6

Differential Topology

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The basic objects studied in differential topology are smooth manifolds, sometimes with boundary, and smooth mappings between such manifolds. Here the word "smooth" is used to mean "differentiable of class C^∞." To give a rough idea of the flavor of this field, let us list a few of its central problems.

The Diffeomorphism Problem. Given two smooth manifolds M and M', how can we decide whether or not there is a diffeomorphism from M to M' (i.e., a smooth homeomorphism with smooth inverse)?

The Cobordism Problem. Given a smooth compact manifold M without boundary, does there exist a smooth compact manifold W whose boundary is equal to M? We may refine this problem by putting extra structures on both M and W. For example, we can require an orientation, or a weakly complex structure, or we may require that M and W should be k-connected (compare [10]).

The Imbedding Problem. Given M and M', does there exist a smooth imbedding $M \to M'$? If so, can we classify all such imbeddings? For example, the problem of classifying imbeddings of the circle in 3-space forms the field of "knot theory."

CHARACTERIZATIONS OF THE n-SPHERE

Let us single out one particular case of the diffeomorphism problem for consideration, namely the problem of characterizing the n-sphere. The many different tools which can be brought to bear on this one question will provide a survey of much of the field of differential topology.

First let us ask what conditions on a smooth manifold guarantee that it is *homeomorphic* to S^n.

The first such characterization is due to Reeb [16]. Let M be a smooth n-dimensional manifold. By a *Morse function* f on M will be meant a smooth real valued function whose critical points are all nondegenerate. Thus in a neighborhood of each critical point we can choose local coordinates $u_1, \ldots, u_n$ so that $f =$ constant $\pm u_1^2 \pm \cdots \pm u_n^2$. (Morse [12, Lemma 4] or [11, §2.2].)

Theorem 1. If M is compact, without boundary, and possesses a Morse function with only two critical points, then M is homeomorphic to S^n.

OUTLINE OF PROOF. (See Figure 1.) Let m_0, m_1 denote the minimum and maximum of f. Thus in a neighborhood of its minimum point we have

$$f = m_0 + u_1^2 + \cdots + u_n^2$$

It follows that the set of points x where $f(x) \leq m_0 + \epsilon$ is diffeomorphic to the disk D^n. Similarly, the set of x with $f(x) \geq m_1 - \epsilon$

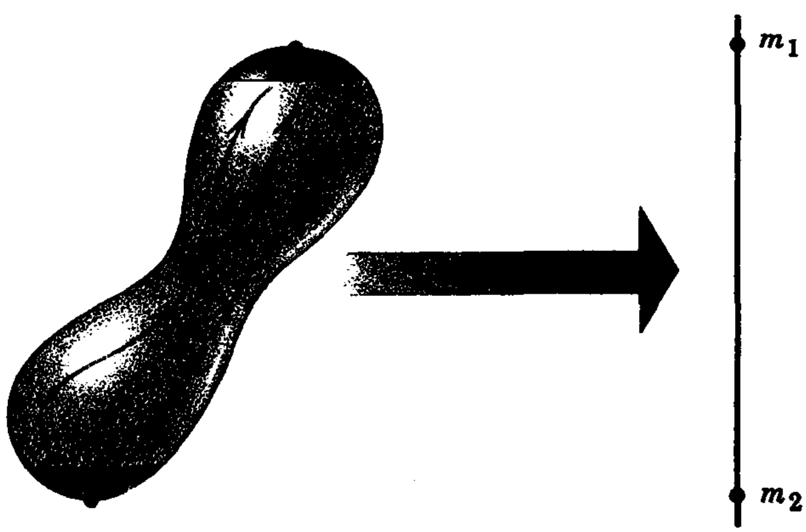


Figure 1

is diffeomorphic to D^n. But, deforming M into itself along the gradient lines of the function f [i.e., along the orthogonal trajectories of the surfaces f^{-1} (constant)] we can slide the disk, $f \leq m_0 + \epsilon$, up so that it covers precisely the disk $f \leq m_1 - \epsilon$. (Compare [7] or [11, §§3, 4].) Thus M can be considered as the union of two imbedded disks which intersect only along their common boundary. But this implies that M is homeomorphic to S^n. To see this, consider first the following.

Lemma 1. Any homeomorphism $h: S^{n-1} \to S^{n-1}$ extends to a homeomorphism $H: D^n \to D^n$.

PROOF. Set $H(tu) = th(u)$ for $0 \leq t \leq 1$, where u denotes an arbitrary unit vector.

Now if M is the union of two topologically imbedded disks $g_0(D^n)$, $g_1(D^n)$ which intersect precisely along their common boundary, we can first choose any homeomorphism from $g_0(S^{n-1}) = g_1(S^{n-1})$ to the equator of S^n, and then extend, using Lemma 1, to a homeomorphism which maps $g_0(D^n)$ to the southern hemisphere and $g_1(S^{n-1})$ to the northern hemisphere. This completes the proof.

Remark. Note that the differentiable structure is destroyed in the course of this proof. The reason is that there is no differentiable analogue of Lemma 1. (Even if $h: S^{n-1} \to S^{n-1}$ is a diffeomorphism, the extension we have constructed is highly nondifferentiable at the origin, unless h happens to be orthogonal.) We will return to this point later, in Section 3.

Following is another partial characterization of S^n (compare [17, 8]).

Theorem 2. Let M be a compact smooth manifold which is the union of two open sets, each diffeomorphic to a euclidean space. Then M is homeomorphic to a sphere.

In fact the proof will show that M with a single point deleted is diffeomorphic to the euclidean space R^n. The proof is based on two lemmas, both of which are interesting in their own right.

Lemma 2 (Palais and Cerf). Let f_1 and f_2 be smooth, orientation preserving imbeddings of the disk D^n into the interior of a connected manifold M^n. Then there exists a diffeomorphism h of M^n onto itself so that $h \circ f_1 = f_2$.

For the proof, which is quite elementary, the reader is referred to [3] or [14].

Lemma 3 (*Brown, Stallings*). *Let M be a paracompact manifold such that every compact subset is contained in an open set diffeomorphic to euclidean space. Then M itself is diffeomorphic to euclidean space.*

PROOF. (Compare [2, 21].) It is not difficult to show that M is a monotone union of disks. That is, we can find submanifolds with boundary

$$W_1 \subset W_2 \subset W_3 \cdots \subset M$$

with union M so that each W_i is diffeomorphic to D^n, and so that each W_i is contained in the interior of W_{i+1}. We wish to compare this sequence with the sequence

$$D_1{}^n \subset D_2{}^n \subset D_3{}^n \subset \cdots \subset R^n$$

where $D_i{}^n$ denotes the disk of radius i in euclidean space. Start with any diffeomorphism $f_1: D_1{}^n \to W_1$. Using Lemma 2, this can be extended to a diffeomorphism $f_2: D_2{}^n \to W_2$, and so on. (To see this, consider the following diagram

$$\begin{tikzcd} W_i \arrow[r, phantom, "\subset"] & W_{i+1} & \\ & & W_{i+1} \arrow[ul, dashed, "h"'] \\ D_i{}^n \arrow[uu, "f_i"] \arrow[r, phantom, "\subset"] & D^n_{i+1} \arrow[ur, "g"] & \end{tikzcd}$$

where g is an arbitrary orientation preserving diffeomorphism. Choosing a diffeomorphism h as in Lemma 2, we can now set f_{i+1} equal to $h \circ g$.) Finally, piecing together all these diffeomorphisms f_i, we obtain the required diffeomorphism $R^n \to M$.

Using these lemmas we can prove a sharpened form of Theorem 1. Let M be compact, without boundary, and let $f: M \to R$ be a smooth function with only two critical points.

Theorem 1′. Even if these critical points are allowed to be degenerate, it still follows that M is homeomorphic to S^n.

PROOF. (See Figure 2.) Let p and q be the critical points and let U be a neighborhood of p which is diffeomorphic to R^n, with $q \notin U$. By deforming M into itself along the gradient lines of f, we can stretch U so that it covers any compact subset of $M - q$.

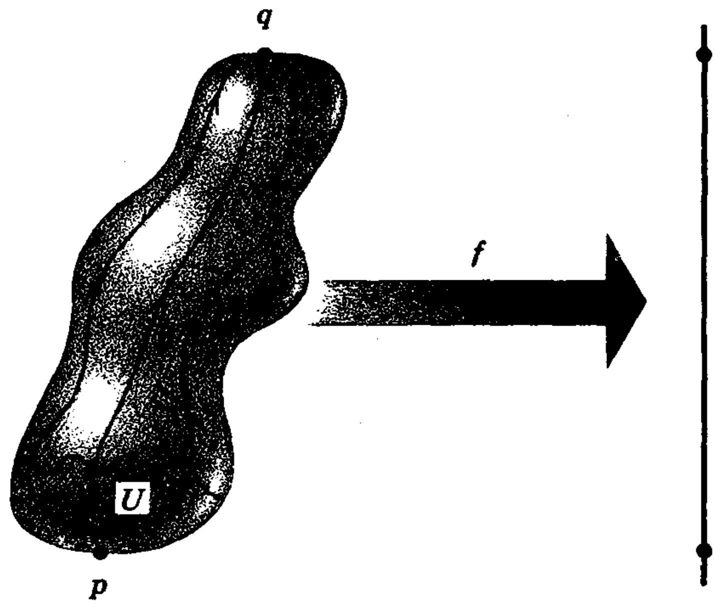


Figure 2

Hence it follows from Lemma 3 that $M - q$ is diffeomorphic to R^n, which clearly completes the proof.

The proof of Theorem 2 is similar. Let M be covered by open sets U and V which are diffeomorphic to R^n (Figure 3). Given $q \in M - U \subset V$ we will show that any compact subset of $M - q$ can be covered by an open set diffeomorphic to U.

Since V is diffeomorphic to R^n, it is easy to construct a diffeomorphism $h: V \to V$ which (1) satisfies $h(q) = q$, (2) shrinks the

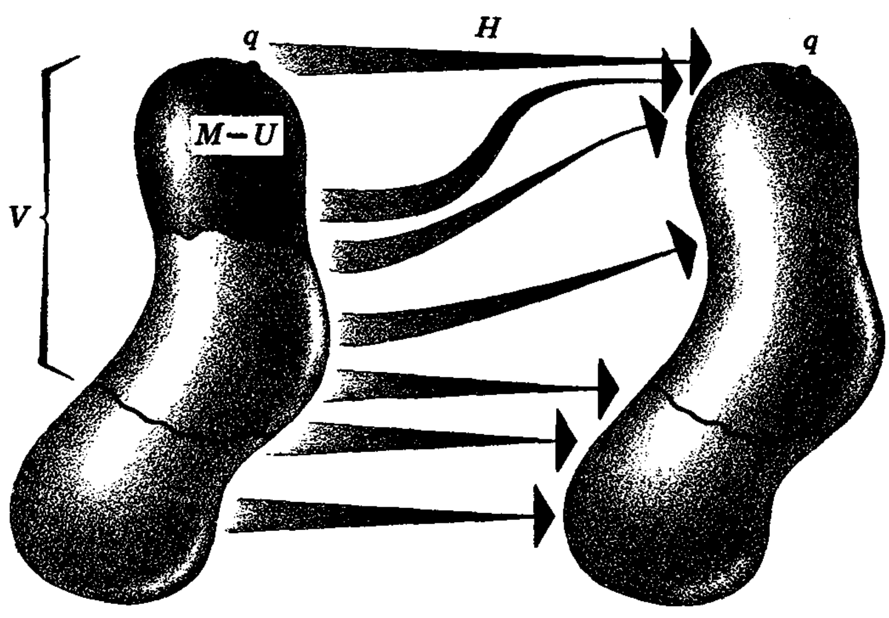


Figure 3

compact set $M - U$ down into an arbitrarily small neighborhood of q, and (3) coincides with the identity outside of a larger compact set. It follows from (3) that h extends to a diffeomorphism H of M. Clearly any compact subset of $M - q$ can be covered by $H(U)$ for suitable choice of the diffeomorphism H. Therefore it follows from Lemma 3 that $M - q$ is diffeomorphic to R^n. This completes the proof of Theorem 2.

One of the most striking properties of the sphere S^n is that the complement of each point is contractible.

Problem 1. Given a smooth manifold M *such that* M − p *is contractible, does it follow that* M *is homeomorphic to a sphere?*

It follows without too much difficulty from this hypothesis that M is compact and has the homotopy type of a sphere. Conversely, if M is compact, without boundary, and has the homotopy type of a sphere, it can be shown that $M - p$ is contractible.

If the dimension n *of* M *is* 0, 1, *or* 2, *then* M *must actually be diffeomorphic to* S^{n}. Here is a proof for $n = 2$. Choose a Riemannian metric and an orientation for M. By a classical theorem (the existence of "isothermal coordinates") each small neighborhood in M can be mapped conformally and diffeomorphically onto a region in the plane. Thus M becomes a Riemann surface. Since M is simply connected, the classical "uniformization theorem" asserts that M is conformally diffeomorphic to either the complex plane, the open unit disk, or the Gauss 2-sphere. But only the last possibility satisfies our hypothesis.

If the dimension n is 3, we have the classical Poincaré problem. So far, all attempts to solve this problem have foundered. For $n = 4$ the problem is also unsolved.

For $n \geq 5$ the problem has been solved affirmatively by Stallings and Zeeman [20, 24] and by Smale. In particular we have the following.

Theorem of Smale [19]. *If M is a smooth homotopy n-sphere of dimension $n \geq 5$, then M admits a Morse function with only two critical points.*

Hence Theorem 1 implies that M is homeomorphic to the n-sphere.

The proof of this theorem is much more difficult than anything we have encountered so far. The basic idea can be outlined very

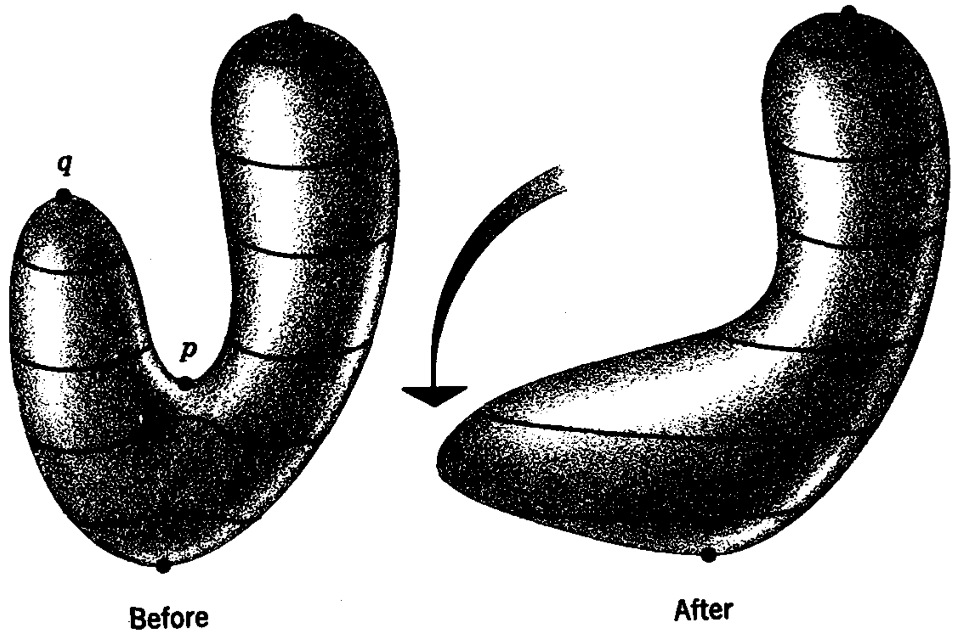


Figure 4

roughly as follows. It is not difficult to construct a Morse function $f: M \to R$ with many critical points. Furthermore f can be chosen so that the critical points occur in the proper order, in the following sense. If

$$f(u_1, \ldots, u_n) = \text{constant} - u_1{}^2 - \cdots - u_\lambda{}^2 + u_{\lambda+1}^2 + \cdots + u_n{}^2$$

in terms of local coordinates $u_1, \ldots, u_n$ near a critical point p, then the integer $\lambda = \lambda(p)$ is called the *index* of this critical point. Now f can be chosen so that $f(p) = \lambda(p)$ for each critical point p. Thus the minimum points, with $\lambda = 0$, occur in $f^{-1}(0)$, the maximum points occur in $f^{-1}(n)$, and the remaining critical points come in between.

The difficult part of the proof now consists in showing that a critical point of index λ and a critical point of index $\lambda + 1$ can sometimes be mutually cancelled. Thus in Figure 4 the critical point p of index 1 and the critical point q of index 2 can be mutually eliminated, by suitably changing the function. Repeating this procedure over and over, we eventually eliminate all critical points which are not essential in order to give the manifold M its proper homology groups. But a homotopy sphere has homology only in dimensions 0 and n. Hence we are left with only two critical points, with indices 0 and n respectively.

2. SOME EXOTIC SPHERES

This section will construct an example of a manifold which is homeomorphic, but not diffeomorphic, to a sphere. The proof will be based on the Hirzebruch signature theorem.

Let V be a compact oriented manifold of dimension $4k$. The signature (or index*) $\sigma(V)$ is defined as follows, following Hermann Weyl [22, p. 41]. Let $\alpha, \beta \in H_{2k}(V; Z)$ be homology classes with integer coefficients. Then the intersection number $\alpha \cdot \beta$ is a well-defined integer. This intersection number is symmetric in α and β since the dimension $2k$ is even. Passing to real coefficients, we can choose a basis $\alpha_1, \ldots, \alpha_r$ for the vector space $H_{2k}(V; R)$ so that the matrix of intersection numbers $(\alpha_i \cdot \alpha_j)$ is diagonal. Now the number of positive diagonal entries minus the number of negative diagonal entries is called the *signature* σ of V.

The following fundamental observation is due to Thom. *Suppose that* V *is the boundary of a compact oriented manifold* W^{4k+1}. *Then the signature* σ(V) *is zero.* Thom's proof is based on the Poincaré duality theorem.

We will also need to make use of the *Pontrjagin classes* of a smooth manifold M. Without attempting a definition, here are some basic properties.

1. To each smooth manifold M there are associated cohomology classes $p_i \in H^{4i}(M; Z)$ for $i = 1, 2, 3, \ldots$.
2. If U is an open subset of M, then $p_i(U)$ is equal to $p_i(M)$ restricted to U.
3. If M is parallelizable, then $p_i(M) = 0$.

Now let V be a closed oriented manifold of dimension $4k$. If $i_1 + i_2 + \cdots + i_r = k$, then the cohomology class $p_{i_1}p_{i_2} \cdots p_{i_r} \in H^{4k}(V; Z)$ gives rise to an integer which is denoted by $p_{i_1} \cdots p_{i_r}[V]$. These integers are called the *Pontrjagin numbers* of V.

If V is a boundary then these Pontrjagin numbers are all zero. Conversely, we have the following:

Thom Cobordism Theorem. If the Pontrjagin numbers $p_{i_1}p_{i_2} \cdots p_{i_r}[V]$ *are all zero, then the m-fold disjoint sum* $V + V + \cdots + V$ *is the boundary of a compact oriented manifold, for some* $m > 0$.

* The term "index" is preferred by Hirzebruch and others. I have substituted "signature" to avoid confusion with the Morse index, as used in Section 1.

An important consequence is the following.

Hirzebruch Signature Theorem. The signature of any closed, oriented $4k$-manifold can be expressed as a linear combination of its Pontrjagin numbers, where the coefficients are rational numbers which depend only on the dimension. In particular:

$$\sigma(V^4) = \frac{p_1[V^4]}{3}$$

$$\sigma(V^8) = \frac{7p_2[V^8] - p_1{}^2[V^8]}{45}$$

$$\sigma(V^{12}) = \frac{62p_3[V^{12}]}{945} + \cdots$$

$$\sigma(V^{16}) = \frac{127p_4[V^{16}]}{4725} + \cdots$$

$$\sigma(V^{20}) = \frac{146p_5[V^{20}]}{13365} + \cdots$$

where the dots indicate terms in $p_1, \ldots, p_{k-1}$.

PROOF. Let V and V' be two manifolds with the same Pontrjagin numbers. Form $V - V'$ (the disjoint sum in which the orientation of V' has been reversed). Then all Pontrjagin numbers of $V - V'$ are zero. Hence some multiple

$$(V - V') + \cdots + (V - V')$$

is a boundary. This implies that the signature

$$\sigma((V - V') + \cdots + (V - V')) = m\sigma(V) - m\sigma(V')$$

is zero, and hence that $\sigma(V) = \sigma(V')$.

Thus $\sigma(V)$ is a function of the Pontrjagin numbers of V. Since both signature and Pontrjagin numbers are integers which behave additively when we form disjoint sums, it is clear that this function must be linear with rational coefficients.

The explicit formulas for $k = 1, 2, 3, 4, 5$ can be obtained by computing the signature and Pontrjagin numbers for suitable examples, and then solving the resulting linear equations. For further details the reader is referred to Hirzebruch [4].

For our purposes we will only need the following. A manifold will be called *almost parallelizable* if it becomes parallelizable when a single point is removed.

Corollary. The signature of a closed, almost parallelizable manifold of dimension 8 is always divisible by 7. Similarly, for a closed, almost parallelizable manifold of dimension 12, 16 or 20, the signature is divisible by 62, 127, or 146 respectively.

PROOF. We have $p_i(V^{4k} - x) = 0$, hence $p_i(V^{4k}) = 0$ for $i < k$. Thus the signature theorem reduces to

$$\sigma(V^8) = \frac{7p_2[V^8]}{45} \equiv 0 \pmod{7}$$

$$\sigma(V^{12}) = \frac{62p_3[V^{12}]}{945} \equiv 0 \pmod{62}$$

and so on.

In higher dimensions we obtain analogous results. (However, sharper results in this direction can be obtained by a different method: see [5].)

Lemma 4. For $k \geq 2$ there exists a compact parallelizable $4k$-dimensional manifold W with signature $+8$, such that the boundary of W is a homotopy sphere.

PROOF. We will construct W in such a way that the matrix of intersection numbers is as follows:

$$\begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix}$$

This remarkable matrix was suggested to the author by Hirzebruch. Note that it is positive definite, with determinant $+1$, and has only even entries on the diagonal.

As basic building block for the manifold W we take a tubular neighborhood T of the diagonal in $S^{2k} \times S^{2k}$. The homology group $H_{2k}(T; Z)$ is infinite cyclic, and the intersection number of a gener-

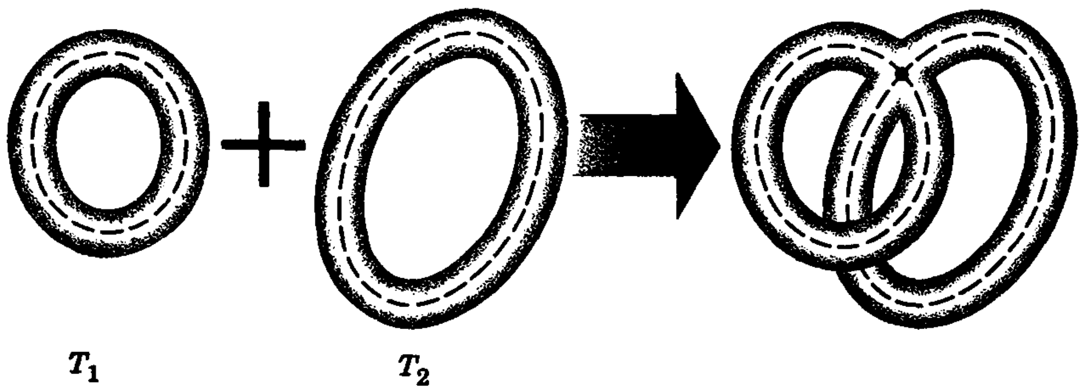


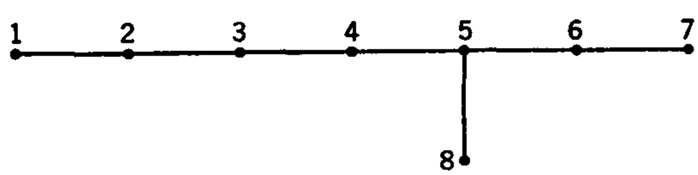
Figure 5

ator with itself is $+2$. [PROOF: if $\alpha, \beta \in H_{2k}(S^{2k} \times S^{2k})$ are the obvious generators, then $(\alpha + \beta) \cdot (\alpha + \beta) = 2\alpha \cdot \beta = 2$.]

Proof that T is parallelizable. Note that $S^{2k} \times S^{2k}$ can be imbedded in R^{4k+1} (as the boundary of a neighborhood of S^{2k}). Hence its tangent bundle is induced from that of the unit sphere by means of the Gauss map $g\colon S^{2k} \times S^{2k} \to S^{4k}$. Since $g|T$ is homotopic to a constant, it follows that T is parallelizable.

Next we introduce the operation of "plumbing together" two copies T_1 and T_2 of T. By this we mean the operation of matching together a region in T_1 and a region in T_2 in such a way that the central $2k$-sphere of T_1 will have intersection number $+1$ with the central $2k$-sphere of T_2. We must then "round off" the corners, so as to obtain a smooth manifold with boundary. For the 2-dimensional case this operation is illustrated in Figure 5.

In practice we need not two but eight copies of T. These are to be plumbed together according to the following schema:



That is, T_2 is to be plumbed together with T_1 and T_3; T_5 is to be plumbed together with T_4, T_6, and T_8; and so on. In this way we obtain a smooth manifold W which clearly has the following properties:

1. The manifold W has the homotopy type of a bouquet consisting of eight copies of S^{2k} which intersect at a single point.

2. If $\alpha_1, \ldots, \alpha_8 \in H_{2k}(W; Z)$ is the corresponding homology basis, then the matrix $(\alpha_i \cdot \alpha_j)$ of intersection numbers is as indicated previously, positive definite with determinant $+1$.

Thus it follows that the signature of W is $+8$.

3. The boundary ∂W is a homology sphere.

For the Poincaré duality theorem implies that

$$H_i(W, \partial W; Z) = \begin{cases} Z+Z+Z+Z+Z+Z+Z+Z & \text{for } i = 2k \\ Z & \text{for } i = 4k \\ 0 & \text{otherwise} \end{cases}$$

Furthermore the homomorphism

$$H_{2k}(W; Z) \to H_{2k}(W, \partial W; Z)$$

corresponds to our intersection matrix, and hence is an isomorphism. Now the homology exact sequence of the pair $(W, \partial W)$ shows that $H_*(\partial W; Z) \cong H_*(S^{4k-1}; Z)$.

4. ∂W is simply connected.

PROOF. Any circle in ∂W bounds a disk in W. By a general position argument, since $2k + 2 < 4k$, this disk can be pushed off of the central $2k$-spheres of the tubes T_i, and hence can be pushed into the boundary of W.

It now follows, by standard arguments in homotopy theory, that ∂W is a homotopy sphere. Since W is clearly parallelizable, this completes the proof of Lemma 4.

Theorem 3. The manifold ∂W is homeomorphic to the sphere S^{4k-1} but is not diffeomorphic to S^{4k-1}.

PROOF. It follows immediately from Smale or Stallings that the homotopy sphere ∂W is homeomorphic to S^{4k-1}. (Recall that the dimension $4k - 1$ is ≥ 7.)

We will only prove the second statement for the special cases $k = 2, 3, 4, 5$. Suppose that ∂W were diffeomorphic to S^{4k-1}. Then by pasting a $4k$-disk onto the boundary of W we would obtain a smooth closed manifold $V = W \cup D^{4k}$. Clearly V is almost parallelizable. Thus, according to the corollary given earlier, the signature of V must be a multiple of 7 or of 62 or 127 or 146 respectively. But $\sigma(V) = \sigma(W) = 8$. This contradiction completes the proof for $2 \leq k \leq 5$. For higher values of k the reader is referred to [6].

3. THE GROUP Γ_n

Since we have seen that exotic spheres exist, it is natural to try to classify them. For this purpose we introduce an abelian group Γ_n which can be described in two different ways.

First Description. Let $\mathrm{Diff}^+(D^n)$ be the group of all orientation preserving diffeomorphisms from a closed n-disk onto itself, and let $\mathrm{Diff}^+(S^{n-1})$ be the corresponding group of diffeomorphisms of its boundary. Consider the restriction homomorphism

$$r: \mathrm{Diff}^+(D^n) \to \mathrm{Diff}^+(S^{n-1})$$

We assert that the image of this homomorphism is a normal subgroup. Hence the quotient group

$$\Gamma_n = \frac{\mathrm{Diff}^+(S^{n-1})}{r\,\mathrm{Diff}^+(D^n)}$$

is defined.

Second Description. A closed oriented manifold M will be called a *twisted sphere* if it admits a Morse function with two critical points. Let Γ'_n denote the set of all oriented-diffeomorphism classes of twisted n-spheres. To make this into a group we introduce the *connected sum* operation.

Given connected oriented n-manifolds M_1, M_2, choose imbeddings

$$f_1: D^n \to M_1, \qquad f_2: D^n \to M_2$$

To make the orientations come out right *it is important that one of these two imbeddings should preserve orientation and the other one should reverse orientation.* The connected sum $M_1 \# M_2$ is now formed from

$$[M_1 - f_1(\mathrm{Int}\, D^n)] \cup [M_2 - f_2(\mathrm{Int}\, D^n)]$$

by pasting together the two boundaries under the diffeomorphism

$$f_2 \circ f_1^{-1}: f_1(S^{n-1}) \to f_2(S^{n-1})$$

Better still, in order to make the differentiable structure clear, extend f_1 and f_2 to imbeddings $F_i: R^n \to M_i$. Then $M_1 \# M_2$ can be formed from

$$[M_1 - F_1(0)] \cup [M_2 - F_2(0)]$$

by identifying each $F_1(x)$ with $F_2(x/\|x\|^2)$. (Note that this correspondence preserves orientation.) Applying Lemma 2 to F_i restricted to some large disk, we can prove that $M_1 \# M_2$ is well defined up to orientation preserving diffeomorphism. (Compare [8, 6].)

To compare the group Γ_n with Γ'_n we will construct a homomorphism

$$\text{Diff}^+(S^{n-1}) \to \Gamma'_n$$

Given $h \in \text{Diff}^+(S^{n-1})$ let $M(h)$ be the twisted sphere which is obtained from two copies of R^n by identifying each $x \in R^n - 0$ in the first with $y = h(x/\|x\|)/\|x\|$ in the second. Then

$$f = \frac{\|x\|^2}{1 + \|x\|^2} = \frac{1}{1 + \|y\|^2}$$

is a Morse function with only two critical points. Taking the orientation of $M(h)$ from the first copy of R^n we obtain a well-defined twisted sphere.

It is easily verified that this construction defines a homomorphism from $\text{Diff}^+(S^{n-1})$ onto Γ'_n. (Compare [7, p. 402].) Let us look at the kernel. Suppose that there is an orientation preserving diffeomorphism g from $M(h_1)$ to $M(h_2)$. According to Lemma 2 we may assume that g carries the point of $M(h_1)$ with coordinate x to the point of $M(h_2)$ with the same coordinate x for all $\|x\| \leq 1$. In terms of the y coordinates, this means that g carries the point with coordinate $y = \text{th}_1(u)$ to the point with coordinate $y = \text{th}_2(u)$ for all $t \geq 1$ and all $u \in S^{n-1}$. Thus we have a diffeomorphism $R^n \to R^n$ which takes S^{n-1} into itself by the diffeomorphism $h_2 \circ h_1^{-1}$. Hence $h_2 \circ h_1^{-1}$ belongs to the image

$$r\, \text{Diff}^+(D^n) \subset \text{Diff}^+(S^{n-1})$$

Conversely it can be shown that any element $h|S^{n-1}$ of the image gives rise to a manifold $M(h|S^{n-1})$ which is diffeomorphic to S^n. (The proof can be based on Munkres [13, §6.1].) *Henceforth we will drop the prime, and identify* Γ_n *with* Γ'_n.

The main properties of these groups can be described as follows.

Theorem 4. The group Γ_n is finite abelian for all n. For $n \leq 6$ we have $\Gamma_n = 0$, but for $n = 7$ the group $\Gamma_7 \cong Z_{28}$ is non-zero.

In fact it follows from Theorem 3 that $\Gamma_{4k-1} \neq 0$ for all $k \geq 2$. The fact that $\Gamma_n = 0$ for $n \leq 3$ is due to Munkres. For $n = 4$ this assertion has recently been announced by J. Cerf. In these cases the proof is based on the first definition, in terms of diffeomorphisms of S^{n-1}. For $n \geq 5$ the proof is based rather on the second definition, in terms of twisted spheres (see [6]). Some indication of the method, for $n \geq 5$, is given in the following section.

4. HOW TO RECOGNIZE AN HONEST SPHERE

Let M be a twisted n-sphere. How can we decide whether or not M is diffeomorphic to the standard n-sphere?

First choose an imbedding $M \subset R^{n+k}$ for some large value of k. This is possible by a well-known theorem of Whitney [23].

Lemma 5. The normal bundle of M is trivial. That is, there exist k continuous linearly independent normal vector fields.

OUTLINE OF PROOF. Since $M - x$ is contractible, it is certainly possible to choose such vector fields in the complement of x. Now the "obstruction" to extending over x is described by an element of the homotopy group $\pi_{n-1}\mathrm{SO}(k)$. These groups (for k large) have been computed by Bott [1].

Case 1. If $n \equiv 3, 5, 6$, or $7 \pmod 8$, then $\pi_{n-1}\mathrm{SO}(k) = 0$, hence there is no obstruction.

Case 2. If $n = 4i$ then the group $\pi_{n-1}\mathrm{SO}(k)$ is infinite cyclic. In this case the obstruction class can be identified with a multiple of the Pontrjagin number $p_i[M]$. (See [5].) But this number is zero by the signature theorem.

Case 3. If $n \equiv 1$ or $2 \pmod 8$, then $\pi_{n-1}\mathrm{SO}(k) \cong Z_2$. The proof in this case is more delicate: the obstruction class $\mathfrak{o}$ satisfies $J(\mathfrak{o}) = 0$ according to [5], but J. F. Adams has shown that the J homomorphism has kernel zero for these values of n. This completes the proof.

Next we must introduce the concept of *framed cobordism*. By a *framing* φ of an imbedded manifold $M^n \subset R^{n+k}$, k large, will be meant a set of k linearly independent normal vector fields. Two framed manifolds (M_0, φ_0) and (M_1, φ_1) in R^{n+k} are called *framed cobordant* if there exists a compact framed manifold (W, ψ) of dimension $n + 1$ in $R^{n+k} \times [0, 1]$ with $\partial W = M_0 \times 0 \cup M_1 \times 1$, where the framing ψ restricts to φ_0 at one end and to φ_1 at the other. Using the disjoint sum as composition operation, the framed cobordism classes of n-manifolds form a group, which will be denoted by Π_n.

There are two fundamental theorems concerning these groups.

Theorem of Pontrjagin [15]. *The framed cobordism group* Π_n *is canonically isomorphic to the stable homotopy group* $\pi_{n+k}(S^k)$, $n < k - 1$.

Theorem of Serre [18]. *The stable homotopy groups* $\pi_{n+k}(S^k)$, $0 < n < k - 1$, *are all finite abelian groups.*

Thus Π_n is finite.

Consider the class of all framed twisted n-spheres in R^{n+k}. Using a suitably defined connected sum operation, and defining an appropriate concept of "isomorphism," we see that *the set of all isomorphism classes of framed twisted* n-*spheres forms an abelian group, which will be denoted by* $\mathrm{F}\Gamma_n$. There is an exact sequence

$$\pi_n \mathrm{SO}(k) \to \mathrm{F}\Gamma_n \to \Gamma_n \to 0$$

PROOF. Every twisted sphere can be framed. The kernel of the homomorphism $\mathrm{F}\Gamma_n \to \Gamma_n$ is obtained by looking at standard spheres with exotic framings. But a framing of the standard $S^n \subset R^{n+k}$ is clearly described by an element of $\pi_n \mathrm{SO}(k)$.

On the other hand, there is clearly a homomorphism

$$j: \mathrm{F}\Gamma_n \to \Pi_n \cong \pi_{n+k}(S^k)$$

Kervaire has shown that this homomorphism j is onto, except possibly when $n \equiv 2 \pmod 4$. Thus *every framed cobordism class contains a twisted sphere,* except in dimensions 2, 6, 10, ...

Consider the kernel of this homomorphism j. Suppose, for example, that $n \equiv 3 \pmod 4$. Then we claim that kernel (j) is infinite cyclic. Let (M, φ) be a framed twisted n-sphere belonging to the kernel of j. Then $(M, \varphi) = \partial(W, \psi)$ where $W \subset R^{n+k} \times [0, 1)$ is a compact framed manifold of dimension $n + 1 = 4i$. Hence the signature $\sigma(W)$ is defined.

This integer $\sigma(W)$ is an invariant of the framed manifold (M, φ).

For if (M, φ) is also the boundary of (W', ψ'), then by placing W in $R^{n+k} \times [0, 1)$ and W' in $R^{n+k} \times (-1, 0]$, we can construct a closed framed manifold

$$W \cup W' \subset R^{n+k} \times R$$

Giving $W \cup W'$ the orientation which is compatible with that of W, we see that

$$\sigma(W \cup W') = \sigma(W) - \sigma(W')$$

But this signature must be zero: the fact that $W \cup W'$ is framed implies that its Pontrjagin classes are all zero and hence, by the signature theorem, that its signature is zero.

Thus $\sigma(W)$ is an invariant $\bar{\sigma}$ of (M, φ). In this way we define a homomorphism

$$\bar{\sigma}\colon \text{kernel } (j) \to Z$$

The construction of Section 2 shows that this homomorphism $\bar{\sigma}$ is nontrivial.*

Finally we claim that the kernel of $\bar{\sigma}$ is zero. That is, if (M, φ) bounds (W, ψ) with $\sigma(W) = 0$, then M is diffeomorphic to S^n, and φ corresponds to the standard framing of S^n. The proof, which is fairly formidable, is divided into two parts. First, using the method of "surgery" [9], we show that all of the homotopy groups of W can be killed. In other words W can be replaced by a contractible manifold W'. Second, a theorem of Smale [19] asserts that such a manifold W' must be diffeomorphic to the $(n + 1)$-disk. No further details are given here.

Thus kernel (j) is infinite cyclic. In other words there is an exact sequence

$$0 \to Z \to F\Gamma_{4i-1} \xrightarrow{j} \Pi_{4i-1} \to 0$$

It follows that $F\Gamma_{4i-1}$ is an infinite abelian group of rank 1. But $\pi_{4i-1}SO(k) \cong Z$ is also a group of rank 1. Using the sequence

$$\pi_{4i-1}SO(k) \to F\Gamma_{4i-1} \to \Gamma_{4i-1} \to 0$$

we finally see that the group Γ_{4i-1} is finite.

* It is curious that $\bar{\sigma}$ behaves somewhat differently in dimension 3 than in higher dimensions. Thus the image of $\bar{\sigma}$ is generated by 8 if $n = 4i - 1 \geq 7$, but is generated by 16 if $n = 3$.

Similar constructions work for other values of n. Thus

$$0 \to F\Gamma_n \to \pi_{n+k}S^k \to 0 \qquad \text{for } n \equiv 0\ (4)$$

$$Z_2 \to F\Gamma_n \to \pi_{n+k}S^k \to 0 \qquad \text{for } n \equiv 1\ (4)$$

$$0 \to F\Gamma_n \to \pi_{n+k}S^k \to Z_2 \qquad \text{for } n \equiv 2\ (4)$$

$$0 \to Z_2 \to F\Gamma_n \to \pi_{n+k}S^k \to 0 \qquad \text{for } n \equiv 3\ (4)$$

For further information on these groups, the reader is referred to [6].

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