Poincare-Hopf theorem, and groupoids

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Abstract

In this note, we present a proof of the classical Poincare-Hopf theorem by localization at critical points. We use the deformation to the normal cone construction of groupoids.

1 Some premilnaries on Groupoids, and index maps

The deformation to the normal cone construction (DNC), was used in noncommutative geometry by A. Connes ([2]), to give a simple proof of Atiyah-Singer index theorem. Many authors have generalised this construction. The construction in the greatest generality was systematicly studied in [3].

We will introduce some vocabulary to facilitate the exposition. A manifold pair (M, V) will mean a smooth manifold M and an injectively immersed non empty manifold $V \subseteq M$. A smooth map $f : (M, V) \to (M', V')$ will mean a smooth map $f : M \to M'$ such that $f(V) \subseteq V'$ and $f|V : V \to V'$ is smooth.

Let (M, V) be a manifold pair. One can construct a manifold $\mathrm{DNC}(M, V)$ called the deformation to the normal cone. As a set it is equal to $M \times \mathbb{R}^* \coprod \mathcal{N}_V^M \times \{0\}$.

Proposition 1.1. *The set* $\mathrm{DNC}(M, V)$ *has naturally the structure of a smooth manifold without boundary.*

Proof. We outline the proof for the sake of completeness. Let $\exp : U \subseteq \mathcal{N}_V^M \to \phi(U)$ be a diffeomorphism given by the tubular neighbourhood theorem. Let $\tilde{U} = \{((x, v, t) \in \mathcal{N}_V^M \times \mathbb{R} : vt \in U)\}$. We define a map as follows

$$\bar{\exp} : \tilde{U} \to \mathrm{DNC}(M, V)$$

$$\bar{\exp}(x, v, 0) = (x, v, 0)$$

$$\bar{\exp}(x, v, t) = (\exp(tv), t), \quad if\ t \neq 0$$

We cover $\mathrm{DNC}(M, V)$ by two sets; the first being $M \times \mathbb{R}^*$ and the second being $\bar{\exp}(\tilde{U})$. We declare the two sets as open sets each with its topology, the first coming from the product topology and the second from the map $\bar{\exp}$. We can verify directly that this topology is well defined and that both the topology and the smooth structure don't depend on the exponential map. □

The smooth structure is generated by three types of smooth functions;

1. The map

$$\begin{aligned} \pi : \mathrm{DNC}(M, V) &\to M \\ (x, t) &\to x \quad \forall x \in M, t \in \mathbb{R}^* \\ (x, X) &\to x \quad \forall x \in V, X \in \mathcal{N}_{V,x}^M \end{aligned}$$

   is smooth.

2. Given a smooth function $f \in C^\infty(M)$ such that $f|V = 0$, then the following function is smooth

$$\begin{aligned} \mathrm{DNC}(M, V) &\to \mathbb{C} \\ (x, t) &\to \frac{f(x)}{t} \quad \forall x \in M, t \in \mathbb{R}^* \\ (x, X) &\to df_x(X) \quad \forall x \in V, X \in \mathcal{N}_{V,x}^M \end{aligned}$$

3. Given a smooth function $f \in C^\infty(\mathbb{R})$, then the following function is smooth

$$\begin{aligned} \mathrm{DNC}(M, V) &\to \mathbb{C} \\ (x, t) &\to f(t) \quad \forall x \in M, t \in \mathbb{R}^* \\ (x, X) &\to f(0) \quad \forall x \in V, X \in \mathcal{N}_{V,x}^M \end{aligned}$$

Proposition 1.2 (Functoriality of DNC)**.** *Given a smooth map $f : (M, V) \to (M', V')$, then the map defined by*

$$\begin{aligned} \mathrm{DNC}(M, V) &\to \mathrm{DNC}(M', V') \\ (x, t) &\to (f(x), t) \quad \forall x \in M, t \in \mathbb{R}^* \\ (x, X) &\to df_x(X) \quad \forall x \in V, X \in \mathcal{N}_{V,x}^M \end{aligned}$$

is a smooth map denoted by $\mathrm{DNC}(f) : \mathrm{DNC}(M, V) \to \mathrm{DNC}(M', V')$.

Proposition 1.3. *Given two smooth maps $f : (M, V) \to (S, T)$ and $g : (M', V') to (S, T)$ that are transversal. By this we mean that the each one of the two pairs of maps $f : M \to S$, $g : M' \to S$ and $f : V \to T$, $g : V' \to T$ is transversal, then there is a canonical diffeomorphism*

$$\mathrm{DNC}(M \times_S M', V \times_T V') \to \mathrm{DNC}(M, V) \times_{\mathrm{DNC}(S,T)} \mathrm{DNC}(M', V')$$

Definition 1.4. A groupoid pair (G, H) is a manifold pair (G, H) such that both G and H are Lie groupoids and that the inclusion $i : H \to G$ is a functor of categories. Notice that this implies that (G^0, H^0) is a manifold pair.

Theorem 1.5. *If (G, H) is a groupoid pair then the manifold $\mathrm{DNC}(G, H) = G \times \mathbb{R}^* \coprod \mathcal{N}_H^G$ is canonically a Lie groupoid over $\mathrm{DNC}(G^0, H^0)$ whose algebroid is equal to $\mathrm{DNC}(\mathfrak{A}G, \mathfrak{A}H)$*

Proof. This is a direct consequence of proposition 1.1 proposition 1.2 and proposition 1.3. The only non trivial part is the injectivety of the natural map $\mathrm{DNC}(i) : \mathrm{DNC}(G^0, H^0) \to \mathrm{DNC}(G, H)$. This is equivalent to the statement $T_xH \cap T_xG^0 = T_xH^0$ for every $x \in H^0$. The last statement is straightforward to verify. □

Remark 1.6. We could have replaced the interval $\mathbb{R}^*$ by $]0, 1]$ or $]0, \infty[$. We will use the three depending on the application we have. If we want to be precise then we will use $\mathrm{DNC}_{\mathbb{R}}(M, V)$, $\mathrm{DNC}_{[0,1]}(M, V)$, $\mathrm{DNC}_{[0,\infty[}(M, V)$ to denote each one respectively.

Using the exact sequence

$$0 \to C^* H \otimes C(]0,1]) \to C^* \operatorname{DNC}(G,H) \xrightarrow{ev_0} C^* \mathcal{N}_H^G \to 0 \qquad (1)$$

we easily deduce the following proposition

Proposition 1.7. *The map* ev_0 *defines is an isomorphism in* K *theory. In particular* $K_*(C^* \operatorname{DNC}(G,H)) = K_*(C^* \mathcal{N}_H^G)$.

Definition 1.8. We call the map

$$\operatorname{ev}_1 \circ \operatorname{ev}_0^{-1} : K_*(C^* \mathcal{N}_H^G) \to K_*(C^* G)$$

the index map of the DNC construction. We will use the notation $\operatorname{Ind}(G,H)$, usually assuming that the inclusion map is clear from the context.

Example 1.9 (Index map for groupoids). Given a groupoid G. The index map

$$\operatorname{Ind}(G, G^0) : K_*(C_0(\mathfrak{A}^* G)) \to K_*(C^* G)$$

is the map sending a symbol to its index. We will denote this element by $\operatorname{Ind}(G)$.

Definition 1.10. Let $\mathfrak{A}G$ be the algebroid of a Lie groupoid G. Let $\sigma(v,x) = c(v) \in \operatorname{End}(\bigwedge \mathfrak{A}^* G \otimes \mathbb{C})$, the Clifford multiplication map. The map σ defines an element in $K_0(C_0(\mathfrak{A}^* G))$. We define the Euler characteristic of a Lie groupoid to be the index of σ which we denote by $\chi(G) \in K_0(C^* G)$.

Proposition 1.11. *Let* G *be a Lie groupoid,* $F \subset G^0$ *a closed saturated subset. The following diagram commutes*

$$\begin{array}{ccc} K_*(C_0(\mathfrak{A}^* G)) & \xrightarrow{\operatorname{res}_F} & K_*(C_0(\mathfrak{A}^*(G_{|F}))) \\ \downarrow{\scriptstyle \operatorname{Ind}(G)} & & \downarrow{\scriptstyle \operatorname{Ind}(G|F)} \\ K_*(C^* G) & \xrightarrow{\operatorname{res}_F} & K_*(C^*(G_{|F})) \end{array}$$

2 Poincare-Hopf index theorem

While the Poincare-Hopf index theorem, is usually stated with vector fields. It is more natural to state it for 1-forms.

Definition 2.1. Let G be a Lie groupoid. By dual vector field on G, we mean a section of $\mathfrak{A}^* G$. We denote by $\operatorname{Crit}(X)$ the set of zeros of X.

Proposition 2.2. *For a generic* X*, the set* $\operatorname{Crit}(X)$ *is an embedded submanifold of* G^0 *of codimension equal to rank of* $\mathfrak{A}G$*. Its normal bundle isomorphic (by the map* dX*) to* $\mathfrak{A}^* G | \operatorname{Crit}(X)$.

Proof. This is a direct application of Thom transversality theorem. □

From now on we will always assume that X is generic, and $Z = \operatorname{Crit}(X)$ is its zeros.

We will regard the groupoid $\operatorname{DNC}_{[0,1]}(G, Z)$. The domain of the index map associated to this Dnc construction is equal to

$$K_0(C^*(\mathfrak{A}G_{|Z} \oplus \mathcal{N}_Z^{G^0})) = K^0(\mathfrak{A}^* G_{|Z} \oplus (\mathcal{N}_Z^{G^0})^*).$$

By proposition 2.2, the vector bundle $\mathfrak{A}^* G_{|Z} \oplus (\mathcal{N}_Z^{G^0})^*$ is isomorphic to $\mathfrak{A}^* G_{|Z} \oplus \mathfrak{A}G_{|Z}$. This vector bundle has a natural symplectic form given by $\omega(\alpha, v) = \alpha(v)$. Hence it is canonicaly K-oriented[1]. By composing the index map with the Thom isomorphism, we obtain a map

$$\operatorname{Ind}(X) : K^0(Z) \to K_0(C^* G).$$

Theorem 2.3. *One has*

$$\mathrm{Ind}(X)([1]) = \chi(G).$$

Proof. One has

$$\mathfrak{A}\,\mathrm{DNC}(G, Z) = \mathrm{DNC}(\mathfrak{A}G, Z).$$

This is a vector bundle over $\mathrm{DNC}(G^0, Z)$ which is naturally isomorphic to $\pi^*\mathfrak{A}G$, where $\pi : \mathrm{DNC}(G^0, Z) \to G^0$ is the smooth map of type 1, on the DNC construction. Similarly, one has

$$\mathfrak{A}^*\,\mathrm{DNC}(G, Z) = \mathrm{DNC}(\mathfrak{A}^*G, Z) = \pi^*\mathfrak{A}^*G.$$

$$h : \mathrm{DNC}(\mathfrak{A}^*G, Z) \to \pi^*\mathfrak{A}^*G \otimes \mathbb{C}$$
$$h(x, \xi, t) = \frac{\xi}{t} + i\frac{X(x)(1-t)}{t}$$
$$h(x, \xi, v) = \xi + idX(x)(v)$$

It is easy to check that $h^{-1}(0) = Z \times [0, 1]$, hence compact. It follows that $c(h)$ the clifford action given by h, defines an element $[c(h)] \in K^0(\mathfrak{A}\,\mathrm{DNC}_{[0,1]}(G, Z))$. Let $\mathrm{Ind}_{\mathrm{DNC}(G,Z)}([c(h)]) \in K_0(C^*\,\mathrm{DNC}(G, Z))$ be the index of $[c(h)]$. By proposition 1.11, one deduces that

$$\mathrm{ev}_0\,\mathrm{Ind}_{\mathrm{DNC}(G,Z)}([c(h)]) = \mathrm{Ind}_{\mathrm{DNC}(G,Z)}(\mathrm{ev}_0([c(h)]))$$

One sees easily that $\mathrm{ev}_1([c(h)])$ is the symbol of the De Rham operator. Hence $\mathrm{Ind}_{\mathrm{DNC}(G,Z)}(\mathrm{ev}_1([c(h)])) = \chi(G)$. Similarly one see that $\mathrm{ev}_1([ch(h)])$ is the Thom element for the Orientation choosen by d_X, and the symplectic form. Hence by the definition of the map $\mathrm{Ind}(X)$, one has

$$\mathrm{Ind}(X)([1]) = \chi(G).$$

□

Now let us regard the classical case $G = M \times M$ and X is a vector field with isolated singularities nondegenerate zeros, $Z = \{x : X(x) = 0\}$.

Lemme 2.4. $\mathrm{Ind}(X) : \oplus_{x\in Z}\mathbb{Z} \to K_0(\mathcal{K}(L^M)) = \mathbb{Z}$ *is the homomorphism sending* (a_x) *to* $\sum_x \mathrm{Ind}_x(X)a_x$.

Proof. The sign comes from the fact that the orientation on $TM \oplus T^*M$ coming from dX, agrees or disagrees with the Bott orientation depending on the determinant of dX_x. □

Corollary 2.5. *(Poincare-Hopf index theorem) Given a vector field X on a manifold M with isolated non-degenerate singularities we have*

$$\sum_x \mathrm{Ind}_x(X) = \chi(M)$$

References

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