Equivariant K-theory and Fredholm operators

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(Communicated by A. Hattori)

§0. Introduction

Let G be a compact Lie group and let $L^2(G, l^2)$ be the complex Hilbert space of square integrable (with respect to a G-invariant Haar measure) functions on G with values in l^2, where l^2 is a separable complex Hilbert space (with a fixed basis). Let $\mathfrak{F}$ be the space of Fredholm operators on $L^2(G, l^2)$. The group G acts on $\mathfrak{F}$ in a natural way. The aim of this note is to prove the following theorem.

MAIN THEOREM. *There is a canonical isomorphism (of groups)*

$$G\text{-index}: \ [X, \mathfrak{F}]_G \xrightarrow{\cong} K_G(X),$$

for any compact G-space X. Here $[\cdot\,,\cdot]_G$ *stands for the set of G-homotopy classes of G-maps.*

This theorem is a generalization to the equivariant case of the result due to M. Atiyah [2] and K. Jänich [9]. We shall get similar representations of $KR_G(X)$ and $KQ_G(X)$ in the last section.

The method of proof follows that of Atiyah in the absolute case. Let $GL(G)$ be the general linear group on $L^2(G, l^2)$. The main step is to prove the G-contractibility of $GL(G)$ instead of its contractibility due to N. Kuiper [10]. Using the Kuiper's theorem, we shall prove first that $GL(G)$ is weakly G-contractible, that is, the H-stationary subgroup of $GL(G)$ is weakly contractible for every closed subgroup H of G.

To conclude the G-contractibility of $GL(G)$ from its weakly G-contractibility, we shall construct a slice covering whose G-nerve dominates $GL(G)$ and then extend a G-map of the G-nerve into $GL(G)$ over a cone of the G-nerve. This process is similar to that which Kuiper used [loc. cit., §3]. The extension of a G-map of the G-nerve is done by a technique of G. Bredon. In particular, when G is a finite group, a G-nerve is itself a G-complex in the sense of Bredon [6]. When G has a positive dimension, we shall need an inductive use of the lemma (3.4). The proof of the lemma (3.4) depends on the concept of the relative G-CW complex which is a generalization of G-complex. But the proof is easy when G is an abelian group. In another paper [11] the author will state further details about a G-CW complex.

Another representation of the equivariant K-theory not by a space but by an object analogous to $BU=\lim BU(n)$ is constructed by S. Araki in [1].

In preparing this note, there appeared a paper of G. Segal [15] which proved the G-contractibility of $GL(G)$ by another method. But our method using the concept of G-CW complexes seems to have other applications, in particular in the calculation of the K_G-group of a G-manifold.

The author wishes to express his warmest thanks to Professor Akio Hattori for his many valuable advices.

§1. Construction of the G-spaces and an exact sequence

In this section we shall construct an exact sequence in the theorem (1.2) which makes clear that the G-contractibility of $GL(G)$ includes the main theorem. The notation, terminology and method follow those of Atiyah in [2] and the proof is omitted.

The G-action on $L^2(G, l^2)$ is defined by

$$(ga)(h)=a(g^{-1}h), \text{ for } a \in L^2(G, l^2),\ g,\ h \in G$$

and continuous with respect to the norm topology.

Let $V_0, V_1, V_2, \cdots$ be the complete sequence of irreducible complex representations of G. Then by the theory of Peter-Weyl, $L^2(G)$ is isomorphic to the Hilbert space $\sum\oplus_{i=0}^{\infty}(\dim V_i)V_i$, where $\sum\oplus$ stands for the direct sum in the category of Hilbert spaces. Moreover $L^2(G, l^2)$ is isomorphic with $\sum\oplus_{i=0}^{\infty}(\dim V_i)V_i\otimes l^2$. And similarly $(\dim V_i)V_i\otimes l^2$ is isomorphic with $V_i\otimes l^2$ by the countability of the basis of l^2. Therefore we get

LEMMA (1.1) $L^2(G, l^2)$ *is isomorphic* with $\sum\oplus_{i=0}^{\infty}V_i\otimes l^2$ *as Hilbert spaces with G-actions. In particular* $L^2(G, l^2)$ *is itself a separable complex Hilbert space.*

Let $\mathfrak{A}$, $\mathfrak{F}$ and $\mathfrak{A}^*=GL$ be the Banach algebra of all bounded operators on $L^2(G, l^2)$, its subalgebra of all Fredholm operators and its unit group = the general linear group on $L^2(G, l^2)$ respectively. The G-action on $\mathfrak{A}$ is defined by

$$T^g(a)=gT(g^{-1}a), \text{ for } T \in \mathfrak{A},\ a \in L^2(G, l^2),\ g \in G$$

and compatible with the algebra structure and the norm topology of $\mathfrak{A}$. We often denote $\mathfrak{A}(G)$ instead of $\mathfrak{A}$ in order to make clear that the G-action on $\mathfrak{A}$ is induced by that of $L^2(G, l^2)$. Evidently $\mathfrak{F}$ and GL are the G-subspaces of $\mathfrak{A}(G)$ and the notations $\mathfrak{F}(G)$ and $GL(G)$ are used for the same purpose as $\mathfrak{A}(G)$.

Since the algebra structure of $\mathfrak{F}(G)$ is compatible with the G-action, $[X, \mathfrak{F}(G)]_G$ has a natural semi-group structure for any G-space X. Likewise $[X, GL(G)]_G$

forms a group. And the equivariant inclusion $GL(G) \to \mathfrak{F}(G)$ induces a natural semi-group homomorphism $[X, GL(G)]_G \to [X, \mathfrak{F}(G)]_G$.

For a G-map $\boldsymbol{T}: X \to \mathfrak{F}(G)$ with $\boldsymbol{T}(x) = T_x$, the class of

$$[\cup \operatorname{Ker} T_x \, (x \in X)] - [\cup \operatorname{Coker} T_x \, (x \in X)]$$

will be expected to give an element of $K_G(X)$. When X is a compact G-space, we can justify the above and define a semi-group homomorphism, G-index: $[X, \mathfrak{F}(G)]_G \to K_G(X)$.

THEOREM (1.2) *For any compact G-space X the following canonical sequence (of semi-groups) is exact:*

$$[X, GL(G)]_G \to [X, \mathfrak{F}(G)]_G \to K_G(X) \to 0 \, .$$

Here the second arrow is the G-index homomorphism.

We prove this theorem by the almost same argument as in the absolute case, using the lemma (1.1) and the existence theorem of the trivial G-vector bundle including a given G-vector bundle [4, p. 2-6]. Especially in proving the surjectivity [2, p. 162], we need to take a projection $\pi_x: \mathrm{L}^2(G, l^2) \to E_x$ instead of $\pi_x: V \to E_x$ and to use the fact that $\mathrm{L}^2(G, l^2) \otimes \mathrm{L}^2(G, l^2) = \mathrm{L}^2(G, l^2 \otimes l^2) \cong \mathrm{L}^2(G, l^2)$.

REMARK. Using the ideal of compact operators, we can prove that the semi-group $[X, \mathfrak{F}(G)]_G$ is a group. Therefore the above sequence is an exact sequence of groups.

If we assume that $GL(G)$ is G-contractible, then we get $[X, GL(G)]_G = 0$ in the above exact sequence of groups and therefore we get the main theorem which we have stated in the introduction, that is, the G-index homomorphism: $[X, \mathfrak{F}(G)]_G \to K_G(X)$ is a group isomorphism.

We have outlined the proof of the G-contractibility of $GL(G)$ in the introduction and we shall carry out the detailed proof in the following three sections.

§2. Weakly G-contractibility of $GL(G)$

In this section we shall prove the lemma (2.3) using the separability condition of l^2 and the lemma (2.4) by the use of the Kuiper's theorem. The weakly G-contractibility of $GL(G)$ (Theorem (2.2)) is easily deduced from these two lemmas.

We begin with the definition of the weakly G-contractibility.

DEFINITION (2.1) A G-space X with a base point x_0 is said to be weakly G-contractible if for every closed subgroup H of G, X^H is arcwise connected and its n-dimensional homotopy group $\pi_n(X^H, x_0)$ vanishes for every $n \geqq 1$. Here X^H stands for the H-pointwise fixed subspace of X.

REMARK. If X is G-contractible, then the equivariant contracting function still works when it is restricted on X^H, and hence X is weakly G-contractible.

THEOREM (2.2) *$GL(G)$ with an identity operator as a base point is weakly G-contractible.*

Let H be a closed subgroup of G. Then H is also a compact Lie group and we shall get

LEMMA (2.3) *$\mathrm{L}^2(G, l^2)$ is isomorphic with $\mathrm{L}^2(H, l^2)$ as Hilbert spaces with H-actions where the H-action on $\mathrm{L}^2(G, l^2)$ is the restriction of its G-action. Hence each pair of $\mathfrak{A}(G)$ and $\mathfrak{A}(H)$, $\mathfrak{F}(G)$ and $\mathfrak{F}(H)$, $GL(G)$ and $GL(H)$ is a pair of H-homeomorphic spaces.*

PROOF OF THE LEMMA. Let W_0, W_1, $W_2, \cdots$ be the complete sequence of irreducible complex representations of H. Then if an irreducible G-representation space V_i is considered to be an H-representation space, V_i is decomposed into $\sum\oplus_{j=0}^{\infty}\alpha_{ij}W_j$ where α_{ij}'s are non-negative integers and the sum is finite. Suppose that W_j is a given irreducible representation of H. Since W_j is contained in the restriction of an irreducible representation of G because of Frobenius reciprocity (For example see [5, p. 172].), at least one of the α_{ij}'s is non-zero. Let us recall that l^2 has a countable basis. Then $\sum\oplus_{i=0}^{\infty}(\alpha_{ij}W_j\otimes l^2)$ is isomorphic with the countable direct sum of the H-representation spaces each of which is isomorphic with W_j. Hence we get

$$\sum\oplus_{i=0}^{\infty}(\alpha_{ij}W_j\otimes l^2)\cong W_j\otimes l^2 .$$

Using this formula and the lemma (1.1) and commuting the order of the sum, we can calculate as follows:

$$\mathrm{L}^2(G, l^2)\cong\sum\oplus_{i=0}^{\infty}V_i\otimes l^2\cong\sum\oplus_{i=0}^{\infty}(\sum\oplus_{j=0}^{\infty}\alpha_{ij}W_j)\otimes l^2\cong\sum\oplus_{i=0}^{\infty}\sum\oplus_{j=0}^{\infty}(\alpha_{ij}W_j\otimes l^2)$$
$$\cong\sum\oplus_{j=0}^{\infty}\sum\oplus_{i=0}^{\infty}(\alpha_{ij}W_j\otimes l^2)\cong\sum\oplus_{j=0}^{\infty}W_j\otimes l^2\cong\mathrm{L}^2(H, l^2) . \qquad \text{q.e.d.}$$

Since $GL(G)$ and $GL(H)$ are H-homeomorphic by this lemma the H-stationary subspaces $GL(G)^H$ and $GL(H)^H$ are homeomorphic. Then writing G instead of H again, we have reduced the proof of the theorem (2.2) to the next lemma.

LEMMA (2.4) *$GL(G)^G$ is contractible for every compact Lie group G.*

Kuiper proved the contractibility of $GL(l^2)$ and then the contractibility of $U(l^2)$ (=the group of all unitary operators on l^2) [10]. Here we shall use the latter result to avoid the divergence difficulty of the norms.

Let $U(G)$ be the group of all unitary operators on $\mathrm{L}^2(G, l^2)$ and $P(G)$ be the space of all bounded, strictly positive definite, hermitian operators on $\mathrm{L}^2(G, l^2)$. (By definition $T\in U(G)$ means $\|Ta\|=\|a\|$ for any $a\in\mathrm{L}^2(G, l^2)$, and $T\in P(G)$ means $T^*=T$ and there exist positive numbers ε and δ such that $\varepsilon<(Ta, a)<\delta$ for any

$a \in L^2(G, l^2)$ with $\|a\|=1$.) Then $U(G)$ and $P(G)$ are G-invariant subspaces of $GL(G)$ and $GL(G)$ has a semi-direct decomposition $U(G) \dot{\times} P(G)$, that is, for any $T \in GL(G)$ there exists a unique pair (u, h) with $u \in U(G)$, $h \in P(G)$ and $T=uh$. (For example see [7, p. 935].) Moreover if $T \in GL(G)^G$, then $uh=T=T^g=u^g h^g$ and hence $u=u^g$ and $h=h^g$. Therefore $GL(G)^G$ also has a semi-direct decomposition $U(G)^G \dot{\times} P(G)^G$.

Define for $h \in P(G)^G$ and $0 \leqq t \leqq 1$, $f_t(h)=(1-t)h+t$. Then $f_t(h) \in P(G)^G$ $(0 \leqq t \leqq 1)$ and $f_0(h)=h$, $f_1(h)=1$. Hence f_t is a contracting function of $P(G)^G$.

On the other hand we shall get

LEMMA (2.5) *$U(G)^G$ is contractible for every compact Lie group G.*

PROOF. In virtue of the lemma (1.1) we shall identify $L^2(G, l^2)$ with $\sum \oplus_{i=0}^{\infty} V_i \otimes l^2$. Then $T \in \mathfrak{A}(G)$ has a matrix-like representation $T=(T_{ij})$ with $T_{ij}: V_j \otimes l^2 \to V_i \otimes l^2$. Moreover if $T \in \mathfrak{A}(G)^G$, that is, T is G-equivariant, then T_{ij} itself has an (infinite) matrix representation (with respect to a basis of l^2) with coefficients in $\mathrm{Hom}_G(V_j, V_i)$, where $\mathrm{Hom}_G(\cdot, \cdot)$ stands for the module of G-equivariant linear homomorphisms. Then Schur's lemma says $T_{ij}=0$ $(i \neq j)$ and T_{ii} is contained in the Banach algebra of all bounded operators on l^2 (in general the separable Hilbert space with coefficients in $\mathrm{Hom}_G(V_i, V_i)$). Hence if $T=(T_{ij}) \in U(G)^G$, then $T_{ij}=0$ $(i \neq j)$ and T_{ii} is contained in the group of all unitary operators on l^2.

Conversely let $T=(T_{ij})$ with $T_{ij}=0$ $(i \neq j)$ and $T_{ii} \in U(l^2)$, then $T \in U(G)^G$. Therefore $U(G)^G$ coincides with the subspace $\Pi_{i=0}^{\infty} U(l^2)$ of $\mathfrak{A}(G)$, where the i-th $U(l^2)$ corresponds to the group of G-equivariant unitary operators on $V_i \otimes l^2$. The subspace $\Pi_{i=0}^{\infty} U(l^2)$ turns out to be a (bounded) metric space with the metric $\|(T_i)-(T'_i)\| = \sup \|T_i - T'_i\|$, in particular, its topology coincides with its product topology. Hence as a direct product of the contractible spaces, $U(G)^G$ is contractible. q.e.d.

This concludes the proof of the lemma (2.4) and hence the theorem of this section.

§ 3. G-contractibility of $GL(G)$

In this section we shall first define the G-nerve induced by a locally finite refined slice covering. Next we shall construct a locally finite refined slice covering of $GL(G)$ and a dominating map of the induced G-nerve explicitly. Lastly assuming the lemma (3.4) we shall construct a G-contracting function of $GL(G)$. The proof of the lemma (3.4) is given in § 4.

3. A. *G-nerve induced by a locally finite refined slice covering*

We first recall the definition of a slice. Let X be a G-space.

DEFINITION (3.1) [13]. A subspace S of X which contains a point x is said to be a slice at x if:

(1) GS is open,

(2) S is closed in GS,

(3) $H_x S = S$,

(4) if g is not contained in H_x then gS is disjoint from S.

Here GS stands for the G-orbit of S and H_x stands for the isotropy subgroup at x.

Let $\mathfrak{S} = \{S_\alpha : \alpha \in \mathrm{A}\}$ be a family of slices where S_α is a slice at x_α such that there is another slice S'_α at x_α with $GS'_\alpha \supset \overline{GS}_\alpha$. Then $\overline{GS}_\alpha = G\bar{S}_\alpha$ where $\bar{S}_\alpha$ stands for the closure of S_α in S'_α (hence in X). Moreover, if $\{GS_\alpha ; \alpha \in \mathrm{A}\}$ covers X and the closed covering $\{\overline{GS}_\alpha ; \alpha \in \mathrm{A}\}$ is locally finite, then $\{S_\alpha : \alpha \in \mathrm{A}\}$ is called to be a locally finite refined slice covering of X.

Let b_α be the point H_α in the coset space G/H_α, where H_α is the isotropy subgroup at x_α. The join of the Gb_α's ($\alpha \in \mathrm{A}$) is defined to be the set

$$\{(t_\alpha \bar{g}_\alpha)_{\alpha \in \mathrm{A}} ; \bar{g}_\alpha \in G/H_\alpha, 0 \leqq t_\alpha \leqq 1, t_\alpha \neq 0 \text{ for only finite } \alpha\text{'s}, \textstyle\sum t_\alpha = 1\}$$

and is denoted by $|\mathrm{A}|$. If we call a join of $(n+1)$ homogenous spaces $G/H_{\alpha i}$ $(0 \leqq i \leqq n)$ a G-n-simplex with G-homogeneous spaces as G-vertexes, then $|\mathrm{A}|$ is the union of all the G-simplexes with Gb_α's as G-vertexes. The topology of $|\mathrm{A}|$ is defined by the weak topology with respect to these G-simplexes.

We shall use the convention $\sum t_\alpha g_\alpha b_\alpha$ $(t_\alpha \neq 0)$ instead of $(t_\alpha \bar{g}_\alpha)$ where $\bar{g}_\alpha = g_\alpha H_\alpha$. And we mean by $g_0 b_{\alpha 0} \circ \cdots \circ g_k b_{\alpha k}$ the subspace $\{\sum_{i=0}^k t_i g_i b_{\alpha i} ; 0 \leqq t_i \leqq 1, \sum t_i = 1\}$ of $|\mathrm{A}|$, which is called a cell.

DEFINITION (3.2) Let $\mathfrak{S} = \{S_\alpha$ a slice at x_α; $\alpha \in \mathrm{A}\}$ be a locally finite refined slice covering of a G-space X. A pair $(|N|, N)$ of an underlying G-space and a collection of cells which are defined as follows is called to be the G-nerve induced by $\mathfrak{S}$. Define

$$N^0 = \{gb_\alpha \subset |\mathrm{A}| ; g \in G\}$$
$$N^1 = \{g_0 b_{\alpha 0} \circ g_1 b_{\alpha 1} \subset |\mathrm{A}| ; g_0 \bar{S}_{\alpha 0} \cap g_1 \bar{S}_{\alpha 1} \neq \phi\}$$
$$\vdots$$
$$N^k = \{g_0 b_{\alpha 0} \circ \cdots \circ g_k b_{\alpha k} \subset |\mathrm{A}| ; g_0 \bar{S}_{\alpha 0} \cap \cdots \cap g_k \bar{S}_{\alpha k} \neq \phi\}$$
$$\vdots$$

and $$N = \cup_{k=0}^\infty N^k.$$

Then the union of all the cells in N forms a G-invariant closed subspace of $|\mathrm{A}|$, which is defined to be the underlying G-space $|N|$ of N.

DEFINITION (3.3) The simplicial G-nerve $(|SN|, SN)$ induced by a locally finite refined slice covering $\mathfrak{S}$ is defined by $SN = \cup_{k=0}^\infty SN^k$ as a collection of G-simplexes where

$$SN^k = \{Gb_{\alpha 0} \circ \cdots \circ Gb_{\alpha k} \subset |\mathrm{A}|\ ;\ g\bar{S}_{\alpha 0} \cdots G\bar{S}_{\alpha k} \neq \phi\}$$

and $|SN|$ is a G-invariant closed subspace of $|\mathrm{A}|$ which is spanned by the G-simplexes in SN.

3.B. *A dominating G-nerve on $GL(G)$*

We need the following theorem due to G. Mostow [12]. See also [13].

THEOREM (Mostow). *Let G be a compact Lie group and X be a G-space which is completely regular. Then there exists a slice at x for every point x of X.*

We recall that $\mathfrak{A}(G)$ is a Banach space with a G-action and its norm is G-equivariant. Moreover $GL(G)$ is an open G-invariant subspace of the G-equivariant Banach space $\mathfrak{A}(G)$. Then there is a 'small' ball with center x for every point x of $GL(G)$, where the ball with radius ε is said to be 'small' if the concentric ball with radius 3ε in $\mathfrak{A}(G)$ is contained in $GL(G)$. Since $GL(G)$ is metrizable and therefore completely regular, we can use the above theorem of Mostow and get a slice S'_x at x. Let $U_x = U(x, \varepsilon(x))$ be a 'small' ball with center x and radius $\varepsilon(x)$. Then since U_x is H_x-invariant, $H_x(S'_x \cap U_x) = S'_x \cap U_x$. On the other hand since $S'_x \cap U_x$ is open in S'_x, $G(S'_x \cap U_x)$ is open. Therefore $S'_x \cap U_x$ is also a slice at x. Henceforth we can assume $S'_x \subset U_x$.

Because the metric on $GL(G)$ is G-invariant, $GL(G)/G$ is a metric space and therefore paracompact. We choose a countable locally finite refined covering of the covering $\{S'_x/G\ ;\ x \in GL(G)\}$ of $GL(G)/G$ and a partition of unity on $GL(G)/G$ attached to the refined covering. This refined covering and the partition of unity induce a countable locally finite refined slice covering $\{S_j$ a slice at $x_j : j$ (natural number)$\}$ of $GL(G)$ and a collection of G-invariant functions φ_j on $GL(G)$ with $\operatorname{supp} \varphi_j \subset GS_j$, $\sum \varphi_j(x) = 1$, $\varphi_j(x) \geqq 0$, $\varphi_j(gx) = \varphi_j(x)$ and $\varphi_j(x) > 0$ for only finite j's.

When $x \in GS_j$, we introduce the notation $x_j(x)$ which stands for the centering point $g(x)x_j$ of the unique slice $g(x)S_j$ which contains x. Here $g(x) \in G$ is determined modulo H_x. Also we shall use the convention $S_j(x)$ instead of $g(x)S_j$. In this notation $x_j(gx) = gx_j(x)$ holds for any $g \in G$ because $gS_j(x) = S_j(gx)$.

Let ε_j denote the radius $\varepsilon(x_j)$ of the 'small' ball with center x_j which has been given for every point x at the beginning. Then

$$S_j \subset S'_j \subset U_j = U(x_j, \varepsilon_j)\ .$$

Therefore if $\varphi_j(x) > 0$ then

$$x \in S_j(x) \subset U(x_j(x), \varepsilon_j)\ .$$

Since $\varphi_j(x) > 0$ for only finite j's, there is an index $m = m(x)$ such that $\varphi_m(x) > 0$

and $\varepsilon_m \geqq \varepsilon_j$ for any j with $\varphi_j(x)>0$. Then since $U(x_j(x), \varepsilon_j)$'s are also 'small' balls, both x and $x_j(x)$'s are contained in $U(x_m(x), 3\varepsilon_m)$. Therefore by the convexity of the ball, for every $x \in GL(G)$ and $0 \leqq t \leqq 1$,

$$\xi_t(x) = (1-t)x + t \sum \varphi_j(x) x_j(x)$$

is contained in $GL(G)$. Here φ_j's are G-invariant and x_j's are G-equivariant. Then ξ_t $(0 \leqq t \leqq 1)$ gives a G-homotopy from the identity map ξ_0 of $GL(G)$ onto $GL(G)$ to $\xi_1 = \sum \varphi_j x_j$.

On the other hand let N be the G-nerve induced by the countable locally finite refined slice covering $\{S_j : j \text{ (natural number)}\}$. Let $b_j(x)$ stand for $g(x)b_j$ where $g=g(x)$ satisfies $x_j(x)=gx_j$. Then the G-equivariantness of $x_j(\cdot)$ induces that of $b_j(\cdot)$. Define two maps,

$$\sigma: GL(G) \to |N| \text{ and } \rho: |N| \to GL(G)$$

by

$$\sigma(x) = \sum \varphi_j(x) b_j(x) \text{ and } \rho(\sum t_j b_j(x)) = \sum t_j x_j(x) .$$

Then these two maps are well-defined, G-equivariant and continuous with respect to the topology of $|N|$. Moreover we have proved that $\rho \circ \sigma = \xi_1$, is G-homotopic to the identity map of $GL(G)$ onto $GL(G)$. This means that $|N|$ dominates $GL(G)$ equivariantly.

3.C. *Equivariant extension of a G-map $\rho: |N| \to GL(G)$ over* C$|N|$

Let b be a coset G in the one point coset space G/G. If the G-map ρ': $|N| \cup b \to GL(G)$, defined by $\rho' \,|\, |N| = \rho$ and $\rho'(b) = 1 \in GL(G)$, is extended equivariantly over $b \circ |N| = \mathrm{C}|N|$, then ρ is G-homotopic to a constant map. Moreover this G-homotopy induces a G-homotopy ξ_t $(1 \leqq t \leqq 2)$ from $\xi_1 = \rho \circ \sigma$ to a constant map $\xi_2 = 1 \in GL(G)$. Then combining with the G-homotopy ξ_t $(0 \leqq t \leqq 1)$ we get a G-homotopy ξ_t $(0 \leqq t \leqq 2)$ from the identity map of $GL(G)$ onto $GL(G)$ to a constant map $1 \in GL(G)$ which gives a G-contracting function of $GL(G)$.

The next lemma is a key lemma to prove the extendability of the G-map.

LEMMA (3.4) *Let X be a weakly G-contractible space which is a G-ANR. Let A be a closed G-invariant subcomplex of a G-simplex $\Delta_G^n = Gb_0 \circ \cdots \circ Gb_n$. Then any G-map $\rho: A \cap \partial\Delta_G \to X$ can be extended equivariantly over A. Here the k-cells in the cell complex structure of Δ_G are $g_{\alpha 0} b_{\alpha 0} \circ \cdots \circ g_{\alpha k} b_{\alpha k}$'s with $\{\alpha_0, \cdots, \alpha_k\} \subset \{0, \cdots, n\}$ and we mean by G-ANR [13] a completely regular G-space X which has a G-map extension property that any G-map of closed G-invariant subspace of a normal G-space to X is equivariantly extendable over a G-invariant neighborhood.*

$GL(G)$ is weakly G-contractible by the theorem (2.2) and a G-ANR because it is an open G-invariant subspace of a G-equivariant Banach space which is easily proved to be a G-ANR. Hence we can take $GL(G)$ as X in the above lemma. And for any $\Delta_G^n \in SN$, $C|N| \cap C\Delta_G$ is a closed G-invariant subcomplex of G-simplex $C\Delta_G^n = Gb \circ Gb_0 \circ \cdots \circ Gb_n$. Then any G-map of $C|N| \cap \partial(C\Delta_G^n) = C|N| \cap (C\partial\Delta_G^n \cup \Delta_G^n)$ to $GL(G)$ is extendable over $C|N| \cap C\Delta_G^n$. Applying this to the G-simplexes Δ_G^0's in SN we get the extension of the G-map of $|N| \cap (\cup \Delta_G^0) = |N^0|$ over $C|N^0|$. Let us assume the G-map has been extended over $C|N^k|$ so as to be compatible with the original map as the induction hypothesis, then the G-map of $C|N^k| \cup |N^{k+1}|$ is extendable over $C|N^{k+1}|$. This induction gives an extension of the G-map over $C|N|$. Therefore assuming the lemma (3.4) we get

THEOREM (3.5) *$GL(G)$ is G-contractible.*

REMARK. When G is a finite group the lemma (3.4) is an easy consequence of Bredon's technique. And in this case we do not need the assumption that X is a G-ANR.

§4. Proof of the Lemma (2.4)

We shall reduce the lemma (3.4) to the lemma (4.3). In general the reduction depends on the proposition (4.4) which says that any G-manifold has a G-CW complex structure. But when G is a finite group the lemma (4.3) is only a generalization of the lemma (3.4) and when G is an abelian group we only need that any free G-manifold has a G-CW complex structure. In this section we use the notation Hx for the isotropy subgroup at x instead of H_x to avoid the complexity of multiple subindices.

4.A. *The case when G is an abelian compact Lie group*

Take the barycentric manifold $M = Gb_0 \times \cdots \times Gb_n = \{\sum t_i g_{\alpha_i} b_{\alpha_i} \in \Delta_G^n ;\ t_1 = t_2 = \cdots = t_n\}$ of a G-simplex $\Delta_G^n = Gb_0 \circ \cdots \circ Gb_n = M \times \Delta^n/\sim$. The weak topology of Δ_G with respect to the closed cells induces a discrete topology on M. So when G is a positive dimensional group we need to change the cell complex structure on M.

When G is an abelian group all the isotropy subgroups at points of $M = Gb_0 \times \cdots \times Gb_n$ is the same group $Hb_0 \cap \cdots \cap Hb_n$ and the induced differentiable $G' = G/(Hb_0 \cap \cdots \cap Hb_n)$-action on M is free. Then the orbit space $M/G = M/G'$ is also a differentiable manifold and has a triangulation by the S. S. Cairns' theorem. Now take a slice covering $\{S_\alpha ;\ \alpha \in A\}$ of M. The slice covering induces an open covering $\{\nu(S_\alpha) ;\ \alpha \in A\}$ of M/G where ν stands for the natural projection $\nu : M \to M/G$. Here we should remark that the restriction of ν on S_α is an into homeomorphism because G'-action on M is free. Take a sufficiently fine subdivision of the triangulation so that the diameter of any simplex is smaller than the Lebesgue number of the open covering. Then any given simplex Δ_λ ($\lambda \in \Lambda$) of the subdivision is contained in the image of a slice $\nu(S_\alpha)$, and an index choosing function $\alpha: \Lambda \to \mathrm{A}$ can be defined so as to be $\Delta_\lambda \subset \nu(S_{\alpha(\lambda)})$. We define a closed cell σ_λ (corresponding to Δ_λ) in M by the homeomorphic inverse image $(\nu|S_{\alpha(\lambda)})^{-1}(\Delta_\lambda)$. Then $M = \cup G\mathring{\sigma}_\lambda$ ($\lambda \in \Lambda$) (disjoint union), where $\mathring{\sigma}_\lambda$ stands for the interior of the cell σ_λ.

The revised cell complex structure on M is defined to be $K = \{\sigma = g_\lambda \sigma_\lambda;\ g_\lambda \in G, \lambda \in \Lambda\}$. The main reason that we do not take the alternative possibility $K' = \{\sigma_\lambda;\ \lambda \in \Lambda\}$ is that the cell components of $\partial\sigma_\lambda$ are not contained in K' because even if Δ_λ is lifted by $\nu|S_{\alpha(\lambda)}$ any simplex components of $\partial\sigma_\lambda$ need not be lifted by the same homeomorphism $\nu|S_{\alpha(\lambda)}$. We shall list up the properties of $(|K| = M, K)$.

PROPERTIES (*) $(|K|, K)$ is a pair of a Hausdorff G-space $|K|$ and a collection of cells K which satisfies:

(a) $|K| = \cup \mathring{\sigma}$ ($\sigma \in K$) (disjoint union),

(b) each cell has its (onto) characteristic map $f_\sigma: \Delta^n \to \sigma^n$ with

(b1) $f_\sigma|(\Delta - \partial\Delta)$ is a homeomorphism onto $\mathring{\sigma}$

(b2) $f_\sigma(\partial\Delta) = \partial\sigma^n \subset |K^{n-1}|$ where $|K^{n-1}|$ is the $(n-1)$ skeleton of K, the union of all cells whose dimensionalities do not exceed $(n-1)$.

REMARK TO (a) AND (b). J. H. C. Whitehead defined the cell complex by the conditions (a) and (b) in [16]. Here we should remark that $\partial\sigma$ is not a union of the lower dimensional cells but is only contained in the union of the lower dimensional cells.

(c) $|K|/G$ is a Hausdorff space,

(d) G acts cellularly, that is, $\sigma \in K$ includes $g\sigma \in K$ for any $g \in G$,

(e) all the isotropy subgroups at interior points of a cell σ are the same subgroup $H\sigma$, in particular, a boundary point of σ is fixed by $H\sigma$,

(f) if g is not contained in $H\sigma$, then $g\mathring{\sigma}$ is disjoint from $\mathring{\sigma}$.

The properties (*) induce a cell complex structure on the orbit space $|K|/G$ by $\{\sigma/G : \sigma \in K\}$. And in the case of M the induced cell complex structure on M/G is the subdivision of the original triangulation, in particular, a CW complex structure. Now we can define a G-CW complex even for a general compact Lie group.

DEFINITION (4.1) Let G be a compact Lie group and $|K|$ satisfies the 1st axiom of countability. The pair $(|K|, K)$ with the properties (*) which induce a CW complex structure on the orbit space is defined to be a G-CW complex.

LEMMA (4.2) *A G-CW complex has a property*: (G-W) $|K|$ *has the weak topology with respect to the G-orbits of closed cells in K.*

We omit the proof of this lemma here. (See [11].) But it is evident that the G-CW complex structure on a G-manifold M has this property.

We have defined the G-CW complex. In the rest of this subsection we shall prove the lemma (3.4) assuming that the barycentric manifold M has a G-CW complex structure. The proof of this assumption when G is not an abelian group nor a finite group is deferred to the proposition (4.4) in the next subsection.

Since

$$\Delta_G^n = M\times\Delta^n/\sim = (M\times\mathring{\Delta}^n)\cup(M\times\partial\Delta^n)/\sim$$

(where $\sim$ stands for a factorization), the revised cell complex structure L on Δ_G is defined by $(\sigma\times\Delta^n)/\sim\in L$ if and only if $\sigma\in K$, where K is the revised cell complex structure of M. But here we should define $|L^{-1}|=\partial\Delta_G$ instead of the ordinary condition $|L^{-1}|=\phi$. The space of this type shall be defined to be a relative G-CW complex.

DEFINITION (4.1′) Let $|L|$ be a Hausdorff G-space which satisfies the 1st axiom of countability. And let L be a collection of non-negative dimensional cells in $|L|$ and a closed subspace $|L^{-1}|$ which is considered to be the (-1)-dimensional skeleton. The triple $(|L|, L, |L^{-1}|)$ is defined to be a relative G-CW complex if it satisfies the same properties as a G-CW complex. (In this case the orbit space has the induced relative CW complex structure.) The only difference is that $|L^{-1}|\neq\phi$.

LEMMA (4.3) *Let X be a weakly G-contractible space. Let $(|L|, L, |L^{-1}|)$ be a relative G-CW complex. Then any G-map $\rho:|L^{-1}|\to X$ can be extended equivariantly over $|L|$.*

PROOF: The proof is done inductively. Assume ρ is already defined on $|L^n|$ $(-1\leqq n)$. Take an $(n+1)$ cell σ^{n+1}. Then $\partial\sigma^{n+1}\subset|L^n|$ and $\rho(\partial\sigma^{n+1})\subset X^{H\sigma}$. Because $\pi_n(X^{H\sigma})$ vanishes, we may extend ρ over σ in such a way $\rho(\sigma)\subset X^{H\sigma}$. Define, for $g\in G$ and $x\in\mathring{\sigma}$, $\rho(gx)=g\rho(x)\in gX^{H\sigma}=X^{Hg\sigma}$. If $g'x'=gx$, then $x'=x$ and $g'=gh$ for some $h\in H\sigma$ so that $g'\rho(x')=g\rho(x)$ (since (f) and $\rho(x)\in X^{H\sigma}$), which shows this definition is valid. And this extension is continuous with respect to the G-weak topology because of the lemma (4.2). The extension over $|L^{n+1}|$ is completed by taking an $(n+1)$ cell from each G-orbit of the $(n+1)$-cells and following the procedure above. Even if the dimension of L is not finite, the induction is valid because $\partial\sigma$ intersects with the only finite G-orbits of cells by the closure finiteness. q.e.d.

PROOF OF THE LEMMA (3.4) UNDER THE ASSUMPTION THAT THE BARYCENTRIC MANIFOLD $M=Gb_0\times\cdots\times Gb_n$ OF THE G-SIMPLEX $\Delta_G^n=Gb_0\circ\cdots\circ Gb_n=M\times\Delta^n/\sim$ HAS A G-CW COMPLEX STRUCTURE WHICH INDUCES A TRIANGULATION ON THE ORBIT SPACE:

Because X is a G-ANR, we may get a G-invariant neighbourhood U of $A\cap\partial\Delta_G$ in $\partial\Delta_G$ on which the G-map is already extended. Let δ be a positive number satisfying $\mathrm{dist}((A\cap\partial\Delta_G), \partial\Delta_G-U)>2\delta$. Let K be a G-CW complex structure on M and take a sufficiently fine subdivision of it so that the diameter of any cell is less than δ. Then the smallest subcomplex K' of K, including the G-orbits of the cells which intersect with A is also a G-CW complex and satisfies $|K'|\supset A\cap M$ and $(|K'|\times\Delta/\sim)\cap\partial\Delta_G\subset U$. Let L' be a subcomplex of L which is defined by $|L'^{-1}|=((|K'|\times\Delta/\sim)\cap\partial\Delta_G)\cup(A\cap\partial\Delta_G)$ and $(\sigma\times\Delta^n)/\sim\in L'$ if and only if $\sigma\in K'$. Then $(|L'|, L', |L'^{-1}|)$ is a relative G-CW complex and we may assume that the G-map is already defined on $|L'^{-1}|$. By the lemma (4.3) we may extend the G-map over $|L'|$. Since A is a subcomplex in the original cell complex structure of Δ_G,

$$A=(A\cap\partial\Delta_G)\cup(A\cap M)\times\Delta^n/\sim$$

and then A is contained in $|L'|$. Therefore restricting this map on A we get a desired extension over A. q.e.d.

4. B. *The case when G is a general compact Lie group*

As we have remarked in §4. A, we finish up the proof of the main theorem with the next proposition.

PROPOSITION (4.4) *Let G be a compact Lie group. Any closed differentiable G-manifold M has a G-CW complex structure which induces a triangulation on the orbit space.*

We use the following theorem due to C. T. Yang. Let $(H)=\{gHg^{-1};\ g\in G\}$ and $M_{(H)}=\{x\in M;\ Hx\in(H)\}$ where H is a closed subgroup of G. When $M_{(H)}\neq\phi$, (H) is called an G-orbit type of M. Then M decomposes into a disjoint union $M=\cup M_{(H)}$ ((H) is an G-orbit type of M).

THEOREM (Yang) [17]. *M/G has a triangulation so that the interior of each simplex is contained in $M_{(H)}/G$ for some (H).*

PROOF OF THE PROPOSITION (4.4): When M has a unique orbit type (H), we can lift every simplex in a sufficiently fine subdivision of the triangulation of M/G into M by a slice covering as we did in §4.A for a manifold with a free action. But we have another lifting of the unsubdivided triangulation. Let Δ^n be a simplex in the triangulation of M/G. Because Δ is contractible, $\nu^{-1}(\Delta)$ is isomorphic to $\Delta\times G/H$ as fibre bundles with G/H as a fibre. Thus we may get a continuous section s of ν over Δ so that all the isotropy subgroups at interior points of $s(\Delta)$ is H. Then $s(\Delta)$ is a desired lifting of Δ.

In the general case we shall lift a barycentric subdivision of the preferred triangulation of the orbit space. Let Δ^n be a simplex in the triangulation. We take a simplex Δ'^n in a barycentric subdivision of Δ^n and consider to lift Δ'^n into $\nu^{-1}(\Delta'^n)$. We may assume that each vertex v_i of Δ'^n is the barycenter of an $(n-i)$-dimensional simplex Δ^{n-i} in Δ^n and $\Delta^n \supset \Delta^{n-1} \supset \cdots \supset \Delta^0$. Then we get $(**): (H_0) \leqq (H_1) \leqq \cdots \leqq (H_n)$ where (H_i) denotes the *unique* orbit type of the interior of Δ^{n-i} which contains v_i. (Uniqueness comes from the theorem of Yang.)

Following to R. Palais [13], the orbit space of a G-space is a $\Sigma(G)$-space. (We do not need the definition of general $\Sigma(G)$-spaces and need only that of the isomorphism between orbit spaces as $\Sigma(G)$-spaces here.) An isomorphism between orbit spaces as $\Sigma(G)$-spaces is defined to be a homeomorphism which preserves the orbit type of each point. The lifting method decomposed into two steps and the following lemma due to R. Palais is very useful.

LEMMA (Palais) [Theorem (2.5.2) loc. cit.]. *Let G be a compact Lie group. Let the orbit space of a locally compact, Hausdorff, second countable G-space X be isomorphic with $Z \times I$ as $\Sigma(G)$-spaces, where Z is the orbit spaces of a G-space and I is a closed interval. Then there exists a pair of G-space Y whose orbit space is isomorphic with Z as $\Sigma(G)$-spaces and a G-homeomorphism $f: Y \times I \to X$ with $\nu_Y \times \mathrm{id} = \nu_X \circ f$.*

1st step: Take a point x_n over v_n and a slice S_n at x_n in the G-space $\nu^{-1}(\Delta'^n)$. Because $\nu(S_n)$ is open in Δ'^n, $\nu(S_n)$ contains a small similar simplex $v_n \circ \Delta_t'^{n-1}$ for some $t \neq 1$ where $\Delta_t'^{n-1} = \{\sum t_i v_i;\ \sum t_i = 1,\ t_i \geqq 0,\ t_n = t\} \subset \Delta'^n$. Since the condition $(**)$ shows that the orbit type of a point $\sum t_i v_i$ of Δ'^n is (H_j) where $j = \min\{i;\ t_i \neq 0\}$, $\cup \Delta_{t'}'^{n-1}$ $(0 \leqq t' \leqq t)$ is isomorphic with $\Delta_t'^{n-1} \times I$ as $\Sigma(G)$-spaces. Then by the lemma of Palais, we may get a G-homeomorphism,

$$f: \nu^{-1}(\Delta_t'^{n-1}) \times I \to \nu^{-1}(\cup \Delta_{t'}'^{n-1}\ (0 \leqq t' \leqq t)) .$$

Using this we define a revised maximal slice S_n' at x_n by

$$S_n' = (S_n \cap \nu^{-1}(v_n \circ \Delta_t'^{n-1})) \cup f(((S_n \cap \nu^{-1}(\Delta_t'^{n-1})) \times I) .$$

Here the essential point is $\nu(S_n') = \Delta'^n$, and in this meaning the slice S_n' is called maximal. This is well-defined because f may be taken to be identity on $\nu^{-1}(\Delta_t'^{n-1})$.

REMARK (n) We may take Hx_n as H_n. Then the slice S_n' is itself an H_n-space. The restriction of the natural projection ν on S_n' is denoted by ν_n. The orbit space $\nu_n(S_n') = S_n'/H_n$ coincides with Δ'^n. The H_n-orbit type of the vertex $v_{i.n} = v_i$ $(i \leqq n)$ is $(H_{i.n} = g_i H_i g_i^{-1})$ for some $g_i \in G$. Moreover the H_n-orbit type of a point $\sum t_i v_{i.n}$ $(i \leqq n)$ is $(H_{j.n})$ where $j = \min\{i;\ t_i \neq 0\}$.

Consider $\Delta'^{n-1} = \Delta_0'^{n-1}$ to be the orbit space of the H_n-space $S_n' \cap \nu^{-1}(\Delta'^{n-1})$. Take a point x_{n-1} of S_n' over v_{n-1} and a slice S_{n-1} at x_{n-1} in this H_n-space. Then the remark (n) above makes us able to get a revised maximal slice S_{n-1}' at

x_{n-1} in the H_n-space $S'_n \cap \nu^{-1}(\Delta'^{n-1})$ by the procedure previous to the remark (n).

Rewriting $(n-1)$ instead of n in and below the remark (n) we may get a revised maximal slice S'_{n-2}. Repeating this procedure we get a sequence of slices,

$$S'_n \supset S'_{n-1} \supset \cdots \supset S'_1 \supset S'_0$$

where S'_k is a slice at x_k over v_k in the $H_{k+1} = Hx_{k+1}$-space $S'_{k+1} \cap \nu^{-1}(\Delta'^k)$ and the orbit space of H_k-space S'_k is Δ'^k. Moreover the H_k-orbit type of a point $\sum t_i v_{i,k}$ of Δ'^k is $(H_{j,k} = Hx_j)$ where $j = \min\{i : t_i \neq 0\}$.

2nd step: Since S'_0 consists of one point, a section s'_0 on Δ'^0 into S'_0 is already given. Assume as an induction hypothesis that we have already got a section s'_k on Δ'^k into S'_k $(k \geq 0)$ so that all the isotropy subgroups at interior points of $s'_k(\Delta'^k)$ is H_0. Regarding s'_k as a section into S'_{k+1}, we may get an extension s_{k+1} over Δ'^k_t $(0 \leq t < 1)$ by the use of the Palais' lemma for H_{k+1}-orbit spaces so that all the isotropy subgroups at interior points of s_{k+1} $(\Delta'^{k+1} - v_{k+1})$ is H_0. Then since $S'_{k+1} \cap \nu^{-1}(v_{k+1})$ consists of one point x_{k+1}, we may define a continuous section s'_{k+1} on Δ'^{k+1} into S'_{k+1} by $s'_{k+1}(v_{k+1}) = x_{k+1}$ and $s'_{k+1}|(\Delta'^{k+1} - v_{k+1}) = s_{k+1}$. This completes an inducting step. Then we may get a section s'_n on Δ'^n into S'_n (hence into X) so that all the isotropy subgroups at interior points of $s'_n(\Delta'^n)$ is H_0. This shows $s'_n(\Delta'^n)$ is a desired lifting of Δ'^n.

Lifting every simplex in the barycentric subdivision by the procedure above we get a G-CW complex structure on M which induces the original barycentric subdivision of the preferred triangulation on the orbit space. q.e.d.

COROLLARY (4.5) *Let G be a compact Lie group. Let X be a locally compact, Hausdorff, second countable G-space whose orbit space has a locally finite triangulation so that the interior of each simplex is contained in $X_{(H)}/G$ for some orbit type (H). Then X has a G-CW complex structure which induces a barycentric subdivision of the original triangulation on the orbit space.*

In the proof of the proposition (4.4) the differentiability is used only to deduce a preferred triangulation on the orbit space by the theorem of Yang. In particular we can omit the closedness condition on M in the proposition (4.4).

§5. Concluding remarks

5.A. Let l^2 be a separable real (quaternionic) Hilbert space instead of a separable complex Hilbert space. Then we also get the G-contractibility of $GL(G) = GL(\mathrm{L}^2(G, l^2))$ using real (quaternionic respectively) representations. There are two points which should be remarked.

1st point: The statements that every irreducible representation is contained in $L^2(G, l^2)$ with countable multiplicity and that any representation of a closed subgroup is contained in a representation of G can be reduced to the complex case. (For example see [14].)

2nd point: If V is an irreducible real (quaternionic) representation of G, the ring $\mathrm{Hom}_G(V, V)$ of G-equivariant real (quaternionic) endomorphism of V is isomorphic with the field of real, complex or quaternion numbers. This is because it is a skew-field and an algebra over the real number field. Then we may use the Kuiper's corresponding results and prove the weakly G-contractibility of $GL(G)$.

5. B. Let X be a 'real' space which means a C_2-space, where C_2 stands for the cyclic group of order 2. Define the C_2-action on $\mathfrak{F}$ by $T^j(a)=jT(j^{-1}a)$ where $j=j^{-1}$ stands for the conjugation. Then we also get a similar exact sequence in the theorem (1.2), changing $K_G(X)$ by $KR(X)$ which is the 'real' K-group in the sense of Atiyah [3]. And C_2-stationary subgroup of GL is the general linear group on a separable real Hilbert space because an operator satisfying $T^j=T$ is an operator with real coefficients. Hence GL is weakly C_2-contractible and we get

MAIN THEOREM II. *For any compact 'real' space X, there is a canonical isomorphism*

$$[X, \mathfrak{F}]_{C_2} \xrightarrow{\cong} KR(X).$$

REMARK. The C_2-stationary subspace of $\mathfrak{F}$ corresponds to the space of Fredholm operators on a separable real Hilbert space and hence this theorem gives also a known representation, $[X, \mathfrak{F}_{\mathrm{real}}] \xrightarrow{\cong} KO(X)$ for any compact space X [9].

Let X be a 'real' G-space which means a $C_2\times G$-space. Since the G-action on $\mathfrak{F}$ is commutable with C_2-action, $\mathfrak{F}$ and GL have $C_2\times G$-action. Because C_2-stationary subspace of GL is the general linear group on separable real Hilbert space it is weakly G-contractible by the result of §5. A. Hence GL is weakly $C_2\times G$-contractible. Therefore constructing the similar exact sequence in the theorem (1.2) we get

MAIN THEOREM II′. *For any compact 'real' G-space X, there is a canonical isomorphism*

$$[X, \mathfrak{F}]_{C_2\times G} \xrightarrow{\cong} KR_G(X)\,.$$

5. C. Let X be a 'real' space. In this subsection we use the notation D_2 for the cyclic group of order 2 acting on X to distinguish from that of the previous subsection. A complex vector bundle over a 'real' space, $E\to X$ is said to be a quaternionic vector bundle if it has a quaternionic structure $j: E\to E$ with $j^2=-1$ and compatible with the conjugations of the coefficient field and the base space. Define $KQ(X)$ to be the Grothendieck group of quaternionic vector bundles over X. If we regard a separable quaternionic Hilbert space as a separable complex Hilbert space, then it has a quaternionic structure j with $j^2=-1$. Define the D_2-action on $\mathfrak{F}$ by $T^j(a)=jT(j^{-1}a)$. Then D_2-stationary subspace of GL is the general linear group on a separable quaternionic Hilbert space and hence weakly contractible. Therefore constructing the similar exact sequence in the theorem (1.2) we get

MAIN THEOREM III. *For any compact 'real' space X, there is a canonical isomorphism*

$$[X, \mathfrak{F}]_{D_2} \xrightarrow{\simeq} KQ(X) .$$

REMARK. The D_2-stationary subspace of $\mathfrak{F}$ corresponds to the space of Fredholm operators on a separable quaternionic Hilbert space and hence this theorem gives also a known representation, $[X, \mathfrak{F}_{\mathrm{quat}}] \xrightarrow{\simeq} KSp(X)$ for any compact space X [9].

We also define $KQ_G(X)$ for a 'real' G-space X. Then by the result of §5. A we get

MAIN THEOREM III'. *For any compact 'real' G-space X, there is a canonical isomorphism*

$$[X, \mathfrak{F}]_{D_2\times G} \xrightarrow{\simeq} KQ_G(X) .$$

REMARK. The j-structure is a real structure when $j^2=1$ and is a quaternionic structure when $j^2=-1$. Thus using the j-structure we may argue both the real and quaternionic cases in a parallel way.

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(Received October 16, 1970)

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