Under consideration for publication in Math. Struct. in Comp. Science

The parametric continuation monad

Paul-André Melliès[†]

Received

Dedicated to Corrado Bohm, on the occasion of his 90th birthday.

Every dialogue category comes equipped with a continuation monad defined by applying the negation functor twice. In this paper, we advocate that this double negation monad should be understood as part of a larger parametric monad (or a lax action) with parameter taken in the opposite of the dialogue category. This alternative point of view has one main conceptual benefit: it reveals that the strength of the continuation monad is the fragment of a more fundamental and symmetric structure — provided by a distributivity law between the parametric continuation monad and the canonical action of the dialogue category over itself. The purpose of this work is to describe the formal properties of this parametric continuation monad and of its distributivity law.

1. Introduction

Origins of tensorial logic. The idea of tensorial logic emerged in Kyoto during a sabbatical stay at the Research Institute in Mathematical Sciences (RIMS). There, in the very first days of the summer 2006, I suddenly realized that the distributivity law of linear logic

$$(A \parr B) \otimes C \quad \longrightarrow \quad A \parr (B \otimes C) \tag{1}$$

could be unified with the tensorial strength of the continuation monad

$$(\neg\neg\, A) \otimes B \quad \longrightarrow \quad \neg\neg\,(A \otimes B) \tag{2}$$

by shifting from linear logic to this more primitive logic of tensor and negation, where negation is not required to be involutive anymore. The very name of "tensorial logic" originates from the observation that the distributivity law

$$\kappa_{X,B,C} \quad : \quad \neg\,(\neg\, B \otimes X) \otimes C \quad \longrightarrow \quad \neg\,(\neg\,(\,B \otimes C\,) \otimes X\,) \tag{3}$$

of tensorial logic can be seen as a refinement of the distributivity law (1) of linear logic and *at the same time* as a parametric version of the tensorial strength (2) with parameter provided by the variable X.

One main purpose of the present article is to clarify the algebraic nature of this primitive and unifying principle of logic (3) starting from the observation that the family

[†] CNRS, Laboratoire PPS, UMR 7126, Université Paris Diderot, Sorbonne Paris Cité, F-75205 Paris, France.
This work has been partly supported by the ANR Project RECRE.

of morphisms κ is canonically defined in every dialogue category. Recall from (Melliès 2009; Melliès-Tabareau 2009) that a dialogue category is a monoidal category $(\mathscr{C}, \otimes, I)$ equipped with an object $\perp$ called its *tensorial pole*, and a pair of natural isomorphisms

$$\begin{array}{lcccc} \varphi_{A,B} & : & \mathscr{C}(A \otimes B, \perp) & \cong & \mathscr{C}(B, A \multimap \perp) \\ \psi_{A,B} & : & \mathscr{C}(A \otimes B, \perp) & \cong & \mathscr{C}(A, \perp \circ\!\!- B) \end{array}$$

providing a representation of the two presheaves

$$A \mapsto \mathscr{C}(A \otimes B, \perp) \quad , \quad B \mapsto \mathscr{C}(A \otimes B, \perp) \quad : \quad \mathscr{C}^{\,op} \longrightarrow Set.$$

The situation is extremely common in logic and in algebra. A typical illustration is provided by the category of (possibly infinite dimensional) vector spaces on a given field k, with the object $\perp$ defined as the field k itself. Another example is provided by any cartesian closed category $\mathscr{C}$ with a fixed object $\perp$ in it.

When the tensor product $\otimes$ of the dialogue category is symmetric, the objects $A \multimap \perp$ and $\perp \circ\!\!- A$ are isomorphic, and are thus often identified and written as $\neg A$ for simplicity. It is well-known that every dialogue category comes equipped with a monad

$$A \mapsto \perp \circ\!\!- (A \multimap \perp) \quad : \quad \mathscr{C} \quad \longrightarrow \quad \mathscr{C} \tag{4}$$

obtained by applying negation twice. This monad is traditionally called the *continuation monad* of the dialogue category in programming language theory because it is related to the continuation-passing style translations used during compilation. Note that the tensorial strength (2) of the continuation monad is defined in any dialogue category $\mathscr{C}$ as the morphism (3) instantiated at the parameter X equal to the tensorial unit I.

Seen from the point of view of tensorial logic, linear logic starts when one decides to force the double-negation monad to coincide with the identity. This step is reflected in categorical terms by the shift from general dialogue categories to the specific case of $*$-autonomous categories. Recall that a $*$-autonomous category is a symmetric dialogue category where the unit

$$A \quad \longrightarrow \quad \neg\neg A$$

of the continuation monad is invertible for every object A. The distributivity law (1) of linear logic is then recovered in any $*$-autonomous category as a special case of the distributivity law (3) of tensorial logic instantiated this time at the parameter $X = \neg A$:

$$\kappa_{\neg A,B,C} \quad : \quad \neg(\neg B \otimes \neg A) \otimes C \quad \longrightarrow \quad \neg(\neg(B \otimes C) \otimes \neg A)$$

The definition of (1) is justified by the fact that the multiplicative disjunction $⅋$ of linear logic is defined as

$$A ⅋ B \quad = \quad \neg(\neg B \otimes \neg A) \tag{5}$$

in any $*$-autonomous category.

This unification of (1) and (2) leads to the methodological question of understanding the algebraic nature of the tensorial principle (3) which underlies both of them. Quite obviously, this mathematical investigation of (3) should shed light on (1) and (2) and benefit at the same time from what is already known about these two well-studied instances. Typically, the fact that the distributivity law (3) is a parametric refinement of the tensorial strength (2) leads us to decompose it in two independent ingredients, for every dialogue category $\mathscr{C}$:

a. a functor

$$\circledast \quad : \quad (X, A) \quad \mapsto \quad X \circledast A = \bot \multimapinv ((A \multimap \bot) \otimes X) \quad : \quad \mathscr{C}^{op} \times \mathscr{C} \quad \longrightarrow \quad \mathscr{C}$$

corresponding to a parametric version of the continuation monad, with the object X as parameter,

b. a natural transformation

$$\kappa_{X,A,B} \quad : \quad (X \circledast A) \otimes B \quad \longrightarrow \quad X \circledast (A \otimes B)$$

generalizing the tensorial strength of the continuation monad.

This decomposition of (3) into a. and b. reduces our original problem to understanding in turn the algebraic nature of this specific functor $\circledast$ and of this specific natural transformation κ. As we will see, the exercise is not particularly difficult in itself — although it should be done with great care — but extremely useful, since it reveals the basic 2-dimensional structures which regulate the logical discourse, and more specifically its use of negation.

Parametric continuation monad. As explained above, the first aim of this paper is to reconstruct the functor $\circledast$ and more precisely to understand in which sense this functor should be understood as a parametric version of the continuation monad. A preliminary step in this direction is to observe that every dialogue category $\mathscr{C}$ comes equipped with an adjunction

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathscr{C}^{op} \arrow[l, bend left, "R"] \end{tikzcd} \tag{6}$$

where L and R denote the expected negation functors:

$$L \quad : \quad a \mapsto a \multimap \bot \qquad\qquad R \quad : \quad b \mapsto \bot \multimapinv b.$$

In order to analyze the algebraic nature of this adjunction, it also appears convenient to rename the monoidal categories $\mathscr{C}$ and $\mathscr{C}^{op}$ in the following way:

— the category $\mathscr{A}$ is the new name for $\mathscr{C}$ and its tensor product and unit are noted $\owedge$ and $\mathtt{true}$ in order to stress the logical interpretation of $\otimes$ and I as a linear conjunction and its neutral element,

— the category $\mathscr{B}$ is the new name for $\mathscr{C}^{op(0,1)}$ whose tensor product and unit are denoted $\ovee$ and $\mathtt{false}$ in order to stress the logical interpretation of $\otimes$ and I as a linear disjunction and its neutral element.

Here, the notation $\mathscr{C}^{op(0,1)}$ means that the orientation of the morphisms (of dimension 1) is reversed in $\mathscr{C}$ as well as the orientation of the tensor product (of dimension 0). This symmetric formulation of dialogue categories leads to the notion of *dialogue chirality* introduced in our companion paper ($\otimes_2$). The interested reader may have a look at the original definition there. However, it will be sufficient in this paper to remember that a dialogue chirality is essentially the same thing as a dialogue category formulated in this 2-sided and symmetric fashion. The specific orientation for the disjunction in the category $\mathscr{B}$ is chosen in order to rewrite the formula (5) as follows:

$$A \parr B \quad = \quad R\,(\,LA \ovee LB\,) \tag{7}$$

and thus to interpret $\ovee$ as a primitive variant of $\parr$, with the functors L and R playing the role of coercions (or shifts) interpreted as identity functors in the case of linear logic. At this point, one should remember that just as in the case of any monoidal category, the monoidal structure of $\mathscr{B}$ defines a (weak) left action

$$* \quad = \quad \ovee \quad : \quad \mathscr{B} \times \mathscr{B} \quad \longrightarrow \quad \mathscr{B} \tag{8}$$

of the monoidal category $(\mathscr{B}, \ovee, \mathbf{false})$ over itself, seen as a category. Recall that by weak left action of a monoidal category $(\mathscr{M}, \otimes, I)$ on a category $\mathscr{X}$, one means a functor

$$* \quad : \quad \mathscr{M} \times \mathscr{X} \quad \longrightarrow \quad \mathscr{X}$$

equipped with natural isomorphisms

$$\mu_{m,n} \;\; : \;\; m \circledast (n * x) \;\; \longrightarrow \;\; (m \otimes n) * x \qquad\qquad \mu_I \;\; : \;\; x \;\; \longrightarrow \;\; I * x$$

satisfying the two expected coherence diagrams:

$$\begin{tikzcd}
(m \otimes n) * (p * x) \arrow[d, "\mu"'] & m * (n * (p * x)) \arrow[l, "\mu"'] \arrow[r, "\mu"] & m * ((n \otimes p) * x) \arrow[d, "\mu"] \\
((m \otimes n) \otimes p) * x \arrow[rr, "\alpha"] & & ((m \otimes n) \otimes p) * x
\end{tikzcd}$$

$$\begin{tikzcd}
m * x \arrow[rr, "\mu"] \arrow[dr, "\rho"'] & & m * (I * x) \arrow[dl, "\mu"] \\
& (m \otimes I) * x &
\end{tikzcd} \tag{9}$$

where α and ρ denote the associativity and unit combinators of the monoidal category $\mathscr{M}$. We will establish in §2 a general *transfer theorem* which states that the weak action (8) may be transported along the adjunction

$$\begin{tikzcd}
\mathscr{A} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathscr{B} \arrow[l, bend left, "R"]
\end{tikzcd} \tag{10}$$

into the lax action defined as

$$\circledast \quad : \quad \mathscr{B} \times \mathscr{A} \quad \xrightarrow{L} \quad \mathscr{B} \times \mathscr{B} \quad \xrightarrow{*} \quad \mathscr{B} \quad \xrightarrow{R} \quad \mathscr{A}.$$

Note that the resulting operation

$$b \circledast a \quad = \quad R\,(\,b \ovee L(a)\,) \tag{11}$$

coincides with the functor we started from. Recall also that a lax action of a monoidal category $(\mathscr{M}, \otimes, I)$ on a category $\mathscr{X}$ is defined as a functor

$$\circledast \quad : \quad \mathscr{M} \times \mathscr{X} \quad \longrightarrow \quad \mathscr{X}$$

equipped with natural morphisms

$$\mu_{m,n} \;\;:\;\; m \circledast (n \circledast x) \;\;\longrightarrow\;\; (m \otimes n) \circledast x \qquad\qquad \mu_I \;\;:\;\; x \;\;\longrightarrow\;\; I \circledast x$$

satisfying the same two coherence diagrams (9) as a weak action. In particular, a weak action is the same thing as a lax action whose morphisms $\mu_{m,n}$ and μ_I are invertible. We will call *parametric monad* in $\mathscr{X}$ with parameters in the monoidal category $(\mathscr{M}, \otimes, I)$ such a lax action on the category $\mathscr{X}$. The terminology is justified by the fact that every such parametric monad $\circledast$ includes a monad in $\mathscr{X}$ defined as $(I \circledast -)$ where I is the unit of the monoidal category $\mathscr{M}$. Typically, starting from the parametric continuation monad (11) defined above, one recovers the continuation monad as

$$\mathbf{false} \circledast a \quad = \quad R\,(\,\mathbf{false} \mathbin{\overline{\vee}} La\,) \quad \cong \quad R \circ L\,(a).$$

We will see moreover in §2 that the coherence diagrams defining a parametric monad are obtained as a direct parametrization of the usual definition of monad.

Commutation between monads. The notion of parametric monad is not only useful in itself: it also leads to a pleasingly symmetric way to think of the notion of tensorial strength. Given a monad T on a monoidal category $(\mathscr{C}, \otimes, I)$, recall that a tensorial strength is defined as a natural family of morphisms

$$\sigma_{A,B} \quad : \quad T(A) \otimes B \longrightarrow T(A \otimes B)$$

regulated by four coherence diagrams. These four diagrams may be organized into two independent series, each of them consisting of two coherence diagrams. The first series of diagrams describes how a single tensor product $\otimes$ interacts with the multiplication and the unit of the monad:

$$\begin{tikzcd}
TT(A) \otimes B \arrow[r, "\sigma"] \arrow[d, "\mu"'] & T(T(A) \otimes B) \arrow[r, "\sigma"] & TT(A \otimes B) \arrow[d, "\mu"] \\
T(A) \otimes B \arrow[rr, "\sigma"] & & T(A \otimes B)
\end{tikzcd}$$

$$\begin{tikzcd}
& A \otimes B \arrow[dl, "\eta"'] \arrow[dr, "\eta"] & \\
T(A) \otimes B \arrow[rr, "\sigma"] & & T(A \otimes B)
\end{tikzcd}$$

The second series of diagrams describes how a single monad T interacts with the associativity and unit law of the tensor product:

$$\begin{tikzcd}
(T(A) \otimes B) \otimes C \arrow[r, "\sigma"] \arrow[d, "\alpha"'] & T(A \otimes B) \otimes C \arrow[r, "\sigma"] & T((A \otimes B) \otimes C) \arrow[d, "\alpha"] \\
T(A) \otimes (B \otimes C) \arrow[rr, "\sigma"] & & T(A \otimes (B \otimes C))
\end{tikzcd}$$

$$\begin{tikzcd}
& TA \arrow[dl, "\rho"'] \arrow[dr, "\rho"] & \\
T(A) \otimes I \arrow[rr, "\sigma"] & & T(A \otimes I)
\end{tikzcd}$$

where α and ρ are the canonical isomorphism of the monoidal category. As we will see, the apparent dissymmetry between the two series of commutative diagrams hides a symmetry which appears when one thinks

— of the monad T as a parametrized $\mathbb{1}$-monad $\circledast$ on the left, with parameters taken in the trivial monoidal category $\mathbb{1}$ with a single object I and a single morphism,

— of the tensor product $\otimes$ as a parametrized $\mathscr{C}$-monad on the right, with parameters taken in the monoidal category $(\mathscr{C}, \otimes, I)$ and multiplication and unit defined as α and ρ.

Here, by parametric monad on the left, we mean a parametric monad in the usual sense, whereas by parametric $\mathscr{M}$-monad on the right, we mean a parametric $\mathscr{M}^{\,op(0)}$-monad where $\mathscr{M}^{\,op(0)}$ denotes the monoidal category $\mathscr{M}$ where the direction of the tensor product has been reversed. This symmetric point of view on tensorial strengths enables to write the tensorial strength as a distributivity law

$$\sigma_{A,B} \quad : \quad (I \circledast A) \otimes B \quad \longrightarrow \quad I \circledast (A \otimes B)$$

between the left and right parametric monads. As we will see, yet another way to think of the category $\mathscr{C}$ equipped with the three data $(\circledast, \otimes, \sigma)$ is to identify it as a lax version of $(\mathbb{1}, \mathscr{C})$-biaction (or bimodule) where the equality

$$(I \circledast A) \otimes B \quad = \quad I \circledast (A \otimes B)$$

has been replaced by a natural transformation σ satisfying the four coherence diagrams recalled above.

Plan of the paper. We introduce the notion of parametric monad in §2 and establish an elementary transfer theorem for parametric monads. We construct in §3 the parametric continuation monad $\circledast$ of a dialogue category, and deduce from the transfer theorem that it indeed defines such a parametric monad in every dialogue chirality. We introduce in §4 the notion of commutator between parametric monads, and show that it generalizes the notion of tensorial strength as well as the notion of distributivity between monads. We conclude the paper in §5 by constructing for every dialogue chirality a double negation commutator between the parametric continuation monad and the action $\oslash$ of the category $\mathscr{A}$ over itself.

Related works. Since the main purpose of the paper is to design a bridge between linear logic and the theory of strong monads, the reader is probably advised to read the original papers (Girard 1987; Girard 1995) about linear logic as well as the seminal papers on strong monads (Kock 1970; Moggi 1991) in algebra and in programming language semantics. It should be mentioned that the distributivity law (1) was originally observed in (Hu & Joyal 1999) and that it was then extensively studied in (Cockett & Seely 1997; Blute & Cockett & Seely & Trimble 1996) in their seminal work on the coherence properties of weakly distributive categories, partially reported in (Melliès 2009). As the reader will see, an important part of the present paper is devoted to the idea that lax algebraic structures may be transported along adjunctions. Although the shift from weak to lax structures plays a fundamental role in our work on tensorial logic, the transfer theorem for lax algebras along adjunctions is only a slight variant of similar transfer theorems along equivalences of categories, most specifically the 2-categorical account developed in (Kelly & Lack 2004) starting from ideas in (Bénabou 1963).

Other parametrized notions of monad. The notion of monad (T, μ, η) is well established today, and it is sufficiently important and primitive to be extended and parametrized in various ways, depending on the situation of interest. Let us mention in particular that two other notions of parametrized monad has been recently introduced for different purposes in (Atkey 2009) and in (Uustalu 2003). Despite the proximity in name, these parametric notions of monad are different, and not immediately related.

Side remark. The reader should be aware that there is an element of choice in picking the double-negation monad (4) instead of the other double-negation monad

$$A \mapsto (\bot \multimapinv A) \multimap \bot \quad : \quad \mathscr{C} \quad \longrightarrow \quad \mathscr{C} \tag{12}$$

also available in any dialogue category, and defined in just the same way as (4) except that the order of negations has been interchanged. However, the choice of (4) against (12) does not really matter because the very notion of dialogue category $\mathscr{C}$ is invariant under the change of orientation

$$\mathscr{C} \quad \mapsto \quad \mathscr{C}^{\,op(0)} \tag{13}$$

of the tensor product. This change of orientation interchanges the left negation and the right negation. From this follows that the double-negation monad (4) taken in the dialogue category $\mathscr{C}^{\,op(0)}$ coincides with the double-negation monad (12) taken in the original dialogue category $\mathscr{C}$. In other words, the two choices are simply equivalent modulo (13). As a matter of fact, the only important point to remember is that the strength studied in the present paper permutes the double-negation monad (4) with the right action of the tensor product:

$$\sigma_{A,B} \quad : \quad \big(\bot \multimapinv (A \multimap \bot)\big) \otimes B \quad \longrightarrow \quad \bot \multimapinv \big((A \otimes B) \multimap \bot\big)$$

whereas the other strength permutes its double-negation monad (12) with the left action of the tensor product:

$$A \otimes \big((\bot \multimapinv B) \multimap \bot \big) \quad \longrightarrow \quad \big(\bot \multimapinv (A \otimes B) \big) \multimap \bot.$$

2. Parametric monads

We start by recalling the formal definition of *adjunction* in a 2-category $\mathscr{W}$ introduced in (Kelly & Street 1974) and then review a series of basic consequences of the definition. In particular, we establish an elementary *transfer theorem* which states that every parametric $\mathscr{J}$-monad on the $\mathscr{B}$-side of an adjunction $L \dashv R$ is transported to a parametric $\mathscr{J}$-monad on its $\mathscr{A}$-side, this for every monoidal category $\mathscr{J}$.

2.1. *Formal adjunctions*

Recall that an adjunction in a 2-category $\mathscr{W}$ consists of a pair $\mathscr{A}$, $\mathscr{B}$ of 0-dimensional cells, of a pair

$$L : \mathscr{A} \longrightarrow \mathscr{B} \qquad\qquad R : \mathscr{B} \longrightarrow \mathscr{A}$$

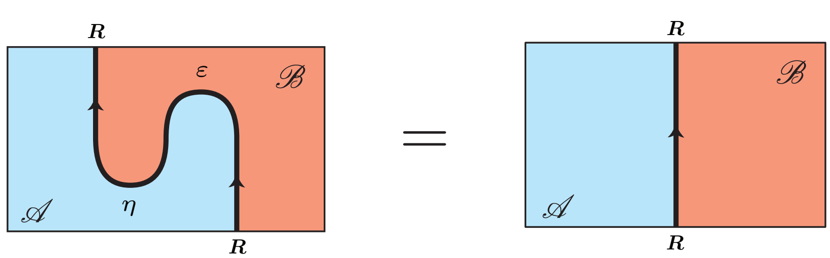
of 1-dimensional cells, and of a pair

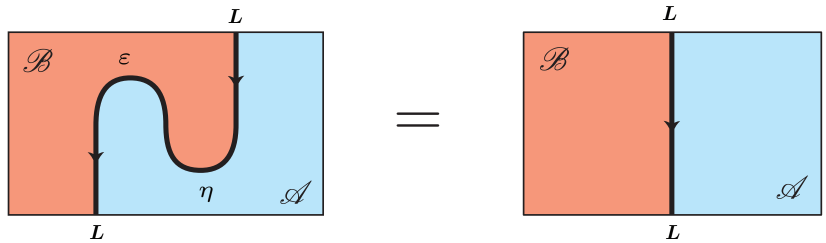
$$\eta : 1_{\mathscr{A}} \Rightarrow R \circ L \qquad\qquad \varepsilon : L \circ R \Rightarrow 1_{\mathscr{B}}$$

of 2-dimensional cells. One requires moreover that the 2-dimensional cells obtained by pasting:

$$\begin{tikzcd}
 & \mathscr{A} \arrow[rr, "1"] \arrow[dr, "L"] \arrow[d, phantom, "\Downarrow\varepsilon"] & \arrow[d, phantom, "\Downarrow\eta"] & \mathscr{A} \\
\mathscr{B} \arrow[ur, "R"] \arrow[rr, "1"'] & & \mathscr{B} \arrow[ur, "R"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{A} \arrow[rr, "1"] \arrow[dr, "L"'] & \arrow[d, phantom, "\Downarrow\eta"] & \mathscr{A} \arrow[dr, "L"] \arrow[d, phantom, "\Downarrow\varepsilon"] & \\
 & \mathscr{B} \arrow[ur, "R"] \arrow[rr, "1"'] & & \mathscr{B}
\end{tikzcd}$$

coincide with the identity on the 1-dimensional cells L and R, respectively. In that case, one writes $L \dashv R$ and one says that the 1-cell L is left adjoint to the 1-cell R, and conversely, that the 1-cell R is right adjoint to the 1-cell L. These equations may be depicted in string diagrams in the following way, with the 0-cell $\mathscr{A}$ coloured blue (or light grey) and the 0-cell $\mathscr{B}$ coloured red (or dark grey).





By convention, the black string representing the functors L and R is oriented downwards when it depicts the functor L and upwards when it depicts the functor R. This specific orientation is justified by the connection with dialogue games exhibited in our companion paper ($\otimes_1$) where the functor R corresponds to the Opponent moves of a dialogue game, and the functor L corresponds to the Player moves. In that case, the orientation of L and R reflects the flow of information and control in the proof.

2.2. *Formal monads*

A monad in a 2-category $\mathscr{W}$ is defined as a 0-cell $\mathscr{A}$ together with a 1-cell

$$T \quad : \quad \mathscr{A} \quad \longrightarrow \quad \mathscr{A}$$

together with a pair of 2-cells

$$\eta \;:\; 1_{\mathscr{A}} \;\longrightarrow\; T \qquad\qquad \mu \;:\; T \circ T \;\longrightarrow\; T$$

making the two diagrams below commute:

$$\begin{tikzcd}
T \circ T \circ T \arrow[r, Rightarrow, "\mu\circ T"] \arrow[d, Rightarrow, "T\circ\mu"'] & T \circ T \arrow[d, Rightarrow, "\mu"] \\
T \circ T \arrow[r, Rightarrow, "\mu"'] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
 & T \circ T \arrow[dr, Rightarrow, bend left, "\mu"] & \\
T \arrow[ur, Rightarrow, bend left, "\eta\circ T"] \arrow[rr, Rightarrow, "\mathrm{id}_T"] \arrow[dr, Rightarrow, bend right, "T\circ\eta"'] & & T \\
 & T \circ T \arrow[ur, Rightarrow, bend right, "\mu"'] &
\end{tikzcd}$$

A comonad in a 2-category $\mathscr{W}$ is defined as a monad in the 2-category $\mathscr{W}^{op^{(2)}}$ obtained by reversing the orientation of the 2-cells in the 2-category $\mathscr{W}$.

2.3. *The external adjunction*

Suppose given a formal adjunction

$$\begin{tikzcd}
\mathscr{A} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathscr{B} \arrow[l, bend left, "R"]
\end{tikzcd} \tag{14}$$

in a 2-category $\mathscr{W}$. It is well-known that every such adjunction induces a monad $R \circ L$ on the 0-cell $\mathscr{A}$ and a comonad $L \circ R$ on the 0-cell $\mathscr{B}$. Less known is the fact that this monad is part of a much broader structure, originally noticed by Jean Bénabou, see (Bénabou

1963) for details, as well as the more recent account in (Kelly & Lack 2004). We describe this broader structure now. Let

$$\mathrm{End}(\mathscr{A}) \quad = \quad \mathscr{W}(\mathscr{A}, \mathscr{A})$$

denote the hom-category of the 0-cells $\mathscr{A}$

— with objects the 1-cells from the 0-cell $\mathscr{A}$ to itself,

— with morphisms the 2-cells between these 1-cells.

The category $\mathrm{End}(\mathscr{A})$ is strict monoidal, with composition $\circ$ as tensor product, and with the identity 1-cell $1_{\mathscr{A}}$ as tensor unit. Note that a monoid in this category $\mathrm{End}(\mathscr{A})$ is the same thing as a monad in $\mathscr{W}$ on the 0-cell $\mathscr{A}$. Similarly, a comonoid in the category $\mathrm{End}(\mathscr{B})$ is the same thing as a comonad in $\mathscr{W}$ on the 0-cell $\mathscr{B}$. Now, the main observation is that the two 1-cells L and R induce in turn two functors

$$\begin{array}{rcccc} [L,R] & : & \mathrm{End}(\mathscr{A}) & \rightarrow & \mathrm{End}(\mathscr{B}) \\ & & F & \mapsto & L \circ F \circ R \\ \\ [R,L] & : & \mathrm{End}(\mathscr{B}) & \rightarrow & \mathrm{End}(\mathscr{A}) \\ & & G & \mapsto & R \circ G \circ L \end{array}$$

defined by pre and postcomposition. The two functors are moreover involved in an adjunction

$$\begin{tikzcd} \mathrm{End}(\mathscr{A}) \arrow[r, bend left, "{[L,R]}"] \arrow[r, phantom, "\perp"] & \mathrm{End}(\mathscr{B}) \arrow[l, bend left, "{[R,L]}"] \end{tikzcd} \tag{15}$$

between the hom-categories. This adjunction is called the *external adjunction* associated to the formal adjunction $L \dashv R$. By external, one means that it is an adjunction between categories $\mathrm{End}(\mathscr{A})$ and $\mathrm{End}(\mathscr{B})$ reflecting to the outside the formal adjunction $L \dashv R$ living inside the 2-category $\mathscr{W}$. The unit and counit of the external adjunction are defined in the expected way:

$$\begin{array}{rcl} [\eta]_F = \eta \circ F \circ \eta & : & F \Rightarrow R \circ L \circ F \circ R \circ L, \\ [\epsilon]_G = \varepsilon \circ G \circ \varepsilon & : & L \circ R \circ G \circ L \circ R \Rightarrow G. \end{array}$$

Looking at it more closely, the adjunction simply says that there exists a one-to-one correspondence between the 2-cells

$$\begin{tikzcd} \mathscr{A} \arrow[r, "F"] \arrow[dr, phantom, "\Downarrow"] & \mathscr{A} \arrow[d, "L"] \\ \mathscr{B} \arrow[u, "R"] \arrow[r, "G"'] & \mathscr{B} \end{tikzcd}$$

and the 2-cells

$$\begin{tikzcd} \mathscr{A} \arrow[r, "F"] \arrow[d, "L"'] \arrow[dr, phantom, "\Downarrow"] & \mathscr{A} \\ \mathscr{B} \arrow[r, "G"'] & \mathscr{B} \arrow[u, "R"'] \end{tikzcd}$$

in the 2-category $\mathscr{W}$, and that this correspondence is natural wrt. the action on F in $\mathrm{End}(\mathscr{A})$ and wrt. the action on G in $\mathrm{End}(\mathscr{B})$. The adjunction may be also seen as an avatar of what Kelly and Street like to call "mate 2-cells" in their 2-categorical theory of adjunctions, see (Kelly & Street 1974) for details.

2.4. *Lax monoidal functors*

Recall that a lax monoidal functor between monoidal categories $(\mathscr{M}, \otimes, I)$ and $(\mathscr{N}, \otimes, I)$ is defined as a functor

$$F \quad : \quad \mathscr{M} \quad \longrightarrow \quad \mathscr{N}$$

equipped with two natural transformations

$$m_{A,B} \;\; : \;\; FA \otimes FB \;\; \longrightarrow \;\; F(A \otimes B) \qquad\qquad m_I \;\; : \;\; I \;\; \longrightarrow \;\; F(I)$$

making the three diagrams below commute:

$$\begin{tikzcd} (FA \otimes FB) \otimes FC \arrow[r, "m"] \arrow[d, "\alpha"] & F(A \otimes B) \otimes FC \arrow[r, "m"] & F((A \otimes B) \otimes C) \arrow[d, "\alpha"] \\ FA \otimes (FB \otimes FC) \arrow[r, "m"] & FA \otimes F(B \otimes C) \arrow[r, "m"] & F(A \otimes (B \otimes C)) \end{tikzcd} \tag{16}$$

$$\begin{tikzcd} & FA \arrow[dl, "m_I"'] \arrow[dr, "\lambda"] & \\ FI \otimes FA \arrow[rr, "m"] & & F(I \otimes A) \end{tikzcd} \qquad \begin{tikzcd} & FA \arrow[dl, "m_I"'] \arrow[dr, "\rho"] & \\ FA \otimes FI \arrow[rr, "m"] & & F(A \otimes I) \end{tikzcd} \tag{17}$$

for all objects A, B, C of the category $\mathscr{M}$, where α, λ and ρ denote the canonical morphisms of the monoidal categories. An important property of lax monoidal functors is that they compose, and in fact define a 2-category with

— monoidal categories as 0-cells,
— lax monoidal functors as 1-cells,
— monoidal natural transformations as 2-cells.

Although we will not really use this notion in the paper, we find useful to recall that a monoidal natural transformation

$$\theta \quad : \quad (F, m) \quad \Rightarrow \quad (G, m) \quad : \quad (\mathscr{M}, \otimes, I) \quad \longrightarrow \quad (\mathscr{N}, \otimes, I)$$

between lax monoidal functors (F, m) and (G, n) is defined as a natural transformation

$$\theta \quad : \quad F \quad \Rightarrow \quad G \quad : \quad \mathscr{M} \quad \longrightarrow \quad \mathscr{N}$$

between the underlying functors, making the diagrams commute

$$\begin{tikzcd} F(A)\otimes F(B) \arrow[r, "\theta\otimes\theta"] \arrow[d, "m"'] & G(A)\otimes G(B) \arrow[d, "n"] \\ F(A\otimes B) \arrow[r, "\theta"] & G(A\otimes B) \end{tikzcd} \qquad \begin{tikzcd} & I \arrow[dl, "m_I"'] \arrow[dr, "n_I"] & \\ F(I) \arrow[rr, "\theta"] & & G(I) \end{tikzcd}$$

A lax monoidal functor is called weak when the coercions m_I and $m_{A,B}$ are isomorphisms for all objects A, B of the category $\mathscr{C}$.

2.5. *Parametric monads*

The notion of lax monoidal functor was introduced by Jean Bénabou, who was guided by his important observation that a monoid in a monoidal category $(\mathscr{M}, \otimes, I)$ is the same thing as a lax monoidal functor

$$\mathbb{1} \quad \longrightarrow \quad (\mathscr{M}, \otimes, I)$$

from the monoidal category $\mathbb{1}$ with a single object and a single morphism. This specific formulation of monoids provides a nice conceptual explanation for the fact that every lax monoidal functor transports monoids to monoids.

Now, it is not difficult to see that a formal monad (T, μ, η) on a 0-cell $\mathscr{A}$ of a 2-category $\mathscr{W}$ is the same thing as a monoid in the monoidal category $\mathrm{End}(\mathscr{A}) = \mathscr{W}(\mathscr{A}, \mathscr{A})$. From this follows that a formal monad is the same thing as a lax monoidal functor

$$\mathbb{1} \quad \longrightarrow \quad \mathrm{End}(\mathscr{A}).$$

The discussion justifies considering a parametric notion of monad, parametrized by a monoidal category $(\mathscr{J}, \otimes, \boldsymbol{e})$ in the following way.

Definition 1 (parametric monad). A parametric $\mathscr{J}$-monad on a 0-cell $\mathscr{A}$ of a 2-category $\mathscr{W}$ is defined as a lax monoidal functor

$$(T, \mu) \quad : \quad \mathscr{J} \quad \longrightarrow \quad \mathrm{End}(\mathscr{A}).$$

The monoidal category $\mathscr{J}$ is called the parameter category of the $\mathscr{J}$-monad; and an object j of the category $\mathscr{J}$ is called a parameter.

The definition is a straightforward application of Bénabou's ideas and we do not claim any originality for it. It is worth mentioning here that in the case of $\mathscr{W} = \boldsymbol{Cat}$, a parametric monad is the same thing as a lax action of the monoidal category $\mathscr{J}$ on the category $\mathscr{A}$. By lax action, one simply means a lax algebra of the 2-monad

$$\mathscr{J} \times - \quad : \quad \boldsymbol{Cat} \quad \longrightarrow \quad \boldsymbol{Cat}$$

on the 2-category of categories. Note also that we use for convenience the greek letter μ rather than the latin letter m in order to denote the coercion maps of the lax monoidal functor T.

At this point, it seems reasonable to give an equivalent and fully explicit description of the notion of parametric monad. A parametric $\mathscr{J}$-monad (T, μ) consists of

— a 1-cell $T_j : \mathscr{A} \longrightarrow \mathscr{A}$ for every parameter j and a 2-cell $T_f : T_j \Rightarrow T_k$ for every morphism $f : j \longrightarrow k$ between such parameters,
— a 2-cell $\mu_{\boldsymbol{e}} : 1_{\mathscr{A}} \Rightarrow T_{\boldsymbol{e}}$ called the unit of the parametric monad,
— a 2-cell $\mu_{j,k} : T_j \circ T_k \Rightarrow T_{j\otimes k}$ called the (j, k)-component of the multiplication of the parametric monad, for every pair of parameters j and k.

These data are moreover required to make a series of coherence diagrams commute in the category $\mathrm{End}(\mathscr{A})$. First, the diagrams

$$\begin{tikzcd} & T_k \arrow[dr, Rightarrow, "T_g"] & \\ T_j \arrow[ur, Rightarrow, "T_f"] \arrow[rr, Rightarrow, "T_{g\circ f}"'] & & T_l \end{tikzcd} \qquad \begin{tikzcd} T_j \arrow[r, Rightarrow, bend left, "\mathrm{id}_{T_j}"] \arrow[r, Rightarrow, bend right, "T_{\mathrm{id}_j}"'] & T_j \end{tikzcd}$$

which express the functoriality of T. Then, the diagrams

$$\begin{tikzcd} T_j \circ T_k \arrow[r, Rightarrow, "T_f \circ T_g"] \arrow[d, Rightarrow, "\mu_{j,k}"'] & T_{j'} \circ T_{k'} \arrow[d, Rightarrow, "\mu_{j',k'}"] \\ T_{j\otimes k} \arrow[r, Rightarrow, "T_{f\otimes g}"] & T_{j'\otimes k'} \end{tikzcd}$$

which express the naturality of μ. Finally, the diagrams

$$\begin{tikzcd} T_j \circ T_k \circ T_l \arrow[rr, Rightarrow, "\mu_{j,k} \circ T_l"] \arrow[d, Rightarrow, "T_j \circ \mu_{k,l}"'] & & T_{j\otimes k} \circ T_l \arrow[d, Rightarrow, "\mu_{j\otimes k,l}"] \\ T_j \circ T_{k\otimes l} \arrow[r, Rightarrow, "\mu_{j,k\otimes l}"] & T_{j\otimes(k\otimes l)} \arrow[r, Rightarrow, "\alpha"] & T_{(j\otimes k)\otimes l} \end{tikzcd}$$

and

$$\begin{tikzcd} & T_{\boldsymbol{e}} \circ T_j \arrow[dr, Rightarrow, "\mu_{\boldsymbol{e},j}"] & \\ T_j \arrow[ur, Rightarrow, "\mu_{\boldsymbol{e}} \circ T_j"] \arrow[rr, Rightarrow, "\mathrm{id}_{T_j}"] \arrow[dr, Rightarrow, "T_j \circ \mu_{\boldsymbol{e}}"'] & & T_j \\ & T_j \circ T_{\boldsymbol{e}} \arrow[ur, Rightarrow, "\mu_{j,\boldsymbol{e}}"'] & \end{tikzcd}$$

which express the monoidality of μ. These diagrams should commute for all indices j, j', k, k', l and all morphisms f, g, h of the parameter category $\mathscr{J}$.

Remark. Note that every parametric $\mathscr{J}$-monad T comes equipped with a morphism $A \rightarrow T_{\boldsymbol{e}} A$ where $\boldsymbol{e}$ is the unit of the monoidal category $\mathscr{J}$. On the other hand, the reader should be careful that is (at least in general) no morphism $A \rightarrow T_j A$ for an object j different of the unit $\boldsymbol{e}$ in the category $\mathscr{J}$.

Parametric comonads. There is also a notion of parametric comonad (K, δ) indexed by a monoidal category $(\mathscr{J}, \otimes, I)$ in a 2-category $\mathscr{W}$ defined by duality as a parametric $\mathscr{J}^{op(1)}$-monad in the 2-category $\mathscr{W}^{op(2)}$. Here, the category $\mathscr{J}^{op(1)}$ is obtained by reversing the orientation of the morphisms of the category $\mathscr{J}^{op(1)}$ and the 2-category $\mathscr{W}^{op(2)}$ by reversing the orientation of the 2-cells. Note in particular that

$$\mathscr{W}^{op(2)}(\mathscr{A}, \mathscr{A}) \quad = \quad \mathscr{W}(\mathscr{A}, \mathscr{A})^{op(1)}$$

and thus that a parametric $\mathscr{J}$-comonad is the same thing as an oplax (rather than lax) monoidal functor

$$(K, \delta) \quad : \quad \mathscr{J} \quad \longrightarrow \quad \mathrm{End}(\mathscr{A}) = \mathscr{W}(\mathscr{A}, \mathscr{A})$$

with the expected notion of oplax monoidal functor between monoidal categories.

2.6. *The transfer theorem*

At this point, we are ready to establish our transfer theorem for parametric monads. To that purpose, we start by considering a formal adjunction (14) in a 2-category $\mathscr{W}$, together with the external adjunction (15) resulting from it. The transfer theorem is based on the key observation that

Proposition 1. The right adjoint functor

$$[R, L] \quad : \quad \mathrm{End}(\mathscr{B}) \quad \longrightarrow \quad \mathrm{End}(\mathscr{A})$$

defines a lax monoidal functor.

In order to establish the property, we need to define a 2-cell

$$m_1 \quad : \quad 1_{\mathscr{A}} \quad \Rightarrow \quad R \circ 1_{\mathscr{B}} \circ L$$

as well as a family of 2-cells

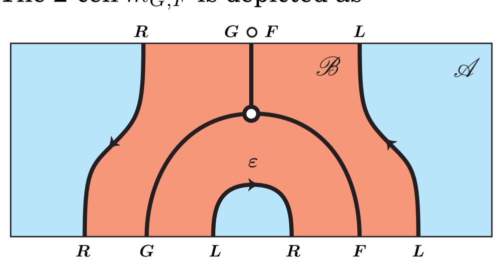
$$m_{G,F} \quad : \quad (R \circ G \circ L) \circ (R \circ F \circ L) \quad \Rightarrow \quad R \circ (G \circ F) \circ L$$

indexed by the 1-cells

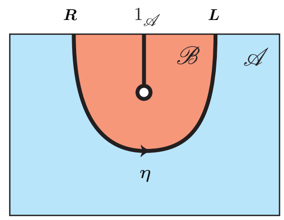
$$F, G \quad : \quad \mathscr{B} \quad \longrightarrow \quad \mathscr{B}$$

and making a series of coherence diagrams commute in the category $\mathrm{End}(\mathscr{A})$. The 2-cells m_1 and $m_{G,F}$ are defined in the expected way, using the unit η and the counit ε of the formal adjunction $L \dashv R$, respectively. The construction may be depicted in the language of string diagrams. The 2-cell $m_{G,F}$ is depicted as

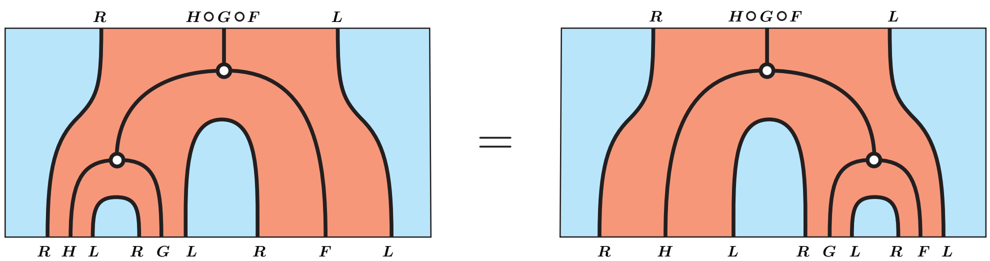
$m_{G,F}$ =



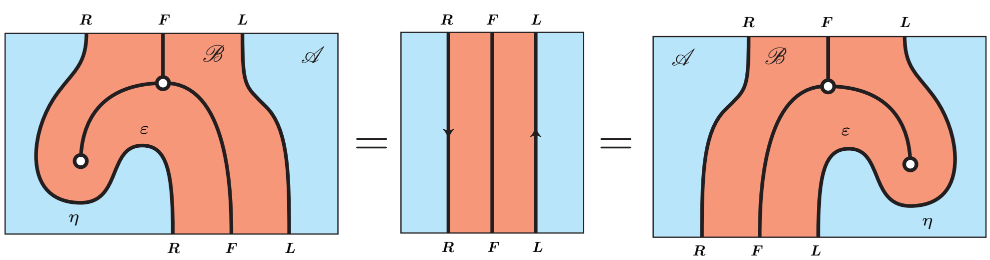
while the 2-cell m_I is depicted as

m_I =


The proof that the coherence diagrams required of a lax monoidal functor commute works exactly in the same way as the proof that the endofunctor $R \circ L$ defines a monoid in the category $\mathrm{End}(\mathscr{A})$. The first coherence diagram is reflected by the pictorial equality



and the second coherence diagrams involving the unit m_I are depicted as the diagrammatic equalities below:



At this point, the transfer theorem below follows from Proposition (1) and the fact that lax monoidal functors compose.

Proposition 2 (transfer theorem). Every parametric $\mathscr{J}$-monad (T, μ) in the 0-cell $\mathscr{B}$ induces a parametric $\mathscr{J}$-monad in the 0-cell $\mathscr{A}$.

The parametric $\mathscr{J}$-monad on the 0-cell $\mathscr{A}$ is simply defined by composing the two lax monoidal functors

$$\mathscr{J} \xrightarrow{\quad T \quad} \mathrm{End}(\mathscr{B}) \xrightarrow{\quad [R,L] \quad} \mathrm{End}(\mathscr{A})$$

The composite functor is lax monoidal and thus defines a parametric $\mathscr{J}$-monad on the 0-cell $\mathscr{A}$. This parametric monad is called the *transferred* parametric monad. Observe that the fact that every formal adjunction $L \dashv R$ defines a formal monad on the 0-cell $\mathscr{A}$ may be seen as a consequence of the transfer theorem. Indeed, the monad on $\mathscr{A}$ is obtained by transferring the identity monad on $\mathscr{B}$ along the adjunction.

3. The parametric continuation monad

As explained in the introduction, the purpose of the transfer theorem is to shed light on the algebraic structure of the continuation monad defined in any dialogue category $(\mathscr{C}, \otimes, I)$. The key idea to deduce the operation (11) defined as

$$b \circledast a \quad = \quad R\,(\, b \ovee L(a)\,)$$

from the action $* = \ovee$ of the monoidal category $(\mathscr{B}, \ovee, \mathbf{false})$ on itself. Here, it is worth recalling that $(\mathscr{B}, \ovee, \mathbf{false})$ is just another name for the opposite $\mathscr{C}^{\,op(0,1)}$ of the original dialogue category $\mathscr{C}$. So, if we think of $\mathscr{A} = \mathscr{C}$ as a category of formulas and *proofs*, it is natural to think of $\mathscr{B}$ as a category of formulas and *refutations*. Accordingly, if we think of the tensor product $\otimes = \owedge$ as a *conjunction* in the original dialogue category $\mathscr{A}$, it is natural to think of the tensor product $\otimes^{\,op(0)} = \ovee$ of the category $\mathscr{B}$ as a *disjunction*.

The important point is that the action of the monoidal category $\mathscr{B}$ on itself may be seen as a parametric $\mathscr{B}$-monad S on the category $\mathscr{B}$, defined by the family of functors

$$S_b \quad : \quad b' \quad \mapsto \quad b \ovee b' \quad : \quad \mathscr{B} \quad \longrightarrow \quad \mathscr{B} \tag{18}$$

where the 2-category $\mathscr{W}$ is taken in that case equal to the 2-category ***Cat*** of categories, functors and natural transformations. Alternatively, the parametric monad S may be formulated as the weak monoidal functor

$$S \quad : \quad \mathscr{B} \quad \longrightarrow \quad \mathrm{End}(\mathscr{B})$$

obtained by currifying the tensor product

$$\ovee \quad : \quad \mathscr{B} \times \mathscr{B} \quad \longrightarrow \quad \mathscr{B}.$$

At this point, the transfer theorem established in Proposition 2 enables us to conclude that:

Proposition 3. In every dialogue chirality, the functor

$$\circledast \quad : \quad \mathscr{B} \times \mathscr{A} \quad \longrightarrow \quad \mathscr{A}$$

defines a parametric monad

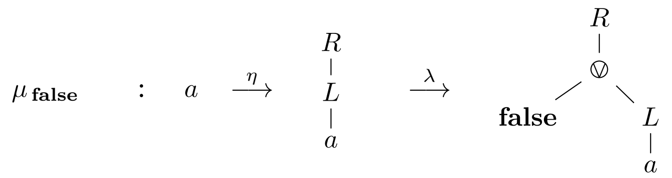
$$T_b \quad : \quad a \quad \mapsto \quad b \circledast a = R\,(\, b \ovee L(a)\,)$$

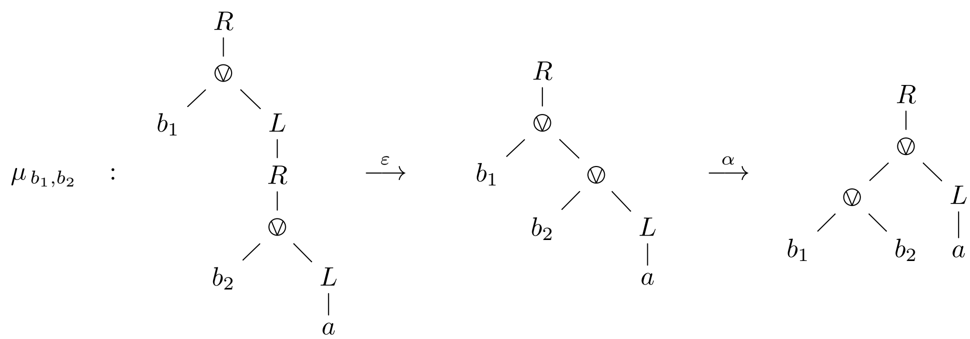
on the category $\mathscr{A}$, parametrized by the monoidal category $(\mathscr{B}, \ovee, \mathbf{false})$.

For convenience and readability, we like to write the functor T_b using the following tree notation:

$$T_b \quad : \quad a \quad \mapsto \quad \begin{tikzcd}[row sep=small, column sep=small] & R \arrow[d, no head] & \\ & \ovee \arrow[dl, no head] \arrow[dr, no head] & \\ b & & L \arrow[d, no head] \\ & & a \end{tikzcd}$$

Using this notation, the natural transformations $\mu_{\mathbf{false}}$ and μ_{b_1, b_2} which equip the parametric monad (T, μ) are defined as:





Although this may be easily checked directly, the fact that (T, μ) satisfies the coherence properties of a parametric monad is a consequence of our general transfer theorem, applied to the parametric monad S defined in (18) and to the adjunction $L \dashv R$.

Remark. The construction of the parametric monad T is not specific to dialogue categories. In particular, it would work for any monoidal category $\mathscr{A} = \mathscr{C}$ equipped with an adjunction $L \dashv R$ with its opposite category $\mathscr{B} = \mathscr{C}^{op(0,1)}$. Even more generally, for any category $\mathscr{A}$ equipped with an adjunction $L \dashv R$ with a monoidal category $(\mathscr{B}, \otimes, \mathbf{false})$.

4. Commutators between parametric monads

At this point, we are ready to introduce the notion of *commutator* between parametric monads, and to establish at the same time that every dialogue chirality is equipped with such a structure. As we will see, the notion of commutator unifies and generalizes the celebrated notions of *tensorial strength* on the one hand, and of *distributivity law* between two monads on the other hand.

4.1. *Definition*

We suppose given a 0-cell $\mathscr{C}$ in a 2-category $\mathscr{W}$ equipped with a parametric $\mathscr{J}$-monad

$$T = \bullet \quad : \quad \mathscr{J} \quad \longrightarrow \quad \mathrm{End}(\mathscr{C})$$

and a parametric $\mathscr{M}^{op(0)}$-monad

$$S = \circ \quad : \quad \mathscr{M}^{op(0)} \quad \longrightarrow \quad \mathrm{End}(\mathscr{C})$$

with parameters taken in the monoidal categories $(\mathscr{J}, \otimes, \boldsymbol{e})$ and $(\mathscr{M}, \otimes, \boldsymbol{u})$.

Definition 2 (commutator). A commutator between two parametric monads $T = \bullet$ and $S = \circ$ is defined as a natural transformation

$$\kappa \quad : \quad S\,T \quad \Rightarrow \quad T\,S \quad : \quad \mathscr{J} \times \mathscr{M}^{op(0)} \quad \longrightarrow \quad \mathrm{End}(\mathscr{C})$$

making the four diagrams below commute

$$
\begin{tikzcd}
\begin{smallmatrix} \circ\, m \\ j\, \bullet \\ k\, \bullet \end{smallmatrix} \arrow[r, "\kappa"] \arrow[d, "\mu\ \bullet"'] & \begin{smallmatrix} j\, \bullet \\ \circ\, m \\ k\, \bullet \end{smallmatrix} \arrow[r, "\kappa"] & \begin{smallmatrix} j\, \bullet \\ k\, \bullet \\ \circ\, m \end{smallmatrix} \arrow[d, "\mu\ \bullet"] \\
\begin{smallmatrix} \circ\, m \\ j\otimes k\, \bullet \end{smallmatrix} \arrow[rr, "\kappa"] & & \begin{smallmatrix} j\otimes k\, \bullet \\ \circ\, m \end{smallmatrix}
\end{tikzcd}
\qquad
\begin{tikzcd}
& \circ\, m \arrow[dl, "\mu\ \bullet"'] \arrow[dr, "\mu\ \bullet"] & \\
\begin{smallmatrix} \circ\, m \\ \boldsymbol{e}\, \bullet \end{smallmatrix} \arrow[rr, "\kappa"'] & & \begin{smallmatrix} \boldsymbol{e}\, \bullet \\ \circ\, m \end{smallmatrix}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\begin{smallmatrix} \circ\, n \\ \circ\, m \\ j\, \bullet \end{smallmatrix} \arrow[r, "\kappa"] \arrow[d, "\mu\ \circ"'] & \begin{smallmatrix} \circ\, n \\ j\, \bullet \\ \circ\, m \end{smallmatrix} \arrow[r, "\kappa"] & \begin{smallmatrix} j\, \bullet \\ \circ\, n \\ \circ\, m \end{smallmatrix} \arrow[d, "\mu\ \circ"] \\
\begin{smallmatrix} \circ\, m\otimes n \\ j\, \bullet \end{smallmatrix} \arrow[rr, "\kappa"] & & \begin{smallmatrix} j\, \bullet \\ \circ\, m\otimes n \end{smallmatrix}
\end{tikzcd}
\qquad
\begin{tikzcd}
& j\, \bullet \arrow[dl, "\mu\ \circ"'] \arrow[dr, "\mu\ \circ"] & \\
\begin{smallmatrix} \circ\, \boldsymbol{u} \\ j\, \bullet \end{smallmatrix} \arrow[rr, "\kappa"'] & & \begin{smallmatrix} j\, \bullet \\ \circ\, \boldsymbol{u} \end{smallmatrix}
\end{tikzcd}
$$

for all objects i, j of the category $\mathscr{J}$ and all objects m, n of the category $\mathscr{M}$.

Remark. In the particular case when $\mathscr{W} = \boldsymbol{Cat}$, a commutator may be alternatively formulated as a natural transformation

$$\kappa \quad : \quad (- \bullet -) \circ - \quad \Rightarrow \quad - \bullet (- \circ -) \quad : \quad \mathscr{J} \times \mathscr{C} \times \mathscr{M} \quad \longrightarrow \quad \mathscr{C}$$

with components

$$\kappa_{j,m,A} \quad : \quad (j \bullet A) \circ m \quad \longrightarrow \quad j \bullet (A \circ m)$$

parametrized by the objects j of the category $\mathscr{J}$, m of the category $\mathscr{M}$ and A of the category $\mathscr{C}$.

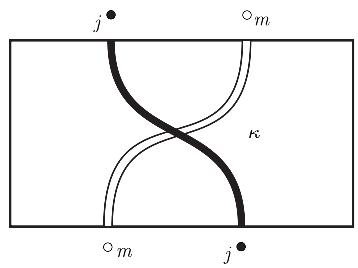
Remark. The question of extending Beck's theorem (Beck 1969) from distributivity laws between monads to general commutators between parametric monads is interesting, but outside the scope of this paper, and we thus prefer to leave it for later work. Let us simply observe at this stage that the existence of a commutator between a left $\mathscr{J}$-monad S and a right $\mathscr{M}$-monad T enables one to construct a $\mathscr{J} \times \mathscr{M}^{op(0)}$-monad noted $T \circ S$ on the 0-cell $\mathscr{A}$ on which the two monads S and T act in the 2-category $\mathscr{W}$. The parametric monad $T \circ S$ is defined as the family of 1-cells

$$(T \circ S)_{(j,m)} \quad = \quad T_m \circ S_j$$

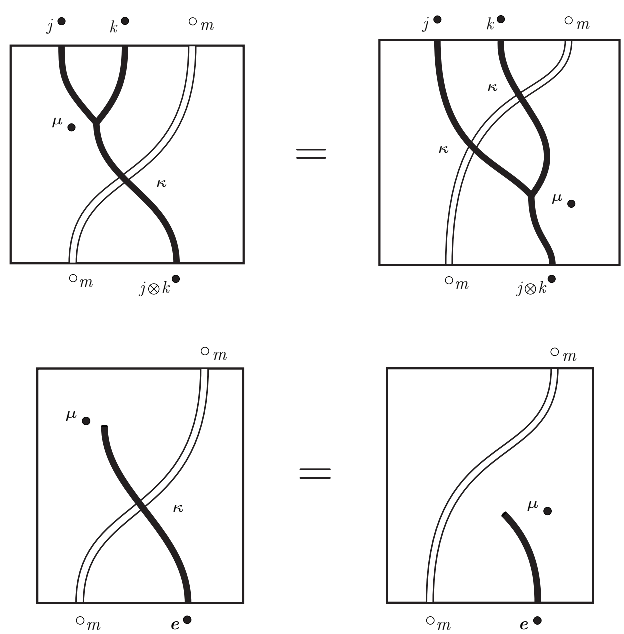
parametrized by the objects (j, m) of the category $\mathscr{J} \times \mathscr{M}^{op(0)}$. The commutator between S and T is used in the definition of the multiplicative structure μ of the parametric monad $T \circ S$. This observation justifies to think of a commutator as a lax notion of bimodule (or biaction).

4.2. *Commutators in string diagrams*

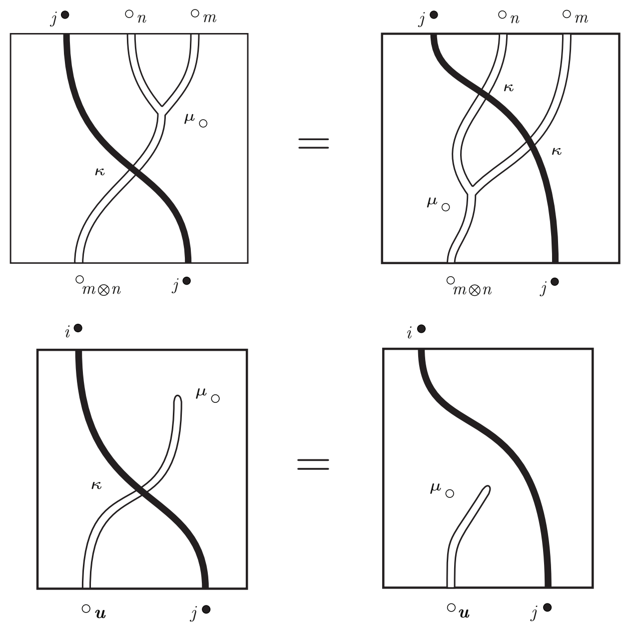
The notion of commutator may be depicted in string diagrams as follows. The basic idea is to depict the commutator κ itself as a braiding



commuting the string representing the action $\bullet$ over the string representing $\circ$. This notation enables to depict the coherence diagrams of the commutator κ as a series of topologically intuitive equations, permuting the multiplication and unit of each parametric monad under or over the string representing the other parametric monad. Typically, the first series of equations in the definition of a commutator "permutes" the operations $\mu_\bullet$ over the string representing the action $\circ$



while the second series of equations "permutes" the operations $\mu_\circ$ under the string representing the action $\bullet$



Remark. The notion of commutator may be easily adapted to the case of a parametric comonad commuting with a parametric comonad, or of a parametric monad commuting with a parametric comonad, with their associated string diagrams.

4.3. *Illustrations*

We show that the notion of commutator is sufficiently general to recover two well-known and apparently disjoint notions of commutation with a monad. The first example is provided by the notion of tensorial strength recalled in the introduction. It is essentially immediate that

Proposition 4. A tensorial strength

$$\sigma_{A,B} \quad : \quad T(A) \otimes B \quad \longrightarrow \quad T(A \otimes B)$$

is the same thing as a commutator between a monad T and the action of the monoidal category $(\mathscr{C}, \otimes, I)$ over itself.

The parametrization is given in that case by $\mathscr{J} = \mathbb{1}$ and $\mathscr{M} = \mathscr{C}$. The second example is provided by the notion of a distributivity law between two monads S and T. Recall that such a distributivity law is defined as a natural transformation

$$\lambda \quad : \quad S\,T \quad \Rightarrow \quad T\,S$$

making the four coherence diagrams

$$
\begin{tikzcd}
S\,S\,T \arrow[r, Rightarrow, "\lambda"] \arrow[d, Rightarrow, "\mu"'] & S\,T\,S \arrow[r, Rightarrow, "\lambda"] & S\,S\,T \arrow[d, Rightarrow, "\mu"] \\
S\,T \arrow[rr, Rightarrow, "\lambda"] & & T\,S
\end{tikzcd}
\qquad
\begin{tikzcd}
& T \arrow[dl, Rightarrow, "\eta"'] \arrow[dr, Rightarrow, "\eta"] & \\
S\,T \arrow[rr, Rightarrow, "\lambda"] & & T\,S
\end{tikzcd}
$$

$$
\begin{tikzcd}
S\,T\,T \arrow[r, Rightarrow, "\lambda"] \arrow[d, Rightarrow, "\mu"'] & T\,S\,T \arrow[r, Rightarrow, "\lambda"] & T\,T\,S \arrow[d, Rightarrow, "\mu"] \\
S\,T \arrow[rr, Rightarrow, "\lambda"] & & T\,S
\end{tikzcd}
\qquad
\begin{tikzcd}
& S \arrow[dl, Rightarrow, "\eta"'] \arrow[dr, Rightarrow, "\eta"] & \\
S\,T \arrow[rr, Rightarrow, "\lambda"] & & T\,S
\end{tikzcd}
$$

commute. Once again, it is essentially straightforward that

Proposition 5. A distributivity law between two monads S and T is the same thing a commutator between them.

The parametrization is given in that case by $\mathscr{J} = \mathbb{1}$ and $\mathscr{M} = \mathbb{1}$.

5. The double negation commutator

In this final section, we conclude the paper and show that in every dialogue chirality $(\mathscr{A}, \mathscr{B})$, the strength

$$\sigma_{a_1,a_2} \quad : \quad RL(a_1) \oslash a_2 \quad \longrightarrow \quad RL(a_1 \oslash a_2)$$

of the continuation monad $T = R \circ L$ is the emerged fragment of a much wider structure, provided by a commutator

$$\kappa_{\,b,a_1,a_2} \quad : \quad (b \circledast a_1) \oslash a_2 \quad \longrightarrow \quad b \circledast (a_1 \oslash a_2)$$

between the parametric continuation monad $T = \circledast$ and the action $S = \oslash$ of the monoidal category $(\mathscr{A}, \oslash, \mathbf{true})$ over itself. Quite obviously, the strength σ_{a_1,a_2} is recovered by instantiating the commutator κ_{b,a_1,a_2} at the specific instance $b = \mathsf{false}$. In order to construct the commutator κ in every dialogue chirality, we find convenient to introduce first the notion of transjunction which provides a pleasant and illuminating shortcut to the construction.

5.1. *Formal transjunctions*

The notion of formal transjunction in a 2-category $\mathscr{W}$ refines the notion of formal adjunction recalled in §2.1 to a situation where the 0-cells $\mathscr{A}$ and $\mathscr{B}$ are themselves replaced by formal adjunctions.

Definition 3 (transjunction). Suppose given a pair of formal adjunctions

$$\begin{tikzcd} \mathscr{A}_1 \arrow[rr, bend left, "L_1"] \arrow[rr, phantom, "\perp"] && \mathscr{B}_1 \arrow[ll, bend left, "R_1"] \end{tikzcd} \qquad\qquad \begin{tikzcd} \mathscr{A}_2 \arrow[rr, bend left, "L_2"] \arrow[rr, phantom, "\perp"] && \mathscr{B}_2 \arrow[ll, bend left, "R_2"] \end{tikzcd}$$

whose units and counits are denoted η_1, η_2 and $\varepsilon_1, \varepsilon_2$ respectively. A formal transjunction $F \dashv\!\!\!\dashv G$ between a pair of 1-cells

$$F \;:\; \mathscr{A}_1 \;\rightarrow\; \mathscr{A}_2 \qquad\qquad G \;:\; \mathscr{B}_2 \;\rightarrow\; \mathscr{B}_1$$

accross the adjunctions $L_1 \dashv R_1$ and $L_2 \dashv R_2$ is defined as a pair of natural transformations

$$\mathbf{axiom} : L_1 \Rightarrow G \circ L_2 \circ F \qquad\qquad \mathbf{cut} : F \circ R_1 \circ G \Rightarrow R_2$$

making the two diagrams

$$\begin{tikzcd} F \circ R_1 \circ L_1 \arrow[r, Rightarrow, "\mathbf{axiom}"] & F \circ R_1 \circ G \circ L_2 \circ F \arrow[d, Rightarrow, "\mathbf{cut}"] \\ F \arrow[u, Rightarrow, "\eta_1"] \arrow[r, Rightarrow, "\eta_2"] \arrow[ur, phantom, "(a)"] & R_2 \circ L_2 \circ F \end{tikzcd} \qquad \begin{tikzcd} G \circ L_2 \circ F \circ R_1 \circ G \arrow[r, Rightarrow, "\mathbf{cut}"] & G \circ L_2 \circ R_2 \arrow[d, Rightarrow, "\varepsilon_2"] \\ L_1 \circ R_1 \circ G \arrow[u, Rightarrow, "\mathbf{axiom}"] \arrow[r, Rightarrow, "\varepsilon_1"] \arrow[ur, phantom, "(b)"] & G \end{tikzcd}$$

commute.

The notion of transjunction is ultimately justified by the following observation, which holds in every 2-category $\mathscr{W}$.

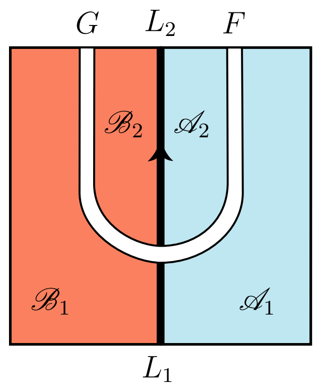
Proposition 6. A transjunction $F \dashv\!\!\!\dashv G$ accross the adjunctions $L_1 \dashv R_1$ and $L_2 \dashv R_2$ is the same thing as a formal adjunction $L_2 \circ F \dashv R_1 \circ G$.

Side remark. Given a pair of 1-cells $F : \mathscr{C}_1 \rightarrow \mathscr{C}_2$ and $G : \mathscr{C}_2 \rightarrow \mathscr{C}_1$ of the 2-category $\mathscr{W}$, a formal adjunction $F \dashv G$ is the same thing as a formal transjunction $F \dashv\!\!\!\dashv G$ accross the identity 1-cells $L_1 = R_1 = id_{\mathscr{C}_1}$ and $L_2 = R_2 = id_{\mathscr{C}_2}$ where $\mathscr{A}_1 = \mathscr{B}_1 = \mathscr{C}_1$ and $\mathscr{A}_2 = \mathscr{B}_2 = \mathscr{C}_2$.

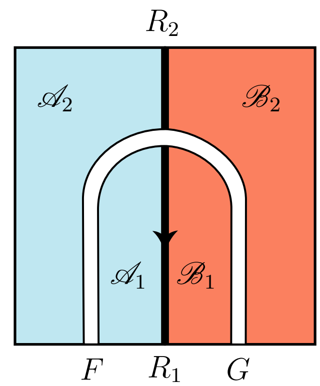
5.2. *Transjunctions in string diagrams*

These various equations between 2-cells may be alternatively depicted as string diagrams living in the ambient 2-category $\mathscr{W}$. First of all, the generators **axiom** and **cut** of a transjunction $F \dashv\!\!\!\dashv G$ accross the adjunctions $L_1 \dashv R_1$ and $L_2 \dashv R_2$ are depicted as

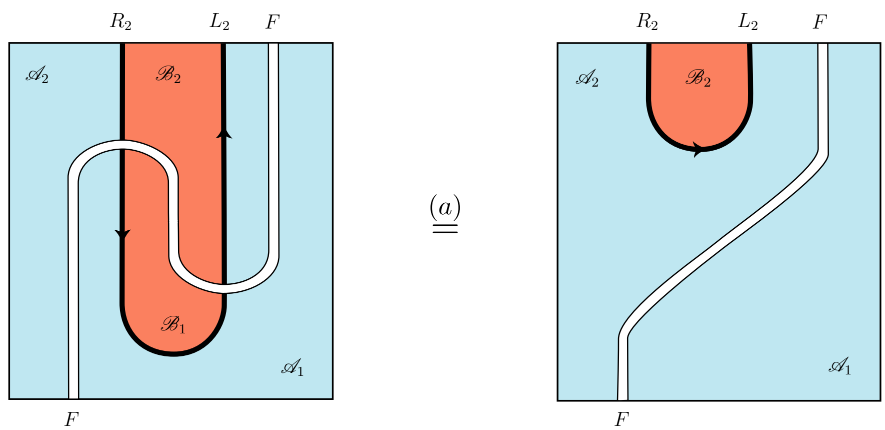
axiom =

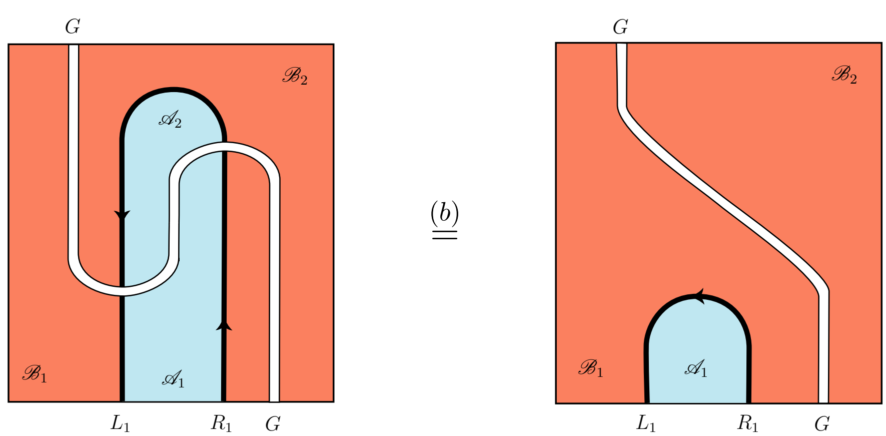


cut =



Then, the two coherence equations (a) and (b) of Definition 3 are depicted as:





Note that one recovers in this way a pair of equations akin to the cut-axiom rule of proof-nets in linear logic. One main difference with linear logic is that the background in tensorial logic (and in transjunctions) is polychromic rather than monochromic — with the 0-cells $\mathscr{A}_i$ in blue (or light grey) and $\mathscr{B}_i$ in red (or dark grey) separated by the oriented boundary defined by the 1-cells L_i and R_i for $i = 1, 2$.

5.3. *Transjunction homomorphism*

It is also useful to consider a notion of homorphism between transjunctions, which gives rise to a category of transjunctions.

Definition 4 (homomorphism). A homomorphism

$$(f, g) \quad : \quad F \mathrel{-\!\!\ni} G \quad \longrightarrow \quad F' \mathrel{-\!\!\ni} G'$$

between two transjunctions $F \mathrel{-\!\!\ni} G$ and $F' \mathrel{-\!\!\ni} G'$ accross the same adjunctions $L_1 \dashv R_1$ and $L_2 \dashv R_2$ is defined as a pair of natural transformations

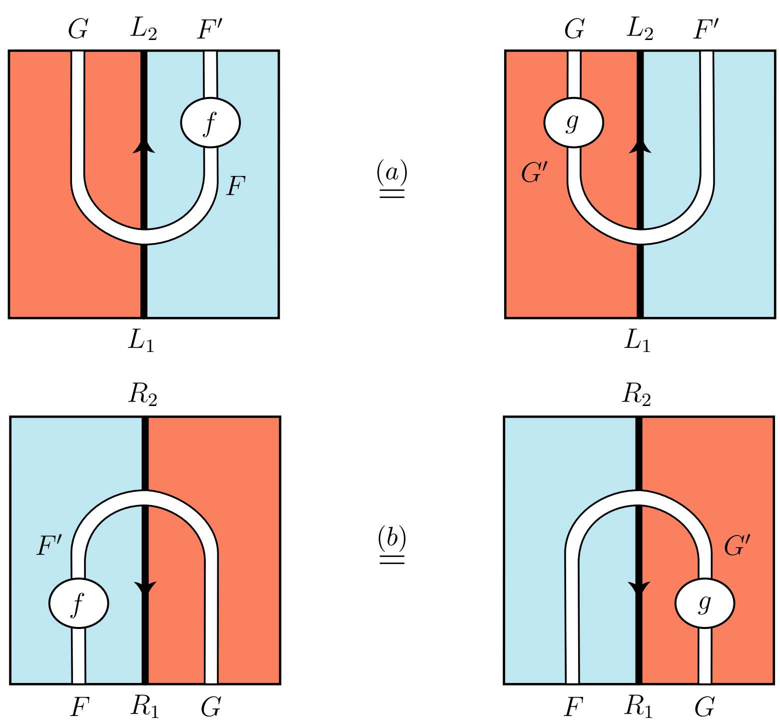
$$f \quad : \quad F \quad \Rightarrow \quad F' \qquad\qquad g \quad : \quad G' \quad \Rightarrow \quad G$$

making the two diagrams

$$\begin{tikzcd}
G \circ L_2 \circ F \arrow[r, Rightarrow, "f"] \arrow[dr, phantom, "(a)"] & G \circ L_2 \circ F' \\
L_1 \arrow[u, Rightarrow, "\textbf{axiom}"] \arrow[r, Rightarrow, "\textbf{axiom}'"] & G' \circ L_2 \circ F' \arrow[u, Rightarrow, "g"']
\end{tikzcd}
\qquad
\begin{tikzcd}
F' \circ R_1 \circ G' \arrow[r, Rightarrow, "\textbf{cut}'"] \arrow[dr, phantom, "(b)"] & R_2 \\
F \circ R_1 \circ G' \arrow[u, Rightarrow, "f"] \arrow[r, Rightarrow, "g"] & F \circ R_1 \circ G \arrow[u, Rightarrow, "\textbf{cut}"']
\end{tikzcd}$$

commute.

Pictorially, such a homomorphism (f, g) is a pair of natural transformations $f : F \Rightarrow F'$ and $g : G' \Rightarrow G$ satisfying the pictorial equalities below:



5.4. *The continuation commutator*

At this final stage of the paper, we are ready to establish that

Proposition 7. Every dialogue chirality is equipped with a commutator

$$\kappa_{\,b,a,m} \quad : \quad (b \circledast a) \owedge m \quad \longrightarrow \quad b \circledast (a \owedge m) \tag{19}$$

between

— the parametric monad $T = \circledast$ acting on the category $\mathscr{A}$, with parameters taken in the monoidal category $(\mathscr{B}, \ovee, \mathbf{false})$,

— the monoidal action $S = \owedge$ of the monoidal category $\mathscr{A}$ over itself.

The construction of the commutator κ is based on the observation that every dialogue chirality is equipped with a family of adjunctions

$$L(- \owedge m) \quad \dashv \quad R(- \ovee m^*) \tag{20}$$

parametrized by the objects m of the category $\mathscr{A}$. Here, the functor

$$(-)^* \quad : \quad \mathscr{A} \quad \longrightarrow \quad \mathscr{B}^{\,op(0,1)}$$

denotes the *change of frame* consisting in transporting an object m in the category $\mathscr{A} = \mathscr{C}$ to the same object m^* seen this time in the opposite category $\mathscr{B} = \mathscr{C}^{\,op(0,1)}$. Note that this adjunction (19) formulated in the style of dialogue chiralities corresponds in the language of dialogue categories to the adjunction

$$(- \otimes A) \multimap \bot \quad \dashv \quad \bot \multimapinv (A \otimes -)$$

which generalizes the adjunction (6) and identifies it as the particular instance where $A = I$ is the tensorial unit of the dialogue category. Each adjunction (20) may be alternatively seen as a transjunction

$$(- \owedge m) \quad \dashv\!\!\!\!\!\!\supset \quad (- \ovee m^*)$$

accross the adjunction $L \dashv R$, presented by the natural transformation

$$\mathbf{axiom}[m] \quad : \quad L(a) \quad \longrightarrow \quad L(a \owedge m) \ovee m^*$$

$$\mathbf{cut}[m] \quad : \quad R(b \ovee m^*) \owedge m \quad \longrightarrow \quad R(b)$$

parametrized by the objects m, a of the category $\mathscr{A}$ and the objects b of the category $\mathscr{B}$. At this point, the morphism

$$\kappa_{\,b,a,m} \quad : \quad R(b \ovee L(a)) \owedge m \quad \longrightarrow \quad R(b \ovee L(a \owedge m))$$

is simply obtained by composing these two combinators and the associativity law of $\mathscr{B}$ in an appropriate fashion:

$$\begin{tikzcd}
R(b \ovee L(a)) \owedge m \arrow[d, "\mathbf{axiom}[m]"'] & R(b \ovee L(a \owedge m)) \\
R(b \ovee (L(a \owedge m) \ovee m^*)) \owedge m \arrow[r, "associativity"] & R((b \ovee L(a \owedge m)) \ovee m^*) \owedge m \arrow[u, "\mathbf{cut}[m]"']
\end{tikzcd}$$

It is not difficult to check that the resulting natural transformation κ satisfies all the coherence diagrams of §4.1 and thus defines a commutator between the parametric continuation monad $T = \circledast$ and the action $S = \oslash$ of the monoidal category $\mathscr{A}$ over itself. This concludes the proof of Proposition 7.

6. Conclusion

The present paper is part of a series of articles ($\otimes_1$; $\otimes_2$; $\otimes_4$; $\otimes_5$) whose purpose is to provide a type-theoretic status to game semantics, based on the algebraic study of negation in dialogue categories. As such, the paper may be also seen as one ingredient in the wider project of adapting to tensorial logic and to dialogue categories the combinatorial presentation of linear logic and of $*$-autonomous categories elaborated in (Cockett & Seely 1997; Blute & Cockett & Seely & Trimble 1996).

References

Robert Atkey. Parameterised notions of computation. Journal of Functional Programming 19(3 & 4), pages 335-376. 2009.

Jon Beck. Distributive laws. Lecture Notes in Mathematics. Lecture Notes in Mathematics 80: 119–140. (1969)

Jean Bénabou. Catégories avec multiplication. C. R. Académie des Sciences, Paris 256 (1963), 1887-1890.

Robin Cockett, Robert Seely. Weakly Distributive Categories. Journal of Pure and Applied Algebra 114 (1997) 2, pp 133-173.

Richard Blute, Robin Cockett, Robert Seely, and Todd Trimble. Natural deduction and coherence for weakly distributive categories. Journal of Pure and Applied Algebra, 113:229-296, 1996.

Jean-Yves Girard. Linear Logic. Theoretical Computer Science, London Mathematical 50:1, pp. 1-102, 1987

——————— Linear Logic, its syntax and semantics. Advances in Linear Logic, eds Girard, Lafont, Regnier, London Mathematical Society Lecture Notes Series 222, Cambridge University Press 1995.

Hongde Hu, André Joyal. Coherence completions of categories. Theoretical Computer Science 01/1999; 2 27:153-184, (1999).

Max Kelly, Stephen Lack. Monoidal functors generated by adjunctions, with applications to transport of structure. Fields Institute Communications 43:319-340, 2004.

Max Kelly, Ross Street. Review of the elements of 2-categories. Proc. Sydney Category Theory Seminar 1972/73, LNM 420, Springer (1974) 75 ? 103.

Anders Kock. On double dualization monads. Math. Scand 27 (1970), 151-165.

Eugenio Moggi. Notions of computation and monads. Information And Computation, 93(1), 1991.

Paul-André Melliès. A transfer theorem along left adjoints. Unpublished manuscript, 2006.

——————— Categorical semantics of linear logic. Published in *Interactive models of computation and program behaviour*. Panoramas et Synthèses 27, Société Mathématique de France, 2009.

—— $\otimes_1$—— Game semantics in string diagrams. Proceedings of the Annual Symposium on Logic in Computer Science, 2012.

—— $\otimes_2$—— Dialogue categories and chiralities. Submitted. Manuscript available on the author's webpage, 2012.

—— $\otimes_4$—— A micrological study of negation. Submitted. Manuscript available on the author's webpage, 2013.
—— $\otimes_5$—— A micrological study of helical negation. Submitted. Manuscript available on the author's webpage, 2013.
Paul-André Melliès, Nicolas Tabareau. Resource modalities in tensorial logic. Annals of Pure and Applied Logic, 2009, vol. 161, number 5, pages 632-653.
Peter Selinger. A survey of graphical languages for monoidal categories New Structures for Physics, Springer Lecture Notes in Physics 813, pp. 289-355, 2011.
Tarmo Uustalu. Generalizing substitution. Theoretical Informatics and Applications, vol 37, number 4, pages 315-336, 2003.

Appendix: a general 2-categorical transfer theorem

In this Appendix, we would like to show that the transfer theorem (Proposition 2) established in §2.6 is a particular case of a more general 2-categorical property — at least when $\mathscr{W}$ coincides with the 2-category $\boldsymbol{Cat}$ of categories, functors and natural transformations. In that case, the 2-functor

$$T \quad : \quad \mathscr{X} \quad \mapsto \quad \mathscr{J} \times \mathscr{X} \quad : \quad \boldsymbol{Cat} \quad \longrightarrow \quad \boldsymbol{Cat} \tag{21}$$

defines a weak 2-monad for every monoidal category $(\mathscr{J}, \otimes, I)$ and a parametric $\mathscr{J}$-monad on a category $\mathscr{C}$ is the same thing as a lax T-algebra

$$* \quad : \quad \mathscr{J} \times \mathscr{C} \quad \longrightarrow \quad \mathscr{C}.$$

Recall that a lax T-algebra for a weak 2-monad (T, m, e) in a 2-category $\mathscr{W}$ is a 1-cell

$$* \quad : \quad T\mathscr{C} \quad \longrightarrow \quad \mathscr{C}$$

together with a pair of 2-cells

$$\begin{tikzcd} \mathscr{C} \arrow[rr, "id"] \arrow[dr, "e"'] & \arrow[d, phantom, "\Downarrow \mu"] & \mathscr{C} \\ & T\mathscr{C} \arrow[ur, "*"'] & \end{tikzcd} \qquad \begin{tikzcd} & T\mathscr{C} \arrow[dr, "*"] & \\ TT\mathscr{C} \arrow[ur, "*"] \arrow[dr, "m"'] & \Downarrow \mu & \mathscr{C} \\ & T\mathscr{C} \arrow[ur, "*"'] & \end{tikzcd}$$

making the expected coherence diagrams commute. Now, a general transfer theorem established in (Melliès 2006) states that given a formal adjunction

$$\begin{tikzcd} \mathscr{A} \arrow[rr, bend left, "L"] \arrow[rr, phantom, "\perp"] & & \mathscr{B} \arrow[ll, bend left, "R"] \end{tikzcd}$$

in a 2-category $\mathscr{W}$ equipped with a weak 2-monad T, every lax T-algebra

$$* \quad : \quad T\mathscr{B} \quad \longrightarrow \quad \mathscr{B}$$

on the 0-cell $\mathscr{B}$ induces a lax T-algebra structure on the 0-cell $\mathscr{A}$, defined as follows:

$$\circledast \quad : \quad T\mathscr{A} \quad \xrightarrow{TL} \quad T\mathscr{B} \quad \xrightarrow{*} \quad \mathscr{B} \quad \xrightarrow{R} \quad \mathscr{A}.$$

In the particular case when $\mathscr{W} = \boldsymbol{Cat}$, one recovers our original transfer theorem (Proposition 2) by applying the result to the weak 2-monad (21). The general transfer theorem may be also applied to the 2-monad

$$T \quad : \quad \boldsymbol{Cat} \quad \longrightarrow \quad \boldsymbol{Cat}$$

which transports every category $\mathscr{C}$ to its free monoidal category $T\mathscr{C}$. In that case, the transfer theorem applied to a dialogue chirality establishes that the monoidal structure $(\mathscr{B}, \oslash, \text{false})$ induces a lax monoidal structure on the category $\mathscr{A}$, provided by the family of n-ary disjunctions

$$[\, A_1 \parr \cdots \parr A_n \,] \quad = \quad R\,(\, LA_1 \oslash \cdots \oslash LA_n \,).$$

This algebraic construction is important because it provides a way to adapt to tensorial logic the familiar definition (7) of the $\parr$ connective in linear logic. The key idea is to replace the binary disjunction of linear logic by a family of n-ary disjunctions. The reason for moving to a family of connectives is that the tensorial version of binary disjunction is not associative — in the sense that the two objects

$$[\,[\, A \parr B \,] \parr C \,] \qquad\qquad [\, A \parr [\, B \parr C \,]\,]$$

are in general not isomorphic in a dialogue category. However, the family of n-ary disjunctions is itself associative in some sense, but in a more subtle and oriented fashion. For instance, there are canonical proofs of tensorial logic connecting the two clusters of binary disjunctions above with the ternary disjunction:

$$[\,[\, A \parr B \,] \parr C \,] \quad \longrightarrow \quad [\, A \parr B \parr C \,] \quad \longleftarrow \quad [\, A \parr [\, B \parr C \,]\,].$$

The point is that these canonical associativity maps are not invertible in general. This purely algebraic analysis clarifies in what sense the linear disjunction $\parr$ living in the dialogue category $\mathscr{A} = \mathscr{C}$ is derived by *deformation* — one should probably say by *adjunction* in that case — from the disjunction $\oslash$ living in the opposite category $\mathscr{B} = \mathscr{C}^{\,op(0,1)}$.