Input for derived algebraic geometry: equivariant multiplicative infinite loop space theory

Peter May

Joint work with
Bertrand Guillou, Mona Merling, and Angelica Osorno

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Nick Gurski, and Mike Shulman

2-category theory ROCKS

especially

Codescent objects and
the formal theory of 2-monads

May 9, 2015: Midwest topology seminar talk:

http://www.math.uchicago.edu/ may/TALKS/Chicago2015.pdf

Most slides: Equivariant May and Segal machines on G-spaces.

All of that is fine and I will not repeat much of it here.

Today's talk is also online:

http://www.math.uchicago.edu/ may/TALKS/Chicago2016.pdf

The subject is infinite loop G-category theory.

OUTLINE

Brief summary: $\mathbb{S}_G$ from $\mathscr{F}$-G-spaces to orthogonal G-spectra

A triviality: B from $\mathscr{F}$-G-categories to $\mathscr{F}$-G-spaces

The rest: from sensible categorical input to $\mathscr{F}$-G-categories

(1) $\mathscr{F}$-categories in **Cat**($\mathscr{V}$) for general $\mathscr{V}$, such as G-spaces
(2) The additive input: symmetric monoidal $\mathscr{V}$-categories
(3) From additive input to $\mathscr{F}$-algebras ($\mathscr{F}$-$\mathscr{V}$-categories)
(4) The multiplicative input: the relevant multicategories
(5) From multiplicative input to **Mult**($\mathscr{F}$-**Alg**) - Start
(6) The formal theory of 2-monads
(7) Codescent objects (2-categorical coequalizers)
(8) From multiplicative input to **Mult**($\mathscr{F}$-**Alg**) - Finish
(9) Controlling the equivariant homotopy theory
(10) Input to the multiplicative input

From $\mathscr{F}$-G-spaces to orthogonal G-spectra

$\mathscr{F}$ is the category of finite based sets $\mathbf{n} = \{0, 1, \cdots, n\}$, basepoint 0. (Alias Γ^{op})

An $\mathscr{F}$-G-space is a functor $X\colon \mathcal{F} \longrightarrow \mathrm{G}-\mathbf{Spaces}$, notation $\mathbf{n} \mapsto X_n$; We assume X is reduced: $X_0 = *$.

$\phi\colon \mathbf{n} \to \mathbf{1}$, $\phi(i) = 1$, $1 \le i \le n$: induces "product" $\phi\colon X_n \longrightarrow X_1$

Segal maps $\delta\colon X_n \longrightarrow X_1^n$; coordinates $\delta_j\colon \mathbf{n} \to \mathbf{1}$, $\delta_j(i) = \delta_{i,j}$.

X is special if $\delta\colon X_n^\Lambda \longrightarrow (X_1^n)^\Lambda$ is a homotopy equivalence for all $\Lambda \subset G \times \Sigma_n$ such that $\Lambda \cap \Sigma_n = \{e\}$. (e.g. $\Lambda = H \subset G$.)

$$X_1^n \xleftarrow[\simeq]{\delta} X_n \xrightarrow{\phi} X_1$$

Theorem (M, Merling, Osorno)

There is a lax symmetric monoidal functor $\mathbb{S}_G$ from $\mathscr{F}$-G-spaces to orthogonal Ω-G-spectra. If X is special, then $\Omega^\infty \mathbb{S}_G X$ is an equivariant group completion of X_1.

Group Completion: group completion on H-fixed points, $H \subset G$.

From (topological) $\mathscr{F}$-G-categories to $\mathscr{F}$-G-spaces

Topological G-category: Object and morphism G-spaces such that Source, Target, Identity, and Composition are maps of G-spaces.

Notation: $G\mathscr{U} = G$-spaces; $\mathbf{Cat(G\mathscr{U})}$ = topological G-categories.

An $\mathscr{F}$-G-category is a functor $\mathscr{X} : \mathscr{F} \longrightarrow \mathbf{Cat(G\mathscr{U})}$.

Special is defined just as for $\mathscr{F}$-G-spaces, via $(-)^\Lambda$.

Theorem (easy)

The classifying space functor B from topological $\mathscr{F}$-G-categories to $\mathscr{F}$-G-spaces is symmetric monoidal, and it takes special $\mathscr{F}$-G-categories to special $\mathscr{F}$-G-spaces.

Generalize: do equivariant theory without working equivariantly.

Separate formal arguments from context specific arguments

$\mathscr{V}$ any bicomplete closed symmetric monoidal category, not just the case $\mathscr{V} = G\mathscr{U}$ of immediate interest.

For derived algebraic geometry, maybe Voevodsky's motivic spaces.

Cat$(\mathscr{V})$ = categories internal to $\mathscr{V}$: object and morphism objects in $\mathscr{V}$; Source, Target, Identity, and Composition maps in $\mathscr{V}$.

Notation $\mathscr{F}$-**Alg** $\equiv$ **Cat**$(\mathscr{V})^{\mathscr{F}}$

This is a 2-category: $\mathscr{V}$-functors $\mathscr{X}\colon \mathscr{F} \longrightarrow \mathbf{Cat}(\mathscr{V})$, $\mathscr{V}$-natural transformations, $\mathscr{V}$-modifications are 0-cells, 1-cells, and 2-cells.

It is symmetric monoidal via Day convolution (left Kan extension)

$$\begin{array}{ccc} \mathscr{F} \times \mathscr{F} & \xrightarrow{\mathscr{X}\times\mathscr{Y}} & \mathbf{Cat}(\mathscr{V}) \\ {\scriptstyle \wedge_{\mathscr{F}}}\big\downarrow & \nearrow_{\mathscr{X}\otimes\mathscr{Y}} & \\ \mathscr{F} & & \end{array}$$

Let $G\mathscr{S}$ = orthogonal G-spectra, symmetric monoidal under $\wedge$.

$$\mathbb{S}_G \circ B\colon \mathbf{Cat}(\mathbf{G}\mathscr{U})^{\mathscr{F}} \longrightarrow \mathbf{G}\mathscr{S}$$

is lax symmetric monoidal.

Goal: categorical machine with additive and multiplicative input (for any $\mathscr{V}$) and additive and multiplicative output in $\mathscr{F}$-**Alg**.

THE ADDITIVE INPUT

Permutativity Operad $\mathscr{P} = \{\mathscr{E}\Sigma_j\}$ in **Cat**.

$\mathscr{E}$ is the chaotic categorification functor from Sets to contractible categories, left adjoint to the object functor.

Permutative categories $\mathscr{A}$: action of $\mathscr{P}$

given by functors $\mathscr{P}(k) \times \mathscr{A}^k \longrightarrow \mathscr{A}$.

Symmetric monoidal categories: pseudoaction of $\mathscr{P}$

given by pseudofunctors $\mathscr{P}(k) \times \mathscr{A}^k \longrightarrow \mathscr{A}$.

"pseudo" means "up to invertible 2-cells", not strict structure.

(Corner-Gurski define operadic pseudoactions carefully)

Permutativity G-Operad $\mathscr{P}_G = \{\mathbf{Cat}(\mathscr{E}\mathbf{G}, \mathscr{E}\mathbf{\Sigma_j})\}$ in $G\mathbf{Cat}$

$\mathscr{G} = \mathbf{Cat}(\mathscr{E}\mathbf{G}, -)$ is the G-ification functor: $\mathbf{Cat} \longrightarrow \mathbf{G}\text{-}\mathbf{Cat}$.

$\mathscr{G}(-)^G$ is Thomason's homotopy fixed point functor.

permutative G-categories $\mathscr{A}$: action of $\mathscr{P}_G$.

Symmetric monoidal G-categories: pseudoaction of $\mathscr{P}_G$.

"Unbiased" structure: defined using all $\mathscr{A}^k$, not just the first few.

Operadic formulation is vital:

no "biased" definitions are known equivariantly.

(Sick Sic: not the same as G-symmetric monoidal category!)

Processing the additive input

$\mathscr{P}_G$-**PsAlg**: $\mathscr{P}_G$-pseudoalgebras and pseudomorphisms.

$\mathscr{D} = \mathscr{D}(\mathscr{P}_G)$: Category of operators generated by $\mathscr{P}_G$

$$\Pi \xrightarrow{\ \iota\ } \mathscr{D} \xrightarrow{\ \xi\ } \mathscr{F}$$

$\Pi \subset \mathscr{F}$: permutations, projections, injections $|\phi^{-1}(j)| \leq 1$ if $j \geq 1$.

$$\mathscr{D}(\mathbf{m}, \mathbf{n}) = \coprod_{\phi\colon \mathbf{m} \longrightarrow \mathbf{n}} \prod_{j=1}^{n} \mathscr{P}_G(|\phi^{-1}(j)|)$$

$\mathscr{D}$-**PsAlg**: $\mathscr{D}$-pseudoalgebras and pseudomorphisms.

$\mathscr{D}$-**AlgPs**: $\mathscr{D}$-algebras (functors) and pseudomorphisms.

$\mathscr{D}$-**AlgSt**: $\mathscr{D}$-algebras and morphisms (transformations)

$$\begin{array}{ccccc} & & \mathscr{D}\text{-}\mathbf{AlgSt} & & \\ & \overset{\mathbb{S}t}{\nearrow} & & \overset{\xi_*}{\searrow} & \\ \mathscr{P}_G\text{-}\mathbf{PsAlg} \xrightarrow{\mathbb{R}} \mathscr{D}\text{-}\mathbf{PsAlg} & & \xrightarrow{\quad \xi_\# \quad} & & \mathscr{F}\text{-}\mathbf{Alg}. \end{array}$$

$\mathbb{R}$: $(\mathbb{R}X)(n) = X^n$ (right adjoint to $\mathbb{L}$, $\mathbb{L}(\mathscr{Y}) = \mathscr{Y}(1)$)

$\mathbb{S}t$: $\mathbb{S}t =$ strictification (Power-Lack) (left adjoint to inclusion $\mathbb{J}$)

ξ_*: $\xi_*(\mathscr{Y}) = \mathscr{F} \otimes_{\mathscr{D}} \mathscr{Y}$ (left adjoint to pull back of action ξ^*)

(I'll come back to the triangle after describing multiplicative input.)

Multicategories = operads with many objects = colored operads
Understood to be symmetric.

For a symmetric monoidal category $(\mathscr{C}, \otimes)$, the multicategory $\mathbf{Mult}(\mathscr{C})$ has k-morphisms the maps $X_1 \otimes \cdots \otimes X_k \longrightarrow Y$ in $\mathscr{C}$. Since $\mathbb{S}_G \circ B$ is lax symmetric monoidal, it gives a multifunctor

$$\mathbb{S}_G \circ B \colon \mathbf{Mult}(\mathbf{Cat}(\mathbf{G}\mathscr{U})^{\mathscr{F}}) \longrightarrow \mathbf{Mult}(\mathbf{G}\mathscr{S}).$$

For any $\mathscr{V}$, the target of our categorical machine is $\mathbf{Mult}(\mathscr{F}\textbf{-Alg})$.

Can form $\mathbf{Mult}(\mathscr{C})$ for some categories that are NOT symmetric monoidal. Same formal structure, data complicated by 2-cells:

$\mathbf{Mult}(\mathscr{O}) \equiv \mathbf{Mult}(\mathscr{O}\textbf{-PsAlg})$ $\qquad$ $\mathbf{Mult}(\mathscr{D}) \equiv \mathbf{Mult}(\mathscr{D}\textbf{-PsAlg})$

for suitable operads $\mathscr{O}$ and categories of operators $\mathscr{D} = \mathscr{D}(\mathscr{O})$.

THE MULTIPLICATIVE INPUT

Mult$(\mathscr{O})$, $\mathscr{O}$ a "pseudocommutative" operad such as $\mathscr{P}$ or $\mathscr{P}_G$

k-morphisms $(F, \delta_i)\colon (\mathscr{A}_1, \cdots, \mathscr{A}_k; \mathscr{B})$ between $\mathscr{O}$-pseudoalgebras:

$$\text{1-cell} \quad F\colon \mathscr{A}_1 \times \cdots \times \mathscr{A}_k \longrightarrow \mathscr{B}$$

Invertible distributivity 2-cells $\delta_i = \{\delta_i(n)\}$, $1 \le i \le k$:

$$\begin{array}{ccc}
\mathscr{O}(n) \times (\mathscr{A}_1 \times \cdots \times \mathscr{A}_k)^n & \xrightarrow{\ \mathrm{id} \times F^n\ } & \mathscr{O}(n) \times \mathscr{B}^n \\
\Big\uparrow{\scriptstyle t_i} & & \\
\mathscr{A}_1 \times \cdots \times \mathscr{O}(n) \times \mathscr{A}_i^n \times \cdots \times \mathscr{A}_k & \Downarrow \delta_i(n) & \Big\downarrow{\scriptstyle \theta(n)} \\
\Big\downarrow{\scriptstyle \mathrm{id} \times \theta(n) \times \mathrm{id}} & & \\
\mathscr{A}_1 \times \cdots \times \mathscr{A}_k & \xrightarrow[\ F\ ]{} & \mathscr{B}
\end{array}$$

t_i from $\Delta\colon \mathscr{A}_j \longrightarrow \mathscr{A}_j^n$, $j \neq i$, and transpositions.

Complicated looking but straightforward coherence data

Mult($\mathscr{D}$), $\mathscr{D}$ a “pseudocommutative” 2-category of operators

k-morphisms $(F,\delta)\colon(\mathscr{X}_1,\cdots,\mathscr{X}_k;\mathscr{Y})$ between $\mathscr{D}$-pseudoalgebras:

$$\text{1-cells}\quad F\colon \mathscr{X}_1(n_1)\times\cdots\times\mathscr{X}_k(n_k)\longrightarrow\mathscr{Y}(n_1\cdots n_k)$$

Invertible distributivity 2-cells δ:

$$\begin{array}{ccc}
\prod_j \mathscr{D}(\mathbf{m}_j,\mathbf{n}_j)\times\prod_j \mathscr{X}_j(m_j) & \xrightarrow{\ \mathrm{id}\times F\ } & \prod_j \mathscr{D}(\mathbf{m}_j,\mathbf{n}_j)\times\mathscr{Y}(\underline{m}) \\
\downarrow{\scriptstyle t} & & \downarrow{\scriptstyle \wedge_{\mathscr{D}}\times\mathrm{id}} \\
\prod_j \mathscr{D}(\mathbf{m}_j,\mathbf{n}_j)\times\mathscr{X}_j(m_j) & \Downarrow\delta & \mathscr{D}(\underline{\mathbf{m}},\underline{\mathbf{n}})\times\mathscr{Y}(\underline{m}) \\
\downarrow{\scriptstyle \prod_j\theta} & & \downarrow{\scriptstyle \theta} \\
\prod_j \mathscr{X}_j(n_j) & \xrightarrow[\ F\ ]{} & \mathscr{Y}(\underline{n}).
\end{array}$$

Here $\underline{m}=m_1\cdots m_k$, $\underline{n}=n_1\cdots n_k$, and $1\leq j\leq k$.

Complicated looking but straightforward coherence data

Processing the multiplicative input

Theorem

If $\mathscr{O}$ is a pseudocommutative operad, then $\mathscr{D} = \mathscr{D}(\mathscr{O})$ is a pseudocommutative category of operators and $\mathbb{R}$ extends to a multifunctor $\textbf{Mult}(\mathscr{O}) \longrightarrow \textbf{Mult}(\mathscr{D})$.

Proof.

Horrible but straightforward checks of coherence. Essential point is that the δ_i in the operadic context work iteratively to construct the single δ in the category of operators context. □

So far this is as in May, 2015, Midwest. The rest is all changed!

(Digression: Frank Adams wrote out the jokes in his talks.)

I once asked Frank Adams for a copy of some work in progress, and his delightful response went as follows:

It is perfectly true that when I last wrote to you I had drafts of sections one and three which I was willing to let people see.

Today I still have the same pieces of paper, but like Mr. Brown, I discern the Capability of Improvement.[1]

The chief rogue (a definition, needless to say) has been marched off to the condemned cell, where he lodges till I determine whether his rival is likely to serve the crown more usefully; he took with him a handful of perfectly valid theorems (humming sadly "we shall not all die, but we shall all be changed")

[1]Refers to Capability Brown, a famous 18th century landscape architect

$$\begin{array}{ccccc} & & \text{Condemned} & & \\ & \overset{\mathbb{S}t}{\nearrow} & & \overset{\xi_*}{\searrow} & \\ \mathbf{Mult}(\mathscr{P}_G) \xrightarrow{\mathbb{R}} \mathbf{Mult}(\mathscr{D}) & & \xrightarrow{\qquad\quad \xi_\# \quad\qquad} & & \mathbf{Mult}(\mathscr{F}\text{-}\mathbf{Alg}). \end{array}$$

The formal theory of 2-monads

Translate problem to monadic avatar:

$$\mathbf{Mult}(\mathscr{D}) \cong \mathbf{Mult}(\mathbb{D}) \xrightarrow{\xi_\#} \mathbf{Mult}(\mathbb{F}\text{-}\mathbf{Alg}) \cong \mathbf{Mult}(\mathscr{F}\text{-}\mathbf{Alg}).$$

$\mathbb{D}$ and $\mathbb{F}$ are 2-monads in the 2-category $\mathscr{K} \equiv \mathbf{Cat}(\mathscr{V})^{\Pi}$.

$$(\mathbb{D}\mathscr{Y})_n = \mathscr{D}(-, \mathbf{n}) \otimes_{\Pi} \mathscr{Y}.$$

(As in May-Thomason on the level of spaces.) Danger?

Colimits don't commute with B. We don't give a damn!

A graded monoid of monads

Monads $\mathbb{D}_k$ on $\mathbf{Cat}(\mathscr{V})^{\mathbf{\Pi^k}}$, $\mathbb{D}_0 = *$,

$$\mathbb{D}_k \mathscr{W} = \mathscr{D}^k \otimes_{\Pi^k} \mathscr{W}$$

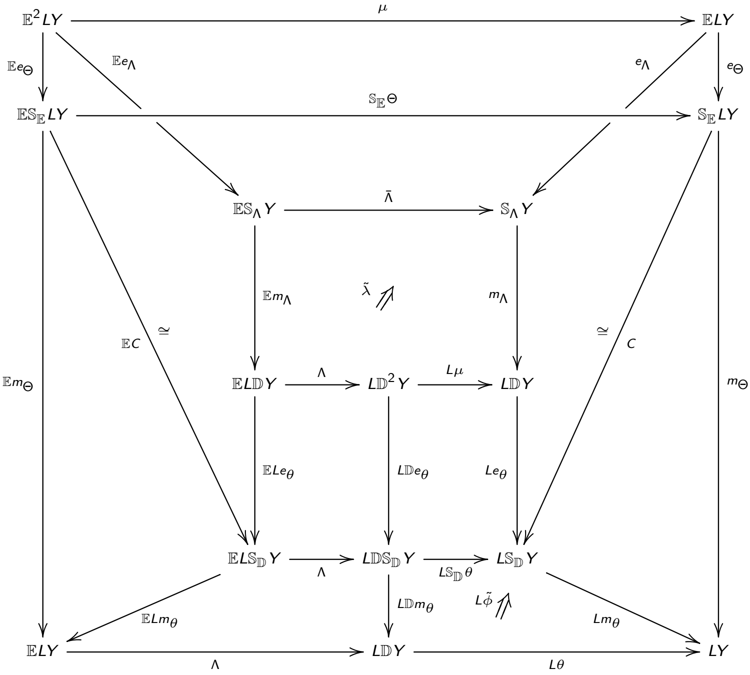
Suitably associative and commutative system of pairings

$$\mathbb{D}_j \times \mathbb{D}_k \longrightarrow \mathbb{D}_{j+k}.$$

Have $\wedge^k_\Pi \colon \Pi^k \longrightarrow \Pi$; $L_k \mathscr{Y} = \mathscr{Y} \circ \wedge^k_\Pi$ for $\mathscr{Y} \colon \Pi \longrightarrow \mathbf{Cat}(\mathscr{V})$.

If $\mathscr{X}_i$, $1 \leq i \leq k$ and $\mathscr{Y}$ are $\mathbb{D}$-pseudoalgebras, then $\mathscr{X}_1 \times \cdots \times \mathscr{X}_k$ and $L_k \mathscr{Y}$ are $\mathbb{D}_k$-pseudoalgebras, and a k-morphism $(\mathscr{X}_1, \cdots, \mathscr{X}_k; \mathscr{Y})$ in $\mathbf{Mult}(\mathbb{D})$ is exactly a pseudomorphism of $\mathbb{D}_k$-pseudoalgebras

$$\mathscr{X}_1 \times \cdots \times \mathscr{X}_k \longrightarrow L_k \mathscr{Y}. \qquad (1)$$



$\mathbb{E}^2LY$
μ
$\mathbb{E}LY$
$\mathbb{E}e_\Theta$
$\mathbb{E}e_\Lambda$
e_Λ
e_Θ
$\mathbb{E}\mathbb{S}_\mathbb{E}LY$
$\mathbb{S}_\mathbb{E}\Theta$
$\mathbb{S}_\mathbb{E}LY$
$\mathbb{E}\mathbb{S}_\Lambda Y$
$\bar{\Lambda}$
$\mathbb{S}_\Lambda Y$
$\mathbb{E}m_\Lambda$
$\tilde{\lambda}$
m_Λ
$\mathbb{E}C$
$\cong$
$\cong$
C
$\mathbb{E}m_\Theta$
$\mathbb{E}L\mathbb{D}Y$
Λ
$L\mathbb{D}^2Y$
$L\mu$
$L\mathbb{D}Y$
m_Θ
$\mathbb{E}Le_\theta$
$L\mathbb{D}e_\theta$
Le_θ
$\mathbb{E}L\mathbb{S}_\mathbb{D}Y$
Λ
$L\mathbb{D}\mathbb{S}_\mathbb{D}Y$
$L\mathbb{S}_\mathbb{D}\theta$
$L\mathbb{S}_\mathbb{D}Y$
$L\mathbb{D}m_\theta$
$L\tilde{\phi}$
$\mathbb{E}Lm_\theta$
Lm_θ
$\mathbb{E}LY$
Λ
$L\mathbb{D}Y$
$L\theta$
LY

The previous slide, a perfectly valid diagram, was smuggled out of the condemned cell. Ignore it. We head towards ξ_*, $\mathbb{S}t$, and $\xi_\#$.

Coequalizer and reflexive coequalizer data:

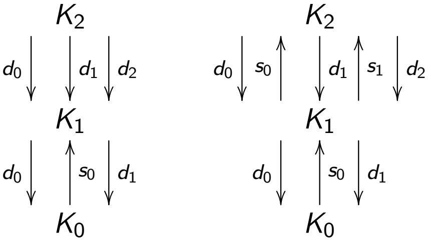
$$\begin{array}{ccc} K_1 & \qquad & K_1 \\ d_0 \Big\downarrow \Big\downarrow d_1 & \qquad & d_0 \Big\downarrow \quad \Big\uparrow s_0 \Big\downarrow d_1 \\ K_0 & \qquad & K_0 \end{array}$$

Monadic example: Let $\xi \colon \mathbb{D} \longrightarrow \mathbb{E}$ be a map of 2-monads in $\mathscr{K}$,

$$\nu = \mu \circ \mathbb{E}\xi \colon \mathbb{E}\mathbb{D} \longrightarrow \mathbb{E}\mathbb{E} \longrightarrow \mathbb{E}.$$

$$\begin{array}{c} \mathbb{E}\mathbb{D}\mathscr{Y} \\ \nu \Big\downarrow \quad \Big\uparrow \mathbb{E}\eta \Big\downarrow \theta \\ \mathbb{E}\mathscr{Y} \\ \Big\downarrow \pi \\ \xi_*\mathscr{Y} = \mathbb{E} \otimes_{\mathbb{D}} \mathscr{Y} \end{array}$$

Codescent and reflexive codescent data:



The identities for compositions of face and degeneracy operators for the 2-skeleton of a simplicial object are replaced by prescribed invertible 2-cells, which are part of the data.

A codescent object for such codescent data is a pair (k, ζ) consisting of a 1-cell k and an invertible 2-cell ζ

$$\begin{array}{c} K_0 \\ \downarrow{\scriptstyle k} \\ K \end{array} \qquad \zeta\colon k \circ d_0 \Longrightarrow k \circ d_1$$

such that certain equalities of pasting diagrams hold, and (k, ζ) is universal with this coherence property.

The universal property is the natural 2-categorical generalization of the existence and uniqueness universal property of coequalizers. Displaying the diagrams[2] would only make simple things look hard.

[2] They are displayed in the Appendix at the end.

Monadic example: Let $\xi\colon \mathbb{D} \longrightarrow \mathbb{E}$ be a map of 2-monads in $\mathscr{K}$,

$$\nu = \mu \circ \mathbb{E}\xi\colon \mathbb{E}\mathbb{D} \longrightarrow \mathbb{E}\mathbb{E} \longrightarrow \mathbb{E}.$$

$$\begin{array}{c} \mathbb{E}\mathbb{D}\mathbb{D}\mathscr{Y} \\ \nu\mathbb{D}\Big\downarrow \quad \mathbb{E}\eta\Big\uparrow \quad \Big\downarrow\mathbb{E}\mu \quad \Big\uparrow\mathbb{E}\mathbb{D}\eta \quad \Big\downarrow\mathbb{E}\mathbb{D}\theta \\ \mathbb{E}\mathbb{D}\mathscr{Y} \\ \nu\Big\downarrow \quad \Big\uparrow\mathbb{E}\eta \quad \Big\downarrow\mathbb{E}\theta \\ \mathbb{E}\mathscr{Y} \end{array}$$

(The resulting codescent object is a 2-truncation of an ∞-categorical 2-sided monadic bar construction.)

With suppressed conventions (all unit data is strict), all but one of the required simplicial identities hold strictly; the only non-identity invertible 2-cell required $(d_1 \circ d_2 \cong d_1 \circ d_1)$ comes from the pseudoaction 2-cell ϕ of $\mathscr{Y}$:

$$\mathbb{E}\phi \colon \mathbb{E}(\theta \circ \mathbb{D}\theta) \Longrightarrow \mathbb{E}(\theta \circ \mu).$$

If $\mathscr{Y}$ is a $\mathbb{D}$-algebra, $\phi = \mathrm{id}$ and we require no non-identity 2-cells. Write

$$\xi_{\#}\mathscr{Y} = \mathbb{E} \boxtimes_{\mathbb{D}} \mathscr{Y}$$

for the resulting codescent object, writing

$$\begin{array}{c} \mathbb{E}\mathscr{Y} \\ \big\downarrow{\scriptstyle \pi} \\ \xi_{\#}\mathscr{Y} \end{array} \qquad \zeta \colon \pi \circ \nu \Longrightarrow \pi \circ \mathbb{E}\theta$$

for the 1-cells and 2-cells witnessing the universality.

The codescent object $\xi_\# \mathscr{Y}$ is a strict $\mathbb{E}$-algebra since our codescent data are in $\mathbb{E}$-**AlgSt** and our codescent objects are constructed there; similarly for morphisms.

Back to processing multiplicative input

Can apply general construction to $\mathsf{id}\colon \mathbb{D} \longrightarrow \mathbb{D}$; strictification is

$$\mathsf{id}_\# \cong \mathbb{S}t\colon \mathbb{D}\text{-}\mathbf{PsAlg} \longrightarrow \mathbb{D}\text{-}\mathbf{AlgSt}.$$

The multicategory associated to the target 2-category is in the condemned cell because the distributivity constraints there would still be unstrictified 2-cells.

Can also apply the general construction to $\xi^k\colon \mathbb{D}_k \longrightarrow \mathbb{F}_k$ to get

$$\xi^k_\#\colon \mathbb{D}_k\text{-}\mathbf{PsAlg} \longrightarrow \mathbb{F}_k\text{-}\mathbf{Alg}, \quad k \geq 1.$$

Let $F\colon \mathscr{X}_1 \times \cdots \times \mathscr{X}_k \longrightarrow L_k\mathscr{Y}$ be a pseudomorphism of $\mathbb{D}_k$-pseudoalgebras. We get a natural transformation of functors $\mathscr{F}^k \longrightarrow \mathbf{Cat}(\mathscr{V})$, ψ coming via the universal property of $\xi^k_\# L_k\mathscr{Y}$:

$$\begin{array}{c} \xi_\#\mathscr{X}_1 \times \cdots \times \xi_\#\mathscr{X}_k \\ \big\downarrow \cong \\ \xi^k_\#(\mathscr{X}_1 \times \cdots \times \mathscr{X}_k) \\ \big\downarrow \xi^k_\# F \\ \xi^k_\# L_k\mathscr{Y} \\ \big\downarrow \psi \\ \xi_\#\mathscr{Y} \circ \wedge^k_{\mathscr{F}} \end{array}$$

By left Kan extension, this is this is the same as a natural transformation of functors $\mathscr{F} \longrightarrow \mathbf{Cat}(\mathscr{V})$

$$\xi_\#\mathscr{X}_1 \otimes \cdots \otimes \xi_\#\mathscr{X}_k \longrightarrow \xi_\#\mathscr{Y},$$

that is a k-morphism in $\mathbf{Mult}(\mathscr{F}\text{-}\mathbf{Alg})$. This gives

$$\xi_\#\colon \mathbf{Mult}(\mathscr{D}) \longrightarrow \mathbf{Mult}(\mathscr{F}\text{-}\mathbf{Alg})$$

Controlling the equivariant homotopy theory

NO equivariant considerations used in this formal theory,

BUT how do we know that $\xi_{\#}$ takes equivalences to equivalences and takes special $\mathbb{D}$-pseudoalgebras to special $\mathscr{F}$-G-categories? That is a question about the underlying additive theory. The nonequivariant specialization is easier.

Equivalence $\mathscr{Y} \longrightarrow \mathscr{Z}$: equivalences $\mathscr{Y}_n^{\Lambda} \longrightarrow \mathscr{Z}_n^{\Lambda}$ for $\Lambda \subset G \times \Sigma_n$ such that $\Lambda \cap \Sigma_n = \{e\}$, as in "special".

Formal theory would see $G \times \Sigma_n$-equivalences, which is too strong. Such a strong notion of specialness would lead only to products of Eilenberg-Mac Lane G-spectra.

$\xi_\#$ cannot give an equivalence in the 2-category $\mathbf{Cat}(G\mathscr{U})^\Pi$.

$\mathscr{F}_G$: finite G-sets; Π_G accordingly.

Categories of operators $\mathscr{D}$ and $\mathscr{D}_G$ from a G-operad $\mathscr{O}$.

Prolongation $\mathbb{P}$ from $\mathbb{D}$-pseudoalgebras to $\mathbb{D}_G$-pseudoalgebras.

Concrete inspection: $B \circ \mathbb{P} \cong \mathbb{P} \circ B$ on strict $\mathbb{D}$-algebras.

Topologically, an $\mathscr{F}$-G-map $X \longrightarrow Y$ is an equivalence if and only if $\mathbb{P}X \longrightarrow \mathbb{P}Y$ is a level G-equivalence. Transports to $\mathbf{Cat}(G\mathscr{U})$.

$$\begin{array}{ccc}
\mathbb{D}_G \boxtimes_{\mathbb{D}_G} \mathbb{P}\mathscr{Y} & \underset{s}{\overset{\xi}{\rightleftarrows}} & \mathbb{F}_G \boxtimes_{\mathbb{D}_G} \mathbb{P}\mathscr{Y} \\
\cong \big\downarrow & & \big\downarrow \cong \\
\mathbb{P}(\mathbb{D} \boxtimes_{\mathbb{D}} \mathscr{Y}) & \xrightarrow[\mathbb{P}\xi]{} & \mathbb{P}(\mathbb{F} \boxtimes_{\mathbb{D}} \mathscr{Y})
\end{array}$$

Work in ground 2-category $\mathbf{Cat}(\mathbf{G}\mathscr{U})^{\mathscr{O}(\mathbf{\Pi_G})}$, which sees only levelwise G-information.

Section $s\colon \mathscr{F}_G \longrightarrow \mathscr{D}_G$, levelwise G-map (ignore Σ_n).

Induces s in diagram such that $\xi \circ s = \mathrm{id}$.

Universal property gives invertible 2-cell $\mathrm{id} \longrightarrow s \circ \xi$, a homotopy on application of B.

Implies $\xi\colon \mathscr{Y} \simeq \mathbb{S}t\mathscr{Y} \longrightarrow \xi_{\#}\mathscr{Y}$ is an equivalence.

Input to the multiplicative input

Little multicategories $\mathscr{Q}$ parametrize algebraic structures

One object = operads: **Ass**, **Com**: monoids, comm. monoids

Two objects: multicategory for monoids acting on objects.

(Think of rings and modules). Many others. Categorify via $\mathscr{E}\mathscr{Q}$.

Big multicategories $\mathscr{M}$, like $\mathrm{Mult}(\mathscr{C}, \otimes)$, are the home for

multiplicative structures given by morphisms of multicategories

$$X \colon \mathscr{Q} \longrightarrow \mathscr{M}.$$

Objects $X(q)$ of $\mathscr{C}$; k-morphisms $\mathscr{Q}(q_1, \cdots, q_k; r)$ induce

$$X(q_1) \otimes \cdots \otimes X(q_k) \longrightarrow X(r).$$

SUMMARY

Multiplicative equivariant infinite loop space theory

transports a $\mathscr{Q}$-structure on $\mathscr{P}_G$-categories $\mathscr{A}(q)$

to a $\mathscr{Q}$-structure on the G-spectra $\mathbb{S}_G B\xi_{\#}\mathbb{R}\mathscr{A}(q)$,

converts G-categorical input to G-spectrum output.

(Elmendorf-Mandell idea when $G = e$,
developed with very different methods)

Free functors give an important class of examples

— but the serious theory is not needed for that.

ALL such nonequivariant structures $X\colon \mathscr{Q} \longrightarrow \mathbf{Mult}(\mathscr{P})$ extend equivariantly by G-ification $\mathscr{G}X\colon \mathscr{G}\mathscr{Q} \longrightarrow \mathbf{Mult}(\mathscr{P}_G)$.

Conjecture

$\mathscr{G}X$ is a global G-structure "of type $\mathscr{Q}$".

Symmetric bimonoidal G-categories $(\oplus, \otimes)$

For $\mathscr{Q} = \mathscr{P}$, $X\colon \mathscr{P} \longrightarrow \mathbf{Mult}(\mathscr{P})$ gives a naive commutative ring structure to a genuine G-spectrum.

For $\mathscr{Q} = \mathscr{P}_G$, $X\colon \mathscr{P}_G \longrightarrow \mathbf{Mult}(\mathscr{P})$ gives a genuine commutative ring structure to a genuine G-spectrum.

There are intermediate kinds of operadic commutative ring structures on genuine G-spectra.

(Kervaire invariant one; Blumberg and Hill)

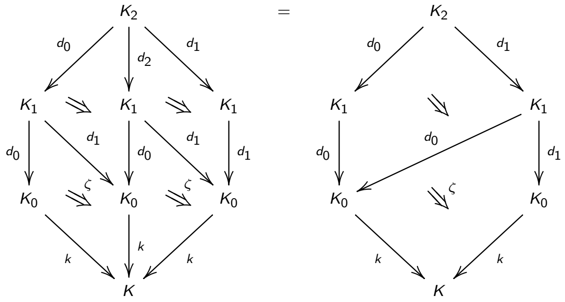
Similarly ring, module, and algebra structures admit variants on genuine G-spectra.

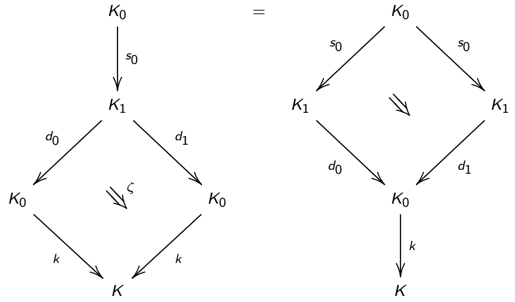
We now know how to recognize such structures on the level of structured G-categories.

They are there. Let's find them and see what they tell us!

I'll end (again) at this beginning.

Appendix: Pasting diagrams for codescent objects

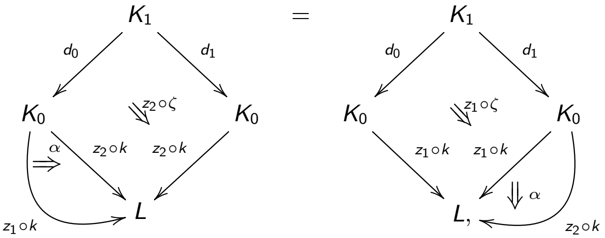




The universality means two things

First, given a pair (ℓ, χ), where $\ell\colon K_0 \longrightarrow L$ is a 1-cell and $\chi\colon \ell \circ d_0 \Longrightarrow \ell \circ d_1$ is an invertible 2-cell which make the evident analogs of the diagrams above commute, there is a unique 1-cell $z\colon K \longrightarrow L$ such that $z \circ k = \ell$ and $z \circ \zeta = \chi$.

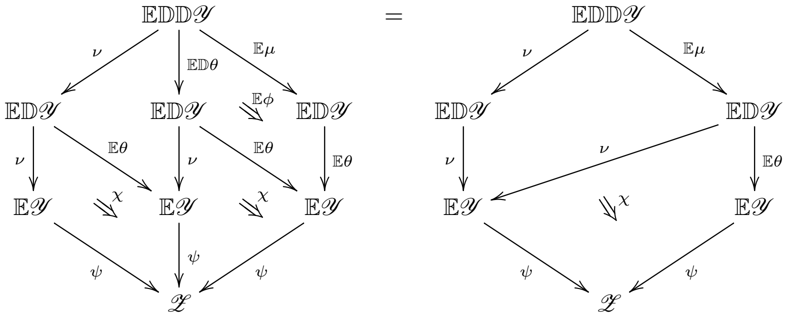
Second, given 1-cells $z_1, z_2\colon K \longrightarrow L$ together with an invertible 2-cell $\alpha\colon z_1 \circ k \Longrightarrow z_2 \circ k$ such that



there is a unique 2-cell $\beta\colon z_1 \Longrightarrow z_2$ such that $\beta \circ k = \alpha$.

The monadic universal property

First, let $\psi\colon \mathbb{E}\mathscr{Y} \longrightarrow \mathscr{Z}$ be a 1-cell in $\mathscr{K}$ and $\chi\colon \psi \circ \nu \Longrightarrow \psi \circ \mathbb{E}\theta$ be an invertible 2-cell such that

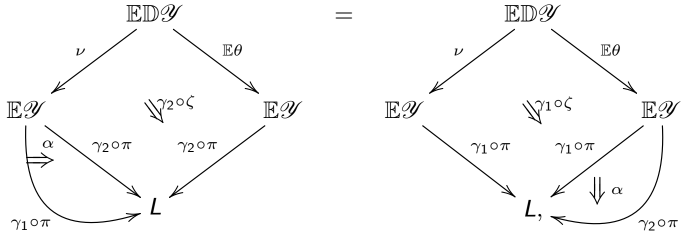


(The other coherence condition holds tautologically in our context).

Then there is a unique 1-cell $\gamma\colon \xi_{\#}\mathscr{Y} \longrightarrow \mathscr{Z}$ such that

$$\gamma \circ \zeta = \psi \quad \text{and} \quad \gamma \circ \pi = \chi.$$

Second, let $\gamma_1, \gamma_2 \colon \xi_{\#}\mathscr{Y} \longrightarrow \mathscr{Z}$ be 1-cells together with an invertible 2-cell $\alpha \colon \gamma_1 \circ \pi \Longrightarrow \gamma_2 \circ \pi$ such that



Then there is a unique 2-cell $\beta \colon \gamma_1 \Longrightarrow \gamma_2$ such that $\beta \circ \pi = \alpha$.