrm{Sh}^{<\infty}(\mathcal{X})$ as the *protruncated shape of* $\mathcal{X}$.

Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a spectral Deligne-Mumford stack. We let $\mathrm{Sh}(\mathfrak{X}) = \mathrm{Sh}(\mathcal{X})$ denote the shape of the underlying ∞-topos of $\mathfrak{X}$. We will refer to $\mathrm{Sh}(\mathfrak{X})$ as the *étale homotopy type* of $\mathfrak{X}$. Similarly, we set

$$\mathrm{Sh}^p(\mathfrak{X}) = \mathrm{Sh}^p(\mathcal{X}) \qquad \mathrm{Sh}^{\mathrm{fc}}(\mathfrak{X}) = \mathrm{Sh}^{\mathrm{fc}}(\mathcal{X}) \qquad \mathrm{Sh}^{<\infty}(\mathfrak{X}) = \mathrm{Sh}^{<\infty}(\mathcal{X}).$$

We refer to $\mathrm{Sh}^p(\mathfrak{X})$, $\mathrm{Sh}^{\mathrm{fc}}(\mathfrak{X})$, and $\mathrm{Sh}^{<\infty}(\mathfrak{X})$ as the *p-profinite*, *profinite*, and *protruncated* étale homotopy types of $\mathfrak{X}$.

Construction 3.6.2. Let R be a connective $\mathbb{E}_\infty$-ring, and let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a spectral Deligne-Mumford stack over R. Let $f_* : \mathcal{X} \to \mathcal{S}$ be the global sections functor, and $f^* : \mathcal{S} \to \mathcal{X}$ its left adjoint. Then the construction

$$K \mapsto \mathcal{O}_{\mathcal{X}}(f^* K)$$

determines a functor $F : \mathcal{S}^{op} \to \mathrm{CAlg}$. There is an evident map

$$\alpha_0 : R \to \Gamma(\mathcal{X}; \mathcal{O}_{\mathcal{X}}) = F(*).$$

Since the functor F is a right Kan extension of its restriction to $\{*\}$, α_0 extends to a natural transformation $\alpha : C^*(\bullet; R) \to F$. Let $\mathbf{1}$ denote the final object of $\mathcal{X}$, and let $\mathrm{Sh}(\mathfrak{X}) = f_* f^*$ denote the étale homotopy type of $\mathfrak{X}$. For every space K, we have a canonical map

$$\mathrm{Sh}(\mathfrak{X})(K) = f_* f^* K \simeq \mathrm{Map}_{\mathcal{X}}(\mathbf{1}, f^* K) \to \mathrm{Map}_{\mathrm{CAlg}_R}(\mathcal{O}_{\mathcal{X}}(f^* K), \mathcal{O}_{\mathcal{X}}(\mathbf{1})) \xrightarrow{\alpha} \mathrm{Map}_{\mathrm{CAlg}_{\mathbf{R}}}(\mathbf{C}^*(\mathbf{K}; \mathbf{R}), \mathbf{\Gamma}(\mathcal{X}; \mathcal{O}_{\mathcal{X}}).$$

Theorem 3.6.3. *Let R be an $\mathbb{E}_\infty$-ring which is p-thin, and let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a p-thin spectral Deligne-Mumford stack over R. Then, for every p-finite space K, the canonical map*

$$\mathrm{Sh}(\mathfrak{X})(K) \to \mathrm{Map}_{\mathrm{CAlg}_R}(C^*(K; R), \Gamma(\mathcal{X}; \mathcal{O}_{\mathcal{X}}))$$

of Construction 3.6.2 is a homotopy equivalence. That is, the p-profinite étale homotopy type of $\mathfrak{X}$ is given by the formula

$$\mathrm{Sh}^{p-\mathrm{fc}}(\mathfrak{X})(K) \simeq \mathrm{Map}_{\mathrm{CAlg}_R}(C^*(K; R), \Gamma(\mathcal{X}; \mathcal{O}_{\mathcal{X}}).$$

Proof. The assertion is local on $\mathfrak{X}$. We may therefore assume without loss of generality that $\mathfrak{X} = \mathrm{Spec}^{\text{ét}} A$ is affine, and Corollary 2.4.18 supplies an equivalence $A \otimes_R C^*(K; R) \simeq C^*(K; A)$. We may therefore replace R by A and thereby reduce to the case where $\mathfrak{X} = \mathrm{Spec}^{\text{ét}} R$. The desired result now follows immediately fro Theorem 2.6.13. □

If X is a normal Noetherian scheme, then theorem of Artin and Mazur asserts that the (protruncated) étale homotopy type of X is profinite. For later reference, we record a proof of their result here. Our treatment follows [1], with a few minor modifications.

Definition 3.6.4. Let $\mathfrak{X}$ be a spectral Deligne-Mumford stack which is locally Noetherian. We will say that $\mathfrak{X}$ is *normal* if, for every étale map $f : \mathrm{Spec}^{\text{ét}} R \to \mathfrak{X}$, the $\mathbb{E}_\infty$-ring R is a discrete and normal (that is, it is equivalent to a finite product of integrally closed integral domains).

Theorem 3.6.5 (Artin-Mazur)**.** *Let $\mathfrak{X}$ be a quasi-compact quasi-separated spectral algebraic space. If $\mathfrak{X}$ is locally Noetherian and normal, then the protruncated étale homotopy type* $\mathrm{Sh}^{<\infty}(\mathfrak{X})$ *is profinite: that is, it belongs to the essential image of the fully faithful embedding* $\mathrm{Pro}(\mathcal{S}^{\mathrm{fc}}) \hookrightarrow \mathrm{Pro}(\mathcal{S}^{<\infty})$.

The proof of Theorem 3.6.5 reduces to the following:

Proposition 3.6.6. *Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a quasi-compact quasi-separated spectral algebraic space which is Noetherian and normal, and let $f_* : \mathcal{X} \to \mathcal{S}$ be the global sections functor. Let Y be a truncated space. For every point $\eta \in f_* f^* Y$, there exists a π-finite space K and a map $K \to Y$ such that η lies in the essential image of the induced map $f_* f^* K \to f_* f^* Y$.*

Let us show that Proposition 3.6.6 implies Theorem 3.6.5.

Proof of Theorem 3.6.5. Let us abuse notation by identifying $\mathrm{Pro}(\mathcal{S}^{\mathrm{fc}})$ and $\mathcal{S}^{<\infty}$ with full subcategories of $\mathrm{Pro}(\mathcal{S}^{<\infty})$. We have an evident map $\mathrm{Sh}^{<\infty}(\mathfrak{X}) \to \mathrm{Sh}^{\mathrm{fc}}(\mathfrak{X})$, and we wish to prove that it is an equivalence in $\mathrm{Pro}(\mathcal{S}^{<\infty})$. For this, it will suffice to show that for every truncated space Y, the induced map

$$\mathrm{Sh}^{\mathrm{fc}}(\mathfrak{X})(Y) \to \mathrm{Sh}^{<\infty}(\mathcal{X})(\mathfrak{X})$$

is a homotopy equivalence. Equivalently, we must show that for each $n \geq 0$, the induced map

$$\theta : \mathrm{Sh}^{\mathrm{fc}}(\mathfrak{X})(Y) \to \mathrm{Sh}^{\mathrm{fc}}(\mathfrak{X})(Y)^{\partial \Delta^n} \times_{\mathrm{Sh}^{<\infty}(\mathfrak{X})(Y)^{\partial \Delta^n}} \mathrm{Sh}^{<\infty}(\mathfrak{X})(Y)$$

is surjective on connected components. Note that every point of the space $\mathrm{Sh}^{\mathrm{fc}}(\mathfrak{X})(Y)^{\partial \Delta^n} \simeq \mathrm{Sh}^{\mathrm{fc}}(\mathfrak{X})(Y^{\partial \Delta^n})$ lies in the essential image of the map $\mathrm{Sh}^{\mathrm{fc}}(\mathfrak{X})(K) \to \mathrm{Sh}^{\mathrm{fc}}(\mathfrak{X})(Y^{\partial \Delta^n})$ for some map $K \to Y^{\partial \Delta^n}$ where K is π-finite. It will therefore suffice to show that for every π-finite space K with a map $K \to Y^{\partial \Delta^n}$, the induced map

$$\mathrm{Sh}^{\mathrm{fc}}(\mathfrak{X})(K) \times_{\mathrm{Sh}^{\mathrm{fc}}(\mathfrak{X})(Y^{\partial \Delta^n})} \mathrm{Sh}^{\mathrm{fc}}(\mathfrak{X})(Y) \to \mathrm{Sh}^{\mathrm{fc}}(\mathfrak{X})(K) \times_{\mathrm{Sh}^{<\infty}(\mathfrak{X})(Y^{\partial \Delta^n})} \mathrm{Sh}^{<\infty}(\mathfrak{X})(Y)$$

is surjective on connected components. Taking $Z = K \times_{Y^{\partial \Delta^n}} Y$, we are reduced to proving that the map

$$\mathrm{Sh}^{\mathrm{fc}}(\mathfrak{X})(Z) \to \mathrm{Sh}^{<\infty}(\mathfrak{X})(Z)$$

is surjective on connected components, which follows immediately from Proposition 3.6.6. $\square$

Lemma 3.6.7. *Let $X_\bullet$ be a simplicial set and $n \geq 2$ an integer. Assume that $X_\bullet$ satisfies the following conditions:*

- (a) *The simplicial set $X_\bullet$ satisfies the Kan extension condition in dimension 2. That is, for $0 \leq i \leq 2$, every map $\Lambda^n_i \to X_\bullet$ extends to a 2-simplex of X.*
- (b) *For each integer k, the set X_k is finite.*
- (c) *For every $x \in X_0$, the homotopy groups $\pi_m(|X_\bullet|, x)$ are finite for $0 < m \leq n$.*

Let G be a finite group acting on $X_\bullet$, and let $Y_\bullet$ be the quotient $X_\bullet/G$. Then $Y_\bullet$ also satisfies condition (a), (b), and (c).

Proof. It is obvious that $Y_\bullet$ satisfies (b). We next prove (a) in the case $i = 1$ (the other cases follow by an essentially identical argument). Suppose we are given a pair of edges $e : y \to y'$ and $e' : y' \to y''$ in the simplicial set $Y_\bullet$. Lift e and e' to edges $\overline{e} : \overline{y} \to \overline{y}'$ and $\widetilde{e}' : \widetilde{y}' \to \widetilde{y}''$ in the simplicial set X. Then $\overline{y}' = g(\widetilde{y}')$ for some $g \in G$. Let $\overline{e}' = g(\widetilde{e}')$. Since $X_\bullet$ satisfies (a), the pair $(\overline{e}, \overline{e}')$ determines a map $\Lambda^2_1 \to X_\bullet$ which extends to a 2-simplex of $X_\bullet$. The image of this 2-simplex in $Y_\bullet$ is the desired extension of (e, e').

We now prove that $Y_\bullet$ satisfies (c). Let $\pi_{\leq 1} Y_\bullet$ denote the fundamental groupoid of $Y_\bullet$. The objects of $\pi_{\leq 1} Y_\bullet$ are given by the elements of Y_0. Using (a), we see that every morphism in $\pi_{\leq 1} Y_\bullet$ is given by an edge of $Y_\bullet$. Since Y_0 and Y_1 are finite, we deduce that the space $|Y_\bullet|$ has finitely many path components, each of which has a finite fundamental group. It will therefore suffice to show that if $Y'_\bullet \to Y_\bullet$ exhibits $Y'_\bullet$ as a universal cover of some path component of $Y_\bullet$, then the homotopy groups $\pi_m |Y'_\bullet|$ are finite for $2 \leq m < n$. Let $X'_\bullet = X_\bullet \times_{Y_\bullet} Y'_\bullet$. Since $Y'_\bullet$ is a finite-sheeted cover of $Y_\bullet$, $X'_\bullet$ is a finite-sheeted cover of $X_\bullet$. It follows that $X'_\bullet$ also satisfies condition (c). We may therefore replace $X_\bullet$ by $X'_\bullet$, and thereby reduce to the case where where $|Y_\bullet|$ is simply connected.

In the simply connected case, the finiteness of the homotopy groups $\pi_m |Y_\bullet|$ for $m \leq n$ is equivalent to the finiteness of the homology groups $\mathrm{H}_m(Y_\bullet; \mathbf{Z})$ for $0 < m < n$. Since each of the sets Y_k is finite, it is easy to see that each $\mathrm{H}_m(Y_\bullet; \mathbf{Z})$ is a finitely generated abelian group for every integer m. It will therefore suffice to show that the rational homology groups $\mathrm{H}_m(Y_\bullet; \mathbf{Q})$ vanish for $0 < m \leq n$. We now observe that $\mathrm{H}_m(Y_\bullet; \mathbf{Q}) = \mathrm{H}_m(X_\bullet/G; \mathbf{Q})$ is the space of coinvariants for the action of G on $\mathrm{H}_m(X_\bullet; \mathbf{Q})$, and therefore trivial (for $0 < m \leq n$) since $X_\bullet$ satisfies condition (c). $\square$

Proof of Proposition 3.6.6. The object $Y \in \mathcal{S}$ is represented by a Kan complex $Y_\bullet$, which we will view as a hypercovering of Y. Let us view $f^*Y_\bullet$ as a hypercovering in the ∞-topos $\mathcal{X}_{/f^*Y}$. Let $\mathbf{1}$ denote the final object of $\mathcal{X}$, so that the point $\eta \in f_* f^* Y$ can be identified with a morphism $\mathbf{1} \to f^*Y$ in $\mathcal{X}$. Let $V_\bullet$ denote the hypercovering of $\mathcal{X}$ given by $V_n = \mathbf{1} \times_{f^*Y} Y_n$.

We now construct a new hypercovering $U_\bullet$ of $\mathcal{X}$ equipped with a natural transformation $U_\bullet \to V_\bullet$, where each U_n is affine. We construct $U_\bullet$ as the union of a family of maps $U_\bullet^{\leq n} : \mathrm{N}(\mathbf{\Delta}_{\leq n})^{op} \to \mathcal{X}$. Assume that $U_\bullet^{\leq n-1}$ has already been constructed, so that we can define latching and matching objects $L_n(U)$ and $M_n(U)$. Since $\mathcal{X}$ is a coherent ∞-topos and U_m is affine for $m < n$, we deduce that $M_n(U)$ is quasi-compact. Since $V_\bullet$ is a hypercovering of $\mathcal{X}$, the map $V_n \to M_n(V)$ is an effective epimorphism. It follows that the map $V_n \times_{M_n(V)} M_n(U) \to M_n(U)$ is an effective epimorphism. We may therefore choose an object $W \in \mathcal{X}$ which is a coproduct of affine objects equipped with a map $W \to V_n \times_{M_n(V)} M_n(U)$ for which the composite map

$$W \to V_n \times_{M_n(V)} M_n(U) \to M_n(U)$$

is an effective epimorphism. Since $M_n(U)$ is quasi-compact, we can assume that W is a finite coproduct of affine objects of $\mathcal{X}$, and is therefore itself affine. Set $U_n = L_n(U) \coprod W$, so that we have an evident commutative diagram

$$\begin{tikzcd}
L_n(U) \arrow[r] \arrow[d] & U_n \arrow[r] \arrow[d] & M_n(U) \arrow[d] \\
L_n(V) \arrow[r] & V_n \arrow[r] & M_n(V).
\end{tikzcd}$$

Using Proposition T.A.2.9.14, we see that this diagram determines a functor $U_\bullet^{\leq n} : \mathrm{N}(\mathbf{\Delta}_{\leq n})^{op} \to \mathcal{X}$ extending $U_\bullet^{\leq n-1}$, together with a natural transformation $U_\bullet^{\leq n} \to V_\bullet | \mathrm{N}(\mathbf{\Delta}_{\leq n})^{op}$. Taking the union of the maps $U_\bullet^{\leq n}$, we obtain a hypercovering $U_\bullet$ with the desired properties.

Since $\mathfrak{X}$ is locally Noetherian, the topological space $|\,\mathfrak{X}\,|$ is Noetherian (Remark XII.1.4.14). Let T denote the finite set of generic points of $|\,\mathfrak{X}\,|$. For each affine object $W \in \mathfrak{X}$, we can identify $\mathcal{O}_{\mathcal{X}}(W)$ with a normal Noetherian commutative ring. We let W^c denote the (finite) set of connected components of the Zariski spectrum $\mathrm{Spec}^{\mathrm{Z}}\, \mathcal{O}_{\mathcal{X}}(W)$. Each of these connected components C has a generic point whose image is a point $q_W(C) \in T$. We regard q_W as a map from W^c to T. Note that a map of affine objects $W \to W'$ induces a map of finite sets $W^c \to W'^c$ fitting into a commutative diagram

$$\begin{tikzcd}
W^c \arrow[rr] \arrow[dr, "q_W"] & & W'^c \arrow[dl, "q_{W'}"] \\
& T. &
\end{tikzcd}$$

We now define a simplicial set $K_\bullet$ by the formula

$$K_n = U_n^c.$$

By construction, we have a compatible family of maps $U_n \to f^*Y_n$, which determines a map of simplicial sets $K_\bullet \to Y_\bullet$. Let $K \in \mathcal{S}$ denote the geometric realization of $K_\bullet$ (that is, K is the space represented by the simplicial set $K_\bullet$). It is clear that the map $U_\bullet \to f^*Y_\bullet$ factors through $f^*K_\bullet$, so that the point $\eta \in f_* f^* Y_\bullet$ lies in the essential image of the map $f_* f^* K \to f_* f^* Y$. Since Y is m-truncated for some m, the map $K \to Y$ factors through $\tau_{\leq m} K$. We will complete the proof by showing that $\tau_{\leq m} K$ is π-finite. Without loss of generality, we may assume that $m \geq 1$.

For each $t \in T$, we let $K_\bullet^t$ denote the simplicial subset of $K_\bullet$ given by $K_n^t = q_{U_n}^{-1}\{t\} \subseteq U_n^c = K_n$. Then $K_\bullet$ is given by the disjoint union $\coprod_{t \in T} K_\bullet^t$, so that K is a disjoint union of spaces $\{K^t = |K_\bullet^t|\}_{t \in T}$. It will therefore suffice to show that $\tau_{\leq m} K^t$ is π-finite. Let κ denote the residue field of $\mathfrak{X}$ at the point t. For each $n \geq 0$, let $\mathfrak{Z}_n$ denote the fiber product

$$\mathrm{Spec}^{\text{ét}}\, \kappa \times_{\mathfrak{X}} (\mathcal{X}_{/U_n}, \mathcal{O}_{\mathcal{X}} | U_n),$$

so that we have a canonical bijection $K^t_n \simeq |\mathfrak{Z}_n|$. Then each $\mathfrak{Z}_n$ is finite and étale over $\mathrm{Spec}^{\text{ét}}\,\kappa$. We may therefore choose a finite Galois extension κ' of κ such that, for $n \leq m+1$, the space $\mathfrak{Z}_n \times_{\mathrm{Spec}^{\text{ét}}\,\kappa} \mathrm{Spec}^{\text{ét}}\,\kappa'$ is a finite disjoint union of copies of $\mathrm{Spec}^{\text{ét}}\,\kappa'$. For each $n \geq 0$, let Z_n denote the set of κ'-points of $\mathfrak{Z}_n$. Then $Z_\bullet$ is a simplicial set which is acted on by the Galois group $G = \mathrm{Gal}(\kappa'/\kappa)$, and we have a canonical map $Z_\bullet/G \to K^t_\bullet$ which induces a bijection $Z_n/G \simeq K^t_n$ for $n \leq m+1$. Since $U_\bullet$ is a hypercovering of $\mathcal{X}$, the simplicial set $Z_\bullet$ is $(m+1)$-connective and satisfies the Kan condition in dimension 2. Using Lemma 3.6.7, we see that the homotopy groups $\pi_k(|Z_\bullet/G|, z)$ are finite for $k \leq m$ and any choice of base point $z \in |Z_\bullet/G|$. From this we immediately deduce that the homotopy groups $\pi_k(|K^t_\bullet|, z)$ are finite for $k \leq m$, so that $\tau_{\leq m}|K^t_\bullet|$ is π-finite. $\square$

References

[1] Artin, M. and B. Mazur. *Étale Homotopy.* Lecture Notes in Mathematics 100, Springer-Verlag, Berlin and New York, 1969.

[2] Atiyah, M. and I. Macdonald. *Introduction to commutative algebra.* Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ontario.

[3] Behrend, K. and B. Fantechi. *The intrinsic normal cone.* Inventiones Mathematicae 128 (1997) no. 1, 45-88.

[4] Ben-Zvi, D., Francis, J., and D. Nadler. *Integral Transforms and and Drinfeld Centers in Derived Algebraic Geometry.*

[5] Bergner, J. *Three models for the homotopy theory of homotopy theories.* Topology 46 (2007), no. 4, 397–436.

[6] Boardman, J. and R. Vogt. *Homotopy Invariant Algebraic Structures on Topological Spaces.* Lecture Notes in Mathematics, 347, Springer-Verlag (1973).

[7] Crane, L. and D.N. Yetter. *Deformations of (bi)tensor categories.* Cahiers Topologie Geom. Differentielle Categ. 39 (1998), no. 3, 163–180.

[8] Deligne, P. *Catégories tannakiennes.* The Grothendieck Festschrift, Vol. II, 111195, Progr. Math., 87, Birkhuser Boston, Boston, MA, 1990.

[9] Grothendieck, A. *Éléments de géométrie algébrique (rédigés avec la collaboration de Jean Dieudonné) : III. Étude cohomologique des faisceaux cohérents, Première partie.* Publications Mathématiques de l'IHÉS, 11 (1961), p. 5-167

[10] Demazure, M. and Grothendieck, A., eds. *Schémas en groupes vol. 1-3.* Séminaire de Géométrie Algbrique du Bois Marie 1962/64 (SGA 3). Lecture Notes in Mathematics, Vol. 151-153 Springer-Verlag, Berlin-New York.

[11] Efimov, A., Lunts, V., and D. Orlov. *Deformation theory of objects in homotopy and derived categories. I: General Theory.* Adv. Math. 222 (2009), no. 2, 359–401.

[12] Efimov, A., Lunts, V., and D. Orlov. *Deformation theory of objects in homotopy and derived categories. II: Pro-representability of the deformation functor.* Available at arXiv:math/0702839v3 .

[13] Efimov, A., Lunts, V., and D. Orlov. *Deformation theory of objects in homotopy and derived categories. III: Abelian categories.* Available as arXiv:math/0702840v3 .

[14] Elmendorf, A.D., Kriz, I. , Mandell, M.A., and J.P. May. *Rings, modules and algebras in stable homotopy theory.* Mathematical Surveys and Monographs 47, American Mathematical Society, 1997.

[15] Etingof, P., Nikshych, D., and V. Ostrik. *On fusion categories.* Ann. of Math. (2) 162 (2005), no. 2, 581–642.

[16] Francis, J. *Derived Algebraic Geometry over $\mathcal{E}_n$-Rings.* Unpublished MIT PhD dissertation.

[17] Frenkel, E., Gaitsgory, D., and K. Vilonen. *Whittaker patterns in the geometry of moduli spaces of bundles on curves.* Ann. of Math. (2) 153 (2001), no. 3, 699–748.

[18] Fresse, B. *Koszul duality of E_n-operads.* Available as arXiv:0904.3123v6 .

[19] Fukuyama, H. and I. Iwanari. *Monoidal Infinity Category of Complexes from Tannakian Viewpoint.* Available at http://arxiv.org/abs/1004.3087.

[20] Fulton, W. *Algebraic curves.* W.A. Benjamin, Inc., New York, 1969.

[21] Fulton, W. and R. Pandharipande. *Notes on stable maps and quantum cohomology.* Algebraic geometry—Santa Cruz 1995, 45–96, Proc. Sympos. Pure Math., 62, Part 2, Amer. Math. Soc., Providence, RI, 1997.

[22] Getzler, E. *Lie theory for L_∞-algebras.* Ann. of Math. (2) 170 (2009), no. 1, 271–301.

[23] Goldman, W. and J. Millson. *Deformations of flat bundles over Kähler manifolds.* Geometry and Topology, Manifolds, Varieties and Knots (C. McCrory and T. Shifrin, eds.), Lecture Notes in Pure and Applied Math., vol. 105, Marcel Dekker, New York/Basel, 1987, pp. 129-145.

[24] Goldman, W. and J. Millson. *The deformation theory of representations of fundamental groups of compact Kähler manifolds.* Bull. Amer. Math. Soc. (N.S.), Volume 18, Number 2 (1988), 153-158.

[25] Goodwillie, T. *Calculus III: Taylor Series.* Geometry and Topology 7 (2003), 645-711.

[26] Grauert, H. and R. Remmert. *Coherent Analytic Sheaves.* Springer, 1984.

[27] Grothendieck, A. *Sur quelques points d'algebra homologique.* Tohoku Math. J. 9, 1957, 119–221.

[28] Hinich, V. *DG coalgebras as formal stacks.* J. Pure Appl. Algebra, 162 (2001), 209-250.

[29] Hinich, V. *Deformations of homotopy algebras.* Communication in Algebra, 32 (2004), 473-494.

[30] Hopkins, M. and J. Lurie. *Vanishing Theorems for Higher Tate Cohomology.* In preparation.

[31] Johnstone, P. *Stone Spaces.* Cambridge University Press, Cambridge, 1982.

[32] Kapranov, M. and V. Ginzburg. *Koszul duality for Operads.* Duke Math. J., 1994.

[33] Keller, B. and W. Lowen. *On Hochschild cohomology and Morita deformations.* Int. Math. Res. Not. IMRN 2009, no. 17, 3221–3235.

[34] Knutson, D. *Algebraic spaces.* Lecture Notes in Mathematics 203, Springer-Verlag, 1971.

[35] Kontsevich, M. and Y. Soibelman. *Deformations of algebras over operads and the Deligne conjecture.* Conference Moshe Flato 1999, Vol. I (Dijon), 255–307, Math. Phys. Stud., 21, Kluwer Acad. Publ., Dordrecht, 2000.

[36] Kontsevich, M. and Y. Soibelman. *Deformation Theory.* Unpublished book available at http://www.math.ksu.edu/s̃oibel/Book-vol1.ps .

[37] Kontsevich, M. and Y. Soibelman. *Notes on A-infinity algebras, A-infinity categories and noncommutative geometry.* Available as math.RA/060624.

[38] Laplaza, M. *Coherence for distributivity.* Coherence in categories, 29-65. Lecture Notes in Mathematics 281, Springer-Verlag, 1972.

[39] Laumon, G. and L. Moret-Bailly. *Champs algebriques.* Springer-Verlag, 2000.

[40] Lowen, W. *Obstruction theory for objects in abelian and derived categories.* Comm. Algebra 33 (2005), no. 9, 3195–3223.

[41] Lowen, W. *Hochschild cohomology, the characteristic morphism, and derived deformations.* Compos. Math. 144 (2008), no. 6, 1557–1580.

[42] Lurie, J. *Tannaka Duality for Geometric Stacks.* Available for download at http://www.math.harvard.edu/ lurie/ .

[43] Lurie, J. *Higher Topos Theory.* Available for download at http://www.math.harvard.edu/ lurie/ .

[44] Lurie, J. *Higher Algebra.* Available for download.

[45] Lurie, J. *Derived Algebraic Geometry V: Structured Spaces.* Available for download.

[46] Lurie, J. *Derived Algebraic Geometry VII: Spectral Schemes.* Available for download.

[47] Lurie, J. *Derived Algebraic Geometry VIII: Quasi-Coherent Sheaves and Tannaka Duality Theorems.* Available for download.

[48] Lurie, J. *Derived Algebraic Geometry IX: Closed Immersions.* Available for download.

[49] Lurie, J. *Derived Algebraic Geometry X: Formal Moduli Problems.* Available for download.

[50] Lurie, J. *Derived Algebraic Geometry XI: Descent Theorems*

[51] Lurie, J. *Derived Algebraic Geometry XII: Proper Morphisms, Completions, and the Grothendieck Existence Theorem.*

[52] Lurie, J. *Derived Algebraic Geometry XIV: Representability Theorems.*

[53] MacLane, S., and I. Moerdijk. *Sheaves in Geometry and Logic.* Springer-Verlag, Berlin and New York, 1992.

[54] Mandell, M. *E_∞-Algebras and p-adic Homotopy Theory.* Topology 40 (2001), no. 1, 43-94.

[55] Manetti, M. *Extended deformation functors.* Int. Math. Res. Not. 2002, no. 14, 719–756.

[56] Matsumura, H. *Commutative algebra, 2nd Edition.* Benjamin-Cummings Pub Co, 1980.

[57] Matsumura, H. *Commutative ring theory.* Cambridge University Press, 1986.

[58] May, P. *The Geometry of Iterated Loop Spaces.* Lectures Notes in Mathematics, Vol. 271. Springer-Verlag, Berlin-New York, 1972. viii+175 pp.

[59] Nisnevich, Y. *The completely decomposed topology on schemes and associated descent spectral sequences in algebraic K-theory.* NATO Advanced Study Institute Series, Ser. C, v. 279, Kluwer, 1989, pp. 241-342.

[60] Pridham, J. *Unifying derived deformation theories.* Adv. Math. 224 (2010), no.3, 772-826.

[61] Quillen, D. *Homotopical Algebra.* Lectures Notes in Mathematics 43, SpringerVerlag, Berlin, 1967.

[62] Raynaud, M. *Anneaux locaux henséliens.* Lecture Notes in Mathematics, 169, Berlin-New York: Springer-Verlag.

[63] Rognes, J. *Galois extensions of structured ring spectra. Stably dualizable groups.* Mem. Amer. Math. Soc. 192 (2008), no. 898, viii+137 pp.

[64] Schlessinger, M. and J. Stasheff. *The Lie algebra structure of tangent cohomology and deformation theory.* Journal of Pure and Applied Algebra 38 (1985), 313-322.

[65] Schlessinger, M. *Functors of Artin Rings.* Trans. Amer. Math. Soc. 130 1968 208–222.

[66] Serre, J-P. *Local algebra.* Springer-Verlag, 2000.

[67] Spivakovsky, M. *A New Proof of D. Popescu's Theorem on Smoothing of Ring Homomorphisms.* Journal of the American Mathematical Society, Volume 12, Number 2, 1999, 381-444.

[68] Thomason, R. W. and T. Trobough. *Higher Algebraic K-theory of schemes and of derived categories.* The Grothendieck Festschrift III, Progress in Math., vol. 88, Birkhuser, 1990, pp. 247435.

[69] Tits, J. *Sur les analogues algbriques des groupes semi-simples complexes.* Colloque dalgebre superieure, tenu a Bruxelles du 19 au 22 decembre 1956, Centre Belge de Recherches Mathematiques Etablissements Ceuterick, Louvain, Paris: Librairie Gauthier-Villars, pp. 261289.

[70] Toën, B. *Champs affines.* Selecta mathematica, new series, 12, 2006.

[71] Toën, B. *Finitude homotopique des dg-algèbres propres et lisses.* Available at math.AT/0609762. Finitude homotopique des dg-algebras

[72] Toën, B. *The homotopy theory of dg-categories and derived Morita theory.* Invent. Math. 167 (2007), no. 3, 615–667.

[73] Toën, B. *Derived Azumaya's algebras and generators for twisted derived categories.*

[74] Toën, B., and M. Vaquié. *Moduli of objects in dg-categories.* Ann. Sci. cole Norm. Sup. (4) 40 (2007), no. 3, 387–444.

[75] Toën, B. and G. Vezzosi. *Brave new algebraic geometry and global derived moduli spaces of ring spectra.* Elliptic cohomology, 325359, London Math. Soc. Lecture Note Ser., 342, Cambridge Univ. Press, Cambridge, 2007.

[76] Toën, B. and G. Vezzosi. *Homotopical algebraic geometry. I. Topos theory.* Adv. Math. 193 (2005), no. 2, 257372.

[77] Toën, B. and G. Vezzosi. *Homotopical algebraic geometry. II. Geometric stacks and applications.* Mem. Amer. Math. Soc. 193 (2008), no. 902, x+224 pp.

[78] Toën, B. and G. Vezzosi. *From HAG to DAG: derived moduli stacks.* Axiomatic, enriched and motivic homotopy theory, 173216, NATO Sci. Ser. II Math. Phys. Chem., 131, Kluwer Acad. Publ., Dordrecht, 2004.

[79] Toën, B. and G. Vezzosi. *Algebraic geometry over model categories.* Available for download: math.AG/0110109.

[80] Quick, G. *Profinite homotopy theory.* Documenta Mathematica, 13, (2008), 585612.

[81] Quillen, D. *Rational homotopy theory.* Ann. of Math. (2) 90 1969 205–295.

[82] Wallbridge, J. *Higher Tannaka duality.* Doctoral dissertation.

[83] Yetter, D. *Braided deformations of monoidal categories and Vassiliev invariants.* Higher category theory (Evanston, IL, 1997), 117–134, Contemp. Math., 230, Amer. Math. Soc., Providence, RI, 1998.